

MAS241 해석학 I

Note

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April 17, 2023

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Chapter 1

Structure of the Real Numbers

1.1 Completeness of the Real Numbers

Definition 1.1.1: Cauchy Sequence

Let X be a space. A sequence $\{x_n\}_{n \in \mathbb{N}}$ is a *Cauchy sequence* if $\|x_n - x_m\| \rightarrow 0$ as $n, m \rightarrow \infty$.

Definition 1.1.2: Completeness

A set X is *complete* if every Cauchy sequence has a limit in X , i.e.,

$$x_n \rightarrow x_\infty \in X.$$

Definition 1.1.3: Boundedness

Let $\emptyset \neq S \subseteq \mathbb{R}$.

- a) S is *bounded above* if $\exists M \in \mathbb{R}, \forall x \in S, x \leq M$.
 - M is called an *upper bound* of S .
- b) S is *bounded below* if $\exists M \in \mathbb{R}, \forall x \in S, x \geq M$.
 - M is called an *lower bound* of S .
- c) S is *bounded* if S is bounded above and below.

Theorem 1.1.1 Archimedes' Principle

Let ε and M be any two positive real numbers. Then, there exists a k in \mathbb{N} such that $M < k\varepsilon$.

The proof of Theorem 1.1.1 can be done by integrating Theorem 1.1.2 and Theorem 1.1.4.

Definition 1.1.4: Supremum and Infimum

- a) Let S be bounded above. Then, the smallest upper bound is called the *supremum* of S , $\sup S$.
- b) Let S be bounded below. Then, the largest lower bound is called the *infimum* of S , $\inf S$.

Example 1.1.1

Let $S = \{(-1)^k(1 - 1/k) \mid k \in \mathbb{N}\}$. It is clear that $-1 < S < 1$; 1 is an upper bound and -1 is a lower bound. We now claim that $\sup S = 1$. To show this, let us assume that $M < 1$ is an upper bound of S . By Archimedes' principle, there exists a natural number k_0 such that $(1 - M)/2 < k_0$, which implies $(-1)^{2k_0}(1 - 1/(2k_0)) > M$; M is not an upper bound. Therefore, 1 is the smallest upper bound. It can be similarly shown that $\inf S = -1$.

Theorem 1.1.2 Completeness Axiom for \mathbb{R}

If $\emptyset \neq S \subseteq \mathbb{R}$ and S is bounded above, then $\sup S$ exists in \mathbb{R} .

Corollary 1.1.1

If $\emptyset \neq S \subseteq \mathbb{R}$ and S is bounded below, then $\inf S$ exists in \mathbb{R} .

Proof. Let $B := \{-x \mid x \in S\}$. Then, $M = \sup S \in \mathbb{R}$ by Theorem 1.1.2. We now claim that $\inf B = -M$.

For all $x \in S$, $-x \in B$, which implies $-x \leq M$, and therefore $x \geq -M$. Thus, $-M$ is a lower bound of B .

Suppose there is a $M_1 > -M$ such that M_1 is a lower bound of S . For all $x \in S$, $x \geq M_1$, which implies $-x \leq -M_1$. Thus, $-M_1$ is an upper bound of B but $-M_1 < M = \sup B$, #.

Therefore, $\inf S = -M \in \mathbb{R}$. □

Example 1.1.2

- $S := \left\{ \sum_{j=0}^k \frac{1}{j!} \mid k \in \mathbb{N} \right\}$. S is bounded above.

$$\sum_{j=0}^k \frac{1}{j!} = 1 + \sum_{j=1}^k \frac{1}{j!} \leq 1 + \sum_{j=1}^k \frac{1}{2^{j-1}} < 3$$

In fact, $e := \sup S$.

- $S := \left\{ \left(1 + \frac{1}{k}\right)^k \mid k \in \mathbb{N} \right\}$. S is bounded above.

$$\left(1 + \frac{1}{k}\right)^k = \sum_{j=0}^k \binom{k}{j} \frac{1}{k^j} \leq \sum_{j=0}^k \frac{1}{j!} \leq e$$

Theorem 1.1.3

Let S be a finite nonempty subset of \mathbb{R} . Then, $\sup S \in S$ and $\inf S \in S$.

Proof. (Induction on $|S|$) For $S = \{x\}$, $x = \inf S = \sup S \in S$.

Take any $k \in \mathbb{N}$ and suppose the statement holds for every S with $|S| = k$. Now, take any $S' \subseteq \mathbb{R}$ such that $|S'| = k + 1$. Let $x \in S'$, $\mu := \sup(S' \setminus \{x\})$, and $\nu := \inf(S' \setminus \{x\})$. By the induction hypothesis, $\mu, \nu \in S' \setminus \{x\}$. Letting $\mu' := \max(\mu, x)$ and $\nu' := \min(\nu, x)$, μ' and ν' are the supremum and infimum of S' , respectively. Moreover, μ' and ν' are elements of S' . □

Theorem 1.1.4

Let $\emptyset \neq S \subseteq \mathbb{R}$.

- If S is bounded above, then “ $\mu = \sup S$ if and only if μ is an upper bound and $\forall \varepsilon \in \mathbb{R}_+, \exists x \in S, \mu - \varepsilon < x \leq \mu$ ”.
- If S is bounded below, then “ $\nu = \inf S$ if and only if ν is a lower bound and $\forall \varepsilon \in \mathbb{R}_+, \exists x \in S, \nu \leq x < \nu + \varepsilon$ ”.

Proof. Let S be bounded above. If there is no $x \in S$ in $(\mu - \varepsilon, \mu]$, then $\mu - \varepsilon$ would be a smaller upper bound.

For the converse, assume M is an upper bound and $M < \mu$. Let $\varepsilon := \mu - M > 0$. Then, there is some $x \in S$ such that $M = \mu - \varepsilon < x \leq \mu$, $\#$ to M is an upper bound. Therefore, μ is the least upper bound.

The same logic may be applied for bounded below S . □

Proof of Theorem 1.1.1. Let $S := \{k\varepsilon \mid k \in \mathbb{N}\}$. Assume S is bounded above and nonempty. Then, by Theorem 1.1.2, there is $\mu = \sup S$. We also know, from Theorem 1.1.4, that there is $k \in \mathbb{N}$ such that $\mu - \varepsilon < k\varepsilon \leq \mu$, which implies $\mu < (k + 1)\varepsilon$. Since $(k + 1)\varepsilon \in S$, μ is not an upper bound of S , which is a contradiction. Therefore, S is not bounded above. In other words, for any $M > 0$, there is some $k \in \mathbb{N}$ such that $M < k\varepsilon$. □

Theorem 1.1.5

Theorem 1.1.1 (Archimedes' principle) is equivalent to the following statement:

$$\forall c \in \mathbb{R}_+, \exists k \in \mathbb{N}, k - 1 \leq c < k.$$

Proof. Assume Archimedes' principle. If $c < 1$, $k = 1$ satisfies, and it is done. Now, let us suppose $c \geq 1$. By Theorem 1.1.1, there is a $k \in \mathbb{N}$ such that $c < k$. We may let $k_0 := \min\{k \in \mathbb{N} \mid k > c\}$ by Well-Ordering of \mathbb{N} . We note that $k_0 - 1 \leq c$ since $k_0 - 1 \in \mathbb{N}$ since $k_0 > 1$. Therefore, $k_0 - 1 \leq c < k_0$.

Now, assume “ $\forall c \in \mathbb{R}_+, \exists k \in \mathbb{N}, k - 1 \leq c < k$ ”. Take any $M > 0$ and $\varepsilon \in \mathbb{R}_+$ and let $c := M/\varepsilon$. The assumption tells the existence of a $k \in \mathbb{N}$ such that $M/\varepsilon = c < k$, which directly implies $M < k\varepsilon$. □

Theorem 1.1.6

Let c and d be real numbers with $c < d$. Then, $\exists x \in \mathbb{Q}, c < x < d$.

Proof. There are three cases: $0 < c < d$, $c \leq 0 < d$, or $c < d \leq 0$.

Case 1) By Archimedes' principle, $\exists q \in \mathbb{N}, 1 < (d - c)q$, which implies $cq + 1 < dq$. By Theorem 1.1.5, $\exists p \in \mathbb{N}, p - 1 \leq cq < p$ since $cq > 0$. To sum up, $p - 1 \leq cq < p \leq cq + 1 < dq$, which implies $c < p/q < d$.

Case 2) By Archimedes' principle, $\exists q \in \mathbb{N}, 1 < dq$. Then, $c \leq 0 < 1/q < d$ holds.

Case 3) By case 1 and 2, there is $r \in \mathbb{Q}$ such that $-d < r < -c$. Then, $c < -r < d$ holds. □

1.2 Neighborhoods and Limit Points

Definition 1.2.1: Neighborhood and Deleted Neighborhood

For each $x \in \mathbb{R}$ and $r \in \mathbb{R}_+$,

$$N(x; r) := \{y \in \mathbb{R} : |y - x| < r\} = (x - r, x + r)$$

is called the *neighborhood* of x with radius r , and

$$N'(x; r) := \{y \in \mathbb{R} : 0 < |y - x| < r\} = N(x; r) \setminus \{x\}$$

is called the *deleted neighborhood* of x with radius r .

Definition 1.2.2: Limit Point and Isolated Point

For $\emptyset \neq S \subseteq \mathbb{R}$, $x \in \mathbb{R}$ is a *limit point* of S if

$$\forall \varepsilon \in \mathbb{R}_+, N'(x, \varepsilon) \cap S \neq \emptyset.$$

If $x \in \mathbb{R}$ is not a limit point of S , then it is called an *isolated point* of S .

Definition 1.2.3: Discrete Set

If $\emptyset \neq S \subseteq \mathbb{R}$ has no limit points, then S is said to be *discrete*.

Example 1.2.1

Let $S := \{(-1)^k(1 + 1/k) \mid k \in \mathbb{N}\}$. Then, 1 and -1 are limit points of S .

To see 1 is a limit point, take any $\varepsilon \in \mathbb{R}_+$ and, using Theorem 1.1.1, choose a $k \in \mathbb{N}$ such that $1 < (2\varepsilon)k$. Then, $1 < 1 + \frac{1}{2k} = (-1)^{2k}(1 + \frac{1}{2k}) < 1 + \varepsilon$; $N'(1, \varepsilon) \cap S \neq \emptyset$. Therefore, 1 is a limit point.

Theorem 1.2.1

Let $\emptyset \neq S \subseteq \mathbb{R}$. Then, $x \in \mathbb{R}$ is a limit point of S if and only if

$$\exists \varepsilon_0 \in \mathbb{R}_+, \forall \varepsilon \in (0, \varepsilon_0), N'(x, \varepsilon) \cap S \neq \emptyset.$$

Proof. Trivial; $0 < \varepsilon_1 < \varepsilon_2$ implies $N'(x, \varepsilon_1) \subsetneq N'(x, \varepsilon_2)$. □

Theorem 1.2.2

Let $\emptyset \neq S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$ be a limit point of S . Then, every deleted neighborhood of x must contain infinitely many points of S .

Proof. Assume $N'(x; \varepsilon) \cap S$ were to contain only finitely many points, namely, $N'(x; \varepsilon) \cap S = \{x_1, x_2, \dots, x_k\}$. Let $S_1 := \{|x - x_i| : i \in [k]\}$. Since S_1 is finite, we may let x_j be an element of $N'(x; \varepsilon) \cap S$ that satisfies $|x - x_j| = \min S_1 = \inf S_1 > 0$. If we let $\varepsilon_0 := |x - x_j|/2$, $N'(x; \varepsilon_0) \cap S = \emptyset$, #. □

Corollary 1.2.1

If S is a finite subset of \mathbb{R} , then S has no limit point.

Example 1.2.2

\mathbb{Z} has no limit point.

Theorem 1.2.3 Bolzano–Weierstrass Theorem

If $S \subseteq \mathbb{R}$ is bounded and has an infinite number of elements, then S has a limit point.

Proof. Since S is bounded, $a_0 := \inf S$ and $b_0 := \sup S$ exist; $S \subseteq [a_0, b_0]$. At least one of $[a_0, (a_0 + b_0)/2]$ and $[(a_0 + b_0)/2, b_0]$ has an infinite number of elements in S , otherwise S must be finite. Choose whichever has an infinite number of elements in S , and let us denote it as $[a_1, b_1]$. Since, $S \cap [a_1, b_1]$ is bounded and has an infinite number of elements, we may find a_2 and b_2 in the same manner. Note that

- (a) for every natural number k , $S \cap [a_k, b_k]$ has an infinite number of elements,
- (b) $\forall k \in \mathbb{N}$, $b_k - a_k = (b_0 - a_0)/2^k > 0$, and
- (c) $\forall k \in \mathbb{N}$, $a_{k-1} \leq a_k < b_k \leq b_{k-1}$.

The sequence $\{a_k\}_{k=0}^{\infty}$ is bounded above by b_0 , and the sequence $\{b_k\}_{k=0}^{\infty}$ is bounded below by a_0 . Therefore, we may let $\alpha := \sup\{a_k\}$ and $\beta := \inf\{b_k\}$.

Since a_j is a lower bound of $\{b_k\}_{k=0}^{\infty}$ for all $j \in \mathbb{N}$, $\forall j \in \mathbb{N}$, $a_j \leq \beta$. This implies β is an upper bound of $\{a_k\}_{k=0}^{\infty}$, therefore $\alpha \leq \beta$. Since $a_j \leq \alpha \leq \beta \leq b_j$ for all $j \in \mathbb{N}$, we get $0 \leq \beta - \alpha \leq b_j - a_j = (b_0 - a_0)/2^j$. Therefore, $\beta - \alpha = 0$.

We now claim that α is a limit point of S . Take any $\varepsilon \in \mathbb{R}_+$. By Theorem 1.1.4, $\exists k_0 \in \mathbb{N}$, $\alpha - \varepsilon < a_{k_0} \leq \alpha$. We may take $k \in \mathbb{N}$ such that $k > k_0$ and $|b_k - a_k| < \varepsilon$ thanks to (b). Since $\alpha \in [a_k, b_k]$, $\alpha - \varepsilon < a_{k_0} \leq a_k \leq \alpha \leq b_k < \alpha + \varepsilon$, which implies $[a_k, b_k] \subseteq N(\alpha; \varepsilon)$.

In conclusion, $S \cap [a_k, b_k]$ has infinitely many elements by (a), and so does $(S \cap [a_k, b_k]) \setminus \{\alpha\}$. $S \cap N'(\alpha; \varepsilon)$ is, therefore, nonempty. \square

Definition 1.2.4: Bolzano–Weierstrass Property

We say that a nonempty set X has the *Bolzano–Weierstrass property* if every bounded, infinite subset S of X has a limit point in X .

1.3 The Limit of a Sequence

Definition 1.3.1: Cluster Point

$c \in \mathbb{R}$ is a *cluster point* of the sequence $\{x_k\}$ if,

$$\forall(\varepsilon, k) \in \mathbb{R}_+ \times \mathbb{N}, \exists k_1 \in \mathbb{N}_{>k}, x_{k_1} \in N(c; \varepsilon).$$

Lemma 1.3.1

$c \in \mathbb{R}$ is a cluster point of $\{x_k\}$ if and only if $\{k \in \mathbb{N} \mid x_k \in N(c; \varepsilon)\}$ is infinite for every $\varepsilon \in \mathbb{R}_+$.

Proof. (\Rightarrow) Suppose $S := \{k \in \mathbb{N} \mid x_k \in N(c; \varepsilon)\}$ is finite for some $\varepsilon \in \mathbb{R}_+$. If S were empty, then, c is not a cluster point by Definition 1.3.1. Therefore, S is nonempty and has a maximum element $k_0 := \max S$ by Theorem 1.1.3. Since c is a cluster point, there is a natural number $k_1 > k_0$ such that $x_{k_1} \in N(c; \varepsilon)$; $k_1 \in S$. This contradicts the maximality of k_0 .

(\Leftarrow) Take any $\varepsilon \in \mathbb{R}_+$ and $k_0 \in \mathbb{N}$. If there is no $k_1 \in \mathbb{N}$ such that $k_1 > k_0$ and $x_{k_1} \in N(c; \varepsilon)$, S will be bounded above by k_0 and finite, which is a contradiction. Therefore, c is a cluster point of S . □

Definition 1.3.2: Convergence and Divergence of a Sequence

The sequence $\{x_k\}$ *converges* to x_0 and x_0 is the *limit* of $\{x_k\}$ if,

$$\forall \varepsilon \in \mathbb{R}_+, \exists k_0 \in \mathbb{N}, \forall k \in \mathbb{N}_{\geq k_0}, x_k \in N(x_0; \varepsilon).$$

We write $\lim_{k \rightarrow \infty} x_k = x_0$. If there is no such x_0 , then $\{x_k\}$ *diverges*.

Lemma 1.3.2

$\lim_{k \rightarrow \infty} x_k = x_0$ if and only if $\{k \in \mathbb{N} \mid x_k \notin N(x_0; \varepsilon)\}$ is finite for every $\varepsilon \in \mathbb{R}_+$.

Proof. (\Rightarrow) Take any $\varepsilon \in \mathbb{R}_+$. There is some $k_0 \in \mathbb{N}$ such that $k \in N(x_0; \varepsilon)$ for all natural numbers $k \geq k_0$. Therefore, $\{k \in \mathbb{N} \mid x_k \notin N(x_0; \varepsilon)\} \subseteq [k_0]$ and thus finite.

(\Leftarrow) Take any $\varepsilon \in \mathbb{R}_+$. Let $k_0 := \max \{k \in \mathbb{N} \mid x_k \notin N(x_0; \varepsilon)\}$. Then, for every natural number k larger than k_0 satisfies $x_k \in N(x_0; \varepsilon)$. □

Lemma 1.3.3

The limit x_0 of a sequence, if it exists, is a cluster point of the sequence.

Theorem 1.3.1 Uniqueness of the Limit

The limit of a convergent sequence of \mathbb{R} is unique.

Proof. Suppose a and b are two limits of a sequence $\{x_k\}$ and $a \neq b$. Let $\varepsilon := |b - a|/2$. Then, by Lemma 1.3.2, $A := \{k \in \mathbb{N} \mid x_k \notin N(a; \varepsilon)\}$ and $B := \{k \in \mathbb{N} \mid x_k \notin N(b; \varepsilon)\}$ are both finite, which means $A \cup B = \mathbb{N}$ is finite, $\#$. □

Theorem 1.3.2

If a sequence has two (or more) cluster points, then it diverges.

Proof. Suppose x_0 is the limit of $\{x_k\}$. Since, by Lemma 1.3.3, x_0 is a cluster point, there is another cluster point c different from x_0 . Let $\varepsilon := |x_0 - c|/2$.

Although $S := \{k \in \mathbb{N} \mid x_k \notin N(x_0; \varepsilon)\}$ should be finite by Lemma 1.3.2, $\{k \in \mathbb{N} \mid x_k \in N(c; \varepsilon)\}$, a subset of S , is infinite by Lemma 1.3.1, $\#$. \square

Theorem 1.3.3

A convergent sequence is bounded.

Proof. Let x_0 is the limit of $\{x_k\}$. There is some $k_0 \in \mathbb{N}$ such that $|x_k - x_0| < 1$ for all $k \in \mathbb{N}_{k_0}$. Let $A := \{x_k \mid k \in \mathbb{N} \text{ and } k \leq k_0\}$ and $B := \{x_k \mid k \in \mathbb{N} \text{ and } k \geq k_0\}$. Then, A is finite and B is bounded above and below by $x_0 + 1$ and $x_0 - 1$, respectively. Therefore, $\{x_k\}$ is bounded above by $\max(\max A, x_0 + 1)$ and below by $\min(\min A, x_0 - 1)$. \square

Corollary 1.3.1

An unbounded sequence diverges.

Lemma 1.3.4

The following hold.

- (i) $\lim_{k \rightarrow \infty} x_k = 0 \iff \lim_{k \rightarrow \infty} |x_k| = 0$
- (ii) $\lim_{k \rightarrow \infty} x_k = x_0 \implies \forall c \in \mathbb{R}, \lim_{k \rightarrow \infty} cx_k = cx_0$

Proof of (ii). If $c = 0$, then it is done; so suppose $c \neq 0$. Take any $\varepsilon \in \mathbb{R}_+$. Then, there is some $k_0 \in \mathbb{N}$ such that $|x_k - x_0| < \varepsilon/|c|$ for all $k \geq k_0$. This directly implies for all $k \geq k_0$, $|cx_k - cx_0| = |c| \cdot |x_k - x_0| < |c| \cdot \varepsilon/|c| = \varepsilon$. \square

Theorem 1.3.4

A bounded, monotone sequence converges.

