

MAS242 선형대수학 Notes

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November 8, 2023

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Chapter 1

Linear Equations

Chapter 2

Vector Spaces

2.1 Bases and Dimension

Theorem 2.1.1

Any subset that is linearly independent can be extended to a basis of V .

Lemma 2.1.1

If W is a subspace of V and $W \subsetneq V$, then $\dim W < \dim V$ provided that V is finite-dimensional.

Proof. Let S_0 be a basis of W . S_0 is linearly independent, so we can enlarge it to get a basis of V . $S' \triangleq S_0 \cup \{v_1, v_2, \dots, v_r\}$ is a basis of V . $|S'| \geq |S_0| + 1$; otherwise $\text{span } S_0 = V$. \square

Theorem 2.1.2 Inclusion/Exclusion Principle for Vector Spaces

If W_1 and W_2 are finite-dimensional subspaces of V , then $W_1 + W_2$ is a finite-dimensional vector space and $\dim W_1 + \dim W_2 = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$.

Proof. Let $a \triangleq \dim W_1$, $b \triangleq \dim W_2$, $c \triangleq \dim(W_1 + W_2)$, and $d \triangleq \dim(W_1 \cap W_2)$. Choose $\{\alpha_1, \alpha_2, \dots, \alpha_d\}$ as a basis for $W_1 \cap W_2$. We may extend this into bases of W_1 and W_2 . Let $\{\alpha_1, \dots, \alpha_d, \beta_{d+1}, \beta_{d+2}, \dots, \beta_a\}$ and $\{\alpha_1, \dots, \alpha_d, \gamma_{d+1}, \gamma_{d+2}, \dots, \gamma_b\}$ be bases for W_1 and W_2 respectively.

We now claim that

$$B \triangleq \{\alpha_1, \dots, \alpha_d, \beta_{d+1}, \dots, \beta_a, \gamma_{d+1}, \dots, \gamma_b\}$$

is a basis of $W_1 + W_2$.

- Let $x \in W_1 + W_2$. Then, $x = w_1 + w_2$ where $w_i \in W_i$. Since $w_1 \in \text{span}\{\alpha_1, \dots, \alpha_d, \beta_{d+1}, \dots, \beta_a\}$ and $w_2 \in \text{span}\{\alpha_1, \dots, \alpha_d, \gamma_{d+1}, \dots, \gamma_b\}$, On the other hand, $B \subseteq W_1 + W_2$. Hence, $\text{span } B = W_1 + W_2$.
- Suppose we have $\sum a_i \alpha_i + \sum b_j \beta_j + \sum c_k \gamma_k = 0$ for some $a_i, b_j, c_k \in F$. Rearranging the terms, we get $\sum a_i \alpha_i + \sum b_j \beta_j = -\sum c_k \gamma_k$, which implies that $\sum c_k \gamma_k \in W_1 \cap W_2$. The fact that γ_k 's are linearly independent from $\{\alpha_i\}$ implies that $c_k = 0$ for all k . Similarly, $b_j = 0$ for all j . Hence, we are left with $\sum a_i \alpha_i = 0$, in which α_i 's are linearly independent; $a_i = 0$. Hence, B is linearly independent.

Therefore, $\dim(W_1 + W_2) = a + b - d$. \square

Definition 2.1.1: Ordered Basis

Let V be a finite-dimensional vector space over F . An *ordered basis* of V is a sequence of vectors that forms a basis.

Note:-

Usually, we emphasize the ordered basis with semicolons like $\{\beta_1; \beta_2\}$.

Lemma 2.1.2

Let V be a finite-dimensional vector space over F . Suppose $B = \{v_1; v_2; \dots; v_n\}$ is an ordered basis of V . Then, for each $x \in V$, there uniquely exists an expression of the form

$$x = x_1 v_1 + x_2 v_2 + \dots + x_n v_n$$

for some $x_i \in F$.

Proof. The existence of the form is obvious since $x \in V = \text{span } B$.

(Uniqueness) Suppose we have two such expressions:

$$x = \sum x_i v_i = \sum y_i v_i$$

where $x_i, y_i \in F$. Then, we have $\sum (x_i - y_i) v_i = 0$. The linear independence of B gives that $x_i - y_i = 0$ for all i . Hence, $x_i = y_i$. \square

Definition 2.1.2: Coordinate Matrix

Let V be a finite-dimensional vector space over F . Let B be an ordered basis of V . Let $x \in V$ and write it as $x = \sum_{i=1}^n x_i v_i$ with $x_i \in F$, $v_i \in B$. Define

$$[x]_B \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

be the *coordinate matrix* of x with respect to the basis B

Theorem 2.1.3

Let V be a finite-dimensional vector space over F . Let B and B' be two ordered bases of V . Then, there uniquely exists an invertible matrix P such that $\forall x \in V$, $[x]_B = P[x]_{B'}$ and $[x]_{B'} = P^{-1}[x]_B$.

Proof. Let $B \triangleq \{\alpha_1; \dots; \alpha_n\}$ and $B' \triangleq \{\alpha'_1; \dots; \alpha'_n\}$. For $\alpha'_j \in B'$, since B is a basis, there are unique $P_{ij} \in F$ ($i \in [n]$) such that $\alpha'_j = \sum_{i=1}^n P_{ij} \alpha_i$.

Let $x \in V$. Write $[x]_B = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $[x]_{B'} = \begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix}$. Then, $x = \sum_{j=1}^n x'_j \alpha'_j = \sum_{j=1}^n \left(\sum_{i=1}^n x'_j P_{ij} \right) \alpha_i$.

By the uniqueness, we have $x_i = \sum_{j=1}^n x'_j P_{ij}$ for each i . In other words, we have

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} P_{11} & \cdots & P_{1n} \\ \vdots & \cdots & \vdots \\ P_{n1} & \cdots & P_{nn} \end{bmatrix} \begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix}$$

Since B and B' are linearly independent, $x = 0 \iff [x]_B = 0 \iff [x]_{B'} = 0$. Hence, P is invertible. \square

Chapter 3

Linear Transformations

3.1 Linear Transformations

Definition 3.1.1: Linear Transformation

Let V_1 and V_2 be vector spaces over F . $T: V_1 \rightarrow V_2$ is said to be a *linear transformation* if

- $\forall x_1, x_2 \in V_1, T(x_1 + x_2) = T(x_1) + T(x_2)$
- $\forall x \in V_1, \forall c \in F, T(cx) = cT(x)$.

Theorem 3.1.1

Let V and W be finite-dimensional vector spaces over F . where $\{\alpha_1, \dots, \alpha_n\}$ is a basis of V . Let $\{\beta_1, \dots, \beta_n\}$ be any given set of vectors of W . Then, there exists a unique transformation $T: V \rightarrow W$ such that $T(\alpha_i) = \beta_i$.

Proof. Let $T_0: V \rightarrow W$ be defined by

$$T_0\left(\sum_{i=1}^n x_i \alpha_i\right) = \sum_{i=1}^n x_i \beta_i.$$

This is a linear transformation indeed.

(Uniqueness) If there is another such $U: V \rightarrow W$, Then, $U\left(\sum_{i=1}^n x_i \alpha_i\right) = \sum_{i=1}^n x_i U(\alpha_i)$. Hence, $U = T_0$. \square

Definition 3.1.2: Null Space and Range Space

Let $T: V \rightarrow W$ be a linear transformation between vector spaces over F .

- $\text{null } T \triangleq \ker T \triangleq \{v \in V \mid T(v) = 0\}$
- $\text{range } T \triangleq \text{Im } T \triangleq \{w \in W \mid \exists v \in V, w = T(v)\}$

Note:-

$\ker T$ and $\text{Im } T$ are subspaces of V and W respectively.

Definition 3.1.3

Let $T: V \rightarrow W$ be a linear transformation between vector spaces over F .

$$\text{nullity}(T) \triangleq \dim \ker(T) \quad \text{and} \quad \text{rank}(T) \triangleq \dim \text{Im}(T)$$

Theorem 3.1.2 Rank-Nullity Theorem

Let $T: V \rightarrow W$ be a linear transformation between vector spaces over F . Then, $\text{rank}(T) + \text{nullity}(T) = \dim V$.

Proof. Let $\{v_1, \dots, v_k\}$ be a basis for $\ker T$ where $k = \text{nullity } T$. Choose $v_{k+1}, \dots, v_n \in V$ such that $\{v_i\}_{i=1}^n$ is a basis of V . We claim that $\{T(v_{k+1}), \dots, T(v_n)\}$ is a basis of $\text{Im } T$.

Suppose $\sum_{i=k+1}^n c_i T(v_i) = 0$ for some $c_i \in F$. Then, we have $T(\sum_{i=k+1}^n c_i v_i) = 0$; hence $\sum_{i=k+1}^n c_i v_i \in \ker T$. Since $\{v_1, \dots, v_k\}$ is a basis of $\ker T$, we have $\sum_{i=k+1}^n c_i v_i = \sum_{i=1}^k a_i v_i$ for some a_i 's. Therefore, since $\{v_1, \dots, v_n\}$ is linearly independent, all c_i 's and a_i 's are zero. This implies that $\{T(v_i)\}_{i=k+1}^n$ is linearly independent.

