MATH 593 - Ring

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Contents

1 Ring homomorphism, Quotient Ring

Definition 1.1 (Ring Homomorphism)**.** *Let X, Y be rings. A* ***Ring Homomorphism*** *is a map $f : X \to Y$ satisfying the following properties:*

- $f(1) = 1$.
- $\forall x_1, x_2 \in X, f(x_1) + f(x_2) = f(x_1 + x_2)$.
- $\forall x_1, x_2 \in X, f(x_1 x_2) = f(x_1)f(x_2)$

Definition 1.2 (Quotient Ring)**.** *Let R be a ring and $I \subseteq R$ a two-sided ideal. The* ***Quotient Ring*** *(R/I) is defined as $(R/\sim)$ with an equivalence relation $\sim$ where $a \sim b$ if and only if $a - b = I$. Elements in (R/I) are denoted as $\bar{a}$, where $\bar{a} = \bar{b}$ if and only if $a \sim b$.*

The natural homomorphism $\pi_I : R \to (R/I)$ is defined as $\pi(a) = \bar{a}$, which satisfies the *universal property of quotient rings*:

Theorem 1.1 (Fundamental Theorem of Ring Homomorphisms)**.** *Let $\varphi : R \to S$ be a ring homomorphism, I a two-sided ideal s.t. $I \subseteq \ker \varphi$, and π be the natural ring homomorphism from R to (R/I). Then there exists a unique ring homomorphism $f : R/I \to S$ s.t. the following diagram commutes, i.e. $\varphi = f \circ \pi$.*

$$\begin{tikzcd}
R \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & S \\
& R/I \arrow[u, "f"']
\end{tikzcd}$$

Proof. It suffices to prove that f exists and is unique, and verify that f is indeed a ring homomorphism.

- **Uniqueness.** By the requirement that f should make the diagram commute, $f(\bar{a}) = \varphi(a),\ \forall a \in R$. Uniqueness of f follows from the fact that φ maps every element in R to a unique element in S.
- **Existence.** It suffices to verify that f is well-defined, i.e. does not vary w.r.t. change of representative in (R/I). For all $a, b \in R$ s.t. $\bar{a} = \bar{b}$, $(a - b) \in I \implies \varphi(a - b) = 0 \implies \varphi(a) = \varphi(b)$ since φ is a ring homomorphism. By the uniqueness of f it is specified that $f(\bar{a}) = \varphi(a)$, which implies that for all $\bar{a} = \bar{b} \in (R/I)$, $f(\bar{a}) = \varphi(a) = \varphi(b) = f(\bar{b})$.
- f **is indeed a homomorphism.** This follows from the fact that φ is a ring homomorphism.

$\square$

2 Ring of Fractions

Definition 2.1 (Multiplicative System). *A subset $S \subseteq R$ for a ring R is a* ***multiplicative system*** *if $1 \in S$, and $\forall s_1, s_2 \in S, s_1 \cdot s_2 \in S$, where $\cdot$ is the multiplication in R.*

Definition 2.2 (Ring of Fractions). *Let R be a commutative ring, with $S \subseteq R$ a multiplicative subset, the* ***ring of fraction*** *$S^{-1}R$ is defined as $R \times S/\sim$, where $(s_1, r_1) \sim (s_2, r_2)$ if and only if there exists $t \in R$ s.t. $t(s_1r_2 - s_2r_1) = 0$. $(s, r) \in S^{-1}R$ is denoted as $\frac{s}{r}$. The definition of operations follows directly from analogy of that in $\mathbb{Q}$.*

The natural homomorphism (inclusion map) from R to $S^{-1}R$ is defined as $r \hookrightarrow \frac{r}{1}$.

Remark 2.1. If R is an integral domain, then $(s_1, r_1) \sim (s_2, r_2)$ iff $s_1r_2 = s_2r_1$, as for $\mathbb{Q}$.

Remark 2.2. If R is not an integral domain, and S contains zero divisors, then the inclusion map ceases to be injective, as choosing t s.t. it satisfies $ts_1 = ts_2 = 0$ for some s_1, s_2 that are zero divisors gives $\varphi(s_1) = \varphi(s_2)$. Changing R to an integral domain guarantees that the inclusion map φ is injective.

Proposition 2.1. *$\sim$ is an equivalence relation.*

Proof. It is clear that $\sim$ is reflexive and symmetric. For transitivity, consider $(s_1, r_1) \sim (s_2, r_2) \wedge (s_2, r_2) \sim (s_3, r_3)$. That is, there exists some $t_1, t_2 \in R$ s.t.

$$\begin{cases} t_1(s_1r_2 - s_2r_1) = 0 \\ t_2(s_2r_3 - s_3r_2) = 0 \end{cases} \implies t_1t_2(s_1r_2s_3 - s_2r_1s_3) = t_1t_2(s_1s_2r_3 - s_2r_1s_3) = t_1t_2s_2(s_1r_3 - s_3r_1) = 0$$

□

Remark 2.3. Notice that if $s \in S, then \frac{s}{a}$ for $a \in R$ is invertible. This tends more to a field, with more elements being "reachable" via multiplying an element from one side. A direct consequence is that less ideals exist in $S^{-1}R$, with ideals in R whose generators differ by a factor that divides s being identified in $S^{-1}R$.

