MATH 593 - Tensor Product

ARessegetes Stery

November 14, 2023

Contents

1	Tensor Product of Modules	2
2	Bimodule	3
3	Extension of Scalar	3
4	Adjoint Property of Tensor Product	3

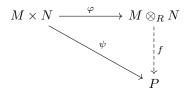
1 Tensor Product of Modules

Definition 1.1 (R-balanced Maps). Let R be a ring, with M a right R-module, N a left R-module and P an abelian group. Then the map $\varphi: M \times N$ is \mathbf{R} -balanced if the followings are satisfied:

- $\varphi(u, v_1 + v_2) = \varphi(u, v_1) + \varphi(u, v_2)$ for all $u \in M, v_1, v_2 \in N$.
- $\varphi(u_1 + u_2, v) = \varphi(u_1, v) + \varphi(u_2, v)$ for all $u_1, u_2 \in M$, $v \in N$.
- $\varphi(ua, v) = \varphi(u, av)$ for all $a \in R, u \in M, b \in N$.

Remark 1.1. The only difference between R-balanced maps and R-linear maps is the third condition: the coefficient in R could be transferred between different positions, but not out of the expression.

Definition 1.2 (Tensor Product). A **tensor product** of M and N is an abelian group $M \otimes_R N$ with an R-balanced map $\varphi : M \times N \to M \otimes_R N$ which is universal w.r.t. the property: i.e. $\forall \psi : M \times N \to P$ which is R-balanced, there exists a unique $f : M \otimes_R N \to P$ s.t. $\psi = f \circ \varphi$ (ψ factors uniquely through φ), i.e. the following diagram commute:



Remark 1.2. If \otimes_R exists, then it is unique up to a canonical isomorphism.

Suppose that for $M, N \in {}_R\underline{\text{Mod}}$, there exists two tensor products T and T'. Denote the canonical map from $M \times N$ to T and T' be φ and φ' , respectively. Then by universal property of tensor product, there exists a unique isomorphism f and f' s.t. $f \circ \varphi = \varphi'$ and $f' \circ \varphi' = \varphi$, which gives $f \circ f' = \operatorname{Id}$.

Proposition 1.1. The tensor product exists.

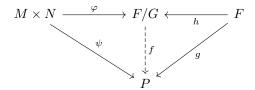
Proof. Proceed to show this via introducing relations on the free group structure. Let $F:=\mathbb{Z}^{M\times N}$ be a free abelian group with basis $\{e_{(u,v)}\mid (u,v)\in M\times N\}$. Quotient out the elements that are claimed to be equivalent by the constraint that the canonical map φ should be R-balanced: consider $G\subseteq F$ to be generated by the following elements:

- $(e_{u_1+u_2,v}-e_{u_1,v}-e_{u_2,v})$, for all $u_1,u_2\in M,v\in N$.
- $(e_{u,v_1+v_2}-e_{u,v_1}-e_{u,v_2})$, for all $u \in M, v_1, v_2 \in N$.
- $(e_{ua,v} e_{u,av})$ for all $u \in M, v \in N, a \in R$.

By construction it is clear that the canonical map $\varphi: M \times N \to M \otimes_R N$ is R-balanced, via specifying $\varphi(u,v) = \overline{e_{u,v}}$.

It suffices to verify that the construction is compatible with the universal property. Consider the R-balanced map $\psi: M \times N \to P$, with the group homomorphism $g: F \to P$ s.t. $g(e_{u,v}) = \psi(u,v)$:

• Existence. Applying the universal property of quotient groups, which implies that there exists a unique f s.t. $f \circ h = g$ where h is the induced group homomorphism of the quotient. This is indeed valid, as ψ is R-linear, which by construction has kernel G.



• Uniqueness. This follows from the result of universal property above; and the fact that φ is surjective.

Remark 1.3. The construction above, together with the fact that tensor products exist uniquely up to isomorphism, implies that for R-modules M and N with their system of generators, (u_i) and (v_i) respectively, for all $x \in M \otimes_R N$, there exists $(d_i) \in \mathbb{Z}$ s.t.

$$x = \sum_{i=1}^{n} d_i (u_i \otimes_R v_i)$$

where the multiplication by integers is simply adding repetitively the elements to itself.

The tensor products could also behave functorially, via composing with the canonical map of tensor product:

Let $f: M \to M'$ a morphism of right R-modules, and $g: N \to N'$ a morphism of left R-modules. Then one could define a map $\psi: M \times N \to M' \otimes_R N'$, where $(u,v) \mapsto f(u) \otimes_R g(v)$. The map is R-balanced since the canonical map of tensor product is R-balanced. Therefore it is valid to apply the universal property of tenbsor product, which gives a unique group homomorphism $f: M \otimes_R N \to M' \otimes_R N'$. This is uniquely determined by f and g; and is often denoted as $f \otimes_R g$.

Remark 1.4. This is also compatible with composition, via applying the universal property twice. Explicitly, for $f:M\to M',f':M'\to M''$ a morphism of right R-modules, and $g:N\to N',g':N'\to N''$ a morphism of left R-modules, we have

$$(f'\otimes g')\circ (f\otimes g)=(f'\circ f)\otimes (g'\circ g)$$

Remark 1.5. In particular the constructions above induces a functor $M \otimes -:_R \underline{\text{Mod}} \to \underline{\text{Ab}}$ for M a right R-module, where

$$N \in \mathrm{Ob}(_{R}\mathrm{Mod}) \mapsto M \otimes N, \qquad f: N \to N' \mapsto \mathrm{Id}_{M} \otimes f$$

Similar to the case of Hom Functors, we seek to lift the functor to $R \bmod A = R \bmod A$. This requires extra structure on the module of interest. Similarly, making R commutative, and restricting M and N to be both R-modules could resolve the issue, but the condition is too strong.

- 2 Bimodule
- 3 Extension of Scalar
- 4 Adjoint Property of Tensor Product