Proof. Suppose $\{x_k\}$ is a monotone increasing sequence. Since it is bounded, $\{x_k\}$ has $\mu := \sup\{x_k \mid k \in \mathbb{N}\}$. Take any $\varepsilon \in \mathbb{R}_+$. By Theorem 1.1.4, there is some $k_0 \in \mathbb{N}$ such that $\mu - \varepsilon < x_{k_0} \leq \mu$. Then, for all $k \in \mathbb{N}_{\geq k_0}$, $\mu - \varepsilon < x_{k_0} \leq x_k \leq \mu$, which implies $|x_k - \mu| < \varepsilon$. Therefore $\lim_{k \rightarrow \infty} x_k = \mu$. \square

Theorem 1.3.5 The Squeeze Play

Let $\{x_k\}$, $\{y_k\}$, and $\{z_k\}$ be sequences that satisfy $x_k \leq y_k \leq z_k$ for $k \in \mathbb{N}$. If both $\{x_k\}$ and $\{z_k\}$ converges to $L \in \mathbb{R}$, then $\{y_k\}$ also converges to L .

Proof. Take any $\varepsilon > 0$. There is $k_1 \in \mathbb{N}$ such that $\forall k \in \mathbb{N}_{\geq k_1}, x_k \in N(L; \varepsilon)$. Similarly, there is $k_2 \in \mathbb{N}$ such that $\forall k \in \mathbb{N}_{\geq k_2}, z_k \in N(L; \varepsilon)$. Then, for all $k \in \mathbb{N}$ not smaller than $\max\{k_1, k_2\}$, $L - \varepsilon < x_k \leq y_k \leq z_k < L + \varepsilon$ holds, which implies $y_k \in N(L; \varepsilon)$. \square

Theorem 1.3.6 Limit is Order Preserving on Convergent Sequences

If both $\{x_k\}$ and $\{y_k\}$ converge and if $x_k \leq y_k$ for each $k \in \mathbb{N}$, then

$$\lim_{k \rightarrow \infty} x_k \leq \lim_{k \rightarrow \infty} y_k.$$

Proof. Let $L_x := \lim_{k \rightarrow \infty} x_k$ and $L_y := \lim_{k \rightarrow \infty} y_k$, and suppose $L_x > L_y$. Let $\varepsilon := (L_x - L_y)/2 > 0$. Then, there is $k \in \mathbb{N}$ such that $x_k \in N(L_x; \varepsilon)$ and $y_k \in N(L_y; \varepsilon)$, which implies $y_k < L_y + \varepsilon = L_x - \varepsilon < x_k$, #. \square

Definition 1.3.3: Subsequence

Let $\{x_k\}$ be any sequence. Choose any strictly monotone increasing sequence $k_1 < k_2 < k_3 < \dots$ of natural numbers. For each $j \in \mathbb{N}$, let $y_j := x_{k_j}$. The sequence $\{y_j\}_{j=1}^{\infty}$ is called an *subsequence* of $\{x_k\}$.

Theorem 1.3.7

The point c is a cluster point of $\{x_k\}$ if and only if there exists a subsequence of $\{x_k\}$ that converges to c .

Proof. (\Rightarrow) Let $\{\varepsilon_k\}$ be an arbitrary sequence of positive real numbers that converges to 0. (e.g. $\varepsilon_k = 1/k$) Define $\{k_j\}_{j=1}^{\infty}$ by the inductive definition below.

- $k_0 := 0$
- For each $j \in \mathbb{N}$, $k_j \in \{k \in \mathbb{N} \mid k > k_{j-1} \text{ and } x_k \in N(c; \varepsilon_j)\}$.

Since c is a cluster point, $\{k \in \mathbb{N} \mid k > k_{j-1} \text{ and } x_k \in N(c; \varepsilon_j)\} \neq \emptyset$ for all $j \in \mathbb{N}$. Therefore, $\{k_j\}$ is well-defined. It is immediate that $\lim_{j \rightarrow \infty} x_{k_j} = c$.

(\Leftarrow) Let $\{x_{k_j}\}_{j=1}^{\infty}$ be a sequence such that $\lim_{j \rightarrow \infty} x_{k_j} = c$. Take any $\varepsilon \in \mathbb{R}_+$ and $k \in \mathbb{N}$. Then, there is some $j_0 \in \mathbb{N}$ such that $\forall j \in \mathbb{N}_{\geq j_0}$, $x_{k_j} \in N(c; \varepsilon)$. Let $k_0 := \min\{k_j \in \mathbb{N} \mid j > j_0 \text{ and } k_j > k\}$. Then, $x_{k_0} \in N(c; \varepsilon)$ and $k_0 > k$. Therefore, c is a cluster point. \square

Theorem 1.3.8

Any bounded sequence $\{x_k\}$ has a cluster point.

Proof. If the set $S := \{x_k \mid k \in \mathbb{N}\}$ is finite, there is some x_{k_0} that is repeated infinitely. Then, x_{k_0} is surely a cluster point.

Now, suppose S is infinite. Then, by Theorem 1.2.3, S has a limit point ℓ . To prove ℓ is a cluster point, take any $\varepsilon \in \mathbb{R}_+$ and $k \in \mathbb{N}$.

Let $S' := \{x_{k'} \mid k' \in \mathbb{N}_{>k}\}$. We first claim that ℓ is a limit point of S' . Take any $\varepsilon' \in \mathbb{R}_+$ less than $m = \min\{|x_{k'} - \ell| \in \mathbb{R}_+ \mid k' \in \mathbb{N}_{\leq k}\}$. (m exists due to Theorem 1.1.3.) Then, $S' \cap N'(\ell; \varepsilon') = S \cap N'(\ell; \varepsilon')$ is nonempty. Therefore, ℓ is a limit point of S' by Theorem 1.2.1.

Finally, we can say $S' \cap N(\ell; \varepsilon)$ is nonempty. This implies there is some $k_0 \in \mathbb{Z}_{>k}$ such that $x_{k_0} \in N(\ell; \varepsilon)$. Therefore, ℓ is a cluster point of $\{x_k\}$. \square

Corollary 1.3.2

If a sequence has no cluster point, then the sequence is unbounded.

Corollary 1.3.3

Any bounded sequence converges if and only if it has exactly one cluster point.

Corollary 1.3.4

A sequence $\{x_k\}$ diverges if and only if at least one of the following conditions holds.

- $\{x_k\}$ has two or more cluster points.
- $\{x_k\}$ is unbounded.

Proof. (\Rightarrow) Suppose $\{x_k\}$ is diverging and bounded. By Theorem 1.3.8, it has at least one cluster point. Also, if it had exactly one cluster point, it would converge by Corollary 1.3.3.

(\Leftarrow) It is direct from Theorem 1.3.2 and Corollary 1.3.1. \square

Theorem 1.3.9

A sequence $\{x_k\}$ converges if and only if every subsequence of $\{x_k\}$ converges.

Proof. (\Rightarrow) Take any subsequence $\{x_{k_i}\}_{i=1}^{\infty}$ of $\{x_k\}_{k=1}^{\infty}$ and $\varepsilon \in \mathbb{R}_+$. There is $i_0 \in \mathbb{N}$ such that $\forall i \in \mathbb{N}_{\geq i_0}, |x_{k_i}| < \varepsilon$. Since $k_i \geq i$ for all natural number i , $\forall i \in \mathbb{N}_{\geq i_0}, |x_{k_i}| < \varepsilon$.

(\Leftarrow) $\{x_k\}$ is a subsequence of itself. \square

Definition 1.3.4: Limit Superior and Inferior

Let $\{x_k\}$ be a sequence and C be a set of cluster points of the sequence.

- $\limsup x_k \triangleq \begin{cases} \sup C & \text{if } \{x_k\} \text{ is bounded} \\ \infty & \text{if } \{x_k\} \text{ is unbounded above} \\ \sup C & \text{if } \{x_k\} \text{ is bounded above but unbounded below and } C \neq \emptyset \\ -\infty & \text{if } \{x_k\} \text{ is bounded above but unbounded below and } C = \emptyset \end{cases}$
is called *limit superior* of $\{x_k\}$.

- $\liminf x_k \triangleq \begin{cases} \inf C & \text{if } \{x_k\} \text{ is bounded} \\ -\infty & \text{if } \{x_k\} \text{ is unbounded below} \\ \inf C & \text{if } \{x_k\} \text{ is bounded below but unbounded above and } C \neq \emptyset \\ \infty & \text{if } \{x_k\} \text{ is bounded below but unbounded above and } C = \emptyset \end{cases}$
is called *limit inferior* of $\{x_k\}$.

Note:-

In all cases, $\liminf x_k \leq \limsup x_k$.

Theorem 1.3.10

- If $\mu = \limsup x_k$ is finite, then μ is in C . ($\mu = \max C$)
- If $\nu = \liminf x_k$ is finite, then ν is in C . ($\nu = \min C$)

Proof. Suppose $\mu = \limsup x_k$ is finite. Take any $\varepsilon \in \mathbb{R}_+$ and $k \in \mathbb{N}$. The finiteness of μ implies $\mu = \sup C$. By Theorem 1.1.4, there is some $c \in C$ such that $\mu - \varepsilon < c \leq \mu$. If $c = \mu$, then we are done. So let $c < \mu$.

Choose any positive ε_1 less than $\min\{c - (\mu - \varepsilon), \mu - c\}$ so $N(c; \varepsilon_1) \subseteq N(\mu; \varepsilon)$. Then, $\{k \in \mathbb{N} \mid x_k \in N(\mu; \varepsilon)\}$ is infinite since it has an infinite set $\{k \in \mathbb{N} \mid x_k \in N(c; \varepsilon_1)\}$ as its subset. (See Lemma 1.3.1.)

The second part can be proven analogously. \square

Theorem 1.3.11

Let $\{x_k\}$ be any bounded sequence in \mathbb{R} . Fix any $\varepsilon \in \mathbb{R}_+$.

- Let $\mu = \limsup x_k$.
 - $\exists k_0 \in \mathbb{N}, \forall k \in \mathbb{N}_{\geq k_0}, x_k < \mu + \varepsilon$.
 - $\forall k \in \mathbb{N}, \exists k_1 \in \mathbb{N}_{>k}, x_{k_1} > \mu - \varepsilon$.
- Let $\nu = \liminf x_k$.
 - $\exists k_0 \in \mathbb{N}, \forall k \in \mathbb{N}_{\geq k_0}, x_k > \nu - \varepsilon$.
 - $\forall k \in \mathbb{N}, \exists k_1 \in \mathbb{N}_{>k}, x_{k_1} < \nu + \varepsilon$.

Proof. Take any $\varepsilon \in \mathbb{R}_+$. Then, $\{k \in \mathbb{N} \mid x_k \geq \mu + \varepsilon\}$ is finite. If it were not, then there would be a cluster point larger than μ since Theorem 1.3.8 implies the existence of a cluster point in a subsequence of $\{x_k\}$ which is composed of x_k 's not smaller than $\mu + \varepsilon$. Therefore, if $k_0 := \max\{k \in \mathbb{N} \mid x_k \geq \mu + \varepsilon\} + 1$, then $x_k < \mu + \varepsilon$ for all k not smaller than k_0 .

Also, since μ is a cluster point by Theorem 1.3.10, $\forall k \in \mathbb{N}, \exists k_1 \in \mathbb{N}_{>k}, x_{k_1} > \mu - \varepsilon$. (See Lemma 1.3.1.)

The second part can be proven analogously. \square

Theorem 1.3.12

Let $\{x_k\}$ be any sequence in \mathbb{R} .

- (i) $\{x_k\}$ converges to x_0 if and only if $\liminf x_k = \limsup x_k = x_0$.
- (ii) $\{x_k\}$ diverges if and only if one of the following holds.
 - Either $\liminf x_k$ or $\limsup x_k$ is infinite.
 - Both $\liminf x_k$ or \limsup are finite and $\liminf x_k < \limsup x_k$.

Proof.

- (i) (\Rightarrow) $C = \{x_0\}$, therefore $\liminf x_k = \limsup x_k = x_0$.
 (\Leftarrow) Take any $\varepsilon \in \mathbb{R}_+$. There are natural numbers k_1 and k_2 such that $\forall k \in \mathbb{N}_{\geq k_1}, x_k < x_0 + \varepsilon$ and $\forall k \in \mathbb{N}_{\geq k_2}, x_k > x_0 - \varepsilon$. Then, for all natural number k not smaller than $k_0 := \max\{k_1, k_2\}$, $x_0 - \varepsilon < x_k < x_0 + \varepsilon$ holds.
- (ii) If it is not $\liminf x_k = \limsup x_k \in \mathbb{R}$, then it is either “One of them is infinite.” or “They are both finite but they are different.” \square

Exercise 1.3.1

Let $\{x_k\}$ be a bounded sequence of positive numbers. For each $k \in \mathbb{N}$ define $y_k := x_{k+1}/x_k$ and $z_k := (x_k)^{1/k}$. Prove that $\liminf y_k \leq \liminf z_k \leq \limsup z_k \leq \limsup y_k$.

Solution: ($\liminf y_k \leq \liminf z_k$) Let $L := \liminf y_k$. Now, we claim that

$$\forall \varepsilon \in \mathbb{R}_+, \exists k_0 \in \mathbb{N}, \forall k \in \mathbb{N}_{\geq k_0}, z_k > L - \varepsilon.$$

If $L = 0$, then it is done. Therefore, suppose $L > 0$. To prove this, take any $\varepsilon \in \mathbb{R}_+$ smaller than L . Then, there is some $k_1 \in \mathbb{N}$ such that $y_k > L - \varepsilon/2$ for all k not smaller than k_1 by

Theorem 1.3.11. Then, for all $k \in \mathbb{N}_{\geq k_1}$, $x_k > (L - \varepsilon/2)^{k-k_1} x_{k_1}$, which is equivalent to

$$z_k = x_k^{1/k} > \left(L - \frac{\varepsilon}{2}\right) \left[\left(L - \frac{\varepsilon}{2}\right)^{-k_1} x_{k_1}\right]^{1/k}.$$

Since $\lim_{k \rightarrow \infty} \left[\left(L - \varepsilon/2\right)^{-k_1} x_{k_1}\right]^{1/k} = 1$, there is some $k_2 \in \mathbb{N}$ such that

$$\left[\left(L - \frac{\varepsilon}{2}\right)^{-k_1} x_{k_1}\right]^{1/k} > 1 - \frac{\varepsilon/2}{L - \varepsilon/2} = \frac{L - \varepsilon}{L - \varepsilon/2}.$$

for all $k \in \mathbb{N}_{\geq k_2}$. Thus, for every natural number k not smaller than $\max\{k_1, k_2\}$,

$$z_k > \left(L - \frac{\varepsilon}{2}\right) \left[\left(L - \frac{\varepsilon}{2}\right)^{-k_1} x_{k_1}\right]^{1/k} > \left(L - \frac{\varepsilon}{2}\right) \cdot \frac{L - \varepsilon}{L - \varepsilon/2} = L - \varepsilon.$$

The claim is now proven.

For the main proof, assume that $\liminf z_k < L$ for the sake of contradiction. Take $\varepsilon_0 := (L - \liminf z_k)/2$. Then, by the previous claim, $\exists k_3 \in \mathbb{N}$, $\forall k \in \mathbb{N}_{\geq k_3}$, $z_k > L - \varepsilon_0 = (L + \liminf z_k)/2$.

Nevertheless, by Theorem 1.3.11, there is some $k_4 \in \mathbb{N}_{>k_3}$ such that $z_{k_4} < \liminf z_k + \varepsilon_0 = (L + \liminf z_k)/2$, which is a contradiction.

$\limsup z_k \leq \limsup y_k$ can be proven analogously.

□

1.4 Cauchy Sequences

Definition 1.4.1: Cauchy Sequence

A sequence $\{x_k\}$ in \mathbb{R} is called a *Cauchy sequence* if

$$\forall \varepsilon \in \mathbb{R}_+, \exists k_0 \in \mathbb{N}, \forall k, m \in \mathbb{N}_{\geq k_0}, |x_k - x_m| < \varepsilon.$$

Theorem 1.4.1

If $\{x_k\}$ is a convergent sequence of real numbers, then $\{x_k\}$ is a Cauchy sequence.

Proof. Let $x_0 := \lim_{k \rightarrow \infty} x_k$. Take any $\varepsilon \in \mathbb{R}_+$. Then, there is some $k_0 \in \mathbb{N}$ such that $|x_k - x_0| < \varepsilon/2$ for all $k \in \mathbb{N}$ not smaller than k_0 . Then, for all $k, m \in \mathbb{N}$ greater than k_0 , $|x_k - x_m| \leq |x_k - x_0| + |x_0 - x_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. □

Theorem 1.4.2

If $\{x_k\}$ is a Cauchy sequence, then $\{x_k\}$ is bounded.

Proof. There is $k_0 \in \mathbb{N}$ such that $|x_k - x_m| < 1$ for all $k, m \in \mathbb{N}_{\geq k_0}$. It implies that $|x_k - x_{k_0}| < 1$, for all $k \in \mathbb{N}_{\geq k_0}$, which implies $|x_k| < |x_{k_0}| + 1$. Therefore, for all $k \in \mathbb{N}$, $|x_k| \leq \max\{|x_1|, |x_2|, \dots, |x_{k_0}|, |x_{k_0}| + 1\}$. □

Theorem 1.4.3

A Cauchy sequence has exactly one cluster point.

Proof. Since a Cauchy sequence is bounded, it has at least one cluster point by Theorem 1.3.8. So, we should prove that the sequence does not have more than one cluster point. Assume c_1 and c_2 are cluster points for the sake of contradiction. Let $\varepsilon := |c_1 - c_2|/3$. Choose $k_0 \in \mathbb{N}$ such that $\forall k, m \in \mathbb{N}_{\geq k_0}, |x_k - x_m| < \varepsilon$. Also, there are $k_1, k_2 \in \mathbb{N}_{> k_0}$ such that $|x_{k_1} - c_1| < \varepsilon$ and $|x_{k_2} - c_2| < \varepsilon$. Note that $|c_1 - c_2| \leq |c_1 - x_{k_1}| + |x_{k_1} - x_{k_2}| + |x_{k_2} - c_2|$. Nevertheless, then

$$\begin{aligned} \varepsilon &> |x_{k_1} - x_{k_2}| \geq |c_1 - c_2| - |c_1 - x_{k_1}| - |c_2 - x_{k_2}| \\ &> 3\varepsilon - \varepsilon - \varepsilon = \varepsilon, \end{aligned}$$

which is a contradiction. □

Theorem 1.4.4 Cauchy Completeness of \mathbb{R}

A sequence in \mathbb{R} is convergent if and only if it is a Cauchy sequence.

Proof. By Corollary 1.3.3, a Cauchy sequence is convergent since it is bounded (Theorem 1.4.2) and has exactly one cluster point (Theorem 1.4.3). A convergent sequence in \mathbb{R} is Cauchy. (Theorem 1.4.1) □

Definition 1.4.2: Cauchy Completeness

A set X is said to be *Cauchy complete* if every Cauchy sequence in X converges to a point of X .

Example 1.4.1

\mathbb{R} is Cauchy complete.

Definition 1.4.3: Contractive Sequence

A sequence $\{x_k\}$ is said to be *contractive* if there exists a constant C , with $0 < C < 1$, such that

$$\forall k \in \mathbb{N}_{>1}, |x_{k+1} - x_k| \leq C|x_k - x_{k-1}|.$$

Theorem 1.4.5

Any contractive sequence in \mathbb{R} is a Cauchy sequence.

Proof. Suppose $0 < C < 1$ and $\forall k \in \mathbb{N}_{>1}, |x_{k+1} - x_k| \leq C|x_k - x_{k-1}|$. If it is trivial when $|x_2 - x_1| = 0$, so suppose $|x_2 - x_1| \neq 0$. By induction, $\forall k \in \mathbb{N}_{>1}, |x_{k+1} - x_k| \leq C^{k-1}|x_2 - x_1|$.

