MATH 593 - Tensor Product

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Contents

1 Tensor Product of Modules

Definition 1.1 (R-balanced Maps)**.** *Let R be a ring, with M a right R-module, N a left R-module and P an abelian group. Then the map $\varphi : M \times N$ is $\mathbf{R}$-**balanced** if the followings are satisfied:*

- $\varphi(u, v_1 + v_2) = \varphi(u, v_1) + \varphi(u, v_2)$ *for all* $u \in M, v_1, v_2 \in N$.
- $\varphi(u_1 + u_2, v) = \varphi(u_1, v) + \varphi(u_2, v)$ *for all* $u_1, u_2 \in M, v \in N$.
- $\varphi(ua, v) = \varphi(u, av)$ *for all* $a \in R, u \in M, b \in N$.

Remark 1.1. The only difference between R-balanced maps and R-linear maps is the third condition: the coefficient in R could be transferred between different positions, but not out of the expression.

Definition 1.2 (Tensor Product)**.** *A **tensor product** of M and N is an abelian group $M \otimes_R N$ with an R-balanced map $\varphi : M \times N \to M \otimes_R N$ which is universal w.r.t. the property: i.e. $\forall \psi : M \times N \to P$ which is R-balanced, there exists a unique $f : M \otimes_R N \to P$ s.t. $\psi = f \circ \varphi$ (ψ factors uniquely through φ), i.e. the following diagram commute:*

$$\begin{tikzcd}
M \times N \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & M \otimes_R N \arrow[d, dashed, "f"] \\
& P
\end{tikzcd}$$

Remark 1.2. If $\otimes_R$ exists, then it is unique up to a canonical isomorphism.

Suppose that for $M, N \in {}_R\underline{\text{Mod}}$, there exists two tensor products T and T'. Denote the canonical map from $M \times N$ to T and T' be φ and φ', respectively. Then by universal property of tensor product, there exists a unique isomorphism f and f' s.t. $f \circ \varphi = \varphi'$ and $f' \circ \varphi' = \varphi$, which gives $f \circ f' = \text{Id}$.

Proposition 1.1. *The tensor product exists.*

Proof. Proceed to show this via introducing relations on the free group structure. Let $F := \mathbb{Z}^{M \times N}$ be a free abelian group with basis $\{e_{(u,v)} \mid (u, v) \in M \times N\}$. Quotient out the elements that are claimed to be equivalent by the constraint that the canonical map φ should be R-balanced: consider $G \subseteq F$ to be generated by the following elements:

- $(e_{u_1+u_2,v} - e_{u_1,v} - e_{u_2,v})$, for all $u_1, u_2 \in M, v \in N$.
- $(e_{u,v_1+v_2} - e_{u,v_1} - e_{u,v_2})$, for all $u \in M, v_1, v_2 \in N$.
- $(e_{ua,v} - e_{u,av})$ for all $u \in M, v \in N, a \in R$.

By construction it is clear that the canonical map $\varphi : M \times N \to M \otimes_R N$ is R-balanced, via specifying $\varphi(u, v) = \overline{e_{u,v}}$.

It suffices to verify that the construction is compatible with the universal property. Consider the R-balanced map $\psi : M \times N \to P$, with the group homomorphism $g : F \to P$ s.t. $g(e_{u,v}) = \psi(u, v)$:

- *Existence.* Applying the universal property of quotient groups, which implies that there exists a unique f s.t. $f \circ h = g$ where h is the induced group homomorphism of the quotient. This is indeed valid, as ψ is R-balanced, which by construction has kernel G.

$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & F/G \arrow[d, dashed, "f"] & F \arrow[l, "h"'] \arrow[dl, "g"] \\ & P & \end{tikzcd}$$

- *Uniqueness.* This follows from the result of universal property above; and the fact that φ is surjective.

$\square$

Remark 1.3. The construction above, together with the fact that tensor products exist uniquely up to isomorphism, implies that for R-modules M and N with their system of generators, (u_i) and (v_i) respectively, for all $x \in M \otimes_R N$, there exists $(d_i) \in \mathbb{Z}$ s.t.

$$x = \sum_{i=1}^{n} d_i (u_i \otimes_R v_i)$$

where the multiplication by integers is simply adding repetitively the elements to itself.

