MAS331 위상수학
Notes

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Contents

Chapter 1

Connectedness and Compactness

1.1 Connected Space

Definition 1.1.1: Separation and Connectedness

Let X be a topological space. A *separation* of X is a pair U and V of subsets of X which satisfy the following.

(i) U and V are open in X.

(ii) $U \cap V = \varnothing$.

(iii) $U \cup V = X$.

The space X is said to be *connected* if there does not exist a separation of X.

Note:-

Connectedness ia a topological property.

Note:-

A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty sets and X itself.

Lemma 1.1.1

If Y is a subspace of X, $A, B \subseteq Y$ is a separation of Y if and only if $A \cap B = \varnothing$, $A \cup B = Y$, and neither A nor B contains a limit point of the other.

Proof. Suppose A and B form a separation of Y. Then, A is both open and closed in Y; thus the closure of A in Y is $\overline{A} \cap Y = A$ by **??**. In other words, $\overline{A} \cap B = \varnothing$. Similarly, $A \cap \overline{B} = \varnothing$. ✓

Suppose A and B are disjoint subsets of Y whose union is Y and $A \cap B' = A' \cap B = \varnothing$. Thus, $A \cap \overline{B} = \overline{A} \cap B = \varnothing$. This implies $\overline{A} \cap Y = A$ and $\overline{B} \cap Y = B$; A and B are closed in Y, and thus they are open in Y as well. □

Lemma 1.1.2

If the sets C and D form a separation of a space X, and if Y is a connected subspace of X, then Y lies entirely within C or D.

Proof. $C \cap Y$ and $D \cap Y$ are open in Y. Also, $(C \cap Y) \cup (D \cap Y) = (C \cup D) \cap Y = Y$. If they were both unempty, they would form a separation of Y. Thus, one of them is empty; Y is entirely in the other. □

Theorem 1.1.1

Let X be a topological space. Let $\{A_\alpha\}_{\alpha\in J}$ be a family of connected subspaces of X. If $\bigcap_{\alpha\in J} A_\alpha \neq \varnothing$, then $\bigcup_{\alpha\in J} A_\alpha$ is connected.

Proof. Take any $p \in \bigcap_{\alpha\in J} A_\alpha$. Suppose C and D form a separation of $Y = \bigcup_{\alpha\in J} A_\alpha$. WLOG, $p \in C$. For each $\alpha \in J$, since $p \in C \cap A_\alpha$, by Lemma 1.1.2, $A_\alpha \subseteq C$. Thus, $\bigcup_{\alpha\in J} A_\alpha \subseteq C$, contradicting that $D \cap Y \neq \varnothing$. □

Theorem 1.1.2

Let A be a connected subspace of X. If $A \subseteq B \subseteq \overline{A}$, then B is also connected.

Proof. Suppose $B = C \cup D$ is a separation of B for the sake of contradiction. By Lemma 1.1.2, WLOG, $A \subseteq C$. Then, $B \subseteq \overline{A} \subseteq \overline{C}$. Since $\overline{C} \cap D = \varnothing$ by Lemma 1.1.1, $B \cap D = \varnothing$, which makes C and D not form a separation, #. □

Theorem 1.1.3 Connected Space and Continuous Map

Let $f : X \to Y$ be a continuous map. If X is connected, then $\operatorname{Im} f$ is connected.

Proof. Note that the surjective map $g : X \to \operatorname{Im} f$ obtained by restricting the codomain of f is also continuous by **??**. Suppose $\operatorname{Im} f = A \cup B$ is a separation of $\operatorname{Im} f$. Then, $g^{-1}(A)$ and $g^{-1}(B)$ are open and disjoint sets in X whose union is X, which is a contradiction to the connectedness of X. □

Theorem 1.1.4 Connected Space and Finite Product

Let $\{X_i\}_{i=1}^n$ be a finite family of connected spaces. then,

$$X = \prod_{i=1}^{n} X_i$$

is connected.

Proof. It is enough to prove for two connected spaces X and Y; extension to finite products can be done inductively. We may assume X and Y are nonempty. Take any $a \times b \in X \times Y$. Let $x \in X$. $X \times \{b\}$ and $\{x\} \times Y$ as subspaces of $X \times Y$ are connected since they are homeomorphic with X and Y, respectively. Thus,

$$T_x = (X \times \{b\}) \cup (\{x\} \times Y)$$

is connected by Theorem 1.1.1, having $x \times b$ as a common point of two spaces. Thus,

$$X \times Y = \bigcup_{x\in X} T_x$$

is connected as they have a point $a \times b$ in common. □

Theorem 1.1.5 Connected Space and Product Topology

Let $\{X_\alpha\}_{\alpha\in J}$ be a family of connected spaces. Then,

$$X = \prod_{\alpha\in J} X_\alpha$$

is connected in the product topology.

Proof. We may assume that $X_\alpha \neq \varnothing$ for each $\alpha \in J$. Let $\mathbf{a} = (a_\alpha)_{\alpha\in J}$ be a fixed point of X.

We first note that, given any finite subset K of J, $X_K \triangleq \{(x_\alpha)_{\alpha\in J} \mid \forall \alpha \in J \setminus K,\ x_\alpha = a_\alpha\}$ is a connected subspace of X as X_K is homeomorphic with $\prod_{\alpha\in K} X_\alpha$, which is connected by Theorem 1.1.4. Note that $Y \triangleq \bigcup\{X_K \mid K \subseteq J \text{ and } K \text{ is finite}\}$ as a subspace of X is connected since $\mathbf{a} \in X_K$ for every finite $K \subseteq J$.

