MATH 593 - Categories and Functors

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Contents

1 Category; Functor

Definition 1.1 (Category). *A **category** $\mathscr{C}$ consists of*

- *A* <u>*class*</u> *of objects $\mathscr{C}$ (which, for example, could contain all sets), denoted* $\mathrm{Ob}(\mathscr{C})$.
- *For all $A, B \in \mathrm{Ob}(\mathscr{C})$, a* <u>*set*</u> $\mathrm{Hom}_{\mathscr{C}}(A, B)$ *the "morphisms in $\mathscr{C}$ from A to B" with map* $\mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(B, C) \to \times \mathrm{Hom}_{\mathscr{C}}(A, C)$ *the "morphism composition" denoted $f \times g \rightsquigarrow (g \circ f)$, satisfying*
  - Existence of an identity. *for all $A \in \mathrm{Ob}(\mathscr{C})$, there exists $1_A \in \mathrm{Hom}_{\mathscr{C}}(A, A)$ s.t.*

$$\begin{cases} 1_A \circ f = f & \forall f \in \mathrm{Hom}_{\mathscr{C}}(A, B) \\ g \circ 1_A = g & \forall g \in \mathrm{Hom}_{\mathscr{C}}(B, A) \end{cases}$$

  - Associativity. *For all $f \in \mathrm{Hom}_{\mathscr{C}}(A, B), g \in \mathrm{Hom}_{\mathscr{C}}(B, C), h \in \mathrm{Hom}_{\mathscr{C}}(C, D)$,*

$$(h \circ g) \circ f = h \circ (g \circ f)$$

Remark 1.1. The definition much resembles previous algebraic structures; but the morphisms and composition laws could be defined in a particularly strange way:

1. Similar to monoids, the definition implies that the identity is unique. Suppose that there are two identities $1_A, 1'_A \in \mathrm{Hom}_{\mathscr{C}}(A, A)$ for $A \in \mathrm{Ob}(\mathscr{C})$, then $1_A = 1_A \circ 1'_A = 1'_A$.
2. The morphism is not necessarily a function; and in such cases composition needs to be re-defined respectively.

Example 1.1. Consider the following categories:

- *Category of Sets* <u>Sets</u>, where the objects are sets, and morphisms are maps between sets.
- *Category of Rings* <u>Rings</u>, where the objects are rings, and morphisms are ring homomorphisms.
- *Category of (left) R-modules* ${}_R\underline{\mathrm{Mod}}$, where objects are lef R-modules, and morphisms R-linear maps.
- Consider the category $\mathscr{C}$ defined on a partially-ordered set $(A, \leq)$ where
  - $\mathrm{Ob}(\mathscr{C})$ consists of elements in A.
  - Morphisms are defined as

$$\mathrm{Hom}_{\mathscr{C}}(A, B) = \begin{cases} \{*\} & A \leq B \\ \emptyset & \text{otherwise} \end{cases}$$

    where the composition of maps is defined as intersection. This is due to the fact that there can be no maps whose image is the empty set.

Definition 1.2 (Functor). *Let $\mathscr{C}$ and $\mathscr{D}$ be categories. The **functor** $F : \mathscr{C} \to \mathscr{D}$ consists of mappings for both objects and morphisms:*

- *For all $A \in \mathrm{Ob}(\mathscr{C})$, $F(A) \in \mathscr{D}$.*
- *For all $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$. $F(f) \in \mathrm{Hom}_{\mathscr{D}}(F(A), F(b))$ s.t.*
  - $F(1_A) = 1_{F(A)}$ *for all $A \in \mathrm{Ob}(\mathscr{C})$.*

– *For all* $f \circ g$ *where* $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathscr{C}}(B, C)$, $F(f \circ g) = F(f) \circ F(g)$.

The composition of functors is conducted in a natural way, i.e. applying consecutively.