Take any $T(v) \in \text{Im } T$. Then, $v = \sum_{i=1}^n c_i v_i$ for some $c_i \in F$. Then, $T(v) = \sum_{i=k+1}^n c_i T(v_i)$. Hence, $\text{Im } T \subseteq \text{span}\{T(v_{k+1}), \dots, T(v_n)\}$

The two paragraphs imply that $\text{rank } T = n - k$. \square

Theorem 3.1.3

Let A be a $m \times n$ matrix. Then $\dim \text{span}(\text{rows}) = \dim \text{span}(\text{columns})$.

Proof. $V = F^n$, $W = F^m$. Then, $\dim \text{span}(\text{columns}) = \dim \text{Im } T = \text{rank } T$, so $\text{nullity } T = n - \text{rank } T = n - \text{colrank } T$.

The number of rows with leading one's in $\text{rref } A$ equals the dimension of the row space of A , which is simply the number of columns with the leading ones. It is equal to the dimension of the column space. Hence, $\text{nullity } T = n - \text{colrank } T$ \square

3.2 The Algebra of Linear Transformations

Definition 3.2.1

Let $T: V \rightarrow W$ be a linear transformation between vector spaces over F . $L(V, W) \triangleq \{T: V \rightarrow W \mid T \text{ is a linear transformation}\}$

Theorem 3.2.1

Let $T: V \rightarrow W$ be a linear transformation between vector spaces over F . Then, $L(V, W)$ is a vector space over F under usual addition and multiplication.

Theorem 3.2.2

Let V and W be n - and m -dimensional vector spaces over F , respectively. Then, $\dim L(V, W) = mn$.

Proof. Let $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ and $\mathcal{B}' = \{\beta_1, \dots, \beta_m\}$ be bases for V and W , respectively. For each $p \in [n]$ and $q \in [m]$, Define

$$E^{p,q}(\alpha_i) = \begin{cases} 0 & \text{if } i \neq p \\ \beta_q & \text{if } i = p \end{cases}.$$

Then,

- These $E^{p,q}$ are linear transformations
- These are linearly independent.

- They span $L(V, W)$.

□

Lemma 3.2.1

Let $T: V \rightarrow W$ and $U: W \rightarrow Z$ be linear transformations between vector spaces over F . Then, $U \circ T \in L(V, Z)$.

Definition 3.2.2: Linear Operator (Endomorphism)

Let $T: V \rightarrow V$ be a linear transformation from a vector space V to itself. Then, T is called a *linear operator*. (Or an *endomorphism*.)

Note:-

For each $T, U \in L(V, V)$, $T \circ U \in L(V, V)$. $(T_1 + T_2) \circ U = T_1 \circ U + T_2 \circ U$. And many more... $(L(V, V), +, \circ)$ is a non-commutative ring.

Definition 3.2.3: Injectivity and Surjectivity

A linear transform $T: V \rightarrow W$ is

- *injective* (or, nonsingular) if $T(v) = 0 \implies v = 0$.
- *surjective* if $T(V) = W$.
- *invertible* if \exists linear transform $U: W \rightarrow V$, $U \circ T = \text{id}_V \wedge T \circ U = \text{id}_W$.

Exercise 3.2.1

$T: V \rightarrow W$ is injective and surjective if and only if T is invertible.

Exercise 3.2.2

If $T: V \rightarrow W$ is a nonsingular linear transformation, then, for any linearly independent subset $S \subseteq V$, $T(S)$ is linearly independent.

Exercise 3.2.3

Suppose V and W are finite-dimensional vector spaces. If $T: V \rightarrow W$ is invertible, then $\dim V = \dim W$.

Theorem 3.2.3

Let V and W be finite-dimensional vector spaces over F with $\dim V = \dim W$. Let $T: V \rightarrow W$ be a linear transform. TFAE

- T is invertible.
- T is injective.
- T is surjective.

Proof. T is injective \iff nullity $T = 0 \iff$ rank $T = n \iff \text{Im } T = W \iff T$ is onto □

Definition 3.2.4: General Linear Group

Let $\text{GL}(V) \triangleq \{ T \in L(V, V) \mid T \text{ is invertible} \}$. Then, $(\text{GL}(V), \circ)$ is called the *general linear group* of V .

Note:-

The general linear group is actually a group.

3.3 Isomorphism

Definition 3.3.1: Isomorphism

Let V and W be vector spaces over F . We say that a linear transformation $T: V \rightarrow W$ is an *isomorphism* if T is an invertible linear transformation.

We say V and W are *isomorphic* if there exists an isomorphism $T: V \rightarrow W$, if V and W are isomorphic, then we write $V \simeq W$.

Theorem 3.3.1

Let V be a vector spaces over F of dimension n . Then, $V \simeq F^n$.

Proof. Let $B = \{\alpha_1; \dots; \alpha_n\}$ be a basis of V . Define $T: V \rightarrow F^n$ by $v \mapsto [v]_B$.

Suppose $T(v) = 0$. Then, $v = 0 \cdot \alpha_1 + \dots + 0 \cdot \alpha_n = 0$. Hence, T is injective. By Theorem 3.2.3, T is isomorphism. \square

3.4 Representation of Transformation by Matrices

Theorem 3.4.1

Let V and W be vector spaces over F with $\dim V = n$ and $\dim W = m$. Let B and B' be bases of V and W , respectively. If $T: V \rightarrow W$ is a linear transformation, then there uniquely exists $m \times n$ matrix A such that $[T(v)]_{B'} = A[v]_B$. We write $[T]_{B,B'} \triangleq A$.

Proof. $A = \begin{bmatrix} [T(v_1)]_{B'} & [T(v_2)]_{B'} & \dots & [T(v_n)]_{B'} \end{bmatrix}$ where v_i is the i^{th} basis vector of B . \square

Theorem 3.4.2

Let $V \xrightarrow{T} W \xrightarrow{U} Z$ be linear transformations. Let $A_1 = [T]_{B,B'}$ and $A_2 = [U]_{B',B''}$. Then, $[U \circ T]_{B,B''} = A_2 A_1$.

Theorem 3.4.3

Let V be finite-dimensional vector space over F with two (possibly different) bases B_1 and B_2 . Let $T \in L(V, V)$. Let P be the matrix such that $[v]_{B_1} = P[v]_{B_2}$. Then, $[T]_{B_i} \triangleq [T]_{B_i, B_i}$ are related by

$$[T]_{B_2} = P^{-1} [T]_{B_1} P.$$

Definition 3.4.1: Similar Matrices

Suppose M and N are $n \times n$ matrices. M and N are *similar* if there exists an invertible P such that $N = P^{-1} M P$.

Proof. $[T(v)]_{B_1} = [T]_{B_1} [v]_{B_1} = [T]_{B_1} P [v]_{B_2}$. $[T(v)]_{B_1} = P [T(v)]_{B_2} = P [T]_{B_2} [v]_{B_2}$.

Since v was arbitrary, $P [T]_{B_2} = [T]_{B_1} P$. \square

Note:-

- A linear transformation $T: V \rightarrow V$ gives varying matrices $[T]_B$ that are all similar when the basis B is changed.
- On linear operators, we will have various definitions.
- Characteristic (eigen) polynomial has $(-1)^{\deg}(\text{constant term})$ as $\det T$ and $-(n - 1 \text{ deg term})$ as $\text{tr } T$.

3.5 Linear Functionals

Definition 3.5.1: Linear Functional

Let V be a vector space over F . A linear transformation $T: V \rightarrow F$ is called a (*linear*) *functional*.

Definition 3.5.2: Dual Vector Space

Let V be a vector space over F . We normally write $V^* \triangleq L(V, F)$ and call it the *dual vector space* of V .

Note:-

By Theorem 3.2.2, we know that $\dim V^* = \dim V$ if V is a finite-dimensional vector space.

Lemma 3.5.1

Let V be a finite-dimensional vector space over F and let $n = \dim V$. Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis for V . Define $f_i \in V^*$ by declaring $f_i(\alpha_j) = \delta_{ij}$. Then, $\{f_1, \dots, f_n\}$ is a basis for V^* .

Proof. Since $\dim V^* = \dim V = n$, we only need to show that the set is linearly independent.

Suppose $\sum_{i=1}^n c_i f_i = 0$ for some $c_i \in F$. Then, for each $j \in [n]$, as $f_i(\alpha_j) = \delta_{ij}$, $0 = (\sum_{i=1}^n c_i f_i)(\alpha_j) = c_j f_j(\alpha_j) = c_j$. Hence, they are linearly independent. \square

Definition 3.5.3: Dual Basis

The set $\{f_1, f_2, \dots, f_n\} \subseteq V^*$ in Lemma 3.5.1 is called the *dual basis* of the basis $\{\alpha_1, \dots, \alpha_n\}$ for V .

Lemma 3.5.2

Let V be a finite-dimensional vector space over F and let $n = \dim V$. Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis for V . Let $\{f_1, \dots, f_n\} \subseteq V^*$ be the dual basis of it.

(i) For each $f \in V^*$, $f = \sum_{i=1}^n f(\alpha_i) f_i$.

(ii) For each $v \in V$, $v = \sum_{i=1}^n f_i(v) \alpha_i$.

Proof.

(i) There exists $x_i \in F$ such that $f = \sum_{i=1}^n x_i f_i$. Evaluating at α_j for each $j \in [n]$, we get $f(\alpha_j) = x_j$.

