MAS242 해석학 II
Notes

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December 7, 2023

Contents

Chapter 1

Differentiation

1.1 Higher order partial derivatives

Definition 1.1.1

Given $f : U \to \mathbb{R}$ where U is an open set in $\mathbb{R}^m$, define $\partial_{ij} \triangleq \partial_i(\partial_j f)(x)$ for each $i, j \in [m]$ to be *2nd order partial derivatives*. Any higher order partial derivatives can be defined inductively.

Definition 1.1.2: C^k-regularity

$f : U \to \mathbb{R}$ is C^k-regular if all partial derivatives up to order k and they are continuous.

Theorem 1.1.1

$f : U(\subseteq \mathbb{R}^2) \to \mathbb{R}$ is C^2 at a point $c \in U$, i.e., $\exists \delta > 0$, f is C^2 in $B_\delta(c)$. Then, $\partial_{12} f(c) = \partial_{21} f(c)$.

Proof. Let $|h| < \delta$. Define $A(h) \triangleq f(c_1 + h_1, c_2 + h_2) - f(c_1 + h_1, c_2) - f(c_1, c_2 + h_2) + f(c_1, c_2)$. Define $u(x_1) \triangleq f(x_1, c_2 + h_2) - f(x_1, c_2)$ and $v(x_2) \triangleq f(c_1 + h_1, x_2) - f(c_1, x_2)$. Note that u and v are differentiable.

Then, $A(h) = u(c_1 + h_1) - u(c_1)$ and $A(h) = v(c_2 + h) - v(c_2)$. By MVT, $\exists c_1^* \in (c_1, c_1 + h_1)$ and $c_2^* \in (c_2, c_2 + h_2)$ s.t. $A(h) = u'(c_1^*)h_1 = h_1\big(\partial_1 f(c_1^*, c_2 + h) - \partial_1 f(c_1^*, c_2)\big) = h_1 h_2 \partial_{21} f(c_1^*, c_2^*)$

Similarly, $\exists c_1^{**}, c_2^{**}$ such that $A(h) = h_1 h_2 \partial_{12} f(c_1^{**}, c_2^{**})$. $\partial_{21} f(c_1^*, c_2^*) = \partial_{12} f(c_1^{**}, c_2^{**})$. Hence, as $|h| \to 0$, due to the continuity, $\partial_{21}(c) = \partial_{12}(c)$. $\square$

Corollary 1.1.1

Suppose $f : U(\subseteq \mathbb{R}^m) \to \mathbb{R}$ is C^k at $c \in U$. Then $\partial_{j_1 j_2 \cdots j_k} f(c) = \partial_{j_1' j_2' \cdots j_k'}$ where $j_1' \cdots$ are a permutation of $j_1 \cdots$.

1.2 Extreme Values of differentiable Functions

Definition 1.2.1: Hessian

Let $f : U(\subseteq \mathbb{R}^m) \to \mathbb{R}$ be C_2 in U. Suppose $p \in U$ is a critical point of f, i.e., $\nabla f(p) = 0$. Define

$$\mathcal{H}f(x) \triangleq \begin{pmatrix} \partial_{11}f(x) & \partial_{21}f(x) & \cdots & \partial_{m1}f(x) \\ \partial_{12}f(x) & \partial_{22}f(x) & \cdots & \partial_{m2}f(x) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{1m}f(x) & \partial_{2m}f(x) & \cdots & \partial_{mm}f(x) \end{pmatrix}.$$

(Sometimes $\mathcal{H}f(x) = D^2 f(x)$.)
Define $D(x) = \det \mathcal{H}f(x)$. (Note that $\mathcal{H}f(x)$ is symmetric when f is C^2 by the theorem above.)

Theorem 1.2.1 2nd-order derivative test for two variable functions.

When $m = 2$ and f is C^2, a critical point p is

- a local maximum if $D(p) > 0$ and $\partial_{11}f(p) > 0$ (or $\partial_{22}f(p) > 0$).
- a local minimum if $D(p) > 0$ and $\partial_{11}f(p) < 0$ (or $\partial_{22}f(p) < 0$).
- a saddle point if $D(p) < 0$.

The test fails when $D(p) = 0$.

Proof. Given a unit vector $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$, $D_{\boldsymbol{u}} f = \nabla f \cdot \boldsymbol{u} = u_1 \partial_1 f + u_2 \partial_2 f$, and thus

$$D_{\boldsymbol{u}}^2 f = (u_1\partial_1 + u_2\partial_2)(u_1\partial_1 f + u_2\partial_2 f) = u_1^2 \partial_{11} f + u_1 u_2 (2\partial_{12} f) + u_2^2 \partial_{22} f.$$

WLOG, $u_1 \neq 0$. Set $z = u_2/u_1$. Then,

$$D_{\boldsymbol{u}}^2 f(p) = u_1^2\big(\partial_{11}f(p) + 2\partial_{12}f(p) z + \partial_{22}f(p) z^2\big).$$

Note that, if $D(p) > 0$, $D_{\boldsymbol{u}}^2 f(p)$ has no real root.

- If $D(p) > 0$ and $\partial_{11}f(p) < 0$, Then, $D^2\boldsymbol{u} < 0$ for all unit vector $\boldsymbol{u}$.
- If $D(p) > 0$ and $\partial_{11}f(p) > 0$, Then, $D^2\boldsymbol{u} > 0$ for all unit vector $\boldsymbol{u}$.
- If $D(p) < 0$, $D_{\boldsymbol{u}}^2 f(p)$ has different signs depending on $\boldsymbol{u}$.

For general m?

$$D_{\boldsymbol{u}}(D_{\boldsymbol{u}} f) = D_{\boldsymbol{u}} \sum_{j=1}^{m} \partial_j f u_j = \sum_{j=1}^{m} \big((\nabla \partial_j f) \cdot \boldsymbol{u}\big) u_j = \sum_{j=1}^{m} \sum_{k=1}^{m} u_k u_j \partial_{kj} f.$$

Hence,

$$D_{\boldsymbol{u}}^2 f(p) = \boldsymbol{u}^{\mathrm{T}} \cdot D^2 f(p) \cdot \boldsymbol{u}$$

Since $D^2 f(p)$ is symmetric, its eigenvalues $\lambda_1, \cdots, \lambda_m$ exists and they are real numbers. Also, there exists an $m \times m$ orthogonal matrix $\mathcal{O}$ such that $D^2 f(p) = \mathcal{O}\Lambda(p)\mathcal{O}^{\mathrm{T}}$ where $\Lambda(p)$ is the diagonal matrix with entries are the eigenvalues.

Then, we can write $D_{\boldsymbol{u}}^2 f(p) = \boldsymbol{u}\mathcal{O}\Lambda(p)\mathcal{O}^{\mathrm{T}}\boldsymbol{u}^{\mathrm{T}} = (\boldsymbol{u}\mathcal{O})\Lambda(p) = (\boldsymbol{u}\mathcal{O})^{\mathrm{T}}$. Since $\mathcal{O}$ is orthogonal, $\boldsymbol{u}\mathcal{O}$ is another arbitrary unit vector. □

Theorem 1.2.2 Generalized 2nd order partial derivatives test

When f is C^2, a critical point p is

- a local maximum if all eigenvalues of $D^2 f(p)$ are negative.

- a local minimum if all eigenvalues of $D^2 f(p)$ are positive.
- a saddle point if there are both negative eigenvalues and positive eigenvalues.

The test fails when there are zero eigenvalues.

Chapter 2

Inverse Function Theorem

2.1 Jacobian

Definition 2.1.1: Jacobian

Let $\boldsymbol{f}: U(\subseteq \mathbb{R}^m) \to \mathbb{R}^n$ be differentiable. The function $J_f: U \to \mathbb{R}$ defined by

$$J_f(\boldsymbol{x}) = \det \begin{bmatrix} \partial_1 f_1(\boldsymbol{x}) & \cdots & \partial_n f_1(\boldsymbol{x}) \\ \vdots & \ddots & \vdots \\ \partial_1 f_n(\boldsymbol{x}) & \cdots & \partial_n f_n(\boldsymbol{x}) \end{bmatrix}$$

is called the *Jacobian* of $\boldsymbol{f}$ at $\boldsymbol{x}$.

Lemma 2.1.1

If $f: V(\subseteq \mathbb{R}^n) - \mathbb{R}$ and $\boldsymbol{g}: U \to V$ are differentiable, then

$$J_{f \circ g}(\boldsymbol{x}) = J_f\big(\boldsymbol{g}(\boldsymbol{x})\big) \cdot J_g(\boldsymbol{x}).$$

Note:-

The linear mapping $d\boldsymbol{f}(c)$ is invertible if and only if $J_f(c)$ is nonzero.

2.2 The Inverse Function Theorem

Lemma 2.2.1 Contraction Mapping Principle

Let (X, d) be a complete metric space. Let $\varphi: X \to X$. Suppose that there exists $M \in [0,1)$ such that $d\big(\varphi(x_1), \varphi(x_2)\big) \leq Md(x_1, x_2)$. (We call it a *contraction mapping*.) Then, there uniquely exists $x_* \in X$ such that $\varphi(x_*) = x_*$.

Proof. Fix any $x_0 \in X$. Since $\{x_j\}_{j \in \mathbb{Z}_+}$, where $x_j = \varphi(x_{j-1})$ for each $j \in \mathbb{Z}_+$, is continuous. It converges to some x_*. As φ is continuous, we have $\varphi(x_*) = x_*$. The uniqueness follows trivially. □

Note:-

- For each $v \in \mathbb{R}^n \setminus \{0\}$, $|Av| = |v| \cdot \left|A\frac{v}{|v|}\right| \leq \|A\|_L \cdot |v|$. The result is trivial when $v = 0$.
- For each $u \in \mathbb{R}^n$ with $|u| = 1$, $|ABu| \leq \|A\|_L |Bu| \leq \|A\|_L \|B\|_L$. Hence, $\|AB\|_L = \|A\|\|B\|$.
- Given invertible $A \in L(\mathbb{R}^n.\mathbb{R}^n)$, $A^{-1}: \mathbb{R}^n \to \mathbb{R}^n$ is linear. Moreover, $\|A\|_L > 0$.

Lemma 2.2.2

Given two linear mappings $A, B : \mathbb{R}^n \to \mathbb{R}^n$ with invertibility of A,

$$\|A-B\|_L \cdot \|A^{-1}\|_L < 1 \implies B \text{ is invertible.}$$

Proof. Let $\|A^{-1}\|_L = 1/\alpha$ and $\|B-A\|_L = \beta$ so that $\beta < \alpha$. Then, for every $\boldsymbol{x} \in \mathbb{R}^n$,

$$\begin{aligned}\alpha|\boldsymbol{x}| &= \alpha|A^{-1}A\boldsymbol{x}| \le \alpha\|A^{-1}\| \cdot |A\boldsymbol{x}| \\ &= |A\boldsymbol{x}| \le |(A-B)\boldsymbol{x}| + |B\boldsymbol{x}| \le \beta|\boldsymbol{x}| + |B\boldsymbol{x}|;\end{aligned}$$

hence $(\alpha-\beta)|\boldsymbol{x}| \le |B\boldsymbol{x}|$ where $\boldsymbol{x} \in \mathbb{R}^n$ is arbitrary. As $\alpha > \beta$, it holds that $B\boldsymbol{x} = 0 \implies \boldsymbol{x} = 0$. □

Corollary 2.2.1

The set $\Omega \subseteq L(\mathbb{R}^n, \mathbb{R}^n)$ of invertible linear transformations is open.

Lemma 2.2.3

The mapping from Ω onto Ω defined by $A \mapsto A^{-1}$ is continuous.

