

MATH 593 - Tensor Product

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Contents

1	Tensor Product of Modules	2
2	Bimodule	3
3	Extension of Scalar	5
4	General Properties of Tensor Product	6

1 Tensor Product of Modules

Definition 1.1 (*R*-balanced Maps). Let R be a ring, with M a right R -module, N a left R -module and P an abelian group. Then the map $\varphi : M \times N \rightarrow P$ is ***R*-balanced** if the followings are satisfied:

- $\varphi(u, v_1 + v_2) = \varphi(u, v_1) + \varphi(u, v_2)$ for all $u \in M, v_1, v_2 \in N$.
- $\varphi(u_1 + u_2, v) = \varphi(u_1, v) + \varphi(u_2, v)$ for all $u_1, u_2 \in M, v \in N$.
- $\varphi(ua, v) = \varphi(u, av)$ for all $a \in R, u \in M, v \in N$.

Remark 1.1. The only difference between R -balanced maps and R -linear maps is the third condition: the coefficient in R could be transferred between different positions, but not out of the expression.

Definition 1.2 (Tensor Product). A **tensor product** of M and N is an abelian group $M \otimes_R N$ with an R -balanced map $\varphi : M \times N \rightarrow M \otimes_R N$ which is universal w.r.t. the property: i.e. $\forall \psi : M \times N \rightarrow P$ which is R -balanced, there exists a unique $f : M \otimes_R N \rightarrow P$ s.t. $\psi = f \circ \varphi$ (ψ factors uniquely through φ), i.e. the following diagram commute:

$$\begin{array}{ccc} M \times N & \xrightarrow{\varphi} & M \otimes_R N \\ & \searrow \psi & \downarrow f \\ & & P \end{array}$$

Remark 1.2. If \otimes_R exists, then it is unique up to a canonical isomorphism.

Suppose that for $M, N \in {}_R\text{Mod}$, there exists two tensor products T and T' . Denote the canonical map from $M \times N$ to T and T' be φ and φ' , respectively. Then by universal property of tensor product, there exists a unique isomorphism f and f' s.t. $f \circ \varphi = \varphi'$ and $f' \circ \varphi' = \varphi$, which gives $f \circ f' = \text{Id}$.

Proposition 1.1. The tensor product exists.

Proof. Proceed to show this via introducing relations on the free group structure. Let $F := \mathbb{Z}^{M \times N}$ be a free abelian group with basis $\{e_{(u,v)} \mid (u,v) \in M \times N\}$. Quotient out the elements that are claimed to be equivalent by the constraint that the canonical map φ should be R -balanced: consider $G \subseteq F$ to be generated by the following elements:

- $(e_{u_1+u_2, v} - e_{u_1, v} - e_{u_2, v})$, for all $u_1, u_2 \in M, v \in N$.
- $(e_{u, v_1+v_2} - e_{u, v_1} - e_{u, v_2})$, for all $u \in M, v_1, v_2 \in N$.
- $(e_{ua, v} - e_{u, av})$ for all $u \in M, v \in N, a \in R$.

By construction it is clear that the canonical map $\varphi : M \times N \rightarrow M \otimes_R N$ is R -balanced, via specifying $\varphi(u, v) = \overline{e_{u,v}}$.

It suffices to verify that the construction is compatible with the universal property. Consider the R -balanced map $\psi : M \times N \rightarrow P$, with the group homomorphism $g : F \rightarrow P$ s.t. $g(e_{u,v}) = \psi(u, v)$:

- *Existence.* Applying the universal property of quotient groups, which implies that there exists a unique f s.t. $f \circ h = g$ where h is the induced group homomorphism of the quotient. This is indeed valid, as ψ is R -linear, which by construction has kernel G .

$$\begin{array}{ccccc}
 M \times N & \xrightarrow{\varphi} & F/G & \xleftarrow{h} & F \\
 & \searrow \psi & \downarrow f & \swarrow g & \\
 & & P & &
 \end{array}$$

- *Uniqueness.* This follows from the result of universal property above; and the fact that φ is surjective.

□

Remark 1.3. The construction above, together with the fact that tensor products exist uniquely up to isomorphism, implies that for R -modules M and N with their system of generators, (u_i) and (v_i) respectively, for all $x \in M \otimes_R N$, there exists $(d_i) \in \mathbb{Z}$ s.t.

$$x = \sum_{i=1}^n d_i (u_i \otimes_R v_i)$$

where the multiplication by integers is simply adding repetitively the elements to itself.