Example 1.2. Functors represent the induced maps w.r.t. a transformation in the structure:

1. Let R be a commutative ring and $S \subseteq R$ a multiplicative system. Consider the functor $F : {}_R\underline{\mathrm{Mod}} \to {}_{S^{-1}R}\underline{\mathrm{Mod}}$ where
   - $F(M) = S^{-1}M$ for all $M \in \mathrm{Ob}({}_R\underline{\mathrm{Mod}})$.
   - For $f : M \to N$, define $F(f) := S^{-1}M \to S^{-1}N$, where $\frac{u}{s} \mapsto \frac{f(u)}{s}$.
2. Let R be a ring, with $I \subseteq R$ a two-sided ideal of R; and M a left R-module. Consider the functor $F : {}_R\underline{\mathrm{Mod}} \to {}_{R/I}\underline{\mathrm{Mod}}$ where
   - $F(M) = M/IM$ for all $M \in \mathrm{Ob}({}_R\underline{\mathrm{Mod}})$.
   - Let $f : M \to N$ be a morphism of left R-modules. Then it induces a map $\bar{f} : M/IM \to N/IN$ s.t. $\bar{f}(\bar{(u)}) = \overline{f(u)}$. Define $F(f) = \bar{f}$.
3. Functors generally can abandon structures. Let M be a left R-module. By definition it is valid to view M as an abelian group. Then functor $F : {}_R\underline{\mathrm{Mod}} \to \underline{\mathrm{Ab}}$ where objects are taken to itself; and morphisms are taken to group homomorphisms. These are called *forgetful functors.*

2 Morphism of Categories

The dual of a category is where the direction of morphisms is inverted. The following gives a formalization of this:

Definition 2.1 (Contravariant Functor)**.** *A* ***contravariant functor*** $F : \mathscr{C} \to \mathscr{D}$ *is a functor which maps composition to that in the inverse order, i.e.*

- *For all* $A \in Ob(\mathscr{C})$, $F(A) \in \mathrm{Ob}(\mathscr{D})$.
- *For all* $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, $F(f) \in \mathrm{Hom}_{\mathscr{C}}(F(B), F(A))$ *s.t.*
  – $F(1_A) = 1_{F(A)}$ *for all* $A \in \mathrm{Ob}(\mathscr{C})$.
  – *For all* $f \circ g$ *where* $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathscr{C}}(B, C)$, $F(f \circ g) = F(g) \circ F(f)$.

Definition 2.2 (Dual Category)**.** *Let* $\mathscr{C}$ *be a category. Then the* ***dual category*** $\mathscr{C}^\circ$ *of* $\mathscr{C}$ *is a category with*

- $\mathrm{Ob}(\mathscr{C}) = \mathrm{Ob}(\mathscr{C}^\circ)$.
- $\mathrm{Hom}_{\mathscr{C}}(A, B) = \mathrm{Hom}_{\mathscr{C}^\circ}(B, A)$.

The composition is compatible as the inversion is done in the functor.

Remark 2.1. Since the dual category is defined on the contravariant of the functor, replacing a functor with its contravariant is equivalent to replacing the category with its dual.

Similar to the case of modules we can define the *Hom Functors*; but as a concept one level up it leaves the image unspecified:

Definition 2.3 (Hom Functor)**.** *Let* $\mathscr{C}$ *be a category, and* $A \in \mathrm{Ob}(\mathscr{C})$. *Then the* ***Hom functor*** $\mathrm{Hom}_{\mathscr{C}}(A, -) : \mathscr{C} \to \underline{\mathrm{Sets}}$ *where*

- *For* $B \in \mathrm{Ob}(\mathscr{C})$*,* $F(B) = \mathrm{Hom}_{\mathscr{C}}(A, B)$*.*
- *For* $f : \mathrm{Hom}_{\mathscr{C}}(B_1, B_2)$*,* $F(g) : \mathrm{Hom}_{\mathscr{C}}(A, B_1) \to \mathrm{Hom}_{\mathscr{C}}(A, B_2)$*,* $g \mapsto f \circ g$*.*

Remark 2.2. Similarly, we can consider the contravariant functor of the Hom functor. $\mathrm{Hom}_{\mathscr{C}^\circ}(-, A) : \mathscr{C}^\circ \to \underline{\mathrm{Sets}}$. By definition $\mathrm{Hom}_{\mathscr{C}}(A, -) = \mathrm{Hom}_{\mathscr{C}^\circ}(-, A)$.