Proof. Let A and B be invertible linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let $\|A^{-1}\| = 1/\alpha$ and $\|B-A\|_L = \beta$. We have $(\alpha-\beta)|\boldsymbol{x}| \le |B\boldsymbol{x}|$ by the same reasoning as in the proof of Lemma 2.2.2. Hence, the following holds.

$$\forall \boldsymbol{y} \in \mathbb{R}^n,\ (\alpha-\beta)|B^{-1}\boldsymbol{y}| \le |BB^{-1}\boldsymbol{y}| = |\boldsymbol{y}|$$

This shows that $\|B^{-1}\|_L \le (\alpha-\beta)^{-1}$.

Hence, we have

$$\|B^{-1}-A^{-1}\|_L \le \|B^{-1}\|_L\|A-B\|_L\|A^{-1}\|_L \le \frac{\beta}{\alpha(\alpha-\beta)}.$$

This implies that $\|B^{-1}-A^{-1}\|_L \to 0$ as $B \to A$. □

Theorem 2.2.1 Inverse Function Theorem

Let $\boldsymbol{f} : E(\subseteq \mathbb{R}^n) \to \mathbb{R}^n$ be C^1 in E and $\boldsymbol{c} \in E$. Suppose that $J_{\boldsymbol{f}}(\boldsymbol{c}) \neq 0$. Then, the following hold.

(i) There exists a neighborhood U of $\boldsymbol{a}$ such that $\boldsymbol{f}\big|_U$ is bijective and $V \triangleq \boldsymbol{f}(U)$ is open.

(ii) The inverse map of $\boldsymbol{f}\big|_U$ is C^1 in V.

Proof. Let $A \triangleq \mathrm{d}\boldsymbol{f}(\boldsymbol{c})$. Define $\lambda \in \mathbb{R}_+$ by $2\lambda\left\|A^{-1}\right\|_L = 1$. Since $\mathrm{d}\boldsymbol{f}$ is continuous, there exists a neighborhood U of $\boldsymbol{c}$ such that $\|\mathrm{d}\boldsymbol{f}(\boldsymbol{x}) - A\|_L < \lambda$ for each $\boldsymbol{x} \in U$.

Given a point $\boldsymbol{y} \in \mathbb{R}^n$, we define $\varphi(\cdot;\boldsymbol{y})$ by

$$\begin{aligned}\varphi(\cdot;\boldsymbol{y}) : B_\delta(\boldsymbol{c}) &\longrightarrow \mathbb{R}^n \\ \boldsymbol{x} &\longmapsto \boldsymbol{x} + A^{-1}(\boldsymbol{y} - \boldsymbol{f}(\boldsymbol{x}))\end{aligned}$$

Note that $\boldsymbol{x}$ is a fixed point of $\varphi(\cdot;\boldsymbol{y})$ if and only if $A^{-1}(\boldsymbol{y}-\boldsymbol{f}(\boldsymbol{x})) = 0$, i.e., $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$. Note also that φ is differentiable and $\mathrm{d}\varphi(\boldsymbol{x};\boldsymbol{y}) = \mathrm{Id} - A^{-1}\,\mathrm{d}\boldsymbol{f}(\boldsymbol{x}) = A^{-1}\big(A - \mathrm{d}\boldsymbol{f}(\boldsymbol{x})\big)$ for each $\boldsymbol{x} \in U$.

Hence, for all $\boldsymbol{x} \in U$,

$$\|\,\mathrm{d}\varphi(\boldsymbol{x};\boldsymbol{y})\|_L = \left\|A^{-1}\bigl(A - \mathrm{d}\boldsymbol{f}(\boldsymbol{x})\bigr)\right\|_L \le \|A^{-1}\|_L \cdot \|A - \mathrm{d}\boldsymbol{f}(\boldsymbol{x})\|_L < 1/(2\lambda)\cdot\lambda = 1/2.$$

Thus, MVT gives

$$|\varphi(\boldsymbol{x}_1;\boldsymbol{y}) - \varphi(\boldsymbol{x}_2;\boldsymbol{y})| \le \frac{1}{2}|\boldsymbol{x}_1 - \boldsymbol{x}_2|$$

whenever $\boldsymbol{x}_1, \boldsymbol{x}_2 \in U$. Note that this implies there is at most one fixed point of $\varphi(\cdot;\boldsymbol{y})$ in U, i.e., $\boldsymbol{f}\big|_U$ is bijective.

Now, we shall show that $V = \boldsymbol{f}(U)$ is open. Take any $\boldsymbol{y}_0 \in V$. There (uniquely) exists $\boldsymbol{x}_0 \in U$ such that $\boldsymbol{y}_0 = \boldsymbol{f}(\boldsymbol{x}_0)$. Fix any $r \in \mathbb{R}_+$ such that $\overline{B} \subseteq U$ where $B = B_r(\boldsymbol{x}_0)$. Take any $\boldsymbol{y} \in B_{\lambda r}(\boldsymbol{y}_0)$. Then,

$$|\varphi(\boldsymbol{x}_0;\boldsymbol{y}) - \boldsymbol{x}_0| = |A^{-1}(\boldsymbol{y} - \boldsymbol{y}_0)| < \|A^{-1}\|_L \lambda r = \frac{r}{2}.$$

Moreover, for any $\boldsymbol{x} \in \overline{B}$,

$$|\varphi(\boldsymbol{x};\boldsymbol{y}) - \boldsymbol{x}_0| \le |\varphi(\boldsymbol{x};\boldsymbol{y}) - \varphi(\boldsymbol{x}_0;\boldsymbol{y})| + |\varphi(\boldsymbol{x}_0;\boldsymbol{y}) - \boldsymbol{x}_0| \le \frac{1}{2}|\boldsymbol{x} - \boldsymbol{x}_0| + \frac{r}{2} < r.$$

This directly implies that $\varphi(\overline{B};\boldsymbol{y}) \subseteq B \subseteq \overline{B}$. Hence, $\varphi(\cdot,\boldsymbol{y})$ is a contraction mapping on a complete metric space $\overline{B}$. By Lemma 2.2.1, there exists a fixed point $\boldsymbol{x} \in \overline{B}$, which satisfies $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$. Thus, $\boldsymbol{y} \in f(\overline{B}) \subseteq f(U) = V$. Hence, $B \subseteq V$, V is open. This proves (i).

Now, let $\boldsymbol{g} : V \to U$ be the local inverse map of $\boldsymbol{f}\big|_U$. Take any $\boldsymbol{y} \in V$ and $\boldsymbol{y} + \boldsymbol{k} \in V$. There are unique $\boldsymbol{x} \in U$ and $\boldsymbol{x} + \boldsymbol{h} \in U$ such that $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{y} + \boldsymbol{k} = \boldsymbol{f}(\boldsymbol{x} + \boldsymbol{h})$. Then, we have

$$\varphi(\boldsymbol{x} + \boldsymbol{h};\boldsymbol{y}) - \varphi(\boldsymbol{x};\boldsymbol{y}) = \boldsymbol{h} + A^{-1}\bigl(\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{x} + \boldsymbol{h})\bigr) = \boldsymbol{h} - A^{-1}\boldsymbol{k},$$

which implies $|\boldsymbol{h} - A^{-1}\boldsymbol{k}| \le |h|/2$. Hence, $|A^{-1}\boldsymbol{k}| \ge |h|/2$ is obtained by the triangle inequality; $|\boldsymbol{h}| \le 2\|A^{-1}\|_L |\boldsymbol{k}| = \lambda^{-1}|\boldsymbol{k}|$.

Then, since $\|\,\mathrm{d}\boldsymbol{f}(\boldsymbol{x}) - A\|_L \|A^{-1}\|_L < \lambda \cdot 1/(2\lambda) = 1/2$, Lemma 2.2.2 implies that $\mathrm{d}\boldsymbol{f}(\boldsymbol{x})$ is invertible. Let $T \triangleq \mathrm{d}\boldsymbol{f}(\boldsymbol{x})$. Then, we have

$$\boldsymbol{g}(\boldsymbol{y} + \boldsymbol{k}) - \boldsymbol{g}(\boldsymbol{y}) - T^{-1}\boldsymbol{k} = \boldsymbol{h} - T^{-1}\boldsymbol{k} = -T^{-1}\bigl(\boldsymbol{f}(\boldsymbol{x} + \boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) - T\boldsymbol{h}\bigr),$$

and thus

$$\frac{|\boldsymbol{g}(\boldsymbol{y} + \boldsymbol{k}) - \boldsymbol{g}(\boldsymbol{y}) - T^{-1}\boldsymbol{k}|}{|\boldsymbol{k}|} \le \frac{\|T^{-1}\|_L}{\lambda} \cdot \frac{|\boldsymbol{f}(\boldsymbol{x} + \boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) - T\boldsymbol{h}|}{|\boldsymbol{h}|}.$$

The equation implies that $\boldsymbol{g}$ is differentiable on V, and that $\mathrm{d}\boldsymbol{g}(\boldsymbol{y}) = T^{-1} = \mathrm{d}\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{y}))^{-1}$. Since $\mathrm{d}\boldsymbol{g}$ is a composition of continuous functions, $\mathrm{d}\boldsymbol{g}$ itself is continuous. □

Corollary 2.2.2

Let $\boldsymbol{f} : E(\subseteq \mathbb{R}^n) \to \mathbb{R}^n$ be C^1 in E and $J_{\boldsymbol{f}}(\boldsymbol{x}) \neq 0$ for all $\boldsymbol{x} \in E$. Then, for every open set $W \subseteq E$, $\boldsymbol{f}(W)$ is open.

Proof. This directly follows from (i) of Theorem 2.2.1. □

2.3 Implicit Function Theorem

Definition 2.3.1

- If $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $\boldsymbol{y} = (y_1, \cdots, y_m) \in \mathbb{R}^m$, let us write $(\boldsymbol{x}, \boldsymbol{y})$ for the point $(x_1, \cdots, x_n, y_1, \cdots, y_m) \in \mathbb{R}^{n+m}$.
- Every $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ can be split into $A_x \in L(\mathbb{R}^n)$ and $A_y \in L(\mathbb{R}^m, \mathbb{R}^n)$ where $A(\boldsymbol{h}, \boldsymbol{k}) = A_x\boldsymbol{h} + A_y\boldsymbol{k}$ for each $\boldsymbol{h} \in \mathbb{R}^n$ and $\boldsymbol{k} \in \mathbb{R}^m$.

Lemma 2.3.1

If $A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$ and if A_x is invertible, then

$$\forall \boldsymbol{k} \in \mathbb{R}^m,\ \exists! \boldsymbol{h} \in \mathbb{R}^n,\ A(\boldsymbol{h}, \boldsymbol{k}) = \boldsymbol{0}.$$

Proof. $A(\boldsymbol{h}, \boldsymbol{k}) = A_x\boldsymbol{h} + A_y\boldsymbol{k} = \boldsymbol{0}$ if and only if $\boldsymbol{h} = -(A_x)^{-1}A_y\boldsymbol{k}$. □

Theorem 2.3.1 Implicit Function Theorem

Let $\boldsymbol{f} : E \to \mathbb{R}^n$ be a C^1 mapping where E is an open set in $\mathbb{R}^{n+m}$. Let $(\boldsymbol{a}, \boldsymbol{b}) \in E$ satisfy $\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{0}$. Let $A = \mathrm{d}\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b})$ and suppose A_x is invertible. Then, there exist open sets $U \subseteq \mathbb{R}^{n+m}$ and $W \subseteq \mathbb{R}^m$ that satisfy the following.

(i) $(\boldsymbol{a}, \boldsymbol{b}) \in U$ and $\boldsymbol{b} \in W$.
(ii) $\forall \boldsymbol{y} \in W,\ \exists! \boldsymbol{x} \in \mathbb{R}^n,\ (\boldsymbol{x}, \boldsymbol{y}) \in U \wedge \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = 0$.
(iii) If the unique $\boldsymbol{x}$ in (ii) is denoted by $\boldsymbol{g}(\boldsymbol{y})$, then $\boldsymbol{g} : W \to \mathbb{R}^n$ is C^1 on W.
(iv) Moreover, $\mathrm{d}\boldsymbol{g}(\boldsymbol{b}) = -(A_x)^{-1}A_y$.

