MAS241 해석학 I
Note

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Contents

Chapter 1

Structure of the Real Numbers

1.1 Completeness of the Real Numbers

Definition 1.1.1: Cauchy Sequence

Let X be a space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is a *Cauchy sequence* if $\|x_n - x_m\| \to 0$ as $n, m \to \infty$.

Definition 1.1.2: Completeness

A set X is *complete* if every Cauchy sequnce has a limit in X, i.e.,

$$x_n \to x_\infty \in X.$$

Definition 1.1.3: Boundedness

Let $\varnothing \neq S \subseteq \mathbb{R}$.

a) S is *bounded above* if $\exists M \in \mathbb{R}$, $\forall x \in S$, $x \leq M$.
   - M is called an *upper bound* of S.

b) S is *bounded below* if $\exists M \in \mathbb{R}$, $\forall x \in S$, $x \geq M$.
   - M is called an *lower bound* of S.

c) S is bounded if S is bounded above and below.

Theorem 1.1.1 Archimedes' Principle

Let ε and M be any two possible real numbers. Then, there exists a k in $\mathbb{N}$ such that $M < k\varepsilon$.

The proof of Theorem 1.1.1 can be done by integrating Theorem 1.1.2 and Theorem 1.1.4.

Definition 1.1.4: Supremum and Infimum

a) Let S be bounded above. Then, the smallest upper bound is called the *supremum* of S, $\sup S$.

b) Let S be bounded below. Then, the largest lower bound is called the *infimum* of S, $\inf S$.

Example 1.1.1

Let $S = \{(-1)^k(1-1/k) \mid k \in \mathbb{N}\}$. It is clear that $-1 < S < 1$; 1 is an upper bound and -1 is a lower bound. We now claim that $\sup S = 1$. To show this, let us assume that $M < 1$ is an upper bound of S. By Archimedes' principle, there exists an natural number k_0 such that $(1-M)/2 < k_0$, which implies $(-1)^{2k_0}(1-1/(2k_0)) > M$; M is not an upper bound. Therefore, 1 is the smallest upper bound. It can be similarly shown that $\inf S = -1$.

Theorem 1.1.2 Completeness Axiom for $\mathbb{R}$

If $\varnothing \neq S \subseteq \mathbb{R}$ and S is bounded above, then $\sup S$ exists in $\mathbb{R}$.

Corollary 1.1.1

If $\varnothing \neq S \subseteq \mathbb{R}$ and S is bounded below, then $\inf S$ exists in $\mathbb{R}$.

Proof. Let $B := \{-x \mid x \in S\}$. Then, $M = \sup S \in \mathbb{R}$ by Theorem 1.1.2. We now claim that $\inf S = -M$.

For all $x \in S$, $-x \in B$, which implies $-x \leq M$, and therefore $x \geq -M$. Thus, $-M$ is a lower bound of S.

Suppose there is a $M_1 > -M$ such that M_1 is a lower bound of S. For all $x \in S$, $x \geq M_1$, which implies $-x \leq -M_1$. Thus, $-M_1$ is an upper bound of B but $-M_1 < M = \sup B$, #.

Therefore, $\inf S = -M \in \mathbb{R}$. $\square$

Example 1.1.2

- $S := \left\{ \sum_{j=0}^{k} \frac{1}{j!} \,\middle|\, k \in \mathbb{N} \right\}$. S is bounded above.

$$\sum_{j=0}^{k} \frac{1}{j!} = 1 + \sum_{j=1}^{k} \frac{1}{j!} \leq 1 + \sum_{j=1}^{k} \frac{1}{2^{j-1}} < 3$$

  In fact, $e := \sup S$.

- $S := \left\{ \left(1 + \frac{1}{k}\right)^k \,\middle|\, k \in \mathbb{N} \right\}$. S is bounded above.

$$\left(1 + \frac{1}{k}\right)^k = \sum_{j=0}^{k} \binom{k}{j} \frac{1}{k^j} \leq \sum_{j=0}^{k} \frac{1}{j!} \leq e$$

Theorem 1.1.3

Let S be a finite nonempty subset of $\mathbb{R}$. Then, $\sup S \in S$ and $\inf S \in S$.

Proof. (Induction on $|S|$) For $S = \{x\}$, $x = \inf S = \sup S \in S$.

Take any $k \in \mathbb{N}$ and suppose the statement holds for every S with $|S| = k$. Now, take any $S' \subseteq \mathbb{R}$ such that $|S'| = k+1$. Let $x \in S'$, $\mu := \sup(S' \setminus \{x\})$, and $\nu := \inf(S' \setminus \{x\})$. By the induction hypothesis, $\mu, \nu \in S' \setminus \{x\}$. Letting $\mu' := \max(\mu, x)$ and $\nu' := \min(\nu, x)$, μ' and ν' are the supremum and infimum of S', respectively. Moreover, μ' and ν' are elements of S'. $\square$

Theorem 1.1.4

Let $\varnothing \neq S \subseteq \mathbb{R}$.

- If S is bounded above, then "$\mu = \sup S$ if and only if μ is an upper bound and $\forall\, \varepsilon \in \mathbb{R}_+,\ \exists\, x \in S,\ \mu - \varepsilon < x \leq \mu$".
- If S is bounded below, then "$\nu = \inf S$ if and only if ν is an lower bound and $\forall\, \varepsilon \in \mathbb{R}_+,\ \exists\, x \in S,\ \nu \leq x < \nu + \varepsilon$".

Proof. Let S be bounded above. If there is no $x \in S$ in $(\mu - \varepsilon, \mu]$, then $\mu - \varepsilon$ would be a smaller upper bound.

For the converse, assume M is an upper bound and $M < \mu$. Let $\varepsilon := \mu - M > 0$. Then, there is some $x \in S$ such that $M = \mu - \varepsilon < x \leq \mu$, # to M is an upper bound. Therfore, μ is the least upper bound.

The same logic may be applied for bounded below S. □

Proof of Theorem 1.1.1. Let $S := \{k\varepsilon \mid k \in \mathbb{N}\}$. Assume S is bounded above and nonempty. Then, by Theorem 1.1.2, there is $\mu = \sup S$. We also know, from Theorem 1.1.4, that there is $k \in \mathbb{N}$ such such that $\mu - \varepsilon < k\varepsilon \leq \mu$, which implies $\mu < (k+1)\varepsilon$. Since $(k+1)\varepsilon \in S$, μ is not an upper bound of S, which is a contradiction. Therefore, S is not bounded above. In other words, for any $M > 0$, there is some $k \in \mathbb{N}$ such that $M < k\varepsilon$. □

Theorem 1.1.5

Theorem 1.1.1 (Archimedes' principle) is equivalent to the following statement:

$$\forall\, c \in \mathbb{R}_+,\ \exists\, k \in \mathbb{N},\ k - 1 \leq c < k.$$

Proof. Assume Archimedes' principle. If $c < 1$, $k = 1$ satisfies, and it is done. Now, let us suppose $c \geq 1$. By Theorem 1.1.1, there is a $k \in \mathbb{N}$ such that $c < k$. We may let $k_0 := \min\{k \in \mathbb{N} \mid k > c\}$ by Well-Ordering of $\mathbb{N}$. We note that $k_0 - 1 \leq c$ since $k_0 - 1 \in \mathbb{N}$ since $k_0 > 1$. Therefore, $k_0 - 1 \leq c < k_0$.

Now, assume "$\forall\, c \in \mathbb{R}_+,\ \exists\, k \in \mathbb{N},\ k - 1 \leq c < k$". Take any $M > 0$ and $\varepsilon \in \mathbb{R}_+$ and let $c := M/\varepsilon$. The assumption tells the existence of a $k \in \mathbb{N}$ such that $M/\varepsilon = c < k$, which directly implies $M < k\varepsilon$. □

Theorem 1.1.6

Let c and d be real numbers with $c < d$. Then, $\exists\, x \in \mathbb{Q},\ c < x < d$.

Proof. There are three cases: $0 < c < d$, $c \leq 0 < d$, or $c < d \leq 0$.

Case 1) By Archimedes' principle, $\exists\, q \in \mathbb{N},\ 1 < (d-c)q$, which implies $cq + 1 < dq$. By Theorem 1.1.5, $\exists q \in \mathbb{N},\ p - 1 \leq cq < p$ since $cq > 0$. To sum up, $p - 1 \leq cq < p \leq cq + 1 < dq$, which implies $c < p/q < d$.

Case 2) By Archimedes' principle, $\exists\, q \in \mathbb{N},\ 1 < dq$. Then, $c \leq 0 < 1/q < d$ holds.

Case 3) By case 1 and 2, there is $r \in \mathbb{Q}$ such that $-d < r < -c$. Then, $c < -r < d$ holds. □

1.2 Neighborhoods and Limit Points

Definition 1.2.1: Neighborhood and Deleted Neighborhood

For each $x \in \mathbb{R}$ and $r \in \mathbb{R}_+$,

$$N(x;r) := \{y \in \mathbb{R} : |y-x| < r\} = (x-r, x+r)$$

is called the *neighborhood* of x with radius r, and

$$N'(x;r) := \{y \in \mathbb{R} : 0 < |y-x| < r\} = N(x;r) \setminus \{x\}$$

is called the *deleted neighborhood* of x with radius r.

Definition 1.2.2: Limit Point and Isolated Point

For $\varnothing \neq S \subseteq \mathbb{R}$, $x \in \mathbb{R}$ is a *limit point* of S if

$$\forall \varepsilon \in \mathbb{R}_+, N'(x,\varepsilon) \cap S \neq \varnothing.$$

If $x \in \mathbb{R}$ is not a limit point of S, then it is called an *isolated point* of S.

Definition 1.2.3: Discrete Set

If $\varnothing \neq S \subseteq \mathbb{R}$ has no limit points, then S is said to be *discrete*.

Example 1.2.1

Let $S := \left\{(-1)^k(1+1/k) \,\middle|\, k \in \mathbb{N}\right\}$. Then, 1 and -1 are limit points of S.
To see 1 is a limit point, take any $\varepsilon \in \mathbb{R}_+$ and, using Theorem 1.1.1, choose a $k \in \mathbb{N}$ such that $1 < (2\varepsilon)k$. Then, $1 < 1 + \frac{1}{2k} = (-1)^{2k}\left(1 + \frac{1}{2k}\right) < 1 + \varepsilon$; $N'(1,\varepsilon) \cap S \neq \varnothing$. Therefore, 1 is a limit point.

Theorem 1.2.1

Let $\varnothing \neq S \subseteq \mathbb{R}$. Then, $x \in \mathbb{R}$ is a limit point of S if and only if

$$\exists \varepsilon_0 \in \mathbb{R}_+, \forall \varepsilon \in (0, \varepsilon_0), N'(x,\varepsilon) \cap S \neq \varnothing.$$

Proof. Trivial; $0 < \varepsilon_1 < \varepsilon_2$ implies $N'(x,\varepsilon_1) \subsetneq N'(x,\varepsilon_2)$. □

Theorem 1.2.2

Let $\varnothing \neq S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$ be a limit point of S. Then, every deleted neighborhood of x must contain infinitely many points of S.

Proof. Assume $N'(x;\varepsilon) \cap S$ were to contain only finitely many points, namely, $N'(x;\varepsilon) \cap S = \{x_1, x_2, \cdots, x_k\}$. Let $S_1 := \left\{|x - x_i| : i \in [k]\right\}$. Since S_1 is finite, we may let x_j be an element of $N'(x;\varepsilon) \cap S$ that satisfies $|x - x_j| = \min S_1 = \inf S_1 > 0$. If we let $\varepsilon_0 := |x - x_j|/2$, $N'(x;\varepsilon_0) \cap S = \varnothing$, #. □

Corollary 1.2.1

If S is a finite subset of $\mathbb{R}$, then S has no limit point.

Example 1.2.2

$\mathbb{Z}$ has no limit point.

Theorem 1.2.3 Bolzano–Weierstrass Theorem

If $S \subseteq \mathbb{R}$ is bounded and has an infinite number of elements, then S has a limit point.

Proof. Since S is bounded, $a_0 := \inf S$ and $b_0 := \sup S$ exist; $S \subseteq [a_0, b_0]$. At least one of $[a_0, (a_0+b_0)/2]$ and $[(a_0+b_0)/2, b_0]$ has an infinite number of elements in S, otherwise S must be finite. Choose whichever has an infinite number of elements in S, and let us denote it as $[a_1, b_1]$. Since, $S \cap [a_1, b_1]$ is bounded and has an infinite number of elements, we may find a_2 and b_2 in the same manner. Note that

(a) for every natural number k, $S \cap [a_k, b_k]$ has an infinite number of elements,

(b) $\forall k \in \mathbb{N}$, $b_k - a_k = (b_0 - a_0)/2^k > 0$, and

(c) $\forall k \in \mathbb{N}$, $a_{k-1} \leq a_k < b_k \leq b_{k-1}$.

The sequence $\{a_k\}_{k=0}^{\infty}$ is bounded above by b_0, and the sequence $\{b_k\}_{k=0}^{\infty}$ is bounded below by a_0. Therefore, we may let $\alpha := \sup\{a_k\}$ and $\beta := \inf\{b_k\}$.

