

# MAS241 해석학 I

## Note

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# CONTENTS

CHAPTER	STRUCTURE OF THE REAL NUMBERS	PAGE 2
	1.1 Completeness of the Real Numbers	2
	1.2 Neighborhoods and Limit Points	5
	1.3 The Limit of a Sequence	7
	1.4 Cauchy Sequences	12
	1.5 The Algebra of Convergent Series	14
	1.6 Cardinality	14
CHAPTER	EUCLIDEAN SPACES	PAGE 15
	2.1 Euclidean $n$ -Space	15
	Sequences in $\mathbb{R}^n$ — 17	
	2.2 Open and Closed Sets	19
	2.3 Completeness	24
	2.4 Relative Topology and Connectedness	26
	2.5 Compactness	29

# Chapter 1

## Structure of the Real Numbers

### 1.1 Completeness of the Real Numbers

#### Definition 1.1.1: Cauchy Sequence

Let  $X$  be a space. A sequence  $\{x_n\}_{n \in \mathbb{N}}$  is a *Cauchy sequence* if  $\|x_n - x_m\| \rightarrow 0$  as  $n, m \rightarrow \infty$ .

#### Definition 1.1.2: Completeness

A set  $X$  is *complete* if every Cauchy sequence has a limit in  $X$ , i.e.,

$$x_n \rightarrow x_\infty \in X.$$

#### Definition 1.1.3: Boundedness

Let  $\emptyset \neq S \subseteq \mathbb{R}$ .

- a)  $S$  is *bounded above* if  $\exists M \in \mathbb{R}, \forall x \in S, x \leq M$ .
  - $M$  is called an *upper bound* of  $S$ .
- b)  $S$  is *bounded below* if  $\exists M \in \mathbb{R}, \forall x \in S, x \geq M$ .
  - $M$  is called an *lower bound* of  $S$ .
- c)  $S$  is *bounded* if  $S$  is bounded above and below.

#### Theorem 1.1.1 Archimedes' Principle

Let  $\varepsilon$  and  $M$  be any two possible real numbers. Then, there exists a  $k$  in  $\mathbb{N}$  such that  $M < k\varepsilon$ .

The proof of Theorem 1.1.1 can be done by integrating Theorem 1.1.2 and Theorem 1.1.4.

#### Definition 1.1.4: Supremum and Infimum

- a) Let  $S$  be bounded above. Then, the smallest upper bound is called the *supremum* of  $S$ ,  $\sup S$ .
- b) Let  $S$  be bounded below. Then, the largest lower bound is called the *infimum* of  $S$ ,  $\inf S$ .

### Example 1.1.1

Let  $S = \{(-1)^k(1 - 1/k) \mid k \in \mathbb{N}\}$ . It is clear that  $-1 < S < 1$ ; 1 is an upper bound and  $-1$  is a lower bound. We now claim that  $\sup S = 1$ . To show this, let us assume that  $M < 1$  is an upper bound of  $S$ . By Archimedes' principle, there exists a natural number  $k_0$  such that  $(1 - M)/2 < k_0$ , which implies  $(-1)^{2k_0}(1 - 1/(2k_0)) > M$ ;  $M$  is not an upper bound. Therefore, 1 is the smallest upper bound. It can be similarly shown that  $\inf S = -1$ .

### Theorem 1.1.2 Completeness Axiom for $\mathbb{R}$

If  $\emptyset \neq S \subseteq \mathbb{R}$  and  $S$  is bounded above, then  $\sup S$  exists in  $\mathbb{R}$ .

### Corollary 1.1.1

If  $\emptyset \neq S \subseteq \mathbb{R}$  and  $S$  is bounded below, then  $\inf S$  exists in  $\mathbb{R}$ .

**Proof.** Let  $B := \{-x \mid x \in S\}$ . Then,  $M = \sup S \in \mathbb{R}$  by Theorem 1.1.2. We now claim that  $\inf B = -M$ .

For all  $x \in S$ ,  $-x \in B$ , which implies  $-x \leq M$ , and therefore  $x \geq -M$ . Thus,  $-M$  is a lower bound of  $B$ .

Suppose there is a  $M_1 > -M$  such that  $M_1$  is a lower bound of  $S$ . For all  $x \in S$ ,  $x \geq M_1$ , which implies  $-x \leq -M_1$ . Thus,  $-M_1$  is an upper bound of  $B$  but  $-M_1 < M = \sup B$ , #.

Therefore,  $\inf S = -M \in \mathbb{R}$ . □

### Example 1.1.2

- $S := \left\{ \sum_{j=0}^k \frac{1}{j!} \mid k \in \mathbb{N} \right\}$ .  $S$  is bounded above.

$$\sum_{j=0}^k \frac{1}{j!} = 1 + \sum_{j=1}^k \frac{1}{j!} \leq 1 + \sum_{j=1}^k \frac{1}{2^{j-1}} < 3$$

In fact,  $e := \sup S$ .

- $S := \left\{ \left(1 + \frac{1}{k}\right)^k \mid k \in \mathbb{N} \right\}$ .  $S$  is bounded above.

$$\left(1 + \frac{1}{k}\right)^k = \sum_{j=0}^k \binom{k}{j} \frac{1}{k^j} \leq \sum_{j=0}^k \frac{1}{j!} \leq e$$

### Theorem 1.1.3

Let  $S$  be a finite nonempty subset of  $\mathbb{R}$ . Then,  $\sup S \in S$  and  $\inf S \in S$ .

**Proof.** (Induction on  $|S|$ ) For  $S = \{x\}$ ,  $x = \inf S = \sup S \in S$ .

Take any  $k \in \mathbb{N}$  and suppose the statement holds for every  $S$  with  $|S| = k$ . Now, take any  $S' \subseteq \mathbb{R}$  such that  $|S'| = k + 1$ . Let  $x \in S'$ ,  $\mu := \sup(S' \setminus \{x\})$ , and  $\nu := \inf(S' \setminus \{x\})$ . By the induction hypothesis,  $\mu, \nu \in S' \setminus \{x\}$ . Letting  $\mu' := \max(\mu, x)$  and  $\nu' := \min(\nu, x)$ ,  $\mu'$  and  $\nu'$  are the supremum and infimum of  $S'$ , respectively. Moreover,  $\mu'$  and  $\nu'$  are elements of  $S'$ . □

### Theorem 1.1.4

Let  $\emptyset \neq S \subseteq \mathbb{R}$ .

- If  $S$  is bounded above, then “ $\mu = \sup S$  if and only if  $\mu$  is an upper bound and  $\forall \varepsilon \in \mathbb{R}_+, \exists x \in S, \mu - \varepsilon < x \leq \mu$ ”.
- If  $S$  is bounded below, then “ $\nu = \inf S$  if and only if  $\nu$  is a lower bound and  $\forall \varepsilon \in \mathbb{R}_+, \exists x \in S, \nu \leq x < \nu + \varepsilon$ ”.

**Proof.** Let  $S$  be bounded above. If there is no  $x \in S$  in  $(\mu - \varepsilon, \mu]$ , then  $\mu - \varepsilon$  would be a smaller upper bound.

For the converse, assume  $M$  is an upper bound and  $M < \mu$ . Let  $\varepsilon := \mu - M > 0$ . Then, there is some  $x \in S$  such that  $M = \mu - \varepsilon < x \leq \mu$ ,  $\#$  to  $M$  is an upper bound. Therefore,  $\mu$  is the least upper bound.

The same logic may be applied for bounded below  $S$ . □

**Proof of Theorem 1.1.1.** Let  $S := \{k\varepsilon \mid k \in \mathbb{N}\}$ . Assume  $S$  is bounded above and nonempty. Then, by Theorem 1.1.2, there is  $\mu = \sup S$ . We also know, from Theorem 1.1.4, that there is  $k \in \mathbb{N}$  such that  $\mu - \varepsilon < k\varepsilon \leq \mu$ , which implies  $\mu < (k + 1)\varepsilon$ . Since  $(k + 1)\varepsilon \in S$ ,  $\mu$  is not an upper bound of  $S$ , which is a contradiction. Therefore,  $S$  is not bounded above. In other words, for any  $M > 0$ , there is some  $k \in \mathbb{N}$  such that  $M < k\varepsilon$ . □

### Theorem 1.1.5

Theorem 1.1.1 (Archimedes' principle) is equivalent to the following statement:

$$\forall c \in \mathbb{R}_+, \exists k \in \mathbb{N}, k - 1 \leq c < k.$$

**Proof.** Assume Archimedes' principle. If  $c < 1$ ,  $k = 1$  satisfies, and it is done. Now, let us suppose  $c \geq 1$ . By Theorem 1.1.1, there is a  $k \in \mathbb{N}$  such that  $c < k$ . We may let  $k_0 := \min\{k \in \mathbb{N} \mid k > c\}$  by Well-Ordering of  $\mathbb{N}$ . We note that  $k_0 - 1 \leq c$  since  $k_0 - 1 \in \mathbb{N}$  since  $k_0 > 1$ . Therefore,  $k_0 - 1 \leq c < k_0$ .

Now, assume “ $\forall c \in \mathbb{R}_+, \exists k \in \mathbb{N}, k - 1 \leq c < k$ ”. Take any  $M > 0$  and  $\varepsilon \in \mathbb{R}_+$  and let  $c := M/\varepsilon$ . The assumption tells the existence of a  $k \in \mathbb{N}$  such that  $M/\varepsilon = c < k$ , which directly implies  $M < k\varepsilon$ . □

### Theorem 1.1.6

Let  $c$  and  $d$  be real numbers with  $c < d$ . Then,  $\exists x \in \mathbb{Q}, c < x < d$ .

**Proof.** There are three cases:  $0 < c < d$ ,  $c \leq 0 < d$ , or  $c < d \leq 0$ .

Case 1) By Archimedes' principle,  $\exists q \in \mathbb{N}, 1 < (d - c)q$ , which implies  $cq + 1 < dq$ . By Theorem 1.1.5,  $\exists q \in \mathbb{N}, p - 1 \leq cq < p$  since  $cq > 0$ . To sum up,  $p - 1 \leq cq < p \leq cq + 1 < dq$ , which implies  $c < p/q < d$ .

Case 2) By Archimedes' principle,  $\exists q \in \mathbb{N}, 1 < dq$ . Then,  $c \leq 0 < 1/q < d$  holds.

Case 3) By case 1 and 2, there is  $r \in \mathbb{Q}$  such that  $-d < r < -c$ . Then,  $c < -r < d$  holds. □

## 1.2 Neighborhoods and Limit Points

### Definition 1.2.1: Neighborhood and Deleted Neighborhood

For each  $x \in \mathbb{R}$  and  $r \in \mathbb{R}_+$ ,

$$N(x; r) := \{y \in \mathbb{R} : |y - x| < r\} = (x - r, x + r)$$

is called the *neighborhood* of  $x$  with radius  $r$ , and

$$N'(x; r) := \{y \in \mathbb{R} : 0 < |y - x| < r\} = N(x; r) \setminus \{x\}$$

is called the *deleted neighborhood* of  $x$  with radius  $r$ .

### Definition 1.2.2: Limit Point and Isolated Point

For  $\emptyset \neq S \subseteq \mathbb{R}$ ,  $x \in \mathbb{R}$  is a *limit point* of  $S$  if

$$\forall \varepsilon \in \mathbb{R}_+, N'(x, \varepsilon) \cap S \neq \emptyset.$$

If  $x \in \mathbb{R}$  is not a limit point of  $S$ , then it is called an *isolated point* of  $S$ .

### Definition 1.2.3: Discrete Set

If  $\emptyset \neq S \subseteq \mathbb{R}$  has no limit points, then  $S$  is said to be *discrete*.

### Example 1.2.1

Let  $S := \{(-1)^k(1 + 1/k) \mid k \in \mathbb{N}\}$ . Then, 1 and  $-1$  are limit points of  $S$ .

To see 1 is a limit point, take any  $\varepsilon \in \mathbb{R}_+$  and, using Theorem 1.1.1, choose a  $k \in \mathbb{N}$  such that  $1 < (2\varepsilon)k$ . Then,  $1 < 1 + \frac{1}{2k} = (-1)^{2k}(1 + \frac{1}{2k}) < 1 + \varepsilon$ ;  $N'(1, \varepsilon) \cap S \neq \emptyset$ . Therefore, 1 is a limit point.

### Theorem 1.2.1

Let  $\emptyset \neq S \subseteq \mathbb{R}$ . Then,  $x \in \mathbb{R}$  is a limit point of  $S$  if and only if

$$\exists \varepsilon_0 \in \mathbb{R}_+, \forall \varepsilon \in (0, \varepsilon_0), N'(x, \varepsilon) \cap S \neq \emptyset.$$

**Proof.** Trivial;  $0 < \varepsilon_1 < \varepsilon_2$  implies  $N'(x, \varepsilon_1) \subsetneq N'(x, \varepsilon_2)$ . □

### Theorem 1.2.2

Let  $\emptyset \neq S \subseteq \mathbb{R}$  and  $x \in \mathbb{R}$  be a limit point of  $S$ . Then, every deleted neighborhood of  $x$  must contain infinitely many points of  $S$ .

**Proof.** Assume  $N'(x; \varepsilon) \cap S$  were to contain only finitely many points, namely,  $N'(x; \varepsilon) \cap S = \{x_1, x_2, \dots, x_k\}$ . Let  $S_1 := \{|x - x_i| : i \in [k]\}$ . Since  $S_1$  is finite, we may let  $x_j$  be an element of  $N'(x; \varepsilon) \cap S$  that satisfies  $|x - x_j| = \min S_1 = \inf S_1 > 0$ . If we let  $\varepsilon_0 := |x - x_j|/2$ ,  $N'(x; \varepsilon_0) \cap S = \emptyset$ , #. □

### Corollary 1.2.1

If  $S$  is a finite subset of  $\mathbb{R}$ , then  $S$  has no limit point.

### Example 1.2.2

$\mathbb{Z}$  has no limit point.

### Theorem 1.2.3 Bolzano–Weierstrass Theorem

If  $S \subseteq \mathbb{R}$  is bounded and has an infinite number of elements, then  $S$  has a limit point.

**Proof.** Since  $S$  is bounded,  $a_0 := \inf S$  and  $b_0 := \sup S$  exist;  $S \subseteq [a_0, b_0]$ . At least one of  $[a_0, (a_0 + b_0)/2]$  and  $[(a_0 + b_0)/2, b_0]$  has an infinite number of elements in  $S$ , otherwise  $S$  must be finite. Choose whichever has an infinite number of elements in  $S$ , and let us denote it as  $[a_1, b_1]$ . Since,  $S \cap [a_1, b_1]$  is bounded and has an infinite number of elements, we may find  $a_2$  and  $b_2$  in the same manner. Note that

- (a) for every natural number  $k$ ,  $S \cap [a_k, b_k]$  has an infinite number of elements,
- (b)  $\forall k \in \mathbb{N}$ ,  $b_k - a_k = (b_0 - a_0)/2^k > 0$ , and
- (c)  $\forall k \in \mathbb{N}$ ,  $a_{k-1} \leq a_k < b_k \leq b_{k-1}$ .

The sequence  $\{a_k\}_{k=0}^{\infty}$  is bounded above by  $b_0$ , and the sequence  $\{b_k\}_{k=0}^{\infty}$  is bounded below by  $a_0$ . Therefore, we may let  $\alpha := \sup\{a_k\}$  and  $\beta := \inf\{b_k\}$ .

