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Image Analysis and Computer Vision
WS 2014/15
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☉_☉ Find any errors? Please send them back, I want to keep them!}

- Image Analysis: "low to mid level", "early" vision
  - image to image (image processing/enhacement)
  - image labeling (segmentation, depht, motion, features)
- Computer Vision: "high level", scene interpretation
  - image to model (3D, texture, lighting)
  - recognition (objects, activities, . . .)

1 Image Types and Discretization

- mapping f from a rectangular domain $\Omega = (0, w) \times (0, h)$ to a co-domain $\mathbb{R}$

$$f : \mathbb{R}^2 \supset \Omega \to \mathbb{R}$$

  - Ω is called *image domain* or *image plane*
- image data only given on rectangular point grid within image domain Ω
- grid point (i, j) is called *pixel* (picture element)
- **Aliasing**: Signals become indistinguishable when sampled
  - Sampling rate must be above *Nyquist rate* (two times the bandwidth of the signal) => aliasing-free result

1.1 Image Noise

1.1.1 Additive Noise

- most important noise
- Gray values and Noise are *independent*

$$f_{i,j} = g_{i,j} + n_{i,j}$$

  where g is the original image, n is the noise and f is the noisy image
- noise n may have different distributions
  - Uniform Distribution (very simple)

$$p(x) = \begin{cases} b - a & \text{for } x \in [a, b] \\ 0 & else \end{cases}$$

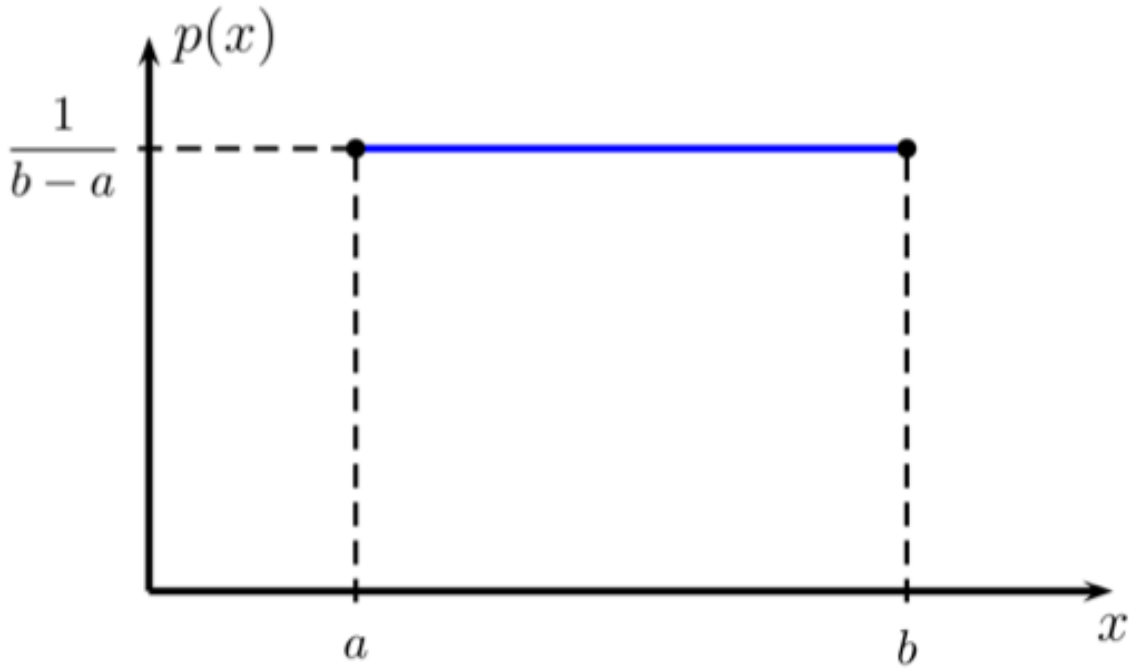


Abbildung 1: Uniformly Distributed Noise

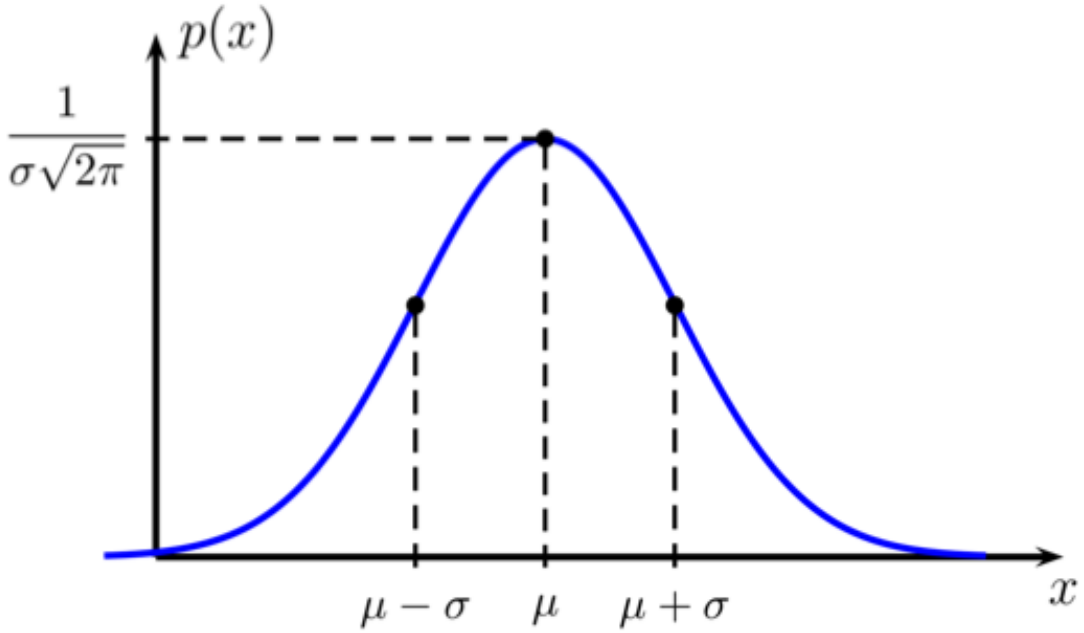


Abbildung 2: Gaussian Noise Distribution

- Gaussian Distribution (very common)
  - most important noise model
  - good approximation for many practical situations
  - density function:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{\frac{(x-\mu)^2}{2\sigma^2}}$$

    where μ is the *mean* and σ is the standard deviation (σ^2 is called the *variance*)
  - $\mu - \sigma \Rightarrow 68\%$
  - $\mu - 2\sigma \Rightarrow 95.5\%$
  - $\mu - 3\sigma \Rightarrow 99.7\%$

1.1.2 Multiplicative Noise

- signal dependent
- often proportional to the gray value:

$$f_{i,j} = g_{i,j} + n_{i,j} \cdot g_{i.j} = (1 + n_{i,j}) \cdot g_{i,j}$$

- often present in radar, ultrasound and tomographic images

1.1.3 Impulse Noise

- degrade image at some pixels only (erroneous grey values are created)
- *unipolar impulse noise* => only one grey value
- *salt-and-pepper noise* => bipolar noise, highest and lowest value inserted

1.2 Correlation and Convolution

- goal is to reduce noise
- take multiple images with same camera (and setup) and average them
- **Image Filtering**: modify pixels based on some function of local neighborhood

Abbildung 3: Image neighborhood

- replace pixel by average of neighborhood

1.2.1 Correlation

- weighted combination of pixels in small neighborhood, e.g. weighted sum:

$$g_{i,j} = \Sigma_{k,l \in \mathbb{Z}} f(i+k, j+l) h(k,l)$$

  where $h(k,l)$ is the **mask** or **weighted kernel** (*filter coefficients*)

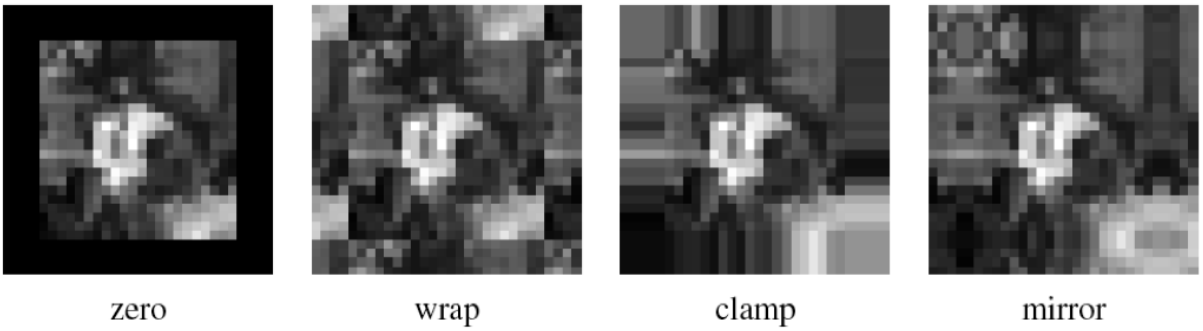


Abbildung 4: Boundary Conditions

- **correlation operator**: $g = f \otimes h$
- *boundary conditions*: What happens on image border.
  - *zero*: everything outside of images is zero
  - *wrap*: move out of image? move back in on the other side!
  - *clamp*: last pixel gets replicated
  - *mirror*: move out of image? move back on the same side, reverse direction
- **Gaussian Correlation**: Removes high-frequency components from the image (low-pass filter)

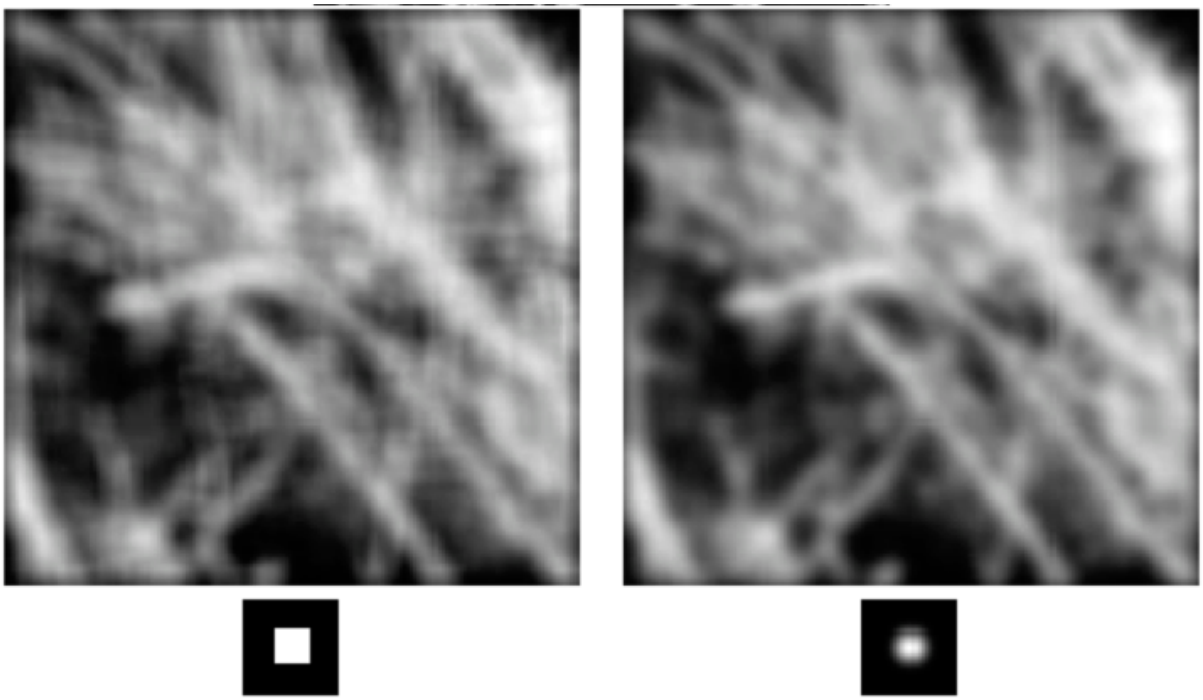


Abbildung 5: Filtering: mean (left), gaussian (right)

  - approximates a 2D Gaussian function
  - Gaussian Function has infinite support, discrete filters use finite kernels
  - *Standard Deviation* determines extent of smoothing

1.2.2 Convolution

- Correlation filters will be flipped by mathematical operation

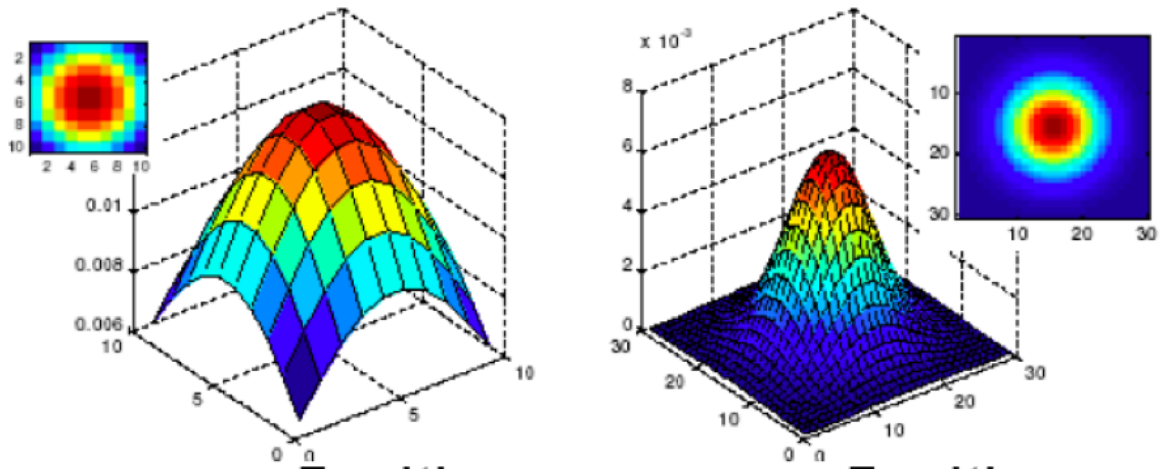


Abbildung 6: Gaussian Kernel $\sigma = 5$: 10×10 kernel (left), 30×30 kernel (right)

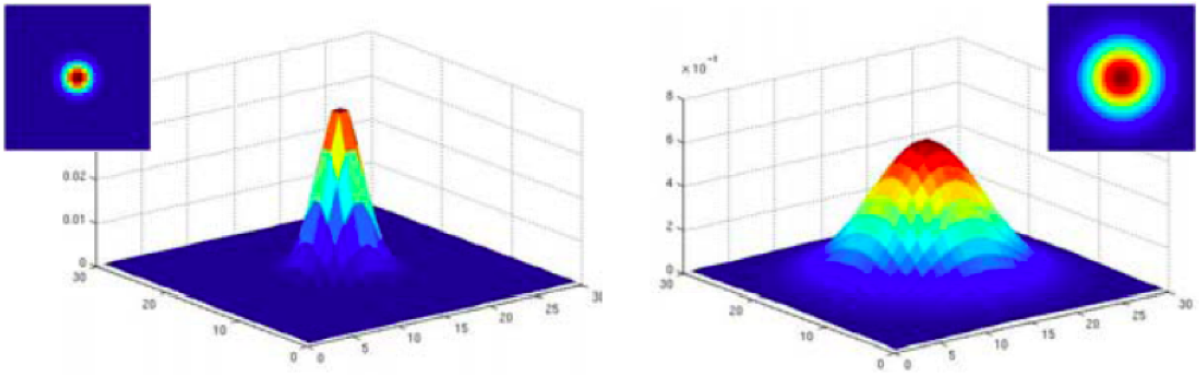


Abbildung 7: Gaussian Kernel: $\sigma = 2$ (left), $\sigma = 5$ (right), 30×30 kernel