Since a_j is a lower bound of $\{b_k\}_{k=0}^{\infty}$ for all $j \in \mathbb{N}$, $\forall j \in \mathbb{N}$, $a_j \leq \beta$. This implies β is an upper bound of $\{a_k\}_{k=0}^{\infty}$, therefore $\alpha \leq \beta$. Since $a_j \leq \alpha \leq \beta \leq b_j$ for all $j \in \mathbb{N}$, we get $0 \leq \beta - \alpha \leq b_j - a_j = (b_0 - a_0)/2^j$. Therefore, $\beta - \alpha = 0$.

We now claim that α is a limit point of S. Take any $\varepsilon \in \mathbb{R}_+$. By Theorem 1.1.4, $\exists k_0 \in \mathbb{N}$, $\alpha - \varepsilon < a_{k_0} \leq \alpha$. We may take $k \in \mathbb{N}$ such that $k > k_0$ and $|b_k - a_k| < \varepsilon$ thanks to (b). Since $\alpha \in [a_k, b_k]$, $\alpha - \varepsilon < a_{k_0} \leq a_k \leq \alpha \leq b_k < \alpha + \varepsilon$, which implies $[a_k, b_k] \subseteq N(\alpha; \varepsilon)$.

In conclusion, $S \cap [a_k, b_k]$ has infinitely many elements by (a), and so does $(S \cap [a_k, b_k]) \setminus \{\alpha\}$. $S \cap N'(\alpha; \varepsilon)$ is, therefore, nonempty. □

Definition 1.2.4: Bolzano–Weierstrass Property

We say that a nonempty set X has the *Bolzano–Weierstrass property* if every bounded, infinite subset S of X has a limit point in X.

1.3 The Limit of a Sequence

Definition 1.3.1: Cluster Point

$c \in \mathbb{R}$ is a *cluster point* of the sequence $\{x_k\}$ if,

$$\forall (\varepsilon, k) \in \mathbb{R}_+ \times \mathbb{N},\ \exists k_1 \in \mathbb{N}_{>k},\ x_{k_1} \in N(c; \varepsilon).$$

Lemma 1.3.1

$c \in \mathbb{R}$ is a cluster point of $\{x_k\}$ if and only if $\left\{k \in \mathbb{N} \,\middle|\, x_k \in N(c; \varepsilon)\right\}$ is infinite for every $\varepsilon \in \mathbb{R}_+$.

Proof. ($\Rightarrow$) Suppose $S := \left\{k \in \mathbb{N} \,\middle|\, x_k \in N(c; \varepsilon)\right\}$ is finite for some $\varepsilon \in \mathbb{R}_+$. If S were empty, then, c is not a cluster point by Definition 1.3.1. Therefore, S is nonempty and has a maximum element $k_0 := \max S$ by Theorem 1.1.3. Since c is a cluster point, there is a natural number $k_1 > k_0$ such that $x_{k_1} \in N(c; \varepsilon)$; $k_1 \in S$. This contradicts the maximality of k_0.

($\Leftarrow$) Take any $\varepsilon \in \mathbb{R}_+$ and $k_0 \in \mathbb{N}$. If there is no $k_1 \in \mathbb{N}$ such that $k_1 > k_0$ and $x_{k_1} \in N(c; \varepsilon)$, S will be bounded above by k_0 and finite, which is a contradiction. Therefore, c is a cluster point of S.

□

Definition 1.3.2: Convergence and Divergence of a Sequnce

The sequnce $\{x_k\}$ *converges* to x_0 and x_0 is the *limit* of $\{x_k\}$ if,

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists k_0 \in \mathbb{N},\ \forall k \in \mathbb{N}_{\geq k_0},\ x_k \in N(x_0; \varepsilon).$$

We write $\lim\limits_{k \to \infty} x_k = x_0$. If there is no such x_0, then $\{x_k\}$ *diverges*.

Lemma 1.3.2

$\lim\limits_{x \to \infty} x_k = x_0$ if and only if $\left\{k \in \mathbb{N} \,\middle|\, x_k \notin N(x_0; \varepsilon)\right\}$ is finite for every $\varepsilon \in \mathbb{R}_+$.

Proof. ($\Rightarrow$) Take any $\varepsilon \in \mathbb{R}_+$. There is some $k_0 \in \mathbb{N}$ such that $k \in N(x_0; \varepsilon)$ for all natural numbers $k \geq k_0$. Therefore, $\left\{k \in \mathbb{N} \,\middle|\, x_k \notin N(x_0; \varepsilon)\right\} \subseteq [k_0]$ and thus finite.

($\Leftarrow$) Take any $\varepsilon \in \mathbb{R}_+$. Let $k_0 := \max\left\{k \in \mathbb{N} \,\middle|\, x_k \notin N(x_0; \varepsilon)\right\}$. Then, for every natural number k larger than k_0 satisfies $x_k \in N(x_0; \varepsilon)$. □

Lemma 1.3.3

The limit x_0 of a sequence, if it exists, is a cluster point of the sequence.

Theorem 1.3.1 Uniqueness of the Limit

The limit of a convergent sequence of $\mathbb{R}$ is unique.

Proof. Suppose a and b are two limits of a sequence $\{x_k\}$ and $a \neq b$. Let $\varepsilon := |b-a|/2$. Then, by Lemma 1.3.2, $A := \{k \in \mathbb{N} \mid x_k \notin N(a; \varepsilon)\}$ and $B := \{k \in \mathbb{N} \mid x_k \notin N(b; \varepsilon)\}$ are both finite, which means $A \cup B = \mathbb{N}$ is finite, #. □

Theorem 1.3.2

If a sequence has two (or more) cluster points, then it diverges.

Proof. Suppose x_0 is the limit of $\{x_k\}$. Since, by Lemma 1.3.3, x_0 is a cluster point, there is another cluster point c different from x_0. Let $\varepsilon := |x_0 - c|/2$.

Although $S := \{k \in \mathbb{N}\ x_k \notin N(x_0; \varepsilon)\}$ should be finite by Lemma 1.3.2, $\{k \in \mathbb{N} \mid x_k \in N(c; \varepsilon)\}$, a subset of S, is infinite by Lemma 1.3.1, #. □

Theorem 1.3.3

A convergent sequence is bounded.

Proof. Let x_0 is the limit of $\{x_k\}$. There is some $k_0 \in \mathbb{N}$ such that $|x_k - x_0| < 1$ for all $k \in \mathbb{N}_{k_0}$. Let $A := \{x_k \mid k \in \mathbb{N} \text{ and } k \leq k_0\}$ and $B := \{x_k \mid k \in \mathbb{N} \text{ and } k \geq k_0\}$. Then, A is finite and B is bounded above and below by $x_0 + 1$ and $x_0 - 1$, respectively. Therefore, $\{x_k\}$ is bounded above by $\max(\max A, x_0 + 1)$ and below by $\min(\min A, x_0 - 1)$. □

Corollary 1.3.1

An unbounded sequence diverges.

Lemma 1.3.4

The following hold.

(i) $\lim_{k\to\infty} x_k = 0 \iff \lim_{k\to\infty} |x_k| = 0$

(ii) $\lim_{k\to\infty} x_k = x_0 \implies \forall c \in \mathbb{R}, \lim_{k\to\infty} cx_k = cx_0$

Proof of (ii). If $c = 0$, then it is done; so suppose $c \neq 0$. Take any $\varepsilon \in \mathbb{R}_+$. Then, there is some $k_0 \in \mathbb{N}$ such that $|x_k - x_0| < \varepsilon/|c|$ for all $k \geq k_0$. This directly implies for all $k \geq k_0$, $|cx_k - cx_0| = |c| \cdot |x_k - x_0| < |c| \cdot \varepsilon/|c| = \varepsilon$. □

Theorem 1.3.4

A bounded, monotone sequence converges.

Proof. Suppose $\{x_k\}$ is a monotone increasing sequence. Since it is bounded, $\{x_k\}$ has $\mu := \sup\{x_k \mid k \in \mathbb{N}\}$. Take any $\varepsilon \in \mathbb{R}_+$. By Theorem 1.1.4, there is some $k_0 \in \mathbb{N}$ such that $\mu - \varepsilon < x_{k_0} \leq \mu$. Then, for all $k \in \mathbb{N}_{\geq k_0}$, $\mu - \varepsilon < x_{k_0} \leq x_k \leq \mu$, which implies $|x_k - \mu| < \varepsilon$. Therefore $\lim_{k\to\infty} x_k = \mu$. □

Theorem 1.3.5 The Squeeze Play

Let $\{x_k\}$, $\{y_k\}$, and $\{z_k\}$ be sequences that satisfy $x_k \leq y_k \leq z_k$ for $k \in \mathbb{N}$. If both $\{x_k\}$ and $\{z_k\}$ converges to $L \in \mathbb{R}$, then $\{y_k\}$ also converges to L.

Proof. Take any $\varepsilon > 0$. There is $k_1 \in \mathbb{N}$ such that $\forall k \in \mathbb{N}_{\geq k_1}$, $x_k \in N(L; \varepsilon)$. Similarly, there is $k_2 \in \mathbb{N}$ such that $\forall k \in \mathbb{N}_{\geq k_2}$, $x_k \in N(L; \varepsilon)$. Then, for all $k \in \mathbb{N}$ not smaller than $\max\{k_1, k_2\}$, $L - \varepsilon < x_k \leq y_k \leq z_k < L + \varepsilon$ holds, which implies $y_k \in N(L; \varepsilon)$. □

Theorem 1.3.6 Limit is Order Preserving on Convergent Sequences

If both $\{x_k\}$ and $\{y_k\}$ converge and if $x_k \le y_k$ for each $k \in \mathbb{N}$, then

$$\lim_{k\to\infty} x_k \le \lim_{k\to\infty} y_k.$$

Proof. Let $L_x := \lim_{k\to\infty} x_k$ and $L_y := \lim_{k\to\infty} y_k$, and suppose $L_x > L_y$. Let $\varepsilon := (L_x - L_y)/2 > 0$. Then, there is $k \in \mathbb{N}$ such that $x_k \in N(L_x;\varepsilon)$ and $y_k \in N(L_y;\varepsilon)$, which implies $y_k < L_y + \varepsilon = L_x - \varepsilon < x_k$, #. □

Definition 1.3.3: Subsequence

Let $\{x_k\}$ be any sequence. Choose any strictly monotone increasing sequnce $k_1 < k_2 < k_3 < \cdots$ of natural numbers. For each $j \in \mathbb{N}$, let $y_j := x_{k_j}$. The sequence $\{y_j\}_{j=1}^{\infty}$ is called an *subsequence* of $\{x_k\}$.

Theorem 1.3.7

The point c is a cluster point of $\{x_k\}$ if and only if there exists a subsequence of $\{x_k\}$ that converges to c.

Proof. ($\Rightarrow$) Let $\{\varepsilon_k\}$ be an arbitrary sequence of positive real numbers that converges to 0. (e.g. $\varepsilon_k = 1/k$) Define $\{k_j\}_{j=1}^{\infty}$ by the inductive definition below.

- $k_0 := 0$
- For each $j \in \mathbb{N}$, $k_j \in \{\, k \in \mathbb{N} \mid k > k_{j-1} \text{ and } x_k \in N(c;\varepsilon_j) \,\}$.

Since c is a cluster point, $\{\, k \in \mathbb{N} \mid k > k_{j-1} \text{ and } x_k \in N(c;\varepsilon_j) \,\} \ne \varnothing$ for all $j \in \mathbb{N}$. Therefore, $\{k_j\}$ is well-defined. It is immediate that $\lim_{j\to\infty} x_{k_j} = c$.

($\Leftarrow$) Let $\{x_{k_j}\}_{j=1}^{\infty}$ be a sequence such that $\lim_{j\to\infty} x_{k_j} = c$. Take any $\varepsilon \in \mathbb{R}_+$ and $k \in \mathbb{N}$. Then, there is some $j_0 \in \mathbb{N}$ such that $\forall j \in \mathbb{N}_{\ge j_0}$, $x_{k_j} \in N(c;\varepsilon)$. Let $k_0 := \min\{\, k_j \in \mathbb{N} \mid j > j_0 \text{ and } k_j > k \,\}$. Then, $x_{k_0} \in N(c;\varepsilon)$ and $k_0 > k$. Therefore, c is a cluster point. □

Theorem 1.3.8

Any bounded sequence $\{x_k\}$ has a cluster point.

Proof. If the set $S := \{\, x_k \mid k \in \mathbb{N} \,\}$ is finite, there is some x_{k_0} that is repeated infinitely. Then, x_{k_0} is surely a cluster point.

Now, suppose S is infinite. Then, by Theorem 1.2.3, S has a limit point ℓ. To prove ℓ is a cluster point, take any $\varepsilon \in \mathbb{R}_+$ and $k \in \mathbb{N}$.

