MAS331 위상수학
Notes

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Contents

Chapter 1

Set Theory and Logic

1.1 Basic Notation

Note:-

- Sets: $A, B, C, \cdots, \mathbb{R}, \mathbb{Q}, \mathbb{Z}$
- Elements: $a, b, c, \cdots, 3, 3/4, \pi$
- $a \in A$, $3 \in \mathbb{Z}$, $3/4 \notin \mathbb{Z}$
- $A \subseteq B$, $A \subsetneq B$, $A \nsubseteq B$
- $\varnothing$: empty set
- $A \times B := \{(a, b) \mid a \in A \text{ and } b \in B\}$ (Cartesian product)

Definition 1.1.1: Function, Restriction, and Composition

A *function* f from a set A to a set B is an assignment of an element of B to each element of A.

- A: Domain
- B: Range or Codomain
- $\operatorname{Im} f := \{f(a) \mid a \in A\}$: Image; $\operatorname{Im} f \subseteq B$

If $A_0 \subseteq A$ and $f : A \to B$ is a function, then the *restriction* of f to A_0 is denoted by $f\big|_{A_0}$ and is defined as

$$f\big|_{A_0}(a_0) := f(a_0)$$

for each $a_0 \in A_0$. If $f : A \to B$ and $g : B \to A$, then the *composite* $g \circ f$ is defined as

$$(g \circ f)(a) := g\big(f(a)\big)$$

for each $a \in A$.

Definition 1.1.2: Injectivity, Surjectivity and Bijectivity

A function $f : A \to B$ is

i) *injective* (or *one-to-one*, 1-1) if $\forall a, a' \in A$, $f(a) = f(a') \implies a = a'$,
ii) *surjective* (or *onto*) if $\forall b \in B$, $\exists a \in A$, $b = f(a)$, and
iii) *bijective* if f is both injective and surjective.

Definition 1.1.3: Inverse Function

If $f : A \to B$ is bijective, then the inverse of f is denoted by

$$f^{-1} : B \to A$$

and is defined as

$$f^{-1}(b) = a$$

for each $b \in B$ where $f(a) = b$.

Example 1.1.1

a) f is bijective $\iff$ f^{-1} is bijective.
b) The inverse is unique.

Solution: Suppose f is bijective. Then,

$$f^{-1}(b_1) = f^{-1}(b_2) \implies b_1 = (f \circ f^{-1})(b_1) = (f \circ f^{-1})(b_2) = b_2.$$

Therefore, f^{-1} is injective.

Take any $a \in A$. Then, $b := f(a) \in B$ satisfies $f^{-1}(b) = a$. Therefore, f^{-1} is surjective.

Now, suppose f^{-1} is bijective. Then,

$$f(a_1) = f(a_2) \implies a_1 = (f^{-1} \circ f)(a_1) = (f^{-1} \circ f)(a_2) = a_2.$$

Therefore, f is injective.

Take any $b \in B$. Then, $a := f^{-1}(a) \in B$ satisfies $f(a) = b$. Therefore, f is surjective; a) is now proven.

Let g and h are inverses of f. Take any $b \in B$. Since f is bijective, $\exists! a \in A,\ f(a) = b$. Therefore, $g(b) = a = h(b)$, which implies $g = h$; b) is now proven.

$\square$

Definition 1.1.4: Image and Preimage of a Set

Let $f : A \to B$ and $A_0 \subseteq A$, $B_0 \subseteq B$.

- $f(A_0) := \{b \mid b = f(a_0) \text{ and } a_0 \in A\}$
- $f^{-1}(B_0) := \{a \mid f(a) \in B_0\}$

Example 1.1.2

a) $A_0 \subseteq f^{-1}\big(f(A_0)\big)$
b) f is injective if and only if $\forall A_0 \subseteq A$, $A_0 = f^{-1}\big(f(A_0)\big)$.
c) $f\big(f^{-1}(B_0)\big) \subseteq B_0$
d) f is surjective if and only if $\forall B_0 \subseteq B$, $B_0 = f\big(f^{-1}(B_0)\big)$.

Solution:

a) For every $a_0 \in A_0$, $f(a_0) \in f(A_0)$, which implies $a_0 \in f^{-1}\big(f(A_0)\big)$. Therefore, $A_0 \subseteq f^{-1}\big(f(A_0)\big)$ holds.

b) Suppose f is injective. Take any $A_0 \subseteq A$ and $a_0 \in f^{-1}\big(f(A_0)\big)$. Then, $f(a_0) \in f(A_0)$. We may take $a_1 \in A_0$ such that $f(a_0) = f(a_1) \in f(A_0)$. Since f is injective, $a_0 = a_1 \in A_0$.

Suppose '$\forall A_0 \subseteq A$, $A_0 = f^{-1}\big(f(A_0)\big)$' holds. Suppose $f(a_1) = f(a_2) = b_0$. Let $A_0 := \{a_1\}$. Then, $A_0 = f^{-1}\big(f(A_0)\big) = f^{-1}\big(\{b_0\}\big) \ni a_2$. This means $a_2 \in \{a_1\}$, which implies $a_1 = a_2$.

c) Take any $b_0 \in f\big(f^{-1}(B_0)\big)$. Then, there is some $a_0 \in f^{-1}(B_0)$ such that $f(a_0) = b_0$. Such a_0 satisfies $f(a_0) \in B_0$, which implies $b_0 = f(a_0) \in B_0$. Therefore, $f\big(f^{-1}(B_0)\big) \subseteq B_0$ holds.

d) Suppose f is surjective. Take any $B_0 subs B$ and $b_0 \in B_0$. Then, there is some $a_0 \in A$ such that $f(a_0) = b_0$, which implies $a_0 \in f^{-1}(B_0)$. Therefore, $b_0 \in f\big(f^{-1}(B_0)\big)$; $B_0 \subseteq \big(f^{-1}(B_0)\big)$.

Suppose '$\forall B_0 \subseteq B$, $B_0 = f\big(f^{-1}(B_0)\big)$' holds. Take any $b_0 \in B$ and let $B_0 := \{b_0\}$. Since $b_0 \in f\big(f^{-1}(B_0)\big)$, There is some $a_0 \in f^{-1}(B_0)$ such that $f(a_0) = b_0$. Therefore, f is surjective.

□

1.2 Relations

Definition 1.2.1: Relation

A relation $\sim$ on a set A is a subset of $A \times A$.

$$x \sim y := (x, y) \in \sim$$

Definition 1.2.2: Equivalence Relation and Equivalence Class

A relation $\sim$ on a set A is an *equivalence relation* if

(1) $x \sim x$ for each $x \in A$ (reflexive)
(2) $x \sim y \implies y \sim x$ (symmetric)
(3) $x \sim y \wedge y \sim z \implies x \sim z$. (transitive)

Moreover, the *equivalence class* of x is defined as

$$\{y \in A \,|\, y \sim x\}.$$

Example 1.2.1 (Partition)
If there are equivalence classes E and E', then they are either $E = E'$ or $E \cap E' = \varnothing$. This implies, if we let $\mathcal{E} := \{E \mid E \text{ is an equivalence class of } x \text{ where } x \in A\}$, $A = \bigcup_{E \in \mathcal{E}} E$.

Solution: Since if $E \cap E' = \varnothing$ it is done, suppose $E \cap E' \neq \varnothing$. There are a and a' such that E and E' are equivalence classes of a and a' respectively. We may take $a_0 \in E \cap E'$. By definition and transitivity, $a \sim a_0 \sim a'$. Therefore, for all $x \in E$, $x \in E'$ since $x \sim a \sim a'$, which implies $E \subseteq E'$. In the same way, $E' \subseteq E$.

□

Definition 1.2.3: Order Relation

A relation $<$ on a set A is an *order relation* if

(1) $x < y$ or $y < x$ for each $x \neq y \in A$
(2) $x \not< x$ for each $x \in A$
(3) $x < y \wedge y < z \implies x < z$.

Also, we define
$$(a, b) := \left\{x \in X \,\middle|\, a < x < b\right\}.$$

Definition 1.2.4: Order Type

Let A and B be sets with order relations $<_A$ and $<_B$, respectively. Then, A and B have the same *order type* if there is a bijection $f : A \to B$ such that $a_1 <_A a_2 \iff f(a_1) <_B f(a_2)$.

Definition 1.2.5: Dictionary Order Relation

Let A, B be sets with order relations $<_A$, $<_B$ respectively. Then, there is an order relation $<_{A\times B}$ on $A \times B$ defined as $(a_1, b_1) <_{A\times B} (a_2, b_2)$ if
$$a_1 <_A a_2 \text{ or } a_1 = a_2 \text{ and } b_2 <_B b_2.$$
This is often called *dictionary order relation* on $A \times B$.