Proof. Define $\boldsymbol{F} : E \to \mathbb{R}^{n+m}$ by $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) \triangleq (\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{y})$. Then, $\boldsymbol{F}$ is C^1. Since $\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{0}$, if $\boldsymbol{r}(\boldsymbol{h}, \boldsymbol{k}) \triangleq \boldsymbol{f}(\boldsymbol{a} + \boldsymbol{h}, \boldsymbol{b} + \boldsymbol{k}) - A(\boldsymbol{h}, \boldsymbol{k})$, we have $\lim_{h,k \to 0} |\boldsymbol{r}(\boldsymbol{h}, \boldsymbol{k})|/|(\boldsymbol{h}, \boldsymbol{k})| = 0$. Hence, from

$$\mathrm{F}(\mathrm{a} + \mathrm{h}, \mathrm{b} + \mathrm{k}) - \mathrm{F}(\mathrm{a}, \mathrm{b}) = (\mathrm{f}(\mathrm{a} + \mathrm{h}, \mathrm{b} + \mathrm{k}), \mathrm{k}) = (A(\mathrm{h}, \mathrm{k}), \mathrm{k}) + (\mathrm{r}(\mathrm{h}, \mathrm{k}), 0),$$

it is obtained that $\mathrm{dF}(\mathrm{a}, \mathrm{b})(\mathrm{h}', \mathrm{k}') = (A(\mathrm{h}', \mathrm{k}'), \mathrm{k}')$ for each $(\boldsymbol{h}', \boldsymbol{k}') \in \mathbb{R}^{n+m}$. If $\mathrm{dF}(\mathrm{a}, \mathrm{b})(\mathrm{h}', \mathrm{k}') = 0$, then $\boldsymbol{k}' = 0$ and $A(\boldsymbol{h}', \boldsymbol{0}) = \boldsymbol{0}$; thus $\boldsymbol{h}' = \boldsymbol{0}$ as A_x is invertible. Hence, $\mathrm{d}\boldsymbol{F}(\boldsymbol{a}, \boldsymbol{b})$ is invertible; Theorem 2.2.1 can be applied to $\boldsymbol{F}$ at $(\boldsymbol{a}, \boldsymbol{b})$.

By Theorem 2.2.1, there exists a neighborhood $U \subseteq E$ of $(\boldsymbol{a}, \boldsymbol{b})$ such that $\boldsymbol{F}\big|_U$ is bijective, $\boldsymbol{F}(U)$ is open, and its inverse is C^1. Let $W \triangleq \{\boldsymbol{y} \in \mathbb{R}^m \mid (\boldsymbol{0}, \boldsymbol{y}) \in \boldsymbol{F}(U)\}$. W is open as $\boldsymbol{F}(U)$ is open. Noting that $\boldsymbol{b} \in W$, we finish the proof for (i).

Take any $\boldsymbol{y} \in W$. Then, there exists $(\boldsymbol{x}, \boldsymbol{y}) \in U$ such that $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{0}, \boldsymbol{y})$; thus $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0}$. If $\boldsymbol{x}, \boldsymbol{x}'$ are two such point corresponding to $\boldsymbol{y}$, then

$$\boldsymbol{F}(\boldsymbol{x}', \boldsymbol{y}) = (\boldsymbol{f}(\boldsymbol{x}', \boldsymbol{y}), \boldsymbol{y}) = (\boldsymbol{0}, \boldsymbol{y}) = (\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}), \boldsymbol{y}) = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}).$$

However, as $\boldsymbol{F}$ being injective, $\boldsymbol{x} = \boldsymbol{x}'$. This proves (ii).

Let $V \triangleq \boldsymbol{F}(U)$. Let $\boldsymbol{G} : V \to U$ be the inverse of $\boldsymbol{F}$, which is C^1 by Theorem 2.2.1. Hence, for each $\boldsymbol{y} \in W$, from $\boldsymbol{F}(\boldsymbol{g}(\boldsymbol{y}), \boldsymbol{y}) = (\boldsymbol{0}, \boldsymbol{y})$, we have $(\boldsymbol{g}(\boldsymbol{y}), \boldsymbol{y}) = \boldsymbol{G}(\boldsymbol{0}, \boldsymbol{y})$. This directly shows that $\boldsymbol{g}$ is C^1 as well. This proves (iii).

Let $\boldsymbol{\Phi} : W \to U$ be defined by $\boldsymbol{\Phi}(\boldsymbol{y}) = \boldsymbol{G}(\boldsymbol{0}, \boldsymbol{y}) = (\boldsymbol{g}(\boldsymbol{y}), \boldsymbol{y})$, which is C^1, indeed. Then, $\mathrm{d}\boldsymbol{\Psi}(\boldsymbol{y}) = (\mathrm{d}\boldsymbol{g}(\boldsymbol{y}), I_m)$. Differentiating both sides of the equality $\boldsymbol{f}(\boldsymbol{\Phi}(\boldsymbol{y})) = \boldsymbol{0}$, we get

$$\mathrm{d}\boldsymbol{f}(\boldsymbol{\Phi}(\boldsymbol{y}))\,\mathrm{d}\boldsymbol{\Phi}(\boldsymbol{y}) = 0.$$

Putting $\boldsymbol{y} := \boldsymbol{b}$, as $\boldsymbol{\Phi}(\boldsymbol{b}) = (\boldsymbol{a}, \boldsymbol{b})$, we get $A\,\mathrm{d}\boldsymbol{\Phi}(\boldsymbol{b}) = 0$, or

$$A_x \,\mathrm{d}\boldsymbol{g}(\boldsymbol{b}) + A_y = 0,$$

i.e., $\mathrm{d}\boldsymbol{g}(\boldsymbol{b}) = -(A_x)^{-1}A_y$. □

Definition 2.3.2: C^1-norm

Suppose $\varphi : \mathbb{R}^n \to \mathbb{R}$ is C^1. Then,

$$\|\varphi\|_{C^0(\overline{\Omega})} \triangleq \sup_{\boldsymbol{x}\in\Omega} |\varphi(x)|$$

$$\|\varphi\|_{C^1(\overline{\Omega})} \triangleq \|\varphi\|_{C^0(\overline{\Omega})} + \sum_{j=1}^{n} \|\partial_j \varphi\|_{C^0(\overline{\Omega})}.$$

This is only for Example 2.3.1.

Example 2.3.1 (Level Sets)

Define $\Omega \triangleq \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_2| \le 1\}$. Given two constants, $a, b \in \mathbb{R}$ with $a < b$, define $\overline{\varphi}(x_1, x_2) = ax_1$ and $\overline{\psi}(x_1, x_2) = bx_1$. Then, $\Gamma_0 = \{\boldsymbol{x} \in \Omega \mid \overline{\varphi}(\boldsymbol{x}) - \overline{\psi}(\boldsymbol{x}) = 0\} = \{\boldsymbol{x} \in \Omega \mid x_1 = 0\}$.

Suppose that $\varphi, \psi : \Omega \to \mathbb{R}$ satisfy

$$\|\varphi - \overline{\varphi}\|_{C^1(\overline{\Omega})} + \|\psi - \overline{\psi}\|_{C^1(\overline{\Omega})} \le \frac{1}{4}|a-b|.$$

Then, what would be the expression for $\Gamma = \{\boldsymbol{x} \in \Omega \mid \varphi(\boldsymbol{x}) - \psi(\boldsymbol{x}) = 0\}$?

Observe that $(\varphi - \psi) = (\varphi - \overline{\varphi}) + (\overline{\varphi} - \overline{\psi}) + (\overline{\psi} - \psi)$ and thus $|(\varphi - \psi)(x_1, x_2) - (a-b)x_1| \le |a-b|/4$. This implies $\lim_{x_1 \to \pm\infty} (\varphi - \psi)(x_1, x_2) = \mp\infty$. Hence, for every $x_2 \in [-1, 1]$, there exists $x_1^* \in \mathbb{R}$ such that $(\varphi - \psi)(x_1^*, x_2) = 0$.

Moreover, $\partial_1(\varphi - \psi) = \partial_1(\varphi - \overline{\varphi}) + (a-b) + \partial_1(\overline{\psi} - \psi)$, and thus $|\partial_1(\varphi - \psi)| \ge \frac{3}{4}|a-b| > 0$. Hence, the x_1^* in the previous paragraph is unique. This means that $\Gamma = \{(f(x_2), x_2) \mid x_2 \in \mathbb{R}\}$ for some f.

$(\varphi - \psi)(f(x_2), x_2) - (\overline{\varphi} - \overline{\psi})(f(x_2), x_2) = -(\overline{\varphi} - \overline{\psi})(f(x_2), x_2) = (b-a)f(x_2)$. Hence,

$$f(x_2) = \frac{(\varphi - \overline{\varphi})(f(x_2), x_2) - (\psi - \overline{\psi})(f(x_2), x_2)}{b-a}.$$

This is the implicit representation of f. Moreover, $|f(x_2)| = \dfrac{|b-a|/4}{|b-a|} = 1/4$.

2.4 Applications of IMFT: Lagrange's Method

Theorem 2.4.1 Optimization Under Multiple Constraints

Let $f, g_1, g_2, \cdots, g_k : E \to \mathbb{R}$ be C^1 where E is an open set in $\mathbb{R}^n$ and $n > k$. Let $Z \triangleq \cap_{j=1}^{k} \{\boldsymbol{z} \in \mathbb{R}^n \mid g_j(\boldsymbol{z}) = 0\}$. Suppose $\boldsymbol{z}_0 \in Z$ is a local maximum point with respect to f

on Z. Suppose also that

$$\Delta \triangleq \det \begin{bmatrix} \partial_1 g_1(\boldsymbol{z}_0) & \cdots & \partial_1 g_k(\boldsymbol{z}_0) \\ \vdots & \ddots & \vdots \\ \partial_k g_1(\boldsymbol{z}_0) & \cdots & \partial_k g_k(\boldsymbol{z}_0) \end{bmatrix} \neq 0.$$

Then, there exists $\lambda_1, \lambda_2, \cdots, \lambda_k \in \mathbb{R}$ such that $\nabla f(\boldsymbol{z}_0) = \sum_{m=1}^k \lambda_m \nabla g_m(\boldsymbol{z}_0)$.

Proof. Since $\Delta \neq 0$, there exists a unique solution $(\lambda_1, \cdots, \lambda_k) \in \mathbb{R}^k$ for the linear system

$$\begin{bmatrix} \partial_1 g_1(\boldsymbol{z}_0) & \cdots & \partial_1 g_k(\boldsymbol{z}_0) \\ \vdots & \ddots & \vdots \\ \partial_k g_1(\boldsymbol{z}_0) & \cdots & \partial_k g_k(\boldsymbol{z}_0) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{bmatrix} = \begin{bmatrix} \partial_1 f(\boldsymbol{z}_0) \\ \vdots \\ \partial_k f(\boldsymbol{z}_0) \end{bmatrix}.$$

For each point $\boldsymbol{z} = (z_1, \cdots, z_n) \in \mathbb{R}^n$, let $\boldsymbol{x} = (z_1, \cdots, z_k)$ and $\boldsymbol{y} = (z_{k+1}, \cdots, z_n)$. Let $\boldsymbol{z}_0 = (\boldsymbol{x}_0, \boldsymbol{y}_0)$. Let $\boldsymbol{g} : E \to \mathbb{R}^k$ be defined by $\boldsymbol{g}(\boldsymbol{z}) = (g_1(\boldsymbol{z}), \cdots, g_k(\boldsymbol{z}))$.

