MATH 594 - Representation of Finite Groups

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1 Complex Representation

The motivation of introducing the representation of G is to have a linearized version of group action on sets. Recall that we have the correspondence between action of G on a set X and group homomorphism $G \to S_x$ where S_x is the group of bijective maps on S, with the operation defined as composition. Explicitly, this is given by

$$\varphi : G \times X \to X \quad \rightsquigarrow \quad G \to S_x,\ g \mapsto \varphi(g, -) \in (X \to X)$$

We now give the formal definition on vector spaces:

Definition 1.1 (Representation). A **(complex) representation** of a group G is a vector space V over $\mathbb{C}$, together with a group homomorphism

$$\rho : G \to \mathrm{GL}(V) := \{\varphi : V \to V \mid \varphi \mathbb{C}\text{-linear isomorphism}\}$$

Equivalently, a representation of G is a vector space over $\mathbb{C}$ with an action of G on V $\rho : G \times V \to V$ s.t. for all $g \in G$, the induced map $\varphi(g, -)$ os $\mathbb{C}$-linear.

Notation. The map $\rho(g, -) : V \to V$ is often abbreviated as ρ_g. The representation is denoted by V or ρ, with V emphasizing the vector space structure.

Definition 1.2 (Dimension of Repr.). The **dimension** of the representation is $\dim_{\mathbb{C}} V$, with the same notation as above.

For most of the time, we will only consider the representation of finite groups on finite-dimensional vector spaces.

Remark 1.3. In general, one can consider representations over other fields than $\mathbb{C}$. The reasons why $\mathbb{C}$ is chosen are the followings:

1) If G is finite, then $|G| \in \mathbb{C}$ is always invertible.

2) $\mathbb{C}$ is algebraically closed. The implications include, for example, every linear map has an eigenvalue.

These specialties will often appear in subsequent proofs.

Definition 1.4 (Morphism of Repr.). Given two representations of G, V and W, a **morphism of representations** (or simply G-**morphism**) is a linear map $f : V \to W$ s.t. $f(gv) = g(fv)$ for all $g \in G$, $v \in V$. This is an **isomorphism** if f is further bijective.

Remark 1.5. Following from the definitions we have the immediate results:

1) If $V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$ are morphisms of representation, then so is $g \circ f$ since $g(f(hv)) = g(hf(v)) = h(g(f(v)))$ for all $h \in G$, $v \in V$. This gives the morphisms of objects, i.e. representations of G give a category.

2) If $f : V \to W$ is an isomorphism of representations, then so is f^{-1} (simply by writing the equation for definition in the inverse order).

3) If V and W are representations of G, then $\{f : V \to W \mid f\ G\text{-morphism}\} \subseteq \mathrm{Hom}_{\mathbb{C}}(V, W)$ gives a $\mathbb{C}$-vector subspace.

This is clear as by the fact that f is linear, V as a representation is closed under addition and scalar multiplication.

Remark 1.6. Given a finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$, choosing a basis $\{e_1, \ldots, e_n\}$ of V gives us an isomorphism $V \simeq \mathbb{C}^n$, i.e. we have the description of representations in matrices

$$\rho : G \to \mathrm{GL}(V) \simeq \mathrm{GL}_n(\mathbb{C}), \qquad g \mapsto \rho_g = (a_{ij}(g))$$

This implies that two representations are isomorphic if and only if there exists some matrix $A \in \mathrm{GL}_n(\mathbb{C})$ s.t. $(a_{ij}(g)) = A(b_{ij}(g))$. In particular, applying the result twice gives that (with identification of representations and its matrix form) $\rho_g = A\rho'_g A^{-1}$, i.e. conjugate representations are isomorphic. Such morphisms of representations (A) are equivariant.

Definition 1.7 (Sub-representation). Given a representation V of G, a **sub-representation** of V is a vector space $W \subseteq V$ s.t. $gv \in W$ for all $v \in W, g \in G$.

Remark 1.8. In particular, for W a sub-representation of V, it is itself a representation with the map ρ' being $\rho(-)|_W$. The inclusion $W \hookrightarrow V$, $x \mapsto X$ is a morphism of representation.

2 Interpretation via the Group Algebra

3 Irreducible Representations

4 Character Theory