Remark 2.4. It is required that R is commutative is to preserve the most structures from R, i.e. ensure that $S^{-1}I$ is an ideal for all ideals in R. This is due to the addition in action:

$$\forall \frac{r_1}{s_1}, \frac{r_2}{s_2} \in S^{-1}R, \qquad \frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1s_2 + s_1r_2}{s_1s_2}$$

which indicates that $S^{-1}I$ is a two-sided ideal if and only if $I \subseteq R$ is a two-sided ideal. For one-sided (left/right) ideal the property is not fully inherited.

Theorem 2.1 (Universal Property of Ring of Fractions). *Suppose R and T are commutative rings, with φ the inclusion of R into $S^{-1}R$. Then for $f : R \to T$ s.t. $\forall s \in S, f(s)$ is invertible in T, there exists a unique ring homomorphism g s.t. $f = g \circ \varphi$, i.e. make the following diagram commute:*

Proof. Adopt the same strategy as in the previous section:

- **Existence.** For all $\frac{a}{s} \in S^{-1}R$, $g(\frac{a}{s}) := f(a)(f(s))^{-1}$ which is well-defined since f is required to map all elements in S to invertible elements. g being a ring homomorphism follows from the fact that f is a ring homomorphism.

$$\begin{tikzcd} R \arrow[r, hook, "\varphi"] \arrow[dr, "f"'] & S^{-1}R \arrow[d, "g"] \\ & T \end{tikzcd}$$

- **Uniqueness.** Follows from specifying $g(\frac{a}{s}) := f(a)(f(s))^{-1}$.

$\square$

Remark 2.5. If $S := R \setminus \{0\}$, then $S^{-1}R$ is the whole field, with localization equivalent to completion of inverse of R.

2.1 Localization of a Ring

Definition 2.3. *A commutative ring $R \neq \{0\}$ is* ***local*** *if it admits a unique maximal ideal M. Local rings are denoted by a pair (R, M).*

Example 2.1. Let R be a commutative ring, with $\mathfrak{p} \subseteq R$ a prime ideal. Let $S = R \setminus p$ be a multiplicative system. Then the ring $S^{-1}R$ is local, with the maximal ideal of it being $S^{-1}\mathfrak{p}$. This results from the fact that $S^{-1}I$ is an ideal if and only if I is an ideal in R. Further since $\mathbb{Z}$ is a PID (see next section), all prime ideals are maximal, $S^{-1}\mathfrak{p}$ is indeed maximal. The fact that there is only one such maximal ideal results from that all other primes are in S, i.e. $S^{-1}\mathfrak{p}' = S^{-1}R$ for all $\mathfrak{p}' \neq \mathfrak{p}$.

Proposition 2.2. *Let $R \neq \{0\}$ be a commutative ring. Then R being local if and only if for all $a \in R$, either a is invertible or $(1-a)$ is invertible. In this case, the maximal ideal M is the set of all non-invertible elements.*

Proof. Proceed by showing implication in both directions:

$\Rightarrow$: Suppose that (R, M) is the local ring of interest. Proceed by showing a contradiction: suppose that both a and $(1-a)$ are non-invertible. Then since R is local $(a) \subseteq M$, $(1-a) \subseteq M$ indicating that $1 \in M$ which is a contradiction. In this case for all a non-invertible, $(a) \subseteq M$, which implies that M is the set of all non-invertible elements.

$\Leftarrow$: Define set $M := \{a \in R \mid \forall x \in R, ax \neq 1\}$. By construction if M is an ideal then it must be maximal, as including an invertible element expands the ideal to the whole ring. Verify that M is indeed an ideal:

- *Closed with addition.* Proceed via showing that the contraposition. Suppose that there exists $a, b \in R$ s.t. both a and b are non-invertible, but there exists some $c \in R$ s.t. $c(a+b) = 1$. Then $ca = 1 - (cb)$ is non-invertible, which implies that $1 - ca$ is invertible. But notice $1 - ca = cb$ is also non-invertible, which is a contradiction.
- *Absorption with multiplication.* This simply results from the fact that a non-invertible element multiplied by a unit is still non-invertible.

