MATH 593 - Module

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Contents

1 Module

Definition 1.1 (R-Module)**.** *An (left)* ***R-Module*** M *is a set with two operations, often denoted as* $(M, +, \times)$*:*

- *Addition* $(+) : M \times M \to M$ *, s.t.* $(M, +)$ *is an abelian group.*
- *Multiplication* $(\times) : R \times M \to M$*, s.t. it has the following properties:*
  - *Identity. For all* $x \in M$*, there exists* $1 \in R$ *s..t* $1 \cdot x = x$.
  - *Associativity. For all* $a, b \in R, x \in M$*,* $a(bx) = (ab)x$.
  - *Distributivity in R. For all* $a_1, a_2 \in R$*,* $(a_1 + a_2)x = a_1x + a_2x$.
  - *Distributivity in* M*. For all* $a \in R, x_1, x_2 \in M$*,* $a(x_1 + x_2) = ax_1 + ax_2$.

Right modules are defined with the same structure, but with $a \times b = b \cdot a$ *for* $a \in R, b \in M$*, where* $\times$ *is the multiplication in* M*, and* $\cdot$ *the multiplication in* R*.*

Definition 1.2 (Submodule)**.** *Let* $(M, +, \times)$ *be an* R*-module.* $N \subseteq M$ *is a* ***R-submodule*** *of* M *if* $(N, +)$ *is a subgroup of* M*; and for all* $n \in N, r \in R, n \times r \in N$.

Remark 1.1. Notice that R itself gives an R-module, just as $\mathbb{K}$ gives a $\mathbb{K}$-vector space. Therefore $\langle S, \varphi \rangle$ an R-algebra induces a two-sided R-module structure. Check that this is indeed the case:

- *Addition.* Adopt the addition in S as a ring.
- *Identity*: Since ring homomorphisms map identity to identity, $\varphi(1_R) = 1_S$, implying that 1_R is the identity for scalar multiplication.
- *Associativity.* Results from the fact that multiplication in S is associative.
- *Distributivity in R/*M. Follows from the fact that φ is a ring homomorphism.

In this sense, module generalizes the algebra structure. Generally one cannot "revert" the structure of a module back to an ideal. Specifically, suppose that R is not commutative, then R is not an R-algebra.

Remark 1.2. (Left) ideals of R are submodules of R taken as an R-submodule.

Remark 1.3. Let M be an abelian group. Making M into a (left) R-module is equivalent to specifying a ring homomorphism $\varphi : R \to \mathrm{End}(M)$, where $\mathrm{End}(\cdot)$ denotes the ring of endomorphisms on the specific structure.

It is worth noticing how the ring of endomorphism structure is defined. Specifically, the multiplication is the composition of endomorphisms on M. This can be viewed in two aspects:

- The associativity for R-modules is essentially stating that multiplication, i.e. elements of R "acting" on those in M is associative. Applying one action after another is the same as applying the composition of action.
- Consider the definition of function as a set of pairs. Then

  $$R \times M \to M \cong (R \to M) \to M \cong R \to (M \to M)$$

  as the application of functions is associative.

In particular, in the consideration of $\mathbb{Z}$-modules, the map $\varphi_{\mathbb{Z}} : \mathbb{Z} \to \operatorname{End}(M)$ is determined uniquely by the requirement that $1 \mapsto 1_M = \mathrm{Id}_M$. Since addition and multiplication should be preserved, $n \mapsto n \cdot \mathrm{Id}_M$ for all $n \in \mathbb{Z}$. With the specification above one could observe the correspondence:

- $\{\mathbb{Z}$ modules$\} \iff \{$Abelian groups$\}$
- $\{\mathbb{Z}/n\mathbb{Z}$ modules$\} \iff \{$Abelian groups M s.t. $nx = 0 \forall x \in M\}$

2 Morphism of R-Modules

Definition 2.1 (Morphism of R-Modules)**.** *A* ***morphism of (left) R-modules*** $f : M \to N$ *is an R-linear map, which satisfies:*

- $f(u_1 + u_2) = f(u_1) + f(u_2)$ *for all* $u_1, u_2 \in M$.
- $f(au) = af(u)$, *for all* $u \in M, a \in R$.

An isomorphism of R-modules $f : M \to N$ is equivalently stating that

- *There exists* $g : N \to M$ *s.t.* $f \circ g = \mathrm{Id}_M$, $g \circ f = \mathrm{Id}_N$.
- f *is a bijection.*

Proposition 2.1. *Let $f : M \to N$ be a morphism of R-modules. Then* $\operatorname{im} f \subseteq M$ *and* $\ker f \subseteq M$ *are submodules; and f is injective if and only if* $\ker f = \{0\}$.

Proof. By the fact that f is R-linear, both the image and kernel should be closed w.r.t. addition and scalar multiplication, i.e. are submodules. For the condition of injectivity, check

$\Rightarrow$: Consider the contraposition. Suppose that $0 \neq a \in \ker f$. Then $f(1) = f(1 + a)$ with $1 \neq 1 + a$ which is a contradiction.