To prove $\{x_k\}$ is a Cauchy sequence, take any $\varepsilon \in \mathbb{R}_+$. Since $\lim_{k \rightarrow \infty} C^{k-1} = 0$,

$$\exists k_0 \in \mathbb{N}, \forall k \in \mathbb{N}_{\geq k_0}, C^{k-1} < \frac{(1-C)\varepsilon}{|x_2 - x_1|}.$$

Then, for any $k, m \in \mathbb{N}$ with $k_0 \leq m < k$,

$$\begin{aligned}
|x_k - x_m| &= \left| \sum_{j=m}^{k-1} (x_{j+1} - x_j) \right| \leq \sum_{j=m}^{k-1} |x_{j+1} - x_j| \\
&\leq \sum_{j=m}^{k-1} C^{j-1} |x_2 - x_1| = C^{m-1} |x_2 - x_1| \sum_{j=0}^{k-m-1} C^j \\
&= C^{m-1} |x_2 - x_1| \frac{1 - C^{k-m}}{1 - C} < \frac{C^{m-1}}{1 - C} |x_2 - x_1| \\
&< \frac{(1 - C)\varepsilon}{|x_2 - x_1|} \cdot \frac{1}{1 - C} |x_2 - x_1| = \varepsilon.
\end{aligned}$$

□

1.5 The Algebra of Convergent Series

Theorem 1.5.1

Let $\{x_k\}$ and $\{y_k\}$ be convergent sequences in \mathbb{R} and $\lim_{k \rightarrow \infty} x_k = x_0$ and $\lim_{k \rightarrow \infty} y_k = y_0$.

- $\lim_{k \rightarrow \infty} (x_k + y_k) = x_0 + y_0$
- $\lim_{k \rightarrow \infty} x_k y_k = x_0 y_0$
- $\lim_{k \rightarrow \infty} \frac{y_k}{x_k} = \frac{y_0}{x_0}$ if $x_0 \neq 0$.

Theorem 1.5.2

Let $\{x_k\}$ and $\{y_k\}$ be convergent sequences in \mathbb{R} and $\lim_{k \rightarrow \infty} x_k = x_0$. Then, if $r \in \mathbb{Q}$, then

$$\lim_{k \rightarrow \infty} x_k^r = x_0^r.$$

Nevertheless, we require $x_0 \neq 0$ if $r < 0$.

1.6 Cardinality

Definition 1.6.1: Dense Set

We say a subset S of T is dense in T if every neighborhood of any point $x \in T$ contains points of S .

Theorem 1.6.1

- \mathbb{N} , \mathbb{Z} , and \mathbb{Q} are countably infinite.
- \mathbb{R} is uncountable.
- \mathbb{Q} is dense in \mathbb{R} .

Chapter 2

Euclidean Spaces

2.1 Euclidean n -Space

Definition 2.1.1: Inner Product

The *inner product* of two vectors $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ in \mathbb{R}^n is

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^n x_j y_j.$$

Theorem 2.1.1

If \mathbf{x} , \mathbf{y} , and \mathbf{z} are arbitrary vectors in \mathbb{R}^n and if a and b are real numbers, then the following hold:

(i) The inner product is *additive* in both its variables:

$$\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$$

$$\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$$

(ii) The inner product is *symmetric*: $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$.

(iii) The inner product is *homogeneous* in both its variables: $\langle a\mathbf{x}, b\mathbf{y} \rangle = ab\langle \mathbf{x}, \mathbf{y} \rangle$.

Definition 2.1.2: Euclidean Norm

The *Euclidean norm* of a vector \mathbf{x} in \mathbb{R}^n is

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

Theorem 2.1.2 The Cauchy-Schwarz Inequality

If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|.$$

Proof. For any $t \in \mathbb{R}$, $0 \leq \|t\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 t^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle t + \|\mathbf{y}\|^2$. Thus, the discriminant $|\langle \mathbf{x}, \mathbf{y} \rangle|^2 - \|\mathbf{x}\|^2 \|\mathbf{y}\|^2$ is nonpositive. \square

Theorem 2.1.3

For vectors \mathbf{x} and \mathbf{y} in \mathbb{R}^n and any $c \in \mathbb{R}$, the Euclidean norm has the following proper-

ties.

- (i) $\|\mathbf{x}\| \geq 0$; $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$. (*Positive Definiteness*)
- (ii) $\|c\mathbf{x}\| = |c| \cdot \|\mathbf{x}\|$. (*Absolute Homogeneity*)
- (iii) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. (*Subadditivity*)

Proof of (iii).

$$\begin{aligned} 0 \leq \|\mathbf{x} + \mathbf{y}\|^2 &= \|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{y}\|^2 \\ &\leq \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2 \end{aligned}$$

□

Definition 2.1.3: Norm

A *norm* on \mathbb{R}^n is any function $n: \mathbb{R}^n \rightarrow \mathbb{R}$ that is positive definite, absolutely homogeneous, and subadditive.

Definition 2.1.4: Metric

A *metric* on \mathbb{R}^n is a function from $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ having the following properties.

- (i) $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{y}) \geq 0$; $d(\mathbf{x}, \mathbf{y}) = 0 \iff \mathbf{x} = \mathbf{y}$. (*Positive Definiteness*)
- (ii) $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$. (*Symmetry*)
- (iii) $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$. (*The Triangle Inequality*)

Definition 2.1.5: Euclidean Metric

The *Euclidean metric* on \mathbb{R}^n is defined by

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \left[\sum_{j=1}^n (x_j - y_j)^2 \right]^{1/2}.$$

Theorem 2.1.4

The Euclidean metric is a metric on \mathbb{R}^n .

Definition 2.1.6: Orthogonality

Two vectors \mathbf{x} and \mathbf{y} in \mathbb{R}^n are said to be *orthogonal* if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$.

Definition 2.1.7: Neighborhood and Deleted Neighborhood

A *neighborhood* $N(\mathbf{x}; r)$ or $\mathbf{x} \in \mathbb{R}^n$ with radius r is the set

$$N(\mathbf{x}; r) = \{\mathbf{y} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{y}\| < r\}.$$

A *deleted neighborhood* $N'(\mathbf{x}, r)$ of \mathbf{x} is $N'(\mathbf{x}, r) = N(\mathbf{x}, r) \setminus \{\mathbf{x}\}$.

Definition 2.1.8: Limit Point

Let S be nonempty subset of \mathbb{R}^n . We say that \mathbf{x} is a *limit point* of S if

$$\forall \varepsilon \in \mathbb{R}_+, N'(\mathbf{x}; \varepsilon) \cap S \neq \emptyset.$$

Theorem 2.1.5

\mathbb{Q}^n is dense in \mathbb{R}^n .

Proof. Take any $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}_+$. For each $j = 1, 2, \dots, n$, choose a rational $x_j \in N(y_j; \varepsilon/\sqrt{n})$ and form $\mathbf{x} \triangleq (x_1, x_2, \dots, x_n) \in \mathbb{Q}^n$. Then,

$$\|\mathbf{x} - \mathbf{y}\|^2 = \sum_{j=1}^n (x_j - y_j)^2 < n(\varepsilon/\sqrt{n})^2 = \varepsilon^2.$$

Therefore \mathbf{y} is a limit point of \mathbb{Q}^n . □

Definition 2.1.9: Boundedness

A subset S of \mathbb{R}^n is said to be *bounded* if

$$\exists M \in \mathbb{R}_+, \forall \mathbf{x} \in S, \|\mathbf{x}\| \leq M.$$

2.1.1 Sequences in \mathbb{R}^n

Definition 2.1.10: Cluster Point

$\mathbf{c} \in \mathbb{R}^n$ is a *cluster point* of the sequence $\{\mathbf{x}_k\}$ if,

$$\forall (\varepsilon, k) \in \mathbb{R}_+ \times \mathbb{N}, \exists k_1 \in \mathbb{N}_{>k}, \mathbf{x}_{k_1} \in N(\mathbf{c}; \varepsilon).$$

Definition 2.1.11: Convergence and Divergence of a Sequence

The sequence $\{\mathbf{x}_k\}$ *converges* to \mathbf{x}_0 and \mathbf{x}_0 is the *limit* of $\{\mathbf{x}_k\}$ if,

$$\forall \varepsilon \in \mathbb{R}_+, \exists k_0 \in \mathbb{N}, \forall k \in \mathbb{N}_{\geq k_0}, \mathbf{x}_k \in N(\mathbf{x}_0; \varepsilon).$$

We write $\lim_{k \rightarrow \infty} \mathbf{x}_k = \mathbf{x}_0$. If there is no such \mathbf{x}_0 , then $\{\mathbf{x}_k\}$ *diverges*.

Theorem 2.1.6

Let $\{\mathbf{x}_k\} = (x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)})$ for each $k \in \mathbb{N}$. Let $\mathbf{x}_0 = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$. The sequence $\{\mathbf{x}_k\}$ converges to \mathbf{x}_0 if and only if, for each $j \in [n]$, the sequence $\{x_j^{(k)}\}$ converges to $\{x_j^{(0)}\}$.

Proof. (\Rightarrow) Take any $\varepsilon \in \mathbb{R}_+$. There there is $k_0 \in \mathbb{N}$ such that

$$\forall k \in \mathbb{N}_{\geq k_0}, \mathbf{x}_k \in N(\mathbf{x}_0; \varepsilon).$$

Then, for each $j \in [n]$,

$$(x_j^{(k)} - x_0^{(k)})^2 \leq \sum_{i=1}^n (x_i^{(k)} - x_0^{(k)})^2 = \|\mathbf{x}_k - \mathbf{x}_0\|^2 < \varepsilon.$$

(\Leftarrow) Take any $\varepsilon \in \mathbb{R}_+$. Then, for each $j \in [n]$, there is some $k_j \in \mathbb{N}$ such that

$$\forall k \in \mathbb{N}_{\geq k_j}, x_j^{(k)} \in N(x_0^{(k)}; \varepsilon/\sqrt{n}).$$

Then, for all natural number k not smaller than $\max_{j \in [n]} k_j$,

$$\|\mathbf{x}_k - \mathbf{x}_0\|^2 = \sum_{j=1}^n (x_j^{(k)} - x_0^{(k)})^2 < n(\varepsilon/\sqrt{n})^2 = \varepsilon^2.$$

□

Definition 2.1.12: Cauchy Sequence

A sequence $\{\mathbf{x}_k\}$ in \mathbb{R}^n is called a *Cauchy sequence* if

$$\forall \varepsilon \in \mathbb{R}_+, \exists k_0 \in \mathbb{N}, \forall k, m \in \mathbb{N}_{\geq k_0}, \|\mathbf{x}_k - \mathbf{x}_m\| < \varepsilon.$$

Theorem 2.1.7 Cauchy's Completeness Theorem in \mathbb{R}^n

A sequence $\{\mathbf{x}_k\}$ in \mathbb{R}^n is Cauchy if and only if it converges. \mathbb{R}^n is Cauchy complete.

Proof. (\Leftarrow) The proof is similar to Theorem 1.4.1.

(\Rightarrow) Let some Cauchy sequence $\{\mathbf{x}_k\}$ in \mathbb{R}^n be given. Take any $\varepsilon \in \mathbb{R}_+$. There is some $k_0 \in \mathbb{N}$ such that for every natural number k and m not smaller than k_0 , $\|\mathbf{x}_k - \mathbf{x}_m\| < \varepsilon$. Then, for each $j \in [n]$, $|x_j^{(k)} - x_j^{(m)}| \leq \|\mathbf{x}_k - \mathbf{x}_m\| < \varepsilon$, which implies each $\{x_j^{(k)}\}_{k \in \mathbb{N}}$ is Cauchy. By Theorem 1.4.4, $\{x_j^{(k)}\}_{k \in \mathbb{N}}$ converges to some number $x_j^{(0)}$. Then, Theorem 2.1.6 ensures that $\lim_{k \rightarrow \infty} \mathbf{x}_k = (x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$. □

Theorem 2.1.8 The Generalized Bolzano–Weierstrass Theorem

Every bounded infinite set in \mathbb{R}^n has a limit point in \mathbb{R}^n .

Proof. Suppose that S is any bounded, infinite set in \mathbb{R}^n . Being bounded, S is contained in some n -cube $C(2M) = [-M, M]^n$ centered at $\mathbf{0}$. Construct C_1, C_2, \dots as following.

- $C_1 \triangleq C(2M) = [a_1^{(1)}, b_1^{(1)}] \times \dots \times [a_n^{(1)}, b_n^{(1)}]$
– Note that $C_1 \cap S = S$ is infinite.
- For each $k \in \mathbb{N}$, C_{k+1} is any cube of the form $[a_1^{(k+1)}, b_1^{(k+1)}] \times \dots \times [a_n^{(k+1)}, b_n^{(k+1)}]$ where each $[a_j^{(k+1)}, b_j^{(k+1)}]$ is either $[a_j^{(k)}, (a_j^{(k)} + b_j^{(k)})/2]$ or $[(a_j^{(k)} + b_j^{(k)})/2, b_j^{(k)}]$ so that $C_{k+1} \cap S$ is infinite.
– This is possible since there is at least one cube among 2^n possible choices that $C_{k+1} \cap S$ is infinite.

Then, the main diagonal d_k of C_k equals to $Mn^{1/2}/2^{k-2}$. Also, note that $C_k \supseteq C_{k+1}$ for all $k \in \mathbb{N}$.

Now, we may construct a sequence $\{\mathbf{x}_k\}_{k \in \mathbb{N}}$ as following.

- \mathbf{x}_1 is any element in $C_1 \cap S$.
- For each $k \in \mathbb{N}$, \mathbf{x}_{k+1} is arbitrarily taken from $C_{k+1} \cap S \setminus \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$.

We claim that $\{\mathbf{x}_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence. To show this, take any $\varepsilon \in \mathbb{R}_+$. There is some $k_0 \in \mathbb{N}$ such that $d_{k_0} = Mn^{1/2}/2^{k_0-2} < \varepsilon$ by Theorem 1.1.1. Then, for all $k, m \in \mathbb{N}_{\geq k_0}$, $\|\mathbf{x}_k - \mathbf{x}_m\| \leq d_{k_0} < \varepsilon$. Therefore, since $\{\mathbf{x}_k\}$ is Cauchy, and therefore convergent by Theorem 2.1.7.

Clearly, $\mathbf{x}_0 \triangleq \lim_{k \rightarrow \infty} \mathbf{x}_k$ is a limit point of S since any deleted neighborhood $N'(\mathbf{x}_0)$ of \mathbf{x}_0 intersects infinitely many points with $\{\mathbf{x}_k\}_{k \in \mathbb{N}} \subseteq S$. □

Definition 2.1.13: Subsequence

Let $\{\mathbf{x}_k\}$ be any sequence in \mathbb{R}^n . Choose any strictly monotone increasing sequence $k_1 < k_2 < k_3 < \dots$ of natural numbers. For each $j \in \mathbb{N}$, let $\mathbf{y}_j := \mathbf{x}_{k_j}$. The sequence $\{\mathbf{y}_j\}_{j=1}^\infty$ is called an *subsequence* of $\{\mathbf{x}_k\}$.

Theorem 2.1.9

The point \mathbf{c} is a cluster point of $\{\mathbf{x}_k\}$ if and only if there exists a subsequence of $\{\mathbf{x}_k\}$ that converges to \mathbf{c} .

Proof. Analogous to Theorem 1.3.7. □

Theorem 2.1.10

Any bounded sequence $\{\mathbf{x}_k\}$ has a cluster point.

Proof. Analogous to Theorem 1.3.8. □

Corollary 2.1.1

If a sequence in \mathbb{R}^n has no cluster point, then the sequence is unbounded.

Corollary 2.1.2

Any bounded sequence in \mathbb{R}^n converges if and only if it has exactly one cluster point.

Corollary 2.1.3

A sequence $\{\mathbf{x}_k\}$ diverges if and only if at least one of the following conditions holds.

- $\{\mathbf{x}_k\}$ has two or more cluster points.
- $\{\mathbf{x}_k\}$ is unbounded.

2.2 Open and Closed Sets

Definition 2.2.1: Interior/Boundary Point and Open/Closed Set

Let S be any subset of \mathbb{R}^n and let \mathbf{x} be any point in \mathbb{R}^n .

- (i) \mathbf{x} is an *interior point* of S if $\exists r \in \mathbb{R}_+, N(\mathbf{x}; r) \subseteq S$.
- (ii) If every point of S is an interior point of S , then S is said to be *open*.
- (iii) We call \mathbf{x} is a *boundary point* of S if $\forall r \in \mathbb{R}_+, N(\mathbf{x}; r) \cap S \neq \emptyset \wedge N(\mathbf{x}; r) \setminus S \neq \emptyset$.
- (iv) If S contains all its boundary points, then S is said to be *closed*.

Definition 2.2.2

Let $S \subseteq \mathbb{R}^n$.

- (i) The *interior* of S , denoted \mathring{S} , is the set of all interior points of S .
- (ii) The *boundary* of S , denoted $\text{bd } S$, is the set of all boundary points of S .
- (iii) The *derived set* of S , denoted S' , is the set of all limit points of S .
- (iv) The *closure* of S , denoted \bar{S} , is the union of S and S' .
- (v) The *complement* of S , denoted S^c , is the set $\mathbb{R}^n \setminus S$.

Note:-

- For $S \subseteq \mathbb{R}^n$, $\mathring{S} \subseteq S \subseteq \bar{S}$.
- For $S \subseteq \mathbb{R}^n$, S is open if and only if $\mathring{S} = S$.
- For $S \subseteq \mathbb{R}^n$, \mathring{S} is open.

Theorem 2.2.1

The union of any collection of open sets in \mathbb{R}^n is open. The intersection of any finite collection of open sets in \mathbb{R}^n is also open.

Proof. To prove the first assertion, suppose that $\{U_\alpha \mid \alpha \in J\}$ is any collection of open sets in \mathbb{R}^n . Let $U \triangleq \bigcup_{\alpha \in J} U_\alpha$. Take any $\mathbf{x} \in U$. Then, there is some $\alpha_0 \in J$ such that $\mathbf{x} \in U_{\alpha_0}$. Since U_{α_0} is open, there is some neighborhood $N(\mathbf{x}; \varepsilon)$ such that $N(\mathbf{x}; \varepsilon) \subseteq U_{\alpha_0}$, which, in turn, $N(\mathbf{x}; \varepsilon) \subseteq U$. Therefore, \mathbf{x} is an interior point of U ; U is open.

To prove the second assertion, let U be the intersection of any finite collection $\{U_1, U_2, \dots, U_k\}$ of open sets and take any $\mathbf{x} \in U$. For each $j \in [k]$, since $\mathbf{x} \in U_j$, there is some $r_j \in \mathbb{R}_+$ such that $N(\mathbf{x}; r_j) \subseteq U_j$. Then, take $r_0 \triangleq \min_{j \in [k]} r_j \in \mathbb{R}_+$. Since, for all $j \in [k]$, $N(\mathbf{x}; r_0) \subseteq U_j$, it is implied that $N(\mathbf{x}; r_0) \subseteq U$. Therefore, \mathbf{x} is an interior point of U ; U is open. \square

Note:-

Intersection of infinitely many open sets may fail to be open. For instance, consider

$$U_k \triangleq N(\mathbf{0}; 1/k),$$

for each $k \in \mathbb{N}$. Then, $\bigcap_{k \in \mathbb{N}} U_k = \{\mathbf{0}\}$, which is not open.

Theorem 2.2.2

A set $C \subseteq \mathbb{R}^n$ is closed if and only if C^c is open.

Proof. (\Rightarrow) Take any $\mathbf{x} \in C^c$. Since C is closed and contains all of its boundary points, \mathbf{x} is not a boundary point of C . Therefore, there is some neighborhood $N(\mathbf{x})$ of \mathbf{x} such that $N(\mathbf{x}) \cap C = \emptyset$ or $N(\mathbf{x}) \cap C^c = \emptyset$. The second case is not possible since $\mathbf{x} \in N(\mathbf{x}) \cap C^c$. Therefore, $N(\mathbf{x}) \cap C = \emptyset$, which implies $N(\mathbf{x}) \subseteq C^c$; \mathbf{x} is an interior point of C^c . Therefore, C^c is open.

(\Leftarrow) Take any boundary point \mathbf{x} of C . Assume $\mathbf{x} \in C^c$ for the sake of contradiction. Since C^c is open, there is a neighborhood $N(\mathbf{x})$ of \mathbf{x} such that $N(\mathbf{x}) \subseteq C^c$. However, that implies $N(\mathbf{x}) \cap C = \emptyset$, which contradicts \mathbf{x} is a boundary point of C . Therefore, $\mathbf{x} \in C$; C contains all of its boundary points. \square

Theorem 2.2.3

The intersection of any collection of closed sets in \mathbb{R}^n is closed. The union of any finite collection of closed sets in \mathbb{R}^n is also closed.

