

MATH 594 - Finite Fields and Galois Theory

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Contents

1	Review of Ring Theory	2
2	Multiplicity of Root	4
3	Characteristic of a Field	4
4	Algebraic Extensions	5
5	The Splitting Field of a Polynomial	15
6	Separable Extensions	15
7	Normal Extensions	15
8	Galois Extensions	15
9	Algebraic Independence & Transcendence Degree*	15
10	The Fundamental Theorem of Galois Theory	15
11	Norm and Trace Maps	15
12	Solvability by Radicals	15

1 Review of Ring Theory

The Galois Theory originates from the question: Given a polynomial with coefficients in a field (e.g. \mathbb{Q}), we want to understand the property of its solutions, and the algebraic structure in which the root lies. This section lays some fundamental notions and results for introducing the whole theory.

Setup the convention. All rings have unit element 1; and all ring homomorphisms map 1 to 1. Inclusion maps, in particular, implies that all subrings of a certain ring must include 1.

Remark 1.1. An immediate result is that finite domains that are not the zero ring are fields.

Recall that a ring R is a domain if and only if zero does not have non-trivial divisors. That is, for all $x, y \in R \setminus \{0\}$, $xy = 0$.

For $a \in R$, $a \neq 0$, consider the map

$$\varphi : R \rightarrow R \quad x \mapsto ax$$

Since R is a domain φ maps nonzero elements to nonzero elements, which implies that there exists some x_a s.t. $ax_a = 1$.

This gives an inverse of a .

Proposition 1.2. If $f : K \rightarrow R$ is a ring homomorphism, and K is a field. Then $R \neq \{0\}$ implies that f is injective.

Proof. Recall that the kernel of a particular ring homomorphism is an ideal. Denote $I = \ker(f) \subseteq K$. Suppose that $a \in I$ s.t. a is nonzero. Then $a^{-1} \in I$ which gives $I = (1) = K \implies f(I) = 0$. But as we require $f(1) = 1_R \implies 1_R = 0_R$, i.e. $R = \{0\}$ which is a contradiction. \square

Corollary 1.3. In particular, ring homomorphisms between fields (field extensions) $K \rightarrow L$ are injective. Not all fields have extensions (ring homomorphisms) between them.

A class of extensions of which we are particularly interested in, is the extension which gives polynomial a root. $f \in K[x]$ may not have a root; and by extending it $K[x] \hookrightarrow L[x]$ we may consider roots of $f \in L[x]$ which may have a root.

Notation. A field extension $k \hookrightarrow K$ is also denoted as K/k . These two notations will be used interchangeably.

Proposition 1.4. Let k be a field, and $R = k[X]$. Let $f \in R \setminus \{0\}$. Then the followings are equivalent:

- 1) f is irreducible.
- 2) (f) is a prime ideal.
- 3) (f) is a maximal ideal.

Proof. Prove the implications cyclically:

- $3) \implies 2)$. It is a general fact that maximal ideals are prime. Prove the contrapositive: suppose that an ideal $(f) \subset R$ is not prime, then there exists $a, b \in R$ s.t. $ab \in (f)$ and neither a and b are in (f) . Then $(f) \subset (a) \subset R$ which implies that (f) is not maximal.
- $2) \implies 1)$. (f) being a prime ideal in R implies that in particular $(f) \neq R$, i.e. f is not invertible. Suppose that there exists g, h not invertible s.t. $f = gh$. Without loss of generality, assume that $g \in (f)$. Then there exists some $u \in R$ s.t. $g = fu$. Multiply on the right by h gives $f = gh = fuh$ which implies that h is invertible, giving a contradiction.

- 3) \implies 1). f being irreducible implies that f is not invertible, i.e. $(f) \neq R$. Suppose that there exists some maximal ideal J s.t. $(f) \subset J \subset R$. Then since maximal ideals are in particular prime, $J = (g)$ for some $g \in R$. But then this implies that $f = gu$ giving that u is invertible, and therefore $(f) = J$, which is a contradiction. \square

Recall that two elements f, g are relative prime if for all $p \in R$ s.t. $p \mid f, p \mid g$, p is invertible. Then we have the following result similar to the case for integer divisibility:

Proposition 1.5. Let k be a field, and $R = k[x]$. For $f, g, h \in R$ s.t. $f \mid gh$, if f and g are relative prime, then $f \mid h$.

Proof. Since k is a field, $k[x]$ is a PID (as every element in the coefficient is invertible, for a, b relative prime $(a, b) = (1) = k[x]$). Consider $I = (f, g) \subseteq R$. Since I is principal, $I = (p)$ for some $p \in k[x]$. Therefore $p \mid f, p \mid g$, which implies that p is invertible. Then $I = R$. This gives that there exists $A, B \in R$ s.t. $Af + Bg = 1$. Multiplying h on the right gives $Afh + Bgh = h$. Since f divides LHS, $f \mid h$. \square

Proposition 1.6. Let k be a field, and $R = k[x]$. If $f \in R \setminus \{0\}$, and denoting $d = \deg f$, then there exists field extensions $k \hookrightarrow L$ and $a_1, \dots, a_d \in L, c \in k$ s.t. $f = c(x - a_1) \cdots (x - a_d)$ in $L[x]$.

Proof. The key step is to show that if f is irreducible in R , then there exists a field extension $k \hookrightarrow k'$ s.t. f has a root in k' .

Consider $k' = k[x]/(f)$. Since f is irreducible, (f) is a maximal ideal in R , and therefore k' is a field. Since elements in k are of degree 0 in $k[x]$. Considering $k \rightarrow k[x] \rightarrow k' := k[x]/(f)$ gives the injective ring homomorphism (field extension). Let $a = \bar{x} \in k'$. Then $f(a) = \overline{f(x)} = 0$, which implies that a is a root of f .

