

Summary for Introduction to Set Theory

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Chapter 1

Sets

1.1 Introduction to Sets

Definition 1.1.1: Set

Every object in the universe of discourse is called a *set*.

1.2 Properties

Definition 1.2.1: Property

Any mathematical sentence^a is called a *property*. If X, Y, \dots, Z are free variables of a property Q , we write $Q(X, Y, \dots, Z)$ and say $Q(X, Y, \dots, Z)$ is a property of X, Y, \dots, Z .

^aRefer to mathematical logic textbook for detailed discussion.

1.3 Axioms

Axiom I The Axiom of Existence

There exists a set which has no elements.

$$\exists A \forall x \neg(x \in A)$$

Note:-

The **Axiom of Existence** guarantees that the universe of discourse is not void.

Axiom II The Axiom of Extensionality

If every element of X is an element of Y and every element of Y is an element of X , then $X = Y$.

$$\forall X \forall Y [\forall x (x \in X \iff x \in Y) \implies X = Y]$$

Note:-

The **Axiom of Extensionality** defines the equality relation with the containment relation(\in).

Lemma 1.3.1

There exists only one set with no elements.

Proof. Let A and B are sets such that $\forall x \neg(x \in A)$ and $\forall x \neg(x \in B)$. Then, we have $\forall x (x \in A \iff x \in B)$. Therefore, by **The Axiom of Extensionality**, $A = B$ is guaranteed. \square

Definition 1.3.2: Empty Set

The unique set with no elements is called the *empty set* and is denoted \emptyset .

Note:-

Definition 1.3.2 is justified by Lemma 1.3.1.

Axiom III The Axiom Schema of Comprehension

Let $P(x)$ be a property of x . For any set A , there exists a set B such that $x \in B$ if and only if $x \in A$ and $P(x)$.

$$\forall A \exists B (x \in B \iff x \in A \wedge P(x))$$

Note:-

Axiom III is a *axiom schema* since it provides unlimited amount of axioms for varying P .

Lemma 1.3.3

Let $P(x)$ be a property of x . For any set A , there uniquely exists a set B such that $x \in B$ if and only if $x \in A$ and $P(x)$.

Proof. Let B' be another set such that $x \in B'$ if and only if $x \in A$ and $P(x)$. Then, for any x , we have $x \in B' \iff x \in A \wedge P(x) \iff x \in B$. Hence, by **The Axiom of Extensionality**, we have $B = B'$. \square

Notation 1.3.4: Set-Builder Notation

Let $P(x)$ be a property of x . Let A be a set. The unique set B such that $x \in B$ if and only if $x \in A$ and $P(x)$ is denoted $\{x \in A \mid P(x)\}$.

Note:-

Notation 1.3.4 is justified by Lemma 1.3.3.

Axiom IV The Axiom of Pair

For any A and B , there exists C such that $x \in C$ if and only if $x = A$ or $x = B$.

$$\forall A \forall B \exists C (x \in C \iff x = A \vee x = B)$$

Note:-

Similarly, the set C such that $x \in C \iff x = A \vee x = B$ is unique by **The Axiom of Extensionality**.

Notation 1.3.5

Let A and B be sets. The unique set C such that $x \in C$ if and only if $x = A$ or $x = B$ is denoted $\{A, B\}$. In particular, if $A = B$, we write $\{A\}$ instead of $\{A, A\}$.

Axiom V The Axiom of Union

For any S , there exists U such that $x \in U$ if and only if $x \in A$ for some $A \in S$.

$$\forall S \exists U (x \in U \iff \exists A x \in A \wedge A \in S)$$

Definition 1.3.6: The Union of System of Sets

Let S be a set. The unique set U such that $x \in U$ if and only if $x \in A$ for some $A \in S$ is denoted $\bigcup S$.

Definition 1.3.7: The Union of Two Sets

Let A and B be sets. Then, $A \cup B$ denotes the unique set $\bigcup \{A, B\}$.

Definition 1.3.8: Subset

Let A and B sets. B is said to be a *subset* of A if $\forall x (x \in B \implies x \in A)$. If B is a subset of A , then we write $B \subseteq A$.

Axiom VI The Axiom of Power Set

For any S , there exists P such that $X \in P$ if and only if $X \subseteq S$.

Note:-

Similarly, the set P is unique by [The Axiom of Extensionality](#).

Definition 1.3.9: Power Set

Let S be a set. The unique set P such that $X \in P$ if and only if $X \subseteq S$ is called the *power set* of S and is denoted $\mathcal{P}(S)$.

Lemma 1.3.10

Let $P(x)$ be a property of x . Let A and A' be sets such that $P(x) \implies x \in A \wedge x \in A'$. Then, $\{x \in A \mid P(x)\} = \{x \in A' \mid P(x)\}$.

Proof. For all x , we have $x \in A \wedge P(x) \iff P(x) \iff x \in A' \wedge P(x)$. Therefore, by [The Axiom of Extensionality](#), the result follows. \square

Notation 1.3.11

Let $P(x)$ be a property of x . If there exists a set A such that $P(x)$ implies $x \in A$, we write $\{x \mid P(x)\} \triangleq \{x \in A \mid P(x)\}$, and it is called *the set of all x with the property $P(x)$* .

Note:-

Notation [1.3.11](#) is justified by Lemma [1.3.10](#).

Selected Problems

Exercise 1.3.1

The set of all x such that $x \in A$ and $x \notin B$ exists.

Proof. We have $x \in A \wedge x \notin B \implies x \in A$. Hence, the set exists and is equal to $\{x \in A \mid x \in A \wedge x \notin B\}$. \square

Exercise 1.3.2

Prove **The Axiom of Existence** only from **The Axiom Schema of Comprehension** and **The Weak Axiom of Existence**.

Weak Axiom of Existence Some set exists.

Proof. Let A be a set known to exist. Then, there exists $B = \{x \in A \mid x \neq x\}$ by **The Axiom Schema of Comprehension**. Since $\forall x (x = x)$, $\forall x (x \notin B)$. \square

Exercise 1.3.3

- (a) Prove that a set of all sets ($\{x \mid \top\}$) does not exist.
- (b) Prove that $\forall A \exists x (x \notin A)$.

Proof.

- (a) Suppose $V = \{x \mid \top\}$ exists. Then, by **The Axiom Schema of Comprehension**, $R = \{x \in V \mid x \notin x\}$ exists. However, we have $R \in R \iff R \notin R$ by definition of R . Hence, V does not exist.
- (b) Suppose $\exists A \forall x (x \in A)$ for the sake of contradiction. Then, A is the set of all sets, which is impossible by (a). \square

Exercise 1.3.6

Prove $\forall X \neg(\mathcal{P}(X) \subseteq X)$.

Proof. Let $Y = \{u \in X \mid u \notin u\}$. Then, by definition, $Y \subseteq X$, and thus $Y \in \mathcal{P}(X)$. Now, suppose $Y \in X$ for the sake of contradiction. Then, $Y \in Y \iff Y \in X \wedge Y \notin Y \iff Y \notin Y$, which is a contradiction. Hence, $Y \notin X$. \square

1.4 Elementary Operations on Sets

Definition 1.4.1: Proper Subset

Let A and B sets. B is said to be a *proper subset* of A if $B \subseteq A$ and $B \neq A$. If B is a proper subset of A , we write $B \subsetneq A$.

Definition 1.4.2: Elementary Operations on Sets

- (i) Intersection
 - The *intersection* of A and B , $A \cap B$, is the set $\{x \mid x \in A \wedge x \in B\}$.
- (ii) Union
 - The *union* of A and B , $A \cup B$, is the set $\{x \mid x \in A \vee x \in B\}$.
- (iii) Difference
 - The *difference* of A and B , $A \setminus B$, is the set $\{x \mid x \in A \wedge x \notin B\}$.
- (iv) Symmetric Difference
 - The *symmetric difference* of A and B , $A \Delta B$, is the set $(A \setminus B) \cup (B \setminus A)$.

Lemma 1.4.3 Simple Properties of Elementary Operations

- (i) Commutativity
 - $A \cap B = B \cap A$
 - $A \cup B = B \cup A$
 - $A \Delta B = B \Delta A$
- (ii) Associativity
 - $(A \cap B) \cap C = A \cap (B \cap C)$
 - $(A \cup B) \cup C = A \cup (B \cup C)$
 - $(A \Delta B) \Delta C = A \Delta (B \Delta C)$
- (iii) Distributivity
 - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
 - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- (iv) De Morgan's Laws
 - $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$
 - $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$
- (v) Miscellaneous
 - $A \cap (B \setminus C) = (A \cap B) \setminus C$
 - $A \setminus B = \emptyset \iff A \subseteq B$
 - $A \Delta B = \emptyset \iff A = B$

Definition 1.4.4: Intersection of System of Sets

Let S be a nonempty set. Then, the *intersection* $\bigcap S$ is the set $\{x \mid \forall A \in S (x \in A)\}$.

Note:-

Note that $\bigcap S$ exists for all nonempty S since $\forall A \in S (x \in A) \implies x \in A_1$ where A_1 is any set such that $A_1 \in S$.

Definition 1.4.5: System of Mutually Disjoint Sets

We say the sets A and B are *disjoint* if $A \cap B = \emptyset$. A set S is a *system of mutually disjoint sets* if $\forall A, B \in S, (A \neq B \implies A \cap B = \emptyset)$.

Selected Problems

Exercise 1.4.2

- (i) $A \setminus B = (A \cup B) \setminus B = A \setminus (A \cap B)$
- (ii) $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$
- (iii) $A \cap B = A \setminus (A \setminus B)$

Proof.

$$\begin{aligned} \text{(i)} \quad x \in A \wedge x \notin B &\iff x \in A \wedge x \notin B \vee x \in B \wedge x \notin B &> \vee\text{-intro} / \vee\text{-syllogism} \\ &\iff (x \in A \vee x \in B) \wedge x \notin B &> \text{Distribution} \end{aligned}$$

$$\begin{aligned} x \in A \wedge x \notin B &\iff x \in A \wedge x \notin A \vee x \in A \wedge x \notin B &> \vee\text{-intro} / \vee\text{-syllogism} \\ &\iff x \in A \wedge (x \notin A \vee x \notin B) &> \text{Distribution} \\ &\iff x \in A \wedge \neg(x \in A \wedge x \in B) &> \text{De Morgan's Law} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad x \in A \wedge \neg(x \in B \wedge x \notin C) &\iff x \in A \wedge (x \notin B \vee x \in C) &> \text{De Morgan's Law} \\ &\iff (x \in A \wedge x \notin B) \vee (x \in A \wedge x \in C) &> \text{Distribution} \end{aligned}$$

$$\text{(iii)} \quad \text{By (ii), } A \setminus (A \setminus B) = (A \setminus A) \cup (A \cap B) = A \cap B. \quad \square$$

Exercise 1.4.4

For any set A , prove that a “complement” of A ($\{x \mid x \notin A\}$) does not exist.

Proof. Let B be the complement of A for the sake of contradiction. Then, $A \cup B$ is the set of all sets, which is impossible by Exercise 1.3.3. \square

Chapter 2

Relations, Function, and Ordering

2.1 Ordered Pairs

Definition 2.1.1: Ordered Pair

$$(a, b) \triangleq \{\{a\}, \{a, b\}\}$$

Theorem 2.1.2

$$(a, b) = (a', b') \iff a = a' \wedge b = b'$$

Proof. (\Leftarrow) is direct.

(\Rightarrow) If $a = b$, we have $\{\{a\}\} = \{\{a'\}, \{a', b'\}\}$, and thus $\{a\} = \{a'\} = \{a', b'\}$, leaving the only option $a = a' = b'$.

If $a \neq b$, we must have $a' \neq b'$ by the argument above. Hence, we have $\{\{a\}, \{a, b\}\} = \{\{a'\}, \{a', b'\}\}$, which implies $\{a\} = \{a'\}$ and $\{a, b\} = \{a', b'\}$. \square

Definition 2.1.3: Ordered Triples and Quadruples

- $(a, b, c) = ((a, b), c)$
- $(a, b, c, d) = ((a, b, c), d)$

Selected Problems

Exercise 2.1.1

If $a, b \in A$, then $(a, b) \in \mathcal{P}(\mathcal{P}(A))$.

Proof. If $a, b \in A$, then $\{a\}, \{a, b\} \in \mathcal{P}(A)$, and thus $(a, b) = \{\{a\}, \{a, b\}\} \subseteq \mathcal{P}(A)$. \square

2.2 Relations

Definition 2.2.1: Binary Relation

A set R is a *binary relation* if all elements of R are ordered pairs.

$$R \text{ is a binary relation} \iff (a \in R \implies \exists x, \exists y, a = (x, y))$$

Notation 2.2.2

If $(x, y) \in R$, we write xRy and say x is in relation R with y .

Definition 2.2.3: Domain, Range, and Field of Binary Relation

Let R be a binary relation.

- $\text{dom}R \triangleq \{x \mid \exists y \, xRy\}$ is called the *domain* of R .
- $\text{ran}R \triangleq \{y \mid \exists x \, xRy\}$ is called the *range* of R .
- $\text{field}R \triangleq \text{dom}R \cup \text{ran}R$ is called the *field* of R .
- If $\text{field}R \subseteq X$, we say that R is a *relation in* X or that R is a relation *between* elements of X .

Lemma 2.2.4

Let R be a binary relation. Then, $\text{dom}R$ and $\text{ran}R$ exist.

Proof. By Exercise 2.2.1, if xRy , then $x, y \in A \triangleq \bigcup(\bigcup R)$. Hence, $\text{dom}R$ and $\text{ran}R$ exist. \square

Definition 2.2.5: Image and Inverse Image

Let R be a binary relation and A be a set.

- $R[A] \triangleq \{y \in \text{ran}R \mid \exists x \in A, xRy\}$ is called the *image* of A under R .
- $R^{-1}[A] \triangleq \{x \in \text{dom}R \mid \exists y \in A, xRy\}$ is called the *inverse image* of A under R .

Notation 2.2.6

We write $\{(x, y) \mid P(x, y)\}$ instead of $\{w \mid \exists x, \exists y, w = (x, y) \wedge P(x, y)\}$.

Definition 2.2.7: Inverse Relation

Let R be a binary relation. The *inverse* of R is the set

$$R^{-1} \triangleq \{(x, y) \mid yRx\}.$$

Definition 2.2.8: Composition

Let R and S be binary relations. The relation

$$S \circ R \triangleq \{(x, z) \mid \exists y, xRy \wedge ySz\}$$

is called the *composition* of R and S .

Definition 2.2.9: Membership Relation and Identity Relation

Let A be a set.

- The *membership relation on A* is defined by

$$\in_A \triangleq \{(a, b) \mid a, b \in A \wedge a \in b\}.$$

- The *identity relation on A* is defined by

$$\text{Id}_A \triangleq \{(a, a) \mid a \in A\}.$$

Definition 2.2.10: Cartesian Product

Let A and B be sets. The set $A \times B \triangleq \{(a, b) \mid a \in A \wedge b \in B\}$ is called the *Cartesian product* of A and B .

Lemma 2.2.11

Let A and B be sets. $A \times B$ exists.

Proof. If $a \in A$ and $b \in B$, by Exercise 2.1.1, we have $(a, b) \in \mathcal{P}(\mathcal{P}(A \cup B))$. □

Corollary 2.2.12

Let R and S be binary relations and A be a set. Then, R^{-1} , $S \circ R$, \in_A , and Id_A exist.

Proof.

- If yRx , then $(x, y) \in (\text{ran } R) \times (\text{dom } R)$.
- If $(x, z) \in S \circ R$, then $(x, z) \in (\text{dom } R) \times (\text{ran } S)$.
- If $a, b \in A$, then $(a, b) \in A \times A$.
- If $a \in A$, then $(a, a) \in A \times A$. □

Lemma 2.2.13

Let R be a binary relation. The inverse image of A under R is equal to the image of A under R^{-1} .

Proof. Note that $\text{dom } R = \{x \mid \exists y \, xRy\} = \{x \mid \exists y \, yR^{-1}x\} = \text{ran } R^{-1}$. Therefore,

$$\begin{aligned} & x \in (\text{the inverse image of } A \text{ under } R) \\ \iff & x \in \text{dom } R \wedge \exists y \in A, \, xRy \\ \iff & x \in \text{ran } R^{-1} \wedge \exists y \in A, \, yR^{-1}x \\ \iff & x \in (\text{the image of } A \text{ under } R^{-1}). \end{aligned}$$
□

Note:-

Lemma 2.2.13 resolves the possible ambiguity on the expression $R^{-1}[A]$.

Notation 2.2.14

We write A^2 instead of $A \times A$.

Selected Problems

Exercise 2.2.1

Let R be a binary relation. Let $A = \bigcup (\bigcup R)$. Prove that $(x, y) \in R$ implies $x \in A$ and $y \in A$.

Proof. If $(x, y) = \{\{x\}, \{x, y\}\} \in R$, Then $\{x, y\} \in \bigcup R$, and thus $x, y \in A$. \square

Exercise 2.2.3

Let R be a binary relation and A and B be sets. Prove:

- (i) $R[A \cup B] = R[A] \cup R[B]$.
- (ii) $R[A \cap B] \subseteq R[A] \cap R[B]$.
- (iii) $R[A \setminus B] \supseteq R[A] \setminus R[B]$.
- (iv) Show by an example that \subseteq and \supseteq in parts (ii) and (iii) cannot be replaced by $=$.
- (v) $R^{-1}[R[A]] \supseteq A \cap \text{dom } R$ and $R[R^{-1}[B]] \supseteq B \cap \text{ran } R$. Give examples where equality does not hold.

Proof.

- (i) $y \in R[A \cup B] \iff \exists x, x \in A \cup B \wedge xRy$
 $\iff \exists x, (x \in A \wedge xRy) \vee (x \in B \wedge xRy)$
 $\iff y \in R[A] \vee y \in R[B] \iff y \in R[A] \cup R[B]$
- (ii) Take any $y \in R[A \cap B]$. Then, there exists $x \in A \cap B$ such that xRy . Hence, $y \in R[A]$ and $y \in R[B]$.
- (iii) Take any $y \in R[A] \setminus R[B]$. Then, there exists $x \in A$ such that xRy . If $x \in B$, it implies that $y \in R[B]$, which is a contradiction. Hence, $x \in A \setminus B$. Therefore, $y \in R[A \setminus B]$.
- (iv) Let a, b , and c be mutually different sets. Let $R = \{(a, a), (b, a), (c, c)\}$. Let $A = \{a, c\}$ and $B = \{b, c\}$. Then, $R[A \cap B] = \{c\} \subsetneq R[A] \cap R[B] = \{a, c\}$, and $R[A] \setminus R[B] = \emptyset \subsetneq R[A \setminus B] = \{a\}$.
- (v) Take any $a \in A \cap \text{dom } R$. Then, there exists b such that aRb . Moreover, $b \in R[A]$. Since $bR^{-1}a$, we conclude that $a \in R^{-1}[R[A]]$.
 Take any $b \in B \cap \text{ran } R$. Then, there exists a such that aRb . Moreover, $a \in R^{-1}[B]$. Hence, $b \in R[R^{-1}[B]]$.

Exercise 2.2.4

Let $R \subseteq X \times Y$. Prove:

- (i) $R[X] = \text{ran } R$ and $R^{-1}[Y] = \text{dom } R$.
- (ii) $\text{dom } R = \text{ran } R^{-1}$ and $\text{ran } R = \text{dom } R^{-1}$.
- (iii) $(R^{-1})^{-1} = R$.
- (iv) $R^{-1} \circ R \supseteq \text{Id}_{\text{dom } R}$ and $R \circ R^{-1} \supseteq \text{Id}_{\text{ran } R}$

Proof.

- (i) We already have $R[X] \subseteq \text{ran } R$ by definition. Take any $y \in \text{ran } R$. There exists x such that $(x, y) \in R$. Since $R \subseteq X \times Y$, $x \in X$. Therefore, $y \in R[X]$; $\text{ran } R \subseteq R[X]$. A similar argument goes for $R^{-1}[Y]$.
- (ii) See the proof of Lemma 2.2.13.
- (iii) For any relation R and for all x and y , we have $xRy \iff yR^{-1}x$. Since R^{-1} is also a relation, we have $xRy \iff yR^{-1}x \iff x(R^{-1})^{-1}y$.
- (iv) Take any $x \in \text{dom } R$. Then, there exists y such that xRy . Then, $yR^{-1}x$, and thus $x(R^{-1} \circ R)x$. A similar argument goes for $R \circ R^{-1}$. \square

Exercise 2.2.8

$A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$.

Proof. (\Rightarrow) If $A \neq \emptyset$ and $B \neq \emptyset$, we have $(a, b) \in A \times B$ where $a \in A$ and $b \in B$, and thus $A \times B \neq \emptyset$.

(\Leftarrow) If $A \times B \neq \emptyset$, then $a \in A$ and $b \in B$ where $(a, b) \in A \times B$. \square

2.3 Functions

Definition 2.3.1: Function

A binary relation F is called a *function* (or *mapping*) if

$$\forall a \forall b_1 \forall b_2 (aFb_1 \wedge aFb_2 \implies b_1 = b_2).$$

For each $a \in \text{dom } F$, the unique b such that aFb is called the *value of F at a* and is denoted $F(a)$ or F_a .

Notation 2.3.2

If F is a function with $\text{dom } F = A$ and $\text{ran } F \subseteq B$, we write $F: A \rightarrow B$, $\langle F(a) \mid a \in A \rangle$, $\langle F_a \mid a \in A \rangle$, $\langle F_a \rangle_{a \in A}$ for the function F . The range of the function F can then be denoted $\{F(a) \mid a \in A\}$ or $\{F_a\}_{a \in A}$.

Lemma 2.3.3

Let F and G be functions. $F = G \iff \text{dom } F = \text{dom } G \wedge \forall x \in \text{dom } F, F(x) = G(x)$.

Proof. (\Rightarrow) is direct.

(\Leftarrow) Take any $(x, F(x)) \in F$. Then, we have $(x, F(x)) = (x, G(x)) \in G$. Therefore, $F \subseteq G$. Similarly, $G \subseteq F$, and thus $F = G$. \square

Definition 2.3.4

Let F be a function and A and B be sets.

- F is a function *on* A if $\text{dom } F = A$.
- F is a function *into* B if $\text{ran } F \subseteq B$.
- F is a function *onto* B if $\text{ran } F = B$.
- The *restriction* of the function F to A is the function $F|_A \triangleq \{(a, b) \in F \mid a \in A\}$. If G is a restriction of F to some A , we say that F is an *extension* of G .

Theorem 2.3.5

Let f and g be functions.

- $g \circ f$ is a function.
- $\text{dom}(g \circ f) = (\text{dom } f) \cap f^{-1}[\text{dom } g]$.
- $\forall x \in \text{dom}(g \circ f), (g \circ f)(x) = g(f(x))$.

Proof.