Since  $a_j$  is a lower bound of  $\{b_k\}_{k=0}^{\infty}$  for all  $j \in \mathbb{N}$ ,  $\forall j \in \mathbb{N}$ ,  $a_j \leq \beta$ . This implies  $\beta$  is an upper bound of  $\{a_k\}_{k=0}^{\infty}$ , therefore  $\alpha \leq \beta$ . Since  $a_j \leq \alpha \leq \beta \leq b_j$  for all  $j \in \mathbb{N}$ , we get  $0 \leq \beta - \alpha \leq b_j - a_j = (b_0 - a_0)/2^j$ . Therefore,  $\beta - \alpha = 0$ .

We now claim that  $\alpha$  is a limit point of  $S$ . Take any  $\varepsilon \in \mathbb{R}_+$ . By Theorem 1.1.4,  $\exists k_0 \in \mathbb{N}$ ,  $\alpha - \varepsilon < a_{k_0} \leq \alpha$ . We may take  $k \in \mathbb{N}$  such that  $k > k_0$  and  $|b_k - a_k| < \varepsilon$  thanks to (b). Since  $\alpha \in [a_k, b_k]$ ,  $\alpha - \varepsilon < a_{k_0} \leq a_k \leq \alpha \leq b_k < \alpha + \varepsilon$ , which implies  $[a_k, b_k] \subseteq N(\alpha; \varepsilon)$ .

In conclusion,  $S \cap [a_k, b_k]$  has infinitely many elements by (a), and so does  $(S \cap [a_k, b_k]) \setminus \{\alpha\}$ .  $S \cap N'(\alpha; \varepsilon)$  is, therefore, nonempty.  $\square$

### Definition 1.2.4: Bolzano–Weierstrass Property

We say that a nonempty set  $X$  has the *Bolzano–Weierstrass property* if every bounded, infinite subset  $S$  of  $X$  has a limit point in  $X$ .

## 1.3 The Limit of a Sequence

### Definition 1.3.1: Cluster Point

$c \in \mathbb{R}$  is a *cluster point* of the sequence  $\{x_k\}$  if,

$$\forall(\varepsilon, k) \in \mathbb{R}_+ \times \mathbb{N}, \exists k_1 \in \mathbb{N}_{>k}, x_{k_1} \in N(c; \varepsilon).$$

### Lemma 1.3.1

$c \in \mathbb{R}$  is a cluster point of  $\{x_k\}$  if and only if  $\{k \in \mathbb{N} \mid x_k \in N(c; \varepsilon)\}$  is infinite for every  $\varepsilon \in \mathbb{R}_+$ .

**Proof.** ( $\Rightarrow$ ) Suppose  $S := \{k \in \mathbb{N} \mid x_k \in N(c; \varepsilon)\}$  is finite for some  $\varepsilon \in \mathbb{R}_+$ . If  $S$  were empty, then,  $c$  is not a cluster point by Definition 1.3.1. Therefore,  $S$  is nonempty and has a maximum element  $k_0 := \max S$  by Theorem 1.1.3. Since  $c$  is a cluster point, there is a natural number  $k_1 > k_0$  such that  $x_{k_1} \in N(c; \varepsilon)$ ;  $k_1 \in S$ . This contradicts the maximality of  $k_0$ .

( $\Leftarrow$ ) Take any  $\varepsilon \in \mathbb{R}_+$  and  $k_0 \in \mathbb{N}$ . If there is no  $k_1 \in \mathbb{N}$  such that  $k_1 > k_0$  and  $x_{k_1} \in N(c; \varepsilon)$ ,  $S$  will be bounded above by  $k_0$  and finite, which is a contradiction. Therefore,  $c$  is a cluster point of  $S$ . □

### Definition 1.3.2: Convergence and Divergence of a Sequence

The sequence  $\{x_k\}$  *converges* to  $x_0$  and  $x_0$  is the *limit* of  $\{x_k\}$  if,

$$\forall \varepsilon \in \mathbb{R}_+, \exists k_0 \in \mathbb{N}, \forall k \in \mathbb{N}_{\geq k_0}, x_k \in N(x_0; \varepsilon).$$

We write  $\lim_{k \rightarrow \infty} x_k = x_0$ . If there is no such  $x_0$ , then  $\{x_k\}$  *diverges*.

### Lemma 1.3.2

$\lim_{k \rightarrow \infty} x_k = x_0$  if and only if  $\{k \in \mathbb{N} \mid x_k \notin N(x_0; \varepsilon)\}$  is finite for every  $\varepsilon \in \mathbb{R}_+$ .

**Proof.** ( $\Rightarrow$ ) Take any  $\varepsilon \in \mathbb{R}_+$ . There is some  $k_0 \in \mathbb{N}$  such that  $k \in N(x_0; \varepsilon)$  for all natural numbers  $k \geq k_0$ . Therefore,  $\{k \in \mathbb{N} \mid x_k \notin N(x_0; \varepsilon)\} \subseteq [k_0]$  and thus finite.

( $\Leftarrow$ ) Take any  $\varepsilon \in \mathbb{R}_+$ . Let  $k_0 := \max \{k \in \mathbb{N} \mid x_k \notin N(x_0; \varepsilon)\}$ . Then, for every natural number  $k$  larger than  $k_0$  satisfies  $x_k \in N(x_0; \varepsilon)$ . □

### Lemma 1.3.3

The limit  $x_0$  of a sequence, if it exists, is a cluster point of the sequence.

### Theorem 1.3.1 Uniqueness of the Limit

The limit of a convergent sequence of  $\mathbb{R}$  is unique.

**Proof.** Suppose  $a$  and  $b$  are two limits of a sequence  $\{x_k\}$  and  $a \neq b$ . Let  $\varepsilon := |b - a|/2$ . Then, by Lemma 1.3.2,  $A := \{k \in \mathbb{N} \mid x_k \notin N(a; \varepsilon)\}$  and  $B := \{k \in \mathbb{N} \mid x_k \notin N(b; \varepsilon)\}$  are both finite, which means  $A \cup B = \mathbb{N}$  is finite,  $\#$ . □



### Theorem 1.3.2

If a sequence has two (or more) cluster points, then it diverges.

**Proof.** Suppose  $x_0$  is the limit of  $\{x_k\}$ . Since, by Lemma 1.3.3,  $x_0$  is a cluster point, there is another cluster point  $c$  different from  $x_0$ . Let  $\varepsilon := |x_0 - c|/2$ .

Although  $S := \{k \in \mathbb{N} \mid x_k \notin N(x_0; \varepsilon)\}$  should be finite by Lemma 1.3.2,  $\{k \in \mathbb{N} \mid x_k \in N(c; \varepsilon)\}$ , a subset of  $S$ , is infinite by Lemma 1.3.1,  $\#$ .  $\square$

### Theorem 1.3.3

A convergent sequence is bounded.

**Proof.** Let  $x_0$  is the limit of  $\{x_k\}$ . There is some  $k_0 \in \mathbb{N}$  such that  $|x_k - x_0| < 1$  for all  $k \in \mathbb{N}_{k_0}$ . Let  $A := \{x_k \mid k \in \mathbb{N} \text{ and } k \leq k_0\}$  and  $B := \{x_k \mid k \in \mathbb{N} \text{ and } k \geq k_0\}$ . Then,  $A$  is finite and  $B$  is bounded above and below by  $x_0 + 1$  and  $x_0 - 1$ , respectively. Therefore,  $\{x_k\}$  is bounded above by  $\max(\max A, x_0 + 1)$  and below by  $\min(\min A, x_0 - 1)$ .  $\square$

### Corollary 1.3.1

An unbounded sequence diverges.

### Lemma 1.3.4

The following hold.

- (i)  $\lim_{k \rightarrow \infty} x_k = 0 \iff \lim_{k \rightarrow \infty} |x_k| = 0$
- (ii)  $\lim_{k \rightarrow \infty} x_k = x_0 \implies \forall c \in \mathbb{R}, \lim_{k \rightarrow \infty} cx_k = cx_0$

**Proof of (ii).** If  $c = 0$ , then it is done; so suppose  $c \neq 0$ . Take any  $\varepsilon \in \mathbb{R}_+$ . Then, there is some  $k_0 \in \mathbb{N}$  such that  $|x_k - x_0| < \varepsilon/|c|$  for all  $k \geq k_0$ . This directly implies for all  $k \geq k_0$ ,  $|cx_k - cx_0| = |c| \cdot |x_k - x_0| < |c| \cdot \varepsilon/|c| = \varepsilon$ .  $\square$

### Theorem 1.3.4

A bounded, monotone sequence converges.

**Proof.** Suppose  $\{x_k\}$  is a monotone increasing sequence. Since it is bounded,  $\{x_k\}$  has  $\mu := \sup\{x_k \mid k \in \mathbb{N}\}$ . Take any  $\varepsilon \in \mathbb{R}_+$ . By Theorem 1.1.4, there is some  $k_0 \in \mathbb{N}$  such that  $\mu - \varepsilon < x_{k_0} \leq \mu$ . Then, for all  $k \in \mathbb{N}_{\geq k_0}$ ,  $\mu - \varepsilon < x_{k_0} \leq x_k \leq \mu$ , which implies  $|x_k - \mu| < \varepsilon$ . Therefore  $\lim_{k \rightarrow \infty} x_k = \mu$ .  $\square$

### Theorem 1.3.5 The Squeeze Play

Let  $\{x_k\}$ ,  $\{y_k\}$ , and  $\{z_k\}$  be sequences that satisfy  $x_k \leq y_k \leq z_k$  for  $k \in \mathbb{N}$ . If both  $\{x_k\}$  and  $\{z_k\}$  converges to  $L \in \mathbb{R}$ , then  $\{y_k\}$  also converges to  $L$ .

**Proof.** Take any  $\varepsilon > 0$ . There is  $k_1 \in \mathbb{N}$  such that  $\forall k \in \mathbb{N}_{\geq k_1}, x_k \in N(L; \varepsilon)$ . Similarly, there is  $k_2 \in \mathbb{N}$  such that  $\forall k \in \mathbb{N}_{\geq k_2}, z_k \in N(L; \varepsilon)$ . Then, for all  $k \in \mathbb{N}$  not smaller than  $\max\{k_1, k_2\}$ ,  $L - \varepsilon < x_k \leq y_k \leq z_k < L + \varepsilon$  holds, which implies  $y_k \in N(L; \varepsilon)$ .  $\square$

### Theorem 1.3.6 Limit is Order Preserving on Convergent Sequences

If both  $\{x_k\}$  and  $\{y_k\}$  converge and if  $x_k \leq y_k$  for each  $k \in \mathbb{N}$ , then

$$\lim_{k \rightarrow \infty} x_k \leq \lim_{k \rightarrow \infty} y_k.$$

**Proof.** Let  $L_x := \lim_{k \rightarrow \infty} x_k$  and  $L_y := \lim_{k \rightarrow \infty} y_k$ , and suppose  $L_x > L_y$ . Let  $\varepsilon := (L_x - L_y)/2 > 0$ . Then, there is  $k \in \mathbb{N}$  such that  $x_k \in N(L_x; \varepsilon)$  and  $y_k \in N(L_y; \varepsilon)$ , which implies  $y_k < L_y + \varepsilon = L_x - \varepsilon < x_k$ , #.  $\square$

### Definition 1.3.3: Subsequence

Let  $\{x_k\}$  be any sequence. Choose any strictly monotone increasing sequence  $k_1 < k_2 < k_3 < \dots$  of natural numbers. For each  $j \in \mathbb{N}$ , let  $y_j := x_{k_j}$ . The sequence  $\{y_j\}_{j=1}^{\infty}$  is called an *subsequence* of  $\{x_k\}$ .

### Theorem 1.3.7

The point  $c$  is a cluster point of  $\{x_k\}$  if and only if there exists a subsequence of  $\{x_k\}$  that converges to  $c$ .

**Proof.**  $(\Rightarrow)$  Let  $\{\varepsilon_k\}$  be an arbitrary sequence of positive real numbers that converges to 0. (e.g.  $\varepsilon_k = 1/k$ ) Define  $\{k_j\}_{j=1}^{\infty}$  by the inductive definition below.

- $k_0 := 0$
- For each  $j \in \mathbb{N}$ ,  $k_j \in \{k \in \mathbb{N} \mid k > k_{j-1} \text{ and } x_k \in N(c; \varepsilon_j)\}$ .

Since  $c$  is a cluster point,  $\{k \in \mathbb{N} \mid k > k_{j-1} \text{ and } x_k \in N(c; \varepsilon_j)\} \neq \emptyset$  for all  $j \in \mathbb{N}$ . Therefore,  $\{k_j\}$  is well-defined. It is immediate that  $\lim_{j \rightarrow \infty} x_{k_j} = c$ .

$(\Leftarrow)$  Let  $\{x_{k_j}\}_{j=1}^{\infty}$  be a sequence such that  $\lim_{j \rightarrow \infty} x_{k_j} = c$ . Take any  $\varepsilon \in \mathbb{R}_+$  and  $k \in \mathbb{N}$ . Then, there is some  $j_0 \in \mathbb{N}$  such that  $\forall j \in \mathbb{N}_{\geq j_0}$ ,  $x_{k_j} \in N(c; \varepsilon)$ . Let  $k_0 := \min\{k_j \in \mathbb{N} \mid j > j_0 \text{ and } k_j > k\}$ . Then,  $x_{k_0} \in N(c; \varepsilon)$  and  $k_0 > k$ . Therefore,  $c$  is a cluster point.  $\square$

### Theorem 1.3.8

Any bounded sequence  $\{x_k\}$  has a cluster point.

**Proof.** If the set  $S := \{x_k \mid k \in \mathbb{N}\}$  is finite, there is some  $x_{k_0}$  that is repeated infinitely. Then,  $x_{k_0}$  is surely a cluster point.

Now, suppose  $S$  is infinite. Then, by Theorem 1.2.3,  $S$  has a limit point  $\ell$ . To prove  $\ell$  is a cluster point, take any  $\varepsilon \in \mathbb{R}_+$  and  $k \in \mathbb{N}$ .

Let  $S' := \{x_{k'} \mid k' \in \mathbb{N}_{>k}\}$ . We first claim that  $\ell$  is a limit point of  $S'$ . Take any  $\varepsilon' \in \mathbb{R}_+$  less than  $m = \min\{|x_{k'} - \ell| \in \mathbb{R}_+ \mid k' \in \mathbb{N}_{\leq k}\}$ . ( $m$  exists due to Theorem 1.1.3.) Then,  $S' \cap N'(\ell; \varepsilon') = S \cap N'(\ell; \varepsilon')$  is nonempty. Therefore,  $\ell$  is a limit point of  $S'$  by Theorem 1.2.1.

Finally, we can say  $S' \cap N(\ell; \varepsilon)$  is nonempty. This implies there is some  $k_0 \in \mathbb{Z}_{>k}$  such that  $x_{k_0} \in N(\ell; \varepsilon)$ . Therefore,  $\ell$  is a cluster point of  $\{x_k\}$ .  $\square$

### Corollary 1.3.2

If a sequence has no cluster point, then the sequence is unbounded.

### Corollary 1.3.3

Any bounded sequence converges if and only if it has exactly one cluster point.

### Corollary 1.3.4

A sequence  $\{x_k\}$  diverges if and only if at least one of the following conditions holds.

- $\{x_k\}$  has two or more cluster points.
- $\{x_k\}$  is unbounded.

**Proof.** ( $\Rightarrow$ ) Suppose  $\{x_k\}$  is diverging and bounded. By Theorem 1.3.8, it has at least one cluster point. Also, if it had exactly one cluster point, it would converge by Corollary 1.3.3.

( $\Leftarrow$ ) It is direct from Theorem 1.3.2 and Corollary 1.3.1.  $\square$

### Theorem 1.3.9

A sequence  $\{x_k\}$  converges if and only if every subsequence of  $\{x_k\}$  converges.

