Summary for Introduction to Set Theory

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Contents

Chapter 1

Sets

Chapter 2

Relations, Function, and Ordering

Chapter 3

Natural Numbers

Chapter 4

Finite, Countable, and Uncountable Sets

4.1 Cardinality of Sets

Definition 4.1.1: Equipotent Sets

Let A and B be sets. A is *equipotent to* B if there is a function $f : A \hookrightarrow\!\!\!\!\rightarrow B$. We write $|A| = |B|$.

Lemma 4.1.2

Let A, B, and C be sets.

(i) $|A| = |A|$.

(ii) If $|A| = |B|$, then $|B| = |A|$.

(iii) If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Proof.

(i) Id_A is an injective function on A onto A.

(ii) If $f : A \hookrightarrow\!\!\!\!\rightarrow B$, then $f^{-1} : B \hookrightarrow\!\!\!\!\rightarrow A$.

(iii) If $f : A \hookrightarrow\!\!\!\!\rightarrow B$, and if $g : B \hookrightarrow\!\!\!\!\rightarrow C$, then $f \circ g : A \hookrightarrow\!\!\!\!\rightarrow C$. □

Note:-

Lemma 4.1.2 essentially says that $|A| = |B|$ behaves like an equivalence relation.

Definition 4.1.3

- We say *the cardinality of A is less than or equal to the cardinality of B* if there is a function $f : A \hookrightarrow B$. We write $|A| \le |B|$.
- We say *the cardinality of A is less than the cardinality of B* if $|A| \le |B|$ and $\neg(|A| = |B|)$. We write $|A| < |B|$.

Lemma 4.1.4

Let A, B, and C be sets.

(i) If $|A| = |B|$, then $|A| \le |B|$.

(ii) $|A| \le |A|$

(iii) If $|A| \le |B|$ and $|B| \le |C|$, then $|A| \le |C|$.

Proof.

(i) If $f : A \hookrightarrow\!\!\!\!\rightarrow B$, then f is injective as well.

(ii) Id_A is an injective function on A into A.

(iii) If $f : A \hookrightarrow B$, and if $g : B \hookrightarrow C$, then $f \circ g : A \hookrightarrow C$. □

Lemma 4.1.5

If $A_1 \subseteq B \subseteq A$ and $|A_1| = |A|$, then $|B| = |A|$.

Note:-

We present two proofs for Lemma 4.1.5. The second proof can be viewed as a more fundamental proof in the sense that it does not depend on **??**.

Proof 1. Let $f : A \hookrightarrow\!\!\!\!\rightarrow A_1$. Define a sequence $\langle A_i \mid i \in \mathbb{N} \rangle$ and $\langle B_i \mid i \in \mathbb{N} \rangle$ recursively by

$$\begin{aligned} A_0 &= A, & B_0 &= B, \\ \forall n \in \mathbb{N},\ A_{n+1} &= f[A_n], & \forall n \in \mathbb{N},\ B_{n+1} &= f[B_n] \end{aligned} \qquad [*]$$

thanks to **??**.

We clearly have $A_1 \subseteq B_0 \subseteq A_0$. If $A_{n+1} \subseteq B_n \subseteq A_n$, then $A_{n+2} = f[A_{n+1}] \subseteq B_{n+1} = f[B_n] \subseteq A_{n+1} = f[A_n]$ by $[*]$. Hence, by $[*]$ and **??**, we have $A_{n+1} \subseteq B_n \subseteq A_n$ for all $n \in \mathbb{N}$.

Let, for each $n \in \mathbb{N}$, $C_n \triangleq A_n \setminus B_n$. Then, by **??**, $C_{n+1} = f[A_n] \setminus f[B_n] = f[A_n \setminus B_n] = f[C_n]$. Let

$$C \triangleq \bigcup_{n=0}^{\infty} C_n \quad \text{and} \quad D \triangleq A \setminus C.$$

Hence, $f[C] = \bigcup_{n=1}^{\infty} C_n \subseteq C$. Now, define a function $g : A \to A$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in C \\ x & \text{if } x \in D. \end{cases}$$

We immediately notice that $g\big|_C = f\big|_C$ and $g\big|_D$ are injective and their ranges—$f[C]$ and D—are disjoint; g is injective.