Let $\mathbf{x} \in X$ and $B = \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i})$ be any basis that contains $\mathbf{x}$ where $\alpha_i \in J$ for each $i \in [n]$. Define $\mathbf{x}' \in X$ be

$$(\mathbf{x}')_\alpha \triangleq \begin{cases} x_\alpha & \text{if } \alpha = \alpha_i \text{ for some } i \in [n] \\ a_\alpha & \text{otherwise.} \end{cases}$$

Then, $\mathbf{x}' \in B \cap Y$. Thus, by **??**, $\overline{Y} = X$. By Theorem 1.1.2, X is connected. □

Example 1.1.1 ($\mathbb{R}^\omega$ in the Box Topology is Disconnected)

Let

$$\begin{aligned} A &= \{\mathbf{x} \in \mathbb{R}^\omega \mid \mathbf{x} \text{ is bounded}\} \text{ and} \\ B &= \{\mathbf{x} \in \mathbb{R}^\omega \mid \mathbf{x} \text{ is unbounded}\}. \end{aligned}$$

If $\mathbf{a}$ is in either A or B, $\prod_{i\in\mathbb{Z}_+}(a_i - 1, a_i + 1)$ is an open set that is contained in either A or B. Thus, each A and B are disjoint open sets in $\mathbb{R}^\omega$ whose union is $\mathbb{R}^\omega$.

1.2 Connected Subspaces of the Real Line

Definition 1.2.1: Linear Continuum

A simply ordered set L having more than one element is called *linear continuum* if the following hold:

(i) L has the least upper bound property.

(ii) $\forall x, y \in L$, $(x < y \implies \exists z \in L,\ x < z < y)$.

Note:-

$\mathbb{R}$ is a linear continuum.

Example 1.2.1 (The Ordered Square is a Linear Continuum)

Let $I = [0,1]$ and $I_0^2 = I \times I$ be the ordered square with the dictionary ordering.

(i) Let $\varnothing \neq A \subseteq I_0^2$ and $\pi_1 : I_0^2 \to I$ be the projection onto its first factor. Then, $\pi_1(A)$ is bounded above by 1. Let $b \triangleq \sup \pi_1(A)$. ($[0,1]$ has l.u.b. property.)
If $b \in A$, it implies that $A \cap (\{b\} \times I) \neq \varnothing$ and is bounded above by 1. Thus, we may let $c \triangleq \sup\big(A \cap (\{b\} \times I)\big)$. One may readily check that $\sup A_0 = b \times c$.
If $b \neq A_0$, then $b \times 0$ is the trivial least upper bound of A_0. ✓

(ii) Suppose $x_1 \times y_1 < x_2 \times y_2$. If $x_1 < x_2$, then $x_1 \times y_1 < (x_1 + x_2)/2 \times 0 < x_2 \times y_2$. If $x_1 = x_2$, then, $x_1 \times y_1 < x_1 \times (y_1 + y_2)/2 < x_2 \times y_2$. ✓

Theorem 1.2.1

If L is a linear continuum in the order topology, any convex subspace of L is connected.

Proof. Let Y be a convex subspace of L. Suppose $Y = A \cup B$ is a separation of Y for the sake of contradiction. Take any $a \in A$ and $b \in B$. WLOG, $a < b$. $[a,b] \subseteq Y$ as Y is convex, and $[a,b]$ as a subspace of Y is exactly $[a,b]$ in the order topology by **??**. Hence,

$$A_0 \triangleq A \cap [a,b] \quad \text{and} \quad B_0 \triangleq B \cap [a,b]$$

form a separation of $[a,b]$.

Let $c \triangleq \sup A_0$. Then, $c \geq a$ as $a \in A_0$, and $c \leq b$ as, if c were larger than b, there would be $z \in L$ such that $b < z < c$, which is an upper bound of A_0 smaller than c. However, we claim that $c \notin A_0 \cup B_0 = [a,b]$, which leads to a contradiction.

($c \notin A_0$) Suppose $c \in A_0$ for the sake of contradiction. Since A_0 is open in $[a,b]$, there must exist $e \in (c,b]$ such that $[c,e) \subseteq A_0$. (e cannot be larger than b as $b \notin A_0$.) As the existence of $e' \in (c,e) \cap L$ is guaranteed and such e' is in A_0, c is no longer an upper bound of A_0, #.

($c \notin B_0$) Suppose $c \in B_0$ for the sake of contradiction. Since B_0 is open in $[a,b]$, there exists $e \in [a,c)$ such that $(e,c] \subseteq B_0$. (e cannot be smaller than a as $a \notin B_0$.) Since, $(c,\infty) \cap A_0 = \varnothing$ as c is the supremum of A_0, e is an upper bound of A_0 that is smaller than c, #. □

Corollary 1.2.1

$\mathbb{R}$ and intervals and rays in $\mathbb{R}$ are connected.

Theorem 1.2.2 Intermediate Value Theorem

Let X be a connected space and Y has an order topology. Let $f : X \to Y$ be a continuous map. Then, if $a, b \in X$ and $r \in Y$ satisfy $f(a) \leq r \leq f(b)$, there exists $c \in X$ such that $f(c) = r$.

Proof. If $f(a) = r$ or $f(b) = r$, then done. So suppose $f(a) < r < f(b)$. $\operatorname{Im} f$ is connected by Theorem 1.1.3. Let

$$A \triangleq \operatorname{Im} f \cap (-\infty, r) \quad \text{and} \quad B \triangleq \operatorname{Im} f \cap (r, \infty).$$

Then, A and B are open in $\operatorname{Im} f$ and $f(a) \in A$ and $f(b) \in B$. Thus, it cannot happen that $\operatorname{Im} f \setminus \{r\} = A \cup B = \operatorname{Im} f$ since $\operatorname{Im} f$ is connected. Therefore, $r \in \operatorname{Im} f$. □

Definition 1.2.2: Path and Path Connectedness

Let X be a space. Given $x, y \in X$, a *path* in X from x to y is a continuous map $f : [a,b] \to X$ where $[a,b]$ is a subspace of $\mathbb{R}$, $f(a) = x$, and $f(b) = y$. The space X is *path connected* if there exists a path in X from x to y for every $x, y \in X$.

