MATH 593 - Multilinear Algebra

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1 The Tensor Algebra

Definition 1.1 (Multilinear). *Let R be a commutative ring, and $M_1, \cdots, M_n, N$ be R-modules. A map $\varphi : M_1 \times \cdots \times M_n \to N$ is **multilinear** if for all $i \in [\![1, n]\!]$, for all $x_j \in M_j$ for $j \neq i$, the map $\varphi(x_1, \cdots, x_{i-1}, -, x_{i+1}, \cdots, x_n) : M_i \to M$ is R-linear.*

Remark 1.1. Via performing induction on n, it can be shown that for a multilinear map $f : M_1 \times \cdots \times M_n \to N$, with the tensor map $\varphi : M_1 \times \cdots \times M_n \to M_1 \otimes_R \cdots \otimes_R M_n$ (which is multilinear), there exists a R-linear map $g : M_1 \otimes_R \cdots \otimes_R M_n \to P$ s.t. $g \circ \varphi = f$.

Definition 1.2 (Tensor Algebra). *Let M be a fixed R-module. Define $T^0(M) := R, T^1(M) = M$; and for $n \geq 2$, define $T^n(M) := \underbrace{M \otimes_R \cdots \otimes_R M}_{n \text{ times}}$. Then the **tensor algebra** is defined as*

$$T(M) := \bigoplus_{i \geq 0} T^i(M) = R \oplus M \oplus (M \otimes_R M) \oplus \cdots$$

Remark 1.2. Since for all i, $T^i(M)$ has an R-module structure, $T(M)$ is also an R-module.

Proposition 1.1. *$T(M)$ also has an R-algebra structure.*

Proof. It suffices to define multiplication for each summand of $T(M)$ and check that it is well-defined. Define

$$\alpha_{ij} : T^i(M) \times T^j(M) \to T^{i+j}(M), \quad (a_1 \otimes \cdots \otimes a_i, b_1 \otimes \cdots \otimes b_j) \mapsto (a_1 \otimes \cdots \otimes a_i \otimes b_1 \otimes \cdots \otimes b_j)$$

This is indeed well-defined, as by applying the universal property of tensor product for i times gives the desired map. Notice that for the case where $i = 0$ or $j = 0$ this is just scalar multiplication, this is just scalar multiplication. $\square$

Remark 1.3. This can be extended to a map $T(M) \times T(M) \to T(M)$, which makes $T(M)$ a ring. The map is given by for $x = \oplus_{i \geq 0} x_i$, $y = \oplus_{j \geq 0} y_j$, the multiplication is defined as $x \cdot y = \oplus_{i,j \geq 0} \alpha_{i,j}(x_i, y_j)$, with $1 \in T^0(M) = R$. Moreover, the inclusion $R = T^0(M) \hookrightarrow T(M)$ is a ring homomorphism which makes $T(M)$ an R-algebra.

Remark 1.4. Notice that this differs from the polynomial ring in that it is not commutative (in terms of the direct summands). Therefore, the terms in $\oplus_{i,j} \alpha_{ij}$ cannot be collected into one. $T^n = M \otimes \cdots \otimes M$ has a basis given by $x_{i_1} \otimes \cdots \otimes x_{i_n}$, where $i_k \in [\![1, d]\!]$ for all $k \in [\![1, n]\!]$. Therefore, for $n \geq 2$, $T^n(M)$ is not commutative.

Proposition 1.2 (Universal Property of $T(M)$). *Consider the (forgetful) functor $F :_R \underline{\text{Alg}} \to {}_R\underline{\text{Mod}}, M \mapsto T(M)$. Let M be an R-module and S an R-algebra. If $f : M \to S$ is a morphism of R-modules, then there exists a unique morphism of R-algebras $g : T(M) \to S$ s.t. $g \mid_{T^1(M)} = f$, i.e. $g \circ F = f$:*

$$\begin{tikzcd} M \arrow[r, hook, "F"] \arrow[dr, "f"'] & T(M) \arrow[d, dashed, "g"] \\ & S \end{tikzcd}$$

Proof. Apply the universal property of tensor product and direct sum.

