MATH 593 - Introduction to Homological Algebra

ARessegetes Stery

December 12, 2023

Contents

1 Exactness

Definition 1.1 (Complex)**.** *A* ***Complex*** *of R-modules is a family of R-modules (M_i) and R-linear maps $d_i : M_i \to M_{i+1}$ s.t. for all i, $d_{i+1} \circ d_i = 0$.*

Remark 1.1. The followings are some specifications on the notations:

- The complex is often denoted by a chain

$$\cdots \xrightarrow{d_{i-2}} M^{i-1} \xrightarrow{d_{i-1}} M^i \xrightarrow{d_i} M^{i+1} \xrightarrow{d_{i+1}} \cdots$$

  or a chain with indices on the bottom with $M^i = M_{-i}$.

- The complex extends to infinity in both ends. If the notation terminated on one side, all modules not written out are the trivial (the zero module).

Remark 1.2. The definition of a complex is the same as stating that $\operatorname{im} d_i \subseteq \ker d_{i+1}$ for all i.

Definition 1.2. *For a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ where f and g are R-linear maps, it is* ***exact at*** $\mathbf{B}$ *if the equality is reached in the remark above, i.e.* $\operatorname{im} f = \ker g$.

A sequence is exact if it is exact at A_i for all i. A complex is exact if it is exact everywhere.

Example 1.1. The sequence $0 \longrightarrow A \xrightarrow{f} B$ is exact implies that $\ker f = \{0\}$, i.e. f is injective. Similarly, $A \xrightarrow{g} B \longrightarrow 0$ implies that g is surjective.

Definition 1.3. *A* ***Short Exact Sequence (SES)*** *is an exact sequence*

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$$

Proposition 1.1. *Given a sequence $(*) : 0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$, the followings are equvalent:*

i) *$(*)$ is a short exact sequence.*

ii) *i is injective, and for $q : M \to \operatorname{coker} i$, there exists a unique isomorphism θ s.t. $\theta \circ q = p$.*

iii) *p is surjective, and for $j : M \to \ker p$, there exists a unique isomorphism η s.t. $i = \eta \circ j$.*

Proof. It suffices to prove the equivalence between i) and ii), as the case with iii) is similar:

- *i) ⇒ ii).* Apply the universal property of cokernel. Since $(*)$ is exact, $p \circ i = 0$, there exists a map θ s.t. the following diagram commutes. The fact that p is surjective, and the diagram should commute gives θ should be surjective. To prove that θ is

$$\begin{tikzcd} M' \arrow[r, "i"] & M \arrow[r, "p"] \arrow[dr, "q"] & M'' \\ & & \operatorname{coker} i \arrow[u, "\theta"] \end{tikzcd}$$

  injective, it suffices to verify that $\theta(b) = 0 \implies b = 0$ for $b \in \operatorname{coker} i$. Since q by definition is surjective, there exists $a \in M$ s.t. $q(a) = b$. This gives $a \in \ker p = \operatorname{im} i$, which implies that $q(a) = 0$ as the cokernel is defined by $M/\operatorname{im} i$.

- *ii)* $\Rightarrow$ *i).* Given that μ is an isomorphism and i is injective, it suffices to verify that p is surjective, and $\operatorname{im} i = \ker p$. μ being surjective implies that p is surjective; and μ being an isomorphism implies that $\ker p = \ker q = \operatorname{im} i$.

$\square$

Proposition 1.2. *Given a short exact sequence* $0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$*, the following statements are equivalent:*

i) There exists $j : M \to M'$ *s.t.* $j \circ i = \mathrm{Id}_{M'}$

ii) There exists $q : M'' \to M$ *s.t.* $p \circ q = \mathrm{Id}_{M''}$

iii) There exists a submodule $N \subseteq M$ *s.t.* M *can be expressed by the internal direct sum* $M = i(M') \oplus N$*; and* p *induces an isomorphism* $N \simeq M''$*.*

Such a short exact sequence is a <u>*split exact sequence*</u>*.*

Proof. It suffices to give the equivalence between i) and iii), as for ii) it is similar.

