MATH 593 - Tensor Product

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Contents

1 Tensor Product of Modules

Definition 1.1 (R-balanced Maps)**.** *Let R be a ring, with M a right R-module, N a left R-module and P an abelian group. Then the map $\varphi : M \times N$ is **$\mathbf{R}$-balanced** if the followings are satisfied:*

- $\varphi(u, v_1 + v_2) = \varphi(u, v_1) + \varphi(u, v_2)$ *for all* $u \in M, v_1, v_2 \in N$.
- $\varphi(u_1 + u_2, v) = \varphi(u_1, v) + \varphi(u_2, v)$ *for all* $u_1, u_2 \in M, v \in N$.
- $\varphi(ua, v) = \varphi(u, av)$ *for all* $a \in R, u \in M, b \in N$.

Remark 1.1. The only difference between R-balanced maps and R-linear maps is the third condition: the coefficient in R could be transferred between different positions, but not out of the expression.

Definition 1.2 (Tensor Product)**.** *A **tensor product** of M and N is an abelian group $M \otimes_R N$ with an R-balanced map $\varphi : M \times N \to M \otimes_R N$ which is universal w.r.t. the property: i.e. $\forall \psi : M \times N \to P$ which is R-balanced, there exists a unique $f : M \otimes_R N \to P$ s.t. $\psi = f \circ \varphi$ (ψ factors uniquely through φ), i.e. the following diagram commute:*

$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & M \otimes_R N \arrow[d, dashed, "f"] \\ & P \end{tikzcd}$$

Remark 1.2. If $\otimes_R$ exists, then it is unique up to a canonical isomorphism.

Suppose that for $M, N \in {}_R\underline{\text{Mod}}$, there exists two tensor products T and T'. Denote the canonical map from $M \times N$ to T and T' be φ and φ', respectively. Then by universal property of tensor product, there exists a unique isomorphism f and f' s.t. $f \circ \varphi = \varphi'$ and $f' \circ \varphi' = \varphi$, which gives $f \circ f' = \text{Id}$.

Proposition 1.1. *The tensor product exists.*

Proof. Proceed to show this via introducing relations on the free group structure. Let $F := \mathbb{Z}^{M \times N}$ be a free abelian group with basis $\{e_{(u,v)} \mid (u, v) \in M \times N\}$. Quotient out the elements that are claimed to be equivalent by the constraint that the canonical map φ should be R-balanced: consider $G \subseteq F$ to be generated by the following elements:

- $(e_{u_1+u_2,v} - e_{u_1,v} - e_{u_2,v})$, for all $u_1, u_2 \in M, v \in N$.
- $(e_{u,v_1+v_2} - e_{u,v_1} - e_{u,v_2})$, for all $u \in M, v_1, v_2 \in N$.
- $(e_{ua,v} - e_{u,av})$ for all $u \in M, v \in N, a \in R$.

By construction it is clear that the canonical map $\varphi : M \times N \to M \otimes_R N$ is R-balanced, via specifying $\varphi(u, v) = \overline{e_{u,v}}$.

It suffices to verify that the construction is compatible with the universal property. Consider the R-balanced map $\psi : M \times N \to P$, with the group homomorphism $g : F \to P$ s.t. $g(e_{u,v}) = \psi(u, v)$:

- *Existence.* Applying the universal property of quotient groups, which implies that there exists a unique f s.t. $f \circ h = g$ where h is the induced group homomorphism of the quotient. This is indeed valid, as ψ is R-linear, which by construction has kernel G.

$$\begin{tikzcd}
M \times N \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & F/G \arrow[d, dashed, "f"] & F \arrow[l, "h"'] \arrow[dl, "g"] \\
& P &
\end{tikzcd}$$

- *Uniqueness.* This follows from the result of universal property above; and the fact that φ is surjective.

□

Remark 1.3. The construction above, together with the fact that tensor products exist uniquely up to isomorphism, implies that for R-modules M and N with their system of generators, (u_i) and (v_i) respectively, for all $x \in M \otimes_R N$, there exists $(d_i) \in \mathbb{Z}$ s.t.

$$x = \sum_{i=1}^{n} d_i (u_i \otimes_R v_i)$$

where the multiplication by integers is simply adding repetitively the elements to itself.

The tensor products could also behave *functorially*, via composing with the canonical map of tensor product:

Let $f : M \to M'$ a morphism of right R-modules, and $g : N \to N'$ a morphism of left R-modules. Then one could define a map $\psi : M \times N \to M' \otimes_R N'$, where $(u, v) \mapsto f(u) \otimes_R g(v)$. The map is R-balanced since the canonical map of tensor product is R-balanced. Therefore it is valid to apply the universal property of tenbsor product, which gives a unique group homomorphism $f : M \otimes_R N \to M' \otimes_R N'$. This is uniquely determined by f and g; and is often denoted as $f \otimes_R g$.