Definition 1.2.6: Boundedness

Let $A_0 \subseteq A$ with an order relation $<_A$.

- The *largest element* of A_0 is $b \in A_0$ if $x \in A_0 \implies x \leq b$.
- The *smallest element* of A_0 is $b \in A_0$ if $x \in A_0 \implies x \geq b$.

- A_0 is *bounded above* by $b \in A$ if $x \in A_0 \implies x \leq b$.
  - The smallest such b is called the *least uppder bound* or the *supremum* of A_0.
- A_0 is *bounded below* by $b \in A$ if $x \in A_0 \implies x \geq b$.
  - The largest such b is called the *greatest lower bound* or the *infimum* of A_0.

- A has *least upper bound property* if every bounded above nonempty set $A_0 \subseteq A$ has a least upper bound.
- A has *greatest lower bound property* if every bounded below nonempty set $A_0 \subseteq A$ has a greatest lower bound.

Theorem 1.2.1

A set A with an order relation $<_A$ has l.u.b. property if and only if A has g.l.b. property.

Proof. Suppose A has l.u.b. property. Let A_0 be any bounded below nonempty subset of A. Let $L := \{a \in A \mid a \text{ is a lower bound of } A_0\}$. Take a $a_0 \in A_0$. Then, since $\ell \leq_A a_0$ for all $\ell \in L$, L is bounded above by a_0. By l.u.b. property of A, there is $\ell_0 := \sup L \in A$.

Take any a_0 in A_0. Since a_0 is an upper bound of L and ℓ_0 is the least upper bound, $\ell_0 \leq_A a_0$. Therefore, ℓ_0 is a lower bound of A_0.

Suppose $\ell_0 <_A \ell_1$ and ℓ_1 is a lower bound of A_0. This implies $\ell_1 \in L$, which contradicts to $\ell_1 \leq_A \sup L = \ell_0$. Therefore, ℓ_0 is the greatest lower bound, and A has g.l.b. property.

The inverse can be proven by the similar reasoning. □

Theorem 1.2.2 Completeness of $\mathbb{R}$

The set of real numbers $\mathbb{R}$ has least upper bound property.

1.3 The Integers and the Real Numbers

Theorem 1.3.1 Well-Ordering Property

Every nonempty subset of $\mathbb{Z}_+$ has a smallest element.

Proof. We first prove that, for each $n \in \mathbb{Z}_+$, every nonempty subset of $[n] := \{1, 2, \cdots, n\}$ has a smallest element, using induction. For the base case, it is known the the only nonempty subset of $[1]$, $\{1\}$, has 1 as its smallest element.

Suppose the statement holds for $n = k$. Now take any nonempty subset S of $[k+1]$. If $S = \{k+1\}$, $k+1$, the only element of S, is a smallest element of S. Otherwise, $S \setminus \{k+1\}$ is nonempty and is a subset of $[k]$; we may let $\mu := \min S$ by the induction hypothesis. Then, μ is also a smallest element of S, regardless of whether it is $k+1 \in S$ or $k+1 \notin S$.

Now, take any $\varnothing \neq T \subseteq \mathbb{Z}_+$ and $m \in T$. Then, by our previous result, since $T \cap [m]$ is a nonempty subset of $[m]$, it has a smallest element, which is also a smallest element of T. □

1.4 Cartesian Products

Definition 1.4.1: Indexing Function and Indexed Family of Sets

Let $\mathcal{A}$ be a nonempty collection of sets. An *indexing function* for $\mathcal{A}$ is a surjective function $f : J \to \mathcal{A}$ where $A_\alpha := f(\alpha)$. An *indexed family* of sets is defined as $\{A_\alpha\}_{\alpha \in J}$. Now, we define

$$\bigcup_{\alpha \in J} A_\alpha := \left\{x \,\middle|\, \exists \alpha \in J,\, x \in A_\alpha\right\}$$
$$\bigcap_{\alpha \in J} A_\alpha := \left\{x \,\middle|\, \forall \alpha \in J,\, x \in A_\alpha\right\}$$
$$\prod_{\alpha \in J} A_\alpha := \left\{f : J \to \textstyle\bigcup_{\alpha \in J} A_\alpha \,\middle|\, \forall \alpha \in J,\, f(\alpha) \in A_\alpha\right\}.$$

1.5 Finite Sets

Definition 1.5.1: Finite Set and Cardinality

A set A is *finite* if there is a bijective $f : A \to [n]$ for some $n \in \mathbb{Z}_+$ or $A = \varnothing$.

- In the former case, we say *cardinality* n or $|A| = n$.
- In the latter case, we say *cardinality* 0 or $|A| = 0$.

Note:-

Let A and B be finite sets. Then, $|A| = |B| = n$ if and only if $\exists$ bijective $f : A \to B$.

Lemma 1.5.1

Let $a_0 \in A$. Then,

$$|A| = n \iff |A \setminus \{a_0\}| = n - 1.$$

Proof. For $n = 1$, it is trivial. So suppose $n \geq 2$.

($\Rightarrow$) There is a bijection $f : A \to [n]$. If $f(a_0) = n$, then $f\big|_{A\setminus\{a_0\}}$ is a bijection from $A \setminus \{a_0\}$ to $[n-1]$, and it's done. Otherwise, let $a_1 := f^{-1}(n)$. Define $g : A \to A$ by

$$g(a) := \begin{cases} a_0 & \text{if } a = a_1 \\ a_1 & \text{if } a = a_0 \\ a & \text{otherwise.} \end{cases}$$

g is bijective. Then, $f \circ g$ is a bijection from A to $[n]$ such that $(f \circ g)(a_0) = n$.

($\Leftarrow$) Trivial. $\square$

Theorem 1.5.1

Let A be a set with $|A| = n$ and $B \subsetneq A$. Then, there is no bijection between B and $[n]$, but (provided $B \neq \varnothing$) there is a bijection between B and $[m]$ for some $m < n$.

Proof by Induction. (Base case) It is trivial for $n = 1$.

(Induction) Suppose it is true for $n \geq 1$. WTS for the case $|A| = n + 1$. Suppose $B \neq \varnothing$ because we have nothing to talk about then. Let $a_0 \in B$. By Lemma 1.5.1, there is a bijection $g : A \setminus \{a_0\} \to [n]$. Since $B \setminus \{a_0\} \subsetneq A \setminus \{a_0\}$, by induction hypothesis, we have two things.

- There is no bijection between $B \setminus \{a_0\}$ and $[n]$.
- As long as $B \neq \{a_0\}$, there is a bijection from $B \setminus \{a_0\}$ to $[m]$ for some $m < n$.

We conclude that there is no bijection from B and $[n+1]$ since, if there were, there would be a trivial bijection from $B \setminus \{a_0\}$ to $[n]$. Moreover, we can construct a bijection between B and $[m+1]$, and $m + 1 < n + 1$. $\square$

Corollary 1.5.1 Uniqueness of Cardinality

The cardinality of a finite set is uniquely determined.

Proof. Let $m < n$ and suppose m and n are cardinalities of a finite set A. Then there are bijections $f : A \to [m]$ and $g : A \to [n]$. Then, $f \circ g^{-1}$ is a bijection from $[m]$ to $[n]$ but it is impossible since $[m] \subsetneq [n]$ and because of Theorem 1.5.1. $\square$

Corollary 1.5.2

$\mathbb{Z}_+$ is not finite.

Proof by Contradiction. Suppose $\mathbb{Z}_+$ is finite and $|\mathbb{Z}_+| = n$. $f : \mathbb{Z}_+ \to \mathbb{Z}_+ \setminus \{1\}$ with $x \mapsto x + 1$ is bijective. Then, by Lemma 1.5.1, $n - 1 = |\mathbb{Z}_+ \setminus \{1\}| = |\mathbb{Z}_+| = n$, #. $\square$

Theorem 1.5.2

Let A be a set. TFAE

(i) $|A| = n$
(ii) $\exists$ surjective $[m] \twoheadrightarrow A$ for some $m \in \mathbb{Z}_+$.
(iii) $\exists$ injective $A \hookrightarrow [m]$ for some $m \in \mathbb{Z}_+$.

Proof. ((i) → (ii)) There is a bijective function from A to $[n]$, and it is also surjective.

((ii) → (iii)) Let f be a surjective function from $[m]$ to A. Since f is surjective, $f^{-1}(\{a\}) \neq \varnothing$ for every $a \in A$. Let $M := \max\{\min f^{-1}(\{a\}) \mid a \in A\}$. M is well defined thanks to Theorem 1.3.1 and the fact that $\varnothing \neq f^{-1}(\{a\}) \subseteq [m]$. Then the function $g : A \to [M]$ defined by $a \mapsto \min f^{-1}(\{a\})$ is injective.

