MAS241 해석학 I
Note

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Chapter 1

Continuity

1.1 Limit and Continuity

Definition 1.1.1: Limit of a Function

Let $f : S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in \overline{S}$. We say f has *limit* L as $\mathbf{x}$ approaches $\mathbf{c}$ provided that, for every neighborhood $N(L)$, there exists a deleted neighborhood $N'(\mathbf{c})$ such that

$$S \cap N'(\mathbf{c}) \subseteq f^{-1}(N(L)).$$

We write $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$.

Note:-

Limit is unique if it exists.

Note:-

Note that $S \cap N'(\mathbf{c}; \delta) = \varnothing$ for sufficiently small δ if $\mathbf{c}$ is an isolated point of S. This implies any real number can be a limit of f as $\mathbf{x}$ approaches $\mathbf{c}$. Somehow, Douglass defined that $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = f(\mathbf{c})$ (since $\mathbf{c} \in S$ in this case). Actually I do not think we should define limit for isolated points.

Note:-

This definition of limit is equivalent to the normal ε-δ definition of limit, except that it defines a limit for isolated points.

Definition 1.1.2: Continuity

Let $f : S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in S$. We say f is *continuous at* $\mathbf{c}$ if

$$\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = f(\mathbf{c}).$$

In other words, for every neighborhood $N(f(\mathbf{c}))$, there exists a neighborhood $N(\mathbf{c})$ such that

$$S \cap N(\mathbf{c}) \subseteq f^{-1}\big(N(f(\mathbf{c}))\big).$$

If f is continuous at every $\mathbf{c} \in S$, then f is said to be *continuous*.

Theorem 1.1.1

Let $f : S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in \overline{S}$ where $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{c}) = L$ exists and. Then, f is locally bounded on some deleted neighborhood of $\mathbf{c}$, that is, there are $M, \delta \in \mathbb{R}_+$ such that

$$\mathbf{x} \in S \cap N'(\mathbf{c};\delta) \implies |f(\mathbf{x})| \le M.$$

Proof. There exists $\delta \in \mathbb{R}_+$ such that $S \cap N'(\mathbf{c};\delta) \subseteq f^{-1}(N(L;1))$. Then, $|f(\mathbf{x})| \le |L| + 1$ if $\mathbf{x} \in S \cap N'(\mathbf{x};\delta)$. $\square$

Theorem 1.1.2

Let $f : S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in \overline{S}$ where $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{c}) = L$ exists and $L \neq 0$. Then, f is locally bounded away from 0 on some deleted neighborhood of $\mathbf{c}$, that is, there are $m, \delta \in \mathbb{R}_+$ such that

$$\mathbf{x} \in S \cap N'(\mathbf{c};\delta) \implies |f(\mathbf{x})| \ge m.$$

Proof. There exists $\delta \in \mathbb{R}_+$ such that $S \cap N'(\mathbf{c};\delta) \subseteq f^{-1}(N(L;|L|/2))$. Then, $|f(\mathbf{x})| \ge |L|/2$ if $\mathbf{x} \in S \cap N'(\mathbf{x};\delta)$. $\square$

Theorem 1.1.3

Let $f_1 : S \to \mathbb{R}$ and $f_2 : S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in \overline{S}$, and suppose $\lim_{\mathbf{x}\to\mathbf{c}} f_1(\mathbf{x}) = L_1$ and $\lim_{\mathbf{x}\to\mathbf{c}} f_2(\mathbf{x}) = L_2$. Then

(i) $\lim_{\mathbf{x}\to\mathbf{c}} \big(f_1(\mathbf{x}) + f_2(\mathbf{x})\big) = L_1 + L_2$.

(ii) For any $a \in \mathbb{R}$, $\lim_{\mathbf{x}\to\mathbf{c}} af(\mathbf{x}) = aL_1$.

(iii) $\lim_{\mathbf{x}\to\mathbf{c}} f_1(\mathbf{x})f_2(\mathbf{x}) = L_1L_2$.

(iv) $\lim_{\mathbf{x}\to\mathbf{c}} f_1(\mathbf{x})/f_2(\mathbf{x}) = L_1/L_2$ provided that $L_2 \neq 0$.

Proof. Proved in MAS102 (Calculus II). $\square$

Theorem 1.1.4 The Squeeze Play

Let f, g, and h be three real-valued functions sharing a common domain $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in \overline{C}$ where $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{c}} h(\mathbf{x}) = L$ exist. Suppose also that, for some $\delta_0 \in \mathbb{R}_+$,

$$\mathbf{x} \in S \cap N'(\mathbf{c};\delta_0) \implies f(\mathbf{x}) \le g(\mathbf{x}) \le h(\mathbf{x})$$

Then, $\lim_{\mathbf{x}\to\mathbf{c}} g(\mathbf{x}) = L$.

Proof. Proved in MAS102 (Calculus II). $\square$

Theorem 1.1.5 Limit is Order Preserving

Let f and g be two real-valued functions sharing a common domain $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in \overline{C}$ where $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L_1$ and $\lim_{\mathbf{x}\to\mathbf{c}} g(\mathbf{x}) = L_2$ exist. Suppose also that, for some $\delta_0 \in \mathbb{R}_+$,

$$\mathbf{x} \in S \cap N'(\mathbf{c};\delta_0) \implies f(\mathbf{x}) \le g(\mathbf{x})$$

Then, $L_1 \le L_2$.

Proof. Proved in MAS102 (Calculus II). $\square$

Theorem 1.1.6

Let S be a nonempty subset of $\mathbb{R}^n$, $\mathbf{c} \in S'$, and $f : S \to \mathbb{R}$. $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$ if and only if, for every Cauchy sequence $\{\mathbf{x}_k\}$ in $S \setminus \{\mathbf{c}\}$ such that $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{c}$, it follows that $\lim_{k\to\infty} f(\mathbf{x}_k) = L$.

Proof. ($\Rightarrow$) Let $\{\mathbf{x}_k\}$ be any of such Cauchy sequences. Take any $\varepsilon \in \mathbb{R}_+$. By continuity, there exists $\delta \in \mathbb{R}_+$ such that $S \cap N'(\mathbf{c};\delta) \subseteq f^{-1}(N(L;\varepsilon))$. On the other hand, by convergence, there exists $k_0 \in \mathbb{N}$ such that $\forall k \in \mathbb{N}$, $(k \geq k_0 \implies \mathbf{x}_k \in N(\mathbf{c};\delta))$. Since $\mathbf{x}_k \neq \mathbf{c}$ for each $k \in \mathbb{N}$, we may say

$$\forall k \in \mathbb{N}, \big(k \geq k_0 \implies \mathbf{x}_k \in N'(\mathbf{c};\delta) \implies \mathbf{x}_k \in f^{-1}(N(L;\varepsilon)) \implies f(\mathbf{x}_k) \in N(L;\varepsilon)\big).$$

Thus, $\lim_{k\to\infty} f(\mathbf{x}_k) = L$ holds.

($\Leftarrow$) Suppose it is not $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$. Then, it is equivalent to say that, there is some neighborhood $N(L;\varepsilon_0)$ such that $S \cap N'(\mathbf{c};\delta) \nsubseteq f^{-1}(N(L;\varepsilon_0))$ for every deleted neighborhood $N'(\mathbf{x};\delta)$. Construct a sequence $\{\mathbf{x}_k\}$ in $S \setminus \{\mathbf{c}\}$ as following.

- $\mathbf{x}_1 \in S \setminus \{\mathbf{c}\} \setminus f^{-1}(N(L;\varepsilon_0))$.
- For each $k \in \mathbb{N}$, $\mathbf{x}_{k+1} \in S \cap N'(\mathbf{x}; |\mathbf{x}_k - \mathbf{c}|/2) \setminus f^{-1}(N(L;\varepsilon_0))$.

