MAS241 해석학 I
Note

한승우

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Contents

Chapter 1

Structure of the Real Numbers

1.1 Completeness of the Real Numbers

Definition 1.1.1: Cauchy Sequence

Let X be a space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is a *Cauchy sequence* if $\|x_n - x_m\| \to 0$ as $n, m \to \infty$.

Definition 1.1.2: Completeness

A set X is *complete* if every Cauchy sequnce has a limit in X, i.e.,

$$x_n \to x_\infty \in X.$$

Definition 1.1.3: Boundedness

Let $\varnothing \neq S \subseteq \mathbb{R}$.

a) S is *bounded above* if $\exists M \in \mathbb{R}$, $\forall x \in S$, $x \leq M$.
   - M is called an *upper bound* of S.

b) S is *bounded below* if $\exists M \in \mathbb{R}$, $\forall x \in S$, $x \geq M$.
   - M is called an *lower bound* of S.

c) S is bounded if S is bounded above and below.

Theorem 1.1.1 Archimedes' Principle

Let ε and M be any two positive real numbers. Then, there exists a k in $\mathbb{N}$ such that $M < k\varepsilon$.

The proof of Theorem 1.1.1 can be done by integrating Theorem 1.1.2 and Theorem 1.1.4.

Definition 1.1.4: Supremum and Infimum

a) Let S be bounded above. Then, the smallest upper bound is called the *supremum* of S, $\sup S$.

b) Let S be bounded below. Then, the largest lower bound is called the *infimum* of S, $\inf S$.

Example 1.1.1

Let $S = \{(-1)^k(1-1/k) \mid k \in \mathbb{N}\}$. It is clear that $-1 < S < 1$; 1 is an upper bound and -1 is a lower bound. We now claim that $\sup S = 1$. To show this, let us assume that $M < 1$ is an upper bound of S. By Archimedes' principle, there exists an natural number k_0 such that $(1-M)/2 < k_0$, which implies $(-1)^{2k_0}(1-1/(2k_0)) > M$; M is not an upper bound. Therefore, 1 is the smallest upper bound. It can be similarly shown that $\inf S = -1$.

Theorem 1.1.2 Completeness Axiom for $\mathbb{R}$

If $\varnothing \neq S \subseteq \mathbb{R}$ and S is bounded above, then $\sup S$ exists in $\mathbb{R}$.

Corollary 1.1.1

If $\varnothing \neq S \subseteq \mathbb{R}$ and S is bounded below, then $\inf S$ exists in $\mathbb{R}$.

Proof. Let $B := \{-x \mid x \in S\}$. Then, $M = \sup S \in \mathbb{R}$ by Theorem 1.1.2. We now claim that $\inf S = -M$.

For all $x \in S$, $-x \in B$, which implies $-x \leq M$, and therefore $x \geq -M$. Thus, $-M$ is a lower bound of S.

Suppose there is a $M_1 > -M$ such that M_1 is a lower bound of S. For all $x \in S$, $x \geq M_1$, which implies $-x \leq -M_1$. Thus, $-M_1$ is an upper bound of B but $-M_1 < M = \sup B$, #.

Therefore, $\inf S = -M \in \mathbb{R}$. $\square$

Example 1.1.2

- $S := \left\{ \sum_{j=0}^{k} \frac{1}{j!} \,\middle|\, k \in \mathbb{N} \right\}$. S is bounded above.
$$\sum_{j=0}^{k} \frac{1}{j!} = 1 + \sum_{j=1}^{k} \frac{1}{j!} \leq 1 + \sum_{j=1}^{k} \frac{1}{2^{j-1}} < 3$$
In fact, $e := \sup S$.
- $S := \left\{ \left(1 + \frac{1}{k}\right)^k \,\middle|\, k \in \mathbb{N} \right\}$. S is bounded above.
$$\left(1 + \frac{1}{k}\right)^k = \sum_{j=0}^{k} \binom{k}{j} \frac{1}{k^j} \leq \sum_{j=0}^{k} \frac{1}{j!} \leq e$$

Theorem 1.1.3

Let S be a finite nonempty subset of $\mathbb{R}$. Then, $\sup S \in S$ and $\inf S \in S$.

Proof. (Induction on $|S|$) For $S = \{x\}$, $x = \inf S = \sup S \in S$.

Take any $k \in \mathbb{N}$ and suppose the statement holds for every S with $|S| = k$. Now, take any $S' \subseteq \mathbb{R}$ such that $|S'| = k+1$. Let $x \in S'$, $\mu := \sup(S' \setminus \{x\})$, and $\nu := \inf(S' \setminus \{x\})$. By the induction hypothesis, $\mu, \nu \in S' \setminus \{x\}$. Letting $\mu' := \max(\mu, x)$ and $\nu' := \min(\nu, x)$, μ' and ν' are the supremum and infimum of S', respectively. Moreover, μ' and ν' are elements of S'. $\square$

Theorem 1.1.4

Let $\varnothing \neq S \subseteq \mathbb{R}$.

- If S is bounded above, then "$\mu = \sup S$ if and only if μ is an upper bound and $\forall \varepsilon \in \mathbb{R}_+, \exists x \in S, \mu - \varepsilon < x \leq \mu$".
- If S is bounded below, then "$\nu = \inf S$ if and only if ν is an lower bound and $\forall \varepsilon \in \mathbb{R}_+, \exists x \in S, \nu \leq x < \nu + \varepsilon$".

Proof. Let S be bounded above. If there is no $x \in S$ in $(\mu - \varepsilon, \mu]$, then $\mu - \varepsilon$ would be a smaller upper bound.

For the converse, assume M is an upper bound and $M < \mu$. Let $\varepsilon := \mu - M > 0$. Then, there is some $x \in S$ such that $M = \mu - \varepsilon < x \leq \mu$, # to M is an upper bound. Therfore, μ is the least upper bound.

The same logic may be applied for bounded below S. □

Proof of Theorem 1.1.1. Let $S := \{k\varepsilon \mid k \in \mathbb{N}\}$. Assume S is bounded above and nonempty. Then, by Theorem 1.1.2, there is $\mu = \sup S$. We also know, from Theorem 1.1.4, that there is $k \in \mathbb{N}$ such such that $\mu - \varepsilon < k\varepsilon \leq \mu$, which implies $\mu < (k+1)\varepsilon$. Since $(k+1)\varepsilon \in S$, μ is not an upper bound of S, which is a contradiction. Therefore, S is not bounded above. In other words, for any $M > 0$, there is some $k \in \mathbb{N}$ such that $M < k\varepsilon$. □

Theorem 1.1.5

Theorem 1.1.1 (Archimedes' principle) is equivalent to the following statement:

$$\forall c \in \mathbb{R}_+, \exists k \in \mathbb{N}, k - 1 \leq c < k.$$

Proof. Assume Archimedes' principle. If $c < 1$, $k = 1$ satisfies, and it is done. Now, let us suppose $c \geq 1$. By Theorem 1.1.1, there is a $k \in \mathbb{N}$ such that $c < k$. We may let $k_0 := \min\{k \in \mathbb{N} \mid k > c\}$ by Well-Ordering of $\mathbb{N}$. We note that $k_0 - 1 \leq c$ since $k_0 - 1 \in \mathbb{N}$ since $k_0 > 1$. Therefore, $k_0 - 1 \leq c < k_0$.

Now, assume "$\forall c \in \mathbb{R}_+, \exists k \in \mathbb{N}, k - 1 \leq c < k$". Take any $M > 0$ and $\varepsilon \in \mathbb{R}_+$ and let $c := M/\varepsilon$. The assumption tells the existence of a $k \in \mathbb{N}$ such that $M/\varepsilon = c < k$, which directly implies $M < k\varepsilon$. □

Theorem 1.1.6

Let c and d be real numbers with $c < d$. Then, $\exists x \in \mathbb{Q}, c < x < d$.

Proof. There are three cases: $0 < c < d$, $c \leq 0 < d$, or $c < d \leq 0$.

Case 1) By Archimedes' principle, $\exists q \in \mathbb{N}, 1 < (d - c)q$, which implies $cq + 1 < dq$. By Theorem 1.1.5, $\exists p \in \mathbb{N}, p - 1 \leq cq < p$ since $cq > 0$. To sum up, $p - 1 \leq cq < p \leq cq + 1 < dq$, which implies $c < p/q < d$.

Case 2) By Archimedes' principle, $\exists q \in \mathbb{N}, 1 < dq$. Then, $c \leq 0 < 1/q < d$ holds.

Case 3) By case 1 and 2, there is $r \in \mathbb{Q}$ such that $-d < r < -c$. Then, $c < -r < d$ holds. □

1.2 Neighborhoods and Limit Points

Definition 1.2.1: Neighborhood and Deleted Neighborhood

For each $x \in \mathbb{R}$ and $r \in \mathbb{R}_+$,

$$N(x;r) := \{y \in \mathbb{R} : |y-x| < r\} = (x-r, x+r)$$

is called the *neighborhood* of x with radius r, and

$$N'(x;r) := \{y \in \mathbb{R} : 0 < |y-x| < r\} = N(x;r) \setminus \{x\}$$

is called the *deleted neighborhood* of x with radius r.

Definition 1.2.2: Limit Point and Isolated Point

For $\varnothing \neq S \subseteq \mathbb{R}$, $x \in \mathbb{R}$ is a *limit point* of S if

$$\forall \varepsilon \in \mathbb{R}_+,\ N'(x,\varepsilon) \cap S \neq \varnothing.$$

If $x \in \mathbb{R}$ is not a limit point of S, then it is called an *isolated point* of S.

Definition 1.2.3: Discrete Set

If $\varnothing \neq S \subseteq \mathbb{R}$ has no limit points, then S is said to be *discrete*.

Example 1.2.1

Let $S := \left\{(-1)^k(1+1/k) \,\middle|\, k \in \mathbb{N}\right\}$. Then, 1 and -1 are limit points of S.

To see 1 is a limit point, take any $\varepsilon \in \mathbb{R}_+$ and, using Theorem 1.1.1, choose a $k \in \mathbb{N}$ such that $1 < (2\varepsilon)k$. Then, $1 < 1 + \frac{1}{2k} = (-1)^{2k}\left(1+\frac{1}{2k}\right) < 1 + \varepsilon$; $N'(1,\varepsilon) \cap S \neq \varnothing$. Therefore, 1 is a limit point.

Theorem 1.2.1

Let $\varnothing \neq S \subseteq \mathbb{R}$. Then, $x \in \mathbb{R}$ is a limit point of S if and only if

$$\exists \varepsilon_0 \in \mathbb{R}_+,\ \forall \varepsilon \in (0, \varepsilon_0),\ N'(x,\varepsilon) \cap S \neq \varnothing.$$

Proof. Trivial; $0 < \varepsilon_1 < \varepsilon_2$ implies $N'(x,\varepsilon_1) \subsetneq N'(x,\varepsilon_2)$. □

Theorem 1.2.2

Let $\varnothing \neq S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$ be a limit point of S. Then, every deleted neighborhood of x must contain infinitely many points of S.

Proof. Assume $N'(x;\varepsilon) \cap S$ were to contain only finitely many points, namely, $N'(x;\varepsilon) \cap S = \{x_1, x_2, \cdots, x_k\}$. Let $S_1 := \left\{|x - x_i| : i \in [k]\right\}$. Since S_1 is finite, we may let x_j be an element of $N'(x;\varepsilon) \cap S$ that satisfies $|x - x_j| = \min S_1 = \inf S_1 > 0$. If we let $\varepsilon_0 := |x - x_j|/2$, $N'(x;\varepsilon_0) \cap S = \varnothing$, #. □

Corollary 1.2.1
If S is a finite subset of $\mathbb{R}$, then S has no limit point.

Example 1.2.2
$\mathbb{Z}$ has no limit point.

Theorem 1.2.3 Bolzano–Weierstrass Theorem
If $S \subseteq \mathbb{R}$ is bounded and has an infinite number of elements, then S has a limit point.

Proof. Since S is bounded, $a_0 := \inf S$ and $b_0 := \sup S$ exist; $S \subseteq [a_0, b_0]$. At least one of $[a_0, (a_0 + b_0)/2]$ and $[(a_0 + b_0)/2, b_0]$ has an infinite number of elements in S, otherwise S must be finite. Choose whichever has an infinite number of elements in S, and let us denote it as $[a_1, b_1]$. Since, $S \cap [a_1, b_1]$ is bounded and has an infinite number of elements, we may find a_2 and b_2 in the same manner. Note that
(a) for every natural number k, $S \cap [a_k, b_k]$ has an infinite number of elements,
(b) $\forall k \in \mathbb{N}$, $b_k - a_k = (b_0 - a_0)/2^k > 0$, and
(c) $\forall k \in \mathbb{N}$, $a_{k-1} \le a_k < b_k \le b_{k-1}$.
The sequence $\{a_k\}_{k=0}^{\infty}$ is bounded above by b_0, and the sequence $\{b_k\}_{k=0}^{\infty}$ is bounded below by a_0. Therefore, we may let $\alpha := \sup\{a_k\}$ and $\beta := \inf\{b_k\}$.

Since a_j is a lower bound of $\{b_k\}_{k=0}^{\infty}$ for all $j \in \mathbb{N}$, $\forall j \in \mathbb{N}$, $a_j \le \beta$. This implies β is an upper bound of $\{a_k\}_{k=0}^{\infty}$, therefore $\alpha \le \beta$. Since $a_j \le \alpha \le \beta \le b_j$ for all $j \in \mathbb{N}$, we get $0 \le \beta - \alpha \le b_j - a_j = (b_0 - a_0)/2^j$. Therefore, $\beta - \alpha = 0$.

We now claim that α is a limit point of S. Take any $\varepsilon \in \mathbb{R}_+$. By Theorem 1.1.4, $\exists k_0 \in \mathbb{N}$, $\alpha - \varepsilon < a_{k_0} \le \alpha$. We may take $k \in \mathbb{N}$ such that $k > k_0$ and $|b_k - a_k| < \varepsilon$ thanks to (b). Since $\alpha \in [a_k, b_k]$, $\alpha - \varepsilon < a_{k_0} \le a_k \le \alpha \le b_k < \alpha + \varepsilon$, which implies $[a_k, b_k] \subseteq N(\alpha; \varepsilon)$.

In conclusion, $S \cap [a_k, b_k]$ has infinitely many elements by (a), and so does $(S \cap [a_k, b_k]) \setminus \{\alpha\}$. $S \cap N'(\alpha; \varepsilon)$ is, therefore, nonempty. □

Definition 1.2.4: Bolzano–Weierstrass Property

We say that a nonempty set X has the *Bolzano–Weierstrass property* if every bounded, infinite subset S of X has a limit point in X.

1.3 The Limit of a Sequence

Definition 1.3.1: Cluster Point

$c \in \mathbb{R}$ is a *cluster point* of the sequence $\{x_k\}$ if,

$$\forall(\varepsilon, k) \in \mathbb{R}_+ \times \mathbb{N},\ \exists k_1 \in \mathbb{N}_{>k},\ x_{k_1} \in N(c;\varepsilon).$$

Lemma 1.3.1

$c \in \mathbb{R}$ is a cluster point of $\{x_k\}$ if and only if $\left\{k \in \mathbb{N} \,\middle|\, x_k \in N(c;\varepsilon)\right\}$ is infinite for every $\varepsilon \in \mathbb{R}_+$.

Proof. ($\Rightarrow$) Suppose $S := \left\{k \in \mathbb{N} \,\middle|\, x_k \in N(c;\varepsilon)\right\}$ is finite for some $\varepsilon \in \mathbb{R}_+$. If S were empty, then, c is not a cluster point by Definition 1.3.1. Therefore, S is nonempty and has a maximum element $k_0 := \max S$ by Theorem 1.1.3. Since c is a cluster point, there is a natural number $k_1 > k_0$ such that $x_{k_1} \in N(c;\varepsilon)$; $k_1 \in S$. This contradicts the maximality of k_0.

($\Leftarrow$) Take any $\varepsilon \in \mathbb{R}_+$ and $k_0 \in \mathbb{N}$. If there is no $k_1 \in \mathbb{N}$ such that $k_1 > k_0$ and $x_{k_1} \in N(c;\varepsilon)$, S will be bounded above by k_0 and finite, which is a contradiction. Therefore, c is a cluster point of S.

□

Definition 1.3.2: Convergence and Divergence of a Sequnce

The sequnce $\{x_k\}$ *converges* to x_0 and x_0 is the *limit* of $\{x_k\}$ if,

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists k_0 \in \mathbb{N},\ \forall k \in \mathbb{N}_{\geq k_0},\ x_k \in N(x_0;\varepsilon).$$

We write $\lim\limits_{k\to\infty} x_k = x_0$. If there is no such x_0, then $\{x_k\}$ *diverges*.

Lemma 1.3.2

$\lim\limits_{x\to\infty} x_k = x_0$ if and only if $\left\{k \in \mathbb{N} \,\middle|\, x_k \notin N(x_0;\varepsilon)\right\}$ is finite for every $\varepsilon \in \mathbb{R}_+$.

Proof. ($\Rightarrow$) Take any $\varepsilon \in \mathbb{R}_+$. There is some $k_0 \in \mathbb{N}$ such that $k \in N(x_0;\varepsilon)$ for all natural numbers $k \geq k_0$. Therefore, $\left\{k \in \mathbb{N} \,\middle|\, x_k \notin N(x_0;\varepsilon)\right\} \subseteq [k_0]$ and thus finite.

($\Leftarrow$) Take any $\varepsilon \in \mathbb{R}_+$. Let $k_0 := \max\left\{k \in \mathbb{N} \,\middle|\, x_k \notin N(x_0;\varepsilon)\right\}$. Then, for every natural number k larger than k_0 satisfies $x_k \in N(x_0;\varepsilon)$. □

Lemma 1.3.3

The limit x_0 of a sequence, if it exists, is a cluster point of the sequence.

Theorem 1.3.1 Uniqueness of the Limit

The limit of a convergent sequence of $\mathbb{R}$ is unique.

Proof. Suppose a and b are two limits of a sequence $\{x_k\}$ and $a \neq b$. Let $\varepsilon := |b-a|/2$. Then, by Lemma 1.3.2, $A := \{k \in \mathbb{N} \mid x_k \notin N(a;\varepsilon)\}$ and $B := \{k \in \mathbb{N} \mid x_k \notin N(b;\varepsilon)\}$ are both finite, which means $A \cup B = \mathbb{N}$ is finite, #. □

Theorem 1.3.2
If a sequence has two (or more) cluster points, then it diverges.

Proof. Suppose x_0 is the limit of $\{x_k\}$. Since, by Lemma 1.3.3, x_0 is a cluster point, there is another cluster point c different from x_0. Let $\varepsilon := |x_0 - c|/2$.

Although $S := \{k \in \mathbb{N} \; x_k \notin N(x_0; \varepsilon)\}$ should be finite by Lemma 1.3.2, $\{k \in \mathbb{N} \mid x_k \in N(c; \varepsilon)\}$, a subset of S, is infinite by Lemma 1.3.1, #. $\square$

Theorem 1.3.3
A convergent sequence is bounded.

Proof. Let x_0 is the limit of $\{x_k\}$. There is some $k_0 \in \mathbb{N}$ such that $|x_k - x_0| < 1$ for all $k \in \mathbb{N}_{k_0}$. Let $A := \{x_k \mid k \in \mathbb{N} \text{ and } k \leq k_0\}$ and $B := \{x_k \mid k \in \mathbb{N} \text{ and } k \geq k_0\}$. Then, A is finite and B is bounded above and below by $x_0 + 1$ and $x_0 - 1$, respectively. Therefore, $\{x_k\}$ is bounded above by $\max(\max A, x_0 + 1)$ and below by $\min(\min A, x_0 - 1)$. $\square$

Corollary 1.3.1
An unbounded sequence diverges.

Lemma 1.3.4
The following hold.
(i) $\lim_{k\to\infty} x_k = 0 \iff \lim_{k\to\infty} |x_k| = 0$
(ii) $\lim_{k\to\infty} x_k = x_0 \implies \forall c \in \mathbb{R}, \lim_{k\to\infty} cx_k = cx_0$

Proof of (ii). If $c = 0$, then it is done; so suppose $c \neq 0$. Take any $\varepsilon \in \mathbb{R}_+$. Then, there is some $k_0 \in \mathbb{N}$ such that $|x_k - x_0| < \varepsilon/|c|$ for all $k \geq k_0$. This directly implies for all $k \geq k_0$, $|cx_k - cx_0| = |c| \cdot |x_k - x_0| < |c| \cdot \varepsilon/|c| = \varepsilon$. $\square$

Theorem 1.3.4
A bounded, monotone sequence converges.

Proof. Suppose $\{x_k\}$ is a monotone increasing sequence. Since it is bounded, $\{x_k\}$ has $\mu := \sup\{x_k \mid k \in \mathbb{N}\}$. Take any $\varepsilon \in \mathbb{R}_+$. By Theorem 1.1.4, there is some $k_0 \in \mathbb{N}$ such that $\mu - \varepsilon < x_{k_0} \leq \mu$. Then, for all $k \in \mathbb{N}_{\geq k_0}$, $\mu - \varepsilon < x_{k_0} \leq x_k \leq \mu$, which implies $|x_k - \mu| < \varepsilon$. Therefore $\lim_{k\to\infty} x_k = \mu$. $\square$

Theorem 1.3.5 The Squeeze Play
Let $\{x_k\}$, $\{y_k\}$, and $\{z_k\}$ be sequences that satisfy $x_k \leq y_k \leq z_k$ for $k \in \mathbb{N}$. If both $\{x_k\}$ and $\{z_k\}$ converges to $L \in \mathbb{R}$, then $\{y_k\}$ also converges to L.

Proof. Take any $\varepsilon > 0$. There is $k_1 \in \mathbb{N}$ such that $\forall k \in \mathbb{N}_{\geq k_1}$, $x_k \in N(L; \varepsilon)$. Similarly, there is $k_2 \in \mathbb{N}$ such that $\forall k \in \mathbb{N}_{\geq k_2}$, $x_k \in N(L; \varepsilon)$. Then, for all $k \in \mathbb{N}$ not smaller than $\max\{k_1, k_2\}$, $L - \varepsilon < x_k \leq y_k \leq z_k < L + \varepsilon$ holds, which implies $y_k \in N(L; \varepsilon)$. $\square$

Theorem 1.3.6 Limit is Order Preserving on Convergent Sequences

If both $\{x_k\}$ and $\{y_k\}$ converge and if $x_k \le y_k$ for each $k \in \mathbb{N}$, then

$$\lim_{k\to\infty} x_k \le \lim_{k\to\infty} y_k.$$

Proof. Let $L_x := \lim_{k\to\infty} x_k$ and $L_y := \lim_{k\to\infty} y_k$, and suppose $L_x > L_y$. Let $\varepsilon := (L_x - L_y)/2 > 0$. Then, there is $k \in \mathbb{N}$ such that $x_k \in N(L_x;\varepsilon)$ and $y_k \in N(L_y;\varepsilon)$, which implies $y_k < L_y + \varepsilon = L_x - \varepsilon < x_k$, #. □

Definition 1.3.3: Subsequence

Let $\{x_k\}$ be any sequence. Choose any strictly monotone increasing sequnce $k_1 < k_2 < k_3 < \cdots$ of natural numbers. For each $j \in \mathbb{N}$, let $y_j := x_{k_j}$. The sequence $\{y_j\}_{j=1}^{\infty}$ is called an *subsequence* of $\{x_k\}$.

