

MAS242 선형대수학 Notes

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Chapter 1

Linear Equations

Chapter 2

Vector Spaces

2.1 Bases and Dimension

Theorem 2.1.1

Any subset that is linearly independent can be extended to a basis of V .

Lemma 2.1.1

If W is a subspace of V and $W \subsetneq V$, then $\dim W < \dim V$ provided that V is finite-dimensional.

Proof. Let S_0 be a basis of W . S_0 is linearly independent, so we can enlarge it to get a basis of V . $S' \triangleq S_0 \cup \{v_1, v_2, \dots, v_r\}$ is a basis of V . $|S'| \geq |S_0| + 1$; otherwise $\text{span } S_0 = V$. \square

Theorem 2.1.2 Inclusion/Exclusion Principle for Vector Spaces

If W_1 and W_2 are finite-dimensional subspaces of V , then $W_1 + W_2$ is a finite-dimensional vector space and $\dim W_1 + \dim W_2 = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$.

Proof. Let $a \triangleq \dim W_1$, $b \triangleq \dim W_2$, $c \triangleq \dim(W_1 + W_2)$, and $d \triangleq \dim(W_1 \cap W_2)$. Choose $\{\alpha_1, \alpha_2, \dots, \alpha_d\}$ as a basis for $W_1 \cap W_2$. We may extend this into bases of W_1 and W_2 . Let $\{\alpha_1, \dots, \alpha_d, \beta_{d+1}, \beta_{d+2}, \dots, \beta_a\}$ and $\{\alpha_1, \dots, \alpha_d, \gamma_{d+1}, \gamma_{d+2}, \dots, \gamma_b\}$ be bases for W_1 and W_2 respectively.

We now claim that

$$B \triangleq \{\alpha_1, \dots, \alpha_d, \beta_{d+1}, \dots, \beta_a, \gamma_{d+1}, \dots, \gamma_b\}$$

is a basis of $W_1 + W_2$.

- Let $x \in W_1 + W_2$. Then, $x = w_1 + w_2$ where $w_i \in W_i$. Since $w_1 \in \text{span}\{\alpha_1, \dots, \alpha_d, \beta_{d+1}, \dots, \beta_a\}$ and $w_2 \in \text{span}\{\alpha_1, \dots, \alpha_d, \gamma_{d+1}, \dots, \gamma_b\}$, On the other hand, $B \subseteq W_1 + W_2$. Hence, $\text{span } B = W_1 + W_2$.
- Suppose we have $\sum a_i \alpha_i + \sum b_j \beta_j + \sum c_k \gamma_k = 0$ for some $a_i, b_j, c_k \in F$. Rearranging the terms, we get $\sum a_i \alpha_i + \sum b_j \beta_j = -\sum c_k \gamma_k$, which implies that $\sum c_k \gamma_k \in W_1 \cap W_2$. The fact that γ_k 's are linearly independent from $\{\alpha_i\}$ implies that $c_k = 0$ for all k . Similarly, $b_j = 0$ for all j . Hence, we are left with $\sum a_i \alpha_i = 0$, in which α_i 's are linearly independent; $a_i = 0$. Hence, B is linearly independent.

Therefore, $\dim(W_1 + W_2) = a + b - d$. \square

Definition 2.1.1: Ordered Basis

Let V be a finite-dimensional vector space over F . An *ordered basis* of V is a sequence of vectors that forms a basis.

Note:-

Usually, we emphasize the ordered basis with semicolons like $\{\beta_1; \beta_2\}$.

Lemma 2.1.2

Let V be a finite-dimensional vector space over F . Suppose $B = \{v_1; v_2; \dots; v_n\}$ is an ordered basis of V . Then, for each $x \in V$, there uniquely exists an expression of the form

$$x = x_1 v_1 + x_2 v_2 + \dots + x_n v_n$$

for some $x_i \in F$.

Proof. The existence of the form is obvious since $x \in V = \text{span } B$.

