

# MATH 593 - Tensor Product

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# 1 Tensor Product of Modules

**Definition 1.1** (*R*-balanced Maps). Let  $R$  be a ring, with  $M$  a right  $R$ -module,  $N$  a left  $R$ -module and  $P$  an abelian group. Then the map  $\varphi : M \times N \rightarrow P$  is ***R*-balanced** if the followings are satisfied:

- $\varphi(u, v_1 + v_2) = \varphi(u, v_1) + \varphi(u, v_2)$  for all  $u \in M, v_1, v_2 \in N$ .
- $\varphi(u_1 + u_2, v) = \varphi(u_1, v) + \varphi(u_2, v)$  for all  $u_1, u_2 \in M, v \in N$ .
- $\varphi(ua, v) = \varphi(u, av)$  for all  $a \in R, u \in M, v \in N$ .

**Remark 1.1.** The only difference between  $R$ -balanced maps and  $R$ -linear maps is the third condition: the coefficient in  $R$  could be transferred between different positions, but not out of the expression.

**Definition 1.2** (Tensor Product). A **tensor product** of  $M$  and  $N$  is an abelian group  $M \otimes_R N$  with an  $R$ -balanced map  $\varphi : M \times N \rightarrow M \otimes_R N$  which is universal w.r.t. the property: i.e.  $\forall \psi : M \times N \rightarrow P$  which is  $R$ -balanced, there exists a unique  $f : M \otimes_R N \rightarrow P$  s.t.  $\psi = f \circ \varphi$  ( $\psi$  factors uniquely through  $\varphi$ ), i.e. the following diagram commute:

$$\begin{array}{ccc} M \times N & \xrightarrow{\varphi} & M \otimes_R N \\ & \searrow \psi & \downarrow f \\ & & P \end{array}$$

**Remark 1.2.** If  $\otimes_R$  exists, then it is unique up to a canonical isomorphism.

Suppose that for  $M, N \in {}_R\text{Mod}$ , there exists two tensor products  $T$  and  $T'$ . Denote the canonical map from  $M \times N$  to  $T$  and  $T'$  be  $\varphi$  and  $\varphi'$ , respectively. Then by universal property of tensor product, there exists a unique isomorphism  $f$  and  $f'$  s.t.  $f \circ \varphi = \varphi'$  and  $f' \circ \varphi' = \varphi$ , which gives  $f \circ f' = \text{Id}$ .

**Proposition 1.1.** The tensor product exists.

*Proof.* Proceed to show this via introducing relations on the free group structure. Let  $F := \mathbb{Z}^{M \times N}$  be a free abelian group with basis  $\{e_{(u,v)} \mid (u,v) \in M \times N\}$ . Quotient out the elements that are claimed to be equivalent by the constraint that the canonical map  $\varphi$  should be  $R$ -balanced: consider  $G \subseteq F$  to be generated by the following elements:

- $(e_{u_1+u_2, v} - e_{u_1, v} - e_{u_2, v})$ , for all  $u_1, u_2 \in M, v \in N$ .
- $(e_{u, v_1+v_2} - e_{u, v_1} - e_{u, v_2})$ , for all  $u \in M, v_1, v_2 \in N$ .
- $(e_{ua, v} - e_{u, av})$  for all  $u \in M, v \in N, a \in R$ .

By construction it is clear that the canonical map  $\varphi : M \times N \rightarrow M \otimes_R N$  is  $R$ -balanced, via specifying  $\varphi(u, v) = \overline{e_{u,v}}$ .

It suffices to verify that the construction is compatible with the universal property. Consider the  $R$ -balanced map  $\psi : M \times N \rightarrow P$ , with the group homomorphism  $g : F \rightarrow P$  s.t.  $g(e_{u,v}) = \psi(u, v)$ :

- *Existence.* Applying the universal property of quotient groups, which implies that there exists a unique  $f$  s.t.  $f \circ h = g$  where  $h$  is the induced group homomorphism of the quotient. This is indeed valid, as  $\psi$  is  $R$ -linear, which by construction has kernel  $G$ .

$$\begin{array}{ccccc}
 M \times N & \xrightarrow{\varphi} & F/G & \xleftarrow{h} & F \\
 & \searrow \psi & \downarrow f & \swarrow g & \\
 & & P & & 
 \end{array}$$

- *Uniqueness.* This follows from the result of universal property above; and the fact that  $\varphi$  is surjective.