$\square$

3 Polynomial Rings

Definition 3.1 (R-algebra)**.** *Let R be a ring. Then a ring S is an R-**algebra** for the specific R mentioned if there exists a ring homomorphism $\varphi : R \to S$ s.t. $\forall r \in R, s \in S, \varphi(r)s = s\varphi(r)$. When the homomorphism needs to be specified, the algebra is often denoted as a pair $\langle S, \varphi \rangle$*

Remark 3.1. An R-algebra is a two-sided R-module, which can be regarded as a generalization of the structure in R. R itself is not necessarily commutative, which implies that the associated homomorphism maps R to the center of S.

Definition 3.2 (Morphism of R-algebras)**.** *Let $\langle R_1, f_1 \rangle, \langle R_2, f_2 \rangle$ be R-algebras. A **Morphism of** R**-algebras** is a ring homomorphism $\varphi : R_1 \to R_2$ s.t. the following diagram commute; i.e. $f_2 = \varphi \circ f_1$:*

$$\begin{tikzcd} R \arrow[r, "f_1"] \arrow[dr, "f_2"'] & R_1 \arrow[d, "\varphi"] \\ & R_2 \end{tikzcd}$$

Definition 3.3 (R-subalgebra)**.** *Let $\langle S, f_s \rangle$ be a R-algebra for R a ring. $\langle T, f_t \rangle$ is a R-**subalgebra** of S if T is a R-algebra, with $f_t(R) \subseteq S$; and there exists a morphism φ from T to S, i.e. φ makes the following diagram commute:*

$$\begin{tikzcd} R \arrow[r, "f_t"] \arrow[dr, "f_s"'] & T \arrow[d, hook, "\varphi"] \\ & S \end{tikzcd}$$

Definition 3.4 (Polynomial Ring)**.** *Let R be a commutative ring. The **polynomial ring of** R, denoted $R[x]$, is defined as*

$$R[x] := \left\{ \sum_{i=0}^{n} c_i x^i \mid n \in \mathbb{N}, c_i \in R \right\}$$

with the addition and multiplication the same as in polynomials over $\mathbb{Z}$. The natural inclusion from R to $R[x]$ is defined as $r \mapsto r$ which is a polynomial of degree 0.

Remark 3.2. If R is a domain, then $R[x]$ is also a domain (consider the product of terms with highest degree); where $\deg(fg) \leq \deg(f) + \deg(g)$.

Theorem 3.1 (Universal Property of Polynomial Ring)**.** *Let R be a ring and $\langle S, f \rangle$ an R-algebra, and φ be the inclusion map from R to $R[x]$. For all $s \in S$, there exists a unique morphism of R-algebra $g : R[x] \to S$ s.t. $g(x) = a$, and the following diagram commutes, i.e. $f = g \circ \varphi$:*

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[dr, hook, "f"'] & R[x] \arrow[d, "g"] \\ & S \end{tikzcd}$$

Proof. Proceed similarly by first determining the form that g takes, and then showing the uniqueness and existence.

- **Uniqueness.** Since it is required that g is a morphism of R-algebras, we have

$$g\left(\sum_{i=0}^{n} a_i x^i\right) = \sum_{i=0}^{n} g(a_i)g(x^i) = \sum_{i=0}^{n} f(a_i)g(x^i) = \sum_{i=0}^{n} f(a_i)a^i$$

  by the requirement that $g(x) = a$. This is the only form that g could take, and thus proves its uniqueness.

- **Existence.** For existence it suffices to check that g is indeed a ring homomorphism. By the uniqueness g is fixed by sending $x \in R[x]$ to $a \in R$. Notice that R is commutative, which indicates that both left and right composition is satisfied; with the addition condition verified in the uniqueness part.

$\square$

Theorem 3.2 (Universal Property of Polynomial Ring of Several Variables)**.** *Let A be a commutative R-algebra and g be the inclusion map from R to $R[x_1, \cdots, x_n]$ with a fixed n. For every R-algebra S and $(a_1, \cdots, a_n) \in S$, there exists a unique homomorphism of R-algebra $h : R[x_1, \cdots, x_n] \to S$ s.t. $h(x_i) = a_i$ for all $i \in [\![1, n]\!]$, and the following diagram commutes, i.e. $f = h \circ g$:*

$$\begin{tikzcd}
R \arrow[r, "f"] \arrow[dr, hook, "g"'] & S \\
& R[x_1, \cdots, x_n] \arrow[u, "h"']
\end{tikzcd}$$

Sketch of Proof. The idea is similarly consider substitution $x_i \mapsto a_i$, and proceed to verify that this is indeed a ring homomorphism. One step that requires caution is that polynomials of several variables are defined in an inductive manner; therefore here proof should also be done inductively, on the number of variables involved. $\square$

Using polynomial of several variables, it is clearer to formalize the "generating set" of a ring via specifying which element each variable maps to:

Definition 3.5 (Finitely Generated R-algebra)**.** *Let R be a commutative ring, with A a commutative R-algebra. Fix $(a_1, \cdots, a_n) \in A$. By the universal property of polynomial of several variables, there exists a unique homomorphism $\varphi : R[x_1, \cdots, x_n]$ s.t. $\varphi(x_i) = a_i$. Then the subalgebra $\operatorname{im}\varphi$ is said to be* ***generated*** *by $\{a_1, \cdots, a_n\}$.*

Remark 3.3. Using the samre-formalization as in the definition above, $\operatorname{im}\varphi$ is smallest R-subalgebra of A that contains $\{a_1, \cdots, a_n\}$.