(Uniqueness) Suppose we have two such expressions:

$$x = \sum x_i v_i = \sum y_i v_i$$

where $x_i, y_i \in F$. Then, we have $\sum (x_i - y_i) v_i = 0$. The linear independence of B gives that $x_i - y_i = 0$ for all i . Hence, $x_i = y_i$. \square

Definition 2.1.2: Coordinate Matrix

Let V be a finite-dimensional vector space over F . Let B be an ordered basis of V . Let $x \in V$ and write it as $x = \sum_{i=1}^n x_i v_i$ with $x_i \in F$, $v_i \in B$. Define

$$[x]_B \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

be the *coordinate matrix* of x with respect to the basis B

Theorem 2.1.3

Let V be a finite-dimensional vector space over F . Let B and B' be two ordered bases of V . Then, there uniquely exists an invertible matrix P such that $\forall x \in V$, $[x]_B = P[x]_{B'}$ and $[x]_{B'} = P^{-1}[x]_B$.

Proof. Let $B \triangleq \{\alpha_1; \dots; \alpha_n\}$ and $B' \triangleq \{\alpha'_1; \dots; \alpha'_n\}$. For $\alpha'_j \in B'$, since B is a basis, there are unique $P_{ij} \in F$ ($i \in [n]$) such that $\alpha'_j = \sum_{i=1}^n P_{ij} \alpha_i$.

Let $x \in V$. Write $[x]_B = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $[x]_{B'} = \begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix}$. Then, $x = \sum_{j=1}^n x'_j \alpha'_j = \sum_{j=1}^n \left(\sum_{i=1}^n x'_j P_{ij} \right) \alpha_i$.

By the uniqueness, we have $x_i = \sum_{j=1}^n x'_j P_{ij}$ for each i . In other words, we have

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} P_{11} & \cdots & P_{1n} \\ \vdots & \cdots & \vdots \\ P_{n1} & \cdots & P_{nn} \end{bmatrix} \begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix}$$

Since B and B' are linearly independent, $x = 0 \iff [x]_B = 0 \iff [x]_{B'} = 0$. Hence, P is invertible. \square

Chapter 3

Linear Transformations

3.1 Linear Transformations

Definition 3.1.1: Linear Transformation

Let V_1 and V_2 be vector spaces over F . $T: V_1 \rightarrow V_2$ is said to be a *linear transformation* if

- $\forall x_1, x_2 \in V_1, T(x_1 + x_2) = T(x_1) + T(x_2)$
- $\forall x \in V_1, \forall c \in F, T(cx) = cT(x)$.

Theorem 3.1.1

Let V and W be finite-dimensional vector spaces over F . where $\{\alpha_1, \dots, \alpha_n\}$ is a basis of V . Let $\{\beta_1, \dots, \beta_n\}$ be any given set of vectors of W . Then, there exists a unique transformation $T: V \rightarrow W$ such that $T(\alpha_i) = \beta_i$.

Proof. Let $T_0: V \rightarrow W$ be defined by

$$T_0\left(\sum_{i=1}^n x_i \alpha_i\right) = \sum_{i=1}^n x_i \beta_i.$$

This is a linear transformation indeed.

(Uniqueness) If there is another such $U: V \rightarrow W$, Then, $U\left(\sum_{i=1}^n x_i \alpha_i\right) = \sum_{i=1}^n x_i U(\alpha_i)$. Hence, $U = T_0$. \square

Definition 3.1.2: Null Space and Range Space

Let $T: V \rightarrow W$ be a linear transformation between vector spaces over F .

- $\text{null } T \triangleq \ker T \triangleq \{v \in V \mid T(v) = 0\}$
- $\text{range } T \triangleq \text{Im } T \triangleq \{w \in W \mid \exists v \in V, w = T(v)\}$

Note:-

$\ker T$ and $\text{Im } T$ are subspaces of V and W respectively.

Definition 3.1.3

Let $T: V \rightarrow W$ be a linear transformation between vector spaces over F .

$$\text{nullity}(T) \triangleq \dim \ker(T) \quad \text{and} \quad \text{rank}(T) \triangleq \dim \text{Im}(T)$$

Theorem 3.1.2 Rank-Nullity Theorem

Let $T: V \rightarrow W$ be a linear transformation between vector spaces over F . Then, $\text{rank}(T) + \text{nullity}(T) = \dim V$.