**Proof.** ( $\Rightarrow$ ) Take any subsequence  $\{x_{k_i}\}_{i=1}^{\infty}$  of  $\{x_k\}_{k=1}^{\infty}$  and  $\varepsilon \in \mathbb{R}_+$ . There is  $i_0 \in \mathbb{N}$  such that  $\forall i \in \mathbb{N}_{\geq i_0}, |x_{k_i}| < \varepsilon$ . Since  $k_i \geq i$  for all natural number  $i$ ,  $\forall i \in \mathbb{N}_{\geq i_0}, |x_{k_i}| < \varepsilon$ .

( $\Leftarrow$ )  $\{x_k\}$  is a subsequence of itself.  $\square$

### Definition 1.3.4: Limit Superior and Inferior

Let  $\{x_k\}$  be a sequence and  $C$  be a set of cluster points of the sequence.

- $\limsup x_k \triangleq \begin{cases} \sup C & \text{if } \{x_k\} \text{ is bounded} \\ \infty & \text{if } \{x_k\} \text{ is unbounded above} \\ \sup C & \text{if } \{x_k\} \text{ is bounded above but unbounded below and } C \neq \emptyset \\ -\infty & \text{if } \{x_k\} \text{ is bounded above but unbounded below and } C = \emptyset \end{cases}$   
is called *limit superior* of  $\{x_k\}$ .

- $\liminf x_k \triangleq \begin{cases} \inf C & \text{if } \{x_k\} \text{ is bounded} \\ -\infty & \text{if } \{x_k\} \text{ is unbounded below} \\ \inf C & \text{if } \{x_k\} \text{ is bounded below but unbounded above and } C \neq \emptyset \\ \infty & \text{if } \{x_k\} \text{ is bounded below but unbounded above and } C = \emptyset \end{cases}$   
is called *limit inferior* of  $\{x_k\}$ .

#### Note:-

In all cases,  $\liminf x_k \leq \limsup x_k$ .

### Theorem 1.3.10

- If  $\mu = \limsup x_k$  is finite, then  $\mu$  is in  $C$ . ( $\mu = \max C$ )
- If  $\nu = \liminf x_k$  is finite, then  $\nu$  is in  $C$ . ( $\nu = \min C$ )

**Proof.** Suppose  $\mu = \limsup x_k$  is finite. Take any  $\varepsilon \in \mathbb{R}_+$  and  $k \in \mathbb{N}$ . The finiteness of  $\mu$  implies  $\mu = \sup C$ . By Theorem 1.1.4, there is some  $c \in C$  such that  $\mu - \varepsilon < c \leq \mu$ . If  $c = \mu$ , then we are done. So let  $c < \mu$ .

Choose any positive  $\varepsilon_1$  less than  $\min\{c - (\mu - \varepsilon), \mu - c\}$  so  $N(c; \varepsilon_1) \subseteq N(\mu; \varepsilon)$ . Then,  $\{k \in \mathbb{N} \mid x_k \in N(\mu; \varepsilon)\}$  is infinite since it has an infinite set  $\{k \in \mathbb{N} \mid x_k \in N(c; \varepsilon_1)\}$  as its subset. (See Lemma 1.3.1.)

The second part can be proven analogously.  $\square$

### Theorem 1.3.11

Let  $\{x_k\}$  be any bounded sequence in  $\mathbb{R}$ . Fix any  $\varepsilon \in \mathbb{R}_+$ .

- Let  $\mu = \limsup x_k$ .
  - $\exists k_0 \in \mathbb{N}, \forall k \in \mathbb{N}_{\geq k_0}, x_k < \mu + \varepsilon$ .
  - $\forall k \in \mathbb{N}, \exists k_1 \in \mathbb{N}_{> k}, x_{k_1} > \mu - \varepsilon$ .
- Let  $\nu = \liminf x_k$ .
  - $\exists k_0 \in \mathbb{N}, \forall k \in \mathbb{N}_{\geq k_0}, x_k > \nu - \varepsilon$ .
  - $\forall k \in \mathbb{N}, \exists k_1 \in \mathbb{N}_{> k}, x_{k_1} < \nu + \varepsilon$ .

**Proof.** Take any  $\varepsilon \in \mathbb{R}_+$ . Then,  $\{k \in \mathbb{N} \mid x_k \geq \mu + \varepsilon\}$  is finite. If it were not, then there would be a cluster point larger than  $\mu$  since Theorem 1.3.8 implies the existence of a cluster point in a subsequence of  $\{x_k\}$  which is composed of  $x_k$ 's not smaller than  $\mu + \varepsilon$ . Therefore, if  $k_0 := \max\{k \in \mathbb{N} \mid x_k \geq \mu + \varepsilon\} + 1$ , then  $x_k < \mu + \varepsilon$  for all  $k$  not smaller than  $k_0$ .

Also, since  $\mu$  is a cluster point by Theorem 1.3.10,  $\forall k \in \mathbb{N}, \exists k_1 \in \mathbb{N}_{> k}, x_{k_1} > \mu - \varepsilon$ . (See Lemma 1.3.1.)

The second part can be proven analogously.  $\square$

### Theorem 1.3.12

Let  $\{x_k\}$  be any sequence in  $\mathbb{R}$ .

- (i)  $\{x_k\}$  converges to  $x_0$  if and only if  $\liminf x_k = \limsup x_k = x_0$ .
- (ii)  $\{x_k\}$  diverges if and only if one of the following holds.
  - Either  $\liminf x_k$  or  $\limsup x_k$  is infinite.
  - Both  $\liminf x_k$  or  $\limsup$  are finite and  $\liminf x_k < \limsup x_k$ .

**Proof.**

- (i)  $(\Rightarrow)$   $C = \{x_0\}$ , therefore  $\liminf x_k = \limsup x_k = x_0$ .  
 $(\Leftarrow)$  Take any  $\varepsilon \in \mathbb{R}_+$ . There are natural numbers  $k_1$  and  $k_2$  such that  $\forall k \in \mathbb{N}_{\geq k_1}, x_k < x_0 + \varepsilon$  and  $\forall k \in \mathbb{N}_{\geq k_2}, x_k > x_0 - \varepsilon$ . Then, for all natural number  $k$  not smaller than  $k_0 := \max\{k_1, k_2\}$ ,  $x_0 - \varepsilon < x_k < x_0 + \varepsilon$  holds.
- (ii) If it is not  $\liminf x_k = \limsup x_k \in \mathbb{R}$ , then it is either “One of them is infinite.” or “They are both finite but they are different.”  $\square$

### Exercise 1.3.1

Let  $\{x_k\}$  be a bounded sequence of positive numbers. For each  $k \in \mathbb{N}$  define  $y_k := x_{k+1}/x_k$  and  $z_k := (x_k)^{1/k}$ . Prove that  $\liminf y_k \leq \liminf z_k \leq \limsup z_k \leq \limsup y_k$ .

**Solution:** ( $\liminf y_k \leq \liminf z_k$ ) Let  $L := \liminf y_k$ . Now, we claim that

$$\forall \varepsilon \in \mathbb{R}_+, \exists k_0 \in \mathbb{N}, \forall k \in \mathbb{N}_{\geq k_0}, z_k > L - \varepsilon.$$

If  $L = 0$ , then it is done. Therefore, suppose  $L > 0$ . To prove this, take any  $\varepsilon \in \mathbb{R}_+$  smaller than  $L$ . Then, there is some  $k_1 \in \mathbb{N}$  such that  $y_k > L - \varepsilon/2$  for all  $k$  not smaller than  $k_1$  by

Theorem 1.3.11. Then, for all  $k \in \mathbb{N}_{\geq k_1}$ ,  $x_k > (L - \varepsilon/2)^{k-k_1} x_{k_1}$ , which is equivalent to

$$z_k = x_k^{1/k} > \left(L - \frac{\varepsilon}{2}\right) \left[\left(L - \frac{\varepsilon}{2}\right)^{-k_1} x_{k_1}\right]^{1/k}.$$

Since  $\lim_{k \rightarrow \infty} \left[\left(L - \varepsilon/2\right)^{-k_1} x_{k_1}\right]^{1/k} = 1$ , there is some  $k_2 \in \mathbb{N}$  such that

$$\left[\left(L - \frac{\varepsilon}{2}\right)^{-k_1} x_{k_1}\right]^{1/k} > 1 - \frac{\varepsilon/2}{L - \varepsilon/2} = \frac{L - \varepsilon}{L - \varepsilon/2}.$$

for all  $k \in \mathbb{N}_{\geq k_2}$ . Thus, for every natural number  $k$  not smaller than  $\max\{k_1, k_2\}$ ,

$$z_k > \left(L - \frac{\varepsilon}{2}\right) \left[\left(L - \frac{\varepsilon}{2}\right)^{-k_1} x_{k_1}\right]^{1/k} > \left(L - \frac{\varepsilon}{2}\right) \cdot \frac{L - \varepsilon}{L - \varepsilon/2} = L - \varepsilon.$$

The claim is now proven.

For the main proof, assume that  $\liminf z_k < L$  for the sake of contradiction. Take  $\varepsilon_0 := (L - \liminf z_k)/2$ . Then, by the previous claim,  $\exists k_3 \in \mathbb{N}$ ,  $\forall k \in \mathbb{N}_{\geq k_3}$ ,  $z_k > L - \varepsilon_0 = (L + \liminf z_k)/2$ .

Nevertheless, by Theorem 1.3.11, there is some  $k_4 \in \mathbb{N}_{>k_3}$  such that  $z_{k_4} < \liminf z_k + \varepsilon_0 = (L + \liminf z_k)/2$ , which is a contradiction.

$\limsup z_k \leq \limsup y_k$  can be proven analogously.

□

## 1.4 Cauchy Sequences

### Definition 1.4.1: Cauchy Sequence

A sequence  $\{x_k\}$  in  $\mathbb{R}$  is called a *Cauchy sequence* if

$$\forall \varepsilon \in \mathbb{R}_+, \exists k_0 \in \mathbb{N}, \forall k, m \in \mathbb{N}_{\geq k_0}, |x_k - x_m| < \varepsilon.$$

### Theorem 1.4.1

If  $\{x_k\}$  is a convergent sequence of real numbers, then  $\{x_k\}$  is a Cauchy sequence.

**Proof.** Let  $x_0 := \lim_{k \rightarrow \infty} x_k$ . Take any  $\varepsilon \in \mathbb{R}_+$ . Then, there is some  $k_0 \in \mathbb{N}$  such that  $|x_k - x_0| < \varepsilon/2$  for all  $k \in \mathbb{N}$  not smaller than  $k_0$ . Then, for all  $k, m \in \mathbb{N}$  greater than  $k_0$ ,  $|x_k - x_m| \leq |x_k - x_0| + |x_0 - x_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ . □

### Theorem 1.4.2

If  $\{x_k\}$  is a Cauchy sequence, then  $\{x_k\}$  is bounded.

**Proof.** There is  $k_0 \in \mathbb{N}$  such that  $|x_k - x_m| < 1$  for all  $k, m \in \mathbb{N}_{\geq k_0}$ . It implies that  $|x_k - x_{k_0}| < 1$ , for all  $k \in \mathbb{N}_{\geq k_0}$ , which implies  $|x_k| < |x_{k_0}| + 1$ . Therefore, for all  $k \in \mathbb{N}$ ,  $|x_k| \leq \max\{|x_1|, |x_2|, \dots, |x_{k_0}|, |x_{k_0}| + 1\}$ . □

### Theorem 1.4.3

A Cauchy sequence has exactly one cluster point.

**Proof.** Since a Cauchy sequence is bounded, it has at least one cluster point by Theorem 1.3.8. So, we should prove that the sequence does not have more than one cluster point. Assume  $c_1$  and  $c_2$  are cluster points for the sake of contradiction. Let  $\varepsilon := |c_1 - c_2|/3$ . Choose  $k_0 \in \mathbb{N}$  such that  $\forall k, m \in \mathbb{N}_{\geq k_0}, |x_k - x_m| < \varepsilon$ . Also, there are  $k_1, k_2 \in \mathbb{N}_{> k_0}$  such that  $|x_{k_1} - c_1| < \varepsilon$  and  $|x_{k_2} - c_2| < \varepsilon$ . Note that  $|c_1 - c_2| \leq |c_1 - x_{k_1}| + |x_{k_1} - x_{k_2}| + |x_{k_2} - c_2|$ . Nevertheless, then

$$\begin{aligned}\varepsilon &> |x_{k_1} - x_{k_2}| \geq |c_1 - c_2| - |c_1 - x_{k_1}| - |c_2 - x_{k_2}| \\ &> 3\varepsilon - \varepsilon - \varepsilon = \varepsilon,\end{aligned}$$

which is a contradiction. □

### Theorem 1.4.4 Cauchy Completeness of $\mathbb{R}$

A sequence in  $\mathbb{R}$  is convergent if and only if it is a Cauchy sequence.

**Proof.** By Corollary 1.3.3, a Cauchy sequence is convergent since it is bounded (Theorem 1.4.2) and has exactly one cluster point (Theorem 1.4.3). A convergent sequence in  $\mathbb{R}$  is Cauchy. (Theorem 1.4.1) □

### Definition 1.4.2: Cauchy Completeness

A set  $X$  is said to be *Cauchy complete* if every Cauchy sequence in  $X$  converges to a point of  $X$ .

### Example 1.4.1

$\mathbb{R}$  is Cauchy complete.

### Definition 1.4.3: Contractive Sequence

A sequence  $\{x_k\}$  is said to be *contractive* if there exists a constant  $C$ , with  $0 < C < 1$ , such that

$$\forall k \in \mathbb{N}_{>1}, |x_{k+1} - x_k| \leq C|x_k - x_{k-1}|.$$

### Theorem 1.4.5

Any contractive sequence in  $\mathbb{R}$  is a Cauchy sequence.

**Proof.** Suppose  $0 < C < 1$  and  $\forall k \in \mathbb{N}_{>1}, |x_{k+1} - x_k| \leq C|x_k - x_{k-1}|$ . If it is trivial when  $|x_2 - x_1| = 0$ , so suppose  $|x_2 - x_1| \neq 0$ . By induction,  $\forall k \in \mathbb{N}_{>1}, |x_{k+1} - x_k| \leq C^{k-1}|x_2 - x_1|$ .

To prove  $\{x_k\}$  is a Cauchy sequence, take any  $\varepsilon \in \mathbb{R}_+$ . Since  $\lim_{k \rightarrow \infty} C^{k-1} = 0$ ,

$$\exists k_0 \in \mathbb{N}, \forall k \in \mathbb{N}_{\geq k_0}, C^{k-1} < \frac{(1-C)\varepsilon}{|x_2 - x_1|}.$$

Then, for any  $k, m \in \mathbb{N}$  with  $k_0 \leq m < k$ ,

$$\begin{aligned}
 |x_k - x_m| &= \left| \sum_{j=m}^{k-1} (x_{j+1} - x_j) \right| \leq \sum_{j=m}^{k-1} |x_{j+1} - x_j| \\
 &\leq \sum_{j=m}^{k-1} C^{j-1} |x_2 - x_1| = C^{m-1} |x_2 - x_1| \sum_{j=0}^{k-m-1} C^j \\
 &= C^{m-1} |x_2 - x_1| \frac{1 - C^{k-m}}{1 - C} < \frac{C^{m-1}}{1 - C} |x_2 - x_1| \\
 &< \frac{(1 - C)\varepsilon}{|x_2 - x_1|} \cdot \frac{1}{1 - C} |x_2 - x_1| = \varepsilon.
 \end{aligned}$$

□

## 1.5 The Algebra of Convergent Series

### Theorem 1.5.1

Let  $\{x_k\}$  and  $\{y_k\}$  be convergent sequences in  $\mathbb{R}$  and  $\lim_{k \rightarrow \infty} x_k = x_0$  and  $\lim_{k \rightarrow \infty} y_k = y_0$ .