Define $f_n : \underbrace{M \times \cdots \times M}_{n \text{ times}} \to S$, where $f_n(x_1, \cdots, x_n) = f(x_1) \cdots f(x_n)$. This is clearly multilinear, which implies that there exists a unique R-linear map $g_n : T^n(M) \to S$ s.t. $g_n(x_1 \otimes \cdots \otimes x_n) = f(x_1) \cdots f(x_n)$. Apply the universal property of direct sum on the superscript gives that there exists a unique R-linear map $g : T(M) \to S$ s.t. $g|_{T^n(M)} = g_n$. Check the followings:

- g is a morphism of R-algebras. Since it is already a morphism of R-modules, it suffices to check that this definition is compatible with multiplication. For $x = \oplus_{i=0}^n x_i, y = \oplus_{j=0}^m y_j$, this gives

$$g(x \cdot y) = g(x_0 \otimes \cdots \otimes x_n \otimes y_0 \otimes \cdots \otimes y_m) = \Pi_{i=0}^n f(x_i) \cdot \Pi_{j=0}^m f(y_j) = g(x_0 \otimes \cdots \otimes x_n) \cdot g(y_0 \otimes \cdots \otimes y_m) = g(x) \cdot g(y)$$

- g is the unique morphism of R-algebras $T(M) \to S$, s.t. $g|_{T^1(M)} = f$. This is clear as defining $g|_{T^1(M)}$ gives the map on $g|_{T^n(M)}$ for all n, as by the definition of the multiplication. Furthermore, the map restricted to $T^0(M)$ is given by the associated morphism with the R-algebra S. Both of which are uniquely determined.

□

Remark 1.5. This makes T a functor, which maps from R-modules to R-algebras. For all R-linear maps $f : M \to N$, there exists a unique morphism of R-algebras $T(f)$ s.t. the following diagram commutes: This makes T a functor as it preserves

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, hook] & N \arrow[d, hook] \\ T(M) \arrow[r, "T(f)"] & T(N) \end{tikzcd}$$

compositions. Further this is the left adjoint of the forgetful functor G which only regards S as an R-module, i.e. we have the following isomorphism

$$\mathrm{Hom}_{R\underline{\mathrm{Alg}}}(T(M), S) \simeq \mathrm{Hom}_R(M, G(S))$$

as by the universal property of tensor algebra the morphism from $T(M)$ to S is uniquely defined by the map $f : M \to S$.

Definition 1.3 (Graded Ring). *A ring R is a* ***graded ring*** *if it comes with a decomposition $R = \oplus_{i \geq 0} R_i$ as abelian groups; and multiplication satisfies the relation $R_p \cdot R_q \subseteq R_{p+q}$ for all $p, q \geq 0$.*

Remark 1.6. Consider the subring $R \subseteq R_0$. If R_0 lies in the center of R, i.e. $R_0 \subseteq \{a \in R \mid ab = ba \forall b \in R\}$, then R becomes an R_0-algebra; and the decomposition $R = \oplus_{i \geq 0} R_i$ is a direct sum of R_0-modules.

Example 1.1. Consider the following examples of graded rings:

1. The tensor algebra $T(M)$ is a graded ring, where $R_0 = T^0(M) = R$
2. The multivariate polynomials $S = R[x_1, \cdots, x_n]$ is a graded ring, where $S_d = \oplus_{\{i_1, \cdots, i_d \mid \sum_k i_k = d\}} R x_1^{i_1} \cdots x_d^{i_d}$.

Definition 1.4 (Homogeneous). *If $R = \oplus_{i \geq 0} R_i$ is a graded ring, the elements of R_n are* ***homogeneous of degree*** *n.*

Definition 1.5 (Morphism of Graded Rings). *If R and S are graded rings, then a morphism of graded rings $f : R \to S$ is a ring homomorphism s.t. $f(R_i) \subseteq S_i$ for all i. Such definition gives the result that graded rings form a category.*

Definition 1.6 (Homogeneous Ideal). *If R is a graded ring, and $I \subseteq R$ an ideal. I is a* ***homogeneous ideal*** *if $I = \oplus_{i \geq 0}(I \cap R_i)$. Equivalently, for all $f \in I$, for all $f_i \in R_i$ s.t. $f = \sum_{i=0}^d f_i$, then $f_j \in I$ for all j.*

Remark 1.7. If further I is two-sided, then $R/I = \oplus_{i\geq 0}(R_i/(R_i \cap I))$ as a direct sum of abelian groups. In this case, R/I is a graded ring, and the quotient $\pi : R \to R/I$ is a morphism of graded rings.

Proposition 1.3. *Let R be a graded ring, and $I \subseteq R$ an ideal. Then I is homogeneous if and only if it can be generated by homogeneous elements.*

Proof. Show implication in two directions:

$\Rightarrow$: Since I is homogeneous, there exists ideals $I_k \subseteq R_k$ s.t. $I = \oplus_{k\geq 0} I_k$. Then it is generated by the generating sets of I_k, which are all homogeneous.

