MATH 593 - Categories and Functors

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1   Category; Functor

Definition 1.1 (Category). *A **category** $\mathcal{C}$ consists of*

- *A <u>class</u> of objects $\mathcal{C}$ (which, for example, could contain all sets), denoted $\mathrm{Ob}(\mathcal{C})$.*

- *For all $A, B \in \mathrm{Ob}(\mathcal{C})$, a <u>set</u> $\mathrm{Hom}_{\mathcal{C}}(A, B)$ the "morphisms in $\mathcal{C}$ from A to B" with map $\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to$ $\times \mathrm{Hom}_{\mathcal{C}}(A, C)$ the "morphism composition" denoted $f \times g \rightsquigarrow (g \circ f)$, satisfying*

  - Existence of an identity. *for all $A \in \mathrm{Ob}(\mathcal{C})$, there exists $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ s.t.*

  $$\begin{cases} 1_A \circ f = f & \forall f \in \mathrm{Hom}_{\mathcal{C}}(A, B) \\ g \circ 1_A = g & \forall g \in \mathrm{Hom}_{\mathcal{C}}(B, A) \end{cases}$$

  - Associativity. *For all $f \in \mathrm{Hom}_{\mathcal{C}}(A, B), g \in \mathrm{Hom}_{\mathcal{C}}(B, C), h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$,*

  $$(h \circ g) \circ f = h \circ (g \circ f)$$

Remark 1.1. The definition much resembles previous algebraic structures; but the morphisms and composition laws could be defined in a particularly strange way:

1. Similar to monoids, the definition implies that the identity is unique. Suppose that there are two identities $1_A, 1_A' \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ for $A \in \mathrm{Ob}(\mathcal{C})$, then $1_A = 1_A \circ 1_A' = 1_A'$.

2. The morphism is not necessarily a function; and in such cases composition needs to be re-defined respectively.

Example 1.1. Consider the following categories:

- *Category of Sets* <u>Sets</u>, where the objects are sets, and morphisms are maps between sets.

- *Category of Rings* <u>Rings</u>, where the objects are rings, and morphisms are ring homomorphisms.

- *Category of (left) R-modules* $_R\underline{\mathrm{Mod}}$, where objects are lef R-modules, and morphisms R-linear maps.

- Consider the category $\mathcal{C}$ defined on a partially-ordered set $(A, \leq)$ where

  - $\mathrm{Ob}(\mathcal{C})$ consists of elements in A.

  - Morphisms are defined as

  $$\mathrm{Hom}_{\mathcal{C}}(A, B) = \begin{cases} \{*\} & A \leq B \\ \emptyset & \text{otherwise} \end{cases}$$

  where the composition of maps is defined as intersection. This is due to the fact that there can be no maps whose image is the empty set.

Definition 1.2 (Functor). *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of mappings for both objects and morphisms:*

- *For all $A \in \mathrm{Ob}(\mathcal{C})$, $F(A) \in \mathcal{D}$.*

- *For all $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(b))$ s.t.*

  - *$F(1_A) = 1_{F(A)}$ for all $A \in \mathrm{Ob}(\mathcal{C})$.*

– *For all $f \circ g$ where $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $F(f \circ g) = F(f) \circ F(g)$.*

The composition of functors is conducted in a natural way, i.e. applying consecutively.

Example 1.2. Functors represent the induced maps w.r.t. a transformation in the structure:

1. Let R be a commutative ring and $S \subseteq R$ a multiplicative system. Consider the functor $F : {}_R\underline{\mathrm{Mod}} \to {}_{S^{-1}R}\underline{\mathrm{Mod}}$ where

   - $F(M) = S^{-1}M$ for all $M \in \mathrm{Ob}({}_R\underline{\mathrm{Mod}})$.

   - For $f : M \to N$, define $F(f) := S^{-1}M \to S^{-1}N$, where $\frac{u}{s} \mapsto \frac{f(u)}{s}$.

2. Let R be a ring, with $I \subseteq R$ a two-sided ideal of R; and M a left R-module. Consider the functor $F : {}_R\underline{\mathrm{Mod}} \to {}_{R/I}\underline{\mathrm{Mod}}$ where

   - $F(M) = M/IM$ for all $M \in \mathrm{Ob}({}_R\underline{\mathrm{Mod}})$.

   - Let $f : M \to N$ be a morphism of left R-modules. Then it induces a map $\bar{f} : M/IM \to N/IN$ s.t. $\bar{f}(\bar{(u)}) = \overline{f(u)}$. Define $F(f) = \bar{f}$.

3. Functors generally can abandon structures. Let M be a left R-module. By definition it is valid to view M as an abelian group. Then functor $F : {}_R\underline{\mathrm{Mod}} \to \underline{\mathrm{Ab}}$ where objects are taken to itself; and morphisms are taken to group homomorphisms. These are called *forgetful functors*.

