Summary for Introduction to Set Theory

SEUNGWOO HAN

Hrbacek, Karel, and Thomas J. Jech. *Introduction to Set Theory, Revised and Expanded*. 3rd ed., CRC Press, 1999.

Contents

Chapter 1

Sets

1.1 Introduction to Sets

Definition 1.1.1: Set

Every object in the universe of discourse is called a *set*.

1.2 Properties

Definition 1.2.1: Property

Any mathematical sentence* is called a *property*. If $X, Y, \cdots, Z$ are free variables of a property $\mathbf{Q}$, we write $\mathbf{Q}(X, Y, \cdots, Z)$ and say $\mathbf{Q}(X, Y, \cdots, Z)$ is a property of $X, Y, \cdots, Z$.

*Refer to mathematical logic textbook for detailed discussion.

1.3 Axioms

Axiom I The Axiom of Existence

There exists a set which has no elements.

$$\exists A \, \forall x \, \neg(x \in A)$$

Note:-

The Axiom of Existence guarantees that the universe of discourse is not void.

Axiom II The Axiom of Extensionality

If every element of X is an element of Y and every element of Y is an element of X, then $X = Y$.

$$\forall X \, \forall Y \, [\forall x \, (x \in X \iff x \in Y) \implies X = Y]$$

Note:-

The Axiom of Extensionality defines the equality relation with the containment relation($\in$).

Lemma 1.3.1

There exists only one set with no elements.

Proof. Let A and B are sets such that $\forall x\, \neg(x \in A)$ and $\forall x\, \neg(x \in B)$. Then, we have $\forall x\, (x \in A \iff x \in B)$. Therefore, by The Axiom of Extensionality, $A = B$ is guaranteed. □

Definition 1.3.2: Empty Set

The unique set with no elements is called the *empty set* and is denoted $\varnothing$.

Note:-
Definition 1.3.2 is justified by Lemma 1.3.1.

Axiom III The Axiom Schema of Comprehension

Let $\mathbf{P}(x)$ be a property of x. For any set A, there exists a set B such that $x \in B$ if and only if $x \in A$ and $\mathbf{P}(x)$.

$$\forall A\, \exists B\, (x \in B \iff x \in A \wedge \mathbf{P}(x))$$

Note:-
Axiom III is a axiom *schema* since it provides unlimited amount of axioms for varying $\mathbf{P}$.

Lemma 1.3.3

Let $\mathbf{P}(x)$ be a property of x. For any set A, there uniquely exists a set B such that $x \in B$ if and only if $x \in A$ and $\mathbf{P}(x)$.

Proof. Let B' be another set such that $x \in B'$ if and only if $x \in A$ and $\mathbf{P}(x)$. Then, for any x, we have $x \in B' \iff x \in A \wedge \mathbf{P}(x) \iff x \in B$. Hence, by The Axiom of Extensionality, we have $B = B'$. □

Notation 1.3.4: Set-Builder Notation

Let $\mathbf{P}(x)$ be a property of x. Let A be a set. The unique set B such that $x \in B$ if and only if $x \in A$ and $\mathbf{P}(x)$ is denoted $\{\, x \in A \mid \mathbf{P}(x) \,\}$.

Note:-
Notation 1.3.4 is justified by Lemma 1.3.3.

Axiom IV The Axiom of Pair

For any A and B, there exists C such that $x \in C$ if and only if $x = A$ or $x = B$.

$$\forall A\, \forall B\, \exists C\, (x \in C \iff x = A \vee x = B)$$

Note:-
Similarly, the set C such that $x \in C \iff x = A \vee x = B$ is unique by The Axiom of Extensionality.

Notation 1.3.5

Let A and B be sets. The unique set C such that $x \in C$ if and only if $x = A$ or $x = B$ is denoted $\{A, B\}$. In particular, if $A = B$, we write $\{A\}$ instead of $\{A, A\}$.

Axiom V The Axiom of Union

For any S, there exists U such that $x \in U$ if and only if $x \in A$ for some $A \in S$.

$$\forall S\, \exists U\, (x \in U \iff \exists A\, x \in A \wedge A \in S)$$

Definition 1.3.6: The Union of System of Sets

Let S be a set. The unique set U such that $x \in U$ if and only if $x \in A$ for some $A \in S$ is denoted $\bigcup S$.

Definition 1.3.7: The Union of Two Sets

Let A and B be sets. Then, $A \cup B$ denotes the unique set $\bigcup\{A, B\}$.

Definition 1.3.8: Subset

Let A and B sets. B is said to be a *subset* of A if $\forall x\, (x \in B \implies x \in A)$. If B is a subset of A, then we write $B \subseteq A$.

Axiom VI The Axiom of Power Set

For any S, there exists P such that $X \in P$ if and only if $X \subseteq S$.

Note:-

Similarly, the set P is unique by The Axiom of Extensionality.

Definition 1.3.9: Power Set

Let S be a set. The unique set P such that $X \in P$ if and only if $X \subseteq S$ is called the *power set* of S and is denoted $\mathcal{P}(S)$.

Lemma 1.3.10

Let $\mathbf{P}(x)$ be a property of x. Let A and A' be sets such that $\mathbf{P}(x) \implies x \in A \wedge x \in A'$. Then, $\{\, x \in A \mid \mathbf{P}(x)\,\} = \{\, x \in A' \mid \mathbf{P}(x)\,\}$.

Proof. For all x, we have $x \in A \wedge \mathbf{P}(x) \iff \mathbf{P}(x) \iff x \in A' \wedge \mathbf{P}(x)$. Therefore, by The Axiom of Extensionality, the result follows. □

Notation 1.3.11

Let $\mathbf{P}(x)$ be a property of x. If there exists a set A such that $\mathbf{P}(x)$ implies $x \in A$, we write $\{\, x \mid \mathbf{P}(x)\,\} \triangleq \{\, x \in A \mid \mathbf{P}(x)\,\}$, and it is called *the set of all x with the property* $\mathbf{P}(x)$.

Note:-

Notation 1.3.11 is justified by Lemma 1.3.10.

Selected Problems

Exercise 1.3.1

The set of all x such that $x \in A$ and $x \notin B$ exists.

Proof. We have $x \in A \wedge x \notin B \implies x \in A$. Hence, the set exists and is equal to $\{ x \in A \mid x \in A \wedge x \notin B \}$. □

Exercise 1.3.2

Prove The Axiom of Existence only from The Axiom Schema of Comprehension and The Weak Axiom of Existence.

Weak Axiom of Existence Some set exists.

Proof. Let A be a set known to exist. Then, there exists $B = \{ x \in A \mid x \neq x \}$ by The Axiom Schema of Comprehension. Since $\forall x\,(x = x)$, $\forall x\,(x \notin B)$. □

Exercise 1.3.3

(a) Prove that a set of all sets($\{ x \mid \top \}$) does not exist.
(b) Prove that $\forall A\, \exists x\,(x \notin A)$.

Proof.

(a) Suppose $V = \{ x \mid \top \}$ exists. Then, by The Axiom Schema of Comprehension, $R = \{ x \in V \mid x \notin x \}$ exists. However, we have $R \in R \iff R \notin R$ by definition of R. Hence, V does not exist.
(b) Suppose $\exists A\, \forall x\,(x \in A)$ for the sake of contradiction. Then, A is the set of all sets, which is impossible by (a). □

Exercise 1.3.6

Prove $\forall X\, \neg(\mathcal{P}(X) \subseteq X)$.

Proof. Let $Y = \{u \in X \mid u \notin u\}$. Then, by definition, $Y \subseteq X$, and thus $Y \in \mathcal{P}(X)$. Now, suppose $Y \in X$ for the sake of contradiction. Then, $Y \in Y \iff Y \in X \wedge Y \notin Y \iff Y \notin Y$, which is a contradiction. Hence, $Y \notin X$. □

1.4 Elementary Operations on Sets

Definition 1.4.1: Proper Subset

Let A and B sets. B is said to be a *proper subset* of A if $B \subseteq A$ and $B \neq A$. If B is a proper subset of A, we write $B \subsetneq A$.

Definition 1.4.2: Elementary Operations on Sets

(i) Intersection
- The *intersection* of A and B, $A \cap B$, is the set $\{x \mid x \in A \wedge x \in B\}$.

(ii) Union
- The *union* of A and B, $A \cup B$, is the set $\{x \mid x \in A \vee x \in B\}$.

(iii) Difference
- The *difference* of A and B, $A \setminus B$, is the set $\{x \mid x \in A \wedge x \notin B\}$.

(iv) Symmetric Difference
- The *symmetric difference* of A and B, $A \triangle B$, is the set $(A \setminus B) \cup (B \setminus A)$.

Lemma 1.4.3 Simple Properties of Elementary Operations

(i) Commutativity
- $A \cap B = B \cap A$
- $A \cup B = B \cup A$
- $A \triangle B = B \triangle A$

(ii) Associativity
- $(A \cap B) \cap C = A \cap (B \cap C)$
- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \triangle B) \triangle C = A \triangle (B \triangle C)$

(iii) Distributivity
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(iv) De Morgan's Laws
- $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$
- $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$

(v) Miscellaneous
- $A \cap (B \setminus C) = (A \cap B) \setminus C$
- $A \setminus B = \varnothing \iff A \subseteq B$
- $A \triangle B = \varnothing \iff A = B$

Definition 1.4.4: Intersection of System of Sets

Let S be a nonempty set. Then, the *intersection* $\bigcap S$ is the set $\{x \mid \forall A \in S\, (x \in A)\}$.

Note:-

Note that $\bigcap S$ exists for all nonempty S since $\forall A \in S\, (x \in A) \implies x \in A_1$ where A_1 is any set such that $A_1 \in S$.

Definition 1.4.5: System of Mutually Disjoint Sets

We say the sets A and B are *disjoint* if $A \cap B = \varnothing$. A set S is a *system of mutually disjoint sets* if $\forall A, B \in S$, $(A \neq B \implies A \cap B = \varnothing)$.

Selected Problems

Exercise 1.4.2

(i) $A \setminus B = (A \cup B) \setminus B = A \setminus (A \cap B)$
(ii) $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$
(iii) $A \cap B = A \setminus (A \setminus B)$

Proof.

(i)
$$\begin{aligned} x \in A \wedge x \notin B &\iff x \in A \wedge x \notin B \vee x \in B \wedge x \notin B && \triangleright \vee\text{-intro / } \vee\text{-syllogism} \\ &\iff (x \in A \vee x \in B) \wedge x \notin B && \triangleright \text{Distribution} \end{aligned}$$

$$\begin{aligned} x \in A \wedge x \notin B &\iff x \in A \wedge x \notin A \vee x \in A \wedge x \notin B && \triangleright \vee\text{-intro / } \vee\text{-syllogism} \\ &\iff x \in A \wedge (x \notin A \vee x \notin B) && \triangleright \text{Distribution} \\ &\iff x \in A \wedge \neg(x \in A \wedge x \in B) && \triangleright \text{De Morgan's Law} \end{aligned}$$

(ii)
$$\begin{aligned} x \in A \wedge \neg(x \in B \wedge x \notin C) &\iff x \in A \wedge (x \notin B \vee x \in C) && \triangleright \text{De Morgan's Law} \\ &\iff (x \in A \wedge x \notin B) \vee (x \in A \wedge x \in C) && \triangleright \text{Distribution} \end{aligned}$$

(iii) By (ii), $A \setminus (A \setminus B) = (A \setminus A) \cup (A \cap B) = A \cap B$. □

Exercise 1.4.4

For any set A, prove that a "complement" of A ($\{x \mid x \notin A\}$) does not exist.

Proof. Let B be the complement of A for the sake of contradiction. Then, $A \cup B$ is the set of all sets, which is impossible by Exercise 1.3.3. □

Chapter 2

Relations, Function, and Ordering

2.1 Ordered Pairs

Definition 2.1.1: Ordered Pair

$(a,b) \triangleq \{\{a\},\{a,b\}\}$

Theorem 2.1.2

$(a,b)=(a',b') \iff a=a' \land b=b'$

Proof. ($\Leftarrow$) is direct.

($\Rightarrow$) If $a=b$, we have $\{\{a\}\}=\{\{a'\},\{a',b'\}\}$, and thus $\{a\}=\{a'\}=\{a',b'\}$, leaving the only option $a=a'=b'$.

If $a \neq b$, we must have $a' \neq b'$ by the argument above. Hence, we have $\{\{a\},\{a,b\}\} = \{\{a'\},\{a',b'\}\}$, which implies $\{a\}=\{a'\}$ and $\{a,b\}=\{a',b'\}$. □

Definition 2.1.3: Ordered Triples and Quadruples

- $(a,b,c)=((a,b),c)$
- $(a,b,c,d)=((a,b,c),d)$

Selected Problems

Exercise 2.1.1

If $a,b\in A$, then $(a,b)\in\mathcal{P}(\mathcal{P}(A))$.

Proof. If $a,b\in A$, then $\{a\},\{a,b\}\in\mathcal{P}(A)$, and thus $(a,b)=\{\{a\},\{a,b\}\}\subseteq\mathcal{P}(A)$. □

2.2 Relations

Definition 2.2.1: Binary Relation

A set R is a *binary relation* if all elements of R are ordered pairs.

$$R \text{ is a binary relation} \iff (a\in R \implies \exists x,\, \exists y,\, a=(x,y))$$

Notation 2.2.2

If $(x, y) \in R$, we write xRy and say *x is in relation R with y*.

Definition 2.2.3: Domain, Range, and Field of Binary Relation

Let R be a binary relation.
- $\operatorname{dom} R \triangleq \{\, x \mid \exists y\ xRy \,\}$ is called the *domain* of R.
- $\operatorname{ran} R \triangleq \{\, y \mid \exists x\ xRy \,\}$ is called the *range* of R.
- $\operatorname{field} R \triangleq \operatorname{dom} R \cup \operatorname{ran} R$ is called the *field* of R.
- If $\operatorname{field} R \subseteq X$, we say that R is a *relation in X* or that R is a relation *between* elements of X.

Lemma 2.2.4

Let R be a binary relation. Then, $\operatorname{dom} R$ and $\operatorname{ran} R$ exist.

Proof. By Exercise 2.2.1, if xRy, then $x, y \in A \triangleq \bigcup\left(\bigcup R\right)$. Hence, $\operatorname{dom} R$ and $\operatorname{ran} R$ exist. □

Definition 2.2.5: Image and Inverse Image

Let R be a binary relation and A be a set.
- $R[A] \triangleq \{\, y \in \operatorname{ran} R \mid \exists x \in A,\ xRy \,\}$ is called the *image* of A under R.
- $R^{-1}[A] \triangleq \{\, x \in \operatorname{dom} R \mid \exists y \in A,\ xRy \,\}$ is called the *inverse image* of A under R.

Notation 2.2.6

We write $\{\,(x, y) \mid \mathbf{P}(x, y)\,\}$ instead of $\{\, w \mid \exists x,\ \exists y,\ w = (x, y) \land \mathbf{P}(x, y)\,\}$.

Definition 2.2.7: Inverse Relation

Let R be a binary relation. The *inverse* of R is the set

$$R^{-1} \triangleq \{\,(x, y) \mid yRx\,\}.$$

Definition 2.2.8: Composition

Let R and S be binary relations. The relation

$$S \circ R \triangleq \{(x, z) \mid \exists y,\ xRy \land ySz\}$$

is called the *composition* of R and S.

Definition 2.2.9: Membership Relation and Identity Relation

Let A be a set.

- The *membership relation on A* is defined by

$$\in_A \triangleq \{(a,b) \mid a,b \in A \wedge a \in b\}.$$

- The *identity relation on A* is defined by

$$\mathrm{Id}_A \triangleq \{(a,a) \mid a \in A\}.$$

Definition 2.2.10: Cartesian Product

Let A and B be sets. The set $A \times B \triangleq \{(a,b) \mid a \in A \wedge b \in B\}$ is called the *Cartesian product* product of A and B.

Lemma 2.2.11

Let A and B be sets. $A \times B$ exists.

Proof. If $a \in A$ and $b \in B$, by Exercise 2.1.1, we have $(a,b) \in \mathcal{P}(\mathcal{P}(A \cup B))$. □

Corollary 2.2.12

Let R and S be binary relations and A be a set. Then, R^{-1}, $S \circ R$, $\in_A$, and Id_A exist.

Proof.

- If yRx, then $(x,y) \in (\operatorname{ran} R) \times (\operatorname{dom} R)$.
- If $(x,z) \in S \circ R$, then $(x,z) \in (\operatorname{dom} R) \times (\operatorname{ran} S)$.
- If $a,b \in A$, then $(a,b) \in A \times A$.
- If $a \in A$, then $(a,a) \in A \times A$. □

Lemma 2.2.13

Let R be a binary relation. The inverse image of A under R is equal to the image of A under R^{-1}.

Proof. Note that $\operatorname{dom} R = \{x \mid \exists y\ xRy\} = \{x \mid \exists y\ yR^{-1}x\} = \operatorname{ran} R^{-1}$. Therefore,

$$\begin{aligned}
& x \in (\text{the inverse image of } A \text{ under } R) \\
\iff & x \in \operatorname{dom} R \wedge \exists y \in A,\ xRy \\
\iff & x \in \operatorname{ran} R^{-1} \wedge \exists y \in A,\ yR^{-1}x \\
\iff & x \in (\text{the image of } A \text{ under } R^{-1}).
\end{aligned}$$

□

Note:-

Lemma 2.2.13 resolves the possible ambiguity on the expression $R^{-1}[A]$.

Notation 2.2.14

We write A^2 instead of $A \times A$.

Selected Problems

Exercise 2.2.1

Let R be a binary relation. Let $A = \bigcup\left(\bigcup R\right)$. Prove that $(x, y) \in R$ implies $x \in A$ and $y \in A$.

Proof. If $(x, y) = \{\{x\}, \{x, y\}\} \in R$, Then $\{x, y\} \in \bigcup R$, and thus $x, y \in A$. $\square$

Exercise 2.2.3

Let R be a binary relation and A and B be sets. Prove:

(i) $R[A \cup B] = R[A] \cup R[B]$.
(ii) $R[A \cap B] \subseteq R[A] \cap R[B]$.
(iii) $R[A \setminus B] \supseteq R[A] \setminus R[B]$.
(iv) Show by an example that $\subseteq$ and $\supseteq$ in parts (ii) and (iii) cannot be replaced by $=$.
(v) $R^{-1}[R[A]] \supseteq A \cap \operatorname{dom} R$ and $R[R^{-1}[B]] \supseteq B \cap \operatorname{ran} R$. Give examples where equality does not hold.

Proof.

(i)
$$\begin{aligned} y \in R[A \cup B] &\iff \exists x,\ x \in A \cup B \land xRy \\ &\iff \exists x,\ (x \in A \land xRy) \lor (x \in B \land xRy) \\ &\iff y \in R[A] \lor y \in R[B] \iff y \in R[A] \cup R[B] \end{aligned}$$
(ii) Take any $y \in R[A \cap B]$. Then, there exists $x \in A \cap B$ such that xRy. Hence, $y \in R[A]$ and $y \in R[B]$.
(iii) Take any $y \in R[A] \setminus R[B]$. Then, there exists $x \in A$ such that xRy. If $x \in B$, it implies that $y \in R[B]$, which is a contradiction. Hence, $x \in A \setminus B$. Therefore, $y \in R[A \setminus B]$.
(iv) Let a, b, and c be mutually different sets. Let $R = \{(a, a), (b, a), (c, c)\}$. Let $A = \{a, c\}$ and $B = \{b, c\}$. Then, $R[A \cap B] = \{c\} \subsetneq R[A] \cap R[B] = \{a, c\}$, and $R[A] \setminus R[B] = \varnothing \subsetneq R[A \setminus B] = \{a\}$.
(v) Take any $a \in A \cap \operatorname{dom} R$. Then, there exists b such that aRb. Moreover, $b \in R[A]$. Since $bR^{-1}a$, we conclude that $a \in R^{-1}[R[A]]$.
Take any $b \in B \cap \operatorname{ran} R$. Then, there exists a such that aRb. Moreover, $a \in R^{-1}[B]$. Hence, $b \in R[R^{-1}[B]]$.

Exercise 2.2.4

Let $R \subseteq X \times Y$. Prove:

(i) $R[X] = \operatorname{ran} R$ and $R^{-1}[Y] = \operatorname{dom} R$.
(ii) $\operatorname{dom} R = \operatorname{ran} R^{-1}$ and $\operatorname{ran} R = \operatorname{dom} R^{-1}$.
(iii) $(R^{-1})^{-1} = R$.
(iv) $R^{-1} \circ R \supseteq \operatorname{Id}_{\operatorname{dom} R}$ and $R \circ R^{-1} \supseteq \operatorname{Id}_{\operatorname{ran} R}$

Proof.

(i) We already have $R[X] \subseteq \operatorname{ran} R$ by definition. Take any $y \in \operatorname{ran} R$. There exists x such that $(x, y) \in R$. Since $R \subseteq X \times Y$, $x \in X$. Therefore, $y \in R[X]$; $\operatorname{ran} R \subseteq R[X]$. A similar argument goes for $R^{-1}[Y]$.
(ii) See the proof of Lemma 2.2.13.
(iii) For any relation R and for all x and y, we have $xRy \iff yR^{-1}x$. Since R^{-1} is also a relation, we have $xRy \iff yR^{-1}x \iff x(R^{-1})^{-1}y$.
(iv) Take any $x \in \operatorname{dom} R$. Then, there exists y such that xRy. Then, $yR^{-1}x$, and thus $x(R^{-1} \circ R)x$. A similar argument goes for $R \circ R^{-1}$. $\square$

Exercise 2.2.8
$A\times B=\varnothing$ if and only if $A=\varnothing$ or $B=\varnothing$.

Proof. ($\Rightarrow$) If $A\neq\varnothing$ and $B\neq\varnothing$, we have $(a,b)\in A\times B$ where $a\in A$ and $b\in B$, and thus $A\times B\neq\varnothing$.

($\Leftarrow$) If $A\times B\neq\varnothing$, then $a\in A$ and $b\in B$ where $(a,b)\in A\times B$. □

2.3 Functions

Definition 2.3.1: Function

A binary relation F is called a *function* (or *mapping*) if

$$\forall a\,\forall b_1\,\forall b_2\,(aFb_1\wedge aFb_2 \implies b_1=b_2).$$

For each $a\in\operatorname{dom}F$, the unique b such that aFb is called the *value of F at a* and is denoted $F(a)$ of F_a.

Notation 2.3.2

If F is a function with $\operatorname{dom}F=A$ and $\operatorname{ran}F\subseteq B$, we write $F:A\to B$, $\langle F(a)\mid a\in A\rangle$, $\langle F_a\mid a\in A\rangle$, $\langle F_a\rangle_{a\in A}$ for the function F. The range of the function F can then be denoted $\{F(a)\mid a\in A\}$ or $\{F_a\}_{a\in A}$.

Lemma 2.3.3
Let F and G be functions. $F=G \iff \operatorname{dom}F=\operatorname{dom}G\wedge\forall x\in\operatorname{dom}F,\ F(x)=G(x)$.

Proof. ($\Rightarrow$) is direct.

($\Leftarrow$) Take any $(x,F(x))\in F$. Then, we have $(x,F(x))=(x,G(x))\in G$. Therefore, $F\subseteq G$. Similarly, $G\subseteq F$, and thus $F=G$. □

Definition 2.3.4

Let F be a function and A and B be sets.

- F is a function *on* A if $\operatorname{dom}F=A$.
- F is a function *into* B if $\operatorname{ran}F\subseteq B$.
- F is a function *onto* B if $\operatorname{ran}F=B$.
- The *restriction* of the function F *to* A is the function $F\big|_A\triangleq\{(a,b)\in F\mid a\in A\}$. If G is a restriction of F to some A, we say that F is an *extension* of G.

Theorem 2.3.5
Let f and g be functions.
(i) $g\circ f$ is a function.
(ii) $\operatorname{dom}(g\circ f)=(\operatorname{dom}f)\cap f^{-1}[\operatorname{dom}g]$.
(iii) $\forall x\in\operatorname{dom}(g\circ f)$, $(g\circ f)(x)=g(f(x))$.

Proof.

(i) Suppose $x(g \circ f)z_1$ and $x(g \circ f)z_2$. There exists y_1 and y_2 such that xfy_1, y_1gz_1, xfy_2, and y_2gz_2. Since f and g are functions, we have $y_1 = y_2$ and $z_1 = z_2$. Therefore, $g \circ f$ is a function.

(ii) $$\begin{aligned} x \in \operatorname{dom}(g \circ f) &\iff \exists z\, x(g \circ f)z \\ &\iff \exists z\, \exists y\, xfy \wedge ygz \\ &\iff x \in \operatorname{dom} f \wedge f(x) \in \operatorname{dom} g \iff x \in \operatorname{dom} f \wedge x \in f^{-1}[\operatorname{dom} g] \quad \square \end{aligned}$$

Definition 2.3.6: Invertible Function

A function f is said to be *invertible* if f^{-1} is a function.

Definition 2.3.7: Injective Function

A function f is said to be *injective* (or *one-to-one*) if

$$\forall a_1, a_2 \in \operatorname{dom} f, (f(a_1) = f(a_2) \implies a_1 = a_2).$$

Notation 2.3.8

Let f be a function.

- If f is a function on A *onto* B, we may write $f : A \twoheadrightarrow B$.
- If f is an *injective* function on A into B, we may write $f : A \hookrightarrow B$.
- If f is an *injective* function on A onto B, we may write $f : A \hookrightarrow\!\!\!\!\twoheadrightarrow B$.
- If f is a function on a *subset* of A into B, we may write $f : A \rightharpoonup B$.

Theorem 2.3.9

Let f be a function.

(i) f is invertible if and only if f is one-to-one.

(ii) If f is invertible, then f^{-1} is also invertible and $(f^{-1})^{-1} = f$.

Proof.

(i) ($\Rightarrow$) Suppose f^{-1} is a function. Then, $f^{-1}(f(a)) = a$ for all $a \in \operatorname{dom} f$. Hence, for all $a_1, a_2 \in \operatorname{dom} f$ such that $f(a_1) = f(a_2)$, it follows that $a_1 = f^{-1}(f(a_1)) = f^{-1}(f(a_2)) = a_2$; f is one-to-one.

($\Leftarrow$) Suppose f is one-to-one. If $yf^{-1}x_1$ and $yf^{-1}x_2$, then x_1fy and x_2fy, i.e., $y = f(x_1) = f(x_2)$. Therefore, $x_1 = x_2$; f^{-1} is a function.

(ii) As f is a relation, by Exercise 2.2.4 (iii), $(f^{-1})^{-1} = f$, and thus f^{-1} is invertible. $\square$

Definition 2.3.10: Compatible Functions

- Functions f and g are called *compatible* if $\forall x \in (\operatorname{dom} f) \cap (\operatorname{dom} g)$, $f(x) = g(x)$.
- A set of functions F is called a *compatible system of functions* if any two functions f and g from F are compatible.

Lemma 2.3.11

Let f and g be functions.

(i) f and g are compatible if and only if $f \cup g$ is a function.

(ii) f and g are compatible if and only if $f\big|_{(\operatorname{dom} f)\cap(\operatorname{dom} g)} = g\big|_{(\operatorname{dom} f)\cap(\operatorname{dom} g)}$.

Proof.

(i) ($\Rightarrow$) Suppose $x(f\cup g)y_1$ and $x(f\cup g)y_2$. WLOG, $(x,y_1)\in f$. If $(x,y_2)\in f$, since f is a function, $y_1=y_2$. If $(x,y_2)\in g$, since f and g are compatible, $y_1=f(x)=g(x)=y_2$. Therefore, $f\cup g$ is a function.
($\Leftarrow$) Take any $x\in(\operatorname{dom} f)\cap(\operatorname{dom} g)$. $(x,f(x))\in f\cup g$ and $(x,g(x))\in f\cup g$. Since $f\cup g$ is a function, we have $f(x)=g(x)$.

(ii) Let $A=(\operatorname{dom} f)\cap(\operatorname{dom} g)$.
($\Rightarrow$) By definition, $\operatorname{dom} f\big|_A=\operatorname{dom} g\big|_A=(\operatorname{dom} f)\cap(\operatorname{dom} g)$. Moreover, for all $x\in(\operatorname{dom} f)\cap(\operatorname{dom} g)$, $f\big|_A(x)=f(x)=g(x)=g\big|_A(x)$. Hence, the result follows by Lemma 2.3.3.
($\Leftarrow$) Take any $x\in A$. Then, $f(x)=f\big|_A(x)=g\big|_A(x)=g(x)$. □

Theorem 2.3.12

If F is a compatible system of functions, then $\bigcup F$ is a function with $\operatorname{dom}\bigcup F=\bigcup\{\operatorname{dom} f\mid f\in F\}$. The function $\bigcup F$ extends all $f\in F$.

Proof. Note that $\bigcup F$ is already a relation. If $(a,b_1),(a,b_2)\in\bigcup F$, then there exist $f_1,f_2\in F$ such that $(a,b_1)\in f_1$ and $(a,b_2)\in f_2$. Since f_1 and f_2 are compatible and $a\in(\operatorname{dom} f_1)\cap(\operatorname{dom} f_2)$, we have $b_1=f_1(a)=f_2(a)=b_2$. Hence, $\bigcup F$ is a function.

