Summary for
Introduction to Set Theory

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Contents

Chapter 1

Sets

1.1 Introduction to Sets

Definition 1.1.1: Set

Every object in the universe of discourse is called a *set*.

1.2 Properties

Definition 1.2.1: Property

Any mathematical sentence[a] is called a *property*. If $X, Y, \cdots, Z$ are free variables of a property $\mathbf{Q}$, we write $\mathbf{Q}(X, Y, \cdots, Z)$ and say $\mathbf{Q}(X, Y, \cdots, Z)$ is a property of $X, Y, \cdots, Z$.

[a] Refer to mathematical logic textbook for detailed discussion.

1.3 Axioms

Axiom I The Axiom of Existence

There exists a set which has no elements.

$$\exists A \, \forall x \, \neg(x \in A)$$

Note:-

The Axiom of Existence guarantees that the universe of discourse is not void.

Axiom II The Axiom of Extensionality

If every element of X is an element of Y and every element of Y is an element of X, then $X = Y$.

$$\forall X \, \forall Y \, [\forall x \, (x \in X \iff x \in Y) \implies X = Y]$$

Note:-

The Axiom of Extensionality defines the equality relation with the containment relation($\in$).

Lemma 1.3.1
There exists only one set with no elements.

Proof. Let A and B are sets such that $\forall x\, \neg(x \in A)$ and $\forall x\, \neg(x \in B)$. Then, we have $\forall x\, (x \in A \iff x \in B)$. Therefore, by The Axiom of Extensionality, $A = B$ is guaranteed. $\square$

Definition 1.3.2: Empty Set

The unique set with no elements is called the *empty set* and is denoted $\varnothing$.

Note:-
Definition 1.3.2 is justified by Lemma 1.3.1.

Axiom III The Axiom Schema of Comprehension

Let $\mathbf{P}(x)$ be a property of x. For any set A, there exists a set B such that $x \in B$ if and only if $x \in A$ and $\mathbf{P}(x)$.

$$\forall A\, \exists B\, (x \in B \iff x \in A \wedge \mathbf{P}(x))$$

Note:-
Axiom III is a axiom *schema* since it provides unlimited amount of axioms for varying $\mathbf{P}$.

Lemma 1.3.3
Let $\mathbf{P}(x)$ be a property of x. For any set A, there uniquely exists a set B such that $x \in B$ if and only if $x \in A$ and $\mathbf{P}(x)$.

Proof. Let B' be another set such that $x \in B'$ if and only if $x \in A$ and $\mathbf{P}(x)$. Then, for any x, we have $x \in B' \iff x \in A \wedge \mathbf{P}(x) \iff x \in B$. Hence, by The Axiom of Extensionality, we have $B = B'$. $\square$

Notation 1.3.4: Set-Builder Notation

Let $\mathbf{P}(x)$ be a property of x. Let A be a set. The unique set B such that $x \in B$ if and only if $x \in A$ and $\mathbf{P}(x)$ is denoted $\{x \in A \mid \mathbf{P}(x)\}$.

Note:-
Notation 1.3.4 is justified by Lemma 1.3.3.

Axiom IV The Axiom of Pair

For any A and B, there exists C such that $x \in C$ if and only if $x = A$ or $x = B$.

$$\forall A\, \forall B\, \exists C\, (x \in C \iff x = A \vee x = B)$$

Note:-
Similarly, the set C such that $x \in C \iff x = A \vee x = B$ is unique by The Axiom of Extensionality.

Notation 1.3.5

Let A and B be sets. The unique set C such that $x \in C$ if and only if $x = A$ or $x = B$ is denoted $\{A, B\}$. In particular, if $A = B$, we write $\{A\}$ instead of $\{A, A\}$.

Axiom V The Axiom of Union

For any S, there exists U such that $x \in U$ if and only if $x \in A$ for some $A \in S$.

$$\forall S\, \exists U\, (x \in U \iff \exists A\, x \in A \wedge A \in S)$$

Definition 1.3.6: The Union of System of Sets

Let S be a set. The unique set U such that $x \in U$ if and only if $x \in A$ for some $A \in S$ is denoted $\bigcup S$.

Definition 1.3.7: The Union of Two Sets

Let A and B be sets. Then, $A \cup B$ denotes the unique set $\bigcup\{A, B\}$.

Definition 1.3.8: Subset

Let A and B sets. B is said to be a *subset* of A if $\forall x\, (x \in B \implies x \in A)$. If B is a subset of A, then we write $B \subseteq A$.

Axiom VI The Axiom of Power Set

For any S, there exists P such that $X \in P$ if and only if $X \subseteq S$.

Note:-
Similarly, the set P is unique by The Axiom of Extensionality.

Definition 1.3.9: Power Set

Let S be a set. The unique set P such that $X \in P$ if and only if $X \subseteq S$ is called the *power set* of S and is denoted $\mathcal{P}(S)$.

Lemma 1.3.10

Let $\mathbf{P}(x)$ be a property of x. Let A and A' be sets such that $\mathbf{P}(x) \implies x \in A \wedge x \in A'$. Then, $\{\, x \in A \mid \mathbf{P}(x)\,\} = \{\, x \in A' \mid \mathbf{P}(x)\,\}$.

