MAS242 해석학 II
Notes

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Contents

Chapter 1

Differentiation

1.1 Higher order partial derivatives

Definition 1.1.1

Given $f : U \to \mathbb{R}$ where U is an open set in $\mathbb{R}^m$, define $\partial_{ij} \triangleq \partial_i(\partial_j f)(x)$ for each $i, j \in [m]$ to be *2nd order partial derivatives*. Any higher order partial derivatives can be defined inductively.

Definition 1.1.2: C^k-regularity

$f : U \to \mathbb{R}$ is C^k-regular if all partial derivatives up to order k and they are continuous.

Theorem 1.1.1

$f : U(\subseteq \mathbb{R}^2) \to \mathbb{R}$ is C^2 at a point $c \in U$, i.e., $\exists \delta > 0$, f is C^2 in $B_\delta(c)$. Then, $\partial_{12} f(c) = \partial_{21} f(c)$.

Proof. Let $|h| < \delta$. Define $A(h) \triangleq f(c_1 + h_1, c_2 + h_2) - f(c_1 + h_1, c_2) - f(c_1, c_2 + h_2) + f(c_1, c_2)$. Define $u(x_1) \triangleq f(x_1, c_2 + h_2) - f(x_1, c_2)$ and $v(x_2) \triangleq f(c_1 + h_1, x_2) - f(c_1, x_2)$. Note that u and v are differentiable.

Then, $A(h) = u(c_1 + h_1) - u(c_1)$ and $A(h) = v(c_2 + h) - v(c_2)$. By MVT, $\exists c_1^* \in (c_1, c_1 + h_1)$ and $c_2^* \in (c_2, c_2 + h_2)$ s.t. $A(h) = u'(c_1^*)h_1 = h_1\big(\partial_1 f(c_1^*, c_2 + h) - \partial_1 f(c_1^*, c_2)\big) = h_1 h_2 \partial_{21} f(c_1^*, c_2^*)$

Similarly, $\exists c_1^{**}, c_2^{**}$ such that $A(h) = h_1 h_2 \partial_{12} f(c_1^{**}, c_2^{**})$. $\partial_{21} f(c_1^*, c_2^*) = \partial_{12} f(c_1^{**}, c_2^{**})$. Hence, as $|h| \to 0$, due to the continuity, $\partial_{21}(c) = \partial_{12}(c)$. $\square$

Corollary 1.1.1

Suppose $f : U(\subseteq \mathbb{R}^m) \to \mathbb{R}$ is C^k at $c \in U$. Then $\partial_{j_1 j_2 \cdots j_k} f(c) = \partial_{j_1' j_2' \cdots j_k'}$ where $j_1' \cdots$ are a permutation of $j_1 \cdots$.

1.2 Extreme Values of differentiable Functions

Definition 1.2.1: Hessian

Let $f : U(\subseteq \mathbb{R}^m) \to \mathbb{R}$ be C_2 in U. Suppose $p \in U$ is a critical point of f, i.e., $\nabla f(p) = 0$. Define

$$\mathcal{H}f(x) \triangleq \begin{pmatrix} \partial_{11}f(x) & \partial_{21}f(x) & \cdots & \partial_{m1}f(x) \\ \partial_{12}f(x) & \partial_{22}f(x) & \cdots & \partial_{m2}f(x) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{1m}f(x) & \partial_{2m}f(x) & \cdots & \partial_{mm}f(x) \end{pmatrix}.$$

(Sometimes $\mathcal{H}f(x) = D^2f(x)$.)
Define $D(x) = \det \mathcal{H}f(x)$. (Note that $\mathcal{H}f(x)$ is symmetric when f is C^2 by the theorem above.)

Theorem 1.2.1 2nd-order derivative test for two variable functions.

When $m = 2$ and f is C^2, a critical point p is

- a local maximum if $D(p) > 0$ and $\partial_{11}f(p) > 0$ (or $\partial_{22}f(p) > 0$).
- a local minimum if $D(p) > 0$ and $\partial_{11}f(p) < 0$ (or $\partial_{22}f(p) < 0$).
- a saddle point if $D(p) < 0$.

The test fails when $D(p) = 0$.

Proof. Given a unit vector $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$, $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = u_1\partial_1 f + u_2\partial_2 f$, and thus

$$D^2_{\mathbf{u}}f = (u_1\partial_1 + u_2\partial_2)(u_1\partial_1 f + u_2\partial_2 f) = u_1^2\partial_{11}f + u_1u_2(2\partial_{12}f) + u_2^2\partial_{22}f.$$

WLOG, $u_1 \neq 0$. Set $z = u_2/u_1$. Then,

$$D^2_{\mathbf{u}}f(p) = u_1^2\big(\partial_{11}f(p) + 2\partial_{12}f(p)z + \partial_{22}f(p)z^2\big).$$

Note that, if $D(p) > 0$, $D^2_{\mathbf{u}}f(p)$ has no real root.