Remark 2.3. Let $\mathscr{C} = {}_R\underline{\mathrm{Mod}}$. Then $\mathrm{Hom}_{\mathscr{C}}(X, -)$ could be lifted to $\mathscr{C} \to \underline{\mathrm{Ab}}$. It can be further lifted to $\mathscr{C} \to {}_R\underline{\mathrm{Mod}}$ if R is commutative, which ensures that the morphisms will be R-linear. In this case this is just the Hom Module of (left) R-modules.

Definition 2.4. *Let* $\mathscr{C}$ *be a category. Then* $u \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ *is an* ***isomorphism*** *if there exists* $v \in \mathrm{Hom}_{\mathscr{C}}(B, A)$ *s.t.* $u \circ v = \mathrm{Id}_B, v \circ u = \mathrm{Id}_A$*.*

Remark 2.4. For a fixed u, such v is unique. Suppose that there exists two distinct vs, we have

$$v = v \circ \mathrm{Id}_B = v \circ (u \circ v') = (v \circ u) \circ v' = v'$$

which is a contradiction.

Remark 2.5. Let $F : \mathscr{C} \to \mathscr{D}$ a functor. Then $u \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ being an isomorphism implies that $F(u)$ is an isomorphism.

This results from the fact that $\mathrm{Id}_{F(B)} = F(\mathrm{Id}_B) = F(u \circ v) = F(u) \circ F(v)$. Result for A is similar; and uniqueness follows from the same reasoning.

Definition 2.5. *Let* $\mathscr{C}$ *be a category:*

- $X \in \mathrm{Ob}(\mathscr{C})$ *is an* ***initial object*** *if* $\forall Y \in \mathrm{Ob}(\mathscr{C}), |\mathrm{Hom}_{\mathscr{C}}(X, Y)| = 1$*.*
- $X \in \mathrm{Ob}(\mathscr{C})$ *is a* ***final object*** *if* $\forall Y \in \mathrm{Ob}(\mathscr{C}), |\mathrm{Hom}_{\mathscr{C}}(Y, X)| = 1$*.*
- $X \in \mathrm{Ob}(\mathscr{C})$ *is a* ***zero-object*** *if it is both an initial object and a final object.*

Remark 2.6. Let $X \in \mathrm{Ob}(\mathscr{C})$ be an initial (final, zero) object. Then X' is initial (final, zero) if and only if there exists an isomorphism between X and X'.

Proof is similar for all three cases. Suppose that X and X' are both initial. Then there exists a unique $f \in \mathrm{Hom}_{\mathscr{C}}(X, X')$ and $f' \in \mathrm{Hom}_{\mathscr{C}}(X', X)$, i.e. $f' \circ f \in \mathrm{Hom}_{\mathscr{C}}(X, X)$. X being initial implies that this is the unique morphism from X to itself, which contains Id_X. Therefore $f' \circ f = \mathrm{Id}_X$. Similar result holds for $f \circ f' = \mathrm{Id}_{X'}$, which implies that X and X' are isomorphic.

Example 2.1. Although if initial/final objects are unique up to isomorphism if they exist, but they actually do not necessarily exist:

1. In ${}_R\underline{\mathrm{Mod}}$, $\{0\}$ is a zero-object.

   The only element in a left R-module that should be preserved in a morphism of R-module is the zero element. For any element $a \neq 0$, there exists two maps that either maps a to 0, or another non-zero element, which indicates that this is not initial.

   Suppose that the final object has (at least) two elements $\{0, a\}$ for $a \neq 0$, then there exists at least two maps from a non-trivial module generated by $(u_1, \cdots, u_r)$ to it: for each u_i it is either mapped to 0 or a, which gives two morphisms.

2. In Sets, $\emptyset$ is initial, and $\{*\}$ (a set containing an arbitrary element) is final.

   Suppose that there exists an element in the initial object, then it could be mapped to any element as morphisms of Setsdo not have constraints.

3. In Rings, $\mathbb{Z}$ is initial; and $\{0\}$ is final.

   $\mathbb{Z}$ being initial results from the fact that ring homomorphisms are required to preserve the 0 and 1 elements; and the maximal ring generated by $(0, 1)$ is isomorphic to $\mathbb{Z}$.