As, $\forall n \geq 1$, $C_n \subseteq A_n \subseteq B_0 = B$, we have $f[C] \subseteq B$. If $x \in D$, then $x \in A \setminus C_0 = A \setminus (A \setminus B) = B$ by **??**.

Now, we shall show $B \subseteq f[C] \cup D$ and thus $B = \operatorname{ran} g$. Take any $y \in B$. Then, $y \in C$ or $y \in D$. If $y \in D$, then it is done; so assume $y \in C$. Then, as $y \notin A \setminus B = C_0$, $y \in f[C]$. Hence, $g : A \hookrightarrow\!\!\!\!\rightarrow B$. □

Proof 2. Let $f : A \hookrightarrow\!\!\!\!\rightarrow A_1$. Let $F : \mathcal{P}(A) \to \mathcal{P}(A)$ be defined by $F(X) = (A \setminus B) \cup f[X]$. If $X \subseteq Y \subseteq A$, then $F(X) = (A \setminus B) \cup f[X] \subseteq (A \setminus B) \cup f[Y] = F(Y)$. Hence, by Exercise 4.1.10, there exists $C \subseteq A$ such that

$$C = (A \setminus B) \cup f[C].$$

Let $D \triangleq A \setminus C$.

Now, define a function $g : A \to A$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in C \\ x & \text{if } x \in D. \end{cases}$$

Then, since $f[C] \subseteq C$, g is injective.

Moreover, $f[C] \subseteq \operatorname{ran} f = A_1 \subseteq B$ and $D = A \setminus C = A \setminus ((A \setminus B) \cup f[C]) \subseteq A \setminus (A \setminus B) = B$, and thus $\operatorname{ran} g \subseteq B$.

Now, take any $y \in B$. If $y \in C$, then, as $y \notin A \setminus B$, $y \in f[C]$. Hence, $B \subseteq f[C] \cup D$. Therefore, $g : A \hookrightarrow\!\!\!\!\rightarrow B$. □

Theorem 4.1.6 Cantor–Bernstein Theorem

If $|X| \le |Y|$ and $|Y| \le |X|$, then $|X| = |Y|$.

Proof. Let $f : X \hookrightarrow Y$ and $g : Y \hookrightarrow X$. Then, $g : Y \hookrightarrow\!\!\!\!\rightarrow g[Y]$, i.e., $|Y| = |g[Y]|$; and $g \circ f : X \hookrightarrow\!\!\!\!\rightarrow (g \circ f)[X]$, i.e., $|X| = |(g \circ f)[X]|$. Moreover, $(g \circ f)[X] \subseteq g[Y] \subseteq X$. Hence, by Lemma 4.1.5, $|g[Y]| = |X|$. We conclude $|X| = |Y|$ from Lemma 4.1.2. □

Assumption 4.1.7

There are sets called *cardinal numbers* (or *cardinals*) with the property that for every set X there is a unique cardinal $|X|$ (the *cardinal number of* X, the *cardinality of* X) and sets X and Y are equipotent if and only if $|X|$ is equal to Y.

Note:-

Assumption 4.1.7 essentially asserts the existence of a unique "representative" for each class of mutually equipotent sets. Assumption 4.1.7 is *harmless* in the sense that we only use it for convenience and we could formulate the theorems without it. We prove Assumption 4.1.7 in **??**: **??**. However, for certain classes of sets, cardinal numbers can be defined without the Axiom of Choice.

Selected Problems

Exercise 4.1.2

Let A, B, and C be sets.

(i) If $|A| < |B|$ and $|B| \le |C|$, then $|A| < |C|$.
(ii) If $|A| \le |B|$ and $|B| < |C|$, then $|A| < |C|$.

Proof.

(i) We already have $|A| \le |C|$ by Lemma 4.1.4 (iii). Let $g : B \hookrightarrow C$. Suppose $f : A \hookrightarrow\!\!\!\!\rightarrow C$ for the sake of contradiction. Then, $f^{-1} \circ g : B \hookrightarrow A$, i.e., $|B| \le |A|$. By Cantor–Bernstein Theorem, we get $|A| = |B|$, which is a contradiction.
(ii) We already have $|A| \le |C|$ by Lemma 4.1.4 (iii). Let $g : A \hookrightarrow B$. Suppose $f : A \hookrightarrow\!\!\!\!\rightarrow C$ for the sake of contradiction. Then, $g \circ f^{-1} : C \hookrightarrow B$, i.e., $|C| \le |B|$. By Cantor–Bernstein Theorem, we get $|B| = |C|$, which is a contradiction.