- (i) Suppose $x(g \circ f)z_1$ and $x(g \circ f)z_2$. There exists y_1 and y_2 such that xfy_1 , y_1gz_1 , xfy_2 , and y_2gz_2 . Since f and g are functions, we have $y_1 = y_2$ and $z_1 = z_2$. Therefore, $g \circ f$ is a function.
- (ii) $x \in \text{dom}(g \circ f) \iff \exists z x(g \circ f)z$
 $\iff \exists z \exists y xfy \wedge ygz$
 $\iff x \in \text{dom } f \wedge f(x) \in \text{dom } g \iff x \in \text{dom } f \wedge x \in f^{-1}[\text{dom } g] \quad \square$

Definition 2.3.6: Invertible Function

A function f is said to be *invertible* if f^{-1} is a function.

Definition 2.3.7: Injective Function

A function f is said to be *injective* (or *one-to-one*) if

$$\forall a_1, a_2 \in \text{dom } f, (f(a_1) = f(a_2) \implies a_1 = a_2).$$

Notation 2.3.8

Let $f : A \rightarrow B$ be a function.

- If f is a function *onto* B , we may write $f : A \twoheadrightarrow B$.
- If f is one-to-one, we may write $f : A \hookrightarrow B$.
- If f is one-to-one and onto B , we may write $f : A \leftrightarrow B$.

Theorem 2.3.9

Let f be a function.

- f is invertible if and only if f is one-to-one.
- If f is invertible, then f^{-1} is also invertible and $(f^{-1})^{-1} = f$.

Proof.

- (\implies) Suppose f^{-1} is a function. Then, $f^{-1}(f(a)) = a$ for all $a \in \text{dom } f$. Hence, for all $a_1, a_2 \in \text{dom } f$ such that $f(a_1) = f(a_2)$, it follows that $a_1 = f^{-1}(f(a_1)) = f^{-1}(f(a_2)) = a_2$; f is one-to-one.
 (\impliedby) Suppose f is one-to-one. If $yf^{-1}x_1$ and $yf^{-1}x_2$, then x_1fy and x_2fy , i.e., $y = f(x_1) = f(x_2)$. Therefore, $x_1 = x_2$; f^{-1} is a function.
- As f is a relation, by Exercise 2.2.4 (iii), $(f^{-1})^{-1} = f$, and thus f^{-1} is invertible. \square

Definition 2.3.10: Compatible Functions

- Functions f and g are called *compatible* if $\forall x \in (\text{dom } f) \cap (\text{dom } g), f(x) = g(x)$.
- A set of functions F is called a *compatible system of functions* if any two functions f and g from F are compatible.

Lemma 2.3.11

Let f and g be functions.

- f and g are compatible if and only if $f \cup g$ is a function.
- f and g are compatible if and only if $f|_{(\text{dom } f) \cap (\text{dom } g)} = g|_{(\text{dom } f) \cap (\text{dom } g)}$.

Proof.

- (i) (\Rightarrow) Suppose $x(f \cup g)y_1$ and $x(f \cup g)y_2$. WLOG, $(x, y_1) \in f$. If $(x, y_2) \in f$, since f is a function, $y_1 = y_2$. If $(x, y_2) \in g$, since f and g are compatible, $y_1 = f(x) = g(x) = y_2$. Therefore, $f \cup g$ is a function.
- (\Leftarrow) Take any $x \in (\text{dom } f) \cap (\text{dom } g)$. $(x, f(x)) \in f \cup g$ and $(x, g(x)) \in f \cup g$. Since $f \cup g$ is a function, we have $f(x) = g(x)$.
- (ii) Let $A = (\text{dom } f) \cap (\text{dom } g)$.
- (\Rightarrow) By definition, $\text{dom } f|_A = \text{dom } g|_A = (\text{dom } f) \cap (\text{dom } g)$. Moreover, for all $x \in (\text{dom } f) \cap (\text{dom } g)$, $f|_A(x) = f(x) = g(x) = g|_A(x)$. Hence, the result follows by Lemma 2.3.3.
- (\Leftarrow) Take any $x \in A$. Then, $f(x) = f|_A(x) = g|_A(x) = g(x)$. \square

Theorem 2.3.12

If F is a compatible system of functions, then $\bigcup F$ is a function with $\text{dom } \bigcup F = \bigcup \{\text{dom } f \mid f \in F\}$. The function $\bigcup F$ extends all $f \in F$.

Proof. Note that $\bigcup F$ is already a relation. If $(a, b_1), (a, b_2) \in \bigcup F$, then there exist $f_1, f_2 \in F$ such that $(a, b_1) \in f_1$ and $(a, b_2) \in f_2$. Since f_1 and f_2 are compatible and $a \in (\text{dom } f_1) \cap (\text{dom } f_2)$, we have $b_1 = f_1(a) = f_2(a) = b_2$. Hence, $\bigcup F$ is a function.

$\text{dom } \bigcup F = \bigcup \{\text{dom } f \mid f \in F\}$ since

$$\begin{aligned} x \in \text{dom } \bigcup F &\iff \exists y, (x, y) \in \bigcup F \\ &\iff \exists y, \exists f \in F, (x, y) \in f \\ &\iff \exists f \in F, x \in \text{dom } f \iff x \in \bigcup \{\text{dom } f \mid f \in F\}. \end{aligned}$$

Take any $f \in F$. As $f \cup \bigcup F = \bigcup F$, f and $\bigcup F$ are compatible by Lemma 2.3.11 (i). Moreover, $\text{dom } f \cap \text{dom } \bigcup F = \text{dom } f$. Hence, by Lemma 2.3.11 (ii), $f = f|_{\text{dom } f} = (\bigcup F)|_{\text{dom } f}$; $\bigcup F$ extends each $f \in F$. \square

Definition 2.3.13

Let A and B be sets. Then, we define

$$B^A \triangleq \{f \mid f \text{ is a function on } A \text{ into } B\}.$$

Definition 2.3.14: Indexed System of Sets

- Let $S = \langle S_i \mid i \in I \rangle$ be a function with domain I . We call the function S an *indexed system of sets* whenever we stress that the values of S are sets.
- We say that a system of sets A is *indexed* by S if $A = \{S_i \mid i \in I\} = \text{ran } S$.

Notation 2.3.15

If A is indexed by $S = \langle S_i \mid i \in I \rangle$, we may write

$$\bigcup \{S_i \mid i \in I\} \quad \text{or} \quad \bigcup_{i \in I} S_i$$

instead of $\bigcup A$. Similarly, we may write $\bigcap \{S_i \mid i \in I\}$ or $\bigcap_{i \in I} S_i$ instead of $\bigcap A$.

Definition 2.3.16: Product of Indexed System of Sets

Let $S = \langle S_i \mid i \in I \rangle$ be an indexed system of sets. We call the set

$$\prod S \triangleq \{f \mid f \text{ is a function on } I \text{ and } \forall i \in I, f_i \in S_i\}$$

the *product* of the indexed system S .

Notation 2.3.17

Other notations for the product of the indexed system $S = \langle S_i \mid i \in I \rangle$ are:

$$\prod \langle S(i) \mid i \in I \rangle, \quad \prod_{i \in I} S(i), \quad \prod_{i \in I} S_i.$$

Note:-

The existence of B^A and $\prod_{i \in I} S_i$ is proved in Exercise 2.3.9.

Note:-

If $A = S_i$ for all $i \in I$, $\prod_{i \in I} S_i = A^I$.

Selected Problems**Exercise 2.3.4**

Let f be a function. If there exists a function g such that $g \circ f = \text{Id}_{\text{dom } f}$, then f is invertible and $f^{-1} = g|_{\text{ran } f}$.

Proof. For $x_1, x_2 \in \text{dom } f$, suppose $f(x_1) = f(x_2)$. Then, $x_1 = (g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = (g \circ f)(x_2) = x_2$. Hence, f is one-to-one and is invertible by Theorem 2.3.9.

Take any $(y, x) \in f^{-1}$. Then, as $x \in \text{dom } f$, we must have $(y, x) \in \text{Id}_{\text{dom } f}$. Hence, $f^{-1} \subseteq g|_{\text{ran } f}$. Now, take any $(y, x) \in g|_{\text{ran } f}$. Since $y \in \text{ran } f$, there exists $x' \in \text{dom } f$ such that $(x', y) \in f$. Since $g \circ f = \text{Id}_{\text{dom } f}$, we have $x = x'$. Therefore, $(y, x) \in f^{-1}$; $g|_{\text{ran } f} \subseteq f^{-1}$. \square

Exercise 2.3.6

Let f be a function.

- (i) $f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B]$
- (ii) $f^{-1}[A \setminus B] = f^{-1}[A] \setminus f^{-1}[B]$

Proof. Thanks to Exercise 2.2.3 (ii) and (iii), we only need to prove the other inclusions.

- (i) Take any $x \in f^{-1}[A] \cap f^{-1}[B]$. Then, there exists $a \in A$ and $b \in B$ such that $xf a$ and $xf b$. Since f is a function, $a = b$, and thus $x \in f^{-1}[A \cap B]$.
- (ii) Take any $x \in f^{-1}[A \setminus B]$. Then, $f(x) \in A \setminus B$. If $x \in f^{-1}[B]$, we would have $f(x) \in B$; thus $x \notin f^{-1}[B]$. Therefore, $x \in f^{-1}[A] \setminus f^{-1}[B]$. \square

Exercise 2.3.8

Every system of sets A can be indexed by a function.

Proof. Let S be the function Id_A so $S_i = i$ for all $i \in A$. Then, $A = \{S_i \mid i \in A\}$; A is indexed by S . \square

Exercise 2.3.9

- (i) Let A and B be sets. Prove that B^A exists.
- (ii) Let $\langle S_i \mid i \in I \rangle$ be an indexed system of sets. Prove that $\prod_{i \in I} S_i$ exists.

Proof.

- (i) If f is a function from A into B , then $f \subseteq A \times B$, i.e., $f \in \mathcal{P}(A \times B)$.
- (ii) If f is a function on I and $f_i \in S_i$ for all $i \in I$, then f is a function onto $\bigcup_{i \in I} S_i$. Hence, $f \in (\bigcup_{i \in I} S_i)^I$. \square

Exercise 2.3.10

Let $\langle F_a \mid a \in \bigcup S \rangle$ be an indexed system of sets.

- (i) $\bigcup_{a \in \bigcup S} F_a = \bigcup_{C \in S} [\bigcup_{a \in C} F_a]$
- (ii) $\bigcap_{a \in \bigcup S} F_a = \bigcap_{C \in S} [\bigcap_{a \in C} F_a]$ if $S \neq \emptyset$ and $\forall C \in S, C \neq \emptyset$.

Proof.

- (i) $x \in \bigcup_{a \in \bigcup S} F_a \iff \exists a \in \bigcup S, x \in F_a$
 $\iff \exists C \in S, \exists a \in C, x \in F_a$
 $\iff \exists C \in S, x \in \bigcup_{a \in C} F_a \iff x \in \bigcup_{C \in S} [\bigcup_{a \in C} F_a]$
- (ii) $x \in \bigcap_{a \in \bigcup S} F_a \iff \forall a \in \bigcup S, x \in F_a$
 $\iff \forall C \in S, \forall a \in C, x \in F_a$
 $\iff \forall C \in S, x \in \bigcap_{a \in C} F_a \iff x \in \bigcap_{C \in S} [\bigcap_{a \in C} F_a]$ \square

Exercise 2.3.11

Let $\langle F_a \mid a \in A \rangle$ be a nonempty indexed system of sets.

- (i) $B \setminus \bigcup_{a \in A} F_a = \bigcap_{a \in A} (B \setminus F_a)$
- (ii) $B \setminus \bigcap_{a \in A} F_a = \bigcup_{a \in A} (B \setminus F_a)$

Proof.

- (i) $x \in B \setminus \bigcup_{a \in A} F_a \iff x \in B \wedge \neg(\exists a \in A, x \in F_a)$
 $\iff x \in B \wedge \forall a \in A, x \notin F_a$
 $\iff \forall a \in A, (x \in B \wedge x \notin F_a) \iff x \in \bigcap_{a \in A} (B \setminus F_a)$
- (ii) $x \in B \setminus \bigcap_{a \in A} F_a \iff x \in B \wedge \neg(\forall a \in A, x \in F_a)$
 $\iff x \in B \wedge \exists a \in A, x \notin F_a$
 $\iff \exists a \in A, (x \in B \wedge x \notin F_a) \iff x \in \bigcup_{a \in A} (B \setminus F_a)$ \square

Exercise 2.3.12

Let R be a relation and let $\langle F_a \mid a \in A \rangle$ be an indexed system of sets.

- (i) $R[\bigcup_{a \in A} F_a] = \bigcup_{a \in A} R[F_a]$
- (ii) $R[\bigcap_{a \in A} F_a] \subseteq \bigcap_{a \in A} R[F_a]$ if $A \neq \emptyset$.
- (iii) $R[\bigcap_{a \in A} F_a] = \bigcap_{a \in A} R[F_a]$ if $A \neq \emptyset$ and R is an injective function.

(iv) $R^{-1}[\bigcap_{a \in A} F_a] = \bigcap_{a \in A} R^{-1}[F_a]$ if $A \neq \emptyset$ and R is a function.

Proof.

- (i) $y \in R[\bigcup_{a \in A} F_a] \iff \exists x \in \bigcup_{a \in A} F_a, xRy$
 $\iff \exists a \in A, \exists x \in F_a, xRy$
 $\iff \exists a \in A, y \in R[F_a] \iff x \in \bigcup_{a \in A} R[F_a]$
- (ii) Take any $y \in R[\bigcap_{a \in A} F_a]$. Then, there exists $x \in \bigcap_{a \in A} F_a$ such that xRy . Hence, for all $a \in A$, $y \in R[F_a]$, i.e., $y \in \bigcap_{a \in A} R[F_a]$.
- (iii) If R is an injective function, then R^{-1} is also a function. Hence, the result follows from (iv) and the fact that $R = (R^{-1})^{-1}$.
- (iv) Thanks to (ii), since R^{-1} is a relation, we only need to prove the other inclusion. Take any $x \in \bigcap_{a \in A} R^{-1}[F_a]$. Fix any $a^* \in A$. Then, there exists $y^* \in F_{a^*}$ such that xRy^* .
 Now, take any $a \in A$. Then, $\exists y \in F_a$ such that xRy . Since R is a function, $y = y^*$; $y^* \in F_a$, i.e., $y^* \in \bigcap_{a \in A} F_a$. Therefore, $x \in R^{-1}[\bigcap_{a \in A} F_a]$. \square

2.4 Equivalences and Partitions

Definition 2.4.1: Equivalence

Let R be a binary relation in A .

- R is called *reflexive* in A if $\forall a \in A, aRa$.
- R is called *symmetric* in A if $\forall a, b \in A, (aRb \implies bRa)$.
- R is called *transitive* in A if $\forall a, b, c \in A, (aRb \wedge bRc \implies aRc)$.
- R is called an *equivalence* on A if it is reflexive, symmetric, and transitive in A .

Definition 2.4.2: Equivalence Class

Let E be an equivalence on A and let $a \in A$. The *equivalence class of a modulo E* is the set

$$[a]_E \triangleq \{x \in A \mid xEa\}.$$

Lemma 2.4.3

Let E be an equivalence on A and let $a, b \in A$.

- (i) $aEb \iff [a]_E = [b]_E$
 (ii) $\neg(aEb) \iff [a]_E \cap [b]_E = \emptyset$

Proof.

- (i) (\implies) Suppose aEb . Take any $c \in [a]_E$. Then, cEa and aEb , and thus cEb by transitivity. Hence, $c \in [b]_E$; $[a]_E \subseteq [b]_E$. $[b]_E \subseteq [a]_E$ can be shown similarly since bEa holds as E is symmetric.
 (\impliedby) Suppose $[a]_E = [b]_E$. Since aEa by reflexivity, we have $a \in [a]_E = [b]_E$. Therefore, aEb .
- (ii) (\implies) Suppose $[a]_E \cap [b]_E \neq \emptyset$. Then, there exists $c \in [a]_E \cap [b]_E$, i.e., cEa and cEb . Then, as E is symmetric, we have aEc , and therefore aEb by transitivity.
 (\impliedby) Suppose aEb . Then, since aEa by reflexivity, we have $a \in [a]_E$. We can see $a \in [b]_E$ from (i). Hence, $[a]_E \cap [b]_E \neq \emptyset$. \square

Definition 2.4.4: Partition

A system S of nonempty sets is called a *partition* of A if

- (i) S is a system of mutually disjoint sets (Definition 1.4.5) and
- (ii) $\bigcup S = A$.

Definition 2.4.5: System of All Equivalence Classes

Let E be an equivalence on A . The *system of all equivalence classes* modulo E is the set

$$A/E \triangleq \{[a]_E \mid a \in A\}.$$

Theorem 2.4.6

Let E be an equivalence on A . Then, A/E is a partition of A .

Proof. If $[a]_E \neq [b]_E$, then by Lemma 2.4.3, we have $[a]_E \cap [b]_E = \emptyset$. Since E is reflexive, $a \in [a]_E$; each $[a]_E$ is nonempty. Therefore, A/E is a system of mutually disjoint nonempty sets.

Take any $a \in A$. Since E is reflexive, $a \in [a]_E \subseteq \bigcup A/E$. Therefore, $A \subseteq \bigcup A/E$. Conversely, since $[a]_E \subseteq A$, we have $\bigcup A/E \subseteq A$. \square

Definition 2.4.7

Let S be a partition of A . The relation E_S in A is defined by

$$E_S \triangleq \{(a, b) \in A \times A \mid \exists C \in S, a \in C \wedge b \in C\}.$$

Theorem 2.4.8

Let S be a partition of A . Then, E_S is an equivalence on A .

Proof.

- Take any $a \in A$. As $A = \bigcup S$, there exists $C \in S$ such that $a \in C$. Therefore, $aE_S a$. E_S is reflexive.
- Assume $aE_S b$. Then, there exists $C \in S$ such that $a, b \in C$. Hence, $bE_S a$. E_S is symmetric.
- Assume $aE_S b$ and $bE_S c$. Then, there exist $C, D \in S$ such that $a, b \in C$ and $b, c \in D$. Then, $C \cap D \neq \emptyset$ as b belongs to both sets. Hence, $C = D$, which implies $aE_S c$. E_S is transitive. \square

Theorem 2.4.9

- (i) If E is an equivalence on A and $S = A/E$, then $E_S = E$.
- (ii) If S is a partition of A , then $A/E_S = S$.

Proof.

- (i) $aE_S b \iff \exists C \in S, a \in C \wedge b \in C \iff \exists c \in A, a \in [c]_E \wedge b \in [c]_E \iff aEb$.
Definition 2.4.7 Lemma 2.4.3
- (ii) Take any $[a]_{E_S} \in A/E_S$. Since S is a partition, there (uniquely) exists C such that $a \in C$. Then, for all b , we have $b \in C \iff aE_S b \iff b \in [a]_{E_S}$; $C = [a]_{E_S}$. Therefore,
Lemma 2.4.3
 $A/E_S \subseteq S$.

For the converse, take any $C \in S$. As C is nonempty, we may take some $a \in C$. Similarly, we have $C = [a]_{E_S}$. Therefore, $C \subseteq A/E_S$. \square

Note:-

Theorem 2.4.9 essentially states that equivalence and partition describe the same “mathematical reality.”

Definition 2.4.10: Set of Representatives

A set $X \subseteq A$ is called a *set of representatives* for the equivalence E_S (or for the partition S of A) if

$$\forall C \in S, \exists a \in C, X \cap C = \{a\}.$$

Selected Problems

Exercise 2.4.2

Let f be a function on A onto B . Define a relation E in A by: aEb if and only if $f(a) = f(b)$.

- (i) Show that E is an equivalence on A .
- (ii) Show that $[a]_E = [a']_E$ implies that $f(a) = f(a')$ so that the function φ on A/E into B defined by $\varphi([a]_E) = f(a)$ is well-defined. Show also that φ is onto B .
- (iii) Let j be the function on A onto A/E given by $j(a) = [a]_E$. Show that $\varphi \circ j = f$.

Proof.

- (i) E can readily be shown to be reflexive, symmetric, and transitive.
- (ii) Assume $[a]_E = [a']_E$. Then, $f(a) = f(a')$ by definition of E . Hence, φ is well-defined. Take any $b \in B$. Since f is onto, there exists $a \in A$ such that $f(a) = b$. Hence, $\varphi([a]_E) = f(a) = b$; φ is onto B .
- (iii) $\text{dom}(\varphi \circ j) = (\text{dom } j) \cap j^{-1}[\text{dom } \varphi] = A = \text{dom } f$ since j is onto. For all $a \in A$, $(\varphi \circ j)(a) = \varphi([a]_E) = f(a)$. Hence, by Lemma 2.3.3, $\varphi \circ j = f$. \square

2.5 Orderings

Definition 2.5.1: Partial Ordering and Strict Ordering

Let R be a binary relation in A .

- R is called *antisymmetric* in A if $\forall a, b \in A, (aRb \wedge bRa \implies a = b)$.
- R is called *asymmetric* in A if $\forall a, b \in A, \neg(aRb \wedge bRa)$.
- R is called a *(partial) ordering* of A if it is reflexive, antisymmetric, and transitive in A .
- R is called a *strict ordering* of A if it is asymmetric and transitive in A .
- If R is a partial ordering of A , then the pair (A, R) is called an *ordered set*.

Example 2.5.2

- Define the relation \subseteq_A in A as follows: $x \subseteq_A y$ if and only if $x, y \in A \wedge x \subseteq y$. Then, (A, \subseteq_A) is an ordered set.
- The relation Id_A is a partial ordering of A .

Theorem 2.5.3

- (i) Let R be a partial ordering of A . Then the relation S in A defined by

$$S \triangleq R \setminus \text{Id}_A$$

is a strict ordering.

- (ii) Let S be a strict ordering of A . Then the relation R in A defined by

$$R \triangleq S \cup \text{Id}_A$$

is a partial ordering.

Proof.

- (i) Suppose aSb and bSa . Since $S \subseteq R$, we have aRb and bRa . As R is antisymmetric, we have aRa , which is impossible since $S \cap \text{Id}_S = \emptyset$. Hence, S is asymmetric in A .
Now, assuming aSb and bSc , we also have aRc since R is transitive. Moreover, a cannot be equal to c since S is shown to be asymmetric. Therefore, aSc ; S is transitive in A .
- (ii) Assume aRb and bRa . If $a \neq b$, then we have aSb and bSa , which is impossible. Therefore, $a = b$; R is antisymmetric.
Assume aRb and bRc . If $a = b$ or $b = c$, then we immediately have aRc . If $a \neq b$ and $b \neq c$, then aSb and bSc , and thus aSc as S is transitive in A ; R is transitive in A).
 R is reflexive in A since $\text{Id}_A \subseteq R$. □

Notation 2.5.4

- If R is a partial ordering, we say $S = R \setminus \text{Id}_A$ corresponds to the partial ordering R .
- If S is a strict ordering, we say $R = S \cup \text{Id}_A$ corresponds to the strict ordering S .

Definition 2.5.5: Comparability

Let $a, b \in A$ and let \leq be a partial ordering of A .