$\operatorname{dom}\bigcup F=\bigcup\{\operatorname{dom} f\mid f\in F\}$ since

$$\begin{aligned}x\in\operatorname{dom}\bigcup F&\iff\exists y,\ (x,y)\in\bigcup F\\&\iff\exists y,\ \exists f\in F,\ (x,y)\in f\\&\iff\exists f\in F,\ x\in\operatorname{dom} f\iff x\in\bigcup\{\operatorname{dom} f\mid f\in F\}.\end{aligned}$$

Take any $f\in F$. As $f\cup\bigcup F=\bigcup F$, f and $\bigcup F$ are compatible by Lemma 2.3.11 (i). Moreover, $\operatorname{dom} f\cap\operatorname{dom}\bigcup F=\operatorname{dom} f$. Hence, by Lemma 2.3.11 (ii), $f=f\big|_{\operatorname{dom} f}=\left(\bigcup F\right)\big|_{\operatorname{dom} f}$; $\bigcup F$ extends each $f\in F$. □

Definition 2.3.13

Let A and B be sets. Then, we define

$$B^A\triangleq\{f\mid f\text{ is a function on }A\text{ into }B\}.$$

Definition 2.3.14: Indexed System of Sets

- Let $S=\langle S_i\mid i\in I\rangle$ be a function with domain I. We call the function S an *indexed system of sets* whenever we stress that the values of S are sets.
- We say that a system of sets A is *indexed* by S if $A=\{S_i\mid i\in I\}=\operatorname{ran} S$.

Notation 2.3.15

If A is indexed by $S=\langle S_i\mid i\in I\rangle$, we may write

$$\bigcup\{S_i\mid i\in I\}\quad\text{or}\quad\bigcup_{i\in I}S_i$$

instead of $\bigcup A$. Similarly, we may write $\bigcap\{S_i\mid i\in I\}$ or $\bigcap_{i\in I}S_i$ instead of $\bigcap A$.

Definition 2.3.16: Product of Indexed System of Sets

Let $S = \langle S_i \mid i \in I \rangle$ be an indexed system of sets. We call the set

$$\prod S \triangleq \{ f \mid f \text{ is a function on } I \text{ and } \forall i \in I,\ f_i \in S_i \}$$

the *product* of the indexed system S.

Notation 2.3.17

Other notations for the product of the indexed system $S = \langle S_i \mid i \in I \rangle$ are:

$$\prod \langle S(i) \mid i \in I \rangle, \quad \prod_{i \in I} S(i), \quad \prod_{i \in I} S_i.$$

Note:-

The existence of B^A and $\prod_{i \in I} S_i$ is proved in Exercise 2.3.9.

Note:-

If $A = S_i$ for all $i \in I$, $\prod_{i \in I} S_i = A^I$.

Selected Problems

Exercise 2.3.4

Let f be a function. If there exists a function g such that $g \circ f = \mathrm{Id}_{\mathrm{dom}\, f}$, then f is invertible and $f^{-1} = g\big|_{\mathrm{ran}\, f}$.

Proof. For $x_1, x_2 \in \mathrm{dom}\, f$, suppose $f(x_1) = f(x_2)$. Then, $x_1 = (g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = (g \circ f)(x_2) = x_2$. Hence, f is one-to-one and is inverible by Theorem 2.3.9.

Take any $(y, x) \in f^{-1}$. Then, as $x \in \mathrm{dom}\, f$, we must have $(y, x) \in \mathrm{Id}_{\mathrm{dom}\, f}$. Hence, $f^{-1} \subseteq g\big|_{\mathrm{ran}\, f}$. Now, take any $(y, x) \in g\big|_{\mathrm{ran}\, f}$. Since $y \in \mathrm{ran}\, f$, there exists $x' \in \mathrm{dom}\, f$ such that $(x', y) \in f$. Since $g \circ f = \mathrm{Id}_{\mathrm{dom}\, f}$, we have $x = x'$. Therefore, $(y, x) \in f^{-1}$; $g\big|_{\mathrm{ran}\, f} \subseteq f^{-1}$. □

Exercise 2.3.6

Let f be a function.

(i) $f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B]$

(ii) $f^{-1}[A \setminus B] = f^{-1}[A] \setminus f^{-1}[B]$

Proof. Thanks to Exercise 2.2.3 (ii) and (iii), we only need to prove the other inclusions.

(i) Take any $x \in f^{-1}[A] \cap f^{-1}[B]$. Then, there exists $a \in A$ and $b \in B$ such that $x f a$ and $x f b$. Since f is a function, $a = b$, and thus $x \in f^{-1}[A \cap B]$.

(ii) Take any $x \in f^{-1}[A \setminus B]$. Then, $f(x) \in A \setminus B$. If $x \in f^{-1}[B]$, we would have $f(x) \in B$; thus $x \notin f^{-1}[B]$. Therefore, $x \in f^{-1}[A] \setminus f^{-1}[B]$. □

Exercise 2.3.8

Every system of sets A can be indexed by a function.

Proof. Let S be the function Id_A so $S_i = i$ for all $i \in A$. Then, $A = \{S_i \mid i \in A\}$; A is indexed by S. $\square$

Exercise 2.3.9

(i) Let A and B be sets. Prove that B^A exists.

(ii) Let $\langle S_i \mid i \in I\rangle$ be an indexed system of sets. Prove that $\prod_{i\in I} S_i$ exists.

Proof.

(i) If f is a function from A into B, then $f \subseteq A \times B$, i.e., $f \in \mathcal{P}(A\times B)$.

(ii) If f is a function on I and $f_i \in S_i$ for all $i \in I$, then f is a function onto $\bigcup_{i\in I} S_i$. Hence, $f \in \left(\bigcup_{i\in I} S_i\right)^I$. $\square$

Exercise 2.3.10

Let $\langle F_a \mid a \in \bigcup S\rangle$ be an indexed system of sets.

(i) $\bigcup_{a\in\bigcup S} F_a = \bigcup_{C\in S}\left[\bigcup_{a\in C} F_a\right]$

(ii) $\bigcap_{a\in\bigcup S} F_a = \bigcap_{C\in S}\left[\bigcap_{a\in C} F_a\right]$ if $S \neq \varnothing$ and $\forall C \in S,\ C \neq \varnothing$.

Proof.

(i)
$$\begin{aligned} x \in \bigcup_{a\in\bigcup S} F_a &\iff \exists a \in \bigcup S,\ x \in F_a \\ &\iff \exists C \in S,\ \exists a \in C,\ x \in F_a \\ &\iff \exists C \in S,\ x \in \bigcup_{a\in C} F_a \iff x \in \bigcup_{C\in S}\left[\bigcup_{a\in C} F_a\right] \end{aligned}$$

(ii)
$$\begin{aligned} x \in \bigcap_{a\in\bigcup S} F_a &\iff \forall a \in \bigcup S,\ x \in F_a \\ &\iff \forall C \in S,\ \forall a \in C,\ x \in F_a \\ &\iff \forall C \in S,\ x \in \bigcap_{a\in C} F_a \iff x \in \bigcap_{C\in S}\left[\bigcap_{a\in C} F_a\right] \end{aligned}$$
$\square$

Exercise 2.3.11

Let $\langle F_a \mid a \in A\rangle$ be an nonempty indexed system of sets.

(i) $B \setminus \bigcup_{a\in A} F_a = \bigcap_{a\in A}(B \setminus F_a)$

(ii) $B \setminus \bigcap_{a\in A} F_a = \bigcup_{a\in A}(B \setminus F_a)$

Proof.

(i)
$$\begin{aligned} x \in B \setminus \bigcup_{a\in A} F_a &\iff x \in B \wedge \neg(\exists a \in A,\ x \in F_a) \\ &\iff x \in B \wedge \forall a \in A,\ x \notin F_a \\ &\iff \forall a \in A,\ (x \in B \wedge x \notin F_a) \iff x \in \bigcap_{a\in A}(B \setminus F_a) \end{aligned}$$

(ii)
$$\begin{aligned} x \in B \setminus \bigcap_{a\in A} F_a &\iff x \in B \wedge \neg(\forall a \in A,\ x \in F_a) \\ &\iff x \in B \wedge \exists a \in A,\ x \notin F_a \\ &\iff \exists a \in A,\ (x \in B \wedge x \notin F_a) \iff x \in \bigcup_{a\in A}(B \setminus F_a) \end{aligned}$$
$\square$

Exercise 2.3.12

Let R be a relation and let $\langle F_a \mid a \in A\rangle$ be an indexed system of sets.

(i) $R\left[\bigcup_{a\in A} F_a\right] = \bigcup_{a\in A} R[F_a]$

(ii) $R\left[\bigcap_{a\in A} F_a\right] \subseteq \bigcap_{a\in A} R[F_a]$ if $A \neq \varnothing$.

(iii) $R\left[\bigcap_{a\in A} F_a\right] = \bigcap_{a\in A} R[F_a]$ if $A \neq \varnothing$ and R is an injective function.

(iv) $R^{-1}\left[\bigcap_{a\in A} F_a\right] = \bigcap_{a\in A} R^{-1}[F_a]$ if $A \neq \varnothing$ and R is a function.

Proof.

(i) $$y \in R\left[\bigcup_{a\in A} F_a\right] \iff \exists x \in \bigcup_{a\in A} F_a,\ xRy \iff \exists a \in A,\ \exists x \in F_a,\ xRy \iff \exists a \in A,\ y \in R[F_a] \iff x \in \bigcup_{a\in A} R[F_a]$$

(ii) Take any $y \in R\left[\bigcap_{a\in A} F_a\right]$. Then, there exists $x \in \bigcap_{a\in A} F_a$ such that xRy. Hence, for all $a \in A$, $y \in R[F_a]$, i.e., $y \in \bigcap_{a\in A} R[F_a]$.

(iii) If R is an injective function, then R^{-1} is also a function. Hence, the result follows from (iv) and the fact that $R = (R^{-1})^{-1}$.

(iv) Thanks to (ii), since R^{-1} is a relation, we only need to prove the other inclusion. Take any $x \in \bigcap_{a\in A} R^{-1}[F_a]$. Fix any $a^* \in A$. Then, there exists $y^* \in F_{a^*}$ such that xRy^*.

Now, take any $a \in A$. Then, $\exists y \in F_a$ such that xRy. Since R is a function, $y = y^*$; $y^* \in F_a$, i.e., $y^* \in \bigcap_{a\in A} F_a$. Therefore, $x \in R^{-1}\left[\bigcap_{a\in A} F_a\right]$. □

2.4 Equivalences and Partitions

Definition 2.4.1: Equivalence

Let R be a binary relation in A.

- R is called *reflexive in A* if $\forall a \in A,\ aRa$.
- R is called *symmetric in A* if $\forall a, b \in A,\ (aRb \implies bRa)$.
- R is called *transitive in A* if $\forall a, b, c \in A,\ (aRb \wedge bRc \implies aRc)$.
- R is called an *equivalence on A* if it is reflexive, symmetric, and transitive in A.

Definition 2.4.2: Equivalence Class

Let E be an equivalence on A and let $a \in A$. The *equivalence class of a modulo E* is the set

$$[a]_E \triangleq \{\, x \in A \mid xEa \,\}.$$

Lemma 2.4.3

Let E be an equivalence on A and let $a, b \in A$.

(i) $aEb \iff [a]_E = [b]_E$

(ii) $\neg(aEb) \iff [a]_E \cap [b]_E = \varnothing$

Proof.

(i) ($\Rightarrow$) Suppose aEb. Take any $c \in [a]_E$. Then, cEa and aEb, and thus cEb by transitivity. Hence, $c \in [b]_E$; $[a]_E \subseteq [b]_E$. $[b]_E \subseteq [a]_E$ can be shown similarly since bEa holds as E is symmetric.

($\Leftarrow$) Suppose $[a]_E = [b]_E$. Since aEa by reflexivity, we have $a \in [a]_E = [b]_E$. Therefore, aEb.

(ii) ($\Rightarrow$) Suppose $[a]_E \cap [b]_E \neq \varnothing$. Then, there exists $c \in [a]_E \cap [b]_E$, i.e., cEa and cEb. Then, as E is symmetric, we have aEc, and therefore aEb by transitivity.

($\Leftarrow$) Suppose aEb. Then, since aEa by reflexivity, we have $a \in [a]_E$. We can see $a \in [b]_E$ from (i). Hence, $[a]_E \cap [b]_E \neq \varnothing$. □

Definition 2.4.4: Partition

A system S of nonempty sets is called a *partition* of A if
(i) S is a system of mutually disjoint sets (Definition 1.4.5) and
(ii) $\bigcup S = A$.

Definition 2.4.5: System of All Equivalence Classes

Let E be an equivalence on A. The *system of all equivalence classes* modulo E is the set

$$A/E \triangleq \{[a]_E \mid a \in A\}.$$

Theorem 2.4.6

Let E be an equivalence on A. Then, A/E is a partition of A.

Proof. If $[a]_E \neq [b]_E$, then by Lemma 2.4.3, we have $[a]_E \cap [b]_E = \varnothing$. Since E is reflexive, $a \in [a]_E$; each $[a]_E$ is nonempty. Therefore, A/E is a system of mutually disjoint nonempty sets.

Take any $a \in A$. Since E is reflexive, $a \in [a]_E \subseteq \bigcup A/E$. Therefore, $A \subseteq \bigcup A/E$. Conversely, since $[a]_E \subseteq A$, we have $\bigcup A/E \subseteq A$. □

Definition 2.4.7

Let S be a partition of A. The relation E_S in A is defined by

$$E_S \triangleq \{(a,b) \in A \times A \mid \exists C \in S,\ a \in C \wedge b \in C\}.$$

Theorem 2.4.8

Let S be a partition of A. Then, E_S is a equivalence on A.

Proof.

- Take any $a \in A$. As $A = \bigcup S$, there exists $C \in S$ such that $a \in C$. Therefore, $aE_S a$. E_S is reflexive.
- Assume $aE_S b$. Then, there exists $C \in S$ such that $a, b \in C$. Hence, $bE_S a$. E_S is symmetric.
- Assume $aE_S b$ and $bE_S c$. Then, there exist $C, D \in S$ such that $a, b \in C$ and $b, c \in D$. Then, $C \cap D \neq \varnothing$ as b belongs to both sets. Hence, $C = D$, which implies $aE_S c$. E_S is transitive. □

Theorem 2.4.9

(i) If E is an equivalence on A and $S = A/E$, then $E_S = E$.
(ii) If S is a partition of A, then $A/E_S = S$.

Proof.

(i) $aE_S b \underbrace{\iff}_{\text{Definition 2.4.7}} \exists C \in S,\ a \in C \wedge b \in C \iff \exists c \in A,\ a \in [c]_E \wedge b \in [c]_E \underbrace{\iff}_{\text{Lemma 2.4.3}} aEb.$

(ii) Take any $[a]_{E_S} \in A/E_S$. Since S is a partition, there (uniquely) exists C such that $a \in C$. Then, for all b, we have $b \in C \iff aE_S b \underbrace{\iff}_{\text{Lemma 2.4.3}} b \in [a]_{E_S}$; $C = [a]_{E_S}$. Therefore, $A/E_S \subseteq S$.

For the converse, take any $C \in S$. As C is nonempty, we may take some $a \in C$. Similarly, we have $C = [a]_{E_S}$. Therefore, $C \subseteq A/E_S$. □

Note:-

Theorem 2.4.9 essentially states that equivalence and partition describe the same "mathematical reality."

Definition 2.4.10: Set of Representatives

A set $X \subseteq A$ is called a *set of representatives* for the equivalence E_S (or for the partition S of A) if
$$\forall C \in S, \exists a \in C, X \cap C = \{a\}.$$

Selected Problems

Exercise 2.4.2

Let f be a function on A onto B. Define a relation E in A by: aEb if and only if $f(a) = f(b)$.

(i) Show that E is an equivalence on A.

(ii) Show that $[a]_E = [a']_E$ implies that $f(a) = f(a')$ so that the function φ on A/E into B defined by $\varphi([a]_E) = f(a)$ is well-defined. Show also that φ is *onto* B.

(iii) Let j be the function on A onto A/E given by $j(a) = [a]_E$. Show that $\varphi \circ j = f$.

Proof.

(i) E can readily be shown to be reflexive, symmetric, and transitive.

(ii) Assume $[a]_E = [a']_E$. Then, $f(a) = f(a')$ by definition of E. Hence, φ is well-defined. Take any $b \in B$. Since f is onto, there exists $a \in A$ such that $f(a) = b$. Hence, $\varphi([a]_E) = f(a) = b$; φ is onto B.

(iii) $\operatorname{dom}(\varphi \circ j) = (\operatorname{dom} j) \cap j^{-1}[\operatorname{dom} \varphi] = A = \operatorname{dom} f$ since j is onto. For all $a \in A$, $(\varphi \circ j)(a) = \varphi([a]_E) = f(a)$. Hence, by Lemma 2.3.3, $\varphi \circ j = f$. □

2.5 Orderings

Definition 2.5.1: Partial Ordering and Strict Ordering

Let R be a binary relation in A.

- R is called *antisymmetric in A* if $\forall a, b \in A$, $(aRb \wedge bRa \implies a = b)$.
- R is called *asymmetric in A* if $\forall a, b \in A$, $\neg(aRb \wedge bRa)$.
- R is called a *(partial) ordering* of A if it is reflexive, antisymmetric, and transitive in A.
- R is called a *strict ordering* of A if it is asymmetric and transitive in A.
- If R is a partial ordering of A, then the pair (A, R) is called an *ordered set*.

Example 2.5.2

- Define the relation $\subseteq_A$ in A as follows: $x \subseteq_A y$ if and only if $x, y \in A \wedge x \subseteq y$. Then, $(A, \subseteq_A)$ is an ordered set.
- The relation Id_A is a partial ordering of A.

Theorem 2.5.3

(i) Let R be a partial ordering of A. Then the relation S in A defined by
$$S \triangleq R \setminus \mathrm{Id}_A$$
is a strict ordering.

(ii) Let S be a strict ordering of A. Then the relation R in A defined by
$$R \triangleq S \cup \mathrm{Id}_A$$
is a partial ordering.

Proof.

(i) Suppose aSb and bSa. Since $S \subseteq R$, we have aRb and bRa. As R is antisymmetric, we have aRa, which is impossible since $S \cap \mathrm{Id}_S = \varnothing$. Hence, S is asymmetric in A.
Now, assuming aSb and bSc, we also have aRc since R is transitive. Moreover, a cannot be equal to c since S is shown to be asymmetric. Therefore, aSc; S is transitive in A.

(ii) Assume aRb and bRa. If $a \neq b$, then we have aSb and bSa, which is impossible. Therefore, $a = b$; R is antisymmetric.
Assume aRb and bRc. If $a = b$ or $b = c$, then we immediately have aRc. If $a \neq b$ and $b \neq c$, then aSb and bSc, and thus aSc as S is transitive in A; R is transitive in A).
R is reflexive in A since $\mathrm{Id}_A \subseteq R$. □

Notation 2.5.4

- If R is a partial ordering, we say $S = R \setminus \mathrm{Id}_A$ *corresponds to the partial ordering* R.
- If S is a strict ordering, we say $R = S \cup \mathrm{Id}_A$ *corresponds to the strict ordering* S.

Definition 2.5.5: Comparability

Let $a, b \in A$ and let $\leq$ be a partial ordering of A.
- We say that a and b are *comparable* in the ordering $\leq$ if $a \leq b$ or $b \leq a$.
- We say that a and b are *incomparable* in the ordering $\leq$ if neither $a \leq b$ nor $b \leq a$.

They can be stated equivalently in terms of the corresponding strict ordering $<$.
- We say that a and b are *comparable* in the ordering $<$ if $a = b$ or $a < b$ or $b < a$.
- We say that a and b are *incomparable* in the ordering $<$ if none of $a = b$, $a < b$, and $b < a$ holds.

Definition 2.5.6: Total Ordering

An ordering $\leq$ (or $<$) is called *linear* or *total* if any two elements of A are comparable. The pair $(A, \leq)$ is then called a *totally ordered set*.

Definition 2.5.7: Chain

Let $(A, \leq)$ be an ordered set and $B \subseteq A$. B is a *chain* in A if any two elements of B are comparable.

Definition 2.5.8: Least/Minimal/Greatest/Maximal Element

Let $(A, \leq)$ be an ordered set and $B \subseteq A$.
- $b \in B$ is the *least element* of B in the ordering $\leq$ if $\forall x \in B,\ b \leq x$.
- $b \in B$ is a *minimal element* of B in the ordering $\leq$ if $\forall x \in B,\ (x \leq b \implies x = b)$.
- $b \in B$ is the *greatest element* of B in the ordering $\leq$ if $\forall x \in B,\ x \leq b$.
- $b \in B$ is a *maximal element* of B in the ordering $\leq$ if $\forall x \in B,\ (b \leq x \implies x = b)$.

Notation 2.5.9

Let $(A, \leq)$ be an ordered set and $B \subseteq A$.
- The least element of B is denoted $\min B$.
- The greatest element of B is denoted $\max B$.

Theorem 2.5.10

Let $(A, \leq)$ be an ordered set and $B \subseteq A$.
(i) B has at most one least element.
(ii) The least element of B—it it exists—is also minimal.
(iii) If B is a chain, then every minimal element of B is also least.

Proof.
(i) If b and b' are least elements of B, then $b \leq b'$ and $b' \leq b$ by the definition. As $\leq$ is antisymmetric, we have $b = b'$.
(ii) Let b be the least element of B (assuming its existence). Take any $x \in B$ and assume $x \leq b$. Then, as b is the least, we have $b \leq x$. As $\leq$ is antisymmetric, $x = b$; b is minimal.
(iii) Let b be a minimal element of B. Take any $x \in B$. Since b and x are comparable, it is $x \leq b$ or $b \leq x$. If $x \leq b$, then $x = b$ as b is minimal. Therefore, b is the least. □

Note:-
Theorem 2.5.10 still holds when 'least' and 'minimal' are replaced by 'greatest' and 'maximal', respectively.

Definition 2.5.11: Lower/Upper Bound and Infimum/Supremum

Let $(A, \leq)$ be an ordered set and $B \subseteq A$.
- $a \in A$ is a *lower bound* of B in the ordered set $(A, \leq)$ if $\forall x \in B,\ a \leq x$.
- $a \in A$ is called an *infimum* (or *greatest lower bound*) of B in the ordered set $(A, \leq)$ if $a = \max\{\, x \in A \mid x \text{ is a lower bound of } B \,\}$.
- $a \in A$ is an *upper bound* of B in the ordered set $(A, \leq)$ if $\forall x \in B,\ x \leq a$.
- $a \in A$ is called an *supremum* (or *least upper bound*) of B in the ordered set $(A, \leq)$ if $a = \min\{\, x \in A \mid x \text{ is an upper bound of } B \,\}$.

Notation 2.5.12

Let $(A, \leq)$ be an ordered set and $B \subseteq A$.
- The infimum of B is denoted $\inf B$.
- The supremum of B is denoted $\sup B$.

Theorem 2.5.13

Let $(A, \le)$ be an ordered set and $B \subseteq A$.

(i) B has at most one infimum.

(ii) If b is the least element of B, then b is the infimum of B.

(iii) If $b \in B$ is the infimum of B, then b is the least element of B.

Proof.

(i) The result follows from the definition and Theorem 2.5.10 (i).

(ii) b is a lower bound of B. If x is a lower bound of B, since $b \in B$, we must have $x \le b$. Therefore, b is the greatest lower bound.

(iii) $b \in B$ is a lower bound of B, and thus b is the least element. □

Note:-

Theorem 2.5.13 still holds when 'least' and 'infimum' are replaced by 'greatest' and 'supremum', respectively.

Definition 2.5.14: Isomorphism Between Ordered Sets

An *isomorphism* between two ordered sets $(P, \le)$ and $(Q, \preceq)$ is a function $f : P \hookrightarrow\!\!\!\!\rightarrow Q$ such that

$$\forall p_1, p_2 \in P, \ (p_1 \le p_2 \iff f(p_1) \preceq f(p_2)).$$

If an isomorphism exists between $(P, \le)$ and $(Q, \preceq)$, then we say $(P, \le)$ and $(Q, \preceq)$ are *isomorphic*. This is justified by Exercise 2.5.13.

Lemma 2.5.15

Let $(P, \le)$ be a totally ordered set and let $(Q, \preceq)$ be an ordered set. Let $h \colon P \hookrightarrow\!\!\!\!\rightarrow Q$ be a function such that

$$\forall p_1, p_2 \in P, \ (p_1 \le p_2 \implies h(p_1) \preceq h(p_2)).$$

Then, h is an isomorphism between $(P, \le)$ and $(Q, \preceq)$, and $(Q, \le)$ is totally ordered.

Proof. Take any $p_1, p_2 \in P$ and assume $h(p_1) \preceq h(p_2)$. Suppose $p_2 < p_1$ for the sake of contradiction. Then, since h is injective, $h(p_1) \ne h(p_2)$, and thus $h(p_1) \prec h(p_2)$. Then, we have $\neg(p_2 \le p_1)$, which is a contradiction. Hence, $\neg(p_2 < p_1)$. Therefore, $p_1 \le p_2$ since $(P, \le)$ is totally ordered.

Take any $q_1, q_2 \in Q$. Then, since h is onto Q, there exist $p_1, p_2 \in P$ such that $q_1 = h(p_1)$ and $p_2 = h(p_2)$. Since P is totally ordered, it is $p_1 \le p_2$ or $p_2 \le p_1$. In either case, we have $q_1 \preceq q_2$ or $p_2 \preceq q_1$. Therefore, $(Q, \preceq)$ is totally ordered. □

Selected Problems

Exercise 2.5.1

(i) Let R be a partial ordering of A and let S be the strict ordering of A corresponding to R. Let R^* be the partial ordering of A corresponding to S. Show that $R^* = R$.

(ii) Let S be a strict ordering of A and let R be the partial ordering of A corresponding to S. Let S^* be the partial ordering of A corresponding to R. Show that $S^* = S$.

Proof.

(i) $R^* = S \cup \mathrm{Id}_A = (R \setminus \mathrm{Id}_A) \cup \mathrm{Id}_A = R$ since $\mathrm{Id}_A \subseteq R$.

(ii) $S^* = R \setminus \mathrm{Id}_A = (S \cup \mathrm{Id}_A) \setminus \mathrm{Id}_A = S$ since $\mathrm{Id}_A \cap S = \varnothing$.

$\square$

Exercise 2.5.6

Let $(A_1, <_1)$ and $(A_2, <_2)$ be strictly ordered sets and let $A_1 \cap A_2 = \varnothing$. Define a relation $\prec$ on $B \triangleq A_1 \cup A_2$ as follows:

$$x \prec y \iff (x <_1 y) \vee (x <_2 y) \vee (x \in A_1 \wedge y \in A_2).$$

Show that $\prec$ is a strict ordering of B and $\prec \cap A_1^2 = <_1$, $\prec \cap A_2^2 = <_2$.

Proof. Note that $\prec \,=\, <_1 \cup <_2 \cup A_1 \times A_2$.

Suppose $x \prec y$ and $y \prec x$. By definition, $x, y \in A_1$ or $x, y \in A_2$. In both cases, we have ($x <_1 y$ and $y <_1 x$) or ($x <_2 y$ and $y <_2 x$), which are impossible as $<_1$ and $<_2$ are asymmetric. Hence, $\prec$ is asymmetric. Transitivity of $\prec$ can be shown easily.

Since $<_1 \cap A_2^2 = <_2 \cap A_1^2 = (A_1 \times A_2) \cap A_1^2 = (A_1 \times A_2) \cap A_2^2 = \varnothing$, we get $\prec \cap A_1^2 = <_1$ and $\prec \cap A_2^2 = <_2$. $\square$

Exercise 2.5.7

Let R be a reflexive and transitive relation in A (R is called a *preordering* of A). Define a relation E in A by

$$aEb \iff aRb \wedge bRa.$$

Show that E is an equivalence on A. Define the relation R/E in A/E by

$$[a]_E R/E [b]_E \iff aRb.$$

Show that R/E is well-defined and that R/E is a partial ordering of A/E.

Proof. Since $aEa \equiv aRa$ and R is reflexive, E is reflexive as well. Since $aEb \equiv bEa$, E is symmetric. Since $aEb \wedge bEc \iff (aRb \wedge bRc) \wedge (cRb \wedge bRa) \implies aRc \wedge cRa \iff aEc$, E is transitive. ✓

Assume $[a]_E = [a']_E$ and $[b]_E = [b']_E$. Then, we have aEa' and bEb' by Lemma 2.4.3, i.e., aRa', $a'Ra$, bRb', and $b'Rb$. By transitivity of R, it follows that $aRb \iff a'Rb'$. Therefore, R/E is well-defined. ✓

It can be shown readily that R/E is reflexive and transitive. To prove R/E is antisymmetric, assume $[a]_E R/E [b]_E$ and $[b]_E R/E [a]_E$. Then, aRb and bRa, which means aEb. Therefore, $[a]_E = [b]_E$ by Lemma 2.4.3. ✓ $\square$

Exercise 2.5.8

Let $A = \mathcal{P}(X)$ where X is a set.

(i) Any $S \subseteq A$ has a supremum in the ordering $\subseteq_A$; $\sup S = \bigcup S$.

(ii) Any $S \subseteq A$ has an infimum in the ordering $\subseteq_A$; $\inf S = \begin{cases} \bigcap S & \text{if } S \neq \varnothing \\ X & \text{if } S = \varnothing \end{cases}$.

Proof.