□

Exercise 4.1.3

If $A \subseteq B$, then $|A| \le |B|$.

Proof. Id_A is an injective function on A into B. □

Exercise 4.1.7

If $S \subseteq T$, then $|A^S| \le |A^T|$. In particular, $|A^m| \le |A^n|$ if $m \le n$.

Proof. If $T = \varnothing$, then $A^S = A^T = \{\varnothing\}$ and it is done.

Assume $T \ne \varnothing$. Fix some $t \in T$. Now, define $f : A^S \hookrightarrow A^T$ by $g \mapsto g \cup \{(x, t) \mid x \in T \setminus S\}$. □

Exercise 4.1.10

Let $F : \mathcal{P}(A) \to \mathcal{P}(A)$ be *monotone*, i.e., if $X \subseteq Y \subseteq A$, then $F(X) \subseteq F(Y)$. Then, F has a least *fixed point* $\overline{X}$, that is to say $F(\overline{X}) = \overline{X}$ and $\forall X \subseteq A$, $(F(X) = X \implies \overline{X} \subseteq X)$.

Proof. Let $T \triangleq \{X \subseteq A \mid F(X) \subseteq X\}$. Then, as $A \in T$, $T \neq \varnothing$; we may let $\overline{X} \triangleq \bigcap T$.

Then, for all $X \in T$, $\overline{X} \subseteq X$; and thus $F(\overline{X}) \subseteq F(X) \subseteq X$. We have $F(\overline{X}) \subseteq \bigcap T = \overline{X}$, i.e., $\overline{X} \in T$.

On the other hand, we have $F(F(\overline{X})) \subseteq F(\overline{X})$, or $F(\overline{X}) \in T$, and thus $\overline{X} = \bigcap T \subseteq F(\overline{X})$. Therefore, $F(\overline{X}) = \overline{X}$. Moreover, if X is a fixed point, then $X \in T$, and thus $\overline{X} = \bigcap T \subseteq X$. □

Exercise 4.1.14

A function $F : \mathcal{P}(A) \to \mathcal{P}(A)$ is *continuous* if, for each sequence $\langle X_i \mid i \in \mathbb{N}\rangle$ of subsets of A such that $\forall i, j \in \mathbb{N}$, $(i \le j \implies X_i \subseteq X_j)$, $F\left(\bigcup_{i\in\mathbb{N}} X_i\right) = \bigcup_{i\in\mathbb{N}} F(X_i)$ holds.

If $\overline{X}$ is the least fixed point of a monotone continuous function, $F : \mathcal{P}(A) \to \mathcal{P}(A)$, then $\overline{X} = \bigcup_{i\in\mathbb{N}} X_i$ where we define recursively $X_0 = \varnothing$, $\forall i \in \mathbb{N}$, $X_{i+1} = F(X_i)$.

Proof. Let $\tilde{X} \triangleq \bigcup_{i\in\mathbb{N}} X_i$. We have $X_0 = \varnothing \subseteq X_1$.

If $X_n \subseteq X_{n+1}$, then $X_{n+1} \subseteq X_{n+2}$ since F is monotone. Hence, $\forall n \in \mathbb{N}$, $X_n \subseteq X_{n+1}$. Therefore, similarly to **??**, we have $X_m \subseteq X_n$ whenever $m \le n$. Hence, $F(\tilde{X}) = \bigcup_{i\in\mathbb{N}} F(X_i) = \bigcup_{i=1}^{\infty} X_i = \tilde{X}$; $\tilde{X}$ is a fixed point of F; hence $\overline{X} \subseteq \tilde{X}$.

We have $X_0 \subseteq \overline{X}$. If $X_n \subseteq \overline{X}$ for $n \in \mathbb{N}$, then $X_{n+1} \subseteq F(\overline{X})) = \overline{X}$. Hence, by **??**, $\tilde{X} \subseteq \overline{X}$. □

4.2 Finite Sets

Definition 4.2.1: Finite Set and Infinite Set

A set S is *finite* if it is equipotent to some natural number $n \in \mathbb{N}$. We then define $|S| = n$ and say *S has n elements*. A set is *infinite* if it is not finite.