Since $\boldsymbol{g}$ is C^1, $\boldsymbol{g}(\boldsymbol{z}_0) = 0$, and $\left(\mathrm{d}\boldsymbol{g}(\boldsymbol{z}_0)\right)_x$ is invertible, by Theorem 2.3.1, there exists an open neighborhood $W \subseteq \mathbb{R}^{n-k}$ of $\boldsymbol{y}_0$ and a C^1 function $\boldsymbol{s} : W \to \mathbb{R}^k$ such that $\boldsymbol{g}(\boldsymbol{s}(\boldsymbol{y}), \boldsymbol{y}) = \boldsymbol{0}$ for each $\boldsymbol{y} \in W$. Note that $\boldsymbol{s}(\boldsymbol{y}_0) = \boldsymbol{x}_0$.

Define $F : W \to \mathbb{R}$ by $\boldsymbol{y} \mapsto f(\boldsymbol{s}(\boldsymbol{y}), \boldsymbol{y})$. As $\boldsymbol{z}_0$ is a local maximum point, so is $\boldsymbol{y}_0$. Hence, $\nabla F(\boldsymbol{y}_0) = \boldsymbol{0}$. For each $j \in [k]$, define $G_j : W \to \mathbb{R}$ by $\boldsymbol{y} \mapsto g_j(\boldsymbol{s}(\boldsymbol{y}), \boldsymbol{y})$. As $(\boldsymbol{s}(\boldsymbol{y}), \boldsymbol{y}) \in Z$, we have $G_j = 0$ for each $j \in [k]$. Thus, $\nabla G_j(\boldsymbol{y}) = \boldsymbol{0}$.

Let $\boldsymbol{s} = (s_1, s_2, \cdots, s_k)$ where each $s_j : W \to \mathbb{R}$. Since

$$\begin{aligned} \nabla F(\boldsymbol{y}) &= \mathrm{d}f(\boldsymbol{s}(\boldsymbol{y}), \boldsymbol{y})\, \mathrm{d}(\boldsymbol{s}(\boldsymbol{y}), \boldsymbol{y}) \\ &= \begin{bmatrix} \partial_1 f(\boldsymbol{s}(\boldsymbol{y}), \boldsymbol{y}) & \cdots & \partial_n f(\boldsymbol{s}(\boldsymbol{y}), \boldsymbol{y}) \end{bmatrix} \begin{bmatrix} \partial_1 s_1(\boldsymbol{y}) & \partial_2 s_1(\boldsymbol{y}) & \cdots & \partial_{n-k} s_1(\boldsymbol{y}) \\ \partial_1 s_2(\boldsymbol{y}) & \partial_2 s_2(\boldsymbol{y}) & \cdots & \partial_{n-k} s_2(\boldsymbol{y}) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 s_k(\boldsymbol{y}) & \partial_2 s_k(\boldsymbol{y}) & \cdots & \partial_{n-k} s_k(\boldsymbol{y}) \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \end{aligned}$$

$\nabla F(\boldsymbol{y}_0) = \boldsymbol{0}$ implies

$$\partial_{k+j} f(\boldsymbol{z}_0) + \sum_{i=1}^k \partial_i f(\boldsymbol{z}_0) \partial_j s_i(\boldsymbol{y}_0) = 0$$

for each $j \in [n-k]$. Similarly, $\nabla G_m(\boldsymbol{y}_0) = \boldsymbol{0}$ for each $m \in [k]$ implies that

$$-\lambda_m \left[\partial_{k+j} g_m(\boldsymbol{z}_0) + \sum_{i=1}^k \partial_i g_m(\boldsymbol{z}_0) \partial_j s_i(\boldsymbol{y}_0) \right] = 0$$

for each $j \in [n-k]$ and $m \in [k]$.

Adding the $k+1$ equations together for each $j \in [n-k]$,

$$0 = \left[\partial_{k+j} f(\boldsymbol{z}_0) - \sum_{m=1}^k \lambda_m \partial_{k+j} g_m(\boldsymbol{z}_0) \right] + \sum_{i=1}^k \left[\partial_i f(\boldsymbol{z}_0) - \sum_{m=1}^k \lambda_m \partial_i g_m(\boldsymbol{z}_0) \right] \partial_j s_i(\boldsymbol{y}_0).$$

By the definition of $\lambda_1, \cdots, \lambda_k$, we are left with only

$$\partial_j f(\boldsymbol{z}_0) = \sum_{m=1}^{k} \lambda_m \partial_j g_m(\boldsymbol{z}_0)$$

for each $j \in \{k+1, \cdots, n\}$. For $j \in [k]$, the same equation holds by the definition of $\lambda_1, \cdots, \lambda_k$. Hence, we have $\nabla f(\boldsymbol{z}_0) = \sum_{m=1}^{k} \lambda_m \nabla g_m(\boldsymbol{z}_0)$. □

Chapter 3

Series of Vectors

3.1 Preliminaries

Definition 3.1.1: Normed Vector Space

Let V be a (real/complex) vector space equipped with a norm $||\cdot||$, i.e., the space $(V, ||\cdot||)$ satisfies the following properties.

(i) $0 \in V$

(ii) $||\boldsymbol{x}|| \geq 0$ for all $x \in V$ and $||\boldsymbol{x}|| = 0$ iff $\boldsymbol{x} = \boldsymbol{0}$. (*positive definiteness*)

(iii) $||\beta \boldsymbol{x}|| = |\beta| \cdot ||\boldsymbol{x}||$ for all $\boldsymbol{x} \in V$ and $\beta \in \mathbb{R}$. (*absolute homogeneity*)

(iv) $||\boldsymbol{x}_1 + \boldsymbol{x}_2|| \leq ||\boldsymbol{x}_1|| + ||\boldsymbol{x}_2||$ for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in V$. (*triangle inequality*)

Note:-

Note that $(V, ||\cdot||)$ is naturally a metric space with the metric function $d(\boldsymbol{x}_1, \boldsymbol{x}_2) = ||\boldsymbol{x}_1 - \boldsymbol{x}_2||$.

Definition 3.1.2: Banach Space

A normed vector space $(V, ||\cdot||)$ is called a *Banach space* if, for every Cauchy sequence $\{x_j\}_{j\in\mathbb{N}}$, there exists a unique $\boldsymbol{x}_* \in V$ such that $\lim_{n\to\infty} ||\boldsymbol{x}_n - \boldsymbol{x}_*|| = 0$.

Example 3.1.1

Let A be a compact subset of $\mathbb{R}^n$. $(V, ||\cdot||)$ where $V = \{f : A \to \mathbb{R} \mid f \text{ is continuous}\}$ and $||f|| = \sup_{x\in A} |f(x)|$ forms a Banach space.

Note:-

A Banach space is a normed vector space whose naturally induced metric space is complete.

Definition 3.1.3: Series

Let $(V, \|\cdot\|)$ be a normed vector space. Given a sequence $\{x_j\}_{j\in\mathbb{N}} \subseteq V$, define $S_k \triangleq \sum_{j=1}^{k} x_j$ for each $k \in \mathbb{N}$. Then, each S_k is called a *partial sum* of $\{x_j\}$. If $\{S_k\}_{k\in\mathbb{N}}$ converges to S_* with respect to $\|\cdot\|$, then we write

$$S_* = \sum_{j=1}^{\infty} x_j.$$

If the limit S_* exists, we symbolically say that "$\sum_{j=1}^{\infty} x_j$ converges."

Lemma 3.1.1

Let $(V, \|\cdot\|)$ be a normed vector space. Let $\{x_j\}_{j\in\mathbb{N}} \subseteq V$ be a sequence. If a series $\sum_{j=1}^{\infty} x_j$ converges, then $\lim_{k\to\infty} \|\boldsymbol{x}_k\| = 0$.

Proof. $\{S_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence. Hence, $\lim_{k\to\infty} \|\boldsymbol{x}_k\| = \lim_{k\to\infty} \|S_{k+1} - S_k\| = 0$. □

Lemma 3.1.2

Let $(V, \|\cdot\|)$ be a Banach space. Let $\{x_j\}_{j\in\mathbb{N}} \subseteq V$ be a sequence. A series $\sum_{j=1}^{\infty} x_j$ converges if and only if $\{S_k\}_{k\in\mathbb{N}}$ is Cauchy.

Proof. The definition of Banach spaces. □

3.2 Finite Dimensional Banach Spaces

Theorem 3.2.1 Comparison Test

Given two real sequence $\{a_j\}$ and $\{b_j\}$, suppose $0 \le a_j \le b_j$ for all $j \ge k_0$ where $k_0 \in \mathbb{N}$ is a fixed constant. Then, if $\sum_{j=1}^{\infty} b_j$ converges, then $\sum_{j=1}^{\infty} a_j$ converges.

Proof. Let $S_k = \sum_{j=k_0}^{k} a_j$ and $T_k = \sum_{j=k_0}^{\infty} b_j$. Then, $0 \le S_n - S_m = \sum_{j=m+1}^{n} a_j \le \sum_{j=m+1}^{n} b_j = T_n - T_m$ whenever $n \ge m \ge k_0$. As $\{T_k\}_{k\in\mathbb{N}}$ is Cauchy, $\{S_k\}_{k\in\mathbb{N}}$ is Cauchy as well. As $(\mathbb{R}, \|\cdot\|)$ is a Banach space, $\sum a_j$ converges. □

Theorem 3.2.2 Absolute Convergence Test

Let $(V, \|\cdot\|)$ be a Banach space. Let $\{\boldsymbol{x}_j\}_{j\in\mathbb{N}} \subseteq V$ be a sequence. If $\sum_{j=1}^{\infty} \|\boldsymbol{x}_j\|$ converges (in $\mathbb{R}$), then $\sum_{j=1}^{\infty} \boldsymbol{x}_j$ converges.

Proof. Let $S_k = \sum_{j=1}^{k} \boldsymbol{x}_j \in V$ and $T_k = \sum_{j=1}^{k} \|\boldsymbol{x}_j\| \in \mathbb{R}$. Then, $\|S_n - S_m\| = \left\|\sum_{j=m+1}^{n} \boldsymbol{x}_j\right\| \le \sum_{j=m+1}^{n} \|\boldsymbol{x}_j\| = T_n - T_m$ whenever $n \ge m$. As $\{T_k\}$ is Cauchy, $\{S_k\}$ is Cauchy as well. Hence, $\sum \boldsymbol{x}_j$ converges. □

Theorem 3.2.3 Summation by Parts

Let $\{a_j\}$ and $\{b_j\}$ be two real sequences. If $\sum a_j$ converges and $\{b_j\}$ is monotonic and convergent, then $\sum_{j=1}^{\infty} a_j b_j$ converges.

Proof. Let $S_k = \sum_{j=1}^{k} a_j b_j \in V$ and $A_k = \sum_{j=1}^{k} a_j \in \mathbb{R}$. ($A_0 = 0$.) Then, $S_k = \sum_{j=1}^{k}(A_j - A_{j-1})b_j = \sum_{j=1}^{k} A_j b_j - \sum_{j=0}^{k} A_0 b_{j+1} + A_k b_{k+1} = A_k b_{k+1} - \sum_{j=1}^{k} A_j (b_{j+1} - b_j)$.

Let $T_k = \sum_{j=1}^{k} |A_j(b_{j+1} - b_j)|$. Then, whenever $n < m$, we have

$$0 \le T_m - T_n \le M \sum_{j=n+1}^{m} |b_{j+1} - b_j| = M|b_{m+1} - b_{n+1}| \to 0,$$

$\{T_k\}$ is Cauchy, and thus converges; $\{S_k\}$ converges as well. □

3.3 Conditional Convergence

Definition 3.3.1: Conditional Convergence

Given a real sequence $\{a_j\}_{j\in\mathbb{N}} \subseteq \mathbb{R}$, if $\sum a_j$ converges, and if $\sum |a_j|$ does not converge, then we say that $\sum a_j$ *converges conditionally*.

Theorem 3.3.1 Alternating Series Test

Let $\{a_j\}_{j\in\mathbb{N}} \subseteq \mathbb{R}$ be a real sequence. If $a_j \ge 0$ for all $j \in \mathbb{N}$, and if $\lim_{j\to\infty} a_j = 0$, then $\sum(-1)^j a_j$ converges.