(i) As $C \subseteq_A \bigcup S$ for all $C \in S$, $\bigcup S$ is an upper bound of S. Let U be any upper bound of S. Take any $x \in \bigcup S$. Then, there exists $C \in S$ such that $x \in C$. Since $C \subseteq_A U$, we have $x \in U$. Therefore, $\bigcup S \subseteq U$; $\bigcup S$ is the least upper bound of S.

(ii) If $S=\varnothing$, then any $C\in A$ is an lower bound of S. Since $\bigcup A = X$—by (i), the supremum of the set of lower bounds of S—is a lower bound of S, X is the infimum of $S=\varnothing$. ✓
If $S\neq\varnothing$, as $\bigcap S\subseteq C$ for all $C\in S$, $\bigcap S$ is a lower bound of S. Let L be any lower bound of S. Take any $x\in L$. Then, $\forall C\in L$, $x\in C$, i.e., $x\in\bigcap S$. Therefore, $L\subseteq_A\bigcap S$; $\bigcap S$ is the infimum of S. ✓ □

Exercise 2.5.9

Let $\mathrm{Fn}(X,Y)$ be the set of all functions mapping a subset of X into Y, i.e., $\mathrm{Fn}(X,Y)=\bigcup_{Z\in\mathcal{P}(X)}Y^Z$. Define a relation $\leq$ in $\mathrm{Fn}(X,Y)$ by

$$f\leq g \iff f\subseteq g.$$

(i) $\leq$ is a partial ordering of $\mathrm{Fn}(X,Y)$.
(ii) Let $F\subseteq\mathrm{Fn}(X,Y)$. $\sup F$ exists if and only if F is a compatible system of functions. Moreover, $\sup F=\bigcup F$ if it exists.

Proof.
(i) $\leq\,=\,\subseteq_{\mathrm{Fn}(X,Y)}$ by definition; $\subseteq_{\mathrm{Fn}(X,Y)}$ is already a partial ordering of $\mathrm{Fn}(X,Y)$.
(ii) ($\Rightarrow$) Assume $h\in\mathrm{Fn}(X,Y)$ is a supremum of F. Then, $\forall f\in F$, $f\subseteq s$. Take any $f,g\in F$. Then, $f\cup g\subseteq h$, and thus $f\cup g$ is a function as h is a function. Therefore, by Lemma 2.3.11, f and g are compatible. Hence, F is a compatible system of functions.
($\Leftarrow$) Assume F is a compatible system of functions. Then, $\bigcup F\in\mathrm{Fn}(X,Y)$ by Theorem 2.3.12, and $f\leq\bigcup F$ for all $f\in F$ by definition; $\bigcup F$ is an upper bound of F. Let U be any upper bound of S. Take any $(x,y)\in\bigcup F$. Then, there exists $f\in S$ such that $(x,y)\in f$. Since $f\subseteq_A U$, we have $x\in U$. Therefore, $\bigcup F\subseteq U$; $\bigcup F$ is the least upper bound of S. □

Exercise 2.5.10

Let $\mathrm{Pt}(A)$ be the set of all partitions of A. Define a relation $\preccurlyeq$ in $\mathrm{Pt}(A)$ by

$$S_1\preccurlyeq S_2 \iff \forall C\in S_1,\ \exists D\in S_2,\ C\subseteq D.$$

(We say that the partition S_1 is a *refinement* of the partition S_2 if $S_1\preccurlyeq S_2$.)
(i) $\preccurlyeq$ is a partial ordering of $\mathrm{Pt}(A)$.
(ii) $\inf T$ exists for all $T\subseteq\mathrm{Pt}(A)$.
(iii) $\sup T$ exists for all $T\subseteq\mathrm{Pt}(A)$.

Proof.
(i) $\preccurlyeq$ is reflexive since, for all $S\in\mathrm{Pt}(A)$ and $C\in S$, $C\subseteq C$, i.e., $S\preccurlyeq S$. ✓
Assume $S_1\preccurlyeq S_2$ and $S_2\preccurlyeq S_1$. Take any $C\in S_1$. Then, there exists $D\in S_2$ such that $C\subseteq D$. In addition, there exists $E\in S_1$ such that $D\subseteq E$. We have $C\subseteq E$ but C is nonempty as S_1 is a partition, which implies $C\cap E\neq\varnothing$. Therefore, as S_1 is a partition, we must have $C=E$ and thus $C=D$. Hence, $S_1\subseteq S_2$. This shows that $\preccurlyeq$ is antisymmetric. ✓
Assume $S_1\preccurlyeq S_2$ and $S_2\preccurlyeq S_3$. Take any $C\in S_1$. There exists $D\in S_2$ such that $C\subseteq D$. There exists $E\in S_3$ such that $D\subseteq E$. Hence, $C\subseteq E$; $S_1\preceq S_3$. This shows that $\preccurlyeq$ is transitive. ✓
(ii) Define a relation E in A by $E\triangleq\{(a,b)\in A^2\mid\forall S\in T,\ \exists C\in S,\ a\in C\wedge b\in C\}$. It can be easily shown that E is an equivalence mimicking the proof of Theorem 2.4.8. Then, $A/E\in\mathrm{Pt}(A)$ by Theorem 2.4.6.

Claim 1. A/E is a lower bound of T.

Proof. If $T = \varnothing$, there is nothing to prove; so assume $T \neq \varnothing$. Take any $S \in T$ and $a \in A$. Then, there exists $C \in S$ such that $a \in S$ since S is a partition of A. Let $b \in [a]_E$. Then, there exists $D \in S$ such that $a, b \in D$, which implies $C = D$. Therefore, $[a]_E \subseteq C$. Hence, $A/E \preccurlyeq S$. □

Claim 2. For each lower bound L of T, $L \preccurlyeq A/E$.

Proof. If $T = \varnothing$, then $A/E = \{A^2\}$ and every partition of A is a lower bound. Since $S \preccurlyeq \{A^2\}$ for all $S \in \mathrm{Pt}(A)$, the result follows.

Now, assume $T \neq \varnothing$. Let L be a lower bound of T. Take any $D \in L$. Fix some $a \in D$. Then, each $d \in D$ has the property that $\forall S \in T, \exists C \in S, \{a, d\} \subseteq D \subseteq C$ as L is a lower bound of T. Therefore, $d \in [a]_E$; $D \subseteq [a]_E$. Hence, $L \preccurlyeq A/E$. □

Claims 1 and 2 say that $\inf T = A/E$. Hence, $\inf T$ exists.

(iii) Let $T' \triangleq \{S' \in \mathrm{Pt}(A) \mid \forall S \in T, S \preccurlyeq S'\}$. By (ii), $S^* \triangleq \inf T'$ exists.

Claim 3. S^* is an upper bound of T.

Proof. In (ii), we showed that $S^* = A/E$ where $E = \{(a, b) \in A^2 \mid \forall S' \in T', \exists C' \in S', a \in C' \wedge b \in C'\}$. Take any $S \in T$ and let $C \in S$. Fix some $c_0 \in C$.

Now, take arbitrary $c \in C$. Then, for all $S' \in T'$, since $S \preccurlyeq S'$, there exists $D' \in S'$ such that $c \in C \subseteq D'$. Hence, we have cEc_0; $C \subseteq [c_0]_E$. Therefore, $S \preccurlyeq S^*$. □

Claim 3 essentially says that $S^* \in T'$. By Theorem 2.5.13 (iii), $S^* = \min T'$, i.e., $S^* = \sup T$. □

Exercise 2.5.13

If h is isomorphism between $(P, \leq)$ and $(Q, \preceq)$, then h^{-1} is an isomorphism between $(Q, \preceq)$ and $(P, \leq)$.

Proof. Take any $q_1, q_2 \in Q$. Then, we have $q_1 \preceq q_2 \iff h(h^{-1}(q_1)) \preceq h(h^{-1}(q_2)) \iff h^{-1}(q_1) \leq h^{-1}(q_2)$. □

Exercise 2.5.14

If f is an isomorphism between $(P_1, \leq_1)$ and $(P_2, \leq_2)$, and if g is an isomorphism between $(P_2, \leq_2)$ and $P_3, \leq_3$, then $g \circ f$ is an isomorphism between $(P_1, \leq_1)$ and $(P_3, \leq_3)$.

Proof. $\mathrm{ran}(g \circ f) = g[\mathrm{ran}\, f] = P_3$. Moreover, $g \circ f$ is one-to-one. Hence, $g \circ f : P_1 \hookrightarrow\!\!\!\!\rightarrow P_3$. For all $p, q \in P_1$, we have $p \leq_1 q \iff f(p) \leq_2 f(q) \iff g(f(p)) \leq_3 \iff g(f(q))$. Hence, $g \circ f$ is an isomorphism between $(P_1, \leq_1)$ and $(P_3, \leq_3)$. □

Chapter 3

Natural Numbers

3.1 The Recursion Theorem

Definition 3.1.1: Sequence

- A *sequence* is a function whose domain is a natural number or $\mathbb{N}$.
- A sequence whose domain is a natural number n is called a *finite sequence of length* n and is denoted

$$\langle\, a_i \mid i < n \,\rangle \quad \text{or} \quad \langle\, a_i \mid i = 0, 1, \cdots, n-1 \,\rangle \quad \text{or} \quad \langle\, a_0, a_1, \cdots, a_{n-1} \,\rangle.$$

  In particular, $\langle\rangle = \varnothing$—the *empty sequence*—is the unique sequence of length 0.

$$\mathrm{Seq}(A) \triangleq \bigcup_{n\in\mathbb{N}} A^n$$

  denote the set of all finite sequence of elements of A.
- A sequence whose domain is $\mathbb{N}$ is called a *infinite sequence* and is denoted

$$\langle\, a_i \mid i \in \mathbb{N} \,\rangle \quad \text{or} \quad \langle\, a_i \mid i = 0, 1, 2, \cdots \,\rangle \quad \text{or} \quad \langle a_i \rangle_{i=0}^{\infty}.$$

  Infinite sequences of elements of A are members of $A^{\mathbb{N}}$. We also use the notation $\{\, a_i \mid i \in \mathbb{N} \,\}$ or $\{a_i\}_{i=0}^{\infty}$, etc., for the range of the sequence $\langle a_i \mid i \in \mathbb{N} \rangle$.

Note:-

- A natural number $n \in \mathbb{N}$ is the set of all natural numbers less than n. See Exercise 3.3.6.
- Since $A^n \in \mathcal{P}(\mathbb{N} \times A)$ for each $n \in \mathbb{N}$, $\mathcal{A} = \{\, w \mid \exists n \in \mathbb{N},\ w = A^n \,\}$ exists, and thus $\mathrm{Seq}(A) = \bigcup \mathcal{A}$ exists.

Theorem 3.1.2 The Recursion Theorem

Let A be a set, $a \in A$, and $g : A \times \mathbb{N} \to A$. Then, there uniquely exists an infinite sequence $f : \mathbb{N} \to A$ such that

(i) $f_0 = a$ and

(ii) $\forall n \in \mathbb{N},\ f_{n+1} = g(f_n, n)$.

Proof. We say $t : (m+1) \to A$ is an *m-step computation based on a and g* if $t_0 = a$ and $\forall k < m,\ t_{k+1} = g(t_k, k)$. Let $F \triangleq \{\, t \in \mathrm{Seq}(A) \mid t \text{ is an } m \text{ step computation for some } m \in \mathbb{N} \,\}$. Let $f \triangleq \bigcup F$.

Claim 1. f is a function.

Proof. We shall show that F is a compatible system of functions so we may conclude f is a function thanks to Theorem 2.3.12. Take any $t, u \in F$. Let $n = \operatorname{dom} t \in \mathbb{N}$ and $m = \operatorname{dom} u \in \mathbb{N}$. WLOG, $n \leq m$ (thanks to $(\mathbb{N}, \leq)$ is Totally Ordered), i.e., $n \subseteq m$. Hence, $(\operatorname{dom} t) \cap (\operatorname{dom} u) = n$. If $n = 0$, then it is done; assume $n > 0$. Then, there exists $n' \in \mathbb{N}$ such that $n' + 1 = n$ by Exercise 3.3.4.

Surely, $t_0 = a = u_0$. Moreover, if $t_k = u_k$ where $k < n'$, then $k + 1 < n' + 1 = n$ (Exercise 3.3.2) and $t_{k+1} = g(t_k, k) = g(u_k, k) = u_{k+1}$. Therefore, by The Finite Induction Principle, we have $\forall k \leq n'$, $t_k = u_k$; t and u are compatible. □

Claim 2. $\operatorname{dom} f = \mathbb{N}$ and $\operatorname{ran} f \subseteq A$.

Proof. We already have $\operatorname{dom} f \subseteq \mathbb{N}$ and $\operatorname{ran} f \subseteq A$ by Theorem 2.3.12. To show $\operatorname{dom} f = \mathbb{N}$, it suffices to show that, for any $n \in \mathbb{N}$, there is an n-step computation based on a and g. Clearly, $t = \{(0, a)\}$ is a 0-step computation.

Assume there exists an n-step computation $t\colon (n+1) \to A$ where $n \in \mathbb{N}$. Then, define $u\colon ((n+1)+1) \to A$ by $u \triangleq t \cup \{(n+1, g(t_n, n))\}$. Then, one may easily verify that u is an $(n+1)$-step computation. Therefore, by The Induction Principle, the result follows. □

We now check if f satisfies the conditions (i) and (ii).

(i) Clearly, $f_0 = a$.

(ii) Take any $n \in \mathbb{N}$. Let t be an $(n+1)$-step computation. Then, $\forall k \leq n$, $f_k = t_k$, and $f_{n+1} = t_{n+1} = g(t_n, n) = g(f_n, n)$.

Now, we are left to show the uniqueness of such f.

Let $h\colon \mathbb{N} \to A$ be a sequence that satisfies the conditions (i) and (ii). Clearly, $f_0 = a = h_0$. And, if $f_n = h_n$, then $f_{n+1} = g(f_n, n) = g(h_n, n) = h_{n+1}$. Therefore, by The Induction Principle, $\forall k \in \mathbb{N}$, $f_k = h_k$, i.e., $f = k$ by Lemma 2.3.3. □

Theorem 3.1.3

Let $(A, \preceq)$ be a nonempty linearly ordered set with the properties:

(i) For every $p \in A$, there exists $q \in A$ such that $p \prec q$.

(ii) Every nonempty subset of A that has a $\preceq$-least element.

(iii) Every nonempty subset of A that has an upper bound has a $\preceq$-greatest element.

Then, $(A, \preceq)$ is isomorphic to $(\mathbb{N}, \leq)$.

Proof. By (i), $\{a \in A \mid x \prec a\} \neq \varnothing$ for each $x \in A$ and it has a $\preceq$-least element. Hence, we may define $g : A \times \mathbb{N} \to A$ by $g(x, n) \triangleq \min\{a \in A \mid x \prec a\}$. Then, The Recursion Theorem guarantees the existence of a function $f : \mathbb{N} \to A$ such that:

- $f_0 = \min A$ ▷ (i) and $A \neq \varnothing$
- $\forall n \in \mathbb{N}$, $f_{n+1} = g(f_n, n) = \min\{a \in A \mid f_n \prec a\}$.

By Exercise 3.1.1, we have $f_m \prec f_n$ whenever $m < n$. This also implies that f is injective.

Claim 1. $\operatorname{ran} f = A$

Proof. Suppose $\operatorname{ran} f \subsetneq A$ for the sake of contradiction. Then, $A \setminus \operatorname{ran} f \neq \varnothing$, and thus we may take $p = \min(A \setminus \operatorname{ran} f)$, which gives $p \neq f_0$ immediately. Hence, $B = \{a \in A \mid a \prec p\} \neq \varnothing$ and p is an upper bound of B. By (iii), $q = \max B$ exists. Since $q \prec p$, we have $q \in \operatorname{ran} f$, i.e., $q = f_m$ for some $m \in \mathbb{N}$.

Suppose there is some $r \in A$ such that $q \prec r \prec p$. Then, $r \in B$, which contradicts the maximality of q. Hence, $p = \min\{ a \in A \mid f_m \prec a \} = f_{m+1}$, which contradicts $p \notin \operatorname{ran} f$. □

We have $f : \mathbb{N} \hookrightarrow\!\!\!\rightarrow A$ by Claim 1. Hence, by $(\mathbb{N}, \leq)$ is Totally Ordered and Lemma 2.5.15, f is an isomorphism between $(\mathbb{N}, \leq)$ and $(A, \preceq)$. □

Theorem 3.1.4 The Recursion Theorem: General Version

Let S be a set and let $g\colon \operatorname{Seq}(S) \to S$. Then, there exists a unique sequence $f : \mathbb{N} \to S$ such that

$$\forall n \in \mathbb{N},\ f_n = g\left(f\big|_n\right) = g(\langle f_0, f_1, \cdots, f_{n-1} \rangle).$$

Proof. Define $G\colon \operatorname{Seq}(S) \times \mathbb{N} \to \operatorname{Seq}(S)$ by

$$G(t,n) = \begin{cases} t \cup \{(n, g(t))\} & \text{if } t \text{ is a sequence of length } n \\ \langle\rangle & \text{otherwise.} \end{cases}$$

Then, by The Recursion Theorem, there exists a sequence $F : \mathbb{N} \to \operatorname{Seq}(S)$ such that:

- $F_0 = \langle\rangle$
- $\forall n \in \mathbb{N},\ F_{n+1} = G(F_n, n)$.

If $F_k \in S^k$, then $F_{k+1} = F_k \cup \{k, g(F_k)\} \in S^{k+1}$. Hence, by The Induction Principle, $\forall n \in \mathbb{N}, F_n \in S^n$. Moreover, since $F_k \subsetneq_{\operatorname{Seq}(S)} F_{k+1}$, by Exercise 3.1.1, $\forall m, n \in \mathbb{N}, (m < n \implies F_m \subsetneq F_n)$; hence $\{ F_n \mid n \in \mathbb{N} \}$ is a compatible system of functions.

Let $f \triangleq \bigcup_{n \in \mathbb{N}} F_n$. Then, we have $f\big|_n = F_n$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, $f_n = F_{n+1}(n) = g(F_n) = g\left(f\big|_n\right)$.

Let $h\colon \mathbb{N} \to S$ be another sequence such that $\forall n \in \mathbb{N},\ h_n = g\left(h\big|_n\right)$. Suppose $\forall k < n, f_k = h_k$. Then, we have $f_n = g\left(f\big|_n\right) = g\left(h\big|_n\right) = h_n$. Therefore, by The Strong Induction Principle, $f = h$. □

Theorem 3.1.5 The Recursion Theorem: Parametric Version

Let $a : P \to A$ and $g : P \times A \times \mathbb{N} \to A$ be functions. Then, there uniquely exists a function $f : P \times \mathbb{N} \to A$ such that

(i) $\forall p \in P,\ f(p,0) = a(p)$

(ii) $\forall n \in \mathbb{N},\ \forall p \in P,\ f(p, n+1) = g(p, f(p,n), n)$.

Proof. Let $G\colon A^P \times \mathbb{N} \to A^P$ be defined by

$$G(x,n)(p) = g(p, x(p), n)$$

for each $x \in A^P$, $p \in P$, and $n \in \mathbb{N}$. Then, by The Recursion Theorem, there exists $F : \mathbb{N} \to A^P$ such that

$$F_0 = a \quad \text{and} \quad \forall n \in \mathbb{N},\ F_{n+1} = G(F_n, n).$$

Now, let $f : P \times \mathbb{N} \to A$ be defined by $f(p,n) = F_n(p)$. We now check if f satisfies the conditions:

(i) For all $p \in P$, we have $f(p,0) = F_0(p) = a(p)$.

(ii) For each $n \in \mathbb{N}$ and $p \in P$, $f(p, n+1) = F_{n+1}(p) = G(F_n, n)(p) = g(p, F_n(p), n) = g(p, f(p,n), n)$.

Let $h\colon P \times \mathbb{N} \to A$ be another function that satisfies (i) and (ii). Clear, we have $\forall p \in P, f(p,0) = a(p) = h(p,0)$. Assuming $\forall p \in P, f(p,n) = h(p,n)$ gives, for all $p \in P$, $f(p,n+1) = g(p, f(p,n), n) = g(p, h(p,n), n) = h(p, n+1)$. Hence, by The Induction Principle, we get $f = h$. □

Selected Problems

Exercise 3.1.1

Let $f : \mathbb{N} \to A$ be an infinite sequence where $(A, \preceq)$ is an ordered set. Then,

$$\forall n \in \mathbb{N},\ f_n \prec f_{n+1} \implies \forall m, n \in \mathbb{N},\ (n < m \implies f_n \prec f_m).$$

Proof. Fix any $n \in \mathbb{N}$ and let $\mathbf{P}(x)$ be the property "$f_n \prec f_x$." $\mathbf{P}(n+1)$ evidently holds. Now, suppose $\mathbf{P}(k)$ holds where $k \in \mathbb{N}$. Then, chaining $f_n \prec f_k$ and $f_k \prec f_{k+1}$ gives $\mathbf{P}(k+1)$. Therefore, by Exercise 3.3.11, we get $\forall m \geq n+1,\ f_n \prec f_m$. □

Exercise 3.1.2

Let $(A, \preceq)$ be a nonempty linearly ordered set. We say that $q \in A$ is a *successor* of $p \in A$ if there is no $r \in A$ such that $p \prec r \prec q$. Assume $(A, \preceq)$ has the following properties:

(i) Every $p \in A$ has a successor.
(ii) Every nonempty subset of A has a $\preceq$-least element.
(iii) If $p \in A$ is not the $\preceq$-least element of A, then p is a successor of some $q \in A$.

Then, $(A, \preceq)$ is isomorphic to $(\mathbb{N}, \leq)$.

Proof. By (i), for each $p \in P$, $\{q \in A \mid p \prec q\} \neq \varnothing$, and thus it has a $\preceq$-least element by (ii). Therefore, by The Recursion Theorem, there exists a sequence $f : \mathbb{N} \to A$ such that $f_0 = \min A$ and $\forall n \in \mathbb{N},\ f_{n+1} = \min\{q \in A \mid f_n \prec q\}$.

> ***Claim 1.*** $\operatorname{ran} f = A$
>
> ***Proof.*** Suppose $X \triangleq A \setminus \operatorname{ran} f \neq \varnothing$ for the sake of contradiction. Then, by (ii), we may take $p = \min X$. Since $\min A = f_0 \in \operatorname{ran} f$, p is not the $\preceq$-least element of A. Hence, by (iii), p is a successor of some $q \in A$. As $q \prec p$, we have $q \in \operatorname{ran} f$ by minimality of q, i.e., $q = f_m$ for some $m \in \mathbb{N}$. Since there is no $r \in A$ such that $q \prec r \prec p$, we have $p = f_{m+1}$ by definition, which contradicts $p \notin \operatorname{ran} f$. □

Since $f_n \prec f_{n+1}$ for all $n \in \mathbb{N}$, by Exercise 3.1.1, $\forall m, n \in \mathbb{N},\ (m < n \implies f_m \prec f_n)$, which means f is injective.

Therefore, together with Claim 1, f is an isomorphism between $(\mathbb{N}, \leq)$ and $(A, \preceq)$ by Lemma 2.5.15. □

Exercise 3.1.5 The Recursion Theorem: Partial Version

Let g be a function such that $\operatorname{dom} g \subseteq A \times \mathbb{N}$ and $\operatorname{ran} g \subseteq A$. Let $a \in A$. Then, there uniquely exists a sequence f of elements of A such that

(i) $f_0 = a$
(ii) $\forall n \in \mathbb{N},\ [n+1 \in \operatorname{dom} f \implies f_{n+1} = g(f_n, n)]$
(iii) f is either an infinite sequence or a finite sequence of length $k+1$ and $(f_k, k) \notin \operatorname{dom} g$.

Proof. Let $\overline{A} = A \cup \{\overline{a}\}$ where $\overline{a} \notin A$. (Such $\overline{a}$ exists by Exercise 1.3.3 (ii).) Define $\overline{g} : \overline{A} \times \mathbb{N} \to \overline{A}$ by

$$\overline{g}(x, n) = \begin{cases} g(x, n) & \text{if } (x, n) \in \operatorname{dom} g \\ \overline{a} & \text{otherwise.} \end{cases}$$

Then, The Recursion Theorem guarantees the existence of $\overline{f} : \mathbb{N} \to \overline{A}$ such that $\overline{f}_0 = a$ and $\forall n \in \mathbb{N}$, $\overline{f}_{n+1} = \overline{g}(\overline{f}_n, n)$. We have two cases: "$\forall n \in \mathbb{N}$, $\overline{f}_n \neq \overline{a}$" and "$\exists n \in \mathbb{N}$, $\overline{f}_n = \overline{a}$." They are resolved by Claims 1 and 2, respectively.

Claim 1. If "$\forall n \in \mathbb{N}$, $\overline{f}_n \neq \overline{a}$," then $\overline{f}$ is an infinite sequence of elements of A that satisfies (i) and (ii).

Proof. The assumption essentially says that $(\overline{f}_n, n) \in \operatorname{dom} g$ and $\overline{f}_{n+1} = g(\overline{f}_n, n) \in A$ for all $n \in \mathbb{N}$, i.e., $\overline{f}$ satisfies (i) and (ii). As $\overline{f}_0 = a \in A$, $\overline{f}$ is an infinite sequence of elements of A. □

Claim 2. If "$\exists n \in \mathbb{N}$, $\overline{f}_n = \overline{a}$," then there exists $k \in \mathbb{N}$ such that $\overline{f}\big|_{k+1}$ satisfies the conditions (i), (ii), and (iii).

Proof. By $(\mathbb{N}, \leq)$ is Well-Ordered, we have $\ell \triangleq \min\{n \in \mathbb{N} \mid \overline{f}_n = \overline{a}\}$. Since $\overline{f}_0 \in A$, we have $\ell \neq 0$, and thus $\ell = k+1$ for some $k \in \mathbb{N}$ by Exercise 3.3.4. It immediately follows that $\forall n \leq k$, $\overline{f}_n \in A$. Hence, $f \triangleq \overline{f}\big|_{k+1}$ is a finite sequence of length $k+1$ of elements of A.

We check if f satisfies the conditions (i), (ii), and (iii):

(i) $f_0 = \overline{f}_0 = a$

(ii) If $n < k$, i.e., $n+1 \in \operatorname{dom} f = k+1$, then $f_{n+1} = \overline{f}_{n+1} = \overline{g}(\overline{f}_n, n) = g(f_n, n)$.

(iii) If $(f_k, k) \in \operatorname{dom} g$, then we would have $\overline{f}_\ell = \overline{g}(\overline{f}_k, k) = \overline{g}(f_k, k) = g(f_k, k) \neq \overline{a}$. Hence, we must have $(f_k, k) \notin \operatorname{dom} g$. □

Now, we prove the uniqueness. Let f and h be two sequences of elements of A that satisfies the conditions (i), (ii), and (iii). WLOG, $\operatorname{dom} h \subseteq \operatorname{dom} f$.

Let $\mathbf{P}(x)$ be the property "$x \in \operatorname{dom} h \wedge f_x = h_x$." $\mathbf{P}(0)$ evidently holds.

Claim 3. $\forall n \in \mathbb{N}$, $(n+1 \in \operatorname{dom} f \wedge \mathbf{P}(n) \implies \mathbf{P}(n+1))$

Proof. Assume $n+1 \in \operatorname{dom} f$ and $\mathbf{P}(n)$. Then, since $(h_n, n) = (f_n, n) \in \operatorname{dom} g$, $n+1 \in \operatorname{dom} h$ and $h_{n+1} = g(h_n, n) = g(f_n, n) = f_{n+1}$. Hence, $\mathbf{P}(n+1)$ holds. □

If f is a finite sequence, Claim 3 and The Finite Induction Principle imply $h = f$. If f is an infinite sequence, Claim 3 and The Induction Principle imply $h = f$. □

Exercise 3.1.6

If $X \subseteq \mathbb{N}$, then there is a one-to-one (finite or infinite) sequence f such that $\operatorname{ran} f = X$.

Proof. If $X = \varnothing$, $\langle\rangle$ is the one we are looking for. Assume $X \neq \varnothing$.

Let $g = \left\{((x, n), y) \in (X \times \mathbb{N}) \times X \;\middle|\; y = \min\{k \in X \mid x < k\}\right\}$. Then, g is a function with $\operatorname{dom} g \subseteq \mathbb{N} \times \mathbb{N}$ and $\operatorname{ran} g \subseteq \mathbb{N}$. By The Recursion Theorem: Partial Version, there exists a sequence f of elements of X such that

(i) $f_0 = \min X$ ▷ $\min X$ exists by $(\mathbb{N}, \leq)$ is Well-Ordered

(ii) $\forall n \in \mathbb{N}$, $(n+1 \in \operatorname{dom} f \implies f_{n+1} = g(f_n, n))$

(iii) f is either an infinite sequence or a finite sequence of length $k+1$ and $(f_k, k) \notin \operatorname{dom} g$.

Note that $\operatorname{dom} g = \{(x, n) \in X \times \mathbb{N} \mid \exists y \in X,\, x < y\}$. Moreover, for each $n \in \mathbb{N}$ such that $n+1 \in \operatorname{dom} f$, we have $f_n < f_{n+1}$; hence $\forall m, n \in \operatorname{dom} f$, $(m < n \implies f_m < f_n)$ (in the similar manner of Exercise 3.1.1), and thus f is injective.

Suppose $Y = X \setminus \operatorname{ran} f \neq \varnothing$ for the sake of contradiction. By $(\mathbb{N}, \leq)$ is Well-Ordered, we may take $y = \min Y$. Then, by Theorem 3.3.8, we may let $z = \max\{x \in X \mid x < y\}$. $z = f_m$ for some $m \in \operatorname{dom} f$. Hence, $y = f_{m+1}$. □

3.2 Introduction to Natural Numbers

Note:-

We cannot prove an existence of an 'infinite' set (in the classical sense) or discuss infinity only from Axioms I to VI.

