MATH 593 - Introduction to Homological Algebra

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Contents

1 Exactness

Definition 1.1 (Complex)**.** *A **Complex** of R-modules is a family of R-modules (M_i) and R-linear maps $d_i : M_i \to M_{i+1}$ s.t. for all i, $d_{i+1} \circ d_i = 0$.*

Remark 1.1. The followings are some specifications on the notations:

- The complex is often denoted by a chain

$$\cdots \xrightarrow{d_{i-2}} M^{i-1} \xrightarrow{d_{i-1}} M^i \xrightarrow{d_i} M^{i+1} \xrightarrow{d_{i+1}} \cdots$$

  or a chain with indices on the bottom with $M^i = M_{-i}$.

- The complex extends to infinity in both ends. If the notation terminated on one side, all modules not written out are the trivial (the zero module).

Remark 1.2. The definition of a complex is the same as stating that $\operatorname{im} d_i \subseteq \ker d_{i+1}$ for all i.

Definition 1.2. *For a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ where f and g are R-linear maps, it is **exact at** $\mathbf{B}$ if the equality is reached in the remark above, i.e.* $\operatorname{im} f = \ker g$.

A sequence is exact if it is exact at A_i for all i. A complex is exact if it is exact everywhere.

Example 1.1. The sequence $0 \longrightarrow A \xrightarrow{f} B$ is exact implies that $\ker f = \{0\}$, i.e. f is injective. Similarly, $A \xrightarrow{g} B \longrightarrow 0$ implies that g is surjective.

Definition 1.3. *A **Short Exact Sequence (SES)** is an exact sequence*

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$$

Proposition 1.1. *Given a sequence $(*) : 0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$, the followings are equvalent:*

i) *$(*)$ is a short exact sequence.*

ii) *i is injective, and for $q : M \to \operatorname{coker} i$, there exists a unique isomorphism μ s.t. $\mu \circ q = p$.*

iii) *p is surjective, and for $j : M \to \ker p$, there exists a unique isomorphism η s.t. $i = \eta \circ j$.*

Proof. It suffices to prove the equivalence between i) and ii), as the case with iii) is similar:

- *i) ⇒ ii).* Apply the universal property of cokernel. Since $(*)$ is exact, $p \circ i = 0$, there exists a map θ s.t. the following diagram commutes. The fact that p is surjective, and the diagram should commute gives θ should be surjective. To prove that θ is

$$\begin{tikzcd} M' \arrow[r, "i"] & M \arrow[r, "p"] \arrow[dr, "q"] & M'' \\ & & \operatorname{coker} i \arrow[u, "\theta"] \end{tikzcd}$$

  injective, it suffices to verify that $\theta(b) = 0 \implies b = 0$ for $b \in \operatorname{coker} i$. Since q by definition is surjective, there exists $a \in M$ s.t. $q(a) = b$. This gives $a \in \ker p = \operatorname{im} i$, which implies that $q(a) = 0$ as the cokernel is defined by $M/\operatorname{im} i$.

- *ii) ⇒ i).* Given that μ is an isomorphism and i is injective, it suffices to verify that p is surjective, and $\operatorname{im} i = \ker p$. μ being surjective implies that p is surjective; and μ being an isomorphism implies that $\ker p = \ker q = \operatorname{im} i$.

□

Proposition 1.2. *Given a short exact sequence* $0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$*, the following statements are equivalent:*

i) There exists $q : M \to M'$ *s.t.* $j \circ i = \mathrm{Id}_{M'}$

ii) There exists $j : M \to M''$ *s.t.* $q \circ p = \mathrm{Id}_{M''}$

iii) There exists a submodule $N \subseteq M$ *s.t.* M *can be expressed by the internal direct sum* $M = i(M') \oplus N$*; and* p *induces an isomorphism* $N \simeq M''$.

Such a short exact sequence is a split exact sequence.

Proof. It suffices to give the equivalence between i) and iii), as for ii) it is similar.

- *i) ⇒ iii).* Let $N = \ker q$. Check that this gives an internal direct sum:
  - $N \cap i(M') = 0$. Let $x \in i(M') \cap N$. $x \in N$ implies $q(x) = 0$, while $q \circ i = \mathrm{Id}_{M'}$ implies that $x = 0$.
  - $N + i(M') = M$. Notice $v - i \circ q(v) \in \ker q$, and by inspection $i \circ q(v) \in \operatorname{im} i$.

  By the first isomorphism theorem, $\operatorname{im} i = \ker p$ implies $M/\operatorname{im} i \simeq N \simeq M''$.

- *iii) ⇒ i).* Define $i : i(M') \oplus N \to i(M') \simeq M'$ since i is injective.