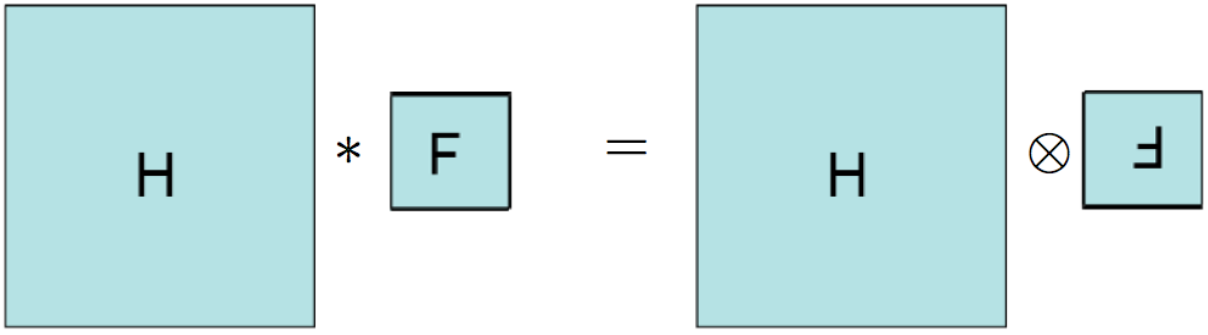


Abbildung 8: Correlation and Convolution

- Convolutions of two images: $g = f * h$ defined as:

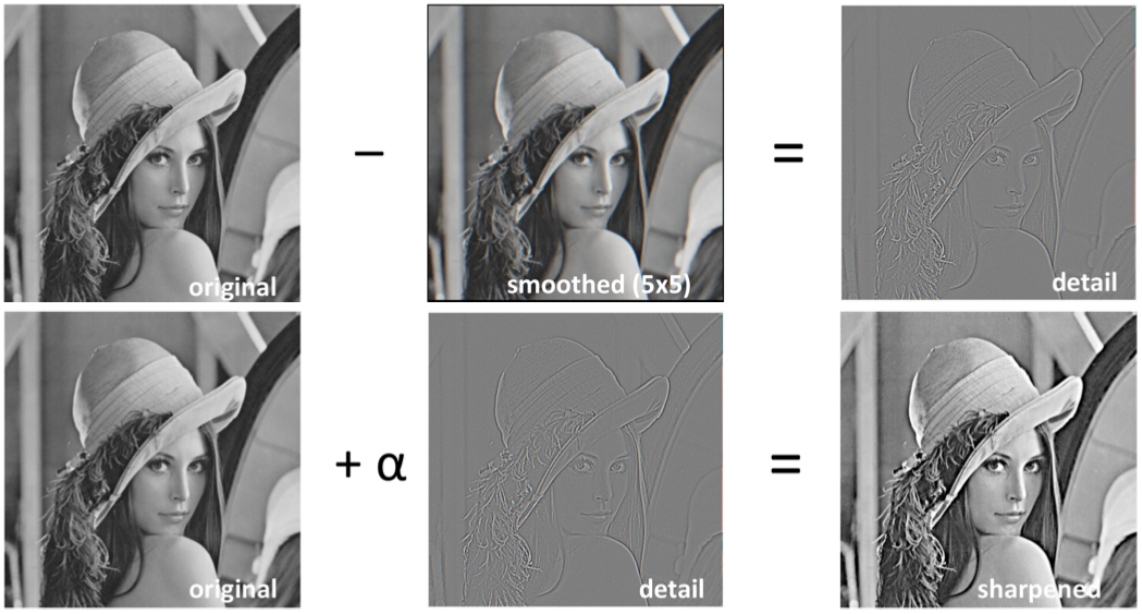
$$g(i,j) = \Sigma_{k,l \in \mathbb{Z}} f(i-k, j-l) h(k,l)$$

- for symmetrical filters (gaussian, box) result is the same
- Visualization of Correlation filters much more intuitive.
- mathematical properties:
  - **commutative**: $f * h = h * f$
  - **associative**: $(f * h) * g = f * (h * g)$
  - **identity**: $\delta * h = h * \delta = h$
  - **linear** (superposition):

$$f * (h_1 + h_2) = f * h_1 + f * h_2$$
$$(f + g) * h = f * h + g * h$$
$$(\alpha f) * h = \alpha (f * h)$$

  - **shift-invariant**: $(shifted\ f) * h = (f * h) shifted$ (shifted image => convolution with shifted impulse)

- **Sharpening**



- Different phenomena in Optics are actually convolutions, e.g. motion blur

1.2.3 Non-Linear filters + denoising

- Measures for denoising quality:
- Mean Square error (MSE):

$$MSE(f,g) = \frac{1}{N} \Sigma_{n=1}^{N} (f_n - g_n)^2$$

- Peak signal-to-noise ratio (PSNR)

$$PSNR(f,g) = 10 \log_{10} \left(\frac{V^2}{MSE(f,g)} \right)$$

- **bilateral filter**:
  - idea: preserve edges, average only over pixels which are nearby *and* have similar color

  - two weights: one for distance, one for intensity, both are gaussian filters
  - parameters $\sigma, \tau > 0$, greyscale image f, *bilateral filter*:

$$g(n) = \frac{1}{K_n} \Sigma_{m=1}^{N} f_m N_\sigma(\|x_n - x_m\|) \cdot N_\tau(|f_n - f_m|)$$
$$K_n = \Sigma_{m=1}^{N} N_\sigma(\|x_n - x_m\|) \cdot N_\tau(|f_n - f_m|)$$

- **non-local means**: images have similat patches, **idea**: when averaging, put most weight on similar patches
  - around each pixel n, define a $k \times k$ window of pixels $\mathcal{W}_\backslash$, which is written as vector of pixel indices
  - define distance
$$d^2_{n,m} := \Sigma_{s=1}^{k^2} \left(f(\mathcal{W}_n(s)) - f(\mathcal{W}_m(s))\right)^2$$
  - kernel weights: $k_{n,m} := e^{-\frac{d^2_{n,m}}{2\sigma^2}}$, where $\sigma > 0$ is a parameter.
  - sums are usually restricted to window around nth pixel
  - very slow filter
- **median filter**: very good for removing outliers => *salt-and-pepper noise*
  - for each pixel n, compute vector f consisting of k^2 grayscale values: $f = (f_{n,1}, \ldots, f_{n,k^2})$ in local $k \times k$ window around n.
  - filtered result is the *median value*
  - usually $k = 3$, better to apply it multiple times (twice)

2 Features and Patterns

2.1 Image Edges and Image Derivates

- Identify sudden changes (discontinuities) in an image
- naive: changes in intensity
  - often does not work: texture, shadows, back- and foreground of similar color, ...
  - => focus on gradient-based edge detection ok for now
- **approximation of the gradient**
  - *interpolate image as function, differentiate, resample*:
$$\delta_x f(x,y) = \lim_{\epsilon \to 0} \frac{f(x+\epsilon, y) - f(x,y)}{\epsilon}$$
  - *difference approximation*: h_x is pixel size in x-direction (usually normalized to 1)
    - *forward differences*: $(\partial_x f)_{i,j} \approx \frac{f(i+1,j)-f(i,j)}{h_x}$
    - *backward differences*: $(\partial_x f)_{i,j} \approx \frac{f(i,j)-f(i-1,j)}{h_x}$
    - *central differences*: $(\partial_x f)_{i,j} \approx \frac{f(i+1,j)-f(i-1,j)}{2h_x}$
    - can be writte as *convolutions*:
      - forward difference kernel:

1	-1	0

      - backward difference kernel:

0	1	-1

      - forward difference kernel:

$\frac{1}{2}$	0	$-\frac{1}{2}$

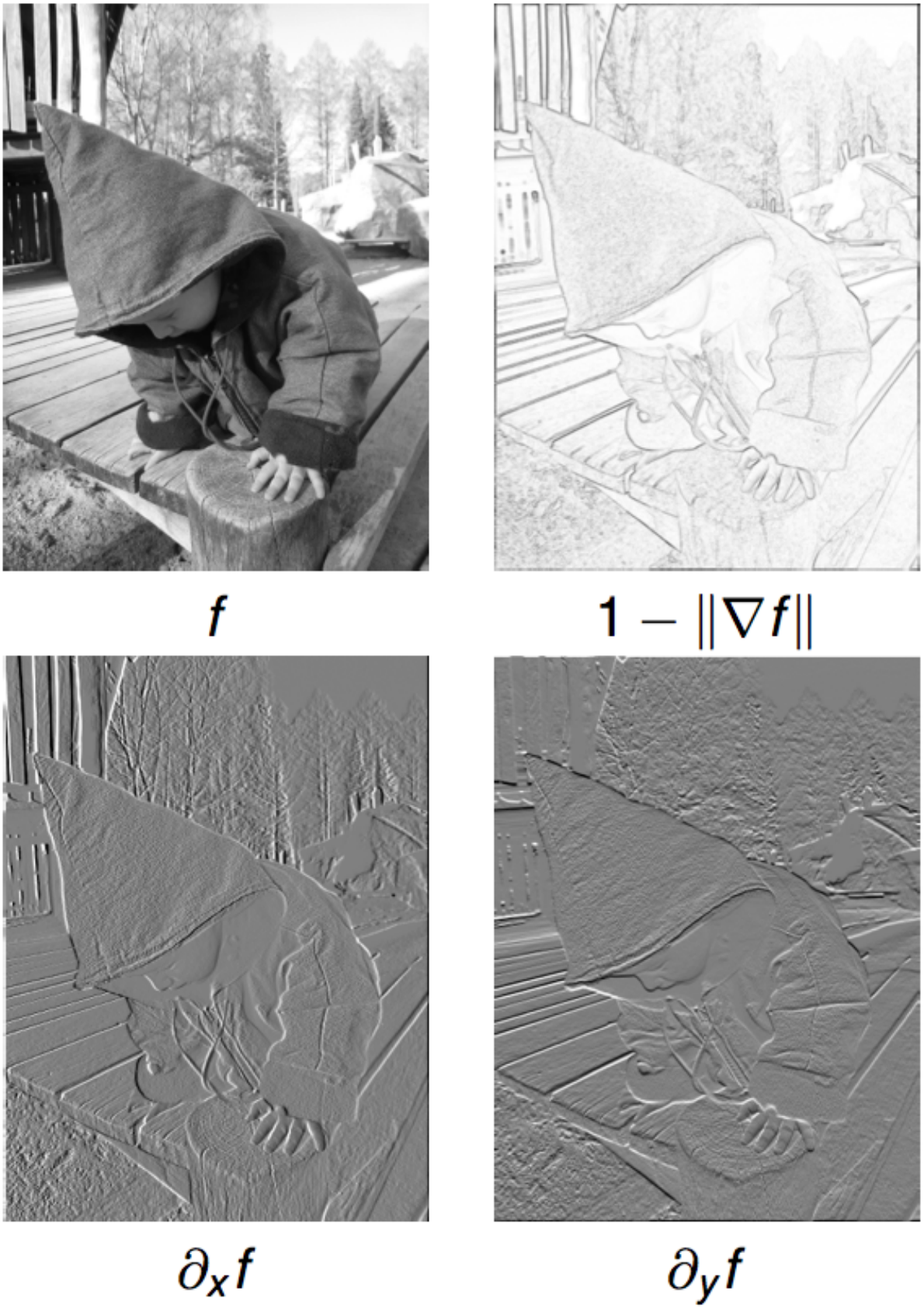


Abbildung 9: Central Difference Edge Detection

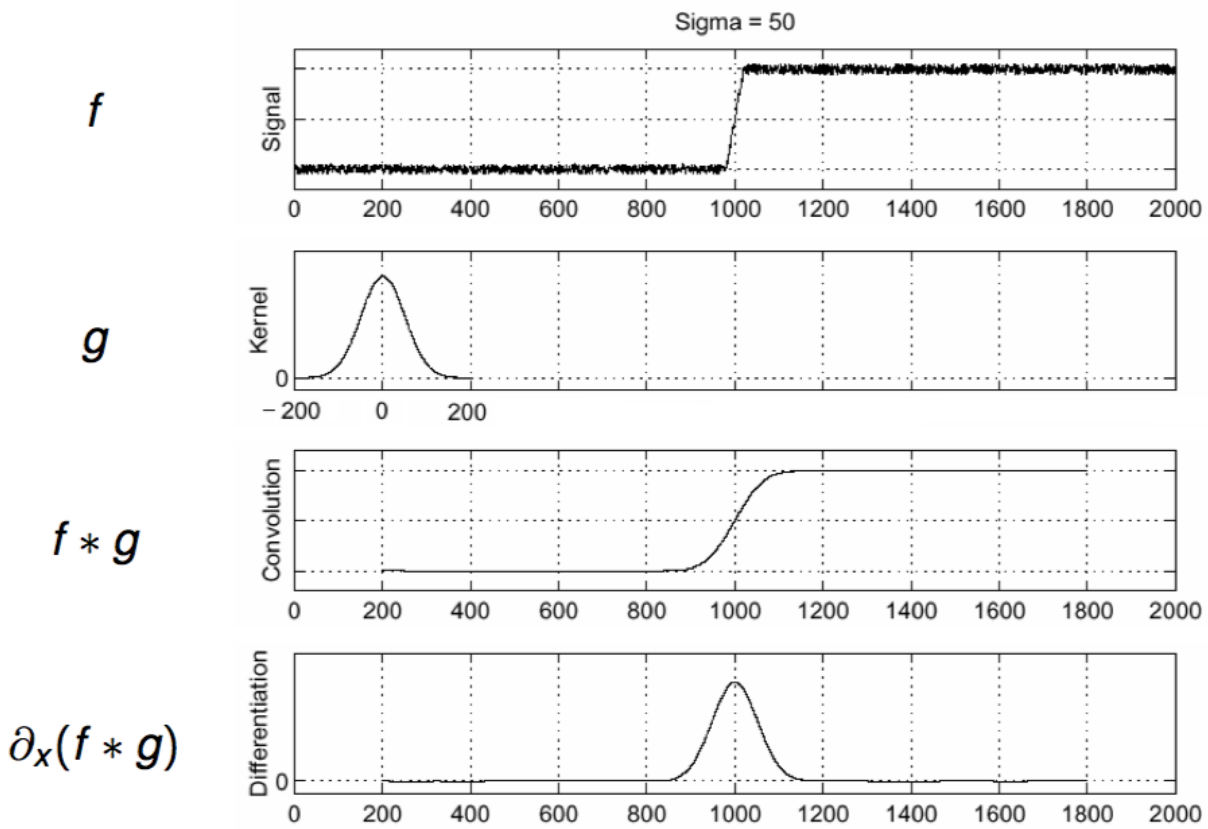


Abbildung 10: Smoothing, then building the derivative

- for greyscaleimage, gradient ∇f is a vector field consisting of partial derivatives: $\nabla f = \begin{bmatrix} \partial_x f \\ \partial_y f \end{bmatrix}$
- the magnitude of the gradient is the length of the vector in every point: $\|\nabla f\| = \sqrt{(\partial_x f)^2 + (\partial_y f)^2}$
- Noise makes this nearly impossible => smooth first, apply gradient later
- Under some technical conditions on f and g, partial derivatives wrt ith variable satisfy

$$\partial_i (f * g) = (\partial_i f) * g = f * (\partial_i g)$$

  * more *efficient*: can be pre-computed, no second convolution necessary
  * more *accurate*: if filter function is known, derivates can be computed analytically
  * normalize to zero after building filter => constant signal

2.2 Canny Edge Detection

- maxima of gradient
- not important for exam

2.3 Feature Detection

- *Detection*: indentify interesting points
- *Description*: Extract Feature descriptor for each point
- *Matching*: Determine correspondences between views
- Goals for detectors:

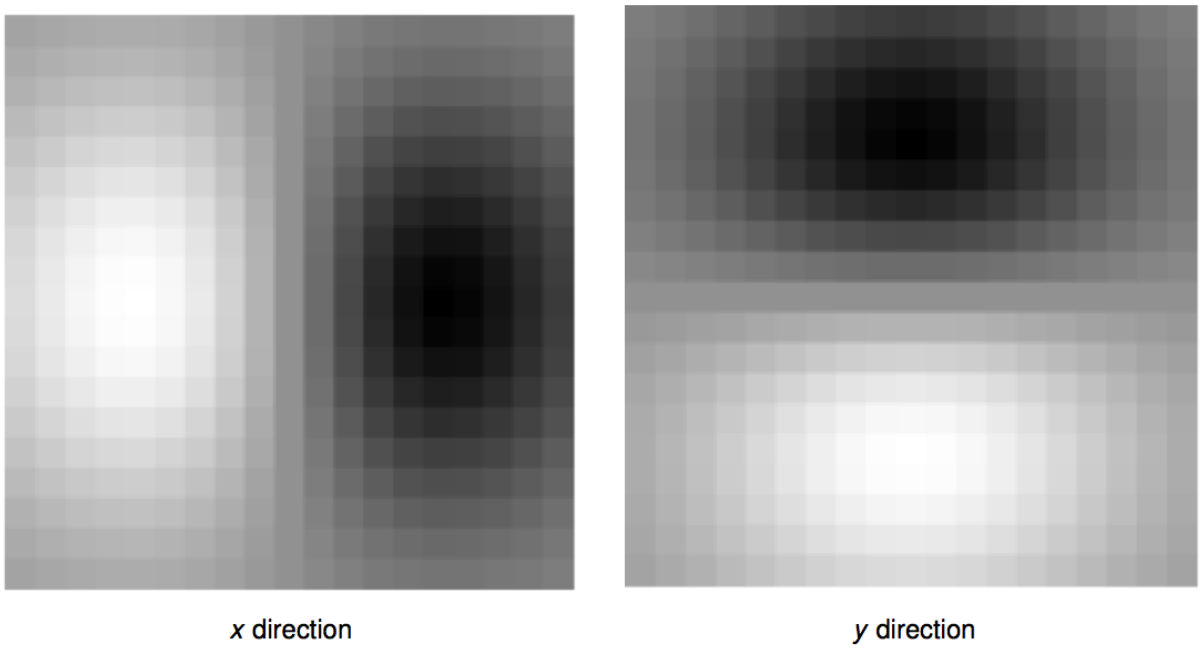


Abbildung 11: Derivatives Of Gaussian

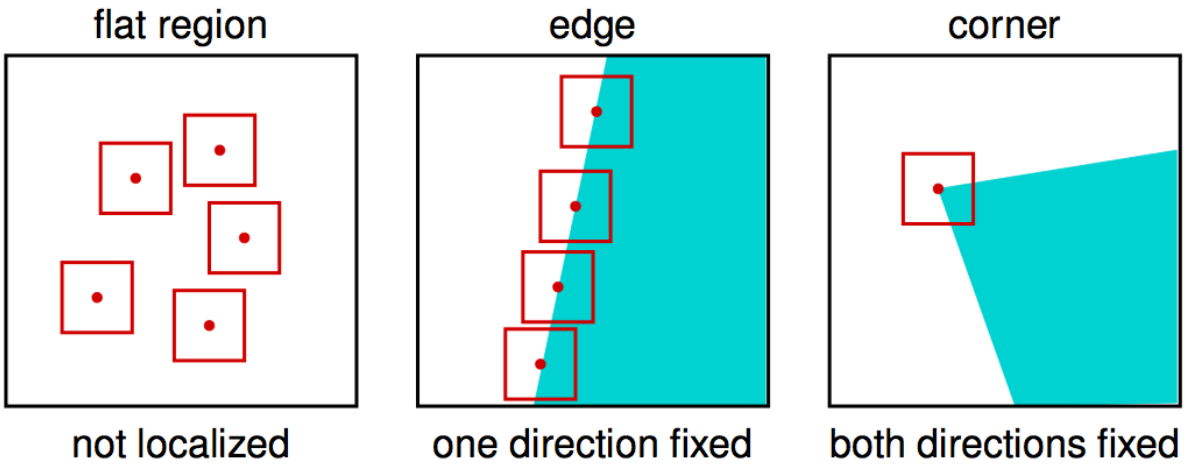


Abbildung 12: Localization

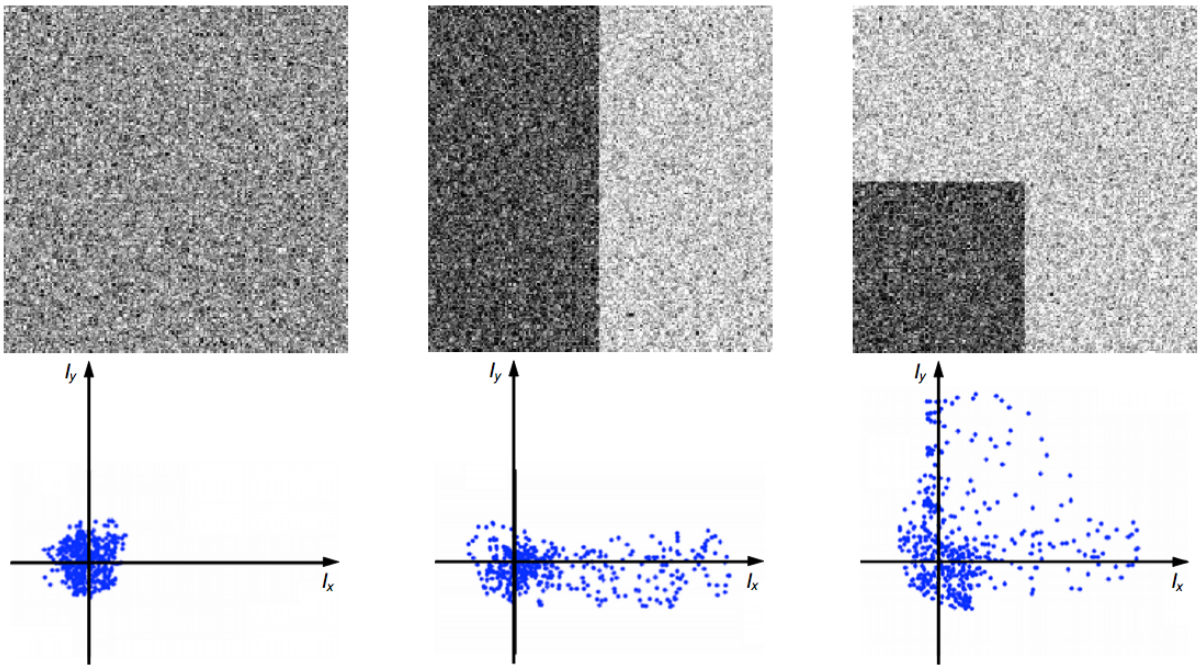


Abbildung 13: Mathematical Localization

  - *locality*: robust to clutter and occlusion
  - *quantity*: quantity: hundreds or thousands in single image
  - *distinctiveness*: differentiate large database of objects
  - *efficiency*: real-time performance?
  - *geometric invariance*: translation, rotation, scale, ...
  - *photometric invariance*: brightness, exposure, ...

- Look for **unsusual** regions => unambiguous matches

2.3.1 Singular Value Decomposition (SVD)

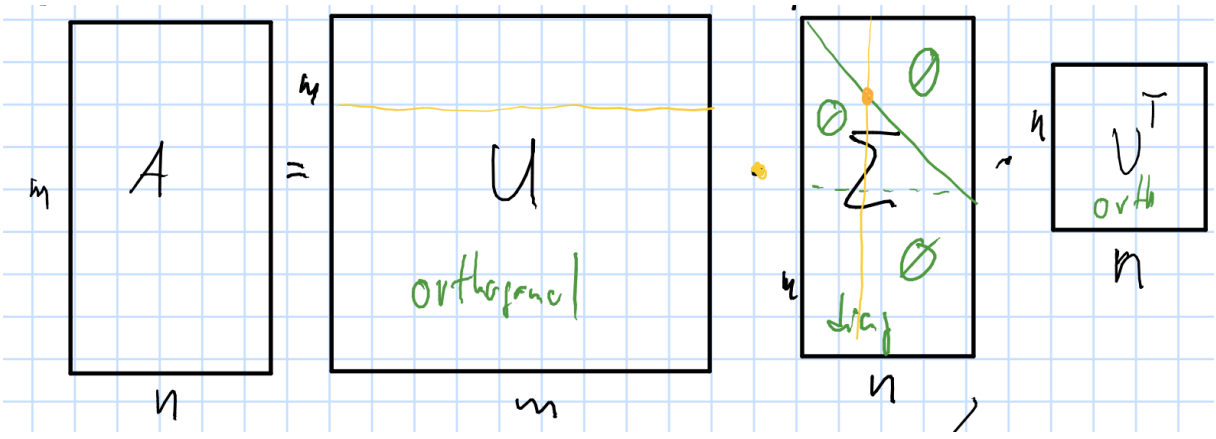


Abbildung 14: Singular Value Decomposition

- If $A \in \mathbb{R}^{m \times n}$ is a $m \times n$ matrix => there exists a factorization of the form $A = U\Sigma V^T$ where

  - $U \in \mathbb{R}^{m \times m}$ is an orthogonal matrix: $UU^T = U^T U = I_m$
  - $V \in \mathbb{R}^{n \times n}$ is also orthogonal

  - $\Sigma \in \mathbb{R}^{m \times n}$ is a diagonal matrix, whose diagonal consists of sortet, non-negative real numbers $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$ => *singular values* of A

- $A = U\Sigma V^T \Leftrightarrow AV = \Sigma U$
  - A maps the columns of V to the columns of U scaled by the singular values

- **properties of SVD**:
  - $rank(A)$ => number of non-zero singular values
  - columns of V are eigenvectors of $A^T A$
  - columns of U are eigenvectors of AA^T

- **PCA**: Characterize joint probability distribution of several variables to find directions with greates variance
  - *sample expectation value*: $E(Z) = \Sigma_{i=1}^{m} p_i z_1$ ($\approx$ weighted average)
  - *sample variance*: $Var(Z) = \Sigma_{i=1}^{m} p_i (Z_i - E(Z))^2$
  - *sample standard deviation*: $\sigma(Z) = \sqrt{Var(Z)}$
  - *covariance*: $cov(X, Y) := \Sigma_{i=m}^{m} p_i (x_i - E(X))(y_i - E(Y))$
    - "centered measurements": $E(X_j) = 0$ => $cov(X_j, Y_k) := \Sigma_{i=m}^{m} p_i \cdot x_{ji} \cdot y_{ki}$
    - intuition: generelly, whenever two values increase together, the covariance is posivite
  - *covariance matrix*: $C(X) := (cov(X_j,X_k))_{j,k=1,\ldots,n}$
    - properties:
      - *symmetric*
      - $cov(X_j, X_j) = var(X_j)$ => diagonal consists of variances
      - *semi-definite*

- $M = U\Sigma V^T$ be SVD of measurement matrix => transform all measurements by V => $M' = MV = U\Sigma$ => covariance matrix: $C' = \Sigma^T U^T U \Sigma = Sigma^2$

$$\left\{ \begin{bmatrix} \sigma_1 \cos(\phi) \\ \sigma_2 \sin(\phi) \end{bmatrix} ; 0 \leq \phi \leq 2\pi \right\}.$$

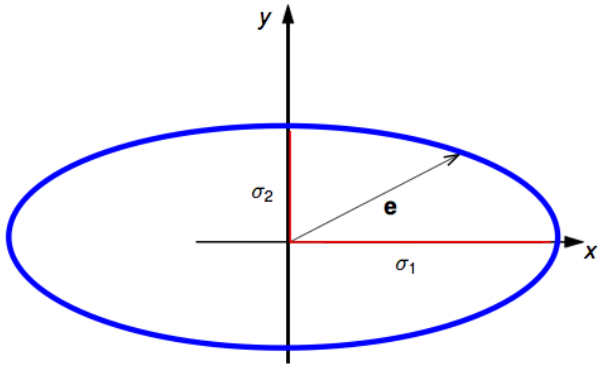


Abbildung 15: Standard Deviation Ellipse

  - The radius of the ellipse yields the standard deviation of the measurements projected onto the corresponding direction.

- PCA can be used to *decorrelate data*

- dimensions corresponding to larger singular values have higher variance => more important

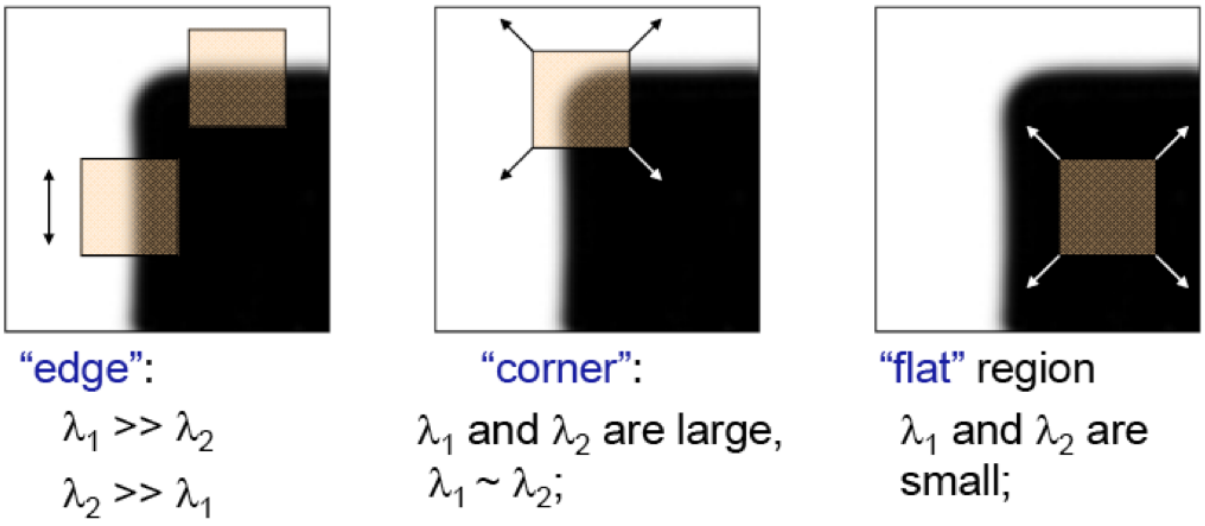


Abbildung 16: Pixel Characterization based on Eigenvalues

2.3.2 Structure Tensor

$$\tau := \begin{bmatrix} G_\tau * (f_x^\sigma)^2 & G_\tau * (f_x^\sigma f_y^\sigma) \\ G_\tau * (f_x^\sigma f_y^\sigma) & G_\tau * (f_y^\sigma)^2 \end{bmatrix} = G_\tau * \begin{bmatrix} (f_x^\sigma)^2 & f_x^\sigma f_y^\sigma \\ f_x^\sigma f_y^\sigma & (f_y^\sigma)^2 \end{bmatrix}$$

- *inner scale parameter* $\sigma > 0$: determines image scale at which derivates are taken
- *outer scale parameter* $\tau > 0$: determines size of the windows on which derivate information is sampled from

=> Harris Corner Detector

2.3.3 Compare Features in different images

- **Sum of Squared Differences** (SSD): Given two rectangular image patches f, g on $\{1, \dots, W\} \times \{1, \dots\}$, the *sum of squared differences distance*:

$$E_{SDD}(f, g) = \Sigma_{i=1}^{W} \Sigma_{j=1}^{H} \left(f(i,j) - g(i,j)\right)^2$$