Proof. It is clear that $\operatorname{im}\varphi$ contains $\{a_1, \cdots, a_n\}$. To see that it is smallest, suppose there is a smaller one A', then there must be some $\sum_{i=0}^{n} a_i x^i \notin A'$, which contradicts with the fact that a ring should be closed. $\square$

Notice that in the definition of polynomial ring it is only required that x could be multiplied with powers of itself. This enables making polynomial a representation of groups:

Definition 3.6 (Group Ring)**.** *Let R a commutative ring, and G a group. A **group ring of** R **on** G is defined as*

$$R[G] := \left\{ \sum_{g \in G} a_g g \mid a_g \in R \right\}$$

with the addition and multiplication the same as that in the polynomial ring.

Remark 3.4. The operation between the ring and the group is not required to be defined and is simply a notation. The polynomial cannot admit any structure that is more complicated (e.g. changing the group to be a ring) as otherwise the addition will not be well-defined.

4 Ideals

Definition 4.1 (Finitely-Generated Ideals)**.** *Let R be a ring. Then*

- *Let (I_α) be a family of ideals for $\alpha \in \Lambda$ the index set, then the **ideal generated by (sum of)** (I_α) is defined as*

$$\sum_{\alpha \in \Lambda' \subseteq \Lambda} I_\alpha := \left\{ \sum_{\alpha \in \Lambda'} a_\alpha \middle| a_\alpha \in I_\alpha, |\Lambda'| \textit{ finite} \right\}$$

- *Alternatively one could consider the **ideal generated by (product of)** two ideals (which can be easily extended to several ideal cases) I and J to be*

$$I \cdot J := \left\{ \sum_{i=1}^{n} a_i b_i \middle| n \in \mathbb{Z}_{>0}, a_i \in I, b_i \in J \forall i \right\}$$

- *Suppose further that R is commutative. Let $\Lambda := \{\lambda_1, \cdots, \lambda_n\}$ be a subset of R. Then the **ideal generated by** Λ is defined as*

$$(\lambda_1, \cdots, \lambda_n) := \left\{ \sum_{k=1}^{n} r_k \lambda_k \middle| r_k \in R \right\}$$

Remark 4.1. Ideals generated by only one element is principal. For finitely generated ideals, the ideal generated by a set of elements is the same as the ideal generated by the corresponding principal ideals of the elements. This simply results from the fact that $(a) = \{ra | r \in R\}$.

Specify R to be a commutative ring, with $I \subseteq R$ an ideal of R. Consider the following special cases of ideals:

Definition 4.2 (Radical Ideal)**.** *$I \subseteq R$ is a **radical ideal** if for all $a \in R$, $\exists n \in \mathbb{Z}_{>0}$ $a^n \in I \implies a \in I$.*

Definition 4.3 (Prime Ideal)**.** *$I \subseteq R$ is a **prime ideal** if $I \neq R$, and for all $a, b \in R$, $ab \in I \implies (a \in I) \vee (b \in I)$.*

Definition 4.4 (Maximal Ideal)**.** *$I \subseteq R$ is a **maximal ideal** if $I \neq R$; and there is no ideal J in R s.t. $I \subsetneq J \subsetneq R$.*

Remark 4.2. Recall that R is a domain if and only if for all $a, b \in R$, $ab = 0 \implies a = 0 \vee b = 0$. This implies that for any ring R with $\mathfrak{p}$ a prime ideal in it, $R/\mathfrak{p}$ is a domain.

Definition 4.5 (Reduced Ring)**.** *A R is a **reduced ring** if and only if it does not have any nilpotent elements, i.e. for all $u \in R$, $u^n = 0 \implies u = 0$ for all $n \in \mathbb{Z}_{>0}$.*

Remark 4.3. For a commutative ring R, I is a radical ideal if and only if R/I is a reduced ring.