Proof. To prove the first assertion, let $\{C_\alpha\}_{\alpha \in J}$ be any collection of closed sets in \mathbb{R}^n . Then, each C_α^c is open by Theorem 2.2.2, and thus $\bigcup_{\alpha \in J} C_\alpha^c$ is open by Theorem 2.2.1. Its complement $(\bigcup_{\alpha \in J} C_\alpha^c)^c$ is closed by Theorem 2.2.2. And note that $(\bigcup_{\alpha \in J} C_\alpha^c)^c = \bigcap_{\alpha \in J} C_\alpha$ by De Morgan's law.

To prove the second assertion, let $\{C_1, C_2, \dots, C_k\}$ be a finite collection of closed sets in \mathbb{R}^n . Then, each C_i^c is open by Theorem 2.2.2, and thus $\bigcap_{i=1}^k C_i^c$ is open by Theorem 2.2.1. Its complement $(\bigcap_{i=1}^k C_i^c)^c$ is closed by Theorem 2.2.2. And note that $(\bigcap_{i=1}^k C_i^c)^c = \bigcup_{i=1}^k C_i$ by De Morgan's law. \square

Theorem 2.2.4

$C \subseteq \mathbb{R}^n$ is closed if and only if $C' \subseteq C$.

Proof. (\Rightarrow) Let $\mathbf{x} \in C'$. Assume $\mathbf{x} \in C^c$ for the sake of contradiction. Since C^c is open by Theorem 2.2.2, there is a neighborhood $N(\mathbf{x})$ of \mathbf{x} such that $N(\mathbf{x}) \subseteq C^c$. Such $N(\mathbf{x})$ satisfies $N(\mathbf{x}) \cap C = \emptyset$, which contradicts $\mathbf{x} \in C'$. Therefore, $\mathbf{x} \in C$; C contains all its limit points.

(\Leftarrow) It is enough to prove C^c is open by Theorem 2.2.2. Take any $\mathbf{x} \in C^c$. \mathbf{x} is not a limit point of C by the hypothesis. Therefore, there is a deleted neighborhood $N'(\mathbf{x})$ of \mathbf{x} such that $N'(\mathbf{x}) \cap C = \emptyset$. Then, $N'(\mathbf{x}) \subseteq C^c$, and thus $N(\mathbf{x}) \subseteq C^c$, which implies \mathbf{x} is an interior point of C^c . Thus, C^c is open. \square

Corollary 2.2.1

$C \subseteq \mathbb{R}^n$ is closed if and only if $\overline{C} = C$.

Theorem 2.2.5

Let $S \subseteq \mathbb{R}^n$. The interior of S is the union of all open sets contained in S .

Proof. Let $\mathcal{U} \triangleq \{U \subseteq S \mid U \text{ is open in } \mathbb{R}^n\}$.

(\subseteq) Let $\mathbf{x} \in \mathring{S}$. Then, there is an open neighborhood $N(\mathbf{x})$ of \mathbf{x} such that $N(\mathbf{x}) \subseteq S$. Noting that $\mathbf{x} \in N(\mathbf{x}) \in \mathcal{U}$, we conclude $\mathring{S} \subseteq \bigcup \mathcal{U}$.

(\supseteq) Take any $\mathbf{x} \in \bigcup \mathcal{U}$. Then, there is an open set U in \mathbb{R}^n such that $\mathbf{x} \in U \subseteq S$. There is a neighborhood $N(\mathbf{x})$ of \mathbf{x} such that $N(\mathbf{x}) \subseteq U$. Therefore, $N(\mathbf{x}) \subseteq S$; \mathbf{x} is an interior point of S . Thus; $\mathring{S} \supseteq \bigcup \mathcal{U}$. \square

Theorem 2.2.6

The closure of S is the intersection of all closed sets that contain S .

Proof. Let $\mathcal{C} \triangleq \{C \subseteq \mathbb{R}^n \mid S \subseteq C \text{ and } C \text{ is closed}\}$.

(\subseteq) Since $S \subseteq \bigcap \mathcal{C}$ is obvious, we only need to show $S' \subseteq \bigcap \mathcal{C}$. Let $\mathbf{x} \in S'$. Then, it is direct that $\forall C \in \mathcal{C}, \mathbf{x} \in C'$ since each $C \in \mathcal{C}$ satisfies $S \subseteq C$. As C is closed and thus $\mathbf{x} \in C' \subseteq C$ by Theorem 2.2.4, Consequently, $\mathbf{x} \in \bigcap \mathcal{C}$; $\overline{S} \subseteq \bigcap \mathcal{C}$.

(\supseteq) It is enough to show that \overline{S} is closed, which, in turn, is sufficient to show that $(\overline{S})' \subseteq \overline{S}$ by Theorem 2.2.4. Let $\mathbf{y} \in (\overline{S})'$ and take any deleted neighborhood $N'(\mathbf{y}; \varepsilon)$ of \mathbf{y} . Then, there is some element \mathbf{z} in $N'(\mathbf{y}; \varepsilon) \cap \overline{S}$. Then, $\mathbf{z} \in S$ or $\mathbf{z} \in S'$.

If $\mathbf{z} \in S$, then $\mathbf{z} \in N'(\mathbf{y}; \varepsilon) \cap S \neq \emptyset$. If $\mathbf{z} \in S'$, take $\varepsilon' \triangleq \min\{\|\mathbf{z} - \mathbf{y}\|, \varepsilon - \|\mathbf{z} - \mathbf{y}\|\}$. Then, $N(\mathbf{z}; \varepsilon') \subseteq N'(\mathbf{y}; \varepsilon)$. Since $\mathbf{z} \in S'$, there is some \mathbf{x} in $N'(\mathbf{z}; \varepsilon') \cap S$. Thus, $\mathbf{x} \in N'(\mathbf{z}; \varepsilon') \cap S \subseteq N'(\mathbf{y}; \varepsilon) \cap S \neq \emptyset$.

In both cases, $N'(\mathbf{y}; \varepsilon) \cap S \neq \emptyset$. Thus, we proved that $\mathbf{y} \in S' \subseteq \overline{S}$; $(\overline{S})' \subseteq \overline{S}$. \square

Corollary 2.2.2

For any $S \subseteq \mathbb{R}^n$, the set \overline{S} is closed.

Corollary 2.2.3

For any $C \subseteq \mathbb{R}^n$, C is closed if and only if $\overline{C} = C$.

Theorem 2.2.7

Let $S \subseteq \mathbb{R}^n$.

- (i) $\overline{\mathring{S}} = \mathring{S}$
- (ii) $\overline{(\overline{S})} = \overline{S}$
- (iii) $\mathring{S} \cap \text{bd } S = \emptyset$

- (iv) $\mathring{S} \cup \text{bd } S = \bar{S}$
- (v) $\bar{S} \cap \bar{S}^c = \text{bd } S$

Proof.

- (i) \mathring{S} is open and an open set is the interior of itself.
- (ii) \bar{S} is closed and a closed set is the closure of itself. (See Corollary 2.2.2 and Corollary 2.2.3).
- (iii) Suppose there is some $\mathbf{x} \in \mathring{S} \cap \text{bd } S$. There is a neighborhood $N(\mathbf{x})$ of \mathbf{x} such that $N(\mathbf{x}) \subseteq S$. Then, $N(\mathbf{x}) \cap S^c = \emptyset$, which contradicts $\mathbf{x} \in \text{bd } S$.
- (iv) (\subseteq) Since it is already $\mathring{S} \subseteq S \subseteq \bar{S}$, we only need to show $\text{bd } S \subseteq \bar{S}$. Let $\mathbf{x} \in \text{bd } S$. If $\mathbf{x} \in S$, then it is done; so suppose $\mathbf{x} \in S^c$. Take any neighborhood $N(\mathbf{x}; \varepsilon)$ of \mathbf{x} . Then, $N(\mathbf{x}; \varepsilon) \cap S \neq \emptyset$. Noting that $N'(\mathbf{x}; \varepsilon) \cap S = N(\mathbf{x}; \varepsilon) \cap S \neq \emptyset$, $\mathbf{x} \in S'$.
 (\supseteq) Let $\mathbf{x} \in \bar{S}$. If $\mathbf{x} \in S$, then it is either “There is a neighborhood $N(\mathbf{x})$ of \mathbf{x} such that $N(\mathbf{x}) \subseteq S$.” or “Every neighborhood $N(\mathbf{x})$ of \mathbf{x} satisfies $N(\mathbf{x}) \cap S^c \neq \emptyset$.” The first case is $\mathbf{x} \in \mathring{S}$ and the latter case is $\mathbf{x} \in \text{bd } S$.
 Now the only left case is if $\mathbf{x} \in S' \setminus S$. Take any deleted neighborhood $N'(\mathbf{x})$ of \mathbf{x} . Then, $N(\mathbf{x}) \cap S = N'(\mathbf{x}) \cap S \neq \emptyset$. Also, $\mathbf{x} \in N(\mathbf{x}) \cap S^c$. Thus, $\mathbf{x} \in \text{bd } S$.
- (v) Using $\bar{S} = \mathring{S} \cup \text{bd } S$, we get

$$\begin{aligned}\bar{S} \cap \bar{S}^c &= (\mathring{S} \cup \text{bd } S) \cap ((\mathring{S}^c) \cup \text{bd } S^c) \\ &= (\mathring{S} \cap (\mathring{S}^c)) \cup (\mathring{S} \cap \text{bd } S^c) \cup (\text{bd } S \cap (\mathring{S}^c)) \cup (\text{bd } S \cap \text{bd } S^c)\end{aligned}$$

$\mathring{S} \cap (\mathring{S}^c) = \emptyset$ since $S \cap S^c = \emptyset$ and $\mathring{S} \subseteq S$ and $\mathring{S}^c \subseteq S^c$.

$\text{bd } S = \text{bd } S^c$ is direct from their definitions. Thus,

$$\begin{aligned}\mathring{S} \cap \text{bd } S^c &= \mathring{S} \cap \text{bd } S = \emptyset \\ \text{bd } S \cap (\mathring{S}^c) &= \text{bd } S^c \cap (\mathring{S}^c) = \emptyset\end{aligned}$$

by (iv). Therefore, $\bar{S} \cap \bar{S}^c = \text{bd } S \cap \text{bd } S^c = \text{bd } S$. □

Definition 2.2.3: Diameter

Let $\emptyset \neq S \subseteq \mathbb{R}^n$ be a bounded set. The *diameter* of S is defined to be

$$d(S) \triangleq \sup\{\|\mathbf{x} - \mathbf{y}\| \mid \mathbf{x}, \mathbf{y} \in S\}.$$

Definition 2.2.4: Distance

Let $\emptyset \neq S \subseteq \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$. The distance from \mathbf{x} to S is defined to be

$$d(\mathbf{x}, S) \triangleq \inf\{\|\mathbf{x} - \mathbf{y}\| \mid \mathbf{y} \in S\}.$$

Exercise 2.2.1

Let S be a nonempty set in \mathbb{R}^n and let \mathbf{x} be a point of \mathbb{R}^n .

- (i) $d(\mathbf{x}, S) = 0$ if and only if $\mathbf{x} \in \bar{S}$.
- (ii) S is closed if and only if $d(\mathbf{x}, S) > 0$ for every $\mathbf{x} \in S^c$.
- (iii) If S is closed, then there exists $\mathbf{y}_0 \in S$ such that $d(\mathbf{x}, S) = \|\mathbf{x} - \mathbf{y}_0\|$.
- (iv) If S is open and if $\mathbf{x} \in S^c$, then there exists no $\mathbf{y} \in S$ such that $d(\mathbf{x}, S) = \|\mathbf{x} - \mathbf{y}\|$.

Solution:

- (i) (\Rightarrow) We shall show that if such \mathbf{x} is not in S , then it is in S' . So, suppose $\mathbf{x} \notin S$. By Theorem 1.1.4, for any $\varepsilon \in \mathbb{R}_+$, there is some $\mathbf{y} \in S$ such that $0 \leq \|\mathbf{x} - \mathbf{y}\| < \varepsilon$. Since $\mathbf{x} \notin S$, $\mathbf{x} \neq \mathbf{y}$, and thus $\mathbf{y} \in N'(\mathbf{x}; \varepsilon) \cap S$, implying \mathbf{x} is a limit point of S .
 (\Leftarrow) Conversely, if $\mathbf{x} \in S' \setminus S$, then for all $\varepsilon \in \mathbb{R}_+$, there is some $\mathbf{z} \in S$ such that $0 < \|\mathbf{x} - \mathbf{z}\| < \varepsilon$. Therefore, $0 \leq d(\mathbf{x}, S) < \varepsilon$. Since ε is arbitrary, $d(\mathbf{x}, S) = 0$. \checkmark
- (ii) (\Rightarrow) $d(\mathbf{x}, S) = 0$ if and only if $\mathbf{x} \in \bar{S} = S$. Therefore, $d(\mathbf{x}, S) > 0$ if and only if $\mathbf{x} \in S^c$.
 (\Leftarrow) For every $\mathbf{x} \in S^c$, $\mathbf{x} \notin \bar{S}$ by (i). Thus, if $\mathbf{x} \in \bar{S}$, then $\mathbf{x} \in S$, or, $\bar{S} \subseteq S$. S is therefore closed. \checkmark
- (iii) If S is finite, then we can easily see $d(\mathbf{x}, S) = \min\{\|\mathbf{x} - \mathbf{y}\| \mid \mathbf{y} \in S\}$.
 Therefore, now suppose S is infinite. Let $\{\varepsilon_k\}_{k \in \mathbb{N}}$ be a sequence defined by $\varepsilon_k = 1/k$ for each $k \in \mathbb{N}$. By Theorem 1.1.4, for each $k \in \mathbb{N}$, we can find $\mathbf{y}_k \in S$ that satisfies

$$d(\mathbf{x}, S) \leq \|\mathbf{x} - \mathbf{y}_k\| < d(\mathbf{x}, S) + \varepsilon_k.$$

If the set $\{\mathbf{y}_k \mid k \in \mathbb{N}\}$ is finite, then there must be some \mathbf{y}_k such that $\|\mathbf{x} - \mathbf{y}_k\| = d(\mathbf{x}, S)$, and we are done.

Suppose $\{\mathbf{y}_k \mid k \in \mathbb{N}\}$ is infinite. Since the set is also bounded ($\|\mathbf{x} - \mathbf{y}_k\| < d(\mathbf{x}, S) + \varepsilon_1$ for each $k \in \mathbb{N}$), by Theorem 2.1.8, there is a convergent subsequence $\{\mathbf{y}_{k_j}\}_{j \in \mathbb{N}}$ of $\{\mathbf{y}_k\}_{k \in \mathbb{N}}$. Let $\mathbf{y}_0 \triangleq \lim_{j \rightarrow \infty} \mathbf{y}_{k_j}$. Since

$$d(\mathbf{x}, S) \leq \|\mathbf{x} - \mathbf{y}_{k_j}\| < d(\mathbf{x}, S) + \varepsilon_{k_j}$$

still holds, it follows that $\|\mathbf{x} - \mathbf{y}_0\| = d(\mathbf{x}, S)$ by Theorem 1.3.5.

$\mathbf{y}_0 \in S$ since S is closed and $\mathbf{y}_0 \in S$ is a limit point of S . \checkmark

- (iv) Suppose there is some $\mathbf{y} \in S$ such that $d(\mathbf{x}, S) = \|\mathbf{x} - \mathbf{y}\|$. $\|\mathbf{x} - \mathbf{y}\| > 0$ since $\mathbf{x} \neq \mathbf{y}$. Since S is open, there is some neighborhood $N(\mathbf{y}; r_0)$ of \mathbf{y} such that $N(\mathbf{y}; r_0) \subseteq S$. It must be $r_0 \leq \|\mathbf{x} - \mathbf{y}\|$. Let

$$\mathbf{z} \triangleq \mathbf{y} + \frac{r_0}{2} \cdot \frac{\mathbf{x} - \mathbf{y}}{\|\mathbf{x} - \mathbf{y}\|}$$

Then,

$$\|\mathbf{z} - \mathbf{y}\| = \frac{r_0}{2} < r_0,$$

thus $\mathbf{z} \in N(\mathbf{y}; r_0) \subseteq S$. However,

$$\|\mathbf{z} - \mathbf{y}\| = \left| 1 - \frac{r_0}{2\|\mathbf{x} - \mathbf{y}\|} \right| \|\mathbf{x} - \mathbf{y}\| < \|\mathbf{x} - \mathbf{y}\|,$$

contradicting the minimality of $\|\mathbf{x} - \mathbf{y}\|$, $\#$. \checkmark

□

Exercise 2.2.2

Let S be a nonempty set in \mathbb{R}^n . Then, $d(S) = d(\bar{S})$.

Solution: Since $S \subseteq \bar{S}$, $d(S) \leq d(\bar{S})$ is direct.

To prove $d(S) = d(\bar{S})$, take any $\varepsilon \in \mathbb{R}_+$. Let \mathbf{x} and \mathbf{y} be any point in \bar{S} . Then, each of $S \cap N(\mathbf{x}; \varepsilon/2)$ and $S \cap N(\mathbf{y}; \varepsilon/2)$ is nonempty. Thus, we take \mathbf{x}' and \mathbf{y}' from each set. Then,

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}\| &\leq \|\mathbf{x} - \mathbf{x}'\| + \|\mathbf{x}' - \mathbf{y}'\| + \|\mathbf{y}' - \mathbf{y}\| \\ &< \varepsilon/2 + d(S) + \varepsilon/2 = d(S) + \varepsilon \end{aligned}$$

Therefore, $d(S) + \varepsilon$ is an upper bound of $\{\|\mathbf{x} - \mathbf{y}\| \mid \mathbf{x}, \mathbf{y} \in \bar{S}\}$. Thus, $d(S) \leq d(\bar{S}) \leq d(S) + \varepsilon$. Since ε was arbitrary, we conclude $d(S) = d(\bar{S})$.

□

2.3 Completeness

Definition 2.3.1: Nested Sets

A sequence $\{S_k\}$ of sets in \mathbb{R}^n such that $S_k \supset S_{k+1}$ for each $k \in \mathbb{N}$ is said to be *nested*.

Theorem 2.3.1 Cantor's Nested Interval Theorem

For each $k \in \mathbb{N}$, let $I_k = [a_k, b_k]$ with $a_k < b_k$. Suppose that $\{I_k\}_{k \in \mathbb{N}}$ is a nested sequence in \mathbb{R} . Then

$$\bigcap_{k=1}^{\infty} I_k = [\alpha, \beta]$$

where $\alpha = \sup\{a_k \mid k \in \mathbb{N}\}$ and $\beta = \inf\{b_k \mid k \in \mathbb{N}\}$.

Proof. Let $A \triangleq \{a_k \mid k \in \mathbb{N}\}$ and $B \triangleq \{b_k \mid k \in \mathbb{N}\}$. Then, A is bounded above by any b_k and B is bounded below by any a_k . Thus, by Theorem 1.1.2, $\alpha = \sup A$ and $\beta = \sup B$ exist.

Any a_k is a lower bound of B , therefore $a_k \leq \beta$ for each $k \in \mathbb{N}$, which implies β is an upper bound of A . Hence $\alpha \leq \beta$.

To prove $\bigcap_{k=1}^{\infty} I_k \supseteq [\alpha, \beta]$, take any $x \in [\alpha, \beta]$. Then, for each $k \in \mathbb{N}$, $a_k \leq \alpha \leq x \leq \beta \leq b_k$, which means $x \in I_k$. Thus, $[\alpha, \beta] \subseteq \bigcap_{k=1}^{\infty} I_k$. Now, to prove the reverse containment, take any $x \in \bigcap_{k=1}^{\infty} I_k$. This means $\forall k \in \mathbb{N}$, $a_k \leq x \leq b_k$; x is an upper bound of A and is a lower bound of B at the same time. Therefore, $\alpha \leq x \leq \beta$, hence $\bigcap_{k=1}^{\infty} I_k \subseteq [\alpha, \beta]$. \square

Another Proof. Since the sequences $\{a_k\}$ and $\{b_k\}$ are bounded and monotone, there are limits $\alpha = \lim_{k \rightarrow \infty} a_k$ and $\beta = \lim_{k \rightarrow \infty} b_k$ by Theorem 1.3.4. By Theorem 1.3.6, $\alpha \leq \beta$.

