

MAS242 해석학 II

Notes

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Chapter 1

Differentiation

1.1 Higher order partial derivatives

Definition 1.1.1

Given $f : U \rightarrow \mathbb{R}$ where U is an open set in \mathbb{R}^m , define $\partial_{ij} \triangleq \partial_i(\partial_j f)(x)$ for each $i, j \in [m]$ to be *2nd order partial derivatives*. Any higher order partial derivatives can be defined inductively.

Definition 1.1.2: C^k -regularity

$f : U \rightarrow \mathbb{R}$ is C^k -regular if all partial derivatives up to order k and they are continuous.

Theorem 1.1.1

$f : U(\subseteq \mathbb{R}^2) \rightarrow \mathbb{R}$ is C^2 at a point $c \in U$, i.e., $\exists \delta > 0$, f is C^2 in $B_\delta(c)$. Then, $\partial_{12}f(c) = \partial_{21}f(c)$.

Proof. Let $|h| < \delta$. Define $A(h) \triangleq f(c_1 + h_1, c_2 + h_2) - f(c_1 + h_1, c_2) - f(c_1, c_2 + h_2) + f(c_1, c_2)$. Define $u(x_1) \triangleq f(x_1, c_2 + h_2) - f(x_1, c_2)$ and $v(x_2) \triangleq f(c_1 + h_1, x_2) - f(c_1, x_2)$. Note that u and v are differentiable.

Then, $A(h) = u(c_1 + h_1) - u(c_1)$ and $A(h) = v(c_2 + h) - v(c_2)$. By MVT, $\exists c_1^* \in (c_1, c_1 + h_1)$ and $c_2^* \in (c_2, c_2 + h_2)$ s.t. $A(h) = u'(c_1^*)h_1 = h_1(\partial_1 f(c_1^*, c_2 + h) - \partial_1 f(c_1^*, c_2)) = h_1 h_2 \partial_{21}f(c_1^*, c_2^*)$

Similarly, $\exists c_1^{**}, c_2^{**}$ such that $A(h) = h_1 h_2 \partial_{12}f(c_1^{**}, c_2^{**})$. $\partial_{21}f(c_1^*, c_2^*) = \partial_{12}f(c_1^{**}, c_2^{**})$. Hence, as $|h| \rightarrow 0$, due to the continuity, $\partial_{21}(c) = \partial_{12}(c)$. \square

Corollary 1.1.1

Suppose $f : U(\subseteq \mathbb{R}^m) \rightarrow \mathbb{R}$ is C^k at $c \in U$. Then $\partial_{j_1 j_2 \dots j_k} f(c) = \partial_{j'_1 j'_2 \dots j'_k} f(c)$ where $j'_1 \dots$ are a permutation of $j_1 \dots$.

1.2 Extreme Values of differentiable Functions

Definition 1.2.1: Hessian

Let $f : U(\subseteq \mathbb{R}^m) \rightarrow \mathbb{R}$ be C_2 in U . Suppose $p \in U$ is a critical point of f , i.e., $\nabla f(p) = 0$. Define

$$\mathcal{H}f(x) \triangleq \begin{pmatrix} \partial_{11}f(x) & \partial_{21}f(x) & \cdots & \partial_{m1}f(x) \\ \partial_{12}f(x) & \partial_{22}f(x) & \cdots & \partial_{m2}f(x) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{1m}f(x) & \partial_{2m}f(x) & \cdots & \partial_{mm}f(x) \end{pmatrix}.$$

(Sometimes $\mathcal{H}f(x) = D^2f(x)$.)

Define $D(x) = \det \mathcal{H}f(x)$. (Note that $\mathcal{H}f(x)$ is symmetric when f is C^2 by the theorem above.)

Theorem 1.2.1 2nd-order derivative test for two variable functions.

When $m = 2$ and f is C^2 , a critical point p is

- a local maximum if $D(p) > 0$ and $\partial_{11}f(p) > 0$ (or $\partial_{22}f(p) > 0$).
- a local minimum if $D(p) > 0$ and $\partial_{11}f(p) < 0$ (or $\partial_{22}f(p) < 0$).
- a saddle point if $D(p) < 0$.

The test fails when $D(p) = 0$.

Proof. Given a unit vector $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$, $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = u_1\partial_1f + u_2\partial_2f$, and thus

$$D_{\mathbf{u}}^2f = (u_1\partial_1 + u_2\partial_2)(u_1\partial_1f + u_2\partial_2f) = u_1^2\partial_{11}f + u_1u_2(2\partial_{12}f) + u_2^2\partial_{22}f.$$

WLOG, $u_1 \neq 0$. Set $z = u_2/u_1$. Then,

$$D_{\mathbf{u}}^2f(p) = u_1^2(\partial_{11}f(p) + 2\partial_{12}f(p)z + \partial_{22}f(p)z^2).$$

Note that, if $D(p) > 0$, $D_{\mathbf{u}}^2f(p)$ has no real root.

- If $D(p) > 0$ and $\partial_{11}f(p) < 0$, Then, $D^2\mathbf{u} < 0$ for all unit vector \mathbf{u} .
- If $D(p) > 0$ and $\partial_{11}f(p) > 0$, Then, $D^2\mathbf{u} > 0$ for all unit vector \mathbf{u} .
- If $D(p) < 0$, $D_{\mathbf{u}}^2f(p)$ has different signs depending on \mathbf{u} .

For general m ?

$$D_{\mathbf{u}}(D_{\mathbf{u}}f) = D_{\mathbf{u}} \sum_{j=1}^m \partial_j f u_j = \sum_{j=1}^m ((\nabla \partial_j f) \cdot \mathbf{u}) u_j = \sum_{j=1}^m \sum_{k=1}^m u_k u_j \partial_{kj} f.$$

Hence,

$$D_{\mathbf{u}}^2f(p) = \mathbf{u}^T \cdot D^2f(p) \cdot \mathbf{u}$$

Since $D^2f(p)$ is symmetric, its eigenvalues $\lambda_1, \dots, \lambda_m$ exists and they are real numbers. Also, there exists an $m \times m$ orthogonal matrix \mathcal{O} such that $D^2f(p) = \mathcal{O}\Lambda(p)\mathcal{O}^T$ where $\Lambda(p)$ is the diagonal matrix with entries are the eigenvalues.