Proof. Let $\{v_1, \dots, v_k\}$ be a basis for $\ker T$ where $k = \text{nullity } T$. Choose $v_{k+1}, \dots, v_n \in V$ such that $\{v_i\}_{i=1}^n$ is a basis of V . We claim that $\{T(v_{k+1}), \dots, T(v_n)\}$ is a basis of $\text{Im } T$.

Suppose $\sum_{i=k+1}^n c_i T(v_i) = 0$ for some $c_i \in F$. Then, we have $T(\sum_{i=k+1}^n c_i v_i) = 0$; hence $\sum_{i=k+1}^n c_i v_i \in \ker T$. Since $\{v_1, \dots, v_k\}$ is a basis of $\ker T$, we have $\sum_{i=k+1}^n c_i v_i = \sum_{i=1}^k a_i v_i$ for some a_i 's. Therefore, since $\{v_1, \dots, v_n\}$ is linearly independent, all c_i 's and a_i 's are zero. This implies that $\{T(v_i)\}_{i=k+1}^n$ is linearly independent.

Take any $T(v) \in \text{Im } T$. Then, $v = \sum_{i=1}^n c_i v_i$ for some $c_i \in F$. Then, $T(v) = \sum_{i=k+1}^n c_i T(v_i)$. Hence, $\text{Im } T \subseteq \text{span}\{T(v_{k+1}), \dots, T(v_n)\}$

The two paragraphs imply that $\text{rank } T = n - k$. \square

Theorem 3.1.3

Let A be a $m \times n$ matrix. Then $\dim \text{span}(\text{rows}) = \dim \text{span}(\text{columns})$.

Proof. $V = F^n$, $W = F^m$. Then, $\dim \text{span}(\text{columns}) = \dim \text{Im } T = \text{rank } T$, so $\text{nullity } T = n - \text{rank } T = n - \text{colrank } T$.

The number of rows with leading one's in $\text{rref } A$ equals the dimension of the row space of A , which is simply the number of columns with the leading ones. It is equal to the dimension of the column space. Hence, $\text{nullity } T = n - \text{colrank } T$ \square

3.2 The Algebra of Linear Transformations

Definition 3.2.1

Let $T: V \rightarrow W$ be a linear transformation between vector spaces over F . $L(V, W) \triangleq \{T: V \rightarrow W \mid T \text{ is a linear transformation}\}$

Theorem 3.2.1

Let $T: V \rightarrow W$ be a linear transformation between vector spaces over F . Then, $L(V, W)$ is a vector space over F under usual addition and multiplication.

Theorem 3.2.2

Let V and W be n - and m -dimensional vector spaces over F , respectively. Then, $\dim L(V, W) = mn$.

Proof. Let $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ and $\mathcal{B}' = \{\beta_1, \dots, \beta_m\}$ be bases for V and W , respectively. For each $p \in [n]$ and $q \in [m]$, Define

$$E^{p,q}(\alpha_i) = \begin{cases} 0 & \text{if } i \neq p \\ \beta_q & \text{if } i = p \end{cases}.$$

Then,

- These $E^{p,q}$ are linear transformations
- These are linearly independent.

- They span $L(V, W)$.

□

Lemma 3.2.1

Let $T: V \rightarrow W$ and $U: W \rightarrow Z$ be linear transformations between vector spaces over F . Then, $U \circ T \in L(V, Z)$.

Definition 3.2.2: Linear Operator (Endomorphism)

Let $T: V \rightarrow V$ be a linear transformation from a vector space V to itself. Then, T is called a *linear operator*. (Or an *endomorphism*.)

Note:-

For each $T, U \in L(V, V)$, $T \circ U \in L(V, V)$. $(T_1 + T_2) \circ U = T_1 \circ U + T_2 \circ U$. And many more... $(L(V, V), +, \circ)$ is a non-commutative ring.

