Summary for Complex Variables I

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Contents

Chapter 1

Preliminaries

1.1 Complex Plane

Definition 1.1.1: Complex Number

$i := \sqrt{-1}$ is called the *imaginary unit*. $\mathbb{C} := \{x + iy \mid x, y \in \mathbb{R}\}$ is the set of complex numbers where $\mathbb{R}$ is the set of real numbers.

Definition 1.1.2: Algebras of $\mathbb{C}$

For $z_k := x_k + iy_k$ where $k \in \mathbb{Z}_+$ and $x_k, y_k \in \mathbb{R}$,

- $z_1 + z_2 := (x_1 + x_2) + i(y_1 + y_2)$
- $z_1 \cdot z_2 := (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$.

Theorem 1.1.3

$\mathbb{C}$ is a field.

Proof. Trivial. □

Note

$z = a + ib$, $a, b \in \mathbb{R}$ with $z \neq 0$. Then, $z^{-1} = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}$.

1.2 Rectangular Representation

Definition 1.2.1

Let $z = x + iy$ where $x, y \in \mathbb{R}$.

(i) $|z| := \sqrt{x^2 + y^2}$ is called *modulus* of z.

(ii) $\overline{z} := x - iy$ is called *conjugate* of z.

(iii) $\Re z = x$ is called the *real part* of z and $\Im z = y$ is called the *imaginary part* of z.

(iv) For $z_1, z_2 \in \mathbb{C}$, $|z_1 - z_2|$ is the *distance* between z_1 and z_2.

Note

- $z + \bar{z} = 2\Re z$
- $z - \bar{z} = 2i\Im z$
- $|z_1 + z_2| \le |z_1| + |z_2|$
- $\big||z_1| - |z_2|\big| \le |z_1 - z_2|$

1.3 Polar Representation

Given $z \in \mathbb{C}$, $|z|$ is unique. $\arg z = \theta + 2k\pi$ $(k \in \mathbb{Z})$ (Or $\arg z = \theta \pmod{2\pi}$)

Definition 1.3.1

If $z = |z| \cdot (\cos\theta + i\sin\theta)$, θ is called an *argument* of z and is written $\arg z = \theta \pmod{2\pi}$ (as $\theta + 2k\pi$ for $k \in \mathbb{Z}$ is an argument of z as well). If $\arg z = \theta^* \pmod{2\pi}$, and if $-\pi < \theta^* \le \pi$, then we define $\operatorname{Arg} z = \theta^*$ and it is called the *principal argument* of z.

Theorem 1.3.2

For $z_1, z_2 \in \mathbb{C}$ with $z_1, z_2 \neq 0$, $\arg z_1 z_2 = \arg z_1 + \arg z_2 \pmod{2\pi}$.

Proof. Let $\arg z_1 = \theta_1 \pmod{2\pi}$ and $\arg z_2 = \theta_2 \pmod{2\pi}$ Then, $z_1 = |z_1|(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = |z_2|(\cos\theta_2 + i\sin\theta_2)$. Now, we have $z_1 \cdot z_2 = |z_1||z_2|\big(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\big)$. $\square$

Chapter 2

Elementary Complex Functions

2.1 Exponential Functions

Definition 2.1.1: Exponential Function

For each $z = x + iy$ where $x, y \in \mathbb{R}$, we define $e^z := e^x \cdot (\cos y + i \sin y)$.

Theorem 2.1.2

For each $z \in \mathbb{C}$, $e^z = \sum_{j=1}^{\infty} \frac{z^j}{j!}$.

Proof. Proved later using complex integral. □

Theorem 2.1.3

For each $z, z' \in \mathbb{C}$,

(a) $e^{z+z'} = e^z \cdot e^{z'}$,

(b) $e^{-z} = \frac{1}{e^z}$, and

(c) $e^{z+2k\pi i} = e^z$ for all $k \in \mathbb{Z}$.