2   Morphism of Categories

The dual of a category is where the direction of morphisms is inverted. The following gives a formalization of this:

Definition 2.1 (Contravariant Functor). *A **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$ is a functor which maps composition to that in the inverse order, i.e.*

- *For all $A \in Ob(\mathcal{C})$, $F(A) \in \mathrm{Ob}(\mathcal{D})$.*

- *For all $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $F(f) \in \mathrm{Hom}_{\mathcal{C}}(F(B), F(A))$ s.t.*

  – *$F(1_A) = 1_{F(A)}$ for all $A \in \mathrm{Ob}(\mathcal{C})$.*

  – *For all $f \circ g$ where $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $F(f \circ g) = F(g) \circ F(f)$.*

Definition 2.2 (Dual Category). *Let $\mathcal{C}$ be a category. Then the **dual category** $\mathcal{C}^{\circ}$ of $\mathcal{C}$ is a category with*

- $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(\mathcal{C}^{\circ})$.

- $\mathrm{Hom}_{\mathcal{C}}(A, B) = \mathrm{Hom}_{\mathcal{C}^{\circ}}(B, A)$.

The composition is compatible as the inversion is done in the functor.

Remark 2.1. Since the dual category is defined on the contravariant of the functor, replacing a functor with its contravariant is equivalent to replacing the category with its dual.

Similar to the case of modules we can define the *Hom Functors*; but as a concept one level up it leaves the image unspecified:

Definition 2.3 (Hom Functor). *Let $\mathcal{C}$ be a category, and $A \in \mathrm{Ob}(\mathcal{C})$. Then the **Hom functor** $\mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \to \underline{\mathrm{Sets}}$ where*

- *For $B \in \mathrm{Ob}(\mathcal{C})$, $F(B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$.*

- *For $f : \mathrm{Hom}_{\mathcal{C}}(B_1, B_2)$, $F(g) : \mathrm{Hom}_{\mathcal{C}}(A, B_1) \to \mathrm{Hom}_{\mathcal{C}}(A, B_2)$, $g \mapsto f \circ g$.*

Remark 2.2. Similarly, we can consider the contravariant functor of the Hom functor. $\mathrm{Hom}_{\mathcal{C}^{\circ}}(-, A) : \mathcal{C}^{\circ} \to \underline{\mathrm{Sets}}$. By definition $\mathrm{Hom}_{\mathcal{C}}(A, -) = \mathrm{Hom}_{\mathcal{C}^{\circ}}(-, A)$.

Remark 2.3. Let $\mathcal{C} = {}_R\underline{\mathrm{Mod}}$. Then $\mathrm{Hom}_{\mathcal{C}}(X, -)$ could be lifted to $\mathcal{C} \to \underline{\mathrm{Ab}}$. It can be further lifted to $\mathcal{C} \to {}_R\underline{\mathrm{Mod}}$ if R is commutative, which ensures that the morphisms will be R-linear. In this case this is just the Hom Module of (left) R-modules.

Definition 2.4. *Let $\mathcal{C}$ be a category. Then $u \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is an **isomorphism** if there exists $v \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ s.t. $u \circ v = \mathrm{Id}_B, v \circ u = \mathrm{Id}_A$.*

Remark 2.4. For a fixed u, such v is unique. Suppose that there exists two distinct vs, we have

$$v = v \circ \mathrm{Id}_B = v \circ (u \circ v') = (v \circ u) \circ v' = v'$$

which is a contradiction.

Remark 2.5. Let $F : \mathcal{C} \to \mathcal{D}$ a functor. Then $u \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ being an isomorphism implies that $F(u)$ is an isomorphism.

This results from the fact that $\mathrm{Id}_{F(B)} = F(\mathrm{Id}_B) = F(u \circ v) = F(u) \circ F(v)$. Result for A is similar; and uniqueness follows from the same reasoning.

Definition 2.5. *Let $\mathcal{C}$ be a category:*

- *$X \in \mathrm{Ob}(\mathcal{C})$ is an **initial object** if $\forall Y \in \mathrm{Ob}(\mathcal{C}), |\mathrm{Hom}_{\mathcal{C}}(X, Y)| = 1$.*

- *$X \in \mathrm{Ob}(\mathcal{C})$ is a **final object** if $\forall Y \in \mathrm{Ob}(\mathcal{C}), |\mathrm{Hom}_{\mathcal{C}}(Y, X)| = 1$.*

- *$X \in \mathrm{Ob}(\mathcal{C})$ is a **zero-object** if it is both an initial object and a final object.*

Remark 2.6. Let $X \in \mathrm{Ob}(\mathcal{C})$ be an initial (final, zero) object. Then X' is initial (final, zero) if and only if there exists an isomorphism between X and X'.