((iii) → (i)) Let f be an injective function from A to $[m]$. Then, $g : A \to \operatorname{Im} f$ defined by $a \mapsto f(a)$ is bijective. A is finite because $\operatorname{Im} f$ is finite by Theorem 1.5.1. □

Exercise 1.5.1

(i) Finite unions of finite sets are finite.
(ii) Finite Cartesian products of finite sets are finite.

Solution: (i) Suppose there are n finite sets $A_1, A_2, \cdots, A_n$ to union. WLOG, $A_i \neq \varnothing$ for each $i \in [n]$. Let $M := \max_{i\in[n]} |A_i|$ and $g_i : [|A_i|] \to A_i$ be a bijective function for each $i \in [n]$. Extend each g_i to $g_i' : [M] \to A_i$ by

$$g_i'(k) = \begin{cases} g_i(k) & \text{if } k \leq |A_i| \\ g_i(1) & \text{otherwise.} \end{cases}$$

for $k \in [M]$. Now, we define $f : [nM] \to \bigcup_{i\in[n]} A_i$ by

$$f\big(n(i-1)+k\big) := g_i'(k)$$

for each $i \in [n]$ and $k \in [M]$. Then, f is surjective. Therefore, $\bigcup_{i\in[n]} A_i$ is finite by Theorem 1.5.2.

(ii) Suppose there are n finite sets $A_1, A_2, \cdots, A_n$ to construct a Cartesian product with. WLOG, $A_i \neq \varnothing$ for each $i \in [n]$. Let $M := \max_{i\in[n]} |A_i|$ and $h_i : A_i \to [|A_i|]$ be a bijective function for each $i \in [n]$. Let p_i be the i^{th} prime. (i.e., $p_1 = 2$, $p_2 = 3$, $p_3 = 5$.) Define a function $f : \prod_{i\in[n]} A_i \to \left[\left(\prod_{i=1}^n p_i\right)^M\right]$ by

$$f(a_1, a_2, \cdots, a_n) := \prod_{i=1}^n p_i^{h_i(a_i)}.$$

f is injective since prime factorization of a natural number is unique. Therefore, $\prod_{i\in[n]} A_i$ is finite by Theorem 1.5.2.

□

1.6 Countable and Uncountable Sets

Definition 1.6.1: Infinite and Countably Infinite

A set A is said to be *infinite* if it is not finite. It is said to be *countably infinite* if there is a bijective correspondence

$$f : A \to \mathbb{Z}_+.$$

Example 1.6.1

$\mathbb{Z}_+$, $\mathbb{Z}$, and $\mathbb{Z}_+ \times \mathbb{Z}_+$ are countably infinite.

Definition 1.6.2: Countability

A set is said to be *countable* if it is either finite or countably infinite. A set that is not countable is said to be *uncountable*.

Lemma 1.6.1

Any subset of $\mathbb{Z}_+$ is countable.

Proof. Let $C \subseteq \mathbb{Z}_+$. If C is finite, then it's done; we now assume C is infinite. Now we want to show that C is countably infinite.

Define $h\colon \mathbb{Z}_+ \to C$ by the following.

(a) $h(1) := \min C$

(b) $h(n+1) := \min\big(C \setminus h([n])\big)$ for each $n \in \mathbb{Z}_+$

h is well defined because $C \setminus h([n])$ is always nonempty. Morever, h is injective since it is $h(m) < h(n)$ whenever $m < n$.

Now, we are going to show h is surjective. To do this, first take any $c \in C$. Since C is infinite and h is injective, $\operatorname{Im} h \not\subseteq [c]$, which means $\exists n \in \mathbb{Z}_+$, $h(n) > c$. From this, we get $m := \min\{\, n \in \mathbb{Z}_+ \mid h(n) \geq c \,\}$ is well-defined. From the definition of m, we also get, for any $1 \leq i < m$, we have $h(i) < c \leq h(m)$. Therefore, $c \notin h([m-1])$. Together with $h(m) = \min\big(C \setminus h([m-1])\big)$, we get $h(m) \leq c \leq h(m)$, which implies $c = h(m)$. $\square$

Theorem 1.6.1

Let $A \neq \varnothing$. TFAE

(i) A is countable.

(ii) $\exists$ surjective $\mathbb{Z}_+ \twoheadrightarrow A$.

(iii) $\exists$ injective $A \hookrightarrow \mathbb{Z}_+$.

Proof. ((i) $\to$ (ii)) Trivial.

((ii) $\to$ (iii)) Let $f : \mathbb{Z}_+ \twoheadrightarrow A$. Define $g : A \to \mathbb{Z}_+$ by $a \mapsto \min f^{-1}(\{a\})$. g is well-defined because $f^{-1}(\{a\}) \neq \varnothing$ for every $a \in A$ and Theorem 1.3.1 holds. g is also injective since $f^{-1}(\{a_1\}) \cap f^{-1}(\{a_2\}) = \varnothing$ if $a_1 \neq a_2 \in A$.

((iii) $\to$ (i)) Let f be an injection from A to $\mathbb{Z}_+$. If we define $g : A \to \operatorname{Im} f$ by $a \mapsto f(a)$, g is a bijection. Since $\operatorname{Im} f \subseteq \mathbb{Z}_+$, A is countable by Lemma 1.6.1. $\square$

Corollary 1.6.1

If $A \subseteq B$ and B is countable, then A is countable.

Proof. $A \xhookrightarrow{\text{trivial injection}} B \xhookrightarrow{\text{injection}} \mathbb{Z}_+$ and Theorem 1.6.1. $\square$

Corollary 1.6.2

$\mathbb{Z}_+ \times \mathbb{Z}_+$ is countably infinite.

Proof. $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$ with $(x, y) \mapsto 2^x 3^y$ is an injection.

Or, $g\colon \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$ with $(x, y) \mapsto \dfrac{(x+y-1)(x+y-2)}{2} + y$ is a bijection. $\square$

Corollary 1.6.3

$\mathbb{Q}$ is countably infinite.

Proof. $f : \mathbb{Z} \times \mathbb{Z}_+ \to \mathbb{Q}$ with $(x, y) \mapsto x/y$ is surjective. $\square$

Exercise 1.6.1

The union of a countable number of countable sets is countable.

Solution: Let $\{A_i\}_{i\in J}$ be an indexed family of sets where J and A_i's are countable. WLOG, $A_i \neq \varnothing$ for each $i \in J$. For each $i \in J$, since A_i is countable, by Theorem 1.6.1, there is a surjection $g_i : \mathbb{Z}_+ \twoheadrightarrow A_i$. Similarly, since J is countable, there is a surjection $h: \mathbb{Z}_+ \twoheadrightarrow J$.

Now, construct a function $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \bigcup_{i\in J} A_i$ by

$$f(i, j) := g_{h(i)}(j).$$

f is naturally surjective by the contruction. Therefore, $\bigcup_{i\in J} A_i$ is countable.

$\square$

Exercise 1.6.2

The Cartesian product of a finite number of countable sets is countable.

Solution: Suppose there are $n \in \mathbb{Z}_+$ sets $A_1, A_2, \cdots, A_n$ to make Cartesian product with and each A_i is countable. WLOG, $A_i \neq \varnothing$ for each $i \in [n]$. For each $i \in [n]$, there is a injection $g_i : A_i \to \mathbb{Z}_+$ by Theorem 1.6.1.

Now, construct a fuction $f : \prod_{i=1}^n A_i \to \mathbb{Z}_+$ by

$$f(a_1, a_2, \cdots, a_n) := \prod_{i=1}^n p_i^{g_i(a_i)},$$

where p_i is the i^{th} prime. Since prime factorization of a natural number is unique, f is injective; therefore $\prod_{i=1}^n A_i$ is countable.

$\square$

Theorem 1.6.2

Let $X_i := \{0, 1\}$ for each $i \in \mathbb{Z}_+$. Then, $\prod_{i\in\mathbb{Z}_+} X_i$ is uncountable.

Proof. Let $f : \mathbb{Z}_+ \to \prod_{i\in\mathbb{Z}_+} X_i$ is any function. Denote $f(n) = (x_{n,1}, x_{n,2}, \cdots) \in \prod_{i\in\mathbb{Z}_+} X_i$ and construct $y = (y_1, y_2, \cdots) \in \prod_{i\in\mathbb{Z}_+} X_i$ by

$$y_i := 1 - x_{i,i}$$

for each $i \in \mathbb{Z}_+$. Then, $y \notin \operatorname{Im} f$; therefore, one cannot construct a surjection from $\mathbb{Z}_+$ to $\prod_{i\in\mathbb{Z}_+} X_i$. $\square$

Corollary 1.6.4

$\mathcal{P}(\mathbb{Z}_+)$ is uncountable.