□

**Remark 1.3.** The construction above, together with the fact that tensor products exist uniquely up to isomorphism, implies that for  $R$ -modules  $M$  and  $N$  with their system of generators,  $(u_i)$  and  $(v_i)$  respectively, for all  $x \in M \otimes_R N$ , there exists  $(d_i) \in \mathbb{Z}$  s.t.

$$x = \sum_{i=1}^n d_i (u_i \otimes_R v_i)$$

where the multiplication by integers is simply adding repetitively the elements to itself.

The tensor products could also behave *functorially*, via composing with the canonical map of tensor product:

Let  $f : M \rightarrow M'$  a morphism of right  $R$ -modules, and  $g : N \rightarrow N'$  a morphism of left  $R$ -modules. Then one could define a map  $\psi : M \times N \rightarrow M' \otimes_R N'$ , where  $(u, v) \mapsto f(u) \otimes_R g(v)$ . The map is  $R$ -balanced since the canonical map of tensor product is  $R$ -balanced. Therefore it is valid to apply the universal property of tensor product, which gives a unique group homomorphism  $f : M \otimes_R N \rightarrow M' \otimes_R N'$ . This is uniquely determined by  $f$  and  $g$ ; and is often denoted as  $f \otimes_R g$ .

**Remark 1.4.** This is also compatible with composition, via applying the universal property twice. Explicitly, for  $f : M \rightarrow M'$ ,  $f' : M' \rightarrow M''$  a morphism of right  $R$ -modules, and  $g : N \rightarrow N'$ ,  $g' : N' \rightarrow N''$  a morphism of left  $R$ -modules, we have

$$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)$$

**Remark 1.5.** In particular the constructions above induces a functor  $M \otimes - : {}_R\mathbf{Mod} \rightarrow \mathbf{Ab}$  for  $M$  a right  $R$ -module, where

$$N \in \mathbf{Ob}({}_R\mathbf{Mod}) \mapsto M \otimes N, \quad f : N \rightarrow N' \mapsto \text{Id}_M \otimes f$$

Similar to the case of Hom Functors, we seek to lift the functor to  ${}_R\mathbf{Mod} \rightarrow {}_R\mathbf{Mod}$ . This requires extra structure on the module of interest. As before, making  $R$  commutative, and restricting  $M$  and  $N$  to be both  $R$ -modules could resolve the issue, but the condition is too strong.

## 2 Bimodule

**Definition 2.1** (Bimodule). Let  $S$  and  $R$  be rings. An **R-S bimodule**  $M$  is given by an abelian group  $M$  that is both a left  $S$ -module and a right  $R$ -module; and module operations is compatible, i.e.

$$(au)b = a(ub) \quad \forall u \in M, a \in S, b \in R$$

**Remark 2.1.** If  $R$  is commutative, then every  $R$ -module is an  $R$ - $R$  bimodule (which is why making  $R$  commutative suffices to ensure the Hom module has an  $R$ -module structure). Specifically,  $R$  is an  $R$ - $R$  bimodule.

**Remark 2.2.** Morphisms between  $S$ - $R$  bimodules inherits from corresponding modules. Compatibility does not interfere with morphisms.

**Proposition 2.1.** Let  $R$  and  $S$  be rings, with  $M$  an  $S$ - $R$  bimodule, and  $N$  a left  $R$ -module. Then there exists a unique left  $S$ -module structure on  $M \otimes N$  s.t.  $\lambda \cdot (u \otimes v) = (\lambda u) \otimes v$ , for all  $\lambda \in S, u \in M, v \in N$ .

*Proof.* Use the universal property, with  $P = M \otimes N$ . Fix  $\lambda \in S$ ; consider  $\varphi : M \times N \rightarrow M \otimes N$  s.t.  $\varphi(u, v) = (\lambda u) \otimes v$ . This map is  $R$ -balanced, as the tensor product on  $R$  is balanced.