(ii) There exists $y_i \in F$ such that $v = \sum_{i=1}^n y_i \alpha_i$. Applying f_j for each $j \in [n]$, we get $f_j(v) = y_j$. \square

Definition 3.5.4: Hyperspace

Let V be a finite-dimensional vector space over F and let $n = \dim V$. A subspace W of V which has the dimension $n - 1$ is called a *hyperspace* in V .

Example 3.5.1

If $f : V \rightarrow F$ is a nonzero functional, then $\ker f$ is an example of a hyperspace in V .

Definition 3.5.5: Annihilator

Let V be a finite-dimensional vector space over F with dimension n . Let $\emptyset \subsetneq S \subseteq V$. The *annihilator* of S , $S^\circ = \text{Ann } S$ is defined to be

$$S^\circ = \{f \in V^* \mid \forall \alpha \in S, f(\alpha) = 0\}.$$

Note:-

- S° is a subspace of V^*
- $\text{Ann } \{0\} = V^*$.
- $\text{Ann } V = \{0\}$.

Theorem 3.5.1

Let V be a finite-dimensional vector space over F with dimension n . Let W be a subspace of V . Then,

$$\dim W + \dim W^\circ = \dim V.$$

Proof. Let $k \triangleq \dim W$ and $\{\alpha_1, \dots, \alpha_k\} \subseteq W$ be a basis for W . We may extend it to the basis for V so that $\{\alpha_1, \dots, \alpha_k, \alpha_{k+1}, \dots, \alpha_n\}$ is a basis for V . Let $\{f_1, \dots, f_k, f_{k+1}, \dots, f_n\}$ be the dual basis of $\{\alpha_1, \dots, \alpha_n\}$.

For each $i \in \{k+1, \dots, n\}$, by the construction of the dual basis, $f_i(\alpha_j) = 0$ for each $j \in [k]$. Hence, $f_{k+1}, \dots, f_n \in W^\circ$.

Take any $f \in W^\circ$. Then, $f = \sum_{i=1}^n f(\alpha_i)f_i$. For each $i \in [k]$, $f(\alpha_i) = 0$. Hence, $f = \sum_{i=k+1}^n f(\alpha_i)f_i$; $\{f_{k+1}, \dots, f_n\}$ spans W° . Therefore, $\{f_{k+1}, \dots, f_n\}$ is a basis for W° . \square

Corollary 3.5.1

Let V be a finite-dimensional vector space over F with dimension n . Let W be a k -dimensional subspace of V . Then, W is the intersection of $n - k$ hyperspaces in V of the form $\ker f_i$ for some $f_i \in V^* \setminus \{0\}$.

Proof. Let $\{\alpha_1, \dots, \alpha_k\}$ be a basis for W and extend it to $\{\alpha_1, \dots, \alpha_n\}$ so that it becomes a basis for V . Let $\{f_1, \dots, f_n\} \subseteq V^*$ be the dual basis of $\{\alpha_1, \dots, \alpha_n\}$. Then, $W = \bigcap_{i=k+1}^n \ker f_i$. \square

Corollary 3.5.2

Let V be a finite-dimensional vector space over F with dimension n . Let W be a hyperspace in V . Then, $W = \ker f$ for some $f \in V^* \setminus \{0\}$.

3.6 The Double Dual

Note:-

Take $\alpha \in V$. Let us define $L_\alpha \in V^{**}$ as follows:

$$\begin{aligned} L_\alpha : V^* &\longrightarrow F \\ f &\longmapsto f(\alpha). \end{aligned}$$

Then, define \mathcal{L} by

$$\begin{aligned} \mathcal{L} : V &\longrightarrow V^{**} \\ \alpha &\longmapsto L_\alpha. \end{aligned}$$

Then, \mathcal{L} is an injective linear transformation.

Theorem 3.6.1

Let V be a finite-dimensional vector space over F with dimension n . Then, $\mathcal{L} : V \rightarrow V^{**}$ is an isomorphism of vector spaces.

Proof. We have $\dim V = \dim V^* = \dim V^{**} = n$ by Theorem 3.2.2. The result follows from Theorem 3.2.3. \square

Definition 3.6.1: Proper Subspace

Let V be a vector space over F . Then, a subspace W of V is a *proper subspace* of V if $W \subsetneq V$.

Definition 3.6.2: Maximal Subspace

A proper subspace W of V is said to be *maximal* if, there exists no subspace Z of V such that $W \subsetneq Z \subsetneq V$.

Definition 3.6.3: Hyperspace

Let V be a vector space over F . A maximal proper subspace W of V is called a *hyperspace* in V .

Note:-

In case of $\dim V = n$, a proper maximal subspace of V is of dimension $n - 1$.

Theorem 3.6.2

Let V be a vector space over F . Let $f \in V^* \setminus \{0\}$. Then, $\ker f$ is a hyperspace in V .

Proof. $\ker f$ is proper, since, otherwise, $f = 0$.

It is enough to show that, for each $\alpha \in V \setminus \ker f$, $\text{span}\{\ker f, \alpha\} = V$. Take any $\beta \in V$. Let $\alpha \in V \setminus \ker f$. Define $c \triangleq f(\alpha)^{-1}f(\beta)$ and $\gamma \triangleq \beta - c\alpha$. Then, $f(\gamma) = f(\beta) - cf(\alpha) = 0$; $\gamma \in \ker f$. Hence, $\beta = \gamma + c\alpha \in \text{span}[\ker f, \alpha]$. \square

Theorem 3.6.3

Let V be a vector space over F . Let W be a hyperspace in V . Then, there exists $f \in$

$V^* \setminus \{0\}$ such that $W = \ker f$.

Proof. There exists $\alpha \in V \setminus W$ such that $\text{span}\{W, \alpha\} = V$. Hence, every $\beta \in V$ can be written as $\beta = \gamma + c\alpha$ where $\gamma \in W$ and $c \in F$. Note that γ and c are uniquely determined by β .

Define $g: V \rightarrow F$ by $g(\beta) = c$. Then, g is a linear functional, and $\ker g = W$ by definition. \square

Note:-

Theorem 3.6.2 and Theorem 3.6.3 together imply that the set of hyperspaces in V and the set of null spaces of functionals have a one-to-one correspondence.

3.7 The Transpose of a Linear Transformation

Definition 3.7.1: Transpose

Let $T: V \rightarrow W$ be a linear transformation. The map $T^t: W^* \rightarrow V^*$ defined by $g \mapsto g \circ T$ is called the *transpose* of T .

Lemma 3.7.1

Let $T: V \rightarrow W$ be a linear transformation. Then, T^t is a linear transformation.

Theorem 3.7.1

Let $T: V \rightarrow W$ be a linear transformation between finite-dimensional vector spaces over F . Fix ordered bases \mathcal{B} and \mathcal{B}' for V and W , respectively. Let \mathcal{B}^* and \mathcal{B}'^* be their dual bases. Let $A \triangleq [T]_{\mathcal{B}, \mathcal{B}'}$ and $A' \triangleq [T^t]_{\mathcal{B}'^*, \mathcal{B}^*}$. Then, $a_{ij} = a'_{ji}$.

Proof. Let $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$, $\mathcal{B}' = \{\beta_1, \dots, \beta_m\}$, $\mathcal{B}^* = \{f_1, \dots, f_n\}$, and $\mathcal{B}'^* = \{g_1, \dots, g_m\}$. Then, we have $T\alpha_j = \sum_{i=1}^m a_{ij}\beta_i$ for each $j \in [n]$ and $T^t g_j = \sum_{i=1}^n b_{ij}f_i$ for each $j \in [m]$.

For each $i \in [n]$ and $j \in [m]$, $(T^t g_j)(\alpha_i) = g_j(T\alpha_i) = g_j\left(\sum_{k=1}^m a_{ki}\beta_k\right) = \sum_{k=1}^m a_{ki}g_j(\beta_k) = a_{ji}$. Hence, since $T^t g_j$ is a linear functional on V , $T^t g_j = \sum_{i=1}^n (T^t g_j)(\alpha_i)f_i = \sum_{i=1}^n a_{ji}f_i$. Therefore, $a_{ij} = b_{ji}$ for each $i \in [n]$ and $j \in [m]$. \square

Theorem 3.7.2

Let $T: V \rightarrow W$ be a linear transformation.

- (i) $\ker T^t = (\text{Im } T)^\circ$.
- (ii) If V and W are finite-dimensional, then $\text{rank } T^t = \text{rank } T$.
- (iii) If V and W are finite-dimensional, then $\text{Im } T^t = (\ker T)^\circ$.

Proof.