Then, we can write $D_{\mathbf{u}}^2f(p) = \mathbf{u}\mathcal{O}\Lambda(p)\mathcal{O}^T\mathbf{u}^T = (\mathbf{u}\mathcal{O})\Lambda(p) = (\mathbf{u}\mathcal{O})^T$. Since \mathcal{O} is orthogonal, $\mathbf{u}\mathcal{O}$ is another arbitrary unit vector. \square

Theorem 1.2.2 Generalized 2nd order partial derivatives test

When f is C^2 , a critical point p is

- a local maximum if all eigenvalues of $D^2f(p)$ are negative.

- a local minimum if all eigenvalues of $D^2f(p)$ are positive.
 - a saddle point if there are both negative eigenvalues and positive eigenvalues.
- The test fails when there are zero eigenvalues.

Chapter 2

Inverse Function Theorem

2.1 Jacobian

Definition 2.1.1: Jacobian

Let $\mathbf{f}: U(\subseteq \mathbb{R}^m) \rightarrow \mathbb{R}^n$ be differentiable. The function $J_{\mathbf{f}}: U \rightarrow \mathbb{R}$ defined by

$$J_{\mathbf{f}}(\mathbf{x}) = \det \begin{bmatrix} \partial_1 f_1(\mathbf{x}) & \cdots & \partial_n f_1(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \partial_1 f_n(\mathbf{x}) & \cdots & \partial_n f_n(\mathbf{x}) \end{bmatrix}$$

is called the *Jacobian* of \mathbf{f} at \mathbf{x} .

Lemma 2.1.1

If $f: V(\subseteq \mathbb{R}^n) \rightarrow \mathbb{R}$ and $\mathbf{g}: U \rightarrow V$ are differentiable, then

$$J_{f \circ \mathbf{g}}(\mathbf{x}) = J_f(\mathbf{g}(\mathbf{x})) \cdot J_{\mathbf{g}}(\mathbf{x}).$$

Note:-

The linear mapping $df(c)$ is invertible if and only if $J_{\mathbf{f}}(c)$ is nonzero.

2.2 The Inverse Function Theorem

Lemma 2.2.1 Contraction Mapping Principle

Let (X, d) be a complete metric space. Let $\varphi: X \rightarrow X$. Suppose that there exists $M \in [0, 1)$ such that $d(\varphi(x_1), \varphi(x_2)) \leq M d(x_1, x_2)$. (We call it a *contraction mapping*.) Then, there uniquely exists $x_* \in X$ such that $\varphi(x_*) = x_*$.

Proof. Fix any $x_0 \in X$. Since $\{x_j\}_{j \in \mathbb{Z}_+}$, where $x_j = \varphi(x_{j-1})$ for each $j \in \mathbb{Z}_+$, is continuous. It converges to some x_* . As φ is continuous, we have $\varphi(x_*) = x_*$. The uniqueness follows trivially. \square

Note:-

- For each $v \in \mathbb{R}^n \setminus \{0\}$, $|Av| = |v| \cdot |A \frac{v}{|v|}| \leq \|A\|_L \cdot |v|$. The result is trivial when $v = 0$.
- For each $u \in \mathbb{R}^n$ with $|u| = 1$, $|ABu| \leq \|A\|_L |Bu| \leq \|A\|_L \|B\|_L$. Hence, $\|AB\|_L = \|A\|_L \|B\|_L$.
- Given invertible $A \in L(\mathbb{R}^n, \mathbb{R}^n)$, $A^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is linear. Moreover, $\|A\|_L > 0$.

Lemma 2.2.2

Given two linear mappings $A, B: \mathbb{R}^n \rightarrow \mathbb{R}^n$ with invertibility of A ,

$$\|A - B\|_L \cdot \|A^{-1}\|_L < 1 \implies B \text{ is invertible.}$$

Proof. Let $\|A^{-1}\|_L = 1/\alpha$ and $\|B - A\|_L = \beta$ so that $\beta < \alpha$. Then, for every $\mathbf{x} \in \mathbb{R}^n$,

$$\begin{aligned} \alpha|\mathbf{x}| &= \alpha|A^{-1}A\mathbf{x}| \leq \alpha\|A^{-1}\| \cdot |A\mathbf{x}| \\ &= |A\mathbf{x}| \leq |(A - B)\mathbf{x}| + |B\mathbf{x}| \leq \beta|\mathbf{x}| + |B\mathbf{x}|; \end{aligned}$$

hence $(\alpha - \beta)|\mathbf{x}| \leq |B\mathbf{x}|$ where $\mathbf{x} \in \mathbb{R}^n$ is arbitrary. As $\alpha > \beta$, it holds that $B\mathbf{x} = 0 \implies \mathbf{x} = 0$. \square

Corollary 2.2.1

The set $\Omega \subseteq L(\mathbb{R}^n, \mathbb{R}^n)$ of invertible linear transformations is open.

Lemma 2.2.3

The mapping from Ω onto Ω defined by $A \mapsto A^{-1}$ is continuous.

Proof. Let A and B be invertible linear transformations from \mathbb{R}^n to \mathbb{R}^n . Let $\|A^{-1}\| = 1/\alpha$ and $\|B - A\|_L = \beta$. We have $(\alpha - \beta)|\mathbf{x}| \leq |B\mathbf{x}|$ by the same reasoning as in the proof of Lemma 2.2.2. Hence, the following holds.

$$\forall \mathbf{y} \in \mathbb{R}^n, (\alpha - \beta)|B^{-1}\mathbf{y}| \leq |BB^{-1}\mathbf{y}| = |\mathbf{y}|$$

This shows that $\|B^{-1}\|_L \leq (\alpha - \beta)^{-1}$.