- We say that a and b are *comparable* in the ordering \leq if $a \leq b$ or $b \leq a$.
 - We say that a and b are *incomparable* in the ordering \leq if neither $a \leq b$ nor $b \leq a$.
- They can be stated equivalently in terms of the corresponding strict ordering $<$.
- We say that a and b are *comparable* in the ordering $<$ if $a = b$ or $a < b$ or $b < a$.
 - We say that a and b are *incomparable* in the ordering $<$ if none of $a = b$, $a < b$, and $b < a$ holds.

Definition 2.5.6: Total Ordering

An ordering \leq (or $<$) is called *linear* or *total* if any two elements of A are comparable. The pair (A, \leq) is then called a *totally ordered set*.

Definition 2.5.7: Chain

Let (A, \leq) be an ordered set and $B \subseteq A$. B is a *chain* in A if any two elements of B are comparable.

Definition 2.5.8: Least/Minimal/Greatest/Maximal Element

Let (A, \leq) be an ordered set and $B \subseteq A$.

- $b \in B$ is the *least element* of B in the ordering \leq if $\forall x \in B, b \leq x$.
- $b \in B$ is a *minimal element* of B in the ordering \leq if $\forall x \in B, (x \leq b \implies x = b)$.
- $b \in B$ is the *greatest element* of B in the ordering \leq if $\forall x \in B, x \leq b$.
- $b \in B$ is a *maximal element* of B in the ordering \leq if $\forall x \in B, (b \leq x \implies x = b)$.

Notation 2.5.9

Let (A, \leq) be an ordered set and $B \subseteq A$.

- The least element of B is denoted $\min B$.
- The greatest element of B is denoted $\max B$.

Theorem 2.5.10

Let (A, \leq) be an ordered set and $B \subseteq A$.

- B has at most one least element.
- The least element of B —if it exists—is also minimal.
- If B is a chain, then every minimal element of B is also least.

Proof.

- If b and b' are least elements of B , then $b \leq b'$ and $b' \leq b$ by the definition. As \leq is antisymmetric, we have $b = b'$.
- Let b be the least element of B (assuming its existence). Take any $x \in B$ and assume $x \leq b$. Then, as b is the least, we have $b \leq x$. As \leq is antisymmetric, $x = b$; b is minimal.
- Let b be a minimal element of B . Take any $x \in B$. Since b and x are comparable, it is $x \leq b$ or $b \leq x$. If $x \leq b$, then $x = b$ as b is minimal. Therefore, b is the least. \square

Note:-

Theorem 2.5.10 still holds when ‘least’ and ‘minimal’ are replaced by ‘greatest’ and ‘maximal’, respectively.

Definition 2.5.11: Lower/Upper Bound and Infimum/Supremum

Let (A, \leq) be an ordered set and $B \subseteq A$.

- $a \in A$ is a *lower bound* of B in the ordered set (A, \leq) if $\forall x \in B, a \leq x$.
- $a \in A$ is called an *infimum* (or *greatest lower bound*) of B in the ordered set (A, \leq) if $a = \max\{x \in A \mid x \text{ is a lower bound of } B\}$.
- $a \in A$ is an *upper bound* of B in the ordered set (A, \leq) if $\forall x \in B, x \leq a$.
- $a \in A$ is called an *supremum* (or *least upper bound*) of B in the ordered set (A, \leq) if $a = \min\{x \in A \mid x \text{ is an upper bound of } B\}$.

Notation 2.5.12

Let (A, \leq) be an ordered set and $B \subseteq A$.

- The infimum of B is denoted $\inf B$.
- The supremum of B is denoted $\sup B$.

Theorem 2.5.13

Let (A, \leq) be an ordered set and $B \subseteq A$.

- (i) B has at most one infimum.
- (ii) If b is the least element of B , then b is the infimum of B .
- (iii) If $b \in B$ is the infimum of B , then b is the least element of B .

Proof.

- (i) The result follows from the definition and Theorem 2.5.10 (i).
- (ii) b is a lower bound of B . If x is a lower bound of B , since $b \in B$, we must have $x \leq b$. Therefore, b is the greatest lower bound.
- (iii) $b \in B$ is a lower bound of B , and thus b is the least element. □

Note:-

Theorem 2.5.13 still holds when ‘least’ and ‘infimum’ are replaced by ‘greatest’ and ‘supremum’, respectively.

Definition 2.5.14: Isomorphism Between Ordered Sets

An *isomorphism* between two ordered sets (P, \leq) and (Q, \preceq) is a function $f : P \hookrightarrow Q$ such that

$$\forall p_1, p_2 \in P, (p_1 \leq p_2 \iff f(p_1) \preceq f(p_2)).$$

If an isomorphism exists between (P, \leq) and (Q, \preceq) , then we say (P, \leq) and (Q, \preceq) are *isomorphic*. This is justified by Exercise 2.5.13.

Lemma 2.5.15

Let (P, \leq) be a totally ordered set and let (Q, \preceq) be an ordered set. Let $h : P \hookrightarrow Q$ be a function such that

$$\forall p_1, p_2 \in P, (p_1 \leq p_2 \implies h(p_1) \preceq h(p_2)).$$

Then, h is an isomorphism between (P, \leq) and (Q, \preceq) , and (Q, \preceq) is totally ordered.

Proof. Take any $p_1, p_2 \in P$ and assume $h(p_1) \preceq h(p_2)$. Suppose $p_2 < p_1$ for the sake of contradiction. Then, since h is injective, $h(p_1) \neq h(p_2)$, and thus $h(p_1) \prec h(p_2)$. Then, we have $\neg(p_2 \leq p_1)$, which is a contradiction. Hence, $\neg(p_2 < p_1)$. Therefore, $p_1 \leq p_2$ since (P, \leq) is totally ordered.

Take any $q_1, q_2 \in Q$. Then, since h is onto Q , there exist $p_1, p_2 \in P$ such that $q_1 = h(p_1)$ and $q_2 = h(p_2)$. Since P is totally ordered, it is $p_1 \leq p_2$ or $p_2 \leq p_1$. In either case, we have $q_1 \preceq q_2$ or $p_2 \preceq q_1$. Therefore, (Q, \preceq) is totally ordered. □

Selected Problems

Exercise 2.5.1

- (i) Let R be a partial ordering of A and let S be the strict ordering of A corresponding to R . Let R^* be the partial ordering of A corresponding to S . Show that $R^* = R$.
- (ii) Let S be a strict ordering of A and let R be the partial ordering of A corresponding to S . Let S^* be the partial ordering of A corresponding to R . Show that $S^* = S$.

Proof.

- (i) $R^* = S \cup \text{Id}_A = (R \setminus \text{Id}_A) \cup \text{Id}_A = R$ since $\text{Id}_A \subseteq R$.

(ii) $S^* = R \setminus \text{Id}_A = (S \cup \text{Id}_A) \setminus \text{Id}_A = S$ since $\text{Id}_A \cap S = \emptyset$. □

Exercise 2.5.6

Let $(A_1, <_1)$ and $(A_2, <_2)$ be strictly ordered sets and let $A_1 \cap A_2 = \emptyset$. Define a relation $<$ on $B \triangleq A_1 \cup A_2$ as follows:

$$x < y \iff (x <_1 y) \vee (x <_2 y) \vee (x \in A_1 \wedge y \in A_2).$$

Show that $<$ is a strict ordering of B and $< \cap A_1^2 = <_1$, $< \cap A_2^2 = <_2$.

Proof. Note that $< = <_1 \cup <_2 \cup A_1 \times A_2$.

Suppose $x < y$ and $y < x$. By definition, $x, y \in A_1$ or $x, y \in A_2$. In both cases, we have $(x <_1 y \text{ and } y <_1 x)$ or $(x <_2 y \text{ and } y <_2 x)$, which are impossible as $<_1$ and $<_2$ are asymmetric. Hence, $<$ is asymmetric. Transitivity of $<$ can be shown easily.

Since $<_1 \cap A_2^2 = <_2 \cap A_1^2 = (A_1 \times A_2) \cap A_1^2 = (A_1 \times A_2) \cap A_2^2 = \emptyset$, we get $< \cap A_1^2 = <_1$ and $< \cap A_2^2 = <_2$. □

Exercise 2.5.7

Let R be a reflexive and transitive relation in A (R is called a *preordering* of A). Define a relation E in A by

$$aEb \iff aRb \wedge bRa.$$

Show that E is an equivalence on A . Define the relation R/E in A/E by

$$[a]_E R/E [b]_E \iff aRb.$$

Show that R/E is well-defined and that R/E is a partial ordering of A/E .

Proof. Since $aEa \equiv aRa$ and R is reflexive, E is reflexive as well. Since $aEb \equiv bEa$, E is symmetric. Since $aEb \wedge bEc \iff (aRb \wedge bRc) \wedge (cRb \wedge bRa) \implies aRc \wedge cRa \iff aEc$, E is transitive. ✓

Assume $[a]_E = [a']_E$ and $[b]_E = [b']_E$. Then, we have aEa' and bEb' by Lemma 2.4.3, i.e., aRa' , $a'Ra$, bRb' , and $b'Rb$. By transitivity of R , it follows that $aRb \iff a'Rb'$. Therefore, R/E is well-defined. ✓

It can be shown readily that R/E is reflexive and transitive. To prove R/E is anti-symmetric, assume $[a]_E R/E [b]_E$ and $[b]_E R/E [a]_E$. Then, aRb and bRa , which means aEb . Therefore, $[a]_E = [b]_E$ by Lemma 2.4.3. ✓ □

Exercise 2.5.8

Let $A = \mathcal{P}(X)$ where X is a set.

(i) Any $S \subseteq A$ has a supremum in the ordering \subseteq_A ; $\sup S = \bigcup S$.

(ii) Any $S \subseteq A$ has an infimum in the ordering \subseteq_A ; $\inf S = \begin{cases} \bigcap S & \text{if } S \neq \emptyset \\ X & \text{if } S = \emptyset \end{cases}$.

Proof.

(i) As $C \subseteq_A \bigcup S$ for all $C \in S$, $\bigcup S$ is an upper bound of S . Let U be any upper bound of S . Take any $x \in \bigcup S$. Then, there exists $C \in S$ such that $x \in C$. Since $C \subseteq_A U$, we have $x \in U$. Therefore, $\bigcup S \subseteq U$; $\bigcup S$ is the least upper bound of S .

- (ii) If $S = \emptyset$, then any $C \in A$ is a lower bound of S . Since $\bigcup A = X$ —by (i), the supremum of the set of lower bounds of S —is a lower bound of S , X is the infimum of $S = \emptyset$. ✓
 If $S \neq \emptyset$, as $\bigcap S \subseteq C$ for all $C \in S$, $\bigcap S$ is a lower bound of S . Let L be any lower bound of S . Take any $x \in L$. Then, $\forall C \in S, x \in C$, i.e., $x \in \bigcap S$. Therefore, $L \subseteq_A \bigcap S$; $\bigcap S$ is the infimum of S . ✓ \square

Exercise 2.5.9

Let $\text{Fn}(X, Y)$ be the set of all functions mapping a subset of X into Y , i.e., $\text{Fn}(X, Y) = \bigcup_{Z \in \mathcal{P}(X)} Y^Z$. Define a relation \leq in $\text{Fn}(X, Y)$ by

$$f \leq g \iff f \subseteq g.$$

- (i) \leq is a partial ordering of $\text{Fn}(X, Y)$.
 (ii) Let $F \subseteq \text{Fn}(X, Y)$. $\sup F$ exists if and only if F is a compatible system of functions. Moreover, $\sup F = \bigcup F$ if it exists.

Proof.

- (i) $\leq = \subseteq_{\text{Fn}(X, Y)}$ by definition; $\subseteq_{\text{Fn}(X, Y)}$ is already a partial ordering of $\text{Fn}(X, Y)$.
 (ii) (\Rightarrow) Assume $h \in \text{Fn}(X, Y)$ is a supremum of F . Then, $\forall f \in F, f \subseteq h$. Take any $f, g \in F$. Then, $f \cup g \subseteq h$, and thus $f \cup g$ is a function as h is a function. Therefore, by Lemma 2.3.11, f and g are compatible. Hence, F is a compatible system of functions.
 (\Leftarrow) Assume F is a compatible system of functions. Then, $\bigcup F \in \text{Fn}(X, Y)$ by Theorem 2.3.12, and $f \leq \bigcup F$ for all $f \in F$ by definition; $\bigcup F$ is an upper bound of F . Let U be any upper bound of S . Take any $(x, y) \in \bigcup F$. Then, there exists $f \in S$ such that $(x, y) \in f$. Since $f \subseteq_A U$, we have $x \in U$. Therefore, $\bigcup F \subseteq U$; $\bigcup F$ is the least upper bound of S . \square

Exercise 2.5.10

Let $\text{Pt}(A)$ be the set of all partitions of A . Define a relation \preceq in $\text{Pt}(A)$ by

$$S_1 \preceq S_2 \iff \forall C \in S_1, \exists D \in S_2, C \subseteq D.$$

(We say that the partition S_1 is a *refinement* of the partition S_2 if $S_1 \preceq S_2$.)

- (i) \preceq is a partial ordering of $\text{Pt}(A)$.
 (ii) $\inf T$ exists for all $T \subseteq \text{Pt}(A)$.
 (iii) $\sup T$ exists for all $T \subseteq \text{Pt}(A)$.

Proof.

- (i) \preceq is reflexive since, for all $S \in \text{Pt}(A)$ and $C \in S, C \subseteq C$, i.e., $S \preceq S$. ✓
 Assume $S_1 \preceq S_2$ and $S_2 \preceq S_1$. Take any $C \in S_1$. Then, there exists $D \in S_2$ such that $C \subseteq D$. In addition, there exists $E \in S_1$ such that $D \subseteq E$. We have $C \subseteq E$ but C is nonempty as S_1 is a partition, which implies $C \cap E \neq \emptyset$. Therefore, as S_1 is a partition, we must have $C = E$ and thus $C = D$. Hence, $S_1 \subseteq S_2$. This shows that \preceq is antisymmetric. ✓
 Assume $S_1 \preceq S_2$ and $S_2 \preceq S_3$. Take any $C \in S_1$. There exists $D \in S_2$ such that $C \subseteq D$. There exists $E \in S_3$ such that $D \subseteq E$. Hence, $C \subseteq E$; $S_1 \preceq S_3$. This shows that \preceq is transitive. ✓
 (ii) Define a relation E in A by $E \triangleq \{(a, b) \in A^2 \mid \forall S \in T, \exists C \in S, a \in C \wedge b \in C\}$. It can be easily shown that E is an equivalence mimicking the proof of Theorem 2.4.8. Then, $A/E \in \text{Pt}(A)$ by Theorem 2.4.6.

Claim 1. A/E is a lower bound of T .

Proof. If $T = \emptyset$, there is nothing to prove; so assume $T \neq \emptyset$. Take any $S \in T$ and $a \in A$. Then, there exists $C \in S$ such that $a \in S$ since S is a partition of A . Let $b \in [a]_E$. Then, there exists $D \in S$ such that $a, b \in D$, which implies $C = D$. Therefore, $[a]_E \subseteq C$. Hence, $A/E \preceq S$. \square

Claim 2. For each lower bound L of T , $L \preceq A/E$.

Proof. If $T = \emptyset$, then $A/E = \{A^2\}$ and every partition of A is a lower bound. Since $S \preceq \{A^2\}$ for all $S \in \text{Pt}(A)$, the result follows.

Now, assume $T \neq \emptyset$. Let L be a lower bound of T . Take any $D \in L$. Fix some $a \in D$. Then, each $d \in D$ has the property that $\forall S \in T, \exists C \in S, \{a, d\} \subseteq D \subseteq C$ as L is a lower bound of T . Therefore, $d \in [a]_E; D \subseteq [a]_E$. Hence, $L \preceq A/E$. \square

Claims 1 and 2 say that $\inf T = A/E$. Hence, $\inf T$ exists.

(iii) Let $T' \triangleq \{S' \in \text{Pt}(A) \mid \forall S \in T, S \preceq S'\}$. By (ii), $S^* \triangleq \inf T'$ exists.

Claim 3. S^* is an upper bound of T .

Proof. In (ii), we showed that $S^* = A/E$ where $E = \{(a, b) \in A^2 \mid \forall S' \in T', \exists C' \in S', a \in C' \wedge b \in C'\}$. Take any $S \in T$ and let $C \in S$. Fix some $c_0 \in C$.

Now, take arbitrary $c \in C$. Then, for all $S' \in T'$, since $S \preceq S'$, there exists $D' \in S'$ such that $c \in C \subseteq D'$. Hence, we have $cEc_0; C \subseteq [c_0]_E$. Therefore, $S \preceq S^*$. \square

Claim 3 essentially says that $S^* \in T'$. By Theorem 2.5.13 (iii), $S^* = \min T'$, i.e., $S^* = \sup T$. \square

Exercise 2.5.13

If h is isomorphism between (P, \leq) and (Q, \preceq) , then h^{-1} is an isomorphism between (Q, \preceq) and (P, \leq) .

Proof. Take any $q_1, q_2 \in Q$. Then, we have $q_1 \preceq q_2 \iff h(h^{-1}(q_1)) \preceq h(h^{-1}(q_2)) \iff h^{-1}(q_1) \leq h^{-1}(q_2)$. \square

Exercise 2.5.14

If f is an isomorphism between (P_1, \leq_1) and (P_2, \leq_2) , and if g is an isomorphism between (P_2, \leq_2) and P_3, \leq_3 , then $g \circ f$ is an isomorphism between (P_1, \leq_1) and (P_3, \leq_3) .

Proof. $\text{ran}(g \circ f) = g[\text{ran } f] = P_3$. Moreover, $g \circ f$ is one-to-one. Hence, $g \circ f : P_1 \hookrightarrow P_3$. For all $p, q \in P_1$, we have $p \leq_1 q \iff f(p) \leq_2 f(q) \iff g(f(p)) \leq_3 g(f(q))$. Hence, $g \circ f$ is an isomorphism between (P_1, \leq_1) and (P_3, \leq_3) . \square

Chapter 3

Natural Numbers

3.1 Introduction to Natural Numbers

Note:-

We cannot prove an existence of an ‘infinite’ set (in the classical sense) or discuss infinity only from Axioms I to VI.

Definition 3.1.1: Successor

The *successor* of a set x is the set $S(x) = x \cup \{x\}$.

Notation 3.1.2: $n + 1$

We write $n + 1$ to denote $S(n)$. There is no implication regarding the classic “addition” in this notation.

Notation 3.1.3: Natural Numbers

- $0 = \emptyset$
- $1 = \{\emptyset\} = S(0) = 0 + 1$
- $2 = \{\emptyset, \{\emptyset\}\} = S(1) = 1 + 1$
- ...

Definition 3.1.4: Inductive Set

A set I is called *inductive* if

$$0 \in I \wedge \forall n \in I, (n + 1) \in I.$$

Axiom VII Axiom of Infinity

An inductive set exists.

Definition 3.1.5: Set of All Natural Numbers

The *set of all natural numbers* is the set

$$\mathbb{N} \triangleq \{x \mid x \in I \text{ for all inductive set } I\}.$$

Note:-

Axiom of Infinity guarantees the existence of \mathbb{N} . For, if A is any inductive set, then $\mathbb{N} = \{x \in A \mid x \in I \text{ for all inductive set } I\}$.

Lemma 3.1.6

\mathbb{N} is inductive. In addition, if I is an inductive set, then $\mathbb{N} \subseteq I$.

Proof. Since $0 \in I$ for all inductive set, $0 \in \mathbb{N}$. If $n \in \mathbb{N}$, then $n \in I$ for all inductive set, and thus $(n+1) \in I$ for all inductive set. Therefore, $(n+1) \in \mathbb{N}$. Hence, \mathbb{N} is inductive.

$\mathbb{N} \subseteq I$ directly follows from the definition of \mathbb{N} . □

Definition 3.1.7

The relation $<$ on \mathbb{N} is defined by: $m < n$ if and only if $m \in n$.

Notation 3.1.8

Although we did not prove $<$ is a strict ordering of \mathbb{N} , we shall use \leq to denote the relation on \mathbb{N} :

$$\leq \triangleq < \cup \text{Id}_{\mathbb{N}}$$

Selected Problems**Exercise 3.1.1**

- (i) $\forall x, x \subseteq S(x)$
- (ii) $\forall x, \neg(\exists z, x \subsetneq z \subsetneq S(x))$

Proof.

- (i) $x \subseteq x \subseteq x \cup \{x\} = S(x)$
- (ii) Take any z such that $x \subseteq z \subseteq S(x) = x \cup \{x\}$. If $z \subseteq x$, then we have $z = x$. If $z \not\subseteq x$, then there exists y such that $y \in z$ and $y \notin x$. However, $y \in x \cup \{x\}$, and thus $y = x$. Therefore, $S(x) \subseteq z$; $z = S(x)$. In conclusion, any z such that $x \subseteq z \subseteq S(x)$ must satisfy $z = x$ or $z = S(x)$. □

3.2 Properties of Natural Numbers**Theorem 3.2.1 The Induction Principle**

Let $P(x)$ be a property (possibly with parameters).

$$P(0) \wedge \forall n \in \mathbb{N}, (P(n) \implies P(n+1)) \implies \forall n \in \mathbb{N}, P(n)$$

Proof. The premise simply says that $A = \{n \in \mathbb{N} \mid P(n)\}$ is inductive. Therefore, $\mathbb{N} \subseteq A$ follows. □

Lemma 3.2.2

- (i) $\forall n \in \mathbb{N}, 0 \leq n$
- (ii) $\forall k, n \in \mathbb{N}, (k < n+1 \iff k < n \vee k = n)$

Proof.

(i) Let $P(x)$ be the property " $0 \leq x$." $P(0)$, i.e., $0 \leq 0$, holds since $0 = 0$.

Now, assume $n \in \mathbb{N}$ and $P(n)$. If $n = 0$, then we have $0 \in S(0) = n + 1$ by definition (Definition 3.1.1). If $0 < n$, then $0 \in n$, and thus $0 \in n \cup \{n\} = S(n)$. Therefore, by **The Induction Principle**, the result follows.

(ii) Note that $k \in n \cup \{n\}$ if and only if $k \in n$ or $k = n$. □

Theorem 3.2.3 (\mathbb{N}, \leq) is Totally Ordered

(\mathbb{N}, \leq) is a totally ordered set.

Proof. We first need to prove that (\mathbb{N}, \leq) is an ordered set.

Claim 1. $<$ is transitive in \mathbb{N} .

Proof. Let $P(x)$ be the property " $\forall k, m \in \mathbb{N}, (k < m \wedge m < x \implies k < x)$." $P(0)$ is true because there is no $m \in \mathbb{N}$ such that $m \in 0 = \emptyset$.

Now assume $n \in \mathbb{N}$ and $P(n)$. Now, let $k, m \in \mathbb{N}$ and $k < m$ and $m < n + 1$. By **Lemma 3.2.2 (ii)**, $m < n$ or $m = n$.

- If $m < n$, then we have $k < n$ as $P(n)$ holds,
- If $m = n$, then we immediately have $k < n$.

In both cases, we have $k < n$; thus $k < n + 1$ by **Lemma 3.2.2 (ii)**. Therefore, the result follows from **The Induction Principle**. □

Claim 2. $<$ is asymmetric in \mathbb{N} .

Proof. Let $P(x)$ be the property " $\neg(x < x)$." $P(0)$ evidently holds since $\emptyset \notin \emptyset$.