- $\lim_{k \rightarrow \infty} (x_k + y_k) = x_0 + y_0$
- $\lim_{k \rightarrow \infty} x_k y_k = x_0 y_0$
- $\lim_{k \rightarrow \infty} \frac{y_k}{x_k} = \frac{y_0}{x_0}$  if  $x_0 \neq 0$ .

### Theorem 1.5.2

Let  $\{x_k\}$  and  $\{y_k\}$  be convergent sequences in  $\mathbb{R}$  and  $\lim_{k \rightarrow \infty} x_k = x_0$ . Then, if  $r \in \mathbb{Q}$ , then

$$\lim_{k \rightarrow \infty} x_k^r = x_0^r.$$

Nevertheless, we require  $x_0 \neq 0$  if  $r < 0$ .

## 1.6 Cardinality

### Definition 1.6.1: Dense Set

We say a subset  $S$  of  $T$  is dense in  $T$  if every neighborhood of any point  $x \in T$  contains points of  $S$ .

### Theorem 1.6.1

- $\mathbb{N}$ ,  $\mathbb{Z}$ , and  $\mathbb{Q}$  are countably infinite.
- $\mathbb{R}$  is uncountable.
- $\mathbb{Q}$  is dense in  $\mathbb{R}$ .

# Chapter 2

## Euclidean Spaces

### 2.1 Euclidean $n$ -Space

#### Definition 2.1.1: Inner Product

The *inner product* of two vectors  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  and  $\mathbf{y} = (y_1, y_2, \dots, y_n)$  in  $\mathbb{R}^n$  is

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^n x_j y_j.$$

#### Theorem 2.1.1

If  $\mathbf{x}$ ,  $\mathbf{y}$ , and  $\mathbf{z}$  are arbitrary vectors in  $\mathbb{R}^n$  and if  $a$  and  $b$  are real numbers, then the following hold:

(i) The inner product is *additive* in both its variables:

$$\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$$

$$\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$$

(ii) The inner product is *symmetric*:  $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ .

(iii) The inner product is *homogeneous* in both its variables:  $\langle a\mathbf{x}, b\mathbf{y} \rangle = ab\langle \mathbf{x}, \mathbf{y} \rangle$ .

#### Definition 2.1.2: Euclidean Norm

The *Euclidean norm* of a vector  $\mathbf{x}$  in  $\mathbb{R}^n$  is

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

#### Theorem 2.1.2 The Cauchy-Schwarz Inequality

If  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ , then

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|.$$

**Proof.** For any  $t \in \mathbb{R}$ ,  $0 \leq \|t\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 t^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle t + \|\mathbf{y}\|^2$ . Thus, the discriminant  $|\langle \mathbf{x}, \mathbf{y} \rangle|^2 - \|\mathbf{x}\|^2 \|\mathbf{y}\|^2$  is nonpositive.  $\square$

#### Theorem 2.1.3

For vectors  $\mathbf{x}$  and  $\mathbf{y}$  in  $\mathbb{R}^n$  and any  $c \in \mathbb{R}$ , the Euclidean norm has the following proper-



ties.

- (i)  $\|\mathbf{x}\| \geq 0$ ;  $\|\mathbf{x}\| = 0$  if and only if  $\mathbf{x} = \mathbf{0}$ . (*Positive Definiteness*)
- (ii)  $\|c\mathbf{x}\| = |c| \cdot \|\mathbf{x}\|$ . (*Absolute Homogeneity*)
- (iii)  $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ . (*Subadditivity*)

**Proof of (iii).**

$$\begin{aligned} 0 \leq \|\mathbf{x} + \mathbf{y}\|^2 &= \|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2 \\ &\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2 \end{aligned}$$

□

### Definition 2.1.3: Norm

A *norm* on  $\mathbb{R}^n$  is any function  $n: \mathbb{R}^n \rightarrow \mathbb{R}$  that is positive definite, absolutely homogeneous, and subadditive.

### Definition 2.1.4: Metric

A *metric* on  $\mathbb{R}^n$  is a function from  $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  having the following properties.

- (i)  $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ ,  $d(\mathbf{x}, \mathbf{y}) \geq 0$ ;  $d(\mathbf{x}, \mathbf{y}) = 0 \iff \mathbf{x} = \mathbf{y}$ . (*Positive Definiteness*)
- (ii)  $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ ,  $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$ . (*Symmetry*)
- (iii)  $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ ,  $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$ . (*The Triangle Inequality*)

### Definition 2.1.5: Euclidean Metric

The *Euclidean metric* on  $\mathbb{R}^n$  is defined by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \left[ \sum_{j=1}^n (x_j - y_j)^2 \right]^{1/2}.$$

### Theorem 2.1.4

The Euclidean metric is a metric on  $\mathbb{R}^n$ .

### Definition 2.1.6: Orthogonality

Two vectors  $\mathbf{x}$  and  $\mathbf{y}$  in  $\mathbb{R}^n$  are said to be *orthogonal* if  $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ .

### Definition 2.1.7: Neighborhood and Deleted Neighborhood

A *neighborhood*  $N(\mathbf{x}; r)$  or  $\mathbf{x} \in \mathbb{R}^n$  with radius  $r$  is the set

$$N(\mathbf{x}; r) = \{\mathbf{y} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{y}\| < r\}.$$

A *deleted neighborhood*  $N'(\mathbf{x}, r)$  of  $\mathbf{x}$  is  $N'(\mathbf{x}; r) = N(\mathbf{x}; r) \setminus \{\mathbf{x}\}$ .

### Definition 2.1.8: Limit Point

Let  $S$  be nonempty subset of  $\mathbb{R}^n$ . We say that  $\mathbf{x}$  is a *limit point* of  $S$  if

$$\forall \varepsilon \in \mathbb{R}_+, N'(\mathbf{x}; \varepsilon) \cap S \neq \emptyset.$$

### Theorem 2.1.5

$\mathbb{Q}^n$  is dense in  $\mathbb{R}^n$ .

**Proof.** Take any  $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$  and  $\varepsilon \in \mathbb{R}_+$ . For each  $j = 1, 2, \dots, n$ , choose a rational  $x_j \in N(y_j; \varepsilon/\sqrt{n})$  and form  $\mathbf{x} \triangleq (x_1, x_2, \dots, x_n) \in \mathbb{Q}^n$ . Then,

$$\|\mathbf{x} - \mathbf{y}\|^2 = \sum_{j=1}^n (x_j - y_j)^2 < n(\varepsilon/\sqrt{n})^2 = \varepsilon^2.$$

Therefore  $\mathbf{y}$  is a limit point of  $\mathbb{Q}^n$ . □

### Definition 2.1.9: Boundedness

A subset  $S$  of  $\mathbb{R}^n$  is said to be *bounded* if

$$\exists M \in \mathbb{R}_+, \forall \mathbf{x} \in S, \|\mathbf{x}\| \leq M.$$

## 2.1.1 Sequences in $\mathbb{R}^n$

### Definition 2.1.10: Cluster Point

$\mathbf{c} \in \mathbb{R}^n$  is a *cluster point* of the sequence  $\{\mathbf{x}_k\}$  if,

$$\forall (\varepsilon, k) \in \mathbb{R}_+ \times \mathbb{N}, \exists k_1 \in \mathbb{N}_{>k}, \mathbf{x}_{k_1} \in N(\mathbf{c}; \varepsilon).$$

### Definition 2.1.11: Convergence and Divergence of a Sequence

The sequence  $\{\mathbf{x}_k\}$  *converges* to  $\mathbf{x}_0$  and  $\mathbf{x}_0$  is the *limit* of  $\{\mathbf{x}_k\}$  if,

$$\forall \varepsilon \in \mathbb{R}_+, \exists k_0 \in \mathbb{N}, \forall k \in \mathbb{N}_{\geq k_0}, \mathbf{x}_k \in N(\mathbf{x}_0; \varepsilon).$$

We write  $\lim_{k \rightarrow \infty} \mathbf{x}_k = \mathbf{x}_0$ . If there is no such  $\mathbf{x}_0$ , then  $\{\mathbf{x}_k\}$  *diverges*.

### Theorem 2.1.6

Let  $\{\mathbf{x}_k\} = (x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)})$  for each  $k \in \mathbb{N}$ . Let  $\mathbf{x}_0 = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$ . The sequence  $\{\mathbf{x}_k\}$  converges to  $\mathbf{x}_0$  if and only if, for each  $j \in [n]$ , the sequence  $\{x_j^{(k)}\}$  converges to  $\{x_j^{(0)}\}$ .

**Proof.** ( $\Rightarrow$ ) Take any  $\varepsilon \in \mathbb{R}_+$ . There there is  $k_0 \in \mathbb{N}$  such that

$$\forall k \in \mathbb{N}_{\geq k_0}, \mathbf{x}_k \in N(\mathbf{x}_0; \varepsilon).$$

Then, for each  $j \in [n]$ ,

$$(x_j^{(k)} - x_0^{(k)})^2 \leq \sum_{i=1}^n (x_i^{(k)} - x_0^{(k)})^2 = \|\mathbf{x}_k - \mathbf{x}_0\|^2 < \varepsilon.$$

( $\Leftarrow$ ) Take any  $\varepsilon \in \mathbb{R}_+$ . Then, for each  $j \in [n]$ , there is some  $k_j \in \mathbb{N}$  such that

$$\forall k \in \mathbb{N}_{\geq k_j}, x_j^{(k)} \in N(x_0^{(k)}; \varepsilon/\sqrt{n}).$$

Then, for all natural number  $k$  not smaller than  $\max_{j \in [n]} k_j$ ,

$$\|\mathbf{x}_k - \mathbf{x}_0\|^2 = \sum_{j=1}^n (x_j^{(k)} - x_0^{(k)})^2 < n(\varepsilon/\sqrt{n})^2 = \varepsilon^2.$$

□

### Definition 2.1.12: Cauchy Sequence

A sequence  $\{\mathbf{x}_k\}$  in  $\mathbb{R}^n$  is called a *Cauchy sequence* if

$$\forall \varepsilon \in \mathbb{R}_+, \exists k_0 \in \mathbb{N}, \forall k, m \in \mathbb{N}_{\geq k_0}, \|\mathbf{x}_k - \mathbf{x}_m\| < \varepsilon.$$

### Theorem 2.1.7 Cauchy's Completeness Theorem in $\mathbb{R}^n$

A sequence  $\{\mathbf{x}_k\}$  in  $\mathbb{R}^n$  is Cauchy if and only if it converges.  $\mathbb{R}^n$  is Cauchy complete.

**Proof.** ( $\Leftarrow$ ) The proof is similar to Theorem 1.4.1.

( $\Rightarrow$ ) Let some Cauchy sequence  $\{\mathbf{x}_k\}$  in  $\mathbb{R}^n$  be given. Take any  $\varepsilon \in \mathbb{R}_+$ . There is some  $k_0 \in \mathbb{N}$  such that for every natural number  $k$  and  $m$  not smaller than  $k_0$ ,  $\|\mathbf{x}_k - \mathbf{x}_m\| < \varepsilon$ . Then, for each  $j \in [n]$ ,  $|x_j^{(k)} - x_j^{(m)}| \leq \|\mathbf{x}_k - \mathbf{x}_m\| < \varepsilon$ , which implies each  $\{x_j^{(k)}\}_{k \in \mathbb{N}}$  is Cauchy. By Theorem 1.4.4,  $\{x_j^{(k)}\}_{k \in \mathbb{N}}$  converges to some number  $x_j^{(0)}$ . Then, Theorem 2.1.6 ensures that  $\lim_{k \rightarrow \infty} \mathbf{x}_k = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$ . □

### Theorem 2.1.8 The Generalized Bolzano–Weierstrass Theorem

Every bounded infinite set in  $\mathbb{R}^n$  has a limit point in  $\mathbb{R}^n$ .

**Proof.** Suppose that  $S$  is any bounded, infinite set in  $\mathbb{R}^n$ . Being bounded,  $S$  is contained in some  $n$ -cube  $C(2M) = [-M, M]^n$  centered at  $\mathbf{0}$ . Construct  $C_1, C_2, \dots$  as following.

- $C_1 \triangleq C(2M) = [a_1^{(1)}, b_1^{(1)}] \times \dots \times [a_n^{(1)}, b_n^{(1)}]$   
– Note that  $C_1 \cap S = S$  is infinite.
- For each  $k \in \mathbb{N}$ ,  $C_{k+1}$  is any cube of the form  $[a_1^{(k+1)}, b_1^{(k+1)}] \times \dots \times [a_n^{(k+1)}, b_n^{(k+1)}]$  where each  $[a_j^{(k+1)}, b_j^{(k+1)}]$  is either  $[a_j^{(k)}, (a_j^{(k)} + b_j^{(k)})/2]$  or  $[(a_j^{(k)} + b_j^{(k)})/2, b_j^{(k)}]$  so that  $C_{k+1} \cap S$  is infinite.  
– This is possible since there is at least one cube among  $2^n$  possible choices that  $C_{k+1} \cap S$  is infinite.

Then, the main diagonal  $d_k$  of  $C_k$  equals to  $Mn^{1/2}/2^{k-2}$ . Also, note that  $C_k \supseteq C_{k+1}$  for all  $k \in \mathbb{N}$ .

Now, we may construct a sequence  $\{\mathbf{x}_k\}_{k \in \mathbb{N}}$  as following.

- $\mathbf{x}_1$  is any element in  $C_1 \cap S$ .
- For each  $k \in \mathbb{N}$ ,  $\mathbf{x}_{k+1}$  is arbitrarily taken from  $C_{k+1} \cap S \setminus \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ .

We claim that  $\{\mathbf{x}_k\}_{k \in \mathbb{N}}$  is a Cauchy sequence. To show this, take any  $\varepsilon \in \mathbb{R}_+$ . There is some  $k_0 \in \mathbb{N}$  such that  $d_{k_0} = Mn^{1/2}/2^{k_0-2} < \varepsilon$  by Theorem 1.1.1. Then, for all  $k, m \in \mathbb{N}_{\geq k_0}$ ,  $\|\mathbf{x}_k - \mathbf{x}_m\| \leq d_{k_0} < \varepsilon$ . Therefore, since  $\{\mathbf{x}_k\}$  is Cauchy, and therefore convergent by Theorem 2.1.7.

Clearly,  $\mathbf{x}_0 \triangleq \lim_{k \rightarrow \infty} \mathbf{x}_k$  is a limit point of  $S$  since any deleted neighborhood  $N'(\mathbf{x}_0)$  of  $\mathbf{x}_0$  intersects infinitely many points with  $\{\mathbf{x}_k\}_{k \in \mathbb{N}} \subseteq S$ . □

### Definition 2.1.13: Subsequence

Let  $\{\mathbf{x}_k\}$  be any sequence in  $\mathbb{R}^n$ . Choose any strictly monotone increasing sequence  $k_1 < k_2 < k_3 < \dots$  of natural numbers. For each  $j \in \mathbb{N}$ , let  $\mathbf{y}_j := \mathbf{x}_{k_j}$ . The sequence  $\{\mathbf{y}_j\}_{j=1}^\infty$  is called an *subsequence* of  $\{\mathbf{x}_k\}$ .

### Theorem 2.1.9

The point  $\mathbf{c}$  is a cluster point of  $\{\mathbf{x}_k\}$  if and only if there exists a subsequence of  $\{\mathbf{x}_k\}$  that converges to  $\mathbf{c}$ .

**Proof.** Analogous to Theorem 1.3.7. □

### Theorem 2.1.10

Any bounded sequence  $\{\mathbf{x}_k\}$  has a cluster point.

**Proof.** Analogous to Theorem 1.3.8. □

### Corollary 2.1.1

If a sequence in  $\mathbb{R}^n$  has no cluster point, then the sequence is unbounded.