Definition 3.2.3: Injectivity and Surjectivity

A linear transform $T: V \rightarrow W$ is

- *injective* (or, nonsingular) if $T(v) = 0 \implies v = 0$.
- *surjective* if $T(V) = W$.
- *invertible* if \exists linear transform $U: W \rightarrow V$, $U \circ T = \text{id}_V \wedge T \circ U = \text{id}_W$.

Exercise 3.2.1

$T: V \rightarrow W$ is injective and surjective if and only if T is invertible.

Exercise 3.2.2

If $T: V \rightarrow W$ is a nonsingular linear transformation, then, for any linearly independent subset $S \subseteq V$, $T(S)$ is linearly independent.

Exercise 3.2.3

Suppose V and W are finite-dimensional vector spaces. If $T: V \rightarrow W$ is invertible, then $\dim V = \dim W$.

Theorem 3.2.3

Let V and W be finite-dimensional vector spaces over F with $\dim V = \dim W$. Let $T: V \rightarrow W$ be a linear transform. TFAE

- T is invertible.
- T is injective.
- T is surjective.

Proof. T is injective \iff nullity $T = 0 \iff$ rank $T = n \iff$ Im $T = W \iff T$ is onto □

Definition 3.2.4: General Linear Group

Let $\text{GL}(V) \triangleq \{ T \in L(V, V) \mid T \text{ is invertible} \}$. Then, $(\text{GL}(V), \circ)$ is called the *general linear group* of V .

Note:-

The general linear group is actually a group.

3.3 Isomorphism

Definition 3.3.1: Isomorphism

Let V and W be vector spaces over F . We say that a linear transformation $T: V \rightarrow W$ is an *isomorphism* if T is an invertible linear transformation.

We say V and W are *isomorphic* if there exists an isomorphism $T: V \rightarrow W$, if V and W are isomorphic, then we write $V \simeq W$.

Theorem 3.3.1

Let V be a vector spaces over F of dimension n . Then, $V \simeq F^n$.

Proof. Let $B = \{\alpha_1; \dots; \alpha_n\}$ be a basis of V . Define $T: V \rightarrow F^n$ by $v \mapsto [v]_B$.

Suppose $T(v) = 0$. Then, $v = 0 \cdot \alpha_1 + \dots + 0 \cdot \alpha_n = 0$. Hence, T is injective. By Theorem 3.1.6, T is isomorphism. \square

3.4 Representation of Transformation by Matrices

Theorem 3.4.1

Let V and W be vector spaces over F with $\dim V = n$ and $\dim W = m$. Let B and B' be bases of V and W , respectively. If $T: V \rightarrow W$ is a linear transformation, then there uniquely exists $m \times n$ matrix A such that $[T(v)]_{B'} = A[v]_B$. We write $[T]_{B,B'} \triangleq A$.

Proof. $A = \begin{bmatrix} [T(v_1)]_{B'} & [T(v_2)]_{B'} & \dots & [T(v_n)]_{B'} \end{bmatrix}$ where v_i is the i^{th} basis vector of B . \square

Theorem 3.4.2

Let $V \xrightarrow{T} W \xrightarrow{U} Z$ be linear transformations. Let $A_1 = [T]_{B,B'}$ and $A_2 = [U]_{B',B''}$. Then, $[U \circ T]_{B,B''} = A_2 A_1$.

Theorem 3.4.3

Let V be finite-dimensional vector space over F with two (possibly different) bases B_1 and B_2 . Let $T \in L(V, V)$. Let P be the matrix such that $[v]_{B_1} = P[v]_{B_2}$. Then, $[T]_{B_i} \triangleq [T]_{B_i, B_i}$ are related by

$$[T]_{B_2} = P^{-1} [T]_{B_1} P.$$

Definition 3.4.1: Similar Matrices

Suppose M and N are $n \times n$ matrices. M and N are *similar* if there exists an invertible P such that $N = P^{-1} M P$.

Proof. $[T(v)]_{B_1} = [T]_{B_1} [v]_{B_1} = [T]_{B_1} P [v]_{B_2}$. $[T(v)]_{B_1} = P [T(v)]_{B_2} = P [T]_{B_2} [v]_{B_2}$.