- If $D(p) > 0$ and $\partial_{11}f(p) < 0$, Then, $D^2\mathbf{u} < 0$ for all unit vector $\mathbf{u}$.
- If $D(p) > 0$ and $\partial_{11}f(p) > 0$, Then, $D^2\mathbf{u} > 0$ for all unit vector $\mathbf{u}$.
- If $D(p) < 0$, $D^2_{\mathbf{u}}f(p)$ has different signs depending on $\mathbf{u}$.

For general m?

$$D_{\mathbf{u}}(D_{\mathbf{u}}f) = D_{\mathbf{u}}\sum_{j=1}^{m}\partial_j f u_j = \sum_{j=1}^{m}\big((\nabla\partial_j f)\cdot\mathbf{u}\big)u_j = \sum_{j=1}^{m}\sum_{k=1}^{m}u_k u_j \partial_{kj}f.$$

Hence,

$$D^2_{\mathbf{u}}f(p) = \mathbf{u}^{\mathrm{T}} \cdot D^2f(p) \cdot \mathbf{u}$$

Since $D^2f(p)$ is symmetric, its eigenvalues $\lambda_1, \cdots, \lambda_m$ exists and they are real numbers. Also, there exists an $m \times m$ orthogonal matrix $\mathcal{O}$ such that $D^2f(p) = \mathcal{O}\Lambda(p)\mathcal{O}^{\mathrm{T}}$ where $\Lambda(p)$ is the diagonal matrix with entries are the eigenvalues.

Then, we can write $D^2_{\mathbf{u}}f(p) = \mathbf{u}\mathcal{O}\Lambda(p)\mathcal{O}^{\mathrm{T}}\mathbf{u}^{\mathrm{T}} = (\mathbf{u}\mathcal{O})\Lambda(p) = (\mathbf{u}\mathcal{O})^{\mathrm{T}}$. Since $\mathcal{O}$ is orthogonal, $\mathbf{u}\mathcal{O}$ is another arbitrary unit vector. □

Theorem 1.2.2 Generalized 2nd order partial derivatives test

When f is C^2, a critical point p is

- a local maximum if all eigenvalues of $D^2f(p)$ are negative.

- a local minimum if all eigenvalues of $D^2 f(p)$ are positive.
- a saddle point if there are both negative eigenvalues and positive eigenvalues.

The test fails when there are zero eigenvalues.

Chapter 2

Inverse Function Theorem

2.1 Jacobian

Definition 2.1.1: Jacobian

Let $\mathbf{f}\colon U(\subseteq \mathbb{R}^m) \to \mathbb{R}^n$ be differentiable. The function $J_f : U \to \mathbb{R}$ defined by

$$J_{\mathbf{f}}(\mathbf{x}) = \det \begin{bmatrix} \partial_1 f_1(\mathbf{x}) & \cdots & \partial_n f_1(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \partial_1 f_n(\mathbf{x}) & \cdots & \partial_n f_n(\mathbf{x}) \end{bmatrix}$$

is called the *Jacobian* of $\mathbf{f}$ at $\mathbf{x}$.

Lemma 2.1.1

If $f : V(\subseteq \mathbb{R}^n) - \mathbb{R}$ and $\mathbf{g}\colon U \to V$ are differentiable, then

$$J_{f \circ \mathbf{g}}(\mathbf{x}) = J_f\big(\mathbf{g}(\mathbf{x})\big) \cdot J_{\mathbf{g}}(\mathbf{x}).$$

Note:-

The linear mapping $\mathrm{d}\mathbf{f}(c)$ is invertible if and only if $J_{\mathbf{f}}(c)$ is nonzero.

2.2 The Inverse Function Theorem

Lemma 2.2.1 Contraction Mapping Principle

Let (X,d) be a complete metric space. Let $\varphi\colon X \to X$. Suppose that there exists $M \in [0,1)$ such that $d\big(\varphi(x_1), \varphi(x_2)\big) \leq M d(x_1, x_2)$. (We call it a *contraction mapping*.) Then, there uniquely exists $x_* \in X$ such that $\varphi(x_*) = x_*$.

Proof. Fix any $x_0 \in X$. Since $\{x_j\}_{j\in\mathbb{Z}_+}$, where $x_j = \varphi(x_{j-1})$ for each $j \in \mathbb{Z}_+$, is continuous. It converges to some x_*. As φ is continuous, we have $\varphi(x_*) = x_*$. The uniqueness follows trivially. ☐

Note:-

- For each $v \in \mathbb{R}^n \setminus \{0\}$, $|Av| = |v| \cdot \left|A\frac{v}{|v|}\right| \leq \|A\|_L \cdot |v|$. The result is trivial when $v = 0$.
- For each $u \in \mathbb{R}^n$ with $|u| = 1$, $|ABu| \leq \|A\|_L |Bu| \leq \|A\|_L \|B\|_L$. Hence, $\|AB\|_L = \|A\|\|B\|$.
- Given invertible $A \in L(\mathbb{R}^n.\mathbb{R}^n)$, $A^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is linear. Moreover, $\|A\|_L > 0$.

