MATH 594 - Representation of Finite Groups

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March 12, 2024

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1 Complex Representation

The motivation of introducing the representation of G is to have a linearized version of group action on sets. Recall that we have the correspondence between action of G on a set X and group homomorphism $G \to S_x$ where S_x is the group of bijective maps on S, with the operation defined as composition. Explicitly, this is given by

$$\varphi : G \times X \to X \quad \rightsquigarrow \quad G \to S_x,\ g \mapsto \varphi(g, -) \in (X \to X)$$

We now give the formal definition on vector spaces:

Definition 1.1 (Representation). A **(complex) representation** of a group G is a vector space V over $\mathbb{C}$, together with a group homomorphism

$$\rho : G \to \mathrm{GL}(V) := \{\varphi : V \to V \mid \varphi\ \mathbb{C}\text{-linear isomorphism}\}$$

Equivalently, a representation of G is a vector space over $\mathbb{C}$ with an action of G on V $\rho : G \times V \to V$ s.t. for all $g \in G$, the induced map $\varphi(g, -)$ os $\mathbb{C}$-linear.

Notation. The map $\rho(g, -) : V \to V$ is often abbreviated as ρ_g. The representation is denoted by V or ρ, with V emphasizing the vector space structure.

Definition 1.2 (Dimension of Repr.). The **dimension** of the representation is $\dim_{\mathbb{C}} V$, with the same notation as above.

For most of the time, we will only consider the representation of finite groups on finite-dimensional vector spaces.

Remark 1.3. In general, one can consider representations over other fields than $\mathbb{C}$. The reasons why $\mathbb{C}$ is chosen are the followings:

1) If G is finite, then $|G| \in \mathbb{C}$ is always invertible.

2) $\mathbb{C}$ is algebraically closed. The implications include, for example, every linear map has an eigenvalue.

These specialties will often appear in subsequent proofs.

Definition 1.4 (Morphism of Repr.). Given two representations of G, V and W, a **morphism of representations** (or simply G**-morphism**) is a linear map $f : V \to W$ s.t. $f(gv) = g(fv)$ for all $g \in G, v \in V$. This is an **isomorphism** if f is further bijective.

Remark 1.5. Following from the definitions we have the immediate results:

1) If $V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$ are morphisms of representation, then so is $g \circ f$ since $g(f(hv)) = g(hf(v)) = h(g(f(v)))$ for all $h \in G, v \in V$. This gives the morphisms of objects, i.e. representations of G give a category.

2) If $f : V \to W$ is an isomorphism of representations, then so is f^{-1} (simply by writing the equation for definition in the inverse order).

3) If V and W are representations of G, then $\{f : V \to W \mid f\ G\text{-morphism}\} \subseteq \mathrm{Hom}_{\mathbb{C}}(V, W)$ gives a $\mathbb{C}$-vector subspace.

This is clear as by the fact that f is linear, V as a representation is closed under addition and scalar multiplication.

Remark 1.6. Given a finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$, choosing a basis $\{e_1, \dots, e_n\}$ of V gives us an isomorphism $V \simeq \mathbb{C}^n$, i.e. we have the description of representations in matrices

$$\rho : G \to \mathrm{GL}(V) \simeq \mathrm{GL}_n(\mathbb{C}), \qquad g \mapsto \rho_g = (a_{ij}(g))$$

This implies that two representations are isomorphic if and only if there exists some matrix $A \in \mathrm{GL}_n(\mathbb{C})$ s.t. $(a_{ij}(g)) = A(b_{ij}(g))$. In particular, applying the result twice gives that (with identification of representations and its matrix form) $\rho_g = A\rho'_g A^{-1}$, i.e. conjugate representations are isomorphic. Such morphisms of representations (A) are equivariant.

Definition 1.7 (Sub-representation)**.** Given a representation V of G, a **sub-representation** of V is a vector space $W \subseteq V$ s.t. $gv \in W$ for all $v \in W, g \in G$.

Remark 1.8. In particular, for W a sub-representation of V, it is itself a representation with the map ρ' being $\rho(-)|_W$. The inclusion $W \hookrightarrow V$, $x \mapsto X$ is a morphism of representation.