- *i)* $\Rightarrow$ *iii).* Let $N = \ker j$. Check that this gives an internal direct sum:
  - $N \cap i(M') = \{0\}$. Let $x \in i(M') \cap N$. Then there exists $u \in M'$ s.t. $i(u) \in \ker j$, i.e. $j \circ i(u) = 0$. But this indicates that $u = 0$ as $j \circ i = \mathrm{Id}_{M'}$. Since i is a morphism of modules, $i(0) = 0$, which indicates that the only element that is in both $i(M')$ and N is 0.
  - $N + i(M') = M$. Notice $v - i \circ j(v) \in \ker q$, and by inspection $i \circ j(v) \in \operatorname{im} i$.

  By the first isomorphism theorem, $\operatorname{im} i = \ker p$ implies $M/\operatorname{im} i \simeq N \simeq M''$.

- *iii)* $\Rightarrow$ *i).* Define $j : i(M') \oplus N \to i(M') \simeq M'$ since i is injective.

$\square$

Remark 1.3. Generally short exact sequences do not split. A counterexample is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} 2\mathbb{Z} \xrightarrow{p} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

where $i(\mathbb{Z}) \simeq \mathbb{Z}$, but the inverse map cannot be extended properly to the whole ring $\mathbb{Z}$. If R is a field, then all short exact sequences split as one can complete a basis in a vector space; and subspaces spanned by a subset of a basis is always a direct summand of the whole space.

The following present a common technique known as "diagram chasing":

Proposition 1.3 (The 5-Lemma). *Consider the following diagram, with blocks commute and rows exact:*

$$\begin{tikzcd}
A_1 \arrow[r, "u_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "u_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "u_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "u_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "v_1"] & B_2 \arrow[r, "v_2"] & B_3 \arrow[r, "v_3"] & B_4 \arrow[r, "v_4"] & B_5
\end{tikzcd}$$

1. *If* f_2, f_4 *are injective,* f_1 *is surjective, then* f_3 *is injective.*

2. *If f_2, f_4 are surjective, f_5 is injective, then f_3 is surjective.*

3. *(Combining i) and ii)) If f_1, f_2, f_4, f_5 are all isomorphisms, then f_3 is an isomorphism.*

Proof. The argument is symmetric, so it suffices to prove the first one. f_3 is injective if and only if $f_3(b) = 0 \implies b = 0$. Following the steps:

- Consider the third square. $v_3 \circ f_3(b) = v_3(0) = 0$, giving $f_4 \circ u_3(b) = 0$. f_4 being injective implies that $u_3(b) = 0$.
- Consider the second square. The top row being exact implies that $b \in \operatorname{im} u_2$, i.e. there exists some $c \in A_2$ s.t. $u_2(c) = b$. Commutativity gives that $v_2 \circ f_2(c) = 0$, i.e. $c' := f_2(c) \in \ker v_1$.
- Consider the first square. The bottom row being exact implies that there exists some $d' \in B_1$ s.t. $v_1(d') = c'$. Since f_1 is surjective, there exists $d \in A_1$ s.t. $f_1(d) = d'$. For the diagram to commute, it is required that $u_1(d) = c$. But this indicates that $c \in \operatorname{im} u_1$, i.e. $c \in \ker u_2$, which gives $b = u_2(c) = 0$.

□

Definition 1.4. *Let R and S be rings, and $F : {}_R\underline{\text{Mod}} \to {}_S\underline{\text{Mod}}$ is an additive functor. Then F is* ***exact*** *if for all short exact sequences of R-modules $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$, the corresponding sequence after applying F is also exact.*

Proposition 1.4. *F is exact if and only if for all exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$, $F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C)$ is also exact.*

Proof. Proceed by showing implication in two directions:

⇐: This holds by definition, where one can consider the particular case where f is injective and g is surjective.