Hence, we have

$$\|B^{-1} - A^{-1}\|_L \leq \|B^{-1}\|_L \|A - B\|_L \|A^{-1}\|_L \leq \frac{\beta}{\alpha(\alpha - \beta)}.$$

This implies that $\|B^{-1} - A^{-1}\|_L \rightarrow 0$ as $B \rightarrow A$. \square

Theorem 2.2.1 Inverse Function Theorem

Let $\mathbf{f}: E(\subseteq \mathbb{R}^n) \rightarrow \mathbb{R}^n$ be C^1 in E and $\mathbf{c} \in E$. Suppose that $J_{\mathbf{f}}(\mathbf{c}) \neq 0$. Then, the following hold.

- (i) There exists a neighborhood U of \mathbf{a} such that $\mathbf{f}|_U$ is bijective and $V \triangleq \mathbf{f}(U)$ is open.
- (ii) The inverse map of $\mathbf{f}|_U$ is C^1 in V .

Proof. Let $A \triangleq d\mathbf{f}(\mathbf{c})$. Define $\lambda \in \mathbb{R}_+$ by $2\lambda\|A^{-1}\|_L = 1$. Since $d\mathbf{f}$ is continuous, there exists a neighborhood U of \mathbf{c} such that $\|d\mathbf{f}(\mathbf{x}) - A\|_L < \lambda$ for each $\mathbf{x} \in U$.

Given a point $\mathbf{y} \in \mathbb{R}^n$, we define $\varphi(\cdot; \mathbf{y})$ by

$$\begin{aligned} \varphi(\cdot; \mathbf{y}) : B_\delta(\mathbf{c}) &\longrightarrow \mathbb{R}^n \\ \mathbf{x} &\longmapsto \mathbf{x} + A^{-1}(\mathbf{y} - \mathbf{f}(\mathbf{x})) \end{aligned}$$

Note that \mathbf{x} is a fixed point of $\varphi(\cdot; \mathbf{y})$ if and only if $A^{-1}(\mathbf{y} - \mathbf{f}(\mathbf{x})) = 0$, i.e., $\mathbf{y} = \mathbf{f}(\mathbf{x})$. Note also that φ is differentiable and $d\varphi(\mathbf{x}; \mathbf{y}) = \text{Id} - A^{-1} d\mathbf{f}(\mathbf{x}) = A^{-1}(A - d\mathbf{f}(\mathbf{x}))$ for each $\mathbf{x} \in U$.

Hence, for all $\mathbf{x} \in U$,

$$\|d\varphi(\mathbf{x}; \mathbf{y})\|_L = \|A^{-1}(A - d\mathbf{f}(\mathbf{x}))\|_L \leq \|A^{-1}\|_L \cdot \|A - d\mathbf{f}(\mathbf{x})\|_L < 1/(2\lambda) \cdot \lambda = 1/2.$$

Thus, MVT gives

$$|\varphi(\mathbf{x}_1; \mathbf{y}) - \varphi(\mathbf{x}_2; \mathbf{y})| \leq \frac{1}{2} |\mathbf{x}_1 - \mathbf{x}_2|$$

whenever $\mathbf{x}_1, \mathbf{x}_2 \in U$. Note that this implies there is at most one fixed point of $\varphi(\cdot; \mathbf{y})$ in U , i.e., $\mathbf{f}|_U$ is bijective.

Now, we shall show that $V = \mathbf{f}(U)$ is open. Take any $\mathbf{y}_0 \in V$. There (uniquely) exists $\mathbf{x}_0 \in U$ such that $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$. Fix any $r \in \mathbb{R}_+$ such that $\bar{B} \subseteq U$ where $B = B_r(\mathbf{x}_0)$. Take any $\mathbf{y} \in B_{\lambda r}(\mathbf{y}_0)$. Then,

$$|\varphi(\mathbf{x}_0; \mathbf{y}) - \mathbf{x}_0| = |A^{-1}(\mathbf{y} - \mathbf{y}_0)| < \|A^{-1}\|_L \lambda r = \frac{r}{2}.$$

Moreover, for any $\mathbf{x} \in \bar{B}$,

$$|\varphi(\mathbf{x}; \mathbf{y}) - \mathbf{x}_0| \leq |\varphi(\mathbf{x}; \mathbf{y}) - \varphi(\mathbf{x}_0; \mathbf{y})| + |\varphi(\mathbf{x}_0; \mathbf{y}) - \mathbf{x}_0| \leq \frac{1}{2} |\mathbf{x} - \mathbf{x}_0| + \frac{r}{2} < r.$$

This directly implies that $\varphi(\bar{B}; \mathbf{y}) \subseteq B \subseteq \bar{B}$. Hence, $\varphi(\cdot, \mathbf{y})$ is a contraction mapping on a complete metric space \bar{B} . By Lemma 2.2.1, there exists a fixed point $\mathbf{x} \in \bar{B}$, which satisfies $\mathbf{y} = \mathbf{f}(\mathbf{x})$. Thus, $\mathbf{y} \in \mathbf{f}(\bar{B}) \subseteq \mathbf{f}(U) = V$. Hence, $B \subseteq V$, V is open. This proves (i).

Now, let $\mathbf{g}: V \rightarrow U$ be the local inverse map of $\mathbf{f}|_U$. Take any $\mathbf{y} \in V$ and $\mathbf{y} + \mathbf{k} \in V$. There are unique $\mathbf{x} \in U$ and $\mathbf{x} + \mathbf{h} \in U$ such that $\mathbf{y} = \mathbf{f}(\mathbf{x})$ and $\mathbf{y} + \mathbf{k} = \mathbf{f}(\mathbf{x} + \mathbf{h})$. Then, we have

$$\varphi(\mathbf{x} + \mathbf{h}; \mathbf{y}) - \varphi(\mathbf{x}; \mathbf{y}) = \mathbf{h} + A^{-1}(\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x} + \mathbf{h})) = \mathbf{h} - A^{-1}\mathbf{k},$$

which implies $|\mathbf{h} - A^{-1}\mathbf{k}| \leq |\mathbf{h}|/2$. Hence, $|A^{-1}\mathbf{k}| \geq |\mathbf{h}|/2$ is obtained by the triangle inequality; $|\mathbf{h}| \leq 2\|A^{-1}\|_L |\mathbf{k}| = \lambda^{-1} |\mathbf{k}|$.

