Summary for Complex Variables I

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Chapter 1

Preliminaries

1.1 Complex Plane

Definition 1.1.1: Complex Number

$i := \sqrt{-1}$ is called the *imaginary unit*. $\mathbb{C} := \{x + iy \mid x, y \in \mathbb{R}\}$ is the set of complex numbers where $\mathbb{R}$ is the set of real numbers.

Definition 1.1.2: Algebras of $\mathbb{C}$

For $z_k := x_k + iy_k$ where $k \in \mathbb{Z}_+$ and $x_k, y_k \in \mathbb{R}$,

- $z_1 + z_2 := (x_1 + x_2) + i(y_1 + y_2)$
- $z_1 \cdot z_2 := (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$.

Theorem 1.1.3

$\mathbb{C}$ is a field.

Proof. Trivial. □

Note

$z = a + ib$, $a, b \in \mathbb{R}$ with $z \neq 0$. Then, $z^{-1} = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}$.

1.2 Rectangular Representation

Definition 1.2.1

Let $z = x + iy$ where $x, y \in \mathbb{R}$.

(i) $|z| := \sqrt{x^2 + y^2}$ is called *modulus* of z.

(ii) $\overline{z} := x - iy$ is called *conjugate* of z.

(iii) $\Re z = x$ is called the *real part* of z and $\Im z = y$ is called the *imaginary part* of z.

(iv) For $z_1, z_2 \in \mathbb{C}$, $|z_1 - z_2|$ is the *distance* between z_1 and z_2.

Note

- $z + \overline{z} = 2\Re z$
- $z - \overline{z} = 2i\Im z$
- $|z_1 + z_2| \le |z_1| + |z_2|$
- $\big||z_1| - |z_2|\big| \le |z_1 - z_2|$

1.3 Polar Representation

Given $z \in \mathbb{C}$, $|z|$ is unique. $\arg z = \theta + 2k\pi$ $(k \in \mathbb{Z})$ (Or $\arg z = \theta \pmod{2\pi}$)

Definition 1.3.1

If $z = |z| \cdot (\cos\theta + i \sin\theta)$, θ is called an *argument* of z and is written $\arg z = \theta \pmod{2\pi}$ (as $\theta + 2k\pi$ for $k \in \mathbb{Z}$ is an argument of z as well). If $\arg z = \theta^* \pmod{2\pi}$, and if $-\pi < \theta^* \le \pi$, then we define $\operatorname{Arg} z = \theta^*$ and it is called the *principal argument* of z.

Theorem 1.3.2

For $z_1, z_2 \in \mathbb{C}$ with $z_1, z_2 \neq 0$, $\arg z_1 z_2 = \arg z_1 + \arg z_2 \pmod{2\pi}$.

Proof. Let $\arg z_1 = \theta_1 \pmod{2\pi}$ and $\arg z_2 = \theta_2 \pmod{2\pi}$ Then, $z_1 = |z_1|(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = |z_2|(\cos\theta_2 + i\sin\theta_2)$. Now, we have $z_1 \cdot z_2 = |z_1||z_2|\big(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\big)$. $\square$

Chapter 2

Elementary Complex Functions

2.1 Exponential Functions

Definition 2.1.1: Exponential Function

For each $z = x + iy$ where $x, y \in \mathbb{R}$, we define $e^z := e^x \cdot (\cos y + i \sin y)$.

Theorem 2.1.2

For each $z \in \mathbb{C}$, $e^z = \sum_{j=1}^{\infty} \frac{z^j}{j!}$.

Proof. Proved later using complex integral. □

Theorem 2.1.3

For each $z, z' \in \mathbb{C}$,

(a) $e^{z+z'} = e^z \cdot e^{z'}$,

(b) $e^{-z} = \frac{1}{e^z}$, and

(c) $e^{z+2k\pi i} = e^z$ for all $k \in \mathbb{Z}$.

Definition 2.1.4

For each $z \in \mathbb{C}$,

(1) $\cos z := \frac{e^{iz}+e^{-iz}}{2}$

(2) $\sin z := \frac{e^{iz}-e^{-iz}}{2i}$

(3) $\cosh z = \frac{e^z+e^{-z}}{2}$

(4) $\sinh z = \frac{e^z-e^{-z}}{2}$

Theorem 2.1.5

For each $z \in \mathbb{C}$, we have $\cosh z = \cos(iz)$ and $\sinh z = -i \sin(iz)$.

Example 2.1.6

Let us solve $\cos z = 2$. Let $t := e^{iz}$ to obtain $t^2 - 4t + 1 = 0$, which gives $t = 2 \pm \sqrt{3}$. Write $z = x + iy$ where $x, y \in \mathbb{R}$ to have $e^{ix}e^{-y} = 2 \pm \sqrt{3}$. Taking modulus to both sides gives $e^{-y} = 2 \pm \sqrt{3}$, i.e., $y = -\ln(2 \pm \sqrt{3})$. Taking argument to both sides gives $x = 2k\pi$

for $k \in \mathbb{Z}$. Thus, $z = 2k\pi - i\ln(2 \pm \sqrt{3})$ for $k \in \mathbb{Z}$.