⇒: Consider the following short exact sequences:

$$\begin{aligned} (1):& \qquad 0 \longrightarrow \ker f \longrightarrow A \xrightarrow{\alpha_1} \operatorname{im} f \longrightarrow 0 \\ (2):& \qquad 0 \longrightarrow \ker g \xrightarrow{\alpha_2} B \xrightarrow{\beta_1} \operatorname{im} g \longrightarrow 0 \\ (3):& \qquad 0 \longrightarrow \operatorname{im} g \xrightarrow{\beta_2} C \longrightarrow \operatorname{coker} g \longrightarrow 0 \end{aligned}$$

where $\operatorname{im} f = \ker g$ as the sequence given is exact. These by construction are all short exact sequences, where applying F gives also short exact sequences. Combining gives the sequence which is still exact after applying F:

$$A \xrightarrow{\alpha_1} \operatorname{im} f \xrightarrow{\alpha_2} B \xrightarrow{\beta_1} \operatorname{im} g \xrightarrow{\beta_2} C$$

where α_1, β_1 are surjective; and α_2, β_2 are injective. What we want to show is $\operatorname{im} F(f) = \ker F(g)$. Since α_1 is surjective, $\operatorname{im} F(f) = \operatorname{im} F(\alpha_2)$; and since β_2 is injective, $\ker F(g) = \ker F(\beta_1)$. From the result of (2) after applying F, we have $\operatorname{im} F(\alpha_2) = \ker F(\beta_1)$.

□

Remark 1.4. "One-sided" exact sequences can be understood functorially:

- Given exact sequence $0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M''$ is the same as saying that i is injective; and M' is the kernel of p.

- Similarly, given exact sequence $M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$ is the same as saying that p is surjective; and M'' is the cokernel of i.

Definition 1.5. *Just as in the remark, one could consider exact functors only on one side. $F : {}_R\underline{\mathrm{Mod}} \to {}_S\underline{\mathrm{Mod}}$ is **left exact** if for all exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C$, the sequence $0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$ is also exact; and the definition is symmetric for right exact functors. Notice that since F is an additive functor, $F(0) = 0$ (as zero morphisms are mapped to zero morphisms).*

Proposition 1.5. *Let M be an R-S bimodule. Then functor $F = \mathrm{Hom}_R(M, -) : {}_R\underline{\mathrm{Mod}} \to {}_S\underline{\mathrm{Mod}}$ is left exact; and the converse is also true, i.e. if $0 \longrightarrow \mathrm{Hom}_R(M, A) \longrightarrow \mathrm{Hom}_R(M, B) \longrightarrow \mathrm{Hom}_R(M, C)$ is exact, then $0 \longrightarrow A \longrightarrow B \longrightarrow C$ is exact.*

Proof. What we first want to show is that if the sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C$ is exact, then the corresponding sequence $0 \longrightarrow \mathrm{Hom}(M, A) \longrightarrow \mathrm{Hom}(M, B) \longrightarrow \mathrm{Hom}(M, C)$ is exact: The natural way to define the functor F is via specifying

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \\ & & M \arrow[ul, dashed, "u"] \arrow[u, "v"'] & \end{tikzcd}$$

$\mathrm{Hom}(M, A) \ni u \mapsto f \circ u$, $\mathrm{Hom}(M, B) \ni v \mapsto g \circ v$. Exactness follows from the universal property of kernel, where for all $v \in \mathrm{Hom}(M, B)$ s.t. $\mathrm{Hom}(M, C) \ni g \circ v = 0$, it factors uniquely through f. Furthermore, since F is an additive functor, it preserves injectivity (via considering elements in $\underline{\mathrm{Sets}}$), which indicates that $F(f)$ is also injective.