The tensor products could also behave *functorially*, via composing with the canonical map of tensor product:

Let $f : M \rightarrow M'$ a morphism of right R -modules, and $g : N \rightarrow N'$ a morphism of left R -modules. Then one could define a map $\psi : M \times N \rightarrow M' \otimes_R N'$, where $(u, v) \mapsto f(u) \otimes_R g(v)$. The map is R -balanced since the canonical map of tensor product is R -balanced. Therefore it is valid to apply the universal property of tensor product, which gives a unique group homomorphism $f : M \otimes_R N \rightarrow M' \otimes_R N'$. This is uniquely determined by f and g ; and is often denoted as $f \otimes_R g$.

Remark 1.4. This is also compatible with composition, via applying the universal property twice. Explicitly, for $f : M \rightarrow M'$, $f' : M' \rightarrow M''$ a morphism of right R -modules, and $g : N \rightarrow N'$, $g' : N' \rightarrow N''$ a morphism of left R -modules, we have

$$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)$$

Remark 1.5. In particular the constructions above induces a functor $M \otimes - : {}_R\mathbf{Mod} \rightarrow \mathbf{Ab}$ for M a right R -module, where

$$N \in \mathbf{Ob}({}_R\mathbf{Mod}) \mapsto M \otimes N, \quad f : N \rightarrow N' \mapsto \text{Id}_M \otimes f$$

Similar to the case of Hom Functors, we seek to lift the functor to ${}_R\mathbf{Mod} \rightarrow {}_R\mathbf{Mod}$. This requires extra structure on the module of interest. As before, making R commutative, and restricting M and N to be both R -modules could resolve the issue, but the condition is too strong.

2 Bimodule

Definition 2.1 (Bimodule). Let S and R be rings. An **R-S bimodule** M is given by an abelian group M that is both a left S -module and a right R -module; and module operations is compatible, i.e.

$$(au)b = a(ub) \quad \forall u \in M, a \in S, b \in R$$

Remark 2.1. If R is commutative, then every R -module is an R - R bimodule (which is why making R commutative suffices to ensure the Hom module has an R -module structure). Specifically, R is an R - R bimodule.

Remark 2.2. Morphisms between S - R bimodules inherits from corresponding modules. Compatibility does not interfere with morphisms.

Proposition 2.1. Let R and S be rings, with M an S - R bimodule, and N a left R -module. Then there exists a unique left S -module structure on $M \otimes N$ s.t. $\lambda \cdot (u \otimes v) = (\lambda u) \otimes v$, for all $\lambda \in S, u \in M, v \in N$.

Proof. Use the universal property, with $P = M \otimes N$. Fix $\lambda \in S$; consider $\varphi : M \times N \rightarrow M \otimes N$ s.t. $\varphi(u, v) = (\lambda u) \otimes v$. This map is R -balanced, as the tensor product on R is balanced.

Then by universal property there exists a unique group homomorphism $f_\lambda : M \otimes N \rightarrow M \otimes N, u \otimes v \mapsto (\lambda u) \otimes v$. This gives the scalar multiplication of λ , which induces an S -module structure on $M \otimes N$. \square

Proposition 2.2. The extra structure on the modules gives extra structure on the morphisms in the universal property:

Let M be a S - R bimodule, N a left R -module, and P a left S -module. Let $\varphi : M \times N \rightarrow M \otimes_R N$ the canonical map of tensor product. Suppose further that the map $\psi : M \times N \rightarrow P$ is S -bilinear. Then there exists a unique morphism of S -modules $f : M \otimes_R N \rightarrow P$.

Proof. By the universal property of tensor product, such morphism f exists, and is uniquely specified by $f(u \otimes v) = \psi(u, v)$. It suffices to check that this is indeed a morphism of S -modules, i.e. for all $a \in S, f((au) \otimes v) = af(u \otimes v)$. It then suffices to check that for certain (set of) fixed u and v , as every element in $M \otimes N$ is of such form. This is clear as

$$f((au) \otimes v) = \psi(au, v) \stackrel{!}{=} a\psi(u, v) = a \cdot f(u \otimes v)$$

Equality (!) requires that ψ is S -bilinear, and M being a bimodule ensures that this is well-formed under the context of S -modules. \square

Remark 2.3. The proposition above lifts the functor $M \otimes -$ to ${}_R\text{Mod} \rightarrow {}_S\text{Mod}$ for all S - R bimodule M .