Definition 2.1.4

For each $z \in \mathbb{C}$,

(1) $\cos z := \frac{e^{iz}+e^{-iz}}{2}$

(2) $\sin z := \frac{e^{iz}-e^{-iz}}{2i}$

(3) $\cosh z = \frac{e^z+e^{-z}}{2}$

(4) $\sinh z = \frac{e^z-e^{-z}}{2}$

Theorem 2.1.5

For each $z \in \mathbb{C}$, we have $\cosh z = \cos(iz)$ and $\sinh z = -i \sin(iz)$.

Example 2.1.6

Let us solve $\cos z = 2$. Let $t := e^{iz}$ to obtain $t^2 - 4t + 1 = 0$, which gives $t = 2 \pm \sqrt{3}$. Write $z = x + iy$ where $x, y \in \mathbb{R}$ to have $e^{ix}e^{-y} = 2 \pm \sqrt{3}$. Taking modulus to both sides gives $e^{-y} = 2 \pm \sqrt{3}$, i.e., $y = -\ln(2 \pm \sqrt{3})$. Taking argument to both sides gives $x = 2k\pi$

for $k \in \mathbb{Z}$. Thus, $z = 2k\pi - i\ln(2 \pm \sqrt{3})$ for $k \in \mathbb{Z}$.

2.2 Mapping Properties

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2.3 Logarithmic "Functions"

Definition 2.3.1: Logarithmic Function

For any $z \in \mathbb{C} \setminus \{0\}$, we define $w = \ln z$ if and only if $e^w = z$.

Note

How to compute $\ln z$? Note that $z = |z| \cdot e^{i(\operatorname{Arg} z + 2k\pi)}$ for $k \in \mathbb{Z}$. Let $w = u + iv$ where $u, v \in \mathbb{R}$ so that $e^w = e^u \cdot e^{iv} = |z| \cdot e^{i(\operatorname{Arg} z + 2k\pi)}$. Hence, we have $u = \ln|z|$ and $v = \operatorname{Arg} z + 2k\pi$. In other words, $\ln z = \ln|z| + i \arg z$. (Note that this is not a "function"!)

Definition 2.3.2: Principal Logarithmic Function

For any $z \in \mathbb{C} \setminus \{0\}$, we define $\operatorname{Log} z := \ln|z| + i \operatorname{Arg} z$ and it is called the *principal value of* $\ln z$.

Definition 2.3.3: Branch of Logarithm

A *branch* of $\ln z$ is a function given by $\omega\colon \ln z$ with $\theta_0 < \arg z \leq \theta_0 + 2\pi$. Here, θ_0 is called a *branch cut*.

Example 2.3.4

$B := \{ z \mid |z+2| < 1 \}$ when mapped with Log is not an open ball but it becomes an open ball when the branch cut is $-\pi/2$.

2.4 Complex Exponents

Definition 2.4.1: Complex Exponents

For $z \in \mathbb{C} \setminus \{0\}$ and $w \in \mathbb{C}$, define

$$z^w := e^{w \ln z}.$$

Note

Complex exponentiation is not a function! If one considers the complex exponentiation as a set of possible values, then $z^{\eta_1} \cdot z^{\eta_2} = z^{\eta_1 + \eta_2}$ may easily fail!

Example 2.4.2

To solve $z^{1-i} = 4$, write $e^{(1-i)\ln z} = e^{\ln 4}$, i.e., $\ln z = (1+i)(\ln 2 + k\pi i)$ for $k \in \mathbb{Z}$. In other words, $\ln|z| + i\arg z = (\ln 2 - k\pi) + i(\ln 2 + k\pi)$. Hence, $|z| = e^{\ln 2 - k\pi}$ and $\arg z = \ln 2 + k\pi \pmod{2\pi}$.