Theorem 1.3.7

The point c is a cluster point of $\{x_k\}$ if and only if there exists a subsequence of $\{x_k\}$ that converges to c.

Proof. ($\Rightarrow$) Let $\{\varepsilon_k\}$ be an arbitrary sequence of positive real numbers that converges to 0. (e.g. $\varepsilon_k = 1/k$) Define $\{k_j\}_{j=1}^{\infty}$ by the inductive definition below.

- $k_0 := 0$
- For each $j \in \mathbb{N}$, $k_j \in \{k \in \mathbb{N} \mid k > k_{j-1} \text{ and } x_k \in N(c;\varepsilon_j)\}$.

Since c is a cluster point, $\{k \in \mathbb{N} \mid k > k_{j-1} \text{ and } x_k \in N(c;\varepsilon_j)\} \ne \varnothing$ for all $j \in \mathbb{N}$. Therefore, $\{k_j\}$ is well-defined. It is immediate that $\lim_{j\to\infty} x_{k_j} = c$.

($\Leftarrow$) Let $\{x_{k_j}\}_{j=1}^{\infty}$ be a sequence such that $\lim_{j\to\infty} x_{k_j} = c$. Take any $\varepsilon \in \mathbb{R}_+$ and $k \in \mathbb{N}$. Then, there is some $j_0 \in \mathbb{N}$ such that $\forall j \in \mathbb{N}_{\ge j_0}$, $x_{k_j} \in N(c;\varepsilon)$. Let $k_0 := \min\{k_j \in \mathbb{N} \mid j > j_0 \text{ and } k_j > k\}$. Then, $x_{k_0} \in N(c;\varepsilon)$ and $k_0 > k$. Therefore, c is a cluster point. □

Theorem 1.3.8

Any bounded sequence $\{x_k\}$ has a cluster point.

Proof. If the set $S := \{x_k \mid k \in \mathbb{N}\}$ is finite, there is some x_{k_0} that is repeated infinitely. Then, x_{k_0} is surely a cluster point.

Now, suppose S is infinite. Then, by Theorem 1.2.3, S has a limit point ℓ. To prove ℓ is a cluster point, take any $\varepsilon \in \mathbb{R}_+$ and $k \in \mathbb{N}$.

Let $S' := \{x_{k'} \mid k' \in \mathbb{N}_{>k}\}$. We first claim that ℓ is a limit point of S'. Take any $\varepsilon' \in \mathbb{R}_+$ less than $m = \min\{|x_{k'} - \ell| \in \mathbb{R}_+ \mid k' \in \mathbb{N}_{\le k}\}$. ($m$ exists due to Theorem 1.1.3.) Then, $S' \cap N'(\ell;\varepsilon') = S \cap N'(\ell;\varepsilon')$ is nonempty. Therefore, ℓ is a limit point of S' by Theorem 1.2.1.

Finally, we can say $S' \cap N(\ell;\varepsilon)$ is nonempty. This implies there is some $k_0 \in \mathbb{Z}_{>k}$ such that $x_{k_0} \in N(\ell;\varepsilon)$. Therefore, ℓ is a cluster point of $\{x_k\}$. □

Corollary 1.3.2

If a sequence has no cluster point, then the sequence is unbounded.

Corollary 1.3.3

Any bounded sequence converges if and only if it has exactly one cluster point.

Corollary 1.3.4

A sequence $\{x_k\}$ diverges if and only if at least one of the following conditions holds.

- $\{x_k\}$ has two or more cluster points.
- $\{x_k\}$ is unbounded.

Proof. ($\Rightarrow$) Suppose $\{x_k\}$ is diverging and bounded. By Theorem 1.3.8, it has at least one cluster point. Also, if it had exactly one cluster point, it would converge by Corollary 1.3.3.

($\Leftarrow$) It is direct from Theorem 1.3.2 and Corollary 1.3.1. $\square$

Theorem 1.3.9

A sequence $\{x_k\}$ converges if and only if every subsequence of $\{x_k\}$ converges.

Proof. ($\Rightarrow$) Take any subsequence $\{x_{k_i}\}_{i=1}^{\infty}$ of $\{x_k\}_{k=1}^{\infty}$ and $\varepsilon \in \mathbb{R}_+$. There is $i_0 \in \mathbb{N}$ such that $\forall i \in \mathbb{N}_{\geq i_0}$, $|x_i| < \varepsilon$. Since $k_i \geq i$ for all natural number i, $\forall i \in \mathbb{N}_{\geq i_0}$, $|x_{k_i}| < \varepsilon$.

($\Leftarrow$) $\{x_k\}$ is a subsequence of itself. $\square$

Definition 1.3.4: Limit Superior and Inferior

Let $\{x_k\}$ be a sequence and C be a set of cluster points of the sequence.

- $$\limsup x_k \triangleq \begin{cases} \sup C & \text{if } \{x_k\} \text{ is bounded} \\ \infty & \text{if } \{x_k\} \text{ is unbounded above} \\ \sup C & \text{if } \{x_k\} \text{ is bounded above but unbounded below and } C \neq \varnothing \\ -\infty & \text{if } \{x_k\} \text{ is bounded above but unbounded below and } C = \varnothing \end{cases}$$
  is called *limit superior* of $\{x_k\}$.
- $$\liminf x_k \triangleq \begin{cases} \inf C & \text{if } \{x_k\} \text{ is bounded} \\ -\infty & \text{if } \{x_k\} \text{ is unbounded below} \\ \inf C & \text{if } \{x_k\} \text{ is bounded below but unbounded above and } C \neq \varnothing \\ \infty & \text{if } \{x_k\} \text{ is bounded below but unbounded above and } C = \varnothing \end{cases}$$
  is called *limit inferior* of $\{x_k\}$.

Note:-

In all cases, $\liminf x_k \leq \limsup x_k$.

Theorem 1.3.10

- If $\mu = \limsup x_k$ is finite, then μ is in C. ($\mu = \max C$)
- If $\nu = \liminf x_k$ is finite, then ν is in C. ($\nu = \min C$)

Proof. Suppose $\mu = \limsup x_k$ is finite. Take any $\varepsilon \in \mathbb{R}_+$ and $k \in \mathbb{N}$. The finiteness of μ implies $\mu = \sup C$. By Theorem 1.1.4, there is some $c \in C$ such that $\mu - \varepsilon < c \leq \mu$. If $c = \mu$, then we are done. So let $c < \mu$.

Choose any positive ε_1 less than $\min\{c-(\mu-\varepsilon), \mu-c\}$ so $N(c;\varepsilon_1) \subseteq N(\mu;\varepsilon)$. Then, $\{k \in \mathbb{N} \mid x_k \in N(\mu;\varepsilon)\}$ is infinite since it has an infinite set $\{k \in \mathbb{N} \mid x_k \in N(c;\varepsilon_1)\}$ as its subset. (See Lemma 1.3.1.)

The second part can be proven analogously. □

Theorem 1.3.11

Let $\{x_k\}$ be any bounded sequence in $\mathbb{R}$. Fix any $\varepsilon \in \mathbb{R}_+$.

- Let $\mu = \limsup x_k$.
  - $\exists k_0 \in \mathbb{N}, \forall k \in \mathbb{N}_{\geq k_0}, x_k < \mu + \varepsilon$.
  - $\forall k \in \mathbb{N}, \exists k_1 \in \mathbb{N}_{>k}, x_{k_1} > \mu - \varepsilon$.
- Let $\nu = \liminf x_k$.
  - $\exists k_0 \in \mathbb{N}, \forall k \in \mathbb{N}_{\geq k_0}, x_k > \nu - \varepsilon$.
  - $\forall k \in \mathbb{N}, \exists k_1 \in \mathbb{N}_{>k}, x_{k_1} < \nu + \varepsilon$.

Proof. Take any $\varepsilon \in \mathbb{R}_+$. Then, $\{k \in \mathbb{N} \mid x_k \geq \mu + \varepsilon\}$ is finite. If it were not, then there would be a cluster point larger than μ since Theorem 1.3.8 implies the existence of a cluster point in a subsequence of $\{x_k\}$ which is composed of x_k's not smaller than $\mu+\varepsilon$. Therefore, if $k_0 := \max\{k \in \mathbb{N} \mid x_k \geq \mu + \varepsilon\} + 1$, then $x_k < \mu + \varepsilon$ for all k not smaller than k_0.

Also, since μ is a cluster point by Theorem 1.3.10, $\forall k \in \mathbb{N}, \exists k_1 \in \mathbb{N}_{>k}, x_{k_1} > \mu - \varepsilon$. (See Lemma 1.3.1.)

The second part can be proven analogously. □

Theorem 1.3.12

Let $\{x_k\}$ be any sequence in $\mathbb{R}$.

(i) $\{x_k\}$ converges to x_0 if and only if $\liminf x_k = \limsup x_k = x_0$.

(ii) $\{x_k\}$ diverges if and only if one of the following holds.

- Either $\liminf x_k$ or $\limsup x_k$ is infinite.
- Both $\liminf x_k$ or $\limsup$ are finite and $\liminf x_k < \limsup x_k$.

Proof.

(i) ($\Rightarrow$) $C = \{x_0\}$, therefore $\liminf x_k = \limsup x_k = x_0$.

($\Leftarrow$) Take any $\varepsilon \in \mathbb{R}_+$. There are natural numbers k_1 and k_2 such that $\forall k \in \mathbb{N}_{\geq k_1}, x_k < x_0 + \varepsilon$ and $\forall k \in \mathbb{N}_{\geq k_0}, x_k > x_0 - \varepsilon$. Then, for all natural number k not smller than $k_0 := \max\{k_1, k_2\}$, $x_0 - \varepsilon < x_k < x_0 + \varepsilon$ holds.

(ii) If it is not $\liminf x_k = \limsup x_k \in \mathbb{R}$, then it is either "One of them is infinite." or "They are both finite but they are different."

□

Exercise 1.3.1

Let $\{x_k\}$ be a bounded sequence of positive numbers. For each $k \in \mathbb{N}$ define $y_k := x_{k+1}/x_k$ and $z_k := (x_k)^{1/k}$. Prove that $\liminf y_k \leq \liminf z_k \leq \limsup z_k \leq \limsup y_k$.

Solution: ($\liminf y_k \leq \liminf z_k$) Let $L := \liminf y_k$. Now, we claim that

$$\forall \varepsilon \in \mathbb{R}_+, \exists k_0 \in \mathbb{N}, \forall k \in \mathbb{N}_{\geq k_0}, z_k > L - \varepsilon.$$

If $L = 0$, then it is done. Therefore, suppose $L > 0$. To prove this, take any $\varepsilon \in \mathbb{R}_+$ smaller than L. Then, there is some $k_1 \in \mathbb{N}$ such that $y_k > L - \varepsilon/2$ for all k not smaller than k_1 by

Theorem 1.3.11. Then, for all $k \in \mathbb{N}_{\geq k_1}$, $x_k > (L-\varepsilon/2)^{k-k_1} x_{k_1}$, which is equivalent to

$$z_k = x_k^{1/k} > \left(L - \frac{\varepsilon}{2}\right)\left[\left(L - \frac{\varepsilon}{2}\right)^{-k_1} x_{k_1}\right]^{1/k}.$$

Since $\lim_{k\to\infty}\left[(L-\varepsilon/2)^{-k_1} x_{k_1}\right]^{1/k} = 1$, there is some $k_2 \in \mathbb{N}$ such that

$$\left[\left(L - \frac{\varepsilon}{2}\right)^{-k_1} x_{k_1}\right]^{1/k} > 1 - \frac{\varepsilon/2}{L-\varepsilon/2} = \frac{L-\varepsilon}{L-\varepsilon/2}.$$

for all $k \in \mathbb{N}_{\geq k_2}$. Thus, for every natural number k not smaller than $\max\{k_1, k_2\}$,

$$z_k > \left(L - \frac{\varepsilon}{2}\right)\left[\left(L - \frac{\varepsilon}{2}\right)^{-k_1} x_{k_1}\right]^{1/k} > \left(L - \frac{\varepsilon}{2}\right) \cdot \frac{L-\varepsilon}{L-\varepsilon/2} = L - \varepsilon.$$

The claim is now proven.

For the main proof, assume that $\liminf z_k < L$ for the sake of contradiction. Take $\varepsilon_0 := (L - \liminf z_k)/2$. Then, by the previous claim, $\exists k_3 \in \mathbb{N}$, $\forall k \in \mathbb{N}_{\geq k_3}$, $z_k > L - \varepsilon_0 = (L + \liminf z_k)/2$.

Nevertheless, by Theorem 1.3.11, there is some $k_4 \in \mathbb{N}_{>k_3}$ such that $z_{k_4} < \liminf z_k + \varepsilon_0 = (L + \liminf z_k)/2$, which is a contradiction.

$\limsup z_k \leq \limsup y_k$ can be proven analogously.

□

1.4 Cauchy Sequences

Definition 1.4.1: Cauchy Seqeunce

A seqeunce $\{x_k\}$ in $\mathbb{R}$ is called a *Cauchy sequence* if

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists k_0 \in \mathbb{N},\ \forall k, m \in \mathbb{N}_{\geq k_0},\ |x_k - x_m| < \varepsilon.$$

Theorem 1.4.1

If $\{x_k\}$ is a convergent sequence of real numbers, then $\{x_k\}$ is a Cauchy sequence.

Proof. Let $x_0 := \lim_{k\to\infty} x_k$. Take any $\varepsilon \in \mathbb{R}_+$. Then, there is some $k_0 \in \mathbb{N}$ such that $|x_k - x_0| < \varepsilon/2$ for all $k \in \mathbb{N}$ not smaller than k_0. Then, for all $k, m \in \mathbb{N}$ greater than k_0, $|x_k - x_m| \leq |x_k - x_0| + |x_k - x_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. □

Theorem 1.4.2

If $\{x_k\}$ is a Cauchy sequence, then $\{x_k\}$ is bounded.

Proof. There is $k_0 \in \mathbb{N}$ such that $|x_k - x_m| < 1$ for all $k, m \in \mathbb{N}_{\geq k_0}$. It implies that $|x_k - x_{k_0}| < 1$, for all $k \in \mathbb{N}_{\geq k_0}$, which impliees $|x_k| < |x_{k_0}| + 1$. Therefore, for all $k \in \mathbb{N}$, $|x_k| \leq \max\left\{|x_1|, |x_2|, \cdots, |x_{k_0}|, |x_{k_0}| + 1\right\}$. □

Theorem 1.4.3

A Cauchy sequence has exactly one cluster point.

Proof. Since a Cauchy sequence is bounded, it has at least one cluster point by Theorem 1.3.8. So, we should prove that the sequence does not have more than one cluster point. Assume c_1 and c_2 are cluster points for the sake of contradiction. Let $\varepsilon := |c_1 - c_2|/3$. Choose $k_0 \in \mathbb{N}$ such that $\forall k, m \in \mathbb{N}_{\geq k_0}$, $|x_k - x_m| < \varepsilon$. Also, there are $k_1, k_2 \in \mathbb{N}_{>k_0}$ such that $|x_{k_1} - c_1| < \varepsilon$ and $|x_{k_2} - c_2| < \varepsilon$. Note that $|c_1 - c_2| \leq |c_1 - x_{k_1}| + |x_{k_1} - x_{k_2}| + |x_{k_2} - c_2|$. Nevertheless, then

$$\begin{aligned}\varepsilon > |x_{k_1} - x_{k_2}| &\geq |c_1 - c_2| - |c_1 - x_{k_1}| - |c_2 - x_{k_2}| \\ &> 3\varepsilon - \varepsilon - \varepsilon = \varepsilon,\end{aligned}$$

which is a contradiction. □

Theorem 1.4.4 Cauchy Completeness of $\mathbb{R}$

A sequence in $\mathbb{R}$ is convergent if and only if it is a Cauchy sequence.

Proof. By Corollary 1.3.3, a Cauchy sequence is convergent since it is bounded (Theorem 1.4.2) and has exactly one cluster point (Theorem 1.4.3). A convergent sequence in $\mathbb{R}$ is Cauchy. (Theorem 1.4.1) □

Definition 1.4.2: Cauchy Completeness

A set X is said to be *Cauchy complete* if every Cauchy sequence in X converges to a point of X.

Example 1.4.1

$\mathbb{R}$ is Cauchy complete.

Definition 1.4.3: Contractive Sequence

A sequence $\{x_k\}$ is said to be *contractive* if there exists a constant C, with $0 < C < 1$, such that

$$\forall k \in \mathbb{N}_{>1},\ |x_{k+1} - x_k| \leq C|x_k - x_{k-1}|.$$

Theorem 1.4.5

Any contractive sequence in $\mathbb{R}$ is a Cauchy sequence.

Proof. Suppose $0 < C < 1$ and $\forall k \in \mathbb{N}_{>1}$, $|x_{k+1} - x_k| \leq C|x_k - s_{k-1}|$. If it is trivial when $|x_2 - x_1| = 0$, so supose $|x_2 - x_1| \neq 0$. By induction, $\forall k \in \mathbb{N}_{>1}$, $|x_{k+1} - x_k| \leq C^{k-1}|x_2 - x_1|$.

To prove $\{x_k\}$ is a Cauchy sequence, take any $\varepsilon \in \mathbb{R}_+$. Since $\lim_{k\to\infty} C^{k-1} = 0$,

$$\exists k_0 \in \mathbb{N},\ \forall k \in \mathbb{N}_{\geq k_0},\ C^{k-1} < \frac{(1-C)\varepsilon}{|x_2 - x_1|}.$$

Then, for any $k, m \in \mathbb{N}$ with $k_0 \le m < k$,

$$\begin{aligned}
|x_k - x_m| &= \left|\sum_{j=m}^{k-1}(x_{j+1} - x_j)\right| \le \sum_{j=m}^{k-1}|x_{j+1} - x_j| \\
&\le \sum_{j=m}^{k-1} C^{j-1}|x_2 - x_1| = C^{m-1}|x_2 - x_1| \sum_{j=0}^{k-m-1} C^j \\
&= C^{m-1}|x_2 - x_1| \frac{1 - C^{k-m}}{1 - C} < \frac{C^{m-1}}{1 - C}|x_2 - x_1| \\
&< \frac{(1 - C)\varepsilon}{|x_2 - x_1|} \cdot \frac{1}{1 - C}|x_2 - x_1| = \varepsilon.
\end{aligned}$$

□

1.5 The Algebra of Convergent Series

Theorem 1.5.1

Let $\{x_k\}$ and $\{y_k\}$ be convergent sequences in $\mathbb{R}$ and $\lim_{k\to\infty} x_k = x_0$ and $\lim_{k\to\infty} y_k = y_0$.

- $\lim_{k\to\infty}(x_k + y_k) = x_0 + y_0$
- $\lim_{k\to\infty} x_k y_k = x_0 y_0$
- $\lim_{k\to\infty} \frac{y_k}{x_k} = \frac{y_0}{x_0}$ if $x_0 \neq 0$.

Theorem 1.5.2

Let $\{x_k\}$ and $\{y_k\}$ be convergent sequences in $\mathbb{R}$ and $\lim_{k\to\infty} x_k = x_0$. Then, if $r \in \mathbb{Q}$, then

$$\lim_{k\to\infty} x_k^r = x_0^r.$$

Nevertheless, we requre $x_0 \neq 0$ if $r < 0$.

1.6 Cardinality

Definition 1.6.1: Dense Set

We say a subset S of T is dense in T if every neighborhood of any point $x \in T$ contains points of S.

Theorem 1.6.1

- $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are countably infinite.
- $\mathbb{R}$ is uncountable.
- $\mathbb{Q}$ is dense in $\mathbb{R}$.

Chapter 2

Euclidean Spaces

2.1 Euclidean n-Space

Definition 2.1.1: Inner Product

The *inner product* of two vectors $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ in $\mathbb{R}^n$ is

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{n} x_j y_j.$$

Theorem 2.1.1

If $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are arbitrary vectors in $\mathbb{R}^n$ and if a and b are real numbers, then the following hold:

(i) The inner product is *additive* in both its variables:

$$\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$$
$$\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$$

(ii) The inner product is *symmetric*: $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.

(iii) The inner product is *homogeneous* in both its variables: $\langle a\mathbf{x}, b\mathbf{y} \rangle = ab\langle \mathbf{x}, \mathbf{y} \rangle$.

Definition 2.1.2: Euclidean Norm

The *Euclidean norm* of a vector $\mathbf{x}$ in $\mathbb{R}^n$ is

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

Theorem 2.1.2 The Cauchy-Schwarz Inequality

If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \le \|\mathbf{x}\| \cdot \|\mathbf{y}\|.$$

Proof. For any $t \in \mathbb{R}$, $0 \le \|t\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 t^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle t + \|\mathbf{y}\|^2$. Thus, the discriminant $|\langle \mathbf{x}, \mathbf{y} \rangle|^2 - \|\mathbf{x}\|^2 \|\mathbf{y}\|^2$ is nonpositive. □

Theorem 2.1.3

For vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ and any $c \in \mathbb{R}$, the Euclidean norm has the following properties.

(i) $||\mathbf{x}|| \geq 0$; $||\mathbf{x}|| = 0$ if and only if $\mathbf{x} = \mathbf{0}$. (*Positive Definiteness*)

(ii) $||c\mathbf{x}|| = |c| \cdot ||\mathbf{x}||$. (*Absolute Homogeneity*)

(iii) $||\mathbf{x}+\mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$. (*Subadditivity*)

Proof of (iii).

$$\begin{aligned} 0 \leq ||\mathbf{x}+\mathbf{y}||^2 &= ||\mathbf{x}||^2 + 2\langle \mathbf{x}, \mathbf{y}\rangle + ||\mathbf{y}||^2 \\ &\leq ||\mathbf{x}||^2 + 2||\mathbf{x}||||\mathbf{y}|| + ||\mathbf{y}||^2 = (||\mathbf{x}|| + ||\mathbf{y}||)^2 \end{aligned}$$

□

Definition 2.1.3: Norm

A *norm* on $\mathbb{R}^n$ is any function $n\colon \mathbb{R}^n \to \mathbb{R}$ that is positive definite, absolutely homogeneous, and subadditive.

Definition 2.1.4: Metric

A *metric* on $\mathbb{R}^n$ is a function from $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ having the following properties.

(i) $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{y}) \geq 0$; $d(\mathbf{x}, \mathbf{y}) = 0 \iff \mathbf{x} = \mathbf{y}$. (*Positive Definiteness*)

(ii) $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{y})$. (*Symmetry*)

(iii) $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$. (*The Triangle Inequality*)

Definition 2.1.5: Euclidean Metric

The *Euclidean metric* on $\mathbb{R}^n$ is defined by

$$d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = \left[\sum_{j=1}^{n} (x_j - y_j)^2\right]^{1/2}.$$

Theorem 2.1.4

The Euclidean metric is a metric on $\mathbb{R}^n$.

