MAS331 위상수학
Notes

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Contents

Chapter 1

Countability and Separation Axioms

1.1 The Countability Axioms

Definition 1.1.1: First Countability Axiom

A topological space X is said to have a *countable basis at* x if there is a countable collection $\mathcal{B}$ of neighborhoods of x in X such that, for each neighborhood U of x, there exists $B \in \mathcal{B}$ with $B \subseteq U$. A space that has a countable basis at each point is said to satisfy the *first countability axiom*, or to be *first-countable*.

Note:-

This definition was already given in **??**. Recall the lemmas **??** and **??**.

Definition 1.1.2: Second Countability Axiom

If a topological space X has a countable basis for its topology, then X is said to satisfy the *second countability axiom*, or to be *second-countable*.

Example 1.1.1

$\mathbb{R}^J$ endowed with the product topology with a countable set J is second-countable;

$$\mathcal{S} \triangleq \bigcup_{\alpha \in J} \left\{ \pi_\alpha^{-1}\big((a,b)\big) \,\middle|\, a, b \in \mathbb{Q} \text{ and } a < b \right\}$$

is a countable subbasis for $\mathbb{R}^J$, which induces a countable basis for $\mathbb{R}^J$.

Note:-

If a topological space X is second-countable with a countable basis $\mathcal{B} = \{B_n\}_{n \in \mathbb{Z}_+}$ and a subspace $A \subseteq X$ with the discrete topology. Then, A must be countable.

Otherwise, for each $a \in A$, there exists $B_a \in \mathcal{B}$ such that $B_a \cap A = \{a\}$. This induces an injection $A \hookrightarrow \mathcal{B}$. Hence, A is countable.

Example 1.1.2 (Uniform Topology and Countability Axioms)

In the uniform topology, $\mathbb{R}^\omega$ is first-countable by **??**. Let $\mathcal{B}$ be a basis of $\mathbb{R}^\omega$. Let

$$A \triangleq \left\{ (x_i)_{i \in \mathbb{Z}_+} \in \mathbb{R}^\omega \,\middle|\, \forall i \in \mathbb{Z}_+,\ x_i \in \{0, 1\} \right\}.$$

Then, A has the discrete topology but A is uncountable. Therefore, $\mathbb{R}^\omega$ with the uniform topology is not second-countable.

Theorem 1.1.1

Let X be a topological space and A be a subspace of X.

- If X is first-countable, then A is first-countable.
- If X is second-countable, then A is second-countable.

Proof.

- Let $a \in A$. Let $\mathcal{B}$ be a countable basis of X at a. Then, $\{B \cap A \mid B \in \mathcal{B}\}$ is a countable basis for the subspace A at a. ✓
- Let $\mathcal{B}$ be a countable basis of X. Then, $\{B \cap A \mid B \in \mathcal{B}\}$ is a countable basis for the subspace A. ✓

□

Theorem 1.1.2

Let $\{X_\alpha\}_{\alpha \in J}$ be a countable family of topological spaces.

- If each X_i is first-countable, then $\prod_{\alpha \in J} X_\alpha$ in the product topology is first-countable.
- If each X_i is second-countable, then $\prod_{\alpha \in J} X_\alpha$ in the product topology is second-countable.

Proof.

- Let $(x_\alpha)_{\alpha \in J} \in \prod_{\alpha \in J} X_\alpha$. Then, for each $\alpha \in J$, there exists a countable basis $\mathcal{B}_\alpha$ of X_α at x_α. Then, $\left\{ \prod_{\alpha \in J} B_\alpha \,\middle|\, \forall \alpha \in J,\, B_\alpha \in \mathcal{B}_\alpha \right\}$ is a countable basis at $(x_\alpha)_{\alpha \in J}$.
- For each $\alpha \in J$, there exists a countable basis $\mathcal{B}_\alpha$ of X_α. Then, $\left\{ \prod_{\alpha \in J} B_\alpha \,\middle|\, \forall \alpha \in J,\, B_\alpha \in \mathcal{B}_\alpha \right\}$ is a countable basis of $\prod_{\alpha \in J} X_\alpha$.

□

Definition 1.1.3: Lindelöf Space

A topological space X is called a *Lindelöf space* if, for every open covering of X, there is a countable subcovering.

Definition 1.1.4: Dense Subset

A subset A of a topological space X is said to be *dense* in X if $\overline{A} = X$.

Definition 1.1.5: Separable Space

A topological space X is said to be *separable* if there is a countable dense subset of X.