  - Smaller value => better fit
  - normalization: divide by $W \cdot H$

- use error threshold θ to compare patches (for every possible matching patch)
- *properties*
  - fast
  - invariant to *translation*
  - *not* invariant to rotation
  - *not* invariant to scaling
  - *not* invariant to contrast/brightness changes
- **Normalized Cross-Correlation** (NCC) Given two rectangular image patches f, g on $\{1, \dots, W\} \times \{1, \dots\}$, the *normalized cross-correlation distance*:

$$E_{NCC}(f, g) = \frac{cov(f, g)}{\sigma(f) \cdot \sigma(g)}$$

  - range is within $[-1, 1]$, larger value is better, value below 0 is opposite of fit.
- *properties*

  - *not* fast
  - invariant to *translation*
  - *not* invariant to rotation
  - *not* invariant to scaling
  - invariant to contrast/brightness changes

- **Autocorrelation** for patch f and shift u (sum runs over all pixels)

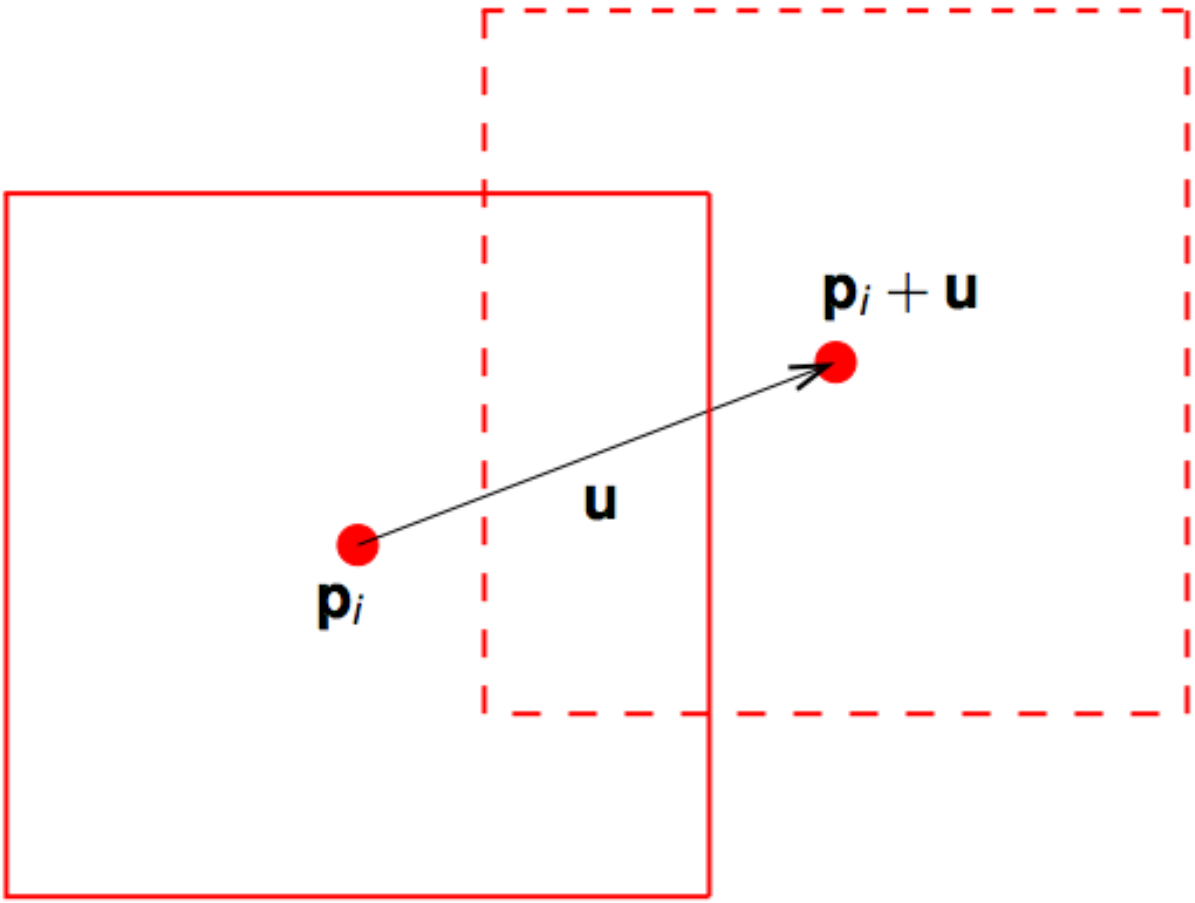
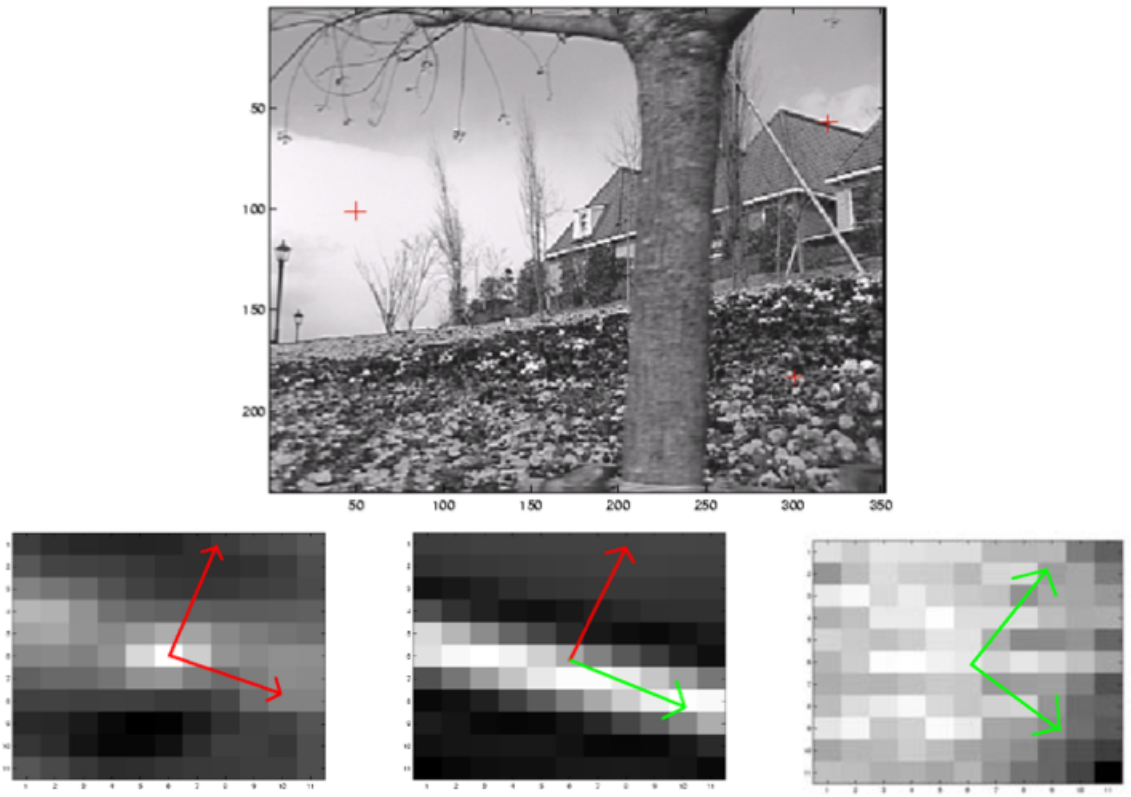


Abbildung 17: Autocorrelation: shifting

$$E_{AC}(f,u) := \Sigma_i w(p_i)(f(p_i+u) - f(p+i))^2$$



- algorithm:
  1. taylor series approximation: $f(p_i + u) \approx f(p_i) + \nabla f(p_i)^T \cdot u$
  2. compute E_{AC} using 1. $E_{AC}(f, u) \approx u^T \left(\Sigma_i w(p_i) \nabla f(p_i) \nabla f(p_i)^T\right) \cdot u$
     * its actually the structure tensor:
     * approximation to *autocorrelation* for small shifts $u \mapsto u^T \tau u$

3 Frequency

- Fouriers idea:

  Any univariate function can be rewritten as a weighted sum of sines and cocsines of different frequencies.

- Elementary building blocks: $f(x, y) = r \cdot \cos(2\pi(\omega_x x + \omega_y y) + \varphi)$
  - *wave number*: $\omega = [\omega_x \omega_y]$
  - *frequency*: $f = |\omega|_2$ => number of peaks per unit length
  - *direction of the wave*: $\frac{\omega}{f}$
  - *phase*: $\varphi \in \mathbb{R}$ => gives distance of the first peak to the origin
  - *amplitude*: $r \geq 0$ => gives maximum peak height

- write waves with complex numbers => formulas become easier to understand
  - $e^{it} = \cos(t) + i \cdot \sin(t)$
  - lies on the unit cirlce
  - angle between e^{it} and real axis is t
  - complex number => $z = r \cdot e^{i\varphi}$
    * *norm* $r = |z|$

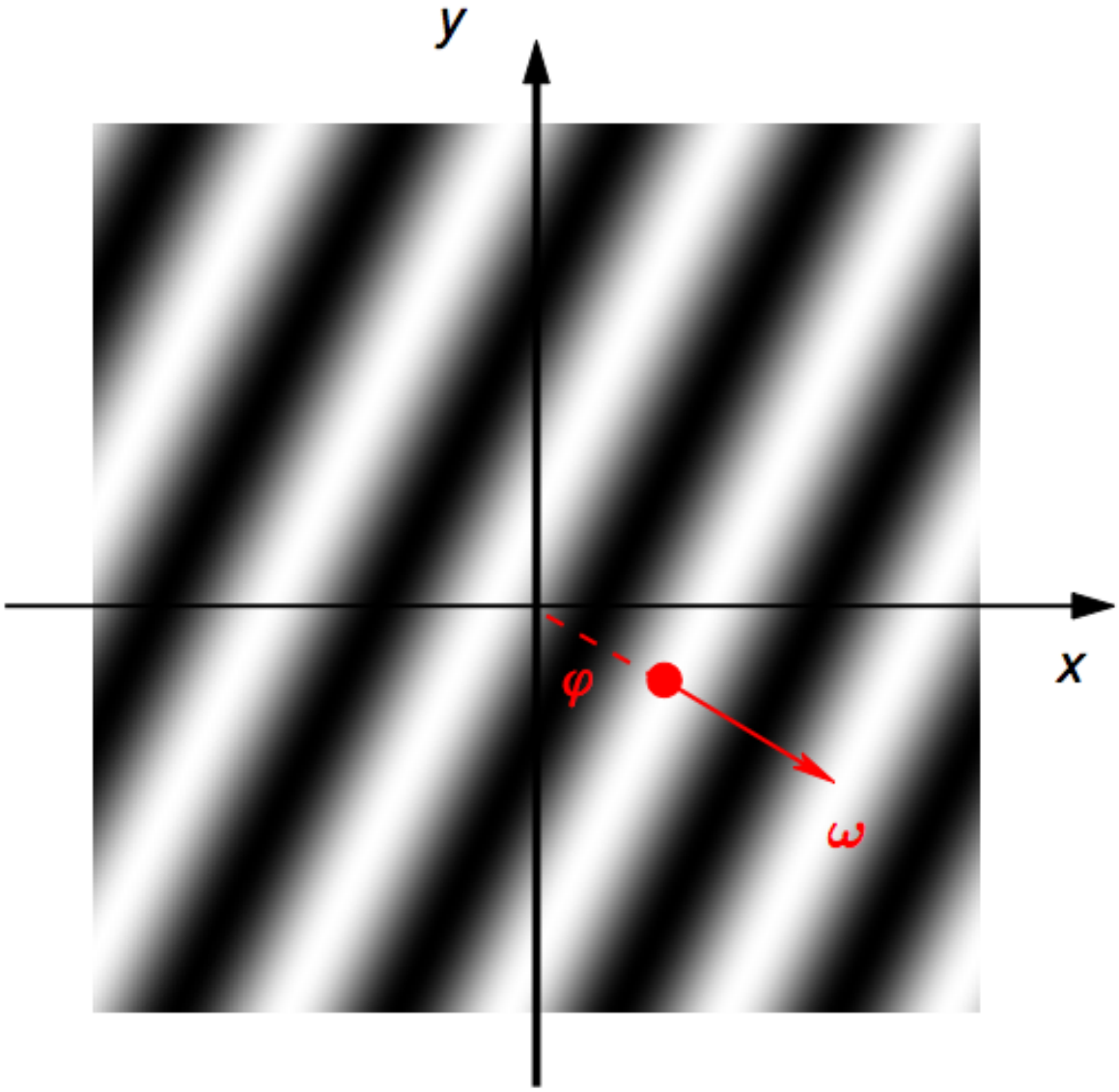


Abbildung 18: Waves in 2D

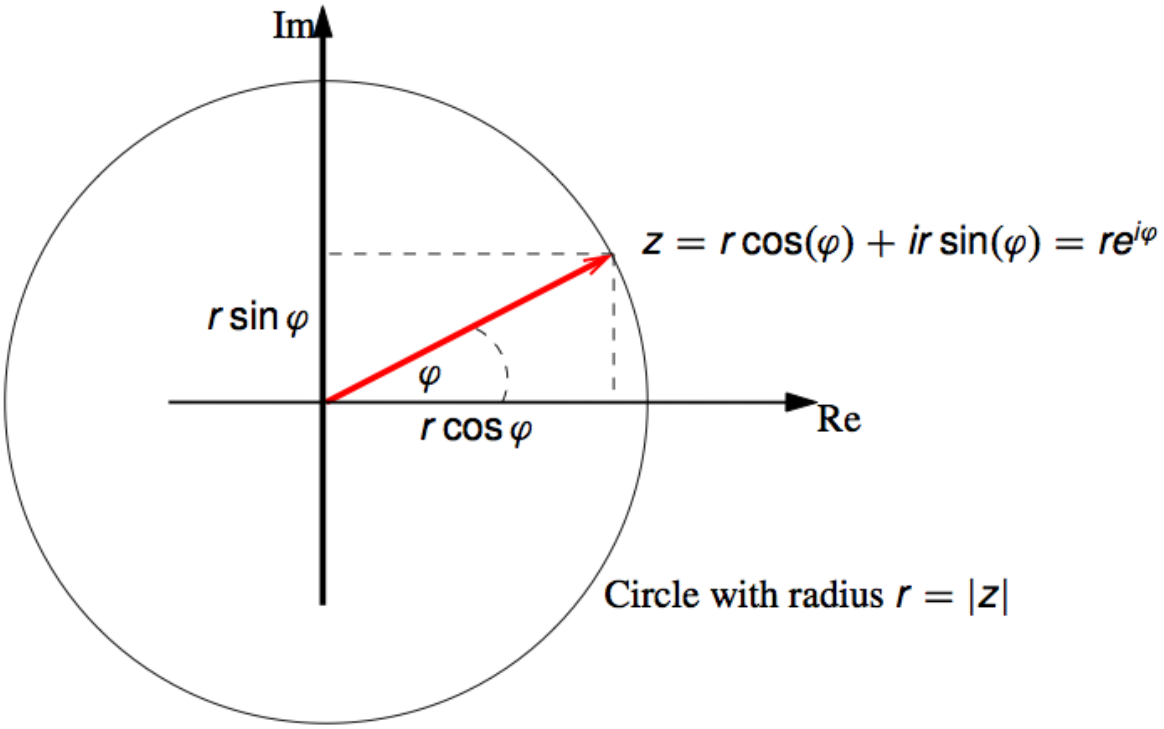


Abbildung 19: Polar representation of complex numbers

  - *argument* φ

- complex elementary wave:

$$\begin{aligned} W_\omega(p) &= e^{2\pi i(\omega \cdot p)} \\ &= \cos(2\pi(\omega \cdot p)) + i\sin(2\pi(\omega \cdot p)) \end{aligned}$$