Definition 2.1.6: Orthogonality

Two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ are said to be *orthogonal* if $\langle \mathbf{x}, \mathbf{y}\rangle = 0$.

Definition 2.1.7: Neighborhood and Deleted Neighborhood

A *neighborhood* $N(\mathbf{x}; r)$ or $\mathbf{x} \in \mathbb{R}^n$ with radius r is the set

$$N(\mathbf{x}; r) = \{\mathbf{y} \in \mathbb{R}^n \mid ||\mathbf{x} - \mathbf{y}|| < r\}.$$

A *deleted neighborhood* $N'(\mathbf{x}, r)$ of $\mathbf{x}$ is $N'(\mathbf{x}; r) = N(\mathbf{x}; r) \setminus \{\mathbf{x}\}$.

Definition 2.1.8: Limit Point

Let S be nonempty subset of $\mathbb{R}^n$. We say that $\mathbf{x}$ is a *limit point* of S if

$$\forall \varepsilon \in \mathbb{R}_+,\ N'(\mathbf{x}; \varepsilon) \cap S \neq \varnothing.$$

Theorem 2.1.5

$\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

Proof. Take any $\mathbf{y} = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}_+$. For each $j = 1, 2, \cdots, n$, choose a rational $x_j \in N(y_j; \varepsilon/\sqrt{n})$ and form $\mathbf{x} \triangleq (x_1, x_2, \cdots, x_n) \in \mathbb{Q}^n$. Then,

$$\|\mathbf{x} - \mathbf{y}\|^2 = \sum_{j=1}^{n} (x_j - y_j)^2 < n(\varepsilon/\sqrt{n})^2 = \varepsilon^2.$$

Therefore $\mathbf{y}$ is a limit point of $\mathbb{Q}^n$. □

Definition 2.1.9: Boundedness

A subset S of $\mathbb{R}^n$ is said to be *bounded* if

$$\exists M \in \mathbb{R}_+, \ \forall \mathbf{x} \in S, \ \|\mathbf{x}\| \le M.$$

2.1.1 Sequences in $\mathbb{R}^n$

Definition 2.1.10: Cluster Point

$\mathbf{c} \in \mathbb{R}^n$ is a *cluster point* of the sequence $\{\mathbf{x}_k\}$ if,

$$\forall (\varepsilon, k) \in \mathbb{R}_+ \times \mathbb{N}, \ \exists k_1 \in \mathbb{N}_{>k}, \ \mathbf{x}_{k_1} \in N(\mathbf{c}; \varepsilon).$$

Definition 2.1.11: Convergence and Divergence of a Sequence

The sequnce $\{\mathbf{x}_k\}$ *converges* to $\mathbf{x}_0$ and $\mathbf{x}_0$ is the *limit* of $\{\mathbf{x}_k\}$ if,

$$\forall \varepsilon \in \mathbb{R}_+, \ \exists k_0 \in \mathbb{N}, \ \forall k \in \mathbb{N}_{\ge k_0}, \ \mathbf{x}_k \in N(\mathbf{x}_0; \varepsilon).$$

We write $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}_0$. If there is no such $\mathbf{x}_0$, then $\{\mathbf{x}_k\}$ *diverges*.

Theorem 2.1.6

Let $\{\mathbf{x}_k\} = (x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)})$ for each $k \in \mathbb{N}$. Let $\mathbf{x}_0 = (x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)})$. The sequence $\{\mathbf{x}_k\}$ converges to $\mathbf{x}_0$ if and only if, for each $j \in [n]$, the sequence $\{x_j^{(k)}\}$ converges to $\{x_j^{(0)}\}$.

Proof. ($\Rightarrow$) Take any $\varepsilon \in \mathbb{R}_+$. There there is $k_0 \in \mathbb{N}$ such that

$$\forall k \in \mathbb{N}_{\ge k_0}, \mathbf{x}_k \in N(\mathbf{x}_0; \varepsilon).$$

Then, for each $j \in [n]$,

$$\left(x_j^{(k)} - x_0^{(k)}\right)^2 \le \sum_{i=1}^{n} \left(x_i^{(k)} - x_0^{(k)}\right)^2 = \|\mathbf{x}_k - \mathbf{x}_0\|^2 < \varepsilon.$$

($\Leftarrow$) Take any $\varepsilon \in \mathbb{R}_+$. Then, for each $j \in [n]$, there is some $k_j \in \mathbb{N}$ such that

$$\forall k \in \mathbb{N}_{\ge k_j}, \ x_j^{(k)} \in N(x_0^{(k)}; \varepsilon/\sqrt{n}).$$

Then, for all natural number k not smaller than $\max_{j\in[n]} k_j$,

$$\|\mathbf{x}_k - \mathbf{x}_0\|^2 = \sum_{j=1}^{n} \left(x_j^{(k)} - x_0^{(k)}\right)^2 < n(\varepsilon/\sqrt{n})^2 = \varepsilon^2.$$

□

Definition 2.1.12: Cauchy Seqeunce

A seqeunce $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$ is called a *Cauchy sequence* if

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists k_0 \in \mathbb{N},\ \forall k, m \in \mathbb{N}_{\geq k_0},\ \|\mathbf{x}_k - \mathbf{x}_m\| < \varepsilon.$$

Theorem 2.1.7 Cauchy's Completeness Theorem in $\mathbb{R}^n$

A sequence $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$ is Cauchy if and only if it converges. $\mathbb{R}^n$ is Cauchy complete.

Proof. ($\Leftarrow$) The proof if similar to Theorem 1.4.1.

($\Rightarrow$) Let some Cauchy sequence $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$ be given. Take any $\varepsilon \in \mathbb{R}_+$. There is some $k_0 \in \mathbb{N}$ such that for every natural number k and m not smaller than k_0, $\|\mathbf{x}_k - \mathbf{x}_m\| < \varepsilon$. Then, for each $j \in [n]$, $|x_j^{(k)} - x_j^{(m)}| \leq \|\mathbf{x}_k - \mathbf{x}_m\| < \varepsilon$, which implies each $\{x_j^{(k)}\}_{k\in\mathbb{N}}$ is Cauchy. By Theorem 1.4.4, $\{x_j^{(k)}\}_{j\in\mathbb{N}}$ converges to some number $x_j^{(0)}$. Then, Theorem 2.1.6 ensures that $\lim_{k\to\infty} \mathbf{x}_k = (x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)})$. □

Theorem 2.1.8 The Generalized Bolzano–Weierstrass Theorem

Every bounded infinite set in $\mathbb{R}^n$ has a limit point in $\mathbb{R}^n$.

Proof. Suppose that S is any bounded, infinite set in $\mathbb{R}^n$. Being bounded, S is contained in some n-cube $C(2M) = [-M, M]^n$ centered at $\mathbf{0}$. Construct $C_1, C_2, \cdots$ as following.

- $C_1 \triangleq C(2M) = [a_1^{(1)}, b_1^{(1)}] \times \cdots \times [a_n^{(1)}, b_n^{(1)}]$
  - Note that $C_1 \cap S = S$ is infinite.
- For each $k \in \mathbb{N}$, C_{k+1} is any cube of the form $[a_1^{(k+1)}, b_1^{(k+1)}] \times \cdots \times [a_n^{(k+1)}, b_n^{(k+1)}]$ where each $[a_j^{(k+1)}, b_j^{(k+1)}]$ is either $[a_j^{(k)}, (a_j^{(k)} + b_j^{(k)})/2]$ or $[(a_j^{(k)} + b_j^{(k)})/2, b_j^{(k)}]$ so that $C_{k+1} \cap S$ is infinite.
  - This is possible since there is at least one cube among 2^n possible choices that $C_{k+1} \cap S$ is infinite.

Then, the main diagonal d_k of C_k equals to $Mn^{1/2}/2^{k-2}$. Also, note that $C_k \supseteq C_{k+1}$ for all $k \in \mathbb{N}$.

Now, we may construct a sequence $\{\mathbf{x}_k\}_{k\in\mathbb{N}}$ as following.

- $\mathbf{x}_1$ is any element in $C_1 \cap S$.
- For each $k \in \mathbb{N}$, $\mathbf{x}_{k+1}$ is arbitrarily taken from $C_{k+1} \cap S \setminus \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\}$.

We claim that $\{\mathbf{x}_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence. To show this, take any $\varepsilon \in \mathbb{R}_+$. There is some $k_0 \in \mathbb{N}$ such that $d_{k_0} = Mn^{1/2}/2^{k_0-2} < \varepsilon$ by Theorem 1.1.1. Then, for all $k, m \in \mathbb{N}_{\geq k_0}$, $\|\mathbf{x}_k - \mathbf{x}_m\| \leq d_{k_0} < \varepsilon$. Therefore, since $\{\mathbf{x}_k\}$ is Cauchy, and therefore convergent by Theorem 2.1.7.

Clearly, $\mathbf{x}_0 \triangleq \lim_{k\to\infty} \mathbf{x}_k$ is a limit point of S since any deleted neighborhood $N'(\mathbf{x}_0)$ of $\mathbf{x}_0$ intersects infinitely many points with $\{\mathbf{x}_k\}_{k\in\mathbb{N}} \subseteq S$. □

Definition 2.1.13: Subsequence

Let $\{\mathbf{x}_k\}$ be any sequence in $\mathbb{R}^n$. Choose any strictly monotone increasing sequnce $k_1 < k_2 < k_3 < \cdots$ of natural numbers. For each $j \in \mathbb{N}$, let $\mathbf{y}_j := \mathbf{x}_{k_j}$. The sequence $\{\mathbf{y}_j\}_{j=1}^{\infty}$ is called an *subsequence* of $\{\mathbf{x}_k\}$.

Theorem 2.1.9

The point $\mathbf{c}$ is a cluster point of $\{\mathbf{x}_k\}$ if and only if there exists a subsequence of $\{\mathbf{x}_k\}$ that converges to $\mathbf{c}$.

Proof. Analogous to Theorem 1.3.7. □

Theorem 2.1.10

Any bounded sequence $\{\mathbf{x}_k\}$ has a cluster point.

Proof. Analogous to Theorem 1.3.8. □

Corollary 2.1.1

If a sequence in $\mathbb{R}^n$ has no cluster point, then the sequence is unbounded.

Corollary 2.1.2

Any bounded sequence in $\mathbb{R}^n$ converges if and only if it has exactly one cluster point.

Corollary 2.1.3

A sequence $\{\mathbf{x}_k\}$ diverges if and only if at least one of the following conditions holds.

- $\{\mathbf{x}_k\}$ has two or more cluster points.
- $\{\mathbf{x}_k\}$ is unbounded.

2.2 Open and Closed Sets

Definition 2.2.1: Interior/Boundary Point and Open/Closed Set

Let S be any subset of $\mathbb{R}^n$ and let $\mathbf{x}$ be any point in $\mathbb{R}^n$.

(i) $\mathbf{x}$ is an *interior point* of S if $\exists r \in \mathbb{R}_+, N(\mathbf{x}; r) \subseteq S$.

(ii) If every point of S is an interior point of S, then S is said to be *open*.

(iii) We call $\mathbf{x}$ is a *boundary point* of S if $\forall r \in \mathbb{R}_+, N(\mathbf{x}; r) \cap S \neq \varnothing \wedge N(\mathbf{x}; r) \setminus S \neq \varnothing$.

(iv) If S continas all its boundary points, then S is said to be *closed*.

Definition 2.2.2

Let $S \subseteq \mathbb{R}^n$.

(i) The *interior* of S, denoted $\mathring{S}$, is the set of all interior points of S.

(ii) The *boundary* of S, denoted $\operatorname{bd} S$, is the set of all boundary points of S.

(iii) The *derived set* of S, denoted S', is the set of all limit points of S.

(iv) The *closure* of S, denoted $\overline{S}$, is the union of S and S'.

(v) The *complement* of S, denoted S^c, is the set $\mathbb{R}^n \setminus S$.

Note:-

- For $S \subseteq \mathbb{R}^n$, $\mathring{S} \subseteq S \subseteq \overline{S}$.
- For $S \subseteq \mathbb{R}^n$, S is open if and only if $\mathring{S} = S$.
- For $S \subseteq \mathbb{R}^n$, $\mathring{S}$ is open.

Theorem 2.2.1

The union of any collection of open sets in $\mathbb{R}^n$ is open. The intersection of any finite collection of open sets in $\mathbb{R}^n$ is also open.

Proof. To prove the first assertion, suppose that $\{ U_\alpha \mid \alpha \in J \}$ is any collection of open sets in $\mathbb{R}^n$. Let $U \triangleq \bigcup_{\alpha\in J} U_\alpha$. Take any $\mathbf{x} \in U$. Then, there is some $\alpha_0 \in J$ such that $\mathbf{x} \in U_{\alpha_0}$. Since U_{α_0} is open, there is some neighborhood $N(\mathbf{x};\varepsilon)$ such that $N(\mathbf{x};\varepsilon) \subseteq U_{\alpha_0}$, which, in turn, $N(\mathbf{x};\varepsilon) \subseteq U$. Therefore, $\mathbf{x}$ is an interior point of U; U is open.

To prove the second assertion, let U be the intersection of any finite collection $\{ U_1, U_2, \cdots, U_k \}$ of open sets and take any $\mathbf{x} \in U$. For each $j \in [k]$, since $\mathbf{x} \in U_j$, there is some $r_j \in \mathbb{R}_+$ such that $N(\mathbf{x}; r_j) \subseteq U_j$. Then, take $r_0 \triangleq \min_{j\in[k]} r_j \in \mathbb{R}_+$. Since, for all $j \in [k]$, $N(\mathbf{x}; r_0) \subseteq U_j$, it is implied that $N(\mathbf{x}; r_0) \subseteq U$. Therefore, $\mathbf{x}$ is an interior point of U; U is open. □

Note:-

Intersection of infinitely many open sets may fail to be open. For instance, consider

$$U_k \triangleq N(\mathbf{0}; 1/k),$$

for each $k \in \mathbb{N}$. Then, $\bigcap_{k\in\mathbb{N}} U_k = \{\mathbf{0}\}$, which is not open.

Theorem 2.2.2

A set $C \subseteq \mathbb{R}^n$ is closed if and only if C^c is open.

Proof. ($\Rightarrow$) Take any $\mathbf{x} \in C^c$. Since C is closed and contains all of its boundary points, $\mathbf{x}$ is not a boundary point of C. Therefore, there is some neighborhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \cap C = \varnothing$ or $N(\mathbf{x}) \cap C^c = \varnothing$. The second case is not possible since $\mathbf{x} \in N(\mathbf{x}) \cap C^c$. Therefore, $N(\mathbf{x}) \cap C = \varnothing$, which implies $N(\mathbf{x}) \subseteq C^c$; $\mathbf{x}$ is an interior point of C^c. Therefore, C^c is open.

($\Leftarrow$) Take any bounddary point $\mathbf{x}$ of C. Assume $\mathbf{x} \in C^c$ for the sake of contradiction. Since C^c is open, there is a neighborhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \subseteq C^c$. However, that implies $N(\mathbf{x}) \cap C = \varnothing$, which contradicts $\mathbf{x}$ is a boundary point of C. Therefore, $\mathbf{x} \in C$; C contains all of its boundary points. □

Theorem 2.2.3

The intersection of any collection of closed sets in $\mathbb{R}^n$ is closed. The union of any finite collection of closed sets in $\mathbb{R}^n$ is also closed.

Proof. To prove the first assertion, let $\{C_\alpha\}_{\alpha\in J}$ be any collection of closed sets in $\mathbb{R}^n$. Then, each ${C_\alpha}^c$ is open by Theorem 2.2.2, and thus $\bigcup_{\alpha\in J} {C_\alpha}^c$ is open by Theorem 2.2.1. Its complement $\left(\bigcup_{\alpha\in J} {C_\alpha}^c\right)^c$ is closed by Theorem 2.2.2. And note that $\left(\bigcup_{\alpha\in J} {C_\alpha}^c\right)^c = \bigcap_{\alpha\in J} C_\alpha$ by De Morgan's law.

To prove the second assertion, let $\{ C_1, C_2, \cdots, C_k \}$ be a finite collection of closed sets in $\mathbb{R}^n$. Then, each ${C_i}^c$ is open by Theorem 2.2.2, and thus $\bigcap_{i=1}^k {C_i}^c$ is open by Theorem 2.2.1. Its complement $\left(\bigcap_{i=1}^k {C_i}^c\right)^c$ is closed by Theorem 2.2.2. And note that $\left(\bigcap_{i=1}^k {C_i}^c\right)^c = \bigcup_{i=1}^k C_i$ by De Morgan's law. □

Theorem 2.2.4

$C \subseteq \mathbb{R}^n$ is closed if and only if $C' \subseteq C$.

Proof. (⇒) Let $\mathbf{x} \in C'$. Assume $\mathbf{x} \in C^c$ for the sake of contradiction. Since C^c is open by Theorem 2.2.2, there is a neighborhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \subseteq C^c$. Such $N(\mathbf{x})$ satisfies $N(\mathbf{x}) \cap C = \varnothing$, which contradicts $\mathbf{x} \in C'$. Therefore, $\mathbf{x} \in C$; C contains all its limit points.

(⇐) It is enough to prove C^c is open by Theorem 2.2.2. Take any $\mathbf{x} \in C^c$. $\mathbf{x}$ is not a limit point of C by the hypothesis. Therefore, there is a deleted neighborhood $N'(\mathbf{x})$ of $\mathbf{x}$ such that $N'(\mathbf{x}) \cap C = \varnothing$. Then, $N'(\mathbf{x}) \subseteq C^c$, and thus $N(\mathbf{x}) \subseteq C^c$, which implies $\mathbf{x}$ is an interior point of C^c. Thus, C^c is open. □

Corollary 2.2.1

$C \subseteq \mathbb{R}^n$ is closed if and only if $\overline{C} = C$.

Theorem 2.2.5

Let $S \subseteq \mathbb{R}^n$. The interior of S is the union of all open sets contained in S.

Proof. Let $\mathcal{U} \triangleq \{U \subseteq S \mid U \text{ is open in } \mathbb{R}^n\}$.

(⊆) Let $\mathbf{x} \in \mathring{S}$. Then, there is an open neighborhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \subseteq S$. Noting that $\mathbf{x} \in N(\mathbf{x}) \in \mathcal{U}$, we conclude $\mathring{S} \subseteq \bigcup \mathcal{U}$.

(⊇) Take any $\mathbf{x} \in \bigcup \mathcal{U}$. Then, there is an open set U in $\mathbb{R}^n$ such that $x \in U \subseteq S$. There is a neighborhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \subseteq U$. Therefore, $N(\mathbf{x}) \subseteq S$; $\mathbf{x}$ is an interior point of S. Thus; $\mathring{S} \supseteq \bigcup \mathcal{U}$. □

Theorem 2.2.6

The closure of S is the intersection of all closed sets that contain S.

Proof. Let $\mathcal{C} \triangleq \{C \subseteq \mathbb{R}^n \mid S \subseteq C \text{ and } C \text{ is closed}\}$.

(⊆) Since $S \subseteq \bigcap \mathcal{C}$ is obvious, we only need to show $S' \subseteq \bigcap \mathcal{C}$. Let $\mathbf{x} \in S'$. Then, it is direct that $\forall C \in \mathcal{C}$, $\mathbf{x} \in C'$ since each $C \in \mathcal{C}$ satisfies $S \subseteq C$. As C is closed and thus $\mathbf{x} \in C' \subseteq C$ by Theorem 2.2.4, Consequently, $\mathbf{x} \in \bigcap \mathcal{C}$; $\overline{S} \subseteq \bigcap \mathcal{C}$.

(⊇) It is enough to show that $\overline{S}$ is closed, which, in turn, is sufficient to show that $(\overline{S})' \subseteq \overline{S}$ by Theorem 2.2.4. Let $\mathbf{y} \in (\overline{S})'$ and take any deleted neighborhood $N'(\mathbf{y}; \varepsilon)$ of $\mathbf{y}$. Then, there is some element $\mathbf{z}$ in $N'(\mathbf{y}; \varepsilon) \cap \overline{S}$. Then, $\mathbf{z} \in S$ or $\mathbf{z} \in S'$.

If $\mathbf{z} \in S$, then $\mathbf{z} \in N'(\mathbf{y}; \varepsilon) \cap S \neq \varnothing$. If $\mathbf{z} \in S'$, take $\varepsilon' \triangleq \min\{\|\mathbf{z} - \mathbf{y}\|, \varepsilon - \|\mathbf{z} - \mathbf{y}\|\}$. Then, $N(\mathbf{z}; \varepsilon') \subseteq N'(\mathbf{y}; \varepsilon)$. Since $\mathbf{z} \in S'$, there is some $\mathbf{x}$ in $N'(\mathbf{z}; \varepsilon') \cap S$. Thus, $\mathbf{x} \in N'(\mathbf{z}; \varepsilon') \cap S \subseteq N'(\mathbf{y}; \varepsilon) \cap S \neq \varnothing$.

In both cases, $N'(\mathbf{y}; \varepsilon) \cap S \neq \varnothing$. Thus, we proved that $\mathbf{y} \in S' \subseteq \overline{S}$; $(\overline{S})' \subseteq \overline{S}$. □

Corollary 2.2.2

For any $S \subseteq \mathbb{R}^n$, the set $\overline{S}$ is closed.

Corollary 2.2.3

For any $C \subseteq \mathbb{R}^n$, C is closed if and only if $\overline{C} = C$.

Theorem 2.2.7

Let $S \subseteq \mathbb{R}^n$.

(i) $\mathring{\mathring{S}} = \mathring{S}$

(ii) $\overline{(\overline{S})} = \overline{S}$

(iii) $\mathring{S} \cap \operatorname{bd} S = \varnothing$

(iv) $\mathring{S} \cup \operatorname{bd} S = \overline{S}$
(v) $\overline{S} \cap \overline{S^c} = \operatorname{bd} S$

Proof.