The tensor products could also behave *functorially*, via composing with the canonical map of tensor product:

Let $f : M \to M'$ a morphism of right R-modules, and $g : N \to N'$ a morphism of left R-modules. Then one could define a map $\psi : M \times N \to M' \otimes_R N'$, where $(u, v) \mapsto f(u) \otimes_R g(v)$. The map is R-balanced since the canonical map of tensor product is R-balanced. Therefore it is valid to apply the universal property of tensor product, which gives a unique group homomorphism $f : M \otimes_R N \to M' \otimes_R N'$. This is uniquely determined by f and g; and is often denoted as $f \otimes_R g$.

Remark 1.4. This is also compatible with composition, via applying the universal property twice. Explicitly, for $f : M \to M', f' : M' \to M''$ a morphism of right R-modules, and $g : N \to N', g' : N' \to N''$ a morphism of left R-modules, we have

$$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)$$

Remark 1.5. In particular the constructions above induces a functor $M \otimes - : {}_R\underline{\text{Mod}} \to \underline{\text{Ab}}$ for M a right R-module, where

$$N \in \text{Ob}({}_R\underline{\text{Mod}}) \mapsto M \otimes N, \qquad f : N \to N' \mapsto \text{Id}_M \otimes f$$

Similar to the case of Hom Functors, we seek to lift the functor to ${}_R\underline{\text{Mod}} \to {}_R\underline{\text{Mod}}$. This requires extra structure on the module of interest. As before, making R commutative, and restricting M and N to be both R-modules could resolve the issue, but the condition is too strong.

2 Bimodule

Definition 2.1 (Bimodule)**.** *Let S and R be rings. An* **R-S** ***bimodule*** *M is given by an an abelian group M that is both a left S-module and a right R-module; and module operations is compatible, i.e.*

$$(au)b = a(ub) \qquad \forall u \in M, a \in S, b \in R$$

Remark 2.1. If R is commutative, then every R-module is an R-R bimodule (which is why making R commutative suffices to ensure the Hom module has an R-module structure). In particular, R is an R-R bimodule.

Remark 2.2. Morphisms between S-R bimodules inherits from corresponding modules. Compatibility does not interfere with morphisms.

Proposition 2.1. *Let R and S be rings, with M an S-R bimodule, and N a left R-module. Then there exists a unique left S-module structure on $M \otimes N$ s.t. $\lambda \cdot (u \otimes v) = (\lambda u) \otimes v$, for all $\lambda \in S, u \in M, v \in N$.*

Proof. Use the universal property, with $P = M \otimes N$. Fix $\lambda \in S$; consider $\varphi : M \times N \to M \otimes N$ s.t. $\varphi(u, v) = (\lambda u) \otimes v$. This map is R-balanced, as the tensor product on R is balanced.

Then by universal property there exists a unique group homomorphism $f_\lambda : M \otimes N \to M \otimes N$, $u \otimes v \mapsto (\lambda u) \otimes v$. This gives the scalar multiplication of λ, which induces an S-module structure on $M \otimes N$. ☐

Proposition 2.2. *The extra structure on the modules gives extra structure on the morphisms in the universal property:*

Let M be a S-R bimodule, N a left R-module, and P a left S-module. Let $\varphi : M \times N \to M \otimes_R N$ the canonical map of tensor product. Suppose further that the map $\psi : M \times N \to P$ is S-bilinear. Then there exists a unique morphism of S-modules $f : M \otimes_R N \to P$.