Let $S' := \{\, x_{k'} \mid k' \in \mathbb{N}_{>k} \,\}$. We first claim that ℓ is a limit point of S'. Take any $\varepsilon' \in \mathbb{R}_+$ less than $m = \min\{\, |x_{k'} - \ell| \in \mathbb{R}_+ \mid k' \in \mathbb{N}_{\le k} \,\}$. ($m$ exists due to Theorem 1.1.3.) Then, $S' \cap N'(\ell;\varepsilon') = S \cap N'(\ell;\varepsilon')$ is nonempty. Therefore, ℓ is a limit point of S' by Theorem 1.2.1.

Finally, we can say $S' \cap N(\ell;\varepsilon)$ is nonempty. This implies there is some $k_0 \in \mathbb{Z}_{>k}$ such that $x_{k_0} \in N(\ell;\varepsilon)$. Therefore, ℓ is a cluster point of $\{x_k\}$. □

Corollary 1.3.2

If a sequence has no cluster point, then the sequence is unbounded.

Corollary 1.3.3

Any bounded sequence converges if and only if it has exactly one cluster point.

Corollary 1.3.4

A sequence $\{x_k\}$ diverges if and only if at least one of the following conditions holds.

- $\{x_k\}$ has two or more cluster points.
- $\{x_k\}$ is unbounded.

Proof. ($\Rightarrow$) Suppose $\{x_k\}$ is diverging and bounded. By Theorem 1.3.8, it has at least one cluster point. Also, if it had exactly one cluster point, it would converge by Corollary 1.3.3.

($\Leftarrow$) It is direct from Theorem 1.3.2 and Corollary 1.3.1. □

Theorem 1.3.9

A sequence $\{x_k\}$ converges if and only if every subsequence of $\{x_k\}$ converges.

Proof. ($\Rightarrow$) Take any subsequence $\{x_{k_i}\}_{i=1}^{\infty}$ of $\{x_k\}_{k=1}^{\infty}$ and $\varepsilon \in \mathbb{R}_+$. There is $i_0 \in \mathbb{N}$ such that $\forall i \in \mathbb{N}_{\geq i_0}$, $|x_i| < \varepsilon$. Since $k_i \geq i$ for all natural number i, $\forall i \in \mathbb{N}_{\geq i_0}$, $|x_{k_i}| < \varepsilon$.

($\Leftarrow$) $\{x_k\}$ is a subsequence of itself. □

Definition 1.3.4: Limit Superior and Inferior

Let $\{x_k\}$ be a sequence and C be a set of cluster points of the sequence.

- $$\limsup x_k \triangleq \begin{cases} \sup C & \text{if } \{x_k\} \text{ is bounded} \\ \infty & \text{if } \{x_k\} \text{ is unbounded above} \\ \sup C & \text{if } \{x_k\} \text{ is bounded above but unbounded below and } C \neq \varnothing \\ -\infty & \text{if } \{x_k\} \text{ is bounded above but unbounded below and } C = \varnothing \end{cases}$$
  is called *limit superior* of $\{x_k\}$.
- $$\liminf x_k \triangleq \begin{cases} \inf C & \text{if } \{x_k\} \text{ is bounded} \\ -\infty & \text{if } \{x_k\} \text{ is unbounded below} \\ \inf C & \text{if } \{x_k\} \text{ is bounded below but unbounded above and } C \neq \varnothing \\ \infty & \text{if } \{x_k\} \text{ is bounded below but unbounded above and } C = \varnothing \end{cases}$$
  is called *limit inferior* of $\{x_k\}$.

Note:-

In all cases, $\liminf x_k \leq \limsup x_k$.

Theorem 1.3.10

- If $\mu = \limsup x_k$ is finite, then μ is in C. ($\mu = \max C$)
- If $\nu = \liminf x_k$ is finite, then ν is in C. ($\nu = \min C$)

Proof. Suppose $\mu = \limsup x_k$ is finite. Take any $\varepsilon \in \mathbb{R}_+$ and $k \in \mathbb{N}$. The finiteness of μ implies $\mu = \sup C$. By Theorem 1.1.4, there is some $c \in C$ such that $\mu - \varepsilon < c \leq \mu$. If $c = \mu$, then we are done. So let $c < \mu$.

Choose any positive ε_1 less than $\min\{c-(\mu-\varepsilon), \mu-c\}$ so $N(c;\varepsilon_1) \subseteq N(\mu;\varepsilon)$. Then, $\{k \in \mathbb{N} \mid x_k \in N(\mu;\varepsilon)\}$ is infinite since it has an infinite set $\{k \in \mathbb{N} \mid x_k \in N(c;\varepsilon_1)\}$ as its subset. (See Lemma 1.3.1.)

The second part can be proven analogously. □

Theorem 1.3.11

Let $\{x_k\}$ be any bounded sequence in $\mathbb{R}$. Fix any $\varepsilon \in \mathbb{R}_+$.

- Let $\mu = \limsup x_k$.
  - $\exists k_0 \in \mathbb{N}$, $\forall k \in \mathbb{N}_{\geq k_0}$, $x_k < \mu + \varepsilon$.
  - $\forall k \in \mathbb{N}$, $\exists k_1 \in \mathbb{N}_{>k}$, $x_{k_1} > \mu - \varepsilon$.
- Let $\nu = \liminf x_k$.
  - $\exists k_0 \in \mathbb{N}$, $\forall k \in \mathbb{N}_{\geq k_0}$, $x_k > \nu - \varepsilon$.
  - $\forall k \in \mathbb{N}$, $\exists k_1 \in \mathbb{N}_{>k}$, $x_{k_1} < \nu + \varepsilon$.

Proof. Take any $\varepsilon \in \mathbb{R}_+$. Then, $\{k \in \mathbb{N} \mid x_k \geq \mu + \varepsilon\}$ is finite. If it were not, then there would be a cluster point larger than μ since Theorem 1.3.8 implies the existence of a cluster point in a subsequence of $\{x_k\}$ which is composed of x_k's not smaller than $\mu + \varepsilon$. Therefore, if $k_0 := \max\{k \in \mathbb{N} \mid x_k \geq \mu + \varepsilon\} + 1$, then $x_k < \mu + \varepsilon$ for all k not smaller than k_0.

Also, since μ is a cluster point by Theorem 1.3.10, $\forall k \in \mathbb{N}$, $\exists k_1 \in \mathbb{N}_{>k}$, $x_{k_1} > \mu - \varepsilon$. (See Lemma 1.3.1.)

The second part can be proven analogously. □

Theorem 1.3.12

Let $\{x_k\}$ be any sequence in $\mathbb{R}$.

(i) $\{x_k\}$ converges to x_0 if and only if $\liminf x_k = \limsup x_k = x_0$.

(ii) $\{x_k\}$ diverges if and only if one of the following holds.
- Either $\liminf x_k$ or $\limsup x_k$ is infinite.
- Both $\liminf x_k$ or $\limsup$ are finite and $\liminf x_k < \limsup x_k$.

Proof.

(i) ($\Rightarrow$) $C = \{x_0\}$, therefore $\liminf x_k = \limsup x_k = x_0$.
($\Leftarrow$) Take any $\varepsilon \in \mathbb{R}_+$. There are natural numbers k_1 and k_2 such that $\forall k \in \mathbb{N}_{\geq k_1}$, $x_k < x_0 + \varepsilon$ and $\forall k \in \mathbb{N}_{\geq k_0}$, $x_k > x_0 - \varepsilon$. Then, for all natural number k not smller than $k_0 := \max\{k_1, k_2\}$, $x_0 - \varepsilon < x_k < x_0 + \varepsilon$ holds.

(ii) If it is not $\liminf x_k = \limsup x_k \in \mathbb{R}$, then it is either "One of them is infinite." or "They are both finite but they are different."

□

Exercise 1.3.1

Let $\{x_k\}$ be a bounded sequence of positive numbers. For each $k \in \mathbb{N}$ define $y_k := x_{k+1}/x_k$ and $z_k := (x_k)^{1/k}$. Prove that $\liminf y_k \leq \liminf z_k \leq \limsup z_k \leq \limsup y_k$.

Solution: ($\liminf y_k \leq \liminf z_k$) Let $L := \liminf y_k$. Now, we claim that

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists k_0 \in \mathbb{N},\ \forall\, k \in \mathbb{N}_{\geq k_0},\ z_k > L - \varepsilon.$$

If $L = 0$, then it is done. Therefore, suppose $L > 0$. To prove this, take any $\varepsilon \in \mathbb{R}_+$ smaller than L. Then, there is some $k_1 \in \mathbb{N}$ such that $y_k > L - \varepsilon/2$ for all k not smaller than k_1 by

Theorem 1.3.11. Then, for all $k \in \mathbb{N}_{\geq k_1}$, $x_k > (L - \varepsilon/2)^{k-k_1} x_{k_1}$, which is equivalent to

$$z_k = x_k^{1/k} > \left(L - \frac{\varepsilon}{2}\right)\left[\left(L - \frac{\varepsilon}{2}\right)^{-k_1} x_{k_1}\right]^{1/k}.$$

Since $\lim_{k\to\infty} \left[(L-\varepsilon/2)^{-k_1} x_{k_1}\right]^{1/k} = 1$, there is some $k_2 \in \mathbb{N}$ such that

$$\left[\left(L - \frac{\varepsilon}{2}\right)^{-k_1} x_{k_1}\right]^{1/k} > 1 - \frac{\varepsilon/2}{L - \varepsilon/2} = \frac{L-\varepsilon}{L-\varepsilon/2}.$$

for all $k \in \mathbb{N}_{\geq k_2}$. Thus, for every natural number k not smaller than $\max\{k_1, k_2\}$,

$$z_k > \left(L - \frac{\varepsilon}{2}\right)\left[\left(L - \frac{\varepsilon}{2}\right)^{-k_1} x_{k_1}\right]^{1/k} > \left(L - \frac{\varepsilon}{2}\right) \cdot \frac{L-\varepsilon}{L-\varepsilon/2} = L - \varepsilon.$$

The claim is now proven.

For the main proof, assume that $\liminf z_k < L$ for the sake of contradiction. Take $\varepsilon_0 := (L - \liminf z_k)/2$. Then, by the previous claim, $\exists k_3 \in \mathbb{N}$, $\forall k \in \mathbb{N}_{\geq k_3}$, $z_k > L - \varepsilon_0 = (L + \liminf z_k)/2$.

Nevertheless, by Theorem 1.3.11, there is some $k_4 \in \mathbb{N}_{>k_3}$ such that $z_{k_4} < \liminf z_k + \varepsilon_0 = (L + \liminf z_k)/2$, which is a contradiction.

$\limsup z_k \leq \limsup y_k$ can be proven analogously.

$\square$

1.4 Cauchy Sequences

Definition 1.4.1: Cauchy Seqeuence

A seqeuence $\{x_k\}$ in $\mathbb{R}$ is called a *Cauchy sequence* if

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists k_0 \in \mathbb{N},\ \forall k, m \in \mathbb{N}_{\geq k_0},\ |x_k - x_m| < \varepsilon.$$

Theorem 1.4.1

If $\{x_k\}$ is a convergent sequence of real numbers, then $\{x_k\}$ is a Cauchy sequence.

Proof. Let $x_0 := \lim_{k\to\infty} x_k$. Take any $\varepsilon \in \mathbb{R}_+$. Then, there is some $k_0 \in \mathbb{N}$ such that $|x_k - x_0| < \varepsilon/2$ for all $k \in \mathbb{N}$ not smaller than k_0. Then, for all $k, m \in \mathbb{N}$ greater than k_0, $|x_k - x_m| \leq |x_k - x_0| + |x_k - x_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. $\square$

Theorem 1.4.2

If $\{x_k\}$ is a Cauchy sequence, then $\{x_k\}$ is bounded.

Proof. There is $k_0 \in \mathbb{N}$ such that $|x_k - x_m| < 1$ for all $k, m \in \mathbb{N}_{\geq k_0}$. It implies that $|x_k - x_{k_0}| < 1$, for all $k \in \mathbb{N}_{\geq k_0}$, which impliees $|x_k| < |x_{k_0}| + 1$. Therefore, for all $k \in \mathbb{N}$, $|x_k| \leq \max\left\{|x_1|, |x_2|, \cdots, |x_{k_0}|, |x_{k_0}| + 1\right\}$. $\square$

Theorem 1.4.3

A Cauchy sequence has exactly one cluster point.

Proof. Since a Cauchy sequence is bounded, it has at least one cluster point by Theorem 1.3.8. So, we should prove that the sequence does not have more than one cluster point. Assume c_1 and c_2 are cluster points for the sake of contradiction. Let $\varepsilon := |c_1 - c_2|/3$. Choose $k_0 \in \mathbb{N}$ such that $\forall k, m \in \mathbb{N}_{\geq k_0}$, $|x_k - x_m| < \varepsilon$. Also, there are $k_1, k_2 \in \mathbb{N}_{>k_0}$ such that $|x_{k_1} - c_1| < \varepsilon$ and $|x_{k_2} - c_2| < \varepsilon$. Note that $|c_1 - c_2| \leq |c_1 - x_{k_1}| + |x_{k_1} - x_{k_2}| + |x_{k_2} - c_2|$. Nevertheless, then

$$\begin{aligned}\varepsilon > |x_{k_1} - x_{k_2}| &\geq |c_1 - c_2| - |c_1 - x_{k_1}| - |c_2 - x_{k_2}| \\ &> 3\varepsilon - \varepsilon - \varepsilon = \varepsilon,\end{aligned}$$

which is a contradiction. □

Theorem 1.4.4 Cauchy Completeness of $\mathbb{R}$

A sequence in $\mathbb{R}$ is convergent if and only if it is a Cauchy sequence.