Then, $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{c}$ indeed holds, but it is not $\lim_{k\to\infty} f(\mathbf{x}_k) = L$ since $f(\mathbf{x}_k) \notin (N(L;\varepsilon_0))$ for each $k \in \mathbb{N}$. □

Theorem 1.1.7

Let S be a nonempty subset of $\mathbb{R}^n$, $\mathbf{c} \in S$, and $f : S \to \mathbb{R}$. f is continuous at $\mathbf{c}$ if and only if, for every Cauchy sequence $\{\mathbf{x}_k\}$ in S such that $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{c}$, it follows that $\lim_{k\to\infty} f(\mathbf{x}_k) = f(\mathbf{c})$.

Proof. ($\Rightarrow$) If at most finitely many $\mathbf{x}_k$ are distinct from $\mathbf{c}$, then $\exists k_0 \in \mathbb{N}$, $\forall k \in \mathbb{N}_{\geq k_0}$, $\mathbf{x}_k = \mathbf{c}$; $\lim_{k\to\infty} f(\mathbf{x}_k) = \mathbf{c}$ is evident.

If there are infinitely many $\mathbf{x}_k$ are distinct from $\mathbf{c}$, then we may extract a subsequence $\{\mathbf{x}_{\mathbf{k}_\mathbf{j}}\}_{j\in\mathbb{N}}$ such that each $\mathbf{x}_{k_j}$ is in $S \setminus \{c\}$. By Theorem 1.1.6, $\lim_{j\to\infty} f(\mathbf{x}_{k_j}) = \mathbf{c}$. This implies $\lim_{k\to\infty} f(\mathbf{x}_k) = \mathbf{c}$, regardless of the number of $\mathbf{x}_k$'s equal to $\mathbf{c}$.

($\Leftarrow$) If $\mathbf{c} \in S'$, then we may directly apply Theorem 1.1.6 since every Cauchy sequence in $S \setminus \{\mathbf{c}\}$ is a Cauchy sequence in S.

If $\mathbf{c} \notin S'$, then $\mathbf{c}$ is an isolated point. Then, $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = f(\mathbf{c})$ by definition. □

Theorem 1.1.8

Let $S \subseteq \mathbb{R}^n$ and $f : S \to \mathbb{R}$. Let $\mathbf{c} = (c_1, \cdots, c_n) \in \mathring{S}$. For $j = 1, 2, \cdots, n$, let

$$g_j(t) = f(c_1, c_2, \cdots, c_{j-1}, t, c_{j+1}, \cdots, c_n).$$

(i) If $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$, then, for each $j \in [n]$, $\lim_{t\to c_j} g_j(t) = L$.
(ii) If f is continuous at $\mathbf{c}$, then, for each $j \in [n]$, g_j is continuous at c_j and $\lim_{t\to c_j} g_j(t) = f(\mathbf{c})$.

Proof.

(i) Take any $j \in [n]$ and $\varepsilon \in \mathbb{R}_+$. By convergence, there exists $\delta_1 \in \mathbb{R}_+$ such that $S \cap N'(\mathbf{c};\delta_1) \subseteq f^{-1}(N(L;\varepsilon))$. Since $\mathbf{x} \in \mathring{S}$, there exists $\delta_2 \in \mathbb{R}_+$ such that $N(\mathbf{c};\delta_2) \subseteq S$. Let $\delta \triangleq \min\{\delta_1, \delta_2\}$. Then, $N'(\mathbf{c};\delta) \subseteq f^{-1}(N(L;\varepsilon))$ and $N(\mathbf{c};\delta) \subseteq S$ hold. Hence, for any $t \in N'(c_j;\delta)$,

$$g_j(t) = f(c_1, \cdots, c_{j-1}, t, c_{j+1}, \cdots, c_n) \in N(L;\varepsilon)$$

as $\|(c_1, \cdots, c_{j-1}, t, c_{j+1}, \cdots, c_n) - \mathbf{c}\| = |t - c_j| < \delta$.

(ii) Since $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = f(\mathbf{c})$, by (a), for each $j \in [n]$, $\lim_{t\to c_j} g(t) = f(\mathbf{c}) = g(c_j)$. $\square$

Note:-
The converse of Theorem 1.1.8 is not true.

1.2 The Topological Description of Continuity

Theorem 1.2.1

A surjective function $f : S \to T$ where $S \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}$ is continuous if and only if $f^{-1}(U)$ is relatively open in S for every relatively open set U in T.

Proof. ($\Rightarrow$) Let U be a relatively open set in T and $\mathbf{c} \in f^{-1}(U)$. Since U is open and $f(\mathbf{c}) \in U$, there is a neighborhood $N(f(\mathbf{c}))$ such that $T \cap N(f(\mathbf{c})) \subseteq U$. By continuity, there is a neighborhood $N(\mathbf{c})$ such that $S \cap N(\mathbf{c}) \subseteq f^{-1}(N(f(\mathbf{c}))) \subseteq f^{-1}(U)$. Therefore, $\mathbf{c}$ is a relative interior point of $f^{-1}(U)$. Since $\mathbf{c}$ was arbitrary, $f^{-1}(U)$ is relatively open in S.

($\Leftarrow$) Take any $\mathbf{c} \in S$ and a neighborhood $N(f(\mathbf{c}))$. Then, $f^{-1}(T \cap N(f(\mathbf{c})))$ is relatively open in S. Since $\mathbf{c} \in f^{-1}(T \cap N(f(\mathbf{c})))$, there is a neighborhood $N(\mathbf{c})$ such that $S \cap N(\mathbf{c}) \subseteq f^{-1}(N(f(\mathbf{c})))$. $\square$

Theorem 1.2.2

If S is a connected subset of $\mathbb{R}^n$ and $f : S \to \mathbb{R}$ is continuous on S, then $T = f(S)$ is also connected.

Proof. Suppose T is disconnected for the sake of contradiction. There exists two disjoint open sets $U, V \subseteq \mathbb{R}$ such that $T \subseteq U \cup V$, $T \cap U \neq \varnothing$, and $T \cap V \neq \varnothing$. Since $T \cap U$ and $T \cap V$ are relatively open in T, $U_1 = f^{-1}(T \cap U)$ and $V_1 = f^{-1}(T \cap V)$ are relatively open in S. Then, $S \subseteq U_1 \cup V_1 = S$, $U_1 \cap V_1 = \varnothing$, $S \cap U_1 \neq \varnothing$, and $S \cap V_1 \neq \varnothing$, which contradicts S is connected, #. $\square$

Theorem 1.2.3

If S is a compact subset of $\mathbb{R}^n$ and $f : S \to \mathbb{R}$ is continuous on S, then $T = f(S)$ is also compact.

Proof. Let $\{U_\alpha\}_{\alpha\in J}$ be an open cover of T. Then, for each $\alpha \in J$, $f^{-1}(U_\alpha)$ is relatively open in S since U_α is open and f is continuous. Because

$$S = f^{-1}(T) = f^{-1}\left(\bigcup_{\alpha\in J} U_\alpha\right) = \bigcup_{\alpha\in J} f^{-1}(U_\alpha),$$

$\{f^{-1}(U_\alpha)\}_{\alpha\in J}$ is a relative open cover of S. Since S is compact, there is a finite subcover $\{f^{-1}(U_{\alpha_i}) \mid i \in [p], \alpha_i \in J\}$ of S. Then,

$$T = f(S) = f\left(\bigcup_{i=1}^{p} f^{-1}(U_{\alpha_i})\right) = \bigcup_{i=1}^{p} f\left(f^{-1}(U_{\alpha_i})\right) \subseteq \bigcup_{i=1}^{p} U_{\alpha_i},$$

implying $\{U_{\alpha_i}\}_{i=1}^{p}$ is a finite subcover of T. $\square$

Theorem 1.2.4

If S is a compact subset of $\mathbb{R}^n$ and $f : S \to \mathbb{R}$ is continous on S, then f has a minimum and a maximum value on S.