Chapter 3

Analytic Functions

3.1 Cauchy–Riemann Equation

Definition 3.1.1: Continuity

For a fixed point $z_0 \in \mathbb{C}$, a function f is said to be continuous at z_0 if

$$\lim_{|z-z_0|\to 0} |f(z)-f(z_0)| = 0.$$

Definition 3.1.2: Differentiability

For a fixed point $z_0 \in \mathbb{C}$, a function f is said to be *continuous at* z_0 if

$$\lim_{\substack{|\omega|\to 0\\ \omega\in\mathbb{C}}} \frac{f(z_0+\omega)-f(z_0)}{\omega}$$

exists. If f is differentiable at z_0, then define the *derivative* of f at z_0 by

$$f'(z_0) := \lim_{\substack{|\omega|\to 0\\ \omega\in\mathbb{C}}} \frac{f(z_0+\omega)-f(z_0)}{\omega}.$$

Example 3.1.3

For each $n \in \mathbb{N}$, one can derive that $f'(z) = nz^{n-1}$ where $f(z) = z^n$.

Theorem 3.1.4

If f is differentiable at z_0, then it is continuous at z_0.

Example 3.1.5

Let us determine differentiability of $f(z) = |z|^2$. Write $z = x + iy$ and $\omega = p + iq$ for $x, y, p, q \in \mathbb{R}$. Then,

$$\frac{f(z+\omega)-f(z)}{\omega} = \frac{2(xp+yq)+|\omega|^2}{\omega}$$

As we know $\lim_{\omega\to 0} \frac{|\omega|^2}{\omega} = 0$, we only need to care if $\lim_{\omega\to 0} \frac{2(xp+yq)}{p+iq}$. Evaluating the limit along the real axis and the imaginary axis gives $2x$ and $-2yi$; hence f is not differentiable at $z \in \mathbb{C} \setminus \{0\}$. At the origin, we have $f'(0) = \lim_{\omega \to 0} \frac{f(0+\omega)-f(0)}{\omega} = 0$.

Theorem 3.1.6

Product, quotient, chain rule still holds in complex derivative.

Theorem 3.1.7 Cauchy–Riemann Equation

If f is differentiable at z, then $f_y(z) = i f_x(z)$ at z, or equivalently,

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

where $u(x,y) := \Re f(x+iy)$ and $v(x,y) := \Im f(x+iy)$ for $x, y \in \mathbb{R}$.

Proof. $f_x(z) = \lim_{\xi \to 0} \dfrac{f(z+\xi)-f(z)}{\xi} = f'(z)$ and $-i f_y(z) = \lim_{\eta \to 0} \dfrac{f(z+i\eta)-f(z)}{i\eta} = f'(z)$. □

Example 3.1.8

Is e^z differentiable in $\mathbb{C}$?

$$\lim_{h \to 0} \frac{e^h - 1}{h} = \lim_{(\xi,\eta)\to 0} \frac{(e^\xi - 1)e^{i\eta} + (e^{i\eta} - 1)}{\xi + i\eta}$$

Note

We may write $f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$. If f is differentiable, we *define*

$$\frac{\partial f}{\partial z} := \left(\frac{1}{2}\partial_x + \frac{1}{2i}\partial_y\right)u + i\left(\frac{1}{2}\partial_x + \frac{1}{2i}\partial_y\right)v = \left(\frac{1}{2}\partial_x + \frac{1}{2i}\partial_y\right)f$$
$$\frac{\partial f}{\partial \bar{z}} := \left(\frac{1}{2}\partial_x - \frac{1}{2i}\partial_y\right)u + i\left(\frac{1}{2}\partial_x - \frac{1}{2i}\partial_y\right)v = \left(\frac{1}{2}\partial_x - \frac{1}{2i}\partial_y\right)f.$$

So that $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)f = 0$ if f is differentiable.

Definition 3.1.9: Domain

A domain is an open and connected subset of $\mathbb{C}$.

Theorem 3.1.10

Any two points in a domain can be connected by polygonal lines parallel to the coordinate axes that lies in the domain.