- (i) $g \in \ker T^t \iff T^t(g) = 0 \iff g \circ T = 0 \iff g \in (\text{Im } T)^\circ$
- (ii) Let $n \triangleq \dim V$ and $m \triangleq \dim W$. Let $r = \text{rank } T$. Then, by Theorem 3.5.1, $\dim(\text{Im } T)^\circ = m - r$. By (i), $(\text{Im } T)^\circ = \ker T^t$; hence $\text{nullity } T^t = m - r$. By the rank-nullity theorem, $\text{rank } T^t = r = \text{rank } T$.
- (iii) Take any $f \in \text{Im } T^t$. Then, there exists $g \in W^*$ such that $f = g \circ T$. Then, for any $\alpha \in \ker T$, $f(\alpha) = g(T(\alpha)) = 0$. Hence, $f \in (\ker T)^\circ$; $\text{Im } T^t \subseteq (\ker T)^\circ$. But since the two spaces have the same dimension, it must be the equality to hold. \square

Chapter 4

Polynomials

4.1 Algebras

Definition 4.1.1: F -algebra

Let F be a field. A vector space \mathcal{A} with a map $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ such that

- (i) $\forall \alpha, \beta, \gamma \in \mathcal{A}, \alpha(\beta\gamma) = (\alpha\beta)\gamma$
- (ii) $\forall \alpha, \beta, \gamma \in \mathcal{A}, \alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ and $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$
- (iii) $\forall c \in F, \forall \alpha, \beta \in \mathcal{A}, c(\alpha\beta) = (c\alpha)\beta = \alpha(c\beta)$

is called a F -algebra or a linear algebra over F .

- If there is an element 1 in \mathcal{A} such that $1\alpha = \alpha 1 = \alpha$ for each $\alpha \in \mathcal{A}$, then we call \mathcal{A} a F -algebra with identity.
- The algebra \mathcal{A} is called *commutative* if $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathcal{A}$.

Definition 4.1.2: Polynomial

Let $F[x]$ be the subspace of F^ω spanned by the vectors $1, x, x^2, \dots$. An element of $F[x]$ is called a *polynomial over F* .

Definition 4.1.3: Degree

For each $f \in F[x] \setminus \{0\}$, $\deg f \triangleq \max\{k \in \mathbb{N} \cup \{0\} \mid f_k \neq 0\}$.

Theorem 4.1.1

Let $f, g \in F[x] \setminus \{0\}$.

- (i) $fg \neq 0$
- (ii) $\deg(fg) = \deg f + \deg g$
- (iii) fg is monic if f and g are monic.
- (iv) fg is scalar polynomial if f and g are scalar polynomials.
- (v) If $f + g \neq 0$, then $\deg(f + g) \leq \max\{\deg f, \deg g\}$.

Theorem 4.1.2 Euclidean Algorithm

Let $f, g \in F[x]$ and $g \neq 0$. Then, there uniquely exists $q, r \in F[x]$ such that

- $f = gq + r$ and
- either $r = 0$ or $\deg r < \deg g$.

Definition 4.1.4: Divisibility

Let $f, g \in F[x]$. If $f = gq$ for some $q \in F[x]$, then we write $g \mid f$.

Lemma 4.1.1

Let $f \in F[x] \setminus \{0\}$ and $c \in F$. Then, $(x - c) \mid f \iff f(c) = 0$.

Proof. There exists $q, r \in F[x]$ such that $f = (x - c)q + r$ with either $r = 0$ or $\deg r = 0$. Note that $f(c) = r$. Hence, $f(c) = 0 \iff (x - c) \mid f$. \square

Definition 4.1.5: Evaluation

Let F be a field. Let $\alpha \in F$ be fixed. Then, the function $\text{ev}_\alpha: F[x] \rightarrow F$ defined by $f \mapsto f(\alpha)$ is called the *evaluation of α in $f(x)$* .

Definition 4.1.6: Ideal

An ideal $M \subseteq F[x]$ is an F -subspace if for every $f \in F[x]$ and $g \in M$, we have $fg \in M$.

Definition 4.1.7: Principal Ideal

An ideal of the form

$$M = \{g_0 h \mid h \in F[x]\} = (g_0)$$

for a fixed g_0 is called a *principal ideal*.

Theorem 4.1.3

Let F be a field. Let $M \subseteq F[x]$ be a nonzero ideal. Then, M is a principal ideal given by a monic polynomial in $F[x]$.

Proof. M does contain nonzero polynomials. Hence, we may let $g_0 \in \arg\min_{g \in M \setminus \{0\}} \deg g$ by the well-orderedness of natural numbers. WLOG, g_0 is monic.

We shall claim that $M = (g_0)$. Take any $f \in M$. By the Euclidean algorithm, $\exists q, r \in F[x]$, $f = g_0 q + r$ with either $r = 0$ or $\deg r < \deg g_0$. If $r \neq 0$, then $r = f - g_0 q \in M$ but $\deg r < \deg g_0$, which contradicts the minimality of $\deg g_0$. Hence, $r = 0$, and thus $f = g_0 q \in (g_0)$. \square

Note:-

By putting “monic” assumption, such g_0 is unique as well.

Corollary 4.1.1

Let $p_1, \dots, p_n \in F[x]$ be a finite number of polynomials where not all of them are zero. Then, there uniquely exists monic $g_0 \in F[x]$ such that

- (i) $p_1 F[x] + p_2 F[x] + \dots + p_n F[x] = (g_0)$
- (ii) $\forall i \in [n], g_0 \mid p_i$
- (iii) $(\forall i \in [n], f \mid p_i) \implies f \mid g_0$

Such g_0 is called the *greatest common divisor* of p_1, \dots, p_n . Sometimes this is denoted by $(p_1, \dots, p_n) = (g_0)$.

Proof. $p_1F[x] + p_2F[x] + \cdots + p_nF[x]$ is an ideal. By Theorem 4.1.3, there uniquely exists monic g_0 that generates it. (ii) directly follows from (i). $g_0 = \sum_{j=1}^n p_j g_j = f \sum_{j=1}^n h_j g_j$. \square

Definition 4.1.8: Relatively Prime

Let p_1, \dots, p_n be nonzero polynomials. We say that they are *relatively prime* if $(p_1, \dots, p_n) = (1)$.

Definition 4.1.9: Reducibility

Let F be a field. We say $f \in F[x] \setminus \{0\}$ is *reducible* if $f = gh$ for some $g, h \in F[x]$ with $\deg g, \deg h \geq 1$. If f is not reducible, we say f is *irreducible*.

Definition 4.1.10: Prime Element

Let F be a field. We say that $f \in F[x]$ is a *prime element* if, for every $g, h \in F[x]$, $f \mid gh \implies (f \mid g \vee f \mid h)$.

Example 4.1.1

- Let F be a field. Then any polynomial over F with degree one is irreducible.
- $F = \mathbb{R}$. $f(x) = x^2 + ax + b$ is irreducible iff $D < 0$.
- $F = \mathbb{F}_p = \mathbb{Z}/p$. There are quite many irreducible polynomial of degree d .

Theorem 4.1.4

Let $p \in F[x] \setminus \{0\}$ be a polynomial. Then, p is irreducible if and only if p is prime.

Proof.

(\implies) Suppose $p \mid gh$ for some $g, h \in F[x]$. If g or h is zero, then it is done. Hence, WMA that $g, h \neq 0$. Let $(p, g) = (d)$. $d \mid p$ implies that $d = 1$ or $d = p$ since p is irreducible. If $d = p$, then $d \mid g$, i.e., $p \mid g$. If $d = 1$, then there exists p_0, g_0 such that $pp_0 + gg_0 = 1$. Hence, $php_0 + ghg_0 = h$. Hence, $p \mid h$.

(\impliedby) Suppose p is reducible. Then, $p = gh$ for some g, h with nonzero degrees. Since p is prime, $p \mid g$ or $p \mid h$. This implies $\deg p \leq \deg g$ or $\deg p \leq \deg h$. This is a contradiction since $\deg p = \deg g + \deg h \leq 2 \deg p$ arises. \square

Theorem 4.1.5 Unique Factorization of Polynomials

Let F be a field. Every non-constant polynomial $f \in F[x]$ factors into a product of irreducible polynomials $f = p_1 p_2 \cdots p_r$. Moreover, the representation is unique up to multiplying nonzero constants and relabeling.

Proof. WLOG, f is monic.

(existence) If $\deg f = 1$, then $f(x) = x - a$ for some $a \in F$, which is itself irreducible.

Suppose $\deg f > 1$. Suppose the theorem holds for all $g \in F[x]$ with $\deg g < \deg f$. If f is itself irreducible, then done. Otherwise, there are $g_1, g_2 \in F[x]$ with $\deg g_i \geq 1$ such that $f = g_1 g_2$. Then, $\deg g_1$ and $\deg g_2$ are less than f . Hence, $g_1 = p_1 p_2 \cdots p_r$ and $g_2 = q_1 q_2 \cdots q_s$ where p_j and q_j are irreducible, yielding $f = p_1 \cdots p_r q_1 \cdots q_s$.

(uniqueness) Suppose we have two factorization $f = p_1 \cdots p_r = q_1 \cdots q_s$. $p_1 \mid q_1 \cdots q_s$. Hence, $p_1 \mid q_j$ for some $j \in [s]$. Since q_j is irreducible, this means p_1 is a nonzero constant

multiple of q_j . Relabeling, $p_1 = q_1$, we have $p_2 \cdots p_r = q_2 \cdots q_s$. Proceeding in this way, we get $r = s$ and $p_j = q_j$ for each j . \square

Definition 4.1.11: (Formal) Derivative

For $f(x) = a_0 + a_1x + \cdots + a_nx^n \in F[x]$, we define

$$f'(x) \triangleq a_1 + 2a_2x + \cdots + na_nx^{n-1}.$$

Note:-

- $(f + g)' = f' + g'$
- $(fg)' = f'g + fg'$

Theorem 4.1.6

f is a product of distinct irreducible polynomials if and only if f and f' are relatively prime.