### Corollary 2.1.2

Any bounded sequence in  $\mathbb{R}^n$  converges if and only if it has exactly one cluster point.

### Corollary 2.1.3

A sequence  $\{\mathbf{x}_k\}$  diverges if and only if at least one of the following conditions holds.

- $\{\mathbf{x}_k\}$  has two or more cluster points.
- $\{\mathbf{x}_k\}$  is unbounded.

## 2.2 Open and Closed Sets

### Definition 2.2.1: Interior/Boundary Point and Open/Closed Set

Let  $S$  be any subset of  $\mathbb{R}^n$  and let  $\mathbf{x}$  be any point in  $\mathbb{R}^n$ .

- (i)  $\mathbf{x}$  is an *interior point* of  $S$  if  $\exists r \in \mathbb{R}_+, N(\mathbf{x}; r) \subseteq S$ .
- (ii) If every point of  $S$  is an interior point of  $S$ , then  $S$  is said to be *open*.
- (iii) We call  $\mathbf{x}$  is a *boundary point* of  $S$  if  $\forall r \in \mathbb{R}_+, N(\mathbf{x}; r) \cap S \neq \emptyset \wedge N(\mathbf{x}; r) \setminus S \neq \emptyset$ .
- (iv) If  $S$  contains all its boundary points, then  $S$  is said to be *closed*.

### Definition 2.2.2

Let  $S \subseteq \mathbb{R}^n$ .

- (i) The *interior* of  $S$ , denoted  $\mathring{S}$ , is the set of all interior points of  $S$ .
- (ii) The *boundary* of  $S$ , denoted  $\text{bd } S$ , is the set of all boundary points of  $S$ .
- (iii) The *derived set* of  $S$ , denoted  $S'$ , is the set of all limit points of  $S$ .
- (iv) The *closure* of  $S$ , denoted  $\bar{S}$ , is the union of  $S$  and  $S'$ .
- (v) The *complement* of  $S$ , denoted  $S^c$ , is the set  $\mathbb{R}^n \setminus S$ .

### Note:-

- For  $S \subseteq \mathbb{R}^n$ ,  $\mathring{S} \subseteq S \subseteq \bar{S}$ .
- For  $S \subseteq \mathbb{R}^n$ ,  $S$  is open if and only if  $\mathring{S} = S$ .
- For  $S \subseteq \mathbb{R}^n$ ,  $\mathring{S}$  is open.

### Theorem 2.2.1

The union of any collection of open sets in  $\mathbb{R}^n$  is open. The intersection of any finite collection of open sets in  $\mathbb{R}^n$  is also open.

**Proof.** To prove the first assertion, suppose that  $\{U_\alpha \mid \alpha \in J\}$  is any collection of open sets in  $\mathbb{R}^n$ . Let  $U \triangleq \bigcup_{\alpha \in J} U_\alpha$ . Take any  $\mathbf{x} \in U$ . Then, there is some  $\alpha_0 \in J$  such that  $\mathbf{x} \in U_{\alpha_0}$ . Since  $U_{\alpha_0}$  is open, there is some neighborhood  $N(\mathbf{x}; \varepsilon)$  such that  $N(\mathbf{x}; \varepsilon) \subseteq U_{\alpha_0}$ , which, in turn,  $N(\mathbf{x}; \varepsilon) \subseteq U$ . Therefore,  $\mathbf{x}$  is an interior point of  $U$ ;  $U$  is open.

To prove the second assertion, let  $U$  be the intersection of any finite collection  $\{U_1, U_2, \dots, U_k\}$  of open sets and take any  $\mathbf{x} \in U$ . For each  $j \in [k]$ , since  $\mathbf{x} \in U_j$ , there is some  $r_j \in \mathbb{R}_+$  such that  $N(\mathbf{x}; r_j) \subseteq U_j$ . Then, take  $r_0 \triangleq \min_{j \in [k]} r_j \in \mathbb{R}_+$ . Since, for all  $j \in [k]$ ,  $N(\mathbf{x}; r_0) \subseteq U_j$ , it is implied that  $N(\mathbf{x}; r_0) \subseteq U$ . Therefore,  $\mathbf{x}$  is an interior point of  $U$ ;  $U$  is open.  $\square$

#### Note:-

Intersection of infinitely many open sets may fail to be open. For instance, consider

$$U_k \triangleq N(\mathbf{0}; 1/k),$$

for each  $k \in \mathbb{N}$ . Then,  $\bigcap_{k \in \mathbb{N}} U_k = \{\mathbf{0}\}$ , which is not open.

### Theorem 2.2.2

A set  $C \subseteq \mathbb{R}^n$  is closed if and only if  $C^c$  is open.

**Proof.** ( $\Rightarrow$ ) Take any  $\mathbf{x} \in C^c$ . Since  $C$  is closed and contains all of its boundary points,  $\mathbf{x}$  is not a boundary point of  $C$ . Therefore, there is some neighborhood  $N(\mathbf{x})$  of  $\mathbf{x}$  such that  $N(\mathbf{x})C = \emptyset$  or  $N(\mathbf{x}) \cap C^c = \emptyset$ . The second case is not possible since  $\mathbf{x} \in N(\mathbf{x}) \cap C^c$ . Therefore,  $N(\mathbf{x}) = \emptyset$ , which implies  $N(\mathbf{x}) \subseteq C^c$ ;  $\mathbf{x}$  is an interior point of  $C^c$ . Therefore,  $C^c$  is open.

( $\Leftarrow$ ) Take any boundary point  $\mathbf{x}$  of  $C$ . Assume  $\mathbf{x} \in C^c$  for the sake of contradiction. Since  $C^c$  is open, there is a neighborhood  $N(\mathbf{x})$  of  $\mathbf{x}$  such that  $N(\mathbf{x}) \subseteq C^c$ . However, that implies  $N(\mathbf{x}) \cap C = \emptyset$ , which contradicts  $\mathbf{x}$  is a boundary point of  $C$ . Therefore,  $\mathbf{x} \in C$ ;  $C$  contains all of its boundary points.  $\square$

### Theorem 2.2.3

The intersection of any collection of closed sets in  $\mathbb{R}^n$  is closed. The union of any finite collection of closed sets in  $\mathbb{R}^n$  is also closed.

**Proof.** To prove the first assertion, let  $\{C_\alpha\}_{\alpha \in J}$  be any collection of closed sets in  $\mathbb{R}^n$ . Then, each  $C_\alpha^c$  is open by Theorem 2.2.2, and thus  $\bigcup_{\alpha \in J} C_\alpha^c$  is open by Theorem 2.2.1. Its complement  $(\bigcup_{\alpha \in J} C_\alpha^c)^c$  is closed by Theorem 2.2.2. And note that  $(\bigcup_{\alpha \in J} C_\alpha^c)^c = \bigcap_{\alpha \in J} C_\alpha$  by De Morgan's law.

To prove the second assertion, let  $\{C_1, C_2, \dots, C_k\}$  be a finite collection of closed sets in  $\mathbb{R}^n$ . Then, each  $C_i^c$  is open by Theorem 2.2.2, and thus  $\bigcap_{i=1}^k C_i^c$  is open by Theorem 2.2.1. Its complement  $(\bigcap_{i=1}^k C_i^c)^c$  is closed by Theorem 2.2.2. And note that  $(\bigcap_{i=1}^k C_i^c)^c = \bigcup_{i=1}^k C_i$  by De Morgan's law.  $\square$

### Theorem 2.2.4

$C \subseteq \mathbb{R}^n$  is closed if and only if  $C' \subseteq C$ .

**Proof.** ( $\Rightarrow$ ) Let  $\mathbf{x} \in C'$ . Assume  $\mathbf{x} \in C^c$  for the sake of contradiction. Since  $C^c$  is open by Theorem 2.2.2, there is a neighborhood  $N(\mathbf{x})$  of  $\mathbf{x}$  such that  $N(\mathbf{x}) \subseteq C^c$ . Such  $N(\mathbf{x})$  satisfies  $N(\mathbf{x}) \cap C = \emptyset$ , which contradicts  $\mathbf{x} \in C'$ . Therefore,  $\mathbf{x} \in C$ ;  $C$  contains all its limit points.

( $\Leftarrow$ ) It is enough to prove  $C^c$  is open by Theorem 2.2.2. Take any  $\mathbf{x} \in C^c$ .  $\mathbf{x}$  is not a limit point of  $C$  by the hypothesis. Therefore, there is a deleted neighborhood  $N'(\mathbf{x})$  of  $\mathbf{x}$  such that  $N'(\mathbf{x}) \cap C = \emptyset$ . Then,  $N'(\mathbf{x}) \subseteq C^c$ , and thus  $N(\mathbf{x}) \subseteq C^c$ , which implies  $\mathbf{x}$  is an interior point of  $C^c$ . Thus,  $C^c$  is open.  $\square$

### Corollary 2.2.1

$C \subseteq \mathbb{R}^n$  is closed if and only if  $\overline{C} = C$ .

### Theorem 2.2.5

Let  $S \subseteq \mathbb{R}^n$ . The interior of  $S$  is the union of all open sets contained in  $S$ .

**Proof.** Let  $\mathcal{U} \triangleq \{U \subseteq S \mid U \text{ is open in } \mathbb{R}^n\}$ .

( $\subseteq$ ) Let  $\mathbf{x} \in \mathring{S}$ . Then, there is an open neighborhood  $N(\mathbf{x})$  of  $\mathbf{x}$  such that  $N(\mathbf{x}) \subseteq S$ . Noting that  $\mathbf{x} \in N(\mathbf{x}) \in \mathcal{U}$ , we conclude  $\mathring{S} \subseteq \bigcup \mathcal{U}$ .

( $\supseteq$ ) Take any  $\mathbf{x} \in \bigcup \mathcal{U}$ . Then, there is an open set  $U$  in  $\mathbb{R}^n$  such that  $\mathbf{x} \in U \subseteq S$ . There is a neighborhood  $N(\mathbf{x})$  of  $\mathbf{x}$  such that  $N(\mathbf{x}) \subseteq U$ . Therefore,  $N(\mathbf{x}) \subseteq S$ ;  $\mathbf{x}$  is an interior point of  $S$ . Thus;  $\mathring{S} \supseteq \bigcup \mathcal{U}$ .  $\square$

### Theorem 2.2.6

The closure of  $S$  is the intersection of all closed sets that contain  $S$ .

**Proof.** Let  $\mathcal{C} \triangleq \{C \subseteq \mathbb{R}^n \mid S \subseteq C \text{ and } C \text{ is closed}\}$ .

( $\subseteq$ ) Since  $S \subseteq \bigcap \mathcal{C}$  is obvious, we only need to show  $S' \subseteq \bigcap \mathcal{C}$ . Let  $\mathbf{x} \in S'$ . Then, it is direct that  $\forall C \in \mathcal{C}, \mathbf{x} \in C'$  since each  $C \in \mathcal{C}$  satisfies  $S \subseteq C$ . As  $C$  is closed and thus  $\mathbf{x} \in C' \subseteq C$  by Theorem 2.2.4, Consequently,  $\mathbf{x} \in \bigcap \mathcal{C}$ ;  $\overline{S} \subseteq \bigcap \mathcal{C}$ .

( $\supseteq$ ) It is enough to show that  $\overline{S}$  is closed, which, in turn, is sufficient to show that  $(\overline{S})' \subseteq \overline{S}$  by Theorem 2.2.4. Let  $\mathbf{y} \in (\overline{S})'$  and take any deleted neighborhood  $N'(\mathbf{y}; \varepsilon)$  of  $\mathbf{y}$ . Then, there is some element  $\mathbf{z}$  in  $N'(\mathbf{y}; \varepsilon) \cap \overline{S}$ . Then,  $\mathbf{z} \in S$  or  $\mathbf{z} \in S'$ .

If  $\mathbf{z} \in S$ , then  $\mathbf{z} \in N'(\mathbf{y}; \varepsilon) \cap S \neq \emptyset$ . If  $\mathbf{z} \in S'$ , take  $\varepsilon' \triangleq \min\{\|\mathbf{z} - \mathbf{y}\|, \varepsilon - \|\mathbf{z} - \mathbf{y}\|\}$ . Then,  $N(\mathbf{z}; \varepsilon') \subseteq N'(\mathbf{y}; \varepsilon)$ . Since  $\mathbf{z} \in S'$ , there is some  $\mathbf{x}$  in  $N'(\mathbf{z}; \varepsilon') \cap S$ . Thus,  $\mathbf{x} \in N'(\mathbf{z}; \varepsilon') \cap S \subseteq N'(\mathbf{y}; \varepsilon) \cap S \neq \emptyset$ .

In both cases,  $N'(\mathbf{y}; \varepsilon) \cap S \neq \emptyset$ . Thus, we proved that  $\mathbf{y} \in S' \subseteq \overline{S}$ ;  $(\overline{S})' \subseteq \overline{S}$ .  $\square$

### Corollary 2.2.2

For any  $S \subseteq \mathbb{R}^n$ , the set  $\overline{S}$  is closed.

### Corollary 2.2.3

For any  $C \subseteq \mathbb{R}^n$ ,  $C$  is closed if and only if  $\overline{C} = C$ .

### Theorem 2.2.7

Let  $S \subseteq \mathbb{R}^n$ .

- (i)  $\overline{\mathring{S}} = \mathring{S}$
- (ii)  $\overline{(\overline{S})} = \overline{S}$
- (iii)  $\mathring{S} \cap \text{bd } S = \emptyset$

- (iv)  $\mathring{S} \cup \text{bd } S = \bar{S}$
- (v)  $\bar{S} \cap \bar{S}^c = \text{bd } S$

**Proof.**

- (i)  $\mathring{S}$  is open and an open set is the interior of itself.
- (ii)  $\bar{S}$  is closed and a closed set is the closure of itself. (See Corollary 2.2.2 and Corollary 2.2.3).
- (iii) Suppose there is some  $\mathbf{x} \in \mathring{S} \cap \text{bd } S$ . There is a neighborhood  $N(\mathbf{x})$  of  $\mathbf{x}$  such that  $N(\mathbf{x}) \subseteq S$ . Then,  $N(\mathbf{x}) \cap S^c = \emptyset$ , which contradicts  $\mathbf{x} \in \text{bd } S$ .
- (iv) ( $\subseteq$ ) Since it is already  $\mathring{S} \subseteq S \subseteq \bar{S}$ , we only need to show  $\text{bd } S \subseteq \bar{S}$ . Let  $\mathbf{x} \in \text{bd } S$ . If  $\mathbf{x} \in S$ , then it is done; so suppose  $\mathbf{x} \in S^c$ . Take any neighborhood  $N(\mathbf{x}; \varepsilon)$  of  $\mathbf{x}$ . Then,  $N(\mathbf{x}; \varepsilon) \cap S \neq \emptyset$ . Noting that  $N'(\mathbf{x}; \varepsilon) \cap S = N(\mathbf{x}; \varepsilon) \cap S \neq \emptyset$ ,  $\mathbf{x} \in S'$ .  
 ( $\supseteq$ ) Let  $\mathbf{x} \in \bar{S}$ . If  $\mathbf{x} \in S$ , then it is either “There is a neighborhood  $N(\mathbf{x})$  of  $\mathbf{x}$  such that  $N(\mathbf{x}) \subseteq S$ .” or “Every neighborhood  $N(\mathbf{x})$  of  $\mathbf{x}$  satisfies  $N(\mathbf{x}) \cap S^c \neq \emptyset$ .” The first case is  $\mathbf{x} \in \mathring{S}$  and the latter case is  $\mathbf{x} \in \text{bd } S$ .  
 Now the only left case is if  $\mathbf{x} \in S' \setminus S$ . Take any deleted neighborhood  $N'(\mathbf{x})$  of  $\mathbf{x}$ . Then,  $N(\mathbf{x}) \cap S = N'(\mathbf{x}) \cap S \neq \emptyset$ . Also,  $\mathbf{x} \in N(\mathbf{x}) \cap S^c$ . Thus,  $\mathbf{x} \in \text{bd } S$ .
- (v) Using  $\bar{S} = \mathring{S} \cup \text{bd } S$ , we get

$$\begin{aligned}\bar{S} \cap \bar{S}^c &= (\mathring{S} \cup \text{bd } S) \cap ((\mathring{S}^c) \cup \text{bd } S^c) \\ &= (\mathring{S} \cap (\mathring{S}^c)) \cup (\mathring{S} \cap \text{bd } S^c) \cup (\text{bd } S \cap (\mathring{S}^c)) \cup (\text{bd } S \cap \text{bd } S^c)\end{aligned}$$

$\mathring{S} \cap (\mathring{S}^c) = \emptyset$  since  $S \cap S^c = \emptyset$  and  $\mathring{S} \subseteq S$  and  $\mathring{S}^c \subseteq S^c$ .