Proof. Theorem 1.2.3 implies $T = f(S) \subseteq \mathbb{R}$ is compact, and thus bounded and closed. Thus, $m = \inf T = \min T$ and $M = \sup T = \max T$ exist. $\square$

Theorem 1.2.5 The Intermediate Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous and c is any number between $f(a)$ and $f(b)$, then there exists an $x \in [a, b]$ such that $f(x) = c$.

Proof. Since $[a, b]$ is connected and compact, Theorem 1.2.2 and Theorem 1.2.3 imply that $f([a, b])$ is connected and compact. Thus, $f([a, b]) = [m, M]$ where

$$m = \min f([a, b]) \leq \min\{f(a), f(b)\}$$

and

$$M = \max f([a, b]) \geq \max\{f(a), f(b)\}.$$

This implies $c \in [m, M] = f([a, b])$, i.e., there exists $x \in [a, b]$ such that $f(x) = c$. $\square$

Theorem 1.2.6 The General Intermediate Value Theorem

If S is any connected and compact subset of $\mathbb{R}^n$ and $f : S \to \mathbb{R}$ is continuous, if $f(\mathbf{x}_1)$ and $f(\mathbf{x}_2)$ are any two values of f on S, and if c is any number between them, then there exists a point $\mathbf{x} \in S$ such that $f(\mathbf{x}) = c$.

Proof. Since S is connected and compact, by Theorem 1.2.2 and Theorem 1.2.3, $f(S)$ is an closed interval $[m, M]$ as in the proof of Theorem 1.2.5. Since $m \leq \min\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}$ and $M \geq \max\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}$, $c \in [m, M] = f(S)$, and thus $\exists \mathbf{x} \in S$, $f(\mathbf{x}) = c$. $\square$

1.2.1 The Composition of Continuous Functions

Theorem 1.2.7

Let $S \subseteq \mathbb{R}^n$, $f : S \to \mathbb{R}$, $f(S) \subseteq T \subseteq \mathbb{R}$, and $g : T \to \mathbb{R}$. If f is continuous at $\mathbf{c} \in S$ and if g is continuous at $f(\mathbf{c}) \in T$, then $g \circ f$ is continuous at $\mathbf{c}$.

Proof. Let $d = (g \circ f)(\mathbf{c})$. Take any neighborhood $N(d)$ of d. By continuity of g at $f(\mathbf{c})$, there exists a neighborhood $N(f(\mathbf{c}))$ such that

$$T \cap N(f(\mathbf{c})) \subseteq g^{-1}(N(d)).$$

By the continuity of f at $\mathbf{c}$, there exists a neighborhood $N(\mathbf{c})$ such that

$$S \cap N(\mathbf{c}) \subseteq f^{-1}(N(f(\mathbf{c}))).$$

These imply $S \cap N(\mathbf{c}) \subseteq f^{-1}(g^{-1}(N(d))) = (g \circ f)^{-1}(N(d))$.

Corollary 1.2.1

Let $S \subseteq \mathbb{R}^n$, $f : S \to \mathbb{R}$, $f(S) \subseteq T \subseteq \mathbb{R}$, and $g : T \to \mathbb{R}$. If f and g are continuous, then

$g \circ f$ is continuous.

Theorem 1.2.8

If $f : [a,b] \to [c,d]$ is strictly monotone, continuous function, then the inverse function f^{-1} is also strictly monotone, continuous, and bijective.

Proof. All are immediate except for the continuity. Denote f^{-1} by g. By Theorem 1.1.7, it suffices to prove that whenever a Cauchy sequence $\{y_k\}$ in $f(S)$ converges to y, then $\{g(y_k)\}$ converges to $g(y)$ in S.

Choose any such sequence and let $x_k \triangleq g(y_k)$ for each $k \in \mathbb{N}$. Since g is bijective, $\{ y_k \mid k \in \mathbb{N} \}$ is finite if and only if $\{ x_k \mid k \in \mathbb{N} \}$ is finite. If they are finite, then $\{y_k\}$ is eventually y, this implies $\{x_k\}$ is eventually $g(y)$, and it is done.

If they are infinite, since domain and codomain are bounded and closed, by **??**, $\{ x_k \mid k \in \mathbb{N} \}$ has a limit point x. But since $[a,b]$ is complete by **??**, $x \in [a,b]$ by (ii) of **??**. x is a cluster point of $\{x_k\}$, thus there is a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $\lim_{j \to \infty} x_{k_j} = x$ by **??**. Now the continuity of f guarantees that

$$\lim_{j \to \infty} f(x_{k_j}) = f(x).$$

At the same time, since $f(x_{k_j}) = y_{k_j}$, $\{f(x_{k_j})\}$ is a subsequence of $\{y_k\}$. As $\{y_k\}$ converges to y, we get

$$\lim_{j \to \infty} f(x_{k_j}) = y.$$

By **??**, $f(x) = y$, or $x = g(y)$.

If there were another limit point x' of $\{ x_k \mid k \in \mathbb{N} \}$, by the same procedure, we get $x = g(y) = x'$; $x = x'$; x is the unique limit point of the set. Thus, $\{x_k\}$ converges to x, i.e., $\{g(y_k)\}$ converges to $g(y)$. □

1.2.2 Limiting Behavior at Infinity

Definition 1.2.1: Function Space $C(S)$ and $C_\infty(S)$

Let $S \neq \varnothing$ be a subset of $\mathbb{R}^n$.

- $C(S)$ is the set of real-valued function on S which is continuous on S.
- $C_\infty(S)$ is the set of real-valued function on S which is bounded and continuous on S.

Note:-

In general, $C_\infty(S) \subseteq C(S)$. If $\varnothing \neq S \subseteq \mathbb{R}^n$ is compact, then $C(S) = C_\infty(S)$.

Definition 1.2.2: Neighborhood of ∞ and $-\infty$

In $\mathbb{R}$,

- $N(\infty; M) \triangleq (M, \infty) = \{ x \in \mathbb{R} \mid x > M \}$
- $N(-\infty, -M) \triangleq (-\infty, -M) = \{ x \in \mathbb{R} \mid x < -M \}$

In $\mathbb{R}^n$,

- $N(\infty; M) \triangleq \{ \mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| > M \}$

Definition 1.2.3: Limit at Infinity

(i) Let S be an unbounded set in $\mathbb{R}$. Let $f : S \to \mathbb{R}$.
- We say f *has limit* L at ∞ if, for all $\varepsilon \in \mathbb{R}_+$, there exists $M \in \mathbb{R}_+$ such that $S \cap N(\infty; M) \subseteq f^{-1}(N(L;\varepsilon))$. We write $\lim_{x\to\infty} f(x) = L$.
- We say f *has limit* L at $-\infty$ if, for all $\varepsilon \in \mathbb{R}_+$, there exists $M \in \mathbb{R}_+$ such that $S \cap N(-\infty; -M) \subseteq f^{-1}(N(L;\varepsilon))$. We write $\lim_{x\to-\infty} f(x) = L$.

(ii) Let S be an unbounded set in $\mathbb{R}^n$. Let $f : S \to \mathbb{R}$. We say that f *has limit* L at ∞, if, for all $\varepsilon \in \mathbb{R}_+$, there exists $M \in \mathbb{R}_+$ such that $S \cap N(\infty; M) \subseteq f^{-1}(N(L;\varepsilon))$. We write $\lim_{\|\mathbf{x}\|\to\infty} f(\mathbf{x}) = L$.

Theorem 1.2.9 The Squeeze Play

Let f, g, and h be three real-valued functions sharing a common unbounded domain $S \subseteq \mathbb{R}^n$. Suppose $\lim_{\|\mathbf{x}\|\to\infty} f(\mathbf{x}) = \lim_{\|\mathbf{x}\|\to\infty} h(\mathbf{x}) = L$. Suppose also that, for some $M \in \mathbb{R}_+$,

$$\mathbf{x} \in S \cap N(\infty; M) \implies f(\mathbf{x}) \le g(\mathbf{x}) \le h(\mathbf{x})$$

Then, $\lim_{\|\mathbf{x}\|\to\infty} g(\mathbf{x}) = L$.