Proof. MAS101. □

Example 3.3.1

$\sum(-1)^j/j$ conditionally converges.

Note:-

Given, a real sequence $\{a_j\}$, we shall use the following definition for now.

For $j \in \mathbb{N}$, define

$$a_j^+ \triangleq \frac{|a_j| + a_j}{2} = \begin{cases} a_j & \text{if } a_j \ge 0 \\ 0 & \text{if } a_j < 0 \end{cases} \quad \text{and} \quad a_j^- \triangleq \frac{|a_j| - a_j}{2} = \begin{cases} 0 & \text{if } a_j \ge 0 \\ -a_j & \text{if } a_j < 0 \end{cases}.$$

Then, $a_j^+, a_j^- \ge 0$, $|a_j| = a_j^+ + a_j^-$, and $a_j = a_j^+ - a_j^-$.

Lemma 3.3.1

Let $\{a_j\}_{j\in\mathbb{N}}$ be a real sequence.

(i) If $\sum a_j$ converges absolutely, then both $\sum a_j^+$ and $\sum a_j^-$ converge. Moreover, $\sum a_j = \sum a_j^+ - \sum a_j^-$.

(ii) If $\sum a_j$ converges conditionally, then both $\sum a_j^+$ and $\sum a_j^-$ diverge.

Proof.

(i) By the definition of a_j^+ and a_j^-.

(ii) If one of $\sum a_j^+$ or $\sum a_j^-$ converges, since $a_j = a_j^+ - a_j^-$, the other converges as well. If they both converge, as $|a_j| = a_j^+ + a_j^-$, $\sum a_j$ converges absolutely.

□

Definition 3.3.2: Rearrangement of Series

Let $\phi : \mathbb{N} \to \mathbb{N}$ be bijective. Given a sequence $\{a_j\}_{j\in\mathbb{N}}$, the series $\sum a_{\phi(j)}$ is called a *rearrangement* of $\sum a_j$.

Theorem 3.3.2 Riemann's Rearrangement Theorem

Let $\{a_j\}_{j\in\mathbb{N}}$ be a conditionally convergent real sequence. Then, for any given $-\infty \leq \alpha \leq \beta \leq \infty$ ($\pm\infty$ is allowed for α and β), there exists a rearrangement $\phi : \mathbb{N} \to \mathbb{N}$ such that $\liminf_{k\to\infty} \sum_{j=1}^{k} a_{\phi(j)} = \alpha$ and $\limsup_{k\to\infty} \sum_{j=1}^{k} a_{\phi(j)} = \beta$.

Proof. Let $\{P_j\}_{j\in\mathbb{N}}$ and $\{Q_j\}_{j\in\mathbb{N}}$ be nonnegative terms and absolute value of negative terms of $\{a_j\}_{j\in\mathbb{N}}$. Then, since they differ from $\{a_j^+\}$ and $\{a_j^-\}$ by zero terms, they are also divergent by Lemma 3.3.1.

Let $\{\alpha_\ell\}_{\ell\in\mathbb{N}}$ and $\{\beta_\ell\}_{\ell\in\mathbb{N}}$ be real sequences such that $\lim_{\ell\to\infty} \alpha_\ell = \alpha$ and $\lim_{\ell\to\infty} \beta_\ell = \beta$. Let $k_1, m_1 \in \mathbb{N}$ be the smallest integers such that

- $S_1 \triangleq P_1 + \cdots + P_{k_1} > \beta_1$ and
- $T_1 \triangleq S_1 - (Q_1 + \cdots + Q_{m_1}) < \alpha_1$.

Inductively, define $\{k_\ell\}_{\ell\in\mathbb{N}}$ and $\{m_\ell\}_{\ell\in\mathbb{N}}$ by

- $k_{\ell+1} \triangleq \min\left\{ k \in \mathbb{N}_{>k_\ell} \;\middle|\; T_\ell + \sum_{j=k_\ell+1}^{k} P_j > \beta_{\ell+1} \right\}$
- $S_{\ell+1} \triangleq T_\ell + \sum_{j=k_\ell+1}^{k_{\ell+1}} P_j$
- $m_{\ell+1} \triangleq \min\left\{ m \in \mathbb{N}_{>m_\ell} \;\middle|\; S_{\ell+1} - \sum_{j=m_\ell+1}^{m} Q_j < \alpha_{\ell+1} \right\}$
- $T_{\ell+1} \triangleq S_{\ell+1} - \sum_{j=m_\ell+1}^{m_{\ell+1}} Q_j$

for each $\ell \in \mathbb{N}$. As $k_\ell \to \infty$ and $m_\ell \to \infty$ as $\ell \to \infty$, this construction gives the natural rearrangement $\phi : \mathbb{N} \to \mathbb{N}$.

By the construction, we have $|S_\ell - \beta_\ell| \leq P_{k_\ell}$ and $|T_\ell - \alpha_\ell| \leq Q_{m_\ell}$ for each $\ell \in \mathbb{N}$. As $P_j, Q_j \to 0$ as $j \to \infty$, we have $S_\ell \to \beta$ and $T_\ell \to \alpha$ as $\ell \to \infty$; α and β are cluster points of $\left\{ \sum_{j=1}^{k} a_{\phi(j)} \right\}_{k\in\mathbb{N}}$ (as long as they are finite).

Moreover, for every sufficiently large $n \in \mathbb{N}$, we have $k_\ell + m_\ell \leq n < k_{\ell+1} + m_{\ell+1}$ for some $\ell \in \mathbb{N}$, and thus $\min\{T_\ell, T_{\ell+1}\} \leq \sum_{j=1}^{n} a_{\phi(j)} \leq S_{\ell+1}$. This, or some more rigorous explanation using arbitrary $\varepsilon \in \mathbb{R}_+$, implies that there do not exist cluster points smaller than α or greater than β. □

3.4 The Cauchy Product

Definition 3.4.1: Cauchy Product

Given two real sequences $\{a_j\}_{j=0}^{\infty}$ and $\{b_j\}_{j=0}^{\infty}$, define

$$C_k \triangleq \sum_{j=0}^{k} a_j b_{k-j}.$$

The series $\sum_{k=1}^{\infty} C_k$ is called the *Cauchy product* of $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} b_j$.

Theorem 3.4.1

Let $\{a_j\}_{j=0}^{\infty}$ and $\{b_j\}_{j=0}^{\infty}$ be two real sequences. Let $\sum_{k=0}^{\infty} C_k$ be the Cauchy product of them.

(i) If $\sum a_j$ converges absolutely, and if $\sum b_j$ converges, then $\sum C_k$ converges to $\left(\sum a_j\right)\left(\sum b_j\right)$.

(ii) If both $\sum a_j$ and $\sum b_j$ converge absolutely, $\sum C_k$ converges absolutely as well.

Proof. (ii) directly follows from the inequality $\sum_{k=0}^{n} |C_k| \le \left(\sum_{j=0}^{n} |a_j|\right)\left(\sum_{j=0}^{n} |b_j|\right)$ as long as (i) is proven.

Let $S_n \triangleq \sum_{k=0}^{n} C_k$, $A_n \triangleq \sum_{j=0}^{n} a_j$, and $B_n \triangleq \sum_{j=0}^{n} b_j$. Let $B \triangleq \lim_{n\to\infty} B_n$ and $\mu_n \triangleq B_n - B$. Then,

$$\begin{aligned} S_n &= \sum_{k=0}^{n} C_k = \sum_{k=0}^{n}\sum_{j=0}^{k} b_{k-j} = \sum_{j=0}^{n} a_j \sum_{k=j}^{n} b_{k-j} \\ &= \sum_{j=0}^{n} a_j B_{n-j} = \sum_{j=0}^{n} a_j (B + \mu_{n-j}) = B\sum_{j=0}^{n} a_j + \sum_{j=0}^{n} a_j \mu_{n-j}. \end{aligned}$$

Claim. $\lim_{n\to\infty} \sum_{j=0}^{n} a_j \mu_{n-j} = 0$.

Take any $\varepsilon \in \mathbb{R}_+$ so there exists $N \in \mathbb{N}$ such that

- $|\mu_n| < \varepsilon$ for all $n \ge N$ (by $\mu_n \to 0$) and
- $\sum_{j=n+1}^{m} |a_j| < \varepsilon$ for all $m > n \ge N$ (by $\sum_{j=0}^{k} |a_j|$ being Cauchy).

As μ_n converges, there exists $\mu^* \triangleq \sup_{n\in\mathbb{N}} |\mu_n|$. Let $K_n \triangleq \sum_{j=0}^{n} a_j \mu_{n-j}$. Whenever $n > 2N$,

$$\begin{aligned} |K_n| &\le \sum_{j=0}^{n} |a_j| \cdot |\mu_{n-j}| = \sum_{j=0}^{N-1} |a_j| \cdot |\mu_{n-j}| + \sum_{j=N}^{n} |a_j| \cdot |\mu_{n-j}| \\ &\le \varepsilon \sum_{j=0}^{N-1} |a_j| + \mu^* \sum_{j=N}^{n} |a_j| \le \varepsilon \left[\sum |a_j| + \mu^*\right]. \end{aligned}$$

Hence, $\lim_{n\to\infty} K_n = 0$; thus $\lim_{n\to\infty} S_n = \left(\sum a_j\right)\left(\sum b_j\right)$. □

3.5 Series on Infinite Dimensional Banach Spaces

Definition 3.5.1: Uniform Convergence of Series

Fix a domain $\Omega \subseteq \mathbb{R}^n$. Given a sequence $\{f_j : \Omega \to \mathbb{R}\}_{j\in\mathbb{N}}$, define $F_n : \Omega \to \mathbb{R}$ by

$$F_n(x) := \sum_{j=1}^{n} f_j(x)$$

for each $x \in \Omega$ and $n \in \mathbb{N}$.

(i) If $\lim_{n\to\infty} F_n(x)$ exists for all $x \in \Omega$, then the series $\sum_{j=1}^{\infty} f_j$ is said to *converge pointwise on* Ω.

(ii) Suppose $\sum_{j=1}^{\infty} f_j(x)$ converges pointwise on Ω and let $F(x) \triangleq \lim_{n\to\infty} F_n(x)$. The series $\sum_{j=1}^{\infty} f_j$ is said to *converge uniformly on* Ω if $\{F_n\}_{n=1}^{\infty}$ uniformly converges to F on Ω.

Theorem 3.5.1

If $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^\Omega$ is a sequence of continuous functions and converges uniformly, then $\lim_{n\to\infty} f_n$ is continuous as well.

Proof. MAS241. □

Definition 3.5.2: Uniform Cauchy

A sequence of function $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^\Omega$ is said to be *uniformly Cauchy on* Ω if

$$\forall \varepsilon \in \mathbb{R}_+,\ \exists N_* \in \mathbb{N},\ \forall n,m \geq N_*,\ \forall x \in \Omega,\ |f_n(x) - f_m(x)| < \varepsilon.$$

Lemma 3.5.1

A sequence of function $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^\Omega$ uniformly converges on Ω if and only if $\{f_n\}_{n\in\mathbb{N}}$ is uniformly Cauchy on Ω.

Proof. ($\Rightarrow$) Let $f(x) = \lim_{n\to\infty} f_n(x)$. Take any $\varepsilon \in \mathbb{R}_+$. Then, there exists $N_* \in \mathbb{N}$ such that, if $n \geq N_*$, then $|f_n(x) - f(x)| < \varepsilon/2$ for all $x \in \Omega$. Consequently, whenever $n, m \geq N_*$, $|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \varepsilon$.

($\Leftarrow$) For each $x \in \mathbb{R}$, the sequence $\{f_n(x)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is Cauchy. As $(\mathbb{R}, |\cdot|)$ is a Banach space, there uniquely exists the limit $f \triangleq \lim_{n\to\infty} f_n$. Take any $\varepsilon \in \mathbb{R}_+$. Then, there exists $N_* \in \mathbb{N}$ such that $|f_m(x) - f_n(x)| < \varepsilon/2$ for all $n, m \geq N_*$ and $x \in \Omega$. From this, we get $f_n(x) - \varepsilon/2 \leq \lim_{m\to\infty} f_m(x) = f(x) \leq f_n(x) + \varepsilon/2$. Hence, $|f_n(x) - f(x)| \leq \varepsilon/2 < \varepsilon$ holds for all $n \geq N_*$ and $x \in \Omega$.