Proposition 4.1. *I is a maximal ideal if and only if R/I is a field.*

Proof. This fact follows directly from the following simple lemma. □

Lemma 4.1. *$R = K$ is a field if and only if it only has two ideals (0) and (1).*

Proof. Consider in both directions:

- $\Rightarrow$: If K is a field, then either there are no invertible elements, which in this case the ideal I can only contain 0 as this is the only non-invertible element in a field; or 1 and therefore every element is in the ideal, as $\forall g \in I, \exists g^{-1} \in K, gg^{-1} = 1 \in I$.
- $\Leftarrow$: If a ring R has only two ideals (0) and (1), then for all $0 \neq u \in R$ consider (u). By hypothesis $(u) = (1)$, i.e. there exists some $u^{-1} \in R$, which implies that R is actually a field.

□

Proposition 4.2. *An ideal being maximal implies that it is prime; and an ideal being prime implies that it is radical.*

Proof. Maximal ideals are prime. Suppose that $I \subseteq R$ is maximal but is not prime, i.e. there exists some $a, b \in R$ s.t. $ab \in R, a \notin R, b \notin R$. By hypothesis $I \cup \{a\} = R$., i.e. there exists some $r \in R, t \in I$ s.t. $a + rt = 1$. But then $b = ba + (br)t \in I$ which is a contradiction.

Prime ideals are radical. Consider inductively on a and a^{n-1}; apply the definition of prime ideals. □

Example 4.1. Consider counterexamples of the converse of the proposition above:

- $\mathbb{Z}_N$ for N not a power of prime is radical, but not prime.
- A trivial case for an ideal being prime but not maximal is (0), where as long as the ring is not a field, it is maximal.
- A more interesting case for an ideal being prime but not maximal is for finitely generated non-PIDs, adding a generator to a prime ideal suffices to create a "larger" ideal. Take the example $(x) \subseteq R[x]$ where R is a domain, which is prime as $R[x]/\langle x \rangle \cong R$ is also a field. But $(x) \subseteq (2, x)$ which is not the whole ring.

5 Noetherian Ring

Lemma 5.1 (Zorn's Lemma)**.** *Suppose that $(P, \leq)$ is an ordered set s.t. every totally order subset $P_0 \subseteq P$ has an upper bound, then P has a maximal element.*

Theorem 5.1. *Let $I \subseteq R$ be an ideal of a commutative ring R. Then there exists some maximal ideal M s.t. $I \subseteq M$.*

Proof. The proof is simply a re-formalization of Zorn's Lemma (Lemma 5.1).

Consider $P := \{J \subseteq R \mid J \text{ ideals}, I \subseteq J, J \neq R\}$, with the order of inclusion. Take $P_0 := \{I_\alpha \mid \alpha \in \Lambda\} \subseteq P$ to be totally ordered. Then $J := \bigcup_\alpha I_\alpha$ is also an ideal. Further $1 \notin J$, otherwise there will exist some $\alpha \in \Lambda$ s.t. $I_\alpha = R$, which contradicts the hypothesis. Therefore J is the upper bound for the family P_0. Applying Zorn's Lemma finishes the proof. □

Definition 5.1 (Noetherian Ring)**.** *A ring R is (left)* ***Noetherian*** *if it satisfies the Ascending Chain Condition (ACC), for (left) ideals, i.e. there is no infinite strictly increasing sequence of (left) ideals:*

$$I_1 \subsetneqq I_2 \subsetneqq \cdots$$

Proposition 5.1. *Let R be a ring, then the followings are equivalent:*

1. *R is (left) Noetherian.*
2. *Let P be a family of (left) ideals in R, then P has a maximal element.*
3. *Every (left) ideal in R is finitely generated.*

Proof.

- (i) being equivalent to (ii) is via simply reformalizing the definition.
- (i) implies (iii). Proceed by proving the contraposition. Suppose that there exists an ideal $I_0 \subseteq R$ that is not finitely generated, then there exists an infinite sequence of generators of I_0 (a_i), $i \in \mathcal{I}$. Then there exists an infinite ACC $(a_1) \subsetneqq (a_1, a_2) \subsetneqq (a_1, \cdots, a_k), \subsetneqq \cdots$.
- (iii) implies (i). Prove by showing a contradiction. Suppose that there exists an infinite ACC $I_1 \subsetneqq I_2 \subsetneqq \cdots \subsetneqq I_k \subsetneqq \cdots$. Then consider $I := \bigcup_{n \geq 1} I_n$. By the hypothesis it is finitely generated, i.e. there exists some $(a_1, \cdots, a_m)$ s.t. $a_i \in I_{n_i}$ for all $i \in [\![1, m]\!]$. Define $n := \max\{n_i \mid i \in [\![1, m]\!]\}$. Then $I_n = I_{n+1}$ which is a contradiction.

□

Theorem 5.2 (Hilbert's Basis Theorem)**.** *Let R be a commutative Noetherian ring. Then $R[x]$ is a Noetherian ring.*

Proof. By proposition 5.1 it suffices to show that every ideal of $R[x]$ is finitely generated.

