MAS242 선형대수학
Notes

한승우

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Vector Spaces

2.1 Bases and Dimension

Theorem 2.1.1
Any subset that is linearly independent can be extended to a basis of V.

Lemma 2.1.1
If W is a subspace of V and $W \subsetneq V$, then $\dim W < \dim V$ provided that V is finite-dimensional.

Proof. Let S_0 be a basis of W. S_0 is linearly independent, so we can enlarge it to a get a basis of V. $S' \triangleq S_0 \cup \{ v_1, v_2, \cdots, v_r \}$ is a basis of V. $|S'| \geq |S_0| + 1$; otherwise $\operatorname{span} S_0 = V$. □

Theorem 2.1.2 Inclusion/Exclusion Principle for Vector Spaces
If W_1 and W_2 are finite-dimensional subspaces of V, then $W_1 + W_2$ is a finite-dimensional vector space and $\dim W_1 + \dim W_2 = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$.

Proof. Let $a \triangleq \dim W_1$, $b \triangleq \dim W_2$, $c \triangleq \dim(W_1 + W_2)$, and $d \triangleq \dim(W_1 \cap W_2)$. Choose $\{ \alpha_1, \alpha_2, \cdots, \alpha_d \}$ as a basis for $W_1 \cap W_2$. We may extend this into bases of W_1 and W_2. Let $\{ \alpha_1, \cdots, \alpha_d, \beta_{d+1}, \beta_{d+2}, \cdots, \beta_a \}$ and $\{ \alpha_1, \cdots, \alpha_d, \gamma_{d+1}, \gamma_{d+2}, \cdots, \gamma_a \}$ be bases for W_1 and W_2 respectively.

We now claim that

$$B \triangleq \{ \alpha_1, \cdots, \alpha_d, \beta_{d+1}, \cdots, \beta_a, \gamma_{d+1}, \cdots, \gamma_b \}$$

is a basis of $W_1 + W_2$.

- Let $x \in W_1 + W_2$. Then, $x = w_1 + w_1$ where $w_i \in W_i$. Since $w_1 \in \operatorname{span}\{ \alpha_1, \cdots, \alpha_d, \beta_{d+1}, \cdots, \beta_a \}$ and $w_1 \in \operatorname{span}\{ \alpha_1, \cdots, \alpha_d, \gamma_{d+1}, \cdots, \gamma_b \}$, On the other hand, $B \subseteq W_1 + W_2$. Hence, $\operatorname{span} B = W_1 + W_2$.
- Suppose we have $\sum a_i \alpha_i + \sum b_j \beta_j + \sum c_k \gamma_k = 0$ for some $a_i, b_j, c_k \in F$. Rearranging the terms, we get $\sum a_i \alpha_i + \sum b_j \beta_j = -\sum c_k \gamma_k$, which implies that $\sum c_k \gamma_k \in W_1 \cap W_2$. The fact that γ_k's are linearly independent from $\{\alpha_i\}$ implies that $c_k = 0$ for all k. Similarly, $b_j = 0$ for all j. Hence, we are left with $\sum a_i \alpha_i = 0$, in which α_i's are linearly independent; $a_i = 0$. Hence, B is linearly independent.

Therefore, $\dim(W_1 + W_2) = a + b - d$. □

Definition 2.1.1: Ordered Basis

Let V be a finite-dimensional vector space over F. An *ordered basis* of V is a sequence of vectors that forms a basis.

Note:-

Usually, we emphasize the ordered basis with semicolons like $\{\beta_1; \beta_2\}$.

Lemma 2.1.2

Let V be a finite-dimensional vector space over F. Suppose $B = \{v_1; v_2; \cdots; v_n\}$ is an ordered basis of V. Then, for each $x \in V$, there uniquely exists an expression of the form

$$x = x_1 v_2 + x_2 v_2 + \cdots + x_n v_n$$

for some $x_i \in F$.

Proof. The existence of the form is obvious since $x \in V = \operatorname{span} B$.