□

Note:-

Lemma 3.5.1 holds for arbitrary sequence of functions from Ω to any Banach space.

Lemma 3.5.2

Let $\{f_j\}_{j\in\mathbb{N}} \subseteq \mathbb{R}^\Omega$ be a series of continuous functions. If $\sum_{j=1}^\infty f_j$ converges uniformly on Ω, then $\sum_{j=1}^\infty f_j$ is continuous on Ω.

Proof. Lemma 3.5.1. □

Chapter 4

Analysis for Series Functions

4.1 Calculus of Series Functions

Theorem 4.1.1

Given a sequence $\{f_j\}_{j\in\mathbb{N}} \subseteq \mathbb{R}^{(a,b)}$ of differentiable functions, suppose the following.

(i) $\{f_j(x_0)\}_{j\in\mathbb{N}} \subseteq \mathbb{R}$ converges for some $x_0 \in (a,b)$.

(ii) $\{f_j'\}_{j\in\mathbb{N}} \subseteq \mathbb{R}^{(a,b)}$ uniformly converges on (a,b).

Then, $f_j \rightrightarrows f$ for some $f : (a,b) \to \mathbb{R}$ on (a,b). Furthermore, f is differentiable on (a,b) and $\forall x \in (a,b)$, $f'(x) = \lim_{j\to\infty} f_j'(x)$.

Proof. We shall first show the uniform convergence of $\{f_j\}$. Take any $\varepsilon \in \mathbb{R}_+$. Then, there exists $N \in \mathbb{N}$ such that, for all $j,k \geq N$,

$$\big(|f_j(x_0) - f_k(x_0)| < \varepsilon/2\big) \wedge \big(\forall x \in (a,b),\ |f_j'(x) - f_k'(x)| < \varepsilon/2(b-a)\big).$$

By MVT, for all $x, \tilde{x} \in (a,b)$ with $x \neq \tilde{x}$, there exists $x_* \in (a,b)$ such that

$$(f_j - f_k)(x) - (f_j - f_k)(\tilde{x}) = (f_j - f_k)'(x_*) \cdot (x - \tilde{x})$$

Hence, $|(f_j - f_k)(x) - (f_j - f_k)(\tilde{x})| < \varepsilon/2$. Therefore, $|(f_j - f_k)(x)| < \varepsilon$ by triangle inequality obtained by setting $\tilde{x} = x_0$. This directly implies that $\{f_j\}$ is uniformly Cauchy and thus uniformly converges by Lemma 3.5.1. ✓

Let $f_j \to f$. Fixing $x \in (a,b)$, define

$$\psi_j(t) \triangleq \frac{f_j(t) - f_j(x)}{t - x} \quad \text{and} \quad \psi(t) \triangleq \frac{f(t) - f(x)}{t - x}$$

for $t \in (a,b)$ and $t \neq x$. Now, we claim that $\{\psi_j\}_{j\in\mathbb{N}}$ is uniformly Cauchy. Take any $\varepsilon \in \mathbb{R}_+$. Then, for $j,k \geq N$,

$$|\psi_j(t) - \psi_k(t)| = \left|\frac{(f_j - f_k)(t) - (f_j - f_k)(x)}{t - x}\right| < \frac{\varepsilon}{2(b-a)}.$$

Hence, $\{\psi_j\}$ uniformly converges by Lemma 3.5.1, and $\psi_j \to \psi$ as $f_j \to f$.

Let $A_j \triangleq \lim_{t\to x} \psi_j(t) = f_j'(x)$. By the supposition (ii), we have convergence of $\{A_j\}_{j\in\mathbb{N}}$. Now, we claim that $\lim_{t\to x} \psi(t) = \lim_{j\to\infty} A_j$. Let $A_j \to A$. Take any $\varepsilon \in \mathbb{R}_+$. There exists $N' \in \mathbb{N}$ such that, if $j \geq N'$, we have $|\psi(t) - \psi_j(t)| < \varepsilon/3$ for all $t \in (a,b)\backslash\{x\}$ ana $|A_j - A| < \varepsilon/3$.

In addition, from the definition of A_j, there exists $\delta \in \mathbb{R}_+$ such that, whenever $0 < |t-x| < \delta$, we have $|\psi_{N'}(t) - A_{N'}| < \varepsilon/3$. Now, we have

$$|\psi(t) - A| \le |\psi(t) - \psi_{N'}(t)| + |\psi_{N'}(t) - A_{N'}| + |A_{N'} - A| < \varepsilon$$

for $0 < |t-x| < \delta$. Hence, $f'(x) = \lim_{t\to x} \psi(t) = \lim_{j\to\infty} f_j'(x)$. □

Corollary 4.1.1 Term-by-Term Differentiation

Given a sequence $\{f_j\}_{j\in\mathbb{N}} \subseteq \mathbb{R}^{(a,b)}$ of differentiable functions, let $F_n = \sum_{j=1}^n f_j$. Suppose the following.

(i) $\{F_n(x_0)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ converges for some $x_0 \in (a,b)$.

(ii) $\{F_n'\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{(a,b)}$ uniformly converges on (a,b).

Then, $\{F_n\}$ converges uniformly to a function $F : (a,b) \to \mathbb{R}$ on (a,b). Furthermore, F is differentiable on (a,b) and $\forall x \in (a,b)$, $F'(x) = \sum_{j=1}^\infty f_j'(x)$.

Example 4.1.1

Let $f_j(x) = \sin(x/j^2)$ for $-1 < x < 1$ and $F_n = \sum_{j=1}^n f_j$.

For $x_0 = 0$, the sequence $\{F_n(x_0)\}_{n\in\mathbb{N}}$ converges (to zero). Now, we have $F_n'(x) = \sum_{j=1}^n \cos(x/j^2)/j^2$. Then, for $n, m \in \mathbb{N}$ with $m \ge n$, $|F_m'(x) - F_n'(x)| \le \sum_{j=n+1}^m 1/j^2 \to 0$ as $n, m \to \infty$. Hence, $\{F_n'\}$ is uniformly Cauchy; and thus it converges uniformly by Lemma 3.5.1. Hence, Corollary 4.1.1 guarantees the uniform convergence and differentiability of $\sum_{j=1}^\infty f_j$.

Theorem 4.1.2

Given a sequence $\{f_j\}_{j\in\mathbb{N}} \subseteq \mathbb{R}^{(a,b)}$ of functions Riemann integrable on (a,b), if $f_j \rightrightarrows f$ on (a,b), then f is Riemann integrable on (a,b). Furthermore, $\int_a^b f(x)\,\mathrm{d}x = \lim_{j\to\infty} \int_a^b f_j(x)\,\mathrm{d}x$.

Proof. □

Corollary 4.1.2 Term-by-Term Integration

Given a sequence $\{f_j\}_{j\in\mathbb{N}} \subseteq \mathbb{R}^{(a,b)}$ of functions Riemann integrable on (a,b), suppose $\sum f_j \rightrightarrows F$ for some $F : (a,b) \to \mathbb{R}$. Then, $\int_a^b F(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_a^b \sum_{j=1}^n f_j(x)\,\mathrm{d}x$.

Theorem 4.1.3

Given a power series $\sum_{j=0}^\infty c_j x^j$, let

$$\alpha \triangleq \limsup_{n\to\infty} \sqrt[n]{|c_n|}, \qquad R \triangleq \frac{1}{\alpha}.$$

(If $\alpha = 0$, put $R = \infty$; if $\alpha = \infty$, put $R = 0$.) Then, $\sum c_j x^j$ converges if $|x| < R$, and diverges if $|x| > R$.

Proof. We have

$$\limsup_{n\to\infty} \sqrt[n]{|c_n x^n|} = \alpha|x|,$$

therefore the result follows from the root test. □

Theorem 4.1.4

Given a power series $P(x) = \sum_{j=0}^{\infty} c_j x^j$, let R be the radius of convergence. Then, for any $\varepsilon \in (0, R)$, $P(x)$ uniformly converges on $[-R+\varepsilon, R-\varepsilon]$.

Note:-

TODO: write proofs for

- Radius of convergence of $P'(x)$ equals the radius of convergence of $P(x)$.
- For all $|x - x_0| < R$, we have $P^{(k)}(x) = \sum_{j=k}^{\infty} j(j-1)\cdots(j-k+1)(x-x_0)^{j-k}$.

Theorem 4.1.5 Taylor's Theorem

Suppose a function $f(x)$ is represented as a power series $f(x) = \sum_{j=0}^{\infty} c_j x^j$ and that the radius of convergence is $R \in [0, \infty]$. Then, for any $x \in (-R, R)$,

$$|x - a| < R - |a| \implies f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(a)}{j!}(x-a)^j.$$

Proof. Fix $a \in (-R, R)$. Suppose that $f(x) = \sum_{j=0}^{\infty} \mu_j (x-a)^j$. By corollary, $f^{(k)}(x) = \sum_{j=k}^{\infty} j(j-1)\cdots(j-k+1)\mu_j(x-x_0)^{j-k}$.

$$\begin{aligned} f(x) &= \sum_{j=0}^{\infty} c_j \big((x-a)+a\big)^j \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{j} c_j \binom{j}{k} a^{j-k}(x-a)^k = \sum_{k=0}^{\infty}\left[\sum_{j=k}^{\infty} c_j \binom{j}{k} a^{j-k}\right](x-a)^k. \end{aligned}$$

The rearrangement is valid when $T(x) = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\left|c_j\binom{j}{k}a^{j-k}(x-a)^k\right| = \sum_{j=0}^{\infty}|c_j|(|x-a|+|a|)^j$ converges, i.e., when $\limsup_{j\to\infty}\left\{|c_j|(|x-a|+|a|)^j\right\}^{1/j} = (|x-a|+|a|)/R < 1$. Hence, $f(x) = \sum_{j=0}^{\infty}\mu_j(x-a)^j$ converges when $|x-a| < R - |a|$. □

Note:-

Theorem 4.1.5 implies that every series function is C^∞ and analytic.

Note:-

We do not have a reliable method to determine the convergence at the boundary points, we have at least a theorem for the situation in which the convergence is given.

Theorem 4.1.6

Let $P(x) = \sum_{j=0}^{\infty} c_j (x-x_0)^j$ be a power series and let $0 < R < \infty$ be its radius of convergence. If $P(x)$ converges at $x = x_0 + R$, then, $P(x)$ uniformly converges on $[x_0, x_0 + R]$.

Proof. For convenience, rescale $P(x)$ by setting $Q\left(\frac{x-x_0}{R}\right) = P(x)$, so $Q(z) = \sum_{j=0}^{\infty} R^j c_j z^j$, and the radius of convergence of Q is 1 and $Q(z)$ converges at $z = 1$. Hence, we are left to prove the uniform convergence of $Q(z)$ on $[0,1]$.