In the case that $I = (0)$, it is finitely generated as R is Noetherian. For the case of that $I \neq (0)$, consider a family of ideals where $f_1 \in I \setminus \{0\}$, with $f_k \in I \setminus (f_1, \cdots, f_k)$ for $k > 1$ s.t. $\deg f_k = \min\{\deg f \mid f \in I \setminus (f_1, \cdots, f_k)\}$. If there exists some k s.t. $(f_1, \cdots, f_k) = I$ then $R[x]$ is by definition Noetherian. Suppose that it is not. Then there exists an infinite ascending chain. Denote $f_n = a_n x^{d_n} + \sum_{k=0}^{d_n - 1} a_k x^k$. From the construction it is clear that $d_1 \leq d_2 \leq \cdots \leq d_n \leq \cdots$.

Define $I := (a_1, \cdots, a_n \mid n \geq 1)$. By hypothesis $I \subseteq R$, which implies that it is finitely generated. Then there exists some k s.t. $I = (a_1, \cdots, a_k)$, with $d_i \geq 1$ (otherwise suppose there exists some $a_0 \in R \setminus (a_1, \cdots, a_k)$, simply add $a_0 x$ to the generators; and do the similar to ensure that the degree of polynomial associated with the corresponding coefficients is at least one. Since R is Noetherian, it is finitely generated, i.e. the process above will terminate, which does not interfere with the condition that the ascending chain does not terminate.)

For f_{k+1}, we know that there exists a family $(c_j)_{j=1}^k$ s.t. $a_{k+1} = \sum\limits_{j=1}^{k} c_j a_j$ since $(a_1, \cdots, a_k)$ are generators. Then consider

$$f = f_{k+1} - \sum_{i=1}^{k} c_i x^{d_{k+1} - d_i} f_i$$

which is a polynomial that is not in $I \setminus (f_1, \cdots, f_n)$, which is a contradiction. □

Corollary 5.1. By induction $R[x_1, \cdots, x_n]$ is also Noetherian if R is Noetherian. Quotient and localization preserves the property that a ring is Noetherian.

6 Euclidean Domain, PIDs and UFDs

Definition 6.1 (Principal Ideal Domain (PID)). *Let R be a integral domain. R is a **Principal Ideal Domain (PID)** if every ideal in R is principal.*

Remark 6.1. If R is a PID, then R is Noetherian, as principal ideals are by definition finitely generated.

Proposition 6.1. *If R is a PID, then every prime ideal in it is maximal.*

Proof. Prove by contradiction. Suppose that $I = (p)$ is a prime ideal that is not maximal. Then by Theorem 5.1 there exists some maximal ideal $x \notin I$ s.t. $I \subseteq (x)$, i.e. there exists some $r \in R$ s.t. $p = xr$. Since $r \notin I, r \in P$. Write $r = pr'$ for $r' \in R$. Then $xr' = 1$, i.e. $(x) = (1)$ which is a contradiction. □

Definition 6.2 (Euclidean Domain). *A **Euclidean Domain** is an integral domain R, for which there exists a function (norm) $N : R \setminus \{0\} \to \mathbb{Z}_{\geq 0}$, s.t. $\forall a, b \in R, \neq 0$, there exists some $q, r \in R$ s.t. $a = bq + r$; and either $r = 0$, or $N(r) < N(b)$.*

Proposition 6.2. *A Euclidean Domain is a PID.*

Proof. Let R be a euclidean domain. Since the domain of the norm is $\mathbb{Z}_{\geq 0}$, there exists some element b s.t. $N(b)$ is minimal. Claim that $R = (b)$.

This is indeed true, as there does not exist any r s.t. $N(r) < N(b)$. Then apply the definition of a Euclidean Domain. □

Definition 6.3. *Let $a, b \in R \setminus \{0\}$. Then a is **associated with** b (denoted $a \sim b$) if there exists some u invertible, s.t. $a = ub$.*

Remark 6.2. $a \sim b$ if and only if $(a) = (b)$.

Definition 6.4 (Greatest Common Divisor). *Let $a, b \in R$ that are not both zero. The **Greatest Common Divisor** of a and b is an element in $R \setminus \{0\}$ s.t. $d \mid a, d \mid b$; and for all $x \in R \setminus \{0\}$, $x \mid a \wedge x \mid b \implies x \mid d$.*

Proposition 6.3. *Let R be a domain, and d be the gcd of a and b. If $(a, b) = (d)$, then $d = \gcd(a, b)$.*

Proof. d is a common divisor of a and b as $a, b \in (d)$. It is the greatest one as since $d \in (a, b)$, there exists some $\lambda, \mu \in R$ s.t. $\lambda a + \mu b = d$. Both sides should divide d, which implies that if there exists some $d' \mid a, d' \mid b$, then $d' \mid d$. □