Since $a_k \leq \alpha \leq \beta \leq b_k$ for each $k \in \mathbb{N}$, $[\alpha, \beta] \subseteq \bigcap_{k=1}^{\infty} I_k$.

Now, take any $x \in \bigcap_{k=1}^{\infty} I_k$. Then, for all $k \in \mathbb{N}$, $a_k \leq x \leq b_k$. If it were $\alpha > x$, there is some $k_0 \in \mathbb{N}$ such that $a_{k_0} > x$. It is similar for the case when $\beta < x$. Therefore, $\alpha \leq x \leq \beta$. We have proven that $\bigcap_{k=1}^{\infty} I_k \subseteq [\alpha, \beta]$. \square

Corollary 2.3.1

If, in the notation of the previous theorem, $\lim_{k \rightarrow \infty} (b_k - a_k) = 0$, then $\bigcap_{k=1}^{\infty} I_k$ is a singleton.

Proof. Take any $\varepsilon \in \mathbb{R}_+$. Then, there is some $k_0 \in \mathbb{N}$ such that $b_{k_0} - a_{k_0} < \varepsilon$.

$$0 \leq \beta - \alpha \leq b_{k_0} - a_{k_0} < \varepsilon$$

holds. This implies that $0 \leq \beta - \alpha < \varepsilon$ for arbitrary $\varepsilon \in \mathbb{R}_+$; therefore $\alpha = \beta$. \square

Theorem 2.3.2 Cantor's Criterion

If $\{C_k\}$ is a nested sequence of closed, bounded, nonempty subsets of \mathbb{R}^n , then

$$\bigcap_{k=1}^{\infty} C_k \neq \emptyset.$$

Furthermore, if $\lim_{k \rightarrow \infty} d(C_k) = 0$, then $\bigcap_{k=1}^{\infty} C_k$ is a singleton.

Proof. If any of C_k is finite, it is trivial. So, we suppose every C_k is infinite. Construct a sequence $\{\mathbf{x}_k\}$ of points in \mathbb{R}^n as following.

- Take any \mathbf{x}_1 in C_1 .
- For each $k \in \mathbb{N}$, take any \mathbf{x}_{k+1} in $C_{k+1} \setminus \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$.

Since $S \subseteq C_1$ is bounded and contains infinitely many points, by Theorem 2.1.8, there is a limit point \mathbf{x}_0 of S in \mathbb{R}^n . We now claim that $\mathbf{x}_0 \in \bigcap_{k=1}^{\infty} C_k$.

Fix any $k \in \mathbb{N}$ and choose any deleted neighborhood $N'(\mathbf{x}_0)$ of \mathbf{x}_0 . Since $N'(\mathbf{x}_0) \cap S$ is infinite, there is some $k_1 \in \mathbb{N}_{>k}$ such that $\mathbf{x}_{k_1} \in N'(\mathbf{x}_0)$. By the construction, $\mathbf{x}_{k_1} \in C_{k_1} \subseteq C_k$. This shows that every deleted neighborhood of \mathbf{x}_0 contains a point in C_k ; $\mathbf{x}_0 \in C'_k$. As each C_k is closed, $\mathbf{x}_0 \in C_k$, and thus $\mathbf{x}_0 \in \bigcap_{k=1}^{\infty} C_k$.

Suppose, in addition, $\lim_{k \rightarrow \infty} d(C_k) = 0$. Assume $\bigcap_{k=1}^{\infty} C_k$ has two distinct points \mathbf{x} and \mathbf{y} for the sake of contradiction. Choose any $\varepsilon \in (0, \|\mathbf{x} - \mathbf{y}\|)$ and then there is some $k \in \mathbb{N}$ with $d(C_k) < \varepsilon$. Nonetheless, $\varepsilon < \|\mathbf{x} - \mathbf{y}\| \leq d(C_k) < \varepsilon$, #. \square

Theorem 2.3.3 Cantor's Criterion in \mathbb{R}^n implies Cantor's Criterion in \mathbb{R}

Cantor's criterion in \mathbb{R}^n implies Cantor's criterion also holds in \mathbb{R} .

Proof. $\mathbb{R} \times \{0\} \times \dots \times \{0\}$ is a closed subset of \mathbb{R}^n . \square

Theorem 2.3.4

Cantor's criterion in \mathbb{R} and Archimedes' principle implies the existence of supremum of any bounded above nonempty subset of \mathbb{R} .

Proof. Let S be a nonempty, bounded above set in \mathbb{R} . Let B denote the set of upper bounds of S and let $A = B^c$. Since $x - 1 \in A$ for all $x \in S$, $A \neq \emptyset$.

We first show that for all $a \in A$ and $b \in B$, $a < b$. If otherwise, i.e., $a \geq b$, $x \leq b \leq a$ for each $x \in S$, which implies $a \in B$, which is a contradiction.

Moreover, $S \cap [a, b] \neq \emptyset$ for each $a \in A$ and $b \in B$. Assume $S \cap [a, b] = \emptyset$ for the sake of contradiction. Since $S \cap (b, \infty) = \emptyset$ as b is an upper bound of S , then it follows $S \subseteq (-\infty, a)$, which implies a is an upper bound of S , which is a contradiction.

Construct a nested sequence $\{[a_k, b_k]\}_{k \in \mathbb{N}}$ of closed interval of which each a_k is in A and b_k is in B .

- Take any a_1 in A and b_1 in B .
- For each $k \in \mathbb{N}$, if $(a_k + b_k)/2 \in A$, then let $a_{k+1} \triangleq (a_k + b_k)/2$ and $b_{k+1} \triangleq b_k$. If $(a_k + b_k)/2 \in B$, then let $a_{k+1} \triangleq a_k$ and $b_{k+1} \triangleq (a_k + b_k)/2$.

Then it is immediate that $\lim_{k \rightarrow \infty} (b_k - a_k) = \lim_{k \rightarrow \infty} 2^{-k+1}(b_1 - a_1) = 0$. Therefore, by Cantor's criterion in \mathbb{R} , $\bigcap_{k=1}^{\infty} [a_k, b_k] = \{x_0\}$ for some $x_0 \in \mathbb{R}$.

We now show that x_0 is an upper bound of S . Assume not for the sake of contradiction, that is, there is some $x \in S$ such that $x > x_0$. Then, we may find some $k \in \mathbb{N}$ such that $b_k - a_k < x - x_0$. Then it follows $b_k - x_0 \leq b_k - a_k < x - x_0$, and therefore $b_k < x_0$. This contradicts that b_k is an upper bound of S . Thus, $x_0 \in B$.

We now claim that x_0 is the least upper bound. Assume to the contrary that there is some $b \in B$ such that $b < x_0$. Then, we may find some $k \in \mathbb{N}$ such that $b_k - a_k < x_0 - b$. It follows $x_0 - a_k \leq b_k - a_k < x_0 - b$, and therefore $b < a_k$. This contradicts that $a_k \in A$. \square

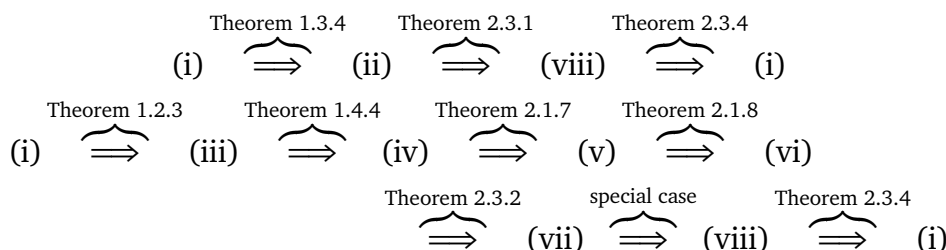
Theorem 2.3.5

Assuming that Archimedes' principle holds in \mathbb{R} , the following are equivalent.

- (i) Every nonempty supset in \mathbb{R} which is bounded above has a supremum in \mathbb{R} .
- (ii) Every bounded monotone sequence in \mathbb{R} converges.
- (iii) \mathbb{R} has Bolzano–Weierstrass property.
- (iv) \mathbb{R} is Cauchy complete.

- (v) \mathbb{R}^n is Cauchy complete.
- (vi) \mathbb{R}^n has Bolzano–Weierstrass property.
- (vii) Cantor’s criterion holds in \mathbb{R}^n .
- (viii) Cantor’s criterion holds in \mathbb{R} .

Proof.



□

Definition 2.3.2: Completeness

When the word *complete* is applied to \mathbb{R}^n , it is assumed that it means any of these statements:

- the existence of least upper bounds in \mathbb{R} ,
- the Monotone Convergence theorem in \mathbb{R}^n ,
- Cantor’s criterion,
- the Bolzano–Weierstrass property, or
- Cauchy completeness.

2.4 Relative Topology and Connectedness

Definition 2.4.1: Relatively Open and Relatively Closed Set

A set S is said to be *relatively open* in X if there exists an open set U in \mathbb{R}^n such that $S = U \cap X$. Likewise, a set S is said to be *relatively closed* if there exists a closed set C in \mathbb{R}^n such that $S = C \cap X$.

Note:-

Every relatively open set in X is open in \mathbb{R}^n if and only if X is open in \mathbb{R}^n . Every relatively closed set in X is closed in \mathbb{R}^n if and only if X is closed in \mathbb{R}^n .

Definition 2.4.2

- A *relative neighborhood* of \mathbf{x} in X is $N(\mathbf{x}; r) \cap X$. A *deleted neighborhood* is $N'(\mathbf{x}; r) \cap X$.
- A sequence $\{\mathbf{x}_k\}$ in X *converges in X* if $\lim_{k \rightarrow \infty} \mathbf{x}_k \in X$.
- The relative closure of S is $\bar{S} \cap X$.
- A point \mathbf{x}_0 in X is a *relative limit point* of S in X if \mathbf{x}_0 is a limit point of S .

Note:-

Depending on X , X may not be complete, i.e., a Cauchy sequence in X may not converge in X , an infinite and bounded subset of X may not have a limit point in X , or a nested sequence of nonempty, bounded, relatively closed subsets of X may have an empty intersection.

Lemma 2.4.1

Let X be a subset of \mathbb{R}^n and C be a subset in X . Then, C is relatively closed in X if and only if C is the relative closure of C .

Proof. The “if” part is trivial; we only prove the “only if” part.

(\subseteq) It is direct since $C \subseteq \overline{C}$.

(\supseteq) There exists a closed set \hat{C} in \mathbb{R}^n such that $C = \hat{C} \cap X$. Since $C \subseteq \hat{C}$ and \hat{C} is closed, $\overline{C} \subseteq \hat{C}$ by Theorem 2.2.6. Thus, $\overline{C} \cap X \subseteq \hat{C} \cap X = C$. \square

Lemma 2.4.2

Let $C \subseteq X \subseteq \mathbb{R}^n$. Then, C is relatively closed in X if and only if C contains all its relative limit points in X .

Proof. (\Rightarrow) If \mathbf{x}_0 is a relative limit point of C in X , then $\mathbf{x}_0 \in \overline{C} \cap X = C$. (Lemma 2.4.1)

(\Leftarrow) It means $\overline{C} \cap X \subseteq C$. And it is already $C \subseteq \overline{C} \cap X$. Therefore, C is relatively closed in X by Lemma 2.4.1. \square

Theorem 2.4.1

Let X be any nonempty subset of \mathbb{R}^n . The following statements are equivalent.

- (i) Every Cauchy sequence $\{\mathbf{x}_k\}$ in X converges to a point of X . Thus X inherits Cauchy completeness from \mathbb{R}^n .
- (ii) If S is a bounded, infinite subset of X , then S has a limit point in X . Thus X inherits the Bolzano–Weierstrass property from \mathbb{R}^n .
- (iii) If $\{C_k\}$ is any nested sequence of nonempty, bounded, relatively closed subsets of X , then $\bigcap_{k=1}^{\infty} C_k \neq \emptyset$. Furthermore, if $\lim_{k \rightarrow \infty} d(C_k) = 0$, then $\bigcap_{k=1}^{\infty} C_k$ is a singleton. Thus X inherits Cantor’s criterion from \mathbb{R}^n .

Proof. We shall first prove (i) \Rightarrow (ii). Let S be any bounded and infinite subset of S . Since S is a bounded and infinite subset of \mathbb{R}^n at the same time, by Theorem 2.1.8, there is a limit point \mathbf{x}_0 of S in \mathbb{R}^n . We may construct a Cauchy sequence in S that converges to \mathbf{x}_0 as we did in the proof of Theorem 1.3.7. Thus, \mathbf{x}_0 , which the Cauchy sequence converges to, is in S by Cauchy completeness of X . Therefore, S has a limit point in X .

We next prove (ii) \Rightarrow (iii). Suppose that X has the Bolzano–Weierstrass property. Take any nested sequence $\{C_k\}$ be bounded, nonempty, and *relatively closed* subsets of X . We assume each C_k is infinite since it is trivial otherwise.

As in the proof of Theorem 2.3.2, recursively choose an infinite set $S = \{\mathbf{x}_k \mid k \in \mathbb{N}\}$ of distinct points such that $\mathbf{x}_k \in C_k$ for each $k \in \mathbb{N}$. By the assumption (ii), there is a limit point $\mathbf{x}_0 \in X$ of S . Since $\mathbf{x}_k \in C_k \subseteq \overline{C_k}$ for each $k \in \mathbb{N}$ and $\{\overline{C_k}\}_{k \in \mathbb{N}}$ is a nested sequence of bounded, nonempty, and closed subsets of \mathbb{R}^n , as in the proof of Theorem 2.3.2, $\mathbf{x}_0 \in \overline{C_k}$ for each k , also. As $C_k = \overline{C_k} \cap X$ for each $k \in \mathbb{N}$ by Lemma 2.4.1, $\mathbf{x}_0 \in C_k$ for each $k \in \mathbb{N}$. Thus, $\mathbf{x}_0 \in \bigcap_{k=1}^{\infty} C_k$.

If, in addition, $\lim_{k \rightarrow \infty} d(C_k) = 0$, then $\lim_{k \rightarrow \infty} d(\overline{C_k}) = 0$ by Exercise 2.2.2. By Theorem 2.3.2, $\bigcap_{k=1}^{\infty} \overline{C_k} = \{\mathbf{x}_0\}$. But since $C_k \subseteq \overline{C_k}$ for each $k \in \mathbb{N}$, $\emptyset \neq \bigcap_{k=1}^{\infty} C_k \subseteq \bigcap_{k=1}^{\infty} \overline{C_k} = \{\mathbf{x}_0\}$. Thus, (ii) implies Cantor’s criterion holds in X .

Finally, it is left to prove (iii) \Rightarrow (i). Let $\{\mathbf{x}_k\}$ be a Cauchy sequence in X . We must show it converges to some point in X . By Cauchy completeness of \mathbb{R}^n , $\{\mathbf{x}_k\}$ converges to some point \mathbf{x}_0 in \mathbb{R}^n . We shall show that $\mathbf{x}_0 \in X$.

Let $\{\varepsilon_k\}_{k \in \mathbb{N}}$ be any sequence of positive numbers that converge monotonically to 0, e.g., $\varepsilon_k = 1/k$. Let $C_k \triangleq \overline{N(\mathbf{x}_0; \varepsilon_k)} \cap X$. Then $\{C_k\}$ is a nested sequence of bounded and relatively closed sets in X .

Take any $k_1 \in \mathbb{N}$. Since $\{\mathbf{x}_k\}$ converges to \mathbf{x}_0 , there is some $k_0 \in \mathbb{N}$ such that $\forall k \in \mathbb{N}_{\geq k_0}$, $\mathbf{x}_k \in N(\mathbf{x}_k; \varepsilon_{k_1})$. And such \mathbf{x}_k 's are also in $N(\mathbf{x}_k; \varepsilon_{k_1}) \cap X$; hence each C_k is nonempty.

Moreover, as $d(C_k) \leq d(N(\mathbf{x}_0; \varepsilon_k)) = 2\varepsilon_k$, $\lim_{k \rightarrow \infty} d(C_k) = 0$. Thus, by assumption (ii), $\bigcap_{k=1}^{\infty} C_k$ is a singleton in X . Then,

$$\bigcap_{k=1}^{\infty} C_k = \bigcap_{k=1}^{\infty} \left(\overline{N(\mathbf{x}_0; \varepsilon_k)} \cap X \right) \subseteq \bigcap_{k=1}^{\infty} \overline{N(\mathbf{x}_0; \varepsilon_k)} = \{\mathbf{x}_0\}.$$

Therefore, $\mathbf{x}_0 \in \bigcap_{k=1}^{\infty} C_k \subseteq X$. □

Definition 2.4.3: Completeness of a Subset of \mathbb{R}^n

Let X be a nonempty subset of \mathbb{R}^n . If any of the equivalent properties of Theorem 2.4.1 hold in the set X , then X is said to be *complete*.

Theorem 2.4.2

A nonempty subset X of \mathbb{R}^n is complete if and only if X is closed in \mathbb{R}^n .

Proof. (\Rightarrow) Let \mathbf{x}_0 be a limit point of X . Then, there is a Cauchy sequence $\{\mathbf{x}_k\}$ that converges to \mathbf{x}_0 . By Cauchy completeness of X , $\mathbf{x}_0 \in X$. Since we have proven that $X' \subseteq X$, X is closed by Theorem 2.2.4.

(\Leftarrow) Let S be any bounded, infinite set in X . By Bolzano–Weierstrass property of \mathbb{R}^n , S has a limit point \mathbf{x}_0 in \mathbb{R}^n . As X being closed, $\mathbf{x}_0 \in X$. Therefore, X has Bolzano–Weierstrass property; X is complete by Theorem 2.4.1. □

Definition 2.4.4: Connectedness

A set S is *disconnected* if there are two open sets U, V such that

- (i) $U \cap V = \emptyset$,
- (ii) $S \subseteq U \cup V$, and
- (iii) $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$.

S is *connected* if S is not disconnected.

Example 2.4.1

- $[0, 1) \cup (1, 2]$ is disconnected. ($U = (-1, 1)$ and $V = (1, 3)$)
- \mathbb{Q} is disconnected. ($U = (-\infty, r)$ and $V = (r, \infty)$ where $r \in \mathbb{R} \setminus \mathbb{Q}$)

Theorem 2.4.3

Any interval of the form of $[a, b]$, $(a, b]$, $[a, b)$, or (a, b) is connected. \mathbb{R} itself is connected.

Proof. Let I be any of these sets. The only fact needed for this proof is that $(u, v) \subseteq I$ for any $u < v$ in I .

Suppose I is disconnected. Then, there are disjoint open sets $U, V \subseteq \mathbb{R}$ such that $I \subseteq U \cup V$, $I \cap U \neq \emptyset$, and $I \cap V \neq \emptyset$. Take any $u \in I \cap U$ and $v \in I \cap V$. WLOG, $u < v$.

Construct sequences $\{u_k\}$ in $I \cap U$ and $\{v_k\}$ in $I \cap V$ as following.

- $u_1 \triangleq u$ and $v_1 \triangleq v$.
- For each $k \in \mathbb{N}$, $u_{k+1} \triangleq \begin{cases} \frac{u_k + v_k}{2} & \frac{u_k + v_k}{2} \in U \\ u_k & \text{otherwise} \end{cases}$ and $v_{k+1} \triangleq \begin{cases} \frac{u_k + v_k}{2} & \frac{u_k + v_k}{2} \in V \\ v_k & \text{otherwise} \end{cases}$.

Then, $\{u_k\}$ and $\{v_k\}$ are bounded, monotone sequences ($u_k < v$ and $v_k > u$ for each $k \in \mathbb{N}$); hence they converge by Theorem 1.3.4.

Since $(u_k + v_k)/2 \in I \subseteq U \cup V$ for each $k \in \mathbb{N}$ and $U \cap V = \emptyset$, $v_{k+1} - u_{k+1} = (v_k - u_k)/2 = (v - u)/2^k$. They converge to the same point x_0 since $\lim_{k \rightarrow \infty} (v_k - u_k) = 0$.

If x_0 is ever equal to u_k , then by openness of U , there is some neighborhood $N(x_0; \varepsilon)$ of x_0 such that $N(x_0; \varepsilon) \subseteq U$, which contradicts $\lim_{k \rightarrow \infty} v_k = x_0$. Thus, x_0 is never equal to u_k ; by the same reason, x_0 is never equal to v_k . We conclude x_0 is a relative limit point in I of both $I \cap U$ and $I \cap V$.