(i) $\mathring{S}$ is open and an open set is the interior of itself.
(ii) $\overline{S}$ is closed and a closed set is the closure of itself. (See Corollary 2.2.2 and Corollary 2.2.3).
(iii) Suppose there is some $\mathbf{x} \in \mathring{S} \cap \operatorname{bd} S$. There is a neighborhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \subseteq S$. Then, $N(\mathbf{x}) \cap C^c = \varnothing$, which contradicts $\mathbf{x} \in \operatorname{bd} S$.
(iv) ($\subseteq$) Since it is already $\mathring{S} \subseteq S \subseteq \overline{S}$, we only need to show $\operatorname{bd} S \subseteq \overline{S}$. Let $\mathbf{x} \in \operatorname{bd} S$. If $\mathbf{x} \in S$, then it is done; so suppose $\mathbf{x} \in S^c$. Take any neighborhood $N(\mathbf{x}; \varepsilon)$ of $\mathbf{x}$. Then, $N(\mathbf{x}; \varepsilon) \cap S \neq \varnothing$. Noting that $N'(\mathbf{x}; \varepsilon) \cap S = N(\mathbf{x}; \varepsilon) \cap S \neq \varnothing$, $\mathbf{x} \in S'$.
($\supseteq$) Let $\mathbf{x} \in \overline{S}$. If $\mathbf{x} \in S$, then it is either "There is a neighborhood $N(\mathbf{x})$ of $\mathbf{x}$ such that $N(\mathbf{x}) \subseteq S$." or "Every neighborhood $N(\mathbf{x})$ of $\mathbf{x}$ satisfies $N(\mathbf{x}) \cap S^c \neq \varnothing$." The first case is $\mathbf{x} \in \mathring{S}$ and the latter case is $\mathbf{x} \in \operatorname{bd} S$.
Now the only left case if $\mathbf{x} \in S' \setminus S$. Take any deleted neighborhood $N'(\mathbf{x})$ of $\mathbf{x}$. Then, $N(\mathbf{x}) \cap S = N'(\mathbf{x}) \cap S \neq \varnothing$. Also, $\mathbf{x} \in N(\mathbf{x}) \cap S^c$. Thus, $\mathbf{x} \in \operatorname{bd} S$.
(v) Using $\overline{S} = \mathring{S} \cup \operatorname{bd} S$, we get

$$\begin{aligned}
\overline{S} \cap \overline{S^c} &= (\mathring{S} \cup \operatorname{bd} S) \cap ((\mathring{S^c}) \cup \operatorname{bd} S^c) \\
&= (\mathring{S} \cap (\mathring{S^c})) \cup (\mathring{S} \cap \operatorname{bd} S^c) \cup (\operatorname{bd} S \cap (\mathring{S^c})) \cup (\operatorname{bd} S \cap \operatorname{bd} S^c)
\end{aligned}$$

$\mathring{S} \cap (\mathring{S^c}) = \varnothing$ since $S \cap S^c = \varnothing$ and $\mathring{S} \subseteq S$ and $\mathring{S^c} \subseteq S^c$.
$\operatorname{bd} S = \operatorname{bd} S^c$ is direct from their definitions. Thus,

$$\mathring{S} \cap \operatorname{bd} S^c = \mathring{S} \cap \operatorname{bd} S = \varnothing$$

$$\operatorname{bd} S \cap (\mathring{S^c}) = \operatorname{bd} S^c \cap (\mathring{S^c}) = \varnothing$$

by (iv). Therefore, $\overline{S} \cap \overline{S^c} = \operatorname{bd} S \cap \operatorname{bd} S^c = \operatorname{bd} S$.

□

Definition 2.2.3: Diameter

Let $\varnothing \neq S \subseteq \mathbb{R}^n$ be a bounded set. The *diameter* of S is defined to be

$$d(S) \triangleq \sup\{\, \|\mathbf{x} - \mathbf{y}\| \mid \mathbf{x}, \mathbf{y} \in S \,\}.$$

Definition 2.2.4: Distance

Let $\varnothing \neq S \subseteq \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$. The distance from $\mathbf{x}$ to S is defined to be

$$d(\mathbf{x}, S) \triangleq \inf\{\, \|\mathbf{x} - \mathbf{y}\| \mid \mathbf{y} \in S \,\}.$$

Exercise 2.2.1

Let S be a nonempty set in $\mathbb{R}^n$ and let $\mathbf{x}$ be a point of $\mathbb{R}^n$.
(i) $d(\mathbf{x}, S) = 0$ if and only if $\mathbf{x} \in \overline{S}$.
(ii) S is closed if and only if $d(\mathbf{x}, S) > 0$ for every $\mathbf{x} \in S^c$.
(iii) If S is closed, then there exists $\mathbf{y}_0 \in S$ such that $d(\mathbf{x}, S) = \|\mathbf{x} - \mathbf{y}_0\|$.
(iv) If S is open and if $\mathbf{x} \in S^c$, then there exists no $\mathbf{y} \in S$ such that $d(\mathbf{x}, S) = \|\mathbf{x} - \mathbf{y}\|$.

Solution:

(i) ($\Rightarrow$) We shall show that if such $\mathbf{x}$ is not in S, then it is in S'. So, suppose $\mathbf{x} \notin S$. By Theorem 1.1.4, for any $\varepsilon \in \mathbb{R}_+$, there is some $\mathbf{y} \in S$ such that $0 \le \|\mathbf{x}-\mathbf{y}\| < \varepsilon$. Since $\mathbf{x} \notin S$, $\mathbf{x} \neq \mathbf{y}$, and thus $\mathbf{y} \in N'(\mathbf{x};\varepsilon) \cap S$, implying $\mathbf{x}$ is a limit point of S.
($\Leftarrow$) Conversely, if $\mathbf{x} \in S' \setminus S$, then for all $\varepsilon \in \mathbb{R}_+$, there is some $\mathbf{z} \in S$ such that $0 < \|\mathbf{x}-\mathbf{z}\| < \varepsilon$. Therefore, $0 \le d(\mathbf{x},S) < \varepsilon$. Since ε is arbitrary, $d(\mathbf{x},S) = 0$. ✓

(ii) ($\Rightarrow$) $d(\mathbf{x},S) = 0$ if and only if $\mathbf{x} \in \overline{S} = S$. Therefore, $d(\mathbf{x},S) > 0$ if and only if $\mathbf{x} \in S^c$.
($\Leftarrow$) For every $\mathbf{x} \in S^c$, $\mathbf{x} \notin \overline{S}$ by (i). Thus, if $\mathbf{x} \in \overline{S}$, then $\mathbf{x} \in S$, or, $\overline{S} \subseteq S$. S is therefore closed. ✓

(iii) If S is finite, then we can easily see $d(\mathbf{x},S) = \min\{\, \|\mathbf{x}-\mathbf{y}\| \mid \mathbf{y} \in S \,\}$.
Therefore, now suppose S is infinite. Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a sequence defined by $\varepsilon_k = 1/k$ for each $k \in \mathbb{N}$. By Theorem 1.1.4, for each $k \in \mathbb{N}$, we can find $\mathbf{y}_k \in S$ that satisfies

$$d(\mathbf{x},S) \le \|\mathbf{x}-\mathbf{y}_k\| < d(\mathbf{x},S) + \varepsilon_k.$$

If the set $\{\, \mathbf{y}_k \mid k \in \mathbb{N} \,\}$ is finite, then there must be some $\mathbf{y}_k$ such that $\|\mathbf{x}-\mathbf{y}_k\| = d(\mathbf{x},S)$, and we are done.
Suppose $\{\, \mathbf{y}_k \mid k \in \mathbb{N} \,\}$ is infinite. Since the set is also bounded ($\|x-y_k\| < d(\mathbf{x},S)+\varepsilon_1$ for each $k \in \mathbb{N}$), by Theorem 2.1.8, there is a convergent subsequence $\{\mathbf{y}_{k_j}\}_{j\in\mathbb{N}}$ of $\{\mathbf{y}_k\}_{k\in\mathbb{N}}$. Let $\mathbf{y}_0 \triangleq \lim_{j\to\infty} \mathbf{y}_{k_j}$. Since

$$d(\mathbf{x},S) \le \|\mathbf{x}-\mathbf{y}_{k_j}\| < d(\mathbf{x},S) + \varepsilon_{k_j}$$

still holds, it follows that $\|\mathbf{x}-\mathbf{y}_0\| = d(\mathbf{x},S)$ by Theorem 1.3.5.
$\mathbf{y}_0 \in S$ since S is closed and $\mathbf{y}_0 \in S$ is a limit point of S. ✓

(iv) Suppose there is some $\mathbf{y} \in S$ such that $d(\mathbf{x},S) = \|\mathbf{x}-\mathbf{y}\|$. $\|\mathbf{x}-\mathbf{y}\| > 0$ since $\mathbf{x} \neq \mathbf{y}$. Since S is open, there is some neighborhood $N(\mathbf{y};r_0)$ of $\mathbf{y}$ such that $N(\mathbf{y};r_0) \subseteq S$. It must be $r_0 \le \|\mathbf{x}-\mathbf{y}\|$. Let

$$\mathbf{z} \triangleq \mathbf{y} + \frac{r_0}{2} \cdot \frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|}$$

Then,

$$\|\mathbf{z}-\mathbf{y}\| = \frac{r_0}{2} < r_0,$$

thus $\mathbf{z} \in N(\mathbf{y};r_0) \subseteq S$. However,

$$\|\mathbf{z}-\mathbf{y}\| = \left| 1 - \frac{r_0}{2\|\mathbf{x}-\mathbf{y}\|} \right| \|\mathbf{x}-\mathbf{y}\| < \|\mathbf{x}-\mathbf{y}\|,$$

contradicting the minimality of $\|\mathbf{x}-\mathbf{y}\|$, #. ✓

□

Exercise 2.2.2

Let S be a nonempty set in $\mathbb{R}^n$. Then, $d(S) = d(\overline{S})$.

Solution: Since $S \subseteq \overline{S}$, $d(S) \le d(\overline{S})$ is direct.

To prove $d(S) = d(\overline{S})$, take any $\varepsilon \in \mathbb{R}_+$. Let $\mathbf{x}$ and $\mathbf{y}$ be any point in $\overline{S}$. Then, each of $S \cap N(\mathbf{x};\varepsilon/2)$ and $S \cap N(\mathbf{y};\varepsilon/2)$ is nonempty. Thus, we take $\mathbf{x}'$ and $\mathbf{y}'$ from each set. Then,

$$\begin{aligned} \|\mathbf{x}-\mathbf{y}\| &\le \|\mathbf{x}-\mathbf{x}'\| + \|\mathbf{x}'-\mathbf{y}'\| + \|\mathbf{y}'-\mathbf{y}\| \\ &< \varepsilon/2 + d(S) + \varepsilon/2 = d(S) + \varepsilon \end{aligned}$$

Therefore, $d(S)+\varepsilon$ is an upper bound of $\{\, \|\mathbf{x}-\mathbf{y}\| \mid \mathbf{x},\mathbf{y} \in \overline{S} \,\}$. Thus, $d(S) \le d(\overline{S}) \le d(S)+\varepsilon$. Since ε was arbitrary, we conclude $d(S) = d(\overline{S})$.

□

2.3 Completeness

Definition 2.3.1: Nested Sets

A sequence $\{S_k\}$ of sets in $\mathbb{R}^n$ such that $S_k \supset S_{k+1}$ for each $k \in \mathbb{N}$ is said to be *nested*.

Theorem 2.3.1 Cantor's Nested Interval Theorem

For each $k \in \mathbb{N}$, let $I_k = [a_k, b_k]$ with $a_k < b_k$. Suppose that $\{I_k\}_{k\in\mathbb{N}}$ is a nested sequence in $\mathbb{R}$. Then

$$\bigcap_{k=1}^{\infty} I_k = [\alpha, \beta]$$

where $\alpha = \sup\{\, a_k \mid k \in \mathbb{N}\,\}$ and $\beta = \inf\{\, b_k \mid k \in \mathbb{N}\,\}$.

Proof. Let $A \triangleq \{\, a_k \mid k \in \mathbb{N}\,\}$ and $B \triangleq \{\, b_k \mid k \in \mathbb{N}\,\}$. Then, A is bounded above by any b_k and B is bounded below by any a_k. Thus, by Theorem 1.1.2, $\alpha = \sup A$ and $\beta = \sup B$ exist.

Any a_k is a lower bound of B, therefore $a_k \le \beta$ for each $k \in \mathbb{N}$, which implies β is an upper bound of A. Hence $\alpha \le \beta$.

To prove $\bigcap_{k=1}^{\infty} I_k \supseteq [\alpha, \beta]$, take any $x \in [\alpha, \beta]$. Then, for each $k \in \mathbb{N}$, $a_k \le \alpha \le x \le \beta \le b_k$, which means $x \in I_k$. Thus, $[\alpha, \beta] \subseteq \bigcap_{k=1}^{\infty} I_k$. Now, to prove the reverse containment, take any $x \in \bigcap_{k=1}^{\infty} I_k$. This means $\forall k \in \mathbb{N}$, $a_k \le x \le b_k$; x is an upper bound of A and is a lower bound of B at the same time. Therefore, $\alpha \le x \le \beta$, hence $\bigcap_{k=1}^{\infty} I_k \subseteq [\alpha, \beta]$. $\square$

Another Proof. Since the sequences $\{a_k\}$ and $\{b_k\}$ are bounded and monotone, there are limits $\alpha = \lim_{k\to\infty} a_k$ and $\beta = \lim_{k\to\infty} b_k$ by Theorem 1.3.4. By Theorem 1.3.6, $\alpha \le \beta$.

Since $a_k \le \alpha \le \beta \le b_k$ for each $k \in \mathbb{N}$, $[a, b] \subseteq \bigcap_{k=1}^{n} I_k$.

Now, take any $x \in \bigcap_{k=1}^{\infty} I_k$. Then, for all $k \in \mathbb{N}$, $a_k \le x \le b_k$. If it were $\alpha > x$, there is some $k_0 \in \mathbb{N}$ such that $a_{k_0} > x$. It is similar for the case when $\beta < x$. Therefore, $\alpha \le x \le \beta$. We have proven that $\bigcap_{k=1}^{\infty} I_k \subseteq [\alpha, \beta]$. $\square$

Corollary 2.3.1

If, in the notation of the previous theorem, $\lim_{k\to\infty}(b_k - a_k) = 0$, then $\bigcap_{k=1}^{\infty} I_k$ is a singleton.

Proof. Take any $\varepsilon \in \mathbb{R}_+$. Then, there is some $k_0 \in \mathbb{N}$ such that $b_{k_0} - a_{k_0} < \varepsilon$.

$$0 \le \beta - \alpha \le b_{k_0} - a_{k_0} < \varepsilon$$

holds. This implies that $0 \le \beta - \alpha < \varepsilon$ for arbitrary $\varepsilon \in \mathbb{R}_+$; therfore $\alpha = \beta$. $\square$

Theorem 2.3.2 Cartor's Criterion

If $\{C_k\}$ is a nested sequence of closed, bounded, nonempty subsets of $\mathbb{R}^n$, then

$$\bigcap_{k=1}^{\infty} C_k \ne \varnothing.$$

Furthermore, if $\lim_{k\to\infty} d(C_k) = 0$, then $\bigcap_{k=1}^{\infty} C_k$ is a singleton.

Proof. If any of C_k is finite, it is trivial. So, we suppose every C_k is infinite. Construct a seqeuence $\{\mathbf{x}_k\}$ of points in $\mathbb{R}^n$ as following.

- Take any $\mathbf{x}_1$ in C_1.
- For each $k \in \mathbb{N}$, take any $\mathbf{x}_{k+1}$ in $C_{k+1} \setminus \{\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\}$.

Since $S \subseteq C_1$ is bounded and contains infinitely many points, by Theorem 2.1.8, there is a limit point $\mathbf{x}_0$ of S in $\mathbb{R}^n$. We now claim that $\mathbf{x}_0 \in \bigcap_{k=1}^{\infty} C_k$.

Fix any $k \in \mathbb{N}$ and choose any deleted neighborhood $N'(\mathbf{x}_0)$ of $\mathbf{x}_0$. Since $N'(\mathbf{x}_0) \cap S$ is infinite, there is some $k_1 \in \mathbb{N}_{>k}$ such that $\mathbf{x}_{k_1} \in N'(\mathbf{x}_0)$. By the construction, $\mathbf{x}_{k_1} \in C_{k_1} \subseteq C_k$. This shows that every deleted neighborhood of $\mathbf{x}_0$ contains a point in C_k; $\mathbf{x}_0 \in C_k'$. As each C_k is closed, $\mathbf{x}_0 \in C_k$, and thus $\mathbf{x}_0 \in \bigcap_{k=1}^{\infty} C_k$.

Suppose, in addition, $\lim_{k\to\infty} d(C_k) = 0$. Assume $\bigcap_{k=1}^{\infty} C_k$ has two distinct points $\mathbf{x}$ and $\mathbf{y}$ for the sake of contradiction. Choose any $\varepsilon \in (0, \|\mathbf{x}-\mathbf{y}\|)$ and then there is some $k \in \mathbb{N}$ with $d(C_k) < \varepsilon$. Nonetheless, $\varepsilon < \|\mathbf{x}-\mathbf{y}\| \le d(C_k) < \varepsilon$, #. □

Theorem 2.3.3 Cantor's Criterion in $\mathbb{R}^n$ implies Cantor's Criterion in $\mathbb{R}$

Cantor's criterion in $\mathbb{R}^n$ implies Cantor's criterion also holds in $\mathbb{R}$.

Proof. $\mathbb{R} \times \{0\} \times \cdots \times \{0\}$ is a closed subset of $\mathbb{R}^n$. □

Theorem 2.3.4

Cantor's criterion in $\mathbb{R}$ and Archimedes' principle implies the existence of supremum of any bounded above nonempty subset of $\mathbb{R}$.

Proof. Let S be a nonempty, bounded above set in $\mathbb{R}$. Let B denote the set of upper bounds of S and let $A = B^c$. Since $x-1 \in A$ for all $x \in S$, $A \neq \varnothing$.

We first show that for all $a \in A$ and $b \in B$, $a < b$. If otherwise, i.e., $a \ge b$, $x \le b \le a$ for each $x \in S$, which implies $a \in B$, which is a contradiction.

Moreover, $S \cap [a,b] \neq \varnothing$ for each $a \in A$ and $b \in B$. Assume $S \cap [a,b] = \varnothing$ for the sake of contradiction. Since $S \cap (b, \infty) = \varnothing$ as b is an upper bound of S, then it follows $S \subseteq (-\infty, a)$, which implies a is an upper bound of S, which is a contradiction.

Construct a nested sequence $\{[a_k, b_k]\}_{k\in\mathbb{N}}$ of closed interval of which each a_k is in A and b_k is in B.

- Take any a_1 in A and b_1 in B.
- For each $k \in \mathbb{N}$, if $(a_k+b_k)/2 \in A$, then let $a_{k+1} \triangleq (a_k+b_k)/2$ and $b_{k+1} \triangleq b_k$. If $(a_k+b_k)/2 \in B$, then let $a_{k+1} \triangleq a_k$ and $b_{k+1} \triangleq (a_k+b_k)/2$.

Then it is immediate that $\lim_{k\to\infty}(b_k - a_k) = \lim_{k\to\infty} 2^{-k+1}(b_1 - a_1) = 0$. Therefore, by Cantor's criterion in $\mathbb{R}$, $\bigcap_{k=1}^{\infty}[a_k, b_k] = \{x_0\}$ for some $x_0 \in \mathbb{R}$.

We now show that x_0 is an upper bound of S. Assume not for the sake of contradiction, that is, there is some $x \in S$ such that $x > x_0$. Then, we may find some $k \in \mathbb{N}$ such that $b_k - a_k < x - x_0$. Then it follows $b_k - x_0 \le b_k - a_k < x - x_0$, and therefore $b_k < x_0$. This contradicts that b_k is an upper bound of S. Thus, $x_0 \in B$.

We now claim that x_0 is the least upper bound. Assume to the contrary that there is some $b \in B$ such that $b < x_0$. Then, we may find some $k \in \mathbb{N}$ such that $b_k - a_k < x_0 - b$. It follows $x_0 - a_k \le b_k - a_k < x_0 - b$, and therefore $b < a_k$. This contradicts that $a_k \in A$. □

Theorem 2.3.5

Assuming that Archimedes' principle holds in $\mathbb{R}$, the following are equivalent.

(i) Eery nonempty supset in $\mathbb{R}$ which is bounded above has a supremum in $\mathbb{R}$.
(ii) Every bounded monotone sequence in $\mathbb{R}$ converges.
(iii) $\mathbb{R}$ has Bolzano–Weierstrass property.
(iv) $\mathbb{R}$ is Cauchy complete.

(v) $\mathbb{R}^n$ is Cauchy complete.
(vi) $\mathbb{R}^n$ has Bolzano–Weierstrass property.
(vii) Cantor's criterion holds in $\mathbb{R}^n$.
(viii) Cantor's criterion holds in $\mathbb{R}$.

Proof.

$$(\mathrm{i}) \overbrace{\Longrightarrow}^{\text{Theorem 1.3.4}} (\mathrm{ii}) \overbrace{\Longrightarrow}^{\text{Theorem 2.3.1}} (\mathrm{viii}) \overbrace{\Longrightarrow}^{\text{Theorem 2.3.4}} (\mathrm{i})$$

$$(\mathrm{i}) \overbrace{\Longrightarrow}^{\text{Theorem 1.2.3}} (\mathrm{iii}) \overbrace{\Longrightarrow}^{\text{Theorem 1.4.4}} (\mathrm{iv}) \overbrace{\Longrightarrow}^{\text{Theorem 2.1.7}} (\mathrm{v}) \overbrace{\Longrightarrow}^{\text{Theorem 2.1.8}} (\mathrm{vi})$$

$$\overbrace{\Longrightarrow}^{\text{Theorem 2.3.2}} (\mathrm{vii}) \overbrace{\Longrightarrow}^{\text{special case}} (\mathrm{viii}) \overbrace{\Longrightarrow}^{\text{Theorem 2.3.4}} (\mathrm{i})$$

□

Definition 2.3.2: Completeness

When the word *complete* is applied to $\mathbb{R}^n$, it is assumed that it means any of these statements:

- the existence of least upper bounds in $\mathbb{R}$,
- the Monotone Convergence theorem in $\mathbb{R}^n$,
- Cantor's criterion,
- the Bolzano–Weierstrass property, or
- Cauchy completeness.

2.4 Relative Topology and Connectedness

Definition 2.4.1: Relatively Open and Relatively Closed Set

A set S is said to be *relatively open* in X if there exists an open set U in $\mathbb{R}^n$ such that $S = U \cap X$. Likewise, a set S is said to be *relatively closed* if there exists a closed set C in $\mathbb{R}^n$ such that $S = C \cap X$.

Note:-

Every relatively open set in X is open in $\mathbb{R}^n$ if and only if X is open in $\mathbb{R}^n$. Every relatively closed set in X is closed in $\mathbb{R}^n$ if and only if X is closed in $\mathbb{R}^n$.

Definition 2.4.2

- A *relative neighborhood* of $\mathbf{x}$ in X is $N(\mathbf{x}; r) \cap X$. A *deleted neighborhood* is $N'(\mathbf{x}; r) \cap X$.
- A sequence $\{\mathbf{x}_k\}$ in X *converges in* X if $\lim_{k\to\infty} \mathbf{x}_k \in X$.
- The relative closure of S is $\overline{S} \cap X$.
- A point $\mathbf{x}_0$ in X is a *relative limit point* of S in X if $\mathbf{x}_0$ is a limit point of S.

Note:-

Depending on X, X may not be complete, i.e., a Cauchy sequence in X may not converge in X, an infinite and bounded subset of X may not have a limit point in X, or a nested sequence of nonempty, bounded, relatively closed subsets of X may have an empty intersection.

Lemma 2.4.1

Let X be a subset of $\mathbb{R}^n$ and C be a subset in X. Then, C is relatively closed in X if and only if C is the relative closure of C.

Proof. The "if" part is trivial; we only prove the "only if" part.

($\subseteq$) It is direct since $C \subseteq \overline{C}$.

($\supseteq$) There exists a closed set $\hat{C}$ in $\mathbb{R}^n$ such that $C = \hat{C} \cap X$. Since $C \subseteq \hat{C}$ and $\hat{C}$ is closed, $\overline{C} \subseteq \hat{C}$ by Theorem 2.2.6. Thus, $\overline{C} \cap X \subseteq \hat{C} \cap X = C$. $\square$

Lemma 2.4.2

Let $C \subseteq X \subseteq \mathbb{R}^n$. Then, C is relatively closed in X if and only if C contains all its relative limit points in X.