Definition 3.2.1: Successor

The *successor* of a set x is the set $S(x) = x \cup \{x\}$.

Notation 3.2.2: $n + 1$

We write $n+1$ to denote $S(n)$. There is no implication regarding the classic "addition" in this notation.

Notation 3.2.3: Natural Numbers

- $0 = \varnothing$
- $1 = \{\varnothing\} = S(0) = 0 + 1$
- $2 = \{\varnothing, \{\varnothing\}\} = S(1) = 1 + 1$
- $\ldots$

Definition 3.2.4: Inductive Set

A set I is called *inductive* if

$$0 \in I \wedge \forall n \in I, (n+1) \in I.$$

Axiom VII Axiom of Infinity

An inductive set exists.

Definition 3.2.5: Set of All Natural Numbers

The *set of all natural numbers* is the set

$$\mathbb{N} \triangleq \{x \mid x \in I \text{ for all inductive set } I\}.$$

Note:-

Axiom of Infinity guarantees the existence of $\mathbb{N}$. For, if A is any inductive set, then $\mathbb{N} = \{x \in A \mid x \in I \text{ for all inductive set } I\}$.

Lemma 3.2.6

$\mathbb{N}$ is inductive. In addition, if I is an inductive set, then $\mathbb{N} \subseteq I$.

Proof. Since $0 \in I$ for all inductive set, $0 \in \mathbb{N}$. If $n \in \mathbb{N}$, then $n \in I$ for all inductive set, and thus $(n+1) \in I$ for all inductive set. Therefore, $(n+1) \in \mathbb{N}$. Hence, $\mathbb{N}$ is inductive.

$\mathbb{N} \subseteq I$ directly follows from the definition of $\mathbb{N}$. □

Definition 3.2.7

The relation $<$ on $\mathbb{N}$ is defined by: $m < n$ if and only if $m \in n$.

Notation 3.2.8

Although we did not prove $<$ is a strict ordering of $\mathbb{N}$, we shall use $\leq$ to denote the relation on $\mathbb{N}$:

$$\leq \triangleq < \cup \mathrm{Id}_{\mathbb{N}}$$

Selected Problems

Exercise 3.2.1

(i) $\forall x,\ x \subseteq S(x)$

(ii) $\forall x,\ \neg(\exists z,\ x \subsetneq z \subsetneq S(x))$

Proof.

(i) $x \subseteq x \subseteq x \cup \{x\} = S(x)$

(ii) Take any z such that such that $x \subseteq z \subseteq S(x) = x \cup \{x\}$. If $z \subseteq x$, then we have $z = x$. If $z \nsubseteq x$, then there exists y such that $y \in z$ and $y \notin x$. However, $y \in x \cup \{x\}$, and thus $y = x$. Therefore, $S(x) \subseteq z$; $z = S(x)$. In conclusion, any z such that $x \subseteq z \subseteq S(x)$ must satisfy $z = x$ or $z = S(x)$. □

3.3 Properties of Natural Numbers

Theorem 3.3.1 The Induction Principle

Let $\mathbf{P}(x)$ be a property (possibly with parameters).

$$\mathbf{P}(0) \wedge \forall n \in \mathbb{N},\ (\mathbf{P}(n) \implies \mathbf{P}(n+1)) \implies \forall n \in \mathbb{N},\ \mathbf{P}(n)$$

Proof. The premise simply says that $A = \{n \in \mathbb{N} \mid \mathbf{P}(n)\}$ is inductive. Therefore, $\mathbb{N} \subseteq A$ follows. □

Lemma 3.3.2

(i) $\forall n \in \mathbb{N},\ 0 \leq n$

(ii) $\forall k, n \in \mathbb{N},\ (k < n+1 \iff k < n \vee k = n)$

Proof.

(i) Let $\mathbf{P}(x)$ be the property "$0 \leq x$." $\mathbf{P}(0)$, i.e., $0 \leq 0$, holds since $0 = 0$.

Now, assume $n \in \mathbb{N}$ and $\mathbf{P}(n)$. If $n = 0$, then we have $0 \in S(0) = n+1$ by definition (Definition 3.2.1). If $0 < n$, then $0 \in n$, and thus $0 \in n \cup \{n\} = S(n)$. Therefore, by The Induction Principle, the result follows.

(ii) Note that $k \in n \cup \{n\}$ if and only if $k \in n$ or $k = n$. □

Theorem 3.3.3 $(\mathbb{N}, \leq)$ is Totally Ordered

$(\mathbb{N}, \leq)$ is a totally ordered set.

Proof. We first need to prove that $(\mathbb{N}, \leq)$ is an ordered set.

Claim 1. $<$ is transitive in $\mathbb{N}$.

Proof. Let $\mathbf{P}(x)$ be the property "$\forall k, m \in \mathbb{N}, (k < m \wedge m < x \implies k < x)$." $\mathbf{P}(0)$ is true because there is no $m \in \mathbb{N}$ such that $m \in 0 = \varnothing$.

Now assume $n \in \mathbb{N}$ and $\mathbf{P}(n)$. Now, let $k, m \in \mathbb{N}$ and $k < m$ and $m < n+1$. By Lemma 3.3.2 (ii), $m < n$ or $m = n$.

- If $m < n$, then we have $k < n$ as $\mathbf{P}(n)$ holds,
- If $m = n$, then we immediately have $k < n$.

In both cases, we have $k < n$; thus $k < n+1$ by Lemma 3.3.2 (ii). Therefore, the result follows from The Induction Principle. □

Claim 2. $<$ is asymmetric in $\mathbb{N}$.

Proof. Let $\mathbf{P}(x)$ be the property "$\neg(x < x)$." $\mathbf{P}(0)$ evidently holds since $\varnothing \notin \varnothing$.

Now, assume $n \in \mathbb{N}$ and $\mathbf{P}(n)$. Suppose $(n+1) < (n+1)$ for the sake of contradiction. By Lemma 3.3.2 (ii), we have $(n+1) = n$ or $(n+1) < n$. In both cases, we have $n < n$ by $n < (n+1)$ (from Lemma 3.3.2 (ii)) and Claim 1, which contradicts $\mathbf{P}(n)$. Therefore, $\mathbf{P}(n+1)$ holds. The result follows from The Induction Principle. □

Hence, $(\mathbb{N}, \leq)$ is an ordered set by Claims 1 and 2 and Theorem 2.5.3. We are left to prove that $\leq$ is a total ordering of $\mathbb{N}$.

Claim 3. $\forall n, m \in \mathbb{N},\ n < m \implies (n+1) \leq m$

Proof. Let $\mathbf{P}(x)$ be the property "$\forall n \in \mathbb{N}, (n < x \implies n+1 \leq x)$." $\mathbf{P}(0)$ holds since there is no $n \in \mathbb{N}$ such that $n < 0$.

Now, assume $m \in \mathbb{N}$ and $\mathbf{P}(m)$. Take any $n \in \mathbb{N}$ such that $n < (m+1)$. Then, by Lemma 3.3.2, we have $n = m$ or $n < m$. If $n = m$, then we have $(n+1) = (m+1)$, which implies $(n+1) \leq (m+1)$. If $n < m$, then $(n+1) \leq m < (m+1)$. Therefore, the result follows from The Induction Principle. □

Claim 4. $<$ is a total ordering of $\mathbb{N}$.

Proof. Let $\mathbf{P}(x)$ be the property "$\forall m \in \mathbb{N},\ m = x \vee m < x \vee x < m$." $\mathbf{P}(0)$ is essentially Lemma 3.3.2 (i).

Assume $n \in \mathbb{N}$ and $\mathbf{P}(n)$. Take any $m \in \mathbb{N}$. If $m < n$ or $m = n$, we have $m < (n+1)$ by Lemma 3.3.2 (ii). If $n < m$, by Claim 3, we have $(n+1) \leq m$. Hence, $\mathbf{P}(n+1)$ holds. Therefore, the result follows from The Induction Principle. □

□

Notation 3.3.4

We may write "$\forall k < n, \mathbf{P}(k)$" instead of "$\forall k \in \mathbb{N}, (k < n \implies \mathbf{P}(k))$" or "$\exists k < n, \mathbf{P}(k)$" instead of "$\exists k \in \mathbb{N},\ k < n \wedge \mathbf{P}(k)$" when no confusion may arise. We may similarly write $(\forall/\exists)k(\leq/>/\geq)n,\ \mathbf{P}(k)$.

Theorem 3.3.5 The Strong Induction Principle

Let $\mathbf{P}(x)$ be a property (possibly with parameters). If, for all $n \in \mathbb{N}$, $\mathbf{P}(k)$ holds for all

$k < n$, then $\mathbf{P}(n)$ holds for all $n \in \mathbb{N}$.

$$\forall n \in \mathbb{N}, [\forall k < n, \implies \mathbf{P}(k) \implies \mathbf{P}(n)] \implies \forall n \in \mathbb{N}, \mathbf{P}(n)$$

Proof. Assume the premise ($\forall n \in \mathbb{N}, [\forall k < n, \implies \mathbf{P}(k) \implies \mathbf{P}(n)]$). Let $\mathbf{Q}(n)$ be the property "$\forall k < n, \mathbf{P}(k)$." $\mathbf{Q}(0)$ holds since there is no $k < 0$.

Now, assume $n \in \mathbb{N}$ and $\mathbf{Q}(n)$. Then, by the premise, we have $\mathbf{P}(n)$. Lemma 3.3.2 (ii) enables us to say that $\forall k \in \mathbb{N}, (k < n+1 \implies P(k))$. Therefore, $\forall n \in \mathbb{N}$, $\mathbf{Q}(n)$ holds by The Induction Principle.

Take any $k \in \mathbb{N}$. Then, we have $k < k+1$ and thus $\mathbf{P}(k)$ holds by $\mathbf{Q}(k+1)$. □

Definition 3.3.6: Well-Ordering

A total ordering $\preceq$ of a set A is a *well-ordering* if every nonempty subset of A has a least element. Then, the ordered set $(A, \preceq)$ is called a *well-ordered set*.

Theorem 3.3.7 $(\mathbb{N}, \leq)$ is Well-Ordered

$(\mathbb{N}, \leq)$ is a well-ordered set.

Proof. Let $X \subseteq \mathbb{N}$ has no least element. For each $n \in \mathbb{N}$, if $\forall k < n$, $k \in \mathbb{N} \setminus X$, we must have $n \in \mathbb{N} \setminus X$ since otherwise $n = \min X$. Then, by The Strong Induction Principle, $\forall n \in \mathbb{N}, n \in \mathbb{N} \setminus X$, i.e., $X = \varnothing$. □

Theorem 3.3.8

Let $\varnothing \subsetneq X \subseteq \mathbb{N}$. If X has an upper bound in the ordering $\leq$, then X has a greatest element.

Proof. Let $Y \triangleq \{k \in \mathbb{N} \mid k \text{ is an upper bound of } X\}$. The assumption says that $Y \neq \varnothing$. By $(\mathbb{N}, \leq)$ is Well-Ordered, $n \triangleq \min Y = \sup X$ exists.

Suppose $n \notin X$ for the sake of contradiction. Then, $\forall m \in X$, $m < n$, which implies $n \neq 0$ as $X \neq \varnothing$. Therefore, $n = k+1$ for some $k \in \mathbb{N}$ by Exercise 3.3.4; and thus $\forall m \in X$, $m \leq k$ by Lemma 3.3.2 (ii). Then, k is an upper bound of A and $k < n$, which is a contradiction to $n = \sup X$. Therefore, $n \in X$, and hence $n = \max X$ by Theorem 2.5.13. □

Selected Problems

Exercise 3.3.2

$\forall m, n \in \mathbb{N}$, $(m < n \implies m+1 < n+1)$. Hence, $S : \mathbb{N} \to \mathbb{N}$ where $n \mapsto n+1$ defines a one-to-one function on $\mathbb{N}$.

Proof. By Claim 3 in the proof of $(\mathbb{N}, \leq)$ is Totally Ordered, we have $m+1 \leq n$. Together with $n < n+1$, we have $m+1 < n+1$.

Now, take any $m, n \in \mathbb{N}$ with $m \neq n$. Then, by $(\mathbb{N}, \leq)$ is Totally Ordered, we have $m < n$ or $n < m$, i.e., $S(m) < S(n)$ or $S(n) < S(m)$. In both cases, $S(m) \neq S(n)$. Therefore, S is one-to-one. □

Exercise 3.3.3

There exists $X \subsetneq \mathbb{N}$ and $f : \mathbb{N} \to X$ such that f is injective.

Proof. Let $S : \mathbb{N} \to \mathbb{N}$ where $n \mapsto n+1$. Then, S is injective by Exercise 3.3.2. Since there exists no $n \in \mathbb{N}$ such that $n \cup \{n\} = \varnothing$, $0 \notin \operatorname{ran} S$; $\operatorname{ran} S \subsetneq \mathbb{N}$. Therefore, $S : \mathbb{N} \to \operatorname{ran} S$ is the function we are looking for. □

Exercise 3.3.4

$\forall n \in \mathbb{N} \setminus \{0\}, \exists! k \in \mathbb{N}, n = k+1$

Proof. Let $\mathbf{P}(x)$ be the property "$x = 0 \vee \exists! k \in \mathbb{N}, x = k+1$." $\mathbf{P}(0)$ holds by definition.

Now, assume $\mathbf{P}(n)$ where $n \in \mathbb{N}$. There exists $k \in \mathbb{N}$ such that $n+1 = k+1$, namely, $k = n$. If k' is another natural number such that $n+1 = k'+1$, then by Exercise 3.3.2, we have $k = k'$. Hence, $\mathbf{P}(n+1)$ holds. The result follows from The Induction Principle. □

Exercise 3.3.6

$\forall n \in \mathbb{N}, n = \{m \in \mathbb{N} \mid m < n\}$

Proof. Let $\mathbf{P}(x)$ be the property "$x = \{m \in \mathbb{N} \mid m < x\}$." We have $\mathbf{P}(0)$ since there exists no $m \in \mathbb{N}$ with $m < 0$.

Now, assume $\mathbf{P}(n)$ where $n \in \mathbb{N}$. Then, $n+1 = \{m \in \mathbb{N} \mid m < n\} \cup \{n\}$. By Lemma 3.3.2 (ii), $m < n+1$ if and only if $m < n$ or $m = n$. Therefore, $\{m \in \mathbb{N} \mid m < n\} \cup \{n\} = \{m \in \mathbb{N} \mid m < n \vee m = n\} = \{m \in \mathbb{N} \mid m < n+1\}$; $\mathbf{P}(n+1)$ holds. The result follows from The Induction Principle. □

Exercise 3.3.8

There is no function $f : \mathbb{N} \to \mathbb{N}$ such that $\forall n \in \mathbb{N}, f(n+1) < f(n)$.

Proof. Let $\mathbf{P}(x)$ be the property "there is no function $f : \mathbb{N} \to \mathbb{N}$ such that $f(0) = x$ and $\forall n \in \mathbb{N}, f(n+1) < f(n)$."

For the sake of induction, assume $\forall k < n, P(k)$ where $n \in \mathbb{N}$. Suppose there exists $f : \mathbb{N} \to \mathbb{N}$ such that $f(0) = n$ and $\forall k \in \mathbb{N}, f(k+1) < f(k)$. Now, define $g : \mathbb{N} \to \mathbb{N}$ by $g(k) = f(k+1)$. Then, $g(0) = f(1) < n$ and $\forall k \in \mathbb{N}, g(k+1) = f((k+1)+1) < f(k+1) = g(k)$. However, by $\mathbf{P}(g(0))$, such g cannot exist; by contradiction, $\mathbf{P}(n)$ holds. Hence, $\forall m \in \mathbb{N}, \mathbf{P}(m)$ by The Strong Induction Principle.

Finally, suppose there exists $f : \mathbb{N} \to \mathbb{N}$ such that $\forall n \in \mathbb{N}, f(n+1) < f(n)$. Then, by $\mathbf{P}(f(0))$, such f may not exist. □

Exercise 3.3.11

Let $\mathbf{P}(x)$ be a property and let $k \in \mathbb{N}$.

$$\mathbf{P}(k) \wedge \forall n \geq k, (\mathbf{P}(n) \implies \mathbf{P}(n+1)) \implies \forall n \geq k, \mathbf{P}(n)$$

Proof. Let $\mathbf{Q}(x)$ be the property "$x < k \vee \mathbf{P}(x)$." If $k = 0$, then $\mathbf{P}(0)$ holds. If $k > 0$, then $0 < k$ holds. Hence, in both cases, $\mathbf{Q}(0)$ holds.

Now assume $\mathbf{Q}(n)$ holds where $n \in \mathbb{N}$. Then, by $(\mathbb{N}, \leq)$ is Totally Ordered, we have $n+1 < k$, $n+1 = k$, or $n+1 > k$. If $n+1 < k$ or $n+1 = k$, we immediately have $\mathbf{Q}(n+1)$. If $n+1 > k$, we have $n \geq k$ by Lemma 3.3.2 (ii). Therefore, $\mathbf{P}(n)$ holds, and thus $\mathbf{P}(n+1)$ holds by assumption. Hence, $\mathbf{Q}(n+1)$. By The Induction Principle, $\forall n \in \mathbb{N}, n < k \vee \mathbf{P}(n)$. In other words, $\forall n \geq k, \mathbf{P}(n)$. □

Exercise 3.3.12 The Finite Induction Principle

Let $\mathbf{P}(x)$ be a property and let $k \in \mathbb{N}$.

$$\mathbf{P}(0) \wedge \forall n < k,\ (\mathbf{P}(n) \implies \mathbf{P}(n+1)) \implies \forall n \leq k,\ \mathbf{P}(n)$$

Proof. Let $\mathbf{Q}(x)$ be the property "$x > k \vee \mathbf{P}(x)$." $\mathbf{Q}(0)$ holds as $\mathbf{P}(0)$.

Now, assume $\mathbf{Q}(n)$ holds where $n \in \mathbb{N}$. Then, by $(\mathbb{N}, \leq)$ is Totally Ordered, we have $n+1 \leq k$ or $n+1 > k$. If $n+1 > k$, then we immediately have $\mathbf{Q}(n+1)$. If $n+1 \leq k$, by Lemma 3.3.2, $n+1 < k+1$. By Exercise 3.3.2 and $(\mathbb{N}, \leq)$ is Totally Ordered, we must have $n < k$. Hence, $\mathbf{P}(n)$ holds, and therefore $\mathbf{P}(n+1)$ holds by the assumption. By The Induction Principle, $\forall n \in \mathbb{N},\ n > k \vee \mathbf{P}(n)$. In other words, $\forall n \leq k,\ \mathbf{P}(n)$. □

Exercise 3.3.13 The Double Induction Principle

Let $\mathbf{P}(x, y)$ be a property.

$$\forall m, n \in \mathbb{N},\ [\forall k, \ell \in \mathbb{N},\ (k < m \vee k = m \wedge \ell < n \implies \mathbf{P}(k, \ell)) \implies \mathbf{P}(m, n)] \qquad [*]$$
$$\implies \forall m, n \in \mathbb{N},\ \mathbf{P}(m, n)$$

Proof. Let $\mathbf{Q}(x)$ be the property "$\forall n \in \mathbb{N},\ \mathbf{P}(x, n)$."

Now, assume $\forall k < m,\ \mathbf{Q}(k)$ where $m \in \mathbb{N}$. For the sake of induction, assume again that $\forall \ell < n,\ \mathbf{P}(m, \ell)$ where $n \in \mathbb{N}$. Now, we have $\mathbf{P}(k, \ell)$ for all $k, \ell \in \mathbb{N}$ such that $k < m$ or $k = m$ and $\ell < n$. Hence, by $[*]$, $\mathbf{P}(m, n)$. By The Strong Induction Principle, we have $\forall n \in \mathbb{N},\ \mathbf{P}(m, n)$. In other words, $\mathbf{Q}(m)$. Again by The Strong Induction Principle, we have $\forall m \in \mathbb{N},\ \mathbf{Q}(m)$, that is to say $\forall m, n \in \mathbb{N},\ \mathbf{P}(m, n)$. □

3.4 Arithmetic of Natural Numbers

Theorem 3.4.1

There uniquely exists a function $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

(i)) $\forall m \in \mathbb{N},\ +(m, 0) = m$

(ii)) $\forall m, n \in \mathbb{N},\ +(m, n+1) = S(+(m, n))$.

Proof. The result directly follows from exploiting The Recursion Theorem: Parametric Version with $A = P = \mathbb{N}$, $a(p) = p$ for all $p \in \mathbb{N}$, and $g(p, x, n) = S(x)$ for all $p, x, n \in \mathbb{N}$. □

Definition 3.4.2: Addition

The function $+$ defined in Theorem 3.4.1 is called the *addition*.

Notation 3.4.3

For all $m \in \mathbb{N}$, we have $+(m, 1) = +(m, 0+1) = +(m, 0) + 1 = m + 1$. Hence, we may write $m+n$ instead of $+(m, n)$ without causing any confusion regarding Notation 3.2.2. We restate the defining properties of the addition for future reference:

$$\forall m \in \mathbb{N},\ m + 0 = m \qquad [1]$$

$$\forall m, n \in \mathbb{N},\ m + (n+1) = (m+n) + 1 \qquad [2]$$

Theorem 3.4.4 + is Commutative
Addition is commutative; that is to say

$$\forall m, n \in \mathbb{N},\ m + n = n + m.$$

Proof. Let $\mathbf{P}(x)$ be the property "$\forall m \in \mathbb{N},\ m + x = x + m$."

Claim 1. $\mathbf{P}(0)$ holds.

Proof. Since $m + 0 = m$ already, we only need to prove $0 + m = m$ for all $m \in \mathbb{N}$. We shall make use of induction. First of all $0 + 0 = 0$ holds by [1].

Suppose $0 + m = m$ where $m \in \mathbb{N}$. Then,

$$\begin{aligned} 0 + (m+1) &= (0+m) + 1 && \triangleright\ [2] \\ &= m + 1. && \triangleright\ 0 + m = m \end{aligned}$$

Hence, by The Induction Principle, $0 + m = m$ for all $m \in \mathbb{N}$. □

Claim 2. $\forall n \in \mathbb{N},\ [\mathbf{P}(n) \implies \mathbf{P}(n+1)]$

Proof. Assume $\mathbf{P}(n)$. We shall show $\mathbf{P}(n+1)$ holds by induction. $0 + (n+1) = (n+1) + 0$ is already shown by Claim 1. Hence, assume $m + (n+1) = (n+1) + m$ for fixed $m \in \mathbb{N}$. Then,

$$\begin{aligned} (m+1) + (n+1) &= ((m+1) + n) + 1 && \triangleright\ [2] \\ &= (n + (m+1)) + 1 && \triangleright\ \mathbf{P}(n) \\ &= ((n+m) + 1) + 1 && \triangleright\ [2] \\ &= ((m+n) + 1) + 1 && \triangleright\ \mathbf{P}(n) \\ &= (m + (n+1)) + 1 && \triangleright\ [2] \\ &= ((n+1) + m) + 1 && \triangleright\ m + (n+1) = (n+1) + m \\ &= (n+1) + (m+1). && \triangleright\ [2] \end{aligned}$$

Hence, by The Induction Principle, $\mathbf{P}(n+1)$ holds. □

From Claim 1, Claim 2, and The Induction Principle, we get $\forall m, n \in \mathbb{N},\ m + n = n + m$. □

Theorem 3.4.5 + is Associative
Addition is associative; that is to say

$$\forall k, m, n \in \mathbb{N},\ (k+m) + n = k + (m+n).$$

Proof. Let $\mathbf{P}(x)$ be the property "$\forall k, m \in \mathbb{N},\ (k+m) + x = k + (m+x)$." $\mathbf{P}(0)$ is direct by [1].

Now, fix any $n \in \mathbb{N}$ and assume $\mathbf{P}(n)$. Then, for all $k, m \in \mathbb{N}$,

$$\begin{aligned} (k+m) + (n+1) &= ((k+m) + n) + 1 && \triangleright\ [2] \\ &= (k + (m+n)) + 1 && \triangleright\ \mathbf{P}(n) \\ &= k + ((m+n) + 1) && \triangleright\ [2] \\ &= k + (m + (n+1)). && \triangleright\ [2] \end{aligned}$$

Hence, by The Induction Principle, the result follows. □

Theorem 3.4.6

There uniquely exists a function $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

(i) $\forall m \in \mathbb{N},\ m \cdot 0 = 0$

(ii) $\forall m, n \in \mathbb{N},\ m \cdot (n+1) = m \cdot n + m$.

Proof. The result directly follows from exploiting The Recursion Theorem: Parametric Version with $A = P = \mathbb{N}$, $a(p) = 0$ for all $p \in \mathbb{N}$, and $g(p, x, n) = x + p$ for all $p, x, n \in \mathbb{N}$. □

Definition 3.4.7: Multiplication

The function $\cdot$ defined in Theorem 3.4.6 is called the *multiplication*.

$$\forall m \in \mathbb{N},\ m \cdot 0 = 0 \tag{3}$$

$$\forall m, n \in \mathbb{N},\ m \cdot (n+1) = m \cdot n + m \tag{4}$$

Theorem 3.4.8 $\cdot$ is Commutative

Multiplication is commutative, i.e.,

$$\forall m, n \in \mathbb{N},\ m \cdot n = n \cdot m.$$

Proof. Let $\mathbf{P}(x)$ be the property "$\forall m \in \mathbb{N},\ m \cdot x = x \cdot m$."

Claim 1. $\mathbf{P}(0)$ holds.

Proof. Since $m \cdot 0 = 0$ already by [3], we only need to prove $0 \cdot m = 0$ for all $m \in \mathbb{N}$. We shall make use of induction. First of all $0 \cdot 0 = 0$ holds by [3].

Suppose $0 \cdot m = 0$ where $m \in \mathbb{N}$. Then,

$$\begin{aligned} 0 \cdot (m+1) &= 0 \cdot m + 0 && \triangleright\ [4] \\ &= 0 + 0 && \triangleright\ 0 \cdot m = 0 \\ &= 0. \end{aligned}$$

Hence, by The Induction Principle, $0 \cdot m = 0$ for all $m \in \mathbb{N}$. □

Claim 2. $\forall n \in \mathbb{N},\ [\mathbf{P}(n) \implies \mathbf{P}(n+1)]$

Proof. Fix any $n \in \mathbb{N}$ and assume $\mathbf{P}(n)$. We shall prove $\mathbf{P}(n+1)$ by induction. We already have $0 \cdot (n+1) = (n+1) \cdot 0$ by Claim 1.

Fix any $m \in \mathbb{N}$ and assume $m \cdot (n+1) = (n+1) \cdot m$. Then,

$$\begin{aligned}
(m+1)\cdot(n+1) &= (m+1)\cdot n + (m+1) && \triangleright \text{ [4]} \\
&= n\cdot(m+1) + (m+1) && \triangleright \ \mathbf{P}(n) \\
&= (n\cdot m + n) + (m+1) && \triangleright \text{ [4]} \\
&= (m\cdot n + n) + (m+1) && \triangleright \ \mathbf{P}(n) \\
&= (m\cdot n + m) + (n+1) && \triangleright \ + \text{ is Commutative, } + \text{ is Associative} \\
&= m\cdot(n+1) + (n+1) && \triangleright \text{ [4]} \\
&= (n+1)\cdot m + (n+1) && \triangleright \ m\cdot(n+1) = (n+1)\cdot m \\
&= (n+1)\cdot(m+1). && \triangleright \text{ [4]}
\end{aligned}$$

Hence, by The Induction Principle, $\mathbf{P}(n+1)$ holds.

From Claim 1, Claim 2, and The Induction Principle, we get $\forall m, n \in \mathbb{N},\ m \cdot n = n \cdot m$. □

Theorem 3.4.9 · Distributes Over +

Multiplication is distributive over addition, i.e.,

$$\begin{aligned}
&\forall k, m, n \in \mathbb{N},\ k\cdot(m+n) = k\cdot m + k\cdot n \quad \text{and} \\
&\forall k, m, n \in \mathbb{N},\ (m+n)\cdot k = m\cdot k + n\cdot k.
\end{aligned}$$

Proof. Let $\mathbf{P}(x)$ be the property "$\forall k, m \in \mathbb{N},\ k\cdot(m+x) = k\cdot m + k\cdot x$." $\mathbf{P}(0)$ holds by [1] and [3].

Fix any $n \in \mathbb{N}$ and assume $\mathbf{P}(n)$. Then, for each $k, m \in \mathbb{N}$,

$$\begin{aligned}
k\cdot(m+(n+1)) &= k\cdot((m+n)+1) && \triangleright \ + \text{ is Associative} \\
&= k\cdot(m+n) + k && \triangleright \text{ [4]} \\
&= (k\cdot m + k\cdot n) + k && \triangleright \ \mathbf{P}(n) \\
&= k\cdot m + (k\cdot n + k) && \triangleright \ + \text{ is Associative} \\
&= k\cdot m + k\cdot(n+1). && \triangleright \text{ [4]}
\end{aligned}$$

Hence, by The Induction Principle, we have $\forall k, m, n \in \mathbb{N},\ k\cdot(m+n) = k\cdot m + k\cdot n$.