Example 1.2.2 (Punctured Euclidean Space)

Define *punctured Euclidean space* to be the space $\mathbb{R}^n \setminus \{\mathbf{0}\}$, where $\mathbf{0}$ is the origin in $\mathbb{R}^n$. If $n > 1$, the space is path connected. We can join $\mathbf{x}$ and $\mathbf{y}$ by the line segment that has x and y as endpoints if the segment does not go through $\mathbf{0}$. Otherwise, we may choose a point $\mathbf{x}'$ by flipping the sign of a coordiate of $\mathbf{x}$. We have a line that connects $\mathbf{x}$ and $\mathbf{x}'$ and other line that connects $\mathbf{x}'$ and $\mathbf{y}$.

Theorem 1.2.3

Every path connected space is connected.

Proof. Let X be a path connected space. If $X = \varnothing$, it is done; let $X \neq \varnothing$. Take $x \in X$. For each $y \in X$, let $f_y : [0,1] \to X$ be a path from x to y. Since $[0,1]$ is connected (Corollary 1.2.1), $\operatorname{Im} f_y$ is connected by Theorem 1.1.3. As $x \in \bigcap_{y \in X} \operatorname{Im} f_y$, $X = \bigcup_{y \in X} \operatorname{Im} f_y$ is connected by Theorem 1.1.1. $\square$

Example 1.2.3 (Connectedness Does Not Imply Path Connectedness)

By Example 1.2.1, I_0^2 is connected. Suppose I_0^2 is path connected for the sake of contradiction. Then, there is a path $f : [0,1] \to I_0^2$ from 0×0 to 1×1. Theorem 1.2.2 says that $\operatorname{Im} f = I_0^2$. For each $x \in I$, let $U_x = f^{-1}(\{x\} \times I)$. Note that $U_x \neq \varnothing$. Since each U_x is open as f is continuous, by the denseness of $\mathbb{Q}$ in $\mathbb{R}$, there exists $q_x \in U_x \cap \mathbb{Q}$ for each $x \in X$. This implies the existence of a injection $g : I \to \mathbb{Q}$ defined by $x \mapsto q_x$, which is a contradiction as I is uncountable. (**??**)

Theorem 1.2.4 Path Connected Space and Continuous Map

Let $f : X \to Y$ be a continuous map. If X is path connected, then $\operatorname{Im} f$ is path connected.

Proof. Take $y_1, y_2 \in \operatorname{Im} f$. There exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X is connected, there exists a continuous map $g : [0,1] \to X$ such that $g(0) = x_1$ and $g(1) = x_2$. Then, $f \circ g : [0,1] \to \operatorname{Im} f$ is a continuous map such that $(f \circ g)(0) = y_1$ and $(f \circ g)(1) = y_2$ by **??**. $\square$

Example 1.2.4 (Unit Sphere)

Define the *unit sphere* S^{n-1} in $\mathbb{R}^n$ by the equation

$$S^{n-1} \triangleq \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| = 1\}.$$

Then, the map $g : \mathbb{R}^n \setminus \{\mathbf{0}\} \to S^{n-1}$ defined by $\mathbf{x} \mapsto \mathbf{x}/\|\mathbf{x}\|$ is a continuous surjective map. Moreover, if $n > 1$, since $\mathbb{R}^n \setminus \{0\}$ is path connected (Example 1.2.2), $S^{n-1} = \operatorname{Im} g$ is also path connected by Theorem 1.2.4.

Example 1.2.5 (Topologist's Sine Curve)

Let

$$S \triangleq \left\{ x \times \sin\frac{1}{x} \in \mathbb{R}^2 \,\middle|\, x \in (0,1] \right\}.$$

Since S is a image of $(0,1]$ under a continuous map $x \mapsto x \times \sin(1/x)$, S is (path) connected. Thus, $\overline{S}$ is connected by Theorem 1.1.2. Note that $S_0 \triangleq \overline{S} \setminus S = \{0\} \times [-1,1]$. ($S_0$ is also closed.)

Suppose $\overline{S}$ is path connected for the sake of contradiction. Then, there is a path $f : [0,1] \to \overline{S}$ from 0×0 to $f(1) \in S$. $f^{-1}(S_0)$ is closed in $[0,1]$ by **??**. Hence $b \triangleq \sup f^{-1}(S_0) \in f^{-1}(S_0)$ and $b \neq 1$. $f(b) \in S_0$ and $f((b,1]) \subseteq S$.

Reparametrize $f : [0,1] \to \overline{S}$ so that $t \mapsto x(t) \times y(t)$; $f(0) \in S_0$ and $f((0,1]) \subseteq S$. ($y(t) = \sin(1/x(t))$) Since $x(t) > 0$ for $t \in (0,1]$, x is continuous, and $x(0) = 0$, we may construct a sequence $\{t_n\}_{n \in \mathbb{Z}_+}$ such that

$$\lim_{n\to\infty} t_n = 0, \quad x(t_n) = \frac{1}{(n+1/2)\pi}, \quad \text{and thus}$$

$$y(t_n) = \sin(1/x(t_n)) = \sin\big((n+1/2)\pi\big) = (-1)^n.$$

However, $\{y(t_n)\}_{n\in\mathbb{Z}_+}$ diverges although y is continuous and $t_n \to 0$. Thus, $\overline{S}$ is not path connected.

1.3 Components and Local Connectedness

Definition 1.3.1: Component

Given a space X, let $\sim$ be a equivalent relation defined by

$$x \sim y \text{ if there is a connected subspace of } X \text{ containing } x \text{ and } y.$$

The equivalence classes of $\sim$ is called *(connected) components* of X.