$\text{bd } S = \text{bd } S^c$  is direct from their definitions. Thus,

$$\begin{aligned}\mathring{S} \cap \text{bd } S^c &= \mathring{S} \cap \text{bd } S = \emptyset \\ \text{bd } S \cap (\mathring{S}^c) &= \text{bd } S^c \cap (\mathring{S}^c) = \emptyset\end{aligned}$$

by (iv). Therefore,  $\bar{S} \cap \bar{S}^c = \text{bd } S \cap \text{bd } S^c = \text{bd } S$ .

□

### Definition 2.2.3: Diameter

Let  $\emptyset \neq S \subseteq \mathbb{R}^n$  be a bounded set. The *diameter* of  $S$  is defined to be

$$d(S) \triangleq \sup\{\|\mathbf{x} - \mathbf{y}\| \mid \mathbf{x}, \mathbf{y} \in S\}.$$

### Definition 2.2.4: Distance

Let  $\emptyset \neq S \subseteq \mathbb{R}^n$  and  $\mathbf{x} \in \mathbb{R}^n$ . The distance from  $\mathbf{x}$  to  $S$  is defined to be

$$d(\mathbf{x}, S) \triangleq \inf\{\|\mathbf{x} - \mathbf{y}\| \mid \mathbf{y} \in S\}.$$

### Exercise 2.2.1

Let  $S$  be a nonempty set in  $\mathbb{R}^n$  and let  $\mathbf{x}$  be a point of  $\mathbb{R}^n$ .

- (i)  $d(\mathbf{x}, S) = 0$  if and only if  $\mathbf{x} \in \bar{S}$ .
- (ii)  $S$  is closed if and only if  $d(\mathbf{x}, S) > 0$  for every  $\mathbf{x} \in S^c$ .
- (iii) If  $S$  is closed, then there exists  $\mathbf{y}_0 \in S$  such that  $d(\mathbf{x}, S) = \|\mathbf{x} - \mathbf{y}_0\|$ .
- (iv) If  $S$  is open and if  $\mathbf{x} \in S^c$ , then there exists no  $\mathbf{y} \in S$  such that  $d(\mathbf{x}, S) = \|\mathbf{x} - \mathbf{y}\|$ .

**Solution:**

- (i) ( $\Rightarrow$ ) We shall show that if such  $\mathbf{x}$  is not in  $S$ , then it is in  $S'$ . So, suppose  $\mathbf{x} \notin S$ . By Theorem 1.1.4, for any  $\varepsilon \in \mathbb{R}_+$ , there is some  $\mathbf{y} \in S$  such that  $0 \leq \|\mathbf{x} - \mathbf{y}\| < \varepsilon$ . Since  $\mathbf{x} \notin S$ ,  $\mathbf{x} \neq \mathbf{y}$ , and thus  $\mathbf{y} \in N'(\mathbf{x}; \varepsilon) \cap S$ , implying  $\mathbf{x}$  is a limit point of  $S$ .  
 ( $\Leftarrow$ ) Conversely, if  $\mathbf{x} \in S' \setminus S$ , then for all  $\varepsilon \in \mathbb{R}_+$ , there is some  $\mathbf{z} \in S$  such that  $0 < \|\mathbf{x} - \mathbf{z}\| < \varepsilon$ . Therefore,  $0 \leq d(\mathbf{x}, S) < \varepsilon$ . Since  $\varepsilon$  is arbitrary,  $d(\mathbf{x}, S) = 0$ .  $\checkmark$
- (ii) ( $\Rightarrow$ )  $d(\mathbf{x}, S) = 0$  if and only if  $\mathbf{x} \in \bar{S} = S$ . Therefore,  $d(\mathbf{x}, S) > 0$  if and only if  $\mathbf{x} \in S^c$ .  
 ( $\Leftarrow$ ) For every  $\mathbf{x} \in S^c$ ,  $\mathbf{x} \notin \bar{S}$  by (i). Thus, if  $\mathbf{x} \in \bar{S}$ , then  $\mathbf{x} \in S$ , or,  $\bar{S} \subseteq S$ .  $S$  is therefore closed.  $\checkmark$
- (iii) If  $S$  is finite, then we can easily see  $d(\mathbf{x}, S) = \min\{\|\mathbf{x} - \mathbf{y}\| \mid \mathbf{y} \in S\}$ .  
 Therefore, now suppose  $S$  is infinite. Let  $\{\varepsilon_k\}_{k \in \mathbb{N}}$  be a sequence defined by  $\varepsilon_k = 1/k$  for each  $k \in \mathbb{N}$ . By Theorem 1.1.4, for each  $k \in \mathbb{N}$ , we can find  $\mathbf{y}_k \in S$  that satisfies

$$d(\mathbf{x}, S) \leq \|\mathbf{x} - \mathbf{y}_k\| < d(\mathbf{x}, S) + \varepsilon_k.$$

If the set  $\{\mathbf{y}_k \mid k \in \mathbb{N}\}$  is finite, then there must be some  $\mathbf{y}_k$  such that  $\|\mathbf{x} - \mathbf{y}_k\| = d(\mathbf{x}, S)$ , and we are done.

Suppose  $\{\mathbf{y}_k \mid k \in \mathbb{N}\}$  is infinite. Since the set is also bounded ( $\|\mathbf{x} - \mathbf{y}_k\| < d(\mathbf{x}, S) + \varepsilon_1$  for each  $k \in \mathbb{N}$ ), by Theorem 2.1.8, there is a convergent subsequence  $\{\mathbf{y}_{k_j}\}_{j \in \mathbb{N}}$  of  $\{\mathbf{y}_k\}_{k \in \mathbb{N}}$ . Let  $\mathbf{y}_0 \triangleq \lim_{j \rightarrow \infty} \mathbf{y}_{k_j}$ . Since

$$d(\mathbf{x}, S) \leq \|\mathbf{x} - \mathbf{y}_{k_j}\| < d(\mathbf{x}, S) + \varepsilon_{k_j}$$

still holds, it follows that  $\|\mathbf{x} - \mathbf{y}_0\| = d(\mathbf{x}, S)$  by Theorem 1.3.5.

$\mathbf{y}_0 \in S$  since  $S$  is closed and  $\mathbf{y}_0 \in S$  is a limit point of  $S$ .  $\checkmark$

- (iv) Suppose there is some  $\mathbf{y} \in S$  such that  $d(\mathbf{x}, S) = \|\mathbf{x} - \mathbf{y}\|$ .  $\|\mathbf{x} - \mathbf{y}\| > 0$  since  $\mathbf{x} \neq \mathbf{y}$ . Since  $S$  is open, there is some neighborhood  $N(\mathbf{y}; r_0)$  of  $\mathbf{y}$  such that  $N(\mathbf{y}; r_0) \subseteq S$ . It must be  $r_0 \leq \|\mathbf{x} - \mathbf{y}\|$ . Let

$$\mathbf{z} \triangleq \mathbf{y} + \frac{r_0}{2} \cdot \frac{\mathbf{x} - \mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|}$$

Then,

$$\|\mathbf{z} - \mathbf{y}\| = \frac{r_0}{2} < r_0,$$

thus  $\mathbf{z} \in N(\mathbf{y}; r_0) \subseteq S$ . However,

$$\|\mathbf{x} - \mathbf{y}\| = \left| 1 - \frac{r_0}{2\|\mathbf{x} - \mathbf{y}\|} \right| \|\mathbf{x} - \mathbf{y}\| < \|\mathbf{x} - \mathbf{y}\|,$$

contradicting the minimality of  $\|\mathbf{x} - \mathbf{y}\|$ ,  $\#$ .  $\checkmark$

□

### Exercise 2.2.2

Let  $S$  be a nonempty set in  $\mathbb{R}^n$ . Then,  $d(S) = d(\bar{S})$ .

**Solution:** Since  $S \subseteq \bar{S}$ ,  $d(S) \leq d(\bar{S})$  is direct.

To prove  $d(S) = d(\bar{S})$ , take any  $\varepsilon \in \mathbb{R}_+$ . Let  $\mathbf{x}$  and  $\mathbf{y}$  be any point in  $\bar{S}$ . Then, each of  $S \cap N(\mathbf{x}; \varepsilon/2)$  and  $S \cap N(\mathbf{y}; \varepsilon/2)$  is nonempty. Thus, we take  $\mathbf{x}'$  and  $\mathbf{y}'$  from each set. Then,

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}\| &\leq \|\mathbf{x} - \mathbf{x}'\| + \|\mathbf{x}' - \mathbf{y}'\| + \|\mathbf{y}' - \mathbf{y}\| \\ &< \varepsilon/2 + d(S) + \varepsilon/2 = d(S) + \varepsilon \end{aligned}$$

Therefore,  $d(S) + \varepsilon$  is an upper bound of  $\{\|\mathbf{x} - \mathbf{y}\| \mid \mathbf{x}, \mathbf{y} \in \bar{S}\}$ . Thus,  $d(S) \leq d(\bar{S}) \leq d(S) + \varepsilon$ . Since  $\varepsilon$  was arbitrary, we conclude  $d(S) = d(\bar{S})$ .

□



## 2.3 Completeness

### Definition 2.3.1: Nested Sets

A sequence  $\{S_k\}$  of sets in  $\mathbb{R}^n$  such that  $S_k \supset S_{k+1}$  for each  $k \in \mathbb{N}$  is said to be *nested*.

### Theorem 2.3.1 Cantor's Nested Interval Theorem

For each  $k \in \mathbb{N}$ , let  $I_k = [a_k, b_k]$  with  $a_k < b_k$ . Suppose that  $\{I_k\}_{k \in \mathbb{N}}$  is a nested sequence in  $\mathbb{R}$ . Then

$$\bigcap_{k=1}^{\infty} I_k = [\alpha, \beta]$$

where  $\alpha = \sup\{a_k \mid k \in \mathbb{N}\}$  and  $\beta = \inf\{b_k \mid k \in \mathbb{N}\}$ .

**Proof.** Let  $A \triangleq \{a_k \mid k \in \mathbb{N}\}$  and  $B \triangleq \{b_k \mid k \in \mathbb{N}\}$ . Then,  $A$  is bounded above by any  $b_k$  and  $B$  is bounded below by any  $a_k$ . Thus, by Theorem 1.1.2,  $\alpha = \sup A$  and  $\beta = \sup B$  exist.

Any  $a_k$  is a lower bound of  $B$ , therefore  $a_k \leq \beta$  for each  $k \in \mathbb{N}$ , which implies  $\beta$  is an upper bound of  $A$ . Hence  $\alpha \leq \beta$ .

To prove  $\bigcap_{k=1}^{\infty} I_k \supseteq [\alpha, \beta]$ , take any  $x \in [\alpha, \beta]$ . Then, for each  $k \in \mathbb{N}$ ,  $a_k \leq \alpha \leq x \leq \beta \leq b_k$ , which means  $x \in I_k$ . Thus,  $[\alpha, \beta] \subseteq \bigcap_{k=1}^{\infty} I_k$ . Now, to prove the reverse containment, take any  $x \in \bigcap_{k=1}^{\infty} I_k$ . This means  $\forall k \in \mathbb{N}$ ,  $a_k \leq x \leq b_k$ ;  $x$  is an upper bound of  $A$  and is a lower bound of  $B$  at the same time. Therefore,  $\alpha \leq x \leq \beta$ , hence  $\bigcap_{k=1}^{\infty} I_k \subseteq [\alpha, \beta]$ .  $\square$

**Another Proof.** Since the sequences  $\{a_k\}$  and  $\{b_k\}$  are bounded and monotone, there are limits  $\alpha = \lim_{k \rightarrow \infty} a_k$  and  $\beta = \lim_{k \rightarrow \infty} b_k$  by Theorem 1.3.4. By Theorem 1.3.6,  $\alpha \leq \beta$ .

Since  $a_k \leq \alpha \leq \beta \leq b_k$  for each  $k \in \mathbb{N}$ ,  $[\alpha, \beta] \subseteq \bigcap_{k=1}^{\infty} I_k$ .

Now, take any  $x \in \bigcap_{k=1}^{\infty} I_k$ . Then, for all  $k \in \mathbb{N}$ ,  $a_k \leq x \leq b_k$ . If it were  $\alpha > x$ , there is some  $k_0 \in \mathbb{N}$  such that  $a_{k_0} > x$ . It is similar for the case when  $\beta < x$ . Therefore,  $\alpha \leq x \leq \beta$ . We have proven that  $\bigcap_{k=1}^{\infty} I_k \subseteq [\alpha, \beta]$ .  $\square$

### Corollary 2.3.1

If, in the notation of the previous theorem,  $\lim_{k \rightarrow \infty} (b_k - a_k) = 0$ , then  $\bigcap_{k=1}^{\infty} I_k$  is a singleton.

**Proof.** Take any  $\varepsilon \in \mathbb{R}_+$ . Then, there is some  $k_0 \in \mathbb{N}$  such that  $b_{k_0} - a_{k_0} < \varepsilon$ .

$$0 \leq \beta - \alpha \leq b_{k_0} - a_{k_0} < \varepsilon$$

holds. This implies that  $0 \leq \beta - \alpha < \varepsilon$  for arbitrary  $\varepsilon \in \mathbb{R}_+$ ; therefore  $\alpha = \beta$ .  $\square$

### Theorem 2.3.2 Cantor's Criterion

If  $\{C_k\}$  is a nested sequence of closed, bounded, nonempty subsets of  $\mathbb{R}^n$ , then

$$\bigcap_{k=1}^{\infty} C_k \neq \emptyset.$$

Furthermore, if  $\lim_{k \rightarrow \infty} d(C_k) = 0$ , then  $\bigcap_{k=1}^{\infty} C_k$  is a singleton.

**Proof.** If any of  $C_k$  is finite, it is trivial. So, we suppose every  $C_k$  is infinite. Construct a sequence  $\{\mathbf{x}_k\}$  of points in  $\mathbb{R}^n$  as following.

- Take any  $\mathbf{x}_1$  in  $C_1$ .
- For each  $k \in \mathbb{N}$ , take any  $\mathbf{x}_{k+1}$  in  $C_{k+1} \setminus \{x_1, x_2, \dots, x_k\}$ .

Since  $S \subseteq C_1$  is bounded and contains infinitely many points, by Theorem 2.1.8, there is a limit point  $\mathbf{x}_0$  of  $S$  in  $\mathbb{R}^n$ . We now claim that  $\mathbf{x}_0 \in \bigcap_{k=1}^{\infty} C_k$ .