Since v was arbitrary, $P [T]_{B_2} = [T]_{B_1} P$. \square

Note:-

- A linear transformation $T: V \rightarrow V$ gives varying matrices $[T]_B$ that are all similar when the basis B is changed.
- On linear operators, we will have various definitions.
- Characteristic (eigen) polynomial has $(-1)^{\deg}(\text{constant term})$ as $\det T$ and $-(n - 1 \text{ deg term})$ as $\text{tr } T$.

3.5 Linear Functionals

Definition 3.5.1: Linear Functional

Let V be a vector space over F . A linear transformation $T: V \rightarrow F$ is called a (*linear*) *functional*.

Definition 3.5.2: Dual Vector Space

Let V be a vector space over F . We normally write $V^* \triangleq L(V, F)$ and call it the *dual vector space* of V .

Note:-

By Theorem 3.1.5, we know that $\dim V^* = \dim V$ if V is a finite-dimensional vector space.

Lemma 3.5.1

Let V be a finite-dimensional vector space over F and let $n = \dim V$. Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis for V . Define $f_i \in V^*$ by declaring $f_i(\alpha_j) = \delta_{ij}$. Then, $\{f_1, \dots, f_n\}$ is a basis for V^* .

Proof. Since $\dim V^* = \dim V = n$, we only need to show that the set is linearly independent.

Suppose $\sum_{i=1}^n c_i f_i = 0$ for some $c_i \in F$. Then, for each $j \in [n]$, as $f_i(\alpha_j) = \delta_{ij}$, $0 = (\sum_{i=1}^n c_i f_i)(\alpha_j) = c_j f_j(\alpha_j) = c_j$. Hence, they are linearly independent. \square

Definition 3.5.3: Dual Basis

The set $\{f_1, f_2, \dots, f_n\} \subseteq V^*$ in Lemma 3.2.1 is called the *dual basis* of the basis $\{\alpha_1, \dots, \alpha_n\}$ for V .

Lemma 3.5.2

Let V be a finite-dimensional vector space over F and let $n = \dim V$. Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis for V . Let $\{f_1, \dots, f_n\} \subseteq V^*$ be the dual basis of it.

(i) For each $f \in V^*$, $f = \sum_{i=1}^n f(\alpha_i) f_i$.

(ii) For each $v \in V$, $v = \sum_{i=1}^n f_i(v) \alpha_i$.

Proof.

(i) There exists $x_i \in F$ such that $f = \sum_{i=1}^n x_i f_i$. Evaluating at α_j for each $j \in [n]$, we get $f(\alpha_j) = x_j$.

(ii) There exists $y_i \in F$ such that $v = \sum_{i=1}^n y_i \alpha_i$. Applying f_j for each $j \in [n]$, we get $f_j(v) = y_j$. \square

Definition 3.5.4: Hyperspace

Let V be a finite-dimensional vector space over F and let $n = \dim V$. A subspace W of V which has the dimension $n - 1$ is called a *hyperspace* in V .

Example 3.5.1

If $f : V \rightarrow F$ is a nonzero functional, then $\ker f$ is an example of a hyperspace in V .

Definition 3.5.5: Annihilator

Let V be a finite-dimensional vector space over F with dimension n . Let $\emptyset \subsetneq S \subseteq V$. The *annihilator* of S , $S^\circ = \text{Ann } S$ is defined to be

$$S^\circ = \{f \in V^* \mid \forall \alpha \in S, f(\alpha) = 0\}.$$

Note:-

- S° is a subspace of V^*
- $\text{Ann } \{0\} = V^*$.
- $\text{Ann } V = \{0\}$.

Theorem 3.5.1

Let V be a finite-dimensional vector space over F with dimension n . Let W be a subspace of V . Then,

$$\dim W + \dim W^\circ = \dim V.$$

Proof. Let $k \triangleq \dim W$ and $\{\alpha_1, \dots, \alpha_k\} \subseteq W$ be a basis for W . We may extend it to the basis for V so that $\{\alpha_1, \dots, \alpha_k, \alpha_{k+1}, \dots, \alpha_n\}$ is a basis for V . Let $\{f_1, \dots, f_k, f_{k+1}, \dots, f_n\}$ be the dual basis of $\{\alpha_1, \dots, \alpha_n\}$.