Proof. ($\Rightarrow$) If $\mathbf{x}_0$ is a relative limit point of C in X, then $\mathbf{x}_0 \in \overline{C} \cap X = C$. (Lemma 2.4.1)

($\Leftarrow$) It means $\overline{C} \cap X \subseteq C$. And it is already $C \subseteq \overline{C} \cap X$. Therefore, C is relatively closed in X by Lemma 2.4.1. $\square$

Theorem 2.4.1

Let X be any nonempty subset of $\mathbb{R}^n$. The following statements are equivalent.

(i) Every Cauchy sequence $\{\mathbf{x}_k\}$ in X converges to a point of X. Thus X inherits Cauchy completeness from $\mathbb{R}^n$.

(ii) If S is a bounded, infinite subset of X, then S has a limit point in X. Thus X inherits the Bolzano–Weierstrass property from $\mathbb{R}^n$.

(iii) If $\{C_k\}$ is any nested sequence of nonempty, bounded, relatively closed subsets of X, then $\bigcap_{k=1}^{\infty} \neq \varnothing$. Furthermore, if $\lim_{k\to\infty} d(C_k) = 0$, then $\bigcap_{k=1}^{\infty}$ is a singleton. Thus X inherits Cantor's criterion from $\mathbb{R}^n$.

Proof. We shall first prove (i) $\Longrightarrow$ (ii). Let S be any bounded and infinite subset of S. Since S is a bounded and infinite subset of $\mathbb{R}^n$ at the same time, by Theorem 2.1.8, there is a limit point $\mathbf{x}_0$ of S in $\mathbb{R}^n$. We may construct a Cauchy sequence in S that converges to $\mathbf{x}_0$ as we did in the proof of Theorem 1.3.7. Thus, $\mathbf{x}_0$, which the Cauchy sequence converges to, is in S by Cauchy completeness of X. Therefore, S has a limit point in X.

We next prove (ii) $\Longrightarrow$ (iii). Suppose that X has the Bolzano–Weierstrass property. Take any nested sequence $\{C_k\}$ be bounded, nonempty, and *relatively closed* subsets of X. We assume each C_k is infinite since it is trivial otherwise.

As in the proof of Theorem 2.3.2, recursively choose an infinite set $S = \{\mathbf{x}_k \mid k \in \mathbb{N}\}$ of distinct points such that $\mathbf{x}_k \in C_k$ for each $k \in \mathbb{N}$. By the assumption (ii), there is a limit point $\mathbf{x}_0 \in X$ of S. Since $\mathbf{x}_k \in C_k \subseteq \overline{C_k}$ for each $k \in \mathbb{N}$ and $\{\overline{C_k}\}_{k\in\mathbb{N}}$ is a nested sequence of bounded, nonempty, and closed subsets of $\mathbb{R}^n$, as in the proof of Theorem 2.3.2, $\mathbf{x}_0 \in \overline{C_k}$ for each k, also. As $C_k = \overline{C_k} \cap X$ for each $k \in \mathbb{N}$ by Lemma 2.4.1, $\mathbf{x}_0 \in C_k$ for each $k \in \mathbb{N}$. Thus, $\mathbf{x}_0 \in \bigcap_{k=1}^{\infty} C_k$.

If, in addition, $\lim_{k\to\infty} d(C_k) = 0$, then $\lim_{k\to\infty} d(\overline{C_k}) = 0$ by Exercise 2.2.2. By Theorem 2.3.2, $\bigcap_{k=1}^{\infty} \overline{C_k} = \{\mathbf{x}_0\}$. But since $C_k \subseteq \overline{C_k}$ for each $k \in \mathbb{N}$, $\varnothing \neq \bigcap_{k=1}^{\infty} C_k \subseteq \bigcap_{k=1}^{\infty} \overline{C_k} = \{\mathbf{x}_0\}$. Thus, (ii) implies Cantor's criterion holds in X.

Finally, it is left to prove (iii) $\Longrightarrow$ (i). Let $\{\mathbf{x}_k\}$ be a Cauchy sequence in X. We must show it converges to some point in X. By Cauchy completeness of $\mathbb{R}^n$, $\{\mathbf{x}_k\}$ converges to some point $\mathbf{x}_0$ in $\mathbb{R}^n$. We shall show that $\mathbf{x}_0 \in X$.

Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be any sequence of positive numbers that converge monotonically to 0, e.g., $\varepsilon_k = 1/k$. Let $C_k \triangleq \overline{N(\mathbf{x}_0;\varepsilon_k)} \cap X$. Then $\{C_k\}$ is a nested sequence of bounded and relatively closed sets in X.

Take any $k_1 \in \mathbb{N}$. Since $\{\mathbf{x}_k\}$ converges to $\mathbf{x}_0$, there is some $k_0 \in \mathbb{N}$ such that $\forall k \in \mathbb{N}_{\geq k_0}$, $\mathbf{x}_k \in N(\mathbf{x}_k;\varepsilon_{k_1})$. And such $\mathbf{x}_k$'s are also in $N(\mathbf{x}_k;\varepsilon_{k_1})\cap X$; hence each C_k is nonempty.

Moreover, as $d(C_k) \leq d\big(N(\mathbf{x}_0;\varepsilon_k)\big) = 2\varepsilon_k$, $\lim_{k\to\infty} d(C_k) = 0$. Thus, by assumption (ii), $\bigcap_{k=1}^{\infty} C_k$ is a singleton in X. Then,

$$\bigcap_{k=1}^{\infty} C_k = \bigcap_{k=1}^{\infty}\left(\overline{N(\mathbf{x}_0;\varepsilon_k)}\cap X\right) \subseteq \bigcap_{k=1}^{\infty}\overline{N(\mathbf{x}_0;\varepsilon_k)} = \{\mathbf{x}_0\}.$$

Therefore, $\mathbf{x}_0 \in \bigcap_{k=1}^{\infty} C_k \subseteq X$. □

Definition 2.4.3: Completeness of a Subset of $\mathbb{R}^n$

Let X be a nonempty subset of $\mathbb{R}^n$. If any of the equivalent properties of Theorem 2.4.1 hold in the set X, then X is said to be *complete*.

Theorem 2.4.2

A nonempty subset X of $\mathbb{R}^n$ is complete if and only if X is closed in $\mathbb{R}^n$.

Proof. ($\Rightarrow$) Let $\mathbf{x}_0$ be a limit point of X. Then, there is a Cauchy sequence $\{\mathbf{x}_k\}$ that converges to $\mathbf{x}_0$. By Cauchy completeness of X, $\mathbf{x}_0 \in X$. Since we have proven that $X' \subseteq X$, X is closed by Theorem 2.2.4.

($\Leftarrow$) Let S be any bounded, infinite set in X. By Bolzano–Weierstrass property of $\mathbb{R}^n$, S has a limit point $\mathbf{x}_0$ in $\mathbb{R}^n$. As X being closed, $\mathbf{x}_0 \in X$. Therefore, X has Bolzano–Weierstrass property; X is complete by Theorem 2.4.1. □

Definition 2.4.4: Connectedness

A set S is *disconnected* if there are two open sets U, V such that
- (i) $U \cap V = \varnothing$,
- (ii) $S \subseteq U \cup V$, and
- (iii) $S \cap U \neq \varnothing$ and $S \cap V \neq \varnothing$.

S is *connected* if S is not disconected.

Example 2.4.1

- $[0,1)\cup(1,2]$ is disconnected. ($U = (-1,1)$ and $V = (1,3)$)
- $\mathbb{Q}$ is disconnected. ($U = (-\infty, r)$ and $V = (r,\infty)$ where $r \in \mathbb{R}\setminus\mathbb{Q}$)

Theorem 2.4.3

Any interval of the form of $[a,b]$, $(a,b]$, $[a,b)$, or (a,b) is connected. $\mathbb{R}$ itself is connected.

Proof. Let I be any of these sets. The only fact needed for this proof is that $(u,v) \subseteq I$ for any $u < v$ in I.

Suppose I is disconnected. Then, there are disjoint open sets $U, V \subseteq \mathbb{R}$ such that $I \subseteq U \cup V$, $I \cap U \neq \varnothing$, and $I \cap V \neq \varnothing$. Take any $u \in I \cap U$ and $v \in I \cap V$. WLOG, $u < v$.

Construct sequences $\{u_k\}$ in $I \cap U$ and $\{v_k\}$ in $I \cap V$ as following.

- $u_1 \triangleq u$ and $v_1 \triangleq v$.
- For each $k \in \mathbb{N}$, $u_{k+1} \triangleq \begin{cases} \frac{u_k+v_k}{2} & \frac{u_k+v_k}{2} \in U \\ u_k & \text{otherwise} \end{cases}$ and $v_{k+1} \triangleq \begin{cases} \frac{u_k+v_k}{2} & \frac{u_k+v_k}{2} \in V \\ v_k & \text{otherwise} \end{cases}$.

Then, $\{u_k\}$ and $\{v_k\}$ are bounded, monotone sequences ($u_k < v$ and $v_k > u$ for each $k \in \mathbb{N}$); hence they converge by Theorem 1.3.4.

Since $(u_k+v_k)/2 \in I \subseteq U \cup V$ for each $k \in \mathbb{N}$ and $U \cap V = \varnothing$, $v_{k+1} - u_{k+1} = (v_k - u_k)/2 = (v-u)/2^k$. They converge to the same point x_0 since $\lim_{k\to\infty}(v_k - u_k) = 0$.

If x_0 is ever equal to u_k, then by openness of U, there is some neighborhood $N(x_0;\varepsilon)$ of x_0 such that $N(x_0;\varepsilon) \subseteq U$, which contradicts $\lim_{k\to\infty} v_k = x_0$. Thus, x_0 is never equal to u_k; by the same reason, x_0 is never equal to v_k. We conclude x_0 is a relative limit point in I of both $I \cap U$ and $I \cap V$.

Since $I \cap U = I \cap V^c$ and V^c is closed, $I \cap U$ is closed and thus any relative limit point of $I \cap U$ in I, x_0, in particular, is in $I \cap U$ by Lemma 2.4.2. Similarly, $x_0 \in I \cap V$, which contradicts $U \cap V = \varnothing$. □

2.5 Compactness

Definition 2.5.1: Open Cover

Let S be any nonempty subset of $\mathbb{R}^n$. An *open cover* of S is any collection $\mathcal{C} = \{U_\alpha \mid \alpha \in A\}$ of open sets such that $S \subseteq \bigcup_{\alpha\in A} U_\alpha$.

Definition 2.5.2: Compactness

A set S in $\mathbb{R}^n$ is said to be *compact* if, for every open cover $\mathcal{C}$ of S, there exists a finite subcover, that is, there is some finite subset A_0 of A such that $S \subseteq \bigcup_{\alpha\in A_0} U_\alpha$.

Theorem 2.5.1 Heine–Borel

Let S be any closed and bounded set in $\mathbb{R}^n$. Then any *countable* open cover of S has a finite subcover.

Proof. Let $\{I_k \mid k \in \mathbb{N}\}$ be any open cover of S. Let $U_k \triangleq \bigcup_{j=1}^{k} I_j$, $C_k \triangleq {U_k}^c$, and $D_k \triangleq C_k \cap S$ for each $k \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$, $U_k \subseteq U_{k+1}$, $C_k \supseteq C_{k+1}$, and $D_k \supseteq D_{k+1}$. Note that each U_k is open and thus C_k and D_k are closed.

Assume that every D_k is nonempty. Then, since $\{D_k\}$ is a nested sequence of bounded, nonempty, closed sets, by Cantor's criterion, there is some $x \in \bigcap_{k=1}^{\infty} D_k$. Then,

$$x \in \bigcap_{k=1}^{\infty}(S \cap C_k) = S \cap \left(\bigcap_{k=1}^{\infty} C_k\right) = S \cap \left(\bigcup_{k=1}^{\infty} U_k\right)^c = S \cap \left(\bigcup_{k=1}^{\infty} I_k\right)^c$$

Thus, x is a point in S that is not covered by $\{I_k \mid k \in \mathbb{N},\}$, which is a contradiction.

Hence, some D_{k_0} is empty, and it means that $S \subseteq \bigcup_{j=1}^{k_0} I_k$. □

Theorem 2.5.2 Lindelöf

Let S be any subset of $\mathbb{R}^n$. Then, every open cover of S has a countable subcover.

Proof. Since the set $\{N(\mathbf{y};r) \mid \mathbf{y} \in \mathbb{Q}^n, r \in \mathbb{Q}\}$ has the trivial one-to-one correspondence with $\mathbb{Q}^{n+1}$, the set is countable. Thus, we may index them as $\{N_k \mid k \in \mathbb{N}\}$.

Let $\mathcal{C} = \{U_\alpha \mid \alpha \in A\}$ be any open cover of S. Take any $\mathbf{x} \in S$. Then there is some U_α such that $\mathbf{x} \in U_\alpha$. Since U_α is open, there is a neighborhood $N(\mathbf{x};\varepsilon)$ such that $N(\mathbf{x};\varepsilon) \subseteq U_\alpha$.

As $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, there is some $\mathbf{y} \in N(\mathbf{x};\varepsilon/2) \cap \mathbb{Q}^n$. As $\mathbb{Q}$ is dense in $\mathbb{R}$, there is some $r \in (\|\mathbf{x}-\mathbf{y}\|, \varepsilon - \|\mathbf{x}-\mathbf{y}\|) \cap \mathbb{Q}$. Then, $\mathbf{x} \in N(\mathbf{y};r) \subseteq N(\mathbf{x};\varepsilon) \subseteq U_\alpha$. The neighborhood $N(\mathbf{y};r)$ is in the set $\{N_k \mid k \in \mathbb{N}\}$.

Let us define $K(\mathbf{x}) \triangleq \{k \in \mathbb{N} \mid \mathbf{x} \in N_k \subseteq U_\alpha \text{ for some } U_\alpha \in \mathcal{C}\}$. Then, by Well-Ordering Property of $\mathbb{N}$, there is $k(\mathbf{x}) \triangleq \min K(\mathbf{x})$.

The collection $\{N_{k(\mathbf{x})} \mid \mathbf{x} \in S\}$ is an open cover of S and is countable since it is a subset of $\{N_k \mid k \in \mathbb{N}\}$.

By construction, for each $N_{k(\mathbf{x})}$, there is $U_\alpha \in \mathcal{C}$ such that $N_{k(\mathbf{x})} \subseteq U_\alpha$. Hence, if we denote one of such sets as $U_{k(\mathbf{x})}$, the collection $\{U_{k(\mathbf{x})} \mid \mathbf{x} \in S\}$ is a countable subcover of S. □

Corollary 2.5.1

If $S \subseteq \mathbb{R}^n$ is closed and bounded, then S is compact.

Proof. Simply combine Theorem 2.5.2 and Theorem 2.5.1. □

Theorem 2.5.3

If S is a compact subset of $\mathbb{R}^n$, then S is closed and bounded.

Proof. Suppose $S \subseteq \mathbb{R}^n$ is not closed. Then S has a limit point $\mathbf{x}_0$ that is not in S. Let $\{\varepsilon_k\}$ be any sequence of positive numbers that monotonically converges to 0. Define $U_k \triangleq \left[\overline{N(\mathbf{x}_0;\varepsilon_k)}\right]^c$.

Then $\mathcal{C} \triangleq \{U_k \mid k \in \mathbb{N}\}$ is an open cover of S. To prove this, take any $\mathbf{x} \in S$. There exists $k_0 \in \mathbb{N}$ such that $\varepsilon_k < \|\mathbf{x} - \mathbf{x}_k\|$ for all $k \in \mathbb{N}_{\geq k_0}$. Thus, for $k \geq k_0$, $\mathbf{x}_0 \notin \overline{N(\mathbf{x}_0;\varepsilon_k)}$, i.e., $\mathbf{x}_0 \in U_k$.

We now claim there is no finite subcover. Take finitely many subsets $\{U_{k_1}, U_{k_2}, \cdots, U_{k_p}\}$ from the open cover. Let $k_0 = \max_{i\in[p]} k_i$. Note that $\bigcup_{i=1}^p U_{k_i} = U_{k_0}$. Take ε from $(0, \varepsilon_{k_0})$. Then, since $\mathbf{x}_0$ is a limit point of S, we may take $\mathbf{x} \in S \cap N'(\mathbf{x}_0;\varepsilon)$. Then, $\mathbf{x} \notin U_{k_0}$. Therefore, $\{U_{k_i}\}_{i\in[p]}$ does not cover S. Thus, we proved that a set which is not closed is not compact.

Now, suppose S is not bounded. Let $U_k = N(\mathbf{0};k)$. Then, $S \subseteq \mathbb{R}^n = \bigcup_{k=1}^\infty U_k$; $\{U_k\}$ is an open cover of S.

Take finitely many subsets $\{U_{k_1}, U_{k_2}, \cdots, U_{k_p}\}$ from the open cover. Let $k_0 = \max_{i\in[p]} U_{k_i}$. Note that $\bigcup_{i=1}^p U_{k_i} = U_{k_0}$. Because S is unbounded, there is some $\mathbf{x} \in S$ such that $\|\mathbf{x}\| > k_0$; $\mathbf{x} \notin U_{k_0} = \bigcup_{i=1}^p U_{k_i}$. Thus, we proved that an unbounded set is not compact. □

Note:-

A set $S \subseteq \mathbb{R}^n$ is complete if and only if S is closed. A set S is compact if and only if S is closed and bounded. Thus, compactness implies completeness, but not vice versa.

Theorem 2.5.4

Let S be a nonempty subset of $\mathbb{R}^n$. Then the following statements are equivalent.

(i) S is closed and bounded.

(ii) S is compact.

(iii) Every infinite subset of S has a limit point which is in S.

Proof. We already know (i) is equivalent to (ii). Therefore, we only need to prove (i) and (iii) are equivalent. (i) $\implies$ (iii) is direct since S is complete by Theorem 2.4.2. So we are left to prove (iii) implies (i).

Suppose S is not bounded. Then we may constrct a sequence $\{\mathbf{x}_k\}$ in S as following.

- Take any $\mathbf{x}_1 \in S$.
- For each $k \in \mathbb{N}$, take $\mathbf{x}_{k+1} \in S \setminus N(\mathbf{0}; \|\mathbf{x}_k\| + 1)$.

The construction is valid since S is unbounded. Then, the set $\{\, \mathbf{x}_k \mid k \in \mathbb{N} \,\}$ is infinite but it does not have a limit point since $\|\mathbf{x}_k - \mathbf{x}_m\| \geq 1$ for each $\{k, m\} \subseteq \mathbb{N}$. Thus, (iii) implies S is bounded.

Now, we shall prove (iii) implies S is closed. Let $\mathbf{x}$ be a limit point of S. Let $\{\varepsilon_k\}$ be a monotonic sequence of positive numbers that converges to 0. Construct a sequence $\{\mathbf{x}_k\}$ as following.

- For each $k \in \mathbb{N}$, take $\mathbf{x}_k \in S \cap N'(\mathbf{x}; \varepsilon_k)$.

The construction is valid since $\mathbf{x}$ is a limit point of S. Then, the set $\{\, \mathbf{x}_k \mid k \in \mathbb{N} \,\}$ has the only limit point $\mathbf{x}$. Then, by (iii), $\mathbf{x} \in S$. We proved that S contains all its limit points, thus S is closed by Theorem 2.2.4. $\square$

Chapter 3

Continuity

3.1 Limit and Continuity

Definition 3.1.1: Limit of a Function

Let $f : S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in \overline{S}$. We say f has *limit* L as $\mathbf{x}$ approaches $\mathbf{c}$ provided that, for every neighborhood $N(L)$, there exists a deleted neighborhood $N'(\mathbf{c})$ such that

$$S \cap N'(\mathbf{c}) \subseteq f^{-1}(N(L)).$$

We write $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$.

Note:-

Limit is unique if it exists.

Note:-

Note that $S \cap N'(\mathbf{c};\delta) = \varnothing$ for sufficiently small δ if $\mathbf{c}$ is an isolated point of S. This implies any real number can be a limit of f as $\mathbf{x}$ approaches $\mathbf{c}$. Somehow, Douglass defined that $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = f(\mathbf{c})$ (since $\mathbf{c} \in S$ in this case). Actually I do not think we should define limit for isolated points.

Note:-

This definition of limit is equivalent to the normal ε-δ definition of limit, except that it defines a limit for isolated points.

Definition 3.1.2: Continuity

Let $f : S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in S$. We say f is *continuous at* $\mathbf{c}$ if

$$\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = f(\mathbf{c}).$$

In other words, for every neighborhood $N(f(\mathbf{c}))$, there exists a neighborhood $N(\mathbf{c})$ such that

$$S \cap N(\mathbf{c}) \subseteq f^{-1}\big(N(f(\mathbf{c}))\big).$$

If f is continuous at every $\mathbf{c} \in S$, then f is said to be *continuous*.

Theorem 3.1.1

Let $f : S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in \overline{S}$ where $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{c}) = L$ exists and. Then, f is locally bounded on some deleted neighborhood of $\mathbf{c}$, that is, there are $M, \delta \in \mathbb{R}_+$ such that

$$\mathbf{x} \in S \cap N'(\mathbf{c}; \delta) \implies |f(\mathbf{x})| \le M.$$

Proof. There exists $\delta \in \mathbb{R}_+$ such that $S \cap N'(\mathbf{c}; \delta) \subseteq f^{-1}(N(L; 1))$. Then, $|f(\mathbf{x})| \le |L| + 1$ if $\mathbf{x} \in S \cap N'(\mathbf{x}; \delta)$. □

Theorem 3.1.2

Let $f : S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in \overline{S}$ where $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{c}) = L$ exists and $L \ne 0$. Then, f is locally bounded away from 0 on some deleted neighborhood of $\mathbf{c}$, that is, there are $m, \delta \in \mathbb{R}_+$ such that

$$\mathbf{x} \in S \cap N'(\mathbf{c}; \delta) \implies |f(\mathbf{x})| \ge m.$$

Proof. There exists $\delta \in \mathbb{R}_+$ such that $S \cap N'(\mathbf{c}; \delta) \subseteq f^{-1}(N(L; |L|/2))$. Then, $|f(\mathbf{x})| \ge |L|/2$ if $\mathbf{x} \in S \cap N'(\mathbf{x}; \delta)$. □

Theorem 3.1.3

Let $f_1 : S \to \mathbb{R}$ and $f_2 : S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in \overline{S}$, and suppose $\lim_{\mathbf{x}\to\mathbf{c}} f_1(\mathbf{x}) = L_1$ and $\lim_{\mathbf{x}\to\mathbf{c}} f_2(\mathbf{x}) = L_2$. Then

(i) $\lim_{\mathbf{x}\to\mathbf{c}} \big(f_1(\mathbf{x}) + f_2(\mathbf{x})\big) = L_1 + L_2$.
(ii) For any $a \in \mathbb{R}$, $\lim_{\mathbf{x}\to\mathbf{c}} af(\mathbf{x}) = aL_1$.
(iii) $\lim_{\mathbf{x}\to\mathbf{c}} f_1(\mathbf{x}) f_2(\mathbf{x}) = L_1 L_2$.
(iv) $\lim_{\mathbf{x}\to\mathbf{c}} f_1(\mathbf{x})/f_2(\mathbf{x}) = L_1/L_2$ provided that $L_2 \ne 0$.