Now, assume $n \in \mathbb{N}$ and $P(n)$. Suppose $(n + 1) < (n + 1)$ for the sake of contradiction. By **Lemma 3.2.2 (ii)**, we have $(n + 1) = n$ or $(n + 1) < n$. In both cases, we have $n < n$ by $n < (n + 1)$ (from **Lemma 3.2.2 (ii)**) and **Claim 1**, which contradicts $P(n)$. Therefore, $P(n + 1)$ holds. The result follows from **The Induction Principle**. □

Hence, (\mathbb{N}, \leq) is an ordered set by **Claims 1 and 2** and **Theorem 2.5.3**. We are left to prove that \leq is a total ordering of \mathbb{N} .

Claim 3. $\forall n, m \in \mathbb{N}, n < m \implies (n + 1) \leq m$

Proof. Let $P(x)$ be the property " $\forall n \in \mathbb{N}, (n < x \implies n + 1 \leq x)$." $P(0)$ holds since there is no $n \in \mathbb{N}$ such that $n < 0$.

Now, assume $m \in \mathbb{N}$ and $P(m)$. Take any $n \in \mathbb{N}$ such that $n < (m + 1)$. Then, by **Lemma 3.2.2**, we have $n = m$ or $n < m$. If $n = m$, then we have $(n + 1) = (m + 1)$, which implies $(n + 1) \leq (m + 1)$. If $n < m$, then $(n + 1) \leq m < (m + 1)$. Therefore, the result follows from **The Induction Principle**. □

Claim 4. $<$ is a total ordering of \mathbb{N} .

Proof. Let $P(x)$ be the property " $\forall m \in \mathbb{N}, m = x \vee m < x \vee x < m$." $P(0)$ is essentially **Lemma 3.2.2 (i)**.

Assume $n \in \mathbb{N}$ and $P(n)$. Take any $m \in \mathbb{N}$. If $m < n$ or $m = n$, we have $m < (n + 1)$ by **Lemma 3.2.2 (ii)**. If $n < m$, by **Claim 3**, we have $(n + 1) \leq m$. Hence, $P(n + 1)$ holds. Therefore, the result follows from **The Induction Principle**. □

□

Notation 3.2.4

We may write “ $\forall k < n, P(k)$ ” instead of “ $\forall k \in \mathbb{N}, (k < n \implies P(k))$ ” or “ $\exists k < n, P(k)$ ” instead of “ $\exists k \in \mathbb{N}, k < n \wedge P(k)$ ” when no confusion may arise. We may similarly write $(\forall/\exists)k(\leq/>/\geq)n, P(k)$.

Theorem 3.2.5 The Strong Induction Principle

Let $P(x)$ be a property (possibly with parameters). If, for all $n \in \mathbb{N}$, $P(k)$ holds for all $k < n$, then $P(n)$ holds for all $n \in \mathbb{N}$.

$$\forall n \in \mathbb{N}, [\forall k < n, \implies P(k) \implies P(n)] \implies \forall n \in \mathbb{N}, P(n)$$

Proof. Assume the premise $(\forall n \in \mathbb{N}, [\forall k < n, \implies P(k) \implies P(n)])$. Let $Q(n)$ be the property “ $\forall k < n, P(k)$.” $Q(0)$ holds since there is no $k < 0$.

Now, assume $n \in \mathbb{N}$ and $Q(n)$. Then, by the premise, we have $P(n)$. Lemma 3.2.2 (ii) enables us to say that $\forall k \in \mathbb{N}, (k < n + 1 \implies P(k))$. Therefore, $\forall n \in \mathbb{N}$, $Q(n)$ holds by The Induction Principle.

Take any $k \in \mathbb{N}$. Then, we have $k < k + 1$ and thus $P(k)$ holds by $Q(k + 1)$. \square

Definition 3.2.6: Well-Ordering

A total ordering \preceq of a set A is a *well-ordering* if every nonempty subset of A has a least element. Then, the ordered set (A, \preceq) is called a *well-ordered set*.

Theorem 3.2.7 (\mathbb{N}, \leq) is Well-Ordered

(\mathbb{N}, \leq) is a well-ordered set.

Proof. Let $X \subseteq \mathbb{N}$ has no least element. For each $n \in \mathbb{N}$, if $\forall k < n, k \in \mathbb{N} \setminus X$, we must have $n \in \mathbb{N} \setminus X$ since otherwise $n = \min X$. Then, by The Strong Induction Principle, $\forall n \in \mathbb{N}, n \in \mathbb{N} \setminus X$, i.e., $X = \emptyset$. \square

Theorem 3.2.8

Let $\emptyset \subsetneq X \subseteq \mathbb{N}$. If X has an upper bound in the ordering \leq , then X has a greatest element.

Proof. Let $Y \triangleq \{k \in \mathbb{N} \mid k \text{ is an upper bound of } X\}$. The assumption says that $Y \neq \emptyset$. By (\mathbb{N}, \leq) is Well-Ordered, $n \triangleq \min Y = \sup X$ exists.

Suppose $n \notin X$ for the sake of contradiction. Then, $\forall m \in X, m < n$, which implies $n \neq 0$ as $X \neq \emptyset$. Therefore, $n = k + 1$ for some $k \in \mathbb{N}$ by Exercise 3.2.4; and thus $\forall m \in X, m \leq k$ by Lemma 3.2.2 (ii). Then, k is an upper bound of A and $k < n$, which is a contradiction to $n = \sup X$. Therefore, $n \in X$, and hence $n = \max X$ by Theorem 2.5.13. \square

Selected Problems

Exercise 3.2.2

$\forall m, n \in \mathbb{N}, (m < n \implies m + 1 < n + 1)$. Hence, $S: \mathbb{N} \rightarrow \mathbb{N}$ where $n \mapsto n + 1$ defines a one-to-one function on \mathbb{N} .

Proof. By Claim 3 in the proof of (\mathbb{N}, \leq) is Totally Ordered, we have $m+1 \leq n$. Together with $n < n+1$, we have $m+1 < n+1$.

Now, take any $m, n \in \mathbb{N}$ with $m \neq n$. Then, by (\mathbb{N}, \leq) is Totally Ordered, we have $m < n$ or $n < m$, i.e., $S(m) < S(n)$ or $S(n) < S(m)$. In both cases, $S(m) \neq S(n)$. Therefore, S is one-to-one. \square

Exercise 3.2.3

There exists $X \subsetneq \mathbb{N}$ and $f: \mathbb{N} \rightarrow X$ such that f is injective.

Proof. Let $S: \mathbb{N} \rightarrow \mathbb{N}$ where $n \mapsto n+1$. Then, S is injective by Exercise 3.2.2. Since there exists no $n \in \mathbb{N}$ such that $n \cup \{n\} = \emptyset$, $0 \notin \text{ran } S$; $\text{ran } S \subsetneq \mathbb{N}$. Therefore, $S: \mathbb{N} \rightarrow \text{ran } S$ is the function we are looking for. \square

Exercise 3.2.4

$\forall n \in \mathbb{N} \setminus \{0\}, \exists! k \in \mathbb{N}, n = k+1$

Proof. Let $P(x)$ be the property “ $x = 0 \vee \exists! k \in \mathbb{N}, x = k+1$.” $P(0)$ holds by definition.

Now, assume $P(n)$ where $n \in \mathbb{N}$. There exists $k \in \mathbb{N}$ such that $n+1 = k+1$, namely, $k = n$. If k' is another natural number such that $n+1 = k'+1$, then by Exercise 3.2.2, we have $k = k'$. Hence, $P(n+1)$ holds. The result follows from The Induction Principle. \square

Exercise 3.2.6

$\forall n \in \mathbb{N}, n = \{m \in \mathbb{N} \mid m < n\}$

Proof. Let $P(x)$ be the property “ $x = \{m \in \mathbb{N} \mid m < x\}$.” We have $P(0)$ since there exists no $m \in \mathbb{N}$ with $m < 0$.

Now, assume $P(n)$ where $n \in \mathbb{N}$. Then, $n+1 = \{m \in \mathbb{N} \mid m < n\} \cup \{n\}$. By Lemma 3.2.2 (ii), $m < n+1$ if and only if $m < n$ or $m = n$. Therefore, $\{m \in \mathbb{N} \mid m < n\} \cup \{n\} = \{m \in \mathbb{N} \mid m < n \vee m = n\} = \{m \in \mathbb{N} \mid m < n+1\}$; $P(n+1)$ holds. The result follows from The Induction Principle. \square

Exercise 3.2.8

There is no function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $\forall n \in \mathbb{N}, f(n+1) < f(n)$.

Proof. Let $P(x)$ be the property “there is no function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $f(0) = x$ and $\forall n \in \mathbb{N}, f(n+1) < f(n)$.”

For the sake of induction, assume $\forall k < n, P(k)$ where $n \in \mathbb{N}$. Suppose there exists $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $f(0) = n$ and $\forall k \in \mathbb{N}, f(k+1) < f(k)$. Now, define $g: \mathbb{N} \rightarrow \mathbb{N}$ by $g(k) = f(k+1)$. Then, $g(0) = f(1) < n$ and $\forall k \in \mathbb{N}, g(k+1) = f((k+1)+1) < f(k+1) = g(k)$. However, by $P(g(0))$, such g cannot exist; by contradiction, $P(n)$ holds. Hence, $\forall m \in \mathbb{N}, P(m)$ by The Strong Induction Principle.

Finally, suppose there exists $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $\forall n \in \mathbb{N}, f(n+1) < f(n)$. Then, by $P(f(0))$, such f may not exist. \square

Exercise 3.2.11

Let $P(x)$ be a property and let $k \in \mathbb{N}$.

$$P(k) \wedge \forall n \geq k, (P(n) \implies P(n+1)) \implies \forall n \geq k, P(n)$$

Proof. Let $Q(x)$ be the property “ $x < k \vee P(x)$.” If $k = 0$, then $P(0)$ holds. If $k > 0$, then $0 < k$ holds. Hence, in both cases, $Q(0)$ holds.

Now assume $Q(n)$ holds where $n \in \mathbb{N}$. Then, by (\mathbb{N}, \leq) is **Totally Ordered**, we have $n + 1 < k$, $n + 1 = k$, or $n + 1 > k$. If $n + 1 < k$ or $n + 1 = k$, we immediately have $Q(n + 1)$. If $n + 1 > k$, we have $n \geq k$ by **Lemma 3.2.2 (ii)**. Therefore, $P(n)$ holds, and thus $P(n + 1)$ holds by assumption. Hence, $Q(n + 1)$. By **The Induction Principle**, $\forall n \in \mathbb{N}, n < k \vee P(n)$. In other words, $\forall n \geq k, P(n)$. \square

Exercise 3.2.12 The Finite Induction Principle

Let $P(x)$ be a property and let $k \in \mathbb{N}$.

$$P(0) \wedge \forall n < k, (P(n) \implies P(n + 1)) \implies \forall n \leq k, P(n)$$

Proof. Let $Q(x)$ be the property “ $x > k \vee P(x)$.” $Q(0)$ holds as $P(0)$.

Now, assume $Q(n)$ holds where $n \in \mathbb{N}$. Then, by (\mathbb{N}, \leq) is **Totally Ordered**, we have $n + 1 \leq k$ or $n + 1 > k$. If $n + 1 > k$, then we immediately have $Q(n + 1)$. If $n + 1 \leq k$, by **Lemma 3.2.2**, $n + 1 < k + 1$. By **Exercise 3.2.2** and (\mathbb{N}, \leq) is **Totally Ordered**, we must have $n < k$. Hence, $P(n)$ holds, and therefore $P(n + 1)$ holds by the assumption. By **The Induction Principle**, $\forall n \in \mathbb{N}, n > k \vee P(n)$. In other words, $\forall n \leq k, P(n)$. \square

Exercise 3.2.13 The Double Induction Principle

Let $P(x, y)$ be a property.

$$\begin{aligned} \forall m, n \in \mathbb{N}, [\forall k, \ell \in \mathbb{N}, (k < m \vee k = m \wedge \ell < n \implies P(k, \ell)) \implies P(m, n)] & \quad [*] \\ \implies \forall m, n \in \mathbb{N}, P(m, n) \end{aligned}$$

Proof. Let $Q(x)$ be the property “ $\forall n \in \mathbb{N}, P(x, n)$.”

Now, assume $\forall k < m, Q(k)$ where $m \in \mathbb{N}$. For the sake of induction, assume again that $\forall \ell < n, P(m, \ell)$ where $n \in \mathbb{N}$. Now, we have $P(k, \ell)$ for all $k, \ell \in \mathbb{N}$ such that $k < m$ or $k = m$ and $\ell < n$. Hence, by $[*]$, $P(m, n)$. By **The Strong Induction Principle**, we have $\forall n \in \mathbb{N}, P(m, n)$. In other words, $Q(m)$. Again by **The Strong Induction Principle**, we have $\forall m \in \mathbb{N}, Q(m)$, that is to say $\forall m, n \in \mathbb{N}, P(m, n)$. \square

3.3 The Recursion Theorem

Definition 3.3.1: Sequence

- A *sequence* is a function whose domain is a natural number or \mathbb{N} .
- A sequence whose domain is a natural number n is called a *finite sequence of length n* and is denoted

$$\langle a_i \mid i < n \rangle \quad \text{or} \quad \langle a_i \mid i = 0, 1, \dots, n-1 \rangle \quad \text{or} \quad \langle a_0, a_1, \dots, a_{n-1} \rangle.$$

In particular, $\langle \rangle = \emptyset$ —the *empty sequence*—is the unique sequence of length 0.

$$\text{Seq}(A) \triangleq \bigcup_{n \in \mathbb{N}} A^n$$

denote the set of all finite sequence of elements of A .

- A sequence whose domain is \mathbb{N} is called a *infinite sequence* and is denoted

$$\langle a_i \mid i \in \mathbb{N} \rangle \quad \text{or} \quad \langle a_i \mid i = 0, 1, 2, \dots \rangle \quad \text{or} \quad \langle a_i \rangle_{i=0}^{\infty}.$$

Infinite sequences of elements of A are members of $A^{\mathbb{N}}$. We also use the notation $\{a_i \mid i \in \mathbb{N}\}$ or $\{a_i\}_{i=0}^{\infty}$, etc., for the range of the sequence $\langle a_i \mid i \in \mathbb{N} \rangle$.

Note:-

- A natural number $n \in \mathbb{N}$ is the set of all natural numbers less than n . See Exercise 3.2.6.
- Since $A^n \in \mathcal{P}(\mathbb{N} \times A)$ for each $n \in \mathbb{N}$, $\mathcal{A} = \{w \mid \exists n \in \mathbb{N}, w = A^n\}$ exists, and thus $\text{Seq}(A) = \bigcup \mathcal{A}$ exists.

Theorem 3.3.2 The Recursion Theorem

Let A be a set, $a \in A$, and $g : A \times \mathbb{N} \rightarrow A$. Then, there uniquely exists an infinite sequence $f : \mathbb{N} \rightarrow A$ such that

- $f_0 = a$ and
- $\forall n \in \mathbb{N}, f_{n+1} = g(f_n, n)$.

Proof. We say $t : (m+1) \rightarrow A$ is an *m -step computation based on a and g* if $t_0 = a$ and $\forall k < m, t_{k+1} = g(t_k, k)$. Let $F \triangleq \{t \in \text{Seq}(A) \mid t \text{ is an } m \text{ step computation for some } m \in \mathbb{N}\}$. Let $f \triangleq \bigcup F$.

Claim 1. f is a function.

Proof. We shall show that F is a compatible system of functions so we may conclude f is a function thanks to Theorem 2.3.12. Take any $t, u \in F$. Let $n = \text{dom } t \in \mathbb{N}$ and $m = \text{dom } u \in \mathbb{N}$. WLOG, $n \leq m$ (thanks to (\mathbb{N}, \leq) is **Totally Ordered**), i.e., $n \subseteq m$. Hence, $(\text{dom } t) \cap (\text{dom } u) = n$. If $n = 0$, then it is done; assume $n > 0$. Then, there exists $n' \in \mathbb{N}$ such that $n' + 1 = n$ by Exercise 3.2.4.

Surely, $t_0 = a = u_0$. Moreover, if $t_k = u_k$ where $k < n'$, then $k+1 < n'+1 = n$ (Exercise 3.2.2) and $t_{k+1} = g(t_k, k) = g(u_k, k) = u_{k+1}$. Therefore, by **The Finite Induction Principle**, we have $\forall k \leq n', t_k = u_k$; t and u are compatible. \square

Claim 2. $\text{dom } f = \mathbb{N}$ and $\text{ran } f \subseteq A$.

Proof. We already have $\text{dom } f \subseteq \mathbb{N}$ and $\text{ran } f \subseteq A$ by Theorem 2.3.12. To show $\text{dom } f = \mathbb{N}$, it suffices to show that, for any $n \in \mathbb{N}$, there is an n -step computation based on a and g . Clearly, $t = \{(0, a)\}$ is a 0-step computation.

Assume there exists an n -step computation $t: (n+1) \rightarrow A$ where $n \in \mathbb{N}$. Then, define $u: ((n+1)+1) \rightarrow A$ by $u \triangleq t \cup \{(n+1, g(t_n, n))\}$. Then, one may easily verify that u is an $(n+1)$ -step computation. Therefore, by The Induction Principle, the result follows. \square

We now check if f satisfies the conditions (i) and (ii).

(i) Clearly, $f_0 = a$.

(ii) Take any $n \in \mathbb{N}$. Let t be an $(n+1)$ -step computation. Then, $\forall k \leq n, f_k = t_k$, and $f_{n+1} = t_{n+1} = g(t_n, n) = g(f_n, n)$.

Now, we are left to show the uniqueness of such f .

Let $h: \mathbb{N} \rightarrow A$ be a sequence that satisfies the conditions (i) and (ii). Clearly, $f_0 = a = h_0$. And, if $f_n = h_n$, then $f_{n+1} = g(f_n, n) = g(h_n, n) = h_{n+1}$. Therefore, by The Induction Principle, $\forall k \in \mathbb{N}, f_k = h_k$, i.e., $f = h$ by Lemma 2.3.3. \square

Theorem 3.3.3

Let (A, \preceq) be a nonempty linearly ordered set with the properties:

- (i) For every $p \in A$, there exists $q \in A$ such that $p \prec q$.
 - (ii) Every nonempty subset of A that has a \preceq -least element.
 - (iii) Every nonempty subset of A that has an upper bound has a \preceq -greatest element.
- Then, (A, \preceq) is isomorphic to (\mathbb{N}, \leq) .

Proof. By (i), $\{a \in A \mid x \prec a\} \neq \emptyset$ for each $x \in A$ and it has a \preceq -least element. Hence, we may define $g: A \times \mathbb{N} \rightarrow A$ by $g(x, n) \triangleq \min\{a \in A \mid x \prec a\}$. Then, The Recursion Theorem guarantees the existence of a function $f: \mathbb{N} \rightarrow A$ such that:

- $f_0 = \min A$ \triangleright (i) and $A \neq \emptyset$
- $\forall n \in \mathbb{N}, f_{n+1} = g(f_n, n) = \min\{a \in A \mid f_n \prec a\}$.

By Exercise 3.3.1, we have $f_m \prec f_n$ whenever $m < n$. This also implies that f is injective.

Claim 1. $\text{ran } f = A$

Proof. Suppose $\text{ran } f \subsetneq A$ for the sake of contradiction. Then, $A \setminus \text{ran } f \neq \emptyset$, and thus we may take $p = \min(A \setminus \text{ran } f)$, which gives $p \neq f_0$ immediately. Hence, $B = \{a \in A \mid a \prec p\} \neq \emptyset$ and p is an upper bound of B . By (iii), $q = \max B$ exists. Since $q \prec p$, we have $q \in \text{ran } f$, i.e., $q = f_m$ for some $m \in \mathbb{N}$.

Suppose there is some $r \in A$ such that $q \prec r \prec p$. Then, $r \in B$, which contradicts the maximality of q . Hence, $p = \min\{a \in A \mid f_m \prec a\} = f_{m+1}$, which contradicts $p \notin \text{ran } f$. \square

We have $f: \mathbb{N} \hookrightarrow A$ by Claim 1. Hence, by (\mathbb{N}, \leq) is Totally Ordered and Lemma 2.5.15, f is an isomorphism between (\mathbb{N}, \leq) and (A, \preceq) . \square

Theorem 3.3.4 The Recursion Theorem: General Version

Let S be a set and let $g: \text{Seq}(S) \rightarrow S$. Then, there exists a unique sequence $f: \mathbb{N} \rightarrow S$ such that

$$\forall n \in \mathbb{N}, f_n = g(f|_n) = g(\langle f_0, f_1, \dots, f_{n-1} \rangle).$$

Proof. Define $G: \text{Seq}(S) \times \mathbb{N} \rightarrow \text{Seq}(S)$ by

$$G(t, n) = \begin{cases} t \cup \{(n, g(t))\} & \text{if } t \text{ is a sequence of length } n \\ \langle \rangle & \text{otherwise.} \end{cases}$$

Then, by **The Recursion Theorem**, there exists a sequence $F: \mathbb{N} \rightarrow \text{Seq}(S)$ such that:

- $F_0 = \langle \rangle$
- $\forall n \in \mathbb{N}, F_{n+1} = G(F_n, n)$.

If $F_k \in S^k$, then $F_{k+1} = F_k \cup \{(k, g(F_k))\} \in S^{k+1}$. Hence, by **The Induction Principle**, $\forall n \in \mathbb{N}, F_n \in S^n$. Moreover, since $F_k \subsetneq_{\text{Seq}(S)} F_{k+1}$, by Exercise 3.3.1, $\forall m, n \in \mathbb{N}, (m < n \implies F_m \subsetneq F_n)$; hence $\{F_n \mid n \in \mathbb{N}\}$ is a compatible system of functions.

Let $f \triangleq \bigcup_{n \in \mathbb{N}} F_n$. Then, we have $f|_n = F_n$ for all $n \in \mathbb{N}$. Therefore, for each $n \in \mathbb{N}$, $f_n = F_{n+1}(n) = g(F_n) = g(f|_n)$.

Let $h: \mathbb{N} \rightarrow S$ be another sequence such that $\forall n \in \mathbb{N}, h_n = g(h|_n)$. Suppose $\forall k < n, f_k = h_k$. Then, we have $f_n = g(f|_n) = g(h|_n) = h_n$. Therefore, by **The Strong Induction Principle**, $f = h$. \square

Theorem 3.3.5 The Recursion Theorem: Parametric Version

Let $a: P \rightarrow A$ and $g: P \times A \times \mathbb{N} \rightarrow A$ be functions. Then, there uniquely exists a function $f: P \times \mathbb{N} \rightarrow A$ such that

- (i) $\forall p \in P, f(p, 0) = a(p)$
- (ii) $\forall n \in \mathbb{N}, \forall p \in P, f(p, n+1) = g(p, f(p, n), n)$.