Fix any  $k \in \mathbb{N}$  and choose any deleted neighborhood  $N'(\mathbf{x}_0)$  of  $\mathbf{x}_0$ . Since  $N'(\mathbf{x}_0) \cap S$  is infinite, there is some  $k_1 \in \mathbb{N}_{>k}$  such that  $\mathbf{x}_{k_1} \in N'(\mathbf{x}_0)$ . By the construction,  $\mathbf{x}_{k_1} \in C_{k_1} \subseteq C_k$ . This shows that every deleted neighborhood of  $\mathbf{x}_0$  contains a point in  $C_k$ ;  $\mathbf{x}_0 \in C'_k$ . As each  $C_k$  is closed,  $\mathbf{x}_0 \in C_k$ , and thus  $\mathbf{x}_0 \in \bigcap_{k=1}^{\infty} C_k$ .

Suppose, in addition,  $\lim_{k \rightarrow \infty} d(C_k) = 0$ . Assume  $\bigcap_{k=1}^{\infty} C_k$  has two distinct points  $\mathbf{x}$  and  $\mathbf{y}$  for the sake of contradiction. Choose any  $\varepsilon \in (0, \|\mathbf{x} - \mathbf{y}\|)$  and then there is some  $k \in \mathbb{N}$  with  $d(C_k) < \varepsilon$ . Nonetheless,  $\varepsilon < \|\mathbf{x} - \mathbf{y}\| \leq d(C_k) < \varepsilon$ , #.  $\square$

### Theorem 2.3.3 Cantor's Criterion in $\mathbb{R}^n$ implies Cantor's Criterion in $\mathbb{R}$

Cantor's criterion in  $\mathbb{R}^n$  implies Cantor's criterion also holds in  $\mathbb{R}$ .

**Proof.**  $\mathbb{R} \times \{0\} \times \dots \times \{0\}$  is a closed subset of  $\mathbb{R}^n$ .  $\square$

### Theorem 2.3.4

Cantor's criterion in  $\mathbb{R}$  and Archimedes' principle implies the existence of supremum of any bounded above nonempty subset of  $\mathbb{R}$ .

**Proof.** Let  $S$  be a nonempty, bounded above set in  $\mathbb{R}$ . Let  $B$  denote the set of upper bounds of  $S$  and let  $A = B^c$ . Since  $x - 1 \in A$  for all  $x \in S$ ,  $A \neq \emptyset$ .

We first show that for all  $a \in A$  and  $b \in B$ ,  $a < b$ . If otherwise, i.e.,  $a \geq b$ ,  $x \leq b \leq a$  for each  $x \in S$ , which implies  $a \in B$ , which is a contradiction.

Moreover,  $S \cap [a, b] \neq \emptyset$  for each  $a \in A$  and  $b \in B$ . Assume  $S \cap [a, b] = \emptyset$  for the sake of contradiction. Since  $S \cap (b, \infty) = \emptyset$  as  $b$  is an upper bound of  $S$ , then it follows  $S \subseteq (-\infty, a)$ , which implies  $a$  is an upper bound of  $S$ , which is a contradiction.

Construct a nested sequence  $\{[a_k, b_k]\}_{k \in \mathbb{N}}$  of closed interval of which each  $a_k$  is in  $A$  and  $b_k$  is in  $B$ .

- Take any  $a_1$  in  $A$  and  $b_1$  in  $B$ .
- For each  $k \in \mathbb{N}$ , if  $(a_k + b_k)/2 \in A$ , then let  $a_{k+1} \triangleq (a_k + b_k)/2$  and  $b_{k+1} \triangleq b_k$ . If  $(a_k + b_k)/2 \in B$ , then let  $a_{k+1} \triangleq a_k$  and  $b_{k+1} \triangleq (a_k + b_k)/2$ .

Then it is immediate that  $\lim_{k \rightarrow \infty} (b_k - a_k) = \lim_{k \rightarrow \infty} 2^{-k+1}(b_1 - a_1) = 0$ . Therefore, by Cantor's criterion in  $\mathbb{R}$ ,  $\bigcap_{k=1}^{\infty} [a_k, b_k] = \{x_0\}$  for some  $x_0 \in \mathbb{R}$ .

We now show that  $x_0$  is an upper bound of  $S$ . Assume not for the sake of contradiction, that is, there is some  $x \in S$  such that  $x > x_0$ . Then, we may find some  $k \in \mathbb{N}$  such that  $b_k - a_k < x - x_0$ . Then it follows  $b_k - x_0 \leq b_k - a_k < x - x_0$ , and therefore  $b_k < x_0$ . This contradicts that  $b_k$  is an upper bound of  $S$ . Thus,  $x_0 \in B$ .

We now claim that  $x_0$  is the least upper bound. Assume to the contrary that there is some  $b \in B$  such that  $b < x_0$ . Then, we may find some  $k \in \mathbb{N}$  such that  $b_k - a_k < x_0 - b$ . It follows  $x_0 - a_k \leq b_k - a_k < x_0 - b$ , and therefore  $b < a_k$ . This contradicts that  $a_k \in A$ .  $\square$

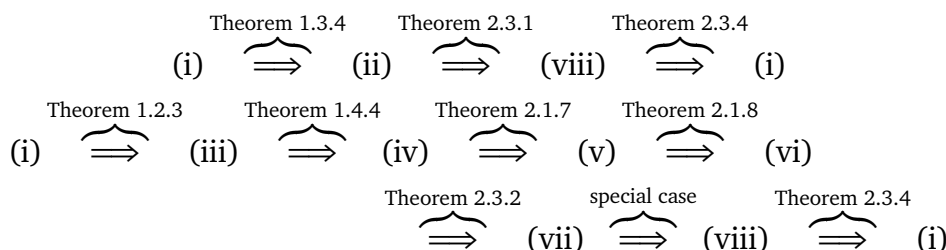
### Theorem 2.3.5

Assuming that Archimedes' principle holds in  $\mathbb{R}$ , the following are equivalent.

- (i) Every nonempty supset in  $\mathbb{R}$  which is bounded above has a supremum in  $\mathbb{R}$ .
- (ii) Every bounded monotone sequence in  $\mathbb{R}$  converges.
- (iii)  $\mathbb{R}$  has Bolzano–Weierstrass property.
- (iv)  $\mathbb{R}$  is Cauchy complete.

- (v)  $\mathbb{R}^n$  is Cauchy complete.
- (vi)  $\mathbb{R}^n$  has Bolzano–Weierstrass property.
- (vii) Cantor’s criterion holds in  $\mathbb{R}^n$ .
- (viii) Cantor’s criterion holds in  $\mathbb{R}$ .

**Proof.**



□

### Definition 2.3.2: Completeness

When the word *complete* is applied to  $\mathbb{R}^n$ , it is assumed that it means any of these statements:

- the existence of least upper bounds in  $\mathbb{R}$ ,
- the Monotone Convergence theorem in  $\mathbb{R}^n$ ,
- Cantor’s criterion,
- the Bolzano–Weierstrass property, or
- Cauchy completeness.

## 2.4 Relative Topology and Connectedness

### Definition 2.4.1: Relatively Open and Relatively Closed Set

A set  $S$  is said to be *relatively open* in  $X$  if there exists an open set  $U$  in  $\mathbb{R}^n$  such that  $S = U \cap X$ . Likewise, a set  $S$  is said to be *relatively closed* if there exists a closed set  $C$  in  $\mathbb{R}^n$  such that  $S = C \cap X$ .

#### Note:-

Every relatively open set in  $X$  is open in  $\mathbb{R}^n$  if and only if  $X$  is open in  $\mathbb{R}^n$ . Every relatively closed set in  $X$  is closed in  $\mathbb{R}^n$  if and only if  $X$  is closed in  $\mathbb{R}^n$ .

### Definition 2.4.2

- A *relative neighborhood* of  $\mathbf{x}$  in  $X$  is  $N(\mathbf{x}; r) \cap X$ . A *deleted neighborhood* is  $N'(\mathbf{x}; r) \cap X$ .
- A sequence  $\{\mathbf{x}_k\}$  in  $X$  *converges in  $X$*  if  $\lim_{k \rightarrow \infty} \mathbf{x}_k \in X$ .
- The relative closure of  $S$  is  $\bar{S} \cap X$ .
- A point  $\mathbf{x}_0$  in  $X$  is a *relative limit point* of  $S$  in  $X$  if  $\mathbf{x}_0$  is a limit point of  $S$ .

#### Note:-

Depending on  $X$ ,  $X$  may not be complete, i.e., a Cauchy sequence in  $X$  may not converge in  $X$ , an infinite and bounded subset of  $X$  may not have a limit point in  $X$ , or a nested sequence of nonempty, bounded, relatively closed subsets of  $X$  may have an empty intersection.

### Lemma 2.4.1

Let  $X$  be a subset of  $\mathbb{R}^n$  and  $C$  be a subset in  $X$ . Then,  $C$  is relatively closed in  $X$  if and only if  $C$  is the relative closure of  $C$ .

**Proof.** The “if” part is trivial; we only prove the “only if” part.

( $\subseteq$ ) It is direct since  $C \subseteq \overline{C}$ .

( $\supseteq$ ) There exists a closed set  $\hat{C}$  in  $\mathbb{R}^n$  such that  $C = \hat{C} \cap X$ . Since  $C \subseteq \hat{C}$  and  $\hat{C}$  is closed,  $\overline{C} \subseteq \hat{C}$  by Theorem 2.2.6. Thus,  $\overline{C} \cap X \subseteq \hat{C} \cap X = C$ .  $\square$

### Lemma 2.4.2

Let  $C \subseteq X \subseteq \mathbb{R}^n$ . Then,  $C$  is relatively closed in  $X$  if and only if  $C$  contains all its relative limit points in  $X$ .

**Proof.** ( $\Rightarrow$ ) If  $\mathbf{x}_0$  is a relative limit point of  $C$  in  $X$ , then  $\mathbf{x}_0 \in \overline{C} \cap X = C$ . (Lemma 2.4.1)

( $\Leftarrow$ ) It means  $\overline{C} \cap X \subseteq C$ . And it is already  $C \subseteq \overline{C} \cap X$ . Therefore,  $C$  is relatively closed in  $X$  by ??  $\square$

### Theorem 2.4.1

Let  $X$  be any nonempty subset of  $\mathbb{R}^n$ . The following statements are equivalent.

1. Every Cauchy sequence  $\{\mathbf{x}_k\}$  in  $X$  converges to a point of  $X$ . Thus  $X$  inherits Cauchy completeness from  $\mathbb{R}^n$ .
2. If  $S$  is a bounded, infinite subset of  $X$ , then  $S$  has a limit point in  $X$ . Thus  $X$  inherits the Bolzano–Weierstrass property from  $\mathbb{R}^n$ .
3. If  $\{C_k\}$  is any nested sequence of nonempty, bounded, relatively closed subsets of  $X$ , then  $\bigcap_{k=1}^{\infty} C_k \neq \emptyset$ . Furthermore, if  $\lim_{k \rightarrow \infty} d(C_k) = 0$ , then  $\bigcap_{k=1}^{\infty} C_k$  is a singleton. Thus  $X$  inherits Cantor’s criterion from  $\mathbb{R}^n$ .

**Proof.** We shall first prove (i)  $\Rightarrow$  (ii). Let  $S$  be any bounded and infinite subset of  $S$ . Since  $S$  is a bounded and infinite subset of  $\mathbb{R}^n$  at the same time, by Theorem 2.1.8, there is a limit point  $\mathbf{x}_0$  of  $S$  in  $\mathbb{R}^n$ . We may construct a Cauchy sequence in  $S$  that converges to  $\mathbf{x}_0$  as we did in the proof of Theorem 1.3.7. Thus,  $\mathbf{x}_0$ , which the Cauchy sequence converges to, is in  $S$  by Cauchy completeness of  $X$ . Therefore,  $S$  has a limit point in  $X$ .

We next prove (ii)  $\Rightarrow$  (iii). Suppose that  $X$  has the Bolzano–Weierstrass property. Take any nested sequence  $\{C_k\}$  be bounded, nonempty, and *relatively closed* subsets of  $X$ . We assume each  $C_k$  is infinite since it is trivial otherwise.

As in the proof of Theorem 2.3.2, recursively choose an infinite set  $S = \{\mathbf{x}_k \mid k \in \mathbb{N}\}$  of distinct points such that  $\mathbf{x}_k \in C_k$  for each  $k \in \mathbb{N}$ . By the assumption (ii), there is a limit point  $\mathbf{x}_0 \in X$  of  $S$ . Since  $\mathbf{x}_k \in C_k \subseteq \overline{C_k}$  for each  $k \in \mathbb{N}$  and  $\{\overline{C_k}\}_{k \in \mathbb{N}}$  is a nested sequence of bounded, nonempty, and closed subsets of  $\mathbb{R}^n$ , as in the proof of Theorem 2.3.2,  $\mathbf{x}_0 \in \overline{C_k}$  for each  $k$ , also. As  $C_k = \overline{C_k} \cap X$  for each  $k \in \mathbb{N}$  by Lemma 2.4.1,  $\mathbf{x}_0 \in C_k$  for each  $k \in \mathbb{N}$ . Thus,  $\mathbf{x}_0 \in \bigcap_{k=1}^{\infty} C_k$ .

If, in addition,  $\lim_{k \rightarrow \infty} d(C_k) = 0$ , then  $\lim_{k \rightarrow \infty} d(\overline{C_k}) = 0$  by ?? By Theorem 2.3.2,  $\bigcap_{k=1}^{\infty} \overline{C_k} = \{\mathbf{x}_0\}$ . But since  $C_k \subseteq \overline{C_k}$  for each  $k \in \mathbb{N}$ ,  $\emptyset \neq \bigcap_{k=1}^{\infty} C_k \subseteq \bigcap_{k=1}^{\infty} \overline{C_k} = \{\mathbf{x}_0\}$ . Thus, (ii) implies Cantor’s criterion holds in  $X$ .

Finally, it is left to prove (iii)  $\Rightarrow$  (i). Let  $\{\mathbf{x}_k\}$  be a Cauchy sequence in  $X$ . We must show it converges to some point in  $X$ . By Cauchy completeness of  $\mathbb{R}^n$ ,  $\{\mathbf{x}_k\}$  converges to some point  $\mathbf{x}_0$  in  $\mathbb{R}^n$ . We shall show that  $\mathbf{x}_0 \in X$ .

Let  $\{\varepsilon_k\}_{k \in \mathbb{N}}$  be any sequence of positive numbers that converge monotonically to 0, e.g.,  $\varepsilon_k = 1/k$ . Let  $C_k \triangleq \overline{N(\mathbf{x}_0; \varepsilon_k)} \cap X$ . Then  $\{C_k\}$  is a nested sequence of bounded and relatively closed sets in  $X$ .

Take any  $k_1 \in \mathbb{N}$ . Since  $\{\mathbf{x}_k\}$  converges to  $\mathbf{x}_0$ , there is some  $k_0 \in \mathbb{N}$  such that  $\forall k \in \mathbb{N}_{\geq k_0}$ ,  $\mathbf{x}_k \in N(\mathbf{x}_k; \varepsilon_{k_1})$ . And such  $\mathbf{x}_k$ 's are also in  $N(\mathbf{x}_k; \varepsilon_{k_1}) \cap X$ ; hence each  $C_k$  is nonempty.