For the converse, take $M = A$. This gives the exact sequence $0 \longrightarrow \mathrm{Hom}(A, A) \xrightarrow{\alpha} \mathrm{Hom}(A, B) \xrightarrow{\beta} \mathrm{Hom}(A, C)$. The natural ways to define the map is via specifying $\alpha = f \circ -$, $\beta = g \circ -$. Observe that the exactness of the sequence gives $\beta \circ \alpha = g \circ f \circ - = 0$, which indicates that $g \circ f = 0$ by associativity. α being injective follows directly from the fact that f is injective. $\square$

Remark 1.5. The dual argument is also true, via applying the universal property of cokernel. That is, the functor $\mathrm{Hom}_R(-, M)$ is right exact. The direction of exactness reverses as the functor is contravariant.

Proposition 1.6. *If M is an S-R bimodule, then $M \otimes_R -$ is right exact.*

Proof. From the right exact version of Proposition 1.5, it suffices to prove that for left R-module N the sequence

$$\mathrm{Hom}(M \otimes A, N) \longrightarrow \mathrm{Hom}(M \otimes B, N) \longrightarrow \mathrm{Hom}(M \otimes C, N) \longrightarrow 0$$

is exact. First make a parenthesis on the generalization of the adjoint property of extension and restriction of scalars:

Parenthesis 1.1 (Prop 4.4 [c6])**.** Let M be an S-R bimodule, C a left R-module, and N a left S-module, then there is a functorial isomorphism

$$\mathrm{Hom}_S(M \otimes_R C, N) \simeq \mathrm{Hom}_R(C, \mathrm{Hom}_S(M, N))$$

Proof. By the universal property of tensor product, it suffices to give every R-balanced map $f : M \times C \to N$ a map in $\mathrm{Hom}_R(C, \mathrm{Hom}_S(M, N))$. Let the isomorphism be F defined via $F(\tilde{f}[u \otimes v \mapsto f(u, v)]) = v \mapsto (u \mapsto f(u, v))$. The inverse exists by inspection. It is well-defined as one can consider the map $u \times v \mapsto f(u, v)$; and use the universal property of tensor product. $\square$

Then apply the parenthesis and Proposition 1.5 gives that it suffices to verify that $0 \longrightarrow \mathrm{Hom}(M, C) \longrightarrow \mathrm{Hom}(M, B) \longrightarrow \mathrm{Hom}(M, A)$ is exact, which holds as M is in particular a right R-module; and in this case the functor $\mathrm{Hom}_R(-, \mathrm{Hom}_S(M, N))$ is a contravariant functor. Recall that for a contravariant functor $\mathrm{Hom}(-, N)$, if the sequence $A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact, then $0 \longrightarrow \mathrm{Hom}(C, N) \longrightarrow \mathrm{Hom}(B, N) \longrightarrow \mathrm{Hom}(A, N)$ is exact. This finishes the proof. $\square$

Remark 1.6. Recall that the above isomorphism holds for all adjoint pairs (F, G). Therefore, the proof applies as long as F is left exact (for G being right exact) or the converse holds.

In general, the above two functors and the contravariant is only left (right) exact instead of exact. Consider the short exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow 2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

- Applying $- \otimes \mathbb{Z}/2\mathbb{Z}$ gives
$$0 \longrightarrow \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \xrightarrow{i} 2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \xrightarrow{p} \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$
where $2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} = 0$ and $\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}$, as the map being R-linear restricts that $f(0, 1) = f(1, 0) = f(0, 0)$. This implies that i is not injective.

- Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/2\mathbb{Z})$ gives
$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{i} \mathrm{Hom}_{\mathbb{Z}}(2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{p} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$
where $\mathrm{Hom}_{\mathbb{Z}}(2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \ni h = 0$ as it is required that $h(1) + h(1) = \bar{1} + \bar{1} = 0$ which indicates that p is not surjective. Similar situations appear in the contravariant case.

It is then of specific interest in which modules are the above functors exact.

