MAS331 위상수학
Notes

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Chapter 1

Set Theory and Logic

1.1 Basic Notation

Note:-

- Sets: $A, B, C, \cdots, \mathbb{R}, \mathbb{Q}, \mathbb{Z}$
- Elements: $a, b, c, \cdots, 3, 3/4, \pi$
- $a \in A$, $3 \in \mathbb{Z}$, $3/4 \notin \mathbb{Z}$
- $A \subseteq B$, $A \subsetneq B$, $A \nsubseteq B$
- $\varnothing$: empty set
- $A \times B := \{(a, b) \mid a \in A \text{ and } b \in B\}$ (Cartesian product)
- $\binom{A}{n} \triangleq \left\{A' \subseteq A \,\middle|\, |A'| = n\right\}$

Definition 1.1.1: Function, Restriction, and Composition

A *function* f from a set A to a set B is an assignment of an element of B to each element of A.

- A: Domain
- B: Range or Codomain
- $\operatorname{Im} f := \{f(a) \mid a \in A\}$: Image; $\operatorname{Im} f \subseteq B$

If $A_0 \subseteq A$ and $f : A \to B$ is a function, then the *restriction* of f to A_0 is denoted by $f\big|_{A_0}$ and is defined as

$$f\big|_{A_0}(a_0) := f(a_0)$$

for each $a_0 \in A_0$. If $f : A \to B$ and $g : B \to C$, then the *composite* $g \circ f$ is defined as

$$(g \circ f)(a) := g\big(f(a)\big)$$

for each $a \in A$.

Definition 1.1.2: Injectivity, Surjectivity and Bijectivity

A function $f : A \to B$ is

i) *injective* (or *one-to-one*, 1-1) if $\forall a, a' \in A,\ f(a) = f(a') \implies a = a'$,
ii) *surjective* (or *onto*) if $\forall b \in B,\ \exists a \in A,\ b = f(a)$, and
iii) *bijective* if f is both injective and surjective.

Definition 1.1.3: Inverse Function

If $f : A \to B$ is bijective, then the inverse of f is denoted by

$$f^{-1} : B \to A$$

and is defined as

$$f^{-1}(b) = a$$

for each $b \in B$ where $f(a) = b$.

Example 1.1.1

a) f is bijective $\iff$ f^{-1} is bijective.
b) The inverse is unique.

Solution: Suppose f is bijective. Then,

$$f^{-1}(b_1) = f^{-1}(b_2) \implies b_1 = (f \circ f^{-1})(b_1) = (f \circ f^{-1})(b_2) = b_2.$$

Therefore, f^{-1} is injective.

Take any $a \in A$. Then, $b := f(a) \in B$ satisfies $f^{-1}(b) = a$. Therefore, f^{-1} is surjective.

Now, suppose f^{-1} is bijective. Then,

$$f(a_1) = f(a_2) \implies a_1 = (f^{-1} \circ f)(a_1) = (f^{-1} \circ f)(a_2) = a_2.$$

Therefore, f is injective.

Take any $b \in B$. Then, $a := f^{-1}(a) \in B$ satisfies $f(a) = b$. Therefore, f is surjective; a) is now proven.

Let g and h are inverses of f. Take any $b \in B$. Since f is bijective, $\exists! a \in A,\ f(a) = b$. Therefore, $g(b) = a = h(b)$, which implies $g = h$; b) is now proven.

□

Definition 1.1.4: Image and Preimage of a Set

Let $f : A \to B$ and $A_0 \subseteq A$, $B_0 \subseteq B$.

- $f(A_0) := \{b \mid b = f(a_0) \text{ and } a_0 \in A\}$
- $f^{-1}(B_0) := \{a \mid f(a) \in B_0\}$

Example 1.1.2

a) $A_0 \subseteq f^{-1}\big(f(A_0)\big)$
b) f is injective if and only if $\forall A_0 \subseteq A$, $A_0 = f^{-1}\big(f(A_0)\big)$.
c) $f\big(f^{-1}(B_0)\big) \subseteq B_0$
d) f is surjective if and only if $\forall B_0 \subseteq B$, $B_0 = f\big(f^{-1}(B_0)\big)$.

Solution:

a) For every $a_0 \in A_0$, $f(a_0) \in f(A_0)$, which implies $a_0 \in f^{-1}\big(f(A_0)\big)$. Therefore, $A_0 \subseteq f^{-1}\big(f(A_0)\big)$ holds.

b) Suppose f is injective. Take any $A_0 \subseteq A$ and $a_0 \in f^{-1}\big(f(A_0)\big)$. Then, $f(a_0) \in f(A_0)$. We may take $a_1 \in A_0$ such that $f(a_0) = f(a_1) \in f(A_0)$. Since f is injective, $a_0 = a_1 \in A_0$.

Suppose '$\forall A_0 \subseteq A$, $A_0 = f^{-1}\big(f(A_0)\big)$' holds. Suppose $f(a_1) = f(a_2) = b_0$. Let $A_0 := \{a_1\}$. Then, $A_0 = f^{-1}\big(f(A_0)\big) = f^{-1}\big(\{b_0\}\big) \ni a_2$. This means $a_2 \in \{a_1\}$, which implies $a_1 = a_2$.

c) Take any $b_0 \in f\big(f^{-1}(B_0)\big)$. Then, there is some $a_0 \in f^{-1}(B_0)$ such that $f(a_0) = b_0$. Such a_0 satisfies $f(a_0) \in B_0$, which implies $b_0 = f(a_0) \in B_0$. Therefore, $f\big(f^{-1}(B_0)\big) \subseteq B_0$ holds.

d) Suppose f is surjective. Take any $B_0 subs B$ and $b_0 \in B_0$. Then, there is some $a_0 \in A$ such that $f(a_0) = b_0$, which implies $a_0 \in f^{-1}(B_0)$. Therefore, $b_0 \in f\big(f^{-1}(B_0)\big)$; $B_0 \subseteq \big(f^{-1}(B_0)\big)$.

Suppose '$\forall B_0 \subseteq B$, $B_0 = f\big(f^{-1}(B_0)\big)$' holds. Take any $b_0 \in B$ and let $B_0 := \{b_0\}$. Since $b_0 \in f\big(f^{-1}(B_0)\big)$, There is some $a_0 \in f^{-1}(B_0)$ such that $f(a_0) = b_0$. Therefore, f is surjective.

□

1.2 Relations

Definition 1.2.1: Relation

A relation $\sim$ on a set A is a subset of $A \times A$.

$$x \sim y :\equiv (x, y) \in \sim$$

Definition 1.2.2: Equivalence Relation and Equivalence Class

A relation $\sim$ on a set A is an *equivalence relation* if

(1) $x \sim x$ for each $x \in A$ (reflexive)
(2) $x \sim y \implies y \sim x$ (symmetric)
(3) $x \sim y \wedge y \sim z \implies x \sim z$. (transitive)

Moreover, the *equivalence class* of x is defined as

$$\big\{y \in A \,\big|\, y \sim x\big\}.$$

Example 1.2.1 (Partition)

If there are equivalence classes E and E', then they are either $E = E'$ or $E \cap E' = \varnothing$. This implies, if we let $\mathcal{E} := \{E \mid E \text{ is an equivalence class of } x \text{ where } x \in A\}$, $A = \bigcup_{E \in \mathcal{E}} E$.

Solution: Since if $E \cap E' = \varnothing$ it is done, suppose $E \cap E' \neq \varnothing$. There are a and a' such that E and E' are equivalence classes of a and a' respectively. We may take $a_0 \in E \cap E'$. By definition and transitivity, $a \sim a_0 \sim a'$. Therefore, for all $x \in E$, $x \in E'$ since $x \sim a \sim a'$, which implies $E \subseteq E'$. In the same way, $E' \subseteq E$.

□

Definition 1.2.3: Order Relation

A relation $<$ on a set A is an *order relation* if

(1) $x < y$ or $y < x$ for each $x \neq y \in A$
(2) $x \not< x$ for each $x \in A$
(3) $x < y \wedge y < z \implies x < z$.

Also, we define
$$(a, b) := \left\{x \in X \,\middle|\, a < x < b\right\}.$$

Definition 1.2.4: Order Type

Let A and B be sets with order relations $<_A$ and $<_B$, respectively. Then, A and B have the same *order type* if there is a bijection $f : A \to B$ such that $a_1 <_A a_2 \iff f(a_1) <_B f(a_2)$.

Definition 1.2.5: Dictionary Order Relation

Let A, B be sets with order relations $<_A$, $<_B$ respectively. Then, there is an order relation $<_{A\times B}$ on $A \times B$ defined as $(a_1, b_1) <_{A\times B} (a_2, b_2)$ if
$$a_1 <_A a_2 \text{ or } a_1 = a_2 \text{ and } b_2 <_B b_2.$$
This is often called *dictionary order relation* on $A \times B$.

Definition 1.2.6: Boundedness

Let $A_0 \subseteq A$ with an order relation $<_A$.

- The *largest element* of A_0 is $b \in A_0$ if $x \in A_0 \implies x \leq b$.
- The *smallest element* of A_0 is $b \in A_0$ if $x \in A_0 \implies x \geq b$.

- A_0 is *bounded above* by $b \in A$ if $x \in A_0 \implies x \leq b$.
  - The smallest such b is called the *least uppder bound* or the *supremum* of A_0.
- A_0 is *bounded below* by $b \in A$ if $x \in A_0 \implies x \geq b$.
  - The largest such b is called the *greatest lower bound* or the *infimum* of A_0.

- A has *least upper bound property* if every bounded above nonempty set $A_0 \subseteq A$ has a least upper bound.
- A has *greatest lower bound property* if every bounded below nonempty set $A_0 \subseteq A$ has a greatest lower bound.

Theorem 1.2.1

A set A with an order relation $<_A$ has l.u.b. property if and only if A has g.l.b. property.

Proof. Suppose A has l.u.b. property. Let A_0 be any bounded below nonempty subset of A. Let $L := \{a \in A \mid a \text{ is a lower bound of } A_0\}$. Take a $a_0 \in A_0$. Then, since $\ell \leq_A a_0$ for all $\ell \in L$, L is bounded above by a_0. By l.u.b. property of A, there is $\ell_0 := \sup L \in A$.

Take any a_0 in A_0. Since a_0 is an upper bound of L and ℓ_0 is the least upper bound, $\ell_0 \leq_A a_0$. Therefore, ℓ_0 is a lower bound of A_0.

Suppose $\ell_0 <_A \ell_1$ and ℓ_1 is a lower bound of A_0. This implies $\ell_1 \in L$, which contradicts to $\ell_1 \leq_A \sup L = \ell_0$. Therefore, ℓ_0 is the greatest lower bound, and A has g.l.b. property.

The inverse can be proven by the similar reasoning. $\square$

Theorem 1.2.2 Completeness of $\mathbb{R}$

The set of real numbers $\mathbb{R}$ has least upper bound property.

1.3 The Integers and the Real Numbers

Theorem 1.3.1 Well-Ordering Property

Every nonempty subset of $\mathbb{Z}_+$ has a smallest element.

Proof. We first prove that, for each $n \in \mathbb{Z}_+$, every nonempty subset of $[n] := \{1, 2, \cdots, n\}$ has a smallest element, using induction. For the base case, it is known the the only nonempty subset of $[1]$, $\{1\}$, has 1 as its smallest element.

Suppose the statement holds for $n = k$. Now take any nonempty subset S of $[k+1]$. If $S = \{k+1\}$, $k+1$, the only element of S, is a smallest element of S. Otherwise, $S \setminus \{k+1\}$ is nonempty and is a subset of $[k]$; we may let $\mu := \min S$ by the induction hypothesis. Then, μ is also a smallest element of S, regardless of whether it is $k+1 \in S$ or $k+1 \notin S$.

Now, take any $\varnothing \neq T \subseteq \mathbb{Z}_+$ and $m \in T$. Then, by our previous result, since $T \cap [m]$ is a nonempty subset of $[m]$, it has a smallest element, which is also a smallest element of T. $\square$

1.4 Cartesian Products

Definition 1.4.1: Indexing Function and Indexed Family of Sets

Let $\mathcal{A}$ be a nonempty collection of sets. An *indexing function* for $\mathcal{A}$ is a surjective function $f : J \to \mathcal{A}$ where $A_\alpha := f(\alpha)$. An *indexed family* of sets is defined as $\{A_\alpha\}_{\alpha \in J}$. Now, we define

$$\bigcup_{\alpha \in J} A_\alpha := \{x \mid \exists \alpha \in J,\ x \in A_\alpha\}$$
$$\bigcap_{\alpha \in J} A_\alpha := \{x \mid \forall \alpha \in J,\ x \in A_\alpha\}$$
$$\prod_{\alpha \in J} A_\alpha := \{f : J \to \textstyle\bigcup_{\alpha \in J} A_\alpha \mid \forall \alpha \in J,\ f(\alpha) \in A_\alpha\}.$$

1.5 Finite Sets

Definition 1.5.1: Finite Set and Cardinality

A set A is *finite* if there is a bijective $f : A \to [n]$ for some $n \in \mathbb{Z}_+$ or $A = \varnothing$.

- In the former case, we say *cardinality* n or $|A| = n$.
- In the latter case, we say *cardinality* 0 or $|A| = 0$.

Note:-

Let A and B be finite sets. Then, $|A| = |B| = n$ if and only if $\exists$ bijective $f : A \to B$.

Lemma 1.5.1

Let $a_0 \in A$. Then,

$$|A| = n \iff |A \setminus \{a_0\}| = n-1.$$

Proof. For $n = 1$, it is trivial. So suppose $n \geq 2$.

($\Rightarrow$) There is a bijection $f : A \to [n]$. If $f(a_0) = n$, then $f\big|_{A\setminus\{a_0\}}$ is a bijection from $A\setminus\{a_0\}$ to $[n-1]$, and it's done. Otherwise, let $a_1 := f^{-1}(n)$. Define $g : A \to A$ by

$$g(a) := \begin{cases} a_0 & \text{if } a = a_1 \\ a_1 & \text{if } a = a_0 \\ a & \text{otherwise.} \end{cases}$$

g is bijective. Then, $f \circ g$ is a bijection from A to $[n]$ such that $(f \circ g)(a_0) = n$.

($\Leftarrow$) Trivial. □

Theorem 1.5.1

Let A be a set with $|A| = n$ and $B \subsetneq A$. Then, there is no bijection between B and $[n]$, but (provided $B \neq \varnothing$) there is a bijection between B and $[m]$ for some $m < n$.

Proof by Induction. (Base case) It is trivial for $n = 1$.

(Induction) Suppose it is true for $n \geq 1$. WTS for the case $|A| = n+1$. Suppose $B \neq \varnothing$ because we have nothing to talk about then. Let $a_0 \in B$. By Lemma 1.5.1, there is a bijection $g : A \setminus \{a_0\} \to [n]$. Since $B \setminus \{a_0\} \subsetneq A \setminus \{a_0\}$, by induction hypothesis, we have two things.

- There is no bijection between $B \setminus \{a_0\}$ and $[n]$.
- As long as $B \neq \{a_0\}$, there is a bijection from $B \setminus \{a_0\}$ to $[m]$ for some $m < n$.

We conclude that there is no bijection from B and $[n+1]$ since, if there were, there would be a trivial bijection from $B \setminus \{a_0\}$ to $[n]$. Moreover, we can construct a bijection between B and $[m+1]$, and $m+1 < n+1$. □

Corollary 1.5.1 Uniqueness of Cardinality

The cardinality of a finite set is uniquely determined.

Proof. Let $m < n$ and suppose m and n are cardinalities of a finite set A. Then there are bijections $f : A \to [m]$ and $g : A \to [n]$. Then, $f \circ g^{-1}$ is a bijection from $[m]$ to $[n]$ but it is impossible since $[m] \subsetneq [n]$ and because of Theorem 1.5.1. □

Corollary 1.5.2

$\mathbb{Z}_+$ is not finite.

Proof by Contradiction. Suppose $\mathbb{Z}_+$ is finite and $|\mathbb{Z}_+| = n$. $f : \mathbb{Z}_+ \to \mathbb{Z}_+ \setminus \{1\}$ with $x \mapsto x+1$ is bijective. Then, by Lemma 1.5.1, $n-1 = |\mathbb{Z}_+ \setminus \{1\}| = |\mathbb{Z}_+| = n$, #. □

Theorem 1.5.2

Let A be a set. TFAE

(i) $|A| = n$
(ii) $\exists$ surjective $[m] \twoheadrightarrow A$ for some $m \in \mathbb{Z}_+$.
(iii) $\exists$ injective $A \hookrightarrow [m]$ for some $m \in \mathbb{Z}_+$.

Proof. ((i) → (ii)) There is a bijective function from A to $[n]$, and it is also surjective.

((ii) → (iii)) Let f be a surjective function from $[m]$ to A. Since f is surjective, $f^{-1}(\{a\}) \neq \varnothing$ for every $a \in A$. Let $M := \max\{\min f^{-1}(\{a\}) \mid a \in A\}$. M is well defined thanks to Theorem 1.3.1 and the fact that $\varnothing \neq f^{-1}(\{a\}) \subseteq [m]$. Then the function $g : A \to [M]$ defined by $a \mapsto \min f^{-1}(\{a\})$ is injective.

((iii) → (i)) Let f be an injective function from A to $[m]$. Then, $g : A \to \operatorname{Im} f$ defined by $a \mapsto f(a)$ is bijective. A is finite because $\operatorname{Im} f$ is finite by Theorem 1.5.1. □

Exercise 1.5.1

(i) Finite unions of finite sets are finite.
(ii) Finite Cartesian products of finite sets are finite.

Solution: (i) Suppose there are n finite sets $A_1, A_2, \cdots, A_n$ to union. WLOG, $A_i \neq \varnothing$ for each $i \in [n]$. Let $M := \max_{i\in[n]} |A_i|$ and $g_i : [|A_i|] \to A_i$ be a bijective function for each $i \in [n]$. Extend each g_i to $g_i' : [M] \to A_i$ by

$$g_i'(k) = \begin{cases} g_i(k) & \text{if } k \leq |A_i| \\ g_i(1) & \text{otherwise.} \end{cases}$$

for $k \in [M]$. Now, we define $f : [nM] \to \bigcup_{i\in[n]} A_i$ by

$$f\big(n(i-1)+k\big) := g_i'(k)$$

for each $i \in [n]$ and $k \in [M]$. Then, f is surjective. Therefore, $\bigcup_{i\in[n]} A_i$ is finite by Theorem 1.5.2.

(ii) Suppose there are n finite sets $A_1, A_2, \cdots, A_n$ to construct a Cartesian product with. WLOG, $A_i \neq \varnothing$ for each $i \in [n]$. Let $M := \max_{i\in[n]} |A_i|$ and $h_i : A_i \to [|A_i|]$ be a bijective function for each $i \in [n]$. Let p_i be the i^{th} prime. (i.e., $p_1 = 2$, $p_2 = 3$, $p_3 = 5$.) Define a function $f : \prod_{i\in[n]} A_i \to \left[\left(\prod_{i=1}^{n} p_i\right)^M\right]$ by

$$f(a_1, a_2, \cdots, a_n) := \prod_{i=1}^{n} p_i^{h_i(a_i)}.$$

f is injective since prime factorization of a natural number is unique. Therefore, $\prod_{i\in[n]} A_i$ is finite by Theorem 1.5.2.

□

1.6 Countable and Uncountable Sets

Definition 1.6.1: Infinite and Countably Infinite

A set A is said to be *infinite* if it is not finite. It is said to be *countably infinite* if there is a bijective correspondence

$$f : A \to \mathbb{Z}_+.$$

Example 1.6.1

$\mathbb{Z}_+$, $\mathbb{Z}$, and $\mathbb{Z}_+ \times \mathbb{Z}_+$ are countably infinite.

Definition 1.6.2: Countability

A set is said to be *countable* if it is either finite or countably infinite. A set that is not countable is said to be *uncountable*.

Lemma 1.6.1

Any subset of $\mathbb{Z}_+$ is countable.

Proof. Let $C \subseteq \mathbb{Z}_+$. If C is finite, then it's done; we now assume C is infinite. Now we want to show that C is countably infinite.

Define $h \colon \mathbb{Z}_+ \to C$ by the following.

(a) $h(1) := \min C$

(b) $h(n+1) := \min\big(C \setminus h([n])\big)$ for each $n \in \mathbb{Z}_+$

h is well defined because $C \setminus h([n])$ is always nonempty. Morever, h is injective since it is $h(m) < h(n)$ whenever $m < n$.

Now, we are going to show h is surjective. To do this, first take any $c \in C$. Since C is infinite and h is injective, $\operatorname{Im} h \nsubseteq [c]$, which means $\exists n \in \mathbb{Z}_+$, $h(n) > c$. From this, we get $m := \min\{n \in \mathbb{Z}_+ \mid h(n) \geq c\}$ is well-defined. From the definition of m, we also get, for any $1 \leq i < m$, we have $h(i) < c \leq h(m)$. Therefore, $c \notin h([m-1])$. Together with $h(m) = \min\big(C \setminus h([m-1])\big)$, we get $h(m) \leq c \leq h(m)$, which implies $c = h(m)$. □

Theorem 1.6.1

Let $A \neq \varnothing$. TFAE

(i) A is countable.

(ii) $\exists$ surjective $\mathbb{Z}_+ \twoheadrightarrow A$.

(iii) $\exists$ injective $A \hookrightarrow \mathbb{Z}_+$.

Proof. ((i) $\to$ (ii)) Trivial.

((ii) $\to$ (iii)) Let $f : \mathbb{Z}_+ \twoheadrightarrow A$. Define $g : A \to \mathbb{Z}_+$ by $a \mapsto \min f^{-1}(\{a\})$. g is well-defined because $f^{-1}(\{a\}) \neq \varnothing$ for every $a \in A$ and Theorem 1.3.1 holds. g is also injective since $f^{-1}(\{a_1\}) \cap f^{-1}(\{a_2\}) = \varnothing$ if $a_1 \neq a_2 \in A$.

((iii) $\to$ (i)) Let f be an injection from A to $\mathbb{Z}_+$. If we define $g : A \to \operatorname{Im} f$ by $a \mapsto f(a)$, g is a bijection. Since $\operatorname{Im} f \subseteq \mathbb{Z}_+$, A is countable by Lemma 1.6.1. □

Corollary 1.6.1

If $A \subseteq B$ and B is countable, then A is countable.

Proof. $A \xhookrightarrow{\text{trivial injection}} B \xhookrightarrow{\text{injection}} \mathbb{Z}_+$ and Theorem 1.6.1. □

Corollary 1.6.2

$\mathbb{Z}_+ \times \mathbb{Z}_+$ is countably infinite.

Proof. $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$ with $(x, y) \mapsto 2^x 3^y$ is an injection.

Or, $g \colon \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$ with $(x, y) \mapsto \dfrac{(x+y-1)(x+y-2)}{2} + y$ is a bijection. □

Corollary 1.6.3

$\mathbb{Q}$ is countably infinite.

Proof. $f : \mathbb{Z} \times \mathbb{Z}_+ \to \mathbb{Q}$ with $(x, y) \mapsto x/y$ is surjective. $\square$

Exercise 1.6.1

The union of a countable number of countable sets is countable.

Solution: Let $\{A_i\}_{i\in J}$ be an indexed family of sets where J and A_i's are countable. WLOG, $A_i \neq \varnothing$ for each $i \in J$. For each $i \in J$, since A_i is countable, by Theorem 1.6.1, there is a surjection $g_i : \mathbb{Z}_+ \twoheadrightarrow A_i$. Similarly, since J is countable, there is a surjection $h : \mathbb{Z}_+ \twoheadrightarrow J$.

Now, construct a function $f : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \bigcup_{i\in J} A_i$ by

$$f(i, j) := g_{h(i)}(j).$$

f is naturally surjective by the contruction. Therefore, $\bigcup_{i\in J} A_i$ is countable.

$\square$

Exercise 1.6.2

The Cartesian product of a finite number of countable sets is countable.

Solution: Suppose there are $n \in \mathbb{Z}_+$ sets $A_1, A_2, \cdots, A_n$ to make Cartesian product with and each A_i is countable. WLOG, $A_i \neq \varnothing$ for each $i \in [n]$. For each $i \in [n]$, there is a injection $g_i : A_i \to \mathbb{Z}_+$ by Theorem 1.6.1.

Now, construct a fuction $f : \prod_{i=1}^n A_i \to \mathbb{Z}_+$ by

$$f(a_1, a_2, \cdots, a_n) := \prod_{i=1}^{n} p_i^{g_i(a_i)},$$

where p_i is the i^{th} prime. Since prime factorization of a natural number is unique, f is injective; therefore $\prod_{i=1}^n A_i$ is countable.

$\square$

Theorem 1.6.2

Let $X_i := \{0, 1\}$ for each $i \in \mathbb{Z}_+$. Then, $\prod_{i\in\mathbb{Z}_+} X_i$ is uncountable.

Proof. Let $f : \mathbb{Z}_+ \to \prod_{i\in\mathbb{Z}_+} X_i$ is any function. Denote $f(n) = (x_{n,1}, x_{n,2}, \cdots) \in \prod_{i\in\mathbb{Z}_+} X_i$ and construct $y = (y_1, y_2, \cdots) \in \prod_{i\in\mathbb{Z}_+} X_i$ by

$$y_i := 1 - x_{i,i}$$

for each $i \in \mathbb{Z}_+$. Then, $y \notin \operatorname{Im} f$; therefore, one cannot construct a surjection from $\mathbb{Z}_+$ to $\prod_{i\in\mathbb{Z}_+} X_i$. $\square$

Corollary 1.6.4

$\mathcal{P}(\mathbb{Z}_+)$ is uncountable.

Proof. $f : \mathcal{P}(\mathbb{Z}_+) \to \prod_{i\in\mathbb{Z}_+} X_i$ defined by

$$S \mapsto (y_1, y_2, \cdots) \text{ where } y_i := \begin{cases} 0 & \text{if } i \in S \\ 1 & \text{if } i \notin S \end{cases}$$

is a bijection, and $\prod_{i\in\mathbb{Z}_+} X_i$ is uncountable by Theorem 1.6.2. □

Theorem 1.6.3

Let A be a set. Then, there is no injection $\mathcal{P}(A) \hookrightarrow A$, and there is no surjection $A \twoheadrightarrow \mathcal{P}(A)$.

Proof. Since a surjective map can be naturally deducted from $f : B \hookrightarrow C$ (by constructing $g : C \to B$ by $g(c) \in f^{-1}(\{c\})$ for $c \in \operatorname{Im} f$ and map c to an arbitrary element in B for $c \notin \operatorname{Im} f$), it suffices to show $A \twoheadrightarrow \mathcal{P}(A)$ does not exist.

Let $f : A \to \mathcal{P}(A)$ be any function, and let $B := \{a \in A \mid a \notin f(a)\} \in \mathcal{P}(A)$. Suppose $B = f(a_0)$ for some $a_0 \in A$. Then, by the definition of B,

$$a_0 \in B \iff a_0 \notin f(a_0) = B,$$

which is a contradiction. Therefore, any such f cannot be surjective. □

1.7 Infinite Sets and the Axiom of Choice

Theorem 1.7.1

Let A be a set. TFAE

(i) A is infinite.
(ii) $\exists$ injection $f : \mathbb{Z}_+ \hookrightarrow A$.
(iii) $\exists$ bijection $g : A \to B$ where $B \subsetneq A$.