Then by universal property there exists a unique group homomorphism  $f_\lambda : M \otimes N \rightarrow M \otimes N, u \otimes v \mapsto (\lambda u) \otimes v$ . This gives the scalar multiplication of  $\lambda$ , which induces an  $S$ -module structure on  $M \otimes N$ .  $\square$

**Proposition 2.2.** The extra structure on the modules gives extra structure on the morphisms in the universal property:

Let  $M$  be a  $S$ - $R$  bimodule,  $N$  a left  $R$ -module, and  $P$  a left  $S$ -module. Let  $\varphi : M \times N \rightarrow M \otimes_R N$  the canonical map of tensor product. Suppose further that the map  $\psi : M \times N \rightarrow P$  is  $S$ -bilinear. Then there exists a unique morphism of  $S$ -modules  $f : M \otimes_R N \rightarrow P$ .

*Proof.* By the universal property of tensor product, such morphism  $f$  exists, and is uniquely specified by  $f(u \otimes v) = \psi(u, v)$ . It suffices to check that this is indeed a morphism of  $S$ -modules, i.e. for all  $a \in S, f((au) \otimes v) = af(u \otimes v)$ . It then suffices to check that for certain (set of) fixed  $u$  and  $v$ , as every element in  $M \otimes N$  is of such form. This is clear as

$$f((au) \otimes v) = \psi(au, v) \stackrel{!}{=} a\psi(u, v) = a \cdot f(u \otimes v)$$

Equality (!) requires that  $\psi$  is  $S$ -bilinear, and  $M$  being a bimodule ensures that this is well-formed under the context of  $S$ -modules.  $\square$

**Remark 2.3.** The proposition above lifts the functor  $M \otimes -$  to  ${}_R\text{Mod} \rightarrow {}_S\text{Mod}$  for all  $S$ - $R$  bimodule  $M$ .

**Remark 2.4.** It may be interesting to consider the following property of bimodules:

1. If  $R$  is commutative, then left or right  $R$ -modules are the same; and in this case  $M \otimes_R N$  is an  $R$ -module.
2. If  $M$  is a  $T$ - $R$  bimodule, and  $N$  is an  $R$ - $S$  bimodule, then  $M \otimes_R N$  is a  $T$ - $S$  bimodule.

For the second remark, it is clear that  $M \otimes_R N$  is both a left  $T$ -module, and a right  $S$ -module, via applying the same proof as in Proposition 2.1. It suffices to prove that they are compatible. This is also clear from the construction in the proposition referred:

$$(a(u \otimes_R v))b = (au \otimes v)b = (au) \otimes (vb) = a(u \otimes (vb)) = a((u \otimes v)b)$$

**Remark 2.5.** Let  $R$  be a ring. Then  $R$  is an  $R$ - $R$  bimodule. Let  $M$  be a left  $R$ -module, which implies that  $R \otimes_R M$  is a left  $R$ -module. Then there exists a functorial isomorphism  $R \otimes_R M \simeq M$  for all  $M \in \text{Ob}({}_R\text{Mod})$ . (This is called functorial as this could be regarded as the property of functor  $R \otimes_R -$ .)

*Proof.* Proof via using the universal property. Consider the morphism of  $R$ -modules  $\alpha : R \times M \rightarrow M$ , where  $\alpha(a, u) = au$  for all  $a \in R, u \in M$ . It is  $R$ -linear, which is by definition  $R$ -balanced. The universal property gives that there exists a unique

$f : R \otimes M \rightarrow M$  s.t.  $f(a \otimes u) = au$ . Designate  $g : M \rightarrow R \otimes M$ ,  $g(u) = 1 \otimes u$  for all  $u \in M$ . This is clearly  $R$ -balanced. This gives an isomorphism as  $g \circ f = \text{Id}_R$ ,  $f \circ g = \text{Id}_{R \otimes M}$ .  $\square$

### 3 Extension of Scalar

Let  $S$  and  $R$  be rings, together with a ring homomorphism  $\varphi : R \rightarrow S$ . Then