Then, since $\|\mathbf{d}\mathbf{f}(\mathbf{x}) - A\|_L \|A^{-1}\|_L < \lambda \cdot 1/(2\lambda) = 1/2$, Lemma 2.2.2 implies that $\mathbf{d}\mathbf{f}(\mathbf{x})$ is invertible. Let $T \triangleq \mathbf{d}\mathbf{f}(\mathbf{x})$. Then, we have

$$\mathbf{g}(\mathbf{y} + \mathbf{k}) - \mathbf{g}(\mathbf{y}) - T^{-1}\mathbf{k} = \mathbf{h} - T^{-1}\mathbf{k} = -T^{-1}(\mathbf{f}(\mathbf{x} + \mathbf{h}) - \mathbf{f}(\mathbf{x}) - T\mathbf{h}),$$

and thus

$$\frac{|\mathbf{g}(\mathbf{y} + \mathbf{k}) - \mathbf{g}(\mathbf{y}) - T^{-1}\mathbf{k}|}{|\mathbf{k}|} \leq \frac{\|T^{-1}\|_L}{\lambda} \cdot \frac{|\mathbf{f}(\mathbf{x} + \mathbf{h}) - \mathbf{f}(\mathbf{x}) - T\mathbf{h}|}{|\mathbf{h}|}.$$

The equation implies that \mathbf{g} is differentiable on V , and that $\mathbf{d}\mathbf{g}(\mathbf{y}) = T^{-1} = \mathbf{d}\mathbf{f}(\mathbf{g}(\mathbf{y}))^{-1}$. Since $\mathbf{d}\mathbf{g}$ is a composition of continuous functions, $\mathbf{d}\mathbf{g}$ itself is continuous. \square

Corollary 2.2.2

Let $\mathbf{f}: E(\subseteq \mathbb{R}^n) \rightarrow \mathbb{R}^n$ be C^1 in E and $J_{\mathbf{f}}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in E$. Then, for every open set $W \subseteq E$, $\mathbf{f}(W)$ is open.

Proof. This directly follows from (i) of Theorem 2.2.1. \square

2.3 Implicit Function Theorem

Definition 2.3.1

- If $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$, let us write (\mathbf{x}, \mathbf{y}) for the point $(x_1, \dots, x_n, y_1, \dots, y_m) \in \mathbb{R}^{n+m}$.
- Every $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ can be split into $A_x \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $A_y \in L(\mathbb{R}^m, \mathbb{R}^n)$ where $A(\mathbf{h}, \mathbf{k}) = A_x \mathbf{h} + A_y \mathbf{k}$ for each $\mathbf{h} \in \mathbb{R}^n$ and $\mathbf{k} \in \mathbb{R}^m$.

Lemma 2.3.1

If $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ and if A_x is invertible, then

$$\forall \mathbf{k} \in \mathbb{R}^m, \exists! \mathbf{h} \in \mathbb{R}^n, A(\mathbf{h}, \mathbf{k}) = \mathbf{0}.$$

Proof. $A(\mathbf{h}, \mathbf{k}) = A_x \mathbf{h} + A_y \mathbf{k} = \mathbf{0}$ if and only if $\mathbf{h} = -(A_x)^{-1} A_y \mathbf{k}$. □

Theorem 2.3.1 Implicit Function Theorem

Let $\mathbf{f}: E \rightarrow \mathbb{R}^n$ be a C^1 mapping where E is an open set in \mathbb{R}^{n+m} . Let $(\mathbf{a}, \mathbf{b}) \in E$ satisfy $\mathbf{f}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$. Let $A = d\mathbf{f}(\mathbf{a}, \mathbf{b})$ and suppose A_x is invertible. Then, there exist open sets $U \subseteq \mathbb{R}^{n+m}$ and $W \subseteq \mathbb{R}^m$ that satisfy the following.

- (i) $(\mathbf{a}, \mathbf{b}) \in U$ and $\mathbf{b} \in W$.
- (ii) $\forall \mathbf{y} \in W, \exists! \mathbf{x} \in \mathbb{R}^n, (\mathbf{x}, \mathbf{y}) \in U \wedge \mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$.
- (iii) If the unique \mathbf{x} in (ii) is denoted by $\mathbf{g}(\mathbf{y})$, then $\mathbf{g}: W \rightarrow \mathbb{R}^n$ is C^1 on W .
- (iv) Moreover, $d\mathbf{g}(\mathbf{b}) = -(A_x)^{-1} A_y$.

Proof. Define $\mathbf{F}: E \rightarrow \mathbb{R}^{n+m}$ by $\mathbf{F}(\mathbf{x}, \mathbf{y}) \triangleq (\mathbf{f}(\mathbf{x}, \mathbf{y}), \mathbf{y})$. Then, \mathbf{F} is C^1 . Since $\mathbf{f}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$, if $\mathbf{r}(\mathbf{h}, \mathbf{k}) \triangleq \mathbf{f}(\mathbf{a} + \mathbf{h}, \mathbf{b} + \mathbf{k}) - A(\mathbf{h}, \mathbf{k})$, we have $\lim_{\mathbf{h}, \mathbf{k} \rightarrow \mathbf{0}} |\mathbf{r}(\mathbf{h}, \mathbf{k})| / |(\mathbf{h}, \mathbf{k})| = 0$. Hence, from

$$\mathbf{F}(\mathbf{a} + \mathbf{h}, \mathbf{b} + \mathbf{k}) - \mathbf{F}(\mathbf{a}, \mathbf{b}) = (\mathbf{f}(\mathbf{a} + \mathbf{h}, \mathbf{b} + \mathbf{k}), \mathbf{k}) = (A(\mathbf{h}, \mathbf{k}), \mathbf{k}) + (\mathbf{r}(\mathbf{h}, \mathbf{k}), \mathbf{0}),$$