$\Leftarrow$: If I can be generated by homogeneous elements, then for all $x \in R$ there exists a decomposition

$$x = \sum_{r\in R} c_r r = \sum_{i\geq 0} \sum_{r\in R_i} c_{ri} r_i$$

where only finitely many c_{ri}s can be non-zero, and $r_i \in I$. Collecting all the terms in the inner summation gives $x = \sum_{i\geq 0} c_i r_i$ for $r_i \in I_i \subseteq R_i$, which satisfies the definition of homogeneous ideals.

□

Remark 1.8. It is not necessary (and also not true) that all the homogeneous elements must (can) have the same degree. For example, it is completely valid to have $R \cdot (I \cap R_0) \subsetneq I \cap R_1$, which prevents any homogeneous generating set of the same degree from existing.

2 Exterior and Symmetric Algebra

Definition 2.1 (Symmetric, Alternating Maps)**.** *Let R be a commutative ring, with M and N R-modules. Let $\varphi : M^n \to N$ be a multilinear map. It is defined to be:*

i) ***Symmetric****, if for all $\sigma \in S_n$, and for all $x_1, \cdots, x_n \in M$, $\varphi(x_1, \cdots, x_n) = \varphi(x_{\sigma(1)}, \cdots, \varphi(x_{\sigma(n)}))$.*

ii) ***Alternating****, if $\varphi(x_1, \cdots, x_n) = 0$ whenever $x_i = x_j$ for $i \neq j$. This is equivalent to stating that φ is* ***skew-symmetric****, where $\varphi(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_n) = -\varphi(x_1, \cdots, x_j, \cdots, x_i, \cdots, x_n)$ for all $i < j$.*

Remark 2.1. Since all elements in S_n (symmetric group) are generated by transpositions, to show that a map is symmetric it suffices to show the equality for all transpositions.

Definition 2.2 (Exterior Algebra)**.** *Let T be the tensor algebra, with J_n an T^n-submodule generated by $\{x_1 \otimes \cdots \otimes x_n \mid x_i = x_j \ \forall i \neq j\}$. Define $\Lambda^n(M) := T^n(M)/J_n$; and the* ***exterior algebra*** *$\Lambda(M) := \oplus_{n\geq 0}\Lambda^n(M)$. The algebra structure is inherited from that of the tensor algebra.*

Remark 2.2. Consider the ideal $J := \oplus_{i\geq 2} J_n \subseteq T(M)$. Here only summands with degree greater than or equal to 2 are taken into consideration as the definition "alternating" makes no sense for the lower degree cases. This is a two-sided ideal as tensor product is R-balanced; and each summand is an element in J_n for some n. It is further homogeneous, as by definition. In this notation the exterior algebra can also be expressed as $\Lambda(M) := T(M)/J$.

Remark 2.3. The equivalence class of $x_1 \otimes \cdots \otimes x_n$ in $\Lambda(M)$ is often denoted as $x_1 \wedge \cdots \wedge x_n$, i.e. the wedge product.

Proposition 2.1 (Universal Property of $\Lambda^n(M)$). *The map $\varphi : M^n \to \lambda^n(M)$, $(x_1, \cdots, x_n) \mapsto x_1 \wedge \cdots \wedge x_n$ is an alternating multilinear map; and for all alternating multilinear map $\psi : M^n \to P$, there exists a unique R-linear map $f : \Lambda^n(M) \to P$ s.t. $f \circ \varphi = \psi$*

Proof. This follows directly from the definition of the exterior algebra, and the universal property of tensor product. ☐

Example 2.1. If $g : M \to P$ is an R-linear map, then this gives an R-linear map $\Lambda^n g : \Lambda^n(M) \to \Lambda^n(P)$ for all n, s.t. $x_1 \wedge \cdots \wedge x_n \mapsto g(x_1) \wedge \cdots \wedge g(x_n)$. The construction implies that combining all $\Lambda^n g$s gives a morphism of graded R-algebra.