Proof. By the universal property of tensor product, such morphism f exists, and is uniquely specified by $f(u \otimes v) = \psi(u, v)$. It suffices to check that this is indeed a morphism of S-modules, i.e. for all $a \in S$, $f((au) \otimes v) = af(u \otimes v)$. It then suffices to check that for certain (set of) fixed u and v, as every element in $M \otimes N$ is of such form. This is clear as

$$f((au) \otimes v) = \psi(au, v) \stackrel{!}{=} a\psi(u, v) = a \cdot f(u \otimes v)$$

Equality (!) requires that ψ is S-bilinear, and M being a bimodule ensures that this is well-formed under the context of S-modules. ☐

Remark 2.3. The proposition above lifts the functor $M \otimes -$ to ${}_R\underline{\text{Mod}} \to {}_S\underline{\text{Mod}}$ for all S-R bimodule M.

Remark 2.4. It may be interesting to consider the following property of bimodules:

1. If R is commutative, then left or right R-modules are the same; and in this case $M \otimes_R N$ is an R-module.
2. If M is a T-R bimodule, and N is an R-S bimodule, then $M \otimes_R N$ is a T-S bimodule.

For the second remark, it is clear that $M \otimes_R N$ is both a left T-module, and a right S-module, via applying the same proof as in Proposition 2.1. It suffices to prove that they are compatible. This is also clear from the construction in the proposition referred:

$$(a(u \otimes_R v))b = (au \otimes v)b = (au) \otimes (vb) = a(u \otimes (vb)) = a((u \otimes v)b)$$

Remark 2.5. Let R be a ring. Then R is an R-R bimodule. Let M be a left R-module, which implies that $R \otimes_R M$ is a left R-module. Then there exists a functorial isomorphism $R \otimes_R M \simeq M$ for all $M \in \text{Ob}({}_R\underline{\text{Mod}})$. (This is called functorial as this could be regarded as the property of functor $R \otimes_R -$.)

Proof. Proof via using the universal property. Consider the morphism of R-modules $\alpha : R \times M \to M$, where $\alpha(a, u) = au$ for all $a \in R$, $u \in M$. It is R-linear, which is by definition R-balanced. The universal property gives that there exists a unique

$f : R \otimes M \to M$ s.t. $f(a \otimes u) = au$. Designate $g : M \to R \otimes M$, $g(u) = 1 \otimes u$ for all $u \in M$. This is clearly R-balanced. This gives an isomorphism as $g \circ f = \mathrm{Id}_R$, $f \circ g = \mathrm{Id}_{R \otimes M}$. □

3 Extension of Scalar

Let S and R be rings, together with a ring homomorphism $\varphi : R \to S$. Then

1. It is clear that there is a *restriction of scalar* functor:

$$F : {}_S\underline{\mathrm{Mod}} \to {}_R\underline{\mathrm{Mod}}, \quad {}_SM \to {}_RM \qquad \text{where } a \cdot u := \varphi(a) \cdot u \;\; (\forall a \in R, u \in {}_SM)$$

2. It is more interesting to consider the *extension of scalar* functor $G : {}_R\underline{\mathrm{Mod}} \to {}_S\underline{\mathrm{Mod}}$. Notice that φ gives S a natural R-module structure, where $rs := \varphi(r)s$ for all $r \in R, s \in S$. This gives a natural extension of scalar functor $(S \otimes_R -)$:
   - For $M \in \mathrm{Ob}({}_R\underline{\mathrm{Mod}})$, this gives $S \otimes_R M$.
   - For $f : M_1 \to M_2$ a morphism of R-modules, this gives $\mathrm{Id}_S \otimes f$.

Example 3.1. Consider the following examples:

- Let $\varphi : R \to R/I$ the canonical quotient map. Then this induces the isomorphism $G(M) \simeq M/IM$.

  Extension of scalar gives $G(M) \simeq R/I \otimes M$. To show that these two left R/I-modules are isomorphic, it suffices to specify maps between them s.t. the composition gives identity. Consider

$$f : R/I \otimes M \to M/IM, \quad \bar{r} \otimes u \mapsto \overline{ru}, \qquad g : M/IM \to R/I \otimes M, \bar{u} \mapsto 1_{R/I} \otimes u$$

  It is clear that $f \circ g = \mathrm{Id}_{R/I \otimes M}$. Notice

$$g \circ f(\bar{r} \otimes u) = g(\overline{ru}) = 1 \otimes \overline{ru} = 1 \otimes \bar{r} \cdot \bar{u} = \bar{r} \otimes \bar{u}$$

  since the canonical map of tensor product is R-balanced.