4. In Fields, there are no initial or final objects.

   This results directly from the fact that $1 \neq 0$ in fields. For every two fields, there exists two maps: one that maps 1 to 1; and the other maps 1 to 0.

Definition 2.6. *Let $\mathscr{C}$ be a category. $u \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ is a **monomorphism** if for all $C \in \mathrm{Ob}(\mathscr{C})$ and $v_1, v_2 \in \mathrm{Hom}_{\mathscr{C}}(C, A)$ s.t. $u \circ v_1 = u \circ v_2$ implies $v_1 = v_2$. $u \in \mathrm{Hom}_{\mathscr{C}}(A, B)$ is an **epimorphism** if for all v_1, v_2 with the same comdition above satisfies $v_1 \circ u = v_2 \circ u$ implies $v_1 = v_2$, i.e. it is a monomorphism in $\mathscr{C}^\circ$.*

Remark 2.7. These are analogies of injective/surjective in the context of category. Since on the category level it is only valid to consider objects or morphisms, such analogies could be only made to morphisms.

Example 2.2. It is not always the case that monomorphisms could correspond to injective maps, and epimorphisms could correspond to surjective maps:

1. In Sets, monomorphisms correspond to injective maps, and epimorphisms correspond to surjective maps.

2. In ${}_R$Mod, such analogy is still true via choosing the v_1, v_2:

$$\ker u \underset{0}{\overset{\text{incl.}}{\rightrightarrows}} A \xrightarrow{u} B \qquad\qquad A \xrightarrow{u} B \underset{0}{\overset{\pi}{\rightrightarrows}} B/\mathrm{im}\, u$$

   - For u being a monomorphism, $u(\ker u) = u(0) = 0$, i.e. the inclusion map from $\ker u$ to A is the same as the zero map, i.e. $\ker u = \{0\}$.
   - For u being an epimorphism, $\pi(u(A)) = 0$ in $B/\mathrm{im}\, u$, i.e. $\pi(B) = 0$ which indicates that $\mathrm{im}\, u = B$.

3. In Rings, monomorphisms are still injective, via for $f : R \to S$ considering $\mathbb{Z}[x] \underset{0}{\overset{x \mapsto u}{\rightrightarrows}} R \xrightarrow{f} S$ for all $u \in \ker f$. This implies that $u = 0$, i.e. $\ker f = \{0\}$.

   But epimorphisms in rings are not necessarily surjective. Take the example $\alpha : \mathbb{Z} \hookrightarrow \mathbb{Q}$ which is the inclusion map. This is an epimorphism as $v_1 \circ \alpha = v_2 \circ \alpha$ if and only if 1 is mapped to the same element; but this always holds as ring homomorphisms preserve the multiplicative unit.

3 Product and Coproduct

Definition 3.1 (Product)**.** *Let $\mathscr{C}$ be a category, with $(X_i)_{i\in I}$ a family of objects in $\mathscr{C}$. Then the **product** of this family is given by an object $\Pi_{i\in I}X_i$ where for all $Y \in \mathrm{Ob}(\mathscr{C})$, with morphisms $q_j : Y \to X_j$ for $j \in I$, there exists a unique morphism f s.t. $q_j = p_j \circ f$ for all j, i.e. the following diagram commute. The p_j is the projection morphism, where $p_j(\Pi_{i\in I}x_i) = x_i$.*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "q_j"'] & \Pi_{i\in I}X_i \arrow[d, "p_j"] \\ & X_j \end{tikzcd}$$

Remark 3.1. Product of a family of objects is unique up to isomorphism. Suppose that there exists another product X' which satisfies the criterion for being a product. Then there exists unique φ and φ' s.t.

$$\varphi : X \to X' \qquad \varphi' : X' \to X$$

since both f and f' are unique. But this gives

$$\begin{cases} p_j = p'_j \circ \varphi' \\ p'_j = p_j \circ \varphi \end{cases} \implies \exists \varphi, \varphi' \text{ s.t. } \begin{cases} \varphi \circ \varphi' = \mathrm{Id}_{X'} \\ \varphi' \circ \varphi = \mathrm{Id}_X \end{cases}$$

By uniqueness this gives that X and X' must be isomorphic.