Since $I \cap U = I \cap V^c$ and V^c is closed, $I \cap U$ is closed and thus any relative limit point of $I \cap U$ in I , x_0 , in particular, is in $I \cap U$ by Lemma 2.4.2. Similarly, $x_0 \in I \cap V$, which contradicts $U \cap V = \emptyset$. \square

2.5 Compactness

Definition 2.5.1: Open Cover

Let S be any nonempty subset of \mathbb{R}^n . An *open cover* of S is any collection $\mathcal{C} = \{U_\alpha \mid \alpha \in A\}$ of open sets such that $S \subseteq \bigcup_{\alpha \in A} U_\alpha$.

Definition 2.5.2: Compactness

A set S in \mathbb{R}^n is said to be *compact* if, for every open cover \mathcal{C} of S , there exists a finite subcover, that is, there is some finite subset A_0 of A such that $S \subseteq \bigcup_{\alpha \in A_0} U_\alpha$.

Theorem 2.5.1 Heine–Borel

Let S be any closed and bounded set in \mathbb{R}^n . Then any *countable* open cover of S has a finite subcover.

Proof. Let $\{I_k \mid k \in \mathbb{N}\}$ be any open cover of S . Let $U_k \triangleq \bigcup_{j=1}^k I_j$, $C_k \triangleq U_k^c$, and $D_k \triangleq C_k \cap S$ for each $k \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$, $U_k \subseteq U_{k+1}$, $C_k \supseteq C_{k+1}$, and $D_k \supseteq D_{k+1}$. Note that each U_k is open and thus C_k and D_k are closed.

Assume that every D_k is nonempty. Then, since $\{D_k\}$ is a nested sequence of bounded, nonempty, closed sets, by Cantor's criterion, there is some $x \in \bigcap_{k=1}^{\infty} D_k$. Then,

$$x \in \bigcap_{k=1}^{\infty} (S \cap C_k) = S \cap \left(\bigcap_{k=1}^{\infty} C_k \right) = S \cap \left(\bigcup_{k=1}^{\infty} U_k \right)^c = S \cap \left(\bigcup_{k=1}^{\infty} I_k \right)^c$$

Thus, x is a point in S that is not covered by $\{I_k \mid k \in \mathbb{N}\}$, which is a contradiction.

Hence, some D_{k_0} is empty, and it means that $S \subseteq \bigcup_{j=1}^{k_0} I_j$. \square

Theorem 2.5.2 Lindelöf

Let S be any subset of \mathbb{R}^n . Then, every open cover of S has a countable subcover.

Proof. Since the set $\{N(\mathbf{y}; r) \mid \mathbf{y} \in \mathbb{Q}^n, r \in \mathbb{Q}\}$ has the trivial one-to-one correspondence with \mathbb{Q}^{n+1} , the set is countable. Thus, we may index them as $\{N_k \mid k \in \mathbb{N}\}$.

Let $\mathcal{C} = \{U_\alpha \mid \alpha \in A\}$ be any open cover of S . Take any $\mathbf{x} \in S$. Then there is some U_α such that $\mathbf{x} \in U_\alpha$. Since U_α is open, there is a neighborhood $N(\mathbf{x}; \varepsilon)$ such that $N(\mathbf{x}; \varepsilon) \subseteq U_\alpha$.

As \mathbb{Q}^n is dense in \mathbb{R}^n , there is some $\mathbf{y} \in N(\mathbf{x}; \varepsilon/2) \cap \mathbb{Q}^n$. As \mathbb{Q} is dense in \mathbb{R} , there is some $r \in (\|\mathbf{x} - \mathbf{y}\|, \varepsilon - \|\mathbf{x} - \mathbf{y}\|) \cap \mathbb{Q}$. Then, $\mathbf{x} \in N(\mathbf{y}; r) \subseteq N(\mathbf{x}; \varepsilon) \subseteq U_\alpha$. The neighborhood $N(\mathbf{y}; r)$ is in the set $\{N_k \mid k \in \mathbb{N}\}$.

Let us define $K(\mathbf{x}) \triangleq \{k \in \mathbb{N} \mid \mathbf{x} \in N_k \subseteq U_\alpha \text{ for some } U_\alpha \in \mathcal{C}\}$. Then, by Well-Ordering Property of \mathbb{N} , there is $k(\mathbf{x}) \triangleq \min K(\mathbf{x})$.

The collection $\{N_{k(\mathbf{x})} \mid \mathbf{x} \in S\}$ is an open cover of S and is countable since it is a subset of $\{N_k \mid k \in \mathbb{N}\}$.

By construction, for each $N_{k(\mathbf{x})}$, there is $U_\alpha \in \mathcal{C}$ such that $N_{k(\mathbf{x})} \subseteq U_\alpha$. Hence, if we denote one of such sets as $U_{k(\mathbf{x})}$, the collection $\{U_{k(\mathbf{x})} \mid \mathbf{x} \in S\}$ is a countable subcover of S . \square

Corollary 2.5.1

If $S \subseteq \mathbb{R}^n$ is closed and bounded, then S is compact.

Proof. Simply combine Theorem 2.5.2 and Theorem 2.5.1. \square

Theorem 2.5.3

If S is a compact subset of \mathbb{R}^n , then S is closed and bounded.

Proof. Suppose $S \subseteq \mathbb{R}^n$ is not closed. Then S has a limit point \mathbf{x}_0 that is not in S . Let $\{\varepsilon_k\}$ be any sequence of positive numbers that monotonically converges to 0. Define $U_k \triangleq \overline{N(\mathbf{x}_0; \varepsilon_k)}^c$.

Then $\mathcal{C} \triangleq \{U_k \mid k \in \mathbb{N}\}$ is an open cover of S . To prove this, take any $\mathbf{x} \in S$. There exists $k_0 \in \mathbb{N}$ such that $\varepsilon_k < \|\mathbf{x} - \mathbf{x}_0\|$ for all $k \in \mathbb{N}_{\geq k_0}$. Thus, for $k \geq k_0$, $\mathbf{x}_0 \notin \overline{N(\mathbf{x}_0; \varepsilon_k)}$, i.e., $\mathbf{x}_0 \in U_k$.

We now claim there is no finite subcover. Take finitely many subsets $\{U_{k_1}, U_{k_2}, \dots, U_{k_p}\}$ from the open cover. Let $k_0 = \max_{i \in [p]} k_i$. Note that $\bigcup_{i=1}^p U_{k_i} = U_{k_0}$. Take ε from $(0, \varepsilon_{k_0})$. Then, since \mathbf{x}_0 is a limit point of S , we may take $\mathbf{x} \in S \cap N'(\mathbf{x}_0; \varepsilon)$. Then, $\mathbf{x} \notin U_{k_0}$. Therefore, $\{U_{k_i}\}_{i \in [p]}$ does not cover S . Thus, we proved that a set which is not closed is not compact.

Now, suppose S is not bounded. Let $U_k = N(\mathbf{0}; k)$. Then, $S \subseteq \mathbb{R}^n = \bigcup_{k=1}^{\infty} U_k$; $\{U_k\}$ is an open cover of S .

Take finitely many subsets $\{U_{k_1}, U_{k_2}, \dots, U_{k_p}\}$ from the open cover. Let $k_0 = \max_{i \in [p]} k_i$. Note that $\bigcup_{i=1}^p U_{k_i} = U_{k_0}$. Because S is unbounded, there is some $\mathbf{x} \in S$ such that $\|\mathbf{x}\| > k_0$; $\mathbf{x} \notin U_{k_0} = \bigcup_{i=1}^p U_{k_i}$. Thus, we proved that an unbounded set is not compact. \square

Note:-

A set $S \subseteq \mathbb{R}^n$ is complete if and only if S is closed. A set S is compact if and only if S is closed and bounded. Thus, compactness implies completeness, but not vice versa.

Theorem 2.5.4

Let S be a nonempty subset of \mathbb{R}^n . Then the following statements are equivalent.

- (i) S is closed and bounded.
- (ii) S is compact.
- (iii) Every infinite subset of S has a limit point which is in S .

Proof. We already know (i) is equivalent to (ii). Therefore, we only need to prove (i) and (iii) are equivalent. (i) \implies (iii) is direct since S is complete by Theorem 2.4.2. So we are left to prove (iii) implies (i).

Suppose S is not bounded. Then we may construct a sequence $\{\mathbf{x}_k\}$ in S as following.

- Take any $\mathbf{x}_1 \in S$.
- For each $k \in \mathbb{N}$, take $\mathbf{x}_{k+1} \in S \setminus N(\mathbf{0}; \|\mathbf{x}_k\| + 1)$.

The construction is valid since S is unbounded. Then, the set $\{\mathbf{x}_k \mid k \in \mathbb{N}\}$ is infinite but it does not have a limit point since $\|\mathbf{x}_k - \mathbf{x}_m\| \geq 1$ for each $\{k, m\} \subseteq \mathbb{N}$. Thus, (iii) implies S is bounded.

Now, we shall prove (iii) implies S is closed. Let \mathbf{x} be a limit point of S . Let $\{\varepsilon_k\}$ be a monotonic sequence of positive numbers that converges to 0. Construct a sequence $\{\mathbf{x}_k\}$ as following.

- For each $k \in \mathbb{N}$, take $\mathbf{x}_k \in S \cap N'(\mathbf{x}; \varepsilon_k)$.

The construction is valid since \mathbf{x} is a limit point of S . Then, the set $\{\mathbf{x}_k \mid k \in \mathbb{N}\}$ has the only limit point \mathbf{x} . Then, by (iii), $\mathbf{x} \in S$. We proved that S contains all its limit points, thus S is closed by Theorem 2.2.4. \square

Chapter 3

Continuity

3.1 Limit and Continuity

Definition 3.1.1: Limit of a Function

Let $f : S \rightarrow \mathbb{R}$ where $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in \bar{S}$. We say f has *limit* L as \mathbf{x} approaches \mathbf{c} provided that, for every neighborhood $N(L)$, there exists a deleted neighborhood $N'(\mathbf{c})$ such that

$$S \cap N'(\mathbf{c}) \subseteq f^{-1}(N(L)).$$

We write $\lim_{\mathbf{x} \rightarrow \mathbf{c}} f(\mathbf{x}) = L$.

Note:-

Limit is unique if it exists.

Note:-

Note that $S \cap N'(\mathbf{c}; \delta) = \emptyset$ for sufficiently small δ if \mathbf{c} is an isolated point of S . This implies any real number can be a limit of f as \mathbf{x} approaches \mathbf{c} . Somehow, Douglass defined that $\lim_{\mathbf{x} \rightarrow \mathbf{c}} f(\mathbf{x}) = f(\mathbf{c})$ (since $\mathbf{c} \in S$ in this case). *Actually I do not think we should define limit for isolated points.*

Note:-

This definition of limit is equivalent to the normal ε - δ definition of limit, except that it defines a limit for isolated points.

Definition 3.1.2: Continuity

Let $f : S \rightarrow \mathbb{R}$ where $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in S$. We say f is *continuous at* \mathbf{c} if

$$\lim_{\mathbf{x} \rightarrow \mathbf{c}} f(\mathbf{x}) = f(\mathbf{c}).$$

In other words, for every neighborhood $N(f(\mathbf{c}))$, there exists a neighborhood $N(\mathbf{c})$ such that

$$S \cap N(\mathbf{c}) \subseteq f^{-1}(N(f(\mathbf{c}))).$$

If f is continuous at every $\mathbf{c} \in S$, then f is said to be *continuous*.

Theorem 3.1.1

Let $f : S \rightarrow \mathbb{R}$ where $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in \bar{S}$ where $\lim_{\mathbf{x} \rightarrow \mathbf{c}} f(\mathbf{c}) = L$ exists and. Then, f is locally bounded on some deleted neighborhood of \mathbf{c} , that is, there are $M, \delta \in \mathbb{R}_+$ such that

$$\mathbf{x} \in S \cap N'(\mathbf{c}; \delta) \implies |f(\mathbf{x})| \leq M.$$

Proof. There exists $\delta \in \mathbb{R}_+$ such that $S \cap N'(\mathbf{c}; \delta) \subseteq f^{-1}(N(L; 1))$. Then, $|f(\mathbf{x})| \leq |L| + 1$ if $\mathbf{x} \in S \cap N'(\mathbf{x}; \delta)$. \square

Theorem 3.1.2

Let $f : S \rightarrow \mathbb{R}$ where $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in \bar{S}$ where $\lim_{\mathbf{x} \rightarrow \mathbf{c}} f(\mathbf{c}) = L$ exists and $L \neq 0$. Then, f is locally bounded away from 0 on some deleted neighborhood of \mathbf{c} , that is, there are $m, \delta \in \mathbb{R}_+$ such that

$$\mathbf{x} \in S \cap N'(\mathbf{c}; \delta) \implies |f(\mathbf{x})| \geq m.$$

Proof. There exists $\delta \in \mathbb{R}_+$ such that $S \cap N'(\mathbf{c}; \delta) \subseteq f^{-1}(N(L; |L|/2))$. Then, $|f(\mathbf{x})| \geq |L|/2$ if $\mathbf{x} \in S \cap N'(\mathbf{x}; \delta)$. \square

Theorem 3.1.3

Let $f_1 : S \rightarrow \mathbb{R}$ and $f_2 : S \rightarrow \mathbb{R}$ where $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in \bar{S}$, and suppose $\lim_{\mathbf{x} \rightarrow \mathbf{c}} f_1(\mathbf{x}) = L_1$ and $\lim_{\mathbf{x} \rightarrow \mathbf{c}} f_2(\mathbf{x}) = L_2$. Then

- (i) $\lim_{\mathbf{x} \rightarrow \mathbf{c}} (f_1(\mathbf{x}) + f_2(\mathbf{x})) = L_1 + L_2$.
- (ii) For any $a \in \mathbb{R}$, $\lim_{\mathbf{x} \rightarrow \mathbf{c}} af(\mathbf{x}) = aL_1$.
- (iii) $\lim_{\mathbf{x} \rightarrow \mathbf{c}} f_1(\mathbf{x})f_2(\mathbf{x}) = L_1L_2$.
- (iv) $\lim_{\mathbf{x} \rightarrow \mathbf{c}} f_1(\mathbf{x})/f_2(\mathbf{x}) = L_1/L_2$ provided that $L_2 \neq 0$.

Proof. Proved in MAS102 (Calculus II). \square

Theorem 3.1.4 The Squeeze Play

Let f, g , and h be three real-valued functions sharing a common domain $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in \bar{C}$ where $\lim_{\mathbf{x} \rightarrow \mathbf{c}} f(\mathbf{x}) = \lim_{\mathbf{x} \rightarrow \mathbf{c}} h(\mathbf{x}) = L$ exist. Suppose also that, for some $\delta_0 \in \mathbb{R}_+$,

$$\mathbf{x} \in S \cap N'(\mathbf{c}; \delta_0) \implies f(\mathbf{x}) \leq g(\mathbf{x}) \leq h(\mathbf{x})$$

Then, $\lim_{\mathbf{x} \rightarrow \mathbf{c}} g(\mathbf{x}) = L$.

Proof. Proved in MAS102 (Calculus II). \square

Theorem 3.1.5 Limit is Order Preserving

Let f and g be two real-valued functions sharing a common domain $S \subseteq \mathbb{R}^n$. Let $\mathbf{c} \in \bar{C}$ where $\lim_{\mathbf{x} \rightarrow \mathbf{c}} f(\mathbf{x}) = L_1$ and $\lim_{\mathbf{x} \rightarrow \mathbf{c}} g(\mathbf{x}) = L_2$ exist. Suppose also that, for some $\delta_0 \in \mathbb{R}_+$,

$$\mathbf{x} \in S \cap N'(\mathbf{c}; \delta_0) \implies f(\mathbf{x}) \leq g(\mathbf{x})$$

Then, $L_1 \leq L_2$.

Proof. Proved in MAS102 (Calculus II). \square

Theorem 3.1.6

Let S be a nonempty subset of \mathbb{R}^n , $\mathbf{c} \in S'$, and $f : S \rightarrow \mathbb{R}$. $\lim_{\mathbf{x} \rightarrow \mathbf{c}} f(\mathbf{x}) = L$ if and only if, for every Cauchy sequence $\{\mathbf{x}_k\}$ in $S \setminus \{\mathbf{c}\}$ such that $\lim_{k \rightarrow \infty} \mathbf{x}_k = \mathbf{c}$, it follows that $\lim_{k \rightarrow \infty} f(\mathbf{x}_k) = L$.

Proof. (\Rightarrow) Let $\{\mathbf{x}_k\}$ be any of such Cauchy sequences. Take any $\varepsilon \in \mathbb{R}_+$. By continuity, there exists $\delta \in \mathbb{R}_+$ such that $S \cap N'(\mathbf{c}; \delta) \subseteq f^{-1}(N(L; \varepsilon))$. On the other hand, by convergence, there exists $k_0 \in \mathbb{N}$ such that $\forall k \in \mathbb{N}, (k \geq k_0 \implies \mathbf{x}_k \in N(\mathbf{c}; \delta))$. Since $\mathbf{x}_k \neq \mathbf{c}$ for each $k \in \mathbb{N}$, we may say

$$\forall k \in \mathbb{N}, (k \geq k_0 \implies \mathbf{x}_k \in N'(\mathbf{c}; \delta) \implies \mathbf{x}_k \in f^{-1}(N(L; \varepsilon)) \implies f(\mathbf{x}_k) \in N(L; \varepsilon)).$$

Thus, $\lim_{k \rightarrow \infty} f(\mathbf{x}_k) = L$ holds.

(\Leftarrow) Suppose it is not $\lim_{\mathbf{x} \rightarrow \mathbf{c}} f(\mathbf{x}) = L$. Then, it is equivalent to say that, there is some neighborhood $N(L; \varepsilon_0)$ such that $S \cap N'(\mathbf{c}; \delta) \not\subseteq f^{-1}(N(L; \varepsilon_0))$ for every deleted neighborhood $N'(\mathbf{x}; \delta)$. Construct a sequence $\{\mathbf{x}_k\}$ in $S \setminus \{\mathbf{c}\}$ as following.

- $\mathbf{x}_1 \in S \setminus \{\mathbf{c}\} \setminus f^{-1}(N(L; \varepsilon_0))$.
- For each $k \in \mathbb{N}$, $\mathbf{x}_{k+1} \in S \cap N'(\mathbf{x}_k; |\mathbf{x}_k - \mathbf{c}|/2) \setminus f^{-1}(N(L; \varepsilon_0))$.

Then, $\lim_{k \rightarrow \infty} \mathbf{x}_k = \mathbf{c}$ indeed holds, but it is not $\lim_{k \rightarrow \infty} f(\mathbf{x}_k) = L$ since $f(\mathbf{x}_k) \notin N(L; \varepsilon_0)$ for each $k \in \mathbb{N}$. \square

Theorem 3.1.7

Let S be a nonempty subset of \mathbb{R}^n , $\mathbf{c} \in S$, and $f : S \rightarrow \mathbb{R}$. f is continuous at \mathbf{c} if and only if, for every Cauchy sequence $\{\mathbf{x}_k\}$ in S such that $\lim_{k \rightarrow \infty} \mathbf{x}_k = \mathbf{c}$, it follows that $\lim_{k \rightarrow \infty} f(\mathbf{x}_k) = f(\mathbf{c})$.

Proof. (\Rightarrow) If at most finitely many \mathbf{x}_k are distinct from \mathbf{c} , then $\exists k_0 \in \mathbb{N}, \forall k \in \mathbb{N}_{\geq k_0}, \mathbf{x}_k = \mathbf{c}$; $\lim_{k \rightarrow \infty} f(\mathbf{x}_k) = f(\mathbf{c})$ is evident.

If there are infinitely many \mathbf{x}_k are distinct from \mathbf{c} , then we may extract a subsequence $\{\mathbf{x}_{k_j}\}_{j \in \mathbb{N}}$ such that each \mathbf{x}_{k_j} is in $S \setminus \{\mathbf{c}\}$. By Theorem 3.1.6, $\lim_{j \rightarrow \infty} f(\mathbf{x}_{k_j}) = f(\mathbf{c})$. This implies $\lim_{k \rightarrow \infty} f(\mathbf{x}_k) = f(\mathbf{c})$, regardless of the number of \mathbf{x}_k 's equal to \mathbf{c} .

(\Leftarrow) If $\mathbf{c} \in S'$, then we may directly apply Theorem 3.1.6 since every Cauchy sequence in $S \setminus \{\mathbf{c}\}$ is a Cauchy sequence in S .