Proof. By Corollary 1.3.3, a Cauchy sequence is convergent since it is bounded (Theorem 1.4.2) and has exactly one cluster point (Theorem 1.4.3). A convergent sequence in $\mathbb{R}$ is Cauchy. (Theorem 1.4.1) □

Definition 1.4.2: Cauchy Completeness

A set X is said to be *Cauchy complete* if every Cauchy sequence in X converges to a point of X.

Example 1.4.1

$\mathbb{R}$ is Cauchy complete.

Definition 1.4.3: Contractive Sequence

A sequence $\{x_k\}$ is said to be *contractive* if there exists a constant C, with $0 < C < 1$, such that

$$\forall k \in \mathbb{N}_{>1},\ |x_{k+1} - x_k| \leq C|x_k - x_{k-1}|.$$

Theorem 1.4.5

Any contractive sequence in $\mathbb{R}$ is a Cauchy sequence.

Proof. Suppose $0 < C < 1$ and $\forall k \in \mathbb{N}_{>1}$, $|x_{k+1} - x_k| \leq C|x_k - s_{k-1}|$. If it is trivial when $|x_2 - x_1| = 0$, so supose $|x_2 - x_1| \neq 0$. By induction, $\forall k \in \mathbb{N}_{>1}$, $|x_{k+1} - x_k| \leq C^{k-1}|x_2 - x_1|$.

To prove $\{x_k\}$ is a Cauchy sequence, take any $\varepsilon \in \mathbb{R}_+$. Since $\lim_{k\to\infty} C^{k-1} = 0$,

$$\exists k_0 \in \mathbb{N},\ \forall k \in \mathbb{N}_{\geq k_0},\ C^{k-1} < \frac{(1-C)\varepsilon}{|x_2 - x_1|}.$$

Then, for any $k, m \in \mathbb{N}$ with $k_0 \le m < k$,

$$\begin{aligned}|x_k - x_m| &= \left|\sum_{j=m}^{k-1}(x_{j+1} - x_j)\right| \le \sum_{j=m}^{k-1}|x_{j+1} - x_j| \\ &\le \sum_{j=m}^{k-1} C^{j-1}|x_2 - x_1| = C^{m-1}|x_2 - x_1| \sum_{j=0}^{k-m-1} C^j \\ &= C^{m-1}|x_2 - x_1|\frac{1 - C^{k-m}}{1 - C} < \frac{C^{m-1}}{1 - C}|x_2 - x_1| \\ &< \frac{(1-C)\varepsilon}{|x_2 - x_1|} \cdot \frac{1}{1 - C}|x_2 - x_1| = \varepsilon.\end{aligned}$$

□

1.5 The Algebra of Convergent Series

Theorem 1.5.1

Let $\{x_k\}$ and $\{y_k\}$ be convergent sequences in $\mathbb{R}$ and $\lim_{k\to\infty} x_k = x_0$ and $\lim_{k\to\infty} y_k = y_0$.

- $\lim_{k\to\infty}(x_k + y_k) = x_0 + y_0$
- $\lim_{k\to\infty} x_k y_k = x_0 y_0$
- $\lim_{k\to\infty} \dfrac{y_k}{x_k} = \dfrac{y_0}{x_0}$ if $x_0 \neq 0$.

Theorem 1.5.2

Let $\{x_k\}$ and $\{y_k\}$ be convergent sequences in $\mathbb{R}$ and $\lim_{k\to\infty} x_k = x_0$. Then, if $r \in \mathbb{Q}$, then

$$\lim_{k\to\infty} x_k^r = x_0^r.$$

Nevertheless, we requre $x_0 \neq 0$ if $r < 0$.

1.6 Cardinality

Definition 1.6.1: Dense Set

We say a subset S of T is dense in T if every neighborhood of any point $x \in T$ contains points of S.

Theorem 1.6.1

- $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are countably infinite.
- $\mathbb{R}$ is uncountable.
- $\mathbb{Q}$ is dense in $\mathbb{R}$.

Chapter 2

Euclidean Spaces

2.1 Euclidean n-Space

Definition 2.1.1: Inner Product

The *inner product* of two vectors $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ in $\mathbb{R}^n$ is

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{n} x_j y_j.$$

Theorem 2.1.1

If $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are arbitrary vectors in $\mathbb{R}^n$ and if a and b are real numbers, then the following hold:

(i) The inner product is *additive* in both its variables:

$$\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$$
$$\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$$

(ii) The inner product is *symmetric*: $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.

(iii) The inner product is *homogeneous* in both its variables: $\langle a\mathbf{x}, b\mathbf{y} \rangle = ab\langle \mathbf{x}, \mathbf{y} \rangle$.

Definition 2.1.2: Euclidean Norm

The *Euclidean norm* of a vector $\mathbf{x}$ in $\mathbb{R}^n$ is

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

Theorem 2.1.2 The Cauchy-Schwarz Inequality

If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \le \|\mathbf{x}\| \cdot \|\mathbf{y}\|.$$

Proof. For any $t \in \mathbb{R}$, $0 \le \|t\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 t^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle t + \|\mathbf{y}\|^2$. Thus, the discriminant $|\langle \mathbf{x}, \mathbf{y} \rangle|^2 - \|\mathbf{x}\|^2 \|\mathbf{y}\|^2$ is nonpositive. □

Theorem 2.1.3

For vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and any $c \in \mathbb{R}$, the Euclidean norm has the following properties.

(i) $||\mathbf{x}|| \geq 0$; $||\mathbf{x}|| = 0$ if and only if $\mathbf{x} = \mathbf{0}$. (*Positive Definiteness*)

(ii) $||c\mathbf{x}|| = |c| \cdot ||\mathbf{x}||$. (*Absolute Homogeneity*)

(iii) $||\mathbf{x}+\mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$. (*Subadditivity*)

Proof of (iii).

$$\begin{aligned} 0 \leq ||\mathbf{x}+\mathbf{y}||^2 &= ||\mathbf{x}||^2 + 2\langle \mathbf{x}, \mathbf{y}\rangle + ||\mathbf{y}||^2 \\ &\leq ||\mathbf{x}||^2 + 2||\mathbf{x}||||\mathbf{y}|| + ||\mathbf{y}||^2 = (||\mathbf{x}|| + ||\mathbf{y}||)^2 \end{aligned}$$

□

Definition 2.1.3: Norm

A *norm* on $\mathbb{R}^n$ is any function $n\colon \mathbb{R}^n \to \mathbb{R}$ that is positive definite, absolutely homogeneous, and subadditive.

Definition 2.1.4: Metric

A *metric* on $\mathbb{R}^n$ is a function from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ having the following properties.

(i) $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{y}) \geq 0$; $d(\mathbf{x}, \mathbf{y}) = 0 \iff \mathbf{x} = \mathbf{y}$. (*Positive Definiteness*)

(ii) $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{y})$. (*Symmetry*)

(iii) $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$. (*The Triangle Inequality*)

Definition 2.1.5: Euclidean Metric

The *Euclidean metric* on $\mathbb{R}^n$ is defined by

$$d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = \left[\sum_{j=1}^{n} (x_j - y_j)^2\right]^{1/2}.$$

Theorem 2.1.4

The Euclidean metric is a metric on $\mathbb{R}^n$.

Definition 2.1.6: Orthogonality

Two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are said to be *orthogonal* if $\langle \mathbf{x}, \mathbf{y}\rangle = 0$.

Definition 2.1.7: Neighborhood and Deleted Neighborhood

A *neighborhood* $N(\mathbf{x}; r)$ or $\mathbf{x} \in \mathbb{R}^n$ with radius r is the set

$$N(\mathbf{x}; r) = \{\mathbf{y} \in \mathbb{R}^n \mid ||\mathbf{x} - \mathbf{y}|| < r\}.$$

A *deleted neighborhood* $N'(\mathbf{x}, r)$ of $\mathbf{x}$ is $N'(\mathbf{x}; r) = N(\mathbf{x}; r) \setminus \{\mathbf{x}\}$.

Definition 2.1.8: Limit Point

Let S be nonempty subset of $\mathbb{R}^n$. We say that $\mathbf{x}$ is a *limit point* of S if

$$\forall \varepsilon \in \mathbb{R}_+, \; N'(\mathbf{x}; \varepsilon) \cap S \neq \varnothing.$$

Theorem 2.1.5

$\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

Proof. Take any $\mathbf{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}_+$. For each $j = 1, 2, \cdots, n$, choose a rational $x_j \in N(y_j; \varepsilon/\sqrt{n})$ and form $\mathbf{x} \triangleq (x_1, x_2, \cdots, x_n) \in \mathbb{Q}^n$. Then,

$$\|\mathbf{x} - \mathbf{y}\|^2 = \sum_{j=1}^{n} (x_j - y_j)^2 < n(\varepsilon/\sqrt{n})^2 = \varepsilon^2.$$

Therefore $\mathbf{y}$ is a limit point of $\mathbb{Q}^n$. $\square$

Definition 2.1.9: Boundedness

A subset S of $\mathbb{R}^n$ is said to be *bounded* if

$$\exists M \in \mathbb{R}_+,\ \forall \mathbf{x} \in S,\ \|\mathbf{x}\| \le M.$$

2.1.1 Sequences in $\mathbb{R}^n$

Definition 2.1.10: Cluster Point

$\mathbf{c} \in \mathbb{R}^n$ is a *cluster point* of the sequence $\{\mathbf{x}_k\}$ if,

$$\forall (\varepsilon, k) \in \mathbb{R}_+ \times \mathbb{N},\ \exists k_1 \in \mathbb{N}_{>k},\ \mathbf{x}_{k_1} \in N(\mathbf{c}; \varepsilon).$$

Definition 2.1.11: Convergence and Divergence of a Sequence

The sequnce $\{\mathbf{x}_k\}$ *converges* to $\mathbf{x}_0$ and $\mathbf{x}_0$ is the *limit* of $\{\mathbf{x}_k\}$ if,

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists k_0 \in \mathbb{N},\ \forall k \in \mathbb{N}_{\ge k_0},\ \mathbf{x}_k \in N(\mathbf{x}_0; \varepsilon).$$

We write $\lim_{k \to \infty} \mathbf{x}_k = \mathbf{x}_0$. If there is no such $\mathbf{x}_0$, then $\{\mathbf{x}_k\}$ *diverges*.

Theorem 2.1.6

Let $\{\mathbf{x}_k\} = (x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)})$ for each $k \in \mathbb{N}$. Let $\mathbf{x}_0 = (x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)})$. The sequence $\{\mathbf{x}_k\}$ converges to $\mathbf{x}_0$ if and only if, for each $j \in [n]$, the sequence $\{x_j^{(k)}\}$ converges to $\{x_j^{(0)}\}$.

Proof. ($\Rightarrow$) Take any $\varepsilon \in \mathbb{R}_+$. There there is $k_0 \in \mathbb{N}$ such that

$$\forall k \in \mathbb{N}_{\ge k_0}, \mathbf{x}_k \in N(\mathbf{x}_0; \varepsilon).$$

Then, for each $j \in [n]$,

$$\left(x_j^{(k)} - x_0^{(k)}\right)^2 \le \sum_{i=1}^{n} \left(x_i^{(k)} - x_0^{(k)}\right)^2 = \|\mathbf{x}_k - \mathbf{x}_0\|^2 < \varepsilon.$$

($\Leftarrow$) Take any $\varepsilon \in \mathbb{R}_+$. Then, for each $j \in [n]$, there is some $k_j \in \mathbb{N}$ such that

$$\forall k \in \mathbb{N}_{\ge k_j},\ x_j^{(k)} \in N(x_0^{(k)}; \varepsilon/\sqrt{n}).$$

Then, for all natural number k not smaller than $\max_{j\in[n]} k_j$,

$$\|\mathbf{x}_k - \mathbf{x}_0\|^2 = \sum_{j=1}^{n} \left(x_j^{(k)} - x_0^{(k)}\right)^2 < n(\varepsilon/\sqrt{n})^2 = \varepsilon^2.$$

□

Definition 2.1.12: Cauchy Seqeunce

A seqeunce $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$ is called a *Cauchy sequence* if

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists k_0 \in \mathbb{N},\ \forall k, m \in \mathbb{N}_{\geq k_0},\ \|\mathbf{x}_k - \mathbf{x}_m\| < \varepsilon.$$

Theorem 2.1.7 Cauchy's Completeness Theorem in $\mathbb{R}^n$

A sequence $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$ is Cauchy if and only if it converges. $\mathbb{R}^n$ is Cauchy complete.