Proof. Let $G: A^P \times \mathbb{N} \rightarrow A^P$ be defined by

$$G(x, n)(p) = g(p, x(p), n)$$

for each $x \in A^P$, $p \in P$, and $n \in \mathbb{N}$. Then, by **The Recursion Theorem**, there exists $F: \mathbb{N} \rightarrow A^P$ such that

$$F_0 = a \quad \text{and} \quad \forall n \in \mathbb{N}, F_{n+1} = G(F_n, n).$$

Now, let $f: P \times \mathbb{N} \rightarrow A$ be defined by $f(p, n) = F_n(p)$. We now check if f satisfies the conditions:

- (i) For all $p \in P$, we have $f(p, 0) = F_0(p) = a(p)$.
- (ii) For each $n \in \mathbb{N}$ and $p \in P$, $f(p, n+1) = F_{n+1}(p) = G(F_n, n)(p) = g(p, F_n(p), n) = g(p, f(p, n), n)$.

Let $h: P \times \mathbb{N} \rightarrow A$ be another function that satisfies (i) and (ii). Clear, we have $\forall p \in P, f(p, 0) = a(p) = h(p, 0)$. Assuming $\forall p \in P, f(p, n) = h(p, n)$ gives, for all $p \in P$, $f(p, n+1) = g(p, f(p, n), n) = g(p, h(p, n), n) = h(p, n+1)$. Hence, by **The Induction Principle**, we get $f = h$. \square

Selected Problems

Exercise 3.3.1

Let $f: \mathbb{N} \rightarrow A$ be an infinite sequence where (A, \preceq) is an ordered set. Then,

$$\forall n \in \mathbb{N}, f_n \prec f_{n+1} \implies \forall m, n \in \mathbb{N}, (n < m \implies f_n \prec f_m).$$

Proof. Fix any $n \in \mathbb{N}$ and let $\mathbf{P}(x)$ be the property “ $f_n \prec f_x$.” $\mathbf{P}(n+1)$ evidently holds. Now, suppose $\mathbf{P}(k)$ holds where $k \in \mathbb{N}$. Then, chaining $f_n \prec f_k$ and $f_k \prec f_{k+1}$ gives $\mathbf{P}(k+1)$. Therefore, by Exercise 3.2.11, we get $\forall m \geq n+1, f_n \prec f_m$. \square

Exercise 3.3.2

Let (A, \preceq) be a nonempty linearly ordered set. We say that $q \in A$ is a *successor* of $p \in A$ if there is no $r \in A$ such that $p \prec r \prec q$. Assume (A, \preceq) has the following properties:

- (i) Every $p \in A$ has a successor.
 - (ii) Every nonempty subset of A has a \preceq -least element.
 - (iii) If $p \in A$ is not the \preceq -least element of A , then p is a successor of some $q \in A$.
- Then, (A, \preceq) is isomorphic to (\mathbb{N}, \leq) .

Proof. By (i), for each $p \in P$, $\{q \in A \mid p \prec q\} \neq \emptyset$, and thus it has a \preceq -least element by (ii). Therefore, by **The Recursion Theorem**, there exists a sequence $f : \mathbb{N} \rightarrow A$ such that $f_0 = \min A$ and $\forall n \in \mathbb{N}$, $f_{n+1} = \min\{q \in A \mid f_n \prec q\}$.

Claim 1. $\text{ran } f = A$

Proof. Suppose $X \triangleq A \setminus \text{ran } f \neq \emptyset$ for the sake of contradiction. Then, by (ii), we may take $p = \min X$. Since $\min A = f_0 \in \text{ran } f$, p is not the \preceq -least element of A . Hence, by (iii), p is a successor of some $q \in A$. As $q \prec p$, we have $q \in \text{ran } f$ by minimality of q , i.e., $q = f_m$ for some $m \in \mathbb{N}$. Since there is no $r \in A$ such that $q \prec r \prec p$, we have $p = f_{m+1}$ by definition, which contradicts $p \notin \text{ran } f$. \square

Since $f_n \prec f_{n+1}$ for all $n \in \mathbb{N}$, by Exercise 3.3.1, $\forall m, n \in \mathbb{N}$, $(m < n \implies f_m \prec f_n)$, which means f is injective.

Therefore, together with Claim 1, f is an isomorphism between (\mathbb{N}, \leq) and (A, \preceq) by Lemma 2.5.15. \square

Exercise 3.3.5 The Recursion Theorem: Partial Version

Let g be a function such that $\text{dom } g \subseteq A \times \mathbb{N}$ and $\text{ran } g \subseteq A$. Let $a \in A$. Then, there uniquely exists a sequence f of elements of A such that

- (i) $f_0 = a$
- (ii) $\forall n \in \mathbb{N}$, $[n + 1 \in \text{dom } f \implies f_{n+1} = g(f_n, n)]$
- (iii) f is either an infinite sequence or a finite sequence of length $k + 1$ and $(f_k, k) \notin \text{dom } g$.

Proof. Let $\bar{A} = A \cup \{\bar{a}\}$ where $\bar{a} \notin A$. (Such \bar{a} exists by Exercise 1.3.3 (ii).) Define $\bar{g} : \bar{A} \times \mathbb{N} \rightarrow \bar{A}$ by

$$\bar{g}(x, n) = \begin{cases} g(x, n) & \text{if } (x, n) \in \text{dom } g \\ \bar{a} & \text{otherwise.} \end{cases}$$

Then, **The Recursion Theorem** guarantees the existence of $\bar{f} : \mathbb{N} \rightarrow \bar{A}$ such that $\bar{f}_0 = a$ and $\forall n \in \mathbb{N}$, $\bar{f}_{n+1} = \bar{g}(\bar{f}_n, n)$. We have two cases: “ $\forall n \in \mathbb{N}$, $\bar{f}_n \neq \bar{a}$ ” and “ $\exists n \in \mathbb{N}$, $\bar{f}_n = \bar{a}$.” They are resolved by Claims 1 and 2, respectively.

Claim 1. If “ $\forall n \in \mathbb{N}$, $\bar{f}_n \neq \bar{a}$,” then \bar{f} is an infinite sequence of elements of A that satisfies (i) and (ii).

Proof. The assumption essentially says that $(\bar{f}_n, n) \in \text{dom } g$ and $\bar{f}_{n+1} = g(\bar{f}_n, n) \in A$ for all $n \in \mathbb{N}$, i.e., \bar{f} satisfies (i) and (ii). As $\bar{f}_0 = a \in A$, \bar{f} is an infinite sequence of elements of A . \square

Claim 2. If “ $\exists n \in \mathbb{N}, \bar{f}_n = \bar{a}$,” then there exists $k \in \mathbb{N}$ such that $\bar{f}|_{k+1}$ satisfies the conditions (i), (ii), and (iii).

Proof. By (\mathbb{N}, \leq) is Well-Ordered, we have $\ell \triangleq \min\{n \in \mathbb{N} \mid \bar{f}_n = \bar{a}\}$. Since $\bar{f}_0 \in A$, we have $\ell \neq 0$, and thus $\ell = k + 1$ for some $k \in \mathbb{N}$ by Exercise 3.2.4. It immediately follows that $\forall n \leq k, \bar{f}_n \in A$. Hence, $f \triangleq \bar{f}|_{k+1}$ is a finite sequence of length $k + 1$ of elements of A .

We check if f satisfies the conditions (i), (ii), and (iii):

- (i) $f_0 = \bar{f}_0 = a$
- (ii) If $n < k$, i.e., $n + 1 \in \text{dom } f = k + 1$, then $f_{n+1} = \bar{f}_{n+1} = \bar{g}(\bar{f}_n, n) = g(f_n, n)$.
- (iii) If $(f_k, k) \in \text{dom } g$, then we would have $\bar{f}_\ell = \bar{g}(\bar{f}_k, k) = \bar{g}(f_k, k) = g(f_k, k) \neq \bar{a}$. Hence, we must have $(f_k, k) \notin \text{dom } g$. \square

Now, we prove the uniqueness. Let f and h be two sequences of elements of A that satisfies the conditions (i), (ii), and (iii). WLOG, $\text{dom } h \subseteq \text{dom } f$.

Let $P(x)$ be the property “ $x \in \text{dom } h \wedge f_x = h_x$.” $P(0)$ evidently holds.

Claim 3. $\forall n \in \mathbb{N}, (n + 1 \in \text{dom } f \wedge P(n) \implies P(n + 1))$

Proof. Assume $n + 1 \in \text{dom } f$ and $P(n)$. Then, since $(h_n, n) = (f_n, n) \in \text{dom } g$, $n + 1 \in \text{dom } h$ and $h_{n+1} = g(h_n, n) = g(f_n, n) = f_{n+1}$. Hence, $P(n + 1)$ holds. \square

If f is a finite sequence, Claim 3 and The Finite Induction Principle imply $h = f$. If f is an infinite sequence, Claim 3 and The Induction Principle imply $h = f$. \square

Exercise 3.3.6

If $X \subseteq \mathbb{N}$, then there is a one-to-one (finite or infinite) sequence f such that $\text{ran } f = X$.

Proof. If $X = \emptyset$, $\langle \rangle$ is the one we are looking for. Assume $X \neq \emptyset$.

Let $g = \{((x, n), y) \in (X \times \mathbb{N}) \times X \mid y = \min\{k \in X \mid x < k\}\}$. Then, g is a function with $\text{dom } g \subseteq \mathbb{N} \times \mathbb{N}$ and $\text{ran } g \subseteq X$. By The Recursion Theorem: Partial Version, there exists a sequence f of elements of X such that

- (i) $f_0 = \min X$ $\triangleright \min X$ exists by (\mathbb{N}, \leq) is Well-Ordered
- (ii) $\forall n \in \mathbb{N}, (n + 1 \in \text{dom } f \implies f_{n+1} = g(f_n, n))$
- (iii) f is either an infinite sequence or a finite sequence of length $k + 1$ and $(f_k, k) \notin \text{dom } g$.

Note that $\text{dom } g = \{(x, n) \in X \times \mathbb{N} \mid \exists y \in X, x < y\}$. Moreover, for each $n \in \mathbb{N}$ such that $n + 1 \in \text{dom } f$, we have $f_n < f_{n+1}$; hence $\forall m, n \in \text{dom } f, (m < n \implies f_m < f_n)$ (in the similar manner of Exercise 3.3.1), and thus f is injective.

Suppose $Y = X \setminus \text{ran } f \neq \emptyset$ for the sake of contradiction. By (\mathbb{N}, \leq) is Well-Ordered, we may take $y = \min Y$. Then, by Theorem 3.2.8, we may let $z = \max\{x \in X \mid x < y\}$. $z = f_m$ for some $m \in \text{dom } f$. Hence, $y = f_{m+1}$. \square

3.4 Arithmetic of Natural Numbers

Theorem 3.4.1

There uniquely exists a function $+: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that

- (i) $\forall m \in \mathbb{N}, +(m, 0) = m$
- (ii) $\forall m, n \in \mathbb{N}, +(m, n + 1) = S(+(m, n))$.

Proof. The result directly follows from exploiting **The Recursion Theorem: Parametric Version** with $A = P = \mathbb{N}$, $a(p) = p$ for all $p \in \mathbb{N}$, and $g(p, x, n) = S(x)$ for all $p, x, n \in \mathbb{N}$. \square

Definition 3.4.2: Addition

The function $+$ defined in Theorem 3.4.1 is called the *addition*.

Notation 3.4.3

For all $m \in \mathbb{N}$, we have $+(m, 1) = +(m, 0 + 1) = +(m, 0) + 1 = m + 1$. Hence, we may write $m + n$ instead of $+(m, n)$ without causing any confusion regarding Notation 3.1.2. We restate the defining properties of the addition for future reference:

$$\forall m \in \mathbb{N}, m + 0 = m \quad [1]$$

$$\forall m, n \in \mathbb{N}, m + (n + 1) = (m + n) + 1 \quad [2]$$

Theorem 3.4.4 $+$ is Commutative

Addition is commutative; that is to say

$$\forall m, n \in \mathbb{N}, m + n = n + m.$$

Proof. Let $P(x)$ be the property “ $\forall m \in \mathbb{N}, m + x = x + m$.”

Claim 1. $P(0)$ holds.

Proof. Since $m + 0 = m$ already, we only need to prove $0 + m = m$ for all $m \in \mathbb{N}$. We shall make use of induction. First of all $0 + 0 = 0$ holds by [1].

Suppose $0 + m = m$ where $m \in \mathbb{N}$. Then,

$$\begin{aligned} 0 + (m + 1) &= (0 + m) + 1 &> [2] \\ &= m + 1. &> 0 + m = m \end{aligned}$$

Hence, by **The Induction Principle**, $0 + m = m$ for all $m \in \mathbb{N}$. \square

Claim 2. $\forall n \in \mathbb{N}, [P(n) \implies P(n + 1)]$

Proof. Assume $P(n)$. We shall show $P(n + 1)$ holds by induction. $0 + (n + 1) = (n + 1) + 0$ is already shown by Claim 1. Hence, assume $m + (n + 1) = (n + 1) + m$ for fixed $m \in \mathbb{N}$. Then,

$$\begin{aligned} (m + 1) + (n + 1) &= ((m + 1) + n) + 1 &> [2] \\ &= (n + (m + 1)) + 1 &> P(n) \\ &= ((n + m) + 1) + 1 &> [2] \\ &= ((m + n) + 1) + 1 &> P(n) \\ &= (m + (n + 1)) + 1 &> [2] \\ &= ((n + 1) + m) + 1 &> m + (n + 1) = (n + 1) + m \\ &= (n + 1) + (m + 1). &> [2] \end{aligned}$$

Hence, by **The Induction Principle**, $P(n + 1)$ holds. \square

From Claim 1, Claim 2, and The Induction Principle, we get $\forall m, n \in \mathbb{N}, m + n = n + m$. \square

Theorem 3.4.5 $+$ is Associative

Addition is associative; that is to say

$$\forall k, m, n \in \mathbb{N}, (k + m) + n = k + (m + n).$$

Proof. Let $P(x)$ be the property “ $\forall k, m \in \mathbb{N}, (k + m) + x = k + (m + x)$.” $P(0)$ is direct by [1].
Now, fix any $n \in \mathbb{N}$ and assume $P(n)$. Then, for all $k, m \in \mathbb{N}$,

$$\begin{aligned} (k + m) + (n + 1) &= ((k + m) + n) + 1 &> [2] \\ &= (k + (m + n)) + 1 &> P(n) \\ &= k + ((m + n) + 1) &> [2] \\ &= k + (m + (n + 1)). &> [2] \end{aligned}$$

Hence, by The Induction Principle, the result follows. \square

Theorem 3.4.6

There uniquely exists a function $\cdot : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that

- (i) $\forall m \in \mathbb{N}, m \cdot 0 = 0$
- (ii) $\forall m, n \in \mathbb{N}, m \cdot (n + 1) = m \cdot n + m$.

Proof. The result directly follows from exploiting The Recursion Theorem: Parametric Version with $A = P = \mathbb{N}$, $a(p) = 0$ for all $p \in \mathbb{N}$, and $g(p, x, n) = x + p$ for all $p, x, n \in \mathbb{N}$. \square

Definition 3.4.7: Multiplication

The function \cdot defined in Theorem 3.4.6 is called the *multiplication*.

$$\forall m \in \mathbb{N}, m \cdot 0 = 0 \quad [3]$$

$$\forall m, n \in \mathbb{N}, m \cdot (n + 1) = m \cdot n + m \quad [4]$$

Theorem 3.4.8 \cdot is Commutative

Multiplication is commutative, i.e.,

$$\forall m, n \in \mathbb{N}, m \cdot n = n \cdot m.$$

Proof. Let $P(x)$ be the property “ $\forall m \in \mathbb{N}, m \cdot x = x \cdot m$.”

Claim 1. $P(0)$ holds.

Proof. Since $m \cdot 0 = 0$ already by [3], we only need to prove $0 \cdot m = 0$ for all $m \in \mathbb{N}$. We shall make use of induction. First of all $0 \cdot 0 = 0$ holds by [3].

Suppose $0 \cdot m = 0$ where $m \in \mathbb{N}$. Then,

$$\begin{aligned} 0 \cdot (m + 1) &= 0 \cdot m + 0 &> [4] \\ &= 0 + 0 &> 0 \cdot m = 0 \\ &= 0. \end{aligned}$$

Hence, by The Induction Principle, $0 \cdot m = 0$ for all $m \in \mathbb{N}$. \square

Claim 2. $\forall n \in \mathbb{N}, [P(n) \implies P(n+1)]$

Proof. Fix any $n \in \mathbb{N}$ and assume $P(n)$. We shall prove $P(n+1)$ by induction. We already have $0 \cdot (n+1) = (n+1) \cdot 0$ by Claim 1.

Fix any $m \in \mathbb{N}$ and assume $m \cdot (n+1) = (n+1) \cdot m$. Then,

$$\begin{aligned}
 (m+1) \cdot (n+1) &= (m+1) \cdot n + (m+1) &> [4] \\
 &= n \cdot (m+1) + (m+1) &> P(n) \\
 &= (n \cdot m + n) + (m+1) &> [4] \\
 &= (m \cdot n + n) + (m+1) &> P(n) \\
 &= (m \cdot n + m) + (n+1) &> + \text{ is Commutative, } + \text{ is Associative} \\
 &= m \cdot (n+1) + (n+1) &> [4] \\
 &= (n+1) \cdot m + (n+1) &> m \cdot (n+1) = (n+1) \cdot m \\
 &= (n+1) \cdot (m+1). &> [4]
 \end{aligned}$$

Hence, by The Induction Principle, $P(n+1)$ holds.

From Claim 1, Claim 2, and The Induction Principle, we get $\forall m, n \in \mathbb{N}, m \cdot n = n \cdot m$. \square

Theorem 3.4.9 · Distributes Over +

Multiplication is distributive over addition, i.e.,

$$\begin{aligned}
 \forall k, m, n \in \mathbb{N}, k \cdot (m+n) &= k \cdot m + k \cdot n \quad \text{and} \\
 \forall k, m, n \in \mathbb{N}, (m+n) \cdot k &= m \cdot k + n \cdot k.
 \end{aligned}$$

Proof. Let $P(x)$ be the property “ $\forall k, m \in \mathbb{N}, k \cdot (m+x) = k \cdot m + k \cdot x$.” $P(0)$ holds by [1] and [3].

Fix any $n \in \mathbb{N}$ and assume $P(n)$. Then, for each $k, m \in \mathbb{N}$,

$$\begin{aligned}
 k \cdot (m + (n+1)) &= k \cdot ((m+n) + 1) &> + \text{ is Associative} \\
 &= k \cdot (m+n) + k &> [4] \\
 &= (k \cdot m + k \cdot n) + k &> P(n) \\
 &= k \cdot m + (k \cdot n + k) &> + \text{ is Associative} \\
 &= k \cdot m + k \cdot (n+1). &> [4]
 \end{aligned}$$

Hence, by The Induction Principle, we have $\forall k, m, n \in \mathbb{N}, k \cdot (m+n) = k \cdot m + k \cdot n$.

Now, we have, for each $k, m, n \in \mathbb{N}$,

$$\begin{aligned}
 (m+n) \cdot k &= k \cdot (m+n) &> \cdot \text{ is Commutative} \\
 &= k \cdot m + k \cdot n \\
 &= m \cdot k + n \cdot k. &> \cdot \text{ is Commutative}
 \end{aligned}$$

\square

Theorem 3.4.10 · is Associative

Multiplication is associative, i.e.,

$$\forall k, m, n \in \mathbb{N}, (k \cdot m) \cdot n = k \cdot (m \cdot n).$$

Proof. Let $P(x)$ be the property “ $\forall k, m \in \mathbb{N}, (k \cdot m) \cdot x = k \cdot (m \cdot x)$.” $P(0)$ is direct from [3].
Fix any $n \in \mathbb{N}$ and assume $P(n)$. Then, for each $k, m \in \mathbb{N}$,

$$\begin{aligned} (k \cdot m) \cdot (n + 1) &= (k \cdot m) \cdot n + k \cdot m &> [4] \\ &= k \cdot (m \cdot n) + k \cdot m &> P(n) \\ &= k \cdot (m \cdot n + m) &> \cdot \text{ Distributes Over } + \\ &= k \cdot (m \cdot (n + 1)). &> [4] \end{aligned}$$

Hence, the result follows by **The Induction Principle**. \square

Lemma 3.4.11

$$\forall m \in \mathbb{N}, m \cdot 1 = m$$

Proof.

$$\begin{aligned} m \cdot 1 &= m \cdot (0 + 1) &> [1], + \text{ is Commutative} \\ &= m \cdot 0 + m &> [4] \\ &= 0 + m &> [3] \\ &= m &> [1], + \text{ is Commutative} \end{aligned}$$

\square

Theorem 3.4.12

There uniquely exists a function $\uparrow: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that

- (i) $\forall m \in \mathbb{N}, m \uparrow 0 = 1$
- (ii) $\forall m, n \in \mathbb{N}, m \uparrow (n + 1) = (m \uparrow n) \cdot m$

Proof. The result directly follows from exploiting **The Recursion Theorem: Parametric Version** with $A = P = \mathbb{N}$, $a(p) = 1$ for all $p \in \mathbb{N}$, and $g(p, x, n) = x \cdot p$ for all $p, x, n \in \mathbb{N}$. \square

Definition 3.4.13: Exponentiation

The function \uparrow defined in Theorem 3.4.12 is called the *exponentiation*. We write m^n instead of $m \uparrow n$.

$$\forall m \in \mathbb{N}, m^0 = 1 \quad [5]$$

$$\forall m, n \in \mathbb{N}, m^{n+1} = m^n \cdot m \quad [6]$$

Theorem 3.4.14 Laws of Exponents

- (i) $\forall m \in \mathbb{N}, m^1 = m$
- (ii) $\forall k, m, n \in \mathbb{N}, k^{m+n} = k^m \cdot k^n$
- (iii) $\forall k, m, n \in \mathbb{N}, (m \cdot n)^k = m^k \cdot n^k$
- (iv) $\forall k, m, n \in \mathbb{N}, (k^m)^n = k^{m \cdot n}$

Proof.

(i) Take any $m \in \mathbb{N}$. Then,

$$\begin{aligned} m^1 &= m^{0+1} &> [1], + \text{ is Commutative} \\ &= m^0 \cdot m &> [6] \\ &= 1 \cdot m &> [5] \\ &= m. &> \cdot \text{ is Commutative, Lemma 3.4.11} \end{aligned}$$

(ii) Let $\mathbf{P}(x)$ be the property “ $\forall k, m \in \mathbb{N}, k^{m+x} = k^m \cdot k^x$.” $\mathbf{P}(0)$ holds since, for each $k, m \in \mathbb{N}$,

$$\begin{aligned} k^{m+0} &= k^m &> [1] \\ &= k^m \cdot 1 &> \text{Lemma 3.4.11} \\ &= k^m \cdot k^0. &> [5] \end{aligned}$$

Now, fix $n \in \mathbb{N}$ and assume $\mathbf{P}(n)$. Then,

$$\begin{aligned} k^{m+(n+1)} &= k^{(m+n)+1} &> + \text{ is Associative} \\ &= k^{m+n} \cdot k &> [6] \\ &= (k^m \cdot k^n) \cdot k &> \mathbf{P}(x) \\ &= k^m \cdot (k^n \cdot k) &> \cdot \text{ is Associative} \\ &= k^m \cdot k^{n+1}. &> [6] \end{aligned}$$

Therefore, by **The Induction Principle**, the result follows.