Proof. (\Leftarrow) Suppose f and f' are relatively prime but $f = p^2h$ for some irreducible polynomial p for the sake of contradiction. Then, $f' = p(2p'h + ph')$, which contradicts $(f, f') = (1)$.
 (\Rightarrow) \square

Definition 4.1.12: Algebraically Closed

A field F is said to be *algebraically closed* if every irreducible polynomial in $F[x]$ is of degree 1.

Note:-

F is algebraically closed.

- \Leftrightarrow Every $f \in F[x]$ with $\deg f \geq 1$ has precisely n roots counting multiplicity.
- \Leftrightarrow Every non-constant $f \in F[x]$ factors into linear polynomials.

Note:-

\mathbb{C} is algebraically closed while \mathbb{R} is not.

Chapter 5

Determinants

5.1 Determinant Functions

Definition 5.1.1: n -linear and Iterating

Let K be a ring. Let $\mathcal{D} : K^{n \times n} \rightarrow K$ be a function. This is considered as a function on n row vectors.

- (i) We say \mathcal{D} is n -linear if \mathcal{D} is a linear function on the i^{th} row while fixing all other rows.

$$\mathcal{D} \begin{bmatrix} \cdots & a_1 + a'_1 & \cdots \\ \cdots & a_2 & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & a_n & \cdots \end{bmatrix} = \mathcal{D} \begin{bmatrix} \cdots & a_1 & \cdots \\ \cdots & a_2 & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & a_n & \cdots \end{bmatrix} + \mathcal{D} \begin{bmatrix} \cdots & a'_1 & \cdots \\ \cdots & a_2 & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & a_n & \cdots \end{bmatrix}$$

- (ii) An n -linear function \mathcal{D} is called *iterating* if $\mathcal{D}(A) = 0$ when two rows are equal.

Note:-

If \mathcal{D} is iterating, and if A' is obtained by switching i^{th} and j^{th} rows of A , then $\mathcal{D}(A') = -\mathcal{D}(A)$.

Definition 5.1.2: Determinant

Let K be a commutative ring with unity. Let $\mathcal{D} : K^{n \times n} \rightarrow K$ be a function. We say \mathcal{D} is a determinant function if

- (i) \mathcal{D} is n -linear,
- (ii) \mathcal{D} is alternating, and
- (iii) $\mathcal{D}(I_n) = 1$.

Definition 5.1.3: Minor Matrix

Let K be a commutative ring with unity. Let $A \in K^{n \times n}$ where $n > 1$. For each $i, j \in [n]$, define $A(i | j)$ be the $(n-1) \times (n-1)$ matrix with the i^{th} row and the j^{th} column are removed. $A(i | j)$ is called (i, j) -minor of A .

Theorem 5.1.1

There exists a determinant function $\mathcal{D} : K^{n \times n} \rightarrow K$.

Proof. We shall prove by exploiting mathematical induction. If $n = 1$, the identity function is a determinant function.

Suppose we have found a function $\mathcal{D}: K^{(n-1) \times (n-1)}$ which is $(n-1)$ -linear and alternating. We shall denote $\mathcal{D}(A(i | j)) = D_{ij}(A)$. Define $E_i(A) \triangleq \sum_{j=1}^n (-1)^{i+j} A_{ij} D_{ij}(A)$ for each $j \in [n]$.

Claim. E_j is an n -linear function on $K^{n \times n}$.

$D_{ij}(A)$ is independent from the entries of the i -th row and the j -th column. Hence, D_{ij} is n -linear as \mathcal{D} is $(n-1)$ -linear. Furthermore, $A \mapsto A_{ij} D_{ij}(A)$ is also n -linear; thus E_j is linear combination of n -linear functions.

Claim. E_j is an alternating function on $K^{n \times n}$.

For the sake of simplicity, suppose A has two equal rows at α_k and α_{k+1} . Hence, when $i \neq k$ and $i \neq k+1$, $A(i | j)$ has two identical rows; thus $D_{ij}(A) = \mathcal{D}(A(i | j)) = 0$. Thus, $E_j(A) = (-1)^{k+j} A_{kj} D_{kj}(A) + (-1)^{k+j+1} A_{(k+1),j} D_{(k+1),j}(A)$.

$$\begin{aligned} E_j(A) &= (-1)^{k+j} A_{kj} D_{kj}(A) + (-1)^{k+j+1} A_{(k+1),j} D_{(k+1),j}(A) \\ &= (-1)^{k+j} (A_{kj} D_{kj}(A) - A_{(k+1),j} D_{(k+1),j}(A)) = 0 \end{aligned}$$

Claim. $E_j(I_n) = 1$.

$$I_n(i | j) = I_{n-1}.$$

□

Corollary 5.1.1

The function defined recursively in the proof of Theorem 5.1.1 is a determinant function.

Definition 5.1.4: Permutation

Let S be a set. A permutation σ of S is a bijective function $\sigma: S \rightarrow S$. S_n is the set of bijective functions from $[n]$ onto $[n]$.

Definition 5.1.5: Transposition

$\tau \in S_n$ is called a *transposition* if it interchanges just the values of two members. A transposition that interchanges i and j is usually written as (i, j) .

Definition 5.1.6: Cycle

A cycle is like:

$$i_1 \mapsto i_2 \mapsto i_3 \mapsto \cdots \mapsto i_n \mapsto i_1.$$

This is written as (i_1, i_2, \dots, i_n) .

Note:-

- Every permutation can be written as a product of disjoint cycles.
- Every cycle can be written as a product of transpositions.
- Every permutation can be written as a product of transpositions.

Theorem 5.1.2

For any permutation $\sigma \in S_n$, the number of transpositions needed to express σ modular 2 is an invariant of σ .

Definition 5.1.7: Sign of Permutation

$$\text{sign}(\sigma) \triangleq \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

Corollary 5.1.2

For $\sigma_1, \sigma_2 \in S_n$, $\text{sign}(\sigma_1 \sigma_2) = \text{sign}(\sigma_1) \text{sign}(\sigma_2)$.

Theorem 5.1.3

There exists a unique determinant function $\mathcal{D}: K^{n \times n} \rightarrow K$, which is equal to

$$\mathcal{D}(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{j \in [n]} A_{j, \sigma(j)}.$$

Proof. Let e_1, \dots, e_n be the rows of I_n . For $A \in K^{n \times n}$, let α_i be the i -th rows of A . Then, $\alpha_i = \sum_{j=1}^n A_{ij} e_j$.

Note that, if $j_i = j_{i'}$, then $\mathcal{D}(e_{j_1}, \dots, e_{j_n}) = 0$. Also, if $\sigma \in S_n$, $\mathcal{D}(e_{\sigma(1)}, \dots, e_{\sigma(n)}) = \text{sign}(\sigma) \mathcal{D}(I_n) = \text{sign}(\sigma)$.

$$\begin{aligned} \mathcal{D}(A) &= \mathcal{D}(\alpha_1, \alpha_2, \dots, \alpha_n) \\ &= \mathcal{D}\left(\sum_{j=1}^n A_{1j} e_j, \alpha_2, \dots, \alpha_n\right) \\ &= \sum_{j=1}^n A_{1j} \mathcal{D}(e_j, \alpha_2, \dots, \alpha_n) \\ &= \dots = \sum_{j_1=1}^n \sum_{j_2=1}^n \dots \sum_{j_n=1}^n A_{1j_1} A_{2j_2} \dots A_{nj_n} \mathcal{D}(e_{j_1}, \dots, e_{j_n}) \\ &= \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{j \in [n]} A_{j, \sigma(j)} \end{aligned}$$

Note that, if \mathcal{D} is a n -linear and alternating, then $\mathcal{D}(A) = \det A \cdot \mathcal{D}(I_n)$. □

Corollary 5.1.3

$$\det(AB) = \det A \cdot \det B$$

Corollary 5.1.4

Any cofactor expansion gives the same value.

Corollary 5.1.5

$$\det A^t = \det A$$

Proof. Theorem 3.7.1 and Theorem 5.1.3. □

Exercise 5.1.1

Let A be $r \times r$ matrix and C be an $s \times s$ matrix. Then,

$$\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \det A \cdot \det C.$$

Hint) Fixing A, B , define $\mathcal{D}(A, B, C)$.

Definition 5.1.8: Adjoint Matrix

Let A be an $n \times n$ matrix. $C_{ij} \triangleq (-1)^{i+j} \det(A(i \mid j))$ for each $i, j \in [n]$ is called the (i, j) -cofactor. Then, $\text{adj}A \triangleq C^t$ where $(C)_{ij} = C_{ij}$ is called the *adjoint* of A .

Corollary 5.1.6

$A \cdot \text{adj}A = (\det A)I_n$. If $\det A \in K$ is invertible, then $A^{-1} = (\det A)^{-1} \text{adj}A$.

Chapter 6

Elementary Canonical Forms

6.1 Eigenvalues

Definition 6.1.1: Eigenvalue

Let V be a vector space over F . Let $T: V \rightarrow V$ be a linear operator.