Note:-

Obvious facts:

- Every compact space is a Lindelöf space.
- The box and product topologies on an finite product of separable spaces is separable. (**??**)
- Every topology on a countable set is Lindelöf and separable.

Theorem 1.1.3

Let X be a second-countable space. Then,

- X is a Lindelöf space.
- X is separable.

Proof. Let $\mathcal{B} = \{B_n\}_{n\in\mathbb{Z}_+}$ be a countable basis for X.

- Let $\mathcal{A}$ be an open covering of X. For each $n \in \mathbb{Z}_+$, there exists $A_n \in \mathcal{A}$ such that $B_n \subseteq A_n$. Then, $\mathcal{A}' \triangleq \{A_n \mid n \in \mathbb{Z}_+\}$ is a countable subcovering of X as $\mathcal{B}$ covers X. ✓
- For each $n \in \mathbb{Z}_+$, choose $x_n \in B_n$. Let $D \triangleq \{x_n \mid n \in \mathbb{Z}_+\}$. Then, for all $x \in X$, every basis element that contains x intersects D; $\overline{D} = X$ by **??**. ✓

□

Example 1.1.3 ($\mathbb{R}_\ell$ and Countability Axioms)

- Given $x \in \mathbb{R}_\ell$, $\{[x, x+1/n) \mid n \in \mathbb{Z}_+\}$ is a countable basis at x. $\mathbb{R}_\ell$ is first-countable.
- $\overline{\mathbb{Q}} = \mathbb{R}_\ell$. $\mathbb{R}_\ell$ is separable.
- Let $\mathcal{B}$ be a basis for $\mathbb{R}_\ell$. Choose, for each $x \in \mathbb{R}_\ell$, an element $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq [x, x+1)$. If $x \neq y$, then $B_x \neq B_y$. Hence $x \mapsto B_x$ is an injection; $\mathcal{B}$ is uncountable. Therefore, $\mathbb{R}_\ell$ is not second-countable.

We now prove $\mathbb{R}_\ell$ is Lindelöf. Thanks to **??**, we only have to prove that, for any open covering $\mathcal{A}$ of $\mathbb{R}_\ell$ by the basis elements, there is a countable subcovering.

Let $\mathcal{A} = \{[a_\alpha, b_\alpha) \mid \alpha \in J\}$ be an open covering of $\mathbb{R}_\ell$. Let $C \triangleq \bigcup_{\alpha\in J}(a_\alpha, b_\alpha)$. We now claim that $\mathbb{R} \setminus C$ is countable. Let $x \in \mathbb{R} \setminus C$. Then $x = a_\beta$ for some $\beta \in J$. Choose $q_x \in \mathbb{Q}$ such that $q_x \in (a_\beta, b_\beta)$. If $x, y \in \mathbb{R} \setminus C$ and $x < y$, then $q_x < q_y$. Hence $x \mapsto q_x$ defines an injection $\mathbb{R} \setminus C \hookrightarrow \mathbb{Q}$. Therefore, $\mathbb{R} \setminus C$ is countable.

Now, let $\mathcal{A}'$ be a countable subcollection of $\mathcal{A}$ that covers $\mathbb{R} \setminus C$. Now, note that $\{(a_\alpha, b_\alpha) \mid \alpha \in J\}$ is an open covering of C as a subspace of $\mathbb{R}$ (with the standard topology). Since $\mathbb{R}$ is second-countable, there exists a finite subcollection $\{(a_{\alpha_1}, b_{\alpha_1}), \cdots, (a_{\alpha_n}, b_{\alpha_n})\}$ covers C. Let $\mathcal{A}'' \triangleq \{[a_{\alpha_1}, b_{\alpha_1}), \cdots, [a_{\alpha_n}, b_{\alpha_n})\}$. Then, $\mathcal{A}' \cup \mathcal{A}''$ is a countble subcovering of $\mathbb{R}_\ell$.

Example 1.1.4 (The Product of Two Lindelöf Spaces Need Not Be Lindelöf)