2.2 Mapping Properties

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2.3 Logarithmic "Functions"

Definition 2.3.1: Logarithmic Function

For any $z \in \mathbb{C} \setminus \{0\}$, we define $w = \ln z$ if and only if $e^w = z$.

Note

How to compute $\ln z$? Note that $z = |z| \cdot e^{i(\operatorname{Arg} z + 2k\pi)}$ for $k \in \mathbb{Z}$. Let $w = u + iv$ where $u, v \in \mathbb{R}$ so that $e^w = e^u \cdot e^{iv} = |z| \cdot e^{i(\operatorname{Arg} z + 2k\pi)}$. Hence, we have $u = \ln|z|$ and $v = \operatorname{Arg} z + 2k\pi$. In other words, $\ln z = \ln|z| + i \arg z$. (Note that this is not a "function"!)

Definition 2.3.2: Principal Logarithmic Function

For any $z \in \mathbb{C} \setminus \{0\}$, we define $\operatorname{Ln} z := \ln|z| + i \operatorname{Arg} z$ and it is called the *principal value of* $\ln z$.

Definition 2.3.3: Branch of Logarithm

A *branch* of $\ln z$ is a function given by $\omega\colon \ln z$ with $\theta_0 < \arg z \leq \theta_0 + 2\pi$. Here, θ_0 is called a *branch cut*.

Example 2.3.4

$B := \{ z \mid |z + 2| < 1 \}$ when mapped with Ln is not an open ball but it becomes an open ball when the branch cut is $-\pi/2$.

2.4 Complex Exponents

Definition 2.4.1: Complex Exponents

For $z \in \mathbb{C} \setminus \{0\}$ and $w \in \mathbb{C}$, define

$$z^w := e^{w \ln z}.$$

Note

Complex exponentiation is not a function! If one considers the complex exponentiation as a set of possible values, then $z^{\eta_1} \cdot z^{\eta_2} = z^{\eta_1 + \eta_2}$ may easily fail!

Example 2.4.2

To solve $z^{1-i} = 4$, write $e^{(1-i)\ln z} = e^{\ln 4}$, i.e., $\ln z = (1+i)(\ln 2 + k\pi i)$ for $k \in \mathbb{Z}$. In other words, $\ln|z| + i \arg z = (\ln 2 - k\pi) + i(\ln 2 + k\pi)$. Hence, $|z| = e^{\ln 2 - k\pi}$ and $\arg z = \ln 2 + k\pi \pmod{2\pi}$.

Chapter 3

Analytic Functions

3.1 Cauchy–Riemann Equation

Definition 3.1.1: Continuity

For a fixed point $z_0 \in \mathbb{C}$, a function f is said to be continuous at z_0 if

$$\lim_{|z-z_0|\to 0} |f(z) - f(z_0)| = 0.$$

Definition 3.1.2: Differentiability

For a fixed point $z_0 \in \mathbb{C}$, a function f is said to be *continuous at* z_0 if

$$\lim_{\substack{|\omega|\to 0\\ \omega\in\mathbb{C}}} \frac{f(z_0+\omega) - f(z_0)}{\omega}$$

exists. If f is differentiable at z_0, then define the *derivative* of f at z_0 by

$$f'(z_0) := \lim_{\substack{|\omega|\to 0\\ \omega\in\mathbb{C}}} \frac{f(z_0+\omega) - f(z_0)}{\omega}.$$

Example 3.1.3

For each $n \in \mathbb{N}$, one can derive that $f'(z) = nz^{n-1}$ where $f(z) = z^n$.

Theorem 3.1.4

If f is differentiable at z_0, then it is continuous at z_0.

Example 3.1.5

Let us determine differentiability of $f(z) = |z|^2$. Write $z = x + iy$ and $\omega = p + iq$ for $x, y, p, q \in \mathbb{R}$. Then,

$$\frac{f(z+\omega) - f(z)}{\omega} = \frac{2(xp+yq) + |\omega|^2}{\omega}$$

As we know $\lim_{\omega\to 0} \frac{|\omega|^2}{\omega} = 0$, we only need to care if $\lim_{\omega\to 0} \frac{2(xp+yq)}{p+iq}$. Evaluating the limit along the real axis and the imaginary axis gives $2x$ and $-2yi$; hence f is not differentiable at $z \in \mathbb{C} \setminus \{0\}$. At the origin, we have $f'(0) = \lim_{\omega\to 0} \frac{f(0+\omega)-f(0)}{\omega} = 0$.

Theorem 3.1.6

Product, quotient, chain rule still holds in complex derivative.

Theorem 3.1.7 Cauchy–Riemann Equation

f is differentiable at z if and only if $f_y(z) = i f_x(z)$ at z, or equivalently,

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

where $f(x+iy) = u(x,y) + iv(x,y)$.

Example 3.1.8

$f(z) = e^x(\cos y + i \sin y)$. $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$ satisfy the Cauchy–Riemann equation for all z; hence it is differentiable everywhere.

Chapter 4

Complex Integration

Chapter 5

Conformal Mapping

End.