2 Flat, Projective, and Injective Modules

Definition 2.1. *Let R be a ring, and M a left R-module. Then:*

- *M is a **flat module** if $- \otimes_R M$ is an exact functor.*
- *M is a **projective module** if $\mathrm{Hom}_R(M, -)$ is an exact functor.*
- *M is an **injective module** if $\mathrm{Hom}_R(-, M)$ is an exact functor.*

Remark 2.1. By comparing with the results obtained in the propositions aforementioned (Proposition 1.5, 1.6), it is clear what is further required by the definitions:

- By Proposition 1.6, a module is flat if and only if for all injective maps $M_1 \to M_2$, the corresponding map $M_1 \otimes M \to M_2 \otimes M$ is injective.
- By Proposition 1.5, a module is projective if and only if for all surjective maps $M_1 \to M_2$, the corresponding map $\mathrm{Hom}_R(M, M_1) \to \mathrm{Hom}_R(M, M_2)$ is surjective; or the corresponding map is injective if the functor is contravariant.

Remark 2.2. The fact that Hom functors in the remark above is surjective is the same as the following definition (for projective modules): M is a projective R-module if and only if for all morphism of R-modules $g : M \to V$ with module U for which there exists a surjective morphism $f : U \to V$, there exists a morphism of R-modules $h : M \to U$ s.t. $g = f \circ h$; that is, making the following diagram commute:

$$\begin{tikzcd} & U \arrow[d, two heads, "f"] \\ M \arrow[ur, dashed, "h"] \arrow[r, "g"] & V \end{tikzcd}$$

In plain words, there exists an embedding into some module (for example, free modules) that could "project" surjectively to V. This embedding (injection) definitely needs not be unique, as for example one could always embed the projective module to a free module with higher rank.

This definition is indeed equivalent with the previous one, as for a surjective morphism $f : U \to V$, for any $g : M \to V$ there exists some $h : M \to U$ s.t. the diagram commute. This indicates that the morphism $\mathrm{Hom}(M, U) \to \mathrm{Hom}(M, V)$ is a surjection.

The dual result holds also for injective modules: for all R-module M being injective, it is equivalent to state that for any morphism $g : M \to V$, there exists some R-module U with an injection $f : V \to U$ and some morphism $h : U \to M$ s.t. $g = h \circ f$, i.e. the following diagram commutes:

$$\begin{tikzcd} M & U \arrow[l, dashed, "h"'] \\ & V \arrow[ul, "g"] \arrow[u, hook, "f"'] \end{tikzcd}$$

Proposition 2.1. *Let M be a left R-module. Then the following statements are equivalent:*

i) M is a projective R-module.

ii) M is a direct summand of a free R-module. That is, there exists a free R-module F and an R-module N s.t. $F \simeq M \oplus N$.

iii) Let P be an R-module, and F be a free R-module. Then every short exact sequence $0 \longrightarrow P \longrightarrow F \longrightarrow M \longrightarrow 0$ is split exact.

Proof.

- *i) $\implies$ iii).* Apply the definition given in Remark 2.2. M being an R-module implies that it admits a system of generators, namely there exists some $(g_i)_{i \in I}$ s.t. $M = (g_i)_{i \in I}$. Then there exist $F = R^{(I)}$ where there exists a surjection $f : F \to M, f(e_i) = g_i$. Let $g : M \to M$ be the identity map. By the alternative definition there exists some $h : M \to F$ s.t. $f \circ h = g = \mathrm{Id}_M$. By Proposition 1.2 this indicates that the sequence of interest splits.

- *iii) $\implies$ ii).* Consider the projection $p : F \to M$ where $M = (g_i)_{i \in I}$ and $F = R^{(I)}$. Therefore, the sequence

  $$0 \longrightarrow \ker p \xrightarrow{i} F \longrightarrow M \longrightarrow 0$$

  is short exact, which gives $F \simeq i(\ker p) \oplus M = \ker p \oplus M$.