- $c \in F$ is said to be an *eigenvalue* (or a *characteristic value*) of T if there exists $v \in V \setminus \{0\}$ such that $T(v) = cv$. Such v is called an *eigenvector* (or a *characteristic vector*) of T associated to c .
- For each $c \in F$, $E_c \triangleq \{v \in V \mid T(v) = cv\}$ is called an *eigenspace* (or a *characteristic space*) associated to c .

Theorem 6.1.1

Let V be a vector space over F . Let $T: V \rightarrow V$ be a linear operator. Then, TFAE.

- (i) $c \in F$ is an eigenvalue of T .
- (ii) $T - cI$ is singular.
- (iii) $\det(T - cI) = 0$.

Proof. The equivalence of (i) and (ii) is trivial. The equivalence of (ii) and (iii) is evident from Corollary 5.1.6. \square

Definition 6.1.2: Characteristic Polynomial

Let A be an $n \times n$ matrix over F . Define $f(x) \triangleq \det(xI - A) \in F[x]$. Then, f is a monic polynomial in x of degree $n = \dim V$.

If there exists a basis \mathcal{B} for V and $A = [T]_{\mathcal{B}}$, then we call $f(x) = \det(xI - A)$ the *characteristic polynomial* of T .

Note:-

The choice of basis does not affect the characteristic polynomial. See Theorem 3.4.3.

Note:-

If f is a characteristic polynomial of T , then $f(c) = 0$ if and only if c is an eigenvalue of T .

Corollary 6.1.1

If T is a linear operator on V , then there are at most n eigenvalues of T .

Proof. Every polynomial of degree n has at most n solutions. □

Definition 6.1.3: Diagonalizability

Let V be a finite-dimensional vector space over F . Let $T \in L(V)$. We say T is *diagonalizable* if there exists a basis \mathcal{B} such that it consists of eigenvectors of T .

Note:-

- If $\mathcal{B} = \{v_1, \dots, v_n\}$ and $Tv_i = c_i v_i$ for each $i \in [n]$, then $[T]_{\mathcal{B}} = \text{diag}(c_1, c_2, \dots, c_n)$.
- If $T \in L(V)$ is diagonalizable, then the characteristic polynomial can be completely decomposed into a product of linear factors.

Lemma 6.1.1

Let V be a finite-dimensional vector space over F . Let $T \in L(V)$. Suppose $c_1, \dots, c_k \in F$ are all the possible distinct characteristic values of T . Let W_i be the eigenspace of c_i , i.e., $W_i = \ker(T - c_i I)$. Then, if \mathcal{B}_i is a basis for W_i for each $i \in [k]$, $\bigcup_{i=1}^k \mathcal{B}_i$ is a basis for $\sum_{i=1}^k W_i$.

Proof. Suppose $\sum \beta_i = 0$ where $\beta_i \in W_i$. Then, applying T, T^2, \dots, T^{k-1} , we get

$$\sum_{i=1}^k c_i^j \beta_i = 0$$

for each $j \in \{0, \dots, k-1\}$. As

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ c_1 & c_2 & \dots & c_k \\ \vdots & \vdots & \ddots & \vdots \\ c_1^{k-1} & c_2^{k-1} & \dots & c_k^{k-1} \end{bmatrix}$$

is invertible since c_i 's are distinct, we get $\beta_i = 0$ for each i . □

Note:-

Lemma 6.1.1 also implies that $\dim(\sum_{i=1}^k W_i) = \sum_{i=1}^k \dim W_i$.

Theorem 6.1.2

Let V be a n -dimensional vector space over F . Let $T \in L(V)$. Suppose $c_1, \dots, c_k \in F$ are all the possible distinct characteristic values of T . Let W_i be the eigenspace of c_i , i.e., $W_i = \ker(T - c_i I)$. TFAE.

- (i) T is diagonalizable.
- (ii) The characteristic polynomial is $p(x) = \prod_{i=1}^k (x - c_i)^{d_i}$ where $d_i = \dim W_i$.
- (iii) $\sum_{i=1}^k d_i = n$.

Proof. ((i) \Rightarrow (ii)) Let \mathcal{B} be the basis for V that consists of eigenvectors of T . If \mathcal{B}_i is the part of \mathcal{B} that only consists of eigenvectors corresponding to c_i , we have $\text{span } \mathcal{B}_i = W_i$. Hence, on

rearranging, $[T]_{\mathcal{B}} = \text{diag}(\overbrace{c_1, \dots, c_1}^{d_1}, \overbrace{c_2, \dots, c_2}^{d_2}, \dots, \overbrace{c_k, \dots, c_k}^{d_k})$.

((ii) \Rightarrow (iii)) A direct consequence of Lemma 6.1.1.

((iii) \Rightarrow (i)) $\dim \sum W_i = \sum \dim W_i = \sum d_i = n$. Hence, $\sum W_i = V$, i.e., V has a basis consisting of characteristic vectors. □

6.2 Annihilating Polynomials

Note:-

Let V be a n -dimensional vector space over F . Let $T \in L(V)$. $\{f \in F[x] \mid f(T) = 0\}$ is a nonzero ideal as $\{I, T, T^2, \dots, T^{n^2}\}$ is linearly dependent.

Definition 6.2.1: Minimal Polynomial

Let V be a n -dimensional vector space over F . Let $T \in L(V)$. The monic generator of the nonzero ideal $\{f \in F[x] \mid f(T) = 0\}$ is called the *minimal polynomial* of T .

Theorem 6.2.1

Let V be a n -dimensional vector space over F . Let $T \in L(V)$. If $p(x)$ is the characteristic polynomial of T and $m(x)$ is the minimal polynomial of T , then $p(x)$ and $m(x)$ has the same solutions in F .

Proof. (\Rightarrow) Suppose $m(c) = 0$. Then, $m(x) = (x - c)q(x)$ for some $q \in F[x]$. As m is minimal, $q(T) \neq 0$. This means that $q(T)(\beta) \neq 0$ for some $\beta \in V$. However, $m(T)(\beta) = ((T - cI)q(T))(\beta) = 0$; hence $q(T)(\beta) \in \ker(T - cI)$, i.e., c is an eigenvalue. This means that $p(c) = 0$.

(\Leftarrow) Suppose $p(c) = 0$, i.e., $T(\alpha) = c\alpha$ for some nonzero $\alpha \in V$. As $T^k(\alpha) = c^k\alpha$ for all $k \in \mathbb{N} \cup \{0\}$, for any polynomial $f \in F[x]$, we have $f(T)(\alpha) = f(c)\alpha$. In particular, $0 = m(T)\alpha = m(c)\alpha$, i.e., $m(c) = 0$. \square

Corollary 6.2.1

Let V be a n -dimensional vector space over F . Let $T \in L(V)$. Suppose $c_1, \dots, c_k \in F$ are all the possible distinct characteristic values of T . If $p(x)$ is the characteristic polynomial of T and $m(x)$ is the minimal polynomial of T , then, $p(x) = \prod_{i=1}^k (x - c_i)^{d_i}$ and $p(x) = \prod_{i=1}^k (x - c_i)^{r_i}$ where $d_i \geq r_i$ for each $i \in [k]$.

Theorem 6.2.2 Cayley-Hamilton

Let V be a n -dimensional vector space over F . Let $T \in L(V)$. If $p(x)$ is the characteristic polynomial of T , then $p(T) = 0$.

Proof. Let $K \triangleq \{h(T) \mid h \in F[x]\}$ be a commutative ring. Let $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ be a basis for V . Let $A \triangleq [T]_{\mathcal{B}}$ so that $T(\alpha_i) = \sum_{j=1}^n A_{ji}\alpha_j$. This is equivalent to $\sum_{j=1}^n (\delta_{ij}T - A_{ji}I)\alpha_j = 0$.

Let $B_{ij} \triangleq \delta_{ij}T - A_{ji}I \in K$ and $B \triangleq [B_{ij}]$. Then, $(\text{adj } B)B = B(\text{adj } B) = (\det B)I$. By construction, $\sum_{j=1}^n (\text{adj } B)_{ki}B_{ij}\alpha_j = 0$ for all $k, i \in [n]$.

Taking sum over i , we have

$$\begin{aligned} 0 &= \sum_{i=1}^n \sum_{j=1}^n (\text{adj } B)_{ki}B_{ij}\alpha_j \\ &= \sum_{j=1}^n \left(\sum_{i=1}^n (\text{adj } B)_{ki}B_{ij} \right) \alpha_j \\ &= \sum_{j=1}^n \delta_{kj}(\det B)\alpha_j = (\det B)\alpha_k \end{aligned}$$

for each $k \in [n]$. As $\{\alpha_1, \dots, \alpha_n\}$ is a basis of V , we have $\det B = 0$, i.e., $p(T) = 0$. \square

6.3 Invariant Subspaces

Definition 6.3.1: T -Invariant Subspace

Let V be a finite-dimensional vector space over F and W be a subspace of V . Let $T \in L(V)$. Then, W is said to be a T -invariant subspace if $T(W) \subseteq W$.

Note:-

If W is a T -invariant subspace of V , then $T|_W$ is a naturally induced linear operator on W .

Example 6.3.1

Let V be a finite-dimensional vector space over F and $T \in L(V)$.

- $W = \{0\}$ is a T -invariant subspace.
- For every eigenvalue c of T , $E_c = \ker(T - cI)$ is a T -invariant subspace.