Lemma 2.2.2

Given two linear mappings $A, B : \mathbb{R}^n \to \mathbb{R}^n$ with invertibility of A,

$$\|A-B\|_L \cdot \|A^{-1}\|_L < 1 \implies B \text{ is invertible.}$$

Proof. Let $\|A^{-1}\|_L = 1/\alpha$ and $\|B-A\|_L = \beta$ so that $\beta < \alpha$. Then, for every $\mathbf{x} \in \mathbb{R}^n$,

$$\begin{aligned}\alpha|\mathbf{x}| &= \alpha|A^{-1}A\mathbf{x}| \le \alpha\|A^{-1}\| \cdot |A\mathbf{x}| \\ &= |A\mathbf{x}| \le |(A-B)\mathbf{x}| + |B\mathbf{x}| \le \beta|\mathbf{x}| + |B\mathbf{x}|;\end{aligned}$$

hence $(\alpha-\beta)|\mathbf{x}| \le |B\mathbf{x}|$ where $\mathbf{x} \in \mathbb{R}^n$ is arbitrary. As $\alpha > \beta$, it holds that $B\mathbf{x} = 0 \implies \mathbf{x} = 0$. □

Corollary 2.2.1

The set $\Omega \subseteq L(\mathbb{R}^n, \mathbb{R}^n)$ of invertible linear transformations is open.

Lemma 2.2.3

The mapping from Ω onto Ω defined by $A \mapsto A^{-1}$ is continuous.

Proof. Let A and B be invertible linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let $\|A^{-1}\| = 1/\alpha$ and $\|B-A\|_L = \beta$. We have $(\alpha-\beta)|\mathbf{x}| \le |B\mathbf{x}|$ by the same reasoning as in the proof of Lemma 2.2.2. Hence, the following holds.

$$\forall \mathbf{y} \in \mathbb{R}^n,\ (\alpha-\beta)|B^{-1}\mathbf{y}| \le |BB^{-1}\mathbf{y}| = |\mathbf{y}|$$

This shows that $\|B^{-1}\|_L \le (\alpha-\beta)^{-1}$.

Hence, we have

$$\|B^{-1} - A^{-1}\|_L \le \|B^{-1}\|_L \|A-B\|_L \|A^{-1}\|_L \le \frac{\beta}{\alpha(\alpha-\beta)}.$$

This implies that $\|B^{-1} - A^{-1}\|_L \to 0$ as $B \to A$. □

Theorem 2.2.1 Inverse Function Theorem

Let $\mathbf{f}$: $E(\subseteq \mathbb{R}^n) \to \mathbb{R}^n$ be C^1 in E and $\mathbf{c} \in E$. Suppose that $J_{\mathbf{f}}(\mathbf{c}) \neq 0$. Then, the following hold.

(i) There exists a neighborhood U of $\mathbf{a}$ such that $\mathbf{f}\big|_U$ is bijective and $V \triangleq \mathbf{f}(U)$ is open.

(ii) The inverse map of $\mathbf{f}\big|_U$ is C^1 in V.

Proof. Let $A \triangleq \mathrm{d}\mathbf{f}(\mathbf{c})$. Define $\lambda \in \mathbb{R}_+$ by $2\lambda\left\|A^{-1}\right\|_L = 1$. Since $\mathrm{d}\mathbf{f}$ is continuous, there exists a neighborhood U of $\mathbf{c}$ such that $\|\mathrm{d}\mathbf{f}(\mathbf{x}) - A\|_L < \lambda$ for each $\mathbf{x} \in U$.

Given a point $\mathbf{y} \in \mathbb{R}^n$, we define $\varphi(\cdot;\mathbf{y})$ by

$$\begin{aligned}\varphi(\cdot;\mathbf{y}) : B_\delta(\mathbf{c}) &\longrightarrow \mathbb{R}^n \\ \mathbf{x} &\longmapsto \mathbf{x} + A^{-1}(\mathbf{y} - \mathbf{f}(\mathbf{x}))\end{aligned}$$

Note that $\mathbf{x}$ is a fixed point of $\varphi(\cdot;\mathbf{y})$ if and only if $A^{-1}(\mathbf{y}-\mathbf{f}(\mathbf{x})) = 0$, i.e., $\mathbf{y} = \mathbf{f}(\mathbf{x})$. Note also that φ is differentiable and $\mathrm{d}\varphi(\mathbf{x};\mathbf{y}) = \mathrm{Id} - A^{-1}\,\mathrm{d}\mathbf{f}(\mathbf{x}) = A^{-1}\big(A - \mathrm{d}\mathbf{f}(\mathbf{x})\big)$ for each $\mathbf{x} \in U$.

