Summary for Complex Variables I

SEUNGWOO HAN

Contents

Chapter 1

Preliminaries

1.1 Complex Plane

> **Definition 1.1.1: Complex Number**
>
> $i := \sqrt{-1}$ is called the *imaginary unit*. $\mathbb{C} := \{ x + iy \mid x, y \in \mathbb{R} \}$ is the set of complex numbers where $\mathbb{R}$ is the set of real numbers.

> **Definition 1.1.2: Algebras of $\mathbb{C}$**
>
> For $z_k := x_k + iy_k$ where $k \in \mathbb{Z}_+$ and $x_k, y_k \in \mathbb{R}$,
> - $z_1 + z_2 := (x_1 + x_2) + i(y_1 + y_2)$
> - $z_1 \cdot z_2 := (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$.

> **Theorem 1.1.3**
> $\mathbb{C}$ is a field.

Proof. Trivial. $\qquad\square$

> **Note**
>
> $z = a + ib$, $a, b \in \mathbb{R}$ with $z \neq 0$. Then, $z^{-1} = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}$.

1.2 Rectangular Representation

> **Definition 1.2.1**
>
> Let $z = x + iy$ where $x, y \in \mathbb{R}$.
> - (i) $|z| := \sqrt{x^2 + y^2}$ is called *modulus* of z.
> - (ii) $\bar{z} := x - iy$ is called *conjugate* of z.
> - (iii) $\Re z = x$ is called the *real part* of z and $\Im z = y$ is called the *imaginary part* of z.
> - (iv) For $z_1, z_2 \in \mathbb{C}$, $|z_1 - z_2|$ is the *distance* between z_1 and z_2.

> **Note**
> - $z + \bar{z} = 2\Re z$
> - $z - \bar{z} = 2i\Im z$
> - $|z_1 + z_2| \leq |z_1| + |z_2|$
> - $\big||z_1| - |z_2|\big| \leq |z_1 - z_2|$

1.3 Polar Representation

Given $z \in \mathbb{C}$, $|z|$ is unique. $\arg z = \theta + 2k\pi$ $(k \in \mathbb{Z})$ (Or $\arg z = \theta \pmod{2\pi}$)

> **Definition 1.3.1**
>
> If $z = |z| \cdot (\cos\theta + i\sin\theta)$, θ is called an *argument* of z and is written $\arg z = \theta \pmod{2\pi}$ (as $\theta + 2k\pi$ for $k \in \mathbb{Z}$ is an argument of z as well). If $\arg z = \theta^* \pmod{2\pi}$, and if $-\pi < \theta^* \leq \pi$, then we define $\operatorname{Arg} z = \theta^*$ and it is called the *principal argument* of z.

> **Theorem 1.3.2**
>
> For $z_1, z_2 \in \mathbb{C}$ with $z_1, z_2 \neq 0$, $\arg z_1 z_2 = \arg z_1 + \arg z_2 \pmod{2\pi}$.

Proof. Let $\arg z_1 = \theta_1 \pmod{2\pi}$ and $\arg z_2 = \theta_2 \pmod{2\pi}$ Then, $z_1 = |z_1|(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = |z_2|(\cos\theta_2 + i\sin\theta_2)$. Now, we have $z_1 \cdot z_2 = |z_1||z_2|\big(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\big)$. $\qquad\square$

Chapter 2

Elementary Complex Functions

2.1 Exponential Functions

Definition 2.1.1: Exponential Function

For each $z = x + iy$ where $x, y \in \mathbb{R}$, we define $e^z := e^x \cdot (\cos y + i \sin y)$.

Theorem 2.1.2

For each $z \in \mathbb{C}$, $e^z = \sum_{j=1}^{\infty} \frac{z^j}{j!}$.

Proof. Proved later using complex integral. □

Theorem 2.1.3

For each $z, z' \in \mathbb{C}$,

(a) $e^{z+z'} = e^z \cdot e^{z'}$,

(b) $e^{-z} = \frac{1}{e^z}$, and

(c) $e^{z+2k\pi i} = e^z$ for all $k \in \mathbb{Z}$.

Definition 2.1.4

For each $z \in \mathbb{C}$,

(1) $\cos z := \frac{e^{iz} + e^{-iz}}{2}$

(2) $\sin z := \frac{e^{iz} - e^{-iz}}{2i}$

(3) $\cosh z = \frac{e^z + e^{-z}}{2}$

(4) $\sinh z = \frac{e^z - e^{-z}}{2}$

Theorem 2.1.5

For each $z \in \mathbb{C}$, we have $\cosh z = \cos(iz)$ and $\sinh z = -i\sin(iz)$.

Example 2.1.6

Let us solve $\cos z = 2$. Let $t := e^{iz}$ to obtain $t^2 - 4t + 1 = 0$, which gives $t = 2 \pm \sqrt{3}$. Write $z = x + iy$ where $x, y \in \mathbb{R}$ to have $e^{ix} e^{-y} = 2 \pm \sqrt{3}$. Taking modulus to both sides gives $e^{-y} = 2 \pm \sqrt{3}$, i.e., $y = -\ln(2 \pm \sqrt{3})$. Taking argument to both sides gives $x = 2k\pi$

for $k \in \mathbb{Z}$. Thus, $z = 2k\pi - i \ln(2 \pm \sqrt{3})$ for $k \in \mathbb{Z}$.