(iii) Let $\mathbf{P}(x)$ be the property “ $\forall m, n \in \mathbb{N}, (m \cdot n)^x = m^x \cdot n^x$.” $\mathbf{P}(0)$ holds since, for each $m, n \in \mathbb{N}$,

$$\begin{aligned} (m \cdot n)^0 &= 1 &> [5] \\ &= 1 \cdot 1 &> \text{Lemma 3.4.11} \\ &= m^0 \cdot n^0. &> [5] \end{aligned}$$

Now, fix $k \in \mathbb{N}$ and assume $\mathbf{P}(k)$. Then,

$$\begin{aligned} (m \cdot n)^{k+1} &= (m \cdot n)^k \cdot (m \cdot n) &> [6] \\ &= (m^k \cdot n^k) \cdot (m \cdot n) &> \mathbf{P}(k) \\ &= (m^k \cdot m) \cdot (n^k \cdot n) &> \cdot \text{ is Commutative, } \cdot \text{ is Associative} \\ &= m^{k+1} \cdot n^{k+1}. &> [6] \end{aligned}$$

Therefore, by **The Induction Principle**, the result follows.

(iv) Let $\mathbf{P}(x)$ be the property “ $\forall k, m \in \mathbb{N}, (k^m)^x = k^{m \cdot x}$.” $\mathbf{P}(0)$ holds since, for each $k, m \in \mathbb{N}$,

$$\begin{aligned} (k^m)^0 &= 1 &> [5] \\ &= k^0 &> [5] \\ &= k^{m \cdot 0}. &> [3] \end{aligned}$$

Now, fix $n \in \mathbb{N}$ and assume $\mathbf{P}(n)$. Then,

$$\begin{aligned} (k^m)^{n+1} &= (k^m)^n \cdot k^m &> [6] \\ &= k^{m \cdot n} \cdot k^m &> \mathbf{P}(n) \\ &= k^{m \cdot n + m} &> \text{Laws of Exponents (ii)} \\ &= k^{m \cdot (n+1)}. &> [4] \end{aligned}$$

Therefore, by **The Induction Principle**, the result follows. □

Theorem 3.4.15

There uniquely exists $\Sigma: \text{Seq}(\mathbb{N}) \rightarrow \mathbb{N}$ such that

(i) $\Sigma(\langle \rangle) = 0$.

(ii) $\Sigma(k) = \Sigma(k|_n) + k_n$ for all $k \in \text{Seq}(\mathbb{N})$ with length $n + 1$.

Proof. Let $g : \text{Seq}(\mathbb{N}) \times \mathbb{N} \times \mathbb{N}$ be defined by

$$g(k, s, n) = \begin{cases} s + k_n & \text{if } n \in \text{dom } k \\ s & \text{otherwise.} \end{cases}$$

Then, by **The Recursion Theorem: Parametric Version**, there exists a function $f : \text{Seq}(\mathbb{N}) \times \mathbb{N} \rightarrow \mathbb{N}$ such that

- (i) $\forall k \in \text{Seq}(\mathbb{N}), f(k, 0) = 0$
(ii) $\forall n \in \mathbb{N}, \forall k \in \text{Seq}(\mathbb{N}), f(k, n + 1) = g(k, f(k, n), n) = \begin{cases} f(k, n) + k_n & \text{if } n \in \text{dom } k \\ f(k, n) & \text{otherwise.} \end{cases} \quad [*]$

Now, define $\Sigma : \text{Seq}(\mathbb{N}) \rightarrow \mathbb{N}$ by $\Sigma(k) = f(k, \text{dom } k)$. (i) evidently holds.

Claim 1. Let $k, \ell \in \text{Seq}(\mathbb{N})$. If $k \subseteq \ell$, then $f(k, \text{dom } k) = f(\ell, \text{dom } k)$.

Proof. Let $P(x)$ be the property

$$\forall k, \ell \in \text{Seq}(\mathbb{N}), [\text{dom } k = x \wedge k \subseteq \ell \implies f(k, x) = f(\ell, x)].$$

$P(0)$ is evident. Now, fix $n \in \mathbb{N}$ and assume $P(n)$.

Fix any $k \in \text{Seq}(\mathbb{N})$ with $\text{dom } k = n + 1$. Then, for any $\ell \in \text{Seq}(\mathbb{N})$ with $k \subseteq \ell$,

$$\begin{aligned} f(\ell, n + 1) &= f(\ell, n) + \ell_n &> [*] \\ &= f(\ell|_n, n) + \ell_n &> P(n) \\ &= f(k|_n, n) + k_n &> k \subseteq \ell \\ &= f(k, n) + k_n &> P(n) \\ &= f(k, n + 1). &> [*] \end{aligned}$$

Hence, by **The Induction Principle**, the result follows. \square

Let $k \in \text{Seq}(\mathbb{N})$ with length $n + 1$. Then, $\Sigma(k) = f(k, n + 1) = f(k, n) + k_n$.

$$\begin{aligned} \Sigma(k) &= f(k, n + 1) \\ &= f(k, n) + k_n &> [*] \\ &= f(k|_n, n) + k_n &> \text{Claim 1} \\ &= \Sigma(k|_n) + k_n. \end{aligned}$$

The uniqueness easily follows. \square

Notation 3.4.16: Summation

For the function Σ defined in Theorem 3.4.15, we write

$$\sum_{0 \leq i < n} k_i \quad \text{or} \quad \sum_{i=0}^{n-1} k_i$$

instead of $\Sigma(\langle k_0, \dots, k_{n-1} \rangle)$.

Selected Problems

Exercise 3.4.2

$$\forall k, m, n \in \mathbb{N}, (m < n \iff m + k < n + k)$$

Proof. Let $P(x)$ be the property “ $\forall m, n \in \mathbb{N}, (m < n \iff m + x < n + x)$.” $P(0)$ is evident from [1].

Now, fix any $k \in \mathbb{N}$ and assume $P(k)$. Then, for all $m, n \in \mathbb{N}$,

$$\begin{aligned} m < n &\iff m + k < n + k &> P(k) \\ &\iff (m + k) + 1 < (n + k) + 1 &> \text{Exercise 3.2.2} \\ &\iff m + (k + 1) < n + (k + 1). &> + \text{ is Associative} \end{aligned}$$

By **The Induction Principle**, the result follows. \square

Exercise 3.4.3

$$\forall m, n \in \mathbb{N}, (m \leq n \iff \exists! k \in \mathbb{N}, n = m + k)$$

Proof. (\Rightarrow) Fix any $m \in \mathbb{N}$ and let $P(x)$ be the property “ $\exists k \in \mathbb{N}, x = m + k$.” $P(m)$ holds since $k = 0$ would satisfy by [1].

Fix any $n \in \mathbb{N}$ such that $m \leq n$ and assume $P(n)$. Then, there exists k such that $n = m + k$, which leads to $n + 1 = m + (k + 1)$ by **+ is Associative**. Hence, $P(n + 1)$ holds. Therefore, $\forall n \geq m, \exists k \in \mathbb{N}, n = m + k$ by Exercise 3.2.11.

To prove the uniqueness, assume $m + k = m + \ell$ where $k, \ell, m \in \mathbb{N}$. WLOG, $k \leq \ell$. If it were $k < \ell$, by Exercise 3.4.2 and **+ is Commutative**, we must have $m + k = k + m < \ell + m = \ell + m$. Hence, $k = \ell$.

(\Leftarrow) Let $P(x)$ be the property “ $\forall m, n \in \mathbb{N}, (n = m + x \implies m \leq n)$.” We have evidently $P(0)$ by [1].

Fix any $k \in \mathbb{N}$ and assume $P(k)$. Then, for each $m, n \in \mathbb{N}$ such that $n = m + (k + 1)$, we have $n = (m + 1) + k$ thanks to **+ is Commutative** and **+ is Associative**, and thus $m < m + 1 \leq n$ by $P(k)$. Hence, by **The Induction Principle**, the result follows. \square

Exercise 3.4.6

$$\forall k, m, n \in \mathbb{N}, [k \neq 0 \implies (m < n \iff m \cdot k < n \cdot k)]$$

Proof. Let $P(x)$ be the property “ $\forall m, n \in \mathbb{N}, (m < n \iff m \cdot k < n \cdot k)$.” $P(1)$ holds since, for all $n \in \mathbb{N}$,

$$\begin{aligned} n \cdot 1 &= n \cdot (0 + 1) &> [1], + \text{ is Commutative} \\ &= n \cdot 0 + n &> [4] \\ &= 0 + n &> [3] \\ &= n. &> [1], + \text{ is Commutative} \end{aligned}$$

Now, fix any $k \in \mathbb{N}$ and assume $P(k)$. Then, for each $m, n \in \mathbb{N}$ with $m < n$,

$$\begin{aligned} m \cdot (k + 1) &= m \cdot k + m &> [4] \\ &< m \cdot k + n &> \text{Exercise 3.4.2} \\ &< n \cdot k + n &> P(k), + \text{ is Commutative, Exercise 3.4.2} \\ &= n \cdot (k + 1). &> [4] \end{aligned}$$

Therefore, by Exercise 3.2.11, the result follows. \square

3.5 Operations and Structures

Definition 3.5.1: Operation

- A *unary operation* on S is a function on a subset of S into S .
- A *binary operation* on S is a function on a subset of S^2 into S .

Notation 3.5.2: Binary Operation

Non-letter symbols such as $+$, \times , $*$, Δ , etc., are often used to denote operations. The value of the operation $*$ at (x, y) is then denoted $x * y$ rather than $*(x, y)$.

Definition 3.5.3: Closedness Under Operation

Let f be a binary operation on S and $A \subseteq S$. A is said to be *closed under the operation* f if $\forall x, y \in A, [(x, y) \in \text{dom } f \implies f(x, y) \in A]$.

Definition 3.5.4: n -Tuple

Let $n \in \mathbb{N}$. An n -tuple is a finite sequence of length n .

Note:-

Let $\langle a_0, \dots, a_{n-1} \rangle$ and $\langle b_0, \dots, b_{n-1} \rangle$ be two n -tuples. We have, by Lemma 2.3.3,

$$\langle a_0, \dots, a_{n-1} \rangle = \langle b_0, \dots, b_{n-1} \rangle \iff \forall i < n, a_i = b_i.$$

This satisfies the usual defining property of n -tuple.

Note:-

- If $\langle A_i \mid 0 \leq i < n \rangle$ is a finite sequence (of sets), then the product of the indexed system of sets $\prod_{0 \leq i < n} A_i$ (Definition 2.3.16) is just the set of all n -tuples $a = \langle a_0, \dots, a_{n-1} \rangle$ such that $\forall i < n, a_i \in A_i$.
- If $\forall i < n, A_i = A$, then $\prod_{0 \leq i < n} A_i = A^n$.
- $A^0 = \{\langle \rangle\}$.

Notation 3.5.5

The ‘ordered pair’ (Definition 2.1.1), $(a_0, a_1) = \{\{a_0\}, \{a_0, a_1\}\}$, is different set from the ‘2-tuple’ (Definition 3.5.4), $\langle a_0, a_1 \rangle = \{(0, a_0), (1, a_1)\}$. Consequently, $A_0 \times A_1$ (Definition 2.2.10) does not generally equal to $\prod_{0 \leq i < 2} A_i$ (Definition 2.3.16).

However, since there is a natural one-to-one correspondence

$$\begin{aligned} \delta : A_0 \times A_1 &\hookrightarrow \prod_{0 \leq i < 2} A_i \\ (a_0, a_1) &\mapsto \langle a_0, a_1 \rangle, \end{aligned}$$

for almost all practical purposes—when only the defining property of n -tuple is needed—it makes so difference which definition one uses.

Therefore, we do not distinguish between ordered pairs and 2-tuples now on. That is to say we use notations

$$\langle a_0, \dots, a_{n-1} \rangle \quad \text{and} \quad (a_0, \dots, a_{n-1})$$

interchangeably from now on.

Definition 3.5.6: n -ary Relation

An n -ary relation R in A is a subset of A^n . We write $R(a_0, a_1, \dots, a_{n-1})$ instead of $\langle a_0, a_1, \dots, a_{n-1} \rangle \in R$.

Definition 3.5.7: n -ary Operation

An n -ary operation F on A is a function on a subset of A^n into A . We write $F(a_0, a_1, \dots, a_{n-1})$ instead of $F(\langle a_0, a_1, \dots, a_{n-1} \rangle)$.

Note:-

- 1-ary relations in A need not be distinguished from subsets of A .
- 1-ary operations on A need not be distinguished from functions on a subset of A into A .
- Nonempty 0-ary operations on A need not be distinguished from A . (A nonempty 0-ary operation is of the form $\{(\langle \rangle, a)\}$ where $a \in A$; a nonempty 0-ary operation is called a *constant*.)

Definition 3.5.8: Structure

- A type τ is an ordered pair $(\langle r_0, \dots, r_{m-1} \rangle, \langle f_0, \dots, f_{n-1} \rangle)$ of finite sequences of natural numbers.
- A structure of type τ is a triple

$$\mathfrak{A} = (A, \langle R_0, \dots, R_{m-1} \rangle, \langle F_0, \dots, F_{n-1} \rangle)$$

where R_i is an r_i -ary relation on A for each $i < m$ and F_j is an f_j -ary operation on A for each $j < n$. In addition, we require $F_j \neq \emptyset$ if $f_j = 0$, i.e., F_j should be constant. A is called the *universe* of the structure \mathfrak{A} .

Example 3.5.9

$\mathfrak{N} = (\mathbb{N}, \langle \leq \rangle, \langle 0, +, \cdot \rangle)$ is a structure of type $(\langle 2 \rangle, \langle 0, 2, 2 \rangle)$.

Notation 3.5.10

We often write the structure of type $(\langle r_0, \dots, r_{m-1} \rangle, \langle f_0, \dots, f_{n-1} \rangle)$ as a $(1+m+n)$ -tuple, for example, $(\mathbb{N}, \leq, 0, +, \cdot)$, when it is understood which symbol represent relations and which operations.

Definition 3.5.11: Isomorphism Between Structures

Let \mathfrak{A} and \mathfrak{A}' be structures of the same type $\tau = (\langle r_0, \dots, r_{m-1} \rangle, \langle f_0, \dots, f_{n-1} \rangle)$. Write $\mathfrak{A} = (A, \langle R_0, \dots, R_{m-1} \rangle, \langle F_0, \dots, F_{n-1} \rangle)$ and $\mathfrak{A}' = (A', \langle R'_0, \dots, R'_{m-1} \rangle, \langle F'_0, \dots, F'_{n-1} \rangle)$. An *isomorphism* between structures \mathfrak{A} and \mathfrak{A}' is a mapping $h: A \hookrightarrow A'$ such that

- (i) $\forall i < m, \forall a \in A^{r_i}, [R_i(a_0, \dots, a_{r_i-1}) \iff R'_i(h(a_0), \dots, h(a_{r_i-1}))]$
- (ii) $\forall j < n, \forall a \in A^{f_j}, [(a_0, \dots, a_{f_j-1}) \in \text{dom } F_j \iff (h(a_0), \dots, h(a_{f_j-1})) \in \text{dom } F'_j]$
- (iii) $\forall j < n, \forall a \in A^{f_j}, [(a_0, \dots, a_{f_j-1}) \in \text{dom } F_j \implies h(F_j(a_0, \dots, a_{f_j-1})) = F'_j(h(a_0), \dots, h(a_{f_j-1}))]$.

Definition 3.5.12: Automorphism

An isomorphism between a structure \mathfrak{A} and itself is called an *automorphism*.

Definition 3.5.13: Closed Set

Fix a structure $\mathfrak{A} = (A, \langle R_0, \dots, R_{m-1} \rangle, \langle F_0, \dots, F_{n-1} \rangle)$. A set $B \subseteq A$ is called *closed* if

$$\forall j < n, \forall a \in B^{f_j}, [(a_0, \dots, a_{f_j-1}) \in \text{dom } F_j \implies F_j(a_0, \dots, a_{f_j-1}) \in B].$$

Definition 3.5.14: Closure

Fix a structure $\mathfrak{A} = (A, \langle R_0, \dots, R_{m-1} \rangle, \langle F_0, \dots, F_{n-1} \rangle)$. Let $C \subseteq A$. The *closure* of C ,

$$\overline{C} \triangleq \bigcap \{B \subseteq A \mid C \subseteq B \text{ and } B \text{ is closed}\},$$

is the least closed set containing all elements of C .

Theorem 3.5.15

Let $\mathfrak{A} = (A, \langle R_0, \dots, R_{m-1} \rangle, \langle F_0, \dots, F_{n-1} \rangle)$ be a structure and let $C \subseteq A$. If the sequence $\langle C_i \mid i \in \mathbb{N} \rangle$ is defined recursively by

$$\begin{aligned} C_0 &= C; \\ \forall i \in \mathbb{N}, C_{i+1} &= C_i \cup \bigcup_{j=0}^{n-1} F_j[C_i^{f_j}], \end{aligned}$$

then $\overline{C} = \bigcup_{i=0}^{\infty} C_i$.

Proof. Note the recursive definition is justified by **The Recursion Theorem**. Let $\tilde{C} \triangleq \bigcup_{i=0}^{\infty} C_i$.

Claim 1. $\overline{C} \subseteq \tilde{C}$

Proof. Since we have $C_0 \subseteq \tilde{C}$, it is enough to show that \tilde{C} is closed.

Take any $j < n$ and $a \in \tilde{C}^{f_j}$. By the definition of \tilde{C} , $\forall r < f_j, \exists i_r \in \mathbb{N}, a_r \in C_{i_r}$. We may take $\bar{i} = \max\{i_r \mid r < f_j\}$ by Exercise 3.5.13. Since $C_i \subseteq C_{i+1}$ for all $i \in \mathbb{N}$, we have $a_r \in C_{i_r} \subseteq C_{\bar{i}}$ for all $r < f_j$. Hence, if $(a_0, \dots, a_{f_j-1}) \in \text{dom } F_j$, we have $F_j(a_0, \dots, a_{f_j-1}) \in F_j[C_{\bar{i}}^{f_j}] \subseteq C_{\bar{i}+1} \subseteq \tilde{C}$. Hence, \tilde{C} is closed. \square

Claim 2. $\tilde{C} \subseteq \bar{C}$

Proof. Clearly $C_0 = C \subseteq \bar{C}$. If $C_i \subseteq \bar{C}$, then, for each $j < n$, $F_j[C_i^{f_j}] \subseteq \bar{C}$ since \bar{C} is closed. Hence, $C_{i+1} \subseteq \bar{C}$. Therefore, by The Induction Principle, $\forall i \in \mathbb{N}, C_i \subseteq \bar{C}$; hence $\tilde{C} \subseteq \bar{C}$. \square

Combining Claims 1 and 2 completes the proof. \square

Theorem 3.5.16 The General Induction Principle

Let $\mathfrak{A} = (A, \langle R_0, \dots, R_{m-1} \rangle, \langle F_0, \dots, F_{n-1} \rangle)$ be a structure and let $C \subseteq A$. Let $P(x)$ be a property. If

- (i) $\forall a \in C, P(a)$
 - (ii) $\forall j < n, \forall a \in A^{f_j}, [(a_0, \dots, a_{f_j-1}) \in \text{dom } F_j \wedge \forall i < f_j, P(a_i) \implies P(F_j(a_0, \dots, a_{f_j-1}))]$
- hold, then $\forall x \in \bar{C}, P(x)$.

Proof. Let $B = \{x \in A \mid P(x)\}$. (i) says $C \subseteq B$ and (ii) says B is closed. Therefore, $\bar{C} \subseteq B$. \square

Note:-

The Induction Principle is a special case of The General Induction Principle for the structure (\mathbb{N}, S) where S is the successor function.

Selected Problems

Exercise 3.5.4

Let $B = \mathcal{P}(A)$. Show that (B, \cup_B, \cap_B) and (B, \cap_B, \cup_B) are isomorphic structures.

Proof. Let $h: B \rightarrow B$ be defined by $h(X) = A \setminus X$. If $A \setminus X = A \setminus Y$, then $X = A \setminus (A \setminus X) = A \setminus (A \setminus Y) = Y$ by Exercise 1.4.2 (iii). Moreover, $h(h(X)) = X$ for all $X \in B$. Hence, $h: B \hookrightarrow B$. \square

Exercise 3.5.7

Let R be a set whose elements are n -tuples. Then, R is an n -ary relation in A for some A .

Proof. Let $a \in R$. Then, $a = \{(0, a_0), \dots, (n-1, a_{n-1})\}$. For each $i < n$, $a_i \in \{i, a_i\} \in (i, a_i) \in a \in R$. Hence, $a_i \in \bigcup [\bigcup (\bigcup R)]$, i.e., R is an n -ary relation in $A = \bigcup [\bigcup (\bigcup R)]$. \square

Exercise 3.5.10

Let A be a sequence of length n . Then, $\prod_{0 \leq i < n} A_i \neq \emptyset \iff \forall i < n, A_i \neq \emptyset$

Proof. Let $P(x)$ be the property “if A is a sequence of length x , then $\prod_{0 \leq i < x} A_i \neq \emptyset \iff \forall i < x, A_i \neq \emptyset$.” $P(0)$ holds since, if A is a function with $\text{dom } A = \emptyset$, then $\prod A = \{\emptyset\}$.

Fix $n \in \mathbb{N}$ and assume $P(n)$ holds. Take any sequence A of length $n+1$.

- Assume $\prod A \neq \emptyset$ and take $a \in \prod A$. Then, for each $i < n + 1$, $a_i \in A_i$; and thus $A_i \neq \emptyset$.
- Assume $\forall i < n + 1$, $A_i \neq \emptyset$. Then, by $\mathbf{P}(n)$, we may take $a' \in \prod_{0 \leq i < n} A_i$. We also may take $b \in A_n$. Then, $a' \cup \{(n, b)\} \in \prod A$.

Hence, $\mathbf{P}(n)$ holds. Thus, the result follows by **The Induction Principle**. \square

Exercise 3.5.13

Let $\langle k_0, \dots, k_{n-1} \rangle$ be a finite sequence of natural numbers of length $n \geq 1$. Then, its range $\{k_0, \dots, k_{n-1}\}$ has a greatest element.

Proof. Let $\mathbf{P}(x)$ be the property “the range of a finite sequence of natural numbers of length x has a greatest element.”

Let $\langle k_0 \rangle$ be a sequence of natural numbers of length 1. Then, $k_0 = \max \text{ran} \langle k_0 \rangle$. Hence, $\mathbf{P}(1)$.

