0.1 Jacobian

Definition 0.1.1: Jacobian

Let $\mathbf{f}\colon U(\subseteq \mathbb{R}^m) \to \mathbb{R}^n$ be differentiable. The function $J_f : U \to \mathbb{R}$ defined by

$$J_{\mathbf{f}}(\mathbf{x}) = \det \begin{bmatrix} \partial_1 f_1(\mathbf{x}) & \cdots & \partial_n f_1(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \partial_1 f_n(\mathbf{x}) & \cdots & \partial_n f_n(\mathbf{x}) \end{bmatrix}$$

is called the *Jacobian* of $\mathbf{f}$ at $\mathbf{x}$.

Lemma 0.1.1

If $f : V(\subseteq \mathbb{R}^n) - \mathbb{R}$ and $\mathbf{g}\colon U \to V$ are differentiable, then

$$J_{f\circ\mathbf{g}}(\mathbf{x}) = J_f\big(\mathbf{g}(\mathbf{x})\big) \cdot J_{\mathbf{g}}(\mathbf{x}).$$

Note:-

The linear mapping $\mathrm{df}(c)$ is invertible if and only if $J_{\mathbf{f}}(c)$ is nonzero.

0.2 The Inverse Function Theorem

Lemma 0.2.1 Contraction Mapping Principle

Let (X, d) be a complete metric space. Let $\varphi\colon X \to X$. Suppose that there exists $M \in [0, 1)$ such that $d\big(\varphi(x_1), \varphi(x_2)\big) \le M d(x_1, x_2)$. (We call it a *contraction mapping*.) Then, there uniquely exists $x_* \in X$ such that $\varphi(x_*) = x_*$.

Proof. Fix any $x_0 \in X$. Since $\{x_j\}_{j\in\mathbb{Z}_+}$, where $x_j = \varphi(x_{j-1})$ for each $j \in \mathbb{Z}_+$, is continuous. It converges to some x_*. As φ is continuous, we have $\varphi(x_*) = x_*$. The uniqueness follows trivially. □

Note:-

- For each $v \in \mathbb{R}^n \setminus \{0\}$, $|Av| = |v| \cdot \left|A\frac{v}{|v|}\right| \le \|A\|_L \cdot |v|$. The result is trivial when $v = 0$.
- For each $u \in \mathbb{R}^n$ with $|u| = 1$, $|ABu| \le \|A\|_L |Bu| \le \|A\|_L \|B\|_L$. Hence, $\|AB\|_L = \|A\|\|B\|$.
- Given invertible $A \in L(\mathbb{R}^n.\mathbb{R}^n)$, $A^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is linear. Moreover, $\|A\|_L > 0$.

Lemma 0.2.2

Given two linear mappings $A, B\colon \mathbb{R}^n \to \mathbb{R}^n$ with invertibility of A,

$$\|A - B\|_L \|A^{-1}\|_L < 1 \implies B \text{ is invertible.}$$

Proof. (Hint: show that $B\mathbf{x} = 0$ has only the trivial solution, i.e., if $\mathbf{x} \neq 0$, then $B\mathbf{x} \neq 0$.) □

Theorem 0.2.1 Inverse Function Theorem

Let $\mathbf{f}\colon E(\subseteq \mathbb{R}^n) \to \mathbb{R}^n$ be C^1 in U, $\mathbf{a} \in E$, and $\mathbf{b} = f(a)$. Suppose that $J_{\mathbf{f}}(a) \neq 0$. Then,

$$\exists \delta \in \mathbb{R}_+,\ \mathbf{f}\big|_{B_\delta(a)} : B_\delta(a) \to \mathbf{f}\big(B_\delta(a)\big) \text{ is invertible.}$$

Moreover, $\mathbf{f}\big(B_\delta(a)\big)$ is an open set, and $\left(\mathbf{f}\big|_{B_\delta(a)}\right)^{-1}$ is C^1.

Proof. Let $A \triangleq \mathrm{d}\mathbf{f}(\mathbf{c})$. Define λ by $\lambda \triangleq \dfrac{1}{2\|A^{-1}\|_L} > 0$ so $2\lambda\left\|A^{-1}\right\|_L = 1$. Since $\mathrm{d}\mathbf{f}$ is continuous, there exists $\delta \in \mathbb{R}_+$ such that $\|\mathrm{d}\mathbf{f}(\mathbf{x}) - \mathrm{d}\mathbf{f}(\mathbf{c})\|_L < \lambda$ for each $B_\delta(\mathbf{c})$.

Given a point $\mathbf{y} \in \mathbb{R}^n$, we define $\varphi(\cdot;\mathbf{y})$ by

$$\begin{aligned}\varphi(\cdot;\mathbf{y}) : B_\delta(\mathbf{c}) &\longrightarrow \mathbb{R}^n \\ \mathbf{x} &\longmapsto \varphi(\mathbf{x};\mathbf{y}) = \mathbf{x} + A^{-1}(\mathbf{y} - \mathbf{f}(\mathbf{x}))\end{aligned}$$

Note that $\mathbf{x}$ is a fixed point of $\varphi(\cdot;\mathbf{y})$ if and only if $A^{-1}(\mathbf{y}-\mathbf{f}(\mathbf{x})) = 0$, i.e., $\mathbf{y} = \mathbf{f}(\mathbf{x})$. Note also that φ is differentiable and $\mathrm{d}\varphi(\mathbf{x};\mathbf{y}) = \mathrm{Id} - A^{-1}\,\mathrm{d}\mathbf{f}(\mathbf{x}) = A^{-1}\big(A - \mathrm{d}\mathbf{f}(\mathbf{x})\big)$ for each $x \in B_\delta(\mathbf{c})$. Let $U \triangleq B_\delta(\mathbf{c})$ and $V \triangleq \mathbf{f}(U)$.