Proof. For all x, we have $x \in A \wedge \mathbf{P}(x) \iff \mathbf{P}(x) \iff x \in A' \wedge \mathbf{P}(x)$. Therefore, by The Axiom of Extensionality, the result follows. □

Notation 1.3.11

Let $\mathbf{P}(x)$ be a property of x. If there exists a set A such that $\mathbf{P}(x)$ implies $x \in A$, we write $\{\, x \mid \mathbf{P}(x)\,\} \triangleq \{\, x \in A \mid \mathbf{P}(x)\,\}$, and it is called *the set of all x with the property* $\mathbf{P}(x)$.

Note:-
Notation 1.3.11 is justified by Lemma 1.3.10.

Selected Problems

Exercise 1.3.1

The set of all x such that $x \in A$ and $x \notin B$ exists.

Proof. We have $x \in A \wedge x \notin B \implies x \in A$. Hence, the set exists and is equal to $\{ x \in A \mid x \in A \wedge x \notin B \}$. $\square$

Exercise 1.3.2

Prove The Axiom of Existence only from The Axiom Schema of Comprehension and The Weak Axiom of Existence.

Weak Axiom of Existence Some set exists.

Proof. Let A be a set known to exist. Then, there exists $B = \{ x \in A \mid x \neq x \}$ by The Axiom Schema of Comprehension. Since $\forall x\,(x = x)$, $\forall x\,(x \notin B)$. $\square$

Exercise 1.3.3

(a) Prove that a set of all sets($\{ x \mid \top \}$) does not exist.
(b) Prove that $\forall A\, \exists x\,(x \notin A)$.

Proof.

(a) Suppose $V = \{ x \mid \top \}$ exists. Then, by The Axiom Schema of Comprehension, $R = \{ x \in V \mid x \notin x \}$ exists. However, we have $R \in R \iff R \notin R$ by definition of R. Hence, V does not exist.
(b) Suppose $\exists A\, \forall x\,(x \in A)$ for the sake of contradiction. Then, A is the set of all sets, which is impossible by (a). $\square$

Exercise 1.3.6

Prove $\forall X\, \neg(\mathcal{P}(X) \subseteq X)$.

Proof. Let $Y = \{ u \in X \mid u \notin u \}$. Then, by definition, $Y \subseteq X$, and thus $Y \in \mathcal{P}(X)$. Now, suppose $Y \in X$ for the sake of contradiction. Then, $Y \in Y \iff Y \in X \wedge Y \notin Y \iff Y \notin Y$, which is a contradiction. Hence, $Y \notin X$. $\square$

1.4 Elementary Operations on Sets

Definition 1.4.1: Proper Subset

Let A and B sets. B is said to be a *proper subset* of A if $B \subseteq A$ and $B \neq A$. If B is a proper subset of A, we write $B \subsetneq A$.

Definition 1.4.2: Elementary Operations on Sets

(i) Intersection
- The *intersection* of A and B, $A \cap B$, is the set $\{x \mid x \in A \wedge x \in B\}$.

(ii) Union
- The *union* of A and B, $A \cup B$, is the set $\{x \mid x \in A \vee x \in B\}$.

(iii) Difference
- The *difference* of A and B, $A \setminus B$, is the set $\{x \mid x \in A \wedge x \notin B\}$.

(iv) Symmetric Difference
- The *symmetric difference* of A and B, $A \triangle B$, is the set $(A \setminus B) \cup (B \setminus A)$.

Lemma 1.4.3 Simple Properties of Elementary Operations

(i) Commutativity
- $A \cap B = B \cap A$
- $A \cup B = B \cup A$
- $A \triangle B = B \triangle A$

(ii) Associativity
- $(A \cap B) \cap C = A \cap (B \cap C)$
- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \triangle B) \triangle C = A \triangle (B \triangle C)$

(iii) Distributivity
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(iv) De Morgan's Laws
- $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$
- $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$

(v) Miscellaneous
- $A \cap (B \setminus C) = (A \cap B) \setminus C$
- $A \setminus B = \varnothing \iff A \subseteq B$
- $A \triangle B = \varnothing \iff A = B$

Definition 1.4.4: Intersection of System of Sets

Let S be a nonempty set. Then, the *intersection* $\bigcap S$ is the set $\{x \mid \forall A \in S\, (x \in A)\}$.

Note:-

Note that $\bigcap S$ exists for all nonempty S since $\forall A \in S\, (x \in A) \implies x \in A_1$ where A_1 is any set such that $A_1 \in S$.

Definition 1.4.5: System of Mutually Disjoint Sets

We say the sets A and B are *disjoint* if $A \cap B = \varnothing$. A set S is a *system of mutually disjoint sets* if $\forall A, B \in S$, $(A \neq B \implies A \cap B = \varnothing)$.

Selected Problems

> **Exercise 1.4.4**
> For any set A, prove that a "complement" of A ($\{x \mid x \notin A\}$) does not exist.

Proof. Let B be the complement of A for the sake of contradiction. Then, $A \cup B$ is the set of all sets, which is impossible by Exercise 1.3.3. $\square$

End.