Theorem 1.2.10

Let S be a closed and unbounded set in $\mathbb{R}^n$ and let $f \in C(S)$. Suppose $\lim_{\|\mathbf{x}\|\to\infty} f(\mathbf{x}) = L$ exists. Then $f \in C_\infty(S)$.

Proof. There exists $M \in \mathbb{R}_+$ such that, for $\mathbf{x} \in S \cap N(\infty; M)$, $|f(\mathbf{x}) - L| < 1$. Thus, for such $\mathbf{x}$, we have $|f(x)| < |L| + 1$.

Since $S \cap \overline{N(\mathbf{0}; M)}$ is a closed, bounded set in $\mathbb{R}^n$, it is compact by ??. Therefore the continuous f is bounded on $S \cap \overline{N(\mathbf{0}; M)}$ by Theorem 1.2.3. In other words, there is some $K \in \mathbb{R}_+$ such that, for $\mathbf{x} \in S \cap N(\mathbf{0}; M)$, we have $|f(x)| \le K$. Thus, $|f(\mathbf{x})| \le \max\{K, |L|+1\}$ for all $\mathbf{x} \in S$. □

1.3 The Algebra of Continuous Functions

Note:-

Let $\varnothing \ne S \subseteq \mathbb{R}^n$. One can easily find that $C(S)$ is a commutative ring and is a vector space.

Theorem 1.3.1

Let $\varnothing \ne S \subseteq \mathbb{R}^n$ and $f_1, f_2 \in C(S)$. Then, the following hold.

(i) $f_1 + f_2 \in C(S)$.
(ii) For any $a \in \mathbb{R}$, $af \in C(S)$.
(iii) $f_1 f_2 \in C(S)$.
(iv) $1/f_2 \in C(S)$, provided that $\forall \mathbf{x} \in S,\ f_2(\mathbf{x}) \ne 0$.
(v) $f_1/f_2 \in C(S)$, provided that $\forall \mathbf{x} \in S,\ f_2(\mathbf{x}) \ne 0$.

Proof. Directly import Theorem 1.1.3. □

Theorem 1.3.2

Suppose f is continous at a point $\mathbf{c}$ in $\mathbb{R}^n$. Then f is locally bounded at $\mathbf{c}$. that is, there are $M, \delta \in \mathbb{R}_+$ such that

$$\mathbf{x} \in S \cap N(\mathbf{c}; \delta) \implies |f(\mathbf{x})| \leq M.$$

Theorem 1.3.3

Suppose f is continous at a point $\mathbf{c}$ in $\mathbb{R}^n$ and $f(\mathbf{c}) \neq 0$. Then f is locally bounded away from 0 at $\mathbf{c}$. that is, there are $m, \delta \in \mathbb{R}_+$ such that

$$\mathbf{x} \in S \cap N(\mathbf{c}; \delta) \implies |f(\mathbf{x})| \geq m.$$

1.4 Uniform Continuity

Definition 1.4.1: Uniform Continuity

A function $f : S \to \mathbb{R}$ with $S \subseteq \mathbb{R}^n$ is said to be *uniformly continuous on* S if,

$$\forall \varepsilon \in \mathbb{R}_+, \exists \delta \in \mathbb{R}_+, \forall \mathbf{c} \in S, S \cap N(\mathbf{c}; \delta) \subseteq f^{-1}(N(f(\mathbf{c}); \varepsilon)).$$

Or, equivalently,

$$\forall \varepsilon \in \mathbb{R}_+, \exists \delta \in \mathbb{R}_+, \forall \mathbf{x}, \mathbf{y} \in S, \left(\|\mathbf{x} - \mathbf{y}\| < \delta \implies |f(x) - f(y)| < \varepsilon\right).$$

Example 1.4.1

$f : [0, b] \to \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous on $[0, b]$. Given any $\varepsilon \in \mathbb{R}_+$, let $\delta \triangleq \varepsilon/2b$. Then, whenever $|x - y| < \delta$ where $x, y \in [0, b]$, $|x^2 - y^2| = |x - y||x + y| < \delta \cdot 2b = \varepsilon$.

Example 1.4.2

$f : (0, M) \to \mathbb{R}$ defined by $f(x) = 1/x$ is not uniformly continuous on $(0, M)$. Let any $\delta \in \mathbb{R}_+$ is given. Let $a \in (0, \min\{\delta, 1/2, M/2\})$. Then, $|a - (2a)| = a < \delta$ but $|f(a) - f(2a)| = |1/a - 1/(2a)| = 1/(2a) > 1$.

This is an example in which f is continuous but the domain is not compact.

Example 1.4.3

$f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ is not uniformly continuous on $[-1, 1]$. Let any $\delta \in \mathbb{R}_+$ is given. Let $a \in (0, \min\{\delta/2, 1\})$. Then, $|a - (-a)| = 2a < \delta$ but $|f(a) - f(-a)| = 1 > 0.5$.

This is an example in which the domain is compact but f is not continuous.

Theorem 1.4.1

Suppose that $f : S \to \mathbb{R}$ is continuous on a compact subset S of $\mathbb{R}^n$. Then f is uniformly continuous on S.

Proof. Let $\varepsilon \in \mathbb{R}_+$ be given. Since f is continuous at each point of S, for each $\mathbf{c}$ of S, we may choose $\delta(\mathbf{c}) \in \mathbb{R}_+$ such that

$$S \cap N(\mathbf{c}; \delta(\mathbf{c})) \subseteq f^{-1}\left(N\left(f(\mathbf{c}); \frac{\varepsilon}{2}\right)\right).$$

Then the set $\mathcal{C} \triangleq \{N(\mathbf{c}; \delta(\mathbf{c})/2) \mid \mathbf{c} \in S\}$ is an open cover of the compact set S. Since S is compact, there is a finite subcover

$$\mathcal{C}_1 = \left\{N\left(\mathbf{c}_1; \frac{\delta(\mathbf{c}_1)}{2}\right), N\left(\mathbf{c}_2; \frac{\delta(\mathbf{c}_2)}{2}\right), \cdots, N\left(\mathbf{c}_k; \frac{\delta(\mathbf{c}_k)}{2}\right)\right\}.$$

Let $\delta_0 \triangleq \min_{i=1}^{k} \delta(\mathbf{c}_i)/2$.

Now, take any $\mathbf{c} \in S$. Since $\mathcal{C}_1$ is an open cover,

$$\exists i \in [k],\ \mathbf{c} \in N\left(\mathbf{c}_i; \frac{\delta(\mathbf{c}_i)}{2}\right).$$

Then, for any $\mathbf{x} \in N(\mathbf{c}; \delta_0)$,

$$\|\mathbf{x} - \mathbf{c}_i\| \leq \|\mathbf{x} - \mathbf{c}\| + \|\mathbf{c} - \mathbf{c}_i\| < \delta_0 + \frac{\delta(\mathbf{c}_i)}{2} \leq \delta(\mathbf{c}_i).$$

Thus, $N(\mathbf{c}; \delta_0) \subseteq N(\mathbf{c}_i; \delta(\mathbf{c}_i))$; or

$$S \cap N(\mathbf{c}; \delta_0) \subseteq S \cap N(\mathbf{c}_i; \delta(\mathbf{c}_i)) \subseteq f^{-1}\left(N\left(f(\mathbf{c}_i); \frac{\varepsilon}{2}\right)\right).$$

Hence, for any $\mathbf{x} \in S \cap N(\mathbf{c}; \delta_0)$,

$$|f(\mathbf{x}) - f(\mathbf{c})| \leq |f(\mathbf{x}) - f(\mathbf{c}_i)| + |f(\mathbf{c}_i) - f(\mathbf{c})| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

as $\mathbf{x}, \mathbf{c} \in S \cap N(\mathbf{c}_i; \delta(\mathbf{c}_i))$. □

1.5 The Uniform Norm: Uniform Convergence

Definition 1.5.1: Function Space $B(S)$

Let $S \neq \varnothing$ be a subset of $\mathbb{R}^n$. $B(S)$ denotes the vector space and ring of all bounded, real-valued functions on S.