Proof. ($\Leftarrow$) The proof if similar to Theorem 1.4.1.

($\Rightarrow$) Let some Cauchy sequence $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$ be given. Take any $\varepsilon \in \mathbb{R}_+$. There is some $k_0 \in \mathbb{N}$ such that for every natural number k and m not smaller than k_0, $\|\mathbf{x}_k - \mathbf{x}_m\| < \varepsilon$. Then, for each $j \in [n]$, $|x_j^{(k)} - x_j^{(m)}| \leq \|\mathbf{x}_k - \mathbf{x}_m\| < \varepsilon$, which implies each $\{x_j^{(k)}\}_{k\in\mathbb{N}}$ is Cauchy. By Theorem 1.4.4, $\{x_j^{(k)}\}_{j\in\mathbb{N}}$ converges to some number $x_j^{(0)}$. Then, Theorem 2.1.6 ensures that $\lim_{k\to\infty} \mathbf{x}_k = (x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)})$. □

Theorem 2.1.8 The Generalized Bolzano–Weierstrass Theorem

Every bounded infinite set in $\mathbb{R}^n$ has a limit point in $\mathbb{R}^n$.

Proof. Suppose that S is any bounded, infinite set in $\mathbb{R}^n$. Being bounded, S is contained in some n-cube $C(2M) = [-M, M]^n$ centered at $\mathbf{0}$. Construct $C_1, C_2, \cdots$ as following.

- $C_1 \triangleq C(2M) = [a_1^{(1)}, b_1^{(1)}] \times \cdots \times [a_n^{(1)}, b_n^{(1)}]$
  - Note that $C_1 \cap S = S$ is infinite.
- For each $k \in \mathbb{N}$, C_{k+1} is any cube of the form $[a_1^{(k+1)}, b_1^{(k+1)}] \times \cdots \times [a_n^{(k+1)}, b_n^{(k+1)}]$ where each $[a_j^{(k+1)}, b_j^{(k+1)}]$ is either $[a_j^{(k)}, (a_j^{(k)} + b_j^{(k)})/2]$ or $[(a_j^{(k)} + b_j^{(k)})/2, b_j^{(k)}]$ so that $C_{k+1} \cap S$ is infinite.
  - This is possible since there is at least one cube among 2^n possible choices that $C_{k+1} \cap S$ is infinite.

Then, the main diagonal d_k of C_k equals to $Mn^{1/2}/2^{k-2}$. Also, note that $C_k \supseteq C_{k+1}$ for all $k \in \mathbb{N}$.

Now, we may construct a sequence $\{\mathbf{x}_k\}_{k\in\mathbb{N}}$ as following.

- $\mathbf{x}_1$ is any element in $C_1 \cap S$.
- For each $k \in \mathbb{N}$, $\mathbf{x}_{k+1}$ is arbitrarily taken from $C_{k+1} \cap S \setminus \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\}$.

We claim that $\{\mathbf{x}_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence. To show this, take any $\varepsilon \in \mathbb{R}_+$. There is some $k_0 \in \mathbb{N}$ such that $d_{k_0} = Mn^{1/2}/2^{k_0-2} < \varepsilon$ by Theorem 1.1.1. Then, for all $k, m \in \mathbb{N}_{\geq k_0}$, $\|\mathbf{x}_k - \mathbf{x}_m\| \leq d_{k_0} < \varepsilon$. Therefore, since $\{\mathbf{x}_k\}$ is Cauchy, and therefore convergent by Theorem 2.1.7.

Clearly, $\mathbf{x}_0 \triangleq \lim_{k\to\infty} \mathbf{x}_k$ is a limit point of S since any deleted neighborhood $N'(\mathbf{x}_0)$ of $\mathbf{x}_0$ intersects infinitely many points with $\{\mathbf{x}_k\}_{k\in\mathbb{N}} \subseteq S$. □

Definition 2.1.13: Subsequence

Let $\{\mathbf{x}_k\}$ be any sequence in $\mathbb{R}^n$. Choose any strictly monotone increasing sequnce $k_1 < k_2 < k_3 < \cdots$ of natural numbers. For each $j \in \mathbb{N}$, let $\mathbf{y}_j := \mathbf{x}_{k_j}$. The sequence $\{\mathbf{y}_j\}_{j=1}^{\infty}$ is called an *subsequence* of $\{\mathbf{x}_k\}$.

Theorem 2.1.9

The point $\mathbf{c}$ is a cluster point of $\{\mathbf{x}_k\}$ if and only if there exists a subsequence of $\{\mathbf{x}_k\}$ that converges to $\mathbf{c}$.

Proof. Analogous to Theorem 1.3.7. □

Theorem 2.1.10

Any bounded sequence $\{\mathbf{x}_k\}$ has a cluster point.

Proof. Analogous to Theorem 1.3.8. □

Corollary 2.1.1

If a sequence in $\mathbb{R}^n$ has no cluster point, then the sequence is unbounded.

Corollary 2.1.2

Any bounded sequence in $\mathbb{R}^n$ converges if and only if it has exactly one cluster point.

Corollary 2.1.3

A sequence $\{\mathbf{x}_k\}$ diverges if and only if at least one of the following conditions holds.

- $\{\mathbf{x}_k\}$ has two or more cluster points.
- $\{\mathbf{x}_k\}$ is unbounded.

2.2 Open and Closed Sets

Definition 2.2.1: Interior/Boundary Point and Open/Closed Set

Let S be any subset of $\mathbb{R}^n$ and let $\mathbf{x}$ be any point in $\mathbb{R}^n$.

(i) $\mathbf{x}$ is an *interior point* of S if $\exists r \in \mathbb{R}_+, N(\mathbf{x}; r) \subseteq S$.

(ii) If every point of S is an interior point of S, then S is said to be *open*.

(iii) We call $\mathbf{x}$ is a *boundary point* of S if $\forall r \in \mathbb{R}_+, N(\mathbf{x}; r) \cap S \neq \varnothing \wedge N(\mathbf{x}; r) \setminus S \neq \varnothing$.

(iv) If S continas all its boundary points, then S is said to be *closed*.

Definition 2.2.2

Let $S \subseteq \mathbb{R}^n$.

(i) The *interior* of S, denoted $\mathring{S}$, is the set of all interior points of S.

(ii) The *boundary* of S, denoted $\operatorname{bd} S$, is the set of all boundary points of S.

(iii) The *derived set* of S, denoted S', is the set of all limit points of S.

(iv) The *closure* of S, denoted $\overline{S}$, is the union of S and S'.

(v) The *complement* of S, denoted S^c, is the set $\mathbb{R}^n \setminus S$.

Note:-

- For $S \subseteq \mathbb{R}^n$, $\mathring{S} \subseteq S \subseteq \overline{S}$.
- For $S \subseteq \mathbb{R}^n$, S is open if and only if $\mathring{S} = S$.
- For $S \subseteq \mathbb{R}^n$, $\mathring{S}$ is open.

Theorem 2.2.1

The union of any collection of open sets in $\mathbb{R}^n$ is open. The intersection of any finite collection of open sets in $\mathbb{R}^n$ is also open.

Proof. To prove the first assertion, suppose that $\{\, U_\alpha \mid \alpha \in J \,\}$ is any collection of open sets in $\mathbb{R}^n$. Let $U \triangleq \bigcup_{\alpha\in J} U_\alpha$. Take any $\mathbf{x} \in U$. Then, there is some $\alpha_0 \in J$ such that $\mathbf{x} \in U_{\alpha_0}$. Since U_{α_0} is open, there is some neighborhood $N(\mathbf{x};\varepsilon)$ such that $N(\mathbf{x};\varepsilon) \subseteq U_{\alpha_0}$, which, in turn, $N(\mathbf{x};\varepsilon) \subseteq U$. Therefore, $\mathbf{x}$ is an interior point of U; U is open.

To prove the second assertion, let U be the intersection of any finite collection $\{\, U_1, U_2, \cdots, U_k \,\}$ of open sets and take any $\mathbf{x} \in U$. For each $j \in [k]$, since $\mathbf{x} \in U_j$, there is some $r_j \in \mathbb{R}_+$ such that $N(\mathbf{x}; r_j) \subseteq U_j$. Then, take $r_0 \triangleq \min_{j\in[k]} r_j \in \mathbb{R}_+$. Since, for all $j \in [k]$, $N(\mathbf{x}; r_0) \subseteq U_j$, it is implied that $N(\mathbf{x}; r_0) \subseteq U$. Therefore, $\mathbf{x}$ is an interior point of U; U is open. □

Note:-

Intersection of infinitely many open sets may fail to be open. For instance, consider

$$U_k \triangleq N(\mathbf{0}; 1/k),$$

for each $k \in \mathbb{N}$. Then, $\bigcap_{k\in\mathbb{N}} U_k = \{\mathbf{0}\}$, which is not open.

Theorem 2.2.2

A set $C \subseteq \mathbb{R}^n$ is closed if and only if C^{c} is open.

Proof. ($\Rightarrow$) Take any $\mathbf{x} \in C^{\mathrm{c}}$. Since C is closed and contains all of its boundary points, $\mathbf{x}$ is not a boundary point of C. Therefore, there is some neighborhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x})C = \varnothing$ or $N(\mathbf{x}) \cap C^{\mathrm{c}} = \varnothing$. The second case is not possible since $\mathbf{x} \in N(\mathbf{x}) \cap C^{\mathrm{c}}$. Therefore, $N(\mathbf{x}) = \varnothing$, which implies $N(\mathbf{x}) \subseteq C^{\mathrm{c}}$; $\mathbf{x}$ is an interior point of C^{c}. Therefore, C^{c} is open.

($\Leftarrow$) Take any bounddary point $\mathbf{x}$ of C. Assume $\mathbf{x} \in C^{\mathrm{c}}$ for the sake of contradiction. Since C^{c} is open, there is a neighborhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \subseteq C^{\mathrm{c}}$. However, that implies $N(\mathbf{x}) \cap C = \varnothing$, which contradicts $\mathbf{x}$ is a boundary point of C. Therefore, $\mathbf{x} \in C$; C contains all of its boundary points. □

Theorem 2.2.3

The intersection of any collection of closed sets in $\mathbb{R}^n$ is closed. The union of any finite collection of closed sets in $\mathbb{R}^n$ is also closed.

Proof. To prove the first assertion, let $\{C_\alpha\}_{\alpha\in J}$ be any collection of closed sets in $\mathbb{R}^n$. Then, each ${C_\alpha}^{\mathrm{c}}$ is open by Theorem 2.2.2, and thus $\bigcup_{\alpha\in J} {C_\alpha}^{\mathrm{c}}$ is open by Theorem 2.2.1. Its complement $\left(\bigcup_{\alpha\in J} {C_\alpha}^{\mathrm{c}}\right)^{\mathrm{c}}$ is closed by Theorem 2.2.2. And note that $\left(\bigcup_{\alpha\in J} {C_\alpha}^{\mathrm{c}}\right)^{\mathrm{c}} = \bigcap_{\alpha\in J} C_\alpha$ by De Morgan's law.

To prove the second assertion, let $\{\, C_1, C_2, \cdots, C_k \,\}$ be a finite collection of closed sets in $\mathbb{R}^n$. Then, each ${C_i}^{\mathrm{c}}$ is open by Theorem 2.2.2, and thus $\bigcap_{i=1}^{k} {C_i}^{\mathrm{c}}$ is open by Theorem 2.2.1. Its complement $\left(\bigcap_{i=1}^{k} {C_i}^{\mathrm{c}}\right)^{\mathrm{c}}$ is closed by Theorem 2.2.2. And note that $\left(\bigcap_{i=1}^{k} {C_i}^{\mathrm{c}}\right)^{\mathrm{c}} = \bigcup_{i=1}^{k} C_i$ by De Morgan's law. □

Theorem 2.2.4

$C \subseteq \mathbb{R}^n$ is closed if and only if $C' \subseteq C$.

Proof. ($\Rightarrow$) Let $\mathbf{x} \in C'$. Assume $\mathbf{x} \in C^c$ for the sake of contradiction. Since C^c is open by Theorem 2.2.2, there is a neighborhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \subseteq C^c$. Such $N(\mathbf{x})$ satisfies $N(\mathbf{x}) \cap C = \varnothing$, which contradicts $\mathbf{x} \in C'$. Therefore, $\mathbf{x} \in C$; C contains all its limit points.

($\Leftarrow$) It is enough to prove C^c is open by Theorem 2.2.2. Take any $\mathbf{x} \in C^c$. $\mathbf{x}$ is not a limit point of C by the hypothesis. Therefore, there is a deleted neighborhood $N'(\mathbf{x})$ of $\mathbf{x}$ such that $N'(\mathbf{x}) \cap C = \varnothing$. Then, $N'(\mathbf{x}) \subseteq C^c$, and thus $N(\mathbf{x}) \subseteq C^c$, which implies $\mathbf{x}$ is an interior point of C^c. Thus, C^c is open. $\square$

Corollary 2.2.1

$C \subseteq \mathbb{R}^n$ is closed if and only if $\overline{C} = C$.