Remark 2.4. It may be interesting to consider the following property of bimodules:

1. If R is commutative, then left or right R -modules are the same; and in this case $M \otimes_R N$ is an R -module.
2. If M is a T - R bimodule, and N is an R - S bimodule, then $M \otimes_R N$ is a T - S bimodule.

For the second remark, it is clear that $M \otimes_R N$ is both a left T -module, and a right S -module, via applying the same proof as in Proposition 2.1. It suffices to prove that they are compatible. This is also clear from the construction in the proposition referred:

$$(a(u \otimes_R v))b = (au \otimes v)b = (au) \otimes (vb) = a(u \otimes (vb)) = a((u \otimes v)b)$$

Remark 2.5. Let R be a ring. Then R is an R - R bimodule. Let M be a left R -module, which implies that $R \otimes_R M$ is a left R -module. Then there exists a functorial isomorphism $R \otimes_R M \simeq M$ for all $M \in \text{Ob}({}_R\text{Mod})$. (This is called functorial as this could be regarded as the property of functor $R \otimes_R -$.)

Proof. Proof via using the universal property. Consider the morphism of R -modules $\alpha : R \times M \rightarrow M$, where $\alpha(a, u) = au$ for all $a \in R, u \in M$. It is R -linear, which is by definition R -balanced. The universal property gives that there exists a unique

$f : R \otimes M \rightarrow M$ s.t. $f(a \otimes u) = au$. Designate $g : M \rightarrow R \otimes M$, $g(u) = 1 \otimes u$ for all $u \in M$. This is clearly R -balanced. This gives an isomorphism as $g \circ f = \text{Id}_R$, $f \circ g = \text{Id}_{R \otimes M}$. \square

3 Extension of Scalar

Let S and R be rings, together with a ring homomorphism $\varphi : R \rightarrow S$. Then

1. It is clear that there is a *restriction of scalar* functor:

$$F : {}_S\text{Mod} \rightarrow {}_R\text{Mod}, \quad {}_S M \rightarrow {}_R M \quad \text{where } a \cdot u := \varphi(a) \cdot u \quad (\forall a \in R, u \in {}_S M)$$

2. It is more interesting to consider the *extension of scalar* functor $G : {}_R\text{Mod} \rightarrow {}_S\text{Mod}$. Notice that φ gives S a natural R -module structure, where $rs := \varphi(r)s$ for all $r \in R, s \in S$. This gives a natural extension of scalar functor $(S \otimes_R -)$:

- For $M \in \text{Ob}({}_R\text{Mod})$, this gives $S \otimes_R M$.
- For $f : M_1 \rightarrow M_2$ a morphism of R -modules, this gives $\text{Id}_S \otimes f$.

Example 3.1. Consider the following examples:

- Let $\varphi : R \rightarrow R/I$ the canonical quotient map. Then this induces the isomorphism $G(M) \simeq M/IM$.

Extension of scalar gives $G(M) \simeq R/I \otimes M$. To show that these two left R/I -modules are isomorphic, it suffices to specify maps between them s.t. the composition gives identity. Consider

$$f : R/I \otimes M \rightarrow M/IM, \quad \bar{r} \otimes u \mapsto \overline{ru}, \quad g : M/IM \rightarrow R/I \otimes M, \quad \bar{u} \mapsto 1_{R/I} \otimes u$$

It is clear that $f \circ g = \text{Id}_{R/I \otimes M}$. Notice

$$g \circ f(\bar{r} \otimes u) = g(\overline{ru}) = 1 \otimes \overline{ru} = 1 \otimes \bar{r} \cdot \bar{u} = \bar{r} \otimes \bar{u}$$

since the canonical map of tensor product is R -balanced.

- Let R be a ring, and $S \subseteq R$ a multiplicative system. Let φ be the canonical map $R \rightarrow S^{-1}R$, $\varphi(a) = \frac{a}{1}$. Then this induces an isomorphism $G(M) \simeq S^{-1}M$.