- Let R be a ring, and $S \subseteq R$ a multiplicative system. Let φ be the canonical map $R \to S^{-1}R$, $\varphi(a) = \frac{a}{1}$. Then this induces an isomorphism $G(M) \simeq S^{-1}M$.

  Apply the similar strategy. It suffices to show that $G(M) = S^{-1}R \otimes M \simeq S^{-1}M$. Consider

$$f : S^{-1}R \otimes M \to S^{-1}M, \quad \frac{r}{s} \otimes u \mapsto \frac{ru}{s}, \qquad g : S^{-1}M \mapsto S^{-1}R \otimes M, \quad \frac{u}{s} \mapsto \frac{1}{s} \otimes u$$

  It is clear $f \circ g(\frac{u}{s}) = f(\frac{1}{s} \otimes u) = \frac{u}{s}$. For the other direction, check

$$g \circ f(\frac{r}{s} \otimes u) = g(\frac{ru}{s}) = \frac{1}{s} \otimes (ru) = \left(\frac{1}{s}\right) \cdot r \otimes u = \frac{1}{s} \cdot \frac{r}{1} \otimes u = \frac{r}{s} \otimes u$$

- Tensor of module and localization of a ring is isomorphic to the localization of the module. Let R be a commutative ring, M be an R-module, and U a multiplicative system in R. Then we have the isomorphism

$$M_U \simeq M \otimes_R R_U$$

Proof is done via constructing concrete maps. Consider the morphisms

$$
\begin{aligned}
f : M_U \to M \otimes_R R_U, &\quad \frac{u}{s} \mapsto u \otimes \frac{1}{s} \\
g : M \otimes_R R_U \to M_U, &\quad u \otimes \frac{r}{s} \mapsto \frac{ru}{s}
\end{aligned}
$$

for all $u \in M, r \in R, s \in R \setminus U$. First verify that these maps are indeed well-defined morphisms of R-modules:

- Consider $\frac{u_1}{s_1} \sim \frac{u_2}{s_2}$ where both of which are in M_U. By the definition of localization this indicates that there exists some $t \in R \setminus U$ s.t. $t(s_1 u_2 - s_2 u_1) = 0$. This gives

$$
f\left(\frac{u_1}{s_1}\right) - f\left(\frac{u_2}{s_2}\right) = (s_2 u_1 - s_1 u_2) \otimes \frac{1}{s_1 s_2} = (t(s_2 u_1 - s_1 u_2)) \otimes \frac{1}{s_1 s_2 t} = 0
$$

  which indicates that the image does not depend on the choice of representative.

- Consider $g' : M \times R_U \to M_U, g'(u, \frac{r}{s}) = \frac{ru}{s}$. It is clear that g' is bilinear as R is commutative, which implies that g' is R-balanced. Then g exists by the universal property of tensor product.

Then verify that the composition gives identity:

$$
f \circ g \left(u \otimes \frac{r}{s}\right) = ru \otimes \frac{1}{s} = u \cdot r \otimes \frac{1}{s} = u \otimes \frac{r}{s} \implies f \circ g = \mathrm{Id}_{M \otimes R_U}, \qquad g \circ f\left(\frac{u}{s}\right) = \frac{u}{s} \implies g \circ f = \mathrm{Id}_{M_U}
$$

This gives the desired isomorphism.

Theorem 3.1 (Universal Property of Extension of Scalars)**.** *Let M be a left R-module, N be a left S-module, together with a ring homomorphism $\varphi : R \to S$. Let $f : M \to N$ a morphism of R-modules (in the sense of restriction of scalars). Then there exists a unique morphism of S-modules $g : S \otimes_R M \to N$ s.t. $g(1 \otimes u) = f(u)$ for all $u \in M$:*

$$
\begin{tikzcd}
M \arrow[r, "G"] \arrow[dr, "f"'] & S \otimes_R M \arrow[d, dashed, "g"] & S \times M \arrow[l, "\otimes"'] \arrow[dl, "\psi"] \\
& N &
\end{tikzcd}
$$