If $\mathbf{c} \notin S'$, then \mathbf{c} is an isolated point. Then, $\lim_{\mathbf{x} \rightarrow \mathbf{c}} f(\mathbf{x}) = f(\mathbf{c})$ by definition. \square

Theorem 3.1.8

Let $S \subseteq \mathbb{R}^n$ and $f : S \rightarrow \mathbb{R}$. Let $\mathbf{c} = (c_1, \dots, c_n) \in \mathring{S}$. For $j = 1, 2, \dots, n$, let

$$g_j(t) = f(c_1, c_2, \dots, c_{j-1}, t, c_{j+1}, \dots, c_n).$$

- If $\lim_{\mathbf{x} \rightarrow \mathbf{c}} f(\mathbf{x}) = L$, then, for each $j \in [n]$, $\lim_{t \rightarrow c_j} g_j(t) = L$.
- If f is continuous at \mathbf{c} , then, for each $j \in [n]$, g_j is continuous at c_j and $\lim_{t \rightarrow c_j} g_j(t) = f(\mathbf{c})$.

Proof.

- Take any $j \in [n]$ and $\varepsilon \in \mathbb{R}_+$. By convergence, there exists $\delta_1 \in \mathbb{R}_+$ such that $S \cap N'(\mathbf{c}; \delta_1) \subseteq f^{-1}(N(L; \varepsilon))$. Since $\mathbf{x} \in \mathring{S}$, there exists $\delta_2 \in \mathbb{R}_+$ such that $N(\mathbf{c}; \delta_2) \subseteq S$. Let $\delta \triangleq \min\{\delta_1, \delta_2\}$. Then, $N'(\mathbf{c}; \delta) \subseteq f^{-1}(N(L; \varepsilon))$ and $N(\mathbf{c}; \delta) \subseteq S$ hold. Hence, for any $t \in N'(c_j; \delta)$,

$$g_j(t) = f(c_1, \dots, c_{j-1}, t, c_{j+1}, \dots, c_n) \in N(L; \varepsilon)$$

- as $\|(c_1, \dots, c_{j-1}, t, c_{j+1}, \dots, c_n) - \mathbf{c}\| = |t - c_j| < \delta$.
- (ii) Since $\lim_{\mathbf{x} \rightarrow \mathbf{c}} f(\mathbf{x}) = f(\mathbf{c})$, by (a), for each $j \in [n]$, $\lim_{t \rightarrow c_j} g(t) = f(\mathbf{c}) = g(c_j)$. □

Note:-

The converse of Theorem 3.1.8 is not true.

3.2 The Topological Description of Continuity

Theorem 3.2.1

A surjective function $f : S \rightarrow T$ where $S \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}$ is continuous if and only if $f^{-1}(U)$ is relatively open in S for every relatively open set U in T .

Proof. (\Rightarrow) Let U be a relatively open set in T and $\mathbf{c} \in f^{-1}(U)$. Since U is open and $f(\mathbf{c}) \in U$, there is a neighborhood $N(f(\mathbf{c}))$ such that $T \cap N(f(\mathbf{c})) \subseteq U$. By continuity, there is a neighborhood $N(\mathbf{c})$ such that $S \cap N(\mathbf{c}) \subseteq f^{-1}(N(f(\mathbf{c}))) \subseteq f^{-1}(U)$. Therefore, \mathbf{c} is a relative interior point of $f^{-1}(U)$. Since \mathbf{c} was arbitrary, $f^{-1}(U)$ is relatively open in S .

(\Leftarrow) Take any $\mathbf{c} \in S$ and a neighborhood $N(f(\mathbf{c}))$. Then, $f^{-1}(T \cap N(f(\mathbf{c})))$ is relatively open in S . Since $\mathbf{c} \in f^{-1}(T \cap N(f(\mathbf{c})))$, there is a neighborhood $N(\mathbf{c})$ such that $S \cap N(\mathbf{c}) \subseteq f^{-1}(N(f(\mathbf{c})))$. □

Theorem 3.2.2

If S is a connected subset of \mathbb{R}^n and f is continuous on S , then $T = f(S)$ is also connected.

Proof. Suppose T is disconnected for the sake of contradiction. There exists two disjoint open sets $U, V \subseteq \mathbb{R}$ such that $T \subseteq U \cup V$, $T \cap U \neq \emptyset$, and $T \cap V \neq \emptyset$. Since $T \cap U$ and $T \cap V$ are relatively open in T , $U_1 = f^{-1}(T \cap U)$ and $V_1 = f^{-1}(T \cap V)$ are relatively open in S . Then, $S \subseteq U_1 \cup V_1 = S$, $U_1 \cap V_1 = \emptyset$, $S \cap U_1 \neq \emptyset$, and $S \cap V_1 \neq \emptyset$, which contradicts S is connected, #. □

Theorem 3.2.3

If S is a compact subset of \mathbb{R}^n and f is continuous on S , then $T = f(S)$ is also compact.

Proof. Let $\{U_\alpha\}_{\alpha \in J}$ be an open cover of T . Then, for each $\alpha \in J$, $f^{-1}(U_\alpha)$ is relatively open in S since U_α is open and f is continuous. Because

$$S = f^{-1}(T) = f^{-1}\left(\bigcup_{\alpha \in J} U_\alpha\right) = \bigcup_{\alpha \in J} f^{-1}(U_\alpha),$$

$\{f^{-1}(U_\alpha)\}_{\alpha \in J}$ is a relative open cover of S . Since S is compact, there is a finite subcover $\{f^{-1}(U_{\alpha_i}) \mid i \in [p], \alpha_i \in J\}$ of S . Then,

$$T = f(S) = f\left(\bigcup_{i=1}^p f^{-1}(U_{\alpha_i})\right) = \bigcup_{i=1}^p f(f^{-1}(U_{\alpha_i})) \subseteq \bigcup_{i=1}^p U_{\alpha_i},$$

implying $\{U_{\alpha_i}\}_{i=1}^p$ is a finite subcover of T . □

Theorem 3.2.4

If S is a compact subset of \mathbb{R}^n and $f : S \rightarrow \mathbb{R}$ is continuous on S , then f has a minimum and a maximum value on S .

Proof. Theorem 3.2.3 implies $T = f(S) \subseteq \mathbb{R}$ is compact, and thus bounded and closed. Thus, $m = \inf T = \min T$ and $M = \sup T = \max T$ exist. \square

Theorem 3.2.5 The Intermediate Value Theorem

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous and c is any number between $f(a)$ and $f(b)$, then there exists an $x \in [a, b]$ such that $f(x) = c$.

Proof. Since $[a, b]$ is connected and compact, Theorem 3.2.2 and Theorem 3.2.3 imply that $f([a, b])$ is connected and compact. Thus, $f([a, b]) = [m, M]$ where

$$m = \min f([a, b]) \leq \min\{f(a), f(b)\}$$

and

$$M = \max f([a, b]) \geq \max\{f(a), f(b)\}.$$

This implies $c \in [m, M] = f([a, b])$, i.e., there exists $x \in [a, b]$ such that $f(x) = c$. \square

Theorem 3.2.6 The General Intermediate Value Theorem

If S is any connected and compact subset of \mathbb{R}^n and $f : S \rightarrow \mathbb{R}$ is continuous, if $f(\mathbf{x}_1)$ and $f(\mathbf{x}_2)$ are any two values of f on S , and if c is any number between them, then there exists a point $\mathbf{x} \in S$ such that $f(\mathbf{x}) = c$.

Proof. Since S is connected and compact, by Theorem 3.2.2 and Theorem 3.2.3, $f(S)$ is an closed interval $[m, M]$ as in the proof of Theorem 3.2.5. Since $m \leq \min\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}$ and $M \geq \max\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}$, $c \in [m, M] = f(S)$, and thus $\exists \mathbf{x} \in S$, $f(\mathbf{x}) = c$. \square

3.2.1 The Composition of Continuous Functions

Theorem 3.2.7

Let $S \subseteq \mathbb{R}^n$, $f : S \rightarrow \mathbb{R}$, $f(S) \subseteq T \subseteq \mathbb{R}$, and $g : T \rightarrow \mathbb{R}$. If f is continuous at $\mathbf{c} \in S$ and if g is continuous at $f(\mathbf{c}) \in T$, then $g \circ f$ is continuous at \mathbf{c} .

Proof. Let $d = (g \circ f)(\mathbf{c})$. Take any neighborhood $N(d)$ of d . By continuity of g at $f(\mathbf{c})$, there exists a neighborhood $N(f(\mathbf{c}))$ such that

$$T \cap N(f(\mathbf{c})) \subseteq g^{-1}(N(d)).$$

By the continuity of f at \mathbf{c} , there exists a neighborhood $N(\mathbf{c})$ such that

$$S \cap N(\mathbf{c}) \subseteq f^{-1}(N(f(\mathbf{c}))).$$

These imply $S \cap N(\mathbf{c}) \subseteq f^{-1}(g^{-1}(N(d))) = (g \circ f)^{-1}(N(d))$.

Corollary 3.2.1

Let $S \subseteq \mathbb{R}^n$, $f : S \rightarrow \mathbb{R}$, $f(S) \subseteq T \subseteq \mathbb{R}$, and $g : T \rightarrow \mathbb{R}$. If f and g are continuous, then

$g \circ f$ is continuous.

Theorem 3.2.8

If $f : [a, b] \rightarrow [c, d]$ is strictly monotone, continuous function, then the inverse function f^{-1} is also strictly monotone, continuous, and bijective.

Proof. All are immediate except for the continuity. Denote f^{-1} by g . By Theorem 3.1.7, it suffices to prove that whenever a Cauchy sequence $\{y_k\}$ in $f(S)$ converges to y , then $\{g(y_k)\}$ converges to $g(y)$ in S .

Choose any such sequence and let $x_k \triangleq g(y_k)$ for each $k \in \mathbb{N}$. Since g is bijective, $\{y_k \mid k \in \mathbb{N}\}$ is finite if and only if $\{x_k \mid k \in \mathbb{N}\}$ is finite. If they are finite, then $\{y_k\}$ is eventually y , this implies $\{x_k\}$ is eventually $g(y)$, and it is done.

If they are infinite, since domain and codomain are bounded and closed, by Theorem 1.2.3, $\{x_k \mid k \in \mathbb{N}\}$ has a limit point x . But since $[a, b]$ is complete by Theorem 2.4.2, $x \in [a, b]$ by (ii) of Theorem 2.4.1. x is a cluster point of $\{x_k\}$, thus there is a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $\lim_{j \rightarrow \infty} x_{k_j} = x$ by Theorem 1.3.7. Now the continuity of f guarantees that

$$\lim_{j \rightarrow \infty} f(x_{k_j}) = f(x).$$

At the same time, since $f(x_{k_j}) = y_{k_j}$, $\{f(x_{k_j})\}$ is a subsequence of $\{y_k\}$. As $\{y_k\}$ converges to y , we get

$$\lim_{j \rightarrow \infty} f(x_{k_j}) = y.$$

By Theorem 1.3.1, $f(x) = y$, or $x = g(y)$.

If there were another limit point x' of $\{x_k \mid k \in \mathbb{N}\}$, by the same procedure, we get $x = g(y) = x'$; $x = x'$; x is the unique limit point of the set. Thus, $\{x_k\}$ converges to x , i.e., $\{g(y_k)\}$ converges to $g(y)$. \square

3.2.2 Limiting Behavior at Infinity

Definition 3.2.1: Function Space $C(S)$ and $C_\infty(S)$

Let $S \neq \emptyset$ be a subset of \mathbb{R}^n .

- $C(S)$ is the set of real-valued function on S which is continuous on S .
- $C_\infty(S)$ is the set of real-valued function on S which is bounded and continuous on S .

Note:-

In general, $C_\infty(S) \subseteq C(S)$. If $\emptyset \neq S \subseteq \mathbb{R}^n$ is compact, then $C(S) = C_\infty(S)$.

Definition 3.2.2: Neighborhood of ∞ and $-\infty$

In \mathbb{R} ,

- $N(\infty; M) \triangleq (M, \infty) = \{x \in \mathbb{R} \mid x > M\}$
- $N(-\infty, -M) \triangleq (-\infty, -M) = \{x \in \mathbb{R} \mid x < -M\}$

In \mathbb{R}^n ,

- $N(\infty; M) \triangleq \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| > M\}$

Definition 3.2.3: Limit at Infinity

- (i) Let S be an unbounded set in \mathbb{R} . Let $f : S \rightarrow \mathbb{R}$.
- We say f has limit L at ∞ if, for all $\varepsilon \in \mathbb{R}_+$, there exists $M \in \mathbb{R}_+$ such that $S \cap N(\infty; M) \subseteq f^{-1}(N(L; \varepsilon))$. We write $\lim_{x \rightarrow \infty} f(x) = L$.
 - We say f has limit L at $-\infty$ if, for all $\varepsilon \in \mathbb{R}_+$, there exists $M \in \mathbb{R}_+$ such that $S \cap N(-\infty; -M) \subseteq f^{-1}(N(L; \varepsilon))$. We write $\lim_{x \rightarrow -\infty} f(x) = L$.
- (ii) Let S be an unbounded set in \mathbb{R}^n . Let $f : S \rightarrow \mathbb{R}$. We say that f has limit L at ∞ , if, for all $\varepsilon \in \mathbb{R}_+$, there exists $M \in \mathbb{R}_+$ such that $S \cap N(\infty; M) \subseteq f^{-1}(N(L; \varepsilon))$. We write $\lim_{\|x\| \rightarrow \infty} f(x) = L$.

Theorem 3.2.9 The Squeeze Play

Let f , g , and h be three real-valued functions sharing a common unbounded domain $S \subseteq \mathbb{R}^n$. Suppose $\lim_{\|x\| \rightarrow \infty} f(x) = \lim_{\|x\| \rightarrow \infty} h(x) = L$. Suppose also that, for some $M \in \mathbb{R}_+$,

$$x \in S \cap N(\infty; M) \implies f(x) \leq g(x) \leq h(x)$$

Then, $\lim_{\|x\| \rightarrow \infty} g(x) = L$.

Theorem 3.2.10

Let S be a closed and unbounded set in \mathbb{R}^n and let $f \in C(S)$. Suppose $\lim_{\|x\| \rightarrow \infty} f(x) = L$ exists. Then $f \in C_\infty(S)$.

Proof. There exists $M \in \mathbb{R}_+$ such that, for $x \in S \cap N(\infty; M)$, $|f(x) - L| < 1$. Thus, for such x , we have $|f(x)| < |L| + 1$.

Since $S \cap \overline{N(0; M)}$ is a closed, bounded set in \mathbb{R}^n , it is compact by Theorem 2.5.3. Therefore the continuous f is bounded on $S \cap \overline{N(0; M)}$ by Theorem 3.2.3. In other words, there is some $K \in \mathbb{R}_+$ such that, for $x \in S \cap \overline{N(0; M)}$, we have $|f(x)| \leq K$. Thus, $|f(x)| \leq \max\{K, |L| + 1\}$ for all $x \in S$. \square

3.3 The Algebra of Continuous Functions

Note:-

Let $\emptyset \neq S \subseteq \mathbb{R}^n$. One can easily find that $C(S)$ is a commutative ring and is a vector space.

Theorem 3.3.1

Let $\emptyset \neq S \subseteq \mathbb{R}^n$ and $f_1, f_2 \in C(S)$. Then, the following hold.

- $f_1 + f_2 \in C(S)$.
- For any $a \in \mathbb{R}$, $af \in C(S)$.
- $f_1 f_2 \in C(S)$.
- $1/f_2 \in C(S)$, provided that $\forall x \in S, f_2(x) \neq 0$.
- $f_1/f_2 \in C(S)$, provided that $\forall x \in S, f_2(x) \neq 0$.

Proof. Directly import Theorem 3.1.3. \square

Theorem 3.3.2

Suppose f is continuous at a point \mathbf{c} in \mathbb{R}^n . Then f is locally bounded at \mathbf{c} . that is, there are $M, \delta \in \mathbb{R}_+$ such that

$$\mathbf{x} \in S \cap N(\mathbf{c}; \delta) \implies |f(\mathbf{x})| \leq M.$$

Theorem 3.3.3

Suppose f is continuous at a point \mathbf{c} in \mathbb{R}^n and $f(\mathbf{c}) \neq 0$. Then f is locally bounded away from 0 at \mathbf{c} . that is, there are $m, \delta \in \mathbb{R}_+$ such that

$$\mathbf{x} \in S \cap N(\mathbf{c}; \delta) \implies |f(\mathbf{x})| \geq m.$$

3.4 Uniform Continuity

Definition 3.4.1: Uniform Continuity

A function $f : S \rightarrow \mathbb{R}$ with $S \subseteq \mathbb{R}^n$ is said to be *uniformly continuous* on S if,

$$\forall \varepsilon \in \mathbb{R}_+, \exists \delta \in \mathbb{R}_+, \forall \mathbf{c} \in S, S \cap N(\mathbf{c}; \delta) \subseteq f^{-1}(N(f(\mathbf{c}); \varepsilon)).$$

Or, equivalently,

$$\forall \varepsilon \in \mathbb{R}_+, \exists \delta \in \mathbb{R}_+, \forall \mathbf{x}, \mathbf{y} \in S, (\|\mathbf{x} - \mathbf{y}\| < \delta \implies |f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon).$$

Example 3.4.1

$f : [0, b] \rightarrow \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous on $[0, b]$. Given any $\varepsilon \in \mathbb{R}_+$, let $\delta \triangleq \varepsilon/2b$. Then, whenever $|x - y| < \delta$ where $x, y \in [0, b]$, $|x^2 - y^2| = |x - y||x + y| < \delta \cdot 2b = \varepsilon$.

Example 3.4.2

$f : (0, M) \rightarrow \mathbb{R}$ defined by $f(x) = 1/x$ is not uniformly continuous on $(0, M)$. Let any $\delta \in \mathbb{R}_+$ is given. Let $a \in (0, \min\{\delta, 1/2, M/2\})$. Then, $|a - (2a)| = a < \delta$ but $|f(a) - f(2a)| = |1/a - 1/(2a)| = 1/(2a) > 1$.

This is an example in which f is continuous but the domain is not compact.

Example 3.4.3

$f : [-1, 1] \rightarrow \mathbb{R}$ defined by $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ is not uniformly continuous on $[-1, 1]$.

Let any $\delta \in \mathbb{R}_+$ is given. Let $a \in (0, \min\{\delta/2, 1\})$. Then, $|a - (-a)| = 2a < \delta$ but $|f(a) - f(-a)| = 1 > 0.5$.

This is an example in which the domain is compact but f is not continuous.

Theorem 3.4.1

Suppose that f is continuous on a compact subset S of \mathbb{R}^n . Then f is uniformly contin-

uous on S .

Proof. Let $\varepsilon \in \mathbb{R}_+$ be given. Since f is continuous at each point of S , for each \mathbf{c} of S , we may choose $\delta(\mathbf{c}) \in \mathbb{R}_+$ such that

$$S \cap N(\mathbf{c}; \delta(\mathbf{c})) \subseteq f^{-1}\left(N\left(f(\mathbf{c}); \frac{\varepsilon}{2}\right)\right).$$

Then the set $\mathcal{C} \triangleq \{N(\mathbf{c}; \delta(\mathbf{c})/2) \mid \mathbf{c} \in S\}$ is an open cover of the compact set S . Since S is compact, there is a finite subcover

$$\mathcal{C}_1 = \left\{N\left(\mathbf{c}_1; \frac{\delta(\mathbf{c}_1)}{2}\right), N\left(\mathbf{c}_2; \frac{\delta(\mathbf{c}_2)}{2}\right), \dots, N\left(\mathbf{c}_k; \frac{\delta(\mathbf{c}_k)}{2}\right)\right\}.$$

Let $\delta_0 \triangleq \min_{i=1}^k \delta(\mathbf{c}_i)/2$.