Proof. ((i) → (ii)) Construct $f : \mathbb{Z}_+ \to A$ recursively as following. Let $c : \mathcal{P}(A) \setminus \{\varnothing\} \to A$ be a function such that $c(A') \in A'$ for every $\varnothing \neq A' \subseteq A$. Its existence is guaranteed by Lemma 1.7.1.
(1) $f(1) := c(A)$
(2) $f(n+1) := c(A \setminus f([n]))$ for each $n \in \mathbb{Z}_+$.

Suppose $A \setminus f([n]) = \varnothing$ for some $n \in \mathbb{Z}_+$. Then, $A \subseteq f([n])$, and $f([n])$ is finite by Theorem 1.5.2; therefore A is finite by Theorem 1.5.1. Thus, f is well-defined and it is injective by definition.

((ii) → (iii)) Let $f : \mathbb{Z}_+ \hookrightarrow A$ be an injectiion. Define $g : A \to A \setminus \{f(1)\}$ by

$$g(a) := \begin{cases} f(n+1) & \text{if } a = f(n) \text{ for some } n \in \mathbb{N}_+ \\ a & \text{if } a \notin \operatorname{Im} f. \end{cases}$$

g is well-defined because f is injective, and it is bijective by definition.

((iii) → (i)) This is just a contrapositive of Theorem 1.5.1. □

Theorem 1.7.2 Axiom of Choice

Given a collection $\mathscr{A}$ of disjoint nonempty sets, there exists a set C such that $C \subseteq \bigcup \mathscr{A}$ and $\forall A \in \mathscr{A}$, $|C \cap A| = 1$.

Lemma 1.7.1 Existence of a Choice Function

Given a collection $\mathscr{B}$ of nonempty sets, there exists a function

$$c\colon \mathscr{B} \to \bigcup \mathscr{B}$$

such that $c(B) \in B$ for each $B \in \mathscr{B}$.

Proof. Let $\mathscr{A} := \{\{(B,x) \mid x \in B\} \mid B \in \mathscr{B}\}$. Then, by Theorem 1.7.2, there exists $c \subseteq \mathscr{A}$ such that $c \subseteq \bigcup \mathscr{A}$ and each $B \in \mathscr{B}$ appears only once in the first coordinate in c. Therefore, c is a function such that $c(B) \in C$ for each $B \in \mathscr{B}$. □

1.8 Well-Ordered Sets

Definition 1.8.1: Well-Ordered

A set A with an order relation is an *well-ordered* set if every nonempty subset of A has a smallest element.

Example 1.8.1

- $\mathbb{Z}_+$ is well-ordered.
- $\{1,2\} \times \mathbb{Z}_+$ is well ordered with respect to the dictionary ordering.

Theorem 1.8.1

Every nonempty finite set has the order type of $[n]$, and thus it is well-ordered.

Proof. We shall first claim that, if A is a nonempty finite set, then it has a largest element. It can be prove by induction on $|A|$. If $|A| = 1$, then it is trivial. Suppose the claim holds for $|A| = n$, and suppose $|A| = n+1$ and $a_0 \in A$. Then, $A \setminus \{a_0\}$ has a largest element a_1. This implies A has a largest element $\max\{a_0, a_1\}$.

Now, we prove there is an order-preserving bijection $f : A \to [n]$. This will also be proven with induction. It is true when $|A| = 1$, so suppose it is true for $|A| = n \in \mathbb{Z}_+$ and let $|A| = n+1$. By above, we may let $a_0 := \max A$. By induction hypothesis, there is an order-preserving bijection $f' : A \setminus \{a_0\} \to [n]$. Define $f : A \to [n+1]$ by

$$f(a) := \begin{cases} f'(a) & \text{if } a \neq a_0 \\ n+1 & \text{if } a = a_0. \end{cases}$$

Then, f is an order-preserving bijection from A to $[n+1]$. □

Theorem 1.8.2

The Cartesian product of finitely many well-ordered sets is well-ordered with respect to the dictionary ordering.

Proof by Induction. We will prove this by induction on the number of sets. If there is one set, then it is trivial.

Assume the theorem holds for n sets. Suppose we have $n+1$ sets $A_1, A_2, \cdots, A_{n+1}$. Then, $\prod_{i=2}^{n+1} A_i$ is well-ordered with respect to a dictionary ordering $<_1$.

Let $<_2$ and $<_3$ be the dictionary order of $A_1 \times \prod_{i=2}^{n+1} A_i$ and $\prod_{i=1}^{n+1} A_i$, respectively. Since $\left(A_1 \times \prod_{i=2}^{n+1} A_i, <_2\right)$ and $\left(\prod_{i=1}^{n+1} A_i, <_3\right)$ has the same order type, we only need to prove that $\left(A_1 \times \prod_{i=2}^{n+1} A_i, <_2\right)$ is well-ordered.

Let $\varnothing \neq S \subseteq A_1 \times \prod_{i=2}^{n+1} A_i$. If we define $S' := \{ a_1 \mid (a_1, b) \in S \} \subseteq A_1$, S' is a nonempty subset of A_1, and therefore has $a_1' := \min S'$. Similarly, if we define $S'' := \{ b_1 \mid (a_1', b_1) \in S \} \subseteq \prod_{i=2}^{n+1} A_i$, S'' is nonempty and has a smallest element b_1'. Then, (a_1', b_1') is a smallest element of $A_1 \times \prod_{i=2}^{n+1} A_i$ with respect to $<_2$. □

Exercise 1.8.1

$\prod_{i \in \mathbb{Z}_+} \mathbb{Z}_+$ is not well-ordered with respect to the dictionary ordering.

Solution: Let $x_{ij} := \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$ for each $i \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_+$. The set $A := \{ (x_{i1}, x_{i2}, \cdots) \mid i \in \mathbb{Z}_+ \} \subseteq \prod_{i \in \mathbb{Z}_+} \mathbb{Z}_+$ has no smallest element.

Theorem 1.8.3 Well-Ordering Theorem

If A is a set, then there exists an order relation on A that is well-ordering.

The proof of Theorem 1.8.3 involves the Axiom of Choice.

Corollary 1.8.1

There exists an uncountable well-ordered set.

Definition 1.8.2: Section

Let X be a well-ordered set. Given $\alpha \in X$, let

$$S_\alpha := \{ x \in X \mid x < \alpha \}.$$

S_α is called the *section* of X by α.

Lemma 1.8.1

There exists a well-ordered set A with the largest element Ω, such that

- section S_Ω of A is uncountable, and,
- for every $\alpha \in A \setminus \{ \Omega \}$, section S_α of A is countable.

Proof. By Corollary 1.8.1, there exists an uncountable well-ordered set B. Let $C := \{ 1, 2 \} \times B$ be a set with a dictionary ordering. C is well-ordered by Theorem 1.8.2.

Let $S := \{ \alpha \in C \mid \text{section } S_\alpha \text{ of } C \text{ is uncountable} \} \subseteq C$. We may let $\Omega := \min S$. Then, the set $\overline{S_\Omega} = S_\Omega \cup \{ \Omega \}$ satisfies the two conditions. □

Theorem 1.8.4

If A is a countable subset of S_Ω (in Lemma 1.8.1), then A has an upper bound in S_Ω.

Proof. For each $a \in A$, the section S_a is countable; therefore, the union $B := \bigcup_{a \in A} S_a$ is also countable by Exercise 1.6.1.

Since S_Ω is uncountable, we may take an $x \in S_\Omega \setminus B$. If it were $x < a$ for some $a \in A$, then x would be contained in S_a, which is a subset of B, $\#$. Therefore, $x \in S_\Omega$ is an upper bound of A. $\square$

Chapter 2

Topological Spaces and Continuous Functions

2.1 Topological Spaces

Definition 2.1.1: Topology and Topological Space

A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X such that

(i) $\varnothing, X \in \mathcal{T}$

(ii) $\{U_i \mid i \in J\} \subseteq \mathcal{T} \implies \bigcup_{i\in J} U_i \in \mathcal{T}$

(iii) $\{U_1, U_2, \cdots, U_n\} \subseteq \mathcal{T} \implies \bigcap_{i=1}^{n} U_i \in \mathcal{T}$

We say $(X, \mathcal{T})$ is a *topological space*, and each element $U \in \mathcal{T}$ is called an *open set*.

Example 2.1.1 (Discrete Topology and Trivial Topology)

- If X is any set, the collection of all subsets of X, $\mathcal{P}(X)$, is a topology on X; it is called the *discrete topology*.
- $\{\varnothing, X\}$ is also an topology on X; we shall call it the *trivial topology*.

Example 2.1.2 (Finite Complement Topology)

Let X be any set. Then, $\mathcal{T} := \{U \subseteq X \mid X \setminus U \text{ is finite}\} \cup \{\varnothing\}$ is a topology.

(i) $\varnothing, X \in \mathcal{T}$ ✓

(ii) If $\{U_\alpha\}_{\alpha\in J} \subseteq \mathcal{T}$, then $X \setminus \bigcup_{\alpha\in J} U_\alpha = \bigcap_{\alpha\in J}(X - U_\alpha)$ is finite. ✓

(iii) If $\{U_1, U_2, \cdots, U_n\} \subseteq \mathcal{T}$, $X \setminus \bigcap_{i=1}^{n} U_\alpha = \bigcup_{i=1}^{n}(X \setminus U_\alpha)$ is finite by Exercise 1.5.1. ✓

The topology is called the *finite complement topology*.

Example 2.1.3

If $X = \{a, b, c\}$, then $\mathcal{T} = \{\varnothing, X, \{a\}, \{a, b\}\}$ is a topology on X.

Definition 2.1.2: Finer and Coarser Topology

Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies of a set X. If $\mathcal{T} \subseteq \mathcal{T}'$, then we say

- $\mathcal{T}'$ is *finer* than $\mathcal{T}$ and
- $\mathcal{T}$ is *coarser* than $\mathcal{T}'$.

Also, $\mathcal{T}$ is *comparable* to $\mathcal{T}'$ if either $\mathcal{T} \supseteq \mathcal{T}'$ or $\mathcal{T} \subseteq \mathcal{T}'$.

2.2 Basis for a Topology

Definition 2.2.1: Basis and Topology Generated by a Basis

A *basis* for X is a collection $\mathcal{B}$ of subsets of X such that:

(i) $\forall x \in X, \exists B \in \mathcal{B}, x \in B$ (i.e., $X = \bigcup \mathcal{B}$) and

(ii) $\forall B_1, B_2 \in \mathcal{B}, \left(x \in B_1 \cap B_2 \implies \exists B_3 \in \mathcal{B}, x \in B_3 \subseteq B_1 \cap B_2\right)$.

The topology $\mathcal{T}$ generated by $\mathcal{B}$ is the collection defined by

$$\mathcal{T} := \{U \subseteq X \mid \forall x \in U, \exists B \in \mathcal{B}, x \in B \subseteq U\}.$$

Note:-

If $\mathcal{B}$ is a basis for X and $\mathcal{T}$ is the topology generated by $\mathcal{B}$, then $\mathcal{B} \subseteq \mathcal{T}$.

Lemma 2.2.1

If $\mathcal{T}$ is the topology generated by basis $\mathcal{B}$ for X, then $\mathcal{T}$ is a topology on X.

Proof.

(i) $\varnothing \in \mathcal{T}$ by vacuous truth, and $X \in \mathcal{T}$ follows directly from (i) in Definition 2.2.1. ✓

(ii) Let $\mathcal{U} := \{U_\alpha\}_{\alpha \in J} \subseteq \mathcal{T}$. Then, $x \in \bigcup \mathcal{U}$ implies $\exists \alpha \in J$, $x \in U_\alpha$. Since $U_\alpha \in \mathcal{T}$, there is $B \in \mathcal{B}$ such that $x \in B \subseteq U_\alpha \subseteq \bigcup \mathcal{U}$. This means $\bigcup \mathcal{U} \subseteq \mathcal{T}$. ✓

(iii) It is enough to prove it for two sets U_1 and U_2 in $\mathcal{T}$. Let $x \in U_1 \cap U_2$. (If $U_1 \cap U_2 = \varnothing$, then it is done.) By the definition of $\mathcal{T}$, there are B_1 and B_2 in $\mathcal{B}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. Since $x \in B_1 \cap B_2$, there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$. Thus, it implies $U_1 \cap U_2 \in \mathcal{T}$. ✓

□

Lemma 2.2.2

If $\mathcal{T}$ is the topology generated by basis $\mathcal{B}$ for X, then $\mathcal{T}$ is the collection of all unions of elements of $\mathcal{B}$. In other words, $\mathcal{T} = \left\{\bigcup \mathcal{U} \,\middle|\, \mathcal{U} \subseteq \mathcal{B}\right\}$.

Proof. Let $\mathcal{T}' := \left\{\bigcup \mathcal{U} \,\middle|\, \mathcal{U} \subseteq \mathcal{B}\right\}$. Since $\mathcal{B} \subseteq \mathcal{T}$ and $\mathcal{T}$ is a topology by Lemma 2.2.1, $\mathcal{T}' \subseteq \mathcal{T}$ follows. (See (ii) in Definition 2.1.1.) Now, we shall prove $\mathcal{T} \subseteq \mathcal{T}'$.

Take any $U \in \mathcal{T}$. Then, for each $x \in U$, there is $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$. Then, $U = \bigcup_{x \in U} B_x \in \mathcal{T}'$, hence $\mathcal{T} \subseteq \mathcal{T}'$. □

Lemma 2.2.3

Let $(X,\mathcal{T})$ be a topological space. If $\mathcal{C}$ is a subset of $\mathcal{T}$ such that

$$\forall U \in \mathcal{T}, (x \in U \implies \exists C \in \mathcal{C}, x \in C \subseteq U),$$

then $\mathcal{C}$ is a basis for X and $\mathcal{T}$ is the topology generated by $\mathcal{C}$.

Proof. We shall prove first $\mathcal{C}$ is a basis for X.

(i) Since $X \in \mathcal{T}$, $\forall x \in X$, $\exists C \in \mathcal{C}$, $x \in C$. ✓

(ii) Let $C_1, C_2 \in \mathcal{C}$ and suppose $x \in C_1 \cap C_2$. Since $C_1 \cap C_2 \in \mathcal{T}$, there is $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$. ✓

Now let $\mathcal{T}'$ be the topology generated by $\mathcal{C}$. We want to show $\mathcal{T} = \mathcal{T}'$.

For $\mathcal{T}' \subseteq \mathcal{T}$, take any $U \in \mathcal{T}'$. Then, by Lemma 2.2.2, $U = \bigcup_{\alpha \in J} C_\alpha$ where each C_α is in $\mathcal{C}$. Now, $U = \bigcup_{\alpha \in J} C_\alpha \in \mathcal{T}$ directly follows. The last inclusion is due to (ii) in Definition 2.1.1 and $\mathcal{C} \subseteq \mathcal{T}$. ✓

For $\mathcal{T} \subseteq \mathcal{T}'$, take any $U \in \mathcal{T}$. Then, for any $x \in U$, there is $C \in \mathcal{C}$ such that $x \in C \subseteq U$, therefore $U \in \mathcal{T}'$ by Definition 2.2.1. □

Lemma 2.2.4

Let $\mathcal{T}$ and $\mathcal{T}'$ are topologies genereated by bases $\mathcal{B}$ and $\mathcal{B}'$, respectively. Then,

$$\mathcal{T} \subseteq \mathcal{T}' \iff \forall B \in \mathcal{B}, \big(x \in B \implies \exists B' \in \mathcal{B}', x \in B' \subseteq B\big).$$

Proof. ($\Leftarrow$) Take any $U \in \mathcal{T}$ and $x \in U$. Since $\mathcal{B}$ generates $\mathcal{T}$, there is $B \in \mathcal{B}$ such that $x \in B \subseteq U$. By the supposition, there is $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B \subseteq U$. This implies we can find $B' \in \mathcal{B}'$ such that $x \in B' \subseteq U$, by definition, $U \in \mathcal{T}'$. ✓

($\Rightarrow$) Take any $B \in \mathcal{B}$ and $x \in B$. Since $B \in \mathcal{T} \subseteq \mathcal{T}'$, by definition of $\mathcal{T}'$, there is $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. ✓ □

Example 2.2.1

Let $\mathcal{B}$ be a set of open region inside a disk, and $\mathcal{B}'$ be a set of open region inside a rectangle. They are bases for $\mathbb{R}^2$, and topologies generated by them are the same by Lemma 2.2.4.

Definition 2.2.2: Common Topologies on $\mathbb{R}$

Define

- $\mathcal{B}_\mathbb{R} := \{(a,b) \subseteq \mathbb{R} \mid a < b\}$
- $\mathcal{B}_\ell := \{[a,b) \subseteq \mathbb{R} \mid a < b\}$

$\mathcal{B}$ and $\mathcal{B}'$ are bases for $\mathbb{R}$. Then,

- $\mathcal{T}_\mathbb{R}$, the topology generated by $\mathcal{B}$, is called the *standard topology* on $\mathbb{R}$, and
- $\mathcal{T}_\ell$, the topology generated by $\mathcal{B}_\ell$, is called the *lower limit topology* on $\mathbb{R}$.

Let $K := \{1/n \mid n \in \mathbb{Z}_+\}$ and $\mathcal{B}_K := \mathcal{B}_\mathbb{R} \cup \{(a,b) \setminus K \mid a < b\}$ Then, $\mathcal{B}''$ is a basis for $\mathbb{R}$ and

- $\mathcal{T}_K$, the topology generated by $\mathcal{B}_K$, is called the *K-topology* on $\mathbb{R}$.

Lemma 2.2.5 Comparison Among the Common Topologies on $\mathbb{R}$

The following holds.

(i) $\mathcal{T}_{\mathbb{R}} \subsetneq \mathcal{T}_\ell$ ($\mathcal{T}_\ell$ is strictly finer than $\mathcal{T}_{\mathbb{R}}$.)
(ii) $\mathcal{T}_{\mathbb{R}} \subsetneq \mathcal{T}_K$ ($\mathcal{T}_K$ is strictly finer than $\mathcal{T}_{\mathbb{R}}$.)
(iii) $\mathcal{T}_\ell$ and $\mathcal{T}_K$ are not comparable.

Proof.

(i) For any $(a,b) \in \mathcal{B}_{\mathbb{R}}$ and $x \in (a,b)$, $[x,b) \in \mathcal{B}_\ell$ and $x \in [x,b) \subseteq (a,b)$. Therefore, by Lemma 2.2.4, $\mathcal{T}_{\mathbb{R}} \subseteq \mathcal{T}_\ell$. ✓
Take any $a \in \mathbb{R}$. a is in the interval $[a,b) \in \mathcal{B}_\ell$ but there are no open interval $(c,d) \in \mathcal{B}_{\mathbb{R}}$ such that $a \in (c,d) \subseteq [a,b)$. Therefore, by Lemma 2.2.4, $\mathcal{T}_\ell \nsubseteq \mathcal{T}_{\mathbb{R}}$. ✓
(ii) $\mathcal{T}_{\mathbb{R}} \subseteq \mathcal{T}_K$ directly follows from $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{B}_K$. ✓
Although $0 \in (-1,1) \setminus K \in \mathcal{T}_K$, there is no $(c,d) \in \mathcal{B}_{\mathbb{R}}$ such that $0 \in (c,d) \in (-1,1) \setminus K$. Therefore, by Lemma 2.2.4, $\mathcal{T}_K \nsubseteq \mathcal{T}_{\mathbb{R}}$. ✓
(iii) The logics in (i) and (ii) can directly imported to prove (iii). ✓

□

Definition 2.2.3: Subbasis

A *subbasis* $\mathcal{S}$ for X is a subset of $\mathcal{P}(X)$ whose union is X, i.e., $\bigcup \mathcal{S} = X$.
The *topology generated by the subbasis* $\mathcal{S}$ is defined to be the collection of all unions of finite intersections of elements of $\mathcal{S}$.

Lemma 2.2.6

Let $\mathcal{S}$ be a subbasis for X. Then, the topology generated by $\mathcal{S}$ is a topology on X.

Proof. By Lemma 2.2.2, it is enough to show that $\mathcal{B} := \left\{ \bigcap_{i=1}^n S_i \mid S_i \in \mathcal{S} \right\}$ is a basis.
(i) Since $\mathcal{S} \subseteq \mathcal{B}$, $X = \bigcup \mathcal{S} \subseteq \bigcup \mathcal{B} \subseteq X$. ✓
(ii) Let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$. Then, $B_1 = \bigcap_{i=1}^n S_i$ and $B_2 = \bigcap_{i=1}^m S_i'$ where $S_i, S_i' \in \mathcal{S}$.
Then, $B_1 \cap B_2 = \left(\bigcap_{i=1}^n S_i\right) \cap \left(\bigcap_{i=1}^m S_i'\right) \in \mathcal{B}$. ✓

□

2.3 The Order Topology

Definition 2.3.1: Intervals

Let X be a set with an order $<$ and $a, b \in X$ with $a < b$ are given.
- $(a,b) := \{x \in X \mid a < x < b\}$ (open interval)
- $[a,b) := \{x \in X \mid a \leq x < b\}$ (half-open interval)
- $(a,b] := \{x \in X \mid a < x \leq b\}$ (half-open interval)
- $[a,b] := \{x \in X \mid a \leq x \leq b\}$ (closed interval)

Definition 2.3.2: Order Topology

Let X has more than one element. Let $\mathcal{B}$ be collection of
- all open intervals (a,b) in X,
- all half-open intervals $[a_0, b)$ where a_0 is the smallest element (if a_0 exists), and
- all half-open intervals $(a, b_0]$ where b_0 is the largest element (if b_0 exists).

Then, $\mathcal{B}$ is a basis and the topology generated by $\mathcal{B}$ is called the *order topology*.

Lemma 2.3.1

The set $\mathcal{B}$ above is a basis.

Proof.

(i) Take any $x \in X$.
- If x is the smallest, then $x \in [x, b)$ where b is some element in $X \setminus \{x\}$.
- If x is the largest, then $x \in (a, x]$ where a is some element in $X \setminus \{x\}$.
- Otherwise, there are some $a, b \in X \setminus \{x\}$ such that $a < x < b$ so $x \in (a, b)$. ✓

(ii) A nonempty intersection of two basis with different types of interval is an open interval. An intersection of two basis with the same type of interval still belongs to the type of interval. ✓

□

Example 2.3.1

The order topology on $\mathbb{Z}_+$ is the discrete topology. $n \in (n-1, n+1) = \{n\}$ if $n > 1$ and $1 \in [1, 2) = \{1\}$.

Example 2.3.2

The order topology on $\mathbb{R}$ is the standard topology on $\mathbb{R}$.

Definition 2.3.3: Ray

Let X be an order set and $a \in X$. There are four types of rays.
- $(a, \infty) := \{ x \in X \mid x > a \}$ (open ray)
- $(-\infty, a) := \{ x \in X \mid x < a \}$ (open ray)
- $[a, \infty) := \{ x \in X \mid x \geq a \}$ (closed ray)
- $(-\infty, a] := \{ x \in X \mid x \leq a \}$ (closed ray)

Note:-

Open rays are open in the order topology.
- If X has a largest element b_0, then $(a, \infty) = (a, b_0]$.
- Otherwise, $(a, \infty) = \bigcup_{a<b}(a, b)$.

Thus, (a, ∞) is open. Similarly, $(-\infty, a)$ is open.

Note:-

Open rays form a subbasis that generates the order topology.

2.4 The Product Topology on $X \times Y$

Definition 2.4.1: Product Topology

Let X, Y be topological spaces. The *product topology* on $X \times Y$ is the topology generated by a basis

$$\mathcal{B} := \{ U \times V \mid U \subseteq X \text{ and } V \subseteq Y \text{ are open} \}.$$

Theorem 2.4.1

Let $\mathcal{B}$ be a basis for X nd $\mathcal{C}$ be a basis for Y. Then

$$\mathcal{D} := \{ B \times C \mid B \in \mathcal{B} \text{ and } C \in \mathcal{C} \}$$

is a basis for the product topology of $X \times Y$.

Proof. We will exploit Lemma 2.2.3. Take any open set $W \subseteq X \times Y$ and $x \times y \in W$. Then, there is a basis element $U \times V$ of the product topology $X \times Y$ such that $x \times y \in U \times V \subsetneq W$. Since U and V are open in X and Y, respectively, and $x \in U$ and $y \in V$, there are $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $x \in B \subsetneq U$ and $y \in C \subsetneq V$.

Here, we find that $x \times y \in B \times C \subsetneq U \times V \subsetneq W$ while $B \times C \in \mathcal{D}$. Therefore, by Lemma 2.2.3, $\mathcal{D}$ generates the product topology. □

Definition 2.4.2: Projection

Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ defined by the equations

$$\pi_1(x, y) = x$$
$$\pi_2(x, y) = y$$

The maps π_1 and π_2 are called the *projections* of $X \times Y$ onto its first and second factors, respectively.

Note:-

If $U \subseteq X$ is open, then $\pi_1^{-1}(U) = U \times Y$ is open. Similarly, if $V \subseteq Y$ is open, then $\pi_2^{-1}(V) = X \times V$ is open.

Theorem 2.4.2

The collection

$$\mathcal{S} := \{ \pi_1^{-1}(U) \mid U \subseteq X \text{ is open} \} \cup \{ \pi_2^{-1}(V) \mid V \subseteq Y \text{ is open} \}$$

is a subbasis for the product topology of $X \times Y$.

Proof. Let $\mathcal{T}$ be the product topology and $\mathcal{T}'$ be the topology generated by $\mathcal{S}$.

- Since $\mathcal{S} \subseteq \mathcal{T}$, every union of finite intersections in $\mathcal{S}$ is in $\mathcal{T}$. Thus, $\mathcal{T}' \subseteq \mathcal{T}$. ✓
- Every open set of $\mathcal{T}$ is a union of elements in $\mathcal{B} := \{ U \times V \mid U \subseteq X \text{ and } V \subseteq Y \text{ are open} \}$. Noting that each $U \times V$ can be expressed as $\pi_1^{-1}(U) \cap \pi_2^{-1}(V)$, which is a finite intersection of elements in $\mathcal{S}$, we may conclude $\mathcal{T} \subseteq \mathcal{T}'$. ✓

□

2.5 The Subspace Topology

Definition 2.5.1: Subspace Topology

Let $(X, \mathcal{T})$ be a topological space. If $Y \subseteq X$, then

$$\mathcal{T}_Y := \{ Y \cap U \mid U \in \mathcal{T} \}$$

is called the *subspace topology* of Y and $(Y, \mathcal{T}_Y)$ is called a *subspace* of $(X, \mathcal{T})$.

Lemma 2.5.1

$(Y, \mathcal{T}_Y)$ is a topological space.

Proof.

(i) $\varnothing = Y \cap \varnothing$ and $Y = Y \cap X$. ✓

(ii) If $U_\alpha \in \mathcal{T}_Y$, $\bigcup_{\alpha \in J} (Y \cap U_\alpha) = Y \cap \left(\bigcup_{\alpha \in J} U_\alpha \right) \in \mathcal{T}_Y$. ✓

(iii) If $U_i \in \mathcal{T}_Y$, $\bigcap_{i=1}^{n} (Y \cap U_i) = Y \cap \left(\bigcap_{i=1}^{n} U_i \right) \in \mathcal{T}_Y$. ✓

□

Lemma 2.5.2

If $\mathcal{B}$ is a basis for $(X, \mathcal{T})$, then

$$\mathcal{B}_Y := \{ Y \cap B \mid B \in \mathcal{B} \}$$

is a basis for the subspace topology on Y.