Apply the similar strategy. It suffices to show that $G(M) = S^{-1}R \otimes M \simeq S^{-1}M$. Consider

$$f : S^{-1}R \otimes M \rightarrow S^{-1}M, \quad \frac{r}{s} \otimes u \mapsto \frac{ru}{s}, \quad g : S^{-1}M \rightarrow S^{-1}R \otimes M, \quad \frac{u}{s} \mapsto \frac{1}{s} \otimes u$$

It is clear $f \circ g(\frac{u}{s}) = f(\frac{1}{s} \otimes u) = \frac{u}{s}$. For the other direction, check

$$g \circ f(\frac{r}{s} \otimes u) = g(\frac{ru}{s}) = \frac{1}{s} \otimes (ru) = \left(\frac{1}{s}\right) \cdot r \otimes u = \frac{1}{s} \cdot \frac{r}{1} \otimes u = \frac{r}{s} \otimes u$$

Theorem 3.1 (Universal Property of Extension of Scalars). *Let M be a left R -module, N be a left S -module, together with a ring homomorphism $\varphi : R \rightarrow S$. Let $f : M \rightarrow N$ a morphism of R -modules (in the sense of restriction of scalars). Then there exists a unique morphism of S -modules $g : S \otimes_R M \rightarrow N$ s.t. $g(1 \otimes u) = f(u)$ for all $u \in M$:*

$$\begin{array}{ccccc}
 M & \xrightarrow{G} & S \otimes_R M & \xleftarrow{\otimes} & S \times M \\
 & \searrow f & \downarrow g & \swarrow \psi & \\
 & & N & &
 \end{array}$$

Proof. Construct using the universal property of tensor product. Since f is a morphism of S -modules, it is in particular S -linear, i.e.

$$f(su) = sf(u) \implies \psi(s, u) = sf(u)$$

There exists a canonical morphism into the setting of universal property, as $G(su) = \otimes(1, su) = \otimes(s, u)$ for all $s \in S, u \in M$; and ψ is R -linear:

$$\psi(sr, u) = (s\varphi(r))f(u) = s(\varphi(r)f(u)) = sf(\varphi(r)u) = \psi(s, ru)$$

$$\psi(ss', u) = ss'f(u) = sf(s'u) = \psi(s, s'u)$$

Then the strengthened version of universal property gives the desired result. \square

Remark 3.1. Functorially, for tensor products we have the natural transformation (given the ring homomorphism $\varphi : R \rightarrow S$)

$$\text{Hom}_S(S \otimes_R M, N) \rightarrow \text{Hom}_R(M, N), \quad g \mapsto (u \mapsto g(1 \otimes u))$$

The universal property gives that this is actually a bijection. Since $\text{Hom}_R(M, N)$ is in essence the restriction of scalar, there is a bijection (which is exactly rephrasing the result above)

$$\text{Hom}_S(G(M), N) \simeq \text{Hom}_R(M, F(N))$$

Definition 3.1 (Adjoint Pair). *Consider functors $F : \mathcal{C} \rightarrow \mathcal{D}, G : \mathcal{D} \rightarrow \mathcal{C}$. They form an **adjoint pair** (F, G) if for all $a \in \text{Ob}(\mathcal{C})$ and $b \in \text{Ob}(\mathcal{D})$, we have a bijection $\text{Hom}_{\mathcal{C}}(G(b), a) \simeq \text{Hom}_{\mathcal{D}}(b, F(a))$ which is functorial w.r.t. both a and b , i.e. there exists a natural transformation from $\text{Hom}_{\mathcal{C}}(G(-), a)$ to $\text{Hom}_{\mathcal{D}}(-, F(a))$.*

Remark 3.2. Extension of scalar functor G and restriction of scalar functor F form an adjoint pair.

4 General Properties of Tensor Product

Proposition 4.1. *Tensor product is commutative, i.e. there exists an isomorphism of abelian groups $M \otimes_R N \simeq N \otimes_R M$. Similarly, if R is commutative, then this is an isomorphism of R -modules.*

Proof. Proceed via using the universal property of tensor product. Consider the map $\varphi : M \times N \rightarrow N \otimes_R M$ given by $\varphi(v, u) = u \otimes_R v$. It is clear φ commutes with addition in either field. To show that φ is indeed R -balanced it suffices to check the third property, which gives

$$\varphi(va, u) = u \otimes_R (va) = u \otimes_R a^{\text{op}}v = ua^{\text{op}} \otimes_R v = (au) \otimes_R v = \varphi(v, au)$$

Similarly there exists an R -balanced map $\tilde{\psi} : N \times M \rightarrow M \otimes_R N$ (as R^{op} modules) which is given by $\psi(u, v) = v \otimes u$. This induces a map $\psi : N \otimes_{R^{\text{op}}} M \rightarrow M \otimes_R N$. It is clear that $\varphi \circ \psi = \text{Id}_{N \otimes_{R^{\text{op}}} M}$, $\psi \circ \varphi = \text{Id}_{M \otimes_R N}$. \square

Remark 4.1. If R is commutative, then the left R -modules are the same as right R -modules, which indicates that $M \otimes_R N \simeq N \otimes_R M$ (as in the commutative setting the opposite ring is the same as the original ring).