Definition 6.5 (Prime; Irreducible)**.** *Let R be a domain, and a a non-zero element. Then*

- *a is a **prime** if (a) is a prime ideal.*
- *a is **irreducible** if for all $b_1, b_2 \in R$ s.t. $a = b_1 b_2$, either b_1 is invertible or b_2 is invertible.*

Proposition 6.4. *Let R be a PID and $r \in R$ a non-zero element. Then r is irreducible if and only if (r) is a maximal ideal.*

Proof. Proceed by showing implication in two directions:

⇒: Let r be an irreducible element. Suppose that there exists an ideal I s.t. $(r) \subsetneqq I \subsetneqq R$. Since R is a PID, there exists some $a \in R$ s.t. $I = (a)$, which indicates that there exists some $x \in R$ s.t. $r = ax$. But since r is irreducible, either a is a unit, i.e. $I = R$, or x is a unit, i.e. $I = (r)$. Both of which lead to a contradiction.

⇐: Proceed by showing the contraposition. Suppose that r is not irreducible, then there exists $p, q \in R$ which are not units s.t. $r = pq$. Then $(r) \subsetneqq (p) \subsetneqq R$ which implies that (r) is not maximal.

□

Proposition 6.5. *If a is prime, then a is irreducible.*

Proof. Let a be a prime. Suppose that there exists $b_1, b_2 \in R$ s.t. $b_1 b_2 = a$. Then $b_1 b_2 \in (a)$. Without loss of generality assume $b_1 \in (a)$, i.e. there exists some $r \in R$ s.t. $b_1 = ar$. This gives $arb_2 = a$, i.e. b_2 is invertible. □

Remark 6.3. The converse is generally not true. Consider in $\mathbb{Z}[\sqrt{5}i]$ which is not a UFD. Then (2) is not prime (as $2 \cdot 3 = (1 + \sqrt{5}i)(1 - \sqrt{5}i)$) but 2 is irreducible.

Definition 6.6 (Unique Factorization Domain (UFD))**.** *A domain R is a **Unique Factorization Domain (UFD)** if for all nonzero $a \in R$ that is not invertible, there exists a decomposition $a = p_1 \cdots p_r$ where $p_1, \cdots, p_r$ are irreducible. For all other families of irreducible elements $q_1, \cdots, q_r \in R$ s.t. $a = q_1 \cdots q_r$, there exists a permutation $\varepsilon : [r+1] \to [r+1]$ s.t. $p_i \sim q_{\varepsilon(i)} \forall i$.*

Proposition 6.6. *Let R be a UFD. Then every irreducible element $p \in R$ is prime.*

Proof. Claim that (p) is a prime ideal given that p is irreducible. Since p is irreducible and R is PID, for all $b_1 b_2 \in (p)$, there exists some irreducible q_is for $i \in I$ s.t. $b_1 b_2 = p \cdot \prod_{i \in I} q_i$. Since factorization unique, at least one of b_1 and b_2 admits a divisor p, which indicates that (p) is a prime ideal. □

Proposition 6.7. *Let R be a domain s.t. every irreducible element is prime. Then R is a UFD.*

Proof. It suffices to prove that factorization is unique up to permutation and multiplication by units. Suppose that p_is and q_is are two irreducible decomposition of a, i.e. $a = p_1 \cdots p_r = q_1 \cdots q_s$. Then either

- $r = 0$. Then a is a unit, which indicates that $s = 0$.

- $r \neq 0$. Then $s \neq 0$. Since p_i is prime for all i, there exists some q_j s.t. $p_i \mid q_j$. this implies that $r \leq s$. Then consider q_is as prime, which implies $s \leq r$ and therefore $s = r$. Further since p_is and q_is are irreducible, for $p_i \mid q_j$ this implies $q_j = p_i u$ for u a unit.

This verifies the definition of a UFD. ☐

Proposition 6.8. *Let R be a Noetherian ring. Then every element $a \in R$ attains an irreducible decomposition $a = p_1 \cdots p_r$ with p_i irreducible for all i.*

Proof. This is simply a re-formalization of the fact that Noetherian rings are finitely generated. Consider the following cases:

- a is irreducible. Then the factorization process is done.
- $a = b_1 b_2$ where b_1 and b_2 are both not units. Then consider separately b_1 and b_2 with this process. This process is sure to terminate at some point as otherwise this gives an ideal of infinite generators.

☐

Remark 6.4. Noetherian rings are generally not UFDs. A simple example is $\mathbb{Z}[\sqrt{5}i]$, the Gaussian Integers.