Now, we have, for each $k, m, n \in \mathbb{N}$,

$$\begin{aligned}
(m+n)\cdot k &= k\cdot(m+n) && \triangleright \ \cdot \text{ is Commutative} \\
&= k\cdot m + k\cdot n \\
&= m\cdot k + n\cdot k. && \triangleright \ \cdot \text{ is Commutative}
\end{aligned}$$

□

Theorem 3.4.10 · is Associative

Multiplication is associative, i.e.,

$$\forall k, m, n \in \mathbb{N},\ (k\cdot m)\cdot n = k\cdot(m\cdot n).$$

Proof. Let $\mathbf{P}(x)$ be the property "$\forall k, m \in \mathbb{N},\ (k\cdot m)\cdot x = k\cdot(m\cdot x)$." $\mathbf{P}(0)$ is direct from [3].

Fix any $n \in \mathbb{N}$ and assume $\mathbf{P}(n)$. Then, for each $k, m \in \mathbb{N}$,

$$\begin{aligned}
(k \cdot m) \cdot (n+1) &= (k \cdot m) \cdot n + k \cdot m && \triangleright \text{[4]} \\
&= k \cdot (m \cdot n) + k \cdot m && \triangleright \mathbf{P}(n) \\
&= k \cdot (m \cdot n + m) && \triangleright \cdot \text{ Distributes Over } + \\
&= k \cdot (m \cdot (n+1)). && \triangleright \text{[4]}
\end{aligned}$$

Hence, the result follows by The Induction Principle. □

Lemma 3.4.11

$\forall m \in \mathbb{N},\ m \cdot 1 = m$

Proof.

$$\begin{aligned}
m \cdot 1 &= m \cdot (0+1) && \triangleright \text{[1]}, + \text{ is Commutative} \\
&= m \cdot 0 + m && \triangleright \text{[4]} \\
&= 0 + m && \triangleright \text{[3]} \\
&= m && \triangleright \text{[1]}, + \text{ is Commutative}
\end{aligned}$$

□

Theorem 3.4.12

There uniquely exists a function $\uparrow : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

(i) $\forall m \in \mathbb{N},\ m \uparrow 0 = 1$

(ii) $\forall m, n \in \mathbb{N},\ m \uparrow (n+1) = (m \uparrow n) \cdot m$

Proof. The result directly follows from exploiting The Recursion Theorem: Parametric Version with $A = P = \mathbb{N}$, $a(p) = 1$ for all $p \in \mathbb{N}$, and $g(p, x, n) = x \cdot p$ for all $p, x, n \in \mathbb{N}$. □

Definition 3.4.13: Exponentiation

The function $\uparrow$ defined in Theorem 3.4.12 is called the *exponentiation*. We write m^n instead of $m \uparrow n$.

$$\forall m \in \mathbb{N},\ m^0 = 1 \qquad [5]$$

$$\forall m, n \in \mathbb{N},\ m^{n+1} = m^n \cdot m \qquad [6]$$

Theorem 3.4.14 Laws of Exponents

(i) $\forall m \in \mathbb{N},\ m^1 = m$

(ii) $\forall k, m, n \in \mathbb{N},\ k^{m+n} = k^m \cdot k^n$

(iii) $\forall k, m, n \in \mathbb{N}, (m \cdot n)^k = m^k \cdot n^k$

(iv) $\forall k, m, n \in \mathbb{N},\ (k^m)^n = k^{m \cdot n}$

Proof.

(i) Take any $m \in \mathbb{N}$. Then,

$$\begin{aligned}
m^1 &= m^{0+1} && \triangleright \text{[1]}, + \text{ is Commutative} \\
&= m^0 \cdot m && \triangleright \text{[6]} \\
&= 1 \cdot m && \triangleright \text{[5]} \\
&= m. && \triangleright \cdot \text{ is Commutative, Lemma 3.4.11}
\end{aligned}$$

(ii) Let $\mathbf{P}(x)$ be the property "$\forall k, m \in \mathbb{N}, k^{m+x} = k^m \cdot k^x$." $\mathbf{P}(0)$ holds since, for each $k, m \in \mathbb{N}$,

$$\begin{aligned} k^{m+0} &= k^m && \triangleright [1] \\ &= k^m \cdot 1 && \triangleright \text{Lemma 3.4.11} \\ &= k^m \cdot k^0. && \triangleright [5] \end{aligned}$$

Now, fix $n \in \mathbb{N}$ and assume $\mathbf{P}(n)$. Then,

$$\begin{aligned} k^{m+(n+1)} &= k^{(m+n)+1} && \triangleright + \text{ is Associative} \\ &= k^{m+n} \cdot k && \triangleright [6] \\ &= (k^m \cdot k^n) \cdot k && \triangleright \mathbf{P}(x) \\ &= k^m \cdot (k^n \cdot k) && \triangleright \cdot \text{ is Associative} \\ &= k^m \cdot k^{n+1}. && \triangleright [6] \end{aligned}$$

Therefore, by The Induction Principle, the result follows.

(iii) Let $\mathbf{P}(x)$ be the property "$\forall m, n \in \mathbb{N}, (m \cdot n)^x = m^x \cdot n^x$." $\mathbf{P}(0)$ holds since, for each $m, n \in \mathbb{N}$,

$$\begin{aligned} (m \cdot n)^0 &= 1 && \triangleright [5] \\ &= 1 \cdot 1 && \triangleright \text{Lemma 3.4.11} \\ &= m^0 \cdot n^0. && \triangleright [5] \end{aligned}$$

Now, fix $k \in \mathbb{N}$ and assume $\mathbf{P}(k)$. Then,

$$\begin{aligned} (m \cdot n)^{k+1} &= (m \cdot n)^k \cdot (m \cdot n) && \triangleright [6] \\ &= (m^k \cdot n^k) \cdot (m \cdot n) && \triangleright \mathbf{P}(k) \\ &= (m^k \cdot m) \cdot (n^k \cdot n) && \triangleright \cdot \text{ is Commutative, } \cdot \text{ is Associative} \\ &= m^{k+1} \cdot n^{k+1}. && \triangleright [6] \end{aligned}$$

Therefore, by The Induction Principle, the result follows.

(iv) Let $\mathbf{P}(x)$ be the property "$\forall k, m \in \mathbb{N}, (k^m)^x = k^{m \cdot x}$." $\mathbf{P}(0)$ holds since, for each $k, m \in \mathbb{N}$,

$$\begin{aligned} (k^m)^0 &= 1 && \triangleright [5] \\ &= k^0 && \triangleright [5] \\ &= k^{m \cdot 0}. && \triangleright [3] \end{aligned}$$

Now, fix $n \in \mathbb{N}$ and assume $\mathbf{P}(n)$. Then,

$$\begin{aligned} (k^m)^{n+1} &= (k^m)^n \cdot k^m && \triangleright [6] \\ &= k^{m \cdot n} \cdot k^m && \triangleright \mathbf{P}(n) \\ &= k^{m \cdot n + m} && \triangleright \text{Laws of Exponents (ii)} \\ &= k^{m \cdot (n+1)}. && \triangleright [4] \end{aligned}$$

Therefore, by The Induction Principle, the result follows. $\square$

Theorem 3.4.15

There uniquely exists $\Sigma\colon \operatorname{Seq}(\mathbb{N}) \to \mathbb{N}$ such that

(i) $\Sigma(\langle\rangle) = 0$.

(ii) $\Sigma(k) = \Sigma\left(k\big|_n\right) + k_n$for all $k \in \operatorname{Seq}(\mathbb{N})$ with length $n+1$.

Proof. Let $g\colon \operatorname{Seq}(\mathbb{N}) \times \mathbb{N} \times \mathbb{N}$ be defined by

$$g(k,s,n) = \begin{cases} s + k_n & \text{if } n \in \operatorname{dom} k \\ s & \text{otherwise.} \end{cases}$$

Then, by The Recursion Theorem: Parametric Version, there exists a function $f\colon \operatorname{Seq}(\mathbb{N}) \times \mathbb{N} \to \mathbb{N}$ such that

(i) $\forall k \in \operatorname{Seq}(\mathbb{N}), f(k,0) = 0$

(ii) $\forall n \in \mathbb{N},\ \forall k \in \operatorname{Seq}(\mathbb{N}), f(k,n+1) = g(k,f(k,n),n) = \begin{cases} f(k,n) + k_n & \text{if } n \in \operatorname{dom} k \\ f(k,n) & \text{otherwise.} \end{cases}$ [$*$]

Now, define $\Sigma\colon \operatorname{Seq}(\mathbb{N}) \to \mathbb{N}$ by $\Sigma(k) = f(k, \operatorname{dom} k)$. (i) evidently holds.

Claim 1. Let $k, \ell \in \operatorname{Seq}(\mathbb{N})$. If $k \subseteq \ell$, then $f(k, \operatorname{dom} k) = f(\ell, \operatorname{dom} k)$.

Proof. Let $\mathbf{P}(x)$ be the property

$$\forall k, \ell \in \operatorname{Seq}(\mathbb{N}),\ [\operatorname{dom} k = x \wedge k \subseteq \ell \implies f(k,x) = f(\ell,x)].$$

$\mathbf{P}(0)$ is evident. Now, fix $n \in \mathbb{N}$ and assume $\mathbf{P}(n)$.

Fix any $k \in \operatorname{Seq}(\mathbb{N})$ with $\operatorname{dom} k = n+1$. Then, for any $\ell \in \operatorname{Seq}(\mathbb{N})$ with $k \subseteq \ell$,

$$\begin{aligned} f(\ell, n+1) &= f(\ell,n) + \ell_n && \triangleright\ [*] \\ &= f\left(\ell\big|_n, n\right) + \ell_n && \triangleright\ \mathbf{P}(n) \\ &= f\left(k\big|_n, n\right) + k_n && \triangleright\ k \subseteq \ell \\ &= f(k,n) + k_n && \triangleright\ \mathbf{P}(n) \\ &= f(k,n+1). && \triangleright\ [*] \end{aligned}$$

Hence, by The Induction Principle, the result follows. □

Let $k \in \operatorname{Seq}(\mathbb{N})$ with length $n+1$. Then, $\Sigma(k) = f(k,n+1) = f(k,n) + k_n$.

$$\begin{aligned} \Sigma(k) &= f(k,n+1) \\ &= f(k,n) + k_n && \triangleright\ [*] \\ &= f\left(k\big|_n, n\right) + k_n && \triangleright\ \text{Claim 1} \\ &= \Sigma\left(k\big|_n\right) + k_n. \end{aligned}$$

The uniqueness easily follows. □

Notation 3.4.16: Summation

For the function Σ defined in Theorem 3.4.15, we write

$$\sum_{0 \le i < n} k_i \quad \text{or} \quad \sum_{i=0}^{n-1} k_i$$

instead of $\Sigma(\langle k_0, \cdots, k_{n-1} \rangle)$.

Selected Problems

Exercise 3.4.2

$\forall k, m, n \in \mathbb{N}, (m < n \iff m + k < n + k)$

Proof. Let $\mathbf{P}(x)$ be the property "$\forall m, n \in \mathbb{N}, (m < n \iff m + x < n + x)$." $\mathbf{P}(0)$ is evident from [1].

Now, fix any $k \in \mathbb{N}$ and assume $\mathbf{P}(k)$. Then, for all $m, n \in \mathbb{N}$,

$$\begin{aligned} m < n &\iff m + k < n + k && \triangleright\ \mathbf{P}(k) \\ &\iff (m + k) + 1 < (n + k) + 1 && \triangleright\ \text{Exercise 3.3.2} \\ &\iff m + (k + 1) < n + (k + 1). && \triangleright\ + \text{ is Associative} \end{aligned}$$

By The Induction Principle, the result follows. □

Exercise 3.4.3

$\forall m, n \in \mathbb{N}, (m \leq n \iff \exists! k \in \mathbb{N}, n = m + k)$

Proof. ($\Rightarrow$) Fix any $m \in \mathbb{N}$ and let $\mathbf{P}(x)$ be the property "$\exists k \in \mathbb{N}, x = m + k$." $\mathbf{P}(m)$ holds since $k = 0$ would satisfy by [1].

Fix any $n \in \mathbb{N}$ such that $m \leq n$ and assume $\mathbf{P}(n)$. Then, there exists k such that $n = m+k$, which leads to $n + 1 = m + (k + 1)$ by $+$ is Associative. Hence, $\mathbf{P}(n + 1)$ holds. Therefore, $\forall n \geq m, \exists k \in \mathbb{N}, n = m + k$ by Exercise 3.3.11.

To prove the uniqueness, assume $m + k = m + \ell$ where $k, \ell, m \in \mathbb{N}$. WLOG, $k \leq \ell$. If it were $k < \ell$, by Exercise 3.4.2 and $+$ is Commutative, we must have $m + k = k + m < \ell + m = \ell + m$. Hence, $k = \ell$.

($\Leftarrow$) Let $\mathbf{P}(x)$ be the property "$\forall m, n \in \mathbb{N}, (n = m + x \implies m \leq n)$." We have evidently $\mathbf{P}(0)$ by [1].

Fix any $k \in \mathbb{N}$ and assume $\mathbf{P}(k)$. Then, for each $m, n \in \mathbb{N}$ such that $n = m + (k + 1)$, we have $n = (m + 1) + k$ thanks to $+$ is Commutative and $+$ is Associative, and thus $m < m + 1 \leq n$ by $\mathbf{P}(k)$. Hence, by The Induction Principle, the result follows. □

Exercise 3.4.6

$\forall k, m, n \in \mathbb{N}, [k \neq 0 \implies (m < n \iff m \cdot k < n \cdot k)]$

Proof. Let $\mathbf{P}(x)$ be the property "$\forall m, n \in \mathbb{N}, (m < n \iff m \cdot k < n \cdot k)$." $\mathbf{P}(1)$ holds since, for all $n \in \mathbb{N}$,

$$\begin{aligned} n \cdot 1 &= n \cdot (0 + 1) && \triangleright\ [1],\ + \text{ is Commutative} \\ &= n \cdot 0 + n && \triangleright\ [4] \\ &= 0 + n && \triangleright\ [3] \\ &= n. && \triangleright\ [1],\ + \text{ is Commutative} \end{aligned}$$

Now, fix any $k \in \mathbb{N}$ and assume $\mathbf{P}(k)$. Then, for each $m, n \in \mathbb{N}$ with $m < n$,

$$\begin{aligned} m \cdot (k + 1) &= m \cdot k + m && \triangleright\ [4] \\ &< m \cdot k + n && \triangleright\ \text{Exercise 3.4.2} \\ &< n \cdot k + n && \triangleright\ \mathbf{P}(k),\ + \text{ is Commutative, Exercise 3.4.2} \\ &= n \cdot (k + 1). && \triangleright\ [4] \end{aligned}$$

Therefore, by Exercise 3.3.11, the result follows. □

3.5 Operations and Structures

Definition 3.5.1: Operation

- A *unary operation* on S is a function $S \rightharpoonup S$.
- A *binary operation* on S is a function $S^2 \rightharpoonup S$.

Notation 3.5.2: Binary Operation

Non-letter symbols such as $+$, $\times$, $*$, $\triangle$, etc., are often used to denote operations. The value of the operation $*$ at (x, y) is then denoted $x * y$ rather than $*(x, y)$.

Definition 3.5.3: Closedness Under Operation

Let f be a binary operation on S and $A \subseteq S$. A is said to be *closed under the operation* f if $\forall x, y \in A$, $[(x, y) \in \operatorname{dom} f \implies f(x, y) \in A]$.

Definition 3.5.4: n-Tuple

Let $n \in \mathbb{N}$. An n-tuple is a finite sequence of length n.

Note:-

Let $\langle a_0, \cdots, a_{n-1} \rangle$ and $\langle b_0, \cdots, b_{n-1} \rangle$ be two n-tuples. We have, by Lemma 2.3.3,

$$\langle a_0, \cdots, a_{n-1} \rangle = \langle b_0, \cdots, b_{n-1} \rangle \iff \forall i < n,\ a_i = b_i.$$

This satisfies the usual defining property of n-tuple.

Note:-

- If $\langle A_i \mid 0 \leq i < n \rangle$ is a finite sequence (of sets), then the product of the indexed system of sets $\prod_{0 \leq i < n} A_i$ (Definition 2.3.16) is just the set of all n-tuples $a = \langle a_0, \cdots, a_{n-1} \rangle$ such that $\forall i < n,\ a_i \in A_i$.
- If $\forall i < n, A_i = A$, then $\prod_{0 \leq i < n} A_i = A^n$.
- $A^0 = \{\langle\rangle\}$.

Notation 3.5.5

The 'ordered pair' (Definition 2.1.1), $(a_0, a_1) = \{\{a_0\}, \{a_0, a_1\}\}$, is different set from the '2-tuple' (Definition 3.5.4), $\langle a_0, a_1\rangle = \{(0, a_0), (1, a_1)\}$. Consequently, $A_0 \times A_1$ (Definition 2.2.10) does not generally equal to $\prod_{0\le i<2} A_i$ (Definition 2.3.16).

However, since there is a natural one-to-one correspondence

$$\begin{aligned} \delta : A_0 \times A_1 &\hookrightarrow\!\!\!\!\rightarrow \textstyle\prod_{0\le i<2} A_i \\ (a_0, a_1) &\longmapsto \langle a_0, a_1\rangle, \end{aligned}$$

for almost all practical purposes—when only the defining property of n-tuple is needed)—it makes so difference which definition one uses.

Therefore, we do not distinguish between ordered pairs and 2-tuples now on. That is to say we use notations

$$\langle a_0, \cdots, a_{n-1}\rangle \quad \text{and} \quad (a_0, \cdots, a_{n-1})$$

interchangeably from now on.

Definition 3.5.6: n-ary Relation

An *n-ary relation* R in A is a subset of A^n. We write $R(a_0, a_1, \cdots, a_{n-1})$ instead of $\langle a_0, a_1, \cdots, a_{n-1}\rangle \in R$.

Definition 3.5.7: n-ary Operation

An *n-ary operation* F on A is a function $A^n \rightharpoonup A$. We write $F(a_0, a_1, \cdots, a_{n-1})$ instead of $F(\langle a_0, a_1, \cdots, a_{n-1}\rangle)$.

Note:-

- 1-ary relations in A need not be distinguished from subsets of A.
- 1-ary operations on A need not be distinguished from functions $A \rightharpoonup A$.
- Nonempty 0-ary operations on A need not be distinguished from A. (A nonempty 0-ary operation is of the form $\{(\langle\rangle, a)\}$ where $a \in A$; a nonempty 0-ary operation is called a *constant*.)

Definition 3.5.8: Structure

- A *type* τ is an ordered pair $(\langle r_0, \cdots, r_{m-1}\rangle, \langle f_0, \cdots, f_{n-1}\rangle)$ of finite sequences of natural numbers.
- A *structure of type* τ is a triple

  $$\mathfrak{A} = (A, \langle R_0, \cdots, R_{m-1}\rangle, \langle F_0, \cdots, F_{n-1}\rangle)$$

  where R_i is an r_i-ary relation on A for each $i < m$ and F_j is an f_j-ary operation on A for each $j < n$. In addition, we require $F_j \neq \varnothing$ if $f_j = 0$, i.e., F_j should be constant. A is called the *universe* of the structure $\mathfrak{A}$.

Example 3.5.9

$\mathfrak{N} = (\mathbb{N}, \langle\le\rangle, \langle 0, +, \cdot\rangle)$ is a structure of type $(\langle 2\rangle, \langle 0, 2, 2\rangle)$.

Notation 3.5.10

We often write the structure of type $(\langle r_0,\cdots,r_{m-1}\rangle,\langle f_0,\cdots,f_{n-1}\rangle)$ as a $(1+m+n)$-tuple, for example, $(\mathbb{N},\leq,0,+,\cdot)$, when it is understood which symbol represent relations and which operations.

Definition 3.5.11: Isomorphism Between Structures

Let $\mathfrak{A}$ and $\mathfrak{A}'$ be structures of the same type $\tau=(\langle r_0,\cdots,r_{m-1}\rangle,\langle f_0,\cdots,f_{n-1}\rangle)$. Write $\mathfrak{A}=(A,\langle R_0,\cdots,R_{m-1}\rangle,\langle F_0,\cdots,F_{n-1}\rangle)$ and $\mathfrak{A}'=(A',\langle R'_0,\cdots,R'_{m-1}\rangle,\langle F'_0,\cdots,F'_{n-1}\rangle)$. An *isomorphism* between structures $\mathfrak{A}$ and $\mathfrak{A}'$ is a mapping $h\colon A\hookrightarrow\!\!\!\!\rightarrow A'$ such that

(i) $\forall i<m,\ \forall a\in A^{r_i},\ [R_i(a_0,\cdots,a_{r_i-1})\iff R'_i(h(a_0),\cdots,h(a_{r_i}-1))]$

(ii) $\forall j<n,\ \forall a\in A^{f_j},\big[(a_0,\cdots,a_{f_j-1})\in\operatorname{dom}F_j\iff(h(a_0),\cdots,h(a_{f_j-1}))\in\operatorname{dom}F'_j\big]$

(iii) $\forall j<n,\ \forall a\in A^{f_j},\big[(a_0,\cdots,a_{f_j-1})\in\operatorname{dom}F_j$

$$\implies h(F_j(a_0,\cdots,a_{f_j-1}))=F'_j(h(a_0),\cdots,h(a_{f_j-1}))\big].$$

Definition 3.5.12: Automorphism

An isomorphism between a structure $\mathfrak{A}$ and itself is called an *automorphism*.

Definition 3.5.13: Closed Set

Fix a structure $\mathfrak{A}=(A,\langle R_0,\cdots,R_{m-1}\rangle,\langle F_0,\cdots,F_{n-1}\rangle)$. A set $B\subseteq A$ is called *closed* if

$$\forall j<n,\ \forall a\in B^{f_j},\ [(a_0,\cdots,a_{f_j-1})\in\operatorname{dom}F_j\implies F_j(a_0,\cdots,a_{f_j-1})\in B].$$

Definition 3.5.14: Closure

Fix a structure $\mathfrak{A}=(A,\langle R_0,\cdots,R_{m-1}\rangle,\langle F_0,\cdots,F_{n-1}\rangle)$. Let $C\subseteq A$. The *closure* of C,

$$\overline{C}\triangleq\bigcap\{B\subseteq A\mid C\subseteq B\text{ and }B\text{ is closed}\},$$

is the least closed set containing all elements of C.

Theorem 3.5.15

Let $\mathfrak{A}=(A,\langle R_0,\cdots,R_{m-1}\rangle,\langle F_0,\cdots,F_{n-1}\rangle)$ be a structure and let $C\subseteq A$. If the sequence $\langle C_i\mid i\in\mathbb{N}\rangle$ is defined recursively by

$$C_0=C;$$
$$\forall i\in\mathbb{N},\ C_{i+1}=C_i\cup\bigcup_{j=0}^{n-1}F_j[C_i^{f_j}],$$

then $\overline{C}=\bigcup_{i=0}^{\infty}C_i$.

Proof. Note the recursive definition is justified by The Recursion Theorem. Let $\tilde{C}\triangleq\bigcup_{i=0}^{\infty}C_i$.

Claim 1. $\overline{C}\subseteq\tilde{C}$

Proof. Since we have $C_0 \subseteq \tilde{C}$, it is enough to show that $\tilde{C}$ is closed.

Take any $j < n$ and $a \in \tilde{C}^{f_j}$. By the definition of $\tilde{C}$, $\forall r < f_j$, $\exists i_r \in \mathbb{N}$, $a_r \in C_{i_r}$. We may take $\bar{\imath} = \max\{ i_r \mid r < f_j \}$ by Exercise 3.5.13. Since $C_i \subseteq C_{i+1}$ for all $i \in \mathbb{N}$, we have $a_r \in C_{i_r} \subseteq C_{\bar{\imath}}$ for all $r < f_j$. Hence, if $(a_0, \cdots, a_{f_j - 1}) \in \operatorname{dom} F_j$, we have $F_j(a_0, \cdots, a_{f_j-1}) \in F_j[C_{\bar{\imath}}^{f_j}] \subseteq C_{\bar{\imath}+1} \subseteq \tilde{C}$. Hence, $\tilde{C}$ is closed. □

Claim 2. $\tilde{C} \subseteq \overline{C}$

Proof. Clearly $C_0 = C \subseteq \overline{C}$. If $C_i \subseteq \overline{C}$, then, for each $j < n$, $F_j[C_i^{f_j}] \subseteq \overline{C}$ since $\overline{C}$ is closed. Hence, $C_{i+1} \subseteq \overline{C}$. Therefore, by The Induction Principle, $\forall i \in \mathbb{N}$, $C_i \subseteq \overline{C}$; hence $\tilde{C} \subseteq \overline{C}$. □

Combining Claims 1 and 2 completes the proof. □

Theorem 3.5.16 The General Induction Principle

Let $\mathfrak{A} = (A, \langle R_0, \cdots, R_{m-1} \rangle, \langle F_0, \cdots, F_{n-1} \rangle)$ be a structure and let $C \subseteq A$. Let $\mathbf{P}(x)$ be a property. If

(i) $\forall a \in C$, $\mathbf{P}(a)$

(ii) $\forall j < n, \forall a \in A^{f_j}, \left[(a_0, \cdots, a_{f_j-1}) \in \operatorname{dom} F_j \wedge \forall i < f_j, \mathbf{P}(a_i) \implies \mathbf{P}(F_j(a_0, \cdots, a_{f_j-1}))\right]$

hold, then $\forall x \in \overline{C}$, $\mathbf{P}(x)$.

Proof. Let $B = \{ x \in A \mid \mathbf{P}(x) \}$. (i) says $C \subseteq B$ and (ii) says B is closed. Therefore, $\overline{C} \subseteq B$. □

Note:-

The Induction Principle is a special case of The General Induction Principle for the structure $(\mathbb{N}, S)$ where S is the successor function.

Selected Problems

Exercise 3.5.4

Let $B = \mathcal{P}(A)$. Show that $(B, \cup_B, \cap_B)$ and $(B, \cap_B, \cup_B)$ are isomorphic structures.

Proof. Let $h \colon B \to B$ be defined by $h(X) = A \setminus X$. If $A \setminus X = A \setminus Y$, then $X = A \setminus (A \setminus X) = A \setminus (A \setminus Y) = Y$ by Exercise 1.4.2 (iii). Moreover, $h(h(X)) = X$ for all $X \in B$. Hence, $h \colon B \hookrightarrow\!\!\!\!\!\rightarrow B$. □

Exercise 3.5.7

Let R be a set whose elements are n-tuples. Then, R is an n-ary relation in A for some A.

Proof. Let $a \in R$. Then, $a = \{(0, a_0), \cdots, (n-1, a_{n-1})\}$. For each $i < n$, $a_i \in \{i, a_i\} \in (i, a_i) \in a \in R$. Hence, $a_i \in \bigcup[\bigcup(\bigcup R)]$, i.e., R is an n-ary relation in $A = \bigcup[\bigcup(\bigcup R)]$. □

Exercise 3.5.10

Let A be a sequence of length n. Then, $\prod_{0 \le i < n} A_i \neq \varnothing \iff \forall i < n, A_i \neq \varnothing$

Proof. Let $\mathbf{P}(x)$ be the property "if A is a sequence of length x, then $\prod_{0 \le i < n} A_i \neq \varnothing \iff \forall i < n, A_i \neq \varnothing$." $\mathbf{P}(0)$ holds since, if A is a function with $\operatorname{dom} A = \varnothing$, then $\prod A = \{\varnothing\}$.

Fix $n \in \mathbb{N}$ and assume $\mathbf{P}(n)$ holds. Take any sequence A of length $n+1$.

- Assume $\prod A \neq \varnothing$ and take $a \in \prod A$. Then, for each $i < n+1$, $a_i \in A_i$; and thus $A_i \neq \varnothing$.
- Assume $\forall i < n+1, A_i \neq \varnothing$. Then, by $\mathbf{P}(n)$, we may take $a' \in \prod_{0 \leq i < n} A_i$. We also may take $b \in A_n$. Then, $a' \cup \{(n, b)\} \in \prod A$.

Hence, $\mathbf{P}(n)$ holds. Thus, the result follows by The Induction Principle. □

Exercise 3.5.13

Let $\langle k_0, \cdots, k_{n-1} \rangle$ be a finite sequence of natural numbers of length $n \geq 1$. Then, its range $\{k_0, \cdots, k_{n-1}\}$ has a greatest element.

Proof. Let $\mathbf{P}(x)$ be the property "the range of a finite sequence of natural numbers of length x has a greatest element."

Let $\langle k_0 \rangle$ be a sequence of natural numbers of length 1. Then, $k_0 = \max \operatorname{ran} \langle k_0 \rangle$. Hence, $\mathbf{P}(1)$.

Fix any $n \in \mathbb{N}$ and assume $\mathbf{P}(n)$. Take any $k \in \operatorname{Seq}(\mathbb{N})$ with length $n+1$. Let $k' = \langle k_0, \cdots, k_{n-1} \rangle$ be another sequence. Then, by $\mathbf{P}(n)$, there exists $m' = \max\{k_0, \cdots, k_{n-1}\}$. Now, let $m = \max\{m', k_n\}$. Then, for all $i < n$, $k_i \leq m' \leq m$, and $k_n \leq m$. Hence, m is an upper bound of $\operatorname{ran} k$; the result follows by Theorem 3.3.8 and Exercise 3.3.11. □

Exercise 3.5.15

Let $R \subseteq A^2$ be a binary relation. Define a binary operation F_R on A^2 by

$$F_R((a_1, a_2), (b_1, b_2)) = \begin{cases} (a_1, b_2) & \text{if } a_2 = b_1 \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then,

(i) The closure of R in (A^2, F_R) is a transitive relation.
(ii) If R is reflexive and symmetric, $\overline{R}$ is also an equivalence.