Theorem 2.2.5

Let $S \subseteq \mathbb{R}^n$. The interior of S is the union of all open sets contained in S.

Proof. Let $\mathcal{U} \triangleq \{U \subseteq S \mid U \text{ is open in } \mathbb{R}^n\}$.

($\subseteq$) Let $\mathbf{x} \in \mathring{S}$. Then, there is an open neighborhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \subseteq S$. Noting that $\mathbf{x} \in N(\mathbf{x}) \in \mathcal{U}$, we conclude $\mathring{S} \subseteq \bigcup \mathcal{U}$.

($\supseteq$) Take any $\mathbf{x} \in \bigcup \mathcal{U}$. Then, there is an open set U in $\mathbb{R}^n$ such that $x \in U \subseteq S$. There is a neighborhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \subseteq U$. Therefore, $N(\mathbf{x}) \subseteq S$; $\mathbf{x}$ is an interior point of S. Thus; $\mathring{S} \supseteq \bigcup \mathcal{U}$. $\square$

Theorem 2.2.6

The closure of S is the intersection of all closed sets that contain S.

Proof. Let $\mathcal{C} \triangleq \{C \subseteq \mathbb{R}^n \mid S \subseteq C \text{ and } C \text{ is closed}\}$.

($\subseteq$) Since $S \subseteq \bigcap \mathcal{C}$ is obvious, we only need to show $S' \subseteq \bigcap \mathcal{C}$. Let $\mathbf{x} \in S'$. Then, it is direct that $\forall C \in \mathcal{C}$, $\mathbf{x} \in C'$ since each $C \in \mathcal{C}$ satisfies $S \subseteq C$. As C is closed and thus $\mathbf{x} \in C' \subseteq C$ by Theorem 2.2.4, Consequently, $\mathbf{x} \in \bigcap \mathcal{C}$; $\overline{S} \subseteq \bigcap \mathcal{C}$.

($\supseteq$) It is enough to show that $\overline{S}$ is closed, which, in turn, is sufficient to show that $(\overline{S})' \subseteq \overline{S}$ by Theorem 2.2.4. Let $\mathbf{y} \in (\overline{S})'$ and take any deleted neighborhood $N'(\mathbf{y};\varepsilon)$ of $\mathbf{y}$. Then, there is some element $\mathbf{z}$ in $N'(\mathbf{y};\varepsilon) \cap \overline{S}$. Then, $\mathbf{z} \in S$ or $\mathbf{z} \in S'$.

If $\mathbf{z} \in S$, then $\mathbf{z} \in N'(\mathbf{y};\varepsilon) \cap S \neq \varnothing$. If $\mathbf{z} \in S'$, take $\varepsilon' \triangleq \min\{\|\mathbf{z}-\mathbf{y}\|, \varepsilon - \|\mathbf{z}-\mathbf{y}\|\}$. Then, $N(\mathbf{z};\varepsilon') \subseteq N'(\mathbf{y};\varepsilon)$. Since $\mathbf{z} \in S'$, there is some $\mathbf{x}$ in $N'(\mathbf{z};\varepsilon') \cap S$. Thus, $\mathbf{x} \in N'(\mathbf{z};\varepsilon') \cap S \subseteq N'(\mathbf{y};\varepsilon) \cap S \neq \varnothing$.

In both cases, $N'(\mathbf{y};\varepsilon) \cap S \neq \varnothing$. Thus, we proved that $\mathbf{y} \in S' \subseteq \overline{S}$; $(\overline{S})' \subseteq \overline{S}$. $\square$

Corollary 2.2.2

For any $S \subseteq \mathbb{R}^n$, the set $\overline{S}$ is closed.

Corollary 2.2.3

For any $C \subseteq \mathbb{R}^n$, C is closed if and only if $\overline{C} = C$.

Theorem 2.2.7

Let $S \subseteq \mathbb{R}^n$.

(i) $\mathring{\mathring{S}} = \mathring{S}$

(ii) $\overline{(\overline{S})} = \overline{S}$

(iii) $\mathring{S} \cap \operatorname{bd} S = \varnothing$

(iv) $\mathring{S} \cup \operatorname{bd} S = \overline{S}$
(v) $\overline{S} \cap \overline{S^c} = \operatorname{bd} S$

Proof.

(i) $\mathring{S}$ is open and an open set is the interior of itself.
(ii) $\overline{S}$ is closed and a closed set is the closure of itself. (See Corollary 2.2.2 and Corollary 2.2.3).
(iii) Suppose there is some $\mathbf{x} \in \mathring{S} \cap \operatorname{bd} S$. There is a neighborhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \subseteq S$. Then, $N(\mathbf{x}) \cap C^c = \varnothing$, which contradicts $\mathbf{x} \in \operatorname{bd} S$.
(iv) ($\subseteq$) Since it is already $\mathring{S} \subseteq S \subseteq \overline{S}$, we only need to show $\operatorname{bd} S \subseteq \overline{S}$. Let $\mathbf{x} \in \operatorname{bd} S$. If $\mathbf{x} \in S$, then it is done; so suppose $\mathbf{x} \in S^c$. Take any neighborhood $N(\mathbf{x};\varepsilon)$ of $\mathbf{x}$. Then, $N(\mathbf{x};\varepsilon) \cap S \neq \varnothing$. Noting that $N'(\mathbf{x};\varepsilon) \cap S = N(\mathbf{x};\varepsilon) \cap S \neq \varnothing$, $\mathbf{x} \in S'$.
($\supseteq$) Let $\mathbf{x} \in \overline{S}$. If $\mathbf{x} \in S$, then it is either "There is a neighborhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \subseteq S$." or "Every neighborhood $N(\mathbf{x})$ of $\mathbf{x}$ satisfies $N(\mathbf{x}) \cap S^c \neq \varnothing$." The first case is $\mathbf{x} \in \mathring{S}$ and the latter case is $\mathbf{x} \in \operatorname{bd} S$.
Now the only left case if $\mathbf{x} \in S' \setminus S$. Take any deleted neighborhood $N'(\mathbf{x})$ of $\mathbf{x}$. Then, $N(\mathbf{x}) \cap S = N'(\mathbf{x}) \cap S \neq \varnothing$. Also, $\mathbf{x} \in N(\mathbf{x}) \cap S^c$. Thus, $\mathbf{x} \in \operatorname{bd} S$.
(v) Using $\overline{S} = \mathring{S} \cup \operatorname{bd} S$, we get

$$\begin{aligned}\overline{S} \cap \overline{S^c} &= (\mathring{S} \cup \operatorname{bd} S) \cap ((\mathring{S^c}) \cup \operatorname{bd} S^c) \\ &= (\mathring{S} \cap (\mathring{S^c})) \cup (\mathring{S} \cap \operatorname{bd} S^c) \cup (\operatorname{bd} S \cap (\mathring{S^c})) \cup (\operatorname{bd} S \cap \operatorname{bd} S^c)\end{aligned}$$

$\mathring{S} \cap (\mathring{S^c}) = \varnothing$ since $S \cap S^c = \varnothing$ and $\mathring{S} \subseteq S$ and $\mathring{S^c} \subseteq S^c$.
$\operatorname{bd} S = \operatorname{bd} S^c$ is direct from their definitions. Thus,

$$\begin{aligned}\mathring{S} \cap \operatorname{bd} S^c = \mathring{S} \cap \operatorname{bd} S = \varnothing \\ \operatorname{bd} S \cap (\mathring{S^c}) = \operatorname{bd} S^c \cap (\mathring{S^c}) = \varnothing\end{aligned}$$

by (iv). Therefore, $\overline{S} \cap \overline{S^c} = \operatorname{bd} S \cap \operatorname{bd} S^c = \operatorname{bd} S$.

□

Definition 2.2.3: Diameter

Let $\varnothing \neq S \subseteq \mathbb{R}^n$ be a bounded set. The *diameter* of S is defined to be

$$d(S) \triangleq \sup\{\, \|\mathbf{x}-\mathbf{y}\| \mid \mathbf{x},\mathbf{y} \in S \,\}.$$

Definition 2.2.4: Distance

Let $\varnothing \neq S \subseteq \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$. The distance from $\mathbf{x}$ to S is defined to be

$$d(\mathbf{x},S) \triangleq \inf\{\, \|\mathbf{x}-\mathbf{y}\| \mid \mathbf{y} \in S \,\}.$$

Exercise 2.2.1

Let S be a nonempty set in $\mathbb{R}^n$ and let $\mathbf{x}$ be a point of $\mathbb{R}^n$.
(i) $d(\mathbf{x},S) = 0$ if and only if $\mathbf{x} \in \overline{S}$.
(ii) S is closed if and only if $d(\mathbf{x},S) > 0$ for every $\mathbf{x} \in S^c$.
(iii) If S is closed, then there exists $\mathbf{y}_0 \in S$ such that $d(\mathbf{x},S) = \|\mathbf{x}-\mathbf{y}_0\|$.
(iv) If S is open and if $\mathbf{x} \in S^c$, then there exists no $\mathbf{y} \in S$ such that $d(\mathbf{x},S) = \|\mathbf{x}-\mathbf{y}\|$.

Solution:

(i) $(\Rightarrow)$ We shall show that if such $\mathbf{x}$ is not in S, then it is in S'. So, suppose $\mathbf{x} \notin S$. By Theorem 1.1.4, for any $\varepsilon \in \mathbb{R}_+$, there is some $\mathbf{y} \in S$ such that $0 \le \|\mathbf{x}-\mathbf{y}\| < \varepsilon$. Since $\mathbf{x} \notin S$, $\mathbf{x} \neq \mathbf{y}$, and thus $\mathbf{y} \in N'(\mathbf{x};\varepsilon) \cap S$, implying $\mathbf{x}$ is a limit point of S.
$(\Leftarrow)$ Conversely, if $\mathbf{x} \in S' \setminus S$, then for all $\varepsilon \in \mathbb{R}_+$, there is some $\mathbf{z} \in S$ such that $0 < \|\mathbf{x}-\mathbf{z}\| < \varepsilon$. Therefore, $0 \le d(\mathbf{x},S) < \varepsilon$. Since ε is arbitrary, $d(\mathbf{x},S) = 0$. ✓

(ii) $(\Rightarrow)$ $d(\mathbf{x},S) = 0$ if and only if $\mathbf{x} \in \overline{S} = S$. Therefore, $d(\mathbf{x},S) > 0$ if and only if $\mathbf{x} \in S^{\mathrm{c}}$.
$(\Leftarrow)$ For every $\mathbf{x} \in S^{\mathrm{c}}$, $\mathbf{x} \notin \overline{S}$ by (i). Thus, if $\mathbf{x} \in \overline{S}$, then $\mathbf{x} \in S$, or, $\overline{S} \subseteq S$. S is therefore closed. ✓

(iii) If S is finite, then we can easily see $d(\mathbf{x},S) = \min\{\, \|\mathbf{x}-\mathbf{y}\| \mid \mathbf{y} \in S \,\}$.
Therefore, now suppose S is infinite. Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a sequence defined by $\varepsilon_k = 1/k$ for each $k \in \mathbb{N}$. By Theorem 1.1.4, for each $k \in \mathbb{N}$, we can find $\mathbf{y}_k \in S$ that satisfies

$$d(\mathbf{x},S) \le \|\mathbf{x}-\mathbf{y}_k\| < d(\mathbf{x},S) + \varepsilon_k.$$

If the set $\{\, \mathbf{y}_k \mid k \in \mathbb{N} \,\}$ is finite, then there must be some $\mathbf{y}_k$ such that $\|\mathbf{x}-\mathbf{y}_k\| = d(\mathbf{x},S)$, and we are done.
Suppose $\{\, \mathbf{y}_k \mid k \in \mathbb{N} \,\}$ is infinite. Since the set is also bounded ($\|x-y_k\| < d(\mathbf{x},S)+\varepsilon_1$ for each $k \in \mathbb{N}$), by Theorem 2.1.8, there is a convergent subsequence $\{\mathbf{y}_{k_j}\}_{j\in\mathbb{N}}$ of $\{\mathbf{y}_k\}_{k\in\mathbb{N}}$. Let $\mathbf{y}_0 \triangleq \lim_{j\to\infty} \mathbf{y}_{k_j}$. Since

$$d(\mathbf{x},S) \le \|\mathbf{x}-\mathbf{y}_{k_j}\| < d(\mathbf{x},S) + \varepsilon_{k_j}$$

still holds, it follows that $\|\mathbf{x}-\mathbf{y}_0\| = d(\mathbf{x},S)$ by Theorem 1.3.5.
$\mathbf{y}_0 \in S$ since S is closed and $\mathbf{y}_0 \in S$ is a limit point of S. ✓

(iv) Suppose there is some $\mathbf{y} \in S$ such that $d(\mathbf{x},S) = \|\mathbf{x}-\mathbf{y}\|$. $\|\mathbf{x}-\mathbf{y}\| > 0$ since $\mathbf{x} \neq \mathbf{y}$. Since S is open, there is some neighborhood $N(\mathbf{y};r_0)$ of $\mathbf{y}$ such that $N(\mathbf{y};r_0) \subseteq S$. It must be $r_0 \le \|\mathbf{x}-\mathbf{y}\|$. Let

$$\mathbf{z} \triangleq \mathbf{y} + \frac{r_0}{2} \cdot \frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|}$$

Then,

$$\|\mathbf{z}-\mathbf{y}\| = \frac{r_0}{2} < r_0,$$

thus $\mathbf{z} \in N(\mathbf{y};r_0) \subseteq S$. However,

$$\|\mathbf{x}-\mathbf{y}\| = \left|1 - \frac{r_0}{2\|\mathbf{x}-\mathbf{y}\|}\right| \|\mathbf{x}-\mathbf{y}\| < \|\mathbf{x}-\mathbf{y}\|,$$

contradicting the minimality of $\|\mathbf{x}-\mathbf{y}\|$, #. ✓

□

Exercise 2.2.2

Let S be a nonempty set in $\mathbb{R}^n$. Then, $d(S) = d(\overline{S})$.