Proof. Let D be a domain and let $z_0 \in D$. Let $A \subseteq D$ be the set of all points in D that can be connected from z_0 by polygonal lines parallel to the coordinate axes. Let $B := D \setminus A$. If $z \in A$ and $r > 0$ satisfy $B_r(z) \subseteq D$, then $B_r(z) \subseteq A$; hence A is open. Similarly, B is open as well. As D is connected, A or B is empty but $z_0 \in A$; hence, $B = \varnothing$. □

Theorem 3.1.11

If $f'(z) \equiv 0$ in a domain D, then f is constant on D.

Proof. $f_x \equiv f_y \equiv 0$; hence $u_x \equiv v_x \equiv u_y \equiv u_x \equiv 0$ on D. Thus, f is contant on every line segment in D parallel to coordinate axes. Hence, f is constant on D. □

Corollary 3.1.12

Let f be differentiable on a domain D.

(1) If $\Re f(z)$ is constant on D, then f is constant on D.

(2) If $\Im f(z)$ is constant on D, then f is constant on D.

(3) If $\operatorname{Arg} f(z)$ is constant on D, then f is constant on D.

Proof.

(1) There is $\omega_0 \in \mathbb{C}$ such that, when g is defined by $g(z) \triangleq f(z) - \omega_0$, we have $\Re g(z) \equiv 0$ and g is differentiable on D.

$$\lim_{\xi \to 0} \frac{f(z+\xi) - f(z)}{\xi} = f'(z) = \lim_{\eta \to 0} \frac{f(z+i\eta) - f(z)}{i\eta}$$

where the left hand side is real and the right hand side is purely imaginary. Therefore, $f'(z) = 0$ for all $z \in D$. The result follows from Theorem 3.1.11.

(2) Let $g(z) = if(z)$ so that g is differentiable on D and $\Re g(z)$ is constant. Therefore, by (1), g is constant and thus f is constant.

(3) There is $\omega_0 \in \mathbb{R}$ such that, when g is defined by $g(z) \triangleq f(z)e^{-i\omega_0}$, we have $\Re g(z)$ is constant and g is differentiable on D. Tehrefore, by (1), g is constant and thus f is constant. □

3.2 Analyticity

Definition 3.2.1: Analytic Function

- For a fixed point $z_0 \in \mathbb{C}$, a function f is *analytic* at z_0 if there is some $r > 0$ such that f is differentiable at every point in $B_r(z_0) \triangleq \{z \in \mathbb{C} \colon |z - z_0| < r\}$.
- A function f is *analytic in domain* D if it is analytic at z for all $z \in D$.
- A function f is *entire* if it is analytic in $\mathbb{C}$.

Theorem 3.2.2

Given a function $f(z) = u(x,y) + iv(x,y)$ in domain D, if

(1) $u(x,y)$ and $v(x,y)$ are C^1 in D, and if

(2) $u(x,y)$ and $v(x,y)$ satisfy the Cauchy–Riemann equations in D,

then f is analytic in D.

Proof. Fix $z = x + iy \in D$ and write $\Delta z := \xi + i\eta$ for $\xi, \eta \in \mathbb{R}$ where Δz is sufficiently small. (This is possible since D is open.) Then,

$$\begin{aligned} f(z+\Delta z) &= (f(z+\xi) - f(z)) - (f(z+\Delta z) - f(z+\xi)) \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(x+t\xi, y)\,\mathrm{d}t + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(x+\xi+i(y+t\eta)))\,\mathrm{d}t \\ &= \xi \int_0^1 f_x(x+t\xi)\,\mathrm{d}t + \eta \int_0^1 f_y(x+\xi+i(y+t\eta))\,\mathrm{d}t \\ &= \xi \int_0^1 f_x(x+t\xi)\,\mathrm{d}t + i\eta \int_0^1 f_x(x+\xi+i(y+t\eta))\,\mathrm{d}t \\ &= f_x(z)\Delta z + \xi \int_0^1 \big(f_x(x+t\xi) - f_x(z)\big)\,\mathrm{d}t + i\eta \int_0^1 \big(f_x(x+\xi+i(y+t\eta)) - f(z)\big)\,\mathrm{d}t \end{aligned}$$