Proof.

(i) Take any $a, b, c \in A$ and assume $a\overline{R}b$ and $b\overline{R}c$. Then, since $\overline{R}$ is closed, $F((a, b), (b, c)) = (a, c) \in \overline{R}$. Hence, $\overline{R}$ is transitive.
(ii) $\mathrm{Id}_A \subseteq R \subseteq \overline{R}$; $\overline{R}$ is reflexive.

Let $\mathbf{P}(x, y)$ be the property "$y\overline{R}x$." As $R \subseteq \overline{R}$, we have $\forall (a, b) \in R$, $\mathbf{P}(a, b)$. Now, take any $(a, b), (b, c) \in A^2$ such that $\mathbf{P}(a, b)$ and $\mathbf{P}(b, c)$. Then, by (i), we have $c\overline{R}a$; $\mathbf{P}(F_R((a, b), (b, c)))$ hold. Therefore, by The General Induction Principle, $b\overline{R}a$ holds for all $(a, b) \in \overline{R}$. □

Chapter 4

Finite, Countable, and Uncountable Sets

4.1 Cardinality of Sets

Definition 4.1.1: Equipotent Sets

Let A and B be sets. A is *equipotent to* B if there is a function $f : A \hookrightarrow\!\!\!\!\rightarrow B$. We write $|A| = |B|$.

Lemma 4.1.2

Let A, B, and C be sets.

(i) $|A| = |A|$.
(ii) If $|A| = |B|$, then $|B| = |A|$.
(iii) If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Proof.

(i) Id_A is an injective function on A onto A.
(ii) If $f : A \hookrightarrow\!\!\!\!\rightarrow B$, then $f^{-1} : B \hookrightarrow\!\!\!\!\rightarrow A$.
(iii) If $f : A \hookrightarrow\!\!\!\!\rightarrow B$, and if $g : B \hookrightarrow\!\!\!\!\rightarrow C$, then $f \circ g : A \hookrightarrow\!\!\!\!\rightarrow C$. $\square$

Note:-

Lemma 4.1.2 essentially says that $|A| = |B|$ behaves like an equivalence relation.

Definition 4.1.3

- We say *the cardinality of A is less than or equal to the cardinality of B* if there is a function $f : A \hookrightarrow B$. We write $|A| \le |B|$.
- We say *the cardinality of A is less than the cardinality of B* if $|A| \le |B|$ and $\neg(|A| = |B|)$. We write $|A| < |B|$.

Lemma 4.1.4

Let A, B, and C be sets.

(i) If $|A| = |B|$, then $|A| \le |B|$.
(ii) $|A| \le |A|$
(iii) If $|A| \le |B|$ and $|B| \le |C|$, then $|A| \le |C|$.

Proof.

(i) If $f : A \hookrightarrow\!\!\!\!\rightarrow B$, then f is injective as well.

(ii) Id_A is an injective function on A into A.

(iii) If $f : A \hookrightarrow B$, and if $g : B \hookrightarrow C$, then $f \circ g : A \hookrightarrow C$. □

Lemma 4.1.5

If $A_1 \subseteq B \subseteq A$ and $|A_1| = |A|$, then $|B| = |A|$.

Note:-

We present two proofs for Lemma 4.1.5. The second proof can be viewed as a more fundamental proof in the sense that it does not depend on Axiom of Infinity.

Proof 1. Let $f : A \hookrightarrow\!\!\!\!\rightarrow A_1$. Define a sequence $\langle A_i \mid i \in \mathbb{N} \rangle$ and $\langle B_i \mid i \in \mathbb{N} \rangle$ recursively by

$$\begin{aligned} A_0 &= A, & B_0 &= B, \\ \forall n \in \mathbb{N},\ A_{n+1} &= f[A_n], & \forall n \in \mathbb{N},\ B_{n+1} &= f[B_n] \end{aligned} \qquad [*]$$

thanks to The Recursion Theorem.

We clearly have $A_1 \subseteq B_0 \subseteq A_0$. If $A_{n+1} \subseteq B_n \subseteq A_n$, then $A_{n+2} = f[A_{n+1}] \subseteq B_{n+1} = f[B_n] \subseteq A_{n+1} = f[A_n]$ by $[*]$. Hence, by $[*]$ and The Induction Principle, we have $A_{n+1} \subseteq B_n \subseteq A_n$ for all $n \in \mathbb{N}$.

Let, for each $n \in \mathbb{N}$, $C_n \triangleq A_n \setminus B_n$. Then, by Exercise 2.3.6 (ii), $C_{n+1} = f[A_n] \setminus f[B_n] = f[A_n \setminus B_n] = f[C_n]$. Let

$$C \triangleq \bigcup_{n=0}^{\infty} C_n \quad \text{and} \quad D \triangleq A \setminus C.$$

Hence, $f[C] = \bigcup_{n=1}^{\infty} C_n \subseteq C$. Now, define a function $g : A \to A$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in C \\ x & \text{if } x \in D. \end{cases}$$

We immediately notice that $g\big|_C = f\big|_C$ and $g\big|_D$ are injective and their ranges—$f[C]$ and D—are disjoint; g is injective.

As, $\forall n \geq 1$, $C_n \subseteq A_n \subseteq B_0 = B$, we have $f[C] \subseteq B$. If $x \in D$, then $x \in A \setminus C_0 = A \setminus (A \setminus B) = B$ by Exercise 1.4.2 (iii).

Now, we shall show $B \subseteq f[C] \cup D$ and thus $B = \operatorname{ran} g$. Take any $y \in B$. Then, $y \in C$ or $y \in D$. If $y \in D$, then it is done; so assume $y \in C$. Then, as $y \notin A \setminus B = C_0$, $y \in f[C]$. Hence, $g : A \hookrightarrow\!\!\!\!\rightarrow B$. □

Proof 2. Let $f : A \hookrightarrow\!\!\!\!\rightarrow A_1$. Let $F : \mathcal{P}(A) \to \mathcal{P}(A)$ be defined by $F(X) = (A \setminus B) \cup f[X]$. If $X \subseteq Y \subseteq A$, then $F(X) = (A \setminus B) \cup f[X] \subseteq (A \setminus B) \cup f[Y] = F(Y)$. Hence, by Exercise 4.1.10, there exists $C \subseteq A$ such that

$$C = (A \setminus B) \cup f[C].$$

Let $D \triangleq A \setminus C$.

Now, define a function $g : A \to A$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in C \\ x & \text{if } x \in D. \end{cases}$$

Then, since $f[C] \subseteq C$, g is injective.

Moreover, $f[C] \subseteq \operatorname{ran} f = A_1 \subseteq B$ and $D = A \setminus C = A \setminus ((A \setminus B) \cup f[C]) \subseteq A \setminus (A \setminus B) = B$, and thus $\operatorname{ran} g \subseteq B$.

Now, take any $y \in B$. If $y \in C$, then, as $y \notin A \setminus B$, $y \in f[C]$. Hence, $B \subseteq f[C] \cup D$. Therefore, $g : A \hookrightarrow\!\!\!\!\rightarrow B$. □

Theorem 4.1.6 Cantor–Bernstein Theorem
If $|X| \le |Y|$ and $|Y| \le |X|$, then $|X| = |Y|$.

Proof. Let $f : X \hookrightarrow Y$ and $g : Y \hookrightarrow X$. Then, $g : Y \hookrightarrow\!\!\!\!\rightarrow g[Y]$, i.e., $|Y| = |g[Y]|$; and $g \circ f : X \hookrightarrow\!\!\!\!\rightarrow (g \circ f)[X]$, i.e., $|X| = |(g \circ f)[X]|$. Moreover, $(g \circ f)[X] \subseteq g[Y] \subseteq X$. Hence, by Lemma 4.1.5, $|g[Y]| = |X|$. We conclude $|X| = |Y|$ from Lemma 4.1.2. □

Assumption 4.1.7
There are sets called *cardinal numbers* (or *cardinals*) with the property that for every set X there is a unique cardinal $|X|$ (the *cardinal number of* X, the *cardinality of* X) and sets X and Y are equipotent if and only if $|X|$ is equal to Y.

Note:-
Assumption 4.1.7 essentially asserts the existence of a unique "representative" for each class of mutually equipotent sets. Assumption 4.1.7 is *harmless* in the sense that we only use it for convenience and we could formulate the theorems without it. We prove Assumption 4.1.7 in Chapter 8: Axiom of Choice. However, for certain classes of sets, cardinal numbers can be defined without the Axiom of Choice.

Selected Problems

Exercise 4.1.2
Let A, B, and C be sets.
(i) If $|A| < |B|$ and $|B| \le |C|$, then $|A| < |C|$.
(ii) If $|A| \le |B|$ and $|B| < |C|$, then $|A| < |C|$.

Proof.
(i) We already have $|A| \le |C|$ by Lemma 4.1.4 (iii). Let $g : B \hookrightarrow C$. Suppose $f : A \hookrightarrow\!\!\!\!\rightarrow C$ for the sake of contradiction. Then, $f^{-1} \circ g : B \hookrightarrow A$, i.e., $|B| \le |A|$. By Cantor–Bernstein Theorem, we get $|A| = |B|$, which is a contradiction.
(ii) We already have $|A| \le |C|$ by Lemma 4.1.4 (iii). Let $g : A \hookrightarrow B$. Suppose $f : A \hookrightarrow\!\!\!\!\rightarrow C$ for the sake of contradiction. Then, $g \circ f^{-1} : C \hookrightarrow B$, i.e., $|C| \le |B|$. By Cantor–Bernstein Theorem, we get $|B| = |C|$, which is a contradiction.

□

Exercise 4.1.3
If $A \subseteq B$, then $|A| \le |B|$.

Proof. Id_A is an injective function on A into B. □

Exercise 4.1.7
If $S \subseteq T$, then $|A^S| \le |A^T|$. In particular, $|A^m| \le |A^n|$ if $m \le n$.

Proof. If $T = \varnothing$, then $A^S = A^T = \{\varnothing\}$ and it is done.

Assume $T \neq \varnothing$. Fix some $t \in T$. Now, define $f : A^S \hookrightarrow A^T$ by $g \mapsto g \cup \{(x, t) \mid x \in T \setminus S\}$. □

Exercise 4.1.10

Let $F : \mathcal{P}(A) \to \mathcal{P}(A)$ be *monotone*, i.e., if $X \subseteq Y \subseteq A$, then $F(X) \subseteq F(Y)$. Then, F has a least *fixed point* $\overline{X}$, that is to say $F(\overline{X}) = \overline{X}$ and $\forall X \subseteq A$, $(F(X) = X \implies \overline{X} \subseteq X)$.

Proof. Let $T \triangleq \{X \subseteq A \mid F(X) \subseteq X\}$. Then, as $A \in T$, $T \neq \varnothing$; we may let $\overline{X} \triangleq \bigcap T$.

Then, for all $X \in T$, $\overline{X} \subseteq X$; and thus $F(\overline{X}) \subseteq F(X) \subseteq X$. We have $F(\overline{X}) \subseteq \bigcap T = \overline{X}$, i.e., $\overline{X} \in T$.

On the other hand, we have $F(F(\overline{X})) \subseteq F(\overline{X})$, or $F(\overline{X}) \in T$, and thus $\overline{X} = \bigcap T \subseteq F(\overline{X})$. Therefore, $F(\overline{X}) = \overline{X}$. Moreover, if X is a fixed point, then $X \in T$, and thus $\overline{X} = \bigcap T \subseteq X$. □

Exercise 4.1.14

A function $F : \mathcal{P}(A) \to \mathcal{P}(A)$ is *continuous* if, for each sequence $\langle X_i \mid i \in \mathbb{N}\rangle$ of subsets of A such that $\forall i, j \in \mathbb{N}$, $(i \leq j \implies X_i \subseteq X_j)$, $F\left(\bigcup_{i\in\mathbb{N}} X_i\right) = \bigcup_{i\in\mathbb{N}} F(X_i)$ holds.

If $\overline{X}$ is the least fixed point of a monotone continuous function, $F : \mathcal{P}(A) \to \mathcal{P}(A)$, then $\overline{X} = \bigcup_{i\in\mathbb{N}} X_i$ where we define recursively $X_0 = \varnothing$, $\forall i \in \mathbb{N}$, $X_{i+1} = F(X_i)$.

Proof. Let $\tilde{X} \triangleq \bigcup_{i\in\mathbb{N}} X_i$. We have $X_0 = \varnothing \subseteq X_1$.

If $X_n \subseteq X_{n+1}$, then $X_{n+1} \subseteq X_{n+2}$ since F is monotone. Hence, $\forall n \in \mathbb{N}$, $X_n \subseteq X_{n+1}$. Therefore, similarly to Exercise 3.1.1, we have $X_m \subseteq X_n$ whenever $m \leq n$. Hence, $F(\tilde{X}) = \bigcup_{i\in\mathbb{N}} F(X_i) = \bigcup_{i=1}^{\infty} X_i = \tilde{X}$; $\tilde{X}$ is a fixed point of F; hence $\overline{X} \subseteq \tilde{X}$.

We have $X_0 \subseteq \overline{X}$. If $X_n \subseteq \overline{X}$ for $n \in \mathbb{N}$, then $X_{n+1} \subseteq F(\overline{X})) = \overline{X}$. Hence, by The Induction Principle, $\tilde{X} \subseteq \overline{X}$. □

4.2 Finite Sets

Definition 4.2.1: Finite Set and Infinite Set

A set S is *finite* if it is equipotent to some natural number $n \in \mathbb{N}$. We then define $|S| = n$ and say *S has n elements*. A set is *infinite* if it is not finite.

Note:-

According to Definition 4.2.1, cardinal numbers of finite sets are the natural numbers. We evidently have $\forall n \in \mathbb{N}$, $|n| = n$.

Lemma 4.2.2

If $n \in \mathbb{N}$ and $X \subsetneq n$, then there is no $f : n \hookrightarrow\!\!\!\!\rightarrow X$.

Proof. If $n = 0$, there is no $X \subsetneq n$; the assertion is true.

Assume the assertion holds for n. Suppose there is some $f : (n+1) \hookrightarrow\!\!\!\!\rightarrow X$ where $X \subsetneq n+1$. There are two cases: $n \in X$ and $n \notin X$.

If $n \notin X$, then $X \subseteq n$, and thus $f\big|_n : n \hookrightarrow\!\!\!\!\rightarrow X \setminus \{f(n)\}$; however $X \setminus \{f(n)\} \subsetneq X \subseteq n$, which is a contradiction.

If $n \in X$, then $n = f(k)$ for some $k \leq n$. Define a function g on n by following:

$$g(i) = \begin{cases} f(n) & \text{if } i = k < n \\ f(i) & \text{otherwise.} \end{cases}$$

Then, $g : n \hookrightarrow\!\!\!\!\rightarrow X \setminus \{n\}$ and $X \setminus \{n\} \subsetneq n$, which is also a contradiction. □

Corollary 4.2.3

(i) If $m \neq n$ where $m, n \in \mathbb{N}$, then there is no $f : m \hookrightarrow\!\!\!\!\!\rightarrow n$.
(ii) If $|S| = m$ and $|S| = n$, then $m = n$.
(iii) $\mathbb{N}$ is infinite.

Proof.
(i) If $n \neq m$, by $(\mathbb{N}, \leq)$ is Totally Ordered, we have $n \subsetneq m$ or $m \subsetneq n$. In either case, we do not have such function by Lemma 4.2.2.
(ii) By Lemma 4.1.2, we have $|m| = |n|$. (i) asserts that $m = n$; otherwise we cannot have $|m| = |n|$.
(iii) By Exercise 3.3.3, there exists $f : \mathbb{N} \hookrightarrow\!\!\!\!\!\rightarrow X$ where $X \subsetneq \mathbb{N}$. If there exists $n \in \mathbb{N}$ and $g : n \hookrightarrow\!\!\!\!\!\rightarrow \mathbb{N}$, $g^{-1} \circ f^{-1} \circ f \circ g$ is a function on n onto a proper subset of n. This contradicts Lemma 4.2.2. $\square$

Theorem 4.2.4

If X is a finite set and $Y \subseteq X$, then Y is finite.

Proof. We may assume $X = \{x_0, \cdots, x_{n-1}\}$, where $\langle x_0, \cdots, x_{n-1} \rangle$ is an injective sequence, and $Y \neq \varnothing$.

Let $g : n \times \mathbb{N} \rightharpoonup n$ be defined by

$$g(a, -) = \begin{cases} \min\{\, j \in n \mid a < j \land x_j \in Y \,\} & \text{if it exists} \\ \text{undefined} & \text{otherwise.} \end{cases} \qquad [*]$$

By The Recursion Theorem: Partial Version, there exists a sequence k of elements in n such that

(i) $k_0 = \min\{\, j \in n \mid x_j \in Y \,\}$. $\triangleright\ Y \neq \varnothing$
(ii) $\forall i \in \mathbb{N}, [i + 1 \in \operatorname{dom} k \implies k_{i+1} = g(k_i, i) = \min\{\, j \in n \mid k_i < j \land x_j \in Y \,\}]$.
(iii) k is either an infinite sequence or a finite sequence of length $\ell + 1$ and $(k_\ell, \ell) \notin \operatorname{dom} g$.

By (ii) and [$*$], $\forall i \in \mathbb{N}, (i + 1 \in \operatorname{dom} k \implies k_i < k_{i+1})$. Hence, k is injective. If k were an infinite sequence, i.e., $k : \mathbb{N} \hookrightarrow n$, then $|\mathbb{N}| \leq |n|$. Together with Exercise 4.1.3 and Cantor–Bernstein Theorem, we get $|\mathbb{N}| = |n|$, which contradicts Corollary 4.2.3 (iii). Hence, k is a finite sequence of length ℓ.

Let $y_i \triangleq x_{k_i}$ for each $i < \ell$. By (i) and (ii), the sequence y is injective and its range is a subset of Y. By the same argument of Claim 1 of Theorem 3.1.3, we have $\operatorname{ran} y = Y$. Therefore, $y : \ell \hookrightarrow\!\!\!\!\!\rightarrow Y$; Y is finite. $\square$

Theorem 4.2.5

If X is finite and f is a function, then $f[X]$ is finite. Moreover, $|f[X]| \leq |X|$.

Proof. If $f[X] = \varnothing$, then it is done; assume $f[X] \neq \varnothing$. WLOG, $X \subseteq \operatorname{dom} f$.

We may assume $X = \{x_0, \cdots, x_{n-1}\}$, where $\langle x_0, \cdots, x_{n-1} \rangle$ is an injective sequence. Let $g : \operatorname{Seq}(n) \rightharpoonup n$ be defined by

$$g(\langle k_0, \cdots, k_{\ell'-1} \rangle) = \begin{cases} 0 & \text{if } \ell' = 0 \\ \min\{\, k \in n \mid k_{\ell'-1} < k \land \forall i < \ell', f(x_k) \neq f(x_{k_i}) \,\} & \text{if it exists and } \ell' > 0 \\ \text{undefined} & \text{otherwise.} \end{cases} \qquad [*]$$

Then, one may modify The Recursion Theorem: General Version to its partial version like The Recursion Theorem: Partial Version to get a sequence k of elements of n such that:

(i) $\forall i \in \operatorname{dom} k$, $k_i = g\left(k|_i\right)$. In particular, $k_0 = 0$.

(ii) k is either an infinite sequence or a finite sequence of length $\ell + 1$ and $k \notin \operatorname{dom} g$.

By (i) and [$*$], $\forall i, j \in \operatorname{dom} k$, $(i \neq j \implies f(x_{k_i}) \neq f(x_{k_j}))$, i.e., the sequence $y = \langle f(x_{k_i}) \mid i \in \operatorname{dom} k \rangle$ is injective and its range is a subset of $f[X]$.

By the similar reason as in the proof of Theorem 4.2.4, k is finite and $\operatorname{ran} y = f[X]$. Finally, we get $|f[X]| \leq |X|$ from $x \circ y^{-1} : f[X] \hookrightarrow X$. □

Lemma 4.2.6

Let X and Y be finite sets.

(i) $X \cup Y$ is finite; moreover, $|X \cup Y| \leq |X| + |Y|$.

(ii) If X and Y are disjoint, then $|X \cup Y| = |X| + |Y|$.

Proof.

(i) Write $X = \{x_0, \cdots, x_{m-1}\}$ and $Y = \{y_0, \cdots, y_{n-1}\}$ where $\langle x_0, \cdots, x_{m-1} \rangle$ and $\langle y_0, \cdots, y_{n-1} \rangle$ are injective sequences.

Now, define $z : (n+m) \twoheadrightarrow X \cup Y$ by

$$z_i = x_i \quad \text{for } 0 \leq i < n \quad \text{and} \quad z_i = y_{i-n} \quad \text{for } n \leq i < n+m.$$

(Here, $i - n$ is the unique $k \in \mathbb{N}$ such that $i = n + k$. See Exercise 3.4.3.) Hence, by Theorem 4.2.5, $X \cup Y$ is finite and $|X \cup Y| \leq n + m$.

(ii) If X and Y are disjoint, then $z : (n+m) \hookrightarrow\!\!\!\!\rightarrow X \cup Y$. Hence, $|X \cup Y| = n + m$. □

Theorem 4.2.7

If S is finite and if every $X \in S$ is finite, then $\bigcup S$ is finite.

Proof. If $|S| = 0$, then it is done.

Assume that the statement is true for all S with $|S| = n$. Let $S = \{X_0, \cdots, X_n\}$ be a set with $n+1$ elements such that each $X_i \in S$ is finite. Then, we have

$$\bigcup S = \left(\bigcup_{i=0}^{n-1} X_i\right) \cup X_n$$

but $\bigcup_{i=0}^{n-1} X_i$ is finite by induction hypothesis and thus $\bigcup S$ is finite by Lemma 4.2.6. Hence, by The Induction Principle, the result follows. □

Theorem 4.2.8

If X is finite, then $\mathcal{P}(X)$ is finite.

Proof. If $|X| = 0$, then $\mathcal{P}(X) = \{\varnothing\}$, which is indeed finite.

Fix any $n \in \mathbb{N}$ and assume that $\mathcal{P}(X)$ is finite for all X with $|X| = n$. Take any Y with $|Y| = n+1$. Let $Y = \{y_0, \cdots, y_n\}$ and $X \triangleq \{y_0, \cdots, y_{n-1}\}$. Note that $\mathcal{P}(Y) = \mathcal{P}(X) \cup U$ where $U = \{u \subseteq Y \mid y_n \in u\}$. Moreover, $f : \mathcal{P}(X) \to U$ defined by $f(x) = x \cup \{y_n\}$ is injective and onto U. Hence, U is finite. By Lemma 4.2.6, $\mathcal{P}(Y)$ is finite. The result follows by The Induction Principle. □

Theorem 4.2.9

If X is infinite, then $|X| > n$ for all $n \in \mathbb{N}$.

Proof. We clearly have $0 \leq |X|$.

For induction, fix any $n \in \mathbb{N}$ and assume $n \leq |X|$, i.e., there exists $f : n \hookrightarrow X$. By Theorem 4.2.5, $\operatorname{ran} f \subsetneq X$; we may take $x \in X \setminus \operatorname{ran} f$. Then, $g \triangleq f \cup \{(n, x)\}$ is an injective function on $n+1$ into X; hence $n+1 \leq |X|$. Therefore, by The Induction Principle, we have $n \geq |X|$ for all $n \in \mathbb{N}$, which is suffices to induce the result. □

Selected Problems

Exercise 4.2.1

If $S = \{X_0, \cdots, X_{n-1}\}$ is a finite set of mutually disjoint sets. Then, $\left|\bigcup S\right| = \sum_{i=0}^{n-1} |X_i|$.

Proof. If $S = \varnothing$, then $\left|\bigcup S\right| = 0 = \sum_{i=0}^{n-1} |X_i|$.

Fix $n \in \mathbb{N}$ and assume the assertion holds for all S with $|S| = n$. Then, take any set T of mutually disjoint sets with $|T| = n+1$. Write $T = \{X_0, \cdots, X_n\}$ and let $S \triangleq= \{X_0, \cdots, X_{n-1}\}$. Then, since $\bigcup T = \left(\bigcup S\right) \cup X_n$, and since $\bigcup S$ and X_n are disjoint, $\left|\bigcup T\right| = \left|\bigcup S\right| + |X_n| = \sum_{i=0}^{n-1} |X_i| + |X_n| = \sum_{i=0}^{n} |X_i|$. Hence, the result follows from The Induction Principle. □

Exercise 4.2.2

If X and Y are finite, then $|X \times Y| = |X| \cdot |Y|$.

Proof. We shall exploit the induction on $|Y|$. If $|Y| = 0$, then

$$\begin{aligned} |X \times Y| &= 0 && \triangleright \text{ Exercise 2.2.8} \\ &= |X| \cdot |Y|. && \triangleright \text{ [3]} \end{aligned}$$

Assume the statement holds for all X and Y with $|Y| = n$. Let $Z = \{z_0, \cdots, z_n\}$ be a set with $|Z| = n+1$. Let $Y \triangleq \{z_0, \cdots, z_{n-1}\}$. Then, for all X, $X \times Z = (X \times Y) \cup (X \times \{z_n\})$. Note that $X \times \{z_n\}$ can be identified with X via $f : X \hookrightarrow\!\!\!\!\rightarrow X \times \{z_n\}$ defined by $x \mapsto (x, z_n)$. Hence, if X is finite,

$$\begin{aligned} |X \times Z| &= |X \times Y| + |X \times \{z_n\}| && \triangleright \text{ Lemma 4.2.6} \\ &= |X \times Y| + |X| && \triangleright\ |X \times \{z_n\}| = |X| \\ &= |X| \cdot |Y| + |X| && \triangleright\ \mathbf{P}(n) \\ &= |X| \cdot (|Y| + 1) && \triangleright \text{ [4]} \\ &= |X| \cdot |Z|. \end{aligned}$$

Therefore, by The Induction Principle, the result follows. □

Exercise 4.2.3

If X is finite, $|\mathcal{P}(X)| = 2^{|X|}$.

Proof. Let $\mathbf{P}(x)$ be the property "$\forall X, (|X| = x \implies |\mathcal{P}(X)| = 2^{|X|})$." $\mathbf{P}(0)$ holds since $|\mathcal{P}(\varnothing)| = |\{\varnothing\}| = 1 = 2^0$.

Fix $n \in \mathbb{N}$ and assume $\mathbf{P}(n)$. Let $Y = \{y_0, \cdots, y_n\}$ be a set with $|Y| = n+1$. Let $X \triangleq \{y_0, \cdots, y_{n-1}\}$. As in the proof of Theorem 4.2.8, $\mathcal{P}(Y) = \mathcal{P}(X) \cup U$ where $U = \{u \subseteq$

$Y \mid y_n \in u\}$. Note that $\mathcal{P}(X) \cap U = \varnothing$ and $f : \mathcal{P}(X) \hookrightarrow\!\!\!\!\rightarrow U$ defined by $x \mapsto x \cup \{y_n\}$ asserts $|\mathcal{P}(X)| = |U|$. Therefore,

$$\begin{aligned}
|\mathcal{P}(Y)| &= |\mathcal{P}(X)| + |U| && \triangleright \text{ Lemma 4.2.6} \\
&= 2^n + 2^n && \triangleright\ |\mathcal{P}(X)| = |U|,\ \mathbf{P}(n) \\
&= 2^n \cdot 1 + 2^n \cdot 1 && \triangleright \text{ Lemma 3.4.11} \\
&= 2^n \cdot 2 && \triangleright \cdot \text{ Distributes Over } + \\
&= 2^{n+1}. && \triangleright\ [6]
\end{aligned}$$

Therefore, by The Induction Principle, the result follows. □

Exercise 4.2.4

If X and Y are finite, then X^Y is finite and $|X^Y| = |X|^{|Y|}$.

Proof. Let $\mathbf{P}(x)$ be the property "if X is finite and $|Y| = x$, then $|X^Y| = |X|^x$." $\mathbf{P}(0)$ holds since $|X^\varnothing| = |\{\varnothing\}| = 1 = |X|^0$ for all X.

Fix $n \in \mathbb{N}$ and assume $\mathbf{P}(n)$. Let $Y = \{y_0, \cdots, y_n\}$ be a set with $|Y| = n+1$. Let $Z \triangleq \{y_0, \cdots, y_{n-1}\}$. Take any finite set X.

We have $|X^Y| = |X^Z \times X|$ since we may define $f : X^Y \hookrightarrow\!\!\!\!\rightarrow X^Z \times X$ by $g \mapsto \left(g\big|_Z, g(y_n)\right)$. Hence,

$$\begin{aligned}
|X^Y| &= |X^Z \times X| \\
&= |X^Z| \cdot |X| && \triangleright \text{ Exercise 4.2.1} \\
&= |X|^n \cdot |X| && \triangleright\ \mathbf{P}(n) \\
&= |X|^{n+1}. && \triangleright\ [6]
\end{aligned}$$

The result follows by The Induction Principle. □

Exercise 4.2.6

X is finite if and only if every $\varnothing \subsetneq U \subseteq \mathcal{P}(X)$ has a $\subseteq$-maximal element.

Proof.

($\Rightarrow$) Let $|X| = n$ and $\varnothing \subsetneq U \subseteq \mathcal{P}(X)$. Since $|Y| \le n$ for all $Y \in U$, by Theorem 3.3.8, we may let $m \triangleq \max\{|Y| \mid Y \in U\}$.