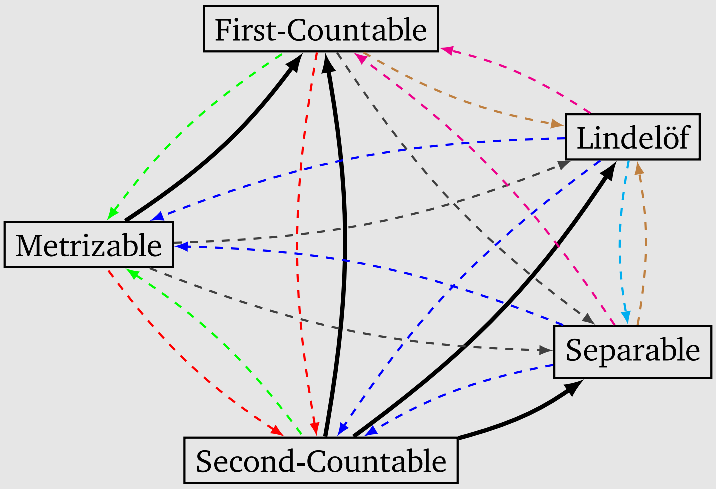
Although $\mathbb{R}_\ell$ is Lindelöf, $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not. Consider the subspace $L \triangleq \{x \times (-x) \mid x \in \mathbb{R}_\ell\}$. Then, L has the discrete topology as $\big([x, x+1) \times [-x, -x+1)\big) \cap L = \{x \times (-x)\}$. Hence, L is not Lindelöf; $\mathbb{R}_\ell^2$ is not Lindelöf.

Example 1.1.5 (A Subspace of a Lindelöf Space Need Not Be Lindelöf)

The ordered square I_o^2 is compact (**??**) and thus is Lindelöf. However, the subspace $A = I \times (0,1)$ is not Lindelöf as an open covering $\{\{x\} \times (0,1) \mid x \in I\}$ does not allow a countable subcovering.

Note:-

Here is the diagram that represents the relations between spaces.



Counterexamples:

- ($\dashrightarrow$) $X = \{0,1\}$ with $\mathcal{T} = \{\varnothing, X, \{0\}\}$ is second-countable but not Hausdorff, thus not metrizable.
- ($\dashrightarrow$) $\mathbb{R}^\omega$ with the uniform topology is metrizable but not second-countable. (Example 1.1.2)
- ($\dashrightarrow$) $\mathbb{R}_\ell$ ($\mathbb{R}$ with the lower limit topology) is first-countable, Lindelöf, and separable; but it is neither second-countable nor metrizable. (Example 1.1.3)
- ($\dashrightarrow$) $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is first countable and separable, but it is not Lindelöf. (Example 1.1.4)
- ($\dashrightarrow$) $\mathbb{R}$ with the discrete topology is first-countable and metrizable; but it is not second-countable, separable, or Lindelöf.
- ($\dashrightarrow$) $\mathbb{R}$ with the finite complement topology is separable and Lindelöf; but it is neither first-countable nor metrizable.
- ($\dashrightarrow$) $\mathbb{R}$ with the countable complement topology is Lindelöf; but it is not first-countable, metrizable, or separable.

1.2 Separation Axioms

Definition 1.2.1: Regular and Normal Space

Let X be a topological space that $\{x\}$ is closed for every $x \in X$. In other words, X is T_1.

- X is said to be T_2 if it is Hausdorff.
- X is said to be *regular*, or T_3, if, for each $x \in X$ and a closed set B disjoint from x, there exist disjoint open sets U and V such that $x \in U$ and $B \subseteq V$.
- X is said to be *normal*, or T_4, if, for each pair A, B of disjoint closed sets in X, there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Note:-

$$T_1 \supseteq T_2 \supseteq T_3 \supseteq T_4$$

Example 1.2.1 (T_2 Does Not Imply T_3)

The space $\mathbb{R}_K$ is T_2 as it is finer than the standard topology. The set $K = \{1/n \mid n \in \mathbb{Z}_+\}$ is closed in $\mathbb{R}_K$ and $0 \notin K$. Suppose there are disjoint open sets U and V such that $0 \in U$ and $K \subseteq V$. Let B be a basis element that $0 \in B \subseteq U$. Then, $B = (a,b) \backslash K$ since any open interval containing 0 intersects K. (It must be $a < 0 < b$.) Let $n \in \mathbb{Z}_+$ such that $1/n < b$. Then, $1/n \in K \subseteq V$. Let B' be a basis element such that $1/n \in B' \subseteq V$. Then, $B' = (c,d)$ for some $c < d$. Let $\max\{c, 1/(n+1)\} < z < 1/n$. Then, $z \in B \cap B' \subseteq U \cap V$. Hence, $\mathbb{R}_K$ is not T_3, #.

Lemma 1.2.1 Another Formulation

Let X be a T_1 space.

(i) X is T_3 if and only if, for each $x \in U$ and a neighborhood U of x, there exists a neighborhood V of x such that $\overline{V} \subseteq U$.

(ii) X is T_4 if and only if, for each closed set A and an open set U containing A, there exists an open set V such that $A \subseteq V$ and $\overline{V} \subseteq U$.