Proceed the rest of the proof by induction:

- $d = 0$. This is the trivial case.
- $d = 1$. Then $f = c(x - a)$ for some $c, a \in k$.
- $d \geq 2$. Apply the above steps iteratively. Then there exists some $a \in k'$ s.t. $(x - a) \mid f$, i.e. in $k'[x]$ we have the decomposition $f = (x - a)g$ for some $g \in k'[x]$ with $\deg g = \deg f - 1$. Applying the inductive hypothesis (the results holds in lower degrees) gives the full decomposition in $L := k'$. \square

Notation. Denote the image of the map $(k[y] \rightarrow k, y \mapsto a)$ by $k[a]$. This is the smallest k -algebra containing a .

Proposition 1.7. Let k be a field, and $R = k[x]$. Let $f \in R \setminus \{0\}$ be irreducible. Suppose that we have the field extension $k \hookrightarrow K$, and $a \in K$ is a root of f . Then $k[a] \simeq k[x]/(f)$. In particular, $k[a]$ is a field.

Proof. Consider the ring homomorphism $\varphi : k[y] \rightarrow K$ s.t. $\varphi(y) = a$. Then by the First Isomorphism Theorem, we have $k[a] = \text{im } \varphi \simeq k[x]/\ker \varphi \simeq k[x]/(f)$ since $f(a) = 0$. \square

Notice that every field extension $k \hookrightarrow K$ is a k -algebra morphism (ring homomorphisms that are k -linear). Since k is a field, this gives K a k -vector space structure.

Definition 1.8 (Degree). The **degree** of the field extension $k \hookrightarrow K$, denoted $[K : k]$, is $\dim_k K \in \mathbb{Z}_{\geq 0}$ or infinite.

Definition 1.9 (Finite). A field extension is **finite** if the degree of it is finite.

Remark 1.10. If $f \in k[x]$ is irreducible, and $K = k[x]/(f)$, then $[K : k] = \deg f$. More generally, if $g \in k[x]$ is a nonzero polynomial, then $\dim_k(k[x]/(g)) = \deg g$.

This can be seen via applying the division algorithm (since $K[x]$ is an Euclidean Domain. This can be seen via computing the division). Then for all $P \in k[x]$, there exists unique $Q, R \in k[x]$ s.t. $P = gQ + R$, with $\deg R < \deg g$. Then since $\overline{P} = \overline{R}$ in $k[x]/(g)$, $\{\overline{1}, \overline{x}, \dots, \overline{x^{\deg g - 1}}\}$ gives a basis of $k[x]/(g)$ over k .

2 Multiplicity of Root

This section provides tools for describing the zeros of a polynomial, and how they in general can look like. The proposition below says that any polynomial can be factored into two parts, with the first part having roots in the field; and the second part requires extension of the field to decompose completely.

Definition 2.1 (Multiplicity). Let $f \in k[x]$ be a nonzero polynomial for k a field, and $a \in R$ a root of f . Then a has **multiplicity** m if $(x - a)^m \mid f$, but $(x - a)^{m+1} \nmid f$.

Proposition 2.2. If $f \in R \setminus \{0\}$, and $a_1, \dots, a_r \in k$ are pairwise distinct roots of f s.t. a_i has multiplicity m_i . Then we have the decomposition of f :

$$f = \prod_{i=1}^r (x - a_i)^{m_i} g, \quad g \in R, g(a_i) \neq 0 \text{ for all } i$$

In particular, $\sum_i m_i \leq \deg f$.

Proof. Apply induction on r :

- *Base case.* Then m_1 is the maximal integer satisfying the condition that $(x - a_1)^{m_1} \mid f$. Then define g be such that $f = (x - a_1)^{m_1} g$.
- *Inductive step.* For $r \geq 2$, denote f_1 be the polynomial s.t. $f = (x - a_1)^{m_1} f_1$. Notice that for all i s.t. $2 \leq i \leq r$, we have $(x - a_i)^{m_i} \mid f$. Then since $(x - a_i)$ and $(x - a_1)$ are relative prime (they are both irreducible) by Proposition 1.5 we have $(x - a_i)^{m_i} \mid f_1$. Then applying inductive hypothesis gives the desired decomposition of f .

□

3 Characteristic of a Field

Recall that in the first section we mentioned that there does not necessarily exist ring homomorphisms between arbitrary fields. This, as we will see in the following, implies some constraints on the structure that a field can have.

Let S be an integral domain. Let $\varphi : \mathbb{Z} \rightarrow S$ s.t. $n \mapsto n \cdot 1_S$. This is the unique ring homomorphism between \mathbb{Z} and S due to the constraint the 1 should be mapped to 1. Since S is a domain, and \mathbb{Z} is a PID, $\ker \varphi = (d)$ for d prime or zero. Then either

- 1) $\ker \varphi = \{0\}$; or
- 2) $\ker \varphi = p\mathbb{Z}$ for some p prime.

In case 1), if we suppose further that $S = k$ which is a field, then for all $n \in \mathbb{Z}$ $\varphi(n)$ is invertible. By the universal property of the quotient ring, this induces a ring homomorphism (which is also a field extension) $\text{Frac}(\mathbb{Z}) = \mathbb{Q} \hookrightarrow S$.

In case 2), we have an injective ring homomorphism $\mathbb{Z}/p\mathbb{Z} \hookrightarrow S$ for some p prime. Defining $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, S becomes an \mathbb{F}_p -algebra by the φ above.