Let $\tilde{c}_j = R^j c_j$ so $Q(z) = \sum_{j=0}^{\infty} \tilde{c}_j z^j$. Let $Q_n(z) = \sum_{j=0}^{n} \tilde{c}_j z^j$ and $S_n = Q_n(1) = \sum_{j=0}^{n} \tilde{c}_j$. Take any $\varepsilon \in \mathbb{R}_+$. Then, there exists $N \in \mathbb{N}$ such that $|S_j - S_k| < \varepsilon/3$ for all $j, k \geq N$. For $n, m \in \mathbb{N}$

with $m > n$,

$$\begin{aligned}
Q_m(z) - Q_n(z) &= \left(S_m z^{m+1} - \textstyle\sum_{j=0}^{m} S_j (z^{j+1} - z^j)\right) - \left(S_n z^{n+1} - \textstyle\sum_{j=0}^{n} S_j (z^{j+1} - z^j)\right) \\
&= \textstyle\sum_{j=n+1}^{m} S_j (z^j - z^{j+1}) + (S_m z^{m+1} - S_n z^{n+1}) \\
&= \textstyle\sum_{j=n+1}^{m} S_j (z^j - z^{j+1}) - S_n \textstyle\sum_{j=n+1}^{m} (z^j - z^{j+1}) + (S_m - S_n) z^{m+1} \\
&= \textstyle\sum_{j=n+1}^{m} (S_j - S_n)(z^j - z^{j+1}) + (S_m - S_n) z^{m+1}.
\end{aligned}$$

Hence, for all $m > n \geq N$ and $z \in [0, 1]$,

$$|Q_m(z) - Q_n(z)| \leq \textstyle\sum_{j=n+1}^{m} (\varepsilon/3)(z^j - z^{j+1}) + \varepsilon/3 = (\varepsilon/3)(z^{n+1} - z^{m+1}) + \varepsilon/3 < \varepsilon.$$

Hence, $Q(z)$ uniformly converges on $[0, 1]$ by Lemma 3.5.1. $\square$

Chapter 5

Applications of Improper Integrals

5.1 Functions Defined by Improper Integrals

Example 5.1.1

Fix a constantt $r > 0$. On $\mathbb{R}$, define

$$F(x) \triangleq \int_0^\infty e^{-rt}\frac{\sin xt}{t}\,\mathrm{d}t = \int_0^\infty f(t,x)\,\mathrm{d}t$$

where $f(t,x) = e^{-rt}\frac{\sin xt}{t}$.

(*Is it well-defined?*) We need to check if $\lim_{R\to\infty}\int_0^R f(t,x)\,\mathrm{d}t$ exists for all $x\in\mathbb{R}$. As $f(t,x)$ is continuous with respect to t, we have $F(x) = \lim_{n\to\infty}F_n(x)$ we may only consider the sequence $F_n(x) = \int_0^n f(t,x)\,\mathrm{d}t$. (Proof?) For $m,n\in\mathbb{N}$ for $m>n$,

$$|F_m(x) - F_n(x)| \le \int_n^m \left|e^{rt}\frac{\sin xt}{t}\right|\mathrm{d}t \le |x|\int_n^m e^{rt}\,\mathrm{d}t \to 0$$

as $m,n\to\infty$. Hence, $\{F_n(x)\}_{n\in\mathbb{N}}$ is Cauchy, and thus is convergent for all $x\in\mathbb{R}$.

(*Is it continuous?*)

$$|F(x_1) - F(x_2)| \le \int_0^\infty \frac{e^{-rt}}{t}|\sin x_1 t - \sin x_2 t|\,\mathrm{d}t \le \frac{|x_1 - x_2|}{r}$$

Hence, F is Lipschitz continuous (and thus uniformly continuous).

(*Is it differentiable?*) If we have differentiability and uniform convergence of F_n, by Theorem 4.1.1, we have differentiability of F and $F' = \lim_{n\to\infty}F_n'$.

$$F_n'(x) \stackrel{?}{=} \int_0^n \frac{\partial}{\partial x}f(t,x)\,\mathrm{d}t = \int_0^n e^{-rt}\cos xt\,\mathrm{d}t$$

Assuming this, we have, for all $m>n$, $|F_m'(x) - F_n'(x)| \le \int_n^m e^{-rt}\,\mathrm{d}t \to 0$, hence $\{F_n'\}_{n\in\mathbb{N}}$ is uniformly convergent. Therefore, by Theorem 4.1.1,

$$F'(x) = \lim_{n\to\infty}\left.\frac{-e^{-rt}\cos(xt)/r + xe^{-rt}\sin(xt)/r^2}{1+(x/r)^2}\right|_{t=0}^{n} = \frac{r}{r^2+x^2}.$$

Moreover, $F(0)=0$; hence $F(x) = \arctan(x/r)$.

Note:-

If $g_h(t) = \frac{f(t,x+h)-f(t,x)}{h}$ converges to $\partial_x f(t,x)$ *uniformly* with respect to $t \in [0,n]$, then $F'(x) = \int_0^n \partial_x f(t,x)\,\mathrm{d}t$.

Example 5.1.2

Fix $x \in \mathbb{R}$ and let $G(r) = \int_0^\infty e^{-rt}\frac{\sin xt}{t}\,\mathrm{d}t$ for $r > 0$. Then,

$$\int_0^\infty \frac{\sin xt}{t}\,\mathrm{d}t = G(0) = \lim_{r\to 0^+} \arctan\left(\frac{x}{r}\right) = \begin{cases} \pi/2 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -\pi/2 & \text{if } x < 0 \end{cases}.$$

Example 5.1.3

Now, repeat with $g(t,x) = t^{x-1}e^{-t}$ and $G(x) = \int_0^1 g(t,x)\,\mathrm{d}t$. Hence, define $G_n(x) = \int_{1/n}^n g(t,x)\,\mathrm{d}t$. For $n \in \mathbb{N}$ and $\sigma \in \mathbb{R}_+$, we have

$$\left| G_n(x) - \int_\sigma^1 t^{x-1}e^{-t} \right| \le \left| \int_{1/n}^\sigma t^{x-1}e^{-t}\,\mathrm{d}x \right| = \frac{\sigma^x - (1/n)^x}{x} \to 0$$

as $n \to \infty$ and $\sigma \to 0^+$. Hence, $G(x) = \lim_{n\to\infty} G_n(x)$. $G(x)$ is well-defined for $0 < x < 1$.

$$G_n'(x) \stackrel{?}{=} \int_{1/n}^1 \partial_x g(t,x)\,\mathrm{d}t = \int_{1/n}^1 t^{x-1}\ln t e^{-t}\,\mathrm{d}t$$

as $\partial_x g(t,x)$ is uniformly continuous on $[1/n, 1]$. (The interchange of limit holds since $(g(t,x+h) - g(t,x))/h \rightrightarrows \partial_x g(t,x)$.)

We claim that, for any fixed $k \in \mathbb{N}$ with $k > 2$, $\{G_n'(x)\}_{n\in\mathbb{N}}$ is uniformly Cauchy on $I_k = [2/k, 1)$. If the claim is proven, then Theorem 4.1.1, $G'(x) = \int_0^1 t^{x-1}\ln t e^{-t}\,\mathrm{d}t$ for all $x \in [2/k, 1)$.

Define an auxiliary function $H_k(t) \triangleq kt^{-1/k} - |\ln t|$ for $0 < t < 1$. Then, $H_k'(t) = t^{-1}(1 - 1/t^{1/k}) < 0$. As $H_k(1) = k$, $H_k(t) > 0$. If $x \in [2/k, 1)$, we have $t^{x-1}|\ln t|e^{-t} \le t^{x-1}\cdot kt^{-1/k} = kt^{x-1/k-1} \le kt^{1/k-1}$. Therefore, for all $x \in I_k$,

$$|G_n'(x) - G_m'(x)| \le \int_{1/n}^{1/m} kt^{1/k-1}\,\mathrm{d}t = k^2\left\{(1/m)^{1/k} - (1/n)^{1/k}\right\} \to 0$$

as $m, n \to \infty$. ($\{G_n'(x)\}_{n\in\mathbb{N}}$ is uniformly Cauchy on I_k.)

Definition 5.1.1: Gamma Function

The function $\Gamma : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t$$

is called the *Gamma function*.

Note:-

(*Well-defined?*) For $x > 1$,

$$|t^{x-1}e^{-t}| = t^{x-1} \cdot \frac{1}{\sum_{j=0}^{\infty} t^j/j!} \le t^{x-1} \cdot \frac{1}{t^{\lceil x \rceil + 1}/(\lceil x \rceil + 1)!}.$$

Theorem 5.1.1 Properties of the Gamma Function

Let Γ be the Gamma function.

(i) $\Gamma(x+1) = x\Gamma(x)$ for each $x \in \mathbb{R}_+$.

(ii) $\Gamma(n+1) = n!$ for each $n \in \mathbb{Z}_{\ge 0}$.

(iii) $\ln \Gamma(x)$ is a convex function.

Proof.

(i)

$$\begin{aligned}\Gamma(x+1) &= \lim_{R\to\infty} \int_0^R t^x e^{-t}\,\mathrm{d}t \\ &= \lim_{R\to\infty} \left[-t^x e^{-t} \Big|_{t=0}^{R} + \int_0^R x t^{x-1} e^{-t}\,\mathrm{d}t \right] = x\Gamma(x)\end{aligned}$$

(ii) Corollary of (i).

(iii) Hölder's Inequality says that $\int |fg|\,\mathrm{d}x \le \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q}$ whenever $1/p + 1/q = 1$. Now, take any $x, y > 0$ and $p, q > 1$ such that $1/p + 1/p = 1$.

$$\begin{aligned}\Gamma\left(\frac{x}{p} + \frac{y}{q}\right) &= \int_0^\infty t^{\frac{x}{p}+\frac{y}{q}-\left(\frac{1}{p}+\frac{1}{q}\right)} e^{-t}\,\mathrm{d}t = \int_0^\infty \left(t^{\frac{x-1}{p}} e^{-t/p}\right)\left(t^{\frac{y-1}{q}} e^{-t/q}\right)\mathrm{d}t \\ &\le \left[\int_0^\infty t^{x-1} e^{-t}\,\mathrm{d}t\right]^{1/p} \left[\int_0^\infty t^{y-1} e^{-t}\,\mathrm{d}t\right]^{1/q} = \Gamma(x)^{1/p}\Gamma(y)^{1/q},\end{aligned}$$

Hence $\ln \Gamma(x/p + y/q) \le (1/p)\Gamma(x) + (1/q)\Gamma(y)$.

□

5.2 The Laplace Transform

Definition 5.2.1: Laplace Transform

For a function $f : \mathbb{R}_{\ge 0} \to \mathbb{R}$ and for $s \in \mathbb{R}$, define

$$\mathcal{L}f(s) \triangleq \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = \lim_{\substack{R\to\infty \\ \sigma\to 0^+}} \int_\sigma^R e^{-st} f(t)\,\mathrm{d}t$$

be the *Laplace transform of f evaluated at s*. The operator $\mathcal{L}\colon f \mapsto \mathcal{L}f$ is called the *Laplace transform operator*.

Example 5.2.1

Take $f(t) = 1$ for all $t \in \mathbb{R}_+$. Then,

$$\mathcal{L}f(s) = \int_0^\infty e^{-st}\,\mathrm{d}t = \begin{cases} 1/s & \text{if } s > 0 \\ \text{undefined} & \text{if } s \le 0 \end{cases}.$$

Example 5.2.2

Take $f(t) = e^{ct}$ for all $t \in \mathbb{R}_+$. Then,

$$\mathcal{L}f(s) = \int_0^\infty e^{-st}\,\mathrm{d}t = \begin{cases} 1/(s-c) & \text{if } s > c \\ \text{undefined} & \text{if } s \le c \end{cases}.$$

Example 5.2.3

Take $f(t) = t^x$ for $x > -1$ and $t > 0$. Then, for $s > 0$,

$$\mathcal{L}f(s) = \int_0^\infty e^{-st} t^x\,\mathrm{d}t = \frac{1}{s^{x+1}} \int_0^\infty e^{-u} u^x\,\mathrm{d}u = \frac{\Gamma(x+1)}{s^{x+1}}.$$

$\mathcal{L}f(s)$ is undefined for $s \le 0$.

Notation 5.1: Translation

For $f : \mathbb{R}_{\ge 0} \to \mathbb{R}$ and $c \in \mathbb{R}_+$, we simply define

$$\tilde{f}(t-c) = \begin{cases} f(t-c) & \text{if } t > c \\ 0 & \text{otherwise} \end{cases}.$$

Lemma 5.2.1

Let $f : \mathbb{R}_{\ge 0} \to \mathbb{R}$ be a continuous function. Suppose that $\mathcal{L}f(s)$ is well-defined for $s > r_0$ for some $r_0 \in \mathbb{R}$. Fix some $c \in \mathbb{R}$.