Note:-

According to Definition 4.2.1, cardinal numbers of finite sets are the natural numbers. We evidently have $\forall n \in \mathbb{N}$, $|n| = n$.

Lemma 4.2.2

If $n \in \mathbb{N}$ and $X \subsetneq n$, then there is no $f : n \hookrightarrow\!\!\!\!\rightarrow X$.

Proof. If $n = 0$, there is no $X \subsetneq n$; the assertion is true.

Assume the assertion holds for n. Suppose there is some $f : (n+1) \hookrightarrow\!\!\!\!\rightarrow X$ where $X \subsetneq n+1$. There are two cases: $n \in X$ and $n \notin X$.

If $n \notin X$, then $X \subseteq n$, and thus $f\big|_n : n \hookrightarrow\!\!\!\!\rightarrow X \setminus \{f(n)\}$; however $X \setminus \{f(n)\} \subsetneq X \subseteq n$, which is a contradiction.

If $n \in X$, then $n = f(k)$ for some $k \le n$. Define a function g on n by following:

$$g(i) = \begin{cases} f(n) & \text{if } i = k < n \\ f(i) & \text{otherwise.} \end{cases}$$

Then, $g : n \hookrightarrow\!\!\!\!\rightarrow X \setminus \{n\}$ and $X \setminus \{n\} \subsetneq n$, which is also a contradiction. □

Corollary 4.2.3

(i) If $m \neq n$ where $m, n \in \mathbb{N}$, then there is no $f : m \hookrightarrow\!\!\!\rightarrow n$.
(ii) If $|S| = m$ and $|S| = n$, then $m = n$.
(iii) $\mathbb{N}$ is infinite.

Proof.

(i) If $n \neq m$, by **??**, we have $n \subsetneq m$ or $m \subsetneq n$. In either case, we do not have such function by Lemma 4.2.2.
(ii) By Lemma 4.1.2, we have $|m| = |n|$. (i) asserts that $m = n$; otherwise we cannot have $|m| = |n|$.
(iii) By **??**, there exists $f : \mathbb{N} \hookrightarrow\!\!\!\rightarrow X$ where $X \subsetneq \mathbb{N}$. If there exists $n \in \mathbb{N}$ and $g : n \hookrightarrow\!\!\!\rightarrow \mathbb{N}$, $g^{-1} \circ f^{-1} \circ f \circ g$ is a function on n onto a proper subset of n. This contradicts Lemma 4.2.2. □

Theorem 4.2.4

If X is a finite set and $Y \subseteq X$, then Y is finite.

Proof. We may assume $X = \{x_0, \cdots, x_{n-1}\}$, where $\langle x_0, \cdots, x_{n-1} \rangle$ is an injective sequence, and $Y \neq \varnothing$.

Let g be a function on a subset of $n \times \mathbb{N}$ into n defined by

$$g(a, -) = \begin{cases} \min\{ j \in n \mid a < j \wedge x_j \in Y \} & \text{if it exists} \\ \text{undefined} & \text{otherwise.} \end{cases} \qquad [*]$$

By **??**, there exists a sequence k of elements in n such that

(i) $k_0 = \min\{ j \in n \mid x_j \in Y \}$. $\quad \triangleright\ Y \neq \varnothing$
(ii) $\forall i \in \mathbb{N}$, $[i + 1 \in \operatorname{dom} k \implies k_{i+1} = g(k_i, i) = \min\{ j \in n \mid k_i < j \wedge x_j \in Y \}]$.
(iii) k is either an infinite sequence or a finite sequence of length $\ell + 1$ and $(k_\ell, \ell) \notin \operatorname{dom} g$.

By (ii) and [$*$], $\forall i \in \mathbb{N}$, $(i + 1 \in \operatorname{dom} k \implies k_i < k_{i+1})$. Hence, k is injective. If k were an infinite sequence, i.e., $k : \mathbb{N} \hookrightarrow n$, then $|\mathbb{N}| \leq |n|$. Together with Exercise 4.1.3 and Cantor–Bernstein Theorem, we get $|\mathbb{N}| = |n|$, which contradicts Corollary 4.2.3 (iii). Hence, k is a finite sequence of length ℓ.