Theorem 6.1. *Every PID is a UFD.*

Proof. Since principal ideals are finitely generated, all PIDs are Noetherian. By proposition 6.8 there exists a decomposition; and by proposition 6.4 and 6.1 irreducible elements are prime. By proposition 6.7 it is a UFD. ☐

Example 6.1. An example where a ring is a UFD but not a PID (where prime ideals are not maximal) is $\mathbb{Z}[x]$, with the ideal $(2, x)$ which is not principal. (x) is prime, but not maximal.

The following proves the theorem:

Theorem 6.2. *Let R be a UFD, then $R[x]$ is also a UFD.*

Definition 6.7 (Primitive; Content)**.** *Let $f \in R[x]$ a nonzero polynomial. Then*

- *The **content** of f, denoted as $c(f)$ is the greatest common divisor of the coefficient of its terms.*
- *$f \in R[x]$ is **primitive** if its content is a unit.*

Lemma 6.1. *Let R be a UFD. Define $K := \operatorname{Frac}(R)$, i.e. $K = S^{-1}R$ for $S := R \setminus \{0\}$. A nonzero element $f \in R[x]$ is irreducible if and only if either of the following holds:*

- *$\deg f = 0$, and f is irreducible in R.*
- *$\deg f \geq 1$, f is primitive and is irreducible in $K[x]$.*

Proof. Consider the following two cases:

- $\deg f = 0$. Since $R \subseteq R[x]$, f irreducible in $R[x]$ implies that it is irreducible in R. For the converse, notice that R is a domain, where the degree of product of two polynomials is at the sum of the degree of the two polynomials, indicating that $f \in \mathbb{R}[x]$ could only attain degree 0 factors. The fact that f is irreducible in R finishes the proof.

- $\deg f \geq 1$. Consider the two directions:
  - $\Rightarrow$: Suppose that f is irreducible in $R[x]$. Notice that for all $g \in K[x]$, $c(g)^{-1}g \in R[x]$. Proceed by showing a contradiction. Suppose that there exists $f_1, f_2 \in K[x]$ of degree at least one s.t. $f = f_1 f_2$ (i.e. f is not irreducible in $K[x]$). Then

$$f = (c(f_1)^{-1}f_1)(c(f_2)^{-1}f_2)c(f_1)c(f_2)$$

    where the four operands for multiplication are all in R. Since f is irreducible in R, either $(c(f_1)^{-1}f_1)$ or $(c(f_2)^{-1}f_2)$ is a unit, which contradicts the hypothesis that $\deg f_1 \geq 1 \wedge \deg f_2 \geq 1$.
  - $\Leftarrow$: Proceed by showing that the contraposition is true. Suppose that $f = f_1 f_2$ where f_1, f_2 are both not units, in R. Then $f = f_1 f_2 \in K[x]$ which is also not irreducible.

□

Lemma 6.2. *Let K be a field. Then $K[x]$ is a PID.*

Proof. Let I be an ideal in $K[x]$. Define $k := \{\deg f \mid f \in I\}$. Such k indeed exists as the degree has a lower bound 0; and k could take only finitely many values with some element $f_0 \in I$ fixed; namely $[\![0, \deg f_0]\!]$. Claim that $I = (x^k)$.

Either $k = 0$, where $I = (1)$; or $k \neq 0$, where for all $f = \sum_{i \mid d_i \geq d} c_i x^{d_i} \sum_i c_i x^{d_i - d} \in K[x]$. □

Proof of Theorem 6.2. Define $K = S^{-1}R$ for $S = R \setminus \{0\}$. From lemma 6.2 we know $K[x]$ is a PID, which is therefore a UFD. The general strategy is to transform the whole problem into $K[x]$ using lemma 6.1, and use the fact that $K[x]$ is a UFD, with elements differ only by a factor in R (which is also a UFD) from those in $R[x]$.

It suffices to show that the decomposition exists and is unique:

- *Existence.* Decompose f in $R[x]$ $f = c(f)g$ s.t. g is primitive. Then $c(g) = u$ where u is some unit in R. Applying the inclusion map gives $g \in K[x]$, where it could be decomposed into $g = g_1 \cdots g_n$ where g_is are irreducible. Denote $g_i = c(g_i)h_i$, which gives $g = \prod_{i=1}^n c(g_i)h_i = c(g)\prod_{i=1}^n h_1 = u \prod_{i=1}^n h_1$. Since $c(f) \in R$ which is a UFD, there exists a decomposition $c(f) = f_1 \cdots f_n$. This gives an irreducible decomposition $f = f_1 \cdots f_n h_1 \cdots h_n$.
- *Uniqueness.* This follows from the fact that both f and $K[x]$ are UFDs, i.e. decomposition of $f \in R$ and $g \in K[x]$ are unique.

  (Alternatively one could prove that irreducible elements in $R[x]$ are also prime, which is essentially the same approach as the content is prime follows from the fact that R is UFD; and the primitive is prime as $K[x]$ is a UFD).

□