1. It is clear that there is a *restriction of scalar* functor:

$$F : {}_S\text{Mod} \rightarrow {}_R\text{Mod}, \quad {}_S M \rightarrow {}_R M \quad \text{where } a \cdot u := \varphi(a) \cdot u \quad (\forall a \in R, u \in {}_S M)$$

2. It is more interesting to consider the *extension of scalar* functor  $G : {}_R\text{Mod} \rightarrow {}_S\text{Mod}$ . Notice that  $\varphi$  gives  $S$  a natural  $R$ -module structure, where  $rs := \varphi(r)s$  for all  $r \in R, s \in S$ . This gives a natural extension of scalar functor  $(S \otimes_R -)$ :

- For  $M \in \text{Ob}({}_R\text{Mod})$ , this gives  $S \otimes_R M$ .
- For  $f : M_1 \rightarrow M_2$  a morphism of  $R$ -modules, this gives  $\text{Id}_S \otimes f$ .

**Example 3.1.** Consider the following examples:

- Let  $\varphi : R \rightarrow R/I$  the canonical quotient map. Then this induces the isomorphism  $G(M) \simeq M/IM$ .

Extension of scalar gives  $G(M) \simeq R/I \otimes M$ . To show that these two left  $R/I$ -modules are isomorphic, it suffices to specify maps between them s.t. the composition gives identity. Consider

$$f : R/I \otimes M \rightarrow M/IM, \quad \bar{r} \otimes u \mapsto \overline{ru}, \quad g : M/IM \rightarrow R/I \otimes M, \quad \bar{u} \mapsto 1_{R/I} \otimes u$$

It is clear that  $f \circ g = \text{Id}_{R/I \otimes M}$ . Notice

$$g \circ f(\bar{r} \otimes u) = g(\overline{ru}) = 1 \otimes \overline{ru} = 1 \otimes \bar{r} \cdot \bar{u} = \bar{r} \otimes \bar{u}$$

since the canonical map of tensor product is  $R$ -balanced.

- Let  $R$  be a ring, and  $S \subseteq R$  a multiplicative system. Let  $\varphi$  be the canonical map  $R \rightarrow S^{-1}R$ ,  $\varphi(a) = \frac{a}{1}$ . Then this induces an isomorphism  $G(M) \simeq S^{-1}M$ .

Apply the similar strategy. It suffices to show that  $G(M) = S^{-1}R \otimes M \simeq S^{-1}M$ . Consider

$$f : S^{-1}R \otimes M \rightarrow S^{-1}M, \quad \frac{r}{s} \otimes u \mapsto \frac{ru}{s}, \quad g : S^{-1}M \rightarrow S^{-1}R \otimes M, \quad \frac{u}{s} \mapsto \frac{1}{s} \otimes u$$

It is clear  $f \circ g(\frac{u}{s}) = f(\frac{1}{s} \otimes u) = \frac{u}{s}$ . For the other direction, check

$$g \circ f(\frac{r}{s} \otimes u) = g(\frac{ru}{s}) = \frac{1}{s} \otimes (ru) = \left(\frac{1}{s}\right) \cdot r \otimes u = \frac{1}{s} \cdot \frac{r}{1} \otimes u = \frac{r}{s} \otimes u$$

**Theorem 3.1** (Universal Property of Extension of Scalars). *Let  $M$  be a left  $R$ -module,  $N$  be a left  $S$ -module, together with a ring homomorphism  $\varphi : R \rightarrow S$ . Let  $f : M \rightarrow N$  a morphism of  $R$ -modules (in the sense of restriction of scalars). Then there exists a unique morphism of  $S$ -modules  $g : S \otimes_R M \rightarrow N$  s.t.  $g(1 \otimes u) = f(u)$  for all  $u \in M$ :*

$$\begin{array}{ccccc}
 M & \xrightarrow{G} & S \otimes_R M & \xleftarrow{\otimes} & S \times M \\
 & \searrow f & \downarrow g & \swarrow \psi & \\
 & & N & & 
 \end{array}$$

*Proof.* Construct using the universal property of tensor product. Since  $f$  is a morphism of  $S$ -modules, it is in particular  $S$ -linear, i.e.