- multiplication with comples numbers:

$$\begin{aligned} W_\omega(p) &= r \cdot e^{i\varphi} \cdot e^{2\pi i(\omega \cdot p)} \\ &= e^{i(2\pi(\omega \cdot p)+\varphi)} \end{aligned}$$

  amplitude r and *phase* φ

- Write function $f : \mathbb{R}^2 \to \mathbb{C}$ as *infinite linear combination* of elementary waves with complex coefficients $\hat{f}(\omega)$:

$$f(p) = \int_{\mathbb{R}^2} \hat{f}(\omega) W_\omega(p) d\omega$$

- Complex valued function $\hat{f} : \mathbb{R}^2 \to \mathbb{C}$ is called **Fourier Transform**
  - For each point in *frequency space*, the value $\hat{f}(\omega) = re^{i\varphi}$ gives amplitude and phase of the elementary wave W_ω

- interpretation:
  - Noise amplifies high frequencies:

- Phase seems more important for the actual image

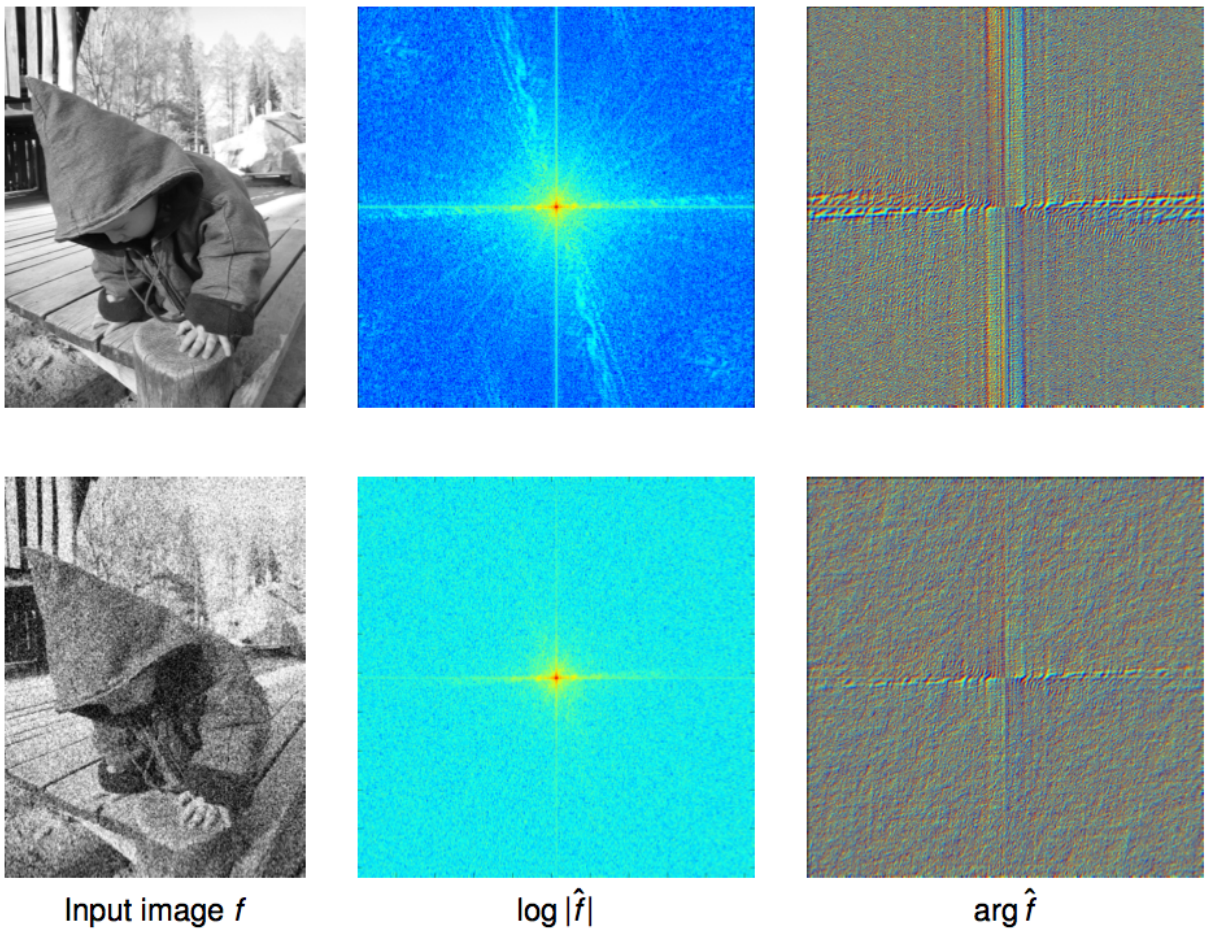


Abbildung 20: Noise and Fourier Transform

3.1 Filtering in Frequency Space

- convolution of elementary wave: $(g * W_\omega)(p) = W_\omega(p)\hat{g}(\omega)$
  - => convolution theorem: $\widehat{f * g} = \hat{f} \cdot \hat{g}$
  - much more efficient than direct convolution

3.1.1 Gaussian in Frequency Space

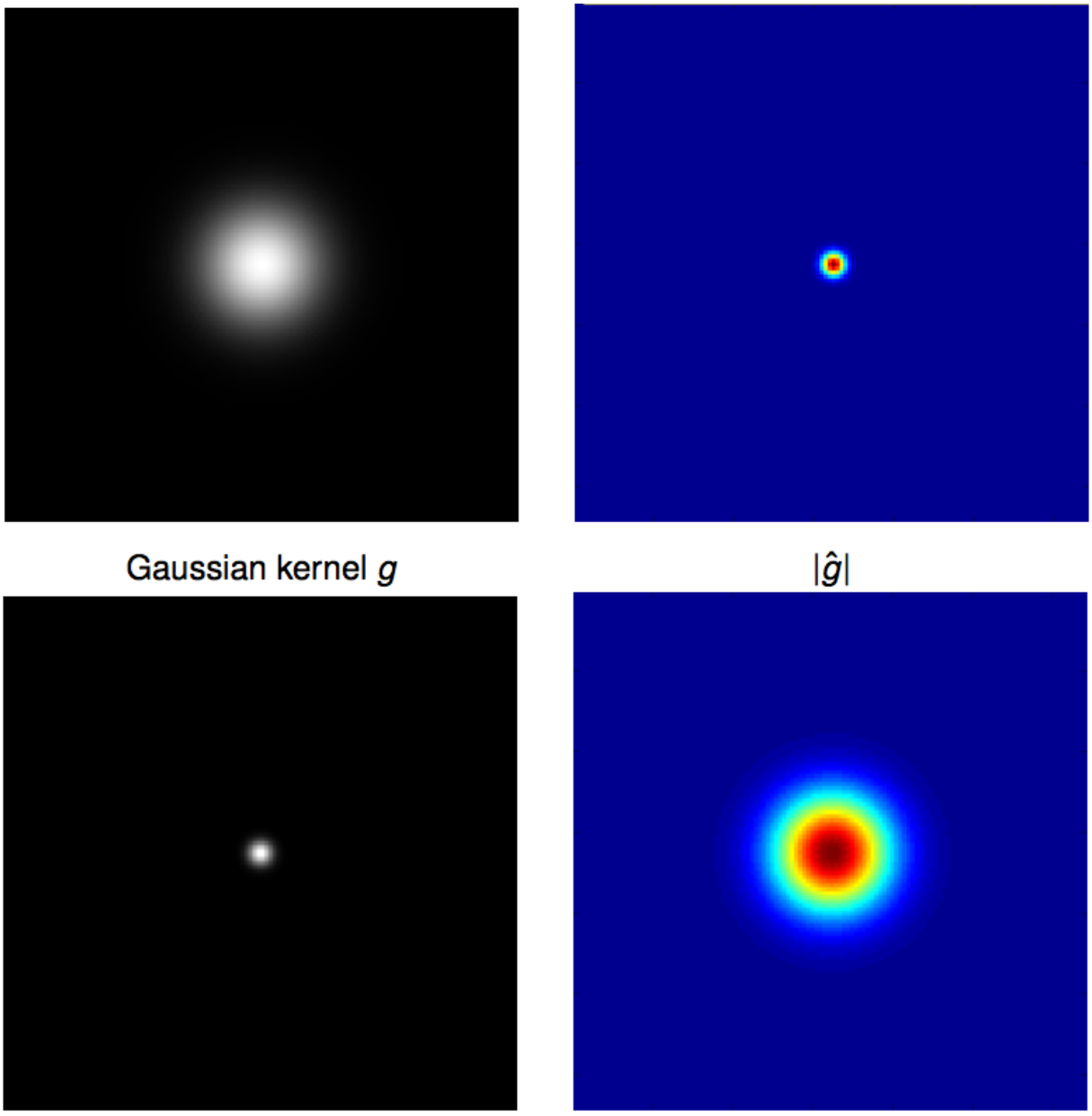


Abbildung 21: Gaussians in Fourier Space

- The Fourier Transform of a Gaussian is a Gaussian-like Function:

$$f(p) = \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{p^2}{2\sigma^2}} \Rightarrow \hat{f}(\omega) = e^{-\frac{4\pi^2\omega^2}{2\sigma^{-2}}}$$

3.1.2 Types of Filters:

- **low-pass filter** filter which leaves low frequencies intact (e.g. gaussian)

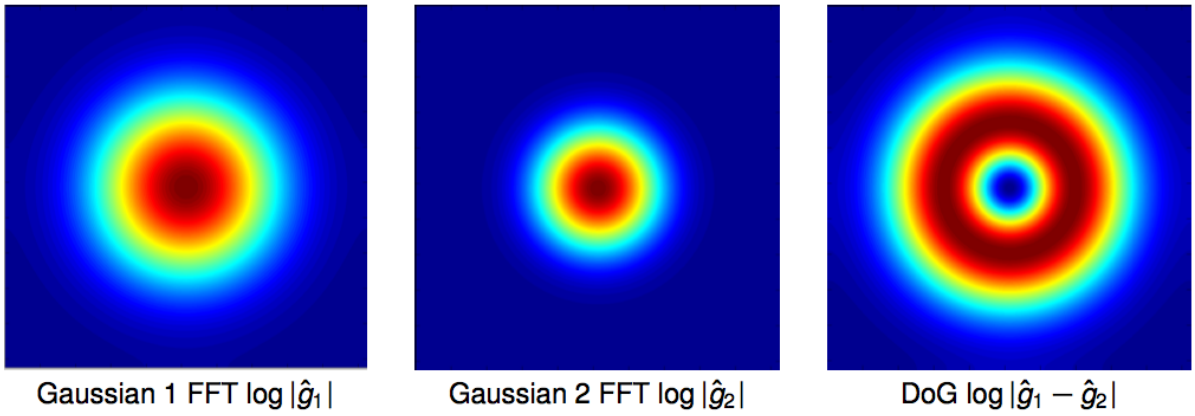


Abbildung 22: Difference of Gaussians

- **high-pass filter** filter which leaves high frequencies intact (e.g. impuls - Gaussian)
- **band-pass filter** filter which leaves a range of frequencies intact while suppressing high and low frequencies (Difference of Gaussians)

3.1.3 Properties of Fourier Transform

- *linearity*: $\widehat{\alpha f + \beta g} = \alpha \hat{f} + \beta \hat{g}$
- *rotation invariance*: when f is rotated, the Fourier transform $\hat{f}$ is rotated by the same angle.
- *shift theorem*: shifting p_0 leads to a phase change according to $\widehat{f(p - p_0)}(\omega) = e^{-2\pi i \omega \cdot p_0} \hat{f}(\omega)$
- *convolution theorem*: $\widehat{f * g} = \hat{f} \cdot \hat{g}$
- *derivates*: $\widehat{\partial_x^n \partial_y^m f}(\omega) = (2\pi i \omega_x)^n (2\pi i \omega_y)^m \hat{f}(\omega)$

3.2 Scale

- If the image has to many pixel compared to the patch => compute difference "scales" of the image and test for each image.

3.2.1 Sampling & Antialiasing

- not important for exam
- blur image with a corresponding gaussian before taking only every *nth* pixel => Gaussian Pyramid

4 Scale Invariant Feature Transform (SIFT)

- **Goals**:
  - detect characteristic feature points in images
  - create detailed descriptos that represents each feature as uniquely as possible
  - ensure invariance with respect to changes in location, scale and orientation
- **Applications**:
  - Finding sparse correspondences between images
  - usefull in computer vision: global relations between images, tracking, structure-from-motion
  - object detection and recognition
- **idea**

1. **detection of characteristic feature points**
   - based on Gaussian scale-space using difference of Gaussians
   - extrema provide location and scale
2. **accurate localization of key points**
   - performs sub-pixel refinement by fitting quadratic functions
   - additionally discards points with high ratio between principal curvatures
3. **assignment of the dominant orientation(s)**
   - based on a histogram of gradients within the local neighbourhood
   - refines orientation by fitting quadratic functions
4. **computation of suitable key point descriptor**
   - unit vector based on accumulated histograms of gradients
   - compensated by location, scale and dominant orientation

4.1 SIFT Feature Detection

- Starting Point: *Gaussian Scale-Space of input image* f
  - discrete levels of smoothing: $\sigma_0, \sigma_1, \ldots, \sigma_t, \ldots, \sigma_{max}$
  - scale-space given by convolution of f with increasing Gaussian G_{σ_t}:

$$f_t = G_{\sigma_t} * f$$

- Difference-of-Gaussian (DoG)

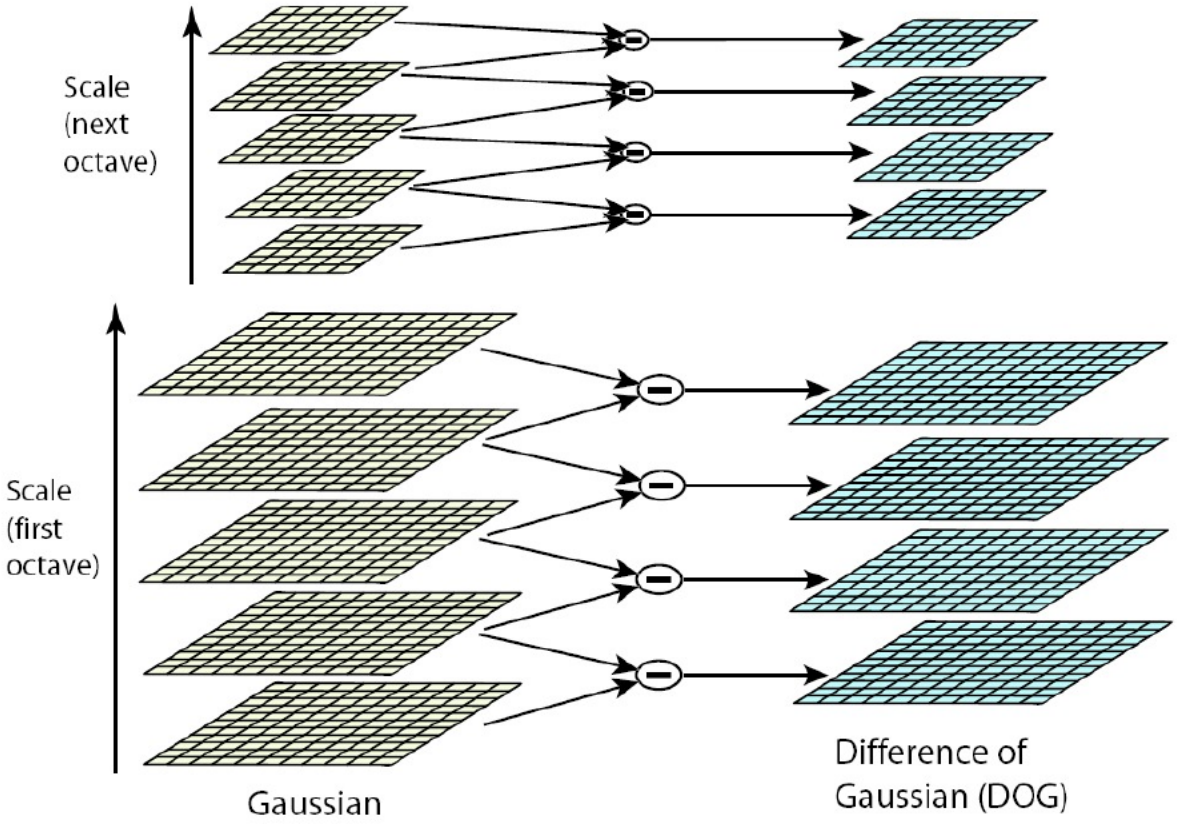


Abbildung 23: SIFT Scale Space Pyramid

  - Difference of two consecutive scales of the Gaussian scale-space:

$$\begin{aligned} D_t &:= f_{t+1} - f_t \\ &= G_{k\sigma_t} * f - G_{\sigma_t} * f \\ &= (G_{k\sigma_t} - G_{\sigma_t}) * f \end{aligned}$$