Proof. $f:\mathcal{P}(\mathbb{Z}_+)\to\prod_{i\in\mathbb{Z}_+}X_i$ defined by

$$S\mapsto(y_1,y_2,\cdots)\text{ where }y_i:=\begin{cases}0 & \text{if } i\in S\\ 1 & \text{if } i\notin S\end{cases}$$

is a bijection, and $\prod_{i\in\mathbb{Z}_+}X_i$ is uncountable by Theorem 1.6.2. □

Theorem 1.6.3

Let A be a set. Then, there is no injection $\mathcal{P}(A)\hookrightarrow A$, and there is no surjection $A\twoheadrightarrow\mathcal{P}(A)$.

Proof. Since a surjective map can be naturally deducted from $f:B\hookrightarrow C$ (by constructing $g:C\to B$ by $g(c)\in f^{-1}(\{c\})$ for $c\in\operatorname{Im} f$ and map c to an arbitrary element in B for $c\notin\operatorname{Im} f$), it suffices to show $A\twoheadrightarrow\mathcal{P}(A)$ does not exist.

Let $f:A\to\mathcal{P}(A)$ be any function, and let $B:=\{a\in A\mid a\notin f(a)\}\in\mathcal{P}(A)$. Suppose $B=f(a_0)$ for some $a_0\in A$. Then, by the definition of B,

$$a_0\in B\iff a_0\notin f(a_0)=B,$$

which is a contradiction. Therefore, any such f cannot be surjective. □

1.7 Infinite Sets and the Axiom of Choice

Theorem 1.7.1

Let A be a set. TFAE

(i) A is infinite.
(ii) $\exists$ injection $f:\mathbb{Z}_+\hookrightarrow A$.
(iii) $\exists$ bijection $g:A\to B$ where $B\subsetneq A$.

Proof. ((i) → (ii)) Construct $f:\mathbb{Z}_+\to A$ recursively as following. Let $c:\mathcal{P}(A)\setminus\{\varnothing\}\to A$ be a function such that $c(A')\in A'$ for every $\varnothing\neq A'\subseteq A$. Its existence is guaranteed by Lemma 1.7.1.

(1) $f(1):=c(A)$
(2) $f(n+1):=c(A\setminus f([n]))$ for each $n\in\mathbb{Z}_+$.

Suppose $A\setminus f([n])=\varnothing$ for some $n\in\mathbb{Z}_+$. Then, $A\subseteq f([n])$, and $f([n])$ is finite by Theorem 1.5.2; therefore A is finite by Theorem 1.5.1. Thus, f is well-defined and it is injective by definition.

((ii) → (iii)) Let $f:\mathbb{Z}_+\hookrightarrow A$ be an injectiion. Define $g:A\to A\setminus\{f(1)\}$ by

$$g(a):=\begin{cases}f(n+1) & \text{if } a=f(n)\text{ for some } n\in\mathbb{N}_+\\ a & \text{if } a\notin\operatorname{Im} f.\end{cases}$$

g is well-defined because f is injective, and it is bijective by definition.

((iii) → (i)) This is just a contrapositive of Theorem 1.5.1. □

Theorem 1.7.2 Axiom of Choice

Given a collection $\mathscr{A}$ of disjoint nonempty sets, there exists a set C such that $C\subseteq\bigcup\mathscr{A}$ and $\forall A\in\mathscr{A}$, $|C\cap A|=1$.

Lemma 1.7.1 Existence of a Choice Function

Given a collection $\mathscr{B}$ of nonempty sets, there exists a function

$$c\colon \mathscr{B} \to \bigcup \mathscr{B}$$

such that $c(B) \in B$ for each $B \in \mathscr{B}$.

Proof. Let $\mathscr{A} := \left\{ \{ (B,x) \mid x \in B \} \,\middle|\, B \in \mathscr{B} \right\}$. Then, by Theorem 1.7.2, there exists $c \subseteq \mathscr{A}$ such that $c \subseteq \bigcup \mathscr{A}$ and each $B \in \mathscr{B}$ appears only once in the first coordinate in c. Therefore, c is a function such that $c(B) \in C$ for each $B \in \mathscr{B}$. $\square$

1.8 Well-Ordered Sets

Definition 1.8.1: Well-Ordered

A set A with an order relation is an *well-ordered* set if every nonempty subset of A has a smallest element.

Example 1.8.1

- $\mathbb{Z}_+$ is well-ordered.
- $\{1,2\} \times \mathbb{Z}_+$ is well ordered with respect to the dictionary ordering.

Theorem 1.8.1

Every nonempty finite set has the order type of $[n]$, and thus it is well-ordered.

Proof. We shall first claim that, if A is a nonempty finite set, then it has a largest element. It can be prove by induction on $|A|$. If $|A| = 1$, then it is trivial. Suppose the claim holds for $|A| = n$, and suppose $|A| = n+1$ and $a_0 \in A$. Then, $A \setminus \{ a_0 \}$ has a largest element a_1. This implies A has a largest element $\max\{ a_0, a_1 \}$.

Now, we prove there is an order-preserving bijection $f : A \to [n]$. This will also be proven with induction. It is true when $|A| = 1$, so suppose it is true for $|A| = n \in \mathbb{Z}_+$ and let $|A| = n+1$. By above, we may let $a_0 := \max A$. By induction hypothesis, there is an order-preserving bijection $f' : A \setminus \{ a_0 \} \to [n]$. Define $f : A \to [n+1]$ by

$$f(a) := \begin{cases} f'(a) & \text{if } a \neq a_0 \\ n+1 & \text{if } a = a_0. \end{cases}$$

Then, f is an order-preserving bijection from A to $[n+1]$. $\square$

Theorem 1.8.2

The Cartesian product of finitely many well-ordered sets is well-ordered with respect to the dictionary ordering.

Proof by Induction. We will prove this by induction on the number of sets. If there is one set, then it is trivial.

Assume the theorem holds for n sets. Suppose we have $n+1$ sets $A_1, A_2, \cdots, A_{n+1}$. Then, $\prod_{i=2}^{n+1} A_i$ is well-ordered with respect to a dictionary ordering $<_1$.

Let $<_2$ and $<_3$ be the dictionary order of $A_1 \times \prod_{i=2}^{n+1} A_i$ and $\prod_{i=1}^{n+1} A_i$, respectively. Since $\left(A_1 \times \prod_{i=2}^{n+1} A_i, <_2\right)$ and $\left(\prod_{i=1}^{n+1} A_i, <_3\right)$ has the same order type, we only need to prove that $\left(A_1 \times \prod_{i=2}^{n+1} A_i, <_2\right)$ is well-ordered.

Let $\varnothing \neq S \subseteq A_1 \times \prod_{i=2}^{n+1} A_i$. If we define $S' := \{ a_1 \mid (a_1, b) \in S \} \subseteq A_1$, S' is a nonempty subset of A_1, and therefore has $a_1' := \min S'$. Similarly, if we define $S'' := \{ b_1 \mid (a_1', b_1) \in S \} \subseteq \prod_{i=2}^{n+1} A_i$, S'' is nonempty and has a smallest element b_1'. Then, (a_1', b_1') is a smallest element of $A_1 \times \prod_{i=2}^{n+1} A_i$ with respect to $<_2$. □

Exercise 1.8.1

$\prod_{i\in\mathbb{Z}_+} \mathbb{Z}_+$ is not well-ordered with respect to the dictionary ordering.

Solution: Let $x_{ij} := \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$ for each $i \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_+$. The set $A := \{ (x_{i1}, x_{i2}, \cdots) \mid i \in \mathbb{Z}_+ \} \subseteq \prod_{i\in\mathbb{Z}_+} \mathbb{Z}_+$ has no smallest element.

Theorem 1.8.3 Well-Ordering Theorem

If A is a set, then there exists an order relation on A that is well-ordering.

The proof of Theorem 1.8.3 involves the Axiom of Choice.

Corollary 1.8.1

There exists an uncountable well-ordered set.

Definition 1.8.2: Section

Let X be a well-ordered set. Given $\alpha \in X$, let

$$S_\alpha := \{ x \in X \mid x < \alpha \}.$$

S_α is called the *section* of X by α.

Lemma 1.8.1

There exists a well-ordered set A with the largest element Ω, such that
- section S_Ω of A is uncountable, and,
- for every $\alpha \in A \setminus \{ \Omega \}$, section S_α of A is countable.