For each $i \in \{k+1, \dots, n\}$, by the construction of the dual basis, $f_i(\alpha_j) = 0$ for each $j \in [k]$. Hence, $f_{k+1}, \dots, f_n \in W^\circ$.

Take any $f \in W^\circ$. Then, $f = \sum_{i=1}^n f(\alpha_i)f_i$. For each $i \in [k]$, $f(\alpha_i) = 0$. Hence, $f = \sum_{i=k+1}^n f(\alpha_i)f_i$; $\{f_{k+1}, \dots, f_n\}$ spans W° . Therefore, $\{f_{k+1}, \dots, f_n\}$ is a basis for W° . \square

Corollary 3.5.1

Let V be a finite-dimensional vector space over F with dimension n . Let W be a k -dimensional subspace of V . Then, W is the intersection of $n - k$ hyperspaces in V of the form $\ker f_i$ for some $f_i \in V^* \setminus \{0\}$.

Proof. Let $\{\alpha_1, \dots, \alpha_k\}$ be a basis for W and extend it to $\{\alpha_1, \dots, \alpha_n\}$ so that it becomes a basis for V . Let $\{f_1, \dots, f_n\} \subseteq V^*$ be the dual basis of $\{\alpha_1, \dots, \alpha_n\}$. Then, $W = \bigcap_{i=k+1}^n \ker f_i$. \square

Corollary 3.5.2

Let V be a finite-dimensional vector space over F with dimension n . Let W be a hyperspace in V . Then, $W = \ker f$ for some $f \in V^* \setminus \{0\}$.

3.6 The Double Dual

Note:-

Take $\alpha \in V$. Let us define $L_\alpha \in V^{**}$ as follows:

$$\begin{aligned} L_\alpha : V^* &\longrightarrow F \\ f &\longmapsto f(\alpha). \end{aligned}$$

Then, define \mathcal{L} by

$$\begin{aligned} \mathcal{L} : V &\longrightarrow V^{**} \\ \alpha &\longmapsto L_\alpha. \end{aligned}$$

Then, \mathcal{L} is an injective linear transformation.

Theorem 3.6.1

Let V be a finite-dimensional vector space over F with dimension n . Then, $\mathcal{L} : V \rightarrow V^{**}$ is an isomorphism of vector spaces.

Proof. We have $\dim V = \dim V^* = \dim V^{**} = n$ by Theorem 3.1.5. The result follows from Theorem 3.1.6. \square

Definition 3.6.1: Proper Subspace

Let V be a vector space over F . Then, a subspace W of V is a *proper subspace* of V if $W \subsetneq V$.

Definition 3.6.2: Maximal Subspace

A proper subspace W of V is said to be *maximal* if, there exists no subspace Z of V such that $W \subsetneq Z \subsetneq V$.

Definition 3.6.3: Hyperspace

Let V be a vector space over F . A maximal proper subspace W of V is called a *hyperspace* in V .

Note:-

In case of $\dim V = n$, a proper maximal subspace of V is of dimension $n - 1$.

Theorem 3.6.2

Let V be a vector space over F . Let $f \in V^* \setminus \{0\}$. Then, $\ker f$ is a hyperspace in V .

Proof. $\ker f$ is proper, since, otherwise, $f = 0$.

It is enough to show that, for each $\alpha \in V \setminus \ker f$, $\text{span}\{\ker f, \alpha\} = V$. Take any $\beta \in V$. Let $\alpha \in V \setminus \ker f$. Define $c \triangleq f(\alpha)^{-1}f(\beta)$ and $\gamma \triangleq \beta - c\alpha$. Then, $f(\gamma) = f(\beta) - cf(\alpha) = 0$; $\gamma \in \ker f$. Hence, $\beta = \gamma + c\alpha \in \text{span}[\ker f, \alpha]$. \square

Theorem 3.6.3

Let V be a vector space over F . Let W be a hyperspace in V . Then, there exists $f \in$

$V^* \setminus \{0\}$ such that $W = \ker f$.