Proof. We will exploit Lemma 2.2.3.

Take any $U \in \mathcal{T}$ and $y \in Y \cap U$. Since $y \in U$, $\exists B \in \mathcal{B}$, $y \in B \subseteq U$, which implies $y \in Y \cap B \subseteq Y \cap U$. □

Note:-

Not all open sets in Y are open in X.
For instance, if $X = \mathbb{R}$ and $Y = [0, 1]$, Y is open in Y but not open in X.

Lemma 2.5.3

All the open sets in Y are open in X if and only if Y is open in X.

Proof. ($\Rightarrow$) Y is open in Y. Hence, Y is open in X.

($\Leftarrow$) Let U be any open set in Y. Then, $U = Y \cap V$ for some open set V in X. Since Y is open in X, U is open in X. □

Theorem 2.5.1

If A is a subspace of X and B is a subspace of Y, then the product topology on $A \times B$ is the same as the the topology $A \times B$ inherits as a subspace of $X \times Y$. In other words, the following two topologies are the same.

$$\text{(i)} \qquad X, Y \xrightarrow{\text{subspace}} A \subseteq X,\ B \subseteq Y \xrightarrow{\text{product}} A \times B$$

$$\text{(ii)} \qquad X, Y \xrightarrow{\text{product}} X \times Y \xrightarrow{\text{subspace}} A \times B \subseteq X \times Y$$

Proof. By Theorem 2.4.1,

$$\{ U \times V \mid U \in \mathcal{B}_X \text{ and } V \in \mathcal{B}_Y \}$$

is a basis for $X \times Y$. Thus,

$$\mathcal{B} := \{ (A \times B) \cap (U \times V) \mid U \in \mathcal{B}_X \text{ and } V \in \mathcal{B}_Y \}$$

is a basis for (ii) by Lemma 2.5.2.

Note that $(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V)$. Also, $\{ A \cap U \mid U \in \mathcal{B}_X \}$ and $\{ B \cap V \mid V \in \mathcal{B}_Y \}$ are bases for A and B. Thus, $\mathcal{B}$ is also a basis for (i) by Theorem 2.4.1. □

Wrong Concept 2.1: Order Topology and Subspace Topology

Unlike product topology and subspace topology, order topology and subspace topology are not associative. Let X be an ordered set and $Y \subseteq X$.

$$\begin{array}{ll} \text{(i)} & \qquad\qquad Y \xrightarrow{\text{order}} Y \\ \text{(ii)} & X \xrightarrow{\text{order}} X \xrightarrow{\text{subspace}} Y \subseteq X \end{array}$$

Then, will those be the same?

Example 1. Consider $X = \mathbb{R}$ and $Y = [0,1]$. Then, the subspace topology of the order topology X has a basis of

$$\mathcal{B}_{[0,1]} = \{ [0,1] \cap (a,b) \mid a < b \},$$

which is in fact the order topology on Y. In this case, (i) = (ii).

Example 2. Consider $X = \mathbb{R}$ and $Y = [0,1) \cup \{2\}$. Then, $\{2\}$ is an open in (ii) since $\{2\} = Y \cap (1.5, 2.5)$. But, there is no basis of the order topology on Y such that contains 2 and is a subset of $\{2\}$. Thus, in this case, (i) $\neq$ (ii).

Example 3. Consider $X = \mathbb{R}^2$ and $Y = I^2$ where $I = [0,1]$. Then, $\{1/2\} \times (1/2, 1]$ is an open set in (ii) since it is $\big(\{1/2\} \times (1/2, 3/2)\big) \cap I^2$. But it is not an open set in (i) since there is no basis that contain $(1/2, 1)$ and is a subset of $\{1/2\} \times (1/2, 1]$.

Definition 2.5.2: Convex Subset

Given an ordered set X and $Y \subseteq X$, Y is called *convex* if

$$\forall a, b \in Y, \big(a < b \implies (a,b) \subseteq Y\big).$$

Theorem 2.5.2

Let X be an ordered set with the ordered topology. If $Y \subseteq X$ is convex, then the order topology on Y is the same as the subspace topology.

Proof. We will make use of the fact that open rays form a subbasis that generates the order topology.

First, every open ray of (i) is an open ray of the subspace (ii).

$$\{ x \in Y \mid x > a \} = \{ x \in X \cap Y \mid x > a \},$$

for example. Therefore, (ii) is finer than (i).

Now, take any open ray in X, $(a,\infty)_X = \{x \in X \mid x > a\}$, for instance. Then, let

$$\begin{aligned} R &\triangleq (a,\infty)_X \cap Y \\ &= \{y \in Y \mid y > a\} = (a,\infty)_Y. \end{aligned}$$

If $a \in Y$, then R is an open ray in Y.

Now consider the case $a \notin Y$. If R is nonempty then there is some $y_0 \in R$. Take any $y \in Y$. If $y_0 < y$, then $y \in R$ since $a < y_0 < y$. If $y < y_0$, it implies $a < y < y_0$ because $y < a < y_0$ with $y, y_0 \in Y$ implies $a \in Y$ by the convexity of Y. Therefore, $y \in R$. So, if $a \notin Y$, it is either $R = \varnothing$ or $R = Y$.

Combining the cases, we get the fact that the intersection of Y and an arbitrary open ray in X is an open ray in Y, an empty set, or the whole Y.

This is the final step. Take any open set U in the ordered topology X. Then, $U = \bigcup_{\alpha\in J} U_\alpha$ where $U_\alpha \neq \varnothing$ is a finite intersection of open rays in X. Noting that $U \cap Y$ is a general form of an open set in Y, we get $U \cap Y = \bigcup_{\alpha\in J}(U_\alpha \cap Y)$, which implies either $U \cap Y = Y$ or $U \cap Y$ is a union of finite intersections of an open ray in Y. □

Corollary 2.5.1

Let X be an ordered set with the ordered topology. The subspace topology of $Y \subseteq X$ is finer than the order topology on Y.

2.6 Closed Sets and Limit Points

2.6.1 Closed Sets

Definition 2.6.1: Closed Set

Let X be a topological space. A subset $A \subseteq X$ is *closed* if $X \setminus A$ is open.

Example 2.6.1

- $[a,b] \subseteq \mathbb{R}$ is closed since $(-\infty, a) \cup (b, \infty)$ is open.
- $[a,b] \times [c,d] \subseteq \mathbb{R}^2$ is closed.
- In discrete topology on X, every subset of X is closed.
- If $Y = [0,1] \cup (2,3) \subseteq \mathbb{R}$, $[0,1]$ and $(2,3)$ are both open and closed in Y.

Theorem 2.6.1

Let X be a topological space. Then the following conditions hold.

(i) $\varnothing$ and X are closed.
(ii) Arbitrary intersections of closed sets are closed.
(iii) Finite unions of closed sets are closed.

Proof.

(i) $X \setminus \varnothing = X$ and $X \setminus X = \varnothing$ are open. ✓
(ii) Let $\{A_\alpha\}_{\alpha\in J}$ be a collection of closed sets. Then,

$$X \setminus \bigcap_{\alpha\in J} A_\alpha = \bigcup_{\alpha\in J}(X \setminus A_\alpha).$$

is open since each $X \setminus A_\alpha$ is open. ✓

(iii) Let $\{A_i\}_{i=1}^n$ be a collection of closed sets. Then,

$$X \setminus \bigcup_{i=1}^n A_i = \bigcap_{i=1}^n (X \setminus A_i).$$

is open since it is a finite intersection of open sets. ✓

□

Theorem 2.6.2

Let X be a topological space and $Y \subseteq X$. Then $A \subseteq Y$ is closed in Y if and only if there is a closed set B in X such that $A = Y \cap B$.

Proof. ($\Leftarrow$) Let B be a closed set of X such that $A = Y \cap B$. Then, $X \setminus B$ is open in X and $Y \cap (X \setminus B) = Y \setminus A$ is open in Y. Thus, A is closed in Y.

($\Rightarrow$) Since $Y \setminus A$ is open in Y, $Y \setminus A = Y \cap U$ for some open set U in X. Then, $A = Y \cap (X \setminus U)$ where $X \setminus U$ is closed in X. □

Theorem 2.6.3

If Y is closed in X, then every closed sets of Y are closed in X if and only if Y is closed in X.

Proof. Proof is analogous to the proof of Lemma 2.5.3. □

Definition 2.6.2: Interior and Closure of a Set

Given a subset A of a topological space $(X, \mathcal{T})$,

- the *interior* of A is $\mathring{A} \triangleq \bigcup\{U \subseteq X \mid U \in \mathcal{T} \text{ and } U \subseteq A\}$, and
- the *closure* of A is $\overline{A} \triangleq \bigcap\{V \subseteq X \mid X \setminus V \in \mathcal{T} \text{ and } A \subseteq V\}$.

Note:-

- $\mathring{A} \subseteq A \subseteq \overline{A}$
- $\mathring{A}$ is open, and $\overline{A}$ is closed.
- $\mathring{A}$ is the largest open set contained in A, and $\overline{A}$ is the smallest closed set containing A.

Theorem 2.6.4

Let Y be a subspace of X and $A \subseteq Y$. Let $\overline{A}$ and $\overline{A}_Y$ denote the closures of A in X and Y, respectively. Then,

$$\overline{A} \cap Y = \overline{A}_Y.$$

Proof. ($\supseteq$) $\overline{A} \cap Y$ is closed in Y by Theorem 2.6.2. Thus, $\overline{A}_Y \subseteq \overline{A} \cap Y$.

($\subseteq$) $\overline{A}_Y = B \cap Y$ for some closed set B in X by Theorem 2.6.2. Also, $\overline{A} \subseteq B$ holds. Therefore, $\overline{A}_Y = B \cap Y \subseteq \overline{A} \cap Y$. □

Definition 2.6.3: Intersection and Neighborhood

- Given two sets A and B, we say A and B *intersect* if $A \cap B \neq \varnothing$.
- An open set containing $x \in X$ is called an open *neighborhood* of x.

Theorem 2.6.5

Let $A \subseteq X$ where X is a topological space. The following hold.

(i) $x \in \overline{A}$ if and only if every neighborhood of x intersects A.

(ii) Let $\mathcal{B}$ be a basis for X. Then, $x \in \overline{A}$ if and only if every $B \in \mathcal{B}$ containing x intersects A.

Proof.

(i) We will prove the contrapositive "$x \notin \overline{A} \iff \exists$ neighborhood U of X, $U \cap A = \varnothing$".

($\Rightarrow$) $U \triangleq X \setminus \overline{A}$ is a neighborhood of x. We find that $U \cap A = \varnothing$ since $A \subseteq \overline{A}$. ✓

($\Leftarrow$) Suppose a neighborhood U of x satisfies $U \cap A = \varnothing$. It implies $A \subseteq X \setminus U$. Since $X \setminus U$ is closed, $\overline{A} \subseteq X \setminus U$ also holds. Since $x \in U$, $x \in \overline{A}$ may never hold. ✓

(ii) ($\Rightarrow$) A basis element that contains x is a neighborhood of x. ✓

($\Leftarrow$) Follows from the definition of basis. (See Definition 2.2.1.) ✓

□

Example 2.6.2

- If $A = (0, 1/2) \subseteq \mathbb{R}$, then $\overline{A} = [0, 1/2]$.
- If $A = \{1/n \mid n \in \mathbb{Z}_+\} \subseteq \mathbb{R}$, then $\overline{A} = A \cup \{0\}$.
- If $A = \mathbb{Q} \subseteq \mathbb{R}$, then $\overline{A} = \mathbb{R}$.
- If $A = \mathbb{Z} \subseteq \mathbb{R}$, then $\overline{A} = \mathbb{Z}$.

2.6.2 Limit Points

Definition 2.6.4: Limit Point

Let $A \subseteq X$ and $x \in X$. The point x is a *limit point* of A if every neighborhood of x intersects A in some point other than x. The set of limit points of A is denoted by A'.

Note:-

Equivalently, x is a limit point of A if $x \in \overline{A \setminus \{x\}}$ thanks to Theorem 2.6.5.

Theorem 2.6.6

Let $A \subseteq X$ where X is a topological space. Then

$$\overline{A} = A \cup A'.$$

Proof. ($\supseteq$) We only need to show $A' \subseteq \overline{A}$. For every $x \in A'$, $x \in \overline{A}$ due to Theorem 2.6.5. ✓

($\subseteq$) Let $x \in \overline{A} \setminus A$. By definition, every neighborhood of x intersects A while x cannot be in the intersection since $x \notin A$. Thus, $x \in A'$. ✓ □

Corollary 2.6.1

Let $A \subseteq X$ where X is a topological space. Then A is closed if and only if $A' \subseteq A$.

Proof. ($\Rightarrow$) $A = \overline{A} = A \cup A'$ and it implies $A' \subseteq A$. ✓

($\Leftarrow$) $\overline{A} = A \cup A' = A$ and $\overline{A}$ is closed. ✓ □

Definition 2.6.5: Convergence of a Sequence

Let X be a topological space. Then, a sequence $\{x_n\}$ in X converges to $x \in X$ if, for every neighborhood U of x, there exists $N \in \mathbb{Z}_+$ such that $x_n \in U$ for all $n \geq N$.

Note:-

The point to which a sequence converges may not be unique in general. If $X = \{a, b, c\}$ and $\mathcal{T} = \{\varnothing, X, \{b\}, \{a, b\}, \{b, c\}\}$, the sequence $x_n = b$ may converge to a, b, or c as any neighborhood of a or c contains b.

2.6.3 Hausdorff Spaces

Definition 2.6.6: Housdorff Space

A topological space $(X, \mathcal{T})$ is called a *Hausdorff space* if for each pair x_1 and x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2, respectively, that are disjoint. In other words,

$$\forall x_1, x_2 \in X, \left(x_1 \neq x_2 \implies \exists U_1, U_2 \in \mathcal{T},\ x_1 \in U_1 \land x_2 \in U_2 \land U_1 \cap U_2 = \varnothing\right).$$

Theorem 2.6.7

Every finite point set in a Hausdorff space X is closed.

Proof. It suffices to prove that every singleton of X is closed since closedness of finite point set will be naturally driven by Theorem 2.6.1.

If $|X| \leq 1$, then it is done. Now, let x and y be distinct elements in X. Then, there are disjoint open sets U and V such that $x \in U$ and $y \in V$. Therefore, x and y are not limit points of each other. Thus, there are at most one limit point of $\{x\}$. (If it exists, it must be x.) Thus, $\{x\}' \subseteq \{x\}$; $\{x\}$ is closed by Corollary 2.6.1. □

Definition 2.6.7: T_1 Axiom

A topological space X is said to satisfy T_1 *axiom* if every singleton in X is closed.

Note:-

Theorem 2.6.7 implies that every Hausdorff space satisfies T_1 axiom.

Note:-

T_1 axiom is strictly weaker than being a Hausdorff space.

- $\mathbb{R}$ in the finite complement topology satisfies T_1 axiom. Every singleton $\{x\}$ is closed since $\mathbb{R} \setminus \{x\}$ is open.
- However, it is not a Hausdorff space. Suppose $x, y \in \mathbb{R}$ with $x \neq y$ and there are disjoint open set U and V such that $x \in U$ and $y \in V$. Then, since $U \cap V = \varnothing$, $\mathbb{R} = \mathbb{R} \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$, which is impossible siunce $X \setminus U$ and $X \setminus V$ are finite.

Theorem 2.6.8

Let X be a space satisfying the T_1 axiom; let $A \subseteq X$. Then $x \in A'$ if and only if every neighborhood of x contains infinitely many points of A.

Proof. ($\Rightarrow$) Let $x \in A'$ and suppose some neighborhood U of x intersects A in finitely many points. Then, it also intersects $A \setminus \{x\}$ in finitely many points; let us denote them $x_1, x_2, \cdots, x_m$. Noting that $\{x_1, x_2, \cdots, x_m\}$ is closed as X satisfies T_1 axiom, $X \setminus \{x_1, x_2, \cdots, x_m\}$ is a neighborhood of x but does not intersect $A \setminus \{x\}$, contradicting that x is a limit point of A.

($\Leftarrow$) Let U be any neighborhood of x. Then, U intersects A in infinitely many points by assumption, and thus it intersects $A \setminus \{x\}$ in infinitely many points. Therefore, x is a limit point of A. □

Theorem 2.6.9

If X is a Hausdorff space, then there is at most one point of X to which a sequence of points of X converges.

Proof. Suppose $\{x_n\}$ is a sequence in X that converges to x. If $y \neq x$, we may find disjoint neighborhoods U and V of x and y, respectively. Then, U has all but finitely many points of x_n, but V cannot. Therefore, y cannot be a point that $\{x_n\}$ converges to. □

Note:-

The finite complement topology on $\mathbb{R}$ is not a Hausdorff.
Let $\{x_n\}$ be a sequence that has no points infinitely repeated in $\{x_n\}$. Then, $\{x_n\}$ converges to every point in $\mathbb{R}^n$.

2.7 Continuous Functions

2.7.1 Continuity of a Function

Definition 2.7.1: Continuity of a Function

Let X and Y be topological spaces. A function $f : X \to Y$ is said to be *continuous* if for each open subset V of Y, $f^{-1}(V)$ is open in X.

Note:-

To prove a function $f : X \to Y$ is continuous, it is enough to prove that every basis of Y has an open preimage in X. Then, for every open $V = \bigcup_{\alpha \in J} B_\alpha \subseteq Y$, it follows that

$$f^{-1}(V) = \bigcup_{\alpha \in J} f^{-1}(B_\alpha)$$

is open in X.
If the topology on Y is given by a subbasis, it is even sufficient to prove every preimage of subbasis element is open. Then, for every basis $B = \bigcap_{i=1}^n S_i$, it follows that

$$f^{-1}(B) = \bigcap_{i=1}^{n} f^{-1}(S_i)$$

is open in X.

Theorem 2.7.1

Let X and Y be topological spaces. TFAE

(i) f is continuous.

(ii) For every subset A of X, $f(\overline{A}) \subseteq \overline{f(A)}$.
(iii) For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.
(iv) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.

Proof. ((i) $\Longrightarrow$ (ii)) Take any $x \in \overline{A}$. Let V be any neighborhood of $f(x)$. Then, $f^{-1}(V)$ is a neighborhood of x. Since $x \in \overline{A}$, by Theorem 2.6.5, $f^{-1}(V)$ intersects A; $A \cap f^{-1}(V) \neq \varnothing$. Therefore, since $\varnothing \neq f(A \cap f^{-1}(V)) = f(A) \cap f(f^{-1}(V)) \subseteq f(A) \cap V$, V intersects $f(A)$; by Theorem 2.6.5, $f(x) \in \overline{f(A)}$ as V was arbitrary. Therefore, $f(\overline{A}) \subseteq \overline{f(A)}$.

((ii) $\Longrightarrow$ (iii)) Let B be closed in Y and let $A \triangleq f^{-1}(B)$. Then, $f(A) = f(f^{-1}(B)) \subseteq B$. Therefore, if $x \in \overline{A}$, $f(x) \in f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{B} = B$; which implies $x \in f^{-1}(B) = A$. This means $\overline{A} \subseteq A$, thus A is closed.

((iii) $\Longrightarrow$ (i)) Let V be an open set of Y. Let $B \triangleq Y \setminus B$. Then

$$f^{-1}(B) = f^{-1}(Y) \setminus f^{-1}(V) = X \setminus f^{-1}(V)$$

is closed as B is closed. Thus, $f^{-1}(V) = X \setminus f^{-1}(B)$ is open.

((i) $\Longrightarrow$ (iv)) For every neighborhood V of $f(x)$, $U = f^{-1}(V)$ is the neighborhood of x that satisfies $f(U) \subseteq V$.

((iv) $\Longrightarrow$ (i)) Let V be an open set of Y. Then, for each $x \in f^{-1}(V)$, since V is a neighborhood of $f(x)$, there exists a neighborhood U_x of x that satisfies $f(U_x) \subseteq V$. Then, $U_x \subseteq f^{-1}(f(U_x)) \subseteq f^{-1}(V)$. Therfore, $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$ is open in X. □

2.7.2 Homeomorphisms

Definition 2.7.2: Homeomorphism

Let X and Y be topological spaces $f : X \to Y$ be a bijection. f is called a *homeomorphism* if both f and f^{-1} are continuous.

Note:-

Since the inverse image under f^{-1} is exactly the image under f, "f^{-1} is continuous" implies "$f(U)$ is open for all open U in X." So, f is a homeomorphism if and only if it is a bijection such that $U \subseteq X$ is open in X if and only if $f(U)$ is open in Y.

Note:-

If f is a homeomorphism between X and Y, then $\mathcal{T}_Y = \{f(U) \mid U \in \mathcal{T}_X\}$ and $\mathcal{T}_X = \{f^{-1}(V) \mid V \in \mathcal{T}_X\}$.
Therefore, any property of X that is entirely expressed in terms of the topology of X yields, via the correspondence f, the corresponding property for the space Y. Such a property of X is called *topological property* of X.

Homeomorphism preserves topological properties.

Definition 2.7.3: Open Map and Closed Map

Let X and Y be topological spaces $f : X \to Y$ be a function.

- f is said to be an *open map* if $f(U)$ is open for all open $U \subseteq X$ in X.
- f is said to be a *closed map* if $f(U)$ is closed for all closed $U \subseteq X$ in X.

Definition 2.7.4: Topological Imbedding

Let X and Y be topological spaces $f : X \hookrightarrow Y$ be an injection. Then, $f' : X \to f(X)$ obtained by restriction is a bijection. If f' is a homeomorphism in which the topology of $\operatorname{Im} f$ is given as the subspace topology, f is said to be a *topological imbedding*, or simply an *imbedding*, of X in Y.

2.7.3 Constructing Continuous Functions

Theorem 2.7.2 Rules for Constructing Continuous Functions

Let X, Y, and Z be topological spaces.

(i) (*Constant Function*) If $f : X \to Y$ has a singleton $f(X)$, f is continuous.

(ii) (*Inclusion*) If A is a subspace of X, the inclusion function $j : A \to X$ is continuous.

(iii) (*Composites*) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f$ is continuous.

(iv) (*Restricting the Domain*) If $f : X \to Y$ is continuous, and if A is a subspace of X, then the restricted function $f\big|_A : A \to Y$ is continuous.

(v) (*Restricting or Expanding the Codomain*) Let $f : X \to Y$ be continuous. If Z is a subspace of Y and $f(X) \subseteq Z$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the function $h : X \to Z$ obtained by expanding the range of f is continuous.

(vi) (*Local Formulation of Continuity*) The map $f : X \to Y$ is continuous if X is a union of open sets U_α such that $f\big|_{U_\alpha}$ is continuous for each α.

Proof.

(i) Let $f(x) = y_0$ for every $x \in X$ for some fixed $y_0 \in Y$. Then, for each (open) set $V \subseteq Y$,

$$f^{-1}(V) = \begin{cases} X & \text{if } y_0 \in V \\ \varnothing & \text{if } y_0 \notin V \end{cases}$$

is always open in X.

(ii) If $U$is open in X, then $f^{-1}(U) = U \cap A$ is open in A (by definition).

(iii) If U is open in Z, then $g^{-1}(U)$ is open in Y, and thus $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open in X.

(iv) $f\big|_A = f \circ j$ where $j : A \to X$ is the inclusion function. Therefore, $f\big|_A$ is continuous by (ii) and (iii).

(v) First, suppose $f(X) \subseteq Z \subseteq Y$. Take any open set $W \subseteq Z$ of Z. Then, $W = V \cap Z$ for some open set V in Y. Because $f(X) \subseteq Z$ and $f(x) = g(x)$ for all $x \in X$,

$$f^{-1}(V) = f^{-1}(V \cap Z) = f^{-1}(W) = g^{-1}(W).$$

Thus, $g^{-1}(W)$ is open in X as f is continuous.

We get h is continuous from noting that $h = j \circ f$ where $j\colon Y \to Z$ is the inclusion function.

(vi) Let $X = \bigcup_{\alpha\in J} U_\alpha$ in which, for each $\alpha \in J$, U_α is an open set in X such that $f\big|_{U_\alpha}$ is continuous. Let V be an open set in Y. Then

$$f^{-1}(V)\cap U_\alpha = \left(f\big|_{U_\alpha}\right)^{-1}(V)$$

for each $\alpha\in J$; $f^{-1}(V)\cap U_\alpha$ is open in X since $f\big|_{U_\alpha}$ is continuous. Therefore,

$$f^{-1}(V) = f^{-1}(V)\cap X = f^{-1}(V)\cap\left(\bigcup_{\alpha\in J} U_\alpha\right) = \bigcup_{\alpha\in J}\left(f^{-1}(V)\cap U_\alpha\right)$$

is open in X.

□

Theorem 2.7.3 The Pasting Lemma

Let $X = A\cup B$ be a topological space, where A and B are closed in B. Let $f : A\to Y$ and $g : B\to Y$ be continuous. If $f(x) = g(x)$ for every $x\in A\cap B$, then the function $h\colon X\to Y$ defined by

$$h(x) \triangleq \begin{cases} f(x) & \text{if } x\in A\\ g(x) & \text{if } x\in B\end{cases}$$

is continuous.

Proof. Let C be a closed subset of Y. Now

$$h^{-1}(C) = f^{-1}(C)\cup g^{-1}(C).$$

Since f and g are continuous and C is closed, $f^{-1}(C)$ and $g^{-1}(C)$ are closed by Theorem 2.7.1. Thus, $h^{-1}(C)$ is closed. Hence, h is continuous. □

Note:-

Theorem 2.7.3 holds if A and B are both open. It is, nonetheless, a special case of (vi) of Theorem 2.7.2.

Note:-

Theorem 2.7.3 does not hold if A is open and B is closed. For instance, the function $h\colon A\cup B\to\mathbb{R}$, where $A = (-\infty, 0)$ and $B = [0,\infty)$, defined by

$$h(x) \triangleq \begin{cases} x-2 & \text{if } x\in A\\ x+2 & \text{if } x\in B\end{cases}$$

is not continuous since $h^{-1}((1,3)) = [0,1)$ is not open.

Theorem 2.7.4 Maps Into Products

Let $f : A\to X\times Y$ be given by

$$f(a) = f_1(a)\times f_2(b).$$

Then f is continuous if and only if the functions

$$f_1 : A \to X \quad \text{and} \quad f_2 : A \to Y$$

are continuous.

Proof. ($\Rightarrow$) We first show that the projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are continuous. For each open sets $U \subseteq X$ and $V \subseteq Y$, $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$ are open; π_1 and π_2 are continuous.

Then, noting that $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$, we conlcude f_1 and f_2 are continuous.