- *ii)* $\implies$ *i)*. Consider F as a projective module. This is true as F can be trivially embedded to itself. It then suffices to prove that if F is projective and $F \simeq M \oplus N$, then M is projective. By Remark 2.1, it suffices to prove that $\mathrm{Hom}_R(M, U) \to \mathrm{Hom}_R(M, V)$ is surjective for all $U \to V$ surjective. F is projective indicates that $\mathrm{Hom}_R(F, U) \to \mathrm{Hom}_R(F, V)$ is surjective. By the fact that $\mathrm{Hom}_R(F, U) \simeq \mathrm{Hom}_R(M, U) \oplus \mathrm{Hom}_R(N, U)$, $\mathrm{Hom}_R(M, U) \to \mathrm{Hom}_R(M, V)$ is surjective.

□

Corollary 2.1. The category of ${}_R\underline{\mathrm{Mod}}$ has enough projective modules. In particular, for $M = (g_i)_{i \in I}$ one can take $F = R^{(I)}$.

Similarly, we would like to prove that there are "enough" injective objects in ${}_R\underline{\mathrm{Mod}}$:

Theorem 2.1. *The category of ${}_R\underline{\mathrm{Mod}}$ has enough injective objects. That is, for all R-module M, there exists an injective embedding $M \hookrightarrow Q$ where Q is an injective module.*

Proposition 2.2 (Baer). *Let M be an R-module. Then M is injective if and only if $\mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(I, M)$ is surjective for all left ideals $I \subseteq R$. That is, for all R-linear maps $f : I \to M$, there exists some $u \in M$ s.t. $f(x) = xu$ for all $x \in I$ (and this gives an extension into R).*

Proof. The implication from M being injective to the condition follows directly from the definition of injective modules, where one considers the map $I \hookrightarrow R$ to be the injection. For the other direction, consider the following construction:

Let $i : M_1 \hookrightarrow M_2$ be an injection, where M_1, M_2 are R-modules; and let $f : M_1 \to M$ be an R-linear map. It suffices to show that there exists some $g : M_2 \to M$ s.t. $g \circ i = f$. Consider the family of modules N_i and corresponding maps h_i s.t. $M_1 \subseteq N_i \subseteq M_2$ for all i; and $h_i|_{M_1} = f$. Define the partial order $(N, h) \leq (N', h')$ if and only if $N \subseteq N'$, and $h'|_N = h$. Notice that such family is non-empty, as in particular M_1 is in the family. Further it is bounded above by M_2, which allows us to apply Zorn's Lemma to retrieve a maximal element $(\bar{N}, \bar{h})$. Handle the cases respectively:

- $\bar{N} = M_2$. Then letting $g = \bar{h}$ finishes the proof.
- $\bar{N} \neq M_2$. Proceed to show that this map can be further extended, which is a contradiction.

  By the inequality there exists some $x \in M_2 - \bar{N}$. Consider $I = \{a \in R \mid ax \in N\}$ which is the submodule of M_2 generated by x. Consider the morphism of R-modules f' where $f'(a) = \bar{h}(ax)$. By hypothesis there exists $u \in M$ s.t. $f'(x) = xu$ for all $x \in I$. Notice that $\bar{h} \circ \bar{i} = f$ where $\bar{i} : M_1 \hookrightarrow \bar{N}$ is the injection. Define the map $\tilde{h} : \bar{N} + I = \bar{N} + Rx \to M, \tilde{h}(a + vx) = \bar{h}(a) + vu$ for $a \in \bar{N}$ and $v \in N$. This is well-defined, as for all v s.t. $vx \in \bar{N}$, $\bar{h}(vx) = h(vx) = f'(v) = vu$ by hypothesis. This indicates that $\bar{N}$ is not maximal as the map can be extended to $\bar{N} + Rx$, which is a contradiction.