Note:-

Reflexivity follows from the fact that $\{x\}$ is a connected subspace of X that contains x. Symmetry is direct.
Let $x, y, z \in X$ and suppose $x \sim y$ and $y \sim z$. There are connected subspaces U and V such that $x, y \in U$ and $y, z \in V$. Then, $U \cup V$ is a connected subspace of X that contains both x and z by Theorem 1.1.1.

Note:-

Let $\{C_\alpha\}_{\alpha\in J}$ be the set of components of X. Then, it is a partition of X (indeed).

Theorem 1.3.1

Let $\{C_\alpha\}_{\alpha\in J}$ be the set of components of X. If $A \subseteq X$ is a connected subspace of X, then $A \subseteq C_\alpha$ for some $\alpha \in J$.

Proof. If $A = \varnothing$, it is done; suppose $A \neq \varnothing$.

Let C_α and C_β be connected components. If $A \cap C_\alpha \neq \varnothing$ and $A \cap C_\beta \neq \varnothing$, we may take $x \in A \cap C_\alpha$ and $y \in A \cap C_\beta$, which makes $x \sim y$. This implies $x_\alpha \sim x_\beta$ for all $x_\alpha \in C_\alpha$ and $x_\beta \in C_\beta$; thus $C_\alpha = C_\beta$.

Now, take any $a \in A$. Since $\{C_\alpha\}_{\alpha\in J}$ is a partition of X, there exists some $\alpha \in J$ such that $a \in C_\alpha$. By the previous result, $A \cap C_\beta = \varnothing$ for all $\beta \in J \setminus \{\alpha\}$. Hence, $A \subseteq C_\alpha$ □

Theorem 1.3.2

Let $\{C_\alpha\}_{\alpha\in J}$ be the set of components of X. Then, for each $\alpha \in J$, C_α is connected.

Proof. Take any $x_0 \in C_\alpha$. Then, for each $x \in C_\alpha$, there exists a connected subspace A_x that contains both x_0 and x. By Theorem 1.3.1, $A_x \subseteq C_\alpha$. Thus, $C_\alpha = \bigcup_{x\in C_\alpha} A_x$, which is connected by Theorem 1.1.1. □

Definition 1.3.2: Path Component

Given a space X, let $\sim$ be a equivalent relation defined by

$$x \sim y \text{ if there is a path in } X \text{ from } x \text{ to } y.$$

The equivalence classes of $\sim$ is called *path components* of X.

Note:-

The relation is reflexive since $f:[0,1]\to X$ defined by $f(t)=x$ is a path from x to x.
The relation is symmetric since, if $f:[a,b]\to X$ is a path from x to y, then $g:[a,b]\to X$ defined by $g(t)=f(a+b-t)$ is a path from y to x.
The relation is transitive since, if $f:[a,b]\to X$ and $g:[c,d]\to X$ are paths from x to y and from y to z, respectively, then $h:[a,b+d-c]$ defined by

$$h(t)=\begin{cases} f(t) & \text{if } a\le t\le b \\ g(t-b+c) & \text{otherwise.} \end{cases}$$

is a path from x to z. h is continuous by **??**.

Theorem 1.3.3

Let $\{P_\alpha\}_{\alpha\in J}$ be the set of path components of X. If $A\subseteq X$ is a path connected subspace of X, then $A\subseteq P_\alpha$ for some $\alpha\in J$.

Proof. Analogous to the proof of Theorem 1.3.1. □

Theorem 1.3.4

Let $\{P_\alpha\}_{\alpha\in J}$ be the set of path components of X. Then, for each $\alpha\in J$, P_α is path connected.

Proof. Analogous to the proof of Theorem 1.3.2. □

Corollary 1.3.1

Every path component is entirely contained in a connected component.

Proof. Every path component is path connected by Theorem 1.3.4, and thus connected by Theorem 1.2.3. By Theorem 1.3.1, it is contained in some connected component. □

Corollary 1.3.2

Every component is closed.

Proof. Let C_α be a connected component of X. Since $\overline{C_\alpha}$ is connected by Theorem 1.1.2, and since $\overline{C_\alpha}\cap C_\alpha\neq\varnothing$, $\overline{C_\alpha}\subseteq C_\alpha$ by Theorem 1.3.1. □

Corollary 1.3.3

If there are a finite number of components, then each component is open.

Proof. Let $X=\bigcup_{i=1}^n C_i$ where each C_i is a component. Then, for each $i\in[n]$, $C_i=X\setminus\bigcup_{j\in[n]\setminus\{i\}} C_j$. C_i is open as $\bigcup_{j\in[n]\setminus\{i\}} C_j$ is closed by Corollary 1.3.2. □

Example 1.3.1 (Path Component Is Not Necessarily Open or Closed)

Let $\overline{S}$ be the topologist's sine curve discussed in Example 1.2.5. Then, S and S_0 are the two path components of $\overline{S}$. S is not closed and S_0 is not open.

Example 1.3.2

Let $A \triangleq S \cup (S_0 \setminus \{0\} \times \mathbb{Q})$. Since $S \subseteq A \subseteq \overline{S}$, A is connected by Theorem 1.1.2. However, $\{0 \times r\}$ for every $r \in [0,1] \setminus \mathbb{Q}$ is a path component. Thus, A has uncountably many path components.

Definition 1.3.3: Locally Connected Space

Let X be a topological space. X is *locally connected at* x if, for any neighborhood U of x, there exists a connected neighborhood V of x such that $x \in V \subseteq U$. X is *locally connected* if X is locally connected at every point of X.

Definition 1.3.4: Locally Path Connected Space

Let X be a topological space. X is *locally path connected at* x if, for any neighborhood U of x, there exists a path connected neighborhood V of x such that $x \in V \subseteq U$. X is *locally path connected* if X is locally path connected at every point of X.