Definition 3.1 (Characteristic). For a field k , the **characteristic** of k is

$$\text{char}(k) = \begin{cases} p, & \text{if } \mathbb{F}_p \hookrightarrow k \text{ (case 1)} \\ 0, & \text{if } \mathbb{Q} \hookrightarrow k \text{ (case 2)} \end{cases}$$

Remark 3.2. If S is an \mathbb{F}_p -algebra (case 2), the map $F : S \rightarrow S, u \mapsto u^p$ is the Frobenius homomorphism. Check that this is indeed a ring homomorphism:

- $F(uv) = F(u)F(v)$. Clear as field is commutative: $(uv)^p = u^p v^p$.
- $F(u + v) = F(u) + F(v)$. Compute:

$$(u + v)^p = u^p + v^p + \underbrace{\sum_{i=1}^{p-1} \binom{p}{i} u^{p-i} v^i}_{\text{divisible by } p}$$

where the last term vanishes, as $\mathbb{F}_p \hookrightarrow S$ should map 0 to 0; and $\bar{p} = \bar{0} \in \mathbb{F}_p$.

4 Algebraic Extensions

The field extensions originating solely from “including the roots of polynomials” are the nice ones and deserve a better name. The discussions formalizes the concept of “algebraic closure” in elementary discussions of polynomials.

Proposition 4.1. If $k \hookrightarrow K \hookrightarrow L$ is a field extension, then $[L : k] = [L : K][K : k]$.

Proof. First consider the cases where one of the degrees is infinite:

- If $[K : k]$ is infinite, then $[L : k]$ is infinite as $K \subseteq L$ is a K -vector subspace of L .
- If $[L : K]$ is infinite, then there exists an infinite set of elements which are linearly independent over K , which are also linearly independent over k since $k \subseteq K$.

Now consider the case where both $[L : K]$ and $[K : k]$ are finite. Denote $m = [L : K]$ and $n = [K : k]$. Denote $\{a_1, \dots, a_m\}$ be a basis of L over K , and $\{b_1, \dots, b_n\}$ be a basis of K over k . Notice that $\{a_i b_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ gives a basis for L over k , as for all $u \in L$, there exists $\lambda_i \in K$, and thus $\mu_{ij} \in k$ s.t.

$$u = \sum_{i=1}^m \lambda_i b_i = \sum_{i,j} \mu_{ij} a_j b_i, \quad \text{for } \lambda_i = \sum_j \mu_{ij} a_j \mu_{ij}$$

which is a decomposition. They are further linearly independent, as for $u = 0$, since b_i s give a basis, $\lambda_i = 0$ for all i , and therefore $\mu_{ij} = 0$ for all i and j . \square

Notation. Let $k \hookrightarrow K$ be a field extension, and $A \subseteq K$ a subset. Then we denote

$$k(A) := \bigcap_{A \subseteq k'} \{k' \mid k \hookrightarrow k' \hookrightarrow K \text{ extension}\}$$

which is the smallest field sub-extension of k inside K containing A .

Remark 4.2. It is worth mentioning that this is different from $k[A]$ which is the smallest k -subalgebra containing A :

- They are related via $k(A) = \text{Frac}(k[A])$. They are equal in some “nice” extensions (see Remark 4.9 below).

By definition we have $k[A] \subseteq k(A)$, as $k[A]$ is only required to be a k -algebra instead of a field extension of k (as field extending k can be seen as k -vector spaces, which are in particular k -algebras). By the universal property of fraction fields, we have $\text{Frac}(k[A]) \subseteq k(A)$, as ring homomorphisms between fields are injective, and by definition for all $f \in k[A]$, f has an inverse in $k(A)$. Further since $A \subseteq \text{Frac}(k[A])$ (also by definition, we have $k(A) = \text{Frac}(k[A])$).

- Considering multiple elements, we can extend the [previous notation](#), by considering for $A = \{a_1, \dots, a_n\}$, then $k[A] = \text{im } \varphi$ for

$$\varphi : k[x_1, \dots, x_n] \rightarrow K, \quad x_i \mapsto a_i$$

Definition 4.3 (Finite Generated). A field extension K/k is **finitely generated** if there exists $a_1, \dots, a_n \in K$ s.t. $k(a_1, \dots, a_n) = K$.

Remark 4.4. If a field extension K/k is finite, then it is also finitely generated, as K/k being finite implies that there exists some finite basis of K over k ; and picking one gives the elements that “finitely generates” K . However, the converse is not true: consider $k \hookrightarrow k(x) = \text{Frac}(k[x])$ is finitely generated (by x) but is not finite (we have the infinite set $\{x^i \mid i \in \mathbb{Z}\}$ whose elements are linearly independent over k)

Definition 4.5 (Algebraic; Transcendental). Let $k \hookrightarrow K$ be a field extension. An element $a \in K$ is **algebraic over k** if there exists $f \in k[x] \setminus \{0\}$ s.t. $f(a) = 0$ in $K[x]$. Otherwise a is **transcendental**. An extension K/k is **algebraic** if for all $a \in K$, it is algebraic over k .

Remark 4.6. Consider the field extensions $k \hookrightarrow K \hookrightarrow L$. Then if $a \in L$ is algebraic over k , then a is also algebraic over K , as a algebraic over k implies that there exists $f \in k[x]$ s.t. $f(a) = 0$; and by definition we also have $f \in K[x]$.

Remark 4.7. Given a field extension $k \hookrightarrow K$, and $a \in K$. Then a is algebraic if and only if the $\varphi : k[x] \rightarrow K, x \mapsto a$ has a non-trivial kernel. This is the direct translation of having a polynomial f with a as its root. Then $\ker \varphi$ is a prime ideal.