($\Leftarrow$) For any basis element $U \times V$ in $X \times Y$,

$$\begin{aligned} f^{-1}(U \times V) &= \{a \in A \mid f(a) \in U \times V\} \\ &= \{a \in A \mid f_1(a) \in U \text{ and } f_2(a) \in V\} \\ &= f_1^{-1}(U) \cap f_2^{-1}(V). \end{aligned}$$

Thus, $f^{-1}(U \times V)$ is open since $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open. $\square$

2.8 The Product Topology

Definition 2.8.1: Box Topology

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of topological spaces. The topology generated by the basis

$$\mathcal{B} = \left\{ \prod_{\alpha \in J} U_\alpha \;\middle|\; \forall \alpha \in J,\ U_\alpha \text{ is open in } X_\alpha \right\}$$

for the product $\prod_{\alpha \in J} X_\alpha$ is called the *box topology*.

Note:-

The collection $\mathcal{B}$ is indeed a basis for $\prod_{\alpha \in J} X_\alpha$. $\bigcup \mathcal{B} = \prod_{\alpha \in J} X_\alpha$ holds since $\prod_{\alpha \in J} X_\alpha \in \mathcal{B}$. Also, an intersection of two basis elements is another basis element. This can be shown by

$$\left(\prod_{\alpha \in J} U_\alpha \right) \cap \left(\prod_{\alpha \in J} V_\alpha \right) = \prod_{\alpha \in J} (U_\alpha \cap V_\alpha).$$

Definition 2.8.2: Projection

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of sets. Let

$$\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$$

be defined by

$$(x_\alpha)_{\alpha \in J} \mapsto x_\beta$$

for each $\beta \in J$. Then, π_β is called the *projection mapping* associated with the index β.

Definition 2.8.3: Product Topology

Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of topological spaces. Let $\mathcal{S}_\beta$ denote the collection

$$\mathcal{S}_\beta = \{\, \pi_\beta^{-1}(U_\beta) \mid U_\beta \text{ is open in } X_\beta \,\}$$

and let

$$\mathcal{S} = \bigcup_{\alpha\in J} \mathcal{S}_\alpha.$$

The topology generated by the subbasis $\mathcal{S}$ for $\prod_{\alpha\in J} X_\alpha$ is called the *product topology*. In this topology, $\prod_{\alpha\in J} X_\alpha$ is called a *product space*.

Note:-

A typical basis of the product topology has a form of

$$B = \pi_{\beta_1}^{-1}(U_{\beta_1}) \cap \pi_{\beta_2}^{-1}(U_{\beta_2}) \cap \cdots \cap \pi_{\beta_n}^{-1}(U_{\beta_n})$$

where $\beta_i \in J$ and U_{β_i} is open in X_{β_i} for each $i \in [n]$. Since $\pi_\beta^{-1}(U_2) \cap \pi_\beta^{-1}(U_2) = \pi_\beta^{-1}(U_1 \cap U_2)$, without loss of generality, β_i's are mutually different. This means,

$$B = \prod_{\alpha\in J} U_\alpha$$

where $U_\alpha = \begin{cases} U_{\beta_i} & \text{if } \alpha = \beta_i \text{ for some } i \in [n] \\ X_\alpha & \text{otherwise.} \end{cases}$ In other words, a basis element is a product of U_α's where U_α is an open set of X_α for finitely many indices and $U_\alpha = X_\alpha$ for the remaining indices.

Note:-

- For finite products, i.e., for finite J, the box topology and the product topology on $\prod_{\alpha\in J} X_\alpha$ are the same.
- In general, the box topology is finer than the product topology since the basis of the box topology contains the basis of the product topology.

Theorem 2.8.1

Suppose the topology on each space X_α is given by a basis $\mathcal{B}_\alpha$. Then,

$$\mathcal{B}_1 = \left\{ \prod_{\alpha\in J} B_\alpha \,\middle|\, \forall \alpha \in J,\, B_\alpha \in \mathcal{B}_\alpha \right\}$$

is a basis for the box topology on $\prod_{\alpha\in J} X_\alpha$.
Moreover,

$$\mathcal{B}_2 = \left\{ \prod_{\alpha\in J} B_\alpha \,\middle|\, B_\alpha \in \mathcal{B}_\alpha \text{ for finitely many } \alpha\text{'s and } B_\alpha = X_\alpha \text{ for remaining indices} \right\}$$

is a basis for the product topology on $\prod_{\alpha\in J} X_\alpha$.

Proof. The basis for the box topology in Definition 2.8.1 has B_1 has a subset. Thus, the box topology is finer than the topology generated by B_1.

Also, for any basis element $\prod_{\alpha\in J} U_\alpha$ of the box topology and $x \in \prod_{\alpha\in J} U_\alpha$, since $x_\alpha \in U_\alpha$, there exists some $B_\alpha \in \mathcal{B}_\alpha$ such that $x_\alpha \in B_\alpha \subseteq U_\alpha$. Thus, $x \in \prod_{\alpha\in J} B_\alpha \subseteq \prod_{\alpha\in J} U_\alpha$; the topology generated by $\mathcal{B}_1$ is finer than the box topology by Lemma 2.2.4.

Every element in $\mathcal{B}_2$ is a basis element of the product topology. Thus, $\mathcal{B}_2$ generates a product which is coarser than the product topology.

Let $B = \prod_{\alpha\in J} U_\alpha$ be a basis of the product topology and $x \in B$. Then, $U_\alpha = X_\alpha$ for all but finitely many many indices; let $\alpha_1, \alpha_2, \cdots, \alpha_n$ denote indices where $U_\alpha \neq X_\alpha$. Then, for each $i \in [n]$, since $x_{\alpha_i} \in U_{\alpha_i}$, there exists bais element $B_{\alpha_i} \in \mathcal{B}_{\alpha_i}$ such that $x_{\alpha_i} \in B_{\alpha_i} \subseteq U_{\alpha_i}$. Thus, $x \in \prod_{\alpha\in J} B_\alpha \subseteq B$ where $B_\alpha = X_\alpha$ if $\alpha \notin \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. □

Theorem 2.8.2

Let A_α be a subspace of X_α for each $\alpha \in J$. Then $\prod_{\alpha\in J} A_\alpha$ is a subspace of $\prod_{\alpha\in J} X_\alpha$, if both products are given in the box topology, or if both products are given in the product topology.

Proof. (*For box topology*) The box topology on $\prod_{\alpha\in J} A_\alpha$ has a basis of

$$\left\{ \prod_{\alpha\in J}(A_\alpha \cap U_\alpha) \,\middle|\, U_\alpha \text{ is open in } X_\alpha \right\},$$

which is exactly equal to the subspace topology of $\prod_{\alpha\in J} A_\alpha$,

$$\left\{ \left(\prod_{\alpha\in J} A_\alpha\right) \cap \left(\prod_{\alpha\in J} U_\alpha\right) \,\middle|\, U_\alpha \text{ is open in } X_\alpha \right\}.$$

(*For product topology*) It is analogous; the theorem comes inherently from the fact that $\prod(A_\alpha \cap U_\alpha) = \left(\prod A_\alpha\right) \cap \left(\prod U_\alpha\right)$. □

Theorem 2.8.3

If each space X_α is a Hausdorff space, then $\prod_{\alpha\in J} X_\alpha$ is a Hausdorff space in both the box and the product topologies.

Proof. Let $x, y \in \prod_{\alpha\in J} X_\alpha$ with $x \neq y$. Then, there is some index $\alpha_0 \in J$ such that $x_{\alpha_0} \neq y_{\alpha_0}$. Then, since X_{α_0} is Hausdorff, there are disjoint neighborhoods U and V in X_{α_0} of x_{α_0} and y_{α_0}, respectively. Then, $x \in \prod_{\alpha\in J} U_\alpha$ and $y \in \prod_{\alpha\in J}$ where

$$U_\alpha \triangleq \begin{cases} U & \text{if } \alpha = \alpha_0 \\ X_\alpha & \text{otherwise} \end{cases} \quad \text{and} \quad V_\alpha \triangleq \begin{cases} V & \text{if } \alpha = \alpha_0 \\ X_\alpha & \text{otherwise.} \end{cases}$$

As $\prod_{\alpha\in J} U_\alpha$ and $\prod_{\alpha\in J} V_\alpha$ are open in both topologies, they are disjoint neighborhoods of x and y in both topologies. □

Theorem 2.8.4

Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of spaces and $A_\alpha \subseteq X_\alpha$ for each $\alpha \in J$. Then

$$\prod_{\alpha\in J} \overline{A_\alpha} = \overline{\prod_{\alpha\in J} A_\alpha}$$

in both the box and the product topologies.

Proof. ($\subseteq$) Let $x \in \prod_{\alpha\in J} \overline{A_\alpha}$. Let $U = \prod_{\alpha\in J} U_\alpha$ be a basis element (for either the box or the product topology) that contains x. For each $\alpha \in J$, since $x_\alpha \in \overline{A_\alpha}$ and U_α is a neighborhood of x, $U_\alpha \cap A_\alpha \neq \varnothing$ by Theorem 2.6.5. This implies

$$\left(\prod_{\alpha\in J} A_\alpha\right) \cap U = \left(\prod_{\alpha\in J} A_\alpha\right) \cap \left(\prod_{\alpha\in J} U_\alpha\right) = \prod_{\alpha\in J} (A_\alpha \cap U_\alpha) \neq \varnothing$$

Since the choice of U was arbitrary, by Theorem 2.6.5, $x \in \overline{\prod_{\alpha\in J} A_\alpha}$.

($\supseteq$) Let $x \in \overline{\prod_{\alpha\in J} A_\alpha}$. Fix any $\alpha_0 \in J$, and let U_{α_0} be a neighborhood of x_{α_0} in X_{α_0}. Since $\pi_{\alpha_0}^{-1}(U_{\alpha_0})$ is a neighborhood of x (in both topologies), $\pi_{\alpha_0}^{-1}(U_{\alpha_0}) \cap \prod_{\alpha\in J} A_\alpha \neq \varnothing$ by Theorem 2.6.5. In particular, at the α_0^{th} index, $U_{\alpha_0} \cap A_{\alpha_0} \neq \varnothing$. Thus, $x_{\alpha_0} \in \overline{A_{\alpha_0}}$.

Therefore, $x \in \prod_{\alpha\in J} \overline{A_\alpha}$. □

Note:-

Theorem 2.8.2, Theorem 2.8.3, and Theorem 2.8.4 illustrate the common property of the box and the product topologies. We are now going to investigate the *differences* that makes the product topology more useful.

Theorem 2.8.5

Let $f : A \to \prod_{\alpha\in J} X_\alpha$ be given by the equation

$$f(a) = \big(f_\alpha(a)\big)_{\alpha\in J},$$

where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod_{\alpha\in J} X_\alpha$ have the product topology. Then f is continuous if and only if each f_α is continuous.

Proof. ($\Rightarrow$) For each $\alpha \in J$, since π_α is continuous, $f_\alpha = \pi_\alpha \circ f$ is continuous by (iii) of Theorem 2.7.2.

($\Leftarrow$) Let $\pi_\alpha^{-1}(U_\alpha)$ be any subbasis element of the product topology. Since $\pi_\alpha \circ f = f_\alpha$, $f^{-1}(\pi_\alpha^{-1}(U_\alpha)) = f_\alpha^{-1}(U_\alpha)$ is open. Thus, f is contiuous. □

Note:-

It still holds in the box topology that, if f is continuous, then each f_α is continuous. The proof is exactly the same.
However, the converse does not hold. If we let $f : \mathbb{R} \to \mathbb{R}^\omega$ (where $\mathbb{R}$ is in the standard topology) defined by

$$f(t) = (t, t, t, \cdots),$$

the coordinate functions $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(t) = t$ are continuous. However, f is not continuous. The set

$$U = \prod_{n\in\mathbb{Z}_+} \left(-\frac{1}{n}, \frac{1}{n}\right)$$

is open in $\mathbb{R}^\omega$ endowed with the box topology. However, its inverse image $f^{-1}(U) = \{0\}$ is not open in $\mathbb{R}$.

2.9 The Metric Topology

Definition 2.9.1: Metric

A *metric* on a set X is a function

$$d: X \times X \to \mathbb{R}$$

having the following properties.

(i) (*Positive Definiteness*) $d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.
(ii) (*Symmetry*) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(iii) (*Triangle Inequality*) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Definition 2.9.2: Epsilon-Ball

Given a metrid d on X and $\varepsilon \in \mathbb{R}_+$, the set

$$B_d(x, \varepsilon) = \{\, y \in X \mid d(x, y) < \varepsilon \,\}$$

is called the *ε-ball centered at x*. Sometimes, we write $B(x, \varepsilon)$ if no confusion arises.

Lemma 2.9.1

Let d be a metric on a set X. If $y \in B(x, \varepsilon)$, then there is some $\delta \in \mathbb{R}_+$ such that $y \in B(y, \delta) \subseteq B(x, \varepsilon)$.

Proof. Let $\delta = \varepsilon - d(x, y)$. ($\delta \in \mathbb{R}_+$, indeed.) Then, if $z \in B(y, \delta)$, $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + (\varepsilon - d(x, y)) = \varepsilon$. Thus, $B(y, \delta) \subseteq B(x, \varepsilon)$. □

Definition 2.9.3: Metric Topology

If d is a metric on the set X, then the topology generated by the basis

$$\mathcal{B} = \{\, B_d(x, \varepsilon) \mid x \in X \text{ and } \varepsilon \in \mathbb{R}_+ \,\}$$

is called the *metric topology induced by d*.

Note:-

$\mathcal{B}$ is actually a basis for X. The first condition can be easily check by noting that $x \in B(x, 1)$ for every $x \in X$.
To check the second condition, let $y \in B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2)$. Then, by Lemma 2.9.1, there are $\delta_1, \delta_2 \in \mathbb{R}_+$ such that $B(y, \delta_1) \subseteq B(x_1, \varepsilon_1)$ and $B(y, \delta_2) \subseteq B(x_2, \varepsilon_2)$. If we take $\delta_0 \triangleq \min\{\delta_1, \delta_2\}$, $y \in B(y, \delta_0) \subseteq B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2)$.

Definition 2.9.4: Metrizability and Metric Space

If X is a topological space, X is said to be *metrizable* if there exists a metric d on X that induces the topology of X. A *metric space* is a metrizable space X together with a specific metric d that gives the topology of X.

Definition 2.9.5: Boundedness

Let (X,d) be a metric space. A subset of A of X is said to be *bounded* if

$$\exists M \in \mathbb{R},\ \forall a_1, a_2 \in A,\ d(a_1,a_2) \le M.$$

Note:-

Boundedness is not a topological property as it depends on the metric. For instance, $\mathbb{R}$ can be metrizable by two metrics:

$$d_1(x,y) = |x-y| \quad \text{and} \quad d_2(x,y) = \min\{|x-y|, 1\}.$$

(Both are metrics and induce the standard topology on $\mathbb{R}$.) However, $\mathbb{R}$ is not bounded with respect to d_1, but is bounded with respect to d_2.

Definition 2.9.6: Diameter

Let (X,d) be a metric space. if $\varnothing \neq A \subseteq X$, the *diameter* of A is defined to be

$$\operatorname{diam} A \triangleq \sup\{\, d(a_1,a_2) \mid a_1, a_2 \in A\}.$$

Theorem 2.9.1

Let (X,d) be a metric space. Define $\overline{d} : X \times X \to \mathbb{R}$ by

$$\overline{d}(x,y) = \min\{d(x,y), 1\}.$$

Then $\overline{d}$ is a metric on X that induces the same topology as d.

Proof. The positive definiteness and the symmetry is direct. Let us check the triangle inequality.

Take any $x,y,z \in X$. Since $\overline{d}(x,z) \le 1$ always holds, we get the triangle inequality in the case of $\overline{d}(x,y) \ge 1$ or $\overline{d}(y,z) \ge 1$.

In the other case, i.e., $\overline{d}(x,y) < 1$ and $\overline{d}(y,z) < 1$, it holds that $\overline{d}(x,y) = d(x,y)$ and $\overline{d}(y,z) = d(y,z)$. This implies

$$\overline{d}(x,z) \le d(x,z) \le d(x,y) + d(y,z) = \overline{d}(x,y) + \overline{d}(y,z),$$

which completes the proof that $\overline{d}$ is a metric on X.

Now, note that, in any metric space,

$$\{\, B_d(x,\varepsilon) \mid x \in X \text{ and } \varepsilon \in \mathbb{R}_+ \,\}$$

and

$$\{\, B_d(x,\varepsilon) \mid x \in X \text{ and } \varepsilon \in (0,1) \,\}$$

generates the same topology. Therefore, it follows that d and $\overline{d}$ generates the same opology on X, because the collections of ε-balls with $\varepsilon < 1$ under these two metrics are the same. $\square$

Definition 2.9.7: Standard Bounded Metric

Let (X,d) be a metric space. Define $\overline{d}: X \times X \to \mathbb{R}$ by

$$\overline{d}(x,y) = \min\{d(x,y), 1\}.$$

Then, $\overline{d}$ is a metric on X and is called the *standard bounded metric corresponding to* d.

Definition 2.9.8: Norm, Euclidean Metric and Square Metric

Given $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, we define the *norm* of $\mathbf{x}$ by the equation.

$$\|\mathbf{x}\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2};$$

and we define the *euclidean metric* d on $\mathbb{R}^n$ by the equation

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \left[(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2\right]^{1/2}.$$

We define the *square metric* ρ on $\mathbb{R}^n$ by the equation

$$\rho(\mathbf{x},\mathbf{y}) = \max\{|x_1 - y_1|, \cdots, |x_n - y_n|\}.$$

Note:-

The proof that ρ is a metric is trivial but for the triangle inequality.
Since, for each $i \in [n]$,

$$|x_i - z_i| \le |x_i - y_i| + |y_i - z_i| \le \rho(\mathbf{x},\mathbf{y}) + \rho(\mathbf{y},\mathbf{z}),$$

it holds that

$$\rho(\mathbf{x},\mathbf{z}) \le \rho(\mathbf{x},\mathbf{y}) + \rho(\mathbf{y},\mathbf{z}).$$

Lemma 2.9.2

Let d and d' be two metrics on the set X; let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they induce, respectively. Then,

$$\mathcal{T} \subseteq \mathcal{T}' \iff \forall (x,\varepsilon) \in X \times \mathbb{R}_+, \exists \delta \in \mathbb{R}_+, B_{d'}(x,\delta) \subseteq B_d(x,\varepsilon).$$

Proof. ($\Rightarrow$) Take any $x \in X$ and $\varepsilon \in \mathbb{R}_+$ Since $B_d(x,\varepsilon)$ is a basis element of $\mathcal{T}$, by Lemma 2.2.4, there is a basis element B' of $\mathcal{T}'$ such that $x \in B' \subseteq B_d(x,\varepsilon)$. By Lemma 2.9.1, there is some $B_{d'}(x,\delta)$ such that $x \in B_{d'}(x,\delta) \subseteq B'$.

($\Leftarrow$) Let $x \in X$; let B be any basis element of $\mathcal{T}$ that contains x. By Lemma 2.9.1, there is some $B_d(x,\varepsilon)$ such that $B_d(x,\varepsilon) \subseteq B$. By supposition, there exists $\delta \in \mathbb{R}_+$ such that $x \in B_{d'}(x,\delta) \subseteq B_d(x,\varepsilon)$. Thus, by Lemma 2.2.4, $\mathcal{T}'$ is finer than $\mathcal{T}$. □

Theorem 2.9.2

The topologies on $\mathbb{R}^n$ induced by d and ρ are the same as the product topology on $\mathbb{R}^n$.

Proof. Let $\mathcal{T}_d$ and $\mathcal{T}_\rho$ be the topologies induced by d and ρ, respectively. Let $\mathcal{T}_{\mathbb{R}^n}$ be the product topology on $\mathbb{R}^n$.

($\mathcal{T}_d = \mathcal{T}_\rho$) Let $\mathbf{x} = (x_1, \cdots, x_n)$ and $\mathbf{y} = (y_1, \cdots, y_n)$. Let $M \in [n]$ such that $|x_M - y_M| = \rho(\mathbf{x},\mathbf{y})$.

Then,

$$\rho(\mathbf{x},\mathbf{y})^2 = |x_M - y_M|^2 \le \sum_{i=1}^{n}(x_i - y_i)^2 = d(\mathbf{x},\mathbf{y})^2$$
$$\le \sum_{i=1}^{n}(x_M - y_M)^2 = n\rho(\mathbf{x},\mathbf{y})^2;$$

thus

$$\rho(\mathbf{x},\mathbf{y}) \le d(\mathbf{x},\mathbf{y}) \le \sqrt{n}\rho(\mathbf{x},\mathbf{y}).$$

Therefore, we get, for every $\mathbf{x} \in \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}_+$,

$$B_d(\mathbf{x},\varepsilon) \subseteq B_\rho(\mathbf{x},\varepsilon) \quad \text{and} \quad B_\rho(\mathbf{x},\varepsilon/\sqrt{n}) \subseteq B_d(\mathbf{x},\varepsilon).$$

By Lemma 2.9.2, one is finer than the other; $\mathcal{T}_d = \mathcal{T}_\rho$.

($\mathcal{T}_\rho = \mathcal{T}_{\mathbb{R}^n}$) $\mathcal{T}_\rho \subseteq \mathcal{T}_{\mathbb{R}^n}$ is direct since every basis element

$$B_\rho(\mathbf{x},\varepsilon) = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon)$$

of $\mathcal{T}_\rho$ is a basis element of $\mathcal{T}_{\mathbb{R}^n}$, by Lemma 2.2.4, $\mathcal{T}_\rho \subseteq \mathcal{T}_{\mathbb{R}^n}$.

To prove the other containment, take any $\mathbf{x} \in \mathbb{R}^n$ and let $B = \prod_{i=i}^{n}(a_i, b_i)$ be a basis element of $\mathcal{T}_{\mathbb{R}^n}$ that contains x. For each $i \in [n]$, let $\varepsilon_i = \min\{x_i - a_i, b_i - x_i\}$. Then, $(x_i - \varepsilon_i, x_i + \varepsilon_i) \subseteq (a_i, b_i)$ for all $i \in [n]$. Thus, it follows that $\mathbf{x} \in B_\rho(\mathbf{x}, \min_{i=1}^{n} \varepsilon_i) \subseteq B$; $\mathcal{T}_{\mathbb{R}^n} \subseteq \mathcal{T}_\rho$ by Lemma 2.2.4. □

Corollary 2.9.1

The product topology on $\mathbb{R}^n$ is metrizable.

Theorem 2.9.3

Given an index set J and given points $\mathbf{x} = (x_\alpha)_{\alpha\in J}$ and $\mathbf{y} = (y_\alpha)_{\alpha\in J}$ of $\mathbb{R}^J$, let us define $\overline{\rho}\colon \mathbb{R}^J \times \mathbb{R}^J \to \mathbb{R}$ by

$$\overline{\rho}(\mathbf{x},\mathbf{y}) = \sup\{\overline{d}(x_\alpha, y_\alpha) \mid \alpha \in J\}$$

where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$. Then, $\overline{\rho}$ is a metric on $\mathbb{R}^J$.

Proof. The positive definiteness and the symmetry is direct. Let us check the triangle inequality.

Let $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}^J$. For each $\alpha \in J$, it holds that

$$\overline{d}(x_\alpha, z_\alpha) \le \overline{d}(x_\alpha, y_\alpha) + \overline{d}(y_\alpha, z_\alpha) \le \overline{\rho}(\mathbf{x},\mathbf{y}) + \overline{\rho}(\mathbf{y},\mathbf{z}).$$

Therefore, $\overline{\rho}(\mathbf{x},\mathbf{z}) \le \overline{\rho}(\mathbf{x},\mathbf{y}) + \overline{\rho}(\mathbf{y},\mathbf{z})$. □

Definition 2.9.9: Uniform Metric and Uniform Topology

Given an index set J, $\overline{\rho}$ in the Theorem 2.9.3 is called the *unifrom metric* on $\mathbb{R}^J$, and the topology it induces is called the *uniform topology*.

Theorem 2.9.4

The uniform topology on $\mathbb{R}^J$ is finer than the product topology and coarser than the box topology. Moreover, they are all strict when J is infinite. In other words,

$$\mathcal{T}_{\text{product}} \subseteq \mathcal{T}_{\text{uniform}} \subseteq \mathcal{T}_{\text{box}}$$

They are strict if J is infinite.

Proof. ($\mathcal{T}_{\text{product}} \subseteq \mathcal{T}_{\text{uniform}}$) Let $B = \prod_{\alpha\in J} U_\alpha$ be a basis element of the product topology and $\mathbf{x} \in B$. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the indices such that $U_{\alpha_i} \neq \mathbb{R}$. Then, for each $i \in [n]$, there exists $\varepsilon_i \in \mathbb{R}_+$ such that $B_{\overline{d}}(x_{\alpha_i}, \varepsilon_i) \subseteq U_{\alpha_i}$. Let $\varepsilon \triangleq \min_{i=1}^n \varepsilon_i$. Then, $B_{\overline{\rho}}(\mathbf{x}, \varepsilon) \subseteq B$. The result follows from Lemma 2.2.4. ✓

($\mathcal{T}_{\text{uniform}} \subseteq \mathcal{T}_{\text{box}}$) Let B be any basis element of the uniform topoloy and $\mathbf{x} \in B$. Then, Lemma 2.9.1 implies that there is some ε-ball centered at $\mathbf{x}$ such that $B_{\overline{\rho}}(\mathbf{x}, \varepsilon) \subseteq B$. Then, $\prod_{\alpha\in J}(x_\alpha - \varepsilon/2, x_\alpha + \varepsilon/2)$ is an open neighborhood of $\mathbf{x}$ which is contained in B. ✓

($\mathcal{T}_{\text{product}} \not\supseteq \mathcal{T}_{\text{uniform}}$ if J is infinite) Let $0 < \varepsilon < 1$ and $\mathbf{x} \in \mathbb{R}^J$. Then, $\mathbf{x} \in B_{\overline{\rho}}(\mathbf{x}, \varepsilon)$ but there is no basis element of the product topology that is contained in $B_{\overline{\rho}}(\mathbf{x}, \varepsilon)$. By Lemma 2.2.4, the product topology is not finer than the uniform topology. ✓

($\mathcal{T}_{\text{uniform}} \not\supseteq \mathcal{T}_{\text{box}}$ if J is infinite.) Let $U \triangleq \prod_{\alpha\in J}(0, 2)$, which is a basis element of the box topology There is an injective function $f : \mathbb{Z}_+ \hookrightarrow J$ by Theorem 1.7.1. Let $\mathbf{x} \in U$ where

$$x_\alpha = \begin{cases} 1/n & \text{if } \exists n \in \mathbb{N}_+,\ f(n) = \alpha \\ 1 & \text{otherwise.} \end{cases}$$

Then, no basis element that contains $\mathbf{x}$ can be contained in U. If otherwise, there is an $B_{\overline{\rho}}(\mathbf{x}, \varepsilon') \subseteq U$ by Lemma 2.9.1. However, there exists $\alpha_0 \in J$ such that $f(n) = \alpha_0$ where $n\varepsilon' > 2$, which implies $x_{\alpha_0} = 1/n < \varepsilon'/2$. ✓

Let $\mathbf{x}' \in \mathbb{R}^J$ defined by

$$x'_\alpha = \begin{cases} x_{\alpha_0} - \varepsilon'/2 & \text{if } \alpha = \alpha_0 \\ x_\alpha & \text{otherwise.} \end{cases}$$

Then, $\mathbf{x}' \in B_{\overline{\rho}}(\mathbf{x}, \varepsilon')$ but $x'_{\alpha_0} - \varepsilon'/2 < 0$; $\mathbf{x}' \notin U$. This contradicts $B_{\overline{\rho}}(\mathbf{x}, \varepsilon') \subseteq U$. ✓ □

Theorem 2.9.5 Countable Product of Metrizable Spaces Is Metrizable

Let X_n be a metric space with metric d_n for each $n \in \mathbb{Z}_+$. Let $\overline{d}_n$ be the standard bounded metric corresponding to d_n. If $\mathbf{x}, \mathbf{y} \in \prod_{i\in\mathbb{Z}_+} X_i$, define

$$D(\mathbf{x}, \mathbf{y}) = \sup\left\{ \frac{\overline{d}_i(x_i, y_i)}{i} \,\middle|\, i \in \mathbb{Z}_+ \right\}.$$

Then D is a metric that induces the product topology on $\prod_{i\in\mathbb{Z}_+} X_i$.