□

Definition 2.2. *Let M be an R-module. Then it is **divisble** if and only if for all $u \in M$, $n \in \mathbb{Z}_{\geq 0}$, there exists $v \in M$ s.t. $nv = u$ where multiplication by $n \in \mathbb{Z}_{\geq 0}$ is adding n copies of the elements to itself.*

Corollary 2.2. If $R = \mathbb{Z}$, then M being an R-module is injective if and only if it is divisible.

Proof. If M is divisible, then for all $n \in \mathbb{Z}_{\geq 0}$, $u \in M$ there exists $v \in M$ s.t. $u = nv$. That is, for all $f : I \to M$ with $n \in I \subseteq \mathbb{Z}$, there exists $v \in M$ s.t. $f(n) = u = nv$. This gives the criterion in Proposition 2.2. The converse holds as the converse holds in the proposition. □

Proof of Theorem 2.1. First prove the theorem with restriction $R = \mathbb{Z}$. From the the previous remark it is clear that $\mathbb{Z}$-modules are injective if and only if it is divisible; and $\mathbb{Q}$ as a $\mathbb{Z}$ module is divisible. Consider the canonical projection $\pi : \mathbb{Z}^{(I)} \to M$ where bases are mapped to generators. Then $M \simeq \mathbb{Z}^{(I)}/\ker\pi$, which embeds into $\mathbb{Q}^{(I)}/\ker\pi$. Since $\ker\pi$ is a submodule of $\mathbb{Z}^{(I)}$, $\mathbb{Q}^{(I)}/\ker\pi$ is divisible and thus injective; which satisfies the condition of interest.

Now consider the general case by reducing to the case of $\mathbb{Z}$-modules. As $\mathbb{Z}$ modules, there exists an injection $h : M \hookrightarrow Q$, where Q is an injective module. Q being injective indicates that the functor $\mathrm{Hom}_{\mathbb{Z}}(-, Q)$ is exact (on R-modules). But notice that by applying the adjoint property this gives

$$\mathrm{Hom}_{\mathbb{Z}}(-, Q) \simeq \mathrm{Hom}_{\mathbb{Z}}(R \otimes_R -, Q) \simeq \mathrm{Hom}_R(-, \mathrm{Hom}_{\mathbb{Z}}(R, Q))$$

which indicates that the functor $\mathrm{Hom}_R(-, \mathrm{Hom}_{\mathbb{Z}}(R, Q))$ is exact, i.e. $\mathrm{Hom}_{\mathbb{Z}}(R, Q)$ is an injective module. It then suffices to give an injective map from M to $\mathrm{Hom}_{\mathbb{Z}}(R, Q)$, which is given by $u \mapsto (x \mapsto (h(xu)))$, where $u \in M$, and $x \in R$. It is well-defined and bilinear by inspection. $\square$

The followings turn to the discussion of flat modules:

Remark 2.3. Since the functor $- \otimes M$ for a given R-module M is right exact, M is flat if and only if for all injective maps $M_1 \to M_2$, $M_1 \otimes M \to M_2 \otimes M$ is also injective.

Proposition 2.3. *Given a family of left R-modules $(M_i)_{i\in I}$, $\bigoplus_{i\in I} M_i$ is flat if and only if M_i is flat for all i.*

Proof. Since direct sum commutes with tensor product, given any injective R-linear map $N_1 \hookrightarrow N_2$, we have the following commutative diagram: Notice f' is an injection if and only if f'_i is an injection for all i by the universal property of coproduct;

$$\begin{tikzcd}
N_1 \otimes (\bigoplus_{i\in I} M_i) \arrow[r, hook, "f"] \arrow[d, "\simeq"] & N_2 \otimes (\bigoplus_{i\in I} M_i) \arrow[d, "\simeq"] \\
\bigoplus_{i\in I}(N_1 \otimes M_i) \arrow[r, hook, "f'"] & \bigoplus_{i\in I}(N_2 \otimes M_i)
\end{tikzcd}$$

and f is an injection if and only if f' is an injection, as the diagram should commute. $\square$

Corollary 2.3. Projective modules are flat.