To prove this, it suffices to show that $k[x]/(\ker \varphi)$ is a domain. This is indeed the case, as $k[x]$ is a domain: for all $g, h \in k[x]$, $g(a) \neq 0$ and $h(a) \neq 0$ implies that $gh(a) \neq 0$, i.e. $gh \notin \ker \varphi$ ($gh \neq \bar{0}$ in $k[x]/(\ker \varphi)$). Therefore, there exists some f s.t. $\ker \varphi = (f)$.

Definition 4.8 (Minimal Polynomial). $f \in k[x]$ is the **minimal polynomial** of $a \in K$ if for $\varphi : k[x] \rightarrow K, x \mapsto a$, $\ker \varphi = (f)$. This is well-defined by the above Remark (Remark 4.7).

Remark 4.9. For a being algebraic, f is a maximal ideal (by the fact that $k[x]/(\ker \varphi)$ is a domain). Then $k[a] = k[x]/(\ker \varphi)$ is a field. This gives $k[a] = k(a)$.

Remark 4.10. Given field extension K/k , if $a \in K$ is transcendental over k , then $k[a] \simeq k[x]$, as since there is no polynomial with root a implies that $k[a]$ can be seen as injecting a formal variable to the field $k[x]$. This further implies $k(a) \simeq k(x)$ (as $k(a) \simeq \text{Frac}(k[a])$; and same for x).

Example 4.11. Suppose that $d \in \mathbb{Z}$ is not a square, and let $a = \sqrt{d}$. Consider the field extension $\mathbb{Q} \rightarrow \mathbb{C}$.

Since a is not in \mathbb{Q} , a cannot be a root of degree 1 polynomials, i.e. the minimal polynomial of a must be of degree at least 2; and we have a as a root of $x^2 - d = 0$, the minimal polynomial of a over k is $x^2 - d = 0$, which also implies that a is algebraic over \mathbb{Q} .

Therefore, $\mathbb{Q}(\sqrt{d})$ can be seen as a \mathbb{Q} -vector space, which has a basis $\{1, \sqrt{d}\}$, i.e. $\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}$. In particular we have $\dim_{\mathbb{Q}} \mathbb{Q}(\sqrt{d}) = 2$ as $\deg f = 2$, resulting from Remark 1.10.

Proposition 4.12. Every finite extension K/k is algebraic.

Proof. Denote $[K : k] = n$ which is finite. Regarding K as a k -vector space, for all $a \in K$ the set of elements $\{1, a, \dots, a^n\}$ are linearly dependent (as there are $(n + 1)$ of them). That is, there exists $c_0, \dots, c_n \in k$ s.t. $c_0 + c_1a + \dots + c_na^n = 0$; and $f = c_0 + c_1x + \dots + c_nx^n \in k[x] \setminus \{0\}$. This gives a polynomial $f \in k[x]$ s.t. $f(a) = 0$, i.e. a is algebraic. Since $a \in K$ can be taken arbitrarily, we have K/k being algebraic. \square

Proposition 4.13. Let $k \hookrightarrow K$ be a field extension, and $a_1, \dots, a_n \in K$ are algebraic over k . Then $k[a_1, \dots, a_n] = k(a_1, \dots, a_n)$; and $k(a_1, \dots, a_n)$ is a finite extension over k .

Proof. Remark 4.9 gives the case for $n = 1$, $k[a] = k(a)$, and Remark 1.10 (using minimal polynomial) gives that the extension is finite.

For $n \geq 2$, repeat the argument with the induction hypothesis. First by definition we have $k[a_1, \dots, a_n] = (k[a_1, \dots, a_{n-1}])[a_n]$; and the same holds for the field extension version (replace brackets with parentheses). Further finite extensions are transitive, by Proposition 4.1. Suppose now that $k' = k[a_1, \dots, a_{n-1}] = k(a_1, \dots, a_{n-1})$ is a finite extension over k . Then $k[a_1, \dots, a_n] = k'[a_n] = k'(a_n) = k(a_1, \dots, a_n)$ is finite over k' ; and by hypothesis we know $k'[a_n]$ is finite over k since k' is. \square

Corollary 4.14. Finitely generated algebraic field extensions are finite.

Proposition 4.15. If $k \hookrightarrow K \hookrightarrow L$ are both algebraic field extensions, then so is $k \hookrightarrow L$.

Proof. Use the above two propositions. Let $a \in L$. Since L/K is algebraic, there exists $f \in K[x] \setminus \{0\}$ s.t. $f(a) = 0$. Let $f = c_0 + c_1x + \cdots + c_nx^n$, and $k' = k(c_0, \dots, c_n)$. Since all $c_0, \dots, c_n \in K$ are algebraic over k , Proposition 4.13 implies that k' is a finite extension over k . Further since a is algebraic over k' , the extension $k' \hookrightarrow k'(a)$ is also finite. Proposition 4.12 implies that the extension $k \hookrightarrow k'(a)$ is algebraic. That is, a is algebraic over k .

Since this holds for all $a \in L$, the extension L/k is algebraic. □

Remark 4.16. Notice that the converse of the statement is also true. If $k \hookrightarrow L$ is algebraic, then for every element $a \in L$ there exists some $f_a \in k[x]$ s.t. $f_a(a) = 0$. But given extension $k \hookrightarrow K \hookrightarrow L$, $k \subseteq K$, which implies f_a is also in $K[x]$; and therefore L/K is algebraic

Proposition 4.17. If $k \hookrightarrow K$ is a field extension, then $k' = \{a \in K \mid a \text{ algebraic over } k\}$ is a subfield of K containing k ,

Proof. To prove this we need to check:

- $k \subseteq k'$. This is clear from the construction of the field.

- k' is closed under additive and multiplicative inverse.