Proof. Construct using the universal property of tensor product. Since f is a morphism of S-modules, it is in particular S-linear, i.e.

$$
f(su) = sf(u) \implies \psi(s, u) = sf(u)
$$

There exists a canonical morphism into the setting of universal property, as $G(su) = \otimes(1, su) = \otimes(s, u)$ for all $s \in S, u \in M$; and ψ is R-linear:

$$
\psi(sr, u) = (s\varphi(r))f(u) = s(\varphi(r)f(u)) = sf(\varphi(r)u) = \psi(s, ru)
$$

$$
\psi(ss', u) = ss'f(u) = sf(s'u) = \psi(s, s'u)
$$

Then the strengthened version of universal property gives the desired result. □

Remark 3.1. Functorially, for tensor products we have the natural transformation (given the ring homomorphism $\varphi : R \to S$)

$$
\mathrm{Hom}_S(S \otimes_R M, N) \to \mathrm{Hom}_R(M, N), \qquad g \mapsto (u \mapsto g(1 \otimes u))
$$

The universal property gives that this is actually a bijection. Since $\mathrm{Hom}_R(M,N)$ is in essence the restriction of scalar, there is a bijection (which is exactly rephrasing the result above)

$$\mathrm{Hom}_S(G(M),N) \simeq \mathrm{Hom}_R(M,F(N))$$

Definition 3.1 (Adjoint Pair)**.** *Consider functors $F:\mathscr{C}\to\mathscr{D}$, $G:\mathscr{D}\to\mathscr{C}$. They form an **adjoint pair** (F,G) if for all $a\in\mathrm{Ob}(\mathscr{C})$ and $b\in\mathrm{Ob}(\mathscr{D})$, we have a bijection $\mathrm{Hom}_{\mathscr{C}}(G(b),a)\simeq\mathrm{Hom}_{\mathscr{D}}(b,F(a))$ which is functorial w.r.t. both a and b, i.e. there exists a natural transformation from $\mathrm{Hom}_{\mathscr{C}}(G(-),a)$ to $\mathrm{Hom}_{\mathscr{D}}(-,F(a))$.*

Remark 3.2. Extension of scalar functor G and restriction of scalar functor F form an adjoint pair.

4 General Properties of Tensor Product

Proposition 4.1. *Tensor product is* commutative*, i.e. there exists an isomorphism of abelian groups $M\otimes_R N\simeq N\otimes_{R^{\mathrm{op}}} M$. Similarly, if R is commutative, then this is an isomorphism of R-modules.*

Proof. Proceed via using the universal property of tensor product. Consider the map $\varphi: M\times N\to N\otimes_{R^{\mathrm{op}}} M$ given by $\varphi(v,u)=u\otimes_{R^{\mathrm{op}}} v$. It is clear φ commutes with addition in either field. To show that φ is indeed R-balanced it suffices to check the third property, which gives

$$\varphi(va,u)=u\otimes_{R^{\mathrm{op}}}(va)=u\otimes_{R^{\mathrm{op}}} a^{\mathrm{op}}v=ua^{\mathrm{op}}\otimes_{R^{\mathrm{op}}} v=(au)\otimes_{R^{\mathrm{op}}} v=\varphi(v,au)$$

Similarly there exists an R-balanced map $\tilde{\psi}: N\times M\to M\otimes_R N$ (as R^{op} modules) which is given by $\psi(u,v)=v\otimes u$. This induces a map $\psi: N\otimes_{R^{\mathrm{op}}} M\to M\otimes_R N$. It is clear that $\varphi\circ\psi=\mathrm{Id}_{N\otimes_{R^{\mathrm{op}}}M}$, $\psi\circ\varphi=\mathrm{Id}_{M\otimes_R N}$. □

Remark 4.1. If R is commutative, then the left R-modules are the same as right R-modules, which indicates that $M\otimes_R N\simeq N\otimes_R M$ (as in the commutative setting the opposite ring is the same as the original ring).