Proof. There exists $\alpha \in V \setminus W$ such that $\text{span}\{W, \alpha\} = V$. Hence, every $\beta \in V$ can be written as $\beta = \gamma + c\alpha$ where $\gamma \in W$ and $c \in F$. Note that γ and c are uniquely determined by β .

Define $g: V \rightarrow F$ by $g(\beta) = c$. Then, g is a linear functional, and $\ker g = W$ by definition. \square

Note:-

Theorem 3.3.2 and Theorem 3.3.3 together imply that the set of hyperspaces in V and the set of null spaces of functionals have a one-to-one correspondence.

3.7 The Transpose of a Linear Transformation

Definition 3.7.1: Transpose

Let $T: V \rightarrow W$ be a linear transformation. The map $T^t: W^* \rightarrow V^*$ defined by $g \mapsto g \circ T$ is called the *transpose* of T .

Lemma 3.7.1

Let $T: V \rightarrow W$ be a linear transformation. Then, T^t is a linear transformation.

Theorem 3.7.1

Let $T: V \rightarrow W$ be a linear transformation between finite-dimensional vector spaces over F . Fix ordered bases \mathcal{B} and \mathcal{B}' for V and W , respectively. Let \mathcal{B}^* and \mathcal{B}'^* be their dual bases. Let $A \triangleq [T]_{\mathcal{B}, \mathcal{B}'}$ and $A' \triangleq [T^t]_{\mathcal{B}'^*, \mathcal{B}^*}$. Then, $a_{ij} = a'_{ji}$.

Proof. Let $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$, $\mathcal{B}' = \{\beta_1, \dots, \beta_m\}$, $\mathcal{B}^* = \{f_1, \dots, f_n\}$, and $\mathcal{B}'^* = \{g_1, \dots, g_m\}$. Then, we have $T\alpha_j = \sum_{i=1}^m a_{ij}\beta_i$ for each $j \in [n]$ and $T^tg_j = \sum_{i=1}^n b_{ij}f_i$ for each $j \in [m]$.

For each $i \in [n]$ and $j \in [m]$, $(T^tg_j)(\alpha_i) = g_j(T\alpha_i) = g_j(\sum_{k=1}^m a_{ki}\beta_k) = \sum_{k=1}^m a_{ki}g_j(\beta_k) = a_{ji}$. Hence, since T^tg_j is a linear functional on V , $T^tg_j = \sum_{i=1}^n (T^tg_j)(\alpha_i)f_i = \sum_{i=1}^n a_{ji}f_i$. Therefore, $a_{ij} = b_{ji}$ for each $i \in [n]$ and $j \in [m]$. \square

Theorem 3.7.2

Let $T: V \rightarrow W$ be a linear transformation.

- (i) $\ker T^t = (\text{Im } T)^\circ$.
- (ii) If V and W are finite-dimensional, then $\text{rank } T^t = \text{rank } T$.
- (iii) If V and W are finite-dimensional, then $\text{Im } T^t = (\ker T)^\circ$.

Proof.

- (i) $g \in \ker T^t \iff T^t(g) = 0 \iff g \circ T = 0 \iff g \in (\text{Im } T)^\circ$
- (ii) Let $n \triangleq \dim V$ and $m \triangleq \dim W$. Let $r = \text{rank } T$. Then, by Theorem 3.2.1, $\dim(\text{Im } T)^\circ = m - r$. By (i), $(\text{Im } T)^\circ = \ker T^t$; hence $\text{nullity } T^t = m - r$. By the rank-nullity theorem, $\text{rank } T^t = r = \text{rank } T$.
- (iii) Take any $f \in \text{Im } T^t$. Then, there exists $g \in W^*$ such that $f = g \circ T$. Then, for any $\alpha \in \ker T$, $f(\alpha) = g(T(\alpha)) = 0$. Hence, $f \in (\ker T)^\circ$; $\text{Im } T^t \subseteq (\ker T)^\circ$. But since the two spaces have the same dimension, it must be the equality to hold. \square

End.