(Uniqueness) Suppose we have two such expressions:

$$x = \sum x_i v_i = \sum y_i v_i$$

where $x_i, y_i \in F$. Then, we have $\sum (x_i - y_i) v_i = 0$. The linear independence of B gives that $x_i - y_i = 0$ for all i. Hence, $x_i = y_i$. $\square$

Definition 2.1.2: Coordinate Matrix

Let V be a finite-dimensional vector space over F. Let B be an ordered basis of V. Let $x \in V$ and write it as $x = \sum_{i=1}^{n} x_i v_i$ with $x_i \in F$, $v_i \in B$. Define

$$[x]_B \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

be the *coordinate matrix of x with respect to the basis B*

Theorem 2.1.3

Let V be a finite-dimensional vector space over F. Let B and B' be two ordered bases of V. Then, there uniquely exists an invertible matrix P such that $\forall x \in V$, $[x]_B = P[x]_{B'}$ and $[x]_{B'} = P^{-1}[x]_B$.

Proof. Let $B \triangleq \{\alpha_1; \cdots; \alpha_n\}$ and $B' \triangleq \{\alpha'_1; \cdots; \alpha'_n\}$ For $\alpha'_j \in B'$, since B is a basis, there are unique $P_{ij} \in F$ $(i \in [n])$ such that $\alpha'_j = \sum_{i=1}^{n} P_{ij} \alpha_i$.

Let $x \in V$. Write $[x]_B = \begin{pmatrix} x_1 \\ \vdots \\ v_n \end{pmatrix}$ and $[x]_{B'} = \begin{bmatrix} x'_1 \\ \vdots \\ v'_n \end{bmatrix}$. Then, $x = \sum_{j=1}^{n} x'_j \alpha_j = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} x'_j P_{ij} \right) \alpha_i$.

By the uniqueness, we have $x_i = \sum_{j=1}^{n} x'_j P_{ij}$ for each i. In other words, we have

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} P_{11} & \cdots & P_{1n} \\ \vdots & \cdots & \vdots \\ P_{n1} & \cdots & P_{nn} \end{bmatrix} \begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix}$$

Since B and B' are linearly independent, $x = 0 \iff [x]_B = 0 \iff [x]_{B'} = 0$. Hence, P is invertible. $\square$

Chapter 3

Linear Transformations

3.1 Linear Transformations

Definition 3.1.1: Linear Transformation

Let V_1 and V_2 be vector spaces over F. $T: V_1 \to V_2$ is said to be a *linear transformation* if

- $\forall x_1, x_2 \in V_1,\ T(x_1 + x_2) = T(x_1) + T(x_2)$
- $\forall x \in V_1,\ \forall c \in F,\ T(cx) = cT(x)$.

Theorem 3.1.1

Let V and W be finite-dimensional vector spaces over F. where $\{\alpha_1, \cdots, \alpha_n\}$ is a basis of V. Let $\{\beta_1, \cdots, \beta_n\}$ be any given set of vectors of W. Then, there exists a unique transformation $T: V \to W$ such that $T(\alpha_i) = \beta_i$.

Proof. Let $T_0: V \to W$ be defined by

$$T_0\big(\textstyle\sum_{i=1}^n x_i\alpha_i\big) = \textstyle\sum_{i=1}^n x_i\beta_i.$$

This is a linear transformation indeed.

(Uniqueness) If there is another such $U: V \to W$, Then, $U\big(\sum_{i=1}^n x_i\alpha_i\big) = \sum_{i=1}^n x_iU(\alpha_i)$. Hence, $U = T_0$. □

Definition 3.1.2: Null Space and Range Space

Let $T: V \to W$ be a linear transformation between vector spaces over F.

- $\operatorname{null} T \triangleq \ker T \triangleq \{v \in V \mid T(v) = 0\}$
- $\operatorname{range} T \triangleq \operatorname{Im} T \triangleq \{w \in W \mid \exists v \in V,\ w = T(v)\}$

Note:-

$\ker T$ and $\operatorname{Im} T$ are subspaces of V and W respectively.

Definition 3.1.3

Let $T: V \to W$ be a linear transformation between vector spaces over F.

$$\operatorname{nullity}(T) \triangleq \dim\ker(T) \quad \text{and} \quad \operatorname{rank}(T) \triangleq \dim\operatorname{Im}(T)$$

Theorem 3.1.2 Rank-Nullity Theorem

Let $T: V \to W$ be a linear transformation between vector spaces over F. Then, $\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim V$.