Proof.

(i) ($\Rightarrow$) $B \triangleq X \setminus U$ is a closed set and $x \notin B$; there exist disjoint open sets V and W such that $x \in V$ and $B \subseteq W$. Then, $\overline{V}$ does not intersect B, i.e., $\overline{V} \subseteq U$. ✓

($\Leftarrow$) Let $x \in X$ and $B \subseteq X$ be a closed set with $x \notin B$. Then, $X \setminus B$ is a neighborhood of x; there exists a neighborhood V of x such that $\overline{V} \subseteq X \setminus B$. Then, V and $X \setminus \overline{V}$ are disjoint open sets that contain x and B, respectively. ✓

(ii) ($\Rightarrow$) $B \triangleq X \setminus U$ is a closed set and $A \cap B = \varnothing$; there exist disjoint open sets V and W such that $A \subseteq V$ and $B \subseteq W$. Then, $\overline{V}$ does not intersect B, i.e., $\overline{V} \subseteq U$. ✓

($\Leftarrow$) Let $A, B \subseteq X$ be disjoint closed sets in X. Then, $X \setminus B$ is an open set that contains A; there exists an open set V such that $A \subseteq V$ and $\overline{V} \subseteq X \setminus B$. Then, V and $X \setminus \overline{V}$ are disjoint open sets that contain A and B, respectively. ✓

□

Theorem 1.2.1

Let X be a topological space and $Y \subseteq X$ be a subspace of X.

(i) If X is T_1, then Y is T_1.

(ii) If X is T_2, then Y is T_2.

(iii) If X is T_3, then Y is T_3.

Proof.

(i) For each $x \in Y$, $\{x\} \cap Y = \{x\}$ is closed.

(ii) Let $x, y \in Y$ with $x \neq y$. Then, there exist disjoint neighborhoods U and V of x and y, respectively, in X. Then, $U \cap Y$ and $V \cap Y$ are disjoint neighborhoods of x and y in Y, respectively.

(iii) Y is already T_1 by (i). Let $x \in Y$ and B be a closed set in Y disjoint from x. Then, $\overline{B} \cap Y = B$ by **??**. Hence, $x \notin \overline{B}$; there are disjoint open sets U and V in X such that $x \in U$ and $\overline{B} \subseteq V$. Then, $U \cap Y$ and $V \cap Y$ are disjoint open sets and $x \in U \cap Y$ and $B \subseteq V \cap Y$.

□

Theorem 1.2.2

Let $\{X_\alpha\}_{\alpha \in J}$ be a family of topological spaces. Let $X \triangleq \prod_{\alpha \in J} X_\alpha$ be endowed with either box or product toplogy.

(i) X is T_1 if and only if each X_α is T_1.

(ii) X is T_2 if and only if each X_α is T_2.

(iii) X is T_3 if and only if each X_α is T_3.

Proof. Let $\mathbf{x} = (x_\alpha)_{\alpha \in J} \in X$. Supopse X is T_1(, T_2, or T_3). Then, For each $\alpha_0 \in J$, X_{α_0} is homeomorphic with the subspace

$$Y \triangleq \{\mathbf{y} \in X \mid \forall \alpha \in J \setminus \{\alpha_0\},\ y_\alpha = x_\alpha\}.$$

Hence, X_{α_0} is T_1(, T_2, or T_3).

(i) ($\Leftarrow$) Let $\mathbf{x} = (x_\alpha)_{\alpha \in J} \in X$. Then, $\{\mathbf{x}\} = \bigcap_{\alpha \in J} \pi_\alpha^{-1}(\{x_\alpha\})$ is closed.

(ii) ($\Leftarrow$) Let $\mathbf{x}, \mathbf{y} \in X$ with $\mathbf{x} \neq \mathbf{y}$. Then, there exists $\alpha_0 \in J$ such that $x_{\alpha_0} \neq y_{\alpha_0}$; there are disjoint neighborhoods U_{α_0} and V_{α_0} of x_{α_0} and y_{α_0} in X_{α_0}. Then, If we define $U, V \subseteq X$ by $U \triangleq \prod_{\alpha \in J} U_\alpha$ and $V \triangleq \prod_{\alpha \in J} V_\alpha$ where

$$U_\alpha \triangleq \begin{cases} U_{\alpha_0} & \text{if } \alpha = \alpha_0 \\ X_\alpha & \text{otherwise} \end{cases} \quad \text{and} \quad V_\alpha \triangleq \begin{cases} V_{\alpha_0} & \text{if } \alpha = \alpha_0 \\ X_\alpha & \text{otherwise,} \end{cases}$$

we find that U and V are disjoint neighborhoods of $\mathbf{x}$ and $\mathbf{y}$ in X.