As f_x is continuous at z, we have

$$\left| \int_0^1 \big(f_x(x+t\xi) - f_x(z)\big)\,\mathrm{d}t \right| \to 0 \text{ and}$$
$$\left| \int_0^1 \big(f_x(x+\xi+i(y+t\eta)) - f(z)\big)\,\mathrm{d}t \right| \to 0$$

as $\Delta z \to 0$. Moreover, since $(\Re z)/z$ and $(\Im z)/z$ are bounded, we have

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = f_x(z).$$

□

Example 3.2.3

(Something's wrong here.) Let $f(x+iy) = x^2 + y^2 + ixy$. $u_x = 2x$, $u_y = 2y$, $v_x = y$, and $v_y = x$. Hence, u and v are C^1 in $\mathbb{R}^2$. $f_x = 2x + yi$ and $-if_y = -i(2y + xi) = x - 2yi$; hence f satisfies the Cauchy–Riemann equation only at $z = 0$. Hence, by Theorem 3.2.2, f is nowhere analytic.

Example 3.2.4 Analyticity of Principal Log

Let $f(z) = \operatorname{Log} z$ and let $D = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ be a domain. Write $\operatorname{Log} z = \ln\sqrt{x^2+y^2} + i \operatorname{Arg} z$ so $u = \ln\sqrt{x^2+y^2}$ and $v = \operatorname{Arg}(x+iy)$. u is obviously C^1 on D. As for v, as z is fixed, one may choose $\operatorname{Arg} z$ from

$$\operatorname{Arg} z = \pm \arccos\left(\frac{x}{\sqrt{x^2+y^2}}\right), \arctan\left(\frac{y}{x}\right)$$

depending on z to argue that $\operatorname{Arg} z$ is C^1 on a local neighborhood of z. Hence, u is C^1 on D.

Write $f(z) = \ln r + i\theta$ in polar coordinates. $f_r = -if_\theta/r$ is the Cauchy–Riemann equation in the polar form. $f_x = f_r \cos\theta - f_\theta \frac{\sin\theta}{r}$; $-if_y = -i\left(f_r \sin\theta + \frac{\cos\theta}{r}\right)$. $f'(z) = f_x = \frac{1}{r}(\cos\theta - i\sin\theta) = \frac{1}{re^{i\theta}} = \frac{1}{z}$.

Similarly, with some fixed branch cut, $\log z$ is also analytic on $\mathbb{C}$ except for the ray of branch cut.

Chapter 4

Power Series

4.1 Quick Review

Some review on definitions of convergence, Cauchy sequence, completeness, Euclidean norm, Banach space, pointwise and uniform convergence, series.

Definition 4.1.1: Power Series

Given a complex sequence $\langle a_n \rangle_{n \in \mathbb{Z}_{>0}}$,

(i) $\sum_{n=0}^{\infty} a_n z^n$ is called a *Maclaurin series*.

(ii) $\sum_{n=0}^{\infty} a_n (z-b)^n$ is called a *Taylor series* centered at b.

Theorem 4.1.2

Let $P(z) = \sum_{n=0}^{\infty} a_n z^n$. If the series converges at $z = z_0$, then $P(z)$ is convergent and analytic in $B_{|z_0|}(0)$.

