MATH 593 - Tensor Product

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Contents

1 Tensor Product of Modules

Definition 1.1 (R-balanced Maps)**.** *Let R be a ring, with M a right R-module, N a left R-module and P an abelian group. Then the map $\varphi : M \times N$ is* $\mathbf{R}$***-balanced*** *if the followings are satisfied:*

- $\varphi(u, v_1 + v_2) = \varphi(u, v_1) + \varphi(u, v_2)$ *for all* $u \in M, v_1, v_2 \in N$.
- $\varphi(u_1 + u_2, v) = \varphi(u_1, v) + \varphi(u_2, v)$ *for all* $u_1, u_2 \in M, v \in N$.
- $\varphi(ua, v) = \varphi(u, av)$ *for all* $a \in R, u \in M, b \in N$.

Remark 1.1. The only difference between R-balanced maps and R-linear maps is the third condition: the coefficient in R could be transferred between different positions, but not out of the expression.

Definition 1.2 (Tensor Product)**.** *A* ***tensor product*** *of M and N is an abelian group $M \otimes_R N$ with an R-balanced map $\varphi : M \times N \to M \otimes_R N$ which is universal w.r.t. the property: i.e. $\forall \psi : M \times N \to P$ which is R-balanced, there exists a unique $f : M \otimes_R N \to P$ s.t. $\psi = f \circ \varphi$ (ψ factors uniquely through φ), i.e. the following diagram commute:*

$$\begin{tikzcd}
M \times N \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & M \otimes_R N \arrow[d, dashed, "f"] \\
& P
\end{tikzcd}$$

Remark 1.2. If $\otimes_R$ exists, then it is unique up to a canonical isomorphism.

Suppose that for $M, N \in {}_R\underline{\text{Mod}}$, there exists two tensor products T and T'. Denote the canonical map from $M \times N$ to T and T' be φ and φ', respectively. Then by universal property of tensor product, there exists a unique isomorphism f and f' s.t. $f \circ \varphi = \varphi'$ and $f' \circ \varphi' = \varphi$, which gives $f \circ f' = \text{Id}$.

Proposition 1.1. *The tensor product exists.*

Proof. Proceed to show this via introducing relations on the free group structure. Let $F := \mathbb{Z}^{M \times N}$ be a free abelian group with basis $\{e_{(u,v)} \mid (u, v) \in M \times N\}$. Quotient out the elements that are claimed to be equivalent by the constraint that the canonical map φ should be R-balanced: consider $G \subseteq F$ to be generated by the following elements:

- $(e_{u_1+u_2,v} - e_{u_1,v} - e_{u_2,v})$, for all $u_1, u_2 \in M, v \in N$.
- $(e_{u,v_1+v_2} - e_{u,v_1} - e_{u,v_2})$, for all $u \in M, v_1, v_2 \in N$.
- $(e_{ua,v} - e_{u,av})$ for all $u \in M, v \in N, a \in R$.

By construction it is clear that the canonical map $\varphi : M \times N \to M \otimes_R N$ is R-balanced, via specifying $\varphi(u, v) = \overline{e_{u,v}}$.

It suffices to verify that the construction is compatible with the universal property. Consider the R-balanced map $\psi : M \times N \to P$, with the group homomorphism $g : F \to P$ s.t. $g(e_{u,v}) = \psi(u, v)$:

- *Existence.* Applying the universal property of quotient groups, which implies that there exists a unique f s.t. $f \circ h = g$ where h is the induced group homomorphism of the quotient. This is indeed valid, as ψ is R-linear, which by construction has kernel G.

$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & F/G \arrow[d, dashed, "f"] & F \arrow[l, "h"'] \arrow[dl, "g"] \\ & P & \end{tikzcd}$$

- *Uniqueness.* This follows from the result of universal property above; and the fact that φ is surjective.

$\square$

Remark 1.3. The construction above, together with the fact that tensor products exist uniquely up to isomorphism, implies that for R-modules M and N with their system of generators, (u_i) and (v_i) respectively, for all $x \in M \otimes_R N$, there exists $(d_i) \in \mathbb{Z}$ s.t.

$$x = \sum_{i=1}^{n} d_i (u_i \otimes_R v_i)$$

where the multiplication by integers is simply adding repetitively the elements to itself.

The tensor products could also behave *functorially*, via composing with the canonical map of tensor product:

Let $f : M \to M'$ a morphism of right R-modules, and $g : N \to N'$ a morphism of left R-modules. Then one could define a map $\psi : M \times N \to M' \otimes_R N'$, where $(u, v) \mapsto f(u) \otimes_R g(v)$. The map is R-balanced since the canonical map of tensor product is R-balanced. Therefore it is valid to apply the universal property of tenbsor product, which gives a unique group homomorphism $f : M \otimes_R N \to M' \otimes_R N'$. This is uniquely determined by f and g; and is often denoted as $f \otimes_R g$.

Remark 1.4. This is also compatible with composition, via applying the universal property twice. Explicitly, for $f : M \to M', f' : M' \to M''$ a morphism of right R-modules, and $g : N \to N', g' : N' \to N''$ a morphism of left R-modules, we have

$$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)$$

Remark 1.5. In particular the constructions above induces a functor $M \otimes - : {}_R\underline{\text{Mod}} \to \underline{\text{Ab}}$ for M a right R-module, where

$$N \in \text{Ob}({}_R\underline{\text{Mod}}) \mapsto M \otimes N, \qquad f : N \to N' \mapsto \text{Id}_M \otimes f$$

Similar to the case of Hom Functors, we seek to lift the functor to ${}_R\underline{\text{Mod}} \to {}_R\underline{\text{Mod}}$. This requires extra structure on the module of interest. Similarly, making R commutative, and restricting M and N to be both R-modules could resolve the issue, but the condition is too strong.

2 Bimodule

3 Extension of Scalar

4 Adjoint Property of Tensor Product