Proof. (D is a metric on $\prod_{i\in\mathbb{Z}_+} X_i$.) The positive definiteness and the symmetry of D is direct. Note that, for each $i \in \mathbb{Z}_+$,

$$\frac{\overline{d}_i(x_i, z_i)}{i} \le \frac{\overline{d}_i(x_i, y_i)}{i} + \frac{\overline{d}_i(y_i, z_i)}{i} \le D(\mathbf{x}, \mathbf{y}) + D(\mathbf{y}, \mathbf{z}).$$

Thus,

$$D(\mathbf{x}, \mathbf{z}) = \sup\left\{ \frac{\overline{d}_i(x_i, z_i)}{i} \,\middle|\, i \in \mathbb{Z}_+ \right\} \le D(\mathbf{x}, \mathbf{y}) + D(\mathbf{y}, \mathbf{z}). ✓$$

($\mathcal{T}_{\text{metric}} \subseteq \mathcal{T}_{\text{product}}$) Let B be any ε'-ball in the metric topology and let $\mathbf{x} \in B$. Then, by Lemma 2.9.1, there exists $\varepsilon \in \mathbb{R}_+$ such that $B_D(\mathbf{x}, \varepsilon) \subseteq B$. Take $N \in \mathbb{Z}_+$ such that $\varepsilon N > 1$. Let V be the basis element for the product topology defined by

$$V \triangleq B_{\overline{d}_1}(x_1, \varepsilon) \times \cdots \times B_{\overline{d}_N}(x_N, \varepsilon) \times X_{n+1} \times X_{n+2} \times \cdots.$$

Note that, given any $\mathbf{y} \in \mathbb{R}^\omega$ and $i \geq N$, $\dfrac{\overline{d}_i(x_i, y_i)}{i} \leq \dfrac{1}{N}$. Thus, when $\mathbf{y} \in V$,

$$D(\mathbf{x}, \mathbf{y}) \leq \max\left\{\frac{\overline{d}_1(x_1, y_1)}{1}, \frac{\overline{d}_2(x_2, y_2)}{2}, \cdots \frac{\overline{d}_N(x_N, y_N)}{N}, \frac{1}{N}\right\} < \varepsilon.$$

Thus, $\mathbf{x} \in V \subseteq B_D(\mathbf{x}, \varepsilon) \subseteq B$. Now, Lemma 2.2.4 tells the result. ✓

($\mathcal{T}_{\text{metric}} \supseteq \mathcal{T}_{\text{product}}$) Let $B = \prod_{i \in \mathbb{Z}_+} U_i$ be a basis element of the product topology and $\mathbf{x} \in B$. Let $i_1, i_2, \cdots, i_n$ be the indices such that $U_{i_k} \neq X_{i_k}$ for each $k \in [n]$.

For each $k \in [n]$, since U_{i_k} is open, there exists $\varepsilon_k \in (0, 1)$ such that $B_{\overline{d}_{i_k}}(x_{i_k}, \varepsilon_k) \subseteq U_{i_k}$. Let $\varepsilon \triangleq \min_{k=1}^n (\varepsilon_k / i_k)$.

Now we claim that $B_D(\mathbf{x}, \varepsilon) \subseteq U$. Let $\mathbf{y} \in B_D(\mathbf{x}, \varepsilon)$. Then, for all $k \in [n]$,

$$\overline{d}_{i_k}(x_{i_k}, y_{i_k}) \leq i_k \cdot D(\mathbf{x}, \mathbf{y}) < i_k \varepsilon \leq \varepsilon_k < 1.$$

It follows that $y_{i_k} \in B_{\overline{d}_{i_k}}(x_{i_k}, \varepsilon_k)$; therefore $\mathbf{y} \in B$. ✓ □

Corollary 2.9.2

$\mathbb{R}^\omega$ with the product topology is metrizable.

2.10 The Metric Topology (continued)

Theorem 2.10.1 The ε-δ Definition of Continuity

Let $f : X \to Y$; let X and Y be metrizable with metrics d_X and d_Y, respectively. Then, f is continuous if and only if

$$\forall x \in X,\ \forall \varepsilon \in \mathbb{R}_+, \exists \delta \in \mathbb{R}_+,\ \forall y \in Y,\ \big(d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \varepsilon\big).$$

Proof. ($\Rightarrow$) Given $x \in X$ and $\varepsilon \in \mathbb{R}_+$, the set $f^{-1}\big(B(f(x), \varepsilon)\big)$ is open and contains x. Thus, there is some δ-ball $B(x, \delta)$ centered at x such that $x \in B(x, \delta) \subseteq f^{-1}\big(B(f(x), \varepsilon)\big)$. ✓

($\Leftarrow$) Let V be open in Y; we claim that $f^{-1}(V)$ is open in X. Let $x \in f^{-1}(V)$. Since $f(x) \in V$, there is some ε-ball $B(f(x), \varepsilon)$ such that $B(f(x), \varepsilon) \subseteq V$. By the supposition, there is some $\delta \in \mathbb{R}_+$ such that $f\big(B(x, \delta)\big) \subseteq B(f(x), \varepsilon)$. Thus, $x \in B(x, \delta) \subseteq f^{-1}(B(f(x), \varepsilon)) \subseteq f^{-1}(V)$. This implies $f^{-1}(V)$ is open by definition. ✓ □

Definition 2.10.1: Local Basis

A space X is said to have a *local basis at the point* $x \in X$ if there is a countable collection $\mathcal{U}$ of open neighborhoods of x such that any neighborhood U of x contains at least one of element of $\mathcal{U}$.

Definition 2.10.2: First Countable Axiom

A space X satisfies the *first countable axiom* if it has countable local basis at each point.

Example 2.10.1 (Every Metrizable Space IS First-Countable)
Any metrizable space satisfies the first countable axiom. For each $x \in X$, $\{B_d(x, 1/n) \mid n \in \mathbb{Z}_+\}$ is a countable local basis at x.

Lemma 2.10.1 The Sequence Lemma
Let X be a topological space; let $A \subseteq X$. If there is a sequence of points in A converging to x, then $x \in \overline{A}$. Moreover, the converse holds if X satisfies the first countable axiom.

Proof. ($\Rightarrow$) Suppose $x_n \to x$ and $x_n \in A$. This means every neighborhood U of x intersects A, so $x \in \overline{A}$ by Theorem 2.6.5. ✓

($\Leftarrow$) Let $\{U_n\}_{n \in \mathbb{Z}_+}$ be a local basis for x. Set $B_n \triangleq \bigcap_{i=1}^{n} U_i$ so that $B_1 \supseteq B_2 \supseteq \cdots$. Since $x \in \overline{A}$ and $x \in B_n$ is open, we may take $x_n \in A \cap B_n$.

We want to show that $x_n \to x$. Take any neighborhood U of x. Then, it contains U_{n_0} for some $n_0 \in \mathbb{Z}_+$. Then, for all $n \geq n_0$, $x_n \in U_{n_0} \in U$. ✓ □

Lemma 2.10.2
Let X and Y be topological spaces. If $f : X \to Y$ is continuous, then for every convergent sequence $x_n \to x$, the sequence $f(x_n)$ converges to $f(x)$. The converse also holds if X satisfies the first countable axiom.

Proof. ($\Rightarrow$) Let V be a neighborhood of $f(x)$ in Y. Then, $f^{-1}(V)$ is a neighborhood of x in X since f is continuous. Since $x_n \to x$, there is some $n_0 \in \mathbb{Z}_+$ such that $x_n \in f^{-1}(V)$ whenever $n \geq n_0$, i.e., $f(x_n) \in V$ whenever $n \geq n_0$. ✓

($\Leftarrow$) We claim that $f(\overline{A}) \subseteq \overline{f(A)}$ for any $A \subseteq X$, and thus f is continuous by Theorem 2.7.1. Let $x \in \overline{A}$. Then, by Lemma 2.10.1, there is a sequence $\{x_n\}_{n \in \mathbb{Z}_+} \subseteq A$ that converges to x. Then, by assumption, the sequence $\{f(x_n)\}_{n \in \mathbb{Z}_+}$ in $f(A)$ converges to $f(x)$. By Lemma 2.10.1, $f(x) \in \overline{f(A)}$. ✓ □

Lemma 2.10.3
The addition, subtraction, and multiplication operations are continuous functions from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$; and the quotient operation is a continuous function from $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$ into $\mathbb{R}$.

Theorem 2.10.2
If X is a topological space, and if $f, g : X \to \mathbb{R}$ are continuous, then $f + g$, $f - g$, and $f \cdot g$ are continuous. If $g(x) \neq 0$ for all x, then f/g is continuous.

Proof. The map $h : X \to \mathbb{R} \times \mathbb{R}$ defined by

$$h(x) = f(x) \times g(x)$$

is continuous by Theorem 2.8.5. The function $f + g$ equals the composite of h and the addition operation

$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R};$$

therefore $f+g$ is continuous by (iii) of Theorem 2.7.2. It is similar for $f-g$, $f\cdot g$, and f/g. $\square$

Definition 2.10.3: Uniform Convergence

Let $\{f_n\} \subseteq X \to Y$ be a sequence of functions from the set X to the metric space Y. Let d be the metric for Y. We say that the sequence $\{f_n\}$ converges uniformly to the function $f : X \to Y$ if

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists N \in \mathbb{Z}_+, \forall n \in \mathbb{Z}_+ \left(n \geq N \implies \forall x \in X,\ d(f_n(x), f(x)) < \varepsilon\right).$$

Note:-

Uniformity of convergence depends not only on the topology of Y but also on its metric.

Theorem 2.10.3 Uniform Limit Theorem

Let $\{f_n\} \subseteq X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If $\{f_n\}$ converges uniformly to f, then f is continuous.

Proof. Let V be open in Y. We want to show that $f^{-1}(V)$ is open. Take any $x_0 \in f^{-1}(V)$. Let $y_0 \triangleq f(x_0) \in V$. Since $f^{-1}(V)$ is open, there exists $\varepsilon \in \mathbb{R}_+$ such that $B(y_0, \varepsilon) \subseteq f^{-1}(V)$. By uniform convergence,

$$\exists N \in \mathbb{Z}_+, \forall x \in X,\ d(f_N(x), f(x)) < \varepsilon/4.$$

where d is the metric on Y. Moreover, since f_N is continuous, $U = f_N^{-1}(B(f_N(x_0), \varepsilon/2))$ is a neighborhood of x_0.

Thus, for each $x \in U$,

$$\begin{aligned} d(y_0, f(x)) &\leq d(f(x_0), f_N(x_0)) + d(f_N(x_0), f_N(x)) + d(f_N(x), f(x)) \\ &< \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = \varepsilon. \end{aligned}$$

Thus, we have $x_0 \in U \subseteq f^{-1}(V)$; $f^{-1}(V)$ is open. $\square$

Theorem 2.10.4

$\{f_n\} \subseteq X \to \mathbb{R}$ converges uniformly to $f : X \to \mathbb{R}$ if and only if $\{f_n\}$ converges to f in the uniform topology.

Proof. ($\Rightarrow$) Let U be any neighborhood of f in the uniform topology. Then, there is an ε-ball $B_{\overline{\rho}}(f, \varepsilon)$ centered at f which is contained in U. By the uniform convergence, there is some $N \in \mathbb{Z}_+$ such that

$$\forall n \in \mathbb{Z}_+, \left(n \geq N \implies \forall x \in X,\ d(f_n(x), f(x)) < \varepsilon/2\right).$$

Thus, for all $n \geq N$, $\overline{\rho}(f_n, f) \leq \varepsilon/2 < \varepsilon$, i.e., $f_n \in B_{\overline{\rho}}(f, \varepsilon) \subseteq U$. ✓

($\Leftarrow$) Take any $\varepsilon \in \mathbb{R}_+$. By the convergence in the uniform topology, there exists some $N \in \mathbb{Z}_+$ such that

$$\forall n \in \mathbb{Z}_+, \left(n \geq N \implies f_n \in B_{\overline{\rho}}(f, \varepsilon)\right).$$

This implies, whenever $n \geq N$, $\forall x \in X$, $d(f_n(x), f(x)) < \varepsilon$. ✓ $\square$

Corollary 2.10.1

$\mathbb{R}^\omega$ with the box topology is not metrizable.

Proof. Let $A = (\mathbb{R}_+)^\omega$ be a subset of $\mathbb{R}^\omega$. Then, $\mathbf{0}$ is a limit point of A. To see this, let

$$B = (a_1, b_1) \times (a_2, b_2) \times \cdots$$

be any basis element that contains $\mathbf{0}$. Then,

$$(b_1/2, b_2/2, \cdots) \in A \cap B.$$

However, there is no sequence of points of A that converge to $\mathbf{0}$. To see this, let $\{\mathbf{a}_n\}_{n\in\mathbb{Z}_+}$ be a sequence of points in A where

$$\mathbf{a}_n = (a_{n1}, a_{n2}, \cdots, a_{in}, \cdots).$$

Let $B' = \prod_{n\in\mathbb{Z}_+}(-a_{nn}, a_{nn})$ is a neighborhood of $\mathbf{0}$ but no $\mathbf{a}_n$ is in B'; $\{\mathbf{a}_n\}$ does not converge to $\mathbf{0}$.

Thus, by Lemma 2.10.1, $\mathbb{R}^\omega$ does not satisfy the first countable axiom, and thus is not metrizable. $\square$

Corollary 2.10.2

$\mathbb{R}^J$ with uncountable J in the product topology is not metrizable.

Proof. Let $A = \{(x_\alpha)_{\alpha\in J} \mid x_\alpha = 1 \text{ for all but finitely many } \alpha\text{'s}\}$.

Let $\prod_{\alpha\in J} U_\alpha$ be a basis that contains $\mathbf{0}$ and suppose $U_\alpha \neq \mathbb{R}$ for $\alpha \in \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Define $(y_\alpha)_{\alpha\in J}$ by

$$y_\alpha \triangleq \begin{cases} 0 & \text{if } \alpha = \alpha_i \text{ for some } i \in [n] \\ 1 & \text{otherwise.} \end{cases}$$

Then, $(y_\alpha)_{\alpha\in J} \in A \cap \prod_{\alpha\in J} U_\alpha$. Hence, $\mathbf{0} \in \bar{A}$ by Theorem 2.6.5.

Now, we shall prove that no sequence in A converges to $\mathbf{0}$. Let $\{\mathbf{a}_n\}_{n\in\mathbb{Z}_+}$ be a sequence in A. For each $n \in \mathbb{Z}_+$, let

$$J_n \triangleq \{\alpha \in J \mid (\mathbf{a}_n)_\alpha \neq 1\}.$$

Since each J_n is finite, and since $\bigcup_{n\in\mathbb{Z}_+} J_n$ is thus countable, we may take $\beta \in J \setminus \left(\bigcup_{n\in\mathbb{Z}_+} J_n\right)$. For such β, it is $(\mathbf{a}_n)_\beta \neq 1$ for all $n \in \mathbb{Z}_+$. This implies that $\mathbf{a}_n \notin \pi_\beta^{-1}\big((-1,1)\big)$ for each $n \in \mathbb{Z}_+$ while $\pi_\beta^{-1}\big((-1,1)\big)$ is a neighborhood of $\mathbf{0}$; $\{\mathbf{a}_n\}_{n\in\mathbb{Z}_+}$ does not converge to $\mathbf{0}$. Thus, $\mathbb{R}^J$ is not metrizable by Lemma 2.10.1. $\square$

2.11 The Quotient Topology

Definition 2.11.1: Quotient Map

Let X and Y be topological spaces. A map $p : X \to Y$ is called a *quotient map* if

(i) p is surjective and

(ii) $V \subseteq Y$ is open in Y $\iff$ $p^{-1}(V)$ is open in X.

Note:-

A quotient map is continuous.

Note:-

(ii) of Definition 2.11.1 is equivalent to

$$C \subseteq Y \text{ is closed in } Y \iff p^{-1}(C) \text{ is closed in } X.$$

as

$$C \text{ is closed in } Y \iff Y \setminus C \text{ is open in } Y \quad \text{and}$$
$$f^{-1}(C) \text{ is closed in } X \iff X \setminus f^{-1}(C) \text{ is closed in } X$$

Definition 2.11.2: Saturated Set

A subset C of X is *saturated* (with respect to the map $p : X \to Y$) if

$$\forall y \in Y, \left(p^{-1}(\{y\}) \cap C \neq \varnothing \implies f^{-1}(\{y\}) \subseteq C\right).$$

In other words, C is saturated if $C = p^{-1}(V)$ for some $V \subseteq Y$.

Note:-

Here is the proof of their equivalence.

- Suppose $C = p^{-1}(V)$ for some $V \subseteq Y$. Let $y \in Y$ and suppose it satisfies $p^{-1}(\{y\}) \cap C \neq \varnothing$. Thus,
$$p^{-1}(\{y\}) \cap p^{-1}(V) = p^{-1}(V \cap \{y\}) \neq \varnothing;$$
$y \in V$. Hence, $p^{-1}(\{y\}) \subseteq p^{-1}(V) = C$.
- For the converse, let
$$\begin{aligned} V &\triangleq \{ y \in V \mid p^{-1}(\{y\}) \cap C \neq \varnothing \} \\ &= \{ y \in V \mid p^{-1}(\{y\}) \subseteq C \} \end{aligned}$$
The second equality follows from the hypothesis.
If $p(x) \in V$ where $x \in X$, by definition of V, $x \in p^{-1}(p(\{x\})) = p^{-1}(\{p(x)\}) \subseteq C$. This proves $p^{-1}(V) \subseteq C$.
For the other containment, let $x \in C$. Then, $\{p(x)\} \cap p(C) \neq \varnothing$, and thus
$$\varnothing \neq p^{-1}\left(\{p(x)\} \cap p(C)\right) = p^{-1}\left(\{p(x)\}\right) \cap p^{-1}\left(p(C)\right) \subseteq p^{-1}\left(\{p(x)\}\right) \cap C$$
is nonempty; $p(x) \in V$ by definition of V. This proves $C \subseteq p^{-1}(V)$. □

Lemma 2.11.1

Let X and Y be topological spaces. A surjective, continuous map $p : X \to Y$ is a quotient map if and only if $p(C)$ is open for every saturated open set $C \subseteq X$.

Proof. The continuity is equivalent to $\Rightarrow$ of Definition 2.11.1 (ii), and 'sending every saturated open set to an open set' is equivalent to $\Leftarrow$ of Definition 2.11.1 (ii). □

Lemma 2.11.2

If $p : X \to Y$ is a map and A is saturated with respect to p, then $p^{-1}(p(A)) = A$.

Proof. It is already $p^{-1}(p(A)) \supseteq A$ by Example 1.1.2.

There exists $V \subseteq Y$ such that $A = p^{-1}(V)$. Then, $p(A) = p(p^{-1}(V)) \subseteq V$; and it implies $p^{-1}(p(A)) \subseteq p^{-1}(V) = A$. □

Lemma 2.11.3

Let X and Y be topological spaces and $p: X \to Y$ be surjective and continuous. Then, if p is an open map or is a closed map, p is a quotient map.

Proof. If p is open, then the result follows directly from Lemma 2.11.1.

Suppose p is closed and let $V \subseteq Y$ such that $p^{-1}(V)$ is open in X. Then, $X \setminus p^{-1}(V)$ is closed, and thus,

$$p(X \setminus p^{-1}(V)) = p(X) \setminus p(p^{-1}(V)) = Y \setminus V$$

is closed in X. The last equality comes from Example 1.1.2. Thus, V is open in X. $\square$

Wrong Concept 2.2: The Converses Do Not Hold

Let $A = \big([0,\infty) \times \mathbb{R}\big) \cup \big(\mathbb{R} \times \{0\}\big)$ be a subspace of $X = \mathbb{R}^2$ endowed with the standard topology. Let $\pi : A \to \mathbb{R}$ be the projection onto its first factor, i.e.,

$$\pi(x \times y) = x.$$

Since π is surjective and $\pi^{-1}(V) = (V \times \mathbb{R}) \cap A$ for each $V \subseteq \mathbb{R}$, π is a quotient map when $\mathbb{R}$ is endowed with the standard topology.

However, it is not open as $\pi\big((\mathbb{R} \times (0,1)) \cap A\big) = [0,\infty)$ is not open. It is also not closed as, if we let $C = \{x \times 1/x \mid x \in \mathbb{R}_+\}$, $p(C) = (0,\infty)$ is not closed although C is closed in A.

This shows that the converses of Lemma 2.11.3 are not true.

Wrong Concept 2.3: Subspaces and Quotient Map

A restriction on a subspace of a quotient map need not be a quotient map.

Let X be the subspace $[0,1] \cup [2,3]$ of $\mathbb{R}$, and let Y be the subspace $[0,2]$ of $\mathbb{R}$. Let $p: X \to Y$ be defined by

$$p(x) = \begin{cases} x & \text{if } x \in [0,1] \\ x-1 & \text{if } x \in [2,3]. \end{cases}$$

p is continuous since

$$p^{-1}\big((a,b) \cap Y\big) = \begin{cases} (a,b) \cap X & \text{if } b \leq 1 \\ (a+1,b+1) \cap X & \text{if } a \geq 1 \\ (a,b+1) \cap X & \text{if } a < 1 < b \end{cases}$$

implies $p^{-1}(V)$ is open in X if V is open in Y.

Also, since id and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x-1$ are closed (homeomorphisms, actually), if C is closed in X,

$$p(C) = p(C \cap [0,1]) \cup p(C \cap [2,3]) = (C \cap [0,1]) \cup g(C \cap [2,3])$$

is closed.

p is surjective, indeed; thus p is a quotient map by Lemma 2.11.3.

Let A be the subspace $[0,1) \cup [2,3]$. Then, the map $q: A \to Y$ obtained by restricting p is continuous and surjective, but it is not a quotient map as $f^{-1}([1,2]) = [2,3]$ is open in A but $[1,2]$ is not open in Y.

Theorem 2.11.1

If X is a space and A is a set and if $p: X \to A$ is a surjective map, then there exists a unique topology $\mathcal{T}$ on A relative to which p is a quotient map. Moreover,

$$\mathcal{T} = \{ V \subseteq A \mid p^{-1}(V) \text{ is open in } X \}.$$

Proof. First, we shall prove that $\mathcal{T}$ is a topology.

(i) $p^{-1}(\varnothing) = \varnothing$ and $p^{-1}(A) = X$ are open in X; thus $\varnothing, A \in \mathcal{T}$. ✓

(ii) For any $\{V_\alpha\}_{\alpha\in J} \subseteq \mathcal{T}$, $p^{-1}\big(\bigcup_{\alpha\in J} V_\alpha\big) = \bigcup_{\alpha\in J} p^{-1}(V_\alpha)$ is open in X. Thus, $\bigcup_{\alpha\in J} U_\alpha \in \mathcal{T}$. ✓

(iii) For any $\{V_i\}_{i=1}^n \subseteq \mathcal{T}$, $p^{-1}\big(\bigcup_{i=1}^n V_\alpha\big) = \bigcup_{i=1}^n p^{-1}(V_i)$ is open in X. Thus, $\bigcup_{i=1}^n V_i \in \mathcal{T}$. ✓

p is a quotient map relative to $\mathcal{T}$. The surjectivity is given by definition, and the continuity is direct from the definition. Moreover, if $p^{-1}(U)$ is open in X where $U \subseteq A$, by the definition of $\mathcal{T}$, $U \in \mathcal{T}$. ✓

To prove the uniqueness, let $\mathcal{T}'$ be a topology on A relative to which p is a quotient map. Then,

$$V \in \mathcal{T} \iff p^{-1}(V) \text{ is open in } X \iff V \in \mathcal{T}';$$

thus $\mathcal{T} = \mathcal{T}'$. ✓ □

Definition 2.11.3: Quotient Topology

Let X be a space and A be a set. Let $p: X \to A$ be a surjective map. Then, according to Theorem 2.11.1,

$$\mathcal{T} = \{ V \subseteq A \mid p^{-1}(V) \text{ is open in } X \}$$

is a unique topology on A relative to which p is a quotient map. Here, $\mathcal{T}$ is called the *quotient topology induced by* p.

Definition 2.11.4: Quotient Space

Let X be a topological space, and let $X^* \subseteq \mathcal{P}(X)$ be a partition of X. Let $p: X \to X^*$ be a function that maps each $x \in X$ to the unique $U \in X$ such that $x \in U$. Then, p is surjective. X^* endowed with the quotient topology induced by p is called a *quotient space* of X.

Note:-

Since $U \subseteq X^*$ is a collection of equivalence classes, it is just $p^{-1}(U) = \bigcup U$.

Lemma 2.11.4

Let X and Y be any sets, and let $p: X \to Y$ be a map. Let A be a subset of X that is saturated with respect to p. Let $q: A \to p(A)$ be the map obtained by restricting p. Then, the following hold.

(i) If $V \subseteq p(A)$, then $p^{-1}(V) = q^{-1}(V)$.

(ii) If $U \subseteq X$, then $p(U \cap A) = p(U) \cap p(A)$.

Proof.

(i) It is direct that

$$q^{-1}(V) = \{ x \in A \mid q(x) \in V \} = \{ x \in A \mid p(x) \in V \} \subseteq \{ x \in X \mid p(x) \in V \} = p^{-1}(V),$$

and it does not require A to be saturated.

For the other direction, let $x \in p^{-1}(V)$. Since A is saturated, $x \in p^{-1}(V) \subseteq p^{-1}(p(A)) = A$ by Lemma 2.11.2. Thus, $x \in q^{-1}(V)$.

(ii) It is already $p(U \cap A) \subseteq p(U) \cap p(A)$ since $p(U \cap A) \subseteq p(U)$ and $p(U \cap A) \subseteq p(A)$. For the reverse inclusion, let $y \in p(U) \cap p(A)$. There exists $u \in U$ and $a \in A$ such that $y = p(u) = p(a)$. Then, $u \in p^{-1}(\{p(u)\}) = p^{-1}(\{p(a)\}) \subseteq A$ since A is saturated. Thus, $u \in U \cap A$; $y = p(u) \in p(U \cap A)$.

□

Theorem 2.11.2

Let X and Y be topological spaces, and let $p: X \to Y$ be a quotient map. Let A be a subspace of X that is saturated with respect to p. Let $q: A \to p(A)$ be the map obtained by restricting p.

(i) If A is either open or closed in X, then q is a quotient map.