Let $y_i \triangleq x_{k_i}$ for each $i < \ell$. By (i) and (ii), the sequence y is injective and its range is a subset of Y. By the same argument of **??** of **??**, we have $\operatorname{ran} y = Y$. Therefore, $y : \ell \hookrightarrow\!\!\!\rightarrow Y$; Y is finite. □

Theorem 4.2.5

If X is finite and f is a function, then $f[X]$ is finite. Moreover, $|f[X]| \leq |X|$.

Proof. We may assume $X = \{x_0, \cdots, x_{n-1}\}$, where $\langle x_0, \cdots, x_{n-1} \rangle$ is an injective sequence. Let g be a function on a subset of $\operatorname{Seq}(n)$ into n defined by

$$g(\langle k_0, \cdots, k_{\ell'-1} \rangle) = \begin{cases} 0 & \text{if } \ell' = 0 \\ \min\{ k \in n \mid k_{\ell'-1} < k \wedge \forall i < \ell', f(x_k) \neq f(x_{k_i}) \} & \text{if it exists and } \ell' > 0 \\ \text{undefined} & \text{otherwise.} \end{cases} \qquad [*]$$

Then, one may modify **??** to its partial version like **??** to get a sequence k of elements of n such that:

(i) $\forall i \in \operatorname{dom} k$, $k_i = g\left(k|_i\right)$. In particular, $k_0 = 0$.

(ii) k is either an infinite sequence or a finite sequence of length $\ell+1$ and $k \notin \operatorname{dom} g$.

By (i) and [$*$], $\forall i, j \in \operatorname{dom} k$, $(i \neq j \implies f(x_{k_i}) \neq f(x_{k_j}))$, i.e., the sequence $y = \langle f(x_{k_i}) \mid i \in \operatorname{dom} k \rangle$ is injective and its range is a subset of $f[X]$.

By the similar reason as in the proof of Theorem 4.2.4, k is finite and $\operatorname{ran} y = f[X]$. Finally, we get $|f[X]| \leq |X|$ from $x \circ y^{-1} \colon f[X] \hookrightarrow X$. □

Lemma 4.2.6

Let X and Y be finite sets.

(i) $X \cup Y$ is finite; moreover, $|X \cup Y| \leq |X| + |Y|$.

(ii) If X and Y are disjoint, then $|X \cup Y| = |X| + |Y|$.

Proof.

(i) Write $X = \{x_0, \cdots, x_{m-1}\}$ and $Y = \{y_0, \cdots, y_{n-1}\}$ where $\langle x_0, \cdots, x_{m-1} \rangle$ and $\langle y_0, \cdots, y_{n-1} \rangle$ are injective sequences.

Now, define $z \colon (n+m) \twoheadrightarrow X \cup Y$ by

$$z_i = x_i \quad \text{for } 0 \leq i < n \quad \text{and} \quad z_i = y_{i-n} \quad \text{for } n \leq i < n+m.$$

(Here, $i-n$ is the unique $k \in \mathbb{N}$ such that $i = n+k$. See **??**.) Hence, by Theorem 4.2.5, $X \cup Y$ is finite and $|X \cup Y| \leq n+m$.

(ii) If X and Y are disjoint, then $z \colon (n+m) \hookrightarrow\!\!\!\!\rightarrow X \cup Y$. Hence, $|X \cup Y| = n+m$. □

Theorem 4.2.7

If S is finite and if every $X \in S$ is finite, then $\bigcup S$ is finite.

Proof. If $|S| = 0$, then it is done.

Assume that the statement is true for all S with $|S| = n$. Let $S = \{X_0, \cdots, X_n\}$ be a set with $n+1$ elements such that each $X_i \in S$ is finite. Then, we have

$$\bigcup S = \left(\bigcup_{i=0}^{n-1} X_i \right) \cup X_n$$

but $\bigcup_{i=0}^{n-1} X_i$ is finite by induction hypothesis and thus $\bigcup S$ is finite by Lemma 4.2.6. Hence, by **??**, the result follows. □

Theorem 4.2.8

If X is finite, then $\mathcal{P}(X)$ is finite.

Proof. If $|X| = 0$, then $\mathcal{P}(X) = \{\varnothing\}$, which is indeed finite.