2.2 Mapping Properties

대충 그래프 그리는 이야기 ㅇㅇ

2.3 Logarithmic "Functions"

Definition 2.3.1: Logarithmic Function

For any $z \in \mathbb{C} \setminus \{0\}$, we define $w = \ln z$ if and only if $e^w = z$.

Note

How to compute $\ln z$? Note that $z = |z| \cdot e^{i(\mathrm{Arg}\, z + 2k\pi)}$ for $k \in \mathbb{Z}$. Let $w = u + iv$ where $u, v \in \mathbb{R}$ so that $e^w = e^u \cdot e^{iv} = |z| \cdot e^{i(\mathrm{Arg}\, z + 2k\pi)}$. Hence, we have $u = \ln|z|$ and $v = \mathrm{Arg}\, z + 2k\pi$. In other words, $\ln z = \ln|z| + i \arg z$. (Note that this is not a "function"!)

Definition 2.3.2: Principal Logarithmic Function

For any $z \in \mathbb{C} \setminus \{0\}$, we define $\mathrm{Ln}\, z := \ln|z| + i\, \mathrm{Arg}\, z$ and it is called the *principal value of* $\ln z$.

Definition 2.3.3: Branch of Logarithm

A *branch* of $\ln z$ is a function given by $\omega \colon \ln z$ with $\theta_0 < \arg z \le \theta_0 + 2\pi$. Here, θ_0 is called a *branch cut*.

Example 2.3.4

$B := \{ z \mid |z + 2| < 1 \}$ when mapped with Ln is not an open ball but it becomes an open ball when the branch cut is $-\pi/2$.

2.4 Complex Exponents

Definition 2.4.1: Complex Exponents

For $z \in \mathbb{C} \setminus \{0\}$ and $w \in \mathbb{C}$, define

$$z^w := e^{w \ln z}.$$

Note

Complex exponentiation is not a function! If one considers the complex exponentiation as a set of possible values, then $z^{\eta_1} \cdot z^{\eta_2} = z^{\eta_1 + \eta_2}$ may easily fail!

> **Example 2.4.2**
>
> To solve $z^{1-i} = 4$, write $e^{(1-i)\ln z} = e^{\ln 4}$, i.e., $\ln z = (1+i)(\ln 2 + k\pi i)$ for $k \in \mathbb{Z}$. In other words, $\ln|z| + i \arg z = (\ln 2 - k\pi) + i(\ln 2 + k\pi)$. Hence, $|z| = e^{\ln 2 - k\pi}$ and $\arg z = \ln 2 + k\pi$ (mod 2π).

Chapter 3

Analytic Functions

3.1 Cauchy–Riemann Equation

Definition 3.1.1: Continuity

For a fixed point $z_0 \in \mathbb{C}$, a function f is said to be continuous at z_0 if

$$\lim_{|z-z_0| \to 0} |f(z) - f(z_0)| = 0.$$

Definition 3.1.2: Differentiability

For a fixed point $z_0 \in \mathbb{C}$, a function f is said to be *continuous at z_0* if

$$\lim_{\substack{|\omega| \to 0 \\ \omega \in \mathbb{C}}} \frac{f(z_0 + \omega) - f(z_0)}{\omega}$$

exists. If f is differentiable at z_0, then define the *derivative* of f at z_0 by

$$f'(z_0) := \lim_{\substack{|\omega| \to 0 \\ \omega \in \mathbb{C}}} \frac{f(z_0 + \omega) - f(z_0)}{\omega}.$$

Example 3.1.3

For each $n \in \mathbb{N}$, one can derive that $f'(z) = nz^{n-1}$ where $f(z) = z^n$.

Theorem 3.1.4

If f is differentiable at z_0, then it is continuous at z_0.

Example 3.1.5

Let us determine differentiability of $f(z) = |z|^2$. Write $z = x + iy$ and $\omega = p + iq$ for $x, y, p, q \in \mathbb{R}$. Then,

$$\frac{f(z + \omega) - f(z)}{\omega} = \frac{2(xp + yq) + |\omega|^2}{\omega}$$

As we know $\lim_{\omega \to 0} \frac{|\omega|^2}{\omega} = 0$, we only need to care if $\lim_{\omega \to 0} \frac{2(xp+yq)}{p+iq}$. Evaluating the limit along the real axis and the imaginary axis gives $2x$ and $-2yi$; hence f is not

differentiable at $z \in \mathbb{C} \setminus \{0\}$. At the origin, we have $f'(0) = \lim_{\omega \to 0} \frac{f(0+\omega)-f(0)}{\omega} = 0$.

Theorem 3.1.6

Product, quotient, chain rule still holds in complex derivative.