Proof. By Corollary 1.8.1, there exists an uncountable well-ordered set B. Let $C := \{1, 2\} \times B$ be a set with a dictionary ordering. C is well-ordered by Theorem 1.8.2.

Let $S := \{ \alpha \in C \mid \text{section } S_\alpha \text{ of } C \text{ is uncountable} \} \subseteq C$. We may let $\Omega := \min S$. Then, the set $\overline{S_\Omega} = S_\Omega \cup \{ \Omega \}$ satisfies the two conditions. □

Theorem 1.8.4

If A is a countable subset of S_Ω (in Lemma 1.8.1), then A has an upper bound in S_Ω.

Proof. For each $a \in A$, the section S_a is countable; therefore, the union $B := \bigcup_{a\in A} S_a$ is also countable by Exercise 1.6.1.

Since S_Ω is uncountable, we may take an $x \in S_\Omega \setminus B$. If it were $x < a$ for some $a \in A$, then x would be contained in S_a, which is a subset of B, $\#$. Therefore, $x \in S_\Omega$ is an upper bound of A. $\square$

Chapter 2

Topological Spaces and Continuous Functions

2.1 Topological Spaces

Definition 2.1.1: Topology and Topological Space

A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X such that

(i) $\varnothing, X \in \mathcal{T}$

(ii) $\{U_i \mid i \in J\} \subseteq \mathcal{T} \implies \bigcup_{i\in J} U_i \in \mathcal{T}$

(iii) $\{U_1, U_2, \cdots, U_n\} \subseteq \mathcal{T} \implies \bigcap_{i=1}^{n} U_i \in \mathcal{T}$

We say $(X, \mathcal{T})$ is a *topological space*, and each element $U \in \mathcal{T}$ is called an *open set*.

Example 2.1.1 (Discrete Topology and Trivial Topology)

- If X is any set, the collection of all subsets of X, $\mathcal{P}(X)$, is a topology on X; it is called the *discrete topology*.
- $\{\varnothing, X\}$ is also an topology on X; we shall call it the *trivial topology*.

Example 2.1.2 (Finite Complement Topology)

Let X be any set. Then, $\mathcal{T} := \{U \subseteq X \mid X \setminus U \text{ is finite}\} \cup \{\varnothing\}$ is a topology.

(i) $\varnothing, X \in \mathcal{T}$ ✓

(ii) If $\{U_\alpha\}_{\alpha\in J} \subseteq \mathcal{T}$, then $X \setminus \bigcup_{\alpha\in J} U_\alpha = \bigcap_{\alpha\in J}(X - U_\alpha)$ is finite. ✓

(iii) If $\{U_1, U_2, \cdots, U_n\} \subseteq \mathcal{T}$, $X \setminus \bigcap_{i=1}^{n} U_\alpha = \bigcup_{i=1}^{n}(X \setminus U_\alpha)$ is finite by Exercise 1.5.1. ✓

The topology is called the *finite complement topology*.

Example 2.1.3

If $X = \{a, b, c\}$, then $\mathcal{T} = \{\varnothing, X, \{a\}, \{a, b\}\}$ is a topology on X.

Definition 2.1.2: Finer and Coarser Topology

Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies of a set X. If $\mathcal{T} \subseteq \mathcal{T}'$, then we say

- $\mathcal{T}'$ is *finer* than $\mathcal{T}$ and
- $\mathcal{T}$ is *coarser* than $\mathcal{T}'$.

Also, $\mathcal{T}$ is *comparable* to $\mathcal{T}'$ if either $\mathcal{T} \supseteq \mathcal{T}'$ or $\mathcal{T} \subseteq \mathcal{T}'$.

2.2 Basis for a Topology

Definition 2.2.1: Basis and Toplogy Generated by a Basis

A *basis* for X is a collection $\mathcal{B}$ of subsets of X such that:

(i) $\forall x \in X, \exists B \in \mathcal{B}, x \in B$ (i.e., $X = \bigcup \mathcal{B}$) and

(ii) $\forall B_1, B_2 \in \mathcal{B}, \left(x \in B_1 \cap B_2 \implies \exists B_3 \in \mathcal{B}, x \in B_3 \subseteq B_1 \cap B_2\right)$.

The topology $\mathcal{T}$ generated by $\mathcal{B}$ is the collection defined by

$$\mathcal{T} := \{U \subseteq X \mid \forall x \in U, \exists B \in \mathcal{B}, x \in B \subseteq U\}.$$

Note:-

If $\mathcal{B}$ is a basis for X and $\mathcal{T}$ is the topology generated by $\mathcal{B}$, then $\mathcal{B} \subseteq \mathcal{T}$.

Lemma 2.2.1

If $\mathcal{T}$ is the topology generated by basis $\mathcal{B}$ for X, then $\mathcal{T}$ is a topology on X.

Proof.

(i) $\varnothing \in \mathcal{T}$ by vacuous truth, and $X \in \mathcal{T}$ follows directly from (i) in Definition 2.2.1. ✓

(ii) Let $\mathcal{U} := \{U_\alpha\}_{\alpha \in J} \subseteq \mathcal{T}$. Then, $x \in \bigcup \mathcal{U}$ implies $\exists \alpha \in J$, $x \in U_\alpha$. Since $U_\alpha \in \mathcal{T}$, there is $B \in \mathcal{B}$ such that $x \in B \subseteq U_\alpha \subseteq \bigcup \mathcal{U}$. This means $\bigcup \mathcal{U} \subseteq \mathcal{T}$. ✓

(iii) It is enough to prove it for two sets U_1 and U_2 in $\mathcal{T}$. Let $x \in U_1 \cap U_2$. (If $U_1 \cap U_2 = \varnothing$, then it is done.) By the definition of $\mathcal{T}$, there are B_1 and B_2 in $\mathcal{B}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. Since $x \in B_1 \cap B_2$, there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$. Thus, it implies $U_1 \cap U_2 \in \mathcal{T}$. ✓

□

Lemma 2.2.2

If $\mathcal{T}$ is the topology generated by basis $\mathcal{B}$ for X, then $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$. In other words, $\mathcal{T} = \left\{\bigcup \mathcal{U} \,\middle|\, \mathcal{U} \subseteq \mathcal{B}\right\}$.

Proof. Let $\mathcal{T}' := \left\{\bigcup \mathcal{U} \,\middle|\, \mathcal{U} \subseteq \mathcal{B}\right\}$. Since $\mathcal{B} \subseteq \mathcal{T}$ and $\mathcal{T}$ is a topology by Lemma 2.2.1, $\mathcal{T}' \subseteq \mathcal{T}$ follows. (See (ii) in Definition 2.1.1.) Now, we shall prove $\mathcal{T} \subseteq \mathcal{T}'$.

Take any $U \in \mathcal{T}$. Then, for each $x \in U$, there is $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. Then, $U = \bigcup_{x \in U} B_x \in \mathcal{T}'$, hence $\mathcal{T} \subseteq \mathcal{T}'$. □

Lemma 2.2.3

Let $(X, \mathcal{T})$ be a topological space. If $\mathcal{C}$ is a subset of $\mathcal{T}$ such that

$$\forall U \in \mathcal{T}, (x \in U \implies \exists C \in \mathcal{C}, x \in C \subseteq U),$$

then $\mathcal{C}$ is a basis for X and $\mathcal{T}$ is the topology generated by $\mathcal{C}$.

Proof. We shall prove first $\mathcal{C}$ is a basis for X.

(i) Since $X \in \mathcal{T}$, $\forall x \in X$, $\exists C \in \mathcal{C}$, $x \in C$. ✓

(ii) Let $C_1, C_2 \in \mathcal{C}$ and suppose $x \in C_1 \cap C_2$. Since $C_1 \cap C_2 \in \mathcal{T}$, there is $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$. ✓

Now let $\mathcal{T}'$ be the topology generated by $\mathcal{C}$. We want to show $\mathcal{T} = \mathcal{T}'$.