Hence, for all $x \in U$,

$$\|\mathrm{d}\varphi(\mathbf{x};\mathbf{y})\|_L = \left\|A^{-1}\big(A - \mathrm{d}\mathbf{f}(\mathbf{x})\big)\right\|_L \le \|A^{-1}\|_L \cdot \|A - \mathrm{d}\mathbf{f}(\mathbf{x})\|_L < 1/(2\lambda)\cdot\lambda = 1/2.$$

Now, fix any $\mathbf{y} \in V$. Fix $\mathbf{x}_1, \mathbf{x}_2 \in U$. Define $\Psi\colon [0,1] \to \mathbb{R}$ by $t \mapsto \varphi(t\mathbf{x}_1 + (1-t)\mathbf{x}_2;\mathbf{y})$. $\Psi(0) = \varphi(\mathbf{x}_2;\mathbf{y})$ and $\Psi(1) = \varphi(\mathbf{x}_1;\mathbf{y})$. Note that Ψ is differentiable on $(0,1)$. By MVT, there exists $t_* \in (0,1)$ such that $\Psi(1) - \Psi(0) = \Psi'(t_*)$. The chain rule gives

$$\Psi'(t_*) = \mathrm{d}\varphi(t_*\mathbf{x}_1 + (1-t_*)\mathbf{x}_2)(\mathbf{x}_1 - \mathbf{x}_2).$$

Hence,

$$|\varphi(\mathbf{x}_1) - \varphi(\mathbf{x}_2)| = |\mathrm{d}\varphi(t_*\mathbf{x}_1 + (1-t_*)\mathbf{x}_2)| \cdot |(\mathbf{x}_1 - \mathbf{x}_2)| \le |\mathbf{x}_1 - \mathbf{x}_2|/2.$$

We want to show that $\mathbf{f}$ is locally invertible. It suffices to show that it is injective. Hence, φ has at most one fixed point, i.e., there exists at most one $\mathbf{x}$ such that $\mathbf{y} = \mathbf{f}(\mathbf{x})$; thus $\mathbf{f}$ is injective on U.

Let $\mathbf{x}_0 \in U$ and $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$. Fix any $r \in \mathbb{R}_+$ such that $\overline{B_r(\mathbf{x}_0)} \subseteq U$. Let $B = B_r(\mathbf{x}_0)$. Take any $\mathbf{y} \in B_{\lambda r}(\mathbf{y}_0)$. Then,

$$|\varphi(\mathbf{x}_0;\mathbf{y}) - \mathbf{x}_0| = |A^{-1}(\mathbf{y} - \mathbf{y}_0)| < \|A^{-1}\|_L \lambda r = \frac{r}{2}.$$

Moreover, for any $\mathbf{x} \in \overline{B}$,

$$|\varphi(\mathbf{x};\mathbf{y}) - \mathbf{x}_0| \le |\varphi(\mathbf{x};\mathbf{y}) - \varphi(\mathbf{x}_0;\mathbf{y})| + |\varphi(\mathbf{x}_0) - \mathbf{x}_0| \le \frac{1}{2}|\mathbf{x} - \mathbf{x}_0| + \frac{r}{2} < r.$$

This directly implies that $\varphi(\overline{B}) \subseteq B \subseteq \overline{B}$. Hence, φ is a contraction mapping on a complete metric space $\overline{B}$. By **??**, there exists a fixed point $\mathbf{x} \in \overline{B}$, which satisfies $\mathbf{y} = \mathbf{f}(\mathbf{x})$. Thus, $\mathbf{y} \in f(\overline{B}) \subseteq f(U) = V$. Hence, $B_{\lambda r}(\mathbf{y}_0) \subseteq V$, V is open.

Now, let $\mathbf{g}\colon V \to U$ be the local inverse of $\mathbf{f}$. Take any $\mathbf{y} \in V$ and $\mathbf{y} + \mathbf{k} \in V$. There are unique $\mathbf{x} \in U$ and $\mathbf{x} + \mathbf{h} \in U$ such that $\mathbf{y} = \mathbf{f}(\mathbf{x})$ and $\mathbf{y} + \mathbf{k} = \mathbf{f}(\mathbf{x} + \mathbf{h})$.

□

Example 0.2.1 (Level Sets)

Define $\Omega \triangleq \{(x_1,x_2) \in \mathbb{R}^2 \mid |x_2| \le 1\}$. Given two constants, $a, b \in \mathbb{R}$ with $a < b$, define $\overline{\varphi}(x_1,x_2) = ax_1$ and $\overline{\psi}(x_1,x_2) = bx_1$. Then, $\Gamma_0 = \{\mathbf{x} \in \Omega \mid \overline{\varphi}(\mathbf{x}) - \overline{\psi}(\mathbf{x}) = 0\} = \{\mathbf{x} \in \Omega \mid x_1 = 0\}$.