$$f(su) = sf(u) \implies \psi(s, u) = sf(u)$$

There exists a canonical morphism into the setting of universal property, as  $G(su) = \otimes(1, su) = \otimes(s, u)$  for all  $s \in S, u \in M$ ; and  $\psi$  is  $R$ -linear:

$$\psi(sr, u) = (s\varphi(r))f(u) = s(\varphi(r)f(u)) = sf(\varphi(r)u) = \psi(s, ru)$$

$$\psi(ss', u) = ss'f(u) = sf(s'u) = \psi(s, s'u)$$

Then the strengthened version of universal property gives the desired result.  $\square$

**Remark 3.1.** Functorially, for tensor products we have the natural transformation (given the ring homomorphism  $\varphi : R \rightarrow S$ )

$$\text{Hom}_S(S \otimes_R M, N) \rightarrow \text{Hom}_R(M, N), \quad g \mapsto (u \mapsto g(1 \otimes u))$$

The universal property gives that this is actually a bijection. Since  $\text{Hom}_R(M, N)$  is in essence the restriction of scalar, there is a bijection (which is exactly rephrasing the result above)

$$\text{Hom}_S(G(M), N) \simeq \text{Hom}_R(M, F(N))$$

**Definition 3.1** (Adjoint Pair). *Consider functors  $F : \mathcal{C} \rightarrow \mathcal{D}, G : \mathcal{D} \rightarrow \mathcal{C}$ . They form an **adjoint pair**  $(F, G)$  if for all  $a \in \text{Ob}(\mathcal{C})$  and  $b \in \text{Ob}(\mathcal{D})$ , we have a bijection  $\text{Hom}_{\mathcal{C}}(G(b), a) \simeq \text{Hom}_{\mathcal{D}}(b, F(a))$  which is functorial w.r.t. both  $a$  and  $b$ , i.e. there exists a natural transformation from  $\text{Hom}_{\mathcal{C}}(G(-), a)$  to  $\text{Hom}_{\mathcal{D}}(-, F(a))$ .*

**Remark 3.2.** Extension of scalar functor  $G$  and restriction of scalar functor  $F$  form an adjoint pair.

## 4 General Properties of Tensor Product

**Proposition 4.1.** *Tensor product is commutative, i.e. there exists an isomorphism of abelian groups  $M \otimes_R N \simeq N \otimes_R M$ . Similarly, if  $R$  is commutative, then this is an isomorphism of  $R$ -modules.*

*Proof.* Proceed via using the universal property of tensor product. Consider the map  $\varphi : M \times N \rightarrow N \otimes_R M$  given by  $\varphi(v, u) = u \otimes_R v$ . It is clear  $\varphi$  commutes with addition in either field. To show that  $\varphi$  is indeed  $R$ -balanced it suffices to check the third property, which gives

$$\varphi(va, u) = u \otimes_R (va) = u \otimes_R a^{\text{op}}v = ua^{\text{op}} \otimes_R v = (au) \otimes_R v = \varphi(v, au)$$

Similarly there exists an  $R$ -balanced map  $\tilde{\psi} : N \times M \rightarrow M \otimes_R N$  (as  $R^{\text{op}}$  modules) which is given by  $\psi(u, v) = v \otimes u$ . This induces a map  $\psi : N \otimes_{R^{\text{op}}} M \rightarrow M \otimes_R N$ . It is clear that  $\varphi \circ \psi = \text{Id}_{N \otimes_{R^{\text{op}}} M}$ ,  $\psi \circ \varphi = \text{Id}_{M \otimes_R N}$ .  $\square$

**Remark 4.1.** If  $R$  is commutative, then the left  $R$ -modules are the same as right  $R$ -modules, which indicates that  $M \otimes_R N \simeq N \otimes_R M$  (as in the commutative setting the opposite ring is the same as the original ring).