For $\mathcal{T}' \subseteq \mathcal{T}$, take any $U \in \mathcal{T}'$. Then, by Lemma 2.2.2, $U = \bigcup_{\alpha \in J} C_\alpha$ where each C_α is in $\mathcal{C}$. Now, $U = \bigcup_{\alpha \in J} C_\alpha \in \mathcal{T}$ directly follows. The last inclusion is due to (ii) in Definition 2.1.1 and $\mathcal{C} \subseteq \mathcal{T}$. ✓

For $\mathcal{T} \subseteq \mathcal{T}'$, take any $U \in \mathcal{T}$. Then, for any $x \in U$, there is $C \in \mathcal{C}$ such that $x \in C \subseteq U$, therefore $U \in \mathcal{T}'$ by Definition 2.2.1. □

Lemma 2.2.4

Let $\mathcal{T}$ and $\mathcal{T}'$ are topologies genereated by bases $\mathcal{B}$ and $\mathcal{B}'$, respectively. Then,

$$\mathcal{T} \subseteq \mathcal{T}' \iff \forall B \in \mathcal{B}, \big(x \in B \implies \exists B' \in \mathcal{B}', x \in B' \subseteq B\big).$$

Proof. $(\Leftarrow)$ Take any $U \in \mathcal{T}$ and $x \in U$. Since $\mathcal{B}$ generates $\mathcal{T}$, there is $B \in \mathcal{B}$ such that $x \in B \subseteq U$. By the supposition, there is $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B \subseteq U$. This implies we can find $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$, by definition, $U \in \mathcal{T}'$. ✓

$(\Rightarrow)$ Take any $B \in \mathcal{B}$ and $x \in B$. Since $B \in \mathcal{T} \subseteq \mathcal{T}'$, by definition of $\mathcal{T}'$, there is $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. ✓ □

Example 2.2.1

Let $\mathcal{B}$ be a set of open region inside a disk, and $\mathcal{B}'$ be a set of open region inside a rectangle. They are bases for $\mathbb{R}^2$, and topologies generated by them are the same by Lemma 2.2.4.

Definition 2.2.2: Common Topologies on $\mathbb{R}$

Define

– $\mathcal{B}_\mathbb{R} := \{ (a,b) \subseteq \mathbb{R} \mid a < b \}$

– $\mathcal{B}_\ell := \{ [a,b) \subseteq \mathbb{R} \mid a < b \}$

$\mathcal{B}$ and $\mathcal{B}'$ are bases for $\mathbb{R}$. Then,

- $\mathcal{T}_\mathbb{R}$, the topology generated by $\mathcal{B}$, is called the *standard topology* on $\mathbb{R}$, and
- $\mathcal{T}_\ell$, the topology generated by $\mathcal{B}_\ell$, is called the *lower limit topology* on $\mathbb{R}$.

Let $K := \{ 1/n \mid n \in \mathbb{Z}_+ \}$ and $\mathcal{B}_K := \mathcal{B}_\mathbb{R} \cup \{ (a,b) \setminus K \mid a < b \}$ Then, $\mathcal{B}''$ is a basis for $\mathbb{R}$ and

- $\mathcal{T}_K$, the topology generated by $\mathcal{B}_K$, is called the *K-topology* on $\mathbb{R}$.

Lemma 2.2.5 Comparison Among the Common Topologies on $\mathbb{R}$

The following holds.

(i) $\mathcal{T}_{\mathbb{R}} \subsetneq \mathcal{T}_\ell$ ($\mathcal{T}_\ell$ is strictly finer than $\mathcal{T}_{\mathbb{R}}$.)
(ii) $\mathcal{T}_{\mathbb{R}} \subsetneq \mathcal{T}_K$ ($\mathcal{T}_K$ is strictly finer than $\mathcal{T}_{\mathbb{R}}$.)
(iii) $\mathcal{T}_\ell$ and $\mathcal{T}_K$ are not comparable.

Proof.

(i) For any $(a,b) \in \mathcal{B}_{\mathbb{R}}$ and $x \in (a,b)$, $[x,b) \in \mathcal{B}_\ell$ and $x \in [x,b) \subseteq (a,b)$. Therefore, by Lemma 2.2.4, $\mathcal{T}_{\mathbb{R}} \subseteq \mathcal{T}_\ell$. ✓
Take any $a \in \mathbb{R}$. a is in the interval $[a,b) \in \mathcal{B}_\ell$ but there are no open interval $(c,d) \in \mathcal{B}_{\mathbb{R}}$ such that $a \in (c,d) \subseteq [a,b)$. Therefore, by Lemma 2.2.4, $\mathcal{T}_\ell \nsubseteq \mathcal{T}_{\mathbb{R}}$. ✓
(ii) $\mathcal{T}_{\mathbb{R}} \subseteq \mathcal{T}_K$ directly follows from $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{B}_K$. ✓
Although $0 \in (-1,1) \setminus K \in \mathcal{T}_K$, there is no $(c,d) \in \mathcal{B}_{\mathbb{R}}$ such that $0 \in (c,d) \in (-1,1) \setminus K$. Therefore, by Lemma 2.2.4, $\mathcal{T}_K \nsubseteq \mathcal{T}_{\mathbb{R}}$. ✓
(iii) The logics in (i) and (ii) can directly imported to prove (iii). ✓

□

Definition 2.2.3: Subbasis

A *subbasis* $\mathcal{S}$ for X is a subset of $\mathcal{P}(X)$ whose union is X, i.e., $\bigcup \mathcal{S} = X$.
The *topology generated by the subbasis* $\mathcal{S}$ is defined to be the collection of all unions of finite intersections of elements of $\mathcal{S}$.

Lemma 2.2.6

Let $\mathcal{S}$ be a subbasis for X. Then, the topology generated by $\mathcal{S}$ is a topology on X.

Proof. By Lemma 2.2.2, it is enough to show that $\mathcal{B} := \left\{ \bigcap_{i=1}^n S_i \mid S_i \in \mathcal{S} \right\}$ is a basis.
(i) Since $\mathcal{S} \subseteq \mathcal{B}$, $X = \bigcup \mathcal{S} \subseteq \bigcup \mathcal{B} \subseteq X$. ✓
(ii) Let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$. Then, $B_1 = \bigcap_{i=1}^n S_i$ and $B_2 = \bigcap_{i=1}^m S_i'$ where $S_i, S_i' \in \mathcal{S}$. Then, $B_1 \cap B_2 = \left(\bigcap_{i=1}^n S_i\right) \cap \left(\bigcap_{i=1}^m S_i'\right) \in \mathcal{B}$. ✓

□

2.3 The Order Topology

Definition 2.3.1: Intervals

Let X be a set with an order $<$ and $a, b \in X$ with $a < b$ are given.

- $(a,b) := \{ x \in X \mid a < x < b \}$ (open interval)
- $[a,b) := \{ x \in X \mid a \le x < b \}$ (half-open interval)
- $(a,b] := \{ x \in X \mid a < x \le b \}$ (half-open interval)
- $[a,b] := \{ x \in X \mid a \le x \le b \}$ (closed interval)

Definition 2.3.2: Order Topology

Let X has more than one element. Let $\mathcal{B}$ be collection of

- all open intervals (a,b) in X,
- all half-open intervals $[a_0, b)$ where a_0 is the smallest element (if a_0 exists), and
- all half-open intervals $(a, b_0]$ where b_0 is the largest element (if b_0 exists).

Then, $\mathcal{B}$ is a basis and the topology generate by $\mathcal{B}$ is called the *order topology*.

Lemma 2.3.1
The set $\mathcal{B}$ above is a basis.

Proof.

(i) Take any $x \in X$.
  - If x is the smallest, then $x \in [x, b)$ where b is some element in $X \setminus \{x\}$.
  - If x is the largest, then $x \in (a, x]$ where a is some element in $X \setminus \{x\}$.
  - Otherwise, there are some $a, b \in X \setminus \{x\}$ such that $a < x < b$ so $x \in (a, b)$. ✓

(ii) A nonempty intersection of two basis with different types of interval is an open interval. An intersection of two basis with the same type of interval still belongs to the type of interval. ✓

□

Example 2.3.1
The order topology on $\mathbb{Z}_+$ is the discrete topology. $n \in (n-1, n+1) = \{n\}$ if $n > 1$ and $1 \in [1, 2) = \{1\}$.

Example 2.3.2
The order topology on $\mathbb{R}$ is the standard topology on $\mathbb{R}$.

Definition 2.3.3: Ray

Let X be an order set and $a \in X$. There are four types of rays.
- $(a, \infty) := \{\, x \in X \mid x > a \,\}$ (open ray)
- $(-\infty, a) := \{\, x \in X \mid x < a \,\}$ (open ray)
- $[a, \infty) := \{\, x \in X \mid x \geq a \,\}$ (closed ray)
- $(-\infty, a] := \{\, x \in X \mid x \leq a \,\}$ (closed ray)

Note:-
Open rays are open in the order topology.
- If X has a largest element b_0, then $(a, \infty) = (a, b_0]$.
- Otherwise, $(a, \infty) = \bigcup_{a<b} (a, b)$.

Thus, (a, ∞) is open. Similarly, $(-\infty, a)$ is open.

Note:-
Open rays form a subbasis that generates the order topology.