Proposition 4.2. *Tensor product is associative, i.e. for M a right R -module, N an R - S bimodule, and P a left S -module, there exists a unique isomorphism $f : (M \otimes_R N) \otimes_S P \rightarrow M \otimes_R (N \otimes_S P)$ s.t. $f((u \otimes_R v) \otimes_S w) = u \otimes_R (v \otimes_S w)$.*

Proof. Apply the universal property of tensor product twice. First consider map $f_z : M \times N \rightarrow M \otimes_R (N \otimes_S P)$ given by $f_z(x, y) = x \otimes_R (y \otimes_S z)$ for some $z \in P$. f_z is R -balanced, as for all $a \in R$,

$$f_z(x, ay) = x \otimes_R (ay \otimes_S z) = x \otimes_R a(y \otimes_S z) = (xa) \otimes_R (y \otimes_S z) = f_z(xa, y)$$

By universal property of tensor product this gives a unique map $\tilde{f}_z : M \otimes_R N \rightarrow M \otimes_R (N \otimes_S P)$, $\tilde{f}_z(x \otimes y) = x \otimes (y \otimes z)$. Now consider the map $f : M \otimes_R N \times P \rightarrow M \otimes_R (N \otimes_S P)$ given by $f(x \otimes y, z) = \tilde{f}_z(x \otimes y)$. This is S -linear, as for all $a \in S$,

$$f((x \otimes y)a, z) = f((x \otimes ya), z) = x \otimes (ya \otimes z) = x \otimes (y \otimes (az)) = f((x \otimes y), az)$$

Similarly this gives a unique map $\tilde{f} : (M \otimes_R N) \otimes_S P \rightarrow M \otimes_R (N \otimes_S P)$. Repeat the process in the converse direction gives the inverse map, and it is clear that the composition of them is identity in the corresponding structure. \square

Proposition 4.3. *Let R be a commutative ring, and $f_1 : R \rightarrow S_1$ and $f_2 : R \rightarrow S_2$ be two R -algebras. Then there is a unique R -algebra structure on $S_1 \otimes_R S_2$ s.t.*

$$(u_1 \otimes v_1) \cdot (u_2 \otimes v_2) = (u_1 u_2) \otimes (v_1 v_2)$$

Proof. Since we are in the commutative setting, by using the associativity and commutativity of tensor product, there is an isomorphism $\Phi : (S_1 \otimes S_2) \otimes (S_1 \otimes S_2) \simeq (S_1 \otimes S_1) \otimes (S_2 \otimes S_2)$. By universal of property of $S_1 \otimes S_1$ and $S_2 \otimes S_2$, there exists a unique morphism of R -module $f_i(a_i \otimes b_i) = a_i b_i$ for all i . This gives $f_1 \otimes f_2 : (S_1 \otimes S_1) \otimes (S_2 \otimes S_2) \rightarrow S_1 \otimes S_2$, which indicates that there is a unique map $f_1 \otimes f_2 \circ \Phi$ that maps $(S_1 \otimes S_2) \otimes (S_1 \otimes S_2) \rightarrow S_1 \otimes S_2$. Composing this with the tensoring of $(S_1 \otimes S_2)$ with itself gives the desired result. \square

Proposition 4.4. *There exists an isomorphism of abelian groups $\Phi : \text{Hom}_S(M \otimes_R N, P) \simeq \text{Hom}_R(N, \text{Hom}_S(M, P))$ for N a left R -module, P a left S -module, and M an S - R bimodule.*

Proof. The only natural way to define this isomorphism is via $\Phi(\varphi) = (N \mapsto (M \mapsto \varphi(M \otimes N)))$, where for $f \in \text{Hom}_S(M, P)$, $a \in R$, $u \in M$, $af(u) := f(ua)$. By construction this is a bijection, and additivity is satisfied in P . \square

Remark 4.2. This indicates that $M \otimes_R -$ and $\text{Hom}_S(M, -)$ form an adjoint pair for M being a S - R bimodule. Furthermore, if $f : R \rightarrow S$ gives an R -algebra structure, then taking $M = S$ gives the adjoint pair of extension/restriction of scalar functors.