There exists $Y \in U$ with $|Y| = m$. Then, for each $Y' \in U$ such that $Y \subseteq Y'$, we have $m \le |Y'|$ by Exercise 4.1.3 and $|Y'| \le m$ by definition of m; thus $|Y'| = |Y| = m$ by Cantor–Bernstein Theorem, which implies we may not have $Y \subsetneq Y'$ by Lemma 4.2.2. Hence, Y is a maximal element of U.

($\Leftarrow$) Assume X is infinite. Let $U = \{Y \subseteq X \mid Y \text{ is finite}\}$. (Note $\varnothing \in U$, hence $U \ne \varnothing$.) Take any $Y \in U$. Since $Y \subsetneq X$, we may take $x \in X \setminus Y$. Then, $Y \subsetneq Y \cup \{x\}$ and $Y \cup \{x\} \in U$ by Lemma 4.2.6. Hence, there is no maximal element of U. □

4.3 Countably Infinite sets

Definition 4.3.1: Countably Infinite Set

- A set S is *countably infinite* if $|S| = |\mathbb{N}|$.
- A set S is *countable* if $|S| \le |\mathbb{N}|$.
- $|\mathbb{N}| = \mathbb{N}$, i.e., the cardinality of countably infinite sets is $\mathbb{N}$.

Note:-

In the book, the author uses the term 'countable' and 'at most countable' for $|S| = |\mathbb{N}|$ and $|S| \le |\mathbb{N}|$, respectively.

Notation 4.3.2: Cardinality of Countably Infinite Sets

We use the symbol $\aleph_0$ (read *aleph-naught*) to denote the cardinality of countably infinite sets, i.e., $\aleph_0 = \mathbb{N}$.

Theorem 4.3.3

A subset of a countably infinite set is countable.

Proof. Assume A is countably infinite and $B \subseteq A$ is infinite. Let $\langle a_i \rangle_{i\in\mathbb{N}}$ be an injective sequence whose range is A.

Let $g\colon \operatorname{Seq}(\mathbb{N}) \to \mathbb{N}$ be defined by

$$g(k) \triangleq \min\left\{ i \in \mathbb{N} \,\middle|\, a_i \in B \setminus \{ a_{k_j} \mid j \in \operatorname{dom} k \} \right\}.$$

Note that g is well-defined since B is infinite. Then, by The Recursion Theorem: General Version, there exists a sequence $\langle k_i \rangle_{i\in\mathbb{N}}$ of natural numbers such that $\forall n \in \mathbb{N}$, $k_n = g\left(k\big|_n\right)$. By construction, $\langle k_i \rangle_{i\in\mathbb{N}}$ is injective, and thus $\langle a_{k_i} \rangle_{i\in\mathbb{N}}$ is an injective sequence whose range is B by the same argument of Claim 1 of Theorem 3.1.3. □

Corollary 4.3.4

A set is countable if and only if it is either finite or countably infinite.

Proof.

($\Rightarrow$) Let S be countable. Let $f : S \hookrightarrow \mathbb{N}$. Then, $|S| = |\operatorname{ran} f|$ and $\operatorname{ran} f$ is a subset of $\mathbb{N}$. Hence, by Theorem 4.3.3, S is countably infinite if it is not finite.

($\Leftarrow$) Theorem 4.2.9 □

Theorem 4.3.5

If X is countably infinite and f is a function, then $f[X]$ is countable.

Proof. If $f[X] = \varnothing$, then it is done; assume $f[X] \ne \varnothing$. WLOG, $X \subseteq \operatorname{dom} f$. Let $\langle x_i \rangle_{i\in\mathbb{N}}$ be an injective sequence whose range is X. Let $g\colon f[X] \to \mathbb{N}$ be defined by

$$g(y) \triangleq \min\{ i \in \mathbb{N} \mid y = f(x_i) \}.$$

g is injective, and thus $|f[X]| \le \aleph_0$. □

Theorem 4.3.6

(i) If A and B are countably infinite, then $A \times B$ is countably infinite.

(ii) If A is countably infinite and $B \ne \varnothing$ is finite, then $A \times B$ is countably infinite.

(iii) If A and B are countable, then $A \times B$ is countable.

Proof.

(i) The function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f(x, y) = 2^x \cdot 3^y$ is injective by elementary number theory. Also, we have an injection $g\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ defined by $g(x) = (x, 0)$. Hence, by Cantor–Bernstein Theorem, we have $|\mathbb{N} \times \mathbb{N}| = \aleph_0$.

(ii) Let $|B| = n$. Then, we have

$$\begin{aligned} |A \times B| &= |\mathbb{N} \times n| \\ &\leq |\mathbb{N} \times \mathbb{N}| && \triangleright \text{ Exercise 4.1.3} \\ &= \aleph_0. && \triangleright \text{ Theorem 4.3.6} \end{aligned}$$

Let $b \in B$. Then, we have

$$\begin{aligned} \aleph_0 &= |A| \\ &= |A \times \{b\}| && \triangleright\ a \mapsto (a, b) \\ &\leq |A \times B|. && \triangleright \text{ Exercise 4.1.3} \end{aligned}$$

Hence, by Cantor–Bernstein Theorem, $|A \times B| = \aleph_0$.

(iii) If one of them is empty, then $A \times B = \varnothing$. If A and B are finite, then $A \times B$ is finite by Exercise 4.2.2. If any of them is countably infinite, and if both are nonempty, then $A \times B$ is countably infinite by (i) and (ii). □

Corollary 4.3.7

Let $\langle A_i \mid i \in n \rangle$ be a system of countably infinite sets where $n > 0$. Then, $\prod_{i=0}^{n-1} A_i$ is countably infinite.

Proof. Let $\mathbf{P}(x)$ be the property "$\prod_{i=0}^{x-1} A_i$ is countably infinite for each system $\langle A_i \mid i \in x \rangle$ of countably infinite sets. $\mathbf{P}(1)$ evidently holds.

Fix $n > 0$ and assume $\mathbf{P}(n)$. Now, take any system $\langle A_i \mid i \in n+1 \rangle$ of countably infinite sets. Then, since we have a natural mapping $f : \prod_{i=0}^{n} A_i \hookrightarrow\!\!\!\!\rightarrow \left(\prod_{i=0}^{n-1} A_i\right) \times A_n$ defined by $\langle a_0, \cdots, a_n \rangle \mapsto (\langle a_0, \cdots, a_{n-1} \rangle, a_n)$, we get

$$\begin{aligned} \left|\prod_{i=0}^{n} A_i\right| &= \left|\left(\prod_{i=0}^{n-1} A_i\right) \times A_n\right| \\ &= |\mathbb{N} \times \mathbb{N}| && \triangleright\ \mathbf{P}(n) \\ &= \aleph_0. && \triangleright \text{ Theorem 4.3.6} \end{aligned}$$

Hence, we have $\mathbf{P}(n+1)$.

Therefore, by Exercise 3.3.11, the result follows. □

Theorem 4.3.8

Let $\langle a_n \mid n \in \mathbb{N} \rangle$ countably infinite system of infinite sequences. Then, $\bigcup_{n\in\mathbb{N}} \operatorname{ran} a_n$ is countable.

Proof. Define $f : \mathbb{N} \times \mathbb{N} \twoheadrightarrow \bigcup_{n\in\mathbb{N}} \operatorname{ran} a_n$ by $f(n, k) = a_n(k)$. The result follows by Theorem 4.3.5 and Theorem 4.3.6. □

Note:-

Note that we cannot yet prove the proposition "the union of countably infinite system of countable sets is countable" since, if $\langle A_n \mid n \in \mathbb{N} \rangle$ is the system, we do not have enough tools to show the existence of $\langle a_n \mid n \in \mathbb{N} \rangle$ such that $\operatorname{ran} a_n = A_n$ for each $n \in \mathbb{N}$.

Theorem 4.3.9

If A is countably infinite, then $\operatorname{Seq}(A)$ is countably infinite.

Proof. It is enough to show $\mathrm{Seq}(\mathbb{N}) = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ is countably infinite. Fix any $f : \mathbb{N} \twoheadrightarrow \mathbb{N} \times \mathbb{N}$. Define $\langle a_n \mid n \in \mathbb{N} \rangle$ recursively by

$$\begin{aligned} \forall i \in \mathbb{N}, \quad & a_0(i) = \langle\rangle \\ \forall n, i \in \mathbb{N}, a_{n+1}(i) &= \langle b_0, \cdots, b_{n-1}, i_2 \rangle \\ & \text{where } g(i) = (i_1, i_2) \text{ and } a_n(i_1) = \langle b_0, \cdots, b_{n-1} \rangle. \end{aligned}$$

The existence is justified by The Recursion Theorem. Then, with The Induction Principle, it is easy to prove that $\operatorname{ran} a_n = \mathbb{N}^n$ for each $n \in \mathbb{N}$. Hence, by Theorem 4.3.8, $\bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ is countably infinite. □

Corollary 4.3.10
The set of all finite subsets of a countably infinite set is countably infinite.

Proof. Let A be countably infinite. Let $f : \mathrm{Seq}(A) \to \mathcal{P}(A)$ by $f(\langle a_0, \cdots, a_{n-1} \rangle) = \{a_0, \cdots, a_{n-1}\}$. Then, $\operatorname{ran} f$ is countable by Theorem 4.3.5 and Theorem 4.3.9. $\operatorname{ran} f$ is countably infinite since we have an injection $a \mapsto \{a\}$. □

Theorem 4.3.11
An equivalence on a countably infinite set has at most countably many equivalence classes.

Proof. Let E be an equivalence on a countably infinite set A. Let $f : A \twoheadrightarrow A/E$ be defined by $a \mapsto [a]_E$. Hence, by Theorem 4.3.5, A/E is countable. □

Theorem 4.3.12
Let $\mathfrak{A} = (A, \langle R_0, \cdots, R_{m-1} \rangle, \langle F_0, \cdots, F_{n-1} \rangle)$ be a structure. If $C \subseteq A$ is countable, then $\overline{C}$ is also countable.

Proof. Theorem 3.5.15 says that $\overline{C} = \bigcup_{i\in\mathbb{N}} C_i$ where $C_0 = C$ and $C_{i+1} = C_i \cup \bigcup_{j=0}^{n-1} F_j[C_i^{f_j}]$.

Let $c : \mathbb{N} \twoheadrightarrow C$. Let $g : \mathbb{N} \twoheadrightarrow (n+1) \times \mathbb{N} \times \mathbb{N}^{f_0} \times \cdots \times \mathbb{N}^{f_{n-1}}$. Now, define $\langle a_i \mid i \in \mathbb{N} \rangle$ recursively by

$$\begin{aligned} \forall k \in \mathbb{N} \qquad & a_0(k) \triangleq c(k) \\ \forall i, k \in \mathbb{N}, \quad & a_{i+1}(k) \triangleq \begin{cases} F_p(a_i(r_p^0), \cdots, a_i(r_p^{f_p - 1})) & \text{if } 0 \le p < n \\ a_i(q) & \text{if } p = n \end{cases} \\ & \text{where } g(k) = \langle p, q, \langle r_0^0, \cdots, r_0^{f_0-1} \rangle, \cdots, \langle r_{n-1}^0, \cdots, r_{n-1}^{f^{n-1}-1} \rangle \rangle. \end{aligned}$$

It is apparent by The Induction Principle that $\operatorname{ran} a_i = C_i$ for each $i \in \mathbb{N}$. Hence, by Theorem 4.3.8, $\overline{C}$ is countable. □

Selected Problems

Exercise 4.3.1
Let $|A_1| = |A_2|$ and $|B_1| = |B_2|$.

(i) If $A_1 \cap B_1 = \varnothing$ and $A_2 \cap B_2 = \varnothing$, then $|A_1 \cup B_1| = |A_2 \cup B_2|$.
(ii) $|A_1 \times B_1| = |A_2 \times B_2|$
(iii) $|\mathrm{Seq}(A_1)| = |\mathrm{Seq}(A_2)|$

Proof.

(i) Let $f_A : A_1 \hookrightarrow\!\!\!\!\rightarrow A_2$ and $f_B : B_1 \hookrightarrow\!\!\!\!\rightarrow B_2$. Then, $f_A \cup f_B : A_1 \cup B_1 \hookrightarrow\!\!\!\!\rightarrow A_2 \cup B_2$.

(ii) Let $f_A : A_1 \hookrightarrow\!\!\!\!\rightarrow A_2$ and $f_B : B_1 \hookrightarrow\!\!\!\!\rightarrow B_2$. We may define $g : A_1 \times B_1 \hookrightarrow\!\!\!\!\rightarrow A_2 \times B_2$ by $(a, b) \mapsto (f_A(a), f_B(b))$.

(iii) Let $f : A_1 \hookrightarrow\!\!\!\!\rightarrow A_2$. We may define $g :\ \mathrm{Seq}(A_1) \hookrightarrow\!\!\!\!\rightarrow \mathrm{Seq}(A_2)$ by

$$\langle a_0, \cdots, a_{n-1} \rangle \mapsto \langle f(a_0), \cdots, f(a_{n-1}) \rangle.$$

□

Exercise 4.3.2

If A is finite and B is countably infinite, then $A \cup B$ is countably infinite.

Proof. Let $f_A : A \hookrightarrow \mathbb{N}$ and $f_B : B \hookrightarrow \mathbb{N}$. Then, we may define $g : A \cup B \hookrightarrow \mathbb{N} \times \mathbb{N}$ by

$$g(x) = \begin{cases} (f_A(x), 0) & \text{if } x \in A \\ (f_B(x), 1) & \text{if } x \in B \setminus A \end{cases}$$

Hence, $|A \cup B| \leq \aleph_0$ by Theorem 4.3.6. Moreover, $\aleph_0 = |B| \leq |A \cup B|$ by Exercise 4.1.3. The result follows from Cantor–Bernstein Theorem. □

Exercise 4.3.4

If A is finite and nonempty, then $\mathrm{Seq}(A)$ is countably infinite.

Proof. Let $B \triangleq A \cup \mathbb{N}$. Then, by Exercise 4.3.2, B is countably infinite and $\mathrm{Seq}(B)$ is countably infinite by Theorem 4.3.9. Hence, as $\mathrm{Seq}(A) \subseteq \mathrm{Seq}(B)$, $|\mathrm{Seq}(A)| \leq \aleph_0$.

Fix any $a \in A$. Let s be the infinite sequence with $\forall i \in \mathbb{N}$, $s_i = a$. Then, we have $f : \mathbb{N} \hookrightarrow \mathrm{Seq}(A)$ defined by $f(n) = s\big|_n$; thus $\aleph_0 \leq |\mathrm{Seq}(A)|$. The result follows from Cantor–Bernstein Theorem. □

Exercise 4.3.5

Let A be countably infinite. The set $[A]^n = \{ S \subseteq A \mid |S| = n \}$ is countably infinite for all $n > 0$.

Proof. It is enough to show that $[\mathbb{N}]^n$ is countably infinite for all $n > 0$. Evidently, $i \mapsto \{i\}$ is an injective mapping on $\mathbb{N}$ onto $[\mathbb{N}]^1$. Hence, $|[\mathbb{N}]^1| = \aleph_0$.

For the sake of induction, fix $n > 0$ and assume $|[\mathbb{N}]^n| = \aleph_0$. We may define $f : [\mathbb{N}]^n \hookrightarrow [\mathbb{N}]^{n+1}$ by

$$f(x) \triangleq x \cup \big\{ \max\{ i \in \mathbb{N} \mid i \in x \} + 1 \big\}.$$

Hence, $\aleph_0 \leq |[\mathbb{N}]^{n+1}|$.

Now, since $|[\mathbb{N}]^n| = |\mathbb{N}^n| = \aleph_0$ by Corollary 4.3.7, there exists an injection $g : [\mathbb{N}]^n \hookrightarrow \mathbb{N}^n$. We define $h : [\mathbb{N}]^{n+1} \hookrightarrow \mathbb{N}^{n+1}$ by

$$h(x) \triangleq g(x \setminus \{i\}) \cup \{(n, i)\}$$
$$\text{where } i = \max x.$$

Hence, $|[\mathbb{N}]^{n+1}| \leq |\mathbb{N}^{n+1}| \leq \aleph_0$. Exercise 3.3.11 assures that $\forall n > 0$, $|[\mathbb{N}]^n| = \aleph_0$. □

Exercise 4.3.6

A sequence $\langle s_n \rangle_{n=0}^{\infty}$ of natural numbers is *eventually constant* if

$$\exists n_0, s \in \mathbb{N},\ \forall n \geq n_0,\ s_n = s.$$

The set of eventually constant sequences of natural numbers is countable.

Proof. Let P be the set of eventually constant sequences of natural numbers. As $P \subseteq Q$ where Q is the set of eventually periodic sequences of natural numbers (see Exercise 4.3.7), we have $|P| \leq \aleph_0$ by Exercises 4.1.3 and 4.3.7.

Moreover, we may define an injective infinite sequence into P recursively by

$$\begin{aligned} g_0 &= \langle 0, 0, \cdots \rangle \\ \forall n \in \mathbb{N}, \quad g_{n+1} &= \langle n+1, a_0, a_1, \cdots \rangle \\ &\qquad \text{where } g_n = \langle a_0, a_1, \cdots \rangle. \end{aligned}$$

Hence, $\aleph_0 \leq |P|$. The result follows by Cantor–Bernstein Theorem. $\square$

Exercise 4.3.7

A sequence $\langle s_n \rangle_{n=0}^{\infty}$ of natural numbers is *eventually periodic* if

$$\exists n_0 \in \mathbb{N},\ \exists p > 0,\ \forall n \geq n_0,\ s_{n+p} = s_n.$$

The set of eventually periodic sequences of natural numbers is countably infinite.

Proof. Let Q be the set of eventually periodic sequences of natural numbers. We may define $f : Q \to \operatorname{Seq}(\mathbb{N}) \times \mathbb{N}$ by

$$\begin{aligned} f(x) &\triangleq \left(x|_{n^*+p^*}, p^* \right) \\ &\qquad \text{where } n^* = \min\{ n_0 \in \mathbb{N} \mid \exists p > 0,\ \forall n \geq n_0,\ s_{n+p} = s_n \} \\ &\qquad \text{and } p^* = \min\{ p > 0 \mid \forall n \geq n^*,\ s_{n+p} = s_n \}. \end{aligned}$$

Then, it can be easily shown that f is injective. Hence, $|Q| \leq |\operatorname{Seq}(\mathbb{N}) \times \mathbb{N}| = \aleph_0$ by Theorems 4.3.6 and 4.3.9.

Moreover, as $P \subseteq Q$ where P is the set of eventually constant sequences of natural numbers and $\aleph_0 \leq |P|$ by Exercise 4.3.6, we have $|Q| = \aleph_0$ by Exercise 4.1.3 and Cantor–Bernstein Theorem. $\square$

Exercise 4.3.10

Let $(S, \preceq)$ be a linearly ordered set and let $\langle A_n \mid n \in \mathbb{N} \rangle$ be an infinite sequence of finite subsets of S. Then, $\bigcup_{n=0}^{\infty} A_n$ is countable.

Proof. WLOG, $A_n \neq \varnothing$ for each $n \in \mathbb{N}$.

Claim 1. For each finite $A \subseteq S$, there uniquely exists a unique isomorphism between $\left(|A|, \leq \cap |A|^2\right)$ and $(A, \preceq \cap A^2)$.

Proof. We have existence for each A by Theorem 4.4.3. Hence, we only prove the uniqueness by induction. If $|A| = 0$, we have only one isomorphism $\varnothing$.

Fix some $n \in \mathbb{N}$ and assume the proposition holds for all A with cardinality n. Take any $A \subseteq S$ with $|A| = n+1$. Let f and g be two isomorphisms between $(n+1, \leq \cap (n+1)^2)$ and $(A, \leq \cap A^2)$. Then, $f(n) = g(n)$ since the greatest element is unique. Let $B = A \setminus \{f(n)\}$. Then, $f\big|_n$ and $g\big|_n$ are isomorphisms between $(n, \leq \cap n^2)$ and $(B, \leq \cap B^2)$. Hence, $f\big|_n = g\big|_n$, and thus $f = g$. The result follows from The Induction Principle.

Claim 1 enables us to guarantee the existence of infinite sequence $\langle a_n \mid n \in \mathbb{N} \rangle$ such that, for each $n \in \mathbb{N}$:

(i) $a_n\big|_{|A_n|}$ is the isomorphism between $\left(|A_n|, \leq\big|_{|A_n|^2}\right)$ and $\left(A_n, \preceq\big|_{A_n^2}\right)$.

(ii) $\forall k \geq |A_n|,\ a_n(k) = a_n(0)$.

Hence, $\operatorname{ran} a_n = A_n$ for each $n \in \mathbb{N}$, and thus $\bigcup_{n=0}^{\infty} A_n$ is countable by Theorem 4.3.8. □

Exercise 4.3.11

Any partition of a countable set has a set of representatives.

Proof. Let A be countable and S be a partition of A. There exists $f : A \hookrightarrow \mathbb{N}$. Then,

$$X \triangleq \{ f^{-1}(\min f[C]) \mid C \in S \}$$

is a set of representatives. □

4.4 Linear Orderings

Definition 4.4.1: Similar Ordered Sets

Totally ordered sets $(A, \leq)$ and $(B, \preceq)$ are *similar* (have *the same order type*) if they are isomorphic. (Definition 2.5.14)

Lemma 4.4.2

Every total ordering on a finite set is a well-ordering.

Proof. Let $(A, \leq)$ be a finite totally ordered set. If $B \subseteq A$ has $|B| = 1$, then the only element of B is $\min B$.

Now, fix $n > 0$ and assume that every $B \subseteq A$ with $|B| = n$ has a least element. Take any $B \subseteq A$ with $|B| = n+1$ and write $B = \{ b_0, \cdots, b_n \}$. Let $C \triangleq \{ b_0, \cdots, b_{n-1} \}$. Then, if $b_n \leq \min C$, then b_n is a least element of B; otherwise, $\min C$ is a least element of B. Hence, by Exercise 3.3.11, every nonempty finite subset of A has a least element, i.e., $(A, \leq)$ is well-ordered. □

Theorem 4.4.3

If $(A_1, \leq_1)$ and $(A_2, \leq_2)$ are finite totally ordered sets with the same cardinality, then $(A_1, \leq_1)$ and $(A_2, \leq_2)$ are similar.

Proof. We shall conduct the induction on the size of the sets. If $A_1 = A_2 = \varnothing$, then they are evidently similar by the isomorphism $\varnothing$.

Fix $n \in \mathbb{N}$ and assume the proposition holds whenever $|A_1| = |A_2| = n$. Take any totally ordered sets $(A_1, \leq_1)$ and $(A_2, \leq_2)$ such that $|A_1| = |A_2| = n+1$. By Lemma 4.4.2, there exist $a_1 = \min A_1$ and $a_2 = \min A_2$. Let $A_1' \triangleq A_1 \setminus \{a_1\}$ and $A_2' \triangleq A_2 \setminus \{a_2\}$. Since $(A_1', \leq_1 \cap A_1'^2)$ and $(A_2', \leq_2 \cap A_2'^2)$ are finite totally ordered sets with $|A_1'| = |A_2'| = n$, there exists an isomorphism

$g: A_1' \hookrightarrow\!\!\!\!\rightarrow A_2'$ by the induction hypothesis. Then, $f \triangleq g \cup \{(a_1, a_2)\}$ is an isomorphism between $(A_1, \leq_1)$ and $(A_2, \leq_2)$. Therefore, the result follows from The Induction Principle. $\square$

Lemma 4.4.4

If $(A, \leq)$ is a totally ordered set, then $(A, \leq^{-1})$ is also a totally ordered set.

Proof. Take any $a, b \in A$. Then, it is $a \leq b$ or $b \leq a$. If $a \leq b$, then $b \leq^{-1} b$. If $b \leq a$, then $a \leq^{-1} b$. Hence, $(A, \leq^{-1})$ is totally ordered. $\square$

Lemma 4.4.5

Let $(A_1, \leq_1)$ and $(A_2, \leq_2)$ be totally ordered sets such that $A_1 \cap A_2 = \varnothing$. The relation $\leq$ on $A = A_1 \cup A_2$ defined by

$$a \leq b \iff (a \leq_1 b) \vee (a \leq_2 b) \vee (a \in A_1 \wedge b \in A_2)$$

is a total ordering.

Proof. Exercise 2.5.6 already shows that $\leq$ is an ordering of A. Totality follows directly by the definition. $\square$

Lemma 4.4.6

Let $(A_1, \leq_1)$ and $(A_2, \leq_2)$ be totally ordered sets. The relation $\leq$ on $A = A_1 \times A_2$ defined by

$$(a_1, a_2) \leq (b_1, b_2) \iff a_1 <_1 b_1 \vee (a_1 = b_1 \wedge a_2 \leq_2 b_2)$$

is a total ordering.

Proof.
- Assume $(a_1, a_2) < (b_1, b_2)$ and $(b_1, b_2) < (c_1, c_2)$. If $a_1 <_1 b_1$, then, we have $a_1 <_1 c_1$ by $b_1 <_1 c_1$. If $b_1 <_1 c_1$, then, we have $a_1 <_1 c_1$ by $a_1 \,{}_1 b_1$. In the only left case, we have $a_1 = b_1 = c_1$ and $a_2 \leq_2 b_2 \leq_2 c_2$. Hence, $(a_1, a_2) < (c_1, c_2)$. Thus $<$ is transitive in A. ✓
- Assume $(a_1, a_2) < (b_1, b_2)$ and $(b_1, b_2) < (a_1, a_2)$. Since $a_1 \leq_1 b_1$ and $b_1 \leq_1 a_1$, by antisymmetry of $\leq_1$, $a_1 = b_1$. The only option now is $a_2 \leq_2 b_2$ and $b_2 \leq_2 a_2$, which implies $a_2 = b_2$ by the antisymmetry of $\leq_2$. Hence, $(a_1, a_2) = (b_1, b_2)$, which is a contradiction. Thus, $<$ is asymmetric in A. ✓
- Let $(a_1, a_2), (b_1, b_2) \in A_1 \times A_2$. As $\leq_1$ is total, WLOG, $a_1 \leq_1 b_1$. If $a_1 < b_1$, then we immediately have $(a_1, a_2) < (b_1, b_2)$. Now, assume $a_1 = b_1$. Then, as $\leq_2$ is total, WLOG, $a_2 \leq_2 b_2$, and thus $(a_1, a_2) \leq (b_1, b_2)$. Hence, $\leq$ is a total ordering. ✓ $\square$

Theorem 4.4.7

Let $\langle (A_i, \leq_i) \mid i \in I \rangle$ be an indexed system of totally ordered sets where $I \subseteq \mathbb{N}$. The relation $\prec$ on $\prod_{i \in I} A_i$ defined by

$$f \prec g \iff \operatorname{diff}(f, g) \triangleq \{ i \in I \mid f_i \neq g_i \} \neq \varnothing \wedge f_{i_0} <_{i_0} g_{i_0}$$
$$\text{where } i_0 = \min\nolimits_{\leq} \operatorname{diff}(f, g)$$

is a total strict ordering of $\prod_{i \in I} A_i$.

Proof.
- Assume $f \prec g$ and $g \prec h$ and let $i_0 = \min \operatorname{diff}(f, g)$ and $j_0 = \min \operatorname{diff}(g, h)$.

- If $i_0 \le j_0$, then $f_{i_0} < g_{i_0} \le h_{i_0}$ and $\operatorname{diff}(f,h) = i_0$.
- If $j_0 < i_0$, then $f_{j_0} = g_{j_0} < h_{j_0}$ and $\operatorname{diff}(f,h) = j_0$.

Hence, $f \prec h$; $\prec$ is transitive in $\prod_{i\in I} A_i$. ✓

- For $f, g \in \prod_{i\in I} A_i$ with $f \ne g$, since $i_0 = \operatorname{diff}(f,g) = \operatorname{diff}(g,f)$, we cannot have $f \prec g$ and $g \prec f$ because of the asymmetry of $<_{i_0}$. ✓
- If $\operatorname{diff}(f,g) = \varnothing$, we have $f = g$. If $i_0 = \min \operatorname{diff}(f,g)$, then we have $f \prec g$ when $f_{i_0} <_{i_0} g_{i_0}$ and $g \prec f$ when $g_{i_0} <_{i_0} f_{i_0}$. Hence, $\prec$ is a total ordering. ✓

Definition 4.4.8: Dense Ordered Set

An ordered set $(X, \le)$ is *dense* if

$$2 \le |X| \wedge \forall a, b \in X, (a < b \implies \exists x \in X, a < x < b).$$

Definition 4.4.9: Endpoints

We now will call the least and greatest elements of a totally ordered set *endpoints* of the set.

Theorem 4.4.10

Let $(P, \preceq)$ and $(Q, \le)$ be countably infinite dense totally ordered sets without endpoints. Then, $(P, \preceq)$ and $(Q, \le)$ are similar.

Proof. Let $\langle p_n \mid n \in \mathbb{N}\rangle$ be an injective sequence onto P. Let $\langle q_n \mid n \in \mathbb{N}\rangle$ be an injective sequence onto Q. Let us call $h\colon P \rightharpoonup Q$ a *partial isomorphism* from P to Q if

$$\forall p, p' \in \operatorname{dom} h, (p \prec p' \iff h(p) < h(p')).$$

Claim 1. If h is a partial isomorphism from P to Q with finite $\operatorname{dom} h$, and if $p \in P$ and $q \in Q$, then there exists a partial isomorphism $h_{p,q}$ from P to Q that extends h such that $p \in \operatorname{dom} h_{p,q}$ and $q \in \operatorname{ran} h_{p,q}$.

