MATH 593 - Module

ARessegetes Stery

October 1, 2023

Contents

1 Module

Definition 1.1 (R-Module)**.** *An (left)* ***R-Module*** M *is a set with two operations, often denoted as* $(M, +, \times)$*:*

- *Addition* $(+) : M \times M \to M$ *, s.t.* $(M, +)$ *is an abelian group.*
- *Multiplication* $(\times) : R \times M \to M$*, s.t. it has the following properties:*
  - *Identity. For all* $x \in M$*, there exists* $1 \in R$ *s..t* $1 \cdot x = x$*.*
  - *Associativity. For all* $a, b \in R, x \in M$*,* $a(bx) = (ab)x$*.*
  - *Distributivity in R. For all* $a_1, a_2 \in R$*,* $(a_1 + a_2)x = a_1x + a_2x$*.*
  - *Distributivity in* M*. For all* $a \in R, x_1, x_2 \in M$*,* $a(x_1 + x_2) = ax_1 + ax_2$*.*

Right modules are defined with the same structure, but with $a \times b = b \cdot a$ *for* $a \in R, b \in M$*, where* $\times$ *is the multiplication in* M*, and* $\cdot$ *the multiplication in* R*.*

Definition 1.2 (Submodule)**.** *Let* $(M, +, \times)$ *be an* R*-module.* $N \subseteq M$ *is a* ***R-submodule*** *of* M *if* $(N, +)$ *is a subgroup of* M*; and for all* $n \in N, r \in R, n \times r \in N$*.*

Remark 1.1. Notice that R itself gives an R-module, just as $\mathbb{K}$ gives a $\mathbb{K}$-vector space. Therefore $\langle S, \varphi \rangle$ an R-algebra induces a two-sided R-module structure. Check that this is indeed the case:

- *Addition.* Adopt the addition in S as a ring.
- *Identity*: Since ring homomorphisms map identity to identity, $\varphi(1_R) = 1_S$, implying that 1_R is the identity for scalar multiplication.
- *Associativity.* Results from the fact that multiplication in S is associative.
- *Distributivity in R/M.* Follows from the fact that φ is a ring homomorphism.

In this sense, module generalizes the algebra structure. Generally one cannot "revert" the structure of a module back to an ideal. Specifically, suppose that R is not commutative, then R is not an R-algebra.

Remark 1.2. (Left) ideals of R are submodules of R taken as an R-submodule.

Remark 1.3. Let M be an abelian group. Making M into a (left) R-module is equivalent to specifying a ring homomorphism $\varphi : R \to \mathrm{End}(M)$, where $\mathrm{End}(\cdot)$ denotes the ring of endomorphisms on the specific structure.

It is worth noticing how the ring of endomorphism structure is defined. Specifically, the multiplication is the composition of endomorphisms on M. This can be viewed in two aspects:

- The associativity for R-modules is essentially stating that multiplication, i.e. elements of R "acting" on those in M is associative. Applying one action after another is the same as applying the composition of action.
- Consider the definition of function as a set of pairs. Then

$$R \times M \to M \cong (R \to M) \to M \cong R \to (M \to M)$$

  as the application of functions is associative.

In particular, in the consideration of $\mathbb{Z}$-modules, the map $\varphi_{\mathbb{Z}} : \mathbb{Z} \to \mathrm{End}(M)$ is determined uniquely by the requirement that $1 \mapsto 1_M = \mathrm{Id}_M$. Since addition and multiplication should be preserved, $n \mapsto n \cdot \mathrm{Id}_M$ for all $n \in \mathbb{Z}$. With the specification above one could observe the correspondence:

- $\{\mathbb{Z} \text{ modules}\} \iff \{\text{Abelian groups}\}$
- $\{\mathbb{Z}/n\mathbb{Z} \text{ modules}\} \iff \{\text{Abelian groups } M s.t. nx = 0 \forall x \in M\}$

2 Morphism of R-Modules

3 Construction of Submodules

4 Free Modules

5 Finiteness Conditions on Modules