it is obtained that $d\mathbf{F}(\mathbf{a}, \mathbf{b})(\mathbf{h}', \mathbf{k}') = (A(\mathbf{h}', \mathbf{k}'), \mathbf{k}')$ for each $(\mathbf{h}', \mathbf{k}') \in \mathbb{R}^{n+m}$. If $d\mathbf{F}(\mathbf{a}, \mathbf{b})(\mathbf{h}', \mathbf{k}') = \mathbf{0}$, then $\mathbf{k}' = \mathbf{0}$ and $A(\mathbf{h}', \mathbf{0}) = \mathbf{0}$; thus $\mathbf{h}' = \mathbf{0}$ as A_x is invertible. Hence, $d\mathbf{F}(\mathbf{a}, \mathbf{b})$ is invertible; Theorem 2.2.1 can be applied to \mathbf{F} at (\mathbf{a}, \mathbf{b}) .

By Theorem 2.2.1, there exists a neighborhood $U \subseteq E$ of (\mathbf{a}, \mathbf{b}) such that $\mathbf{F}|_U$ is bijective, $\mathbf{F}(U)$ is open, and its inverse is C^1 . Let $W \triangleq \{\mathbf{y} \in \mathbb{R}^m \mid (\mathbf{0}, \mathbf{y}) \in \mathbf{F}(U)\}$. W is open as $\mathbf{F}(U)$ is open. Noting that $\mathbf{b} \in W$, we finish the proof for (i).

Take any $\mathbf{y} \in W$. Then, there exists $(\mathbf{x}, \mathbf{y}) \in U$ such that $\mathbf{F}(\mathbf{x}, \mathbf{y}) = (\mathbf{0}, \mathbf{y})$; thus $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$. If \mathbf{x}, \mathbf{x}' are two such point corresponding to \mathbf{y} , then

$$\mathbf{F}(\mathbf{x}', \mathbf{y}) = (\mathbf{f}(\mathbf{x}', \mathbf{y}), \mathbf{y}) = (\mathbf{0}, \mathbf{y}) = (\mathbf{f}(\mathbf{x}, \mathbf{y}), \mathbf{y}) = \mathbf{F}(\mathbf{x}, \mathbf{y}).$$

However, as \mathbf{F} being injective, $\mathbf{x} = \mathbf{x}'$. This proves (ii).

Let $V \triangleq \mathbf{F}(U)$. Let $\mathbf{G}: V \rightarrow U$ be the inverse of \mathbf{F} , which is C^1 by Theorem 2.2.1. Hence, for each $\mathbf{y} \in W$, from $\mathbf{F}(\mathbf{g}(\mathbf{y}), \mathbf{y}) = (\mathbf{0}, \mathbf{y})$, we have $(\mathbf{g}(\mathbf{y}), \mathbf{y}) = \mathbf{G}(\mathbf{0}, \mathbf{y})$. This directly shows that \mathbf{g} is C^1 as well. This proves (iii).

Let $\Phi: W \rightarrow U$ be defined by $\Phi(\mathbf{y}) = \mathbf{G}(\mathbf{0}, \mathbf{y}) = (\mathbf{g}(\mathbf{y}), \mathbf{y})$, which is C^1 , indeed. Then, $d\Phi(\mathbf{y}) = (d\mathbf{g}(\mathbf{y}), I_m)$. Differentiating both sides of the equality $\mathbf{f}(\Phi(\mathbf{y})) = \mathbf{0}$, we get

$$d\mathbf{f}(\Phi(\mathbf{y})) d\Phi(\mathbf{y}) = \mathbf{0}.$$

Putting $\mathbf{y} := \mathbf{b}$, as $\Phi(\mathbf{b}) = (\mathbf{a}, \mathbf{b})$, we get $A d\Phi(\mathbf{b}) = \mathbf{0}$, or

$$A_x d\mathbf{g}(\mathbf{b}) + A_y = \mathbf{0},$$

i.e., $d\mathbf{g}(\mathbf{b}) = -(A_x)^{-1} A_y$. □

Definition 2.3.2: C^1 -norm

Suppose $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is C^1 . Then,

$$\begin{aligned}\|\varphi\|_{C^0(\bar{\Omega})} &\triangleq \sup_{\mathbf{x} \in \bar{\Omega}} |\varphi(\mathbf{x})| \\ \|\varphi\|_{C^1(\bar{\Omega})} &\triangleq \|\varphi\|_{C^0(\bar{\Omega})} + \sum_{j=1}^n \|\partial_j \varphi\|_{C^0(\bar{\Omega})}.\end{aligned}$$

This is only for Example 2.3.1.

Example 2.3.1 (Level Sets)

Define $\Omega \triangleq \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_2| \leq 1\}$. Given two constants, $a, b \in \mathbb{R}$ with $a < b$, define $\bar{\varphi}(x_1, x_2) = ax_1$ and $\bar{\psi}(x_1, x_2) = bx_1$. Then, $\Gamma_0 = \{\mathbf{x} \in \Omega \mid \bar{\varphi}(\mathbf{x}) - \bar{\psi}(\mathbf{x}) = 0\} = \{\mathbf{x} \in \Omega \mid x_1 = 0\}$.

Suppose that $\varphi, \psi : \Omega \rightarrow \mathbb{R}$ satisfy

$$\|\varphi - \bar{\varphi}\|_{C^1(\bar{\Omega})} + \|\psi - \bar{\psi}\|_{C^1(\bar{\Omega})} \leq \frac{1}{4}|a - b|.$$

Then, what would be the expression for $\Gamma = \{\mathbf{x} \in \Omega \mid \varphi(\mathbf{x}) - \psi(\mathbf{x}) = 0\}$?