Hence, for all $\mathbf{x} \in U$,

$$\|\mathrm{d}\varphi(\mathbf{x};\mathbf{y})\|_L = \left\|A^{-1}\big(A - \mathrm{d}\mathbf{f}(\mathbf{x})\big)\right\|_L \le \|A^{-1}\|_L \cdot \|A - \mathrm{d}\mathbf{f}(\mathbf{x})\|_L < 1/(2\lambda) \cdot \lambda = 1/2.$$

Thus, MVT gives

$$|\varphi(\mathbf{x}_1;\mathbf{y})-\varphi(\mathbf{x}_2;\mathbf{y})| \le \frac{1}{2}|\mathbf{x}_1-\mathbf{x}_2|$$

whenever $\mathbf{x}_1, \mathbf{x}_2 \in U$. Note that this implies there is at most one fixed point of $\varphi(\cdot;\mathbf{y})$ in U, i.e., $\mathbf{f}\big|_U$ is bijective.

Now, we shall show that $V = \mathbf{f}(U)$ is open. Take any $\mathbf{y}_0 \in V$. There (uniquely) exists $\mathbf{x}_0 \in U$ such that $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$. Fix any $r \in \mathbb{R}_+$ such that $\overline{B} \subseteq U$ where $B = B_r(\mathbf{x}_0)$. Take any $\mathbf{y} \in B_{\lambda r}(\mathbf{y}_0)$. Then,

$$|\varphi(\mathbf{x}_0;\mathbf{y})-\mathbf{x}_0| = |A^{-1}(\mathbf{y}-\mathbf{y}_0)| < \|A^{-1}\|_L \lambda r = \frac{r}{2}.$$

Moreover, for any $\mathbf{x} \in \overline{B}$,

$$|\varphi(\mathbf{x};\mathbf{y})-\mathbf{x}_0| \le |\varphi(\mathbf{x};\mathbf{y})-\varphi(\mathbf{x}_0;\mathbf{y})| + |\varphi(\mathbf{x}_0;\mathbf{y})-\mathbf{x}_0| \le \frac{1}{2}|\mathbf{x}-\mathbf{x}_0| + \frac{r}{2} < r.$$

This directly implies that $\varphi(\overline{B};\mathbf{y}) \subseteq B \subseteq \overline{B}$. Hence, $\varphi(\cdot,\mathbf{y})$ is a contraction mapping on a complete metric space $\overline{B}$. By Lemma 2.2.1, there exists a fixed point $\mathbf{x} \in \overline{B}$, which satisfies $\mathbf{y} = \mathbf{f}(\mathbf{x})$. Thus, $\mathbf{y} \in f(\overline{B}) \subseteq f(U) = V$. Hence, $B \subseteq V$, V is open. This proves (i).

Now, let $\mathbf{g}\colon V \to U$ be the local inverse map of $\mathbf{f}\big|_U$. Take any $\mathbf{y} \in V$ and $\mathbf{y}+\mathbf{k} \in V$. There are unique $\mathbf{x} \in U$ and $\mathbf{x}+\mathbf{h} \in U$ such that $\mathbf{y} = \mathbf{f}(\mathbf{x})$ and $\mathbf{y}+\mathbf{k} = \mathbf{f}(\mathbf{x}+\mathbf{h})$. Then, we have

$$\varphi(\mathbf{x}+\mathbf{h};\mathbf{y})-\varphi(\mathbf{x};\mathbf{y}) = \mathbf{h}+A^{-1}\big(\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x}+\mathbf{h})\big) = \mathbf{h}-A^{-1}\mathbf{k},$$

which implies $|\mathbf{h}-A^{-1}\mathbf{k}| \le |h|/2$. Hence, $|A^{-1}\mathbf{k}| \ge |h|/2$ is obtained by the triangle inequality; $|\mathbf{h}| \le 2\|A^{-1}\|_L|\mathbf{k}| = \lambda^{-1}|\mathbf{k}|$.

Then, since $\|\,\mathrm{d}\mathbf{f}(\mathbf{x})-A\|_L\|A^{-1}\|_L < \lambda\cdot 1/(2\lambda) = 1/2$, Lemma 2.2.2 implies that $\mathrm{d}\mathbf{f}(\mathbf{x})$ is invertible. Let $T \triangleq \mathrm{d}\mathbf{f}(\mathbf{x})$. Then, we have

$$\mathbf{g}(\mathbf{y}+\mathbf{k})-\mathbf{g}(\mathbf{y})-T^{-1}\mathbf{k} = \mathbf{h}-T^{-1}\mathbf{k} = -T^{-1}\big(\mathbf{f}(\mathbf{x}+\mathbf{h})-\mathbf{f}(\mathbf{x})-T\mathbf{h}\big),$$

and thus

$$\frac{|\mathbf{g}(\mathbf{y}+\mathbf{k})-\mathbf{g}(\mathbf{y})-T^{-1}\mathbf{k}|}{|\mathbf{k}|} \le \frac{\|T^{-1}\|_L}{\lambda}\cdot\frac{|\mathbf{f}(\mathbf{x}+\mathbf{h})-\mathbf{f}(\mathbf{x})-T\mathbf{h}|}{|\mathbf{h}|}.$$