Proof. Let $\{v_1, \cdots, v_k\}$ be a basis for $\ker T$ where $k = \operatorname{nullity} T$. Choose $v_{k+1}, \cdots, v_n \in V$ such that $\{v_i\}_{i=1}^n$ is a basis of V. We claim that $\{T(v_{k+1}), \cdots, T(v_n)\}$ is a basis of $\operatorname{Im} T$.

Suppose $\sum_{i=k+1}^n c_i T(v_i) = 0$ for some $c_i \in F$. Then, we have $T\left(\sum_{i=k+1}^n c_i v_i\right) = 0$; hence $\sum_{i=k+1}^n c_i v_i \in \ker T$. Since $\{v_1, \cdots, v_k\}$ is a basis of $\ker T$, we have $\sum_{i=k+1}^n c_i v_i = \sum_{i=1}^k a_i v_i$ for some a_i's. Therefore, since $\{v_1, \cdots, v_n\}$ is linearly independent, all c_i's and a_i's are zero. This implies that $\{T(v_i)\}_{i=k+1}^n$ is linearly independent.

Take any $T(v) \in \operatorname{Im} T$. Then, $v = \sum_{i=1}^n c_i v_i$ for some $c_i \in F$. Then, $T(v) = \sum_{i=k+1}^n c_i T(v_i)$. Hence, $\operatorname{Im} T \subseteq \operatorname{span}\{T(v_{k+1}), \cdots, T(v_n)\}$

The two paragraphs imply that $\operatorname{rank} T = n - k$. □

Theorem 3.1.3

Let A be a $m \times n$ matrix. Then $\dim \operatorname{span}(\text{rows}) = \dim \operatorname{span}(\text{columns})$.

Proof. $V = F^n$, $W = F^m$. Then, $\dim \operatorname{span}(\text{columns}) = \dim \operatorname{Im} T = \operatorname{rank} T$, so $\operatorname{nullity} T = n - \operatorname{rank} T = n - \operatorname{colrank} T$.

The number of rows with leading one's in rref$A$ equals the dimension of the row space of A, which is simply the number of columns with the leading ones. It is equal to the dimension of the column space. Hence, $\operatorname{nullity} T = n - \operatorname{colrank} T$ □

Definition 3.1.4

Let $T: V \to W$ be a linear transformation between vector spaces over F. $L(V, W) \triangleq \{T: V \to W \mid T \text{ is a linear transformation}\}$

Theorem 3.1.4

Let $T: V \to W$ be a linear transformation between vector spaces over F. Then, $L(V, W)$ is a vector space over F under usual addition and multiplication.

Theorem 3.1.5

Let V and W be n- and m-dimensional vector spaces over F, respectively. Then, $\dim L(V, W) = mn$.

Proof. Let $\mathcal{B} = \{\alpha_1, \cdots, \alpha_n\}$ and $\mathcal{B}' = \{\beta_1, \cdots, \beta_m\}$ be bases for V and W, respectively. For each $p \in [n]$ and $q \in [m]$, Define

$$E^{p,q}(\alpha_i) = \begin{cases} 0 & \text{if } i \neq q \\ \beta_p & \text{if } i = q \end{cases}.$$

Then,

- These $E^{p,q}$ are linear transformations
- These are linearly independent.
- They span $L(V, W)$.

□

Lemma 3.1.1

Let $T : V \to W$ and $U : W \to Z$ be linear transformations between vector spaces over F. Then, $U \circ T \in L(V, Z)$.

Definition 3.1.5: Linear Operator (Endomorphism)

Let $T : V \to V$ be a linear transformation from a vector space V to itself. Then, T is called a *linear operator*. (Or an *endomorphism*.)

Note:-

For each $T, U \in L(V,V)$, $T \circ U \in L(V,V)$. $(T_1 + T_2) \circ U = T_1 \circ U + T_2 \circ U$. And many more... $(L(V,V), +, \circ)$ is a non-commutative ring.

Definition 3.1.6: Injectivity and Surjectivity

A linear transform $T : V \to W$ is

- *injective* (or, nonsingular) if $T(v) = 0 \implies v = 0$.
- *surjective* if $T(V) = W$.
- *invertible* if $\exists$linear transform $U : W \to V$, $U \circ T = \mathrm{id}_V \land T \circ U = \mathrm{id}_W$.

Exercise 3.1.1

$T : V \to W$ is injective and surjective if and only if T is invertible.