Observe that $(\varphi - \psi) = (\varphi - \bar{\varphi}) + (\bar{\varphi} - \bar{\psi}) + (\bar{\psi} - \psi)$ and thus $|(\varphi - \psi)(x_1, x_2) - (a - b)x_1| \leq |a - b|/4$. This implies $\lim_{x_1 \rightarrow \pm\infty} (\varphi - \psi)(x_1, x_2) = \mp\infty$. Hence, for every $x_2 \in [-1, 1]$, there exists $x_1^* \in \mathbb{R}$ such that $(\varphi - \psi)(x_1^*, x_2) = 0$.

Moreover, $\partial_1(\varphi - \psi) = \partial_1(\varphi - \bar{\varphi}) + (a - b) + \partial_1(\bar{\psi} - \psi)$, and thus $|\partial_1(\varphi - \psi)| \geq \frac{3}{4}|a - b| > 0$. Hence, the x_1^* in the previous paragraph is unique. This means that $\Gamma = \{(f(x_2), x_2) \mid x_2 \in \mathbb{R}\}$ for some f .

$(\varphi - \psi)(f(x_2), x_2) - (\bar{\varphi} - \bar{\psi})(f(x_2), x_2) = -(\bar{\varphi} - \bar{\psi})(f(x_2), x_2) = (b - a)f(x_2)$. Hence,

$$f(x_2) = \frac{(\varphi - \bar{\varphi})(f(x_2), x_2) - (\psi - \bar{\psi})(f(x_2), x_2)}{b - a}.$$

This is the implicit representation of f . Moreover, $|f(x_2)| = \frac{|b - a|/4}{|b - a|} = 1/4$.

2.4 Applications of IMFT: Lagrange's Method

Theorem 2.4.1 Optimization Under Multiple Constraints

Let $f, g_1, g_2, \dots, g_k : E \rightarrow \mathbb{R}$ be C^1 where E is an open set in \mathbb{R}^n and $n > k$. Let $Z \triangleq \bigcap_{j=1}^k \{\mathbf{z} \in \mathbb{R}^n \mid g_j(\mathbf{z}) = 0\}$. Suppose $\mathbf{z}_0 \in Z$ is a local maximum point with respect to f on Z . Suppose also that

$$\Delta \triangleq \det \begin{bmatrix} \partial_1 g_1(\mathbf{z}_0) & \cdots & \partial_1 g_k(\mathbf{z}_0) \\ \vdots & \ddots & \vdots \\ \partial_k g_1(\mathbf{z}_0) & \cdots & \partial_k g_k(\mathbf{z}_0) \end{bmatrix} \neq 0.$$

Then, there exists $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{R}$ such that $\nabla f(\mathbf{z}_0) = \sum_{m=1}^k \lambda_m \nabla g_m(\mathbf{z}_0)$.

Proof. Since $\Delta \neq 0$, there exists a unique solution $(\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$ for the linear system

$$\begin{bmatrix} \partial_1 g_1(\mathbf{z}_0) & \cdots & \partial_1 g_k(\mathbf{z}_0) \\ \vdots & \ddots & \vdots \\ \partial_k g_1(\mathbf{z}_0) & \cdots & \partial_k g_k(\mathbf{z}_0) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{bmatrix} = \begin{bmatrix} \partial_1 f(\mathbf{z}_0) \\ \vdots \\ \partial_k f(\mathbf{z}_0) \end{bmatrix}.$$

For each point $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{R}^n$, let $\mathbf{x} = (z_1, \dots, z_k)$ and $\mathbf{y} = (z_{k+1}, \dots, z_n)$. Let $\mathbf{z}_0 = (\mathbf{x}_0, \mathbf{y}_0)$. Let $\mathbf{g}: E \rightarrow \mathbb{R}^k$ be defined by $\mathbf{g}(\mathbf{z}) = (g_1(\mathbf{z}), \dots, g_k(\mathbf{z}))$.

Since \mathbf{g} is C^1 , $\mathbf{g}(\mathbf{z}_0) = \mathbf{0}$, and $(d\mathbf{g}(\mathbf{z}_0))_{\mathbf{x}}$ is invertible, by Theorem 2.3.1, there exists an open neighborhood $W \subseteq \mathbb{R}^{n-k}$ of \mathbf{y}_0 and a C^1 function $\mathbf{s}: W \rightarrow \mathbb{R}^k$ such that $\mathbf{g}(\mathbf{s}(\mathbf{y}), \mathbf{y}) = \mathbf{0}$ for each $\mathbf{y} \in W$. Note that $\mathbf{s}(\mathbf{y}_0) = \mathbf{x}_0$.

Define $F: W \rightarrow \mathbb{R}$ by $\mathbf{y} \mapsto f(\mathbf{s}(\mathbf{y}), \mathbf{y})$. As \mathbf{z}_0 is a local maximum point, so is \mathbf{y}_0 . Hence, $\nabla F(\mathbf{y}_0) = \mathbf{0}$. For each $j \in [k]$, define $G_j: W \rightarrow \mathbb{R}$ by $\mathbf{y} \mapsto g_j(\mathbf{s}(\mathbf{y}), \mathbf{y})$. As $(\mathbf{s}(\mathbf{y}), \mathbf{y}) \in Z$, we have $G_j = 0$ for each $j \in [k]$. Thus, $\nabla G_j(\mathbf{y}) = \mathbf{0}$.