- DoG is related to scale derivative via approximation:

$$\partial_\sigma \approx \frac{G_{k\sigma} - G_\sigma}{k_\sigma - \sigma}$$

  - The analytical scale derivative is the scaled *Laplacian-of-Gaussian* (LoG):

$$\partial_\sigma G_\sigma = \sigma \delta G_\sigma = \sigma(\partial_x^2 G_\sigma + \partial_y^2 G_\sigma)$$

  - DoG is an *approximation* of the LoG:

$$G_{k\sigma} - G_\sigma \approx (k-1)\sigma^2 \delta G_\sigma$$

Abbildung 24: The normalized Laplacian of Gaussian

    - $(k-1)$ can be neglected (independent of scale σ)
    - $\sigma^2 \delta G_\sigma$ is the *normalized Laplacian-of-Gaussian*
  - filters detect shapes that look like them => the LoG detects *"blobs"*

- Structures live only at certain scale => appear at certain scale, then vanish again (bandpass property of DoG)
  - *characteristic scale* => magnitude of normalized LoG => extremal value (min or max)
  - provides *scale* and *location*

4.2 Accurate Localization of Key Points

- not important for exam

4.3 Assignement of Dominant Orientations

- histograms of gradients (HoG) (computed from gradients of corresponding scale $\hat{\sigma}_i$)
- not important for exam

4.4 SIFT Descriptor

- histograms
- not important for exam

4.5 SIFT properties

- **invariant** to:
  - **rotation**
  - **translation**
  - **scaling**
- **robust** wrt "general changes" (brightness, contrast, ...)
- downside: computation intensive

5 Coordinate Transformation

- Image warping (not important for exam)

5.1 Linear Transformations

- linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a map of the form: $T(p) = Ap$ with $A \in \mathbb{R}^{2\times 2}$
- **scaling**: $A = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}$ with $s_x, s_y \neq 0$
- **rotation**: $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ with $\cos\theta \in \mathbb{R}$
- $A = U\Sigma V^T \Leftrightarrow AV = U\Sigma$, A maps the columns of V onto scaled columns of U (SVD)
  - any linear transformation can be written as rotation V^T followed by scaling Σ followed by another rotation U
- **properties**:
  - origin maps to origin
  - lines map to lines
  - parallel lines remain parallel
  - ratios are preserved
  - closed under composition

5.2 Affine Transformations

- extend linear transformations with **translations**:

$$T(x) = A \cdot x + t,\ A \in \mathbb{R}^{2\times 2}, t \in \mathbb{R}^2$$

- **properties**:
  - origin does *not necessariliy* map to origin (only when $t = 0$)
  - lines map to lines
  - parallel lines remain parallel
  - ratios are preserved
  - closed under composition

- Special cases of affine transformations:
  - *pure translation*: $A = I_2$
  - *Euclidean*: (rigid motion): A is pure rotation (no scaling)
  - *Similarity*: $A = sR$, where R is rotation and $s > 0$, rotation and scaling
  - *Area preserving*: $\det(A) = 1$, five degrees of freedom

5.3 Homogenous Transformations => Homographies

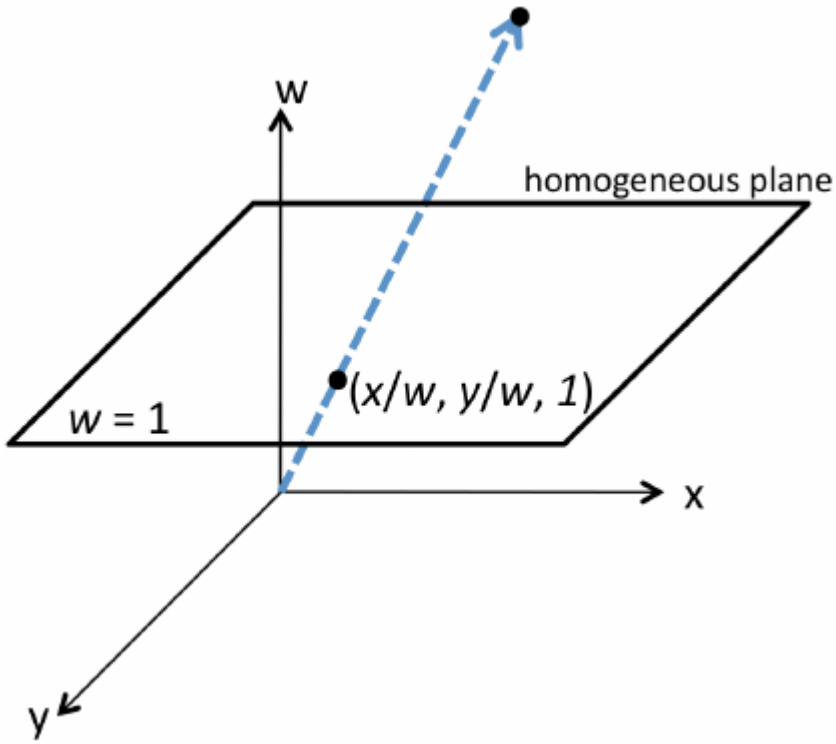


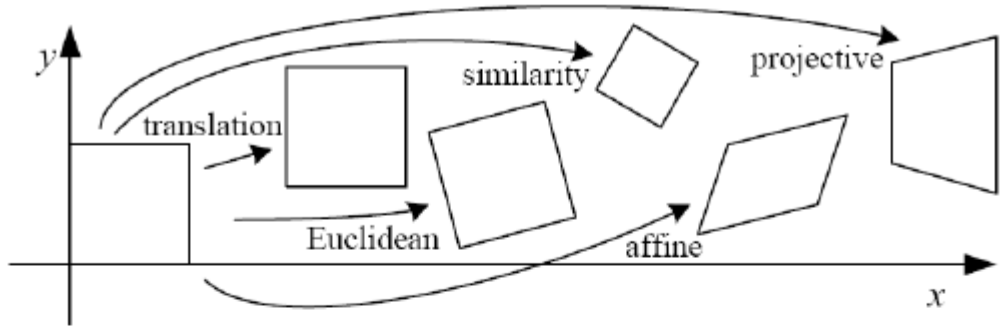
Abbildung 25: Homogenous Coordinates

- moving to *homogenous coordinates*: $p \in \mathbb{R}^2 \Longrightarrow \hat{p} = \widehat{\begin{bmatrix} x \\ y \end{bmatrix}} := \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \in \mathbb{P}^2$
- $\mathbb{P}^2$ is called *projective space*, coordinates are *homogenous coordinates*
- Reverse projection to image coordinates is given by:

$$\pi\left(\begin{bmatrix} x \\ y \\ w \end{bmatrix}\right) := \begin{bmatrix} x/w \\ y/w \end{bmatrix}$$

- *affine transformations* can now be written as matrices: $\widehat{T(p)} = \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix}$
- **Homography**: $H = \begin{bmatrix} A & t \\ b^T & 1 \end{bmatrix}, A \in \mathbb{R}^{2\times 2}, \det(A) \neq 0, t, b \in \mathbb{R}^2$
- Special case: $H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{f} & 0 & 1 \end{bmatrix}$ with $f \neq 0$

- image point of $\begin{bmatrix} x \\ y \end{bmatrix}$ is $\begin{bmatrix} x \\ y \\ -\frac{x}{f}+1 \end{bmatrix} \sim \begin{bmatrix} \frac{fx}{f-x} \\ \frac{fy}{f-x} \end{bmatrix}$
- at $x = f$ the denominator becomes 0 => points go to infinity
- horizontal lines are mapped onto lines which "vanish" in $\begin{bmatrix} -f \\ 0 \end{bmatrix}$ => vanishing point
- two vanishing points: $H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{f_x} & -\frac{1}{f_y} & 1 \end{bmatrix}$ with $f \neq 0$



Transformation	Matrix	# DoF	Preserves	Icon
translation	$\left[\boldsymbol{I} \mid \boldsymbol{t} \right]_{2\times3}$	2	orientation	
rigid (Euclidean)	$\left[\boldsymbol{R} \mid \boldsymbol{t} \right]_{2\times3}$	3	lengths	
similarity	$\left[s\boldsymbol{R} \mid \boldsymbol{t} \right]_{2\times3}$	4	angles	
affine	$\left[\boldsymbol{A} \right]_{2\times3}$	6	parallelism	
projective	$\left[\tilde{\boldsymbol{H}} \right]_{3\times3}$	8	straight lines	

Abbildung 26: Coordinate Transformations

5.4 Model Fitting 1: Least squares

- estimating Homographies can be done from *correspondence information* => feature detection + matching
- want to find: $H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & 1 \end{bmatrix}$ => eight unknowns
  - projective coordinates are only correct up to a scale factor:

$$x_i' = h_{11}x_i + h_{12}y_i + h_{13}$$
$$y_i' = h_{21}x_i + h_{22}y_i + h_{23}$$
$$w_i' = h_{31}x_i + h_{32}y_i + 1$$

– normalizing with w and rearranging this yields linear system.

$$\begin{bmatrix} & & & & \vdots & & & \\ -x_i & -y_i & -1 & 0 & 0 & 0 & \frac{x_i'}{w_i'}x_i & \frac{x_i'}{w_i'}y_i \\ 0 & 0 & 0 & -x_i & -y_i & -1 & \frac{y_i'}{w_i'}x_i & \frac{y_i'}{w_i'}y_i \\ & & & & \vdots & & & \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} = \begin{bmatrix} \vdots \\ -\frac{x_i'}{w_i'} \\ -\frac{y_i'}{w_i'} \\ \vdots \end{bmatrix}$$

Abbildung 27: Linear System for m correspondences

– System is *over-determined* with linearly dependent equations => very improbable
– naive way: If A^TA has an inverse, normal equations can be solved as a normal linear system => inefficient and not very robust.
– better way => SVD: $A = U\Sigma V^T$ => normal equations reduce to: $\Sigma^2 V^T \hat{x} = \Sigma U^T b$
  * Use *pseudo-inverse* Σ^+ to solve: $\hat{x} = V\Sigma^+ U^T b$

5.4.1 Weighted Least Squares

- Often: knowledge about which features are better (e.g. from SIFT) => assign weight $\sqrt{w_i} > 0$ to each row of $Ax = b$ => adjust minimization problem:

$$argmin_{x \in \mathbb{R}^n} 1/2 \Sigma_{i=1}^m w_i (Ax - b)_i^2$$

- Solve as before: $A' = W^{1/2}A$ and $b' = W^{1/2}b$
- very sensible to outliers

5.5 Model Fitting 2: RANdom SAmple Consensus (RANSAC)

1. Repeat N times:
   1. select random subset of samples to fit the model
   2. compute model from selected subset of samples
   3. compute error between samples and estimated model
   4. count *inliers*: points with error below a threshold
2. Refit model using largest inlier set found during any of the above N iterations

- parameters:
  – *number of s points selected to fit the model*: always choose minimum number => fast, allows more trials in given time, might not get good initialization from sample size
  – *threshold for inliers*: choose in a way, that inliers have high probability to stay below threshold (e.g. 95%)

  - *number N of trials*: choose N s.t. with high probability p at least one random sample is free of outliers:
$$N = \frac{\log(1-p)}{\log(1-(1-e)^s)}$$

- In practice, RANSAC works well for outlier ratios below 50%
- e is usually unknown in advance
  - compute e from data during RANSAC:

$$e \leftarrow \min\left(e, 1 - \frac{\text{number of inliers}}{\text{total number of points}}\right) N \qquad \leftarrow \frac{\log(1-p)}{\log(1-(1-e)^s)}$$

- this might run for a *long time* => use maximum number of iterations
- **Pro**
  - simple and general
  - applicability to different problems
  - works well in practice
- **Con**
  - several tunable parameters
  - not deterministic
  - iterative with unknown number of iterations
  - does not work well for low inlier ratios
  - can always get good initialization based on minimum number of samples

5.6 Model Fitting 3: Hough transform

- another voting algorithm for fitting low-parameter models
- simple objeects represented by *very few parameters*
- not important for exam

6 Light, Opctics and Color

- not important for exam

7 Cameras

- **Pinhole Camera Model**: very simple, but often sufficient model of a camera system.
- perspective projection of 3D space onto 2D image plane => maps 3D camera coordinates $M = (X, Y, Z)$ with origin C to 2D image coordinates $m = (x, y)$
- *notation*:
  - M: *scene point* in 3d Space
  - C: *focal point*, *optical centre*: location of pinhole
  - m: *image point*
  - I: *image plane*
  - F: *focal plane*: coplanar to image plane, contains focal point C
  - *optical axis*: orthogonal to image plane, passes through C
  - *optical ray*: passes through m, C, M

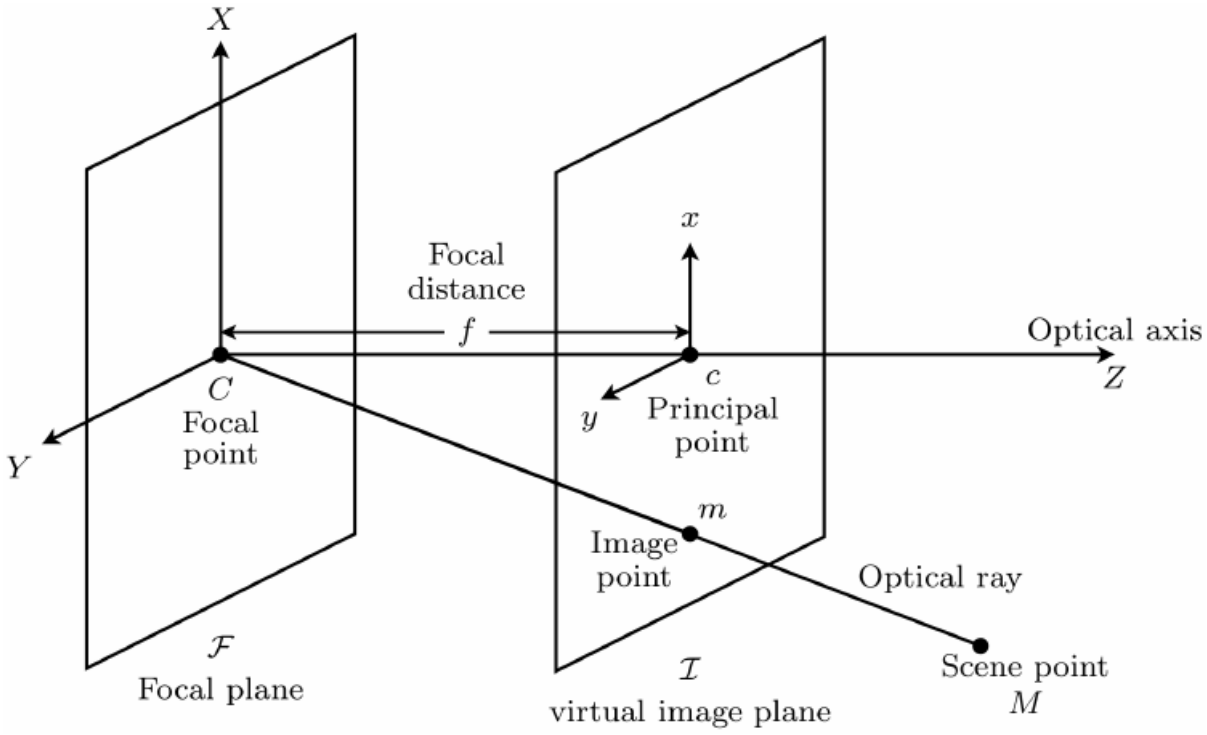


Abbildung 28: Pinhole Camera

  - f: focal distance: distance I to C
  - c: *principal point*: intersection between image plane and optical axis