□

Remark 1.3. Generally short exact sequences do not split. A counterexample is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} 2\mathbb{Z} \xrightarrow{p} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

where $i(\mathbb{Z}) \simeq \mathbb{Z}$, but the map cannot be extended properly to the whole $\mathbb{Z}$. If R is a field, then all short exact sequences split as one can complete a basis in a vector space; and subspaces spanned by a subset of a basis is always a direct summand of the whole space.

The following present a common technique known as "diagram chasing":

Proposition 1.3 (The 5-Lemma)**.** *Consider the following diagram, with blocks commute and rows exact:*

$$\begin{tikzcd}
A_1 \arrow[r, "u_1"] \arrow[d, "f_1"] & A_2 \arrow[r, "u_2"] \arrow[d, "f_2"] & A_3 \arrow[r, "u_3"] \arrow[d, "f_3"] & A_4 \arrow[r, "u_4"] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r, "v_1"] & B_2 \arrow[r, "v_2"] & B_3 \arrow[r, "v_3"] & B_4 \arrow[r, "v_4"] & B_5
\end{tikzcd}$$

1. *If* f_2, f_4 *are injective,* f_1 *is surjective, then* f_3 *is injective.*
2. *If* f_2, f_4 *are surjective,* f_5 *is injective, then* f_3 *is surjective.*

3. *(Combining i) and ii)) If f_1, f_2, f_4, f_5 are all isomorphisms, then f_3 is an isomorphism.*

Proof. The argument is symmetric, so it suffices to prove the first one. f_3 is injective if and only if $f_3(b) = 0 \implies b = 0$. Following the steps:

- Consider the third square. $v_3 \circ f_3(b) = v_3(0) = 0$, giving $f_4 \circ u_3(b) = 0$. f_4 being injective implies that $u_3(b) = 0$.
- Consider the second square. The top row being exact implies that $b \in \mathrm{im}\ u_2$, i.e. there exists some $c \in A_2$ s.t. $u_2(c) = b$. Commutativity gives that $v_2 \circ f_2(c) = 0$, i.e. $c' := f_2(c) \in \ker v_1$.
- Consider the first square. The bottom row being exact implies that there exists some $d' \in B_1$ s.t. $v_1(d') = c'$. Since f_1 is surjective, there exists $d \in A_1$ s.t. $f_1(d) = d'$. For the diagram to commute, it is required that $u_1(d) = c$. But this indicates that $c \in \mathrm{im}\ u_1$, i.e. $c \in \ker u_2$, which gives $b = u_2(c) = 0$.

□

Definition 1.4. *Let R and S be rings, and $F : {}_R\underline{\mathrm{Mod}} \to {}_S\underline{\mathrm{Mod}}$ is an additive functor. Then F is **exact** if for all short exact sequences of R-modules $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$, the corresponding sequence after applying F is also exact.*

Proposition 1.4. *F is exact if and only if for all exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$, $F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C)$ is also exact.*

Proof. Proceed by showing implication in two directions:

$\Leftarrow$: This holds by definition, where f is injective and g is surjective.

$\Rightarrow$: Consider the following short exact sequences:

$$\begin{aligned}
(1): &\quad 0 \longrightarrow \ker f \longrightarrow A \xrightarrow{\alpha_1} \mathrm{im}\ f \longrightarrow 0 \\
(2): &\quad 0 \longrightarrow \ker g \xrightarrow{\alpha_2} B \xrightarrow{\beta_1} \mathrm{im}\ g \longrightarrow 0 \\
(3): &\quad 0 \longrightarrow \mathrm{im}\ g \xrightarrow{\beta_2} C \longrightarrow \mathrm{coker}\ g \longrightarrow 0
\end{aligned}$$

where $\mathrm{im}\ f = \ker g$ as the sequence given is exact. These by construction are all short exact sequences, where applying F gives also short exact sequences. Combining gives the sequence which is still exact after applying F:

$$A \xrightarrow{\alpha_1} \mathrm{im}\ f \xrightarrow{\alpha_2} B \xrightarrow{\beta_1} \mathrm{im}\ g \xrightarrow{\beta_2} C$$

where α_1, β_1 are surjective; and α_2, β_2 are injective. What we want to show is $\mathrm{im}\ F(f) = \ker F(g)$. Since α_1 is surjective, $\mathrm{im}\ F(f) = \mathrm{im}\ F(\alpha_2)$; and since β_2 is injective, $\ker F(g) = \ker F(\beta_1)$. From the result of (2) after applying F, we have $\mathrm{im}\ F(\alpha_2) = \ker F(\beta_1)$.

□

Remark 1.4. "One-sided" exact sequences can be understood functorially:

- Given exact sequence $0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M''$ is the same as saying that i is injective; and M' is the kernel of p.
- Similarly, given exact sequence $M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$ is the same as saying that p is surjective; and M'' is the cokernel of i.