Let $\mathbf{s} = (s_1, s_2, \dots, s_k)$ where each $s_j: W \rightarrow \mathbb{R}$. Since

$$\begin{aligned} \nabla F(\mathbf{y}) &= df(\mathbf{s}(\mathbf{y}), \mathbf{y}) d(\mathbf{s}(\mathbf{y}), \mathbf{y}) \\ &= \begin{bmatrix} \partial_1 f(\mathbf{s}(\mathbf{y}), \mathbf{y}) & \cdots & \partial_n f(\mathbf{s}(\mathbf{y}), \mathbf{y}) \end{bmatrix} \begin{bmatrix} \partial_1 s_1(\mathbf{y}) & \partial_2 s_1(\mathbf{y}) & \cdots & \partial_{n-k} s_1(\mathbf{y}) \\ \partial_1 s_2(\mathbf{y}) & \partial_2 s_2(\mathbf{y}) & \cdots & \partial_{n-k} s_2(\mathbf{y}) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 s_k(\mathbf{y}) & \partial_2 s_k(\mathbf{y}) & \cdots & \partial_{n-k} s_k(\mathbf{y}) \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \end{aligned}$$

$\nabla F(\mathbf{y}_0) = \mathbf{0}$ implies

$$\partial_{k+j} f(\mathbf{z}_0) + \sum_{i=1}^k \partial_i f(\mathbf{z}_0) \partial_j s_i(\mathbf{y}_0) = 0$$

for each $j \in [n-k]$. Similarly, $\nabla G_m(\mathbf{y}_0) = \mathbf{0}$ for each $m \in [k]$ implies that

$$-\lambda_m \left[\partial_{k+j} g_m(\mathbf{z}_0) + \sum_{i=1}^k \partial_i g_m(\mathbf{z}_0) \partial_j s_i(\mathbf{y}_0) \right] = 0$$

for each $j \in [n-k]$ and $m \in [k]$.

Adding the $k+1$ equations together for each $j \in [n-k]$,

$$0 = \left[\partial_{k+j} f(\mathbf{z}_0) - \sum_{m=1}^k \lambda_m \partial_{k+j} g_m(\mathbf{z}_0) \right] + \sum_{i=1}^k \left[\partial_i f(\mathbf{z}_0) - \sum_{m=1}^k \lambda_m \partial_i g_m(\mathbf{z}_0) \right] \partial_j s_i(\mathbf{y}_0).$$

By the definition of $\lambda_1, \dots, \lambda_k$, we are left with only

$$\partial_j f(\mathbf{z}_0) = \sum_{m=1}^k \lambda_m \partial_j g_m(\mathbf{z}_0)$$

for each $j \in \{k+1, \dots, n\}$. For $j \in [k]$, the same equation holds by the definition of $\lambda_1, \dots, \lambda_k$. Hence, we have $\nabla f(\mathbf{z}_0) = \sum_{m=1}^k \lambda_m \nabla g_m(\mathbf{z}_0)$. \square

Chapter 3

Series of Vectors

3.1 Preliminaries

Definition 3.1.1: Normed Vector Space

Let V be a (real/complex) vector space equipped with a norm $\|\cdot\|$, i.e., the space $(V, \|\cdot\|)$ satisfies the following properties.

- (i) $0 \in V$
- (ii) $\|\mathbf{x}\| \geq 0$ for all $x \in V$ and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = \mathbf{0}$. (*positive definiteness*)
- (iii) $\|\beta \mathbf{x}\| = |\beta| \cdot \|\mathbf{x}\|$ for all $\mathbf{x} \in V$ and $\beta \in \mathbb{R}$. (*absolute homogeneity*)
- (iv) $\|\mathbf{x}_1 + \mathbf{x}_2\| \leq \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$ for all $\mathbf{x}_1, \mathbf{x}_2 \in V$. (*triangle inequality*)

Note:-

Note that $(V, \|\cdot\|)$ is naturally a metric space with the metric function $d(\mathbf{x}_1, \mathbf{x}_2) = \|\mathbf{x}_1 - \mathbf{x}_2\|$.

Definition 3.1.2: Banach Space

A normed vector space $(V, \|\cdot\|)$ is called a *Banach space* if, for every Cauchy sequence $\{\mathbf{x}_j\}_{j \in \mathbb{N}}$, there exists a unique $\mathbf{x}_* \in V$ such that $\lim_{n \rightarrow \infty} \|\mathbf{x}_n - \mathbf{x}_*\| = 0$.

Example 3.1.1

Let A be a compact subset of \mathbb{R}^n . $(V, \|\cdot\|)$ where $V = \{f : A \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$ and $\|f\| = \sup_{x \in A} |f(x)|$ forms a Banach space.

Note:-

A Banach space is a normed vector space whose naturally induced metric space is complete.

Definition 3.1.3: Series

Let $(V, \|\cdot\|)$ be a normed vector space. Given a sequence $\{x_j\}_{j \in \mathbb{N}} \subseteq V$, define $S_k \triangleq \sum_{j=1}^k x_j$ for each $k \in \mathbb{N}$. Then, each S_k is called a *partial sum* of $\{x_j\}$. If $\{S_k\}_{k \in \mathbb{N}}$ converges to S_* with respect to $\|\cdot\|$, then we write

$$S_* = \sum_{j=1}^{\infty} x_j.$$

If the limit S_* exists, we symbolically say that “ $\sum_{j=1}^{\infty} x_j$ converges.”

Lemma 3.1.1

Let $(V, \|\cdot\|)$ be a normed vector space. Let $\{x_j\}_{j \in \mathbb{N}} \subseteq V$ be a sequence. If a series $\sum_{j=1}^{\infty} x_j$ converges, then $\lim_{k \rightarrow \infty} \|x_k\| = 0$.

Proof. $\{S_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence. Hence, $\lim_{k \rightarrow \infty} \|x_k\| = \lim_{k \rightarrow \infty} \|S_{k+1} - S_k\| = 0$. \square

Lemma 3.1.2

Let $(V, \|\cdot\|)$ be a Banach space. Let $\{x_j\}_{j \in \mathbb{N}} \subseteq V$ be a sequence. A series $\sum_{j=1}^{\infty} x_j$ converges if and only if $\{S_k\}_{k \in \mathbb{N}}$ is Cauchy.

Proof. The definition of Banach spaces. \square

3.2 Finite Dimensional Banach Spaces

Theorem 3.2.1 Comparison Test

Given two real sequence $\{a_j\}$ and $\{b_j\}$, suppose $0 \leq a_j \leq b_j$ for all $j \geq k_0$ where $k_0 \in \mathbb{N}$ is a fixed constant. Then, if $\sum_{j=1}^{\infty} b_j$ converges, then $\sum_{j=1}^{\infty} a_j$ converges.