The equation implies that $\mathbf{g}$ is differentiable on V, and that $\mathrm{d}\mathbf{g}(\mathbf{y}) = T^{-1} = \mathrm{d}\mathbf{f}(\mathbf{g}(\mathbf{y}))^{-1}$. Since $\mathrm{d}\mathbf{g}$ is a composition of continuous functions, $\mathrm{d}\mathbf{g}$ itself is continuous. □

Corollary 2.2.2

Let $\mathbf{f}\colon E(\subseteq \mathbb{R}^n) \to \mathbb{R}^n$ be C^1 in E and $J_{\mathbf{f}}(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in E$. Then, for every open set $W \subseteq E$, $\mathbf{f}(W)$ is open.

Proof. This directly follows from (i) of Theorem 2.2.1. □

2.3 Implicit Function Theorem

Definition 2.3.1

- If $\mathbf{x} = (x_1,\cdots,x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1,\cdots,y_m) \in \mathbb{R}^m$, let us write $(\mathbf{x},\mathbf{y})$ for the point $(x_1,\cdots,x_n,y_1,\cdots,y_m) \in \mathbb{R}^{n+m}$.
- Every $A \in L(\mathbb{R}^{n+m},\mathbb{R}^n)$ can be split into $A_x \in L(\mathbb{R}^n)$ and $A_y \in L(\mathbb{R}^m,\mathbb{R}^n)$ where $A(\mathbf{h},\mathbf{k}) = A_x\mathbf{h}+A_y\mathbf{k}$ for each $\mathbf{h} \in \mathbb{R}^n$ and $\mathbf{k} \in \mathbb{R}^m$.

Lemma 2.3.1

If $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ and if A_x is invertible, then

$$\forall \mathbf{k} \in \mathbb{R}^m,\ \exists! \mathbf{h} \in \mathbb{R}^n,\ A(\mathbf{h}, \mathbf{k}) = \mathbf{0}.$$

Proof. $A(\mathbf{h}, \mathbf{k}) = A_x \mathbf{h} + A_y \mathbf{k} = \mathbf{0}$ if and only if $\mathbf{h} = -(A_x)^{-1} A_y \mathbf{k}$. □

Theorem 2.3.1 Implicit Function Theorem

Let $\mathbf{f}\colon E \to \mathbb{R}^n$ be a C^1 mapping where E is an open set in $\mathbb{R}^{n+m}$. Let $(\mathbf{a}, \mathbf{b}) \in E$ satisfy $\mathbf{f}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$. Let $A = \mathrm{d}\mathbf{f}(\mathbf{a}, \mathbf{b})$ and suppose A_x is invertible. Then, there exist open sets $U \subseteq \mathbb{R}^{n+m}$ and $W \subseteq \mathbb{R}^m$ that satisfy the following.

(i) $(\mathbf{a}, \mathbf{b}) \in U$ and $\mathbf{b} \in W$.

(ii) $\forall \mathbf{y} \in W,\ \exists! \mathbf{x} \in \mathbb{R}^n,\ (\mathbf{x}, \mathbf{y}) \in U \wedge \mathbf{f}(\mathbf{x}, \mathbf{y}) = 0$.

(iii) If the unique $\mathbf{x}$ in (ii) is denoted by $\mathbf{g}(\mathbf{y})$, then $\mathbf{g}\colon W \to \mathbb{R}^n$ is C^1 on W.

(iv) Moreover, $\mathrm{d}\mathbf{g}(\mathbf{b}) = -(A_x)^{-1} A_y$.

Proof. Define $\mathbf{F}\colon E \to \mathbb{R}^{n+m}$ by $\mathbf{F}(\mathbf{x}, \mathbf{y}) \triangleq (\mathbf{f}(\mathbf{x}, \mathbf{y}), \mathbf{y})$. Then, $\mathbf{F}$ is C^1. Since $\mathbf{f}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$, if $\mathbf{r}(\mathbf{h}, \mathbf{k}) \triangleq \mathbf{f}(\mathbf{a} + \mathbf{h}, \mathbf{b} + \mathbf{k}) - A(\mathbf{h}, \mathbf{k})$, we have $\lim_{\mathbf{h}, \mathbf{k} \to \mathbf{0}} |\mathbf{r}(\mathbf{h}, \mathbf{k})| / |(\mathbf{h}, \mathbf{k})| = 0$. Hence, from

$$\mathbf{F}(\mathbf{a} + \mathbf{h}, \mathbf{b} + \mathbf{k}) - \mathbf{F}(\mathbf{a}, \mathbf{b}) = (\mathbf{f}(\mathbf{a} + \mathbf{h}, \mathbf{b} + \mathbf{k}), \mathbf{k}) = (A(\mathbf{h}, \mathbf{k}), \mathbf{k}) + (\mathbf{r}(\mathbf{h}, \mathbf{k}), \mathbf{0}),$$