Definition 3.2 (Coproduct)**.** *Given a family $(X_i)_{i\in I} \in \mathrm{Ob}(\mathscr{C})$, the coproduct $\amalg_{i\in I}X_i$ is the product in the dual category, i.e. with all the arrow reversed. That is, for $Y \in \mathrm{Ob}(\mathscr{C})$ and $f_j : X_j \to Y$, denote α_j to be the natural embedding of X_j into the product $\amalg_{i\in I}X_i$, then there exists a unique f s.t. $f \circ \alpha_j = f_j$ for all j, i.e. the following diagram commute:*

$$\begin{tikzcd} Y & \amalg_{i\in I}X_i \arrow[l, "f"'] \\ & X_j \arrow[ul, "f_j"] \arrow[u, "\alpha_j"'] \end{tikzcd}$$

Remark 3.2. In Sets, the product is given by Cartesian product, and the coproduct is given by disjoint union. Notice the difference: projection has no corresponding morphism from disjoint union; and so is natural embedding into Cartesian product.

In ${}_R$Mod, the coproduct is given by the direct sum; and the product is given by the direct product.

Remark 3.3. Product in the context of categories provides a generalization of Cartesian product, where specifying morphisms into the product gives the morphisms into each of its components. Coproduct, being the dual notion of product, simply "reverses the arrows", i.e. specifying morphisms from the coproduct gives morphisms from each of its components.

Definition 3.3 (Preadditive Category)**.** *A **preadditive category** $\mathscr{C}$ is a category s.t. its morphisms form an abelian group, and is bilinear w.r.t. composition.*

Remark 3.4. Rings is not a preadditive category, as its morphisms do not have a zero element (since ring homomorphisms are required to map 1 to 1.) ${}_R$Mod, with out such constraint, is a preadditive category.

Definition 3.4 (Additive Functor)**.** *Let $\mathscr{C}$ and $\mathscr{D}$ be preadditive categories. $F : \mathscr{C} \to \mathscr{D}$ is an **additive functor** if for all $A, B \in \mathrm{Ob}(\mathscr{C})$, F w.r.t. morphisms is a group homomorphism.*

Remark 3.5. The definition of "linear", or "group homomorphism" implicitly requires that the underlying structure should have a valid operation. Those who don't, for example Sets, are naturally excluded from such discussion.

Example 3.1. Let R be a commutative ring and $I \subseteq R$ an ideal. Then the functor

$$F : {}_R\underline{\mathrm{Mod}} \to {}_R\underline{\mathrm{Mod}}, \qquad M \mapsto M/IM$$

is an additive functor. This results from the fact that morphism of R-modules $M \to N$ and the quotient morphism are both R-linear.

Remark 3.6. Let $\mathscr{C}$ be a preadditive category. Then

1. If $X \in \mathrm{Ob}(\mathscr{C})$ is an initial/final object, then it is a zero object.

   Consider $\mathrm{Hom}_{\mathscr{C}}(X, X)$. Since $\mathscr{C}$ is preadditive, this forms a group of one element, which is zero. Therefore for all $f \in \mathrm{Hom}_{\mathscr{C}}(X, X)$ this gives $f = 1_X \circ f = 0 \circ f = 0$. This immediately implies that X is a zero object, as for all $g \in \mathrm{Hom}_{\mathscr{C}}(Y, X)$, $h \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$
   $$g = g \circ 1_X = g \circ 0 = 0, \qquad h = 1_X \circ h = 0 \circ h = 0$$

2. $\mathscr{C}$ being preadditive implies that $\mathscr{C}^\circ$ is preadditive, as reversing the arrow does not interfere with the group structure, or the additive property.

3. Let $F : \mathscr{C} \to \mathscr{D}$ an additive functor, with $\mathscr{C}$ and $\mathscr{D}$ preadditive categories. Then for $0 \in \mathrm{Ob}(\mathscr{C})$ the zero object in $\mathscr{C}$, $F(0)$ is also the zero object in $\mathscr{D}$.