(ii) If p is either an open map or a closed map, then q is a quotient map.

Proof. Note that, q is already surjective and continuous by Theorem 2.7.2. Let $V \subseteq p(A)$ and assume $q^{-1}(V)$ is open in A. $q^{-1}(V) = p^{-1}(V)$ by Lemma 2.11.4.

(i) Suppose A is open. Then, $q^{-1}(V) = p^{-1}(V)$, which is open in A, is open in X. Since p is a quotient map, V is open in X. Thus, $V = V \cap p(A)$ is also open in $p(A)$.

(ii) Suppose p is open. Since $p^{-1}(V)$ is open in A, $p^{-1}(V) = U \cap A$ for some open set U in X. Since p is surjective,

$$V = p(p^{-1}(V)) = p(U \cap A) = p(U) \cap p(A).$$

The last equation comes from Lemma 2.11.4. Since $p(U)$ is open in Y, V is also open in $p(A)$.

Replace "open" with "closed" to get the proof for closed A and closed p. □

Theorem 2.11.3

Let X, Y, and Z be topological spaces, and let $p: X \to Y$ and $q: Y \to Z$ be quotient maps. Then, $q \circ p: X \to Z$ is a quotient map.

Proof. $q \circ p$ is surjective and continuous by Theorem 2.7.2. Also, if $(q \circ p)^{-1}(V)$ is open in X, since $(q \circ p)^{-1}(V) = p^{-1}(q^{-1}(V))$, $q^{-1}(V)$ is open, and thus V is open. □

Wrong Concept 2.4: Products and Quotient Map

The product of two quotient maps need not be a quotient map.

Let $X = \mathbb{R}$ and X^* be obtained by

$$X^* = \left\{ \{x\} \,\middle|\, x \in \mathbb{R} \setminus \mathbb{Z}_+ \right\} \cup \{\mathbb{Z}_+\},$$

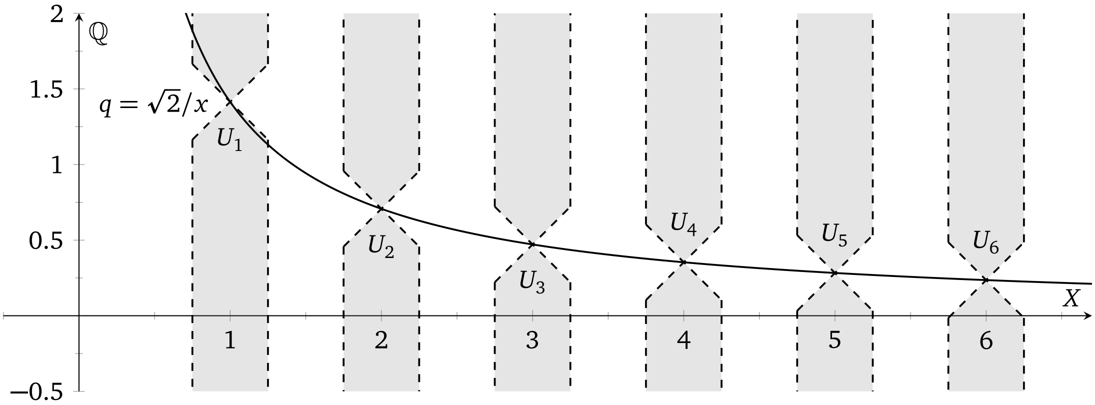
i.e., identifying $\mathbb{Z}_+$ to one point $b = \mathbb{Z}_+$. Let $p: X \to X^*$ be the quotient map. Let $\mathbb{Q}$ be the subspace of $\mathbb{R}$ endowed with the standard topology; let $i: \mathbb{Q} \to \mathbb{Q}$ be the identity map. We show that

$$p \times i: X \times \mathbb{Q} \to X^* \times \mathbb{Q}$$

it not a quotient map.

Let $c_n = \sqrt{2}/n$ where $n \in \mathbb{Z}_+$. For each $n \in \mathbb{Z}_+$, let

$$U_n \triangleq \left\{ (x, q) \in X \times \mathbb{Q} \,\middle|\, |x - n| < 1/4 \text{ and } |q - c_n| > |x - n| \right\}.$$



Then, it is easy to see that each U_n is open; so

$$U \triangleq \bigcup_{n \in \mathbb{Z}_+} U_n$$

is open. Moreover, U is saturated with respect to $p \times i$ as $\mathbb{Z}_+ \times \{q\} \subseteq U$ (a potential source that makes U not saturated) for all $q \in \mathbb{Q}$.

Suppose $U' \triangleq (p \times i)(U)$ is open for the sake of contradiction. Since $\mathbb{Z}_+ \times \{0\} \subseteq U$, $b \times 0 \in U'$ by definition. Hence, U' contains an open set $W \times I_\delta$ where W is a neighborhood of b in X^* and $I_\delta = (-\delta, \delta) \cap \mathbb{Q}$. Then, we have

$$p^{-1}(W) \times I_\delta = (p \times i)^{-1}(W \times I_\delta) \subseteq (p \times i)^{-1}(U') = U.$$

(The last equation follows from Lemma 2.11.2.)

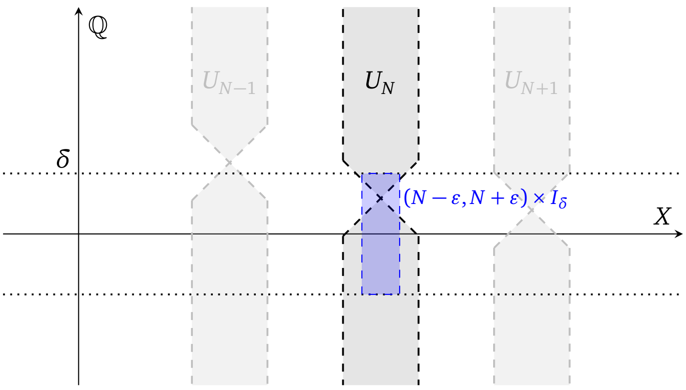
There exists $N \in \mathbb{Z}_+$ such that $c_N = \sqrt{2}/N < \delta$. Since p is continuous, $p^{-1}(W)$ is open in X and contains $\mathbb{Z}_+$. Thus, there exists $\varepsilon \in (0, 1/4)$ so that $(N - \varepsilon, N + \varepsilon) \subseteq p^{-1}(W)$. This implies

$$(N - \varepsilon, N + \varepsilon) \times I_\delta \subseteq U,$$

but this is impossible since, if we let $c'_N \in (c_N - \varepsilon/2, c_N + \varepsilon/2) \cap I_\delta$,

$$(N + \varepsilon/2) \times c'_N \in (N - \varepsilon, N + \varepsilon) \times I_\delta$$

but $(N + \varepsilon/2) \times c'_N \notin U$, #. Thus, $U' = (p \times i)(U)$ is not open while U is saturated; $p \times i$ is not a quotient map.



Theorem 2.11.4

Let $p: X \to Y$ be quotient map. Let Z be a space and let $g: X \to Z$ be a map that is constant on each set $p^{-1}(\{y\})$, $y \in Y$. Then, g induces a map $f: Y \to Z$ such that $f \circ p = g$. Moreover, the following hold.

(i) f is continuous if and only if g is continuous.

(ii) f is a quotient map if and only if g is a quotient map.

Proof. For each $y \in Y$, the set $g(p^{-1}(\{y\}))$ is a one-point set in Z as we assumed g is constant on $p^{-1}(\{y\})$. Define $f(y)$ to be the only element of it. Then, $f(p(x))$ is the only elemnt of $A = g\big(p^{-1}(p(\{x\}))\big)$ while $g(x) \in A$. Thus, $f(p(x)) = g(x)$ for each $x \in X$; $f \circ p = g$.

(i) If f is continuous, $g = f \circ p$ is continuous by Theorem 2.7.2. Suppose g is continuous. Let V be open in Z. Then, $g^{-1}(V)$ is open in X as g is continuous. Noting that $g^{-1}(V) = p^{-1}(f^{-1}(V))$ and p is a quotient map, we get $f^{-1}(V)$ is also open in Y. ✓

(ii) If f is a quotient map, $g = f \circ p$ is a quotient map by Theorem 2.11.3. Suppose g is a quotient map. f is already surjective by basic set theory and continuous by (i). Let V be open in Z and suppose $f^{-1}(V)$ is open in Y. $p^{-1}(f^{-1}(V)) = g^{-1}(V)$ is open since p is continuous. Because g is a quotient map, V is open. Thus, f is a quotient map.

□

Corollary 2.11.1

Let $g: X \to Z$ be a surjective contiuous map. Let X^* be defined by

$$X^* \triangleq \{\, g^{-1}(\{z\}) \subseteq X \mid z \in Z \,\}.$$

Give X^* the quotient topology. Then, the following hold.

(i) The map g induces a bijective continuous map $f: X^* \to Z$, which is a homeomorphism if and only if g is a quotient map.

(ii) If Z is Hausdorff, so is X^*.

Proof.

(i) Let $p: X \to X^*$ be the quotient map that induces the quotient topology on X^*. Then, by Theorem 2.11.4, the induced $f: X^* \to Z$ is continuous. f is surjective since g and p are surjective. f is injective since $f(g^{-1}(\{z\})) = z$ for each $z \in Z$. ✓
Suppose f is a homeomorphism. Then both f and p are quotient maps; thus $g = f \circ p$ is a quotient map. Suppose g is a quotient map. Then, by Theorem 2.11.4, f is a quotient map. Since f is already bijective, f is a homeomorphism. ✓

(ii) Suppose Z is Hausdorff. Given distinct points $a, b \in X^*$, $f(a) \neq f(b)$ since f is injective. Thus, there are disjoint neighborhoods U and V in Z of $f(a)$ and $f(b)$, respectively. Then, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint neighborhoods of a and b as f is continuous. Thus, X^* is Hausdorff. ✓

□

Chapter 3

Connectedness and Compactness

3.1 Connected Space

Definition 3.1.1: Separation and Connectedness

Let X be a topological space. A *separation* of X is a pair U and V of subsets of X which satisfy the following.

(i) U and V are open in X.

(ii) $U \cap V = \varnothing$.

(iii) $U \cup V = X$.

The space X is said to be *connected* if there does not exist a separation of X.

Note:-

Connectedness ia a topological property.

Note:-

A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty sets and X itself.

Lemma 3.1.1

If Y is a subspace of X, $A, B \subseteq Y$ is a separation of Y if and only if $A \cap B = \varnothing$, $A \cup B = Y$, and neither A nor B contains a limit point of the other.

Proof. Suppose A and B form a separation of Y. Then, A is both open and closed in Y; thus the closure of A in Y is $\overline{A} \cap Y = A$ by Theorem 2.6.4. In other words, $\overline{A} \cap B = \varnothing$. Similarly, $A \cap \overline{B} = \varnothing$. ✓

Suppose A and B are disjoint subsets of Y whose union is Y and $A \cap B' = A' \cap B = \varnothing$. Thus, $A \cap \overline{B} = \overline{A} \cap B = \varnothing$. This implies $\overline{A} \cap Y = A$ and $\overline{B} \cap Y = B$; A and B are closed in Y, and thus they are open in Y as well. □

Lemma 3.1.2

If the sets C and D form a separation of a space X, and if Y is a connected subspace of X, then Y lies entirely within C or D.

Proof. $C \cap Y$ and $D \cap Y$ are open in Y. Also, $(C \cap Y) \cup (D \cap Y) = (C \cup D) \cap Y = Y$. If they were both unempty, they would form a separation of Y. Thus, one of them is empty; Y is entirely in the other. □

Theorem 3.1.1

Let X be a topological space. Let $\{A_\alpha\}_{\alpha\in J}$ be a family of connected subspaces of X. If $\bigcap_{\alpha\in J} A_\alpha \neq \varnothing$, then $\bigcup_{\alpha\in J} A_\alpha$ is connected.

Proof. Take any $p \in \bigcap_{\alpha\in J} A_\alpha$. Suppose C and D form a separation of $Y = \bigcup_{\alpha\in J} A_\alpha$. WLOG, $p \in C$. For each $\alpha \in J$, since $p \in C \cap A_\alpha$, by Lemma 3.1.2, $A_\alpha \subseteq C$. Thus, $\bigcup_{\alpha\in J} A_\alpha \subseteq C$, contradicting that $D \cap Y \neq \varnothing$. □

Theorem 3.1.2

Let A be a connected subspace of X. If $A \subseteq B \subseteq \overline{A}$, then B is also connected.

Proof. Suppose $B = C \cup D$ is a separation of B for the sake of contradiction. By Lemma 3.1.2, WLOG, $A \subseteq C$. Then, $B \subseteq \overline{A} \subseteq \overline{C}$. Since $\overline{C} \cap D = \varnothing$ by Lemma 3.1.1, $B \cap D = \varnothing$, which makes C and D not form a separation, #. □

Theorem 3.1.3 Connected Space and Continuous Map

Let $f : X \to Y$ be a continuous map. If X is connected, then $\operatorname{Im} f$ is connected.

Proof. Note that the surjective map $g : X \to \operatorname{Im} f$ obtained by restricting the codomain of f is also continuous by Theorem 2.7.2. Suppose $\operatorname{Im} f = A \cup B$ is a separation of $\operatorname{Im} f$. Then, $g^{-1}(A)$ and $g^{-1}(B)$ are open and disjoint sets in X whose union is X, which is a contradiction to the connectedness of X. □

Theorem 3.1.4 Connected Space and Finite Product

Let $\{X_i\}_{i=1}^{n}$ be a finite family of connected spaces. then,

$$X = \prod_{i=1}^{n} X_i$$

is connected.

Proof. It is enough to prove for two connected spaces X and Y; extension to finite products can be done inductively. We may assume X and Y are nonempty. Take any $a \times b \in X \times Y$. Let $x \in X$. $X \times \{b\}$ and $\{x\} \times Y$ as subspaces of $X \times Y$ are connected since they are homeomorphic with X and Y, respectively. Thus,

$$T_x = (X \times \{b\}) \cup (\{x\} \times Y)$$

is connected by Theorem 3.1.1, having $x \times b$ as a common point of two spaces. Thus,

$$X \times Y = \bigcup_{x\in X} T_x$$

is connected as they have a point $a \times b$ in common. □

Theorem 3.1.5 Connected Space and Product Topology

Let $\{X_\alpha\}_{\alpha\in J}$ be a family of connected spaces. Then,

$$X = \prod_{\alpha\in J} X_\alpha$$

is connected in the product topology.

Proof. We may assume that $X_\alpha \neq \varnothing$ for each $\alpha \in J$. Let $\mathbf{a} = (a_\alpha)_{\alpha\in J}$ be a fixed point of X.

We first note that, given any finite subset K of J, $X_K \triangleq \{(x_\alpha)_{\alpha\in J} \mid \forall \alpha \in J \setminus K,\ x_\alpha = a_\alpha\}$ is a connected subspace of X as X_K is homeomorphic with $\prod_{\alpha\in K} X_\alpha$, which is connected by Theorem 3.1.4. Note that $Y \triangleq \bigcup\{X_K \mid K \subseteq J$ and K is finite$\}$ as a subspace of X is connected since $\mathbf{a} \in X_K$ for every finite $K \subseteq J$.

Let $\mathbf{x} \in X$ and $B = \bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(U_{\alpha_i})$ be any basis that contains $\mathbf{x}$ where $\alpha_i \in J$ for each $i \in [n]$. Define $\mathbf{x}' \in X$ be

$$(\mathbf{x}')_\alpha \triangleq \begin{cases} x_\alpha & \text{if } \alpha = \alpha_i \text{ for some } i \in [n] \\ a_\alpha & \text{otherwise.} \end{cases}$$

Then, $\mathbf{x}' \in B \cap Y$. Thus, by Theorem 2.6.5, $\overline{Y} = X$. By Theorem 3.1.2, X is connected. □

Example 3.1.1 ($\mathbb{R}^\omega$ in the Box Topology is Disconnected)

Let

$$A = \{\mathbf{x} \in \mathbb{R}^\omega \mid \mathbf{x} \text{ is bounded}\} \text{ and}$$
$$B = \{\mathbf{x} \in \mathbb{R}^\omega \mid \mathbf{x} \text{ is unbounded}\}.$$

If $\mathbf{a}$ is in either A or B, $\prod_{i\in\mathbb{Z}_+}(a_i - 1, a_i + 1)$ is an open set that is contained in either A or B. Thus, each A and B are disjoint open sets in $\mathbb{R}^\omega$ whose union is $\mathbb{R}^\omega$.

3.2 Connected Subspaces of the Real Line

Definition 3.2.1: Linear Continuum

A simply ordered set L having more than one element is called *linear continuum* if the following hold:

(i) L has the least upper bound property.

(ii) $\forall x, y \in L,\ (x < y \implies \exists z \in L,\ x < z < y)$.

Note:-

$\mathbb{R}$ is a linear continuum.

Example 3.2.1 (The Ordered Square is a Linear Continuum)

Let $I = [0,1]$ and $I_0^2 = I \times I$ be the ordered square with the dictionary ordering.

(i) Let $\varnothing \neq A \subseteq I_0^2$ and $\pi_1 : I_0^2 \to I$ be the projection onto its first factor. Then, $\pi_1(A)$ is bounded above by 1. Let $b \triangleq \sup \pi_1(A)$. ($[0,1]$ has l.u.b. property.)

If $b \in A$, it implies that $A \cap (\{b\} \times I) \neq \varnothing$ and is bounded above by 1. Thus, we may let $c \triangleq \sup\big(A \cap (\{b\} \times I)\big)$. One may readily check that $\sup A_0 = b \times c$.

If $b \neq A_0$, then $b \times 0$ is the trivial least upper bound of A_0. ✓

(ii) Suppose $x_1 \times y_1 < x_2 \times y_2$. If $x_1 < x_2$, then $x_1 \times y_1 < (x_1 + x_2)/2 \times 0 < x_2 \times y_2$. If $x_1 = x_2$, then, $x_1 \times y_1 < x_1 \times (y_1 + y_2)/2 < x_2 \times y_2$. ✓

Theorem 3.2.1

If L is a linear continuum in the order topology, any convex subspace of L is connected.

Proof. Let Y be a convex subspace of L. Suppose $Y = A \cup B$ is a separation of Y for the sake of contradiction. Take any $a \in A$ and $b \in B$. WLOG, $a < b$. $[a, b] \subseteq Y$ as Y is convex, and $[a, b]$ as a subspace of Y is exactly $[a, b]$ in the order topology by Theorem 2.5.2. Hence,

$$A_0 \triangleq A \cap [a, b] \quad \text{and} \quad B_0 \triangleq B \cap [a, b]$$

form a separation of $[a, b]$.

Let $c \triangleq \sup A_0$. Then, $c \geq a$ as $a \in A_0$, and $c \leq b$ as, if c were larger than b, there would be $z \in L$ such that $b < z < c$, which is an upper bound of A_0 smaller than c. However, we claim that $c \notin A_0 \cup B_0 = [a, b]$, which leads to a contradiction.

($c \notin A_0$) Suppose $c \in A_0$ for the sake of contradiction. Since A_0 is open in $[a, b]$, there must exist $e \in (c, b]$ such that $[c, e) \subseteq A_0$. (e cannot be larger than b as $b \notin A_0$.) As the existence of $e' \in (c, e) \cap L$ is guaranteed and such e' is in A_0, c is no longer an upper bound of A_0, #.

($c \notin B_0$) Suppose $c \in B_0$ for the sake of contradiction. Since B_0 is open in $[a, b]$, there exists $e \in [a, c)$ such that $(e, c] \subseteq B_0$. (e cannot be smaller than a as $a \notin B_0$.) Since, $(c, \infty) \cap A_0 = \varnothing$ as c is the supremum of A_0, e is an upper bound of A_0 that is smaller than c, #. □

Corollary 3.2.1

$\mathbb{R}$ and intervals and rays in $\mathbb{R}$ are connected.

Theorem 3.2.2 Intermediate Value Theorem

Let X be a connected space and Y has an order topology. Let $f : X \to Y$ be a continuous map. Then, if $a, b \in X$ and $r \in Y$ satisfy $f(a) \leq r \leq f(b)$, there exists $c \in X$ such that $f(c) = r$.

Proof. If $f(a) = r$ or $f(b) = r$, then done. So suppose $f(a) < r < f(b)$. $\operatorname{Im} f$ is connected by Theorem 3.1.3. Let

$$A \triangleq \operatorname{Im} f \cap (-\infty, r) \quad \text{and} \quad B \triangleq \operatorname{Im} f \cap (r, \infty).$$

Then, A and B are open in $\operatorname{Im} f$ and $f(a) \in A$ and $f(b) \in B$. Thus, it cannot happen that $\operatorname{Im} f \setminus \{r\} = A \cup B = \operatorname{Im} f$ since $\operatorname{Im} f$ is connected. Therefore, $r \in \operatorname{Im} f$. □

Definition 3.2.2: Path and Path Connectedness

Let X be a space. Given $x, y \in X$, a *path* in X from x to y is a continuous map $f : [a, b] \to X$ where $[a, b]$ is a subspace of $\mathbb{R}$, $f(a) = x$, and $f(b) = y$. The space X is *path connected* if there exists a path in X from x to y for every $x, y \in X$.

Example 3.2.2 (Punctured Euclidean Space)

Define *punctured Euclidean space* to be the space $\mathbb{R}^n \setminus \{\mathbf{0}\}$, where $\mathbf{0}$ is the origin in $\mathbb{R}^n$. If $n > 1$, the space is path connected. We can join $\mathbf{x}$ and $\mathbf{y}$ by the line segment that has x and y as endpoints if the segment does not go through $\mathbf{0}$. Otherwise, we may choose a point $\mathbf{x}'$ by flipping the sign of a coordiate of $\mathbf{x}$. We have a line that connects $\mathbf{x}$ and $\mathbf{x}'$ and other line that connects $\mathbf{x}'$ and $\mathbf{y}$.

Theorem 3.2.3

Every path connected space is connected.

Proof. Let X be a path connected space. If $X = \varnothing$, it is done; let $X \neq \varnothing$. Take $x \in X$. For each $y \in X$, let $f_y : [0,1] \to X$ be a path from x to y. Since $[0,1]$ is connected (Corollary 3.2.1), $\operatorname{Im} f_y$ is connected by Theorem 3.1.3. As $x \in \bigcap_{y \in X} \operatorname{Im} f_y$, $X = \bigcup_{y \in X} \operatorname{Im} f_y$ is connected by Theorem 3.1.1. □

Example 3.2.3 (Connectedness Does Not Imply Path Connectedness)

By Example 3.2.1, I_0^2 is connected. Suppose I_0^2 is path connected for the sake of contradiction. Then, there is a path $f : [0,1] \to I_0^2$ from 0×0 to 1×1. Theorem 3.2.2 says that $\operatorname{Im} f = I_0^2$. For each $x \in I$, let $U_x = f^{-1}(\{x\} \times I)$. Note that $U_x \neq \varnothing$. Since each U_x is open as f is continuous, by the denseness of $\mathbb{Q}$ in $\mathbb{R}$, there exists $q_x \in U_x \cap \mathbb{Q}$ for each $x \in X$. This implies the existence of a injection $g : I \to \mathbb{Q}$ defined by $x \mapsto q_x$, which is a contradiction as I is uncountable. (Theorem 1.6.1)

Theorem 3.2.4 Path Connected Space and Continuous Map

Let $f : X \to Y$ be a continuous map. If X is path connected, then $\operatorname{Im} f$ is path connected.

Proof. Take $y_1, y_2 \in \operatorname{Im} f$. There exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X is connected, there exists a continuous map $g : [0,1] \to X$ such that $g(0) = x_1$ and $g(1) = x_2$. Then, $f \circ g : [0,1] \to \operatorname{Im} f$ is a continuous map such that $(f \circ g)(0) = y_1$ and $(f \circ g)(1) = y_2$ by Theorem 2.7.2. □

Example 3.2.4 (Unit Sphere)

Define the *unit sphere* S^{n-1} in $\mathbb{R}^n$ by the equation

$$S^{n-1} \triangleq \{\, \mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| = 1 \,\}.$$

Then, the map $g : \mathbb{R}^n \setminus \{\mathbf{0}\} \to S^{n-1}$ defined by $\mathbf{x} \mapsto \mathbf{x}/\|\mathbf{x}\|$ is a continuous surjective map. Moreover, if $n > 1$, since $\mathbb{R}^n \setminus \{0\}$ is path connected (Example 3.2.2), $S^{n-1} = \operatorname{Im} g$ is also path connected by Theorem 3.2.4.

Example 3.2.5 (Topologist's Sine Curve)

Let

$$S \triangleq \left\{ x \times \sin\frac{1}{x} \in \mathbb{R}^2 \,\middle|\, x \in (0,1] \right\}.$$

Since S is a image of $(0,1]$ under a continuous map $x \mapsto x \times \sin(1/x)$, S is (path) connected. Thus, $\overline{S}$ is connected by Theorem 3.1.2. Note that $S_0 \triangleq \overline{S} \setminus S = \{0\} \times [-1,1]$. ($S_0$ is also closed.)

Suppose $\overline{S}$ is path connected for the sake of contradiction. Then, there is a path $f : [0,1] \to \overline{S}$ from 0×0 to $f(1) \in S$. $f^{-1}(S_0)$ is closed in $[0,1]$ by Theorem 2.7.1. Hence $b \triangleq \sup f^{-1}(S_0) \in f^{-1}(S_0)$ and $b \neq 1$. $f(b) \in S_0$ and $f((b,1]) \subseteq S$.

Reparametrize $f : [0,1] \to \overline{S}$ so that $t \mapsto x(t) \times y(t)$; $f(0) \in S_0$ and $f((0,1]) \subseteq S$. ($y(t) = \sin(1/x(t))$) Since $x(t) > 0$ for $t \in (0,1]$, x is continuous, and $x(0) = 0$, we may construct a sequence $\{t_n\}_{n \in \mathbb{Z}_+}$ such that

$$\lim_{n\to\infty} t_n = 0, \quad x(t_n) = \frac{1}{(n+1/2)\pi}, \quad \text{and thus}$$

$$y(t_n) = \sin(1/x(t_n)) = \sin\big((n+1/2)\pi\big) = (-1)^n.$$

However, $\{y(t_n)\}_{n\in\mathbb{Z}_+}$ diverges although y is continuous and $t_n \to 0$. Thus, $\overline{S}$ is not path connected.

3.3 Components and Local Connectedness

Definition 3.3.1: Component

Given a space X, let $\sim$ be a equivalent relation defined by

$$x \sim y \text{ if there is a connected subspace of } X \text{ containing } x \text{ and } y.$$

The equivalence classes of $\sim$ is called *(connected) components* of X.

Note:-

Reflexivity follows from the fact that $\{x\}$ is a connected subspace of X that contains x.
Symmetry is direct.
Let $x, y, z \in X$ and suppose $x \sim y$ and $y \sim z$. There are connected subspaces U and V such that $x, y \in U$ and $y, z \in V$. Then, $U \cup V$ is a connected subspace of X that contains both x and z by Theorem 3.1.1.

Note:-

Let $\{C_\alpha\}_{\alpha\in J}$ be the set of components of X. Then, it is a partition of X (indeed).