Solution: Since $S \subseteq \overline{S}$, $d(S) \le d(\overline{S})$ is direct.

To prove $d(S) = d(\overline{S})$, take any $\varepsilon \in \mathbb{R}_+$. Let $\mathbf{x}$ and $\mathbf{y}$ be any point in $\overline{S}$. Then, each of $S \cap N(\mathbf{x};\varepsilon/2)$ and $S \cap N(\mathbf{y};\varepsilon/2)$ is nonempty. Thus, we take $\mathbf{x}'$ and $\mathbf{y}'$ from each set. Then,

$$\begin{aligned}\|\mathbf{x}-\mathbf{y}\| &\le \|\mathbf{x}-\mathbf{x}'\| + \|\mathbf{x}'-\mathbf{y}'\| + \|\mathbf{y}'-\mathbf{y}\| \\ &< \varepsilon/2 + d(S) + \varepsilon/2 = d(S) + \varepsilon\end{aligned}$$

Therefore, $d(S)+\varepsilon$ is an upper bound of $\{\, \|\mathbf{x}-\mathbf{y}\| \mid \mathbf{x},\mathbf{y} \in \overline{S} \,\}$. Thus, $d(S) \le d(\overline{S}) \le d(S)+\varepsilon$. Since ε was arbitrary, we conclude $d(S) = d(\overline{S})$.

□

2.3 Completeness

Definition 2.3.1: Nested Sets

A sequence $\{S_k\}$ of sets in $\mathbb{R}^n$ such that $S_k \supset S_{k+1}$ for each $k \in \mathbb{N}$ is said to be *nested*.

Theorem 2.3.1 Cantor's Nested Interval Theorem

For each $k \in \mathbb{N}$, let $I_k = [a_k, b_k]$ with $a_k < b_k$. Suppose that $\{I_k\}_{k\in\mathbb{N}}$ is a nested sequence in $\mathbb{R}$. Then

$$\bigcap_{k=1}^{\infty} I_k = [\alpha, \beta]$$

where $\alpha = \sup\{\, a_k \mid k \in \mathbb{N}\,\}$ and $\beta = \inf\{\, b_k \mid k \in \mathbb{N}\,\}$.

Proof. Let $A \triangleq \{\, a_k \mid k \in \mathbb{N}\,\}$ and $B \triangleq \{\, b_k \mid k \in \mathbb{N}\,\}$. Then, A is bounded above by any b_k and B is bounded below by any a_k. Thus, by Theorem 1.1.2, $\alpha = \sup A$ and $\beta = \sup B$ exist.

Any a_k is a lower bound of B, therefore $a_k \leq \beta$ for each $k \in \mathbb{N}$, which implies β is an upper bound of A. Hence $\alpha \leq \beta$.

To prove $\bigcap_{k=1}^{\infty} I_k \supseteq [\alpha, \beta]$, take any $x \in [\alpha, \beta]$. Then, for each $k \in \mathbb{N}$, $a_k \leq \alpha \leq x \leq \beta \leq b_k$, which means $x \in I_k$. Thus, $[\alpha, \beta] \subseteq \bigcap_{k=1}^{\infty} I_k$. Now, to prove the reverse containment, take any $x \in \bigcap_{k=1}^{\infty} I_k$. This means $\forall k \in \mathbb{N}$, $a_k \leq x \leq b_k$; x is an upper bound of A and is a lower bound of B at the same time. Therefore, $\alpha \leq x \leq \beta$, hence $\bigcap_{k=1}^{\infty} I_k \subseteq [\alpha, \beta]$. □

Another Proof. Since the sequences $\{a_k\}$ and $\{b_k\}$ are bounded and monotone, there are limits $\alpha = \lim_{k\to\infty} a_k$ and $\beta = \lim_{k\to\infty} b_k$ by Theorem 1.3.4. By Theorem 1.3.6, $\alpha \leq \beta$.

Since $a_k \leq \alpha \leq \beta \leq b_k$ for each $k \in \mathbb{N}$, $[a, b] \subseteq \bigcap_{k=1}^{n} I_k$.

Now, take any $x \in \bigcap_{k=1}^{\infty} I_k$. Then, for all $k \in \mathbb{N}$, $a_k \leq x \leq b_k$. If it were $\alpha > x$, there is some $k_0 \in \mathbb{N}$ such that $a_{k_0} > x$. It is similar for the case when $\beta < x$. Therefore, $\alpha \leq x \leq \beta$. We have proven that $\bigcap_{k=1}^{\infty} I_k \subseteq [\alpha, \beta]$. □

Corollary 2.3.1

If, in the notation of the previous theorem, $\lim_{k\to\infty}(b_k - a_k) = 0$, then $\bigcap_{k=1}^{\infty} I_k$ is a singleton.

Proof. Take any $\varepsilon \in \mathbb{R}_+$. Then, there is some $k_0 \in \mathbb{N}$ such that $b_{k_0} - a_{k_0} < \varepsilon$.

$$0 \leq \beta - \alpha \leq b_{k_0} - a_{k_0} < \varepsilon$$

holds. This implies that $0 \leq \beta - \alpha < \varepsilon$ for arbitrary $\varepsilon \in \mathbb{R}_+$; therfore $\alpha = \beta$. □

Theorem 2.3.2 Cartor's Criterion

If $\{C_k\}$ is a nested sequence of closed, bounded, nonempty subsets of $\mathbb{R}^n$, then

$$\bigcap_{k=1}^{\infty} C_k \neq \varnothing.$$

Furthermore, if $\lim_{k\to\infty} d(C_k) = 0$, then $\bigcap_{k=1}^{\infty} C_k$ is a singleton.

Proof. If any of C_k is finite, it is trivial. So, we suppose every C_k is infinite. Construct a seqeuence $\{\mathbf{x}_k\}$ of points in $\mathbb{R}^n$ as following.

- Take any $\mathbf{x}_1$ in C_1.
- For each $k \in \mathbb{N}$, take any $\mathbf{x}_{k+1}$ in $C_{k+1} \setminus \{x_1, x_2, \cdots, x_k\}$.

Since $S \subseteq C_1$ is bounded and contains infinitely many points, by Theorem 2.1.8, there is a limit point $\mathbf{x}_0$ of S in $\mathbb{R}^n$. We now claim that $\mathbf{x}_0 \in \bigcap_{k=1}^{\infty} C_k$.

Fix any $k \in \mathbb{N}$ and choose any deleted neighborhood $N'(\mathbf{x}_0)$ of $\mathbf{x}_0$. Since $N'(\mathbf{x}_0) \cap S$ is infinite, there is some $k_1 \in \mathbb{N}_{>k}$ such that $\mathbf{x}_{k_1} \in N'(\mathbf{x}_0)$. By the construction, $\mathbf{x}_{k_1} \in C_{k_1} \subseteq C_k$. This shows that every deleted neighborhood of $\mathbf{x}_0$ contains a point in C_k; $\mathbf{x}_0 \in C_k'$. As each C_k is closed, $\mathbf{x}_0 \in C_k$, and thus $\mathbf{x}_0 \in \bigcap_{k=1}^{\infty} C_k$.

Suppose, in addition, $\lim_{k\to\infty} d(C_k) = 0$. Assume $\bigcap_{k=1}^{\infty} C_k$ has two distinct points $\mathbf{x}$ and $\mathbf{y}$ for the sake of contradiction. Choose any $\varepsilon \in (0, \|\mathbf{x}-\mathbf{y}\|)$ and then there is some $k \in \mathbb{N}$ with $d(C_k) < \varepsilon$. Nonetheless, $\varepsilon < \|\mathbf{x}-\mathbf{y}\| \le d(C_k) < \varepsilon$, #. □

Theorem 2.3.3 Cantor's Criterion in $\mathbb{R}^n$ implies Cantor's Criterion in $\mathbb{R}$

Cantor's criterion in $\mathbb{R}^n$ implies Cantor's criterion also holds in $\mathbb{R}$.

Proof. $\mathbb{R} \times \{0\} \times \cdots \times \{0\}$ is a closed subset of $\mathbb{R}^n$. □

Theorem 2.3.4

Cantor's criterion in $\mathbb{R}$ and Archimedes' principle implies the existence of supremum of any bounded above nonempty subset of $\mathbb{R}$.

Proof. Let S be a nonempty, bounded above set in $\mathbb{R}$. Let B denote the set of upper bounds of S and let $A = B^c$. Since $x - 1 \in A$ for all $x \in S$, $A \neq \varnothing$.

We first show that for all $a \in A$ and $b \in B$, $a < b$. If otherwise, i.e., $a \ge b$, $x \le b \le a$ for each $x \in S$, which implies $a \in B$, which is a contradiction.

Moreover, $S \cap [a, b] \neq \varnothing$ for each $a \in A$ and $b \in B$. Assume $S \cap [a, b] = \varnothing$ for the sake of contradiction. Since $S \cap (b, \infty) = \varnothing$ as b is an upper bound of S, then it follows $S \subseteq (-\infty, a)$, which implies a is an upper bound of S, which is a contradiction.

Construct a nested sequence $\{[a_k, b_k]\}_{k\in\mathbb{N}}$ of closed interval of which each a_k is in A and b_k is in B.

- Take any a_1 in A and b_1 in B.
- For each $k \in \mathbb{N}$, if $(a_k+b_k)/2 \in A$, then let $a_{k+1} \triangleq (a_k+b_k)/2$ and $b_{k+1} \triangleq b_k$. If $(a_k+b_k)/2 \in B$, then let $a_{k+1} \triangleq a_k$ and $b_{k+1} \triangleq (a_k + b_k)/2$.

Then it is immediate that $\lim_{k\to\infty}(b_k - a_k) = \lim_{k\to\infty} 2^{-k+1}(b_1 - a_1) = 0$. Therefore, by Cantor's criterion in $\mathbb{R}$, $\bigcap_{k=1}^{\infty}[a_k, b_k] = \{x_0\}$ for some $x_0 \in \mathbb{R}$.

We now show that x_0 is an upper bound of S. Assume not for the sake of contradiction, that is, there is some $x \in S$ such that $x > x_0$. Then, we may find some $k \in \mathbb{N}$ such that $b_k - a_k < x - x_0$. Then it follows $b_k - x_0 \le b_k - a_k < x - x_0$, and therefore $b_k < x_0$. This contradicts that b_k is an upper bound of S. Thus, $x_0 \in B$.

We now claim that x_0 is the least upper bound. Assume to the contrary that there is some $b \in B$ such that $b < x_0$. Then, we may find some $k \in \mathbb{N}$ such that $b_k - a_k < x_0 - b$. It follows $x_0 - a_k \le b_k - a_k < x_0 - b$, and therefore $b < a_k$. This contradicts that $a_k \in A$. □

Theorem 2.3.5

Assuming that Archimedes' principle holds in $\mathbb{R}$, the following are equivalent.

(i) Eery nonempty supset in $\mathbb{R}$ which is bounded above has a supremum in $\mathbb{R}$.
(ii) Every bounded monotone sequence in $\mathbb{R}$ converges.
(iii) $\mathbb{R}$ has Bolzano–Weierstrass property.
(iv) $\mathbb{R}$ is Cauchy complete.

(v) $\mathbb{R}^n$ is Cauchy complete.
(vi) $\mathbb{R}^n$ has Bolzano–Weierstrass property.
(vii) Cantor's criterion holds in $\mathbb{R}^n$.
(viii) Cantor's criterion holds in $\mathbb{R}$.