Note:-

- For each $f \in B(S)$, $\sup\{|f(\mathbf{x})| \mid \mathbf{x} \in S\}$ exists.
- $C_\infty(S) = C(S) \cap B(S)$

Definition 1.5.2: Uniform Norm

The *uniform norm* of $f \in B(S)$ is defined to be

$$\|f\|_\infty = \sup\{|f(\mathbf{x})| \mid \mathbf{x} \in S\}.$$

Theorem 1.5.1

The uniform norm is a norm.

Proof. The positive definiteness and the absolute homogeneity is direct.

Take any $f, g \in f$. Then, for any $\mathbf{x} \in S$,

$$|(f+g)(\mathbf{x})| \leq |f(\mathbf{x})| + |g(\mathbf{x})| \leq \|f\|_\infty + \|g\|_\infty.$$

Thus, $\|f+g\|_\infty = \sup\{\, |(f+g)(\mathbf{x})| \mid \mathbf{x} \in S \,\} \leq \|f\|_\infty + \|g\|_\infty$; $\|\cdot\|_\infty$ satisfies the subadditivity. □

Definition 1.5.3: Uniform Metric

The *uniform metric* on $B(S)$ is

$$d_\infty(f,g) = \|f-g\|_\infty.$$

Note:-

The uniform metric is naturally a metric since the uniform norm is a norm.

Definition 1.5.4: (Deleted) Uniform Neighborhood

The *(uniform) neighborhood* $N(f;r)$ of f with radius r is the set

$$N(f;r) \triangleq \{\, g \in B(S) \mid d_\infty(f,g) < r \,\}.$$

The *deleted (uniform) neighborhood* $N'(f;r)$ of f with radius r is the set

$$N'(f;r) \triangleq \{\, g \in B(S) \mid 0 < d_\infty(f,g) < r \,\}.$$

Definition 1.5.5: Limit Point of a Set of Functions

A function $f_0 \in B(S)$ is said to be a *(uniform) limit point* of a set $F \subseteq B(S)$ if

$$\forall \varepsilon \in \mathbb{R}_+,\ F \cap N'(f_0;\varepsilon) \neq \varnothing.$$

Definition 1.5.6: Convergence of a Sequence of Functions

- A sequence $\{f_k\}_{k\in\mathbb{N}}$ in $B(S)$ is said to *converge uniformly* to $f_0 \in S \to \mathbb{R}$ on S if

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists k_0 \in \mathbb{N},\ \forall k \in \mathbb{N},\ \big(k \geq k_0 \implies f_k \in N(f_0;\varepsilon)\big).$$

  We write

$$\lim_{k\to\infty} f_k = f_0 \text{ [uniformly]}.$$

- A sequence $\{f_k\}_{k\in\mathbb{N}}$ in $B(S)$ is said to *converge pointwise* to $f_0 : S \to \mathbb{R}$ on S if

$$\forall (\mathbf{c},\varepsilon) \in S \times \mathbb{R}_+,\ \exists k_0 \in \mathbb{N},\ \forall k \in \mathbb{N},\ \big(k \geq k_0 \implies f_k(\mathbf{c}) \in N(f_0(\mathbf{c});\varepsilon)\big).$$

  We write

$$\lim_{k\to\infty} f_k = f_0 \text{ [pointwise]}.$$

- A sequence $\{f_k\}_{k\in\mathbb{N}}$ in $B(S)$ is said to be *(uniformly) Cauchy* if

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists k_0 \in \mathbb{N},\ \forall k,m \in \mathbb{N},\ \big(k,m \geq k_0 \implies \|f_m - f_k\|_\infty < \varepsilon\big).$$

Note:-
A pointwise convergent sequence in $C_\infty(S)$ may fail to have a limit that is in $C_\infty(S)$.

Note:-
If $\{f_k\}$ in $B(S)$ converges uniformly, then it converges pointwise.

Theorem 1.5.2
Let $\varnothing \neq S \subseteq \mathbb{R}^n$. Suppose that $\{f_k\}$ is a sequence in $C(S)$ and it converges uniformly to $f_0 : S \to \mathbb{R}$ on S. Then $f_0 \in C(S)$.

Proof. Take any $\mathbf{c} \in S$ and $\varepsilon \in \mathbb{R}_+$. By uniform convergence, there exists $k \in \mathbb{N}$ such that

$$\|f_k - f_0\|_\infty < \frac{\varepsilon}{4}.$$

Since f_k is continuous, there exists $\delta \in \mathbb{R}_+$ such that

$$S \cap N(\mathbf{c};\delta) \subseteq f_k^{-1}\left(N\left(f_k(\mathbf{c});\frac{\varepsilon}{2}\right)\right).$$

Thus, for any $\mathbf{x} \in S \cap N(\mathbf{c};\delta)$,

$$\begin{aligned}|f_0(\mathbf{x}) - f_0(\mathbf{c})| &\leq |f_0(\mathbf{x}) - f_k(\mathbf{x})| + |f_k(\mathbf{x}) - f_k(\mathbf{c})| + |f_k(\mathbf{c}) - f_0(\mathbf{c})| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.\end{aligned}$$

This exactly means that f_0 is continuous at $\mathbf{c}$. Since $\mathbf{c}$ is arbitrary, f_0 is continous on S. □

Theorem 1.5.3
Let $\varnothing \neq S \subseteq \mathbb{R}^n$. A Cauchy sequence $\{f_k\}$ in $B(S)$ is bounded. That is, $\exists M \in \mathbb{R}_+$, $\forall k \in \mathbb{N}$, $\|f_k\|_\infty \leq M$.

Proof. Immitate the proof of **??**. □

Theorem 1.5.4

$C_\infty(S)$ is complete. That is, given any Cauchy sequence $\{f_k\}$ in $C_\infty(S)$, there exists $f_0 \in C_\infty(S)$ such that $\lim_{k\to\infty} f_k = f_0$ [uniformly] on S.

Proof. Since, for each $\mathbf{c} \in S$, $\{f_k(\mathbf{c})\}_{k\in\mathbb{N}}$ is Cauchy, by **??**, $\{f_k(\mathbf{c})\}_{k\in\mathbb{N}}$ converges in $\mathbb{R}$. Thus we may define $f_0 \colon S \to \mathbb{R}$ by

$$f_0(\mathbf{c}) = \lim_{k\to\infty} f_k(\mathbf{c}).$$

Now, we first claim that $\lim_{k\to\infty} f_k = f_0$ [uniformly]. Take any $\varepsilon \in \mathbb{R}_+$. Since $\{f_k\}$ is Cauchy,

$$\exists k_0 \in \mathbb{N},\ \forall k, m \in \mathbb{N},\ \big(k, m \geq k_0 \implies \|f_k - f_m\|_\infty < \varepsilon/2\big).$$

Then, for each $k \in \mathbb{N}_{\geq k_0}$,

$$\forall \mathbf{c} \in S,\ |f_k(\mathbf{c}) - f_0(\mathbf{c})| = \lim_{m\to\infty} |f_k(\mathbf{c}) - f_m(\mathbf{c})| \leq \varepsilon/2.$$

This means $\|f_k - f_0\|_\infty \leq \varepsilon/2 < \varepsilon$; $\{f_k\}$ converges to f_0 uniformly on S. It directly follows from Theorem 1.5.2 that f_0 is continuous on S.