2 Interpretation via the Group Algebra

Similar to the case of group action where we interpreted the structure of group action by the group homomorphism $G \to S_x$, we would like to have some equivalence to structures that are more explicit, and easier to analyze. This introduces the following definitions:

Definition 2.1 (Group Algebra)**.** Let G be a group. Then the **group algebra over** $\mathbb{C}$, denoted $\mathbb{C}[G]$ is a vector space with a basis $\{\alpha(g) \mid g \in G\}$ in bijection with elements in G (formally). Endow it with a multiplication $\alpha(g) \cdot \alpha(g) = \alpha(gh)$ compatible with the group structure gives the desired ring structure.

Remark 2.2. Verifying the ring axioms, we have the fact that the identity in $\mathbb{C}[G]$ to be $\alpha(e)$. This is in fact a $\mathbb{C}$-algebra, with the associated morphism given by $\mathbb{C} \to \mathbb{C}[G]$. Since the image of it are scalars, it is clearly in the center of the group.

Notice that G is not necessarily a finite group. Therefore the vector space can be infinite-dimensional, which we have imposed the requirement that every element should be a finite sum of linear combination of basis. In the following deduction, denote $\sum'$ to be the finite sum.

Proposition 2.3. The group algebra is well-defined.

Proof. This is clear for the cases where G is finite. Consider the case where G is infinite. Then by definition of the group algebra, for all $u, v \in \mathbb{C}[G]$, we have their decomposition into elements in the basis:

$$u = \sum_{g \in G}{}' a_g \alpha(g), \qquad v \in \sum_{g \in G}{}' b_g \alpha(g)$$

Multiplying these two terms together gives

$$u \cdot v = \sum_{g \in G} \left(\sum_{g_1 g_2 = g} (a_{g_1} b_{g_2}) \right) \alpha(g)$$

Furthermore there are only finitely many such a_gs and b_gs being nonzero, implying that there are only finitely many nonzero such products. □

Notation. If G is abelian, and the correspondence of elements in G and in $\mathbb{C}[G]$ is written additively. Instead of $\alpha(g)$ one usually writes χ^g (with the convention that $\chi^g \cdot \chi^h = \chi^{g+h}$).

Remark 2.4. $\mathbb{C}[G]$ is a commutative ring if and only if G is an abelian group. "Only if" is clear as if $\mathbb{C}[G]$ is commutative implies for all $g, h \in G$, they commute. "If" results from the fact that for every element in $x \in \mathbb{C}[G]$ there exists a scalar λ s.t. $\lambda x = \alpha(g)$ for some $g \in G$ as $\mathbb{C}$ is a field.

Example 2.5. If $G = (\mathbb{Z}, +)$, identifying $x \leftrightarrow \chi^x$ for $x \in \mathbb{Z}$, we have $\mathbb{C}[G] \simeq \bigoplus_{m]in\mathbb{Z}} \mathbb{C}\chi^m \simeq S^{-1}\mathbb{C}[x]$ for $S = \langle x \rangle = \{1, x, x^2, \ldots\}$. These are the Laurent Polynomials.

If $G = (\mathbb{Z}/n\mathbb{Z}, +)$, we have the identification $x^n = 1$, giving $\mathbb{C}[G] \simeq \mathbb{C}[x]/(x^n - 1)$.

Proposition 2.6. We have the identification between representations of G and $\mathbb{C}[G]$-modules. Morphisms and sub-objects (sub-representations and submodules) are also in correspondence.

Proof. It suffices to verify 1), as identifications in 2) and 3) are induced by 1).

Suppose that V is a representation of G, Then V has a structure of $\mathbb{C}[G]$-module, whose addition is the same as in the vector space, and scalar multiplication is given by

$$\left({\sum_{g \in G}}' (a_g \cdot \alpha(g)) \right) \cdot v = {\sum_{g \in G}}' (a_g \cdot \alpha(gv))$$

Conversely, if M is a $\mathbb{C}[G]$-module, then it has a vector space structure via considering the action $\mathbb{C} \hookrightarrow \mathbb{C}[G]$ which acts on M; and the linear map is given by $(g, -)$, where $(g, x) \mapsto \alpha(g) \cdot x$ as specified by the $\mathbb{C}[G]$ module. □

Remark 2.7. In general, for a representation over a field $\mathbb{F}$ of G, it can be identified with $\mathbb{F}[G]$.

3 Examples of Representations

4 Irreducible Representations

5 Character Theory