Note:-

If a topological space X is locally path connected, then it is locally connected as well.

Theorem 1.3.5

A topological space X is locally connected if and only if, for every open set U in X, each connected component of U is open.

Proof. ($\Rightarrow$) Let U be open in X and let $\{C_\alpha\}_{\alpha \in J}$ be the set of components of U. Take any C_α and let $x \in C_\alpha$. Since X is locally connected at x, there exists a connected neighborhood V of x such that $x \in V \subseteq U$. By Theorem 1.3.1, $x \in V \subseteq C_\alpha$. This proves that C_α is open.

($\Leftarrow$) Let $x \in X$ and U be a neighborhood of x. Let $\{C_\alpha\}_{\alpha \in J}$ be the components of U. There exists some $\alpha_0 \in J$ such that $x \in C_{\alpha_0}$. Since C_{α_0} is open by assumption, C_{α_0} is a connected neighborhood of x and satisfies $x \in C_{\alpha_0} \subseteq U$. $\square$

Theorem 1.3.6

A topological space X is locally path connected if and only if, for every open set U in X, each path component of U is open.

Proof. Analogous to Theorem 1.3.5. $\square$

Theorem 1.3.7

Let X be a locally path connected space. Then, the connected components and the path components are the same.

Proof. Let C be a connected component of X. C is open by Theorem 1.3.5 as X is locally connected. Let $x \in C$ and let P be the path component which x is contained in. Then, $P \subseteq C$ by Corollary 1.3.1.

Suppose $P \subsetneq C$ for the sake of contradiction. Let

$$Q \triangleq \bigcup \{\, \hat{P} \subseteq C \mid \hat{P} \text{ is a path component of } X \text{ and } \hat{P} \neq P \,\}.$$

Since path component of an open set, especially, C, is open by Theorem 1.3.6, P and Q are open. Moreover, since $C = P \cup Q$, they form a separation of C, which is a contradiction, #. $\square$

1.4 Compact Spaces

Definition 1.4.1: Open Cover

A collection $\mathcal{A}$ of subsets of a space X is said to *cover* X, or to be a *covering* of X, if $\bigcup \mathcal{A} = X$. It is called an *open covering* if A is open in X for each $A \in \mathcal{A}$.

Definition 1.4.2: Compactness

A space X is said to be *compact* if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Example 1.4.1 (ℝ Is Not Compact)

The open cover $\mathcal{A} \triangleq \{(n, n+2) \mid n \in \mathbb{Z}\}$ does not have a finite subcollection that covers $\mathbb{R}$. Thus, $\mathbb{R}$ is not compact.

Lemma 1.4.1

Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.

Proof. ($\Rightarrow$) Let $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is a covering of Y by open sets in X. Then, the collection

$$\{A_\alpha \cap Y \mid \alpha \in J\}$$

is an open covering of Y. Thus, there exists a finite subcollection

$$\{A_{\alpha_1} \cap Y, \cdots, A_{\alpha_n} \cap Y\}$$

that covers Y. Then, $\{A_{\alpha_1}, \cdots, A_{\alpha_n}\}$ is a finite subcollection of $\mathcal{A}$ that covers Y.

($\Leftarrow$) Let $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ be an open covering of Y. For each $\alpha \in J$, there is an open set $\hat{A}_\alpha$ in X such that $A_\alpha = \hat{A}_\alpha \cap Y$. Then, the collection $\{\hat{A}_\alpha\}_{\alpha \in J}$ composed of open sets in X that covers Y; by the assumption, there exists a fintie subcollection

$$\{\hat{A}_{\alpha_1}, \cdots, \hat{A}_{\alpha_n}\}$$

that covers Y. Then, $\{A_{\alpha_1}, \cdots, A_{\alpha_n}\}$ is a finite subcollection of $\mathcal{A}$ that covers Y. □

Theorem 1.4.1

Let X be a compact space. If Y is a closed subset of X, then Y as a subspace of X is compact.

Proof. If $Y = \varnothing$, then it is done. So, suppose $Y \neq \varnothing$. Let $\mathcal{A}$ be a covering of Y composed of sets open in X.

$$\mathcal{B} \triangleq \mathcal{A} \cup \{X \setminus Y\}$$

is an open covering of X. Thus, it has a finite subcollection

$$\{A_1, A_2, \cdots, A_n, X \setminus Y\}$$

that covers X where $A_i \in \mathcal{A}$ for each $i \in [n]$. (WLOG, $X \setminus Y$ is in the subcollection since we may just add $X \setminus Y$ and does not affect its finiteness.) Then, $\{A_i\}_{i \in [n]}$ is a finite subcollection of $\mathcal{A}$ that covers Y. □

Theorem 1.4.2

Let X be a Hausdorff space. If $Y \subseteq X$ is a compact subspace of X, then Y is closed in X.

Proof. If $Y = \varnothing$ or $Y = X$, then it is done; suppose $\varnothing \neq Y \subsetneq X$. Let $x_0 \in X \setminus Y$. For each $y \in Y$, there are disjoint neighborhoods U_y and V_y of x_0 and y in X. Then, $\{V_y\}_{y \in Y}$ is an open covering of Y. Thus there exists a finite subcollection of it

$$\{V_{y_1}, V_{y_2}, \cdots, V_{y_n}\}$$

that covers Y.

Let

$$V \triangleq \bigcup_{i=1}^{n} V_{y_i} \quad \text{and} \quad U \triangleq \bigcap_{i=1}^{n} U_{y_i}.$$

Then, U is a neighborhood of x_0 and does not intersect V, which covers Y. Thus, $U \subseteq X \setminus Y$. Hence, $X \setminus Y$ is open; Y is closed. □

Example 1.4.2 (Being Hausdorff Is Needed)

Let $X = \mathbb{R}$ be endowed with the finite complement topology. Then, every subset of X is compact. To see this, suppose $\mathcal{A}$ is a collection of open sets in X that covers $Y \subseteq X$. Then, take any $A \in \mathcal{A}$ and it will cover all but finitely many points in Y. For each remaining point, choose an open set in $\mathcal{A}$ that contains the point. Thus, we get a finite collection of $\mathcal{A}$ that covers Y. However, only closed sets are finite subsets of X and $\mathbb{R}$.