Now, we are left to show f_0 is bounded on S. Since $\{f_k\}$ uniformly converges to f_0,

$$\exists k_0 \in \mathbb{N},\ \forall k \in \mathbb{N},\ \big(k \geq k_0 \implies \|f_k - f_0\|_\infty < 1\big).$$

This implies $\forall \mathbf{c} \in S$, $|f_{k_0}(\mathbf{c}) - f_0(\mathbf{c})| < 1$. Hence, $\forall \mathbf{c} \in S$, $|f_0(\mathbf{c})| < |f_{k_0}(\mathbf{c})| + 1 \leq \|f_{k_0}\|_\infty + 1$. Thus, $\{\, |f_0(\mathbf{c})| \mid \mathbf{c} \in S \,\}$ is bounded above by $\|f_{k_0}\|_\infty + 1$. $\square$

Corollary 1.5.1

If $\varnothing \neq S \subseteq \mathbb{R}^n$ is compact, then $C(S)$ is complete.

Definition 1.5.7: Unifrom Denseness

Let $\varnothing \neq S \subseteq \mathbb{R}^n$. A collection $F \subseteq C_\infty(S)$ is *uniformly dense* in $C_\infty(S)$ if, for all $f_0 \in C_\infty(S)$ and its neighborhood $N(f_0)$, $F \cap N(f_0) \neq \varnothing$.

Definition 1.5.8: Polynomial Space

Let $\varnothing \neq S \subseteq \mathbb{R}$. Let $P(S)$ denote the set of polynomial functions $f : S \to \mathbb{R}$ in the single variable x.

Note:-

If $S \neq \varnothing$ is a compact subset of $\mathbb{R}$, then $P(S)$ is certainly a subset of $C_\infty(S) = C(S)$.

Theorem 1.5.5 The Weierstrass Approximation Theorem

If S is a compact subset of $\mathbb{R}$, then $P(S)$ is uniformly dense in $C(S)$.

Definition 1.5.9: Bernstein Polynomial

Given a continuous function on $[0,1]$, for each $k \in \mathbb{N}$, the k^{th} Bernstein polynomial for f, $B_k(x)$, is defined as follows.

$$B_k(x) \triangleq \sum_{j=0}^{k} \binom{k}{j} f\left(\frac{j}{k}\right) x^j (1-x)^{k-j}$$

where $\binom{k}{j}$ is the binomial coefficient $\dfrac{k!}{j!(k-j)!}$.

Note:-
The following lemmas are for the proof of Theorem 1.5.5.

Lemma 1.5.1

For any $k \in \mathbb{N}$, $\displaystyle\sum_{j=0}^{k} \binom{k}{j} x^j (1-x)^{k-j} = 1$.

Proof. Expand $1 = \big(x + (1-x)\big)^k$ via the binomial theorem. □

Lemma 1.5.2

For any $k \in \mathbb{N}$, $\displaystyle\sum_{j=0}^{k} \binom{k}{j} \frac{j}{k} x^j (1-x)^{k-j} = x$.

Proof. Using Lemma 1.5.1, we get

$$\begin{aligned}
x &= x\left[\sum_{j=0}^{k-1} \binom{k-1}{j} x^j (1-x)^{(k-1)-j}\right] \\
&= \sum_{j=0}^{k-1} \binom{k-1}{j} x^{j+1} (1-x)^{k-(j+1)} = \sum_{j=1}^{k} \binom{k-1}{j-1} x^j (1-x)^{k-j} \\
&= \sum_{j=1}^{k} \binom{k}{j} \frac{j}{k} x^j (1-x)^{k-j} = \sum_{j=0}^{k} \binom{k}{j} \frac{j}{k} x^j (1-x)^{k-j}.
\end{aligned}$$

□

Lemma 1.5.3

For any $k \in \mathbb{N}$, $\displaystyle\sum_{j=0}^{k} \binom{k}{j} (j^2 - j) x^j (1-x)^{k-j} = k(k-1)x^2$.

Proof. Note that, for each $j \in \{2, 3, \cdots, k\}$,

$$\binom{k}{j}(j^2 - j) = \binom{k-2}{j-2} k(k-1).$$

Therefore, using Lemma 1.5.1, we get

$$\begin{aligned}
\sum_{j=0}^{k}\binom{k}{j}(j^2-j)x^j(1-x)^{k-j} &= \sum_{j=2}^{k}\binom{k}{j}(j^2-j)x^j(1-x)^{k-j}\\
&= x^2\sum_{j=2}^{k}\binom{k}{j}(j^2-j)x^{j-2}(1-x)^{(k-2)-(j-2)}\\
&= x^2\sum_{j=2}^{k}\binom{k-2}{j-2}k(k-1)x^{j-2}(1-x)^{(k-2)-(j-2)}\\
&= k(k-1)x^2\sum_{j=0}^{k-2}\binom{k-2}{j}x^j(1-x)^{(k-2)-j}\\
&= k(k-1)x^2
\end{aligned}$$

$\square$

Lemma 1.5.4

For any $k \in \mathbb{N}$, $\displaystyle\sum_{j=0}^{k}\binom{k}{j}\left(\frac{j}{k}\right)^2 x^j(1-x)^{k-j} = \left(1-\frac{1}{k}\right)x^2+\frac{x}{k}$.

Proof. Using Lemma 1.5.2 and Lemma 1.5.3, we get

$$\begin{aligned}
\left(1-\frac{1}{k}\right)x^2+\frac{x}{k} &= \frac{k(k-1)x^2}{k^2}+\frac{x}{k}\\
&= \frac{1}{k^2}\sum_{j=0}^{k}\binom{k}{j}(j^2-j)x^j(1-x)^{k-j}+\frac{1}{k}\sum_{j=0}^{k}\binom{k}{j}\frac{j}{k}x^j(1-x)^{k-j}\\
&= \sum_{j=0}^{k}\binom{k}{j}\left(\frac{j^2-j}{k^2}+\frac{j}{k^2}\right)x^j(1-x)^{k-j}\\
&= \sum_{j=0}^{k}\binom{k}{j}\left(\frac{j}{k}\right)^2 x^j(1-x)^{k-j}.
\end{aligned}$$

$\square$

Lemma 1.5.5

For any $k \in \mathbb{N}$ and any $x \in [0,1]$, $\displaystyle\sum_{j=0}^{k}\binom{k}{j}\left(x-\frac{j}{k}\right)^2 x^j(1-x)^{k-j} = \frac{x(1-x)}{k} \leq \frac{1}{4k}$.

Proof. Using Lemma 1.5.1, Lemma 1.5.2, and Lemma 1.5.4, we get

$$\begin{aligned}\sum_{j=0}^{k}\binom{k}{j}\left(x-\frac{j}{k}\right)^2 x^j(1-x)^{k-j} &= \sum_{j=0}^{k}\binom{k}{j}\left[x^2-2x\left(\frac{j}{k}\right)-\left(\frac{j}{k}\right)^2\right]x^j(1-x)^{k-j}\\ &= x^2\sum_{j=0}^{k}\binom{k}{j}x^j(1-x)^{k-j}+2x\sum_{j=0}^{k}\binom{k}{j}\left(\frac{j}{k}\right)x^j(1-x)^{k-j}\\ &\quad+\sum_{j=0}^{k}\binom{k}{j}\left(\frac{j}{k}\right)^2 x^j(1-x)^{k-j}\\ &= x^2-2x^2+\left(1-\frac{1}{k}\right)x^2+\frac{x}{k}=\frac{x(1-x)}{k}.\end{aligned}$$

□

Theorem 1.5.6 Bernstein's Approximation Theorem

Let $f \in C([0,1])$. Then, for any $\varepsilon \in \mathbb{R}_+$, there exists $k_0 \in \mathbb{N}$ such that, for $k \geq k_0$, $\|f-B_k\|_\infty < \varepsilon$ where B_k is the k^{th} Bernstein polynomial for f.

Proof. Take any $\varepsilon \in \mathbb{R}_+$. Since f is continuous on a compact set $[0,1]$, then f is uniformly continuous on $[0,1]$ by Theorem 1.4.1. Therefore,

$$\exists\delta \in \mathbb{R}_+, \forall x,y \in [0,1], \big(|x-y|<\delta \implies |f(x)-f(y)|<\varepsilon/4\big).$$

Also, f is bounded on $[0,1]$ by Theorem 1.2.3. Thus, by **??**, there exists $k_0 \in \mathbb{N}$ such that $2\|f\|_\infty < \varepsilon\delta^2 k_0$. Now, suppose $k \geq k_0$.