- $\frac{x}{X} = \frac{y}{Y} = \frac{f}{Z}$
- All points on the optical ray $(wX, wY, wZ), w > 0$ map onto same image point $m = (x, y)$ => depth information is lost
- **pinhole projection matrix**: describe 3D points in 4D projective coordinates:

$$\hat{m} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \hat{=} \begin{bmatrix} Zx \\ Zy \\ Z \end{bmatrix} = \begin{bmatrix} fX \\ fY \\ Z \end{bmatrix} = \underbrace{\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{P} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

  - P is the *projection matrix*

- **extrinsic parameters**: denote position of world coordinate system relative to camera coordinate system: (only depending on camera orientation => extrinsic)
  - *translation*:

$$T = \begin{bmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  - *rotation*:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  with $RR^T = R^T R = I_4$

- *combined matrix for rotation and translation*: generally does not commute

$$TR = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \neq RT$$

- **intrinsic parameters** characterize geometry of image plane inside the camera

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} k & 0 & u_0 \\ 0 & \ell & v_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x//y//1 \end{bmatrix}$$

  - generalisation: $H = \begin{bmatrix} k & -k\cot\theta & u_0 \\ 0 & \ell/\sin\theta & v_0 \\ 0 & 0 & 1 \end{bmatrix}$
    - skew is generally assumed to be zero $=> \theta = 90°$

- matrix H and projection P are often multiplied into a single **intrinsic matrix**:

$$K = \underbrace{\begin{bmatrix} k & -k\cot\theta & u_0 \\ 0 & \ell/\sin\theta & v_0 \\ 0 & 0 & 1 \end{bmatrix}}_{H} \cdot \underbrace{\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{P} = \begin{bmatrix} kf & -kf\cot\theta & u_0 & 0 \\ 0 & \ell f/\sin\theta & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

  - f is usually given in some physical unit (i.e. meters), k, ℓ are also given (i.e. pixels per meter)

- 3×4 **perspective projection matrix**:

$$\Pi = KTR = \begin{bmatrix} \pi_{11} & \pi 12 & \pi 13 & \pi_{14} \\ \pi_{21} & \pi 22 & \pi 23 & \pi_{24} \\ \pi_{31} & \pi 22 & \pi 33 & \pi_{34} \end{bmatrix}$$

  - Approximated by **affine projection matrix**

$$\Pi_{affine} = \begin{bmatrix} \pi_{11} & \pi 12 & \pi 13 & \pi_{14} \\ \pi_{21} & \pi 22 & \pi 23 & \pi_{24} \\ 0 & 0 & 0 & Z_{const} \end{bmatrix}$$

7.1 Properties of Camera projection

- **many to one**: all points along same visual ray map to same image point
- **points map to points**, but projection on focal plane is undefined
- **lines map to lines**, but lines through focal point (visual rays) project to a point
- convex sets in 3D are mapped to convex sets in 2D

7.2 Vanishing Points

- each direction in space has a vanishing point

7.3 Homographies

- rotation of camera leads to homography in image plane

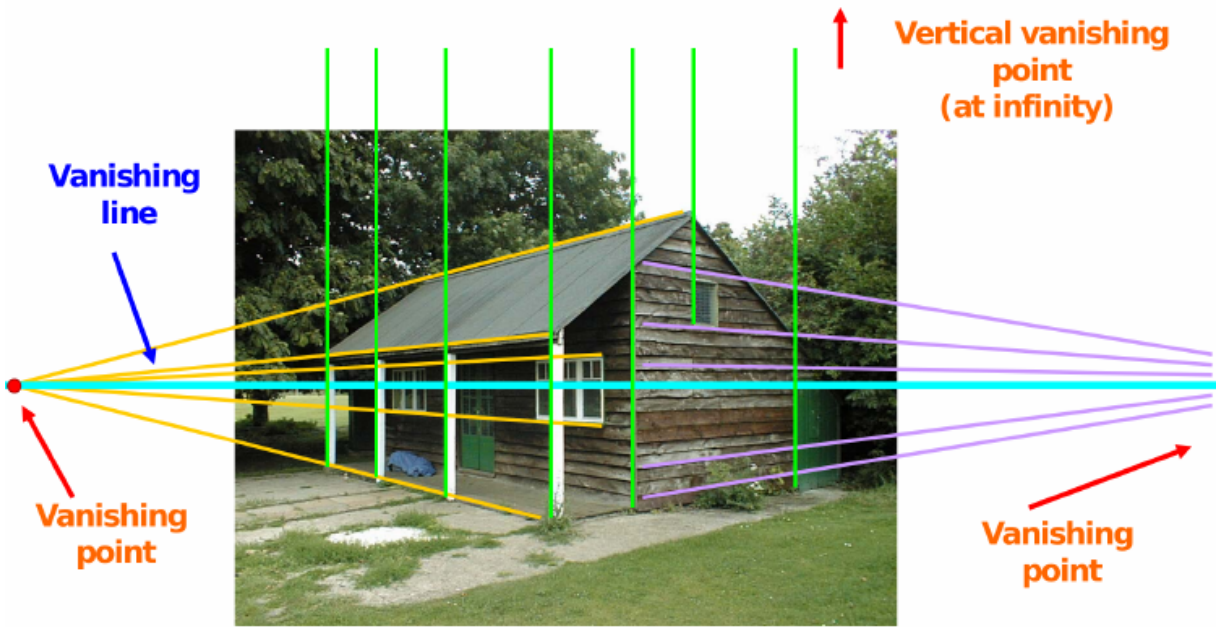


Abbildung 29: Vanishing points and lines

$$\begin{bmatrix} -X_w^i & -Y_w^i & -Z_w^i & -1 & 0 & 0 & 0 & 0 & u^i X_w^i & u^i Y_w^i & u^i Z_w^i & u^i \\ 0 & 0 & 0 & 0 & -X_w^i & -Y_w^i & -Z_w^i & -1 & v^i X_w^i & v^i Y_w^i & v^i Z_w^i & v^i \end{bmatrix} \begin{bmatrix} \pi_{11} \\ \pi_{12} \\ \pi_{13} \\ \pi_{14} \\ \pi_{21} \\ \pi_{22} \\ \pi_{23} \\ \pi_{24} \\ \pi_{31} \\ \pi_{32} \\ \pi_{33} \\ \pi_{34} \end{bmatrix} = 0$$

Abbildung 30: Camera Calibration Equations

7.4 Camera Calibration

- basic equations:

$$\widehat{\begin{bmatrix} u^i \\ v^i \end{bmatrix}} = \begin{bmatrix} u^i \\ v^i \\ 1 \end{bmatrix} \hat{=} \begin{bmatrix} Z^i u^i \\ Z^i v^i \\ Z^i \end{bmatrix} = \begin{bmatrix} \pi_{11} & \pi 12 & \pi 13 & \pi_{14} \\ \pi_{21} & \pi 22 & \pi 23 & \pi_{24} \\ \pi_{31} & \pi 22 & \pi 33 & \pi_{34} \end{bmatrix} \begin{bmatrix} X_w^i \\ Y_w^i \\ Z_w^i \\ 1 \end{bmatrix}$$

  - Depth Z_i was lost during projection
  - Solve camera calibration equation $Ax = 0$: find $argmin_{x \in \mathbb{R}^n, \|x\|=1} \|Ax\|^2$
  - => Use SVD, use last column of V
  - Image-to-world correspondences through some *calibration object* with known geometry

7.5 Lens Distortion

- Usually, images have some kind of distortion (radial + tangential)
- not important for exam

8 Geometry of Two Views

- Now consider two cameras P and P', X is a point in space => $x = PX$, and $x' = P'X$ are projections of two views.
- **Correspondence geometry**: Given x in the first view, what are constraints on corresponding point in x' in second view.
- **Camera geometry** (motion): Given a set of corresponding image points: What can we learn about the projections P, P'
- **Scene Geometry** (structure): Given corrspondence $x \leftrightarrow x'$ and two cameras, what is X

8.1 Generalization of Points in Projective Space

- $\mathbb{P}^2$ is the space of vectors with three components $0 \neq (a, b, c) \in \mathbb{R}^3$ with:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \hat{=} \begin{bmatrix} wa \\ wb \\ wc \end{bmatrix} \text{ for all } w \neq 0$$

- $\mathbb{R}^2$ is embedded in $\mathbb{P}^2$ by defining for $m = (x, y) \in \mathbb{R}^2$

$$\hat{m} := \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \in \mathbb{P}^2$$

- If $c \neq 0$ a point from $\mathbb{P}^2$ can be projected back into $\mathbb{R}^2$ by normalizing a, b with c
- Points in $\mathbb{P}^2$ with $c = 0$ => Point lies "*at infinity*" or *ideal points*

8.2 Lines in Projective space

- A line in 2D space is given by all points (x, y):

$$ax + by + c = 0$$

- (a, b, c) determines the line, but is not unique: multiplication with $w \neq 0$ => same line
- If $a^2 + b^2 = 1$ => line is in normal form, c is distance to origin

- Almost one-to-one correspondence between *lines* in $\mathbb{R}^2$ and *points* in $\mathbb{P}^2$
- **line at infinity** $I_\infty := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
- **Intersection of Lines**: $x = \ell \times \ell$

8.2.1 Cross-Product

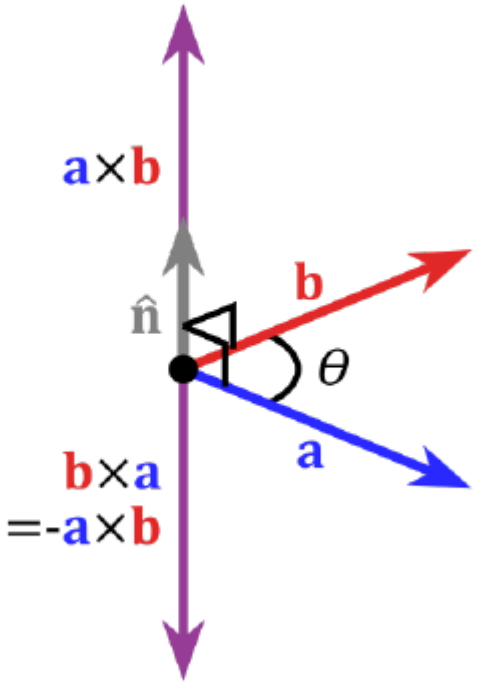


Abbildung 31: Cross Product

- either as you are used to
- or by **cross-product matrix**: $a \times b \hat{=} [a]_x b = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & \\ -a_1 & & \\ -a_2 & a_1 0 & \end{bmatrix}$
- A line connecting points x and x' is given by $\ell = x \times x'$
- lines and points are *dual partners*

8.2.2 Infinity

- If lines ℓ and ℓ' have the same direction d (i.e. they are parallel), the point of intersection is $\begin{bmatrix} d_x \\ d_y \\ 0 \end{bmatrix}$
- if ℓ has direction d, it intersects I_∞ in $\begin{bmatrix} d_x \\ d_y \\ 0 \end{bmatrix}$
- If x and x' are points at infinity, the line through them is ℓ_∞

8.3 Homogprahies of lines

- The projection of a line is: $\ell' = H^{-1 \cdot T} = H^{-T}$ (multiplication with the *inverse-transpose*)

8.4 Fundamental Matrix

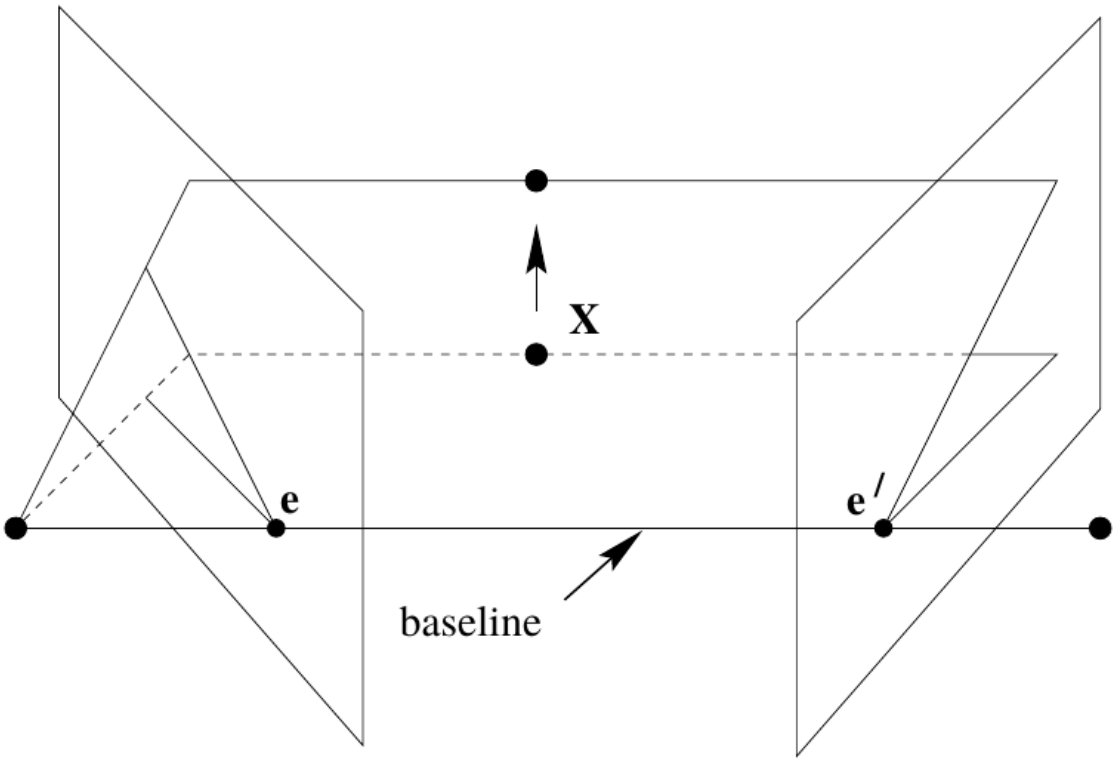


Abbildung 32: Epipolar Geometry

- C and C' are *cameras*
- x and x' are *corresponding* points, X is the world point
- e and e' are the *epipoles* (location of the other camera)
- ℓ and ℓ' are the *epipolar lines* (lines connecting) x' and e'
- π is the *epipolar plane* (plane between e, e' and X)
- the *baseline* is the line through C, C' (and e, e')
- The **fundamental matrix** for two view: projects one view onto the other
  - *project epipolar line onto point*: $\ell' = Fx$
  - any pair of correspondences satisfies: $x'^T Fx = 0$
  - => The fundamental matrix can be computed using correspondences alone
  - F is the fundamental matrix for (P, P') => F^T is the fundamental matrix for (P', P)
  - the epipole e' is the *left nullspace* of F: $e'^T F = 0$ and $Fe = 0$ => Compute epipoles when F is found
  - F has only rank two
  - F has seven degrees of freedom (9 Entries, -1 for homogenousness, -1 for $\det(F) = 0$)