Proposition 2.2. *If $(x_1, \cdots, x_d)$ is a system of generators of M, then $\Lambda^n(M)$ is generated as an R-module by $\{x_{i_1} \wedge \cdots \wedge x_{i_n} \mid 1 \le i_1 < \cdots < i_n \le d\}$. In particular, for all $n > d$, $\Lambda^n(M) = 0$, which implies that $\Lambda(M)$ is a finitely generated R-module.*

Proof. For the cases where $n \le d$, the result follows from the fact that the tensor of several modules is generated by the tensor of the generators of the corresponding modules; and multilinear maps into the tensor product is alternating. ☐

Proposition 2.3. *M is a free, finitely generated R-module, with basis $(x_1, \cdots, x_d)$. Then for all $n \le d$, $\Lambda^n(M)$ is free with basis given by $\{x_{i_1} \wedge \cdots \wedge x_{i_n} \mid 1 \le i_1 < \cdots < i_n \le d\}$. Its rank is $\binom{d}{n}$.*

Proof. By the previous proposition, we only need to show that all the elements are linearly independent over R. Since the tensor product is commutative, fix the representation for e_I for $I \subseteq \{1, \cdots, d\}$ to be $e_I = x_{i_1} \wedge \cdots \wedge x_{i_n}$ for $i_1 < \cdots < i_n$.

Now consider the Is respectively. For a fixed I, define $\bar{I} := \{1, \cdots, d\} \backslash I$. By the fact that maps into the exterior algebra is alternating, we have

$$0 = e_{\bar{I}} \wedge \sum_{|J|=n} a_J e_J = \sum_{|J|=n} a_J (e_{\bar{I}} \wedge e_J) = a_I (x_1 \wedge \cdots \wedge x_n) \qquad (*)$$

We now seek to prove that $a_I = 0$ for all I, via apply a transformation into R. Consider the map $\psi : M^d \to R$ s.t. $\psi(u_1, \cdots, u_d) = \det(a_{ij})$, where $u_i = \sum_{i=1}^n a_{ij} x_j$, which is multilinear and alternating by construction. Then, by the universal property of exterior algebra, there exists a map $f : \Lambda^d(M) \to R$ s.t. $\psi(u_1, \cdots, u_d) = f(u_1 \wedge \cdots \wedge u_d)$. In particular, $f(x_1 \wedge \cdots \wedge x_d) = 1$. Applying f to $(*)$ gives $a_I = 0$, which gives as a consequence the linear independence. ☐

Definition 2.3 (Symmetric Algebra). *Define I as the two-sided ideal generated by elements in the form of $\{x \otimes y - y \otimes x \mid x, y \in M\}$. The* ***symmetric algebra*** *$S(M) := T(M)/I$. This is a commutative R-algebra.*

Remark 2.4. By construction I is generated by homogeneous elements of degree 2, which is a homogeneous ideal (with $I_0 = I_1 = \{0\}$). This gives an alternative expression for the symmetric algebra

$$S(M) = \bigoplus_{n \ge 0} \frac{T^n(M)}{I \cap T^n(M)}$$

which indicates that this is a graded ring. The denominator $I \cap T^n(M)$ is often denoted $S^n(M)$ or $\mathrm{Sym}^n(M)$.

Proposition 2.4 (Universal Property of $S(M)$). *$S(M)$ is a commutative R-algebra; and we have an inclusion $M \hookrightarrow S(M)$ which gives an isomorphism $M \simeq S^1(M)$. If S is a commutative R-algebra, and $\beta : M \to S$ is a R-linear map, then there exists a unique R-algebra homomorphism $f : S(M) \to S$ s.t. $f \circ \alpha = \beta$.*

Proof. By the universal property of $T(M)$, there exists a unique R-algebra homomorphism $\tilde{\beta} : T(M) \to S$ s.t. $\tilde{\beta}\mid_M = \alpha$. Since S is commutative, $I \subseteq \ker(\tilde{\beta})$. By the universal property of quotient, there exists a unique morphism $\beta : S(M) \to S$ s.t. $\beta\mid_M = \alpha$. □

Example 2.2. Let M be a free R-module, with basis $x_1, \cdots, x_n$. The above universal property gives the isomorphisms where S is a commutative R-algebra:

$$\{R\text{-algebra homomorphisms } S(M) \to S\} \simeq \{R\text{-linear maps } M \to S\} \simeq \{\text{maps } \{x_1, \cdots, x_n\} \to S\}$$

which implies that $S(M)$ satisfies the universal property of multivariate polynomials, i.e. we have the isomorphism $S(M) \simeq R[x_1, \cdots, x_n]$.

Example 2.3. Let $\varphi : M^p \to T^p(M) \to S^p(M)$ be a symmetric multilinear map. For every symmetric multilinear map $\psi : M^p \to N$, there exists a unique R-linear map $f : S^p(M) \to N$ s.t. $\psi = f \circ \varphi$. This can be proved similarly using the universal property of quotient rings.

3 Symmetric, Alternating and Hermitian Forms

4 The Spectral Theorem