Proof. Note that $|a_n z^n|$ is bounded, say, by $M > 0$. Thus, for any z with $|z| < |z_0|$,

$$\sum_{n=0}^{\infty} |a_n z^n| = \sum_{n=0}^{\infty} |a_n||z_0|^n \left(\frac{|z|}{|z_0|} \right)^n \leq M \sum_{n=0}^{\infty} \left(\frac{|z|}{|z_0|} \right)^n;$$

hence $\sum_{n=0}^{\infty} a_n z^n$ (absolutely) converges. Similarly, fixing $\varepsilon \in (0, |z_0|)$, for $0 < n < m$ and $|z| \leq |z_0| - \varepsilon$, we have

$$\left| \sum_{j=n+1}^{m} a_n z^n \right| \leq M \sum_{j=n+1}^{m} \left(\frac{|z|}{|z_0|} \right)^j \leq M \sum_{j=n+1}^{m} \left(1 - \frac{\varepsilon}{|z_0|} \right)^j \to 0$$

as $n, m \to \infty$. Hence, $P(z)$ uniformly converges on $D = \{ z \in \mathbb{C} \colon |z| \leq |z_0| - \varepsilon \}$.

We now prove the analyticity. Fix z_1 with $|z_1| < |z_0|$ and any $\varepsilon \in \mathbb{R}_{>0}$.

$$L(z) := \frac{P(z) - P(z_0)}{z - z_0} - \sum_{n=1}^{\infty} a_n z^{n-1}.$$

Take any $\varepsilon \in \mathbb{R}_{>0}$. Let $P_k(z)$ be the kth partial sum of $P(z)$. Then, we have

$$L(z) = \underbrace{\frac{P_k(z) - P_k(z_1)}{z - z_0} - P_k'(z_1)}_{\mu_k(z)} + \underbrace{\sum_{n=k+1}^{\infty} a_n \left(\frac{z^n - z_1^n}{z - z_1} - n z_1^{n-1} \right)}_{\omega_k(z)}.$$

for any k. Now, we have

$$|\omega_k(z)| = \left| \sum_{n=k+1}^{\infty} a_n \left(\sum_{\ell=0}^{n-1} z^\ell z_1^{n-1-\ell} - n z_1^{n-1} \right) \right| \leq \sum_{n=k+1}^{\infty} |a_n| \cdot 2n(\max\{z, z_1\})^{n-1}$$

$\square$

Corollary 4.1.3

Let $P(z) = \sum_{n=0}^{\infty} a_n z^n$. If the series diverges at $z = z_0$, then $P(z)$ is divergent at z for all $|z| > |z_0|$.

Corollary 4.1.4

Let $P(z) = \sum_{n=0}^{\infty} a_n z^n$. Let

$$R \triangleq \sup \left\{ |z| : \sum_{n=0}^{\infty} |a_n||z|^n \text{ converges} \right\}.$$

Then,

(i) $P(z)$ converges absolutely in $|z| < R$;

(ii) $P(z)$ converges unifomrly in $|z| \leq r$ for $0 < r < R$;

(iii) $P(z)$ diverges for $|z| > R$.

Example 4.1.5

Given a power series $P(z) = \sum_{n=0}^{\infty} a_n z^n$, if the radius of the convergence is R, what is the radius of convergence of

$$\sum_{n=1}^{\infty} n a_n z^{n-1}?$$

We have

$$\begin{aligned} \limsup_{n\to\infty} \sqrt[n]{n|a_n||z|^{n-1}} &= \limsup_{n\to\infty} \sqrt[n]{n}\sqrt[n]{|a_n|}|z|^{1-1/n} \\ &= |z| \limsup_{n\to\infty} \sqrt[n]{|a_n|}. \end{aligned}$$

Hence, by the root test, the radius of convergence is R.

Corollary 4.1.6

If $P(z) = \sum_{n=0}^{\infty} a_n z^n$ has the radius of convergence $R > 0$, then

$$a_n = \frac{P^{(n)}(0)}{n!}$$

for $n \geq 0$.

Corollary 4.1.7

If $\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n$ in some open neighborhood of 0, then $a_n = b_n$ for all $n \geq 0$.

Chapter 5

Complex Integration

Chapter 6

Conformal Mapping

End.