(iii) ($\Leftarrow$) Let $\mathbf{x} \in X$ and let U be a neighborhood of $\mathbf{x}$ in X. Choose a basis element $B = \prod_{\alpha \in J} U_\alpha$ so that $\mathbf{x} \in B \subseteq U$. For each $\alpha \in J$, let $V_\alpha = X_\alpha$ if $U_\alpha = X_\alpha$. Otherwise, by Lemma 1.2.1, let V_α be a neighborhood of x_α in X such that $\overline{V_\alpha} \subseteq U_\alpha$. Then, $V = \prod_{\alpha \in J} V_\alpha$ is a neighborhood of $\mathbf{x}$ and $\overline{V} = \prod_{\alpha \in J} \overline{V_\alpha} \subseteq B \subseteq U$. By Lemma 1.2.1, X is T_3.

□

Example 1.2.2 ($\mathbb{R}_\ell$ Is T_4)

$\mathbb{R}_\ell$ is T_1 as it is finer than the standard topology. Suppose A and B are disjoint closed sets in $\mathbb{R}_\ell$. For each $a \in A$ choose a basis element $[a, x_a)$ not intersecting B. This is possible since $\mathbb{R} \setminus B$ is open in $\mathbb{R}_\ell$. Similarly, for each $b \in B$, choose a basis element $[b, x_b)$ not intersecting A. Then,

$$U \triangleq \bigcup_{a \in A} [a, x_a) \quad \text{and} \quad V \triangleq \bigcup_{b \in B} [b, x_b)$$

are disjoint open sets such that $A \subseteq U$ and $B \subseteq V$.

Example 1.2.3 ($\mathbb{R}_\ell^2$ is not T_4)

The space $\mathbb{R}_\ell$ is T_3; hence $\mathbb{R}_\ell^2$ is T_3 by Theorem 1.2.2.

Suppose $\mathbb{R}_\ell^2$ is normal for the sake of contradiction. Let L be a subspace of $\mathbb{R}_\ell^2$ where $L \triangleq \{x \times (-x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$. Here are some facts:

- L has the discrete topology. Thus, every subset of L is closed in L, especially.
- L is closed in $\mathbb{R}_\ell^2$ as it is closed in $\mathbb{R}^2$, which is coarser than $\mathbb{R}_\ell^2$.
- Every subset A of L is closed in $\mathbb{R}_\ell^2$.
- For every $\varnothing \neq A \subsetneq L$, there are disjoint open sets U_A and V_A in $\mathbb{R}_\ell^2$ containing A and $L \setminus A$, respectively.

Here, we define a function $\theta : \mathcal{P}(L) \to \mathcal{P}(\mathbb{Q}^2)$ by

$$A \mapsto \begin{cases} \mathbb{Q}^2 \cap U_A & \text{if } \varnothing \subsetneq A \subsetneq L \\ \varnothing & \text{if } A = \varnothing \\ \mathbb{Q}^2 & \text{if } A = L. \end{cases}$$

To show θ is injective, let $\varnothing \subsetneq A, B \subsetneq L$ with $A \neq B$. WLOG, $A \nsubseteq B$; let $x \in A \setminus B$. Then, since $x \in L \setminus B$, $x \in U_A \cap V_B$. Since $\mathbb{Q}^2$ is dense in $\mathbb{R}_\ell^2$ and $U_A \cap V_B$ is open and nonempty, there exists $q \in \mathbb{Q}^2 \cap U_A \cap V_B$. Hence, $\mathbb{Q}^2 \cap U_A \nsubseteq \mathbb{Q}^2 \cap U_B$. Therefore, θ is injective.

Also, the map $\psi : \mathcal{P}(\mathbb{Z}_+) \to \mathbb{R}$ defined by

$$S \mapsto \sum_{i=1}^{\infty} \frac{a_i}{10^i}$$

is injective where $a_i = 1$ if $i \in S$ and $a_i = 0$ if $i \notin S$. Thus, there exists an injective map $\psi' : \mathcal{P}(\mathbb{Q}^2) \to L$. Then, $\psi' \circ \theta$ is an injective map from $\mathcal{P}(L)$ to L, #. (**??**)

This shows that

(i) A product of T_4 spaces need not be T_4.

(ii) A T_3 space need not be T_4.

Note:-

$$T_1 \supsetneq T_2 \supsetneq T_3 \supsetneq T_4$$