# Summary for Complex Variables I

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# Chapter 1

# Preliminaries

## 1.1 Complex Plane

**Definition 1.1.1: Complex Number**

$i := \sqrt{-1}$ is called the *imaginary unit*. $\mathbb{C} := \{x + iy \mid x, y \in \mathbb{R}\}$ is the set of complex numbers where $\mathbb{R}$ is the set of real numbers.

**Definition 1.1.2: Algebras of $\mathbb{C}$**

For $z_k := x_k + iy_k$ where $k \in \mathbb{Z}_+$ and $x_k, y_k \in \mathbb{R}$,

- $z_1 + z_2 := (x_1 + x_2) + i(y_1 + y_2)$
- $z_1 \cdot z_2 := (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$.

**Theorem 1.1.3**

$\mathbb{C}$ is a field.

***Proof.*** Trivial. □

**Note**

$z = a + ib$, $a, b \in \mathbb{R}$ with $z \neq 0$. Then, $z^{-1} = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}$.

## 1.2 Rectangular Representation

**Definition 1.2.1**

Let $z = x + iy$ where $x, y \in \mathbb{R}$.

(i) $|z| := \sqrt{x^2 + y^2}$ is called *modulus* of $z$.

(ii) $\overline{z} := x - iy$ is called *conjugate* of $z$.

(iii) $\Re z = x$ is called the *real part* of $z$ and $\Im z = y$ is called the *imaginary part* of $z$.

(iv) For $z_1, z_2 \in \mathbb{C}$, $|z_1 - z_2|$ is the *distance* between $z_1$ and $z_2$.

**Note**

- $z + \bar{z} = 2\Re z$
- $z - \bar{z} = 2i\Im z$
- $|z_1 + z_2| \le |z_1| + |z_2|$
- $\big||z_1| - |z_2|\big| \le |z_1 - z_2|$

## 1.3 Polar Representation

Given $z \in \mathbb{C}$, $|z|$ is unique. $\arg z = \theta + 2k\pi$ $(k \in \mathbb{Z})$ (Or $\arg z = \theta \pmod{2\pi}$)

**Definition 1.3.1**

If $z = |z| \cdot (\cos\theta + i\sin\theta)$, $\theta$ is called an *argument* of $z$ and is written $\arg z = \theta \pmod{2\pi}$ (as $\theta + 2k\pi$ for $k \in \mathbb{Z}$ is an argument of $z$ as well). If $\arg z = \theta^* \pmod{2\pi}$, and if $-\pi < \theta^* \le \pi$, then we define $\operatorname{Arg} z = \theta^*$ and it is called the *principal argument* of $z$.

**Theorem 1.3.2**

For $z_1, z_2 \in \mathbb{C}$ with $z_1, z_2 \neq 0$, $\arg z_1 z_2 = \arg z_1 + \arg z_2 \pmod{2\pi}$.

***Proof.*** Let $\arg z_1 = \theta_1 \pmod{2\pi}$ and $\arg z_2 = \theta_2 \pmod{2\pi}$ Then, $z_1 = |z_1|(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = |z_2|(\cos\theta_2 + i\sin\theta_2)$. Now, we have $z_1 \cdot z_2 = |z_1||z_2|\big(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\big)$. $\square$

# Chapter 2

# Elementary Complex Functions

## 2.1 Exponential Functions

**Definition 2.1.1: Exponential Function**

For each $z = x + iy$ where $x, y \in \mathbb{R}$, we define $e^z := e^x \cdot (\cos y + i \sin y)$.

**Theorem 2.1.2**

For each $z \in \mathbb{C}$, $e^z = \sum_{j=1}^{\infty} \frac{z^j}{j!}$.

***Proof.*** Proved later using complex integral. □

**Theorem 2.1.3**

For each $z, z' \in \mathbb{C}$,

(a) $e^{z+z'} = e^z \cdot e^{z'}$,

(b) $e^{-z} = \frac{1}{e^z}$, and

(c) $e^{z+2k\pi i} = e^z$ for all $k \in \mathbb{Z}$.

**Definition 2.1.4**

For each $z \in \mathbb{C}$,

(1) $\cos z := \frac{e^{iz}+e^{-iz}}{2}$

(2) $\sin z := \frac{e^{iz}-e^{-iz}}{2i}$

(3) $\cosh z = \frac{e^{z}+e^{-z}}{2}$

(4) $\sinh z = \frac{e^{z}-e^{-z}}{2}$

**Theorem 2.1.5**

For each $z \in \mathbb{C}$, we have $\cosh z = \cos(iz)$ and $\sinh z = -i \sin(iz)$.

**Example 2.1.6**

Let us solve $\cos z = 2$. Let $t := e^{iz}$ to obtain $t^2 - 4t + 1 = 0$, which gives $t = 2 \pm \sqrt{3}$. Write $z = x + iy$ where $x, y \in \mathbb{R}$ to have $e^{ix}e^{-y} = 2 \pm \sqrt{3}$. Taking modulus to both sides gives $e^{-y} = 2 \pm \sqrt{3}$, i.e., $y = -\ln(2 \pm \sqrt{3})$. Taking argument to both sides gives $x = 2k\pi$

for $k \in \mathbb{Z}$. Thus, $z = 2k\pi - i\ln(2 \pm \sqrt{3})$ for $k \in \mathbb{Z}$.

## 2.2 Mapping Properties

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## 2.3 Logarithmic "Functions"

**Definition 2.3.1: Logarithmic Function**

For any $z \in \mathbb{C} \setminus \{0\}$, we define $w = \ln z$ if and only if $e^w = z$.

**Note**

How to compute $\ln z$? Note that $z = |z| \cdot e^{i(\operatorname{Arg} z + 2k\pi)}$ for $k \in \mathbb{Z}$. Let $w = u + iv$ where $u, v \in \mathbb{R}$ so that $e^w = e^u \cdot e^{iv} = |z| \cdot e^{i(\operatorname{Arg} z + 2k\pi)}$. Hence, we have $u = \ln|z|$ and $v = \operatorname{Arg} z + 2k\pi$. In other words, $\ln z = \ln|z| + i \arg z$. (Note that this is not a "function"!)

**Definition 2.3.2: Principal Logarithmic Function**

For any $z \in \mathbb{C} \setminus \{0\}$, we define $\operatorname{Ln} z := \ln|z| + i \operatorname{Arg} z$ and it is called the *principal value of* $\ln z$.

# Chapter 3

# Analytic Functions

# Chapter 4

# Complex Integration

# Chapter 5

# Conformal Mapping

*End.*