MATH 593 - Ring

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September 24, 2023

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1 Ring homomorphism, Quotient Ring

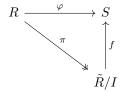
Definition 1 (Ring Homomorphism). Let X, Y be rings. A **Ring Homomorphism** is a map $f: X \to Y$ satisfying the following properties:

- f(1) = 1.
- $\forall x_1, x_2 \in X, f(x_1) + f(x_2) = f(x_1 + x_2).$
- $\forall x_1, x_2 \in X, f(x_1x_2) = f(x_1)f(x_2)$

Definition 2 (Quotient Ring). Let R be a ring and $I \subseteq R$ a two-sided ideal. The **Quotient Ring** (R/I) is defined as (R/\sim) with an equivalence relation \sim where $a \sim b$ if and only if a - b = I. Elements in (R/I) are denoted as \bar{a} , where $\bar{a} = \bar{b}$ if and only if $a \sim b$.

The natural homomorphism $\pi_I: R \to (R/I)$ is defined as $\pi(a) = \bar{a}$, which satisfies the *universal property of quotient rings*:

Theorem 1 (Fundamental Theorem of Ring Homomorphisms). Let $\varphi: R \to S$ be a ring homomorphism, I a two-sided ideal s.t. $I \subseteq \ker \varphi$, and π be the natural ring homomorphism from R to (R/I). Then there exists a unique ring homomorphism $f: R/I \to S$ s.t. the following diagram commutes, i.e. $\varphi = f \circ \pi$.



Proof. It suffices to prove that f exists and is unique, and verify that f is indeed a ring homomorphism.

- Uniqueness. By the requirement that f should make the diagram commute, $f(\bar{a}) = \varphi(a), \ \forall a \in R$. Uniqueness of f follows from the fact that φ maps every element in R to a unique element in S.
- Existence. It suffices to verify that f is well-defined, i.e. does not vary w.r.t. change of representative in (R/I). For all $a,b\in R$ s.t. $\bar{a}=\bar{b}, (a-b)\in I \implies \varphi(a-b)=0 \implies \varphi(a)=\varphi(b)$ since φ is a ring homomorphism. By the uniqueness of f it is specified that $f(\bar{a})=\varphi(a)$, which implies that for all $\bar{a}=\bar{b}\in (R/I), f(\bar{a})=\varphi(a)=\varphi(b)=f(\bar{b})$.

• f is indeed a homomorphism. This follows from the fact that φ is a ring homomorphism.

2 Ring of Fractions

Definition 3 (Multiplicative System). A subset $S \subseteq R$ for a ring R is a **multiplicative system** if $1 \in S$, and $\forall s_1, s_2 \in S$, where \cdot is multiplication in R.

Definition 4 (Ring of Fractions). Let R be a commutative ring, with $S \subseteq R$ a multiplicative subset, the **ring of fraction** $S^{-1}R$ is defined as $R \times S / \sim$, where $(s_1, r_1) \sim (s_2, r_2)$ if and only if there exists $t \in R$ s.t. $t(s_1r_2 - s_2r_1) = 0$. $(s, r) \in S^{-1}R$ is denoted as $\frac{s}{r}$.

Remark 1. If R is an integral domain, then $(s_1, r_1) \sim (s_2, r_2)$ iff $s_1 r_2 = s_2 r_1$, as for \mathbb{Q} .

Proposition 1. \sim is an equivalence relation.

Proof. It is clear that \sim is reflexive and symmetric. For transitivity, consider $(s_1, r_1) \sim (s_2, r_2) \wedge (s_2, r_2) \sim (s_3, r_3)$. That is, there exists some $t_1, t_2 \in R$ s.t.

$$\begin{cases} t_1(s_1r_2 - s_2r_1) = 0 \\ t_2(s_2r_3 - s_3r_2) = 0 \end{cases} \implies t_1t_2(s_1r_2s_3 - s_2r_1s_3) = t_1t_2(s_1s_2r_3 - s_2r_1s_3) = t_1t_2s_2(s_1r_3 - s_3r_1) = 0$$

2.1 Localization of a Ring

- 3 Polynomial Rings
- 4 Ideals
- 5 Euclidean Domain, PIDs and UFDs