   This results directly from the fact that group homomorphisms map 0 to 0; and by the first point in the remark, an object X is zero if and only if $1_X = 0$ in $\mathrm{Hom}_{\mathscr{C}}(X, X)$.

4 Kernel and Cokernel

Throughout the discussion, fix $\mathscr{C}$ to be a preadditive category.

Definition 4.1 (Kernel)**.** *Let $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$. The **kernel** of f is a morphism $i : K \to A$ s.t.*

- $f \circ i = 0$.
- *f is universal with this property (factors uniquely through i), i.e. for $i' : L \to A$, there exists a unique morphism $g : L \to K$ s.t. $i' = i \circ g$:*

$$\begin{tikzcd}
K \arrow[r, "i"] & A \arrow[r, "f"] & B \\
L \arrow[u, "g"] \arrow[ur, "i'"] & &
\end{tikzcd}$$

Remark 4.1. Using exactly the same reasoning as in products, the kernel of a morphism is unique up to isomorphism. Adopting the same notation as above, consider i' to be the kernel. Then there exists a unique $g' : K \to L$ s.t. the diagram commute, which gives $g \circ g' = \mathrm{Id}$.

Remark 4.2. Let $i : K \to A$ be the kernel. Then i is a monomorphism.

This follows directly from the uniqueness in the universal property. Suppose that there exists $f_1, f_2 : V \to K$ s.t. $i \circ f_1 = i \circ f_2 = g$, then $g : V \to F$ factors uniquely through i, i.e. there exists a unique h s.t. $g = i \circ h$. This implies that $f_1 = f_2 = h$.

Example 4.1. The kernel often comes with an associated K, which is the kernel in the algebraic sense.

Take $M, N \in \mathrm{Ob}({}_R\underline{\mathrm{Mod}})$, and $f : M \to N$ is a morphism of R-modules. Then naturally one can construct the following s.t. $g = \ker f$.

$$\ker f \overset{g}{\hookrightarrow} M \xrightarrow{f} N$$

Definition 4.2 (Cokernel)**.** *The **cokernel** of f is the kernel in the dual category, which effectively annihilates a morphism.*

Explicitly, let $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, $p \in \mathrm{Hom}_{\mathscr{C}}(B, C)$. p is the cokernel of f if

- $p \circ f = 0$.
- *For all $g : B \to C'$, there exists a unique morphism $h : C \to C'$ s.t. $g = h \circ p$; i.e. any cokernel of f factors uniquely through p:*

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "p"] \arrow[dr, "g"] & C \arrow[d, "h"] \\
& & C'
\end{tikzcd}$$

Remark 4.3. The dual argument w.r.t. kernel holds. That is, if cokernel exists, it is unique up to isomorphism; and cokernel is always an epimorphism.

Example 4.2. Let $f : M \to N$ be a morphism of ${}_R\underline{\mathrm{Mod}}$. Then the cokernel of f is given via $\pi : N \to N/\mathrm{im}\, f$ which is the quotient map.

It is in particular interesting to put in juxtaposition of the two concepts. Let $\mathscr{C}$ be a preadditive category, and for all morphisms in $\mathscr{C}$ its kernel and cokernel exist. Then consider the following commutative diagram:

$$\begin{tikzcd}
\ker f \arrow[r, "i"] & A \arrow[r, "f"] \arrow[d, "q"] \arrow[dr, dashed, "g"] & B \arrow[r, "p"] & \mathrm{coker}\, f \\
& \mathrm{coker}\, i \arrow[r, dashed, "h"] \arrow[ur, dashed, "r"] & \ker p \arrow[u, "j"] &
\end{tikzcd}$$

The universal property of kernel and cokernel gives:

- f factors uniquely through q, i.e. there exists a unique r s.t. $f = r \circ q$.
- f factors uniquely through j, i.e. there exists a unique g s.t. $f = j \circ g$.
- g factors uniquely through q, i.e. there exists a unique h s.t. $g = h \circ q$.

Combining these gives the uniqueness of h s.t. $f = j \circ h \circ q$.