$a \in k'$ implies $-a \in k'$ as $-a$ is the root of a polynomial via considering the minimal polynomial and inverting the corresponding coefficients. $a \in k' \setminus \{0\}$ implies $a^{-1} \in k' \setminus \{0\}$ since $k[a] = k(a)$ as $k(a)$ is finitely generated, and therefore finite and algebraic. $k(a)$ is a field, implying that $a^{-1} \in k(a)$.

- k' is closed under addition and multiplication. That is, for all $a, b \in k'$, $a + b \in k'$ and $ab \in k'$.

Consider the field $k(a, b)$. Since both a and b are algebraic over k , $k(a, b)/k$ is finite by Proposition 4.13; and by Proposition 4.12 the extension is algebraic. By definition $k(a, b) \subseteq k'$; and is the smallest field containing both a and b ; and therefore both $a + b$ and ab are in k' . □

Definition 4.18 (Algebraic Closure). For a field k with field extension $k \hookrightarrow K$, the field $k' = \{a \in K \mid a \text{ algebraic over } k\}$ is the **algebraic closure** of k in K . By the above Proposition 4.17, this is indeed a field.

Definition 4.19 (Algebraically Closed). A field k is **algebraically closed** if every nonzero polynomial over k has a root in k .

Remark 4.20. By induction, if k is algebraically closed and f is a nonzero polynomial in $k[x]$ with degree n_i then there exists $a_1, \dots, a_n \in k$, $c \in k^*$ s.t. $f = c(x - a_1) \cdots (x - a_k)$. That is, every irreducible polynomial in an algebraically closed field has degree 1.

Notation. For a field k , the set of invertible (nonzero) elements in it are often denoted as k^* or k^\times .

Proposition 4.21. A field k is algebraically closed if and only if for all field extensions $k \hookrightarrow K$, for all $a \in K \setminus k$, a is transcendental over k .

Proof. Prove implication in two directions:

\Rightarrow : Suppose that k is algebraically closed, and $a \in K \setminus k$ algebraic over k . Then by definition there exists an irreducible polynomial $f \in k[x]$ s.t. $f(a) = 0$. Since k is algebraically closed, the irreducible polynomials are of degree 1, i.e. $\deg f = 1$. Then $a \in k$, which is a contradiction.

\Leftarrow : Suppose that for all $a \in K \setminus k$, a is transcendental. Proceed to prove that k is algebraically closed by showing that every irreducible polynomial has a root.

Consider $f \in k[x] \setminus \{0\}$. Consider $K = k[x]/(f)$. Then since $f(\bar{x}) = 0$, \bar{x} is algebraic over k . But since all elements in $K \setminus k$ are transcendental, $\bar{x} \in k$. Then in K we have $\bar{x} - a = 0$, for some $a \in k$. Then $f(x) = x - a$ in $K[x]$ which has a root a ; and as this holds for all f , k is algebraically closed.

□

In summary, a field k being algebraically closed is equivalent to the following conditions:

- 1) For all field extensions $k \hookrightarrow K$, either $k \simeq K$, or the extension is not algebraic.
- 2) For all $f \in k[x] \setminus \{0\}$, f factors as a product of polynomials of degree 1.
- 3) Every irreducible $f \in k[x] \setminus \{0\}$ has a root in k .

Recall that we have the [algebraic closure in a specific field](#). The following theorem seeks to construct such closure without any ambient structure:

Theorem 4.22. Given any field k , there exists an algebraic extension $k \hookrightarrow \bar{k}$ s.t. \bar{k} is algebraically closed. Such an extension is an algebraic closure of k .

Parenthesis 4.23 (Direct Limit (of rings)). To provide the construction of an algebraic closure, the main step is to iteratively include the roots of the some irreducible polynomials; and we need to ensure that this process terminates. That is, there exists a field containing all the intermediate fields on which we conducted the extension. This parenthesis formalizes this idea.

Definition 4.24 (Directed Set). (I, \leq) is a **directed set** if for all $i, j \in I$ there exists $k \in I$ s.t. $i \leq k$ and $j \leq k$.

Definition 4.25 (Direct System). Given a directed set I , a **direct system** $(R_i)_{i \in I}$ is a family of rings R_i satisfying:

- For all $i \leq j$ in I , there exists a ring homomorphism $\varphi_{ij} : R_i \rightarrow R_j$; and $\varphi_{ii} = \text{Id}_{R_i}$.
- For all $i \leq j \leq k$ in I , the ring homomorphisms above satisfy $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$.

Definition 4.26 (Direct Limit). Given a direct system $(R_i)_{i \in I}$, the **direct limit** of the system, denoted $R = \varinjlim R_i$, together with a family of ring homomorphisms $f_i : R_i \rightarrow R$ s.t.

- $f_j \circ \varphi_{ij} = f_i$.
- The pair $(R, (f_i)_{i \in I})$ is universal with this property, i.e. every ring homomorphism from R_i for all i factors uniquely through R . That is, for any ring T and family of ring homomorphisms $g_i : R_i \rightarrow T$ s.t. $g_j \circ \varphi_{ij} = g_i$ (the g_i s are compatible w.r.t. the system), there exists a unique ring homomorphism $g : R \rightarrow T$ s.t. $g \circ f_i = g_i$ for all i .

The direct limit exists, by considering the class of elements that are closed along the morphisms of R_i s: define $R := (\bigsqcup_{i \in I} R_i) / \sim$, where the equivalence relation \sim is given by $R_i \ni x_i \sim \varphi_{ij}(x_i)$. The operations are given by for $a_i \in R_i$ and $b_j \in R_j$, finding k s.t. $i \leq k$ and $j \leq k$ (which exists as I is a directed set), and define:

$$a_i + b_j := \varphi_{ik}(a_i) + \varphi_{jk}(b_j) \in R_k, \quad a_i b_j := \varphi_{ik}(a_i) \cdot \varphi_{jk}(b_j) \in R_k$$

Existence of inverse, and distributivity are guaranteed by the ring structure of R_j . Further see that this is well-defined, as for $k' \neq k$ that we have picked s.t. $i \leq k'$ and $j \leq k'$, since the set is directed there exists ℓ s.t. $k \leq \ell$ and $k' \leq \ell$. Now use the equivalence relation with $\varphi_{k\ell}$ and $\varphi_{k'\ell}$; and the fact that \sim is an equivalence relation and is therefore transitive.