Fix any $n \in \mathbb{N}$ and assume $\mathbf{P}(n)$. Take any $k \in \text{Seq}(\mathbb{N})$ with length $n + 1$. Let $k' = \langle k_0, \dots, k_{n-1} \rangle$ be another sequence. Then, by $\mathbf{P}(n)$, there exists $m' = \max\{k_0, \dots, k_{n-1}\}$. Now, let $m = \max\{m', k_n\}$. Then, for all $i < n$, $k_i \leq m' \leq m$, and $k_n \leq m$. Hence, m is an upper bound of $\text{ran } k$; the result follows by Theorem 3.2.8 and Exercise 3.2.11. \square

Exercise 3.5.15

Let $R \subseteq A^2$ be a binary relation. Define a binary operation F_R on A^2 by

$$F_R((a_1, a_2), (b_1, b_2)) = \begin{cases} (a_1, b_2) & \text{if } a_2 = b_1 \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then,

- The closure of R in (A^2, F_R) is a transitive relation.
- If R is reflexive and symmetric, \bar{R} is also an equivalence.

Proof.

- Take any $a, b, c \in A$ and assume $a\bar{R}b$ and $b\bar{R}c$. Then, since \bar{R} is closed, $F((a, b), (b, c)) = (a, c) \in \bar{R}$. Hence, \bar{R} is transitive.
- $\text{Id}_A \subseteq R \subseteq \bar{R}$; \bar{R} is reflexive.

Let $\mathbf{P}(x, y)$ be the property “ $y\bar{R}x$.” As $R \subseteq \bar{R}$, we have $\forall (a, b) \in R$, $\mathbf{P}(a, b)$. Now, take any $(a, b), (b, c) \in A^2$ such that $\mathbf{P}(a, b)$ and $\mathbf{P}(b, c)$. Then, by (i), we have $c\bar{R}a$; $\mathbf{P}(F_R((a, b), (b, c)))$ hold. Therefore, by **The General Induction Principle**, $b\bar{R}a$ holds for all $(a, b) \in \bar{R}$. \square

Chapter 4

Finite, Countable, and Uncountable Sets

4.1 Cardinality of Sets

Definition 4.1.1: Equipotent Sets

Let A and B be sets. A is *equipotent* to B if there is a function $f : A \hookrightarrow B$. We write $|A| = |B|$.

Lemma 4.1.2

Let A , B , and C be sets.

- (i) $|A| = |A|$.
- (ii) If $|A| = |B|$, then $|B| = |A|$.
- (iii) If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Proof.

- (i) Id_A is an injective function on A onto A .
- (ii) If $f : A \hookrightarrow B$, then $f^{-1} : B \hookrightarrow A$.
- (iii) If $f : A \hookrightarrow B$, and if $g : B \hookrightarrow C$, then $f \circ g : A \hookrightarrow C$. □

Note:-

Lemma 4.1.2 essentially says that $|A| = |B|$ behaves like an equivalence relation.

Definition 4.1.3

- We say *the cardinality of A is less than or equal to the cardinality of B* if there is a function $f : A \hookrightarrow B$. We write $|A| \leq |B|$.
- We say *the cardinality of A is less than the cardinality of B* if $|A| \leq |B|$ and $\neg(|A| = |B|)$. We write $|A| < |B|$.

Lemma 4.1.4

Let A , B , and C be sets.

- (i) If $|A| = |B|$, then $|A| \leq |B|$.
- (ii) $|A| \leq |A|$
- (iii) If $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

Proof.

- (i) If $f : A \hookrightarrow B$, then f is injective as well.

- (ii) Id_A is an injective function on A into A .
 (iii) If $f : A \hookrightarrow B$, and if $g : B \hookrightarrow C$, then $f \circ g : A \hookrightarrow C$. □

Lemma 4.1.5

If $A_1 \subseteq B \subseteq A$ and $|A_1| = |A|$, then $|B| = |A|$.

Note:-

We present two proofs for Lemma 4.1.5. The second proof can be viewed as a more fundamental proof in the sense that it does not depend on **Axiom of Infinity**.

Proof 1. Let $f : A \hookrightarrow A_1$. Define a sequence $\langle A_i \mid i \in \mathbb{N} \rangle$ and $\langle B_i \mid i \in \mathbb{N} \rangle$ recursively by

$$\begin{aligned} A_0 &= A, & B_0 &= B, \\ \forall n \in \mathbb{N}, A_{n+1} &= f[A_n], & \forall n \in \mathbb{N}, B_{n+1} &= f[B_n] \end{aligned} \quad [*]$$

thanks to **The Recursion Theorem**.

We clearly have $A_1 \subseteq B_0 \subseteq A_0$. If $A_{n+1} \subseteq B_n \subseteq A_n$, then $A_{n+2} = f[A_{n+1}] \subseteq B_{n+1} = f[B_n] \subseteq A_{n+1} = f[A_n]$ by $[\ast]$. Hence, by $[\ast]$ and **The Induction Principle**, we have $A_{n+1} \subseteq B_n \subseteq A_n$ for all $n \in \mathbb{N}$.

Let, for each $n \in \mathbb{N}$, $C_n \triangleq A_n \setminus B_n$. Then, by **Exercise 2.3.6 (ii)**, $C_{n+1} = f[A_n] \setminus f[B_n] = f[A_n \setminus B_n] = f[C_n]$. Let

$$C \triangleq \bigcup_{n=0}^{\infty} C_n \quad \text{and} \quad D \triangleq A \setminus C.$$

Hence, $f[C] = \bigcup_{n=1}^{\infty} C_n \subseteq C$. Now, define a function $g : A \rightarrow A$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in C \\ x & \text{if } x \in D. \end{cases}$$

We immediately notice that $g|_C = f|_C$ and $g|_D$ are injective and their ranges— $f[C]$ and D —are disjoint; g is injective.

As, $\forall n \geq 1, C_n \subseteq A_n \subseteq B_0 = B$, we have $f[C] \subseteq B$. If $x \in D$, then $x \in A \setminus C_0 = A \setminus (A \setminus B) = B$ by **Exercise 1.4.2 (iii)**.

Now, we shall show $B \subseteq f[C] \cup D$ and thus $B = \text{ran } g$. Take any $y \in B$. Then, $y \in C$ or $y \in D$. If $y \in D$, then it is done; so assume $y \in C$. Then, as $y \notin A \setminus B = C_0$, $y \in f[C]$. Hence, $g : A \hookrightarrow B$. □

Proof 2. Let $f : A \hookrightarrow A_1$. Let $F : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ be defined by $F(X) = (A \setminus B) \cup f[X]$. If $X \subseteq Y \subseteq A$, then $F(X) = (A \setminus B) \cup f[X] \subseteq (A \setminus B) \cup f[Y] = F(Y)$. Hence, by **Exercise 4.1.10**, there exists $C \subseteq A$ such that

$$C = (A \setminus B) \cup f[C].$$

Let $D \triangleq A \setminus C$.

Now, define a function $g : A \rightarrow A$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in C \\ x & \text{if } x \in D. \end{cases}$$

Then, since $f[C] \subseteq C$, g is injective.

Moreover, $f[C] \subseteq \text{ran } f = A_1 \subseteq B$ and $D = A \setminus C = A \setminus ((A \setminus B) \cup f[C]) \subseteq A \setminus (A \setminus B) = B$, and thus $\text{ran } g \subseteq B$.

Now, take any $y \in B$. If $y \in C$, then, as $y \notin A \setminus B$, $y \in f[C]$. Hence, $B \subseteq f[C] \cup D$. Therefore, $g : A \hookrightarrow B$. □

Theorem 4.1.6 Cantor–Bernstein Theorem

If $|X| \leq |Y|$ and $|Y| \leq |X|$, then $|X| = |Y|$.

Proof. Let $f : X \hookrightarrow Y$ and $g : Y \hookrightarrow X$. Then, $g : Y \hookrightarrow g[Y]$, i.e., $|Y| = |g[Y]|$; and $g \circ f : X \hookrightarrow (g \circ f)[X]$, i.e., $|X| = |(g \circ f)[X]|$. Moreover, $(g \circ f)[X] \subseteq g[Y] \subseteq X$. Hence, by Lemma 4.1.5, $|g[Y]| = |X|$. We conclude $|X| = |Y|$ from Lemma 4.1.2. \square

Assumption 4.1.7

There are sets called *cardinal numbers* (or *cardinals*) with the property that for every set X there is a unique cardinal $|X|$ (the *cardinal number of X* , the *cardinality of X*) and sets X and Y are equipotent if and only if $|X|$ is equal to $|Y|$.

Note:-

Assumption 4.1.7 essentially asserts the existence of a unique “representative” for each class of mutually equipotent sets. Assumption 4.1.7 is *harmless* in the sense that we only use it for convenience and we could formulate the theorems without it. We prove Assumption 4.1.7 in Chapter 8: **Axiom of Choice**. However, for certain classes of sets, cardinal numbers can be defined without the Axiom of Choice.

Selected Problems**Exercise 4.1.2**

Let A , B , and C be sets.

- (i) If $|A| < |B|$ and $|B| \leq |C|$, then $|A| < |C|$.
- (ii) If $|A| \leq |B|$ and $|B| < |C|$, then $|A| < |C|$.

Proof.

- (i) We already have $|A| \leq |C|$ by Lemma 4.1.4 (iii). Let $g : B \hookrightarrow C$. Suppose $f : A \hookrightarrow C$ for the sake of contradiction. Then, $f^{-1} \circ g : B \hookrightarrow A$, i.e., $|B| \leq |A|$. By Cantor–Bernstein Theorem, we get $|A| = |B|$, which is a contradiction.
- (ii) We already have $|A| \leq |C|$ by Lemma 4.1.4 (iii). Let $g : A \hookrightarrow B$. Suppose $f : A \hookrightarrow C$ for the sake of contradiction. Then, $g \circ f^{-1} : C \hookrightarrow B$, i.e., $|C| \leq |B|$. By Cantor–Bernstein Theorem, we get $|B| = |C|$, which is a contradiction. \square

Exercise 4.1.3

If $A \subseteq B$, then $|A| \leq |B|$.

Proof. Id_A is an injective function on A into B . \square

Exercise 4.1.7

If $S \subseteq T$, then $|A^S| \leq |A^T|$. In particular, $|A^m| \leq |A^n|$ if $m \leq n$.

Proof. If $T = \emptyset$, then $A^S = A^T = \{\emptyset\}$ and it is done.

Assume $T \neq \emptyset$. Fix some $t \in T$. Now, define $f : A^S \hookrightarrow A^T$ by $g \mapsto g \cup \{(x, t) \mid x \in T \setminus S\}$. \square

Exercise 4.1.10

Let $F: \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ be *monotone*, i.e., if $X \subseteq Y \subseteq A$, then $F(X) \subseteq F(Y)$. Then, F has a least fixed point \bar{X} , that is to say $F(\bar{X}) = \bar{X}$ and $\forall X \subseteq A, (F(X) = X \implies \bar{X} \subseteq X)$.

Proof. Let $T \triangleq \{X \subseteq A \mid F(X) \subseteq X\}$. Then, as $A \in T$, $T \neq \emptyset$; we may let $\bar{X} \triangleq \bigcap T$.

Then, for all $X \in T$, $\bar{X} \subseteq X$; and thus $F(\bar{X}) \subseteq F(X) \subseteq X$. We have $F(\bar{X}) \subseteq \bigcap T = \bar{X}$, i.e., $\bar{X} \in T$.

On the other hand, we have $F(F(\bar{X})) \subseteq F(\bar{X})$, or $F(\bar{X}) \in T$, and thus $\bar{X} = \bigcap T \subseteq F(\bar{X})$. Therefore, $F(\bar{X}) = \bar{X}$. Moreover, if X is a fixed point, then $X \in T$, and thus $\bar{X} = \bigcap T \subseteq X$. \square

Exercise 4.1.14

A function $F: \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ is *continuous* if, for each sequence $\langle X_i \mid i \in \mathbb{N} \rangle$ of subsets of A such that $\forall i, j \in \mathbb{N}, (i \leq j \implies X_i \subseteq X_j)$, $F(\bigcup_{i \in \mathbb{N}} X_i) = \bigcup_{i \in \mathbb{N}} F(X_i)$ holds.

If \bar{X} is the least fixed point of a monotone continuous function, $F: \mathcal{P}(A) \rightarrow \mathcal{P}(A)$, then $\bar{X} = \bigcup_{i \in \mathbb{N}} X_i$ where we define recursively $X_0 = \emptyset$, $\forall i \in \mathbb{N}, X_{i+1} = F(X_i)$.

Proof. Let $\tilde{X} \triangleq \bigcup_{i \in \mathbb{N}} X_i$. We have $X_0 = \emptyset \subseteq X_1$.

If $X_n \subseteq X_{n+1}$, then $X_{n+1} \subseteq X_{n+2}$ since F is monotone. Hence, $\forall n \in \mathbb{N}, X_n \subseteq X_{n+1}$. Therefore, similarly to Exercise 3.3.1, we have $X_m \subseteq X_n$ whenever $m \leq n$. Hence, $F(\tilde{X}) = \bigcup_{i \in \mathbb{N}} F(X_i) = \bigcup_{i \in \mathbb{N}} X_{i+1} = \tilde{X}$; \tilde{X} is a fixed point of F ; hence $\bar{X} \subseteq \tilde{X}$.

We have $X_0 \subseteq \bar{X}$. If $X_n \subseteq \bar{X}$ for $n \in \mathbb{N}$, then $X_{n+1} \subseteq F(\bar{X}) = \bar{X}$. Hence, by **The Induction Principle**, $\tilde{X} \subseteq \bar{X}$. \square

4.2 Finite Sets

Definition 4.2.1: Finite Set and Infinite Set

A set S is *finite* if it is equipotent to some natural number $n \in \mathbb{N}$. We then define $|S| = n$ and say S has n elements. A set is *infinite* if it is not finite.

Note:-

According to Definition 4.2.1, cardinal numbers of finite sets are the natural numbers. We evidently have $\forall n \in \mathbb{N}, |n| = n$.

Lemma 4.2.2

If $n \in \mathbb{N}$ and $X \subsetneq n$, then there is no $f: n \hookrightarrow X$.

Proof. If $n = 0$, there is no $X \subsetneq n$; the assertion is true.

Assume the assertion holds for n . Suppose there is some $f: (n+1) \hookrightarrow X$ where $X \subsetneq n+1$. There are two cases: $n \in X$ and $n \notin X$.

If $n \notin X$, then $X \subseteq n$, and thus $f|_n: n \hookrightarrow X \setminus \{f(n)\}$; however $X \setminus \{f(n)\} \subsetneq X \subseteq n$, which is a contradiction.

If $n \in X$, then $n = f(k)$ for some $k \leq n$. Define a function g on n by following:

$$g(i) = \begin{cases} f(n) & \text{if } i = k < n \\ f(i) & \text{otherwise.} \end{cases}$$

Then, $g: n \hookrightarrow X \setminus \{n\}$ and $X \setminus \{n\} \subsetneq n$, which is also a contradiction. \square

Corollary 4.2.3

- (i) If $m \neq n$ where $m, n \in \mathbb{N}$, then there is no $f : m \hookrightarrow n$.
- (ii) If $|S| = m$ and $|S| = n$, then $m = n$.
- (iii) \mathbb{N} is infinite.

Proof.

- (i) If $n \neq m$, by (\mathbb{N}, \leq) is **Totally Ordered**, we have $n \subsetneq m$ or $m \subsetneq n$. In either case, we do not have such function by Lemma 4.2.2.
- (ii) By Lemma 4.1.2, we have $|m| = |n|$. (i) asserts that $m = n$; otherwise we cannot have $|m| = |n|$.
- (iii) By Exercise 3.2.3, there exists $f : \mathbb{N} \hookrightarrow X$ where $X \subsetneq \mathbb{N}$. If there exists $n \in \mathbb{N}$ and $g : n \hookrightarrow \mathbb{N}$, $g^{-1} \circ f^{-1} \circ f \circ g$ is a function on n onto a proper subset of n . This contradicts Lemma 4.2.2. \square

Theorem 4.2.4

If X is a finite set and $Y \subseteq X$, then Y is finite.

Proof. We may assume $X = \{x_0, \dots, x_{n-1}\}$, where $\langle x_0, \dots, x_{n-1} \rangle$ is an injective sequence, and $Y \neq \emptyset$.

Let g be a function on a subset of $n \times \mathbb{N}$ into n defined by

$$g(a, -) = \begin{cases} \min\{j \in n \mid a < j \wedge x_j \in Y\} & \text{if it exists} \\ \text{undefined} & \text{otherwise.} \end{cases} \quad [*]$$

By **The Recursion Theorem: Partial Version**, there exists a sequence k of elements in n such that

- (i) $k_0 = \min\{j \in n \mid x_j \in Y\}$. $\triangleright Y \neq \emptyset$
- (ii) $\forall i \in \mathbb{N}, [i + 1 \in \text{dom } k \implies k_{i+1} = g(k_i, i) = \min\{j \in n \mid k_i < j \wedge x_j \in Y\}]$.
- (iii) k is either an infinite sequence or a finite sequence of length $\ell + 1$ and $(k_\ell, \ell) \notin \text{dom } g$.

By (ii) and $[*]$, $\forall i \in \mathbb{N}, (i + 1 \in \text{dom } k \implies k_i < k_{i+1})$. Hence, k is injective. If k were an infinite sequence, i.e., $k : \mathbb{N} \hookrightarrow n$, then $|\mathbb{N}| \leq |n|$. Together with Exercise 4.1.3 and **Cantor–Bernstein Theorem**, we get $|\mathbb{N}| = |n|$, which contradicts **Corollary 4.2.3 (iii)**. Hence, k is a finite sequence of length ℓ .

Let $y_i \triangleq x_{k_i}$ for each $i < \ell$. By (i) and (ii), the sequence y is injective and its range is a subset of Y . By the same argument of Claim 1 of Theorem 3.3.3, we have $\text{ran } y = Y$. Therefore, $y : \ell \hookrightarrow Y$; Y is finite. \square

Theorem 4.2.5

If X is finite and f is a function, then $f[X]$ is finite. Moreover, $|f[X]| \leq |X|$.

Proof. If $f[X] = \emptyset$, then it is done; assume $f[X] \neq \emptyset$. WLOG, $X \subseteq \text{dom } f$.

We may assume $X = \{x_0, \dots, x_{n-1}\}$, where $\langle x_0, \dots, x_{n-1} \rangle$ is an injective sequence. Let g be a function on a subset of $\text{Seq}(n)$ into n defined by

$$g(\langle k_0, \dots, k_{\ell'-1} \rangle) = \begin{cases} 0 & \text{if } \ell' = 0 \\ \min\{k \in n \mid k_{\ell'-1} < k \wedge \forall i < \ell', f(x_{k_i}) \neq f(x_k)\} & \text{if it exists and } \ell' > 0 \\ \text{undefined} & \text{otherwise.} \end{cases} \quad [*]$$

Then, one may modify **The Recursion Theorem: General Version** to its partial version like **The Recursion Theorem: Partial Version** to get a sequence k of elements of n such that:

(i) $\forall i \in \text{dom } k, k_i = g(k|_i)$. In particular, $k_0 = 0$.

(ii) k is either an infinite sequence or a finite sequence of length $\ell + 1$ and $k \notin \text{dom } g$.

By (i) and $[*]$, $\forall i, j \in \text{dom } k, (i \neq j \implies f(x_{k_i}) \neq f(x_{k_j}))$, i.e., the sequence $y = \langle f(x_{k_i}) \mid i \in \text{dom } k \rangle$ is injective and its range is a subset of $f[X]$.

By the similar reason as in the proof of Theorem 4.2.4, k is finite and $\text{ran } y = f[X]$. Finally, we get $|f[X]| \leq |X|$ from $x \circ y^{-1}: f[X] \hookrightarrow X$. \square

Lemma 4.2.6

Let X and Y be finite sets.

(i) $X \cup Y$ is finite; moreover, $|X \cup Y| \leq |X| + |Y|$.

(ii) If X and Y are disjoint, then $|X \cup Y| = |X| + |Y|$.

Proof.

(i) Write $X = \{x_0, \dots, x_{m-1}\}$ and $Y = \{y_0, \dots, y_{n-1}\}$ where $\langle x_0, \dots, x_{m-1} \rangle$ and $\langle y_0, \dots, y_{n-1} \rangle$ are injective sequences.

Now, define $z: (n+m) \rightarrow X \cup Y$ by

$$z_i = x_i \quad \text{for } 0 \leq i < n \quad \text{and} \quad z_i = y_{i-n} \quad \text{for } n \leq i < n+m.$$

(Here, $i-n$ is the unique $k \in \mathbb{N}$ such that $i = n+k$. See Exercise 3.4.3.) Hence, by Theorem 4.2.5, $X \cup Y$ is finite and $|X \cup Y| \leq n+m$.

(ii) If X and Y are disjoint, then $z: (n+m) \hookrightarrow X \cup Y$. Hence, $|X \cup Y| = n+m$. \square

Theorem 4.2.7

If S is finite and if every $X \in S$ is finite, then $\bigcup S$ is finite.

Proof. If $|S| = 0$, then it is done.

Assume that the statement is true for all S with $|S| = n$. Let $S = \{X_0, \dots, X_n\}$ be a set with $n+1$ elements such that each $X_i \in S$ is finite. Then, we have

$$\bigcup S = \left(\bigcup_{i=0}^{n-1} X_i \right) \cup X_n$$

but $\bigcup_{i=0}^{n-1} X_i$ is finite by induction hypothesis and thus $\bigcup S$ is finite by Lemma 4.2.6. Hence, by The Induction Principle, the result follows. \square

Theorem 4.2.8

If X is finite, then $\mathcal{P}(X)$ is finite.

Proof. If $|X| = 0$, then $\mathcal{P}(X) = \{\emptyset\}$, which is indeed finite.

Fix any $n \in \mathbb{N}$ and assume that $\mathcal{P}(X)$ is finite for all X with $|X| = n$. Take any Y with $|Y| = n+1$. Let $Y = \{y_0, \dots, y_n\}$ and $X \triangleq \{y_0, \dots, y_{n-1}\}$. Note that $\mathcal{P}(Y) = \mathcal{P}(X) \cup U$ where $U = \{u \subseteq Y \mid y_n \in u\}$. Moreover, $f: \mathcal{P}(X) \rightarrow U$ defined by $f(x) = x \cup \{y_n\}$ is injective and onto U . Hence, U is finite. By Lemma 4.2.6, $\mathcal{P}(Y)$ is finite. The result follows by The Induction Principle. \square

Theorem 4.2.9

If X is infinite, then $|X| > n$ for all $n \in \mathbb{N}$.

Proof. We clearly have $0 \leq |X|$.

For induction, fix any $n \in \mathbb{N}$ and assume $n \leq |X|$, i.e., there exists $f: n \hookrightarrow X$. By Theorem 4.2.5, $\text{ran } f \subsetneq X$; we may take $x \in X \setminus \text{ran } f$. Then, $g \triangleq f \cup \{(n, x)\}$ is an injective function on $n + 1$ into X ; hence $n + 1 \leq |X|$. Therefore, by The Induction Principle, we have $n \geq |X|$ for all $n \in \mathbb{N}$, which suffices to induce the result. \square

Selected Problems

Exercise 4.2.1

If $S = \{X_0, \dots, X_{n-1}\}$ is a finite set of mutually disjoint sets. Then, $|\bigcup S| = \sum_{i=0}^{n-1} |X_i|$.

Proof. If $S = \emptyset$, then $|\bigcup S| = 0 = \sum_{i=0}^{n-1} |X_i|$.