Fix any $n \in \mathbb{N}$ and assume that $\mathcal{P}(X)$ is finite for all X with $|X| = n$. Take any Y with $|Y| = n+1$. Let $Y = \{y_0, \cdots, y_n\}$ and $X \triangleq \{y_0, \cdots, y_{n-1}\}$. Note that $\mathcal{P}(Y) = \mathcal{P}(X) \cup U$ where $U = \{u \subseteq Y \mid y_n \in u\}$. Moreover, $f \colon \mathcal{P}(X) \to U$ defined by $f(x) = x \cup \{y_n\}$ is injective and onto U. Hence, U is finite. By Lemma 4.2.6, $\mathcal{P}(Y)$ is finite. The result follows by **??**. □

Theorem 4.2.9

If X is infinite, then $|X| > n$ for all $n \in \mathbb{N}$.

Proof. We clearly have $0 \leq |X|$.

For induction, fix any $n \in \mathbb{N}$ and assume $n \leq |X|$, i.e., there exists $f \colon n \hookrightarrow X$. By Theorem 4.2.5, $\operatorname{ran} f \subsetneq X$; we may take $x \in X \setminus \operatorname{ran} f$. Then, $g \triangleq f \cup \{(n, x)\}$ is an injective function on $n+1$ into X; hence $n+1 \leq |X|$. Therefore, by **??**, we have $n \geq |X|$ for all $n \in \mathbb{N}$, which is suffices to induce the result. □

Selected Problems

Exercise 4.2.1

If $S = \{X_0, \cdots, X_{n-1}\}$ is a finite set of mutually disjoint sets. Then, $\left|\bigcup S\right| = \sum_{i=0}^{n-1} |X_i|$.

Proof. If $S = \varnothing$, then $\left|\bigcup S\right| = 0 = \sum_{i=0}^{n-1} |X_i|$.

Fix $n \in \mathbb{N}$ and assume the assertion holds for all S with $|S| = n$. Then, take any set T of mutually disjoint sets with $|T| = n+1$. Write $T = \{X_0, \cdots, X_n\}$ and let $S \triangleq= \{X_0, \cdots, X_{n-1}\}$. Then, since $\bigcup T = \left(\bigcup S\right) \cup X_n$, and since $\bigcup S$ and X_n are disjoint, $\left|\bigcup T\right| = \left|\bigcup S\right| + |X_n| = \sum_{i=0}^{n-1} |X_i| + |X_n| = \sum_{i=0}^{n} |X_i|$. Hence, the result follows from **??**. $\square$

Exercise 4.2.2

If X and Y are finite, then $|X \times Y| = |X| \cdot |Y|$.

Proof. We shall exploit the induction on $|Y|$. If $|Y| = 0$, then

$$\begin{aligned} |X \times Y| &= 0 && \triangleright \textbf{??} \\ &= |X| \cdot |Y|. && \triangleright [\textbf{??}] \end{aligned}$$

Assume the statement holds for all X and Y with $|Y| = n$. Let $Z = \{z_0, \cdots, z_n\}$ be a set with $|Z| = n+1$. Let $Y \triangleq \{z_0, \cdots, z_{n-1}\}$. Then, for all X, $X \times Z = (X \times Y) \cup (X \times \{z_n\})$. Note that $X \times \{z_n\}$ can be identified with X via $f : X \hookrightarrow\!\!\!\!\rightarrow X \times \{z_n\}$ defined by $x \mapsto (x, z_n)$. Hence, if X is finite,

$$\begin{aligned} |X \times Z| &= |X \times Y| + |X \times \{z_n\}| && \triangleright \text{Lemma 4.2.6} \\ &= |X \times Y| + |X| && \triangleright |X \times \{z_n\}| = |X| \\ &= |X| \cdot |Y| + |X| && \triangleright \mathbf{P}(n) \\ &= |X| \cdot (|Y| + 1) && \triangleright [\textbf{??}] \\ &= |X| \cdot |Z|. \end{aligned}$$

Therefore, by **??**, the result follows. $\square$

Exercise 4.2.3

If X is finite, $|\mathcal{P}(X)| = 2^{|X|}$.