Proof. Let $S_k = \sum_{j=k_0}^k a_j$ and $T_k = \sum_{j=k_0}^k b_j$. Then, $0 \leq S_n - S_m = \sum_{j=m+1}^n a_j \leq \sum_{j=m+1}^n b_j = T_n - T_m$ whenever $n \geq m \geq k_0$. As $\{T_k\}_{k \in \mathbb{N}}$ is Cauchy, $\{S_k\}_{k \in \mathbb{N}}$ is Cauchy as well. As $(\mathbb{R}, \|\cdot\|)$ is a Banach space, $\sum a_j$ converges. \square

Theorem 3.2.2 Absolute Convergence Test

Let $(V, \|\cdot\|)$ be a Banach space. Let $\{x_j\}_{j \in \mathbb{N}} \subseteq V$ be a sequence. If $\sum_{j=1}^{\infty} \|x_j\|$ converges (in \mathbb{R}), then $\sum_{j=1}^{\infty} x_j$ converges.

Proof. Let $S_k = \sum_{j=1}^k x_j \in V$ and $T_k = \sum_{j=1}^k \|x_j\| \in \mathbb{R}$. Then, $\|S_n - S_m\| = \|\sum_{j=m+1}^n x_j\| \leq \sum_{j=m+1}^n \|x_j\| = T_n - T_m$ whenever $n \geq m$. As $\{T_k\}$ is Cauchy, $\{S_k\}$ is Cauchy as well. Hence, $\sum x_j$ converges. \square

Theorem 3.2.3 Summation by Parts

Let $\{a_j\}$ and $\{b_j\}$ be two real sequences. If $\sum a_j$ converges and $\{b_j\}$ is monotonic and convergent, then $\sum_{j=1}^{\infty} a_j b_j$ converges.

Proof. Let $S_k = \sum_{j=1}^k a_j b_j \in V$ and $A_k = \sum_{j=1}^k a_j \in \mathbb{R}$. ($A_0 = 0$.) Then, $S_k = \sum_{j=1}^k (A_j - A_{j-1}) b_j = \sum_{j=1}^k A_j b_j - \sum_{j=0}^k A_0 b_{j+1} + A_k b_{k+1} = A_k b_{k+1} - \sum_{j=1}^k A_j (b_{j+1} - b_j)$.

Let $T_k = \sum_{j=1}^k |A_j (b_{j+1} - b_j)|$. Then, whenever $n < m$, we have

$$0 \leq T_m - T_n \leq M \sum_{j=n+1}^m |b_{j+1} - b_j| = M |b_{m+1} - b_{n+1}| \rightarrow 0,$$

$\{T_k\}$ is Cauchy, and thus converges; $\{S_k\}$ converges as well. \square

Definition 3.2.1: Conditional Convergence

Given a real sequence $\{a_j\}_{j \in \mathbb{N}} \subseteq \mathbb{R}$, if $\sum a_j$ converges, and if $\sum |a_j|$ does not converge, then we say that $\sum a_j$ converges conditionally.

Theorem 3.2.4 Alternating Series Test

Let $\{a_j\}_{j \in \mathbb{N}} \subseteq \mathbb{R}$ be a real sequence. If $a_j \geq 0$ for all $j \in \mathbb{N}$, and if $\lim_{j \rightarrow \infty} a_j = 0$, then $\sum (-1)^j a_j$ converges.

Proof. MAS101. \square

Example 3.2.1

$\sum (-1)^j / j$ conditionally converges.

Note:-

Given, a real sequence $\{a_j\}$, we shall use the following definition for now.

For $j \in \mathbb{N}$, define

$$a_j^+ \triangleq \frac{|a_j| + a_j}{2} = \begin{cases} a_j & \text{if } a_j \geq 0 \\ 0 & \text{if } a_j < 0 \end{cases} \quad \text{and} \quad a_j^- \triangleq \frac{|a_j| - a_j}{2} = \begin{cases} 0 & \text{if } a_j \geq 0 \\ -a_j & \text{if } a_j < 0 \end{cases}.$$

Then, $a_j^+, a_j^- \geq 0$, $|a_j| = a_j^+ + a_j^-$, and $a_j = a_j^+ - a_j^-$.

Lemma 3.2.1

Let $\{a_j\}_{j \in \mathbb{N}}$ be a real sequence.

- (i) If $\sum a_j$ converges absolutely, then both $\sum a_j^+$ and $\sum a_j^-$ converge. Moreover, $\sum a_j = \sum a_j^+ - \sum a_j^-$.
- (ii) If $\sum a_j$ converges conditionally, then both $\sum a_j^+$ and $\sum a_j^-$ diverge.

Proof.

(i) By the definition of a_j^+ and a_j^- .

(ii) If one of $\sum a_j^+$ or $\sum a_j^-$ converges, since $a_j = a_j^+ - a_j^-$, the other converges as well. If they both converge, as $|a_j| = a_j^+ + a_j^-$, $\sum a_j$ converges absolutely. \square

Definition 3.2.2: Rearrangement of Series

Let $\phi: \mathbb{N} \rightarrow \mathbb{N}$ be bijective. Given a sequence $\{a_j\}_{j \in \mathbb{N}}$, the series $\sum a_{\phi(j)}$ is called a rearrangement of $\sum a_j$.

Theorem 3.2.5 Riemann's Rearrangement Theorem

Let $\{a_j\}_{j \in \mathbb{N}}$ be a conditionally convergent real sequence. Then, for any given $-\infty \leq \alpha \leq \beta \leq \infty$ ($\pm\infty$ is allowed for α and β), there exists a rearrangement $\phi: \mathbb{N} \rightarrow \mathbb{N}$ such that $\liminf_{k \rightarrow \infty} \sum_{j=1}^k a_{\phi(j)} = \alpha$ and $\limsup_{k \rightarrow \infty} \sum_{j=1}^k a_{\phi(j)} = \beta$.

Proof.

□

End.