Fix some $x \in [0,1]$ and let $A_1 \triangleq \{\, j \in \mathbb{Z} \mid 0 \leq j \leq k \text{ and } |x-j/k|<\delta \,\}$ and $A_2 \triangleq \{\, j \in \mathbb{Z} \mid 0 \leq j \leq k \text{ and } |x-j/k| \geq \delta \,\}$. Now, we are ready to prove $\|f-B_k\|_\infty < \varepsilon$.

Using Lemma 1.5.1 at the first step, we get

$$\begin{aligned}|f(x)-B_k(x)| &= \left|\sum_{j=0}^{k}\binom{k}{j}f(x)x^j(1-x)^{k-j}-\sum_{j=0}^{k}\binom{k}{j}f\left(\frac{j}{k}\right)x^j(1-x)^{k-j}\right|\\ &= \left|\sum_{j=0}^{k}\binom{k}{j}\left[f(x)-f\left(\frac{j}{k}\right)\right]x^j(1-x)^{k-j}\right|\\ &\leq \sum_{j=0}^{k}\binom{k}{j}\left|f(x)-f\left(\frac{j}{k}\right)\right|x^j(1-x)^{k-j}\\ &= \sum_{j\in A_1}\binom{k}{j}\left|f(x)-f\left(\frac{j}{k}\right)\right|x^j(1-x)^{k-j}+\sum_{j\in A_2}\binom{k}{j}\left|f(x)-f\left(\frac{j}{k}\right)\right|x^j(1-x)^{k-j}\\ &= S_1+S_2\end{aligned}$$

where S_1 and S_2 are the sums over A_1 and A_2, respectively.

Since $\left|x-\dfrac{j}{k}\right| < \delta$ for $j \in A_1$, the following holds.

$$\begin{aligned}S_1 &\triangleq \sum_{j\in A_1}\binom{k}{j}\left|f(x)-f\left(\frac{j}{k}\right)\right|x^j(1-x)^{k-j}\\ &< \sum_{j\in A_1}\binom{k}{j}\left(\frac{\varepsilon}{4}\right)x^j(1-x)^{k-j} \leq \frac{\varepsilon}{4}\sum_{j=0}^{k}\binom{k}{j}x^j(1-x)^{k-j}=\frac{\varepsilon}{2}\end{aligned}$$

We now investigate S_2. Note that $\dfrac{1}{|x-j/k|} \leq \dfrac{1}{\delta}$ for $j \in A_2$. Using Lemma 1.5.5, we get

$$\begin{aligned}
S_2 &\triangleq \sum_{j\in A_2} \binom{k}{j} \left| f(x) - f\left(\frac{j}{k}\right) \right| x^j (1-x)^{k-j} \\
&\leq 2\|f\|_\infty \sum_{j\in A_2} \binom{k}{j} \left(x - \frac{j}{k}\right)^2 \frac{1}{\left(x-\frac{j}{k}\right)^2} x^j (1-x)^{k-j} \\
&\leq \frac{2\|f\|_\infty}{\delta^2} \sum_{j\in A_2} \binom{k}{j} \left(x - \frac{j}{k}\right)^2 x^j (1-x)^{k-j} \\
&\leq \frac{2\|f\|_\infty}{\delta^2} \sum_{j=0}^{k} \binom{k}{j} \left(x - \frac{j}{k}\right)^2 x^j (1-x)^{k-j} \\
&\leq \frac{2\|f\|_\infty}{\delta^2} \frac{1}{4k} \leq \frac{\|f\|_\infty}{2k\delta^2} \leq \frac{\|f\|_\infty}{2k_0\delta^2} < \frac{\varepsilon}{4}.
\end{aligned}$$

Thus, we have $|f(x) - B_k(x)| \leq S_1 + S_2 < \varepsilon/4 + \varepsilon/4 = \varepsilon/2$. Since x is arbitrary, $\|f - B_k\|_\infty \leq \varepsilon/2 < \varepsilon$. □

1.6 Vector-Valued Functions on $\mathbb{R}^n$

Definition 1.6.1: Component Function

Let $\mathbf{f}$ be a function with domain $S \subseteq \mathbb{R}^n$ and codomain $T \subseteq \mathbb{R}^m$. For $\mathbf{x} \in S$, we write

$$\mathbf{f}(\mathbf{x}) = \mathbf{y} = (y_1, y_2, \cdots, y_m).$$

Then, for each $j \in [m]$, there is a real-valued function $f_i : S \to \mathbb{R}$ defined by $f_j(\mathbf{x}) = y_j$. The functions $f_1, f_2, \cdots, f_m$ are called *component functions* of $\mathbf{f}$. We write

$$\mathbf{f} = (f_1, f_2, \cdots, f_m).$$

Definition 1.6.2: Limit and Continuity of Vector-Valued Functions

Let $\varnothing \neq S \subseteq \mathbb{R}^n$ and $\mathbf{f}\colon S \to \mathbb{R}^m$.

- Let $\mathbf{c} \in \overline{S}$. We say that $\mathbf{f}$ has *limit* $\mathbf{v}$ as $\mathbf{x}$ approaches $\mathbf{c}$, and we write

  $$\lim_{\mathbf{x}\to\mathbf{c}} \mathbf{f}(\mathbf{x}) = \mathbf{v}$$

  if, for every neighborhood $N(\mathbf{v})$, there exists a deleted neighborhood $N'(\mathbf{c})$ such that

  $$S \cap N'(\mathbf{c}) \subseteq \mathbf{f}^{-1}(N(\mathbf{v})).$$

- Let $\mathbf{c} \in S$. We say that $\mathbf{f}$ is *continuous at* $\mathbf{c}$ if

  $$\lim_{\mathbf{x}\to\mathbf{c}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{c}).$$

- We say that $\mathbf{f}$ is *continuous on* S if $\mathbf{f}$ is contiuous at every point of S.

Theorem 1.6.1

Let $\varnothing \neq S \subseteq \mathbb{R}^n$, $\mathbf{f}\colon S \to \mathbb{R}^m$, and $\mathbf{f} = (f_1, f_2, \cdots, f_m)$.

(i) Let $\mathbf{c} \in \overline{S}$. Then $\lim_{\mathbf{x}\to\mathbf{c}} \mathbf{f}(\mathbf{x}) = \mathbf{v} = (v_1, v_2, \cdots, v_m)$ if and only if, for each $j \in [m]$, $\lim_{\mathbf{x}\to\mathbf{c}} f_j(\mathbf{x}) = v_j$.

(ii) Let $\mathbf{c} \in S$. Then $\mathbf{f}$ is continuous at $\mathbf{c}$ if and only if, for each $j \in [m]$, f_j is continuous at $\mathbf{c}$.

(iii) $\mathbf{f}$ is continuous on S if and only if, for each $j \in [m]$, f_j is continuous on S.

Proof. (ii) and (iii) directly follows from (i). So, we only need to prove (i).

($\Rightarrow$) Take any $\varepsilon \in \mathbb{R}_+$. Then, there exists $\delta \in \mathbb{R}_+$ such that

$$\forall \mathbf{x} \in S,\ \big(0 < \|\mathbf{x} - \mathbf{c}\| < \delta \implies \|\mathbf{f}(\mathbf{x}) - \mathbf{v}\| < \varepsilon\big).$$

Then, for each $j \in [m]$, whenever $\mathbf{x} \in S \cap N'(\mathbf{c}; \delta)$,

$$|f_j(\mathbf{x}) - v_j|^2 \leq \sum_{i=1}^{m} |f_j(\mathbf{x}) - v_i|^2 = \|\mathbf{f}(\mathbf{x}) - \mathbf{v}\|^2 < \varepsilon^2,$$

which implies $S \cap N'(\mathbf{c}; \delta) \subseteq f_j^{-1}(N(v_j; \varepsilon))$; $\lim_{\mathbf{x}\to\mathbf{c}} f_j(\mathbf{x}) = v_j$.