Proof. Proved in MAS102 (Calculus II). □

Theorem 3.1.4 The Squeeze Play

Let f, g, and h be three real-valued functions sharing a common domain $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in \overline{C}$ where $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = \lim_{\mathbf{x}\to\mathbf{c}} h(\mathbf{x}) = L$ exist. Suppose also that, for some $\delta_0 \in \mathbb{R}_+$,

$$\mathbf{x} \in S \cap N'(\mathbf{c}; \delta_0) \implies f(\mathbf{x}) \le g(\mathbf{x}) \le h(\mathbf{x})$$

Then, $\lim_{\mathbf{x}\to\mathbf{c}} g(\mathbf{x}) = L$.

Proof. Proved in MAS102 (Calculus II). □

Theorem 3.1.5 Limit is Order Preserving

Let f and g be two real-valued functions sharing a common domain $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in \overline{C}$ where $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L_1$ and $\lim_{\mathbf{x}\to\mathbf{c}} g(\mathbf{x}) = L_2$ exist. Suppose also that, for some $\delta_0 \in \mathbb{R}_+$,

$$\mathbf{x} \in S \cap N'(\mathbf{c}; \delta_0) \implies f(\mathbf{x}) \le g(\mathbf{x})$$

Then, $L_1 \le L_2$.

Proof. Proved in MAS102 (Calculus II). □

Theorem 3.1.6

Let S be a nonempty subset of $\mathbb{R}^n$, $\mathbf{c} \in S'$, and $f : S \to \mathbb{R}$. $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$ if and only if, for every Cauchy sequence $\{\mathbf{x}_k\}$ in $S \setminus \{\mathbf{c}\}$ such that $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{c}$, it follows that $\lim_{k\to\infty} f(\mathbf{x}_k) = L$.

Proof. ($\Rightarrow$) Let $\{\mathbf{x}_k\}$ be any of such Cauchy sequences. Take any $\varepsilon \in \mathbb{R}_+$. By continuity, there exists $\delta \in \mathbb{R}_+$ such that $S \cap N'(\mathbf{c};\delta) \subseteq f^{-1}(N(L;\varepsilon))$. On the other hand, by convergence, there exists $k_0 \in \mathbb{N}$ such that $\forall k \in \mathbb{N}$, $(k \geq k_0 \implies \mathbf{x}_k \in N(\mathbf{c};\delta))$. Since $\mathbf{x}_k \neq \mathbf{c}$ for each $k \in \mathbb{N}$, we may say

$$\forall k \in \mathbb{N}, \big(k \geq k_0 \implies \mathbf{x}_k \in N'(\mathbf{c};\delta) \implies \mathbf{x}_k \in f^{-1}(N(L;\varepsilon)) \implies f(\mathbf{x}_k) \in N(L;\varepsilon)\big).$$

Thus, $\lim_{k\to\infty} f(\mathbf{x}_k) = L$ holds.

($\Leftarrow$) Suppose it is not $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$. Then, it is equivalent to say that, there is some neighborhood $N(L;\varepsilon_0)$ such that $S \cap N'(\mathbf{c};\delta) \nsubseteq f^{-1}(N(L;\varepsilon_0))$ for every deleted neighborhood $N'(\mathbf{x};\delta)$. Construct a sequence $\{\mathbf{x}_k\}$ in $S \setminus \{\mathbf{c}\}$ as following.

- $\mathbf{x}_1 \in S \setminus \{\mathbf{c}\} \setminus f^{-1}(N(L;\varepsilon_0))$.
- For each $k \in \mathbb{N}$, $\mathbf{x}_{k+1} \in S \cap N'(\mathbf{x}; |\mathbf{x}_k - \mathbf{c}|/2) \setminus f^{-1}(N(L;\varepsilon_0))$.

Then, $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{c}$ indeed holds, but it is not $\lim_{k\to\infty} f(\mathbf{x}_k) = L$ since $f(\mathbf{x}_k) \notin (N(L;\varepsilon_0))$ for each $k \in \mathbb{N}$. □

Theorem 3.1.7

Let S be a nonempty subset of $\mathbb{R}^n$, $\mathbf{c} \in S$, and $f : S \to \mathbb{R}$. f is continuous at $\mathbf{c}$ if and only if, for every Cauchy sequence $\{\mathbf{x}_k\}$ in S such that $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{c}$, it follows that $\lim_{k\to\infty} f(\mathbf{x}_k) = f(\mathbf{c})$.

Proof. ($\Rightarrow$) If at most finitely many $\mathbf{x}_k$ are distinct from $\mathbf{c}$, then $\exists k_0 \in \mathbb{N}$, $\forall k \in \mathbb{N}_{\geq k_0}$, $\mathbf{x}_k = \mathbf{c}$; $\lim_{k\to\infty} f(\mathbf{x}_k) = \mathbf{c}$ is evident.

If there are infinitely many $\mathbf{x}_k$ are distinct from $\mathbf{c}$, then we may extract a subsequence $\{\mathbf{x}_{\mathbf{k}_\mathbf{j}}\}_{j\in\mathbb{N}}$ such that each $\mathbf{x}_{k_j}$ is in $S \setminus \{c\}$. By Theorem 3.1.6, $\lim_{j\to\infty} f(\mathbf{x}_{k_j}) = \mathbf{c}$. This implies $\lim_{k\to\infty} f(\mathbf{x}_k) = \mathbf{c}$, regardless of the number of $\mathbf{x}_k$'s equal to $\mathbf{c}$.

($\Leftarrow$) If $\mathbf{c} \in S'$, then we may directly apply Theorem 3.1.6 since every Cauchy sequence in $S \setminus \{\mathbf{c}\}$ is a Cauchy sequence in S.

If $\mathbf{c} \notin S'$, then $\mathbf{c}$ is an isolated point. Then, $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = f(\mathbf{c})$ by definition. □

Theorem 3.1.8

Let $S \subseteq \mathbb{R}^n$ and $f : S \to \mathbb{R}$. Let $\mathbf{c} = (c_1, \cdots, c_n) \in \mathring{S}$. For $j = 1, 2, \cdots, n$, let

$$g_j(t) = f(c_1, c_2, \cdots, c_{j-1}, t, c_{j+1}, \cdots, c_n).$$

(i) If $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x}) = L$, then, for each $j \in [n]$, $\lim_{t\to c_j} g_j(t) = L$.
(ii) If f is continuous at $\mathbf{c}$, then, for each $j \in [n]$, g_j is continuous at c_j and $\lim_{t\to c_j} g_j(t) = f(\mathbf{c})$.

Proof.

(i) Take any $j \in [n]$ and $\varepsilon \in \mathbb{R}_+$. By convergence, there exists $\delta_1 \in \mathbb{R}_+$ such that $S \cap N'(\mathbf{c};\delta_1) \subseteq f^{-1}(N(L;\varepsilon))$. Since $\mathbf{x} \in \mathring{S}$, there exists $\delta_2 \in \mathbb{R}_+$ such that $N(\mathbf{c};\delta_2) \subseteq S$. Let $\delta \triangleq \min\{\delta_1, \delta_2\}$. Then, $N'(\mathbf{c};\delta) \subseteq f^{-1}(N(L;\varepsilon))$ and $N(\mathbf{c};\delta) \subseteq S$ hold. Hence, for any $t \in N'(c_j;\delta)$,
$$g_j(t) = f(c_1, \cdots, c_{j-1}, t, c_{j+1}, \cdots, c_n) \in N(L;\varepsilon)$$

as $\|(c_1,\cdots,c_{j-1},t,c_{j+1},\cdots,c_n)-\mathbf{c}\|=|t-c_j|<\delta$.

(ii) Since $\lim_{\mathbf{x}\to\mathbf{c}} f(\mathbf{x})=f(\mathbf{c})$, by (a), for each $j\in[n]$, $\lim_{t\to c_j} g(t)=f(\mathbf{c})=g(c_j)$.

$\square$

Note:-

The converse of Theorem 3.1.8 is not true.

3.2 The Topological Description of Continuity

Theorem 3.2.1

A surjective function $f:S\to T$ where $S\subseteq\mathbb{R}^n$ and $T\subseteq\mathbb{R}$ is continuous if and only if $f^{-1}(U)$ is relatively open in S for every relatively open set U in T.

Proof. ($\Rightarrow$) Let U be a relatively open set in T and $\mathbf{c}\in f^{-1}(U)$. Since U is open and $f(\mathbf{c})\in U$, there is a neighborhood $N(f(\mathbf{c}))$ such that $T\cap N(f(\mathbf{c}))\subseteq U$. By continuity, there is a neighborhood $N(\mathbf{c})$ such that $S\cap N(\mathbf{c})\subseteq f^{-1}(N(f(\mathbf{c})))\subseteq f^{-1}(U)$. Therefore, $\mathbf{c}$ is a relative interior point of $f^{-1}(U)$. Since $\mathbf{c}$ was arbitrary, $f^{-1}(U)$ is relatively open in S.

($\Leftarrow$) Take any $\mathbf{c}\in S$ and a neighborhood $N(f(\mathbf{c}))$. Then, $f^{-1}(T\cap N(f(\mathbf{c})))$ is relatively open in S. Since $\mathbf{c}\in f^{-1}(T\cap N(f(\mathbf{c})))$, there is a neighborhood $N(\mathbf{c})$ such that $S\cap N(\mathbf{c})\subseteq f^{-1}(N(f(\mathbf{c})))$. $\square$

Theorem 3.2.2

If S is a connected subset of $\mathbb{R}^n$ and $f:S\to\mathbb{R}$ is continuous on S, then $T=f(S)$ is also connected.

Proof. Suppose T is disconnected for the sake of contradiction. There exists two disjoint open sets $U,V\subseteq\mathbb{R}$ such that $T\subseteq U\cup V$, $T\cap U\neq\varnothing$, and $T\cap V\neq\varnothing$. Since $T\cap U$ and $T\cap V$ are relatively open in T, $U_1=f^{-1}(T\cap U)$ and $V_1=f^{-1}(T\cap V)$ are relatively open in S. Then, $S\subseteq U_1\cup V_1=S$, $U_1\cap V_1=\varnothing$, $S\cap U_1\neq\varnothing$, and $S\cap V_1\neq\varnothing$, which contradicts S is connected, #. $\square$

Theorem 3.2.3

If S is a compact subset of $\mathbb{R}^n$ and $f:S\to\mathbb{R}$ is continuous on S, then $T=f(S)$ is also compact.

Proof. Let $\{U_\alpha\}_{\alpha\in J}$ be an open cover of T. Then, for each $\alpha\in J$, $f^{-1}(U_\alpha)$ is relatively open in S since U_α is open and f is continuous. Because

$$S=f^{-1}(T)=f^{-1}\left(\bigcup_{\alpha\in J}U_\alpha\right)=\bigcup_{\alpha\in J}f^{-1}(U_\alpha),$$

$\{f^{-1}(U_\alpha)\}_{\alpha\in J}$ is a relative open cover of S. Since S is compact, there is a finite subcover $\{f^{-1}(U_{\alpha_i})\mid i\in[p],\alpha_i\in J\}$ of S. Then,

$$T=f(S)=f\left(\bigcup_{i=1}^{p}f^{-1}(U_{\alpha_i})\right)=\bigcup_{i=1}^{p}f\left(f^{-1}(U_{\alpha_i})\right)\subseteq\bigcup_{i=1}^{p}U_{\alpha_i},$$

implying $\{U_{\alpha_i}\}_{i=1}^{p}$ is a finite subcover of T. $\square$

Theorem 3.2.4

If S is a compact subset of $\mathbb{R}^n$ and $f : S \to \mathbb{R}$ is continous on S, then f has a minimum and a maximum value on S.

Proof. Theorem 3.2.3 implies $T = f(S) \subseteq \mathbb{R}$ is compact, and thus bounded and closed. Thus, $m = \inf T = \min T$ and $M = \sup T = \max T$ exist. □

Theorem 3.2.5 The Intermediate Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous and c is any number between $f(a)$ and $f(b)$, then there exists an $x \in [a, b]$ such that $f(x) = c$.

Proof. Since $[a, b]$ is connected and compact, Theorem 3.2.2 and Theorem 3.2.3 imply that $f([a, b])$ is connected and compact. Thus, $f([a, b]) = [m, M]$ where

$$m = \min f([a, b]) \le \min\{f(a), f(b)\}$$

and

$$M = \max f([a, b]) \ge \max\{f(a), f(b)\}.$$

This implies $c \in [m, M] = f([a, b])$, i.e., there exists $x \in [a, b]$ such that $f(x) = c$. □

Theorem 3.2.6 The General Intermediate Value Theorem

If S is any connected and compact subset of $\mathbb{R}^n$ and $f : S \to \mathbb{R}$ is continuous, if $f(\mathbf{x}_1)$ and $f(\mathbf{x}_2)$ are any two values of f on S, and if c is any number between them, then there exists a point $\mathbf{x} \in S$ such that $f(\mathbf{x}) = c$.

Proof. Since S is connected and compact, by Theorem 3.2.2 and Theorem 3.2.3, $f(S)$ is an closed interval $[m, M]$ as in the proof of Theorem 3.2.5. Since $m \le \min\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}$ and $M \ge \max\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}$, $c \in [m, M] = f(S)$, and thus $\exists \mathbf{x} \in S$, $f(\mathbf{x}) = c$. □

3.2.1 The Composition of Continuous Functions

Theorem 3.2.7

Let $S \subseteq \mathbb{R}^n$, $f : S \to \mathbb{R}$, $f(S) \subseteq T \subseteq \mathbb{R}$, and $g : T \to \mathbb{R}$. If f is continuous at $\mathbf{c} \in S$ and if g is continuous at $f(\mathbf{c}) \in T$, then $g \circ f$ is continuous at $\mathbf{c}$.

Proof. Let $d = (g \circ f)(\mathbf{c})$. Take any neighborhood $N(d)$ of d. By continuity of g at $f(\mathbf{c})$, there exists a neighborhood $N(f(\mathbf{c}))$ such that

$$T \cap N(f(\mathbf{c})) \subseteq g^{-1}(N(d)).$$

By the continuity of f at $\mathbf{c}$, there exists a neighborhood $N(\mathbf{c})$ such that

$$S \cap N(\mathbf{c}) \subseteq f^{-1}(N(f(\mathbf{c}))).$$

These imply $S \cap N(\mathbf{c}) \subseteq f^{-1}(g^{-1}(N(d))) = (g \circ f)^{-1}(N(d))$.

Corollary 3.2.1

Let $S \subseteq \mathbb{R}^n$, $f : S \to \mathbb{R}$, $f(S) \subseteq T \subseteq \mathbb{R}$, and $g : T \to \mathbb{R}$. If f and g are continuous, then

$g \circ f$ is continuous.

Theorem 3.2.8

If $f : [a,b] \to [c,d]$ is strictly monotone, continuous function, then the inverse function f^{-1} is also strictly monotone, continuous, and bijective.

Proof. All are immediate except for the continuity. Denote f^{-1} by g. By Theorem 3.1.7, it suffices to prove that whenever a Cauchy sequence $\{y_k\}$ in $f(S)$ converges to y, then $\{g(y_k)\}$ converges to $g(y)$ in S.

Choose any such sequence and let $x_k \triangleq g(y_k)$ for each $k \in \mathbb{N}$. Since g is bijective, $\{y_k \mid k \in \mathbb{N}\}$ is finite if and only if $\{x_k \mid k \in \mathbb{N}\}$ is finite. If they are finite, then $\{y_k\}$ is eventually y, this implies $\{x_k\}$ is eventually $g(y)$, and it is done.

If they are infinite, since domain and codomain are bounded and closed, by Theorem 1.2.3, $\{x_k \mid k \in \mathbb{N}\}$ has a limit point x. But since $[a,b]$ is complete by Theorem 2.4.2, $x \in [a,b]$ by (ii) of Theorem 2.4.1. x is a cluster point of $\{x_k\}$, thus there is a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $\lim_{j\to\infty} x_{k_j} = x$ by Theorem 1.3.7. Now the continuity of f guarantees that

$$\lim_{j\to\infty} f(x_{k_j}) = f(x).$$

At the same time, since $f(x_{k_j}) = y_{k_j}$, $\{f(x_{k_j})\}$ is a subsequence of $\{y_k\}$. As $\{y_k\}$ converges to y, we get

$$\lim_{j\to\infty} f(x_{k_j}) = y.$$

By Theorem 1.3.1, $f(x) = y$, or $x = g(y)$.

If there were another limit point x' of $\{x_k \mid k \in \mathbb{N}\}$, by the same procedure, we get $x = g(y) = x'$; $x = x'$; x is the unique limit point of the set. Thus, $\{x_k\}$ converges to x, i.e., $\{g(y_k)\}$ converges to $g(y)$. □

3.2.2 Limiting Behavior at Infinity

Definition 3.2.1: Function Space $C(S)$ and $C_\infty(S)$

Let $S \neq \varnothing$ be a subset of $\mathbb{R}^n$.

- $C(S)$ is the set of real-valued function on S which is continuous on S.
- $C_\infty(S)$ is the set of real-valued function on S which is bounded and continuous on S.

Note:-

In general, $C_\infty(S) \subseteq C(S)$. If $\varnothing \neq S \subseteq \mathbb{R}^n$ is compact, then $C(S) = C_\infty(S)$.

Definition 3.2.2: Neighborhood of ∞ and $-\infty$

In $\mathbb{R}$,

- $N(\infty; M) \triangleq (M, \infty) = \{x \in \mathbb{R} \mid x > M\}$
- $N(-\infty, -M) \triangleq (-\infty, -M) = \{x \in \mathbb{R} \mid x < -M\}$

In $\mathbb{R}^n$,

- $N(\infty; M) \triangleq \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| > M\}$

Definition 3.2.3: Limit at Infinity

(i) Let S be an unbounded set in $\mathbb{R}$. Let $f : S \to \mathbb{R}$.
- We say f *has limit* L at ∞ if, for all $\varepsilon \in \mathbb{R}_+$, there exists $M \in \mathbb{R}_+$ such that $S \cap N(\infty; M) \subseteq f^{-1}(N(L;\varepsilon))$. We write $\lim_{x\to\infty} f(x) = L$.
- We say f *has limit* L at $-\infty$ if, for all $\varepsilon \in \mathbb{R}_+$, there exists $M \in \mathbb{R}_+$ such that $S \cap N(-\infty; -M) \subseteq f^{-1}(N(L;\varepsilon))$. We write $\lim_{x\to-\infty} f(x) = L$.

(ii) Let S be an unbounded set in $\mathbb{R}^n$. Let $f : S \to \mathbb{R}$. We say that f *has limit* L at ∞, if, for all $\varepsilon \in \mathbb{R}_+$, there exists $M \in \mathbb{R}_+$ such that $S \cap N(\infty; M) \subseteq f^{-1}(N(L;\varepsilon))$. We write $\lim_{\|\mathbf{x}\|\to\infty} f(\mathbf{x}) = L$.

Theorem 3.2.9 The Squeeze Play

Let f, g, and h be three real-valued functions sharing a common unbounded domain $S \subseteq \mathbb{R}^n$. Suppose $\lim_{\|\mathbf{x}\|\to\infty} f(\mathbf{x}) = \lim_{\|\mathbf{x}\|\to\infty} h(\mathbf{x}) = L$. Suppose also that, for some $M \in \mathbb{R}_+$,

$$\mathbf{x} \in S \cap N(\infty; M) \implies f(\mathbf{x}) \le g(\mathbf{x}) \le h(\mathbf{x})$$

Then, $\lim_{\|\mathbf{x}\|\to\infty} g(\mathbf{x}) = L$.

Theorem 3.2.10

Let S be a closed and unbounded set in $\mathbb{R}^n$ and let $f \in C(S)$. Suppose $\lim_{\|\mathbf{x}\|\to\infty} f(\mathbf{x}) = L$ exists. Then $f \in C_\infty(S)$.

Proof. There exists $M \in \mathbb{R}_+$ such that, for $\mathbf{x} \in S \cap N(\infty; M)$, $|f(\mathbf{x}) - L| < 1$. Thus, for such $\mathbf{x}$, we have $|f(x)| < |L| + 1$.

Since $S \cap \overline{N(\mathbf{0}; M)}$ is a closed, bounded set in $\mathbb{R}^n$, it is compact by Theorem 2.5.3. Therefore the continuous f is bounded on $S \cap \overline{N(\mathbf{0}; M)}$ by Theorem 3.2.3. In other words, there is some $K \in \mathbb{R}_+$ such that, for $\mathbf{x} \in S \cap \overline{N(\mathbf{0}; M)}$, we have $|f(x)| \le K$. Thus, $|f(\mathbf{x})| \le \max\{K, |L| + 1\}$ for all $\mathbf{x} \in S$. □

3.3 The Algebra of Continuous Functions

Note:-

Let $\varnothing \ne S \subseteq \mathbb{R}^n$. One can easily find that $C(S)$ is a commutative ring and is a vector space.

Theorem 3.3.1

Let $\varnothing \ne S \subseteq \mathbb{R}^n$ and $f_1, f_2 \in C(S)$. Then, the following hold.

(i) $f_1 + f_2 \in C(S)$.
(ii) For any $a \in \mathbb{R}$, $af \in C(S)$.
(iii) $f_1 f_2 \in C(S)$.
(iv) $1/f_2 \in C(S)$, provided that $\forall \mathbf{x} \in S,\ f_2(\mathbf{x}) \ne 0$.
(v) $f_1/f_2 \in C(S)$, provided that $\forall \mathbf{x} \in S,\ f_2(\mathbf{x}) \ne 0$.

Proof. Directly import Theorem 3.1.3. □

Theorem 3.3.2

Suppose f is continous at a point $\mathbf{c}$ in $\mathbb{R}^n$. Then f is locally bounded at $\mathbf{c}$. that is, there are $M, \delta \in \mathbb{R}_+$ such that

$$\mathbf{x} \in S \cap N(\mathbf{c}; \delta) \implies |f(\mathbf{x})| \le M.$$

Theorem 3.3.3

Suppose f is continous at a point $\mathbf{c}$ in $\mathbb{R}^n$ and $f(\mathbf{c}) \neq 0$. Then f is locally bounded away from 0 at $\mathbf{c}$. that is, there are $m, \delta \in \mathbb{R}_+$ such that

$$\mathbf{x} \in S \cap N(\mathbf{c}; \delta) \implies |f(\mathbf{x})| \ge m.$$

3.4 Uniform Continuity

Definition 3.4.1: Uniform Continuity

A function $f : S \to \mathbb{R}$ with $S \subseteq \mathbb{R}^n$ is said to be *uniformly continuous on* S if,

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists \delta \in \mathbb{R}_+,\ \forall \mathbf{c} \in S,\ S \cap N(\mathbf{c}; \delta) \subseteq f^{-1}(N(f(\mathbf{c}); \varepsilon)).$$

Or, equivalently,

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists \delta \in \mathbb{R}_+,\ \forall \mathbf{x}, \mathbf{y} \in S,\ \big(\|\mathbf{x} - \mathbf{y}\| < \delta \implies |f(x) - f(y)| < \varepsilon\big).$$

Example 3.4.1

$f : [0, b] \to \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous on $[0, b]$. Given any $\varepsilon \in \mathbb{R}_+$, let $\delta \triangleq \varepsilon/2b$. Then, whenever $|x - y| < \delta$ where $x, y \in [0, b]$, $|x^2 - y^2| = |x - y||x + y| < \delta \cdot 2b = \varepsilon$.