Corollary 1.4.1

Let X be a Hausdorff space. If $Y \subseteq X$ is a compact subspace of X, then, given any $x_0 \in X \setminus Y$, there are disjoint open sets U and V in X containing x_0 and Y, respectively.

Proof. U and V defined in the proof of Theorem 1.4.2 are those. □

Theorem 1.4.3

Let X be a compact space. Let $f : X \to Y$ be a continuous map. Then, $\operatorname{Im} f$ as a subspace of Y is compact.

Proof. Let $\mathcal{A}$ be a covering of the set $\operatorname{Im} f$ by sets open in Y. Then, the collection

$$\{f^{-1}(A) \mid A \in \mathcal{A}\}$$

is an open covering of X as f is continuous. Hence, there are a finite subcollection $\{A_1, \cdots, A_n\}$ of $\mathcal{A}$ such that $\{f^{-1}(A_i)\}_{i \in [n]}$ covers X. The sets $\{A_1, \cdots, A_n\}$ covers $\operatorname{Im} f$. □

Theorem 1.4.4

Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. We only need to prove f^{-1} is continuous. Let $A \subseteq X$ is closed in X. Then, A is compact by Theorem 1.4.1. Thus, since $f\big|_A : A \to Y$ is continuous (**??**), $f(A)$ is compact by Theorem 1.4.3. By Theorem 1.4.2, $f(A)$ is closed. Hence, we proved that $f(A)$ is closed for each closed subset A of X; f^{-1} is continuous by **??**. □

Lemma 1.4.2 The Tube Lemma

Let X and Y be topological spaces and Y is compact. Given any $x_0 \in X$ and an open set N in $X \times Y$ that contains $\{x_0\} \times Y$, there exists a neighborhood W of x_0 in X such that $W \times Y \subseteq N$.

Proof. For each $y \in Y$, there exists a basis element $U_y \times V_y$ in the product topology such that $x_0 \times y \in U_y \times V_y \subseteq N$. Then, $\mathcal{A} \triangleq \{ U_y \times V_y \mid y \in Y \}$ is a covering of $\{x_0\} \times Y$ by open sets in $X \times Y$. Since $\{x_0\} \times Y$, being homeomorphic with Y, is compact, there is a finite subcollection

$$\mathcal{A}' = \{ U_{y_1} \times V_{y_1}, \cdots, U_{y_n} \times V_{y_n} \}$$

of $\mathcal{A}$ that covers $\{x_0\} \times Y$. Note that $\{x_0\} \times Y \subseteq \bigcup_{i=1}^{n} (U_{y_i} \times V_{y_i}) \subseteq N$. Let

$$W \triangleq \bigcap_{i=1}^{n} U_{y_i}.$$

Then, W is a neighborhood of x_0 in X.

Now, take $x \times y \in W \times Y$. There exists some $i \in [n]$ such that $y \in V_{y_i}$; $x \times y \in U_{y_i} \times V_{y_i} \subseteq N$. This shows $W \times Y \subseteq N$. □

Note:-

The set $W \times Y$ is often called a *tube* about $x_0 \times Y$.

Note:-

Lemma 1.4.2 may not hold if Y is not compact. If $X = Y = \mathbb{R}$, the open set

$$N \triangleq \left\{ x \times y \in \mathbb{R}^2 \;\middle|\; |x| < \frac{1}{y^2 + 1} \right\}$$

does contain $\{0\} \times Y$ but there is no open neighborhood W of 0 in X such that $W \times Y \subseteq N$.

Theorem 1.4.5

Let $X_1, X_2, \cdots, X_n$ be topological spaces. Then, $\prod_{i=1}^{n} X_i$ is compact if and only if X_i is compact for each $i \in [n]$.

Proof. It is enough to prove for two topological spaces X and Y.

($\Rightarrow$) It is enough to prove X is compact. Let $\mathcal{A}$ be an open covering of X. Then, $\{A \times Y \mid A \in \mathcal{A}\}$ is an open covering of $X \times Y$; there exists a finite subcollection

$$\{A_1 \times Y, A_2 \times Y, \cdots, A_n \times Y\}$$

that covers $X \times Y$. Thus, $\{A_i \mid i \in [n]\}$ is a finite subcollection of $\mathcal{A}$ that covers X.

($\Leftarrow$) Let $\mathcal{A}$ be an open covering of $X \times Y$. For each $x \in X$, since $\{x\} \times Y$ is compact, there are finite subcollection $\{A_1, A_2, \cdots, A_{n_x}\} \subseteq \mathcal{A}$ that covers $\{x\} \times Y$. Then, $N_x \triangleq \bigcup_{i=1}^{n_x} A_i$ is an open set in $X \times Y$ that contains $\{x\} \times Y$. Thus, by Lemma 1.4.2, there exists a tube $W_x \times Y$ such that $\{x\} \times Y \subseteq W_x \times Y \subseteq N_x$.

Noting that $\{ W_x \mid x \in X \}$ is an open covering of X, there are finite subcover $\{ W_{x_1}, W_{x_2}, \cdots, W_{x_k} \}$ that covers X. Hence, $\{ W_{x_i} \times Y \mid i \in [k]\}$ covers $X \times Y$ and each element of it is covered by finite elements in $\mathcal{A}$. □

Note:-

Theorem 1.4.5 holds for an arbitrary product but is is slightly technical.