**Proposition 4.2.** *Tensor product is associative, i.e. for  $M$  a right  $R$ -module,  $N$  an  $R$ - $S$  bimodule, and  $P$  a left  $S$ -module, there exists a unique isomorphism  $f : (M \otimes_R N) \otimes_S P \rightarrow M \otimes_R (N \otimes_S P)$  s.t.  $f((u \otimes_R v) \otimes_S w) = u \otimes_R (v \otimes_S w)$ .*

*Proof.* Apply the universal property of tensor product twice. First consider map  $f_z : M \times N \rightarrow M \otimes_R (N \otimes_S P)$  given by  $f_z(x, y) = x \otimes_R (y \otimes_S z)$  for some  $z \in P$ .  $f_z$  is  $R$ -balanced, as for all  $a \in R$ ,

$$f_z(x, ay) = x \otimes_R (ay \otimes_S z) = x \otimes_R a(y \otimes_S z) = (xa) \otimes_R (y \otimes_S z) = f_z(xa, y)$$

By universal property of tensor product this gives a unique map  $\tilde{f}_z : M \otimes_R N \rightarrow M \otimes_R (N \otimes_S P)$ ,  $\tilde{f}_z(x \otimes y) = x \otimes (y \otimes z)$ . Now consider the map  $f : M \otimes_R N \times P \rightarrow M \otimes_R (N \otimes_S P)$  given by  $f(x \otimes y, z) = \tilde{f}_z(x \otimes y)$ . This is  $S$ -linear, as for all  $a \in S$ ,

$$f((x \otimes y)a, z) = f((x \otimes ya), z) = x \otimes (ya \otimes z) = x \otimes (y \otimes (az)) = f((x \otimes y), az)$$

Similarly this gives a unique map  $\tilde{f} : (M \otimes_R N) \otimes_S P \rightarrow M \otimes_R (N \otimes_S P)$ . Repeat the process in the converse direction gives the inverse map, and it is clear that the composition of them is identity in the corresponding structure.  $\square$

**Proposition 4.3.** *Let  $R$  be a commutative ring, and  $f_1 : R \rightarrow S_1$  and  $f_2 : R \rightarrow S_2$  be two  $R$ -algebras. Then there is a unique  $R$ -algebra structure on  $S_1 \otimes_R S_2$  s.t.*

$$(u_1 \otimes v_1) \cdot (u_2 \otimes v_2) = (u_1 u_2) \otimes (v_1 v_2)$$

*Proof.* Since we are in the commutative setting, by using the associativity and commutativity of tensor product, there is an isomorphism  $\Phi : (S_1 \otimes S_2) \otimes (S_1 \otimes S_2) \simeq (S_1 \otimes S_1) \otimes (S_2 \otimes S_2)$ . By universal property of  $S_1 \otimes S_1$  and  $S_2 \otimes S_2$ , there exists a unique morphism of  $R$ -module  $f_i(a_i \otimes b_i) = a_i b_i$  for all  $i$ . This gives  $f_1 \otimes f_2 : (S_1 \otimes S_1) \otimes (S_2 \otimes S_2) \rightarrow S_1 \otimes S_2$ , which indicates that there is a unique map  $f_1 \otimes f_2 \circ \Phi$  that maps  $(S_1 \otimes S_2) \otimes (S_1 \otimes S_2) \rightarrow S_1 \otimes S_2$ . Composing this with the tensoring of  $(S_1 \otimes S_2)$  with itself gives the desired result.  $\square$

**Proposition 4.4.** *There exists an isomorphism of abelian groups  $\Phi : \text{Hom}_S(M \otimes_R N, P) \simeq \text{Hom}_R(N, \text{Hom}_S(M, P))$  for  $N$  a left  $R$ -module,  $P$  a left  $S$ -module, and  $M$  an  $S$ - $R$  bimodule.*

*Proof.* The only natural way to define this isomorphism is via  $\Phi(\varphi) = (N \mapsto (M \mapsto \varphi(M \otimes N)))$ , where for  $f \in \text{Hom}_S(M, P)$ ,  $a \in R$ ,  $u \in M$ ,  $af(u) := f(ua)$ . By construction this is a bijection, and additivity is satisfied in  $P$ .  $\square$

**Remark 4.2.** This indicates that  $M \otimes_R -$  and  $\text{Hom}_S(M, -)$  form an adjoint pair for  $M$  being a  $S$ - $R$  bimodule. Furthermore, if  $f : R \rightarrow S$  gives an  $R$ -algebra structure, then taking  $M = S$  gives the adjoint pair of extension/restriction of scalar functors.