Proof is similar for all three cases. Suppose that X and X' are both initial. Then there exists a unique $f \in \mathrm{Hom}_{\mathcal{C}}(X, X')$ and $f' \in \mathrm{Hom}_{\mathcal{C}}(X', X)$, i.e. $f' \circ f \in \mathrm{Hom}_{\mathcal{C}}(X, X)$. X being initial implies that this is the unique morphism from X to itself, which contains Id_X. Therefore $f' \circ f = \mathrm{Id}_X$. Similar result holds for $f \circ f' = \mathrm{Id}_{X'}$, which implies that X and X' are isomorphic.

Example 2.1. Although if initial/final objects are unique up to isomorphism if they exist, but they actually do not necessarily exist:

1. In ${}_R\underline{\mathrm{Mod}}$, $\{0\}$ is a zero-object.

   The only element in a left R-module that should be preserved in a morphism of R-module is the zero element. For any element $a \neq 0$, there exists two maps that either maps a to 0, or another non-zero element, which indicates that this is not initial.

   Suppose that the final object has (at least) two elements $\{0, a\}$ for $a \neq 0$, then there exists at least two maps from a non-trivial module generated by $(u_1, \cdots, u_r)$ to it: for each u_i it is either mapped to 0 or a, which gives two morphisms.

2. In <u>Sets</u>, $\emptyset$ is initial, and $\{*\}$ (a set containing an arbitrary element) is final.

   Suppose that there exists an element in the initial object, then it could be mapped to any element as morphisms of <u>Sets</u>do not have constraints.

3. In <u>Rings</u>, $\mathbb{Z}$ is initial; and $\{0\}$ is final.

   $\mathbb{Z}$ being initial results from the fact that ring homomorphisms are required to preserve the 0 and 1 elements; and the maximal ring generated by $(0, 1)$ is isomorphic to $\mathbb{Z}$.

4. In <u>Fields</u>, there are no initial or final objects.

   This results directly from the fact that $1 \neq 0$ in fields. For every two fields, there exists two maps: one that maps 1 to 1; and the other maps 1 to 0.

Definition 2.6. *Let $\mathcal{C}$ be a category. $u \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is a **monomorphism** if for all $C \in \mathrm{Ob}(\mathcal{C})$ and $v_1, v_2 \in \mathrm{Hom}_{\mathcal{C}}(C, A)$ s.t. $u \circ v_1 = u \circ v_2$ implies $v_1 = v_2$. $u \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is an **epimorphism** if for all v_1, v_2 with the same comdition above satisfies $v_1 \circ u = v_2 \circ u$ implies $v_1 = v_2$, i.e. it is a monomorphism in $\mathcal{C}^{\circ}$.*

Remark 2.7. These are analogies of injective/surjective in the context of category. Since on the category level it is only valid to consider objects or morphisms, such analogies could be only made to morphisms.

Example 2.2. It is not always the case that monomorphisms could correspond to injective maps, and epimorphisms could correspond to surjective maps:

1. In <u>Sets</u>, monomorphisms correspond to injective maps, and epimorphisms correspond to surjective maps.

2. In $_R\underline{\mathrm{Mod}}$, such analogy is still true via choosing the v_1, v_2:

$$\ker u \underset{0}{\overset{\mathrm{incl.}}{\rightrightarrows}} A \overset{u}{\longrightarrow} B \qquad\qquad\qquad A \overset{u}{\longrightarrow} B \underset{0}{\overset{\pi}{\rightrightarrows}} B/\mathrm{im}\, u$$

   - For u being a monomorphism, $u(\ker u) = u(0) = 0$, i.e. the inclusion map from $\ker u$ to A is the same as the zero map, i.e. $\ker u = \{0\}$.

   - For u being an epimorphism, $\pi(u(A)) = 0$ in $B/\mathrm{im}\, u$, i.e. $\pi(B) = 0$ which indicates that $\mathrm{im}\, u = B$.

3. In <u>Rings</u>, monomorphisms are still injective, via for $f : R \to S$ considering $\mathbb{Z}[x] \underset{0}{\overset{x \mapsto u}{\rightrightarrows}} R \overset{f}{\longrightarrow} S$ for all $u \in \ker f$. This implies that $u = 0$, i.e. $\ker f = \{0\}$.

   But epimorphisms in rings are not necessarily surjective. Take the example $\alpha : \mathbb{Z} \hookrightarrow \mathbb{Q}$ which is the inclusion map. This is an epimorphism as $v_1 \circ \alpha = v_2 \circ \alpha$ if and only if 1 is mapped to the same element; but this always holds as ring homomorphisms preserve the multiplicative unit.

3 Products and Coproducts

4 Kernels and Cokernels

For the motivation of kernel one could refer to this article.

5 Natural Transformations of Functors