Lemma 6.3.1

Let V be a finite-dimensional vector space over F and $T \in L(V)$. Let W be a T -invariant subspace of V . Then, $m_W \mid m$ and $f_W \mid f$ where m_W and m are minimal polynomials of $T|_W$ and T , and f_W and f are characteristic polynomials of $T|_W$ and T .

Proof. Let $\mathcal{B}' = \{\alpha_1, \dots, \alpha_k\}$ be a basis for W , and extend it to $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ so \mathcal{B} is a basis for V . As W is T -invariant, we have

$$M \triangleq [T]_{\mathcal{B}} = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where $A = [T|_W]_{\mathcal{B}'}$. Hence, $f(x) = \det(xI - M) = \det(xI - A) \det(xI - C) = f_W(x) \det(xI - C)$.

Now, noting that $M^r = \begin{bmatrix} A^r & * \\ 0 & C^r \end{bmatrix}$, whenever $p(x) \in F[x]$ satisfies $p(M) = 0$, we always have $p(A) = 0$ as A is invertible; $m(A) = 0$. By the definition of m_W , we have $m_W \mid m$. \square

Definition 6.3.2: T -conductor

Let V be a finite-dimensional vector space over F and $T \in L(V)$. Let W be a T -invariant subspace of V . Then, for each $\alpha \in V$, the set

$$S_T(\alpha; W) \triangleq \{g \in F[x] \mid g(T)\alpha \in W\}$$

is called the T -conductor of α to W .

Lemma 6.3.2

Let V be a finite-dimensional vector space over F and $T \in L(V)$. Let W be a T -invariant subspace of V . Then, for each $\alpha \in V$, $S_T(\alpha; W)$ is a nonzero ideal.

Proof. $S_T(\alpha; W)$ is nonzero as the characteristic polynomial is contained in the set by Theorem 6.2.2.

It is evident that $S_T(\alpha; W)$ is a subspace of $F[x]$. Now, take any $h \in F[x]$ and $g \in S_T(\alpha; W)$. Then, $(hg)(T)\alpha = h(T)g(T)\alpha \in W$ as W is T -invariant and $g(T)\alpha \in W$. \square

Definition 6.3.3: T -conductor

Due to Lemma 6.3.2 and Theorem 4.1.3, there uniquely exists the monic generator $g_{T,\alpha,W}$ of $S_T(\alpha, W)$. $g_{T,\alpha,W}$ is also often called the T -conductor of α to W .

Note:-

Since $m(T) = f(T) = 0$ where m and f are minimal and characteristic polynomials of T , they are elements of $S_T(\alpha, W)$ for any α, W . Hence,

$$g_{T,\alpha,W} \mid m \mid f.$$

Definition 6.3.4: Triangulable Matrix

Let V be a finite-dimensional vector space over F and $T \in L(V)$. T is said to be *triangulable* if there exists basis \mathcal{B} for V such that $[T]_{\mathcal{B}}$ is upper triangular matrix.

Note:-

If T is diagonalizable, then T is triangulable.

Lemma 6.3.3

Let V be a finite-dimensional vector space over F . Let $T : V \rightarrow V$ be a linear operator on V such that the minimal polynomial m of T has the form of

$$m(x) = \prod_{i=1}^k (x - c_i)^{r_i}.$$

If W is a proper subspace of V , then there exists $\alpha \in V \setminus W$ and an eigenvalue $c \in F$ such that $(T - cI)\alpha \in W$. In other words, $x - c$ is the T -conductor of α on W .

Proof. Take $\beta \in V \setminus W$. Then, $g \triangleq g_{T,\beta,W} \mid m$, i.e.,

$$g(x) = \prod_{i=1}^k (x - c_i)^{e_i}.$$

By the definition of g , and since $\beta \notin W$, there exists $j \in [k]$ such that $e_j \geq 1$. $g(x) = (x - c_j)h(x)$ for some $h \in F[x]$. By the minimality of g , $\alpha \triangleq h(T)\beta \notin V \setminus W$ but $(T - c_jI)\alpha = (T - c_jI)h(T)\beta = g(T)\beta \in W$. \square

Note:-

For $\alpha \notin W$ and $T \in L(V)$, TFAE.

- (i) $(T - cI)\alpha \in W$ for some $c \in F$.
- (ii) $x - c$ is the T -conductor of α on W for some $c \in F$.
- (iii) $T\alpha \in \text{span}\{W, \alpha\}$.

Theorem 6.3.1

Let V be a finite-dimensional vector space over F . Let $T : V \rightarrow V$ be a linear operator on V . Then, T is triangulable if and only if the minimal polynomial of T is a product of linear polynomials over F .

Proof. (\Rightarrow) Since T is triangulable, there exists a basis \mathcal{B} such that $A = [T]_{\mathcal{B}}$ is upper triangular. Hence, the characteristic polynomial is $\det(xI - A) = \prod_{i=1}^n (x - (A)_{ii})$. The result follows due to Theorem 6.2.1.

(\Leftarrow) Suppose $m(x) = \prod_{i=1}^k (x - c_i)^{r_i}$. We shall make use of Lemma 6.3.3 repeatedly over different choices of W . With $W = \{0\}$, we have $\alpha \in V \setminus \{0\}$ such that $(T - d_1 I)\alpha_1 = 0$ for some eigenvalue d_1 . Inductively define α_i by:

- $W_i = \text{span}\{\alpha_1, \dots, \alpha_i\}$.
- Thanks to Lemma 6.3.3, take $\alpha_{i+1} \in V \setminus W_i$ such that $(T - d_{i+1} I)\alpha_{i+1} \in W_i$ where d_{i+1} is an eigenvalue.

Then, by construction, $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ is a basis for V and $[T]_{\mathcal{B}}$ is an upper triangular matrix since $T\alpha_{i+1} \in \text{span}\{\alpha_1, \dots, \alpha_i\} + d_{i+1}\alpha_{i+1}$. \square

Corollary 6.3.1

Let V be a n -dimensional vector space over an algebraically closed field F . Then, every linear operator on V is triangulable.

Theorem 6.3.2

Let V be a n -dimensional vector space over F . Let $T \in L(V)$. Then, T is diagonalizable if and only if the minimal polynomial is $m(x) = \prod_{i=1}^k (x - c_i)$ where c_1, \dots, c_k are all the distinct eigenvalues of T .

Proof. (\Rightarrow) By Theorem 6.1.2 and Theorem 6.2.1, we already have $m(x) = \prod_{i=1}^k (x - c_i)^{e_i}$ where $e_i \geq 1$. Now, we claim that $S \triangleq \prod_{i=1}^k (T - c_i I) = 0$.

From assumption, there exists a basis $\{\alpha_1, \dots, \alpha_n\}$ for V which consists of eigenvectors. Let α_j corresponds to the eigenvalue $c_{i(j)}$. Then, for each $j \in [n]$, $(T - c_{i(j)} I)\alpha_j = 0$, i.e., $S\alpha_j = 0$. Therefore, $S = 0$.

(\Leftarrow) Let W be the subspace spanned by eigenvectors of T . For the sake of contradiction, suppose $W \subsetneq V$. As W is T -invariant, by Lemma 6.3.3, there exists $\alpha \in V \setminus W$ and an eigenvalue $c_j \in F$ such that $\beta \triangleq (T - c_j I)\alpha \in W$.

Write $m(x) = (x - c_j)h(x)$ so h does not have $x - c_j$ as a factor of it. As $h(x) - h(c_j)$ has $x = c_j$ as a root, $h(x) - h(c_j) = (x - c_j)q(x)$ for some q . Then, we have

$$h(T)\alpha - h(c_j)\alpha = q(T)(T - c_j I)\alpha = q(T)\beta \in W$$

since W is T -invariant.

Moreover, $0 = m(T)\alpha = (T - c_j I)h(T)\alpha$ and thus $h(T)\alpha \in E_{c_j} \subseteq W$. This implies that $h(c_j)\alpha \in W$ but $\alpha \notin W$; thus $h(c_j) = 0$, implying the multiplicity of $x - c_j$ in the minimal polynomial. \square

6.4 Simultaneous Triangulation and Diagonalization

Definition 6.4.1: Commuting Family of Linear Operators

Let V be a n -dimensional vector space over F . A set of linear operators \mathcal{F} is said to be a *commuting family* of linear operators if $T_1 T_2 = T_2 T_1$ for each $T_1, T_2 \in \mathcal{F}$.

Definition 6.4.2: \mathcal{F} -invariant

Let V be a n -dimensional vector space over F . A subspace W of V is said to be \mathcal{F} -*invariant* if it is T -invariant for all $T \in \mathcal{F}$.

Lemma 6.4.1

Let V be a n -dimensional vector space over F . Suppose \mathcal{F} is a commuting family of triangulable linear operators on V . Suppose a proper subspace W of V is \mathcal{F} -invariant. Then, there exists $\alpha \in V \setminus W$ such that $\forall T \in \mathcal{F}, T\alpha \in \text{span}\{W, \alpha\}$.

Proof. Suppose $\{T_1, \dots, T_r\}$ is a basis for the subspace spanned by \mathcal{F} . Note that $\text{span } \mathcal{F}$ is still a commuting family of triangulable linear operators.