(i) $\mathcal{L}\big(e^{ct} f(t)\big)(s) = \mathcal{L}f(s-c)$ for $s > r_0 + c$.

(ii) $\mathcal{L}\big(\tilde{f}(t-c)\big)(s) = e^{-cs}\mathcal{L}f(s)$ for $s > r_0$.

(iii) For $c > 0$, $\mathcal{L}\big(f(ct)\big)(s) = (1/c)\mathcal{L}f(s/c)$ for $s > r_0$.

Proof. Simple calculation. □

Lemma 5.2.2

Given two functions $f_1, f_2 \in \mathbb{R}_{\ge 0} \to \mathbb{R}$, suppose that $\mathcal{L}f_1(s)$ and $\mathcal{L}f_2(s)$ are well-defined for $s > r_0$ for some $r_0 \in \mathbb{R}$. Then, $\mathcal{L}\big(c_1 f_1 + c_2 f_2\big)(s) = c_1\mathcal{L}f_1(s) + c_2\mathcal{L}f_2(s)$. That is, $\mathcal{L}$ is a linear operator.

Note:-

Suppose that $f : \mathbb{R}_{\ge 0} \to \mathbb{R}$ is k times differentiable and that $\forall t \ge 0$, $|f^{(k)}(t)| \le Ae^{Rt}$

for some $A, R > 0$. Then,

$$|f^{(k-1)}(t)| \leq |f^{(k-1)}(0)| + \int_0^t Ae^{R\tau}\, d\tau.$$

Thus, there exists $\tilde{A} > 0$ such that $|f^{(k-1)}(t)| \leq \tilde{A}e^{Rt}$ for all $t \geq 0$. By induction, we have, for each $j \in \{0, 1, \cdots, k-1\}$, there exists $A_j \in \mathbb{R}_+$ such that $|f^{(j)}(t)| \leq A_j e^{Rt}$ for all $t \geq 0$. Hence, $\mathcal{L}(f^{(j)})(s)$ is well-defined for $s > R$.

Lemma 5.2.3

Suppose that $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is differentiable and that $\forall t \geq 0$, $|f'(t)| \leq Ae^{Rt}$ for some $A, R > 0$. Then, we have $\mathcal{L}(f')(s) = s\mathcal{L}f(s) - f(0)$ for $s > R$.

Proof. Integration by parts. □

Corollary 5.2.1

Suppose that $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is k times differentiable and that $\forall t \geq 0$, $|f^{(k)}(t)| \leq Ae^{Rt}$ for some $A, R > 0$. Then, $\mathcal{L}(f^{(k)})(s) = s^k \mathcal{L}f(s) - \sum_{j=0}^{k-1} s^{k-1-j} f^{(j)}(0)$ for $s > R$.

Proof. Induction using Lemma 5.2.3. □

Example 5.2.4

Solve $y'' - y' - 2y = 0$, $y(0) = 2$, $y'(0) = 3$ for y.

Let $\eta(s) \triangleq \mathcal{L}y(s)$. Applying the Laplace transform to the both sides (without justifying the well-definedness), we get

$$\begin{aligned} 0 &= \mathcal{L}(y'') - \mathcal{L}(y') - 2\mathcal{L}(y) \\ &= s^2\eta - (2s+3) - (s\eta - 2) - 2\eta. \end{aligned}$$

Thus,

$$\mathcal{L}y(s) = \eta(s) = \frac{2s+1}{(s-2)(s+1)} = \frac{5}{3} \cdot \frac{1}{s-2} + \frac{1}{3} \cdot \frac{1}{s+1}$$

and it is well-defined for $s > 2$. From Example 5.2.2,

$$\mathcal{L}y(s) = \mathcal{L}\left(\frac{5}{3}e^{2t} + \frac{1}{3}e^{-t}\right).$$

Now, we shall discuss the *injectivity* of $\mathcal{L}$.

Theorem 5.2.1

Given a continuous function $f \in \mathbb{R}_{\geq 0} \to \mathbb{R}$, suppose that $\mathcal{L}f(s) = 0$ for all $s > R$ for some $R \in \mathbb{R}$. Then, $f = 0$.

Proof. The proof is so sophisticated that it is not discussed in MAS242. :(□

Note:-

Actually, the restrictions on the functions in Theorem 5.2.1 can be relaxed to not requiring continuity.

Definition 5.2.2: Convolution

- Given two functions $\phi, \psi \colon \mathbb{R} \to \mathbb{R}$, we define $\phi * \psi$ by

$$(\phi * \psi)(t) \triangleq \int_{-\infty}^{\infty} \phi(x)\psi(t-x)\,\mathrm{d}x.$$

  $\phi * \psi$ is called the *convolution of* ϕ *and* ψ.
- Given two functions $\Phi, \Psi \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$, we define $\Phi * \Psi \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ by

$$(\Phi * \Psi)(t) \triangleq= \int_{0}^{t} \Phi(x)\Psi(t-x)\,\mathrm{d}x.$$

  $\Phi * \Psi$ is called the *convolution of* Φ *and* Ψ.

Note:-
Convolution is commutative, i.e., $\phi * \psi = \psi * \phi$.

Theorem 5.2.2
Given two continuous functions $f, g \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$, suppose that

$$\exists A, R > 0,\ \forall t \geq 0,\ \max\{\,|\phi(t)|, |\psi(t)|\,\} \leq Ae^{Rt}.$$

Then,

$$\forall s > R,\ \mathcal{L}(\phi * \psi)(s) = \mathcal{L}\phi(s) \cdot \mathcal{L}\psi(s).$$

Heuristic Proof. Extend ϕ, ψ to $\mathbb{R}$ where $\phi(t) = \psi(t) = 0$ for $t < 0$.

$$\begin{aligned}
\mathcal{L}(\phi * \psi)(s) &= \int_0^{\infty} e^{-st}(\phi * \psi)(t)\,\mathrm{d}t \\
&= \int_0^{\infty} e^{-st} \int_0^{t} \phi(x)\psi(t-x)\,\mathrm{d}x\,\mathrm{d}t = \int_0^{\infty} e^{-st} \int_0^{\infty} \phi(x)\psi(t-x)\,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{\kappa_1 \to \infty} \int_0^{\kappa_1} e^{-st} \lim_{\kappa_2 \to \infty} \int_0^{\kappa_2} \phi(x)\psi(t-x)\,\mathrm{d}x\,\mathrm{d}t \\
&\overset{?}{=} \lim_{\kappa_1 \to \infty} \lim_{\kappa_2 \to \infty} \int_0^{\kappa_1} e^{-st} \int_0^{\kappa_2} \phi(x)\psi(t-x)\,\mathrm{d}x\,\mathrm{d}t \\
&= \lim_{\kappa_1 \to \infty} \lim_{\kappa_2 \to \infty} \int_0^{\kappa_2} \int_0^{\kappa_1} e^{-st}\phi(x)\psi(t-x)\,\mathrm{d}t\,\mathrm{d}x \\
&= \lim_{\kappa_1 \to \infty} \lim_{\kappa_2 \to \infty} \int_0^{\kappa_2} e^{-sx}\phi(x) \int_0^{\kappa_1} e^{-s(t-x)}\psi(t-x)\,\mathrm{d}t\,\mathrm{d}x \\
&= \lim_{\kappa_2 \to \infty} \int_0^{\kappa_2} e^{-sx}\phi(x) \lim_{\kappa_1 \to \infty} \int_x^{\kappa_1} e^{-s(t-x)}\psi(t-x)\,\mathrm{d}t\,\mathrm{d}x \\
&= \lim_{\kappa_2 \to \infty} \int_0^{\kappa_2} e^{-sx}\phi(x)\big[\mathcal{L}\psi(s)\big]\,\mathrm{d}x = \mathcal{L}\phi(s) \cdot \mathcal{L}\psi(s)
\end{aligned}$$

□

Example 5.2.5

Solve $y'' - y' - 2y = f(t)$, $y(0) = 2$, $y'(0) = 3$ for y. In the same way as in Example 5.2.4, we have

$$\mathcal{L}y(s) = \mathcal{L}\left(\frac{5}{3}e^{2t} + \frac{1}{3}e^{-t}\right) + \mathcal{L}f(s) \cdot \mathcal{L}\left(\frac{1}{3}e^{2t} + \frac{1}{3}e^{-t}\right).$$

By Theorem 5.2.2, we get

$$y(t) = \frac{5}{3}e^{2t} + \frac{1}{3}e^{-t} + \int_0^t f(x)\left[\frac{1}{3}e^{2(t-x)} - \frac{1}{3}e^{-(t-x)}\right] \mathrm{d}x.$$

5.3 Applications of Laplace Transforms

Note:-

Reference: *Partial Differential Equations*, Walter Strauss

Definition 5.3.1: Laplace Transform

Let $u\colon I \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ where I is an interval on $\mathbb{R}$. We define

$$\mathcal{L}u(x,s) \triangleq \int_0^\infty e^{-st}u(x,t)\,\mathrm{d}t$$

be the *Laplace transform of u*.

Notation 5.2

$$U(x,s) = \mathcal{L}u(x,s)$$

Question 1: 1-D Heat Equation

Solve the following partial differential equation

$$\partial_t u = \partial_{xx} u \quad \text{for } 0 < x < 1 \text{ and } t > 0$$

where

- $u(0,t) = u(1,t) = 1$ for all $t \geq 0$ and
- $u(x,0) = 1 + \sin \pi x$ for all $0 \leq x \leq 1$.

Solution: By Lemma 5.2.3,

$$\mathcal{L}(\partial_{xx}u)(x,s) = \mathcal{L}(\partial_t u)(x,s) = s\mathcal{L}u(x,s) - u(x,0) = sU(x,s) - (1 + \sin \pi x).$$

Also,

$$\mathcal{L}(\partial_{xx}u)(x,s) = \int_0^\infty e^{-st}\partial_{xx}u(x,t)\,\mathrm{d}t \overset{?}{=} \partial_{xx}\int_0^\infty e^{-st}u(x,t)\,\mathrm{d}t = \partial_{xx}U(x,s).$$

From the boundary condition, we have $U(0,s) = U(1,s) = 1/s$. We now notice that we are left with a non-homogeneous second-order ordinary differential equation for $U(x,s)$ with respect to x:

$$U_{xx} - sU = -(1 + \sin \pi x).$$

Let U^h be the homogeneous solution and U^p be the particular solution for the ODE (as usual) so $U = U^h + U^p$ is the general solution. (Assume $s > 0$. Why?)

It is known that $U^h = c_1 \cosh(\sqrt{s}x) + c_2 \sinh(\sqrt{s}x)$ where $c_1, c_2 \in \mathbb{R}$ is the homogeneous solution from the functional analysis. And, from some calculation, we get

$$U^p = \frac{1}{s} + \frac{1}{\pi^2 + s} \sin \pi x.$$

Hence,

$$U(x,s) = c_1 \cosh(\sqrt{s}x) + c_2 \sinh(\sqrt{s}x) + \frac{1}{s} + \frac{1}{\pi^2 + s} \sin \pi x$$

is the general solution. From the boundary condition, we have

$$\begin{aligned} \frac{1}{s} &= U(0,s) = c_1 + \frac{1}{s} \\ \frac{1}{s} &= U(1,s) = c_2 \sinh(\sqrt{s}) + \frac{1}{s}; \end{aligned}$$

hence $c_1 = 0$ and $c_2 = 0$, we get

$$U(x,s) = \frac{1}{s} + \frac{1}{\pi^2 + s} \sin \pi x = \mathcal{L}\left(1 + e^{-\pi^2 t} \sin \pi x\right)(s),$$

i.e., $u(x,t) = 1 + e^{-\pi^2 t} \sin \pi x$. ✓

End.