Further the direct limit of a direct system is unique up to isomorphism. Suppose we have two direct limits R and R' , there exists $g : R \rightarrow R'$ and $g' : R' \rightarrow R$, by the universal property of the direct limit s.t. $g \circ f_i = f'_i, g' \circ f'_i = f_i$. This implies that $g \circ g' = \text{Id}_{R'}$, and $g' \circ g = \text{Id}_R$ by the symmetric argument. This gives the isomorphism.

The last thing we need to notice is that if all R_i s are fields, then R is a field, as in the definition of the operation having multiplicative inverse in R_k induces the multiplicative inverse in R .

Now we can use the above tools to describe the polynomial with variables indexed by a (possibly infinite) set I . Given a commutative ring R , a set I , the polynomial ring $R[x_i \mid i \in I]$ is defined as follows:

For $J \subset I$ finite subset, denote $R_J := R[x_i \mid i \in J]$. Define $\mathcal{P} := \{\text{finite subsets of } I\}$, ordered by inclusion. This is a directed set, as for all $M, N \in \mathcal{P}$, we have $M \subseteq I$ and $N \subseteq I$ by definition.

For $J_1 \subseteq J_2$ in \mathcal{P} , we have a morphism of R -algebras $\varphi_{J_1 J_2} : R_{J_1} \rightarrow R_{J_2}, x_i \mapsto x_i$ (embeddings). This gives a direct system of R -algebras; and we can define $R[x_i \mid i \in I] := \varinjlim (R_J)_{J \in \mathcal{P}}$. Notice:

- All $\varphi_{J_1 J_2}$ are injective (as they are embeddings), and therefore the map $R_J \rightarrow R[x_i \mid i \in I]$ is also injective.
- Observe that we have $\bigcup_{J \in \mathcal{P}} R_J$ is a direct limit of the system $(R_J)_{J \in \mathcal{P}}$; and since the direct limit is unique up to isomorphisms for a particular directed system, we have $R[x_i \mid i \in I] = \bigcup_{J \in \mathcal{P}} R_J$.

Now prove the theorem:

Proof of Theorem 4.22. As we have mentioned, the idea of the proof is to construct a direct system of fields, where each φ (which is a field extension here) includes the roots of irreducible polynomials in the base field; then taking the direct limit of the system gives the desired algebraic closure.

The first step is to construct a field extension $k \hookrightarrow k_1$ s.t. for all $f \in k[x]$ irreducible, there exists $a \in k_1$ s.t. $f(a) = 0$. Define $A = \{f \in k[x] \mid f \text{ irreducible}\}$. Define $R = k[y_f \mid f \in A]$ and $\underline{a} = (f(y_f) \mid f \in A)$ where y_f are formal variables indexed by polynomials in A ; and the parenthesis in \underline{a} implies that this is an ideal generated by such elements.

Claim that $\underline{a} \neq R$. First observe that by definition $\underline{a} \subseteq R$, as in particular we have $f \in k[y_f] \subseteq k[y_f \mid f \in A] =: R$. Suppose that $\underline{a} = R$. Then the condition is translated to:

$$\exists r \in \mathbb{Z}_{>0}, f_1, \dots, f_n \in A, g_1, \dots, g_n \in R, \sum_{i=1}^r \underbrace{g_i}_{\in R} \underbrace{f_i(y_{f_i})}_{\in \underline{a}} = 1 \quad (*)$$

as the ideal being equal to the whole ring is the same as the ideal containing 1. By Proposition 1.7, for all i there exists field extension $k \hookrightarrow k_i$ s.t. there exists $a_i \in k_i$ satisfying $f_i(a_i) = 0$. Since there finitely many (exactly r) polynomials, we can do this iteratively and get a field extension $k \hookrightarrow K$ s.t. there exists field extensions $k_i \hookrightarrow K$ for all i .

Let $J \subseteq A$ be the finite subset containing all the f_i s, and also containing all polynomials g s.t. y_g appears in some $g_i \in k$. Define $\varphi : k[y_g \mid g \in J] \hookrightarrow K$. $y_{f_i} \mapsto a_i$ for all i , and $y_g \mapsto \varepsilon \in K$ which is some value we do not care about. Recall that a_i s are defined s.t. $f_i(a_i) = 0$. This is a k -algebra morphism.

Now apply φ to Eq. (*). Since φ is k -linear, we have

$$0 = \sum_{i=1}^r \varphi(g_i) f_i(a_i) = \varphi(1) = 1$$

which is a contradiction, as for a field we require $0 \neq 1$. Therefore $\underline{a} \neq R$. Further since \underline{a} is an ideal in R , there exists a maximal ideal M in R s.t. $\underline{a} \subseteq M$. Define $k' = R/M$ which is not trivial since $\underline{a} \neq R$. Denote $k_1 := \{a \in k' \mid a \text{ algebraic over } k\}$. By Proposition 4.17 this is a subfield of k' , and by definition is algebraic over k . Now consider $f \in k[x]$ irreducible, and $\overline{y_f} \in k' = R/M$ a formal variable. By definition of algebraic closure, $f(\overline{y_f}) = 0$ in \bar{k} ; and since $f(y_f) \in \underline{a} \subseteq M$, $\overline{f(y_f)} = f(\overline{y_f}) = 0 \in R/M$, which implies that $\overline{y_f} \in k_1$. Since $k' = R/M$ with R extending elements in the form of y_f , every element in k' is in the same as $\overline{y_f}$ for some $f \in A$; and by the previous argument we know $k \hookrightarrow k'$ is algebraic. That is, every $f \in k[x]$ irreducible has a root in k_1 .