($\Leftarrow$) Take any $\varepsilon \in \mathbb{R}_+$. Then, for each $j \in [m]$, there exists $\delta_j \in \mathbb{R}_+$ such that

$$\forall \mathbf{x} \in S,\ \big(0 < \|\mathbf{x} - \mathbf{c}\| < \delta_j \implies |f_j(\mathbf{x}) - v_j| < \varepsilon/\sqrt{m}\big).$$

Then, whenever $\mathbf{x} \in S \cap N'(\mathbf{c}; \delta_0)$ where $\delta_0 \triangleq \min_{j=1}^{m} \delta_j$,

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{v}\|^2 = \sum_{i=1}^{m} |f_j(\mathbf{x}) - v_j|^2 < \sum_{i=1}^{m} \left(\frac{\varepsilon}{\sqrt{m}}\right)^2 = \varepsilon^2,$$

which implies $S \cap N'(\mathbf{c}; \delta_0) \subseteq \mathbf{f}^{-1}(N(\mathbf{v}; \varepsilon))$; $\lim_{\mathbf{x}\to\mathbf{c}} \mathbf{f}(\mathbf{x}) = \mathbf{v}$. $\square$

Theorem 1.6.2

Let $\varnothing \neq S \subseteq \mathbb{R}^n$ and $\mathbf{f}\colon S \to \mathbb{R}^m$.

(i) Let $\mathbf{c}$ be a point in $\overline{S}$. Then $\lim_{\mathbf{x}\to\mathbf{c}} \mathbf{f}(\mathbf{x}) = \mathbf{v}$ if and only if, for every sequence $\{\mathbf{x}_k\}$ in $S \setminus \{\mathbf{c}\}$ such that $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{c}$, we have $\lim_{k\to\infty} \mathbf{f}(\mathbf{x}_k) = \mathbf{v}$.

(ii) Let $\mathbf{c}$ be a point in S. Then $\mathbf{f}$ is continuous at $\mathbf{c}$ if and only if, for every sequence $\{\mathbf{x}_k\}$ in S such that $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{c}$, we have $\lim_{k\to\infty} \mathbf{f}(\mathbf{x}_k) = \mathbf{f}(\mathbf{c})$.

Proof. By Theorem 1.6.1,

$$\lim_{\mathbf{x}\to\mathbf{c}} \mathbf{f}(\mathbf{x}) = \mathbf{v} \iff \forall j \in [m], \lim_{\mathbf{x}\to\mathbf{c}} f_j(\mathbf{x}) = v_j$$

where $\mathbf{v} = (v_1, v_2, \cdots, v_m)$. By Theorem 1.1.6, for each $j \in [m]$,

$$\lim_{\mathbf{x}\to\mathbf{c}} f_j(\mathbf{x}) = v_j \iff \forall \{\mathbf{x}_k\} \in (S \setminus \{\mathbf{c}\})^\omega,\ \big(\lim_{k\to\infty} \mathbf{x}_k = \mathbf{c} \implies \lim_{k\to\infty} f_j(\mathbf{x}_k) = v_j\big).$$

By **??**,

$$\lim_{k\to\infty} \mathbf{f}(\mathbf{x}_k) = \mathbf{v} \iff \forall j \in [m],\ \lim_{k\to\infty} f_j(\mathbf{x}_k) = v_j.$$

Thus, (i) is proven, and (ii) can be proven similarly with the aid of Theorem 1.1.7. $\square$

Theorem 1.6.3

Let $f : S \to T$ where $S \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}^m$. Suppose f is surjective. Then $\mathbf{f}$ is continuous on S if and only if the inverse image of every relatively open set in T is relatively open in S.

Proof. Repeat the proof of Theorem 1.2.1 verbatim. □

Theorem 1.6.4

If S is a connected subset of $\mathbb{R}^n$ and $\mathbf{f}\colon S \to \mathbb{R}^m$ is continuous on S, then $T = \mathbf{f}(S)$ is also connected.

Proof. Repeat the proof of Theorem 1.2.2 verbatim. □

Theorem 1.6.5

If S is a compact subset of $\mathbb{R}^n$ and $\mathbf{f}\colon S \to \mathbb{R}^m$ is continuous on S, then $T = \mathbf{f}(S)$ is also compact.

Proof. Repeat the proof of Theorem 1.2.3 verbatim. □

Theorem 1.6.6

Let $S \subseteq \mathbb{R}^n$, $f : S \to \mathbb{R}^m$, $f(S) \subseteq T \subseteq \mathbb{R}^m$, and $g : T \to \mathbb{R}^p$. If f is continuous at $\mathbf{c} \in S$ and if g is continuous at $f(\mathbf{c}) \in T$, then $g \circ f$ is continuous at $\mathbf{c}$.

Proof. Repeat the proof of Theorem 1.2.7 verbatim. □

Theorem 1.6.7

Let $\varnothing \neq S \subseteq \mathbb{R}^n$ is compact and $\varnothing \neq T \subseteq \mathbb{R}^m$. Let $\mathbf{f}\colon S \to T$ is continuous on S and bijective. Then, $\mathbf{f}^{-1}$ is also continuous on $\mathbf{f}(S)$.

Proof. Repeat the proof of Theorem 1.2.8 verbatim. □

Definition 1.6.3: Uniform Continuity

A function $\mathbf{f}\colon S \to \mathbb{R}^m$ with $S \subseteq \mathbb{R}^n$ is said to be *uniformly continuous on S* if,

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists \delta \in \mathbb{R}_+,\ \forall \mathbf{c} \in S,\ S \cap N(\mathbf{c};\delta) \subseteq \mathbf{f}^{-1}(N(\mathbf{f}(\mathbf{c});\varepsilon)).$$

Or, equivalently,

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists \delta \in \mathbb{R}_+,\ \forall \mathbf{x},\mathbf{y} \in S,\ \big(\|\mathbf{x}-\mathbf{y}\| < \delta \implies \|\mathbf{f}(x)-\mathbf{f}(y)\| < \varepsilon\big).$$

Theorem 1.6.8

Let $\mathbf{f} = (f_1, f_2, \cdots, f_m)$ be a function $\mathbf{f}\colon S \to \mathbb{R}^m$ where $S \subseteq \mathbb{R}^n$. Then, $\mathbf{f}$ is uniformly continuous if and only if f_j is uniformly continuous on S for each $j \in [m]$.

Proof. (⇒) Take any $j \in [m]$ and $\varepsilon \in \mathbb{R}_+$. Then, there exist $\delta \in \mathbb{R}_+$ such that,

$$\forall \mathbf{x}, \mathbf{y} \in S, \left(\|\mathbf{x} - \mathbf{y}\| < \delta \implies \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| < \varepsilon \right).$$

Since $|f_j(\mathbf{x}) - f_j(\mathbf{y})| \leq \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\|$, f_j is uniformly continuous.

(⇐) Take any $\varepsilon \in \mathbb{R}_+$. Then, for each $j \in [m]$, there exists $\delta_j \in \mathbb{R}_+$ such that

$$\forall \mathbf{x}, \mathbf{y} \in S, \left(\|\mathbf{x} - \mathbf{y}\| < \delta_j \implies |f_j(\mathbf{x}) - f_j(\mathbf{y})| < \varepsilon / \sqrt{m} \right).$$

Let $\delta \triangleq \min_{j \in [m]} \delta_j$. Then,

$$\forall \mathbf{x}, \mathbf{y} \in S, \left(\|\mathbf{x} - \mathbf{y}\| < \delta \implies \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| < \varepsilon \right).$$

□

Theorem 1.6.9

Suppose that $f : S \to \mathbb{R}^m$ is continuous on a compact subset S of $\mathbb{R}^n$. Then f is uniformly continuous on S.

Proof. Repeat the proof of Theorem 1.4.1 verbatim. □