Exercise 3.1.2

If $T : V \to W$ is a nonsingular linear transformation, then, for any linearly independent subset $S \subseteq V$, $T(S)$ is linearly independent.

Exercise 3.1.3

Suppose V and W are finite-dimensional vector spaces. If $T : V \to W$ is invertible, then $\dim V = \dim W$.

Theorem 3.1.6

Let V and W be finite-dimensional vector spaces over F with $\dim V = \dim W$. Let $T : V \to W$ be a linear transform. TFAE

(i) T is invertible.

(ii) T is injective.

(iii) T is surjective.

Proof. T is injective $\iff$ $\operatorname{nullity} T = 0 \iff \operatorname{rank} T = n \iff \operatorname{Im} T = W \iff T$ is onto □

Definition 3.1.7: General Linear Group

Let $\mathrm{GL}(V) \triangleq \{ T \in L(V,V) \mid T \text{ is invertible} \}$. Then, $(\mathrm{GL}(V), \circ)$ is called the *general linear group of* V.

Note:-

The general linear group is actually a group.

Definition 3.1.8: Isomorphism

Let V and W be vector spaces over F. We say that a linear transformation $T: V \to W$ is an *isomorphism* if T is an invertible linear transformation.
We say V and W are *isomorphic* if there exists an isomorphism $T: V \to W$, if V and W are isomorphic, then we write $V \simeq W$.

Theorem 3.1.7

Let V be a vector spaces over F of dimension n. Then, $V \simeq F^n$.

Proof. Let $B = \{\alpha_1; \cdots; \alpha_n\}$ be a basis of V. Define $T: V \to F^n$ by $v \mapsto [v]_B$.
Suppose $T(v) = 0$. Then, $v = 0 \cdot \alpha_1 + \cdots 0 \cdot \alpha_n = 0$. Hence, T is injective. By Theorem 3.1.6, T is isomorphism. □

Theorem 3.1.8

Let V and W be vector spaces over F with $\dim V = n$ and $\dim W = m$. Let B and B' be bases of V and W, respectively. If $T: V \to W$ is a linear transformation, then there uniquely exists $m \times n$ matrix A such that $[T(v)]_{B'} = A[v]_B$. We write $[T]_{B,B'} \triangleq A$.

Proof. $A = \begin{bmatrix} [T(v_1)]_{B'} & [T(v_2)]_{B'} & \cdots & [T(v_n)]_{B'} \end{bmatrix}$ where v_i is the i^{th} basis vector of B. □

Theorem 3.1.9

Let $V \xrightarrow{T} W \xrightarrow{U} Z$ be linear transformations. Let $A_1 = [T]_{B,B'}$ and $A_2 = [U]_{B',B''}$. Then, $[U \circ T]_{B,B''} = A_2 A_1$.

Theorem 3.1.10

Let V be finite-dimensional vector space over F with two (possibly different) bases B_1 and B_2. Let $T \in L(V, V)$. Let P be the matrix such that $[v]_{B_1} = P[v]_{B_2}$. Then, $[T]_{B_i} \triangleq [T]_{B_i,B_i}$ are related by

$$[T]_{B_2} = P^{-1}[T]_{B_1} P.$$

Definition 3.1.9: Similar Matrices

Suppose M and N are $n \times n$ matrices. M and N are *similar* if there exists an invertible P such that $N = P^{-1}MP$.

Proof. $[T(v)]_{B_1} = [T]_{B_1}[v]_{B_1} = [T]_{B_1} P[v]_{B_2}$. $[T(v)]_{B_1} = P[T(v)]_{B_2} = P[T]_{B_2}[v]_{B_2}$.
Since v was arbitrary, $P[T]_{B_2} = [T]_{B_1} P$. □

Note:-

- A linear transformation $T: V \to V$ gives varying matrices $[T]_B$ that are all similar when the basis B is changed.
- On linear operators, we will have various definitions.
- Characteristic (eigen) polynomial has $(-1)^{\deg}$(constant term) as $\det T$ and $-(n-1$ deg term) as $\operatorname{tr} T$.

Definition 3.1.10: Linear Functional

Let V be a vector space over F. A linear transformation $T : V \to F$ is called a *(linear) functional*.

Definition 3.1.11: Dual Vector Space

Let V be a vector space over F. We normally write $V^* \triangleq L(V, F)$ and call it the *dual vector space* of V.

End.