Proposition 4.2. *Tensor product is* associative*, i.e. for M a right R-module, N an R-S bimodule, and P a left S-module, there exists a unique isomorphism $f:(M\otimes_R N)\otimes_S P\to M\otimes_R(N\otimes_S P)$ s.t. $f((u\otimes_R v)\otimes_S w)=u\otimes_R(v\otimes_S w)$.*

Proof. Apply the universal property of tensor product twice. First consider map $f_z: M\times N\to M\otimes_R(N\otimes_S P)$ given by $f_z(x,y)=x\otimes_R(y\otimes_S z)$ for some $z\in P$. f_z is R-balanced, as for all $a\in R$,

$$f_z(x,ay)=x\otimes_R(ay\otimes_S z)=x\otimes_R a(y\otimes_S z)=(xa)\otimes_R(y\otimes_S z)=f_z(xa,y)$$

By universal property of tensor product this gives a unique map $\tilde{f}_z: M\otimes_R N\to M\otimes_R(N\otimes_S P)$, $f_z(x\otimes y)=x\otimes(y\otimes z)$. Now consider the map $f: M\otimes_R N\times P\to M\otimes_R(N\otimes_S P)$ given by $f(x\otimes y,z)=f_z(x,y)$. This is S-linear, as for all $a\in S$,

$$f((x\otimes y)a,z)=f((x\otimes ya),z)=x\otimes(ya\otimes z)=x\otimes(y\otimes(az))=f((x\otimes y),az)$$

Similarly this gives a unique map $\tilde{f}:(M\otimes_R N)\otimes_S P\to M\otimes_R(N\otimes_S P)$. Repeat the process in the converse direction gives the inverse map, and it is clear that the composition of them is identity in the corresponding structure. □

Proposition 4.3. *Let R be a commutative ring, and $f_1 : R \to S_1$ and $f_2 : R \to S_2$ be two R-algebras. Then there is a unique R-algebra structure on $S_1 \otimes_R S_2$ s.t.*

$$(u_1 \otimes v_1) \cdot (u_2 \otimes v_2) = (u_1 u_2) \otimes (v_1 v_2)$$

Proof. Since we are in the commutative setting, by using the associativity and commutativity of tensor product, there is an isomorphism $\Phi : (S_1 \otimes S_2) \otimes (S_1 \otimes S_2) \simeq (S_1 \otimes S_1) \otimes (S_2 \otimes S_2)$. By universal of property of $S_1 \otimes S_1$ and $S_2 \otimes S_2$, there exists a unique morphism of R-module $f_i(a_i \otimes b_i) = a_i b_i$ for all $a_i, b_i \in S_i$ with $i \in \{1, 2\}$. This gives $f_1 \otimes f_2 : (S_1 \otimes S_1) \otimes (S_2 \otimes S_2) \to S_1 \otimes S_2$, which indicates that there is a unique map $f_1 \otimes f_2 \circ \Phi$ that maps $(S_1 \otimes S_2) \otimes (S_1 \otimes S_2) \to S_1 \otimes S_2$. Composing this with the tensoring of $(S_1 \otimes S_2)$ with itself gives the desired result. $\square$

Proposition 4.4. *There exists an isomorphism of abelian groups $\Phi : \mathrm{Hom}_S(M \otimes_R N, P) \simeq \mathrm{Hom}_R(N, \mathrm{Hom}_S(M, P))$ for N a left R-module, P a left S-module, and M an S-R bimodule.*

Proof. The only natural way to define this isomorphism is via $\Phi(\varphi) = (N \mapsto (M \mapsto \varphi(M \otimes N)))$, where for $f \in \mathrm{Hom}_S(M, P)$, $a \in R, u \in M$, $af(u) := f(ua)$. By construction this is a bijection, and additivity is satisfied in P. $\square$

Remark 4.2. This indicates that $M \otimes_R -$ and $\mathrm{Hom}_S(M, -)$ form an adjoint pair for M being a S-R bimodule. Furthermore, if $f : R \to S$ gives an R-algebra structure, then taking $M = S$ gives the adjoint pair of extension/restriction of scalar functors.