Proof. Let $\mathbf{P}(x)$ be the property "$\forall X, (|X| = x \implies |\mathcal{P}(X)| = 2^{|X|})$." $\mathbf{P}(0)$ holds since $|\mathcal{P}(\varnothing)| = |\{\varnothing\}| = 1 = 2^0$.

Fix $n \in \mathbb{N}$ and assume $\mathbf{P}(n)$. Let $Y = \{y_0, \cdots, y_n\}$ be a set with $|Y| = n+1$. Let $X \triangleq \{y_0, \cdots, y_{n-1}\}$. As in the proof of Theorem 4.2.8, $\mathcal{P}(Y) = \mathcal{P}(X) \cup U$ where $U = \{u \subseteq Y \mid y_n \in u\}$. Note that $\mathcal{P}(X) \cap U = \varnothing$ and $f : \mathcal{P}(X) \hookrightarrow\!\!\!\!\rightarrow U$ defined by $x \mapsto x \cup \{y_n\}$ asserts $|\mathcal{P}(X)| = |U|$. Therefore,

$$\begin{aligned} |\mathcal{P}(Y)| &= |\mathcal{P}(X)| + |U| && \triangleright \text{Lemma 4.2.6} \\ &= 2^n + 2^n && \triangleright |\mathcal{P}(X)| = |U|, \mathbf{P}(n) \\ &= 2^n \cdot 1 + 2^n \cdot 1 && \triangleright \textbf{??} \\ &= 2^n \cdot 2 && \triangleright \textbf{??} \\ &= 2^{n+1}. && \triangleright [\textbf{??}] \end{aligned}$$

Therefore, by **??**, the result follows. $\square$

Exercise 4.2.4

If X and Y are finite, then X^Y is finite and $|X^Y| = |X|^{|Y|}$.

Proof. Let $\mathbf{P}(x)$ be the property "if X is finite and $|Y| = x$, then $|X^Y| = |X|^x$." $\mathbf{P}(0)$ holds since $|X^\varnothing| = |\{\varnothing\}| = 1 = |X|^0$ for all X.

Fix $n \in \mathbb{N}$ and assume $\mathbf{P}(n)$. Let $Y = \{y_0, \cdots, y_n\}$ be a set with $|Y| = n+1$. Let $Z \triangleq \{y_0, \cdots, y_{n-1}\}$. Take any finite set X.

We have $|X^Y| = |X^Z \times X|$ since we may define $f : X^Y \hookrightarrow\!\!\!\!\rightarrow X^Z \times X$ by $g \mapsto \left(g\big|_Z, g(y_n)\right)$. Hence,

$$\begin{aligned}
|X^Y| &= |X^Z \times X| \\
&= |X^Z| \cdot |X| && \triangleright \text{ Exercise 4.2.2} \\
&= |X|^n \cdot |X| && \triangleright\ \mathbf{P}(n) \\
&= |X|^{n+1}. && \triangleright\ [\mathbf{??}]
\end{aligned}$$

The result follows by **??**. □

Exercise 4.2.6

X is finite if and only if every $\varnothing \subsetneq U \subseteq \mathcal{P}(X)$ has a $\subseteq$-maximal element.

Proof.

($\Rightarrow$) Let $|X| = n$ and $\varnothing \subsetneq U \subseteq \mathcal{P}(X)$. Since $|Y| \le n$ for all $Y \in U$, by **??**, we may let $m \triangleq \max\{|Y| \mid Y \in U\}$.

There exists $Y \in U$ with $|Y| = m$. Then, for each $Y' \in U$ such that $Y \subseteq Y'$, we have $m \le |Y'|$ by Exercise 4.1.3 and $|Y'| \le m$ by definition of m; thus $|Y'| = |Y| = m$ by Cantor–Bernstein Theorem, which implies we may not have $Y \subsetneq Y'$ by Lemma 4.2.2. Hence, Y is a maximal element of U.

($\Leftarrow$) Assume X is infinite. Let $U = \{Y \subseteq X \mid Y \text{ is finite}\}$. (Note $\varnothing \in U$, hence $U \ne \varnothing$.) Take any $Y \in U$. Since $Y \subsetneq X$, we may take $x \in X \setminus Y$. Then, $Y \subsetneq Y \cup \{x\}$ and $Y \cup \{x\} \in U$ by Lemma 4.2.6. Hence, there is no maximal element of U. □

End.