Fix $n \in \mathbb{N}$ and assume the assertion holds for all S with $|S| = n$. Then, take any set T of mutually disjoint sets with $|T| = n + 1$. Write $T = \{X_0, \dots, X_n\}$ and let $S \triangleq \{X_0, \dots, X_{n-1}\}$. Then, since $\bigcup T = (\bigcup S) \cup X_n$, and since $\bigcup S$ and X_n are disjoint, $|\bigcup T| = |\bigcup S| + |X_n| = \sum_{i=0}^{n-1} |X_i| + |X_n| = \sum_{i=0}^n |X_i|$. Hence, the result follows from The Induction Principle. \square

Exercise 4.2.2

If X and Y are finite, then $|X \times Y| = |X| \cdot |Y|$.

Proof. We shall exploit the induction on $|Y|$. If $|Y| = 0$, then

$$\begin{aligned} |X \times Y| &= 0 &> \text{Exercise 2.2.8} \\ &= |X| \cdot |Y|. &> [3] \end{aligned}$$

Assume the statement holds for all X and Y with $|Y| = n$. Let $Z = \{z_0, \dots, z_n\}$ be a set with $|Z| = n + 1$. Let $Y \triangleq \{z_0, \dots, z_{n-1}\}$. Then, for all X , $X \times Z = (X \times Y) \cup (X \times \{z_n\})$. Note that $X \times \{z_n\}$ can be identified with X via $f: X \hookrightarrow X \times \{z_n\}$ defined by $x \mapsto (x, z_n)$. Hence, if X is finite,

$$\begin{aligned} |X \times Z| &= |X \times Y| + |X \times \{z_n\}| &> \text{Lemma 4.2.6} \\ &= |X \times Y| + |X| &> |X \times \{z_n\}| = |X| \\ &= |X| \cdot |Y| + |X| &> \mathbf{P}(n) \\ &= |X| \cdot (|Y| + 1) &> [4] \\ &= |X| \cdot |Z|. \end{aligned}$$

Therefore, by The Induction Principle, the result follows. \square

Exercise 4.2.3

If X is finite, $|\mathcal{P}(X)| = 2^{|X|}$.

Proof. Let $\mathbf{P}(x)$ be the property “ $\forall X, (|X| = x \implies |\mathcal{P}(X)| = 2^{|X|})$.” $\mathbf{P}(0)$ holds since $|\mathcal{P}(\emptyset)| = |\{\emptyset\}| = 1 = 2^0$.

Fix $n \in \mathbb{N}$ and assume $\mathbf{P}(n)$. Let $Y = \{y_0, \dots, y_n\}$ be a set with $|Y| = n + 1$. Let $X \triangleq \{y_0, \dots, y_{n-1}\}$. As in the proof of Theorem 4.2.8, $\mathcal{P}(Y) = \mathcal{P}(X) \cup U$ where $U = \{u \subseteq$

$Y \mid y_n \in u\}$. Note that $\mathcal{P}(X) \cap U = \emptyset$ and $f: \mathcal{P}(X) \hookrightarrow U$ defined by $x \mapsto x \cup \{y_n\}$ asserts $|\mathcal{P}(X)| = |U|$. Therefore,

$$\begin{aligned}
 |\mathcal{P}(Y)| &= |\mathcal{P}(X)| + |U| &> \text{Lemma 4.2.6} \\
 &= 2^n + 2^n &> |\mathcal{P}(X)| = |U|, \mathbf{P}(n) \\
 &= 2^n \cdot 1 + 2^n \cdot 1 &> \text{Lemma 3.4.11} \\
 &= 2^n \cdot 2 &> \cdot \text{Distributes Over } + \\
 &= 2^{n+1}. &> [6]
 \end{aligned}$$

Therefore, by **The Induction Principle**, the result follows. \square

Exercise 4.2.4

If X and Y are finite, then X^Y is finite and $|X^Y| = |X|^{|Y|}$.

Proof. Let $\mathbf{P}(x)$ be the property “if X is finite and $|Y| = x$, then $|X^Y| = |X|^x$.” $\mathbf{P}(0)$ holds since $|X^\emptyset| = |\{\emptyset\}| = 1 = |X|^0$ for all X .

Fix $n \in \mathbb{N}$ and assume $\mathbf{P}(n)$. Let $Y = \{y_0, \dots, y_n\}$ be a set with $|Y| = n + 1$. Let $Z \triangleq \{y_0, \dots, y_{n-1}\}$. Take any finite set X .

We have $|X^Y| = |X^Z \times X|$ since we may define $f: X^Y \hookrightarrow X^Z \times X$ by $g \mapsto (g|_Z, g(y_n))$. Hence,

$$\begin{aligned}
 |X^Y| &= |X^Z \times X| \\
 &= |X^Z| \cdot |X| &> \text{Exercise 4.2.1} \\
 &= |X|^n \cdot |X| &> \mathbf{P}(n) \\
 &= |X|^{n+1}. &> [6]
 \end{aligned}$$

The result follows by **The Induction Principle**. \square

Exercise 4.2.6

X is finite if and only if every $\emptyset \subsetneq U \subseteq \mathcal{P}(X)$ has a \subseteq -maximal element.

Proof.

(\Rightarrow) Let $|X| = n$ and $\emptyset \subsetneq U \subseteq \mathcal{P}(X)$. Since $|Y| \leq n$ for all $Y \in U$, by Theorem 3.2.8, we may let $m \triangleq \max\{|Y| \mid Y \in U\}$.

There exists $Y \in U$ with $|Y| = m$. Then, for each $Y' \in U$ such that $Y \subseteq Y'$, we have $m \leq |Y'|$ by Exercise 4.1.3 and $|Y'| \leq m$ by definition of m ; thus $|Y'| = |Y| = m$ by **Cantor–Bernstein Theorem**, which implies we may not have $Y \subsetneq Y'$ by Lemma 4.2.2. Hence, Y is a maximal element of U .

(\Leftarrow) Assume X is infinite. Let $U = \{Y \subseteq X \mid Y \text{ is finite}\}$. (Note $\emptyset \in U$, hence $U \neq \emptyset$.) Take any $Y \in U$. Since $Y \subsetneq X$, we may take $x \in X \setminus Y$. Then, $Y \subsetneq Y \cup \{x\}$ and $Y \cup \{x\} \in U$ by Lemma 4.2.6. Hence, there is no maximal element of U . \square

4.3 Countably Infinite sets

Definition 4.3.1: Countably Infinite Set

- A set S is *countably infinite* if $|S| = |\mathbb{N}|$.
- A set S is *countable* if $|S| \leq |\mathbb{N}|$.
- $|\mathbb{N}| = \aleph_0$, i.e., the cardinality of countably infinite sets is \aleph_0 .

Note:-

In the book, the author uses the term ‘countable’ and ‘at most countable’ for $|S| = |\mathbb{N}|$ and $|S| \leq |\mathbb{N}|$, respectively.

Notation 4.3.2: Cardinality of Countably Infinite Sets

We use the symbol \aleph_0 (read *aleph-naught*) to denote the cardinality of countably infinite sets, i.e., $\aleph_0 = \aleph$.

Theorem 4.3.3

A subset of a countably infinite set is countable.

Proof. Assume A is countably infinite and $B \subseteq A$ is infinite. Let $\langle a_i \rangle_{i \in \mathbb{N}}$ be an injective sequence whose range is A .

Let $g : \text{Seq}(\mathbb{N}) \rightarrow \mathbb{N}$ be defined by

$$g(k) \triangleq \min \{ i \in \mathbb{N} \mid a_i \in B \setminus \{ a_{k_j} \mid j \in \text{dom } k \} \}.$$

Note that g is well-defined since B is infinite. Then, by **The Recursion Theorem: General Version**, there exists a sequence $\langle k_i \rangle_{i \in \mathbb{N}}$ of natural numbers such that $\forall n \in \mathbb{N}, k_n = g(k|_n)$. By construction, $\langle k_i \rangle_{i \in \mathbb{N}}$ is injective, and thus $\langle a_{k_i} \rangle_{i \in \mathbb{N}}$ is an injective sequence whose range is B by the same argument of Claim 1 of Theorem 3.3.3. \square

Corollary 4.3.4

A set is countable if and only if it is either finite or countably infinite.

Proof.

(\Rightarrow) Let S be countable. Let $f : S \hookrightarrow \mathbb{N}$. Then, $|S| = |\text{ran } f|$ and $\text{ran } f$ is a subset of \mathbb{N} . Hence, by Theorem 4.3.3, S is countably infinite if it is not finite.

(\Leftarrow) Theorem 4.2.9 \square

Theorem 4.3.5

If X is countably infinite and f is a function, then $f[X]$ is countable.

Proof. If $f[X] = \emptyset$, then it is done; assume $f[X] \neq \emptyset$. WLOG, $X \subseteq \text{dom } f$. Let $\langle x_i \rangle_{i \in \mathbb{N}}$ be an injective sequence whose range is X . Let $g : f[X] \rightarrow \mathbb{N}$ be defined by

$$g(y) \triangleq \min \{ i \in \mathbb{N} \mid y = f(x_i) \}.$$

g is injective, and thus $|f[X]| \leq \aleph_0$. \square

Theorem 4.3.6

- (i) If A and B are countably infinite, then $A \times B$ is countably infinite.
- (ii) If A is countably infinite and $B \neq \emptyset$ is finite, then $A \times B$ is countably infinite.
- (iii) If A and B are countable, then $A \times B$ is countable.

Proof.

- (i) The function $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ defined by $f(x, y) = 2^x \cdot 3^y$ is injective by elementary number theory. Also, we have an injection $g : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ defined by $g(x) = (x, 0)$. Hence, by **Cantor–Bernstein Theorem**, we have $|\mathbb{N} \times \mathbb{N}| = \aleph_0$.

(ii) Let $|B| = n$. Then, we have

$$\begin{aligned} |A \times B| &= |\mathbb{N} \times n| \\ &\leq |\mathbb{N} \times \mathbb{N}| &> \text{Exercise 4.1.3} \\ &= \aleph_0. &> \text{Theorem 4.3.6} \end{aligned}$$

Let $b \in B$. Then, we have

$$\begin{aligned} \aleph_0 &= |A| \\ &= |A \times \{b\}| &> a \mapsto (a, b) \\ &\leq |A \times B|. &> \text{Exercise 4.1.3} \end{aligned}$$

Hence, by **Cantor–Bernstein Theorem**, $|A \times B| = \aleph_0$.

(iii) If one of them is empty, then $A \times B = \emptyset$. If A and B are finite, then $A \times B$ is finite by Exercise 4.2.2. If any of them is countably infinite, and if both are nonempty, then $A \times B$ is countably infinite by (i) and (ii). \square

Corollary 4.3.7

Let $\langle A_i \mid i \in n \rangle$ be a system of countably infinite sets where $n > 0$. Then, $\prod_{i=0}^{n-1} A_i$ is countably infinite.

Proof. Let $P(x)$ be the property “ $\prod_{i=0}^{x-1} A_i$ is countably infinite for each system $\langle A_i \mid i \in x \rangle$ of countably infinite sets. $P(1)$ evidently holds.

Fix $n > 0$ and assume $P(n)$. Now, take any system $\langle A_i \mid i \in n+1 \rangle$ of countably infinite sets. Then, since we have a natural mapping $f : \prod_{i=0}^n A_i \hookrightarrow \left(\prod_{i=0}^{n-1} A_i \right) \times A_n$ defined by $\langle a_0, \dots, a_n \rangle \mapsto (\langle a_0, \dots, a_{n-1} \rangle, a_n)$, we get

$$\begin{aligned} \left| \prod_{i=0}^n A_i \right| &= \left| \left(\prod_{i=0}^{n-1} A_i \right) \times A_n \right| \\ &= |\mathbb{N} \times \mathbb{N}| &> P(n) \\ &= \aleph_0. &> \text{Theorem 4.3.6} \end{aligned}$$

Hence, we have $P(n+1)$.

Therefore, by Exercise 3.2.11, the result follows. \square

Theorem 4.3.8

Let $\langle a_n \mid n \in \mathbb{N} \rangle$ countably infinite system of infinite sequences. Then, $\bigcup_{n \in \mathbb{N}} \text{ran } a_n$ is countable.

Proof. Define $f : \mathbb{N} \times \mathbb{N} \rightarrow \bigcup_{n \in \mathbb{N}} \text{ran } a_n$ by $f(n, k) = a_n(k)$. The result follows by Theorem 4.3.5 and Theorem 4.3.6. \square

Note:-

Note that we cannot yet prove the proposition “the union of countably infinite system of countable sets is countable” since, if $\langle A_n \mid n \in \mathbb{N} \rangle$ is the system, we do not have enough tools to show the existence of $\langle a_n \mid n \in \mathbb{N} \rangle$ such that $\text{ran } a_n = A_n$ for each $n \in \mathbb{N}$.

Theorem 4.3.9

If A is countably infinite, then $\text{Seq}(A)$ is countably infinite.

Proof. It is enough to show $\text{Seq}(\mathbb{N}) = \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$ is countably infinite. Fix any $f : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$. Define $\langle a_n \mid n \in \mathbb{N} \rangle$ recursively by

$$\begin{aligned} \forall i \in \mathbb{N}, \quad a_0(i) &= \langle \rangle \\ \forall n, i \in \mathbb{N}, \quad a_{n+1}(i) &= \langle b_0, \dots, b_{n-1}, i_2 \rangle \\ &\text{where } g(i) = (i_1, i_2) \text{ and } a_n(i_1) = \langle b_0, \dots, b_{n-1} \rangle. \end{aligned}$$

The existence is justified by **The Recursion Theorem**. Then, with **The Induction Principle**, it is easy to prove that $\text{ran } a_n = \mathbb{N}^n$ for each $n \in \mathbb{N}$. Hence, by Theorem 4.3.8, $\bigcup_{n \in \mathbb{N}} \mathbb{N}^n$ is countably infinite. \square

Corollary 4.3.10

The set of all finite subsets of a countably infinite set is countably infinite.

Proof. Let A be countably infinite. Let $f : \text{Seq}(A) \rightarrow \mathcal{P}(A)$ by $f(\langle a_0, \dots, a_{n-1} \rangle) = \{a_0, \dots, a_{n-1}\}$. Then, $\text{ran } f$ is countable by Theorem 4.3.5 and Theorem 4.3.9. $\text{ran } f$ is countably infinite since we have an injection $a \mapsto \{a\}$. \square

Theorem 4.3.11

An equivalence on a countably infinite set has at most countably many equivalence classes.

Proof. Let E be an equivalence on a countably infinite set A . Let $f : A \rightarrow A/E$ be defined by $a \mapsto [a]_E$. Hence, by Theorem 4.3.5, A/E is countable. \square

Theorem 4.3.12

Let $\mathfrak{A} = (A, \langle R_0, \dots, R_{m-1} \rangle, \langle F_0, \dots, F_{n-1} \rangle)$ be a structure. If $C \subseteq A$ is countable, then \overline{C} is also countable.

Proof. Theorem 3.5.15 says that $\overline{C} = \bigcup_{i \in \mathbb{N}} C_i$ where $C_0 = C$ and $C_{i+1} = C_i \cup \bigcup_{j=0}^{n-1} F_j[C_i^{f_j}]$.

Let $c : \mathbb{N} \rightarrow C$. Let $g : \mathbb{N} \rightarrow (n+1) \times \mathbb{N} \times \mathbb{N}^{f_0} \times \dots \times \mathbb{N}^{f_{n-1}}$. Now, define $\langle a_i \mid i \in \mathbb{N} \rangle$ recursively by

$$\begin{aligned} \forall k \in \mathbb{N} \quad a_0(k) &\triangleq c(k) \\ \forall i, k \in \mathbb{N}, \quad a_{i+1}(k) &\triangleq \begin{cases} F_p(a_i(r_p^0), \dots, a_i(r_p^{f_p-1})) & \text{if } 0 \leq p < n \\ a_i(q) & \text{if } p = n \end{cases} \\ &\text{where } g(k) = \langle p, q, \langle r_0^0, \dots, r_0^{f_0-1} \rangle, \dots, \langle r_{n-1}^0, \dots, r_{n-1}^{f_{n-1}-1} \rangle \rangle. \end{aligned}$$

It is apparent by **The Induction Principle** that $\text{ran } a_i = C_i$ for each $i \in \mathbb{N}$. Hence, by Theorem 4.3.8, \overline{C} is countable. \square

Selected Problems

Exercise 4.3.1

Let $|A_1| = |A_2|$ and $|B_1| = |B_2|$.

- (i) If $A_1 \cap B_1 = \emptyset$ and $A_2 \cap B_2 = \emptyset$, then $|A_1 \cup B_1| = |A_2 \cup B_2|$.
- (ii) $|A_1 \times B_1| = |A_2 \times B_2|$
- (iii) $|\text{Seq}(A_1)| = |\text{Seq}(A_2)|$

Proof.

- (i) Let $f_A: A_1 \hookrightarrow A_2$ and $f_B: B_1 \hookrightarrow B_2$. Then, $f_A \cup f_B: A_1 \cup B_1 \hookrightarrow A_2 \cup B_2$.
- (ii) Let $f_A: A_1 \hookrightarrow A_2$ and $f_B: B_1 \hookrightarrow B_2$. We may define $g: A_1 \times B_1 \hookrightarrow A_2 \times B_2$ by $(a, b) \mapsto (f_A(a), f_B(b))$.
- (iii) Let $f: A_1 \hookrightarrow A_2$. We may define $g: \text{Seq}(A_1) \hookrightarrow \text{Seq}(A_2)$ by

$$\langle a_0, \dots, a_{n-1} \rangle \mapsto \langle f(a_0), \dots, f(a_{n-1}) \rangle.$$

□

Exercise 4.3.2

If A is finite and B is countably infinite, then $A \cup B$ is countably infinite.

Proof. Let $f_A: A \hookrightarrow \mathbb{N}$ and $f_B: B \hookrightarrow \mathbb{N}$. Then, we may define $g: A \cup B \hookrightarrow \mathbb{N} \times \mathbb{N}$ by

$$g(x) = \begin{cases} (f_A(x), 0) & \text{if } x \in A \\ (f_B(x), 1) & \text{if } x \in B \setminus A \end{cases}$$

Hence, $|A \cup B| \leq \aleph_0$ by Theorem 4.3.6. Moreover, $\aleph_0 = |B| \leq |A \cup B|$ by Exercise 4.1.3. The result follows from Cantor–Bernstein Theorem. □

Exercise 4.3.4

If A is finite and nonempty, then $\text{Seq}(A)$ is countably infinite.

Proof. Let $B \triangleq A \cup \mathbb{N}$. Then, by Exercise 4.3.2, B is countably infinite and $\text{Seq}(B)$ is countably infinite by Theorem 4.3.9. Hence, as $\text{Seq}(A) \subseteq \text{Seq}(B)$, $|\text{Seq}(A)| \leq \aleph_0$.

Fix any $a \in A$. Let s be the infinite sequence with $\forall i \in \mathbb{N}, s_i = a$. Then, we have $f: \mathbb{N} \hookrightarrow \text{Seq}(A)$ defined by $f(n) = s|_n$; thus $\aleph_0 \leq |\text{Seq}(A)|$. The result follows from Cantor–Bernstein Theorem. □

Exercise 4.3.5

Let A be countably infinite. The set $[A]^n = \{S \subseteq A \mid |S| = n\}$ is countably infinite for all $n > 0$.

Proof. It is enough to show that $[\mathbb{N}]^n$ is countably infinite for all $n > 0$. Evidently, $i \mapsto \{i\}$ is an injective mapping on \mathbb{N} onto $[\mathbb{N}]^1$. Hence, $|[\mathbb{N}]^1| = \aleph_0$.

For the sake of induction, fix $n > 0$ and assume $|[\mathbb{N}]^n| = \aleph_0$. We may define $f: [\mathbb{N}]^n \hookrightarrow [\mathbb{N}]^{n+1}$ by

$$f(x) \triangleq x \cup \{ \max\{i \in \mathbb{N} \mid i \in x\} + 1 \}.$$

Hence, $\aleph_0 \leq |[\mathbb{N}]^{n+1}|$.

Now, since $|[\mathbb{N}]^n| = |\mathbb{N}^n| = \aleph_0$ by Corollary 4.3.7, there exists an injection $g: [\mathbb{N}]^n \hookrightarrow \mathbb{N}^n$. We define $h: [\mathbb{N}]^{n+1} \hookrightarrow \mathbb{N}^{n+1}$ by

$$h(x) \triangleq g(x \setminus \{i\}) \cup \{(n, i)\} \\ \text{where } i = \max x.$$

Hence, $|[\mathbb{N}]^{n+1}| \leq |\mathbb{N}^{n+1}| = \aleph_0$. Exercise 3.2.11 assures that $\forall n > 0, |[\mathbb{N}]^n| = \aleph_0$. □

Exercise 4.3.10

Let (S, \preceq) be a linearly ordered set and let $\langle A_n \mid n \in \mathbb{N} \rangle$ be an infinite sequence of finite subsets of S . Then, $\bigcup_{n=0}^{\infty} A_n$ is countable.

Proof. WLOG, $A_n \neq \emptyset$ for each $n \in \mathbb{N}$.

Claim 1. For each finite $A \subseteq S$, there uniquely exists a unique isomorphism between $(|A|, \leq_{|A|^2})$ and (A, \preceq_{A^2}) .

Proof. We have existence for each A by ???. Hence, we only prove the uniqueness by induction. If $|A| = 0$, we have only one isomorphism \emptyset .

Fix some $n \in \mathbb{N}$ and assume the proposition holds for all A with cardinality n . Take any $A \subseteq S$ with $|A| = n + 1$. Let f and g be two isomorphisms between $(n + 1, \leq_{(n+1)^2})$ and (A, \preceq_{A^2}) . Then, $f(n) = g(n)$ since the greatest element is unique. Let $B = A \setminus \{f(n)\}$. Then, $f|_B$ and $g|_B$ are isomorphisms between (n, \leq_{n^2}) and (B, \preceq_{B^2}) . Hence, $f|_B = g|_B$, and thus $f = g$. The result follows from **The Induction Principle**.

Claim 1 enables us to guarantee the existence of infinite sequence $\langle a_n \mid n \in \mathbb{N} \rangle$ such that, for each $n \in \mathbb{N}$:

- (i) $a_n|_{|A_n|}$ is the isomorphism between $(|A_n|, \leq_{|A_n|^2})$ and $(A_n, \preceq_{A_n^2})$.
- (ii) $\forall k \geq |A_n|, a_n(k) = a_n(0)$.

Hence, $\text{ran } a_n = A_n$ for each $n \in \mathbb{N}$, and thus $\bigcup_{n=0}^{\infty} A_n$ is countable by Theorem 4.3.8. \square

Exercise 4.3.11

Any partition of a countable set has a set of representatives.

Proof. Let A be countable and S be a partition of A . There exists $f : A \hookrightarrow \mathbb{N}$. Then,

$$X \triangleq \{f^{-1}(\min f[C]) \mid C \in S\}$$

is a set of representatives. \square

Chapter 5

Cardinal Numbers

Chapter 6

Ordinal Numbers

Chapter 7

Alephs

Chapter 8

Axiom of Choice

End.