Theorem 3.3.1

Let $\{C_\alpha\}_{\alpha\in J}$ be the set of components of X. If $A \subseteq X$ is a connected subspace of X, then $A \subseteq C_\alpha$ for some $\alpha \in J$.

Proof. If $A = \varnothing$, it is done; suppose $A \neq \varnothing$.

Let C_α and C_β be connected components. If $A \cap C_\alpha \neq \varnothing$ and $A \cap C_\beta \neq \varnothing$, we may take $x \in A \cap C_\alpha$ and $y \in A \cap C_\beta$, which makes $x \sim y$. This implies $x_\alpha \sim x_\beta$ for all $x_\alpha \in C_\alpha$ and $x_\beta \in C_\beta$; thus $C_\alpha = C_\beta$.

Now, take any $a \in A$. Since $\{C_\alpha\}_{\alpha\in J}$ is a partition of X, there exists some $\alpha \in J$ such that $a \in C_\alpha$. By the previous result, $A \cap C_\beta = \varnothing$ for all $\beta \in J \setminus \{\alpha\}$. Hence, $A \subseteq C_\alpha$ □

Theorem 3.3.2

Let $\{C_\alpha\}_{\alpha\in J}$ be the set of components of X. Then, for each $\alpha \in J$, C_α is connected.

Proof. Take any $x_0 \in C_\alpha$. Then, for each $x \in C_\alpha$, there exists a connected subspace A_x that contains both x_0 and x. By Theorem 3.3.1, $A_x \subseteq C_\alpha$. Thus, $C_\alpha = \bigcup_{x\in C_\alpha} A_x$, which is connected by Theorem 3.1.1. □

Definition 3.3.2: Path Component

Given a space X, let $\sim$ be a equivalent relation defined by

$$x \sim y \text{ if there is a path in } X \text{ from } x \text{ to } y.$$

The equivalence classes of $\sim$ is called *path components* of X.

Note:-

The relation is reflexive since $f : [0,1] \to X$ defined by $f(t) = x$ is a path from x to x.
The relation is symmetric since, if $f : [a,b] \to X$ is a path from x to y, then $g : [a,b] \to X$ defined by $g(t) = f(a+b-t)$ is a path from y to x.
The relation is transitive since, if $f : [a,b] \to X$ and $g : [c,d] \to X$ are paths from x to y and from y to z, respectively, then $h : [a, b+d-c]$ defined by

$$h(t) = \begin{cases} f(t) & \text{if } a \le t \le b \\ g(t-b+c) & \text{otherwise.} \end{cases}$$

is a path from x to z. h is continuous by Theorem 2.7.3.

Theorem 3.3.3

Let $\{P_\alpha\}_{\alpha \in J}$ be the set of path components of X. If $A \subseteq X$ is a path connected subspace of X, then $A \subseteq P_\alpha$ for some $\alpha \in J$.

Proof. Analogous to the proof of Theorem 3.3.1. □

Theorem 3.3.4

Let $\{P_\alpha\}_{\alpha \in J}$ be the set of path components of X. Then, for each $\alpha \in J$, P_α is path connected.

Proof. Analogous to the proof of Theorem 3.3.2. □

Corollary 3.3.1

Every path component is entirely contained in a connected component.

Proof. Every path component is path connected by Theorem 3.3.4, and thus connected by Theorem 3.2.3. By Theorem 3.3.1, it is contained in some connected component. □

Corollary 3.3.2

Every component is closed.

Proof. Let C_α be a connected component of X. Since $\overline{C_\alpha}$ is connected by Theorem 3.1.2, and since $\overline{C_\alpha} \cap C_\alpha \neq \varnothing$, $\overline{C_\alpha} \subseteq C_\alpha$ by Theorem 3.3.1. □

Corollary 3.3.3

If there are a finite number of components, then each component is open.

Proof. Let $X = \bigcup_{i=1}^{n} C_i$ where each C_i is a component. Then, for each $i \in [n]$, $C_i = X \setminus \bigcup_{j \in [n] \setminus \{i\}} C_j$. C_i is open as $\bigcup_{j \in [n] \setminus \{i\}} C_j$ is closed by Corollary 3.3.2. □

Example 3.3.1 (Path Component Is Not Necessarily Open or Closed)

Let $\overline{S}$ be the topologist's sine curve discussed in Example 3.2.5. Then, S and S_0 are the two path components of $\overline{S}$. S is not closed and S_0 is not open.

Example 3.3.2

Let $A \triangleq S \cup (S_0 \setminus \{0\} \times \mathbb{Q})$. Since $S \subseteq A \subseteq \overline{S}$, A is connected by Theorem 3.1.2. However, $\{0 \times r\}$ for every $r \in [0,1] \setminus \mathbb{Q}$ is a path component. Thus, A has uncountably many path components.

Definition 3.3.3: Locally Connected Space

Let X be a topological space. X is *locally connected at* x if, for any neighborhood U of x, there exists a connected neighborhood V of x such that $x \in V \subseteq U$. X is *locally connected* if X is locally connected at every point of X.

Definition 3.3.4: Locally Path Connected Space

Let X be a topological space. X is *locally path connected at* x if, for any neighborhood U of x, there exists a path connected neighborhood V of x such that $x \in V \subseteq U$. X is *locally path connected* if X is locally path connected at every point of X.

Note:-

If a topological space X is locally path connected, then it is locally connected as well.

Theorem 3.3.5

A topological space X is locally connected if and only if, for every open set U in X, each connected component of U is open.

Proof. ($\Rightarrow$) Let U be open in X and let $\{C_\alpha\}_{\alpha \in J}$ be the set of components of U. Take any C_α and let $x \in C_\alpha$. Since X is locally connected at x, there exists a connected neighborhood V of x such that $x \in V \subseteq U$. By Theorem 3.3.1, $x \in V \subseteq C_\alpha$. This proves that C_α is open.

($\Leftarrow$) Let $x \in X$ and U be a neighborhood of x. Let $\{C_\alpha\}_{\alpha \in J}$ be the components of U. There exists some $\alpha_0 \in J$ such that $x \in C_{\alpha_0}$. Since C_{α_0} is open by assumption, C_{α_0} is a connected neighborhood of x and satisfies $x \in C_{\alpha_0} \subseteq U$. $\square$

Theorem 3.3.6

A topological space X is locally path connected if and only if, for every open set U in X, each path component of U is open.

Proof. Analogous to Theorem 3.3.5. $\square$

Theorem 3.3.7

Let X be a locally path connected space. Then, the connected components and the path components are the same.

Proof. Let C be a connected component of X. C is open by Theorem 3.3.5 as X is locally connected. Let $x \in C$ and let P be the path component which x is contained in. Then, $P \subseteq C$ by Corollary 3.3.1.

Suppose $P \subsetneq C$ for the sake of contradiction. Let

$$Q \triangleq \bigcup \{\, \hat{P} \subseteq C \mid \hat{P} \text{ is a path component of } X \text{ and } \hat{P} \neq P \,\}.$$

Since path component of an open set, especially, C, is open by Theorem 3.3.6, P and Q are open. Moreover, since $C = P \cup Q$, they form a separation of C, which is a contradiction, #. $\square$

3.4 Compact Spaces

Definition 3.4.1: Open Cover

A collection $\mathcal{A}$ of subsets of a space X is said to *cover* X, or to be a *covering* of X, if $\bigcup \mathcal{A} = X$. It is called an *open covering* if A is open in X for each $A \in \mathcal{A}$.

Definition 3.4.2: Compactness

A space X is said to be *compact* if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Example 3.4.1 ($\mathbb{R}$ Is Not Compact)

The open cover $\mathcal{A} \triangleq \{(n, n+2) \mid n \in \mathbb{Z}\}$ does not have a finite subcollection that covers $\mathbb{R}$. Thus, $\mathbb{R}$ is not compact.

Lemma 3.4.1

Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.

Proof. ($\Rightarrow$) Let $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is a covering of Y by open sets in X. Then, the collection

$$\{A_\alpha \cap Y \mid \alpha \in J\}$$

is an open covering of Y. Thus, there exists a finite subcollection

$$\{A_{\alpha_1} \cap Y, \cdots, A_{\alpha_n} \cap Y\}$$

that covers Y. Then, $\{A_{\alpha_1}, \cdots, A_{\alpha_n}\}$ is a finite subcollection of $\mathcal{A}$ that covers Y.

($\Leftarrow$) Let $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ be an open covering of Y. For each $\alpha \in J$, there is an open set $\hat{A}_\alpha$ in X such that $A_\alpha = \hat{A}_\alpha \cap Y$. Then, the collection $\{\hat{A}_\alpha\}_{\alpha \in J}$ composed of open sets in X that covers Y; by the assumption, there exists a fintie subcollection

$$\{\hat{A}_{\alpha_1}, \cdots, \hat{A}_{\alpha_n}\}$$

that covers Y. Then, $\{A_{\alpha_1}, \cdots, A_{\alpha_n}\}$ is a finite subcollection of $\mathcal{A}$ that covers Y. □

Theorem 3.4.1

Let X be a compact space. If Y is a closed subset of X, then Y as a subspace of X is compact.

Proof. If $Y = \varnothing$, then it is done. So, suppose $Y \neq \varnothing$. Let $\mathcal{A}$ be a covering of Y composed of sets open in X.

$$\mathcal{B} \triangleq \mathcal{A} \cup \{X \setminus Y\}$$

is an open covering of X. Thus, it has a finite subcollection

$$\{A_1, A_2, \cdots, A_n, X \setminus Y\}$$

that covers X where $A_i \in \mathcal{A}$ for each $i \in [n]$. (WLOG, $X \setminus Y$ is in the subcollection since we may just add $X \setminus Y$ and does not affect its finiteness.) Then, $\{A_i\}_{i \in [n]}$ is a finite subcollection of $\mathcal{A}$ that covers Y. □

Theorem 3.4.2

Let X be a Hausdorff space. If $Y \subseteq X$ is a compact subspace of X, then Y is closed in X.

Proof. If $Y = \varnothing$ or $Y = X$, then it is done; suppose $\varnothing \neq Y \subsetneq X$. Let $x_0 \in X \setminus Y$. For each $y \in Y$, there are disjoint neighborhoods U_y and V_y of x_0 and y in X. Then, $\{V_y\}_{y\in Y}$ is an open covering of Y. Thus there exists a finite subcollection of it

$$\{V_{y_1}, V_{y_2}, \cdots, V_{y_n}\}$$

that covers Y.

Let

$$V \triangleq \bigcup_{i=1}^{n} V_{y_i} \quad \text{and} \quad U \triangleq \bigcap_{i=1}^{n} U_{y_i}.$$

Then, U is a neighborhood of x_0 and does not intersect V, which covers Y. Thus, $U \subseteq X \setminus Y$. Hence, $X \setminus Y$ is open; Y is closed. □

Example 3.4.2 (Being Hausdorff Is Needed)

Let $X = \mathbb{R}$ be endowed with the finite complement topology. Then, every subset of X is compact. To see this, suppose $\mathcal{A}$ is a collection of open sets in X that covers $Y \subseteq X$. Then, take any $A \in \mathcal{A}$ and it will cover all but finitely many points in Y. For each remaining point, choose an open set in $\mathcal{A}$ that contains the point. Thus, we get a finite collection of $\mathcal{A}$ that covers Y. However, only closed sets are finite subsets of X and $\mathbb{R}$.

Corollary 3.4.1

Let X be a Hausdorff space. If $Y \subseteq X$ is a compact subspace of X, then, given any $x_0 \in X \setminus Y$, there are disjoint open sets U and V in X containing x_0 and Y, respectively.

Proof. U and V defined in the proof of Theorem 3.4.2 are those. □

Theorem 3.4.3

Let X be a compact space. Let $f : X \to Y$ be a continuous map. Then, $\operatorname{Im} f$ as a subspace of Y is compact.

Proof. Let $\mathcal{A}$ be a covering of the set $\operatorname{Im} f$ by sets open in Y. Then, the collection

$$\{f^{-1}(A) \mid A \in \mathcal{A}\}$$

is an open covering of X as f is continuous. Hence, there are a finite subcollection $\{A_1, \cdots, A_n\}$ of $\mathcal{A}$ such that $\{f^{-1}(A_i)\}_{i\in[n]}$ covers X. The sets $\{A_1, \cdots, A_n\}$ covers $\operatorname{Im} f$. □

Theorem 3.4.4

Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. We only need to prove f^{-1} is continuous. Let $A \subseteq X$ is closed in X. Then, A is compact by Theorem 3.4.1. Thus, since $f\big|_A : A \to Y$ is continuous (Theorem 2.7.2), $f(A)$ is compact by Theorem 3.4.3. By Theorem 3.4.2, $f(A)$ is closed. Hence, we proved that $f(A)$ is closed for each closed subset A of X; f^{-1} is continuous by Theorem 2.7.1. □

Lemma 3.4.2 The Tube Lemma

Let X and Y be topological spaces and Y is compact. Given any $x_0 \in X$ and an open set N in $X \times Y$ that contains $\{x_0\} \times Y$, there exists a neighborhood W of x_0 in X such that $W \times Y \subseteq N$.

Proof. For each $y \in Y$, there exists a basis element $U_y \times V_y$ in the product topology such that $x_0 \times y \in U_y \times V_y \subseteq N$. Then, $\mathcal{A} \triangleq \{U_y \times V_y \mid y \in Y\}$ is a covering of $\{x_0\} \times Y$ by open sets in $X \times Y$. Since $\{x_0\} \times Y$, being homeomorphic with Y, is compact, there is a finite subcollection

$$\mathcal{A}' = \{U_{y_1} \times V_{y_1}, \cdots, U_{y_n} \times V_{y_n}\}$$

of $\mathcal{A}$ that covers $\{x_0\} \times Y$. Note that $\{x_0\} \times Y \subseteq \bigcup_{i=1}^{n}(U_{y_i} \times V_{y_i}) \subseteq N$. Let

$$W \triangleq \bigcap_{i=1}^{n} U_{y_i}.$$

Then, W is a neighborhood of x_0 in X.

Now, take $x \times y \in W \times Y$. There exists some $i \in [n]$ such that $y \in V_{y_i}$; $x \times y \in U_{y_i} \times V_{y_i} \subseteq N$. This shows $W \times Y \subseteq N$. □

Note:-

The set $W \times Y$ is often called a *tube* about $x_0 \times Y$.

Note:-

Lemma 3.4.2 may not hold if Y is not compact. If $X = Y = \mathbb{R}$, the open set

$$N \triangleq \left\{ x \times y \in \mathbb{R}^2 \;\middle|\; |x| < \frac{1}{y^2+1} \right\}$$

does contain $\{0\} \times Y$ but there is no open neighborhood W of 0 in X such that $W \times Y \subseteq N$.

Theorem 3.4.5

Let $X_1, X_2, \cdots, X_n$ be topological spaces. Then, $\prod_{i=1}^{n} X_i$ is compact if and only if X_i is compact for each $i \in [n]$.

Proof. It is enough to prove for two topological spaces X and Y.

($\Rightarrow$) It is enough to prove X is compact. Let $\mathcal{A}$ be an open covering of X. Then, $\{A \times Y \mid A \in \mathcal{A}\}$ is an open covering of $X \times Y$; there exists a finite subcollection

$$\{A_1 \times Y, A_2 \times Y, \cdots, A_n \times Y\}$$

that covers $X \times Y$. Thus, $\{A_i \mid i \in [n]\}$ is a finite subcollection of $\mathcal{A}$ that covers X.

($\Leftarrow$) Let $\mathcal{A}$ be an open covering of $X \times Y$. For each $x \in X$, since $\{x\} \times Y$ is compact, there are finite subcollection $\{A_1, A_2, \cdots, A_{n_x}\} \subseteq \mathcal{A}$ that covers $\{x\} \times Y$. Then, $N_x \triangleq \bigcup_{i=1}^{n_x} A_i$ is an open set in $X \times Y$ that contains $\{x\} \times Y$. Thus, by Lemma 3.4.2, there exists a tube $W_x \times Y$ such that $\{x\} \times Y \subseteq W_x \times Y \subseteq N_x$.

Noting that $\{W_x \mid x \in X\}$ is an open covering of X, there are finite subcover $\{W_{x_1}, W_{x_2}, \cdots, W_{x_k}\}$ that covers X. Hence, $\{W_{x_i} \times Y \mid i \in [k]\}$ covers $X \times Y$ and each element of it is covered by finite elements in $\mathcal{A}$. □

Note:-

Theorem 3.4.5 holds for an arbitrary product but is is slightly technical.

Definition 3.4.3: Finite Intersection Property

A collection $\mathcal{C}$ of subsets of X is said to have *finite intersection property* if, for any finite subcollection

$$\{C_1, C_2, \cdots, C_n\} \subseteq \mathcal{C}$$

of $\mathcal{C}$, we have

$$\bigcap_{i=1}^{n} C_i \neq \varnothing.$$

In other words,

$$\forall n \in \mathbb{Z}_+,\ \forall \mathcal{C}' \in \binom{\mathcal{C}}{n},\ \bigcap \mathcal{C}' \neq \varnothing.$$

Theorem 3.4.6

Let X be a topological space. Then X is compact if and only if, for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap \mathcal{C}$ is nonempty.

Proof. Given a collection $\mathcal{A}$ of subsets of X, let

$$\mathcal{C} \triangleq \{X \setminus A \mid A \in \mathcal{A}\}.$$

Then the following hold.

- $\mathcal{A}$ is a collection of open sets if and only if $\mathcal{C}$ is a collection of closed sets.
- $\bigcup \mathcal{A} = X$ if and only if $\bigcap \mathcal{C} = \varnothing$.
- The finite subcollection $\{A_1, \cdots, A_n\}$ covers X if and only if $\bigcap_{i=1}^{n}(X \setminus A_i) = \varnothing$.

Therefore, these are equivalent.

(i) Every open covering of X allows a finite subcover.

(ii) A collection of open sets in X that does not allow a finite subcover does not cover X. - contrapositive of (i)

(iii) A collection of closed sets in X that does not allow a nonempty intersection of finite subcollection does not have a nonempty intersection.

□

Definition 3.4.4: Nested Sequence

A sequence of sets $\{C_n\}_{n\in\mathbb{Z}_+}$ is called a *nested sequence* if $C_n \supseteq C_{n+1}$ for each $n \in \mathbb{Z}_+$.

Corollary 3.4.2

Let X be a compact space. Let $\{C_n\}_{n\in\mathbb{Z}_+}$ be a nested sequence of nonempty closed sets in X. Then,

$$\bigcap_{n\in\mathbb{Z}_+} C_n \neq \varnothing.$$

Proof. Let $\mathcal{C} \triangleq \{C_n \mid n \in \mathbb{Z}_+\}$. Then, $\mathcal{C}$ satisfies the finite intersection property as

$$C_{n_1} \cap C_{n_2} \cap \cdots \cap C_{n_k} = C_{\max_{i=1}^{k} n_i} \neq \varnothing.$$

The result follows from Theorem 3.4.6. □

3.5 Compact Subspaces of the Real Line

Theorem 3.5.1

Let X be a simply ordered set having the least upper bound property. In the order topology, every closed interval $[a,b]$ in X is compact.

Proof. Let $\mathcal{A}$ be an open covering of $[a,b]$.

We claim that, given any $x \in [a,b)$, there exists $y \in (x,b]$ such that $[x,y]$ can be covered by at most two elements of $\mathcal{A}$.

(i) If there exists an immediate successor $y \in (x,b]$ of x, then $[x,y] = \{x,y\}$. Pick two open sets in $\mathcal{A}$ that contain x and y, respectively.

(ii) Otherwise, let $A \in \mathcal{A}$ with $x \in A$. Then, $[x,c) \subseteq A$ for some $c \in (x,b]$ and $|[x,c)| = \infty$. Take any $y \in (x,c) \subseteq (x,b]$, then $[x,y] \subseteq [x,c) \subseteq A$. ✓

Let

$$C \triangleq \left\{ y \in (a,b] \,\middle|\, [a,y] \text{ can be covered by finitely many elements of } \mathcal{A} \right\}.$$

By the previous claim, $C \neq \varnothing$, and C is bounded above by b. Thus, we may let $c \triangleq \sup C$. ($a \le c \le b$, indeed.) ✓

Suppose $c \notin C$ for the sake of contradiction. Choose $A \in \mathcal{A}$ that contains c. Then, there exists $d \in [a,c)$ such that $(d,c] \subseteq A$. Hence, there exists $z \in C \cap (d,c]$. Since $z \in C$, the interval $[a,z]$ can be covered by finitely many, say n, elements of $\mathcal{A}$, then, since $[a,c] = [a,z] \cup [z,c]$ and $[z,c] \subseteq (d,c] \subseteq A$, $[a,c]$ can be covered by at most $n+1$ elements of $\mathcal{A}$, which is contradicting to $c \notin C$, #. ✓

Suppose $c < b$ for the sake of contradiction. Then, there exists $y \in (c,b]$ such that $[c,y]$ can be covered by finitely many elements of $\mathcal{A}$ by the previous claim. Hence, $[a,y] = [a,c] \cup [c,y]$ can be covered by finitely many elements of $\mathcal{A}$ since $c \in C$. This implies $y \in C$, contradicting that c is an upper bound of C, #. ✓ □

Example 3.5.1

The ordered square $I_o^2 = [0 \times 0, 1 \times 1]$ is compact.

Corollary 3.5.1

Every closed interval in $\mathbb{R}$ is compact.

Theorem 3.5.2

A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and it is bounded in the Euclidean metrid d or the square metric ρ.

Proof. It suffices to prove only for ρ as A is bounded in d if and only if A is bounded in ρ. (See the proof of Theorem 2.9.2.)

($\Rightarrow$) By Theorem 3.4.2, A is closed. ✓

The collection

$$\{ B_\rho(\mathbf{0}, m) \mid m \in \mathbb{Z}_+ \}$$

is an open covering of A. Thus, $A \subseteq B_\rho(\mathbf{0}, M)$ for some M. Therefore, $\rho(\mathbf{x},\mathbf{y}) \le 2M$ for each $\mathbf{x},\mathbf{y} \in A$. Thus, A is bounded. ✓

($\Leftarrow$) There exists $M \in \mathbb{R}_+$ such that $\rho(\mathbf{x},\mathbf{y}) \le M$ for each $\mathbf{x},\mathbf{y} \in A$. Choose a point $\mathbf{x}_0 \in A$, and let $b \triangleq \rho(\mathbf{x}_0, \mathbf{0})$. Then, $\rho(\mathbf{x},\mathbf{0}) \le P \triangleq N + b$ for every $\mathbf{x} \in A$. Thus, $A \subseteq [-P,P]^n$. $[-P,P]^n$ is compact by Corollary 3.5.1, Theorem 3.4.5, and Theorem 2.5.1. Since A is closed in $[-P,P]^n$ and $[-P,P]^n$ is compact, A is compact by Theorem 3.4.1. $\square$

Theorem 3.5.3 Extreme Value Theorem

Let X be a compact set and Y be an ordered set endowed by the order topology. Let $f : X \to y$ be a continuous map. Then, there exist $c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in X$.

Proof. Suppose $\operatorname{Im} f$ does not have a maximum. Then,

$$\{(-\infty, a) \subseteq \mathbb{R} \mid a \in \operatorname{Im} f\}$$

is an open covering of $\operatorname{Im} f$. Since $\operatorname{Im} f$ is compact by Theorem 3.4.3, $\operatorname{Im} f \subseteq (-\infty, a)$ for some $a \in \operatorname{Im} f$, #. $\square$

Definition 3.5.1: Distance From a Point to a Set

Let (X,d) be a metric space and let $\varnothing \neq A \subseteq X$. For each $x \in X$, we define the *distance from x to A* by the equation

$$d(x,A) \triangleq \inf\{d(x,a) \mid a \in A\}.$$

Definition 3.5.2: Uniform Continuity

A function $f : X \to Y$ from the metrix space (X, d_X) to the metric space (Y, d_Y) is said to be *uniformly continuous* if

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists \delta \in \mathbb{R}_+,\ \forall x_1, x_2 \in X,\ \big(d_X(x_1,x_2) < \delta \implies d_Y(f(x_1), f(x_2))\big).$$

Theorem 3.5.4

Let (X,d) be a metric space and let $\varnothing \neq A \subseteq X$. Then, $f : X \to \mathbb{R}$ defined by

$$f(x) \triangleq d(x,A)$$

is uniformly continuous.

Proof. Take any $\varepsilon \in \mathbb{R}_+$ and let $\delta \triangleq \varepsilon$. For any $x, y \in X$ and $a \in A$ with $d(x,y) < \varepsilon$, we have $d(x,A) \leq (x,a) \leq d(x,y) + d(y,a)$. Thus,

$$d(x,A) - d(x,y) \leq \inf_{a \in A} d(y,a) = d(y,A),$$

which implies $|d(x,A) - d(y,A)| \leq d(x,y) < \delta = \varepsilon$. $\square$

Lemma 3.5.1 The Lebesgue Number Lemma

Let (X,d) be a compact metric space. Then, for each open covering $\mathcal{A}$ of X,

$$\exists \delta \in \mathbb{R}_+,\ \forall B \in \mathcal{P}(X) \setminus \{\varnothing\},\ \big(\operatorname{diam} B < \delta \implies \exists A \in \mathcal{A},\ B \subseteq A\big).$$

The number δ is called a *Lebesgue number* for the covering $\mathcal{A}$.

Proof. If $X \in \mathcal{A}$, then every $\delta \in \mathbb{R}_+$ satisfies the condition. Therefore, we may suppose $X \notin \mathcal{A}$.

Choose a finite subcollection $\{A_1, A_2, \cdots, A_n\}$ of $\mathcal{A}$ that covers X. For each $i \in [n]$, let $C_i \triangleq X \setminus A_i$. We define $f : X \to \mathbb{R}$ by

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i).$$

Take any $x \in X$. Then, there exists some $i \in [n]$ such that $x \in A_i$. Since A_i is open, there exists some $\varepsilon \in \mathbb{R}_+$ such that $B(x, \varepsilon) \subseteq A_i$; $d(x, C_i) \geq \varepsilon$. Hence, $f(x) \geq \varepsilon / n$. We just showed that $f(x) > 0$ for all $x \in X$.

Since f is continuous, there exists a minimum of $\operatorname{Im} f$, say δ, by Theorem 3.5.3. We claim that δ is a Lebesgue number for $\mathcal{A}$. Let $\varnothing \neq B \subseteq X$ with $\operatorname{diam} B < \delta$. Take $x_0 \in B$. Then $B \subseteq B(x_0, \delta)$. Then,

$$\delta \leq f(x_0) \leq \max_{i \in [n]} d(x_0, C_i) = d(x_0, C_m).$$

where $m \in [n]$. Then, $B \subseteq B(x_0, \delta) \subseteq A_m$. □

Theorem 3.5.5 Uniform Continuity Theorem

Let (X, d_X) be a compact metric space; let (Y, d_Y) be a metric space. If $f : X \to Y$ is a continuous map, then f is uniformly continuous.