it is obtained that $\mathrm{d}\mathbf{F}(\mathbf{a}, \mathbf{b})(\mathbf{h}', \mathbf{k}') = (A(\mathbf{h}', \mathbf{k}'), \mathbf{k}')$ for each $(\mathbf{h}', \mathbf{k}') \in \mathbb{R}^{n+m}$. If $\mathrm{d}\mathbf{F}(\mathbf{a}, \mathbf{b})(\mathbf{h}', \mathbf{k}') = \mathbf{0}$, then $\mathbf{k}' = 0$ and $A(\mathbf{h}', \mathbf{0}) = \mathbf{0}$; thus $\mathbf{h}' = \mathbf{0}$ as A_x is invertible. Hence, $\mathrm{d}\mathbf{F}(\mathbf{a}, \mathbf{b})$ is invertible; Theorem 2.2.1 can be applied to $\mathbf{F}$ at $(\mathbf{a}, \mathbf{b})$.

By Theorem 2.2.1, there exists a neighborhood $U \subseteq E$ of $(\mathbf{a}, \mathbf{b})$ such that $\mathbf{F}\big|_U$ is bijective, $\mathbf{F}(U)$ is open, and its inverse is C^1. Let $W \triangleq \{\mathbf{y} \in \mathbb{R}^m \mid (\mathbf{0}, \mathbf{y}) \in \mathbf{F}(U)\}$. W is open as $\mathbf{F}(U)$ is open. Noting that $\mathbf{b} \in W$, we finish the proof for (i).

Take any $\mathbf{y} \in W$. Then, there exists $(\mathbf{x}, \mathbf{y}) \in U$ such that $\mathbf{F}(\mathbf{x}, \mathbf{y}) = (\mathbf{0}, \mathbf{y})$; thus $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$. If $\mathbf{x}, \mathbf{x}'$ are two such point corresponding to $\mathbf{y}$, then

$$\mathbf{F}(\mathbf{x}', \mathbf{y}) = (\mathbf{f}(\mathbf{x}', \mathbf{y}), \mathbf{y}) = (\mathbf{0}, \mathbf{y}) = (\mathbf{f}(\mathbf{x}, \mathbf{y}), \mathbf{y}) = \mathbf{F}(\mathbf{x}, \mathbf{y}).$$

However, as $\mathbf{F}$ being injective, $\mathbf{x} = \mathbf{x}'$. This proves (ii).

Let $V \triangleq \mathbf{F}(U)$. Let $\mathbf{G}\colon V \to U$ be the inverse of $\mathbf{F}$, which is C^1 by Theorem 2.2.1. Hence, for each $\mathbf{y} \in W$, from $\mathbf{F}(\mathbf{g}(\mathbf{y}), \mathbf{y}) = (\mathbf{0}, \mathbf{y})$, we have $(\mathbf{g}(\mathbf{y}), \mathbf{y}) = \mathbf{G}(\mathbf{0}, \mathbf{y})$. This directly shows that $\mathbf{g}$ is C^1 as well. This proves (iii).

Let $\mathbf{\Phi}\colon W \to U$ be defined by $\mathbf{\Phi}(\mathbf{y}) = \mathbf{G}(\mathbf{0}, \mathbf{y}) = (\mathbf{g}(\mathbf{y}), \mathbf{y})$, which is C^1, indeed. Then, $\mathrm{d}\mathbf{\Psi}(\mathbf{y}) = (\mathrm{d}\mathbf{g}(\mathbf{y}), I_m)$. Differentiating both sides of the equality $\mathbf{f}(\mathbf{\Phi}(\mathbf{y})) = \mathbf{0}$, we get

$$\mathrm{d}\mathbf{f}(\mathbf{\Phi}(\mathbf{y}))\, \mathrm{d}\mathbf{\Phi}(\mathbf{y}) = 0.$$

Putting $\mathbf{y} := \mathbf{b}$, as $\mathbf{\Phi}(\mathbf{b}) = (\mathbf{a}, \mathbf{b})$, we get $A\, \mathrm{d}\mathbf{\Phi}(\mathbf{b}) = 0$, or

$$A_x\, \mathrm{d}\mathbf{g}(\mathbf{b}) + A_y = 0,$$

i.e., $\mathrm{d}\mathbf{g}(\mathbf{b}) = -(A_x)^{-1} A_y$. □

Definition 2.3.2: C^1-norm

Suppose $\varphi : \mathbb{R}^n \to \mathbb{R}$ is C^1. Then,

$$\|\varphi\|_{C^0(\overline{\Omega})} \triangleq \sup_{\mathbf{x}\in\Omega} |\varphi(x)|$$

$$\|\varphi\|_{C^1(\overline{\Omega})} \triangleq \|\varphi\|_{C^0(\overline{\Omega})} + \sum_{j=1}^{n} \|\partial_j \varphi\|_{C^0(\overline{\Omega})}.$$

This is only for Example 2.3.1.