Theorem 3.1.7 Cauchy–Riemann Equation

If f is differentiable at z, then $f_y(z) = if_x(z)$ at z, or equivalently,

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

where $u(x, y) := \mathfrak{R}f(x + iy)$ and $v(x, y) := \mathfrak{I}f(x + iy)$ for $x, y \in \mathbb{R}$.

Proof. $f_x(z) = \lim_{\xi \to 0} \frac{f(z+\xi)-f(z)}{\xi} = f'(z)$ and $-if_y(z) = \lim_{\eta \to 0} \frac{f(z+i\eta)-f(z)}{i\eta} = f'(z)$. □

Example 3.1.8

Is e^z differentiable in $\mathbb{C}$?

$$\lim_{h \to 0} \frac{e^h - 1}{h} = \lim_{(\xi,\eta) \to 0} \frac{(e^\xi - 1)e^{i\eta} + (e^{i\eta} - 1)}{\xi + i\eta}$$

Note

We may write $f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$. If f is differentiable, we *define*

$$\frac{\partial f}{\partial z} := \left(\frac{1}{2}\partial_x + \frac{1}{2i}\partial_y\right)u + i\left(\frac{1}{2}\partial_x + \frac{1}{2i}\partial_y\right)v = \left(\frac{1}{2}\partial_x + \frac{1}{2i}\partial_y\right)f$$

$$\frac{\partial f}{\partial \bar{z}} := \left(\frac{1}{2}\partial_x - \frac{1}{2i}\partial_y\right)u + i\left(\frac{1}{2}\partial_x - \frac{1}{2i}\partial_y\right)v = \left(\frac{1}{2}\partial_x - \frac{1}{2i}\partial_y\right)f.$$

So that $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)f = 0$ if f is differentiable.

Definition 3.1.9: Domain

A domain is an open and connected subset of $\mathbb{C}$.

Theorem 3.1.10

Any two points in a domain can be connected by polygonal lines parallel to the coordinate axes that lies in the domain.

Proof. Let D be a domain and let $z_0 \in D$. Let $A \subseteq D$ be the set of all points in D that can be connected from z_0 by polygonal lines parallel to the coordinate axes. Let $B := D \setminus A$. If $z \in A$ and $r > 0$ satisfy $B_r(z) \subseteq D$, then $B_r(z) \subseteq A$; hence A is open. Similarly, B is open as well. As D is connected, A or B is empty but $z_0 \in A$; hence, $B = \varnothing$. □

> **Theorem 3.1.11**
>
> If $f'(z) \equiv 0$ in a domain D, then f is constant on D.

Proof. $f_x \equiv f_y \equiv 0$; hence $u_x \equiv v_x \equiv u_y \equiv u_x \equiv 0$ on D. Thus, f is contant on every line segment in D parallel to coordinate axes. Hence, f is constant on D. $\qquad\square$

> **Corollary 3.1.12**
>
> Let f be differentiable on a domain D.
>
> (1) If $\Re f(z)$ is constant on D, then f is constant on D.
> (2) If $\Im f(z)$ is constant on D, then f is constant on D.
> (3) If $\operatorname{Arg} f(z)$ is constant on D, then f is constant on D.

Proof.

(1) There is $\omega_0 \in \mathbb{C}$ such that, when g is defined by $g(z) \triangleq f(z) - \omega_0$, we have $\Re g(z) \equiv 0$ and g is differentiable on D.

$$\lim_{\xi \to 0} \frac{f(z+\xi) - f(z)}{\xi} = f'(z) = \lim_{\eta \to 0} \frac{f(z+i\eta) - f(z)}{i\eta}$$

where the left hand side is real and the right hand side is purely imaginary. Therefore, $f'(z) = 0$ for all $z \in D$. The result follows from Theorem 3.1.11.

(2) Let $g(z) = if(z)$ so that g is differentiable on D and $\Re g(z)$ is constant. Therefore, by (1), g is constant and thus f is constant.

(3) There is $\omega_0 \in \mathbb{R}$ such that, when g is defined by $g(z) \triangleq f(z)e^{-i\omega_0}$, we have $\Re g(z)$ is constant and g is differentiable on D. Tehrefore, by (1), g is constant and thus f is constant. $\qquad\square$

3.2 Analyticity

> **Definition 3.2.1: Analytic Function**
>
> - For a fixed point $z_0 \in \mathbb{C}$, a function f is *analytic* at z_0 if there is some $r > 0$ such that f is differentiable at every point in $B_r(z_0) \triangleq \{z \in \mathbb{C} : |z - z_0| < r\}$.
> - A function f is *analytic in domain D* if it is analytic at z for all $z \in D$.
> - A function f is *entire* if it is analytic in $\mathbb{C}$.

> **Theorem 3.2.2**
>
> Given a function $f(z) = u(x,y) + iv(x,y)$ in domain D, if
>
> (1) $u(x,y)$ and $v(x,y)$ are C^1 in D, and if
> (2) $u(x,y)$ and $v(x,y)$ satisfy the Cauchy–Riemann equations in D,
>
> then f is analytic in D.

Chapter 4

Complex Integration

Chapter 5

Conformal Mapping

End.