Now repeat the process with k replace by k_1 . Conducting induction gives that for all $f \in k_i[x]$, there exists $a \in k_{i+1}$, $f(a) = 0$ in $k_{i+1}[x]$. This gives algebraic extensions

$$k \hookrightarrow k_1 \hookrightarrow k_2 \hookrightarrow \dots$$

Now take $\bar{k} := \varinjlim k_i$. This is a field by Parenthesis 4.23 satisfying $k \hookrightarrow k_i \hookrightarrow \bar{k}$, and by the following remark $\bar{j} = \bigcup_{i \geq 1} k_i$. Check:

- \bar{k} is algebraic over k . This results from the fact that each of the intermediate extension is algebraic; and the result follows from Proposition 4.15.
- \bar{k} is algebraically closed. For all $f \in \bar{k}[x]$, there exists i s.t. $f \in k_i[x]$. Then f has a root in k_{i+1} which can be extended to \bar{k} by universal property of direct limit.

□

Remark 4.27. If the field k is infinite, the algebraic closure \bar{k} is of the same cardinality as k .

To address the uniqueness of \bar{k} , we prove the following theorem that is more general:

Theorem 4.28. Given two field extensions $k \hookrightarrow K$ and $k \hookrightarrow L$ s.t. K/k is algebraic, and L is algebraically closed. Then there is a morphism of k -extensions (as k -algebras) $K \hookrightarrow L$.

To prove the theorem we need Zorn's Lemma:

Lemma 4.29 (Zorn). Given a nonempty ordered set (A, \leq) s.t. every chain in A has an upper bound in A . Then a has a maximal element. The terminologies are:

- A chain is a totally ordered subset.
- An upper bound for $B \subseteq A$ is an element $a \in A$ s.t. for all $b \in B$, $b \leq a$.
- A maximal element in A is some $a \in A$ s.t. for all a' s.t. $a \leq a'$, we have $a' = a$.

Proof of Theorem 4.28. Consider the set $A = \{(k', \varphi)\}$ where k' is a subfield of K for which there exists extensions $k \hookrightarrow k' \hookrightarrow K$; and for $i : k \rightarrow k'$, $j : k \rightarrow L$ we have $\varphi \circ i = j$. Define the partial order on A be such that $(K', \varphi) \leq (K'', \psi)$ if and only if $K' \subseteq K''$, and $\psi|_{K'} = \varphi$.

Now apply Zorn. Check the followings:

- A is non empty. In particular, $(k, j) \in A$.
- Suppose that $B = \{(K_i, \varphi_i) \mid i \in I\}$ is a chain (totally ordered subset) in A . Then B has a maximal element (which also serves as an upper bound) (K, φ) given by $K = \bigcup_{i \in I} K_i$ and φ be such that $\varphi|_{K_i} = \varphi_i$. Such φ exists as B is totally ordered, i.e. for any subset of B the maximal element gives the corresponding φ .

Then Zorn's Lemma gives that there exists a maximal element $(K', \varphi) \in A$. Then either:

- $K' = K$. This gives the desired result.
- $K' \neq K$. We seek to find a contradiction. Since $K' \subseteq K$, there exists $a \in K \setminus K'$. Since K/k is algebraic by hypothesis, by Remark 4.16, K/K' is also algebraic. Let $f \in K'[x]$ be the minimal polynomial of a over K' . Try to extend on a :

$$K' \hookrightarrow K'[a] \simeq K'(a) \simeq K'[x]/(f)$$

where $K'[a] = K'(a)$ since a is algebraic over k' . Since L is algebraically closed, f (considering it in $L[x]$) has a root $b \in L$. Then there exists a unique K' -algebra morphism $\varphi' : K'(a) \rightarrow L$, $a \mapsto b$, which is indeed a K' -algebra morphism as any $g \in K[x]$ s.t. $g(a) = 0$ satisfies $f \mid g$, which has b also as a root. But then $(K'(a), \varphi')$ satisfies the condition, which contradicts with the maximality of (K', φ) .

Therefore $K' = K$ and we have the desired result. □

Corollary 4.30. The algebraic closure of a given field k is unique up to isomorphism.

Suppose that we have field extensions $k \hookrightarrow K$ and $k \hookrightarrow L$, with both K and L algebraically closed. Then applying the above theorem twice gives field extensions $K \hookrightarrow L$ and $L \hookrightarrow K$, which are both injective by Proposition 1.2. This gives the desired isomorphism.

Notation. Since the algebraic closures of a specific field k are isomorphic, we can simply refer to it as *the* algebraic closure, and denote it with \bar{k} .

Remark 4.31. We have the following immediate results:

- 1) If $k \hookrightarrow K$ is algebraic, and $K \hookrightarrow \bar{K}$ algebraic closure, then $k \hookrightarrow \bar{K}$ is also an algebraic closure (by considering $f \in k[x]$ as elements of $K[x]$).
- 2) If we have $k \hookrightarrow K$ a field extension, and K is algebraically closed. Then for $k' := \{a \in K \mid a \text{ algebraic over } k\}$ we have k'/k an algebraic closure. This is clearly algebraic by definition; and it is closed as if $f \in k'[x] \setminus \{0\}$, there exists $a \in K$ s.t. $f(a) = 0$ as K is algebraically closed. Then a is algebraic over $k \implies k' \hookrightarrow k'(a)$ finite $\implies k' \hookrightarrow k'(a)$ algebraic $\implies k \hookrightarrow k' \hookrightarrow k'(a)$ algebraic. But this gives the fact that a is algebraic over k , and since $a \in K$ by definition $a \in k'$.