Example 2.3.1 (Level Sets)

Define $\Omega \triangleq \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_2| \le 1\}$. Given two constants, $a, b \in \mathbb{R}$ with $a < b$, define $\overline{\varphi}(x_1, x_2) = ax_1$ and $\overline{\psi}(x_1, x_2) = bx_1$. Then, $\Gamma_0 = \{\mathbf{x} \in \Omega \mid \overline{\varphi}(\mathbf{x}) - \overline{\psi}(\mathbf{x}) = 0\} = \{\mathbf{x} \in \Omega \mid x_1 = 0\}$.

Suppose that $\varphi, \psi : \Omega \to \mathbb{R}$ satisfy

$$\|\varphi - \overline{\varphi}\|_{C^1(\overline{\Omega})} + \|\psi - \overline{\psi}\|_{C^1(\overline{\Omega})} \le \frac{1}{4}|a-b|.$$

Then, what would be the expression for $\Gamma = \{\mathbf{x} \in \Omega \mid \varphi(\mathbf{x}) - \psi(\mathbf{x}) = 0\}$?

Observe that $(\varphi - \psi) = (\varphi - \overline{\varphi}) + (\overline{\varphi} - \overline{\psi}) + (\overline{\psi} - \psi)$ and thus $|(\varphi - \psi)(x_1, x_2) - (a-b)x_1| \le |a-b|/4$. This implies $\lim_{x_1 \to \pm\infty} (\varphi - \psi)(x_1, x_2) = \mp\infty$. Hence, for every $x_2 \in [-1, 1]$, there exists $x_1^* \in \mathbb{R}$ such that $(\varphi - \psi)(x_1^*, x_2) = 0$.

Moreover, $\partial_1(\varphi - \psi) = \partial_1(\varphi - \overline{\varphi}) + (a-b) + \partial_1(\overline{\psi} - \psi)$, and thus $|\partial_1(\varphi - \psi)| \ge \frac{3}{4}|a-b| > 0$. Hence, the x_1^* in the previous paragraph is unique. This means that $\Gamma = \{(f(x_2), x_2) \mid x_2 \in \mathbb{R}\}$ for some f.

$(\varphi - \psi)(f(x_2), x_2) - (\overline{\varphi} - \overline{\psi})(f(x_2), x_2) = -(\overline{\varphi} - \overline{\psi})(f(x_2), x_2) = (b-a)f(x_2)$. Hence,

$$f(x_2) = \frac{(\varphi - \overline{\varphi})(f(x_2), x_2) - (\psi - \overline{\psi})(f(x_2), x_2)}{b-a}.$$

This is the implicit representation of f. Moreover, $|f(x_2)| = \dfrac{|b-a|/4}{|b-a|} = 1/4$.

2.4 Applications of IMFT: Lagrange's Method

Theorem 2.4.1 Optimization Under Multiple Constraints

Let $f, g_1, g_2, \cdots, g_k : E \to \mathbb{R}$ be C^1 where E is an open set in $\mathbb{R}^n$ and $n > k$. Let $Z \triangleq \cap_{j=1}^{k} \{\mathbf{z} \in \mathbb{R}^n \mid g_j(\mathbf{z}) = 0\}$. Suppose $\mathbf{z}_0 \in Z$ is a local maximum point with respect to f on Z. Suppose also that

$$\Delta \triangleq \det \begin{bmatrix} \partial_1 g_1(\mathbf{z}_0) & \cdots & \partial_1 g_k(\mathbf{z}_0) \\ \vdots & \ddots & \vdots \\ \partial_k g_1(\mathbf{z}_0) & \cdots & \partial_k g_k(\mathbf{z}_0) \end{bmatrix} \ne 0.$$

Then, there exists $\lambda_1, \lambda_2, \cdots, \lambda_k \in \mathbb{R}$ such that $\nabla f(\mathbf{z}_0) = \sum_{m=1}^{k} \lambda_m \nabla g_m(\mathbf{z}_0)$.

Proof. Since $\Delta \neq 0$, there exists a unique solution $(\lambda_1, \cdots, \lambda_k) \in \mathbb{R}^k$ for the linear system

$$\begin{bmatrix} \partial_1 g_1(\mathbf{z}_0) & \cdots & \partial_1 g_k(\mathbf{z}_0) \\ \vdots & \ddots & \vdots \\ \partial_k g_1(\mathbf{z}_0) & \cdots & \partial_k g_k(\mathbf{z}_0) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{bmatrix} = \begin{bmatrix} \partial_1 f(\mathbf{z}_0) \\ \vdots \\ \partial_k f(\mathbf{z}_0) \end{bmatrix}.$$

For each point $\mathbf{z} = (z_1, \cdots, z_n) \in \mathbb{R}^n$, let $\mathbf{x} = (z_1, \cdots, z_k)$ and $\mathbf{y} = (z_{k+1}, \cdots, z_n)$. Let $\mathbf{z}_0 = (\mathbf{x}_0, \mathbf{y}_0)$. Let $\mathbf{g}\colon E \to \mathbb{R}^k$ be defined by $\mathbf{g}(\mathbf{z}) = (g_1(\mathbf{z}), \cdots, g_k(\mathbf{z}))$.