Remark 4.4. This resembles the first isomorphism theorem.

In ${}_R\underline{\text{Mod}}$, it is given that $\operatorname{coker} i = A/\ker f$, and $\ker p = B/(B/\operatorname{im} f) \simeq \operatorname{im} f$, which is exactly the first isomorphism theorem.

Definition 4.3 (Additive Category)**.** *An* ***additive category*** $\mathscr{C}$ *is a preadditive category s.t.*

- $\mathscr{C}$ *has a zero object.*
- $\forall X, Y \in \mathrm{Ob}(\mathscr{C})$*, the product* $X \times Y$ *exists (and it's isomorphic to their coproduct).*

Definition 4.4 (Abelian Category)**.** *An* ***abelian category*** *is an additive category* $\mathscr{C}$ *s.t.*

- *Every morphism has a kernel and a cokernel.*
- *"The first isomorphism theorem holds", i.e. for* f *considered as above, the morphism* $h : \operatorname{coker} i \to \ker p$ *is an isomorphism.*

5 Natural Transformation of Functors

Definition 5.1 (Natural Transformation)**.** *Let* $F, G : \mathscr{C} \to \mathscr{D}$ *be functors. Then the* ***natural transformation*** *from* F *to* G $T : F \Rightarrow G$ *is given by* $T_U : F(U) \to G(U)$ *for all* $U \in \mathrm{Ob}(\mathscr{C})$ *s.t. for all* $f : A \to B$ *in* $\mathscr{C}$ *the following diagram commutes:*

$$\begin{tikzcd} F(A) \arrow[r, "T_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "T_B"] & G(B) \end{tikzcd}$$

Example 5.1. In ${}_R\underline{\text{Mod}}$ where R is commutative, one can consider ideals $I \subseteq J$ the transformation which maps $u \pmod{IM}$ to

$$\begin{tikzcd} M/IM \arrow[r, "T_A"] \arrow[d, "F(f)"'] & M/JM \arrow[d, "G(f)"] \\ N/IN \arrow[r, "T_B"] & N/JN \end{tikzcd}$$

$u \pmod{JM}$. This is possible as $I \subseteq J$, where no information is missing.

Remark 5.1. The trivial natural transformation is simply the identity $\mathrm{Id}_F : F \Rightarrow F$. Via composing the commutative diagrams, it is easy to see that natural transformations is valid through composition.

Definition 5.2. *A natural transformation $T : \mathscr{C} \to \mathscr{D}$ is an* ***isomorphism of functors*** *if for all $A \in \mathscr{C}$, T_A is an isomorphism (in $\mathscr{D}$).*

Remark 5.2. A natural transformation T is an isomorphism if there exists $T' : G \Rightarrow F$ s.t. $T \circ T' = \mathrm{Id}_F, T' \circ T = \mathrm{Id}_G$.

This results from the fact that every isomorphism has an inverse in $\mathscr{D}$; and two natural transformations are equal iff they match on all instances of objects in $\mathscr{D}$.

Example 5.2. Consider two functors $F, G : {}_R\underline{\mathrm{Mod}} \to \underline{\mathrm{Ab}}$, where F is the forgetful functor, and $G = \mathrm{Hom}_{{}_R\underline{\mathrm{Mod}}}(R, -)$. Since a morphism of R-modules from R suffices to specify where 1 is mapped to, $\mathrm{Hom}_{{}_R\underline{\mathrm{Mod}}}(R, M) \simeq M$, which implies that F and G are isomorphic functors.

Definition 5.3. *Two categories $\mathscr{C}$ and $\mathscr{D}$ are* ***isomorphic*** *if there exists a functor isomorphism $F : \mathscr{C} \to \mathscr{D}$ s.t. there exists $G : \mathscr{D} \to \mathscr{C}$ s.t. $F \circ G = \mathrm{Id}_{\mathscr{D}}, G \circ F = \mathrm{Id}_{\mathscr{C}}$. They are* ***equivalent*** *if instead $F \circ G \simeq \mathrm{Id}_{\mathscr{D}}, G \circ F \simeq \mathrm{Id}_{\mathscr{C}}$.*