Notice that we cannot use the same trick in the proof for Theorem 4.22 as in that case the extensions $k \hookrightarrow k_1$ is not necessarily algebraic.

Now we can use the above results to classify finite fields:

Let K be a finite field. Since $\mathbb{Z} \rightarrow K$ cannot be injective, $\text{char}(K) = p$ for some prime p . This gives a field extension $\mathbb{F}_p \hookrightarrow K$; and since $e = [K : \mathbb{F}_p]$ is finite (since K is finite), $|K| = p^e$ for some e . Denote \bar{K} to be the algebraic closure of K . By Remark 4.31 this is also the algebraic closure of \mathbb{F}_p .

Claim 4.32. $K = \{u \in \bar{K} \mid u^{p^e} = u\}$. This gives the existence of finite fields with order p^e .

Proof. Notice that (K^\times, \cdot) is a finite group with $(p^e - 1)$ elements. Therefore, for all $u \in K$, $u^{p^e - 1} = 1$, which implies that $u^{p^e} = u$ i.e. $K \subseteq \text{RHS}$. Now consider the polynomial $x^{p^e} - x$ in $\bar{K}[x]$, which has p^e roots since \bar{K} is algebraically closed. $|K| = p^e$ gives the desired equality. \square

Proposition 4.33. If K_1 and K_2 are both fields with p^e elements, then $K_1 \simeq K_2$. Without ambiguity we denote such fields as \mathbb{F}_{p^e} .

Proof. Consider the extensions:

$$\begin{array}{ccc} \mathbb{F}_p & \hookrightarrow & K_1 \\ \downarrow & & \downarrow \varphi \\ K_2 & \hookrightarrow & \bar{K}_2 \end{array}$$

This existence of φ is guaranteed by Theorem 4.28. Since φ is a ring homomorphism between fields, it is injective; and therefore $|\varphi(K_1)| = |K_1| = p^e \implies \varphi(K_1) = \{u \in \bar{K}_2 \mid u^{p^e} = u\}$. But this then coincides with K_2 . \square

Example 4.34. There exists a ring homomorphism $\mathbb{F}_{p^e} \rightarrow \mathbb{F}_{p^f}$ if and only if $e \mid f$.

Proof. Verify both implications:

\Rightarrow : Since we have the field extension $\mathbb{F}_{p^e} \hookrightarrow \mathbb{F}_{p^f}$, we can view \mathbb{F}_{p^f} as a \mathbb{F}_{p^e} -vector space. As further we have two fields being finite, $|\mathbb{F}_{p^f}| = (|\mathbb{F}_{p^e}|)^{\dim_{\mathbb{F}_{p^e}} \mathbb{F}_{p^f}}$. In particular this implies that $e \mid f$.

\Leftarrow : By Claim 4.32 we know that the elements in the field \mathbb{F}_{p^f} are those in the set $\{u \in \overline{\mathbb{F}_{p^f}} \mid u^{p^f} = u\}$. Now Consider the roots of the polynomial $f = x^{p^f} - x$ in $\overline{\mathbb{F}_{p^f}}$. Since \mathbb{F}_{p^f} embeds naturally into its algebraic closure, the image is exactly the roots of f . Since $e \mid f$, there exists n s.t. $f = ne$. Notice then that f factors as follows:

$$f = x^{p^f} - x = (x^{p^e} - x) \left(\sum_{i=1}^{n-1} p^{i(p^e-1)} \right)$$

which implies that elements satisfying the relation $u^{p^e} = u$ in particular also satisfy $u^{p^f} = u$; and by Claim 4.32 these give the elements in field \mathbb{F}_{p^e} . This gives the field extension $\mathbb{F}_{p^e} \hookrightarrow \mathbb{F}_{p^f} \hookrightarrow \overline{\mathbb{F}_{p^f}}$ (and also by the uniqueness of algebraic closure (Corollary 4.30) we have $\overline{\mathbb{F}_{p^e}} = \overline{\mathbb{F}_{p^f}}$).

□

Example 4.35. Inside the algebraic closure of \mathbb{F}_p , $\overline{\mathbb{F}_p}$, for all $e \geq 1$ we have a unique copy of \mathbb{F}_{p^e} ; and $\overline{\mathbb{F}_p} = \bigcup_{e \geq 1} \mathbb{F}_{p^e}$.

Proof. Proceed to verify the inclusion in both directions:

\subseteq For any element $u \in \overline{\mathbb{F}_p}$, consider its minimal polynomial over \mathbb{F}_p . Let $f_u \in \mathbb{F}_p[x]$ be the polynomial of u . Then $u \in \mathbb{F}_p(u) \simeq \mathbb{F}_p[x]/(f_u)$. By Remark 1.10 we have $|\mathbb{F}_p(u)| = p^{\deg f} \implies u \in \mathbb{F}_{p^{\deg f}} \subseteq \bigcup_{e \geq 1} \mathbb{F}_{p^e}$.

\supseteq By Example 4.34 we have the embedding of \mathbb{F}_{p^e} to its algebraic closure; and also such closures can be identified by the uniqueness of algebraic closure and viewing \mathbb{F}_{p^n} as elements satisfying relation $u^{p^n} = u$ by Claim 4.32.

□

5 The Splitting Field of a Polynomial**6 Separable Extensions****7 Normal Extensions****8 Galois Extensions****9 Algebraic Independence & Transcendence Degree*****10 The Fundamental Theorem of Galois Theory****11 Norm and Trace Maps****12 Solvability by Radicals**