2.4 The Product Topology on $X \times Y$

Definition 2.4.1: Product Topology

Let X, Y be topological spaces. The *product topology* on $X \times Y$ is the toplogy generated by a basis

$$\mathcal{B} := \{\, U \times V \mid U \subseteq X \text{ and } V \subseteq Y \text{ are open} \,\}.$$

Theorem 2.4.1

Let $\mathcal{B}$ be a basis for X nd $\mathcal{C}$ be a basis for Y. Then

$$\mathcal{D} := \{ B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{C} \}$$

is a basis for the product topology of $X \times Y$.

Proof. We will exploit Lemma 2.2.3. Take any open set $W \subseteq X \times Y$ and $x \times y \in W$. Then, there is a basis element $U \times V$ of the product topology $X \times Y$ such that $x \times y \in U \times V \subseteq W$. Since U and V are open in X and Y, respectively, and $x \in U$ and $y \in V$, there are $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $x \in B \subseteq U$ and $y \in C \subseteq V$.

Here, we find that $x \times y \in B \times C \subseteq U \times V \subseteq W$ while $B \times C \in \mathcal{D}$. Therefore, by Lemma 2.2.3, $\mathcal{D}$ generates the product topology. $\square$

Definition 2.4.2: Projection

Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ defined by the equations

$$\pi_1(x, y) = x$$
$$\pi_2(x, y) = y$$

The maps π_1 and π_2 are called the *projections* of $X \times Y$ onto its first and second factors, respectively.

Note:-

If $U \subseteq X$ is open, then $\pi_1^{-1}(U) = U \times Y$ is open. Similarly, if $V \subseteq Y$ is open, then $\pi_2^{-1}(V) = X \times V$ is open.

Theorem 2.4.2

The collection

$$\mathcal{S} := \{ \pi_1^{-1}(U) \mid U \subseteq X \text{ is open} \} \cup \{ \pi_2^{-1}(V) \mid V \subseteq Y \text{ is open} \}$$

is a subbasis for the product topology of $X \times Y$.

Proof. Let $\mathcal{T}$ be the product topology and $\mathcal{T}'$ be the toplogy generated by $\mathcal{S}$.

- Since $\mathcal{S} \subseteq \mathcal{T}$, every union of finite intersections in $\mathcal{S}$ is in $\mathcal{T}$. Thus, $\mathcal{T}' \subseteq \mathcal{T}$. ✓
- Every open set of $\mathcal{T}$ is a union of elements in $\mathcal{B} := \{ U \times V \mid U \subseteq X \text{ and } V \subseteq Y \text{ are open} \}$. Noting that each $U \times V$ can be expressed as $\pi_1^{-1}(U) \cap \pi_2^{-1}(V)$, which is a finite intersection of elements in $\mathcal{S}$, we may conclude $\mathcal{T} \subseteq \mathcal{T}'$. ✓

$\square$

2.5 The Subspace Topology

Definition 2.5.1: Subspace Topology

Let $(X, \mathcal{T})$ be a topological space. If $Y \subseteq X$, then

$$\mathcal{T}_Y := \{Y \cap U \mid U \in \mathcal{T}\}$$

is called the *subspace toplogy* of Y and $(Y, \mathcal{T}_Y)$ is called a *subspace* of $(X, \mathcal{T})$.

Lemma 2.5.1

$(Y, \mathcal{T}_Y)$ is a topological space.

Proof.

(i) $\varnothing = Y \cap \varnothing$ and $Y = Y \cap X$. ✓

(ii) If $U_\alpha \in \mathcal{T}_Y$, $\bigcup_{\alpha\in J}(Y \cap U_\alpha) = Y \cap \left(\bigcup_{\alpha\in J} U_\alpha\right) \in \mathcal{T}_Y$. ✓

(iii) If $U_i \in \mathcal{T}_Y$, $\bigcap_{i=1}^{n}(Y \cap U_i) = Y \cap \left(\bigcap_{i=1}^{n} U_i\right) \in \mathcal{T}_Y$. ✓

□

Lemma 2.5.2

If $\mathcal{B}$ is a basis for $(X, \mathcal{T})$, then

$$\mathcal{B}_Y := \{Y \cap B \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y.

Proof. We will exploit Lemma 2.2.3.

Take any $U \in \mathcal{T}$ and $y \in Y \cap U$. Since $y \in U$, $\exists B \in \mathcal{B}$, $y \in B \subseteq U$, which implies $y \in Y \cap B \subseteq Y \cap U$. □

Note:-

Not all open sets in Y are open in X.
For instance, if $X = \mathbb{R}$ and $Y = [0, 1]$, Y is open in Y but not open in X.

Lemma 2.5.3

All the open sets in Y are open in X if and only if Y is open in X.

Proof. ($\Rightarrow$) Y is open in Y. Hence, Y is open in X.

($\Leftarrow$) Let U be any open set in Y. Then, $U = Y \cap V$ for some open set V in X. Since Y is open in X, U is open in X. □

Theorem 2.5.1

If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the the topology $A \times B$ inherits as a subspace of $X \times Y$. In other words, the following two topologies are the same.

$$\text{(i)} \qquad X, Y \xrightarrow{\text{subspace}} A \subseteq X,\, B \subseteq Y \xrightarrow{\text{product}} A \times B$$
$$\text{(ii)} \qquad X, Y \xrightarrow{\text{product}} X \times Y \xrightarrow{\text{subspace}} A \times B \subseteq X \times Y$$

Proof. By Theorem 2.4.1,

$$\{U \times V \mid U \in \mathcal{B}_X \text{ and } V \in \mathcal{B}_Y\}$$

is a basis for $X \times Y$. Thus,

$$\mathcal{B} := \{(A \times B) \cap (U \times V) \mid U \in \mathcal{B}_X \text{ and } V \in \mathcal{B}_Y\}$$

is a basis for (ii) by Lemma 2.5.2.

Note that $(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V)$. Also, $\{A \cap U \mid U \in \mathcal{B}_X\}$ and $\{B \cap V \mid V \in \mathcal{B}_Y\}$ are bases for A and B. Thus, $\mathcal{B}$ is also a basis for (i) by Theorem 2.4.1. □

Wrong Concept 2.1: Order Topology and Subspace Topology

Unlike product topology and subspace topology, order topology and subspace topology are not associative. Let X be an ordered set and $Y \subseteq X$.

$$\begin{array}{rl} \text{(i)} & Y \xrightarrow{\text{order}} Y \\ \text{(ii)} & X \xrightarrow{\text{order}} X \xrightarrow{\text{subspace}} Y \subseteq X \end{array}$$

Then, will those be the same?

Example 1. Consider $X = \mathbb{R}$ and $Y = [0,1]$. Then, the subspace topology of the order topology X has a basis of

$$\mathcal{B}_{[0,1]} = \{[0,1] \cap (a,b) \mid a < b\},$$

which is in fact the order topology on Y. In this case, (i) = (ii).

Example 2. Consider $X = \mathbb{R}$ and $Y = [0,1) \cup \{2\}$. Then, $\{2\}$ is an open in (ii) since $\{2\} = Y \cap (1.5, 2.5)$. But, there is no basis of the order topology on Y such that contains 2 and is a subset of $\{2\}$. Thus, in this case, (i) $\neq$ (ii).

Example 3. Consider $X = \mathbb{R}^2$ and $Y = I^2$ where $I = [0,1]$. Then, $\{1/2\} \times (1/2, 1]$ is an open set in (ii) since it is $\big(\{1/2\} \times (1/2, 3/2)\big) \cap I^2$. But it is not an open set in (i) since there is no basis that contain $(1/2, 1)$ and is a subset of $\{1/2\} \times (1/2, 1]$.

Definition 2.5.2: Convex Subset

Given an ordered set X and $Y \subseteq X$, Y is called *convex* if

$$\forall a, b \in Y, \big(a < b \implies (a,b) \subseteq Y\big).$$

Theorem 2.5.2

Let X be an ordered set with the ordered topology. If $Y \subseteq X$ is convex, then the order topology on Y is the same as the subspace topology.

Proof. We will make use of the fact that open rays form a subbasis that generates the order topology.

First, every open ray of (i) is an open ray of the subspace (ii).

$$\{x \in Y \mid x > a\} = \{x \in X \cap Y \mid x > a\},$$

for example. Therefore, (ii) is finer than (i).

Now, take any open ray in X, $(a,\infty)_X = \{x \in X \mid x > a\}$, for instance. Then, let

$$\begin{aligned} R &\triangleq (a,\infty)_X \cap Y \\ &= \{y \in Y \mid y > a\} = (a,\infty)_Y. \end{aligned}$$

If $a \in Y$, then R is an open ray in Y.