Moreover, as  $d(C_k) \leq d(N(\mathbf{x}_0; \varepsilon_k)) = 2\varepsilon_k$ ,  $\lim_{k \rightarrow \infty} d(C_k) = 0$ . Thus, by assumption (ii),  $\bigcap_{k=1}^{\infty} C_k$  is a singleton in  $X$ . Then,

$$\bigcap_{k=1}^{\infty} C_k = \bigcap_{k=1}^{\infty} \left( \overline{N(\mathbf{x}_0; \varepsilon_k)} \cap X \right) \subseteq \bigcap_{k=1}^{\infty} \overline{N(\mathbf{x}_0; \varepsilon_k)} = \{\mathbf{x}_0\}.$$

Therefore,  $\mathbf{x}_0 \in \bigcap_{k=1}^{\infty} C_k \subseteq X$ . □

### Definition 2.4.3: Completeness of a Subset of $\mathbb{R}^n$

Let  $X$  be a nonempty subset of  $\mathbb{R}^n$ . If any of the equivalent properties of Theorem 2.4.1 hold in the set  $X$ , then  $X$  is said to be *complete*.

### Theorem 2.4.2

A nonempty subset  $X$  of  $\mathbb{R}^n$  is complete if and only if  $X$  is closed in  $\mathbb{R}^n$ .

**Proof.** ( $\Rightarrow$ ) Let  $\mathbf{x}_0$  be a limit point of  $X$ . Then, there is a Cauchy sequence  $\{\mathbf{x}_k\}$  that converges to  $\mathbf{x}_0$ . By Cauchy completeness of  $X$ ,  $\mathbf{x}_0 \in X$ . Since we have proven that  $X' \subseteq X$ ,  $X$  is closed by Theorem 2.2.4.

( $\Leftarrow$ ) Let  $S$  be any bounded, infinite set in  $X$ . By Bolzano–Weierstrass property of  $\mathbb{R}^n$ ,  $S$  has a limit point  $\mathbf{x}_0$  in  $\mathbb{R}^n$ . As  $X$  being closed,  $\mathbf{x}_0 \in X$ . Therefore,  $X$  has Bolzano–Weierstrass property;  $X$  is complete by Theorem 2.4.1. □

### Definition 2.4.4: Connectedness

A set  $S$  is *disconnected* if there are two open sets  $U, V$  such that

- (i)  $U \cap V = \emptyset$ ,
  - (ii)  $S \subseteq U \cup V$ , and
  - (iii)  $S \cap U \neq \emptyset$  and  $S \cap V \neq \emptyset$ .
- $S$  is *connected* if  $S$  is not disconnected.

### Example 2.4.1

- $[0, 1) \cup (1, 2]$  is disconnected. ( $U = (-1, 1)$  and  $V = (1, 3)$ )
- $\mathbb{Q}$  is disconnected. ( $U = (-\infty, r)$  and  $V = (r, \infty)$  where  $r \in \mathbb{R} \setminus \mathbb{Q}$ )

### Theorem 2.4.3

Any interval of the form of  $[a, b]$ ,  $(a, b]$ ,  $[a, b)$ , or  $(a, b)$  is connected.  $\mathbb{R}$  itself is connected.

**Proof.** Let  $I$  be any of these sets. The only fact needed for this proof is that  $(u, v) \subseteq I$  for any  $u < v$  in  $I$ .

Suppose  $I$  is disconnected. Then, there are disjoint open sets  $U, V \subseteq \mathbb{R}$  such that  $I \subseteq U \cup V$ ,  $I \cap U \neq \emptyset$ , and  $I \cap V \neq \emptyset$ . Take any  $u \in I \cap U$  and  $v \in I \cap V$ . WLOG,  $u < v$ .

Construct sequences  $\{u_k\}$  in  $I \cap U$  and  $\{v_k\}$  in  $I \cap V$  as following.

- $u_1 \triangleq u$  and  $v_1 \triangleq v$ .
- For each  $k \in \mathbb{N}$ ,  $u_{k+1} \triangleq \begin{cases} \frac{u_k + v_k}{2} & \frac{u_k + v_k}{2} \in U \\ u_k & \text{otherwise} \end{cases}$  and  $v_{k+1} \triangleq \begin{cases} \frac{u_k + v_k}{2} & \frac{u_k + v_k}{2} \in V \\ v_k & \text{otherwise} \end{cases}$ .

Then,  $\{u_k\}$  and  $\{v_k\}$  are bounded, monotone sequences ( $u_k < v$  and  $v_k > u$  for each  $k \in \mathbb{N}$ ); hence they converge by Theorem 1.3.4.

Since  $(u_k + v_k)/2 \in I \subseteq U \cup V$  for each  $k \in \mathbb{N}$  and  $U \cap V = \emptyset$ ,  $v_{k+1} - u_{k+1} = (v_k - u_k)/2 = (v - u)/2^k$ . They converge to the same point  $x_0$  since  $\lim_{k \rightarrow \infty} (v_k - u_k) = 0$ .

If  $x_0$  is ever equal to  $u_k$ , then by openness of  $U$ , there is some neighborhood  $N(x_0; \varepsilon)$  of  $x_0$  such that  $N(x_0; \varepsilon) \subseteq U$ , which contradicts  $\lim_{k \rightarrow \infty} v_k = x_0$ . Thus,  $x_0$  is never equal to  $u_k$ ; by the same reason,  $x_0$  is never equal to  $v_k$ . We conclude  $x_0$  is a relative limit point in  $I$  of both  $I \cap U$  and  $I \cap V$ .

Since  $I \cap U = I \cap V^c$  and  $V^c$  is closed,  $I \cap U$  is closed and thus any relative limit point of  $I \cap U$  in  $I$ ,  $x_0$ , in particular, is in  $I \cap U$  by Lemma 2.4.2. Similarly,  $x_0 \in I \cap V$ , which contradicts  $U \cap V = \emptyset$ .  $\square$

## 2.5 Compactness

### Definition 2.5.1: Open Cover

Let  $S$  be any nonempty subset of  $\mathbb{R}^n$ . An *open cover* of  $S$  is any collection  $\mathcal{C} = \{U_\alpha \mid \alpha \in A\}$  of open sets such that  $S \subseteq \bigcup_{\alpha \in A} U_\alpha$ .

### Definition 2.5.2: Compactness

A set  $S$  in  $\mathbb{R}^n$  is said to be *compact* if, for every open cover  $\mathcal{C}$  of  $S$ , there exists a finite subcover, that is, there is some finite subset  $A_0$  of  $A$  such that  $S \subseteq \bigcup_{\alpha \in A_0} U_\alpha$ .

### Theorem 2.5.1 Heine–Borel

Let  $S$  be any closed and bounded set in  $\mathbb{R}^n$ . Then any *countable* open cover of  $S$  has a finite subcover.

**Proof.** Let  $\{I_k \mid k \in \mathbb{N}\}$  be any open cover of  $S$ . Let  $U_k \triangleq \bigcup_{j=1}^k I_j$ ,  $C_k \triangleq U_k^c$ , and  $D_k \triangleq C_k \cap S$  for each  $k \in \mathbb{N}$ . Then, for each  $k \in \mathbb{N}$ ,  $U_k \subseteq U_{k+1}$ ,  $C_k \supseteq C_{k+1}$ , and  $D_k \supseteq D_{k+1}$ . Note that each  $U_k$  is open and thus  $C_k$  and  $D_k$  are closed.

Assume that every  $D_k$  is nonempty. Then, since  $\{D_k\}$  is a nested sequence of bounded, nonempty, closed sets, by Cantor's criterion, there is some  $x \in \bigcap_{k=1}^{\infty} D_k$ . Then,

$$x \in \bigcap_{k=1}^{\infty} (S \cap C_k) = S \cap \left( \bigcap_{k=1}^{\infty} C_k \right) = S \cap \left( \bigcup_{k=1}^{\infty} U_k \right)^c = S \cap \left( \bigcup_{k=1}^{\infty} I_k \right)^c$$

Thus,  $x$  is a point in  $S$  that is not covered by  $\{I_k \mid k \in \mathbb{N}\}$ , which is a contradiction.

Hence, some  $D_{k_0}$  is empty, and it means that  $S \subseteq \bigcup_{j=1}^{k_0} I_j$ .  $\square$

### Theorem 2.5.2 Lindelöf

Let  $S$  be any subset of  $\mathbb{R}^n$ . Then, every open cover of  $S$  has a countable subcover.

**Proof.** Since the set  $\{N(\mathbf{y}; r) \mid \mathbf{y} \in \mathbb{Q}^n, r \in \mathbb{Q}\}$  has the trivial one-to-one correspondence with  $\mathbb{Q}^{n+1}$ , the set is countable. Thus, we may index them as  $\{N_k \mid k \in \mathbb{N}\}$ .

Let  $\mathcal{C} = \{U_\alpha \mid \alpha \in A\}$  be any open cover of  $S$ . Take any  $\mathbf{x} \in S$ . Then there is some  $U_\alpha$  such that  $\mathbf{x} \in U_\alpha$ . Since  $U_\alpha$  is open, there is a neighborhood  $N(\mathbf{x}; \varepsilon)$  such that  $N(\mathbf{x}; \varepsilon) \subseteq U_\alpha$ .

As  $\mathbb{Q}^n$  is dense in  $\mathbb{R}^n$ , there is some  $\mathbf{y} \in N(\mathbf{x}; \varepsilon/2) \cap \mathbb{Q}^n$ . As  $\mathbb{Q}$  is dense in  $\mathbb{R}$ , there is some  $r \in (\|\mathbf{x} - \mathbf{y}\|, \varepsilon - \|\mathbf{x} - \mathbf{y}\|) \cap \mathbb{Q}$ . Then,  $\mathbf{x} \in N(\mathbf{y}; r) \subseteq N(\mathbf{x}; \varepsilon) \subseteq U_\alpha$ . The neighborhood  $N(\mathbf{y}; r)$  is in the set  $\{N_k \mid k \in \mathbb{N}\}$ .

Let us define  $K(\mathbf{x}) \triangleq \{k \in \mathbb{N} \mid \mathbf{x} \in N_k \subseteq U_\alpha \text{ for some } U_\alpha \in \mathcal{C}\}$ . Then, by Well-Ordering Property of  $\mathbb{N}$ , there is  $k(\mathbf{x}) \triangleq \min K(\mathbf{x})$ .

The collection  $\{N_{k(\mathbf{x})} \mid \mathbf{x} \in S\}$  is an open cover of  $S$  and is countable since it is a subset of  $\{N_k \mid k \in \mathbb{N}\}$ .

By construction, for each  $N_{k(\mathbf{x})}$ , there is  $U_\alpha \in \mathcal{C}$  such that  $N_{k(\mathbf{x})} \subseteq U_\alpha$ . Hence, if we denote one of such sets as  $U_{k(\mathbf{x})}$ , the collection  $\{U_{k(\mathbf{x})} \mid \mathbf{x} \in S\}$  is a countable subcover of  $S$ .  $\square$

### Corollary 2.5.1

If  $S \subseteq \mathbb{R}^n$  is closed and bounded, then  $S$  is compact.

**Proof.** Simply combine Theorem 2.5.2 and Theorem 2.5.1.  $\square$

### Theorem 2.5.3

If  $S$  is a compact subset of  $\mathbb{R}^n$ , then  $S$  is closed and bounded.

**Proof.** Suppose  $S \subseteq \mathbb{R}^n$  is not closed. Then  $S$  has a limit point  $\mathbf{x}_0$  that is not in  $S$ . Let  $\{\varepsilon_k\}$  be any sequence of positive numbers that monotonically converges to 0. Define  $U_k \triangleq \overline{N(\mathbf{x}_0; \varepsilon_k)}^c$ .

Then  $\mathcal{C} \triangleq \{U_k \mid k \in \mathbb{N}\}$  is an open cover of  $S$ . To prove this, take any  $\mathbf{x} \in S$ . There exists  $k_0 \in \mathbb{N}$  such that  $\varepsilon_k < \|\mathbf{x} - \mathbf{x}_0\|$  for all  $k \in \mathbb{N}_{\geq k_0}$ . Thus, for  $k \geq k_0$ ,  $\mathbf{x}_0 \notin \overline{N(\mathbf{x}_0; \varepsilon_k)}$ , i.e.,  $\mathbf{x}_0 \in U_k$ .

We now claim there is no finite subcover. Take finitely many subsets  $\{U_{k_1}, U_{k_2}, \dots, U_{k_p}\}$  from the open cover. Let  $k_0 = \max_{i \in [p]} k_i$ . Note that  $\bigcup_{i=1}^p U_{k_i} = U_{k_0}$ . Take  $\varepsilon$  from  $(0, \varepsilon_{k_0})$ . Then, since  $\mathbf{x}_0$  is a limit point of  $S$ , we may take  $\mathbf{x} \in S \cap N'(\mathbf{x}_0; \varepsilon)$ . Then,  $\mathbf{x} \notin U_{k_0}$ . Therefore,  $\{U_{k_i}\}_{i \in [p]}$  does not cover  $S$ . Thus, we proved that a set which is not closed is not compact.

Now, suppose  $S$  is not bounded. Let  $U_k = N(\mathbf{0}; k)$ . Then,  $S \subseteq \mathbb{R}^n = \bigcup_{k=1}^{\infty} U_k$ ;  $\{U_k\}$  is an open cover of  $S$ .

Take finitely many subsets  $\{U_{k_1}, U_{k_2}, \dots, U_{k_p}\}$  from the open cover. Let  $k_0 = \max_{i \in [p]} k_i$ . Note that  $\bigcup_{i=1}^p U_{k_i} = U_{k_0}$ . Because  $S$  is unbounded, there is some  $\mathbf{x} \in S$  such that  $\|\mathbf{x}\| > k_0$ ;  $\mathbf{x} \notin U_{k_0} = \bigcup_{i=1}^p U_{k_i}$ . Thus, we proved that an unbounded set is not compact.  $\square$

### Note:-

A set  $S \subseteq \mathbb{R}^n$  is complete if and only if  $S$  is closed. A set  $S$  is compact if and only if  $S$  is closed and bounded. Thus, compactness implies completeness, but not vice versa.

### Theorem 2.5.4

Let  $S$  be a nonempty subset of  $\mathbb{R}^n$ . Then the following statements are equivalent.

- (i)  $S$  is closed and bounded.
- (ii)  $S$  is compact.
- (iii) Every infinite subset of  $S$  has a limit point which is in  $S$ .



**Proof.** We already know (i) is equivalent to (ii). Therefore, we only need to prove (i) and (iii) are equivalent. (i)  $\implies$  (iii) is direct since  $S$  is complete by Theorem 2.4.2. So we are left to prove (iii) implies (i).

Suppose  $S$  is not bounded. Then we may construct a sequence  $\{\mathbf{x}_k\}$  in  $S$  as following.

- Take any  $\mathbf{x}_1 \in S$ .
- For each  $k \in \mathbb{N}$ , take  $\mathbf{x}_{k+1} \in S \setminus N(\mathbf{0}; \|\mathbf{x}_k\| + 1)$ .

The construction is valid since  $S$  is unbounded. Then, the set  $\{\mathbf{x}_k \mid k \in \mathbb{N}\}$  is infinite but it does not have a limit point since  $\|\mathbf{x}_k - \mathbf{x}_m\| \geq 1$  for each  $\{k, m\} \subseteq \mathbb{N}$ . Thus, (iii) implies  $S$  is bounded.

Now, we shall prove (iii) implies  $S$  is closed. Let  $\mathbf{x}$  be a limit point of  $S$ . Let  $\{\varepsilon_k\}$  be a monotonic sequence of positive numbers that converges to 0. Construct a sequence  $\{\mathbf{x}_k\}$  as following.

- For each  $k \in \mathbb{N}$ , take  $\mathbf{x}_k \in S \cap N'(\mathbf{x}; \varepsilon_k)$ .

The construction is valid since  $\mathbf{x}$  is a limit point of  $S$ . Then, the set  $\{\mathbf{x}_k \mid k \in \mathbb{N}\}$  has the only limit point  $\mathbf{x}$ . Then, by (iii),  $\mathbf{x} \in S$ . We proved that  $S$  contains all its limit points, thus  $S$  is closed by Theorem 2.2.4.  $\square$