Now, take any $\mathbf{c} \in S$. Since \mathcal{C}_1 is an open cover,

$$\exists i \in [k], \mathbf{c} \in N\left(\mathbf{c}_i; \frac{\delta(\mathbf{c}_i)}{2}\right).$$

Then, for any $\mathbf{x} \in N(\mathbf{c}; \delta_0)$,

$$\|\mathbf{x} - \mathbf{c}_i\| \leq \|\mathbf{x} - \mathbf{c}\| + \|\mathbf{c} - \mathbf{c}_i\| < \delta_0 + \frac{\delta(\mathbf{c}_i)}{2} \leq \delta(\mathbf{c}_i).$$

Thus, $N(\mathbf{c}; \delta_0) \subseteq N(\mathbf{c}_i; \delta(\mathbf{c}_i))$; or

$$S \cap N(\mathbf{c}; \delta_0) \subseteq S \cap N(\mathbf{c}_i; \delta(\mathbf{c}_i)) \subseteq f^{-1}\left(N\left(f(\mathbf{c}_i); \frac{\varepsilon}{2}\right)\right).$$

Hence, for any $\mathbf{x} \in S \cap N(\mathbf{c}; \delta_0)$,

$$|f(\mathbf{x}) - f(\mathbf{c})| \leq |f(\mathbf{x}) - f(\mathbf{c}_i)| + |f(\mathbf{c}_i) - f(\mathbf{c})| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

as $\mathbf{x}, \mathbf{c} \in S \cap N(\mathbf{c}_i; \delta(\mathbf{c}_i))$. □

3.5 The Uniform Norm: Uniform Convergence

Definition 3.5.1: Function Space $B(S)$

Let $S \neq \emptyset$ be a subset of \mathbb{R}^n . $B(S)$ denotes the vector space and ring of all bounded, real-valued functions on S .

Note:-

- For each $f \in B(S)$, $\sup\{|f(\mathbf{x})| \mid \mathbf{x} \in S\}$ exists.
- $C_\infty(S) = C(S) \cap B(S)$

Definition 3.5.2: Uniform Norm

The *uniform norm* of $f \in B(S)$ is defined to be

$$\|f\|_\infty = \sup\{|f(\mathbf{x})| \mid \mathbf{x} \in S\}.$$

Theorem 3.5.1

The uniform norm is a norm.

Proof. The positive definiteness and the absolute homogeneity is direct.

Take any $f, g \in f$. Then, for any $\mathbf{x} \in S$,

$$|(f + g)(\mathbf{x})| \leq |f(\mathbf{x})| + |g(\mathbf{x})| \leq \|f\|_\infty + \|g\|_\infty.$$

Thus, $\|f + g\|_\infty = \sup\{|(f + g)(\mathbf{x})| \mid \mathbf{x} \in S\} \leq \|f\|_\infty + \|g\|_\infty$; $\|\cdot\|_\infty$ satisfies the subadditivity. \square

Definition 3.5.3: Uniform Metric

The *uniform metric* on $B(S)$ is

$$d_\infty(f, g) = \|f - g\|_\infty.$$

Note:-

The uniform metric is naturally a metric since the uniform norm is a norm.

Definition 3.5.4: (Deleted) Uniform Neighborhood

The *(uniform) neighborhood* $N(f; r)$ of f with radius r is the set

$$N(f; r) \triangleq \{g \in B(S) \mid d_\infty(f, g) < r\}.$$

The *deleted (uniform) neighborhood* $N'(f; r)$ of f with radius r is the set

$$N'(f; r) \triangleq \{g \in B(S) \mid 0 < d_\infty(f, g) < r\}.$$

Definition 3.5.5: Limit Point of a Set of Functions

A function $f_0 \in B(S)$ is said to be a *(uniform) limit point* of a set $F \subseteq B(S)$ if

$$\forall \varepsilon \in \mathbb{R}_+, F \cap N'(f_0; \varepsilon) \neq \emptyset.$$

Definition 3.5.6: Convergence of a Sequence of Functions

- A sequence $\{f_k\}_{k \in \mathbb{N}}$ in $B(S)$ is said to *converge uniformly* to $f_0 \in S \rightarrow \mathbb{R}$ on S if

$$\forall \varepsilon \in \mathbb{R}_+, \exists k_0 \in \mathbb{N}, \forall k \in \mathbb{N}, (k \geq k_0 \implies f_k \in N(f_0; \varepsilon)).$$

We write

$$\lim_{k \rightarrow \infty} f_k = f_0 \text{ [uniformly]}.$$

- A sequence $\{f_k\}_{k \in \mathbb{N}}$ in $B(S)$ is said to *converge pointwise* to $f_0: S \rightarrow \mathbb{R}$ on S if

$$\forall (\mathbf{c}, \varepsilon) \in S \times \mathbb{R}_+, \exists k_0 \in \mathbb{N}, \forall k \in \mathbb{N}, (k \geq k_0 \implies f_k(\mathbf{c}) \in N(f_0(\mathbf{c}); \varepsilon)).$$

We write

$$\lim_{k \rightarrow \infty} f_k = f_0 \text{ [pointwise]}.$$

- A sequence $\{f_k\}_{k \in \mathbb{N}}$ in $B(S)$ is said to be *(uniformly) Cauchy* if

$$\forall \varepsilon \in \mathbb{R}_+, \exists k_0 \in \mathbb{N}, \forall k, m \in \mathbb{N}, (k, m \geq k_0 \implies \|f_m - f_k\|_\infty < \varepsilon).$$

Note:-

A pointwise convergent sequence in $C_\infty(S)$ may fail to have a limit that is in $C_\infty(S)$.

Note:-

If $\{f_k\}$ in $B(S)$ converges uniformly, then it converges pointwise.

Theorem 3.5.2

Let $\emptyset \neq S \subseteq \mathbb{R}^n$. Suppose that $\{f_k\}$ is a sequence in $C(S)$ and it converges uniformly to $f_0: S \rightarrow \mathbb{R}$ on S . Then $f_0 \in C(S)$.

Proof. Take any $\mathbf{c} \in S$ and $\varepsilon \in \mathbb{R}_+$. By uniform convergence, there exists $k \in \mathbb{N}$ such that

$$\|f_k - f_0\|_\infty < \frac{\varepsilon}{4}.$$

Since f_k is continuous, there exists $\delta \in \mathbb{R}_+$ such that

$$S \cap N(\mathbf{c}; \delta) \subseteq f_k^{-1}\left(N\left(f_k(\mathbf{c}); \frac{\varepsilon}{2}\right)\right).$$

Thus, for any $\mathbf{x} \in S \cap N(\mathbf{c}; \delta)$,

$$\begin{aligned} |f_0(\mathbf{x}) - f_0(\mathbf{c})| &\leq |f_0(\mathbf{x}) - f_k(\mathbf{x})| + |f_k(\mathbf{x}) - f_k(\mathbf{c})| + |f_k(\mathbf{c}) - f_0(\mathbf{c})| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

This exactly means that f_0 is continuous at \mathbf{c} . Since \mathbf{c} is arbitrary, f_0 is continuous on S . □

Theorem 3.5.3

Let $\emptyset \neq S \subseteq \mathbb{R}^n$. A Cauchy sequence $\{f_k\}$ in $B(S)$ is bounded. That is, $\exists M \in \mathbb{R}_+, \forall k \in \mathbb{N}, \|f_k\|_\infty \leq M$.

Proof. Immitate the proof of Theorem 1.4.2. □

Theorem 3.5.4

$C_\infty(S)$ is complete. That is, given any Cauchy sequence $\{f_k\}$ in $C_\infty(S)$, there exists $f_0 \in C_\infty(S)$ such that $\lim_{k \rightarrow \infty} f_k = f_0$ [uniformly] on S .

Proof. Since, for each $\mathbf{c} \in S$, $\{f_k(\mathbf{c})\}_{k \in \mathbb{N}}$ is Cauchy, by Theorem 1.4.4, $\{f_k(\mathbf{c})\}_{k \in \mathbb{N}}$ converges in \mathbb{R} . Thus we may define $f_0: S \rightarrow \mathbb{R}$ by

$$f_0(\mathbf{c}) = \lim_{k \rightarrow \infty} f_k(\mathbf{c}).$$

Now, we first claim that $\lim_{k \rightarrow \infty} f_k = f_0$ [uniformly]. Take any $\varepsilon \in \mathbb{R}_+$. Since $\{f_k\}$ is Cauchy,

$$\exists k_0 \in \mathbb{N}, \forall k, m \in \mathbb{N}, (k, m \geq k_0 \implies \|f_k - f_m\|_\infty < \varepsilon/2).$$

Then, for each $k \in \mathbb{N}_{\geq k_0}$,

$$\forall \mathbf{c} \in S, |f_k(\mathbf{c}) - f_0(\mathbf{c})| = \lim_{m \rightarrow \infty} |f_k(\mathbf{c}) - f_m(\mathbf{c})| \leq \varepsilon/2.$$

This means $\|f_k - f_0\|_\infty \leq \varepsilon/2 < \varepsilon$; $\{f_k\}$ converges to f_0 uniformly on S . It directly follows from Theorem 3.5.2 that f_0 is continuous on S .

Now, we are left to show f_0 is bounded on S . Since $\{f_k\}$ uniformly converges to f_0 ,

$$\exists k_0 \in \mathbb{N}, \forall k \in \mathbb{N}, (k \geq k_0 \implies \|f_k - f_0\|_\infty < 1).$$

This implies $\forall \mathbf{c} \in S, |f_{k_0}(\mathbf{c}) - f_0(\mathbf{c})| < 1$. Hence, $\forall \mathbf{c} \in S, |f_0(\mathbf{c})| < |f_{k_0}(\mathbf{c})| + 1 \leq \|f_{k_0}\|_\infty + 1$. Thus, $\{|f_0(\mathbf{c})| \mid \mathbf{c} \in S\}$ is bounded above by $\|f_{k_0}\|_\infty + 1$. \square

Corollary 3.5.1

If $\emptyset \neq S \subseteq \mathbb{R}^n$ is compact, then $C(S)$ is complete.

Definition 3.5.7: Uniform Denseness

Let $\emptyset \neq S \subseteq \mathbb{R}^n$. A collection $F \subseteq C_\infty(S)$ is *uniformly dense* in $C_\infty(S)$ if, for all $f_0 \in C_\infty(S)$ and its neighborhood $N(f_0)$, $F \cap N(f_0) \neq \emptyset$.

Definition 3.5.8: Polynomial Space

Let $\emptyset \neq S \subseteq \mathbb{R}$. Let $P(S)$ denote the set of polynomial functions $f: S \rightarrow \mathbb{R}$ in the single variable x .

Note:-

If $S \neq \emptyset$ is a compact subset of \mathbb{R} , then $P(S)$ is certainly a subset of $C_\infty(S) = C(S)$.

Theorem 3.5.5 The Weierstrass Approximation Theorem

If S is a compact subset of \mathbb{R} , then $P(S)$ is uniformly dense in $C(S)$.

Definition 3.5.9: Bernstein Polynomial

Given a continuous function on $[0, 1]$, for each $k \in \mathbb{N}$, the k^{th} Bernstein polynomial for f , $B_k(x)$, is defined as follows.

$$B_k(x) \triangleq \sum_{j=0}^k \binom{k}{j} f\left(\frac{j}{k}\right) x^j (1-x)^{k-j}$$

where $\binom{k}{j}$ is the binomial coefficient $\frac{k!}{j!(k-j)!}$.

Note:-

The following lemmas are for the proof of Theorem 3.5.5.

Lemma 3.5.1

For any $k \in \mathbb{N}$, $\sum_{j=0}^k \binom{k}{j} x^j (1-x)^{k-j} = 1$.

Proof. Expand $1 = (x + (1-x))^k$ via the binomial theorem. □

Lemma 3.5.2

For any $k \in \mathbb{N}$, $\sum_{j=0}^k \binom{k}{j} \frac{j}{k} x^j (1-x)^{k-j} = x$.

Proof. Using Lemma 3.5.1, we get

$$\begin{aligned} x &= x \left[\sum_{j=0}^{k-1} \binom{k-1}{j} x^j (1-x)^{(k-1)-j} \right] \\ &= \sum_{j=0}^{k-1} \binom{k-1}{j} x^{j+1} (1-x)^{k-(j+1)} = \sum_{j=1}^k \binom{k-1}{j-1} x^j (1-x)^{k-j} \\ &= \sum_{j=1}^k \binom{k}{j} \frac{j}{k} x^j (1-x)^{k-j} = \sum_{j=0}^k \binom{k}{j} \frac{j}{k} x^j (1-x)^{k-j}. \end{aligned}$$

□

Lemma 3.5.3

For any $k \in \mathbb{N}$, $\sum_{j=0}^k \binom{k}{j} (j^2 - j) x^j (1-x)^{k-j} = k(k-1)x^2$.

Proof. Note that, for each $j \in \{2, 3, \dots, k\}$,

$$\binom{k}{j} (j^2 - j) = \binom{k-2}{j-2} k(k-1).$$

Therefore, using Lemma 3.5.1, we get

$$\begin{aligned}
\sum_{j=0}^k \binom{k}{j} (j^2 - j) x^j (1-x)^{k-j} &= \sum_{j=2}^k \binom{k}{j} (j^2 - j) x^j (1-x)^{k-j} \\
&= x^2 \sum_{j=2}^k \binom{k}{j} (j^2 - j) x^{j-2} (1-x)^{(k-2)-(j-2)} \\
&= x^2 \sum_{j=2}^k \binom{k-2}{j-2} k(k-1) x^{j-2} (1-x)^{(k-2)-(j-2)} \\
&= k(k-1) x^2 \sum_{j=0}^{k-2} \binom{k-2}{j} x^j (1-x)^{(k-2)-j} \\
&= k(k-1) x^2
\end{aligned}$$

□

Lemma 3.5.4

For any $k \in \mathbb{N}$, $\sum_{j=0}^k \binom{k}{j} \left(\frac{j}{k}\right)^2 x^j (1-x)^{k-j} = \left(1 - \frac{1}{k}\right) x^2 + \frac{x}{k}$.

Proof. Using Lemma 3.5.2 and Lemma 3.5.3, we get

$$\begin{aligned}
\left(1 - \frac{1}{k}\right) x^2 + \frac{x}{k} &= \frac{k(k-1)x^2}{k^2} + \frac{x}{k} \\
&= \frac{1}{k^2} \sum_{j=0}^k \binom{k}{j} (j^2 - j) x^j (1-x)^{k-j} + \frac{1}{k} \sum_{j=0}^k \binom{k}{j} \frac{j}{k} x^j (1-x)^{k-j} \\
&= \sum_{j=0}^k \binom{k}{j} \left(\frac{j^2 - j}{k^2} + \frac{j}{k^2}\right) x^j (1-x)^{k-j} \\
&= \sum_{j=0}^k \binom{k}{j} \left(\frac{j}{k}\right)^2 x^j (1-x)^{k-j}.
\end{aligned}$$

□

Lemma 3.5.5

For any $k \in \mathbb{N}$ and any $x \in [0, 1]$, $\sum_{j=0}^k \binom{k}{j} \left(x - \frac{j}{k}\right)^2 x^j (1-x)^{k-j} = \frac{x(1-x)}{k} \leq \frac{1}{4k}$.

Proof. Using Lemma 3.5.1, Lemma 3.5.2, and Lemma 3.5.4, we get

$$\begin{aligned}
\sum_{j=0}^k \binom{k}{j} \left(x - \frac{j}{k}\right)^2 x^j (1-x)^{k-j} &= \sum_{j=0}^k \binom{k}{j} \left[x^2 - 2x \left(\frac{j}{k}\right) + \left(\frac{j}{k}\right)^2 \right] x^j (1-x)^{k-j} \\
&= x^2 \sum_{j=0}^k \binom{k}{j} x^j (1-x)^{k-j} + 2x \sum_{j=0}^k \binom{k}{j} \left(\frac{j}{k}\right) x^j (1-x)^{k-j} \\
&\quad + \sum_{j=0}^k \binom{k}{j} \left(\frac{j}{k}\right)^2 x^j (1-x)^{k-j} \\
&= x^2 - 2x^2 + \left(1 - \frac{1}{k}\right)x^2 + \frac{x}{k} = \frac{x(1-x)}{k}.
\end{aligned}$$

□

Theorem 3.5.6 Bernstein's Approximation Theorem

Let $f \in C([0, 1])$. Then, for any $\varepsilon \in \mathbb{R}_+$, there exists $k_0 \in \mathbb{N}$ such that, for $k \geq k_0$, $\|f - B_k\|_\infty < \varepsilon$ where B_k is the k^{th} Bernstein polynomial for f .

Proof. Take any $\varepsilon \in \mathbb{R}_+$. Since f is continuous on a compact set $[0, 1]$, then f is uniformly continuous on $[0, 1]$ by Theorem 3.4.1. Therefore,

$$\exists \delta \in \mathbb{R}_+, \forall x, y \in [0, 1], (|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon/4).$$

Also, f is bounded on $[0, 1]$ by Theorem 3.2.3. Thus, by Theorem 1.1.1, there exists $k_0 \in \mathbb{N}$ such that $2\|f\|_\infty < \varepsilon \delta^2 k_0$. Now, suppose $k \geq k_0$.

Fix some $x \in [0, 1]$ and let $A_1 \triangleq \{j \in \mathbb{Z} \mid 0 \leq j \leq k \text{ and } |x - j/k| < \delta\}$ and $A_2 \triangleq \{j \in \mathbb{Z} \mid 0 \leq j \leq k \text{ and } |x - j/k| \geq \delta\}$. Now, we are ready to prove $\|f - B_k\|_\infty < \varepsilon$.

Using Lemma 3.5.1 at the first step, we get

$$\begin{aligned}
|f(x) - B_k(x)| &= \left| \sum_{j=0}^k \binom{k}{j} f(x) x^j (1-x)^{k-j} - \sum_{j=0}^k \binom{k}{j} f\left(\frac{j}{k}\right) x^j (1-x)^{k-j} \right| \\
&= \left| \sum_{j=0}^k \binom{k}{j} \left[f(x) - f\left(\frac{j}{k}\right) \right] x^j (1-x)^{k-j} \right| \\
&\leq \sum_{j=0}^k \binom{k}{j} \left| f(x) - f\left(\frac{j}{k}\right) \right| x^j (1-x)^{k-j} \\
&= \sum_{j \in A_1} \binom{k}{j} \left| f(x) - f\left(\frac{j}{k}\right) \right| x^j (1-x)^{k-j} + \sum_{j \in A_2} \binom{k}{j} \left| f(x) - f\left(\frac{j}{k}\right) \right| x^j (1-x)^{k-j} \\
&= S_1 + S_2
\end{aligned}$$

where S_1 and S_2 are the sums over A_1 and A_2 , respectively.

Since $\left|x - \frac{j}{k}\right| < \delta$ for $j \in A_1$, the following holds.

$$\begin{aligned}
S_1 &\triangleq \sum_{j \in A_1} \binom{k}{j} \left| f(x) - f\left(\frac{j}{k}\right) \right| x^j (1-x)^{k-j} \\
&< \sum_{j \in A_1} \binom{k}{j} \left(\frac{\varepsilon}{4}\right) x^j (1-x)^{k-j} \leq \frac{\varepsilon}{4} \sum_{j=0}^k \binom{k}{j} x^j (1-x)^{k-j} = \frac{\varepsilon}{2}
\end{aligned}$$

We now investigate S_2 . Note that $\frac{1}{|x - j/k|} \leq \frac{1}{\delta}$ for $j \in A_2$. Using Lemma 3.5.5, we get

$$\begin{aligned}
S_2 &\triangleq \sum_{j \in A_2} \binom{k}{j} \left| f(x) - f\left(\frac{j}{k}\right) \right| x^j (1-x)^{k-j} \\
&\leq 2\|f\|_\infty \sum_{j \in A_2} \binom{k}{j} \left(x - \frac{j}{k}\right)^2 \frac{1}{\left(x - \frac{j}{k}\right)^2} x^j (1-x)^{k-j} \\
&\leq \frac{2\|f\|_\infty}{\delta^2} \sum_{j \in A_2} \binom{k}{j} \left(x - \frac{j}{k}\right)^2 x^j (1-x)^{k-j} \\
&\leq \frac{2\|f\|_\infty}{\delta^2} \sum_{j=0}^k \binom{k}{j} \left(x - \frac{j}{k}\right)^2 x^j (1-x)^{k-j} \\
&\leq \frac{2\|f\|_\infty}{\delta^2} \frac{1}{4k} \leq \frac{\|f\|_\infty}{2k\delta^2} \leq \frac{\|f\|_\infty}{2k_0\delta^2} < \frac{\varepsilon}{4}.
\end{aligned}$$

Thus, we have $|f(x) - B_k(x)| \leq S_1 + S_2 < \varepsilon/4 + \varepsilon/4 = \varepsilon/2$. Since x is arbitrary, $\|f - B_k\|_\infty \leq \varepsilon/2 < \varepsilon$. \square