Proof. Write $h = \{(p_{i_0}, q_{i_0}), \cdots, (p_{i_k}, q_{i_k})\}$ where $p_{i_0} \prec p_{i_1} \prec \cdots \prec p_{i_k}$ and thus $q_{i_0} < q_{i_1} < \cdots < q_{i_k}$. (This is justified by Theorem 4.4.3.) We have four cases:

- Assume $p \in \operatorname{dom} h$. Then, let $h' \triangleq h$.
- Assume $p \prec p_{i_0}$. Then, as Q has no least element, $n = \min\{ i \in \mathbb{N} \mid q_i < q_{i_0}\}$ exists. Let $h' \triangleq h \cup \{(p, q_n)\}$.
- Assume there exists $e < k$ such that $p_{i_e} \prec p \prec p_{i_{e+1}}$. Then, as Q is dense, $n = \min\{ i \in \mathbb{N} \mid q_{i_e} < q_i < q_{i_{e+1}}\}$ exists. Let $h' \triangleq h \cup \{(p, q_n)\}$.
- Assume $p_{i_k} \prec p$. Then, as Q has no greatest element, $n = \min\{ i \in \mathbb{N} \mid q_{i_k} < q_i\}$ exists. Let $h' \triangleq h \cup \{(p, q_n)\}$.

Then, h' is a partial isomorphism from P to Q and $p \in \operatorname{dom} h'$. Similarly, one may extend h' in the same way so it has q in its range. □

Now, we may create a sequence of compatible partial isomorphisms from P to Q recursively by

$$h_0 = \varnothing$$
$$\forall n \in \mathbb{N}, \quad h_{n+1} = (h_n)_{p_n, q_n}$$

where $(h_n)_{p_n,q_n}$ is the extension of h_n (provided by the steps in the proof of Claim 1) such that $p_n \in \operatorname{dom}[(h_n)_{p_n,q_n}]$ and $q_n \in \operatorname{ran}[(h_n)_{p_n,q_n}]$. Then, $h \triangleq \bigcup_{n\in\mathbb{N}} h_n$ is a function by Theorem 2.3.12. It is easy to check if $h\colon P \hookrightarrow\!\!\!\!\rightarrow Q$ is a desired isomorphism. □

Theorem 4.4.11

Let $(P,\preceq)$ be a countably infinite totally ordered set, and let $(Q,\leq)$ be a countably infinite dense totally ordered set without endpoints. Then, there exists $h\colon P \hookrightarrow Q$ such that

$$\forall p,p' \in P,\ (p \prec p' \implies h(p) < h(p')).$$

Proof. This is essentially the one-sided version of Theorem 4.4.10. Let $\langle p_n \mid n \in \mathbb{N}\rangle$ be an injective sequence onto P. In a similar way as Claim 1 in the proof of Theorem 4.4.10, if f is a partial isomorphism from P to Q with finite $\operatorname{dom} f$, and if $p \in P$, there exists another partial isomorphism f_p from P to Q that extends f such that $p \in \operatorname{dom} f_p$.

Then, one is able to make a sequence of compatible partial isomorphisms from P to Q recursively by

$$h_0 = \varnothing$$
$$\forall n \in \mathbb{N},\quad h_{n+1} = (h_n)_{p_n}$$

where $(h_n)_{p_n}$ is the extension of h_n such that $p_n \in \operatorname{dom}[(h_n)_{p_n}]$. The rest is the same as the proof of Theorem 4.4.10. $\square$

Selected Problems

Exercise 4.4.1

Assume that $(A_1,\leq_1)$ is similar to $(B_1,\preceq_1)$ and $(A_2,\leq_2)$ is similar to $(B_2,\preceq_2)$.

(i) Assuming $A_1 \cap A_2 = B_1 \cap B_2 = \varnothing$, the sum of $(A_1,\leq_1)$ and $(A_2,\leq_2)$ is similar to the sum of $(B_1,\preceq_1)$ and $(B_2,\preceq_2)$. (See Lemma 4.4.5.)

(ii) The lexicographic product of $(A_1,\leq_1)$ and $(A_2,\leq_2)$ is similar to the lexicographic product of $(B_1,\preceq_1)$ and $(B_2,\preceq_2)$.

Proof. Let f_1 be an isomorphism between $(A_1,\leq_1)$ and $(B_1,\preceq_1)$, and let f_2 be an isomorphism between $(A_2,\leq_2)$ and $(B_2,\preceq_2)$.

(i) $g \triangleq f_1 \cup f_2$ is an isomorphism between 'the sum of $(A_1,\leq_1)$ and $(A_2,\leq_2)$' and 'the sum of $(B_1,\preceq_1)$ and $(B_2,\preceq_2)$'.

(ii) $g\colon A_1 \times A_2 \hookrightarrow\!\!\!\!\rightarrow B_1 \times B_2$ defined by $(a_1,a_2) \mapsto (f_1(a_1), f_2(a_2))$ is an isomorphism between 'the lexicographic product of $(A_1,\leq_1)$ and $(A_2,\leq_2)$' and 'the lexicographic product of $(B_1,\preceq_1)$ and $(B_2,\preceq_2)$'. $\square$

Exercise 4.4.2

Give an example of linear orderings $(A_1,\leq_1)$ and $(A_2,\leq_2)$ such that the sum of $(A_1,\leq_1)$ and $(A_2,\leq_2)$ does not have the same order type as the sum of $(A_2,\leq_2)$ and $(A_1,\leq_1)$. Do the same for lexicographic product.

Proof. Let $A = \{a,b\}$ where $a \neq b$ and $a,b \notin \mathbb{N}$ with $\preceq = \{(a,a),(a,b),(b,b)\}$.

- The sum of $(\mathbb{N},\leq)$ and $(A,\preceq)$ has a greatest element b but the sum of $(A,\preceq)$ and $(\mathbb{N},\leq)$ does not have a greatest element. Hence, they are not similar.
- The lexicographic product of $(\mathbb{N},\leq)$ and $(A,\preceq)$ is isomorphic to $(\mathbb{N},\leq)$ via the isomorphism $(n,x) \mapsto \begin{cases} 2\cdot n & \text{if } x = a \\ 2\cdot n+1 & \text{if } x = b. \end{cases}$ However, $\{a\}\times\mathbb{N}$, a subset of $A\times\mathbb{N}$, is bounded above by $(b,0)$ but it does not have a greatest element in the lexicographic product of $(A,\preceq)$ and $(\mathbb{N},\leq)$. Hence, they are not similar. $\square$

Exercise 4.4.3

The sum and the lexicographic product of two well-orderings are well-orderings.

Proof. Let $(A_1, \le_1)$ and $(A_2, \le_2)$ be well-orderings.

- Assume $A \cap B = \varnothing$. Let $(B, \le)$ be the sum of $(A_1, \le_1)$ and $(A_2, \le_2)$. Take any $C \subseteq B$ with $C \neq \varnothing$. If $C \cap A_1 = \varnothing$, then $\min_{\le} C = \min_{\le_2} C$. Otherwise, $\min_{\le} C = \min_{\le_1}(C \cap A_1)$.
- Let $(A_1 \times A_2, \le)$ be the lexicographic product of $(A_1, \le_1)$ and $(A_2, \le_2)$. Take any $R \subseteq A_1 \times A_2$ with $R \neq \varnothing$. Let $a_1 \triangleq \min_{\le_1} \operatorname{dom} R$ and $a_2 \triangleq \min_{\le_2} R[\{a_1\}]$. Then, $(a_1, a_2) = \min_{\le} R$. □

Exercise 4.4.4

If $\langle A_i \mid i \in \mathbb{N} \rangle$ is an infinite sequence of totally ordered sets of natural numbers and $|A_i| \ge 2$ for all $i \in \mathbb{N}$, then the lexicographic ordering of $\prod_{i\in\mathbb{N}} A_i$ is *not* a well-ordering.

Proof. For each $i \in \mathbb{N}$, justified by $|A_i| \ge 2$, let $a_i \triangleq \min A_i$ and $b_i \triangleq \min(A_i \setminus \{a_i\})$. Define $X \subseteq \prod_{i\in\mathbb{N}} A_i$ by

$$X \triangleq \left\{ f \in \textstyle\prod_{i\in\mathbb{N}} A_i \,\middle|\, \exists i \in \mathbb{N}, [f(i) = b_i \wedge \forall j \in \mathbb{N}, (i \neq j \implies f(j) = a_j)] \right\}.$$

Then, X, being similar to $(\mathbb{N}, \le^{-1})$, does not have a least element. □

Exercise 4.4.5

Let $\langle (A_i, \le_i) \mid i \in I \rangle$ be an indexed system of mutually disjoint totally ordered sets where $I \subseteq \mathbb{N}$. The relation $\prec$ on $\bigcup_{i\in I} A_i$ defined by

$$a \prec b \iff (\exists i \in I,\ a <_i b) \vee (\exists i, j \in I,\ i < j \wedge a \in A_i \wedge b \in A_j)$$

is a strict total ordering. Moreover, if all $\le_i$ are well-orderings, so is $\preceq$.

Proof.

- Assume $a \prec b$ and $b \prec c$. There exist $i, j, k \in I$ such that $a \in A_i$, $b \in A_j$ and $c \in A_k$. Then, we have $i \le j \le k$. If $i < j$ or $j < k$, we immediately have $a \prec c$ by definition. If $i = j = k$, by transitivity of $<_i$, we have $a \prec c$. Hence, $\prec$ is transitive in $\bigcup_{i\in I} A_i$. ✓
- Suppose $a \prec b$ and $b \prec a$ for the sake of contradiction. There exists $i, j \in I$ such that $a \in A_i$ and $b \in A_j$. We cannot have $i < j$ or $j < i$ as it contradicts one of $a \prec b$ and $b \prec a$. Hence, $i = j$ by totality of $(\mathbb{N}, \le)$. Then, we have $a <_i b$ and $b <_i a$, which is impossible by asymmetry of $<_i$. Hence, $\prec$ is asymmetric in $\bigcup_{i\in I} A_i$. ✓
- Take any $a, b \in \bigcup_{i\in I} A_i$. There exists $i, j \in I$ such that $a \in A_i$ and $b \in A_j$. If $i \neq j$, we immediately have $a \prec b$ or $b \prec a$. If $i = j$, as $\le_i$ is total, we have $a \le_i b$ or $b \le_i a$. Thus, $a \preceq b$ or $b \preceq a$. Hence, $\preceq$ is total. ✓
- Assume $\le_i$ is a well-ordering for each $i \in \mathbb{N}$. Take any $X \subseteq \bigcup_{i\in I} A_i$ with $X \neq \varnothing$. Let $i_0 \triangleq \min\{i \in I \mid A_i \cap X \neq \varnothing\}$, thanks to $(\mathbb{N}, \le)$ is Well-Ordered and X being nonempty. Then, let $a \triangleq \min_{\le_{i_0}}(A_{i_0} \cap X)$, which exists as $\le_{i_0}$ is a well-ordering. Then, $a = \min_{\preceq} X$. ✓ □

Exercise 4.4.7

Let $\preceq$ be the lexicographical ordering of $\mathbb{N}^{\mathbb{N}}$ (where $\mathbb{N}$ is ordered in the usual way) and let $P \subseteq \mathbb{N}^{\mathbb{N}}$ be the set of all eventually periodic, but not eventually constant, sequences of natural numbers. (See Exercises 4.3.6 and 4.3.7 for the definitions.) Then, $(P, \preceq \cap P^2)$

is a countably infinite dense totally ordered set without endpoints.

Proof. We already have $\preceq \cap P^2$ is a total ordering.

Claim 1. P is countably infinite.

Proof. As P is a subset of the set of all eventually periodic sequences, by Exercises 4.1.3 and 4.3.7, $|P| \leq \aleph_0$. We may define an injective infinite sequence into P recursively by

$$\begin{aligned} g_0 &= \langle 0, 1, 0, 1, \cdots \rangle \\ \forall n \in \mathbb{N}, \quad g_{n+1} &= \langle n+1, a_0, a_1, \cdots \rangle \\ &\quad \text{where } g_n = \langle a_0, a_1, \cdots \rangle. \end{aligned}$$

Hence, $\aleph_0 \leq |P|$; thus $|P| = \aleph_0$ by Cantor–Bernstein Theorem. □

Claim 2. $(P, \preceq \cap P^2)$ is dense.

Proof. Take any $f, g \in P$ with $f \prec g$. Then, $f_{i_0} < g_{i_0}$ where $i_0 = \min\{ i \in \mathbb{N} \mid f_i \neq g_i \}$. Define $h \in \mathbb{N}^{\mathbb{N}}$ by

$$h_i \triangleq \begin{cases} f_i & \text{if } i \leq i_0 \\ f_i + 1 & \text{otherwise.} \end{cases}$$

Then, it is evidently $f \prec h \prec g$.

There exist $n_0 \in \mathbb{N}$ and $p > 0$ such that $\forall n \geq n_0$, $f_{n+p} = f_n$. Let $n_0' \triangleq \max\{ n_0, i_0 + 1 \}$. Then, for each $n \geq n_0'$, we have $h_{n+p} = f_{n+p} + 1 = f_n + 1 = h_n$; thus $h \in P$. Hence, $(P, \preceq \cap P^2)$ is dense. □

Claim 3. $(P, \preceq \cap P^2)$ has no endpoints.

Proof. Fix any $f \in P$. Then, we may define $g \in \mathbb{N}^{\mathbb{N}}$ by $\forall n \in \mathbb{N}$, $g_n = f_n + 1$. It is easy to show that $f \prec g$ and $g \in P$. Hence, $(P, \preceq \cap P^2)$ has no greatest element.

We may also define $h \in \mathbb{N}^{\mathbb{N}}$ by

$$\forall n \in \mathbb{N}, \; h_n = \begin{cases} 0 & \text{if } n = 0 \\ f_{n-1} & \text{otherwise.} \end{cases}$$

As f is not eventually constant, $f \neq h$ and we may let $i_0 \triangleq \min\{ i \in \mathbb{N} \mid f_i \neq 0 \}$. As $\forall i < i_0$, $f_i = 0$, $\min \operatorname{diff}(f, h) = i_0$. Since $0 = h_{i_0} < f_{i_0}$, $h \prec f$. Thus, $(P, \preceq \cap P^2)$ has no least element. □

Combining Claims 1 to 3 gives the desired result. □

Exercise 4.4.11

Let $(A, \preceq)$ be a dense totally ordered set. Show that for all $a, b \in A$ such that $a \prec b$, the *closed interval* $[a, b] \triangleq \{ x \in A \mid a \preceq x \preceq b \}$ is infinite.

Proof. Assume there are some $a, b \in A$ such that $a \prec b$ and $I = [a, b]$ is finite. Let $|I| = n$. (Since $a, b \in I$, $n \geq 2$.) Then, by Theorem 4.4.3, there exists an isomorphism $h\colon n \hookrightarrow\!\!\!\!\rightarrow I$ between $(n, \leq \cap n^2)$ and $(I, \preceq \cap I^2)$. Then, $\exists i < n$, $h(0) \prec h(i) \prec h(1)$ as $(A, \preceq)$ is dense. However, there does not exist $i \in \mathbb{N}$ with $0 < i < 1$ by Exercise 3.2.1, which is a contradiction.
□

Exercise 4.4.12

Let $(P, \preceq)$ and $(Q, \leq)$ be countably infinite dense totally ordered sets with both endpoints. Then, $(P, \preceq)$ and $(Q, \leq)$ are similar.

Proof. Define the following:

$$\begin{aligned} p &\triangleq \min_{\preceq} P & p' &\triangleq \max_{\preceq} P \\ q &\triangleq \min_{\leq} Q & q' &\triangleq \max_{\leq} Q \\ P_0 &\triangleq P \setminus \{p, p'\} & Q_0 &\triangleq Q \setminus \{q, q'\}. \end{aligned}$$

Then, $(P_0, \preceq \cap P_0^2)$ and $(Q_0, \preceq \cap Q_0^2)$ are countably infinite dense totally ordered sets without endpoints. Hence, by Theorem 4.4.10, we have an isomorphism $h \colon P_0 \hookrightarrow\!\!\!\!\rightarrow Q_0$ between them. Then, $h' \cup \{(p,q),(p',q')\}$ is an isomorphism between $(P, \preceq)$ and $(Q, \leq)$. □

4.5 Complete Linear Ordering

Definition 4.5.1: Cut, Dedekind Cut, and Gap

Let $(P, \leq)$ be a totally ordered set.

- A *cut* is a pair (A, B) of sets such that
  - (i) $\{A, B\}$ is a partition of P.
  - (ii) $\forall a \in A,\ \forall b \in B,\ a < b$
- A *Dedekind cut* is a cut (A, B) such that $\max A$ does not exist.
- A *gap* is a cut (A, B) such that $\max A$ and $\min B$ do not exist.

Definition 4.5.2

Let $(P, \leq)$ be a totally ordered set and let $\varnothing \subsetneq A \subseteq P$.

- A is *bounded* if A has both lower and upper bounds.
- A is *bounded from below* if it has a lower bound.
- A is *bounded from above* if it has an upper bound.

Lemma 4.5.3

Let $(P, \leq)$ be a totally ordered set. Every nonempty $S \subseteq P$ bounded from above has a supremum if and only if $(P, \leq)$ has no gap.

Proof.

($\Rightarrow$) Suppose (A, B) is a gap of $(P, \leq)$. Then, as any $b \in B$ is a supremum of A, A is bounded from above. Hence, by assumption, there exists $\mu = \sup A$. There are two cases: $\mu \in A$ and $\mu \in B$.

If $\mu \in A$, then by Theorem 2.5.13 (iii), $\mu = \max A$, which contradicts the fact that (A, B) is a gap. If $\mu \in B$, as any $b \in B$ is an upper bound of A, $\forall b \in B$, $\mu \leq b$, i.e., $\mu = \min B$, which is a contradiction as well.

($\Leftarrow$) Suppose there is nonempty $S \subseteq P$ such that S is bounded above and S has no supremum. Let

$$A \triangleq \{x \in P \mid \exists s \in S,\ x \leq s\},$$
$$B \triangleq \{x \in P \mid \forall s \in S,\ x > s\}.$$

> ***Claim 1.*** $\{A, B\}$ is a partition of P.
>
> ***Proof.*** We have $A \cap B = \varnothing$ and $A \cup B = P$. As $A = P$ implies the existence of $\max S$, which contradicts the nonexistence of $\sup S$ by Item Theorem 2.5.13 (ii), we have $B \neq \varnothing$; moreover, as $B = P$ is impossible by $S \cap P = \varnothing$, we have $A \neq \varnothing$. Hence, $\{A, B\}$ is a partition of P. □

> ***Claim 2.*** $\forall a \in A$, $\forall b \in B$, $a < b$.
>
> ***Proof.*** Take any $a \in A$ and $b \in B$. Then, there exists $s \in S$ such that $a \leq s$. As $b \in B$, we have $a \leq s < b$. Hence, $a < b$. □

> ***Claim 3.*** $\max A$ and $\min B$ do not exist.
>
> ***Proof.*** Suppose $m = \min B$ exists for the sake of contradiction. Let m' be an upper bound of S. If $m' \in S$, then $m' = \sup S$, which is a contradiction. Hence, we have $\forall s \in S$, $m' > s$, i.e., $m' \in B$. This implies $m \leq m'$, that is to say $m = \sup S$. Hence, $\min B$ does not exist.
>
> Suppose $M = \max A$ exists for the sake of contradiction. Then, as $S \subseteq A$, M is an upper bound of S. Let M' be another upper bound of S. Then, as $\forall s \in S$, $M \leq s \leq M'$, $M = \sup S$, which is a contradiction. Hence, $\max A$ does not exist. □

Combining Claims 1 to 3 gives the result. □

Definition 4.5.4: Complete Dense Totally Ordered Set

Let $(P, \leq)$ be a dense totally ordered set. $(P, \leq)$ is said to be *complete* if every nonempty $S \subseteq P$ bounded from above has a supremum, i.e., if $(P, \leq)$ has no gap. (See Lemma 4.5.3.)

Theorem 4.5.5 Completion

Let $(P, \leq)$ be a dense totally ordered set without endpoints. Then, there exists a complete totally ordered set $(C, \preceq)$ such that

(i) $P \subseteq C$.
(ii) $\forall p, q \in P$, $(p < q \iff p \prec q)$.
(iii) $\forall c, d \in C$, $(c \prec d \implies \exists p \in P,\ c \prec p \prec d)$, i.e., P is dense in C.
(iv) C does not have endpoints.

Moreover, such $(C, \preceq)$ is unique up to isomorphism h with $\mathrm{Id}_P \subseteq h$.[†] The complete totally ordered set $(C, \preceq)$ is called the *completion* of $(P, \leq)$.

[†]In other words, if $(C, \preceq)$ and $(C^*, \preceq^*)$ satisfy all the requirements, then there exists an isomorphism h between $(C, \preceq)$ and $(C^*, \preceq^*)$ such that $\forall p \in P$, $h(p) = p$.

Proof. For each $B \subseteq P$, one may easily verify that $\min B$ exists if and only if there exists $p \in P$ such that $B = \{x \in P \mid x \geq p\}$. Hence, there are two types of Dedekind cuts—a Dedekind cut (A, B) such that:

(i) There (uniquely) exists $p \in P$ such that $B = \{x \in P \mid x \geq p\}$; in this case we write $(A, B) = [p]$.
(ii) (A, B) is a gap.

Note that $[p]$ is a Dedekind cut of $(P, \leq)$ for all $p \in P$.

Now, define C and a relation $\preceq$ on C by:

$$C \triangleq \{(A,B) \mid (A,B) \text{ is a Dedekind cut of } (P,\leq)\}$$

and

$$(A,B) \preceq (A',B') \iff A \subseteq A'.$$

Claim 1. $(C,\preceq)$ is a totally ordered set.

Proof. It is evident that $(C,\preceq)$ is an ordered set. Take any $(A,B),(A',B') \in C$. Suppose they are incomparable for the sake of contradiction, i.e., $A \setminus A' = A \cap B' \neq \varnothing$ and $A' \setminus A = A' \cap B \neq \varnothing$. Let $a \in A \cap B'$ and $a' \in A' \cap B$. Then, $a < a'$ and $a' < a$, which is impossible by asymmetry of $<$. $\square$

If $p < q$ where $p,q \in P$, then we have $[p] \prec [q]$. Hence, $(P', \preceq \cap P'^2)$ is isomorphic to $(P,\leq)$ where $P' \triangleq \{[x] \mid x \in P\}$. Thus, we simply prove that $(C,\preceq)$ is a completion of $(P', \preceq \cap P'^2)$.

Claim 2. $\forall c,d \in C, (c \prec d \implies \exists p \in P, c \prec [p] \prec d)$.

Proof. Take any $c,d \in C$ with $c \prec d$. In other words, $c = (A,B)$ and $d = (A',B')$ with $A \subsetneq A'$. Take $p \in A' \setminus A$. As $(P,\leq)$ has no greatest element, WLOG, we may assume p is not a least element of B. Let $[p] = (A'',B'')$.

- As $(P,\leq)$ is totally ordered, the fact that p is not a least element of B asserts the existence of $b \in B$ such that $b < p$. Hence, $\forall x \in B''$, $\forall a \in A$, $a < b < p \leq x$. Thus, $B'' \subsetneq B$; $(A,B) \prec [p]$. ✓
- As A' has no greatest element and $(P,\leq)$ is totally ordered, there exists $a' \in A'$ such that $p < a'$. Then, we have $\forall x \in A''$, $\forall b' \in B'$, $x < p < a' < b'$, i.e., $A'' \subsetneq A'$. Hence, $[p] \prec (A',B')$. ✓ $\square$

Claim 2 also shows that $(C,\preceq)$ is a densely ordered set.

Claim 3. $(C,\preceq)$ has no endpoints.

Proof. Take any $(A,B) \in C$.

- In the same way as in the proof of Claim 2, any $p \in B$ which is not a least element of B (which exists) satisfies $(A,B) \prec [p]$. Hence, C has no greatest element. ✓
- Take any $p \in A$. As A has no greatest element, there exists $a \in A$ such that $p < a$. Hence, $[p] \prec (A,B)$. ✓ $\square$

Claim 4. $(C,\preceq)$ is complete.

Proof. Let $\varnothing \subsetneq S \subseteq C$ be bounded from above. Let (A_0,B_0) be an upper bound of S. Define

$$A_S \triangleq \bigcup\{A \mid (A,B) \in S\} \quad \text{and} \quad B_S \triangleq P \setminus A_S = \bigcap\{B \mid (A,B) \in S\}.$$

As $B_0 \subseteq B_S$, (A_S,B_S) is a cut. Moreover, if x is a greatest element of A_S, then x is a greatest element of some A where $(A,B) \in S$. Hence, A_S has no greatest element; $(A_S,B_S) \in C$.

As $A \subseteq A_S$ for all $(A,B) \in S$, (A_S,B_S) is an upper bound of S. If (A',B') is another upper bound of S, then $\forall (A,B) \in S$, $A \subseteq A'$, i.e., $A_S \subseteq A'$. Hence, $(A_S,B_S) \preceq (A',B')$. Thus, $(A_S,B_S) = \sup_{\preceq} S$. $\square$

Claims 1 to 4 shows that $(C,\preceq)$ satisfies the requirements of the theorem; hence the existence part is done. We now prove the uniqueness.

Claim 5. Let $(C, \preceq)$ and $(C^*, \preceq^*)$ be two complete totally ordered sets that satisfies (i)–(iv). Then, there exists an isomorphism h between $(C, \preceq)$ and $(C^*, \preceq^*)$ such that $\forall p \in P$, $h(p) = p$.

Proof. For each $c \in C$ and $c^* \in C^*$ define

$$S_c \triangleq \{\, p \in P \mid p \prec c \,\} \quad \text{and} \quad S_{c^*} \triangleq \{\, p \in P \mid p \prec^* c^* \,\}.$$

Note that:

(I) $\sup_{\preceq} S_c = c$ and $\sup_{\preceq^*} S_{c^*} = c^*$ by (iii).

(II) Take any $c \in C$. By (iii) and (iv), there exists $q \in P$ such that $c \prec q$. Hence, $\forall p \in S_c$, $p \prec c \prec q$, and thus, $\forall p \in S_c$, $p \prec^* q$ by (ii). Therefore, every S_c is bounded from above in $(C^*, \preceq^*)$. By symmetry, every S_{c^*} is bounded from above in $(C, \preceq)$.

(III) For $p \in P$ and $\varnothing \subsetneq X \subseteq P$ such that both $\sup_{\preceq} X$ and $\sup_{\preceq^*} X$ exist, we have

$$p \prec \sup_{\preceq} X \iff \exists x \in X,\ p \prec x \iff \exists x \in X,\ p \prec^* x \iff p \prec^* \sup_{\preceq^*} X.$$

Now, define $h : C \to C^*$ by $c \mapsto \sup_{\preceq^*} S_c$. h is well-defined by (II) and $(C^*, \preceq^*)$ being complete. We now show that h is a desired isomorphism.

- Take any $c^* \in C^*$. Let $c \triangleq \sup_{\preceq} S_{c^*}$. Then,

$$\begin{aligned} h(c) &= \sup_{\preceq^*} S_c \\ &= \sup_{\preceq^*} \{\, p \in P \mid p \prec \sup_{\preceq} S_{c^*} \,\} \\ &= \sup_{\preceq^*} \{\, p \in P \mid p \prec^* \sup_{\preceq^*} S_{c^*} \,\} && \triangleright \text{(III)} \\ &= \sup_{\preceq^*} \{\, p \in P \mid p \prec^* c^* \,\} && \triangleright \text{(I)} \\ &= \sup_{\preceq^*} S_{c^*} = c^*. && \triangleright \text{(I)} \end{aligned}$$

  Hence, h is onto C^*. ✓
- Take any $c, d \in C$ with $c \prec d$. Then, by (iii), there exist $p_1, p_2 \in P$ such that $c \prec p_1 \prec p_2 \prec d$. We then have $\sup_{\preceq^*} S_c \preceq^* p_1 \prec^* p_2 \preceq^* \sup_{\preceq^*} S_d \prec$. Hence, $h(c) \prec^* h(d)$; h is an isomorphism. ✓
- For each $p \in P \subseteq C \cap C^*$, by (I), $h(p) = \sup_{\preceq^*} S_p = p$. ✓

Thus the theorem is now proven. □

Selected Problems

Exercise 4.5.4

A dense totally ordered set $(P, \preceq)$ is complete if and only if every nonempty $S \subseteq P$ bounded from below has an infimum.

Proof. We notice that (A, B) is a gap of $(P, \preceq)$ if and only if (B, A) is a gap of $(P, \preceq^{-1})$. Hence, by Lemma 4.5.3, $(P, \preceq)$ is complete if and only if every nonempty set $S \subseteq P$ bounded from above in $(P, \preceq^{-1})$ has an supremum in $(P, \preceq^{-1})$.

Note that $S \subseteq P$ is bounded from above in $(P, \preceq^{-1})$ if and only if S is bounded from below in $(P, \preceq)$, and $\sup_{\preceq^{-1}} S = \inf_{\preceq} S$. □

Chapter 5

Cardinal Numbers

Chapter 6

Ordinal Numbers

Chapter 7

Alephs

Chapter 8

Axiom of Choice

End.