Proof.

$$\text{(i)} \overbrace{\Longrightarrow}^{\text{Theorem 1.3.4}} \text{(ii)} \overbrace{\Longrightarrow}^{\text{Theorem 2.3.1}} \text{(viii)} \overbrace{\Longrightarrow}^{\text{Theorem 2.3.4}} \text{(i)}$$

$$\text{(i)} \overbrace{\Longrightarrow}^{\text{Theorem 1.2.3}} \text{(iii)} \overbrace{\Longrightarrow}^{\text{Theorem 1.4.4}} \text{(iv)} \overbrace{\Longrightarrow}^{\text{Theorem 2.1.7}} \text{(v)} \overbrace{\Longrightarrow}^{\text{Theorem 2.1.8}} \text{(vi)}$$

$$\overbrace{\Longrightarrow}^{\text{Theorem 2.3.2}} \text{(vii)} \overbrace{\Longrightarrow}^{\text{special case}} \text{(viii)} \overbrace{\Longrightarrow}^{\text{Theorem 2.3.4}} \text{(i)}$$

□

Definition 2.3.2: Completeness

When the word *complete* is applied to $\mathbb{R}^n$, it is assumed that it means any of these statements:

- the existence of least upper bounds in $\mathbb{R}$,
- the Monotone Convergence theorem in $\mathbb{R}^n$,
- Cantor's criterion,
- the Bolzano–Weierstrass property, or
- Cauchy completeness.

2.4 Relative Topology and Connectedness

Definition 2.4.1: Relatively Open and Relatively Closed Set

A set S is said to be *relatively open* in X if there exists an open set U in $\mathbb{R}^n$ such that $S = U \cap X$. Likewise, a set S is said to be *relatively closed* if there exists a closed set C in $\mathbb{R}^n$ such that $S = C \cap X$.

Note:-

Every relatively open set in X is open in $\mathbb{R}^n$ if and only if X is open in $\mathbb{R}^n$. Every relatively closed set in X is closed in $\mathbb{R}^n$ if and only if X is closed in $\mathbb{R}^n$.

Definition 2.4.2

- A *relative neighborhood* of $\mathbf{x}$ in X is $N(\mathbf{x}; r) \cap X$. A *deleted neighborhood* is $N'(\mathbf{x}; r) \cap X$.
- A sequence $\{\mathbf{x}_k\}$ in X *converges in* X if $\lim_{k\to\infty} \mathbf{x}_k \in X$.
- The relative closure of S is $\overline{S} \cap X$.
- A point $\mathbf{x}_0$ in X is a *relative limit point* of S in X if $\mathbf{x}_0$ is a limit point of S.

Note:-

Depending on X, X may not be complete, i.e., a Cauchy sequence in X may not converge in X, an infinite and bounded subset of X may not have a limit point in X, or a nested sequence of nonempty, bounded, relatively closed subsets of X may have an empty intersection.

Lemma 2.4.1

Let X be a subset of $\mathbb{R}^n$ and C be a subset in X. Then, C is relatively closed in X if and only if C is the relative closure of C.

Proof. The "if" part is trivial; we only prove the "only if" part.

($\subseteq$) It is direct since $C \subseteq \overline{C}$.

($\supseteq$) There exists a closed set $\hat{C}$ in $\mathbb{R}^n$ such that $C = \hat{C} \cap X$. Since $C \subseteq \hat{C}$ and $\hat{C}$ is closed, $\overline{C} \subseteq \hat{C}$ by Theorem 2.2.6. Thus, $\overline{C} \cap X \subseteq \hat{C} \cap X = C$. □

Lemma 2.4.2

Let $C \subseteq X \subseteq \mathbb{R}^n$. Then, C is relatively closed in X if and only if C contains all its relative limit points in X.

Proof. ($\Rightarrow$) If $\mathbf{x}_0$ is a relative limit point of C in X, then $\mathbf{x}_0 \in \overline{C} \cap X = C$. (Lemma 2.4.1)

($\Leftarrow$) It means $\overline{C} \cap X \subseteq C$. And it is already $C \subseteq \overline{C} \cap X$. Therefore, C is relatively closed in X by **??**. □

Theorem 2.4.1

Let X be any nonempty subset of $\mathbb{R}^n$. The following statements are equivalent.

1. Every Cauchy sequence $\{\mathbf{x}_k\}$ in X converges to a point of X. Thus X inherits Cauchy completeness from $\mathbb{R}^n$.
2. If S is a bounded, infinite subset of X, then S has a limit point in X. Thus X inherits the Bolzano–Weierstrass property from $\mathbb{R}^n$.
3. If $\{C_k\}$ is any nested sequence of nonempty, bounded, relatively closed subsets of X, then $\bigcap_{k=1}^{\infty} \neq \varnothing$. Furthermore, if $\lim_{k\to\infty} d(C_k) = 0$, then $\bigcap_{k=1}^{\infty}$ is a singleton. Thus X inherits Cantor's criterion from $\mathbb{R}^n$.

Proof. We shall first prove (i) $\implies$ (ii). Let S be any bounded and infinite subset of S. Since S is a bounded and infinite subset of $\mathbb{R}^n$ at the same time, by Theorem 2.1.8, there is a limit point $\mathbf{x}_0$ of S in $\mathbb{R}^n$. We may construct a Cauchy sequence in S that converges to $\mathbf{x}_0$ as we did in the proof of Theorem 1.3.7. Thus, $\mathbf{x}_0$, which the Cauchy sequence converges to, is in S by Cauchy completeness of X. Therefore, S has a limit point in X.

We next prove (ii) $\implies$ (iii). Suppose that X has the Bolzano–Weierstrass property. Take any nested sequence $\{C_k\}$ be bounded, nonempty, and *relatively closed* subsets of X. We assume each C_k is infinite since it is trivial otherwise.

As in the proof of Theorem 2.3.2, recursively choose an infinite set $S = \{\mathbf{x}_k \mid k \in \mathbb{N}\}$ of distinct points such that $\mathbf{x}_k \in C_k$ for each $k \in \mathbb{N}$. By the assumption (ii), there is a limit point $\mathbf{x}_0 \in X$ of S. Since $\mathbf{x}_k \in C_k \subseteq \overline{C_k}$ for each $k \in \mathbb{N}$ and $\{\overline{C_k}\}_{k\in\mathbb{N}}$ is a nested sequence of bounded, nonempty, and closed subsets of $\mathbb{R}^n$, as in the proof of Theorem 2.3.2, $\mathbf{x}_0 \in \overline{C_k}$ for each k, also. As $C_k = \overline{C_k} \cap X$ for each $k \in \mathbb{N}$ by Lemma 2.4.1, $\mathbf{x}_0 \in C_k$ for each $k \in \mathbb{N}$. Thus, $\mathbf{x}_0 \in \bigcap_{k=1}^{\infty} C_k$.

If, in addition, $\lim_{k\to\infty} d(C_k) = 0$, then $\lim_{k\to\infty} d(\overline{C_k}) = 0$ by **??**. By Theorem 2.3.2, $\bigcap_{k=1}^{\infty} \overline{C_k} = \{\mathbf{x}_0\}$. But since $C_k \subseteq \overline{C_k}$ for each $k \in \mathbb{N}$, $\varnothing \neq \bigcap_{k=1}^{\infty} C_k \subseteq \bigcap_{k=1}^{\infty} \overline{C_k} = \{\mathbf{x}_0\}$. Thus, (ii) implies Cantor's criterion holds in X.

Finally, it is left to prove (iii) $\implies$ (i). Let $\{\mathbf{x}_k\}$ be a Cauchy sequence in X. We must show it converges to some point in X. By Cauchy completeness of $\mathbb{R}^n$, $\{\mathbf{x}_k\}$ converges to some point $\mathbf{x}_0$ in $\mathbb{R}^n$. We shall show that $\mathbf{x}_0 \in X$.

Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be any sequence of positive numbers that converge monotonically to 0, e.g., $\varepsilon_k = 1/k$. Let $C_k \triangleq \overline{N(\mathbf{x}_0;\varepsilon_k)} \cap X$. Then $\{C_k\}$ is a nested sequence of bounded and relatively closed sets in X.

Take any $k_1 \in \mathbb{N}$. Since $\{\mathbf{x}_k\}$ converges to $\mathbf{x}_0$, there is some $k_0 \in \mathbb{N}$ such that $\forall k \in \mathbb{N}_{\geq k_0}$, $\mathbf{x}_k \in N(\mathbf{x}_k;\varepsilon_{k_1})$. And such $\mathbf{x}_k$'s are also in $N(\mathbf{x}_k;\varepsilon_{k_1}) \cap X$; hence each C_k is nonempty.

Moreover, as $d(C_k) \leq d\big(N(\mathbf{x}_0;\varepsilon_k)\big) = 2\varepsilon_k$, $\lim_{k\to\infty} d(C_k) = 0$. Thus, by assumption (ii), $\bigcap_{k=1}^{\infty} C_k$ is a singleton in X. Then,

$$\bigcap_{k=1}^{\infty} C_k = \bigcap_{k=1}^{\infty}\left(\overline{N(\mathbf{x}_0;\varepsilon_k)} \cap X\right) \subseteq \bigcap_{k=1}^{\infty}\overline{N(\mathbf{x}_0;\varepsilon_k)} = \{\mathbf{x}_0\}.$$

Therefore, $\mathbf{x}_0 \in \bigcap_{k=1}^{\infty} C_k \subseteq X$. □

Definition 2.4.3: Completeness of a Subset of $\mathbb{R}^n$

Let X be a nonempty subset of $\mathbb{R}^n$. If any of the equivalent properties of Theorem 2.4.1 hold in the set X, then X is said to be *complete*.

Theorem 2.4.2

A nonempty subset X of $\mathbb{R}^n$ is complete if and only if X is closed in $\mathbb{R}^n$.

Proof. ($\Rightarrow$) Let $\mathbf{x}_0$ be a limit point of X. Then, there is a Cauchy sequence $\{\mathbf{x}_k\}$ that converges to $\mathbf{x}_0$. By Cauchy completeness of X, $\mathbf{x}_0 \in X$. Since we have proven that $X' \subseteq X$, X is closed by Theorem 2.2.4.

($\Leftarrow$) Let S be any bounded, infinite set in X. By Bolzano–Weierstrass property of $\mathbb{R}^n$, S has a limit point $\mathbf{x}_0$ in $\mathbb{R}^n$. As X being closed, $\mathbf{x}_0 \in X$. Therefore, X has Bolzano–Weierstrass property; X is complete by Theorem 2.4.1. □

Definition 2.4.4: Connectedness

A set S is *disconnected* if there are two open sets U, V such that
(i) $U \cap V = \varnothing$,
(ii) $S \subseteq U \cup V$, and
(iii) $S \cap U \neq \varnothing$ and $S \cap V \neq \varnothing$.
S is *connected* if S is not disconected.

Example 2.4.1

- $[0,1) \cup (1,2]$ is disconnected. ($U = (-1,1)$ and $V = (1,3)$)
- $\mathbb{Q}$ is disconnected. ($U = (-\infty, r)$ and $V = (r,\infty)$ where $r \in \mathbb{R} \setminus \mathbb{Q}$)

Theorem 2.4.3

Any interval of the form of $[a,b]$, $(a,b]$, $[a,b)$, or (a,b) is connected. $\mathbb{R}$ itself is connected.

Proof. Let I be any of these sets. The only fact needed for this proof is that $(u,v) \subseteq I$ for any $u < v$ in I.

Suppose I is disconnected. Then, there are disjoint open sets $U, V \subseteq \mathbb{R}$ such that $I \subseteq U \cup V$, $I \cap U \neq \varnothing$, and $I \cap V \neq \varnothing$. Take any $u \in I \cap U$ and $v \in I \cap V$. WLOG, $u < v$.

Construct sequences $\{u_k\}$ in $I \cap U$ and $\{v_k\}$ in $I \cap V$ as following.

- $u_1 \triangleq u$ and $v_1 \triangleq v$.
- For each $k \in \mathbb{N}$, $u_{k+1} \triangleq \begin{cases} \dfrac{u_k + v_k}{2} & \dfrac{u_k + v_k}{2} \in U \\ u_k & \text{otherwise} \end{cases}$ and $v_{k+1} \triangleq \begin{cases} \dfrac{u_k + v_k}{2} & \dfrac{u_k + v_k}{2} \in V \\ v_k & \text{otherwise} \end{cases}$.

Then, $\{u_k\}$ and $\{v_k\}$ are bounded, monotone sequences ($u_k < v$ and $v_k > u$ for each $k \in \mathbb{N}$); hence they converge by Theorem 1.3.4.

Since $(u_k + v_k)/2 \in I \subseteq U \cup V$ for each $k \in \mathbb{N}$ and $U \cap V = \varnothing$, $v_{k+1} - u_{k+1} = (v_k - u_k)/2 = (v-u)/2^k$. They converge to the same point x_0 since $\lim_{k\to\infty}(v_k - u_k) = 0$.

If x_0 is ever equal to u_k, then by openness of U, there is some neighborhood $N(x_0;\varepsilon)$ of x_0 such that $N(x_0;\varepsilon) \subseteq U$, which contradicts $\lim_{k\to\infty} v_k = x_0$. Thus, x_0 is never equal to u_k; by the same reason, x_0 is never equal to v_k. We conclude x_0 is a relative limit point in I of both $I \cap U$ and $I \cap V$.

Since $I \cap U = I \cap V^{\mathrm{c}}$ and V^{c} is closed, $I \cap U$ is closed and thus any relative limit point of $I \cap U$ in I, x_0, in particular, is in $I \cap U$ by Lemma 2.4.2. Similarly, $x_0 \in I \cap V$, which contradicts $U \cap V = \varnothing$. $\square$