$\Leftarrow$: Consider the contraposition. Suppose that there exists $a \neq b \in R$ s.t. $f(a) = f(b)$, i.e. f is not injective; then $f(a - b) = 0$ which indicates that $0 \neq (a - b) \in \ker$.

□

Definition 2.2 (Quotient Module)**.** *Let $N \subseteq M$ be a R-submodule. Define the equivalence relation $\sim$: $a \sim b$ if and only if $a - b \in N$. Then $M/N := M/\sim$ is a* ***quotient module****, with $\pi : m \to M/N$ the induced morphism of R-modules.*

Theorem 2.1 (Universal Property of Quotient Modules)**.** *Let $f : M \to P$ be a morphism of R-modules. Let N be a submodule of M, with π the induced morphism of R-modules. Further suppose that $N \subseteq \ker f$. Then there exists a unique $g : M/N \to P$ s.t. $f = g \circ \pi$, i.e. the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, two heads, "\pi"] \arrow[dr, "f"'] & M/N \arrow[d, dashed, "g"] \\ & P \end{tikzcd}$$

Proof. It suffices to verify that such map exists and is unique.

- *Uniqueness.* Since the diagram is required to commute, if such function exists, it is fixed by $f(x) = g(\pi(x)) = g(\bar{x})$.
- *Existence.* Then it suffices to check that g such defined is indeed a morphism of R-modules. This is indeed the case as f is a morphism of R-modules.

$\square$

Theorem 2.2 (First Isomorphism)**.** *Let $f : M \to N$ be a surjective morphism of R-modules. Define $K := \ker f$. If there exists a morphism of R-modules $\bar{f} : M/K \to N$ s.t. it is R-linear and $\bar{f} \circ \pi = f$, i.e. the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "\pi"'] & N \\ & M/K \arrow[u, dashed, "\bar{f}"'] \end{tikzcd}$$

Then $\bar{f}$ is an isomorphism.

Proof. By the universal property of morphism of R-modules (Theorem 2.1), a morphism $f : M/K \to N$ s.t. the diagram above commutes exists. It suffices to verify that $\bar{f}$ is bijective. It is surjective as f is surjective; and is injective as $f(x) - f(y) = 0$ if and only if $(x - y) \in K$. $\square$

Definition 2.3 (Direct Product; Direct Sum)**.** *Let $(R_i)_{i \in I}$ be a family (potentially infinite) of modules. Then*

- *The **direct product** of them is the cartesian product $\prod_{i \in I} R_i$, where addition and multiplication is defined element-wise.*
- *The **direct sum** is a sub-ring of the direct sum $\bigoplus_{i \in I} R_i$ where only finitely many elements can be non-zero.*

Theorem 2.3 (Universal Property of Direct Product)**.** *Let P be an R-module, $(M_i)_{i \in I}$ be a family of R-modules, with $f_j : P \to M_j$ a morphism of R-modules. Further let $p_j : \prod_{i \in I} M_i \to M_j$ the projection map s.t. $p_j(x) = x_j$ which is the j-th entry of the input. Then there exists a unique morphism of R-modules $f : P \to \prod_{i \in I} M_i$ s.t. $f(x) = (f_1(x), \cdots, f_n(x), \cdots)$; i.e. the following diagram commutes:*

$$\begin{tikzcd} P \arrow[r, dashed, "f"] \arrow[dr, "f_j"'] & \prod_{i \in I} M_i \arrow[d, "p_j"] \\ & M_j \end{tikzcd}$$

Proof. Uniqueness follows from the fact that $p_j \circ f$ should commute with f_j for all j. Existence holds as f_j is itself a morphism of R-modules. $\square$

Theorem 2.4 (Universal Property of Direct Sum). *Let $(M_i)_{i\in I}$ be a family of modules, with $f_j : M_j \to Q$ a family of morphism of R-algebras. Denote α_j to be the natrual embedding s.t.*

$$\alpha_j : M_j \to \bigoplus_{i\in I} M_i, \qquad \alpha_j(x) = (x_i)i, \quad \textit{where } x_i = \begin{cases} x, & i = j \\ 0, & \textit{otherwise} \end{cases}$$

Then there exists a unique R-linear map $f : \bigoplus_{i\in I} M_i \to Q$ s.t. $f \circ \alpha_j = f_j$ for all j, i.e. the following diagram commutes:

$$\begin{tikzcd} M_j \arrow[r, hook, "\alpha_j"] \arrow[dr, "f_j"'] & \bigoplus_{i\in I} M_i \arrow[d, dashed, "f"] \\ & Q \end{tikzcd}$$

Proof. Since f is required to be a morphism of R-modules, for all $x = (x_i)_{i\in I} \in \bigoplus_{i\in I} M_i$ it should satisfy the following conditions:

$$f(x) = f\left(\sum_{k\in I} \alpha_k(p_k(x))\right) = \sum_{k\in I} f(\alpha_k(p_k(x))) = \sum_{k\in I} f_k(p_k(x))$$

which is unique as f_ks and p_ks are uniquely defined. Since both f_k and p_k are homomorphisms, the composition is also a homomorphism. □

3 Construction of Submodules

4 Free Modules

5 Finiteness Conditions on Modules