Example 3.4.2

$f : (0, M) \to \mathbb{R}$ defined by $f(x) = 1/x$ is not uniformly continuous on $(0, M)$. Let any $\delta \in \mathbb{R}_+$ is given. Let $a \in (0, \min\{\delta, 1/2, M/2\})$. Then, $|a - (2a)| = a < \delta$ but $|f(a) - f(2a)| = |1/a - 1/(2a)| = 1/(2a) > 1$.

This is an example in which f is continuous but the domain is not compact.

Example 3.4.3

$f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$ is not uniformly continuous on $[-1, 1]$. Let any $\delta \in \mathbb{R}_+$ is given. Let $a \in (0, \min\{\delta/2, 1\})$. Then, $|a - (-a)| = 2a < \delta$ but $|f(a) - f(-a)| = 1 > 0.5$.

This is an example in which the domain is compact but f is not continuous.

Theorem 3.4.1

Suppose that $f : S \to \mathbb{R}$ is continuous on a compact subset S of $\mathbb{R}^n$. Then f is uniformly continuous on S.

Proof. Let $\varepsilon \in \mathbb{R}_+$ be given. Since f is continuous at each point of S, for each $\mathbf{c}$ of S, we may choose $\delta(\mathbf{c}) \in \mathbb{R}_+$ such that

$$S \cap N(\mathbf{c}; \delta(\mathbf{c})) \subseteq f^{-1}\left(N\left(f(\mathbf{c}); \frac{\varepsilon}{2}\right)\right).$$

Then the set $\mathcal{C} \triangleq \{N(\mathbf{c}; \delta(\mathbf{c})/2) \mid \mathbf{c} \in S\}$ is an open cover of the compact set S. Since S is compact, there is a finite subcover

$$\mathcal{C}_1 = \left\{N\left(\mathbf{c}_1; \frac{\delta(\mathbf{c}_1)}{2}\right), N\left(\mathbf{c}_2; \frac{\delta(\mathbf{c}_2)}{2}\right), \cdots, N\left(\mathbf{c}_k; \frac{\delta(\mathbf{c}_k)}{2}\right)\right\}.$$

Let $\delta_0 \triangleq \min_{i=1}^{k} \delta(\mathbf{c}_i)/2$.

Now, take any $\mathbf{c} \in S$. Since $\mathcal{C}_1$ is an open cover,

$$\exists i \in [k],\ \mathbf{c} \in N\left(\mathbf{c}_i; \frac{\delta(\mathbf{c}_i)}{2}\right).$$

Then, for any $\mathbf{x} \in N(\mathbf{c}; \delta_0)$,

$$\|\mathbf{x} - \mathbf{c}_i\| \leq \|\mathbf{x} - \mathbf{c}\| + \|\mathbf{c} - \mathbf{c}_i\| < \delta_0 + \frac{\delta(\mathbf{c}_i)}{2} \leq \delta(\mathbf{c}_i).$$

Thus, $N(\mathbf{c}; \delta_0) \subseteq N(\mathbf{c}_i; \delta(\mathbf{c}_i))$; or

$$S \cap N(\mathbf{c}; \delta_0) \subseteq S \cap N(\mathbf{c}_i; \delta(\mathbf{c}_i)) \subseteq f^{-1}\left(N\left(f(\mathbf{c}_i); \frac{\varepsilon}{2}\right)\right).$$

Hence, for any $\mathbf{x} \in S \cap N(\mathbf{c}; \delta_0)$,

$$|f(\mathbf{x}) - f(\mathbf{c})| \leq |f(\mathbf{x}) - f(\mathbf{c}_i)| + |f(\mathbf{c}_i) - f(\mathbf{c})| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

as $\mathbf{x}, \mathbf{c} \in S \cap N(\mathbf{c}_i; \delta(\mathbf{c}_i))$. $\square$

3.5 The Uniform Norm: Uniform Convergence

Definition 3.5.1: Function Space $B(S)$

Let $S \neq \varnothing$ be a subset of $\mathbb{R}^n$. $B(S)$ denotes the vector space and ring of all bounded, real-valued functions on S.

Note:-

- For each $f \in B(S)$, $\sup\{|f(\mathbf{x})| \mid \mathbf{x} \in S\}$ exists.
- $C_\infty(S) = C(S) \cap B(S)$

Definition 3.5.2: Uniform Norm

The *uniform norm* of $f \in B(S)$ is defined to be

$$\|f\|_\infty = \sup\{|f(\mathbf{x})| \mid \mathbf{x} \in S\}.$$

Theorem 3.5.1

The uniform norm is a norm.

Proof. The positive definiteness and the absolute homogeneity is direct.

Take any $f, g \in f$. Then, for any $\mathbf{x} \in S$,

$$|(f+g)(\mathbf{x})| \leq |f(\mathbf{x})| + |g(\mathbf{x})| \leq \|f\|_\infty + \|g\|_\infty.$$

Thus, $\|f+g\|_\infty = \sup\{\,|(f+g)(\mathbf{x})| \mid \mathbf{x} \in S\,\} \leq \|f\|_\infty + \|g\|_\infty$; $\|\cdot\|_\infty$ satisfies the subadditivity. □

Definition 3.5.3: Uniform Metric

The *uniform metric* on $B(S)$ is

$$d_\infty(f, g) = \|f - g\|_\infty.$$

Note:-

The uniform metric is naturally a metric since the uniform norm is a norm.

Definition 3.5.4: (Deleted) Uniform Neighborhood

The *(uniform) neighborhood* $N(f; r)$ of f with radius r is the set

$$N(f; r) \triangleq \{\, g \in B(S) \mid d_\infty(f, g) < r\,\}.$$

The *deleted (uniform) neighborhood* $N'(f; r)$ of f with radius r is the set

$$N'(f; r) \triangleq \{\, g \in B(S) \mid 0 < d_\infty(f, g) < r\,\}.$$

Definition 3.5.5: Limit Point of a Set of Functions

A function $f_0 \in B(S)$ is said to be a *(uniform) limit point* of a set $F \subseteq B(S)$ if

$$\forall \varepsilon \in \mathbb{R}_+,\ F \cap N'(f_0; \varepsilon) \neq \varnothing.$$

Definition 3.5.6: Convergence of a Sequence of Functions

- A sequence $\{f_k\}_{k\in\mathbb{N}}$ in $B(S)$ is said to *converge uniformly* to $f_0 \in S \to \mathbb{R}$ on S if

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists k_0 \in \mathbb{N},\ \forall k \in \mathbb{N},\ \big(k \geq k_0 \implies f_k \in N(f_0;\varepsilon)\big).$$

  We write
$$\lim_{k\to\infty} f_k = f_0 \text{ [uniformly]}.$$

- A sequence $\{f_k\}_{k\in\mathbb{N}}$ in $B(S)$ is said to *converge pointwise* to $f_0 : S \to \mathbb{R}$ on S if

$$\forall (\mathbf{c},\varepsilon) \in S \times \mathbb{R}_+,\ \exists k_0 \in \mathbb{N},\ \forall k \in \mathbb{N},\ \big(k \geq k_0 \implies f_k(\mathbf{c}) \in N(f_0(\mathbf{c});\varepsilon)\big).$$

  We write
$$\lim_{k\to\infty} f_k = f_0 \text{ [pointwise]}.$$

- A sequence $\{f_k\}_{k\in\mathbb{N}}$ in $B(S)$ is said to be *(uniformly) Cauchy* if

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists k_0 \in \mathbb{N},\ \forall k,m \in \mathbb{N},\ \big(k,m \geq k_0 \implies \|f_m - f_k\|_\infty < \varepsilon\big).$$

Note:-
A pointwise convergent sequence in $C_\infty(S)$ may fail to have a limit that is in $C_\infty(S)$.

Note:-
If $\{f_k\}$ in $B(S)$ converges uniformly, then it converges pointwise.

Theorem 3.5.2
Let $\varnothing \neq S \subseteq \mathbb{R}^n$. Suppose that $\{f_k\}$ is a sequence in $C(S)$ and it converges uniformly to $f_0 : S \to \mathbb{R}$ on S. Then $f_0 \in C(S)$.

Proof. Take any $\mathbf{c} \in S$ and $\varepsilon \in \mathbb{R}_+$. By uniform convergence, there exists $k \in \mathbb{N}$ such that

$$\|f_k - f_0\|_\infty < \frac{\varepsilon}{4}.$$

Since f_k is continuous, there exists $\delta \in \mathbb{R}_+$ such that

$$S \cap N(\mathbf{c};\delta) \subseteq f_k^{-1}\left(N\left(f_k(\mathbf{c});\frac{\varepsilon}{2}\right)\right).$$

Thus, for any $\mathbf{x} \in S \cap N(\mathbf{c};\delta)$,

$$\begin{aligned}|f_0(\mathbf{x}) - f_0(\mathbf{c})| &\leq |f_0(\mathbf{x}) - f_k(\mathbf{x})| + |f_k(\mathbf{x}) - f_k(\mathbf{c})| + |f_k(\mathbf{c}) - f_0(\mathbf{c})| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.\end{aligned}$$

This exactly means that f_0 is continuous at $\mathbf{c}$. Since $\mathbf{c}$ is arbitrary, f_0 is continous on S. □

Theorem 3.5.3
Let $\varnothing \neq S \subseteq \mathbb{R}^n$. A Cauchy sequence $\{f_k\}$ in $B(S)$ is bounded. That is, $\exists M \in \mathbb{R}_+$, $\forall k \in \mathbb{N}$, $\|f_k\|_\infty \leq M$.

Proof. Immitate the proof of Theorem 1.4.2. □

Theorem 3.5.4

$C_\infty(S)$ is complete. That is, given any Cauchy sequence $\{f_k\}$ in $C_\infty(S)$, there exists $f_0 \in C_\infty(S)$ such that $\lim_{k\to\infty} f_k = f_0$ [uniformly] on S.

Proof. Since, for each $\mathbf{c} \in S$, $\{f_k(\mathbf{c})\}_{k\in\mathbb{N}}$ is Cauchy, by Theorem 1.4.4, $\{f_k(\mathbf{c})\}_{k\in\mathbb{N}}$ converges in $\mathbb{R}$. Thus we may define $f_0 : S \to \mathbb{R}$ by

$$f_0(\mathbf{c}) = \lim_{k\to\infty} f_k(\mathbf{c}).$$

Now, we first claim that $\lim_{k\to\infty} f_k = f_0$ [uniformly]. Take any $\varepsilon \in \mathbb{R}_+$. Since $\{f_k\}$ is Cauchy,

$$\exists k_0 \in \mathbb{N},\ \forall k, m \in \mathbb{N},\ \big(k, m \geq k_0 \implies \|f_k - f_m\|_\infty < \varepsilon/2\big).$$

Then, for each $k \in \mathbb{N}_{\geq k_0}$,

$$\forall \mathbf{c} \in S,\ |f_k(\mathbf{c}) - f_0(\mathbf{c})| = \lim_{m\to\infty} |f_k(\mathbf{c}) - f_m(\mathbf{c})| \leq \varepsilon/2.$$

This means $\|f_k - f_0\|_\infty \leq \varepsilon/2 < \varepsilon$; $\{f_k\}$ converges to f_0 uniformly on S. It directly follows from Theorem 3.5.2 that f_0 is continuous on S.

Now, we are left to show f_0 is bounded on S. Since $\{f_k\}$ uniformly converges to f_0,

$$\exists k_0 \in \mathbb{N},\ \forall k \in \mathbb{N},\ \big(k \geq k_0 \implies \|f_k - f_0\|_\infty < 1\big).$$

This implies $\forall \mathbf{c} \in S$, $|f_{k_0}(\mathbf{c}) - f_0(\mathbf{c})| < 1$. Hence, $\forall \mathbf{c} \in S$, $|f_0(\mathbf{c})| < |f_{k_0}(\mathbf{c})| + 1 \leq \|f_{k_0}\|_\infty + 1$. Thus, $\{\, |f_0(\mathbf{c})| \mid \mathbf{c} \in S \,\}$ is bounded above by $\|f_{k_0}\|_\infty + 1$. □

Corollary 3.5.1

If $\varnothing \neq S \subseteq \mathbb{R}^n$ is compact, then $C(S)$ is complete.

Definition 3.5.7: Unifrom Denseness

Let $\varnothing \neq S \subseteq \mathbb{R}^n$. A collection $F \subseteq C_\infty(S)$ is *uniformly dense* in $C_\infty(S)$ if, for all $f_0 \in C_\infty(S)$ and its neighborhood $N(f_0)$, $F \cap N(f_0) \neq \varnothing$.

Definition 3.5.8: Polynomial Space

Let $\varnothing \neq S \subseteq \mathbb{R}$. Let $P(S)$ denote the set of polynomial functions $f : S \to \mathbb{R}$ in the single variable x.

Note:-

If $S \neq \varnothing$ is a compact subset of $\mathbb{R}$, then $P(S)$ is certainly a subset of $C_\infty(S) = C(S)$.

Theorem 3.5.5 The Weierstrass Approximation Theorem

If S is a compact subset of $\mathbb{R}$, then $P(S)$ is uniformly dense in $C(S)$.

Definition 3.5.9: Bernstein Polynomial

Given a continuous function on $[0,1]$, for each $k \in \mathbb{N}$, the k^{th} Bernstein polynomial for f, $B_k(x)$, is defined as follows.

$$B_k(x) \triangleq \sum_{j=0}^{k} \binom{k}{j} f\left(\frac{j}{k}\right) x^j (1-x)^{k-j}$$

where $\binom{k}{j}$ is the binomial coefficient $\dfrac{k!}{j!(k-j)!}$.

Note:-
The following lemmas are for the proof of Theorem 3.5.5.

Lemma 3.5.1

For any $k \in \mathbb{N}$, $\sum_{j=0}^{k} \binom{k}{j} x^j (1-x)^{k-j} = 1$.

Proof. Expand $1 = \big(x + (1-x)\big)^k$ via the binomial theorem. □

Lemma 3.5.2

For any $k \in \mathbb{N}$, $\sum_{j=0}^{k} \binom{k}{j} \frac{j}{k} x^j (1-x)^{k-j} = x$.

Proof. Using Lemma 3.5.1, we get

$$\begin{aligned}
x &= x\left[\sum_{j=0}^{k-1} \binom{k-1}{j} x^j (1-x)^{(k-1)-j}\right] \\
&= \sum_{j=0}^{k-1} \binom{k-1}{j} x^{j+1} (1-x)^{k-(j+1)} = \sum_{j=1}^{k} \binom{k-1}{j-1} x^j (1-x)^{k-j} \\
&= \sum_{j=1}^{k} \binom{k}{j} \frac{j}{k} x^j (1-x)^{k-j} = \sum_{j=0}^{k} \binom{k}{j} \frac{j}{k} x^j (1-x)^{k-j}.
\end{aligned}$$

□

Lemma 3.5.3

For any $k \in \mathbb{N}$, $\sum_{j=0}^{k} \binom{k}{j} (j^2 - j) x^j (1-x)^{k-j} = k(k-1)x^2$.

Proof. Note that, for each $j \in \{2, 3, \cdots, k\}$,

$$\binom{k}{j}(j^2 - j) = \binom{k-2}{j-2} k(k-1).$$

Therefore, using Lemma 3.5.1, we get

$$\begin{aligned}
\sum_{j=0}^{k}\binom{k}{j}(j^2-j)x^j(1-x)^{k-j} &= \sum_{j=2}^{k}\binom{k}{j}(j^2-j)x^j(1-x)^{k-j} \\
&= x^2\sum_{j=2}^{k}\binom{k}{j}(j^2-j)x^{j-2}(1-x)^{(k-2)-(j-2)} \\
&= x^2\sum_{j=2}^{k}\binom{k-2}{j-2}k(k-1)x^{j-2}(1-x)^{(k-2)-(j-2)} \\
&= k(k-1)x^2\sum_{j=0}^{k-2}\binom{k-2}{j}x^j(1-x)^{(k-2)-j} \\
&= k(k-1)x^2
\end{aligned}$$

□

Lemma 3.5.4

For any $k \in \mathbb{N}$, $\displaystyle\sum_{j=0}^{k}\binom{k}{j}\left(\frac{j}{k}\right)^2 x^j(1-x)^{k-j} = \left(1-\frac{1}{k}\right)x^2+\frac{x}{k}$.

Proof. Using Lemma 3.5.2 and Lemma 3.5.3, we get

$$\begin{aligned}
\left(1-\frac{1}{k}\right)x^2+\frac{x}{k} &= \frac{k(k-1)x^2}{k^2}+\frac{x}{k} \\
&= \frac{1}{k^2}\sum_{j=0}^{k}\binom{k}{j}(j^2-j)x^j(1-x)^{k-j}+\frac{1}{k}\sum_{j=0}^{k}\binom{k}{j}\frac{j}{k}x^j(1-x)^{k-j} \\
&= \sum_{j=0}^{k}\binom{k}{j}\left(\frac{j^2-j}{k^2}+\frac{j}{k^2}\right)x^j(1-x)^{k-j} \\
&= \sum_{j=0}^{k}\binom{k}{j}\left(\frac{j}{k}\right)^2 x^j(1-x)^{k-j}.
\end{aligned}$$

□

Lemma 3.5.5

For any $k \in \mathbb{N}$ and any $x \in [0,1]$, $\displaystyle\sum_{j=0}^{k}\binom{k}{j}\left(x-\frac{j}{k}\right)^2 x^j(1-x)^{k-j} = \frac{x(1-x)}{k} \le \frac{1}{4k}$.

Proof. Using Lemma 3.5.1, Lemma 3.5.2, and Lemma 3.5.4, we get

$$\begin{aligned}
\sum_{j=0}^{k}\binom{k}{j}\left(x-\frac{j}{k}\right)^2 x^j(1-x)^{k-j} &= \sum_{j=0}^{k}\binom{k}{j}\left[x^2-2x\left(\frac{j}{k}\right)-\left(\frac{j}{k}\right)^2\right]x^j(1-x)^{k-j} \\
&= x^2\sum_{j=0}^{k}\binom{k}{j}x^j(1-x)^{k-j}+2x\sum_{j=0}^{k}\binom{k}{j}\left(\frac{j}{k}\right)x^j(1-x)^{k-j} \\
&\quad+\sum_{j=0}^{k}\binom{k}{j}\left(\frac{j}{k}\right)^2 x^j(1-x)^{k-j} \\
&= x^2-2x^2+\left(1-\frac{1}{k}\right)x^2+\frac{x}{k}=\frac{x(1-x)}{k}.
\end{aligned}$$

□

Theorem 3.5.6 Bernstein's Approximation Theorem

Let $f \in C([0,1])$. Then, for any $\varepsilon \in \mathbb{R}_+$, there exists $k_0 \in \mathbb{N}$ such that, for $k \geq k_0$, $\|f-B_k\|_\infty < \varepsilon$ where B_k is the k^{th} Bernstein polynomial for f.

Proof. Take any $\varepsilon \in \mathbb{R}_+$. Since f is continuous on a compact set $[0,1]$, then f is uniformly continuous on $[0,1]$ by Theorem 3.4.1. Therefore,

$$\exists\delta \in \mathbb{R}_+, \forall x,y \in [0,1], \big(|x-y|<\delta \implies |f(x)-f(y)|<\varepsilon/4\big).$$

Also, f is bounded on $[0,1]$ by Theorem 3.2.3. Thus, by Theorem 1.1.1, there exists $k_0 \in \mathbb{N}$ such that $2\|f\|_\infty < \varepsilon\delta^2 k_0$. Now, suppose $k \geq k_0$.

Fix some $x \in [0,1]$ and let $A_1 \triangleq \{\, j \in \mathbb{Z} \mid 0 \leq j \leq k \text{ and } |x-j/k|<\delta \,\}$ and $A_2 \triangleq \{\, j \in \mathbb{Z} \mid 0 \leq j \leq k \text{ and } |x-j/k| \geq \delta \,\}$. Now, we are ready to prove $\|f-B_k\|_\infty < \varepsilon$.

Using Lemma 3.5.1 at the first step, we get

$$\begin{aligned}
|f(x)-B_k(x)| &= \left|\sum_{j=0}^{k}\binom{k}{j}f(x)x^j(1-x)^{k-j}-\sum_{j=0}^{k}\binom{k}{j}f\left(\frac{j}{k}\right)x^j(1-x)^{k-j}\right| \\
&= \left|\sum_{j=0}^{k}\binom{k}{j}\left[f(x)-f\left(\frac{j}{k}\right)\right]x^j(1-x)^{k-j}\right| \\
&\leq \sum_{j=0}^{k}\binom{k}{j}\left|f(x)-f\left(\frac{j}{k}\right)\right|x^j(1-x)^{k-j} \\
&= \sum_{j\in A_1}\binom{k}{j}\left|f(x)-f\left(\frac{j}{k}\right)\right|x^j(1-x)^{k-j}+\sum_{j\in A_2}\binom{k}{j}\left|f(x)-f\left(\frac{j}{k}\right)\right|x^j(1-x)^{k-j} \\
&= S_1+S_2
\end{aligned}$$

where S_1 and S_2 are the sums over A_1 and A_2, respectively.

Since $\left|x-\dfrac{j}{k}\right|<\delta$ for $j \in A_1$, the following holds.

$$\begin{aligned}
S_1 &\triangleq \sum_{j\in A_1}\binom{k}{j}\left|f(x)-f\left(\frac{j}{k}\right)\right|x^j(1-x)^{k-j} \\
&< \sum_{j\in A_1}\binom{k}{j}\left(\frac{\varepsilon}{4}\right)x^j(1-x)^{k-j} \leq \frac{\varepsilon}{4}\sum_{j=0}^{k}\binom{k}{j}x^j(1-x)^{k-j}=\frac{\varepsilon}{2}
\end{aligned}$$

We now investigate S_2. Note that $\dfrac{1}{|x-j/k|} \le \dfrac{1}{\delta}$ for $j \in A_2$. Using Lemma 3.5.5, we get

$$\begin{aligned}
S_2 &\triangleq \sum_{j\in A_2} \binom{k}{j} \left| f(x) - f\left(\frac{j}{k}\right) \right| x^j (1-x)^{k-j} \\
&\le 2\|f\|_\infty \sum_{j\in A_2} \binom{k}{j} \left(x - \frac{j}{k}\right)^2 \frac{1}{(x-\frac{j}{k})^2} x^j (1-x)^{k-j} \\
&\le \frac{2\|f\|_\infty}{\delta^2} \sum_{j\in A_2} \binom{k}{j} \left(x - \frac{j}{k}\right)^2 x^j (1-x)^{k-j} \\
&\le \frac{2\|f\|_\infty}{\delta^2} \sum_{j=0}^{k} \binom{k}{j} \left(x - \frac{j}{k}\right)^2 x^j (1-x)^{k-j} \\
&\le \frac{2\|f\|_\infty}{\delta^2} \frac{1}{4k} \le \frac{\|f\|_\infty}{2k\delta^2} \le \frac{\|f\|_\infty}{2k_0\delta^2} < \frac{\varepsilon}{4}.
\end{aligned}$$

Thus, we have $|f(x) - B_k(x)| \le S_1 + S_2 < \varepsilon/4 + \varepsilon/4 = \varepsilon/2$. Since x is arbitrary, $\|f - B_k\|_\infty \le \varepsilon/2 < \varepsilon$. ☐

3.6 Vector-Valued Functions on $\mathbb{R}^n$

Definition 3.6.1: Component Function

Let $\mathbf{f}$ be a function with domain $S \subseteq \mathbb{R}^n$ and codomain $T \subseteq \mathbb{R}^m$. For $\mathbf{x} \in S$, we write

$$\mathbf{f}(\mathbf{x}) = \mathbf{y} = (y_1, y_2, \cdots, y_m).$$

Then, for each $j \in [m]$, there is a real-valued function $f_i : S \to \mathbb{R}$ defined by $f_j(\mathbf{x}) = y_j$. The functions $f_1, f_2, \cdots, f_m$ are called *component functions* of $\mathbf{f}$. We write

$$\mathbf{f} = (f_1, f_2, \cdots, f_m).$$

Definition 3.6.2: Limit and Continuity of Vector-Valued Functions

Let $\varnothing \neq S \subseteq \mathbb{R}^n$ and $\mathbf{f}: S \to \mathbb{R}^m$.