Definition 1.4.3: Finite Intersection Property

A collection $\mathcal{C}$ of subsets of X is said to have *finite intersection property* if, for any finite subcollection

$$\{C_1, C_2, \cdots, C_n\} \subseteq \mathcal{C}$$

of $\mathcal{C}$, we have

$$\bigcap_{i=1}^{n} C_i \neq \varnothing.$$

In other words,

$$\forall n \in \mathbb{Z}_+,\ \forall \mathcal{C}' \in \binom{\mathcal{C}}{n},\ \bigcap \mathcal{C}' \neq \varnothing.$$

Theorem 1.4.6

Let X be a topological space. Then X is compact if and only if, for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap \mathcal{C}$ is nonempty.

Proof. Given a collection $\mathcal{A}$ of subsets of X, let

$$\mathcal{C} \triangleq \{X \setminus A \mid A \in \mathcal{A}\}.$$

Then the following hold.

- $\mathcal{A}$ is a collection of open sets if and only if $\mathcal{C}$ is a collection of closed sets.
- $\bigcup \mathcal{A} = X$ if and only if $\bigcap \mathcal{C} = \varnothing$.
- The finite subcollection $\{A_1, \cdots, A_n\}$ covers X if and only if $\bigcap_{i=1}^{n} (X \setminus A_i) = \varnothing$.

Therefore, these are equivalent.

(i) Every open covering of X allows a finite subcover.

(ii) A collection of open sets in X that does not allow a finite subcover does not cover X. - contrapositive of (i)

(iii) A collection of closed sets in X that does not allow a nonempty intersection of finite subcollection does not have a nonempty intersection.

□

Definition 1.4.4: Nested Sequence

A sequence of sets $\{C_n\}_{n\in\mathbb{Z}_+}$ is called a *nested sequence* if $C_n \supseteq C_{n+1}$ for each $n \in \mathbb{Z}_+$.

Corollary 1.4.2

Let X be a compact space. Let $\{C_n\}_{n\in\mathbb{Z}_+}$ be a nested sequence of nonempty closed sets in X. Then,

$$\bigcap_{n\in\mathbb{Z}_+} C_n \neq \varnothing.$$

Proof. Let $\mathcal{C} \triangleq \{C_n \mid n \in \mathbb{Z}_+\}$. Then, $\mathcal{C}$ satisfies the finite intersection property as

$$C_{n_1} \cap C_{n_2} \cap \cdots \cap C_{n_k} = C_{\max_{i=1}^{k} n_i} \neq \varnothing.$$

The result follows from Theorem 1.4.6. □

1.5 Compact Subspaces of the Real Line

Theorem 1.5.1

Let X be a simply ordered set having the least upper bound property. In the order topology, every closed interval $[a, b]$ in X is compact.

Proof. Let $\mathcal{A}$ be an open covering of $[a, b]$.

We claim that, given any $x \in [a, b)$, there exists $y \in (x, b]$ such that $[x, y]$ can be covered by at most two elements of $\mathcal{A}$.

(i) If there exists an immediate successor $y \in (x, b]$ of x, then $[x, y] = \{x, y\}$. Pick two open sets in $\mathcal{A}$ that contain x and y, respectively.

(ii) Otherwise, let $A \in \mathcal{A}$ with $x \in A$. Then, $[x, c) \subseteq A$ for some $c \in (x, b]$ and $|[x, c)| = \infty$. Take any $y \in (x, c) \subseteq (x, b]$, then $[x, y] \subseteq [x, c) \subseteq A$. ✓

Let

$$C \triangleq \left\{ y \in (a, b] \,\middle|\, [a, y] \text{ can be covered by finitely many elements of } \mathcal{A} \right\}.$$

By the previous claim, $C \neq \varnothing$, and C is bounded above by b. Thus, we may let $c \triangleq \sup C$. ($a \leq c \leq b$, indeed.) ✓

Suppose $c \notin C$ for the sake of contradiction. Choose $A \in \mathcal{A}$ that contains c. Then, there exists $d \in [a, c)$ such that $(d, c] \subseteq A$. Hence, there exists $z \in C \cap (d, c]$. Since $z \in C$, the interval $[a, z]$ can be covered by finitely many, say n, elements of $\mathcal{A}$, then, since $[a, c] = [a, z] \cup [z, c]$ and $[z, c] \subseteq (d, c] \subseteq A$, $[a, c]$ can be covered by at most $n + 1$ elements of $\mathcal{A}$, which is contradicting to $c \notin C$, #. ✓

Suppose $c < b$ for the sake of contradiction. Then, there exists $y \in (c, b]$ such that $[c, y]$ can be covered by finitely many elements of $\mathcal{A}$ by the previous claim. Hence, $[a, y] = [a, c] \cup [c, y]$ can be covered by finitely many elements of $\mathcal{A}$ since $c \in C$. This implies $y \in C$, contradicting that c is an upper bound of C, #. ✓ □

Corollary 1.5.1

Every closed interval in $\mathbb{R}$ is compact.

Theorem 1.5.2

A subspace A of $\mathbb{R}^n$ is compact if and only if it is closed and it is bounded in the Euclidean metrid d or the square metric ρ.

Proof. It suffices to prove only for ρ as A is bounded in d if and only if A is bounded in ρ. (See the proof of **??**.)