8.4.1 Computing the Fundamental Matrix

- a single 3D world point X_w projects onto x => $X_w = P^+ x$
  - project X_w into second image via P'

- epipolar line connects epipole e' with $P'X_w$:

$$\ell' = [e']_x P' X_w = \underbrace{[e']_x P' P^+ x}_{=F}$$

- SVD: $P = U\Sigma V^T \Rightarrow P^+ = V\Sigma^+ U^T$, where Σ^+ is Σ with reciprocal of all non-zero diagonal elements
- Pseudo-Inverse is *left-inverse*: $A^+A = I_m$

- Computing the fundamental matrix:

$$F = [e']_x P' P^+$$

with $e' = P'C_w$ and $PC_w = 0$

- If rotation and translation are known: $C_w = T^{-1}R^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

8.4.2 Projection matrix

$P = K[R|t]$

8.4.3 Fundamental Matrix for translational motion (pure translation)

- Choose $P = K[I|0]$ (world-CO and camera-CO were the same) => second camera translated only $P' = K[I|t]$ (translation, no rotation) => $P^+ = \begin{bmatrix} K^{-1} \\ 0 \end{bmatrix}$
  - => $P'P^+ = KK^{-1} = I_3$

8.4.4 Horizontal motion

- Motion parallel to x-axis: epipole is at $e' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
- $F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$
  - correspondence equation reduces to: $x'^T Fx = 0 \;\leftrightarrow\; y = y'$
  - also goal of *rectification*

8.4.5 Estimating Fundamental Matrix from Correspondences

$$\begin{bmatrix} x_i x'_i & y - i x'_i & x'_i & x_i y'_i & y_i y'_i & y'_i & x_i & y_i & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0$$

- homogenous linear system with $n \times 9$ matrix A => interested in non-trivial solution corresponding to smalles singular value (rightmost column of V from the SVD)

- enforce rank 2

  1. inital solution $\tilde{F}$
  2. SVD: $\tilde{F} = V \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} V^T, \quad \lambda_1 \geq \lambda_2 \geq \lambda_3$
  3. $F = V \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T$ (which obviously has rank 2)

8.4.6 Increasing robustness with RANSAC

- important to remove outliers
- 8 points are sufficient for estimation
- *inliers* are feature matches, that satisfy the epipolar constraint within given accuracy: $x'^T F x \leq \epsilon$

8.5 Ambiguity of Reconstruction

8.5.1 Projective reconstruction

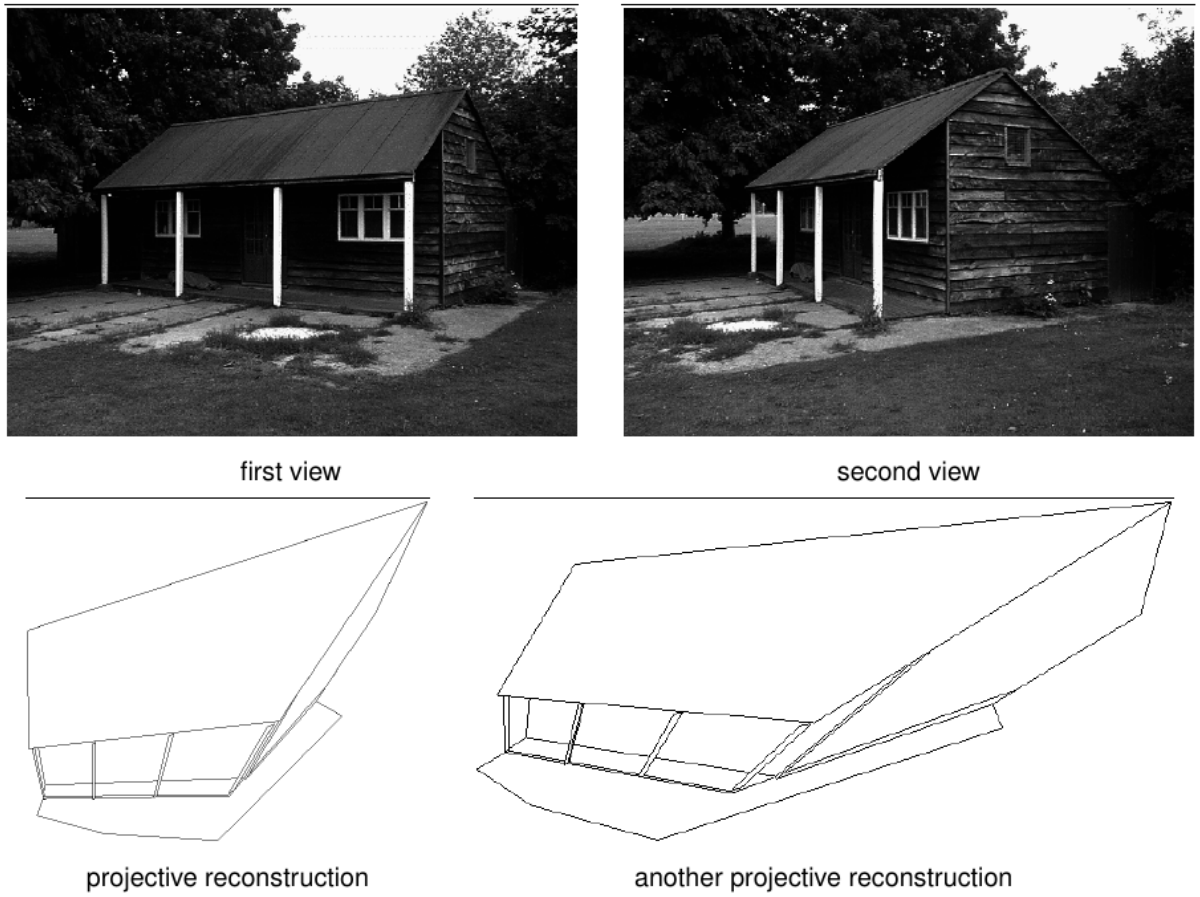


Abbildung 33: Projective Reconstruction

- Given a set of correspondences $x_i \leftrightarrow x'_i$ => projection matrices: P and P' and set of world points: X_i
- only accurate up to a homography
- possible from feature matches alone

8.5.2 Affine Reconstruction

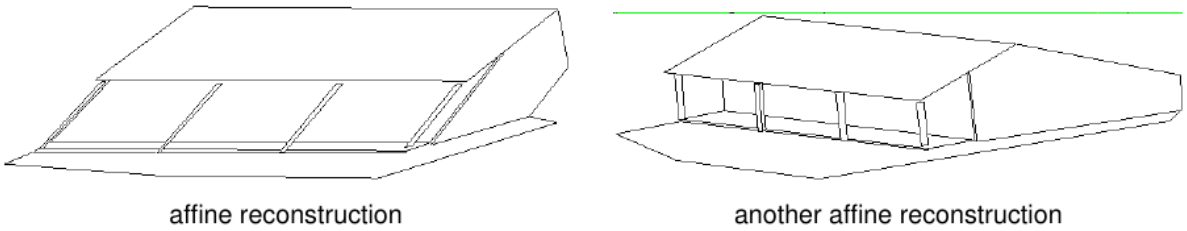


Abbildung 34: Affine Reconstruction

- **affine reconstruction**: the scene is correctly reproduced up to an *affine transformation*:
  - parallel lines are parallel
  - angles are *not* necessarily correct
- need to find plane at infinity π_∞ ("2D boundary of 3D space")
  - e.g. three vanishing points needed in both images => three sets of parallel lines

8.5.3 Metric Reconstruction

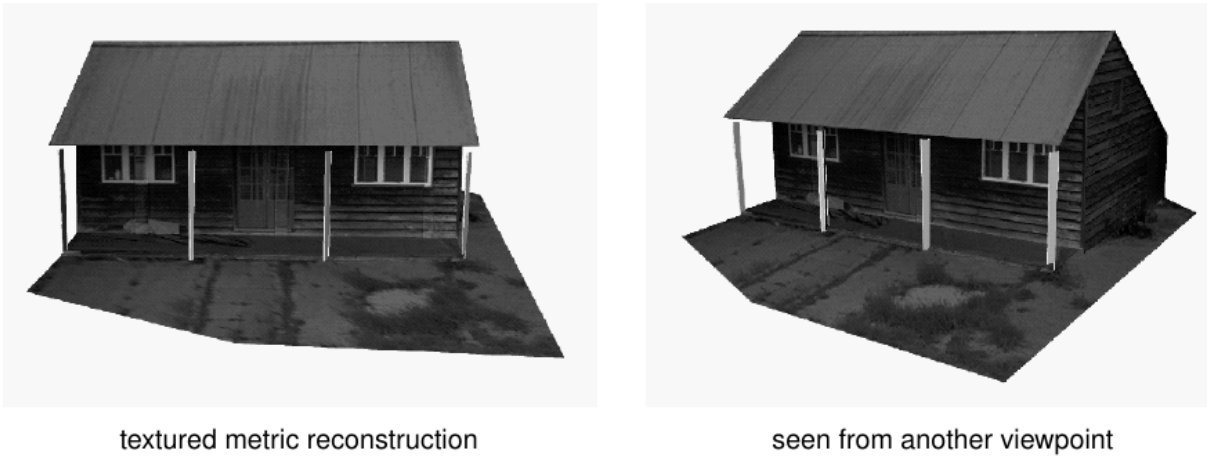
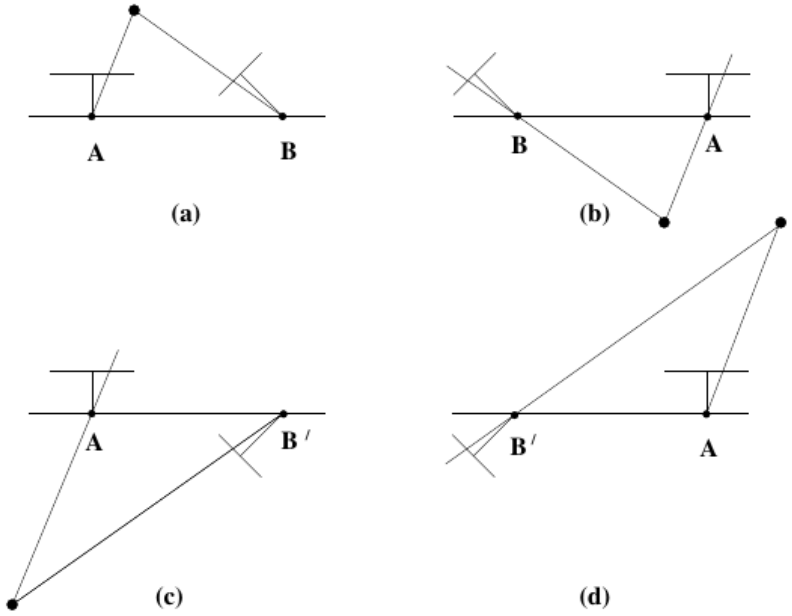


Abbildung 35: Metric Reconstruction

- **metric reconstruction**: scene is correctly reproduced up to similarity transformation_
  - parallel lines are parallel
  - angles are correct
  - scale is *not* necessarily correct
    - inserting a segment with known length identifiable in both images can help here
- need to have some information about *absolute angles* in scene (=> intrinsics of camera)
- => computing the *Image of the Absolute Coniac* (IAC): $\omega = (KK^T)^{-1}$
  - e.g. assumption of square pixels plus set of three vanishing points

8.6 Pose Estimation: The Essential Matrix

- **essential matrix** E is a specialization of the fundamental matrix
  - pre-calibrated camera => intrinsic parameters (i.e. matrix K) are known
- immediately obtain *metric reconstruction*
- given $P = [I|0]$ and $P' = [R|t]$ => $E = [t]_x R$
  - decompose E into rotation and translation: SVD



  - four possible solutions

8.7 Triangulation: Structure from Motion

- in 3D space two rays do not necessarily meet, even if they are not parallel
- *Direct Linear Triangulation* (DLT): Recover X as $AX = 0$, where $A = \begin{bmatrix} x(p^3) - p^1 \\ y(p^3) - p^1 \\ x'(p'^3) - p'^1 \\ y'(p'^3) - p'^2 \end{bmatrix}$
  - solution is last column in V of SVD

8.8 Bundler: Practical example

9 Dense Correpondences

- not important for exam

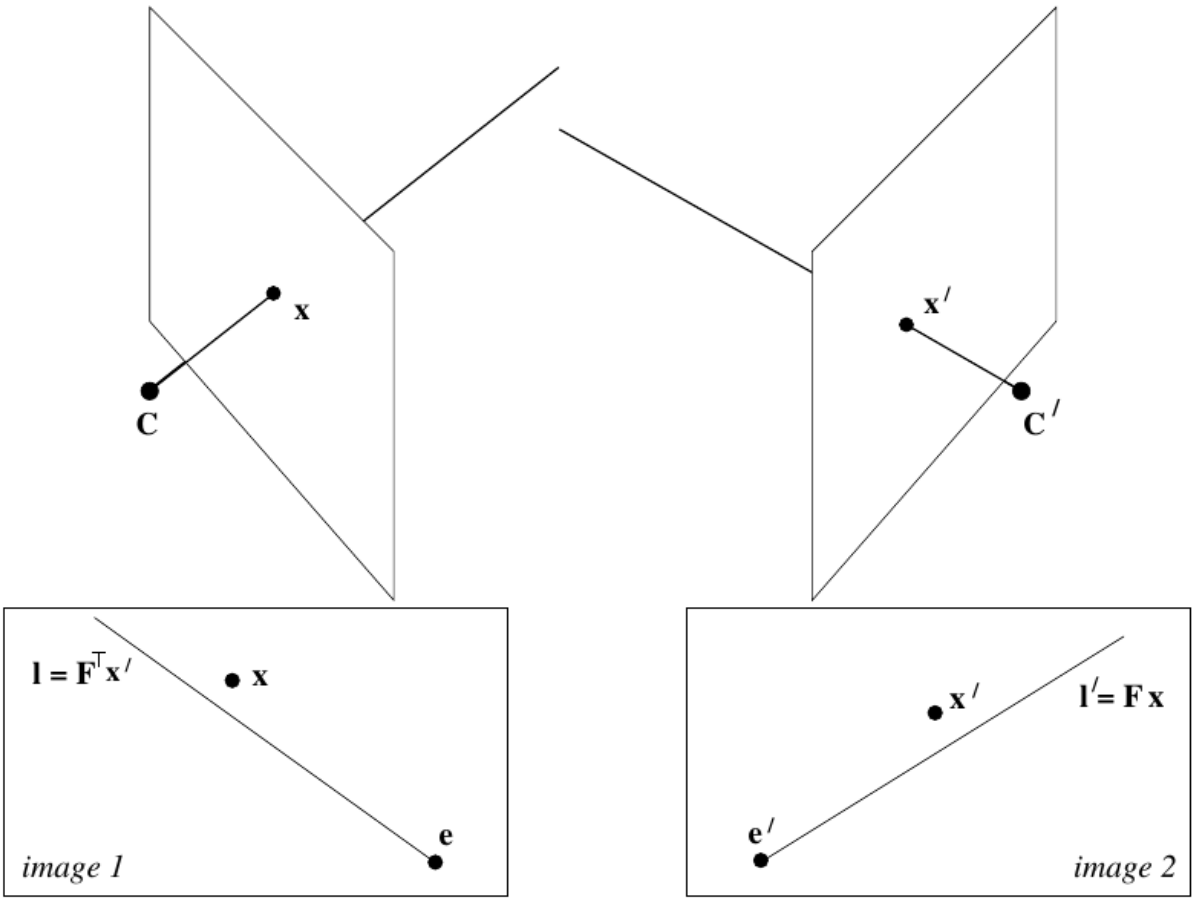


Abbildung 36: Triangulation from imperfect points