- Let $\mathbf{c} \in \overline{S}$. We say that $\mathbf{f}$ has *limit* $\mathbf{v}$ as $\mathbf{x}$ approaches $\mathbf{c}$, and we write

  $$\lim_{\mathbf{x}\to\mathbf{c}} \mathbf{f}(\mathbf{x}) = \mathbf{v}$$

  if, for every neighborhood $N(\mathbf{v})$, there exists a deleted neighborhood $N'(\mathbf{c})$ such that

  $$S \cap N'(\mathbf{c}) \subseteq \mathbf{f}^{-1}(N(\mathbf{v})).$$

- Let $\mathbf{c} \in S$. We say that $\mathbf{f}$ is *continuous at* $\mathbf{c}$ if

  $$\lim_{\mathbf{x}\to\mathbf{c}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{c}).$$

- We say that $\mathbf{f}$ is *continuous on* S if $\mathbf{f}$ is contiuous at every point of S.

Theorem 3.6.1

Let $\varnothing \neq S \subseteq \mathbb{R}^n$, $\mathbf{f}\colon S \to \mathbb{R}^m$, and $\mathbf{f} = (f_1, f_2, \cdots, f_m)$.

(i) Let $\mathbf{c} \in \overline{S}$. Then $\lim_{\mathbf{x}\to\mathbf{c}} \mathbf{f}(\mathbf{x}) = \mathbf{v} = (v_1, v_2, \cdots, v_m)$ if and only if, for each $j \in [m]$, $\lim_{\mathbf{x}\to\mathbf{c}} f_j(\mathbf{x}) = v_j$.

(ii) Let $\mathbf{c} \in S$. Then $\mathbf{f}$ is continuous at $\mathbf{c}$ if and only if, for each $j \in [m]$, f_j is continuous at $\mathbf{c}$.

(iii) $\mathbf{f}$ is continuous on S if and only if, for each $j \in [m]$, f_j is continuous on S.

Proof. (ii) and (iii) directly follows from (i). So, we only need to prove (i).

($\Rightarrow$) Take any $\varepsilon \in \mathbb{R}_+$. Then, there exists $\delta \in \mathbb{R}_+$ such that

$$\forall \mathbf{x} \in S, \left(0 < \|\mathbf{x} - \mathbf{c}\| < \delta \implies \|\mathbf{f}(\mathbf{x}) - \mathbf{v}\| < \varepsilon\right).$$

Then, for each $j \in [m]$, whenever $\mathbf{x} \in S \cap N'(\mathbf{c}; \delta)$,

$$|f_j(\mathbf{x}) - v_j|^2 \leq \sum_{i=1}^{m} |f_j(\mathbf{x}) - v_i|^2 = \|\mathbf{f}(\mathbf{x}) - \mathbf{v}\|^2 < \varepsilon^2,$$

which implies $S \cap N'(\mathbf{c}; \delta) \subseteq f_j^{-1}(N(v_j; \varepsilon))$; $\lim_{\mathbf{x}\to\mathbf{c}} f_j(\mathbf{x}) = v_j$.

($\Leftarrow$) Take any $\varepsilon \in \mathbb{R}_+$. Then, for each $j \in [m]$, there exists $\delta_j \in \mathbb{R}_+$ such that

$$\forall \mathbf{x} \in S, \left(0 < \|\mathbf{x} - \mathbf{c}\| < \delta_j \implies |f_j(\mathbf{x}) - v_j| < \varepsilon/\sqrt{m}\right).$$

Then, whenever $\mathbf{x} \in S \cap N'(\mathbf{c}; \delta_0)$ where $\delta_0 \triangleq \min_{j=1}^{m} \delta_j$,

$$\|\mathbf{f}(\mathbf{x}) - \mathbf{v}\|^2 = \sum_{i=1}^{m} |f_j(\mathbf{x}) - v_j|^2 < \sum_{i=1}^{m} \left(\frac{\varepsilon}{\sqrt{m}}\right)^2 = \varepsilon^2,$$

which implies $S \cap N'(\mathbf{c}; \delta_0) \subseteq \mathbf{f}^{-1}(N(\mathbf{v}; \varepsilon))$; $\lim_{\mathbf{x}\to\mathbf{c}} \mathbf{f}(\mathbf{x}) = \mathbf{v}$. □

Theorem 3.6.2

Let $\varnothing \neq S \subseteq \mathbb{R}^n$ and $\mathbf{f}\colon S \to \mathbb{R}^m$.

(i) Let $\mathbf{c}$ be a point in $\overline{S}$. Then $\lim_{\mathbf{x}\to\mathbf{c}} \mathbf{f}(\mathbf{x}) = \mathbf{v}$ if and only if, for every sequence $\{\mathbf{x}_k\}$ in $S \setminus \{\mathbf{c}\}$ such that $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{c}$, we have $\lim_{k\to\infty} \mathbf{f}(\mathbf{x}_k) = \mathbf{v}$.

(ii) Let $\mathbf{c}$ be a point in S. Then $\mathbf{f}$ is continuous at $\mathbf{c}$ if and only if, for every sequence $\{\mathbf{x}_k\}$ in S such that $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{c}$, we have $\lim_{k\to\infty} \mathbf{f}(\mathbf{x}_k) = \mathbf{f}(\mathbf{c})$.

Proof. By Theorem 3.6.1,

$$\lim_{\mathbf{x}\to\mathbf{c}} \mathbf{f}(\mathbf{x}) = \mathbf{v} \iff \forall j \in [m], \lim_{\mathbf{x}\to\mathbf{c}} f_j(\mathbf{x}) = v_j$$

where $\mathbf{v} = (v_1, v_2, \cdots, v_m)$. By Theorem 3.1.6, for each $j \in [m]$,

$$\lim_{\mathbf{x}\to\mathbf{c}} f_j(\mathbf{x}) = v_j \iff \forall \{\mathbf{x}_k\} \in (S \setminus \{\mathbf{c}\})^{\omega}, \left(\lim_{k\to\infty} \mathbf{x}_k = \mathbf{c} \implies \lim_{k\to\infty} f_j(\mathbf{x}_k) = v_j\right).$$

By Theorem 2.1.6,

$$\lim_{k\to\infty} \mathbf{f}(\mathbf{x}_k) = \mathbf{v} \iff \forall j \in [m], \lim_{k\to\infty} f_j(\mathbf{x}_k) = v_j.$$

Thus, (i) is proven, and (ii) can be proven similarly with the aid of Theorem 3.1.7. □

Theorem 3.6.3

Let $f : S \to T$ where $S \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}^m$. Suppose f is surjective. Then $\mathbf{f}$ is continuous on S if and only if the inverse image of every relatively open set in T is relatively open in S.

Proof. Repeat the proof of Theorem 3.2.1 verbatim. □

Theorem 3.6.4

If S is a connected subset of $\mathbb{R}^n$ and $\mathbf{f}\colon S \to \mathbb{R}^m$ is continuous on S, then $T = \mathbf{f}(S)$ is also connected.

Proof. Repeat the proof of Theorem 3.2.2 verbatim. □

Theorem 3.6.5

If S is a compact subset of $\mathbb{R}^n$ and $\mathbf{f}\colon S \to \mathbb{R}^m$ is continuous on S, then $T = \mathbf{f}(S)$ is also compact.

Proof. Repeat the proof of Theorem 3.2.3 verbatim. □

Theorem 3.6.6

Let $S \subseteq \mathbb{R}^n$, $f : S \to \mathbb{R}^m$, $f(S) \subseteq T \subseteq \mathbb{R}^m$, and $g : T \to \mathbb{R}^p$. If f is continuous at $\mathbf{c} \in S$ and if g is continuous at $f(\mathbf{c}) \in T$, then $g \circ f$ is continuous at $\mathbf{c}$.

Proof. Repeat the proof of Theorem 3.2.7 verbatim. □

Theorem 3.6.7

Let $\varnothing \neq S \subseteq \mathbb{R}^n$ is compact and $\varnothing \neq T \subseteq \mathbb{R}^m$. Let $\mathbf{f}\colon S \to T$ is continuous on S and bijective. Then, $\mathbf{f}^{-1}$ is also continuous on $\mathbf{f}(S)$.

Proof. Repeat the proof of Theorem 3.2.8 verbatim. □

Definition 3.6.3: Uniform Continuity

A function $\mathbf{f}\colon S \to \mathbb{R}^m$ with $S \subseteq \mathbb{R}^n$ is said to be *uniformly continuous on S* if,

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists \delta \in \mathbb{R}_+,\ \forall \mathbf{c} \in S,\ S \cap N(\mathbf{c};\delta) \subseteq \mathbf{f}^{-1}(N(\mathbf{f}(\mathbf{c});\varepsilon)).$$

Or, equivalently,

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists \delta \in \mathbb{R}_+,\ \forall \mathbf{x},\mathbf{y} \in S,\ \big(\|\mathbf{x}-\mathbf{y}\| < \delta \implies \|\mathbf{f}(x)-\mathbf{f}(y)\| < \varepsilon\big).$$

Theorem 3.6.8

Let $\mathbf{f} = (f_1, f_2, \cdots, f_m)$ be a function $\mathbf{f}\colon S \to \mathbb{R}^m$ where $S \subseteq \mathbb{R}^n$. Then, $\mathbf{f}$ is uniformly continuous if and only if f_j is uniformly continuous on S for each $j \in [m]$.

Proof. ($\Rightarrow$) Take any $j \in [m]$ and $\varepsilon \in \mathbb{R}_+$. Then, there exist $\delta \in \mathbb{R}_+$ such that,

$$\forall \mathbf{x}, \mathbf{y} \in S, \big(\|\mathbf{x}-\mathbf{y}\| < \delta \implies \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| < \varepsilon\big).$$

Since $|f_j(\mathbf{x}) - f_j(\mathbf{y})| \le \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\|$, f_j is uniformly continuous.

($\Leftarrow$) Take any $\varepsilon \in \mathbb{R}_+$. Then, for each $j \in [m]$, there exists $\delta_j \in \mathbb{R}_+$ such that

$$\forall \mathbf{x}, \mathbf{y} \in S, \big(\|\mathbf{x}-\mathbf{y}\| < \delta_j \implies |f_j(\mathbf{x}) - f_j(\mathbf{y})| < \varepsilon/\sqrt{m}\big).$$

Let $\delta \triangleq \min_{j\in[m]} \delta_j$. Then,

$$\forall \mathbf{x}, \mathbf{y} \in S, \big(\|\mathbf{x}-\mathbf{y}\| < \delta \implies \|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\| < \varepsilon\big).$$

□

Theorem 3.6.9

Suppose that $f : S \to \mathbb{R}^m$ is continuous on a compact subset S of $\mathbb{R}^n$. Then f is uniformly continuous on S.

Proof. Repeat the proof of Theorem 3.4.1 verbatim. □

Chapter 4

Differentiation

4.1 The Derivative

Definition 4.1.1: Derivative

Let f be defined on an interval I on $\mathbb{R}$. Let c be a point in I. The *derivative* of f at c is defined to be

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

provided this limit exists. The derivative of f at c is denoted by $f'(c)$. We say that f is *differentiable at* c if $f'(c)$ exists. We say that f is *differentiable on* I if $f'(x)$ exists for each $x \in I$.

Definition 4.1.2: Differential

Let f be a real-valued function defined on an interval I in $\mathbb{R}$. Suppose that f is differentiable at a $c \in I$. The function $\mathrm{d}f(c;\cdot)\colon \mathbb{R}\to\mathbb{R}$ defined by

$$\mathrm{d}f(c;t) = f'(c)t$$

is called the *differential* of f at c.

Theorem 4.1.1

Suppose f is differentiable at c in its domain. Then, for any $\varepsilon \in \mathbb{R}_+$, there exists a deleted neighborhood $N'(0)$ such that,

$$|f(c+t)-f(c)-\mathrm{d}f(c;t)| < \varepsilon|t|,$$

for all $t \in N'(0)$.

Proof. There is a deleted neighborhood $N'(0)$ such that

$$\left|\frac{f(c+t)-f(c)}{t} - f'(c)\right| < \varepsilon$$

for all $t \in N'(0)$. □

Theorem 4.1.2

If f is differentiable at c, then f is continuous at c.

Proof.

$$\lim_{x\to c}\big(f(x)-f(c)\big)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\cdot(x-c)=f'(c)\cdot 0=0$$

Hence, $\lim_{x\to c} f(x)=f(c)$; f is continuous at c. □

Corollary 4.1.1

If f is differentiable on I, then $f\in C(I)$.

Theorem 4.1.3

Suppose that f and g are two functions each differentiable at c and that $a\in\mathbb{R}$.

(i) $f+g$ is differentiable at c and

$$(f+g)'(c)=f'(c)+g'(c).$$

(ii) af is differentiable at c and

$$(af)'(c)=af'(c).$$

(iii) fg is differentiable at c and

$$(fg)'(c)=f'(c)g(c)+f(c)g'(c).$$

(iv) If $g(c)\neq 0$, then $1/g$ is differentiable at c and

$$\left(\frac{1}{g}\right)'(c)=-\frac{g'(c)}{g(c)^2}.$$

(v) If $g(c)\neq 0$, then f/g is differentiable at c and

$$\left(\frac{f}{g}\right)'(c)=\frac{f'(c)g(c)-f(c)g'(c)}{g(c)^2}.$$

Proof.

(i) $\displaystyle\lim_{x\to c}\frac{(f+g)(x)-(f+g)(c)}{x-c}=\lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}+\frac{g(x)-g(c)}{x-c}\right]=f'(c)+g'(c).$

(ii) $\displaystyle\lim_{x\to c}\frac{(af)(x)-(af)(c)}{x-c}=\lim_{x\to c}a\cdot\frac{f(x)-f(c)}{x-c}=af'(c).$

(iii) First, note that

$$\begin{aligned}(fg)(x)-(fg)(c)&=f(x)g(x)-f(c)g(x)+f(c)g(x)-f(c)g(c)\\&=\big(f(x)-f(c)\big)g(x)+\big(g(x)-g(c)\big)f(c).\end{aligned}$$

Since g is continuous at c,

$$\begin{aligned}\lim_{x\to c}\frac{(fg)(x)-(fg)(c)}{x-c}&=\lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}g(x)+\frac{g(x)-g(c)}{x-c}f(c)\right]\\&=f'(c)g(c)+f(c)g'(c).\end{aligned}$$

(iv)

$$\lim_{x\to c}\frac{1/g(x)-1/g(c)}{x-c}=\lim_{x\to c}\left[\frac{g(x)-g(c)}{(x-c)}\cdot\frac{1}{g(x)g(c)}\right]=-\frac{g'(c)}{g(c)^2}.$$

(v)
$$\left(\frac{f}{g}\right)'(c) = f'(c)\left(\frac{1}{g}\right)(c) + f(c)\left(\frac{1}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}.$$

□

Definition 4.1.3: kth Derivative

Suppose $f' = f^{(1)}$, $f'' = f^{(2)}$, $\cdots$, $f^{(k)}$ is defined on a neighborhood of c and if

$$f^{(k+1)}(c) \triangleq \lim_{h\to 0} \frac{f^{(k)}(c+h) - f^{(k)}(c)}{h}$$

exists, then $f^{(k)}(c)$ is called the kth derivative of f at c and f is said to be k *times differentiable* at c.
If $f^{(k)}(c)$ exists for all $k \in \mathbb{N}$, then f is said to have *derivatives of all orders* at c. If $f^{(k)}$ exists for all $k \in \mathbb{N}$ and at all $x \in I$, then f is said to have *derivatives of all orders* on I.

4.2 Composition of Functions: The Chain Rule

Theorem 4.2.1 The Chain Rule

Suppose the following.

- c is an interior point of an interval I.
- g is differentiable at c.
- $d = g(c)$ is an interior point of $g(I)$.
- f is defined on $g(I)$.
- f is differentiable at d.

Then, $f \circ g$ is differentiable at c and $(f \circ g)'(c) = f'(g(c))g'(c)$.

Proof. For y in $g(I)$, define

$$h(y) = \begin{cases} \dfrac{f(y) - f(d)}{y - d} - f'(d) & \text{if } y \neq d \\ 0 & \text{if } y = d. \end{cases}$$

Since $\lim_{y\to d} h(y) = 0 = h(d)$, h is continuous at d. Also, since h is defined on $g(I)$, we may composite it with g;

$$(h \circ g)(x) = \begin{cases} \dfrac{f(g(x)) - f(g(c))}{g(x) - g(c)} - f'(g(c)) & \text{if } g(x) \neq g(c) \\ 0 & \text{if } g(x) = g(c) = d. \end{cases}$$

Multiplying by $g(x) - g(c)$, we get

$$f(g(x)) - f(g(c)) = \{(h \circ g)(x) + f'(g(c))\}\{g(x) - g(c)\}.$$

Hence,

$$\begin{aligned} \lim_{x\to c} \frac{f(g(x)) - f(g(c))}{x - c} &= \lim_{x\to c} \{(h \circ g)(x) + f'(g(c))\} \frac{g(x) - g(c)}{x - c} \\ &= \{0 + f'(g(c))\} g'(c) = f'(g(c))g'(c). \end{aligned}$$

□

4.3 The Mean Value Theorem

Definition 4.3.1: Local Minimum and Maximum

Let f be a real-valued function defined on the interval I and c is an interior point of I.

- f has a *local maximum* at c if there exists a neighborhood $N(c)$ such that $\forall x \in N(c)$, $f(x) \leq f(c)$.
- f has a *local minimum* at c if there exists a neighborhood $N(c)$ such that $\forall x \in N(c)$, $f(x) \geq f(c)$.

Theorem 4.3.1

If f has either a local maximum or a local minimum at an interior point c of I and if f is differentiable at c, then $f'(c) = 0$.

Proof. Let $N(c)$ be a neighborhood of c such that $\forall x \in N(c)$, $f(x) \geq f(c)$. Then, since

$$\frac{f(x)-f(c)}{x-c} \geq 0 \quad \text{on } N(c)\cap(c,\infty) \text{ and}$$

$$\frac{f(x)-f(c)}{x-c} \leq 0 \quad \text{on } N(c)\cap(-\infty,c),$$

It follows that

$$f'(c) = \lim_{x\to c^+}\frac{f(x)-f(c)}{x-c} \geq 0 \quad \text{and} \quad f'(c) = \lim_{x\to c^-}\frac{f(x)-f(c)}{x-c} \leq 0.$$

Hence, $f'(c) = 0$. □

Definition 4.3.2: Critical Point

If a function f is differentiable at a point c and if $f'(c) = 0$, then c is called a *critical point* of f.

Theorem 4.3.2 Rolle's Theorem

Suppose that f is continuous on $[a, b]$ and is differentiable on (a, b). Suppose further that $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof. If $f(x) = f(a)$ for all $x \in [a, b]$, then $f' = 0$ on $[a, b]$, and it is done.

Otherwise, f has a maximum value or a minimum value at some point c in (a, b) by Theorem 3.2.4. c is either a local maximum or a local minimum of f as well. Hence, by Theorem 4.3.1, $f'(c) = 0$. □

Theorem 4.3.3 The Mean Value Theorem

Suppose f is continuous on $[a, b]$ and and is differentiable on (a, b). Then,

$$\exists c \in (a,b),\ \frac{f(b)-f(a)}{b-a} = f'(c).$$

Proof. Let

$$h(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}(x-a)+f(a)\right].$$

Then, h is continuous on $[a,b]$, differentiable on (a,b), and $h(a) = h(b) = 0$. Hence, by Theorem 4.3.2, there exists $c \in (a,b)$ such that

$$h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0.$$

□

Corollary 4.3.1

Suppose f is continuous on $[a,b]$ and is differentiable on (a,b). If $f'(x) = 0$ for each $x \in (a,b)$, then f is constant on $[a,b]$.

Proof. Take any $x \in (a,b]$. By Theorem 4.3.3, there exists $c \in (a,x)$ such that

$$f'(c) = \frac{f(x)-f(a)}{x-a}.$$

Since $f'(c) = 0$, it follows that $f(x) = f(a)$. Since x was arbitrary, f is constant on $[a,b]$. □

Corollary 4.3.2

Suppose f and g are continuous on $[a,b]$ and are differentiable on (a,b). If $f'(x) = g'(x)$ for each $x \in (a,b)$, then f and g differ by a constant.

Proof. Let $h(x) = f(x) - g(x)$ and apply Corollary 4.3.1. □

Corollary 4.3.3

Suppose f is continuous on $[a,b]$ and is differentiable on (a,b).

- If $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing.
- If $f'(x) < 0$ for all $x \in (a,b)$, then f is strictly decreasing.

Proof. Suppose f' is positive on (a,b). Take any $x_1, x_2 \in [a,b]$ with $x_1 < x_2$. By Theorem 4.3.3, there exists $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1} > 0.$$

Hence, $f(x_2) > f(x_1)$; f is strictly increasing. It is analogous to show for negative f'. □

Theorem 4.3.4 Cauchy's Generalized Mean Value Theorem

Suppose f and g are continuous on $[a,b]$ and are differentiable on (a,b). Then, there exists $c \in (a,b)$ such that

$$f'(c)\{g(b)-g(a)\} = g'(c)\{f(b)-f(a)\}.$$

Proof. Let $h(x) = f(x)\{g(b)-g(a)\} - g(x)\{f(b)-f(a)\}$. Then h is continuous on $[a,b]$ and is differentiable on (a,b). A calculation will verify that $h(a) = h(b)$. Hence, by Theorem 4.3.2, there exists $c \in (a,b)$ such that

$$h'(c) = f'(c)\{g(b)-g(a)\} - g'(c)\{f(b)-f(a)\} = 0.$$

□