Remark 1.4. This is also compatible with composition, via applying the universal property twice. Explicitly, for $f : M \to M', f' : M' \to M''$ a morphism of right R-modules, and $g : N \to N', g' : N' \to N''$ a morphism of left R-modules, we have

$$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)$$

Remark 1.5. In particular the constructions above induces a functor $M \otimes - : {}_R\underline{\text{Mod}} \to \underline{\text{Ab}}$ for M a right R-module, where

$$N \in \text{Ob}({}_R\underline{\text{Mod}}) \mapsto M \otimes N, \qquad f : N \to N' \mapsto \text{Id}_M \otimes f$$

Similar to the case of Hom Functors, we seek to lift the functor to ${}_R\underline{\text{Mod}} \to {}_R\underline{\text{Mod}}$. This requires extra structure on the module of interest. Similarly, making R commutative, and restricting M and N to be both R-modules could resolve the issue, but the condition is too strong.

2 Bimodule

Definition 2.1 (Bimodule). *Let S and R be rings. An* **R-S** ***bimodule*** *M is given by an an abelian group M that is both a left S-module and a right R-module; and module operations is compatible, i.e.*

$$(au)b = a(ub) \qquad \forall u \in M, a \in S, b \in R$$

Remark 2.1. If R is commutative, then every R-module is an R-R bimodule (which is why making R commutative suffices to ensure the Hom module has an R-module structure). Specifically, R is an R-R bimodule.

Remark 2.2. Morphisms between S-R bimodules inherits from corresponding modules. Compatibility does not interfere with morphisms.

Proposition 2.1. *Let R and S be rings, with M an S-R bimodule, and N a left R-module. Then there exists a unique left S-module structure on $M \otimes N$ s.t. $\lambda \cdot (u \otimes v) = (\lambda u) \otimes v$, for all $\lambda \in S, u \in M, v \in N$.*

Proof. Use the universal property, with $P = M \otimes N$. Fix $\lambda \in S$; consider $\varphi : M \times N \to M \otimes N$ s.t. $\varphi(u, v) = (\lambda u) \otimes v$. This map is R-balanced, as the tensor product on R is balanced.

Then by universal property there exists a unique group homomorphism $f_\lambda : M \otimes N \to M \otimes N$, $u \otimes v \mapsto (\lambda u) \otimes v$. This gives the scalar multiplication of λ, which induces an S-module structure on $M \otimes N$. □

Proposition 2.2. *The extra structure on the modules gives extra structure on the morphisms in the universal property.*

Let M be a S-R bimodule, N a left R-module, and P a left S-module. Let $\varphi : M \times N \to M \otimes_R N$ the canonical map of tensor product. Suppose further that the map $\psi : M \times N \to P$ is S-bilinear. Then there exists a unique morphism of S-modules $f : M \otimes_R N \to P$.

Proof. By the universal property of tensor product, such morphism f exists, and is uniquely specified by $f(u \otimes v) = \psi(u, v)$. It suffices to check that this is indeed a morphism of S-modules, i.e. for all $a \in S$, $f((au) \otimes v) = af(u \otimes v)$. It then suffices to check that for certain (set of) fixed u and v, as every element in $M \otimes N$ is of such form. This is clear as

$$f((au) \otimes v) = \psi(au, v) \overset{!}{=} a\psi(u, v) = a \cdot f(u \otimes v)$$

Equality (!) requires that ψ is S-bilinear, and M being a bimodule ensures that this is well-formed under the context of S-modules. □

Remark 2.3. The proposition above lifts the functor $M \otimes -$ to ${}_R\underline{\text{Mod}} \to {}_S\underline{\text{Mod}}$ for all S-R bimodule M.

Remark 2.4. It may be interesting to consider the following property of bimodules:

1. If R is commutative, then left or right R-modules are the same; and in this case $M \otimes_R N$ is an R-module.
2. If M is a T-R bimodule, and N is an R-S bimodule, then $M \otimes_R N$ is a T-S bimodule.

For the second remark, it is clear that $M \otimes_R N$ is both a left T-module, and a right S-module, via applying the same proof as in Proposition 2.1. It suffices to prove that they are compatible. This is also clear from the construction in the proposition referred:

$$(a(u \otimes_R v))b = (au \otimes v)b = (au) \otimes (vb) = a(u \otimes (vb)) = a((u \otimes v)b)$$

Remark 2.5. Let R be a ring. Then R is an R-R bimodule. Let M be a left R-module, which implies that $R \otimes_R M$ is a left R-module. Then there exists a functorial isomorphism $R \otimes_R M \simeq M$ for all $M \in \text{Ob}({}_R\underline{\text{Mod}})$. (This is called functorial as this could be regarded as the property of functor $R \otimes_R -$.)

Proof. Proof via using the universal property. Consider the morphism of R-modules $\alpha : R \times M \to M$, where $\alpha(a, u) = au$ for all $a \in R$, $u \in M$. It is R-linear, which is by definition R-balanced. The universal property gives that there exists a unique

$f : R \otimes M \to M$ s.t. $f(a \otimes u) = au$. Designate $g : M \to R \otimes M$, $g(u) = 1 \otimes u$ for all $u \in M$. This is clearly R-balanced. This gives an isomorphism as $g \circ f = \mathrm{Id}_R$, $f \circ g = \mathrm{Id}_{R \otimes M}$. $\square$

3 Extension of Scalar

4 Adjoint Property of Tensor Product