MATH 593 - Categories and Functors

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November 5, 2023

Contents

1 Category; Functor

Definition 1.1 (Category). *A **category** $\mathcal{C}$ consists of*

- *A <u>class</u> of objects $\mathcal{C}$ (which, for example, could contain all sets), denoted* $\mathrm{Ob}(\mathcal{C})$.
- *For all $A, B \in \mathrm{Ob}(\mathcal{C})$, a <u>set</u> $\mathrm{Hom}_{\mathcal{C}}(A, B)$ the "morphisms in $\mathcal{C}$ from A to B" with map $\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \times \mathrm{Hom}_{\mathcal{C}}(A, C)$ the "morphism composition" denoted $f \times g \rightsquigarrow (g \circ f)$, satisfying*
  - Existence of an identity. *for all $A \in \mathrm{Ob}(\mathcal{C})$, there exists $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ s.t.*

$$\begin{cases} 1_A \circ f = f & \forall f \in \mathrm{Hom}_{\mathcal{C}}(A, B) \\ g \circ 1_A = g & \forall g \in \mathrm{Hom}_{\mathcal{C}}(B, A) \end{cases}$$

  - Associativity. *For all $f \in \mathrm{Hom}_{\mathcal{C}}(A, B), g \in \mathrm{Hom}_{\mathcal{C}}(B, C), h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$,*

$$(h \circ g) \circ f = h \circ (g \circ f)$$

Remark 1.1. The definition much resembles previous algebraic structures; but the morphisms and composition laws could be defined in a particularly strange way:

1. Similar to monoids, the definition implies that the identity is unique. Suppose that there are two identities $1_A, 1'_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ for $A \in \mathrm{Ob}(\mathcal{C})$, then $1_A = 1_A \circ 1'_A = 1'_A$.
2. The morphism is not necessarily a function; and in such cases composition needs to be re-defined respectively.

Example 1.1. Consider the following categories:

- *Category of Sets* <u>Sets</u>, where the objects are sets, and morphisms are maps between sets.
- *Category of Rings* <u>Rings</u>, where the objects are rings, and morphisms are ring homomorphisms.
- *Category of (left) R-modules* ${}_R\underline{\mathrm{Mod}}$, where objects are lef R-modules, and morphisms R-linear maps.
- Consider the category $\mathcal{C}$ defined on a partially-ordered set $(A, \leq)$ where
  - $\mathrm{Ob}(\mathcal{C})$ consists of elements in A.
  - Morphisms are defined as

$$\mathrm{Hom}_{\mathcal{C}}(A, B) = \begin{cases} \{*\} & A \leq B \\ \emptyset & \text{otherwise} \end{cases}$$

    where the composition of maps is defined as intersection. This is due to the fact that there can be no maps whose image is the empty set.

Definition 1.2 (Functor). *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of mappings for both objects and morphisms:*

- *For all $A \in \mathrm{Ob}(\mathcal{C})$, $F(A) \in \mathcal{D}$.*
- *For all $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(b))$ s.t.*
  - $F(1_A) = 1_{F(A)}$ *for all $A \in \mathrm{Ob}(\mathcal{C})$.*

  - *For all* $f \circ g$ *where* $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $F(f \circ g) = F(f) \circ F(g)$.

The composition of functors is conducted in a natural way, i.e. applying consecutively.

Example 1.2. Functors represent the induced maps w.r.t. a transformation in the structure:

1. Let R be a commutative ring and $S \subseteq R$ a multiplicative system. Consider the functor $F : {}_R\underline{\text{Mod}} \to {}_{S^{-1}R}\underline{\text{Mod}}$ where
   - $F(M) = S^{-1}M$ for all $M \in \mathrm{Ob}({}_R\underline{\text{Mod}})$.
   - For $f : M \to N$, define $F(f) := S^{-1}M \to S^{-1}N$, where $\frac{u}{s} \mapsto \frac{f(u)}{s}$.
2. Let R be a ring, with $I \subseteq R$ a two-sided ideal of R; and M a left R-module. Consider the functor $F : {}_R\underline{\text{Mod}} \to {}_{R/I}\underline{\text{Mod}}$ where
   - $F(M) = M/IM$ for all $M \in {}_R\underline{\text{Mod}}$.
   - Let $f : M \to N$ be a morphism of left R-modules. Then it induces a map $\bar{f} : M/IM \to N/IN$ s.t. $\bar{f}(\bar{(u)}) = \overline{f(u)}$. Define $F(f) = \bar{f}$.
3. Functors generally can abandon structures. Let M be a left R-module. By definition it is valid to view M as an abelian group. Then functor $F : {}_R\underline{\text{Mod}} \to \underline{\text{Ab}}$ where objects are taken to itself; and morphisms are taken to group homomorphisms. These are called forgetful functors.

2 Morphism of Categories

3 Products and Coproducts

4 Kernels and Cokernels

For the motivation of kernel one could refer to this article.

5 Natural Transformations of Functors