Proof. Take any $\varepsilon \in \mathbb{R}_+$. Let

$$\mathcal{A} \triangleq \left\{ f^{-1}\big(B(y, \varepsilon/2)\big) \,\middle|\, y \in Y \right\}$$

be an open covering of X. Let δ be a Lebesgue number for $\mathcal{A}$. Then, for each $x_1, x_2 \in X$ such that $d_X(x_1, x_2) < \delta$, since $\operatorname{diam}\{x_1, x_2\} = d_X(x_1, x_2) < \delta$, there exists $y \in Y$ such that $\{f(x_1), f(x_2)\} \subseteq B(y, \varepsilon/2)$. Then, $d_Y(f(x_1), f(x_2)) < \varepsilon$. □

Definition 3.5.3: Isolated Point

If X is a topological space, a point $x \in X$ is said to be an *isolated point* of X if $\{x\}$ is open in X.

Lemma 3.5.2

Let X be a nonempty Hausdorff space which has no isolated points. Then, for any nonempty open set U of X and $x \in X$, there exists a nonempty open set V contained in U such that $x \notin \overline{V}$.

Proof. Take any $y \in U \setminus \{x\}$. (This is possible since $U \neq \{x\}$.) Choose disjoint neighborhoods W_1 and W_2 of x and y, respectively. Then, $V = W_2 \cap U$ is the open set we are looking for. V is empty, nonempty as $y \in V$, and its closure does not contain x by Theorem 2.6.5. □

Theorem 3.5.6

Let X be nonempty compact Hausdorff space. If X has no isolated points, then X is uncountable.

Proof. Now let $f : \mathbb{Z}_+ \to X$ be any function. Let $V_0 = X$. Construct $V_1, V_2, \cdots$ as following.

- For each $n \in \mathbb{Z}_+$, choose V_n to be a nonempty set such that $V_n \subseteq V_{n-1}$ and $f(n) \notin \overline{V_n}$.

This is possible thanks to Lemma 3.5.2.

Now, we have a nested sequence $\{\overline{V_n}\}_{n \in \mathbb{Z}_+}$ of closed sets in X. By Corollary 3.4.2, there exists $x \in \bigcap_{n \in \mathbb{Z}_+} \overline{V_n}$. Then, $x \neq f(n)$ for all $n \in \mathbb{Z}_+$ as $x \in \overline{V_n}$ and $f(n) \notin \overline{V_n}$. □

Corollary 3.5.2

Every closed interval in $\mathbb{R}$ is uncountable.

3.6 Limit Point Compactness

Definition 3.6.1: Limit Point Compactness

A topological space X is said to be *limit point compact* if every infinite subset A of X has $A' \neq \varnothing$.

Theorem 3.6.1

If a topological space X is compact, then it is limit point compact.

Proof. Suppose A has no limit point. Then, by Corollary 2.6.1, A is closed in X. Moreover, for each $a \in A$, there exists a neighborhood U_a of a such that $U_a \cap A = \{a\}$ by the definition of limit point. Then,

$$\mathcal{A} \triangleq \{X \setminus A\} \cup \{U_a \mid a \in A\}$$

is an open covering of X. Since X is compact, there is a finite subcollection of $\mathcal{A}$ that covers X. As $X \setminus A$ does not intersect A, only finite number of open sets of the form U_a covers A, which means A is finite. □

Example 3.6.1 (Limit Point Compactness Does Not Imply Compactness)

Let $Y = \{a, b\}$ and give Y the trivial topology. Any nonempty subset A of X has a limit point, for if $(n, a) \in A$ or $(n, b) \in A$, (n, b) or (n, a) is a limit point of A. Thus, the space $X \triangleq \mathbb{Z}_+ \times Y$ is limit point compact. However, it is not compact as the open covering

$$\mathcal{A} \triangleq \{\{n\} \times Y \mid n \in \mathbb{Z}_+\}$$

does not have a finite subcover.

Definition 3.6.2: Sequentially Compact

A topological space X is said to be *sequentially compact* if any sequence $\{x_n\}_{n \in \mathbb{Z}_+}$ in X has a convergent subsequence.

Lemma 3.6.1 The Lebesgue Number Lemma For Sequentially Compact Spaces

Let (X, d) be a sequentially compact metric space. Then, for each open covering $\mathcal{A}$ of X,

$$\exists \delta \in \mathbb{R}_+, \forall B \in \mathcal{P}(X) \setminus \{\varnothing\}, \big(\operatorname{diam} B < \delta \implies \exists A \in \mathcal{A}, B \subseteq A\big).$$

Proof. Suppose to the contrary that there does not exist such δ. Let $\mathcal{A}$ be an open covering of X. Therefore, for each $n \in \mathbb{Z}_+$, there exists a nonempty subset C_n of X such that $\operatorname{diam} C_n < 1/n$ and there is no $A \in \mathcal{A}$ such that $C_n \subseteq A$.

Choose a point $x_n \in C_n$. Since X is sequentially compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}_{i \in \mathbb{Z}_+}$. Let x be a point to which the subsequence converges. (Such point is unique by Theorem 2.6.9.)

Then, $x \in A$ for some $A \in \mathcal{A}$. Thus, there exists $\varepsilon \in \mathbb{R}_+$ such that $x \in B(x,\varepsilon) \subsetneq A$. By the convergence, there exists $i \in \mathbb{Z}_+$ such that $x_{n_i} \in B(x,\varepsilon/2)$ and $1/n_i < \varepsilon/2$. Then, $\operatorname{diam} C_{n_i} < 1/n_i < \varepsilon/2$; hence $C_{n_i} \subseteq B(x_{n_i},\varepsilon/2) \subseteq B(x,\varepsilon) \subseteq A$, #. ☐

Lemma 3.6.2

Let (X,d) be a sequentially compact metric space. Then, for each $\varepsilon \in \mathbb{R}_+$, there exists a finite subset A of X such that $\{B(a,\varepsilon) \mid a \in A\}$ covers X.

Proof. Suppose there does not exist such finite cover for the sake of contradiction. Construct a sequence $\{x_n\}_{n\in\mathbb{Z}_+}$ in X as following.

- $x_1 \in X$
- For each $n \in \mathbb{Z}_+$, $x_{n+1} \in X \setminus \bigcup_{i=1}^n B(x_i,\varepsilon)$.

This is possible as $\bigcup_{i=1}^n B(x_i,\varepsilon) \subsetneq X$ for every $n \in \mathbb{Z}_+$. However, since $d(x_m,x_n) \geq \varepsilon$ for every distinct $n,m \in \mathbb{Z}_+$ by construction, every $\varepsilon/2$-ball may contain at most one x_n. Hence, $\{x_n\}$ has no convergent subsequence. ☐

Theorem 3.6.2

Let X be a metrizable space. TFAE

(i) X is compact.
(ii) X is limit point compact.
(iii) X is sequentially compact.

Proof. (i) $\Rightarrow$ (ii) is already proved by Theorem 3.6.1. ✓

((ii) $\Rightarrow$ (iii)) Let $\{x_n\}_{n\in\mathbb{Z}_+}$ be a sequence in X. Consider the set $A \triangleq \{x_n \mid n \in \mathbb{Z}_+\}$.

If A is finite, there exists $x \in X$ such that $x_n = x$ for infinitely many $n \in \mathbb{Z}_+$. In this case, $\{x_n\}_{n\in\mathbb{Z}_+}$ has a constant, thus convergent, subsequence.

If A is infinite, by the limit point compactness, A has a limit point x. Let $n_0 = 1$. Construct $\{n_i\}_{i\in\mathbb{Z}_+}$ as following.

- For each $i \in \mathbb{Z}_+$, choose $n_i \in \mathbb{Z}_+$ so that $x_{n_i} \in B(x,1/i)$ and $n_i > n_{i-1}$.

This is possible since $B(x,\varepsilon) \cap A$ is infinite for every $\varepsilon \in \mathbb{R}_+$ by Theorem 2.6.8. The sequence $\{x_{n_i}\}_{i\in\mathbb{Z}_+}$ converges to x. ✓

((iii) $\Rightarrow$ (i)) Let $\mathcal{A}$ be an open covering of X. By Lemma 3.6.1, there exists a Lebesgue number δ for $\mathcal{A}$. Let $\varepsilon \triangleq \delta/3$. By Lemma 3.6.2, there exists a finite open covering $\mathcal{A}'$ by ε-balls. Since every $A' \in \mathcal{A}'$ has $\operatorname{diam} A \leq 2\delta/3 < \varepsilon$, $A' \subseteq A$ for some $A \in \mathcal{A}$. The collection of such A consists of a finite subcollection of $\mathcal{A}$ that covers X. ✓ ☐

3.7 Local Compactness

Definition 3.7.1: Local Compactness

Let X be a topological space. X is said to be *locally compact at* x if there exist a compact subspace C and an open set U of X such that $x \in U \subseteq C$. X is said to be *locally compact* if it is locally compact at every point.

Note:-

If X is a compact space, X is locally compact.

Theorem 3.7.1

Let X be a topological space. Then X is locally compact Hausdorff if and only if there exists a space Y which satisfies the following.

(i) X is a subspace of Y.

(ii) $|Y \setminus X| = 1$.

(iii) Y is a compact Hausdorff space.

Moreover, such Y is unique up to homeomorphism. (In other words, if Y and Y' satisfy the three conditions, then they are homeomorphic with each other.)

Proof. ($\Rightarrow$) Let $Y = X \cup \{\infty\}$ where $\infty \notin X$ is a new point. Give Y the topology $\mathcal{T}_Y \triangleq \mathcal{T}_X \cup \mathcal{T}'$ where

$$\mathcal{T}' \triangleq \{ Y \setminus C \mid C \subseteq X \text{ is compact subspace of } X \}$$

$\mathcal{T}_Y$ is actually a topology:

(i) For intersections,

$$\begin{aligned} U_1 \cap U_2 &\in \mathcal{T}_X \\ (Y \setminus C_1) \cap (Y \setminus C_2) = Y \setminus (C_1 \cup C_2) &\in \mathcal{T}' \\ U_1 \cap (Y \setminus C_1) = U_1 \cap (X \setminus C_1) &\in \mathcal{T}_X. \checkmark \end{aligned}$$

(ii) For unions,

$$\begin{aligned} \textstyle\bigcup_{\alpha\in J} U_\alpha &\in \mathcal{T}_X \\ \textstyle\bigcup_{\beta\in I}(Y \setminus C_\beta) = Y \setminus \bigcap_{\beta\in I} C_\beta &\in \mathcal{T}' \\ \left(\textstyle\bigcup_{\alpha\in J} U_\alpha\right) \cup \left(\textstyle\bigcup_{\beta\in I}(Y \setminus C_\beta)\right) = U \cup (Y \setminus C) = Y \setminus (C \setminus U) &\in \mathcal{T}'. \end{aligned}$$

$C \setminus U$ is compact since it is a closed subspace of a compact space C. $\checkmark$

X is a subspace of Y since $X \cap (Y \setminus C) = X \setminus C \in \mathcal{T}_X$.

Now, we claim that Y endowed with $\mathcal{T}_Y$ is **compact**. Let $\mathcal{A}$ be an open covering of Y. It must be $\mathcal{A} \cap \mathcal{T}' \neq \varnothing$ since $\infty \notin X = \bigcup \mathcal{T}_X$. Take any $Y \setminus C \in \mathcal{A} \cap \mathcal{T}'$. Then, consider

$$\mathcal{A}' \triangleq \left\{ A \cap X \,\middle|\, A \in \mathcal{A} \setminus \{ Y \setminus C \} \right\},$$

which is a covering of C by open sets in X. Hence, there is a finite subcollection

$$\{A_1 \cap X, A_2 \cap X, \cdots, A_n \cap X\}$$

of $\mathcal{A}'$ that covers C. Then,

$$\{ Y \setminus C, A_1, A_2, \cdots, A_n \}$$

is a finite subcover of $\mathcal{A}$.

To show Y is **Hausdorff**, let x and y be two different points of Y. If $x, y \in X$, then it is done by X being a Hausdorff space. If $x \in X$ and $y = \infty$, then since X is locally compact, there exists a compact subspace C of X that contains a neighborhood U of x in X. Then, U and $Y \setminus C$ are disjoint neighborhoods of x and y, respectively.

We now prove the **uniqueness**. Let Y and Y' be the spaces that satisfy the conditions. Let $\{p\} = Y \setminus X$ and $\{q\} = Y' \setminus X$. Note that X is open in both Y and Y' as $\{p\}$ and $\{q\}$ are closed in Y and Y', respectively, by Theorem 2.6.7. Define a map $f : Y \to Y'$ by

$$x \mapsto \begin{cases} x & \text{if } x \in X \\ q & \text{if } x = p. \end{cases}$$

f is naturally a bijection, and we only need to prove f is an open map thanks to the symmetry.

If U is an open set in Y that does not contain p, U is open in X and thus $f(U) = U$ is open in Y'. If U is an open set in Y that contains p, then $C = Y \setminus U$ is closed in a compact space Y; C is compact by Theorem 3.4.1. C, being a compact subspace of X, is also a compact subspace of Y', which is Hausdorff. Hence C is closed in Y' as well by Theorem 3.4.2. Hence, $h(U) = Y' \setminus C$ is open in Y'.

($\Leftarrow$) X is Hausdorff because it is a subspace of a Hausdorff space. To show X is locally compact, take any $x \in X$. Choose disjoint neighborhoods U and V of x and the single point in $Y \setminus X$, respectively, in Y. Then, the set $C = Y \setminus V$ is closed in Y, so it is a compact subspace of Y. Since $C \subseteq X$, it is a compact subspace of X that contains the neighborhood U of x. $\square$

Note:-

In Theorem 3.7.1, if X is already compact, then $\mathcal{T}'$ contains the singleton $\{\infty\}$, which makes ∞ an isolated point. Therefore, $\overline{X} = X$; X is closed in Y.
However, if X is not compact, then every neighborhood of ∞ intersects X, which means ∞ is a limit point of X. Hence, $\overline{X} = Y$.

Note:-

In either case, every open set in X is still open in Y. Moreover, every open set in Y that does not contain ∞ is open in X.

Definition 3.7.2: One-point Compactification

If Y is a compact Hausdorff space and $X \subsetneq Y$ is a proper subspace of Y such that $\overline{X} = Y$, then Y is said to be a *compactification* of X. If $|Y \setminus X| = 1$, then Y is called the *one-point compactification* of X.

Corollary 3.7.1

A topological space X has a one-point compactification if and only if X is a locally compact Hausdorff space that is not itself compact.

Theorem 3.7.2

Let X be a Hausdorff space. Then X is locally compact if and only if, for any given $x \in X$ and neighborhood U of x, there exists a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subseteq U$.

Proof. ($\Rightarrow$) Take the one-point compactification Y of X, and let $C \triangleq Y \setminus U$. Since C is closed in Y, it is compact by Theorem 3.4.1. By Corollary 3.4.1, there are disjoint open sets V and W in Y such that $x \in V$ and $C \subsetneq W$. Then, $\overline{V}$, being closed in a compact set Y, is compact by Theorem 3.4.1. Furthermore, $\overline{V} \cap W = \varnothing$; otherwise, Theorem 2.6.5 ensures the nonempty intersection of V and W. This implies $\overline{V} \subseteq Y \setminus W \subseteq Y \setminus C = U$, as desired. Hence, V is an open set in X.

($\Leftarrow$) $C = \overline{V}$ is a compact subspace of X that contains a neighborhood V of x in X. $\square$

Corollary 3.7.2

Let X be a topological space and A be a subspace of X.

- If X is locally compact and A is closed in X, then A is locally compact.
- If X is locally compact *Hausdorff* and A is open in X, then A is locally compact.

Proof.

- Suppose A is closed in X. Given $x \in A$, let U be an open set and C be a compact subspace such that $x \in U \subseteq C$. Then, $C \cap A$ is closed in C and thus compact by Theorem 3.4.1, and it contains a neighborhood $U \cap A$ of x in A. ✓
- Suppose A is open in X. Given $x \in A$, by Theorem 3.7.2, there exists a neighborhood V of x in X such that $\overline{V}$ is compact and $\overline{V} \subseteq A$. Then, $\overline{V}$ is a compact subspace of A containing the neighborhood V of x in A. ✓

□

Corollary 3.7.3

A topological space X is homeomorphic to an open subspace of a compact Hausdorff space if and only if X is locally compact Hausdorff.

Proof. (⇒) Theorem 3.7.1. (⇐) Corollary 3.7.2. □

Chapter 4

Countability and Separation Axioms

4.1 The Countability Axioms

Definition 4.1.1: First Countability Axiom

A topological space X is said to have a *countable basis at* x if there is a countable collection $\mathcal{B}$ of neighborhoods of x in X such that, for each neighborhood U of x, there exists $B \in \mathcal{B}$ with $B \subseteq U$. A space that has a countable basis at each point is said to satisfy the *first countability axiom*, or to be *first-countable*.

Note:-

This definition was already given in Definition 2.10.2. Recall the lemmas Lemma 2.10.1 and Lemma 2.10.2.

Definition 4.1.2: Second Countability Axiom

If a topological space X has a countable basis for its topology, then X is said to satisfy the *second countability axiom*, or to be *second-countable*.

Example 4.1.1

$\mathbb{R}^J$ endowed with the product topology with a countable set J is second-countable;

$$\mathcal{S} \triangleq \bigcup_{\alpha \in J} \left\{ \pi_\alpha^{-1}\big((a,b)\big) \,\middle|\, a, b \in \mathbb{Q} \text{ and } a < b \right\}$$

is a countable subbasis for $\mathbb{R}^J$, which induces a countable basis for $\mathbb{R}^J$.

Note:-

If a topological space X is second-countable with a countable basis $\mathcal{B} = \{B_n\}_{n \in \mathbb{Z}_+}$ and a subspace $A \subseteq X$ with the discrete topology. Then, A must be countable.
Otherwise, for each $a \in A$, there exists $B_a \in \mathcal{B}$ such that $B_a \cap A = \{a\}$. This induces an injection $A \hookrightarrow \mathcal{B}$. Hence, A is countable.

Example 4.1.2 (Uniform Topology and Countability Axioms)

In the uniform topology, $\mathbb{R}^\omega$ is first-countable by Example 2.10.1. Let $\mathcal{B}$ be a basis of

$\mathbb{R}^\omega$. Let

$$A \triangleq \left\{ (x_i)_{i\in\mathbb{Z}_+} \in \mathbb{R}^\omega \,\middle|\, \forall i \in \mathbb{Z}_+,\ x_i \in \{0,1\} \right\}.$$

Then, A has the discrete topology but A is uncountable. Therefore, $\mathbb{R}^\omega$ with the uniform topology is not second-countable.

Theorem 4.1.1

Let X be a topological space and A be a subspace of X.

- If X is first-countable, then A is first-countable.
- If X is second-countable, then A is second-countable.

Proof.

- Let $a \in A$. Let $\mathcal{B}$ be a countable basis of X at a. Then, $\{B \cap A \mid B \in \mathcal{B}\}$ is a countable basis for the subspace A at a. ✓
- Let $\mathcal{B}$ be a countable basis of X. Then, $\{B \cap A \mid B \in \mathcal{B}\}$ is a countable basis for the subspace A. ✓

□

Theorem 4.1.2

Let $\{X_\alpha\}_{\alpha\in J}$ be a countable family of topological spaces.

- If each X_i is first-countable, then $\prod_{\alpha\in J} X_\alpha$ in the product topology is first-countable.
- If each X_i is second-countable, then $\prod_{\alpha\in J} X_\alpha$ in the product topology is second-countable.

Proof.

- Let $(x_\alpha)_{\alpha\in J} \in \prod_{\alpha\in J} X_\alpha$. Then, for each $\alpha \in J$, there exists a countable basis $\mathcal{B}_\alpha$ of X_α at x_α. Then, $\left\{ \prod_{\alpha\in J} B_\alpha \,\middle|\, \forall \alpha \in J,\ B_\alpha \in \mathcal{B}_\alpha \right\}$ is a countable basis at $(x_\alpha)_{\alpha\in J}$.
- For each $\alpha \in J$, there exists a countable basis $\mathcal{B}_\alpha$ of X_α. Then, $\left\{ \prod_{\alpha\in J} B_\alpha \,\middle|\, \forall \alpha \in J,\ B_\alpha \in \mathcal{B}_\alpha \right\}$ is a countable basis of $\prod_{\alpha\in J} X_\alpha$.

□

Definition 4.1.3: Lindelöf Space

A topological space X is called a *Lindelöf space* if, for every open covering of X, there is a countable subcovering.

Definition 4.1.4: Dense Subset

A subset A of a topological space X is said to be *dense* in X if $\overline{A} = X$.

Definition 4.1.5: Separable Space

A topological space X is said to be *separable* if there is a countable dense subset of X.

Note:-

Obvious facts:

- Every compact space is a Lindelöf space.
- The box and product topologies on an finite product of separable spaces is separable.

(Theorem 2.8.4)
- Every topology on a countable set is Lindelöf and separable.

Theorem 4.1.3

Let X be a second-countable space. Then,
- X is a Lindelöf space.
- X is separable.

Proof. Let $\mathcal{B} = \{B_n\}_{n\in\mathbb{Z}_+}$ be a countable basis for X.
- Let $\mathcal{A}$ be an open covering of X. For each $n \in \mathbb{Z}_+$, there exists $A_n \in \mathcal{A}$ such that $B_n \subseteq A_n$. Then, $\mathcal{A}' \triangleq \{A_n \mid n \in \mathbb{Z}_+\}$ is a countable subcovering of X as $\mathcal{B}$ covers X. ✓
- For each $n \in \mathbb{Z}_+$, choose $x_n \in B_n$. Let $D \triangleq \{x_n \mid n \in \mathbb{Z}_+\}$. Then, for all $x \in X$, every basis element that contains x intersects D; $\overline{D} = X$ by Theorem 2.6.5. ✓

□

Example 4.1.3 ($\mathbb{R}_\ell$ and Countability Axioms)
- Given $x \in \mathbb{R}_\ell$, $\{[x, x+1/n) \mid n \in \mathbb{Z}_+\}$ is a countable basis at x. $\mathbb{R}_\ell$ is first-countable.
- $\overline{\mathbb{Q}} = \mathbb{R}_\ell$. $\mathbb{R}_\ell$ is separable.
- Let $\mathcal{B}$ be a basis for $\mathbb{R}_\ell$. Choose, for each $x \in \mathbb{R}_\ell$, an element $B_x \in \mathcal{B}$ such that $x \in B_x \subsetneq [x, x+1)$. If $x \neq y$, then $B_x \neq B_y$. Hence $x \mapsto B_x$ is an injection; $\mathcal{B}$ is uncountable. Therefore, $\mathbb{R}_\ell$ is not second-countable.

We now prove $\mathbb{R}_\ell$ is Lindelöf. Thanks to Lemma 2.2.2, we only have to prove that, for any open covering $\mathcal{A}$ of $\mathbb{R}_\ell$ by the basis elements, there is a countable subcovering.
Let $\mathcal{A} = \{[a_\alpha, b_\alpha) \mid \alpha \in J\}$ be an open covering of $\mathbb{R}_\ell$. Let $C \triangleq \bigcup_{\alpha\in J}(a_\alpha, b_\alpha)$. We now claim that $\mathbb{R}\setminus C$ is countable. Let $x \in \mathbb{R}\setminus C$. Then $x = a_\beta$ for some $\beta \in J$. Choose $q_x \in \mathbb{Q}$ such that $q_x \in (a_\beta, b_\beta)$. If $x, y \in \mathbb{R}\setminus C$ and $x < y$, then $q_x < q_y$. Hence $x \mapsto q_x$ defines an injection $\mathbb{R}\setminus C \hookrightarrow \mathbb{Q}$. Therefore, $\mathbb{R}\setminus C$ is countable.
Now, let $\mathcal{A}'$ be a countable subcollection of $\mathcal{A}$ that covers $\mathbb{R}\setminus C$. Now, note that $\{(a_\alpha, b_\alpha) \mid \alpha \in J\}$ is an open covering of C as a subspace of $\mathbb{R}$ (with the standard topology). Since $\mathbb{R}$ is second-countable, there exists a finite subcollection $\{(a_{\alpha_1}, b_{\alpha_1}), \cdots, (a_{\alpha_n}, b_{\alpha_n})\}$ covers C. Let $\mathcal{A}'' \triangleq \{[a_{\alpha_1}, b_{\alpha_1}), \cdots, [a_{\alpha_n}, b_{\alpha_n})\}$. Then, $\mathcal{A}' \cup \mathcal{A}''$ is a countble subcovering of $\mathbb{R}_\ell$.

Example 4.1.4 (The Product of Two Lindelöf Spaces Need Not Be Lindelöf)

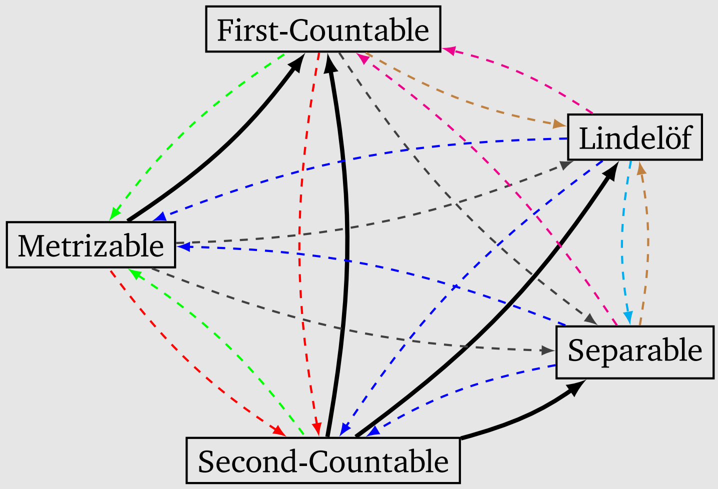
Although $\mathbb{R}_\ell$ is Lindelöf, $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not. Consider the subspace $L \triangleq \{x \times (-x) \mid x \in \mathbb{R}_\ell\}$. Then, L has the discrete topology as $\big([x, x+1)\times[-x, -x+1)\big)\cap L = \{x \times (-x)\}$. Hence, L is not Lindelöf; $\mathbb{R}_\ell^2$ is not Lindelöf.

Example 4.1.5 (A Subspace of a Lindelöf Space Need Not Be Lindelöf)

The ordered square I_o^2 is compact (Example 3.5.1) and thus is Lindelöf. However, the subspace $A = I \times (0,1)$ is not Lindelöf as an open covering $\{\{x\} \times (0,1) \mid x \in I\}$ does not allow a countable subcovering.

Note:-

Here is the diagram that represents the relations between spaces.



Counterexamples:

- (⇢) $X = \{0,1\}$ with $\mathcal{T} = \{\varnothing, X, \{0\}\}$ is second-countable but not Hausdorff, thus not metrizable.
- (⇢) $\mathbb{R}^\omega$ with the uniform topology is metrizable but not second-countable. (Example 4.1.2)
- (⇢) $\mathbb{R}_\ell$ ($\mathbb{R}$ with the lower limit topology) is first-countable, Lindelöf, and separable; but it is neither second-countable nor metrizable. (Example 4.1.3)
- (⇢) $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is first countable and separable, but it is not Lindelöf. (Example 4.1.4)
- (⇢) $\mathbb{R}$ with the discrete topology is first-countable and metrizable; but it is not second-countable, separable, or Lindelöf.
- (⇢) $\mathbb{R}$ with the finite complement topology is separable and Lindelöf; but it is neither first-countable nor metrizable.
- (⇢) $\mathbb{R}$ with the countable complement topology is Lindelöf; but it is not first-countable, metrizable, or separable.