Proof. By Proposition 2.1, an R-module M is projective if and only if there exists R-modules F and N where F is free s.t. $F \simeq M \oplus N$. By the previous proposition, it suffices to prove that F is flat. Since $F \simeq \oplus_{i\in I} R^{(I)}$, it suffices to prove that R is flat. This is indeed the case as $- \otimes_R R = \mathrm{Id}_R$, which is trivially exact. $\square$

Definition 2.3. *Let R be a commutative ring, and $\varphi : R \to S$ specifies an R-algebra. Then S is a **flat $\mathbf{R}$-algebra** if it is flat as an R-module.*

Remark 2.4. This implies that the extension of scalar functor is exact, as by the fact that tensor product should be R-balanced, multiplication is indeed scalar multiplication on the R-module itself, which preserves injection.

Example 2.1. Consider the following flat structures:

- $R \to R[x_1, \cdots, x_n]$ is a flat R-algebra, as $R[x_1, \cdots, x_n]$ has a free R-module structure, where the basis is all monomials.

- Let $S \subseteq R$ be a multiplicative system. Then $R \to S^{-1}R$ is a flat R-algebra. It suffices to verify that $S^{-1}R \otimes -$ is exact. Notice $S^{-1}R \otimes M \simeq S^{-1}M$, this is the same as stating that $S^{-1}(-)$ is exact.

  Since $N \subseteq M \implies S^{-1}N \subseteq S^{-1}M$, $S^{-1}(-)$ preserves injections; and since $S^{-1}(M/N) \simeq S^{-1}M/S^{-1}N$, $S^{-1}(-)$ preserves surjection. Thus verifies the exactness and flatness.

Proposition 2.4. *Let $(R, \mathfrak{m})$ be a local Noetherian ring, and M a finitely generated R-module. Then M is projective if and only if M is free.*

Proof. It suffices to verify that M is free if it is projective. By Nakayama's Lemma, $(u_1, \cdots, u_m)$ forms a minimal system of generators of M if and only if $(\bar{u}_1, \cdots, \bar{u}_m)$ forms a basis in $M/\mathfrak{m}$, This is indeed the case, as denoting $N = (u_1, \cdots, u_m)$, this gives $N + \mathfrak{m}M = M$, which indicates that $N = M$. Minimality is given by the minimality of cardinality of basis. Choose $F = R^m$ with $\varphi : F \to M$ s.t. $\varphi(e_i) = u_i$ for all i. Since M is projective, consider the short exact sequence that splits:

$$0 \longrightarrow K \longrightarrow F \xrightarrow{\varphi} M \longrightarrow 0$$

To prove that M is free, it suffices to show that $K = 0$. Note that if $(a_1, \cdots, a_m) \in K$, then $\sum_{i=1}^m a_i u_i = 0 \implies \sum_{i=1}^m \bar{a}_i \bar{u}_i = 0 \implies \bar{a}_i = 0 \implies a_i \in \mathfrak{m}$ for all i since $\bar{u}_i$s give a basis. That is, $K \subseteq \mathfrak{m}R^m$. Now apply the functor $- \otimes R/\mathfrak{m}$. This is an additive functor, which preserves morphisms as it acts as a group homomorphism, i.e. split exact sequences remain split exact after applying the functor. This gives the sequence

$$0 \longrightarrow K/\mathfrak{m}K \xrightarrow{\alpha} F/\mathfrak{m}F \longrightarrow M/\mathfrak{m}M \longrightarrow 0$$

which is also split. Since $K \subseteq \mathfrak{m}R^m$, $K \subseteq \mathfrak{m}F$, which indicates that $\alpha = 0$. Since α is injective as the sequence is exact, $K/\mathfrak{m}K = 0 \implies K = 0$. The sequence being split exact gives $F \simeq M \oplus K$, which implies $F \simeq M$. $\square$

3 Complexes*

4 Projective and Injective Resolution*

5 Derived Functors*