Since $\mathbf{g}$ is C^1, $\mathbf{g}(\mathbf{z}_0) = 0$, and $\left(\mathrm{d}\mathbf{g}(\mathbf{z}_0)\right)_x$ is invertible, by Theorem 2.3.1, there exists an open neighborhood $W \subseteq \mathbb{R}^{n-k}$ of $\mathbf{y}_0$ and a C^1 function $\mathbf{s}\colon W \to \mathbb{R}^k$ such that $\mathbf{g}(\mathbf{s}(\mathbf{y}), \mathbf{y}) = \mathbf{0}$ for each $\mathbf{y} \in W$. Note that $\mathbf{s}(\mathbf{y}_0) = \mathbf{x}_0$.

Define $F\colon W \to \mathbb{R}$ by $\mathbf{y} \mapsto f(\mathbf{s}(\mathbf{y}), \mathbf{y})$. As $\mathbf{z}_0$ is a local maximum point, so is $\mathbf{y}_0$. Hence, $\nabla F(\mathbf{y}_0) = \mathbf{0}$. For each $j \in [k]$, define $G_j\colon W \to \mathbb{R}$ by $\mathbf{y} \mapsto g_j(\mathbf{s}(\mathbf{y}), \mathbf{y})$. As $(\mathbf{s}(\mathbf{y}), \mathbf{y}) \in Z$, we have $G_j = 0$ for each $j \in [k]$. Thus, $\nabla G_j(\mathbf{y}) = \mathbf{0}$.

Let $\mathbf{s} = (s_1, s_2, \cdots, s_k)$ where each $s_j\colon W \to \mathbb{R}$. Since

$$\begin{aligned} \nabla F(\mathbf{y}) &= \mathrm{d}f(\mathbf{s}(\mathbf{y}), \mathbf{y})\,\mathrm{d}(\mathbf{s}(\mathbf{y}), \mathbf{y}) \\ &= \begin{bmatrix} \partial_1 f(\mathbf{s}(\mathbf{y}), \mathbf{y}) & \cdots & \partial_n f(\mathbf{s}(\mathbf{y}), \mathbf{y}) \end{bmatrix} \begin{bmatrix} \partial_1 s_1(\mathbf{y}) & \partial_2 s_1(\mathbf{y}) & \cdots & \partial_{n-k} s_1(\mathbf{y}) \\ \partial_1 s_2(\mathbf{y}) & \partial_2 s_2(\mathbf{y}) & \cdots & \partial_{n-k} s_2(\mathbf{y}) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 s_k(\mathbf{y}) & \partial_2 s_k(\mathbf{y}) & \cdots & \partial_{n-k} s_k(\mathbf{y}) \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \end{aligned}$$

$\nabla F(\mathbf{y}_0) = \mathbf{0}$ implies

$$\partial_{k+j} f(\mathbf{z}_0) + \sum_{i=1}^{k} \partial_i f(\mathbf{z}_0) \partial_j s_i(\mathbf{y}_0) = 0$$

for each $j \in [n-k]$. Similarly, $\nabla G_m(\mathbf{y}_0) = \mathbf{0}$ for each $m \in [k]$ implies that

$$-\lambda_m \left[\partial_{k+j} g_m(\mathbf{z}_0) + \sum_{i=1}^{k} \partial_i g_m(\mathbf{z}_0) \partial_j s_i(\mathbf{y}_0) \right] = 0$$

for each $j \in [n-k]$ and $m \in [k]$.

Adding the $k+1$ equations together for each $j \in [n-k]$,

$$0 = \left[\partial_{k+j} f(\mathbf{z}_0) - \sum_{m=1}^{k} \lambda_m \partial_{k+j} g_m(\mathbf{z}_0) \right] + \sum_{i=1}^{k} \left[\partial_i f(\mathbf{z}_0) - \sum_{m=1}^{k} \lambda_m \partial_i g_m(\mathbf{z}_0) \right] \partial_j s_i(\mathbf{y}_0).$$

By the definition of $\lambda_1, \cdots, \lambda_k$, we are left with only

$$\partial_j f(\mathbf{z}_0) = \sum_{m=1}^{k} \lambda_m \partial_j g_m(\mathbf{z}_0)$$

for each $j \in \{k+1, \cdots, n\}$. For $j \in [k]$, the same equation holds by the definition of $\lambda_1, \cdots, \lambda_k$. Hence, we have $\nabla f(\mathbf{z}_0) = \sum_{m=1}^{k} \lambda_m \nabla g_m(\mathbf{z}_0)$. $\square$

End.