Now consider the case $a \notin Y$. If R is nonempty then there is some $y_0 \in R$. Take any $y \in Y$. If $y_0 < y$, then $y \in R$ since $a < y_0 < y$. If $y < y_0$, it implies $a < y < y_0$ because $y < a < y_0$ with $y, y_0 \in Y$ implies $a \in Y$ by the convexity of Y. Therefore, $y \in R$. So, if $a \notin Y$, it is either $R = \varnothing$ or $R = Y$.

Combining the cases, we get the fact that the intersection of Y and an arbitrary open ray in X is an open ray in Y, an empty set, or the whole Y.

This is the final step. Take any open set U in the ordered topology X. Then, $U = \bigcup_{\alpha \in J} U_\alpha$ where $U_\alpha \neq \varnothing$ is a finite intersection of open rays in X. Noting that $U \cap Y$ is a general form of an open set in Y, we get $U \cap Y = \bigcup_{\alpha \in J}(U_\alpha \cap Y)$, which implies either $U \cap Y = Y$ or $U \cap Y$ is a union of finite intersections of an open ray in Y. □

Corollary 2.5.1

Let X be an ordered set with the ordered topology. The subspace topology of $Y \subseteq X$ is finer than the order topology on Y.

2.6 Closed Sets and Limit Points

2.6.1 Closed Sets

Definition 2.6.1: Closed Set

Let X be a topological space. A subset $A \subseteq X$ is *closed* if $X \setminus A$ is open.

Example 2.6.1

- $[a,b] \subseteq \mathbb{R}$ is closed since $(-\infty, a) \cup (b, \infty)$ is open.
- $[a,b] \times [c,d] \subseteq \mathbb{R}^2$ is closed.
- In discrete topology on X, every subset of X is closed.
- If $Y = [0,1] \cup (2,3) \subseteq \mathbb{R}$, $[0,1]$ and $(2,3)$ are both open and closed in Y.

Theorem 2.6.1

Let X be a topological space. Then the following conditions hold.

(i) $\varnothing$ and X are closed.
(ii) Arbitrary intersections of closed sets are closed.
(iii) Finite unions of closed sets are closed.

Proof.

(i) $X \setminus \varnothing = X$ and $X \setminus X = \varnothing$ are open. ✓
(ii) Let $\{A_\alpha\}_{\alpha \in J}$ be a collection of closed sets. Then,

$$X \setminus \bigcap_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in J}(X \setminus A_\alpha).$$

is open since each $X \setminus A_\alpha$ is open. ✓

(iii) Let $\{A_i\}_{i=1}^n$ be a collection of closed sets. Then,

$$X \setminus \bigcup_{i=1}^n A_i = \bigcap_{i=1}^n (X \setminus A_i).$$

is open since it is a finite intersection of open sets. ✓

□

Theorem 2.6.2

Let X be a topological space and $Y \subseteq X$. Then $A \subseteq Y$ is closed in Y if and only if there is a closed set B in X such that $A = Y \cap B$.

Proof. ($\Leftarrow$) Let B be a closed set of X such that $A = Y \cap B$. Then, $X \setminus B$ is open in X and $Y \cap (X \setminus B) = X \setminus Y$ is open in Y. Thus, A is closed in Y.

($\Rightarrow$) Since $Y \setminus A$ is open in Y, $Y \setminus A = Y \cap U$ for some open set U in X. Then, $A = Y \cap (X \setminus U)$ where $X \setminus U$ is closed in X. □

Theorem 2.6.3

If Y is closed in X, then every closed sets of Y are closed in X if and only if Y is closed in X.

Proof. Proof is analogous to the proof of Lemma 2.5.3. □

Definition 2.6.2: Interior and Closure of a Set

Given a subset A of a topological space $(X, \mathcal{T})$,

- the *interior* of A is $\mathring{A} \triangleq \bigcup \{U \subseteq X \mid U \in \mathcal{T} \text{ and } U \subseteq A\}$, and
- the *closure* of A is $\overline{A} \triangleq \bigcap \{V \subseteq X \mid X \setminus V \in \mathcal{T} \text{ and } A \subseteq V\}$.

Note:-

- $\mathring{A} \subseteq A \subseteq \overline{A}$
- $\mathring{A}$ is open, and $\overline{A}$ is closed.
- $\mathring{A}$ is the largest open set contained A, and $\overline{A}$ is the smallest closed set containing A.

Theorem 2.6.4

Let Y be a subspace of X and $A \subseteq Y$. Let $\overline{A}$ and $\overline{A}_Y$ denote the closures of A in X and Y, respectively. Then,

$$\overline{A} \cap Y = \overline{A}_Y.$$

Proof. ($\supseteq$) $\overline{A} \cap Y$ is closed in Y by Theorem 2.6.2. Thus, $\overline{A}_Y \subseteq \overline{A} \cap Y$.

($\subseteq$) $\overline{A}_Y = B \cap Y$ for some closed set B in X by Theorem 2.6.2. Also, $\overline{A} \subseteq B$ holds. Therefore, $\overline{A}_Y = B \cap Y \subseteq \overline{A} \cap Y$. □

Definition 2.6.3: Intersection and Neighborhood

- Given two sets A and B, we say A and B *intersect* if $A \cap B \neq \varnothing$.
- An open set containing $x \in X$ is called an open *neighborhood* of x.

Theorem 2.6.5

Let $A \subseteq X$ where X is a topological space. The following hold.

(i) $x \in \overline{A}$ if and only if every neighborhood of x intersects A.

(ii) Let $\mathcal{B}$ be a basis for X. Then, $x \in \overline{A}$ if and only if every $B \in \mathcal{B}$ containing x intersects A.

Proof.

(i) We will prove the contrapositive "$x \notin \overline{A} \iff \exists$ neighborhood U of X, $U \cap A = \varnothing$".
(⇒) $U \triangleq X \setminus \overline{A}$ is a neighborhood of x. We find that $U \cap A = \varnothing$ since $A \subseteq \overline{A}$. ✓
(⇐) Suppose a neighborhood U of x satisfies $U \cap A = \varnothing$. It implies $A \subseteq X \setminus U$. Since $X \setminus U$ is closed, $\overline{A} \subseteq X \setminus U$ also holds. Since $x \in U$, $x \in \overline{A}$ may never hold. ✓

(ii) (⇒) A basis element that contains x is a neighborhood of x. ✓
(⇐) Follows from the definition of basis. (See Definition 2.2.1.) ✓

□

Example 2.6.2

- If $A = (0, 1/2) \subseteq \mathbb{R}$, then $\overline{A} = [0, 1/2]$.
- If $A = \{1/n \mid n \in \mathbb{Z}_+\} \subseteq \mathbb{R}$, then $\overline{A} = A \cup \{0\}$.
- If $A = \mathbb{Q} \subseteq \mathbb{R}$, then $\overline{A} = \mathbb{R}$.
- If $A = \mathbb{Z} \subseteq \mathbb{R}$, then $\overline{A} = \mathbb{Z}$.

2.6.2 Limit Points

Definition 2.6.4: Limit Point

Let $A \subseteq X$ and $x \in X$. The point x is a *limit point* of A if every neighborhood of x intersects A in some point other than x. The set of limit points of A is denoted by A'.

Note:-

Equivalently, x is a limit point of A if $x \in \overline{A \setminus \{x\}}$ thanks to Theorem 2.6.5.

Theorem 2.6.6

Let $A \subseteq X$ where X is a topological space. Then

$$\overline{A} = A \cup A'.$$

Proof. (⊇) We only need to show $A' \subseteq \overline{A}$. For every $x \in A'$, $x \in \overline{A}$ due to Theorem 2.6.5. ✓
(⊆) Let $x \in \overline{A} \setminus A$. By definition, every neighborhood of x intersects A while x cannot be in the intersection since $x \notin A$. Thus, $x \in A'$. ✓ □

Corollary 2.6.1

Let $A \subseteq X$ where X is a topological space. Then A is closed if and only if $A' \subseteq A$.

Proof. (⇒) $A = \overline{A} = A \cup A'$ and it implies $A' \subseteq A$. ✓
(⇐) $\overline{A} = A \cup A' = A$ and $\overline{A}$ is closed. ✓ □

2.6.3 Hausdorff Spaces

Definition 2.6.5: Housdorff Space

A topological space X is called a *Hausdorff space* if for each pair x_1 and x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2, respectively, that are disjoint. In other words,

$$\forall x_1, x_2 \in X, \left(x_1 \neq x_2 \implies \exists U_1, U_2 \in \mathcal{T},\ x_1 \in U_1 \wedge x_2 \in U_2 \wedge U_1 \cap U_2 = \varnothing\right).$$