($\Rightarrow$) By Theorem 1.4.2, A is closed. ✓

The collection

$$\{ B_\rho(\mathbf{0}, m) \mid m \in \mathbb{Z}_+ \}$$

is an open covering of A. Thus, $A \subseteq B_\rho(\mathbf{0}, M)$ for some M. Therefore, $\rho(\mathbf{x}, \mathbf{y}) \leq 2M$ for each $\mathbf{x}, \mathbf{y} \in A$. Thus, A is bounded. ✓

($\Leftarrow$) There exists $M \in \mathbb{R}_+$ such that $\rho(\mathbf{x}, \mathbf{y}) \leq M$ for each $\mathbf{x}, \mathbf{y} \in A$. Choose a point $\mathbf{x}_0 \in A$, and let $b \triangleq \rho(\mathbf{x}_0, \mathbf{0})$. Then, $\rho(\mathbf{x}, \mathbf{0}) \leq P \triangleq N + b$ for every $\mathbf{x} \in A$. Thus, $A \subseteq [-P, P]^n$. $[-P, P]^n$ is compact by Corollary 1.5.1, Theorem 1.4.5, and **??**. Since A is closed in $[-P, P]^n$ and $[-P, P]^n$ is compact, A is compact by Theorem 1.4.1. □

Theorem 1.5.3 Extreme Value Theorem

Let X be a compact set and Y be an ordered set endowed by the order topology. Let $f: X \to y$ be a continuous map. Then, there exist $c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in X$.

Proof. Suppose $\operatorname{Im} f$ does not have a maximum. Then,

$$\{(-\infty, a) \subseteq \mathbb{R} \mid a \in \operatorname{Im} f\}$$

is an open covering of $\operatorname{Im} f$. Since $\operatorname{Im} f$ is compact by Theorem 1.4.3, $\operatorname{Im} f \subsetneq (-\infty, a)$ for some $a \in \operatorname{Im} f$, #. □

Definition 1.5.1: Distance From a Point to a Set

Let (X, d) be a metric space and let $\varnothing \neq A \subseteq X$. For each $x \in X$, we define the *distance from x to A* by the equation

$$d(x, A) \triangleq \inf\{d(x, a) \mid a \in A\}.$$

Definition 1.5.2: Uniform Continuity

A function $f: X \to Y$ from the metrix space (X, d_X) to the metric space (Y, d_Y) is said to be *uniformly continuous* if

$$\forall \varepsilon \in \mathbb{R}_+, \exists \delta \in \mathbb{R}_+, \forall x_1, x_2 \in X, \big(d_X(x_1, x_2) < \delta \implies d_Y(f(x_1), f(x_2))\big).$$

Theorem 1.5.4

Let (X, d) be a metric space and let $\varnothing \neq A \subseteq X$. Then, $f: X \to \mathbb{R}$ defined by

$$f(x) \triangleq d(x, A)$$

is uniformly continuous.

Proof. Take any $\varepsilon \in \mathbb{R}_+$ and let $\delta \triangleq \varepsilon$. For any $x, y \in X$ and $a \in A$ with $d(x, y) < \varepsilon$, we have $d(x, A) \leq (x, a) \leq d(x, y) + d(y, a)$. Thus,

$$d(x, A) - d(x, y) \leq \inf_{a \in A} d(y, a) = d(y, A),$$

which implies $|d(x, A) - d(y, A)| \leq d(x, y) < \delta = \varepsilon$. □

Lemma 1.5.1 The Lebesgue Number Lemma

Let (X, d) be a compact metric space. Then, for each open covering $\mathcal{A}$ of X,

$$\exists \delta \in \mathbb{R}_+, \forall B \in \mathcal{P}(X) \setminus \{\varnothing\}, \big(\operatorname{diam} B < \delta \implies \exists A \in \mathcal{A}, B \subseteq A\big).$$

The number δ is called a *Lebesgue number* for the covering $\mathcal{A}$.

Proof. If $X \in \mathcal{A}$, then every $\delta \in \mathbb{R}_+$ satisfies the condition. Therefore, we may suppose $X \notin \mathcal{A}$.

Choose a finite subcollection $\{A_1, A_2, \cdots, A_n\}$ of $\mathcal{A}$ that covers X. For each $i \in [n]$, let $C_i \triangleq X \setminus A_i$. We define $f : X \to \mathbb{R}$ by

$$f(x) = \frac{1}{n}\sum_{i=1}^{n} d(x, C_i).$$

Take any $x \in X$. Then, there exists some $i \in [n]$ such that $x \in A_i$. Since A_i is open, there exists some $\varepsilon \in \mathbb{R}_+$ such that $B(x, \varepsilon) \subseteq A_i$; $d(x, C_i) \geq \varepsilon$. Hence, $f(x) \geq \varepsilon/n$. We just showed that $f(x) > 0$ for all $x \in X$.

Since f is continuous, there exists a minimum of $\operatorname{Im} f$, say δ, by Theorem 1.5.3. We claim that δ is a Lebesgue number for $\mathcal{A}$. Let $\neq \varnothing B \subseteq X$ with $\operatorname{diam} B < \delta$. Take $x_0 \in B$. Then $B \subseteq B(x_0, \delta)$. Then,

$$\delta \leq f(x_0) \leq \max_{i \in [n]} d(x_0, C_i) = d(x_0, C_m).$$

where $m \in [n]$. Then, $B \subseteq B(x_0, \delta) \subseteq A_m$. □

Theorem 1.5.5 Uniform Continuity Theorem

Let (X, d_X) be a compact metric space; let (Y, d_Y) be a metric space. If $f : X \to Y$ is a continuous map, then f is uniformly continuous.

Proof. Take any $\varepsilon \in \mathbb{R}_+$. Let

$$\mathcal{A} \triangleq \left\{ f^{-1}\big(B(y, \varepsilon/2)\big) \,\middle|\, y \in Y \right\}$$

be an open covering of X. Let δ be a Lebesgue number for $\mathcal{A}$. Then, for each $x_1, x_2 \in X$ such that $d_X(x_1, x_2) < \delta$, since $\operatorname{diam}\{x_1, x_2\} = d_X(x_1, x_2) < \delta$, there exists $y \in Y$ such that $\{f(x_1), f(x_2)\} \subseteq B(y, \varepsilon/2)$. Then, $d_Y(f(x_1), f(x_2)) < \varepsilon$. □