MATH 593 - Ring

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Contents

1 Ring homomorphism, Quotient Ring

Definition 1 (Ring Homomorphism)**.** *Let X, Y be rings. A **Ring Homomorphism** is a map $f : X \to Y$ satisfying the following properties:*

- $f(1) = 1$.
- $\forall x_1, x_2 \in X, f(x_1) + f(x_2) = f(x_1 + x_2)$.
- $\forall x_1, x_2 \in X, f(x_1 x_2) = f(x_1) f(x_2)$

Definition 2 (Quotient Ring)**.** *Let R be a ring and $I \subseteq R$ a two-sided ideal. The **Quotient Ring** (R/I) is defined as $(R/\sim)$ with an equivalence relation $\sim$ where $a \sim b$ if and only if $a - b = I$. Elements in (R/I) are denoted as $\bar{a}$, where $\bar{a} = \bar{b}$ if and only if $a \sim b$.*

The natural homomorphism $\pi_I : R \to (R/I)$ is defined as $\pi(a) = \bar{a}$, which satisfies the *universal property of quotient rings*:

Theorem 1 (Fundamental Theorem of Ring Homomorphisms)**.** *Let $\varphi : R \to S$ be a ring homomorphism, I a two-sided ideal s.t. $I \subseteq \ker\varphi$, and π be the natural ring homomorphism from R to (R/I). Then there exists a unique ring homomorphism $f : R/I \to S$ s.t. the following diagram commutes, i.e. $\varphi = f \circ \pi$.*

$$\begin{tikzcd}
R \arrow[r, "\varphi"] \arrow[dr, "\pi"] & S \\
& \tilde{R}/I \arrow[u, "f"']
\end{tikzcd}$$

Proof. It suffices to prove that f exists and is unique, and verify that f is indeed a ring homomorphism.

- **Uniqueness.** By the requirement that f should make the diagram commute, $f(\bar{a}) = \varphi(a),\ \forall a \in R$. Uniqueness of f follows from the fact that φ maps every element in R to a unique element in S.
- **Existence.** It suffices to verify that f is well-defined, i.e. does not vary w.r.t. change of representative in (R/I). For all $a, b \in R$ s.t. $\bar{a} = \bar{b}$, $(a - b) \in I \implies \varphi(a - b) = 0 \implies \varphi(a) = \varphi(b)$ since φ is a ring homomorphism. By the uniqueness of f it is specified that $f(\bar{a}) = \varphi(a)$, which implies that for all $\bar{a} = \bar{b} \in (R/I)$, $f(\bar{a}) = \varphi(a) = \varphi(b) = f(\bar{b})$.
- f **is indeed a homomorphism.** This follows from the fact that φ is a ring homomorphism.

$\square$

2 Ring of Fractions

Definition 3 (Multiplicative System)**.** *A subset $S \subseteq R$ for a ring R is a **multiplicative system** if $1 \in S$, and $\forall s_1, s_2 \in S, s_1 \cdot s_2 \in S$, where $\cdot$ is multiplication in R.*

Definition 4 (Ring of Fractions)**.** *Let R be a commutative ring, with $S \subseteq R$ a multiplicative subset, the **ring of fraction** $S^{-1}R$ is defined as $R \times S/\sim$, where $(s_1, r_1) \sim (s_2, r_2)$ if and only if there exists $t \in R$ s.t. $t(s_1 r_2 - s_2 r_1) = 0$. $(s, r) \in S^{-1}R$ is denoted as $\frac{s}{r}$.*

Remark 1. *If R is an integral domain, then $(s_1, r_1) \sim (s_2, r_2)$ iff $s_1 r_2 = s_2 r_1$, as for $\mathbb{Q}$.*

Proposition 1. *$\sim$ is an equivalence relation.*

Proof. It is clear that $\sim$ is reflexive and symmetric. For transitivity, consider $(s_1, r_1) \sim (s_2, r_2) \wedge (s_2, r_2) \sim (s_3, r_3)$. That is, there exists some $t_1, t_2 \in R$ s.t.

$$\begin{cases} t_1(s_1 r_2 - s_2 r_1) = 0 \\ t_2(s_2 r_3 - s_3 r_2) = 0 \end{cases} \implies t_1 t_2 (s_1 r_2 s_3 - s_2 r_1 s_3) = t_1 t_2 (s_1 s_2 r_3 - s_2 r_1 s_3) = t_1 t_2 s_2 (s_1 r_3 - s_3 r_1) = 0$$

□

2.1 Localization of a Ring

3 Polynomial Rings

4 Ideals

5 Euclidean Domain, PIDs and UFDs