Let $V_0 = V$. Construct V_1, \dots, V_r and β_1, \dots, β_r as follows. For each $i \in [r]$,

- (i) Let $U_i = T_i|_{V_{i-1}}$. Then, $U_i \in L(V_{i-1})$ by (iii)-(c).
- (ii) Take $\beta_i \in V_{i-1} \setminus W$ and $c_i \in F$ such that $(U_i - c_i I)\beta_i \in W$. Their existence is guaranteed by Lemma 6.3.3 and (iii)-(b).
- (iii) Let $V_i \triangleq \{\beta \in V_{i-1} \mid (T_i - c_i I)\beta \in W\}$. Then, by construction, the following hold.
 - (a) $\beta_i \in V_i \setminus W$
 - (b) $W \subsetneq V_i \subseteq V_{i-1}$
 - (c) V_i is \mathcal{F} -invariant as, for each $T \in \mathcal{F}$ and $\beta \in V_i$, $(T_i - c_i I)(T\beta) = T(T_i - c_i I)\beta \in W$, i.e., $T\beta \in V_i$.

Then, β_r satisfies $T_i \beta_r \in \text{span}\{W, \beta_r\}$ for each $i \in [r]$. \square

Corollary 6.4.1

Let V be a n -dimensional vector space over F . Let \mathcal{F} be a commuting family of *triangulable* linear operators on V . Then, there exists a basis \mathcal{B} for V such that $[T]_{\mathcal{B}}$ is an *upper triangular* matrix for all $T \in \mathcal{F}$.

Proof. Take any $\alpha_1 \in V$. Now, construct $\alpha_2, \dots, \alpha_n$ as following. For each $i \in [n-1]$,

- Let $W_i \triangleq \text{span}\{\alpha_1, \dots, \alpha_i\}$.
- Take $\alpha_{i+1} \in V \setminus W_i$ such that $T\alpha_{i+1} \in \text{span}\{\alpha_1, \dots, \alpha_{i+1}\}$ for each $T \in \mathcal{F}$. The existence is guaranteed by Lemma 6.4.1.

Then, $\mathcal{B} = \{\alpha_1; \dots; \alpha_n\}$ is the ordered basis we are looking for. \square

Theorem 6.4.1

Let V be a n -dimensional vector space over F . Let \mathcal{F} be a commuting family of *diagonalizable* linear operators on V . Then, there exists a basis \mathcal{B} for V such that $[T]_{\mathcal{B}}$ is a *diagonal* matrix for all $T \in \mathcal{F}$.

Proof. We will apply the mathematical induction over $\dim V$. If $\dim V = 1$, there is nothing to prove. Hence, suppose the theorem holds for any finite-dimensional vector space V over F with dimension less than n .

If \mathcal{F} only consists of multiples of identity, it is done. So we may assume the existence of $T \in \mathcal{F}$ which is not a multiple of identity. Let c_1, \dots, c_k be its distinct characteristic values. For each $i \in [k]$, let $\mathcal{F}_i \triangleq \{T|_{W_i} \in L(W_i, V) : T \in \mathcal{F}\}$ where W_i is the eigenspace associated to c_i . Then:

- (i) As T is not a multiple of identity, $k > 1$ and $\dim W_i < n$.
- (ii) As W_i is \mathcal{F} -invariant, $\mathcal{F}_i \subseteq L(W_i)$.
- (iii) For all $T' \in \mathcal{F}$, if m_i and m are minimal polynomials of $T'|_{W_i}$ and T' , $m_i \mid m$ thanks to Lemma 6.3.1.
- (iv) By (iii) and Theorem 6.3.2, every linear operator in \mathcal{F}_i is diagonalizable.
- (v) By (i), (iv), and the induction hypothesis, there exists a basis \mathcal{B}_i for W_i that simultaneously diagonalize all linear operators in \mathcal{F}_i .

Now, $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_k)$ is an ordered basis for V due to Lemma 6.1.1, and \mathcal{B} is the basis we are looking for. \square

Corollary 6.4.2

Let V be a n -dimensional vector space over an *algebraically closed field* F . Let \mathcal{F} be a commuting family of linear operators on V . Then, there exists a basis \mathcal{B} for V such that $[T]_{\mathcal{B}}$ is a diagonal matrix for all $T \in \mathcal{F}$.

6.5 Direct-Sum Decompositions

Definition 6.5.1: Independent Subspaces

Let V be a n -dimensional vector space over F . We say subspaces W_1, \dots, W_k of V are *independent* if, whenever $a_1 + \dots + a_k = 0$ where $a_i \in W_i$, $a_i = 0$ for all $i \in [k]$.

Definition 6.5.2: Direct Sum

Let V be a n -dimensional vector space over F . Let W_1, \dots, W_k be the finite number of subspaces of V . Then, we say that the sum $W = \sum_{i=1}^k W_i$ is *direct* if W_1, \dots, W_k are independent. We write $W = W_1 \oplus \dots \oplus W_k = \oplus_{i=1}^k W_i$ if the sum is direct.

Definition 6.5.3: Projection

Let V be a vector space over F . A linear operator $E \in L(V)$ such that $E^2 = E$ is called a *projection*.

Example 6.5.1

Suppose $V = V_1 \oplus V_2$. Then, $P_1 \in L(V)$ defined by $v_1 + v_2 \mapsto v_1$ where $v_1 \in V_1$ and $v_2 \in V_2$ is a projection.

Lemma 6.5.1

Let V be a vector space over F . Let $E \in L(V)$ be a projection. Then, $V = V_1 \oplus V_2$ for some subspaces V_1 and V_2 of V such that E can be represented by $E(v_1 + v_2) = v_1$ where $v_1 \in V_1$ and $v_2 \in V_2$.

Proof. Take $V_1 = \text{Im } E$ and $V_2 = \ker E$. Take any $v \in V$. Then, $v = Ev + (v - Ev)$ while $Ev \in V_1$ and $v - Ev \in \ker E$. Hence, $V = V_1 + V_2$.

Take any $v_1 \in \text{Im } E$ and $v_2 \in \ker E$ and suppose $v_1 + v_2 = 0$. Then, there exists $v'_1 \in V$ such that $v_1 = E(v'_1)$. Then, $0 = E(v_1 + v_2) = E(v_1) = E^2(v'_1) = E(v'_1) = v_1$. Hence, the sum is direct. It is also shown that $E(v_1 + v_2) = E(v_1) = v_1$. \square

Theorem 6.5.1

Let V be a n -dimensional vector space over F . Suppose $V = \oplus_{i=1}^k W_i$ for some subspaces W_i of V . Then, for each $i \in [k]$, there exists $E_i \in L(V)$ such that

- (i) E_i is a projection for each $i \in [k]$,
- (ii) $E_i E_j = 0$ if $i \neq j$.

- (iii) $I = \sum_{i=1}^k E_i$.
- (iv) $\text{Im } E_i = W_i$ for each $i \in [k]$.

Proof. All $v \in V$ can be uniquely written as $v = \sum_{i=1}^k v_i$ where $v_i \in W_i$ for each $i \in [k]$. Hence, define $E_i: V \rightarrow V$ by $v \mapsto v_i$. Then, E_i 's satisfy the four constraints. \square

Theorem 6.5.2

Let V be a n -dimensional vector space over F . Let $T \in L(V)$. Suppose $V = \oplus_{i=1}^k W_i$ for some subspaces W_i of V . Let E_1, \dots, E_k be the projections in Theorem 6.5.1. Then, W_i is T -invariant for all $i \in [k]$ if and only if T commutes with all E_i 's.

Proof. (\Rightarrow) Suppose W_i is T -invariant for each $i \in [k]$. Take any $\alpha \in V$ and write $\alpha = \sum_{i=1}^k \alpha_i$ where $\alpha_i \in W_i$ for each $i \in [k]$. Then, $E_i T \alpha = \sum_{j=1}^k E_i T \alpha_j = T \alpha_i = T E_i \alpha$.
 (\Leftarrow) Suppose $T E_i = E_i T$. Take any $\alpha_i \in W_i$. Then, $T \alpha_i = T E_i \alpha_i = E_i (T \alpha_i) \in W_i$ by the definition of E_i . Hence, W_i is T -invariant. \square

Theorem 6.5.3

Let V be a n -dimensional vector space over F . Let $T \in L(V)$. If T is diagonalizable and c_1, \dots, c_k are all the distinct eigenvalues, we have projections $E_i \in L(V)$ for each $i \in [k]$ on $W_i = E_{c_i}$ such that $T = \sum_{i=1}^k c_i E_i$ and $V = \oplus_{i=1}^k W_i$ with $I = \sum_{i=1}^k E_i$ and $E_i E_j = \delta_{ij} E_i$.

Note:-

The converse of Theorem 6.5.3 also holds.

Theorem 6.5.4 Primary Decomposition Theorem

Let V be a n -dimensional vector space over F . Let $T \in L(V)$ and $m \in F[x]$ be its minimal polynomial. Write $m(x) = \prod_{i=1}^k p_i^{r_i}$ where p_i 's are irreducible polynomials in $F[x]$ and $r_i \geq 1$. Let $W_i \triangleq \ker(p_i(T)^{r_i})$. Then,

- (i) $V = \oplus_{i=1}^k W_i$.
- (ii) Each W_i is T -invariant.
- (iii) The minimal polynomial of $T_i = T|_{W_i}$ is $p_i^{r_i}$.

End.