Definition 1.5. *Just as the remark, one could consider exact functors only on one side. $F : {}_R\underline{\mathrm{Mod}} \to {}_S\underline{\mathrm{Mod}}$ is* ***left exact*** *if for all exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C$, the sequence $0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$ is also exact; and the definition is symmetric for right exact functors. Notice that since F is an additive functor, $F(0) = 0$ (as zero morphisms are mapped to zero morphisms).*

Proposition 1.5. *Let M be a left R-module. Then functor $F = \mathrm{Hom}_R(M, -) : {}_R\underline{\mathrm{Mod}} \to {}_S\underline{\mathrm{Mod}}$ is exact; and the coverse is also true, i.e. if $0 \longrightarrow \mathrm{Hom}_R(M, A) \longrightarrow \mathrm{Hom}_R(M, B) \longrightarrow \mathrm{Hom}_R(M, C)$ is exact, then $0 \longrightarrow A \longrightarrow B \longrightarrow C$ is exact.*

Proof. What we want to show is that if the sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C$ is exact, then the corresponding sequence $0 \longrightarrow \mathrm{Hom}(M, A) \longrightarrow \mathrm{Hom}(M, B) \longrightarrow \mathrm{Hom}(M, C)$ is exact: The natural way to define the functor F is via specifying $\mathrm{Hom}(M, A) \ni$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \\
 & & M \arrow[u, "v"'] \arrow[ul, dashed, "u"] &
\end{tikzcd}$$

$u \mapsto f \circ u$, $\mathrm{Hom}(M, B) \ni v \mapsto g \circ v$. Exactness follows from the universal property of kernel, where for all $v \in \mathrm{Hom}(M, B)$ s.t. $\mathrm{Hom}(M, C) \ni g \circ v = 0$, it factors uniquely through f.

For the converse, take $M = A$, which gives $F(g) \circ F(f) = 0$, i.e. $F(f) = \ker F(g)$. But $F(f) = f$ by construction, which gives the exactness. □

Remark 1.5. The symmetric argument is also true, via applying the universal property of cokernel.

Proposition 1.6. *If M is an S-R bimodule, then $M \otimes_R -$ is right exact.*

Proof. From the right exact version of 1.5, it suffices to prove that for left R-module N the sequence

$$\mathrm{Hom}(M \otimes A, N) \longrightarrow \mathrm{Hom}(M \otimes B, N) \longrightarrow \mathrm{Hom}(M \otimes C, N) \longrightarrow 0$$

is exact. First make a parenthesis on the generalization of the adjoint property of extension and restriction of scalars:

Parenthesis 1.1 (Prop 4.4 [c6])**.** Let M be an S-R bimodule, C a left R-module, and N a left S-module, then there is a functorial isomorphism

$$\mathrm{Hom}_S(M \otimes_R C, N) \simeq \mathrm{Hom}_R(C, \mathrm{Hom}_S(M, N))$$

Proof. By the universal property of tensor product, it suffices to give every R-balanced map $f : M \times C \to N$ a map in $\mathrm{Hom}_R(C, \mathrm{Hom}_S(M, N))$. Denote the isomorphism to be F. Let $F(\tilde{f}[u \otimes v \mapsto f(u, v)]) = v \mapsto (u \mapsto f(u, v))$. The inverse exists by inspection. □

Then apply the parenthesis and 1.5 gives that it suffices to verify that $\mathrm{Hom}(M, A) \longrightarrow \mathrm{Hom}(M, B) \longrightarrow \mathrm{Hom}(M, C) \longrightarrow 0$ is exact, which holds as M is in particular a right R-module. □

Remark 1.6. Recall that the above isomorphism holds for all adjoint pairs (F, G). Therefore, the proof applies as long as F is left exact (for G being right exact) or the converse holds.

In general, the above two functors and the contravariant is only left (right) exact instead of exact. Consider the short exact sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow 2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

- Applying $- \otimes \mathbb{Z}/2\mathbb{Z}$ gives

  $$0 \longrightarrow \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \xrightarrow{i} 2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \xrightarrow{p} \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

  where $2\mathbb{Z}\otimes\mathbb{Z}/2\mathbb{Z} = 0$ and $\mathbb{Z}\otimes\mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}\otimes\mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}$, as the map being R-linear restricts that $f(0,1) = f(1,0) = f(0,0)$. This implies that i is not injective.

- Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/2\mathbb{Z})$ gives

  $$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{i} \mathrm{Hom}_{\mathbb{Z}}(2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{p} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \longrightarrow 0$$

  where $\mathrm{Hom}_{\mathbb{Z}}(2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \ni h = 0$ as it is required that $h(1) + h(1) = \bar{1} + \bar{1} = 0$ which indicates that p is not surjective. Similar situations appear in the contravariant case.

It is then of specific interest in which modules are the above functors exact.

2 Flat/Projective/Injective Modules