MATH 594 - Representation of Finite Groups

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1 Complex Representation

The motivation of introducing the representation of G is to have a linearized version of group action on sets. Recall that we have the correspondence between action of G on a set X and group homomorphism $G \to S_x$ where S_x is the group of bijective maps on S, with the operation defined as composition. Explicitly, this is given by

$$\varphi : G \times X \to X \quad \rightsquigarrow \quad G \to S_x,\ g \mapsto \varphi(g, -) : (X \to X)$$

We now give the formal definition on vector spaces:

Definition 1.1 (Representation). A **(complex) representation** of a group G is a vector space V over $\mathbb{C}$, together with a group homomorphism

$$\rho : G \to \mathrm{GL}(V) := \{\varphi : V \to V \mid \varphi\ \mathbb{C}\text{-linear isomorphism}\}$$

Equivalently, a representation of G is a vector space over $\mathbb{C}$ with an action of G on V $\rho : G \times V \to V$ s.t. for all $g \in G$, the induced map $\varphi(g, -)$ os $\mathbb{C}$-linear.

Notation. The map $\rho(g, -) : V \to V$ is often abbreviated as ρ_g. The representation is denoted by V or ρ, with V emphasizing the vector space structure.

Definition 1.2 (Degree of Repr.). The **degree** of the representation is $\dim_{\mathbb{C}} V$, with the same notation as above.

For most of the time, we will only consider the representation of finite groups on finite-dimensional vector spaces.

Remark 1.3. In general, one can consider representations over other fields than $\mathbb{C}$. The reasons why $\mathbb{C}$ is chosen are the followings:

1) If G is finite, then $|G| \in \mathbb{C}$ is always invertible.

2) $\mathbb{C}$ is algebraically closed. The implications include, for example, every linear map has an eigenvalue.

These specialties will often appear in subsequent proofs.

Definition 1.4 (Morphism of Repr.). Given two representations of G, V and W, a **morphism of representations** (or simply G-**morphism**) is a linear map $f : V \to W$ s.t. $f(gv) = g(fv)$ for all $g \in G, v \in V$. This is an **isomorphism** if f is further bijective.

Remark 1.5. Following from the definitions we have the immediate results:

1) If $V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$ are morphisms of representation, then so is $g \circ f$ since $g(f(hv)) = g(hf(v)) = h(g(f(v)))$ for all $h \in G, v \in V$. This gives the morphisms of objects, i.e. representations of G give a category.

2) If $f : V \to W$ is an isomorphism of representations, then so is f^{-1} (simply by writing the equation for definition in the inverse order).

3) If V and W are representations of G, then $\{f : V \to W \mid f\ G\text{-morphism}\} \subseteq \mathrm{Hom}_{\mathbb{C}}(V, W)$ gives a $\mathbb{C}$-vector subspace.

This is clear as by the fact that f is linear, V as a representation is closed under addition and scalar multiplication.

Remark 1.6. Given a finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$, choosing a basis $\{e_1, \dots, e_n\}$ of V gives us an isomorphism $V \simeq \mathbb{C}^n$, i.e. we have the description of representations in matrices

$$\rho : G \to \mathrm{GL}(V) \simeq \mathrm{GL}_n(\mathbb{C}), \qquad g \mapsto \rho_g = (a_{ij}(g))$$

Let A be the matrix representation of a morphism of representations f. Since it is required that a morphism of representations should be compatible with application of $g \in G$, we have $A \circ \rho_g = \rho'_g \circ A$. This implies that $\rho'_g = A \circ \rho_g \circ A^{-1}$, i.e. two representations are isomorphic if and only if they are conjugate in matrix presentation; and the matrix that describes the conjugation is the same for all elements $g \in G$.

Definition 1.7 (Sub-representation). Given a representation V of G, a **sub-representation** of V is a vector space $W \subseteq V$ s.t. $gv \in W$ for all $v \in W, g \in G$.

Remark 1.8. In particular, for W a sub-representation of V, it is itself a representation with the map ρ' being $\rho(-)|_W$. The inclusion $W \hookrightarrow V$, $x \mapsto X$ is a morphism of representation.

2 Interpretation via the Group Algebra

Similar to the case of group action where we interpreted the structure of group action by the group homomorphism $G \to S_x$, we would like to have some equivalence to structures that are more explicit, and easier to analyze. This introduces the following definitions:

Definition 2.1 (Group Algebra). Let G be a group. Then the **group algebra over** $\mathbb{C}$, denoted $\mathbb{C}[G]$ is a vector space with a basis $\{\alpha(g) \mid g \in G\}$ in bijection with elements in G (formally). Endow it with a multiplication $\alpha(g) \cdot \alpha(h) = \alpha(gh)$ compatible with the group structure gives the desired ring structure.

Remark 2.2. Verifying the ring axioms, we have the fact that the identity in $\mathbb{C}[G]$ to be $\alpha(e)$. This is in fact a $\mathbb{C}$-algebra, with the associated morphism given by $\mathbb{C} \to \mathbb{C}[G]$. Since the image of it are scalars, it is clearly in the center of the group.

Notice that G is not necessarily a finite group. Therefore the vector space can be infinite-dimensional, which we have imposed the requirement that every element should be a finite sum of linear combination of basis. In the following deduction, denote $\sum'$ to be the finite sum.

Proposition 2.3. The group algebra is well-defined.

Proof. This is clear for the cases where G is finite. Consider the case where G is infinite. Then by definition of the group algebra, for all $u, v \in \mathbb{C}[G]$, we have their decomposition into elements in the basis:

$$u = \sum_{g \in G}' a_g \alpha(g), \qquad v \in \sum_{g \in G}' b_g \alpha(g)$$

Multiplying these two terms together gives

$$u \cdot v = \sum_{g \in G} \left(\sum_{g_1 g_2 = g} (a_{g_1} b_{g_2}) \right) \alpha(g)$$

Furthermore there are only finitely many such a_gs and b_gs being nonzero, implying that there are only finitely many nonzero such products. □

Notation. If G is abelian, and the correspondence of elements in G and in $\mathbb{C}[G]$ is written additively. Instead of $\alpha(g)$ one usually writes χ^g (with the convention that $\chi^g \cdot \chi^h = \chi^{g+h}$).

Remark 2.4. $\mathbb{C}[G]$ is a commutative ring if and only if G is an abelian group. "Only if" is clear as if $\mathbb{C}[G]$ is commutative implies for all $g, h \in G$, they commute. "If" results from the fact that for every element in $x \in \mathbb{C}[G]$ there exists a scalar λ s.t. $\lambda x = \alpha(g)$ for some $g \in G$ as $\mathbb{C}$ is a field.

Example 2.5. If $G = (\mathbb{Z}, +)$, identifying $x \leftrightarrow \chi^x$ for $x \in \mathbb{Z}$, we have $\mathbb{C}[G] \simeq \bigoplus_{m]in\mathbb{Z}} \mathbb{C}\chi^m \simeq S^{-1}\mathbb{C}[x]$ for $S = \langle x \rangle = \{1, x, x^2, \dots\}$. These are the Laurent Polynomials.

If $G = (\mathbb{Z}/n\mathbb{Z}, +)$, we have the identification $x^n = 1$, giving $\mathbb{C}[G] \simeq \mathbb{C}[x]/(x^n - 1)$.

Proposition 2.6. We have the identification between representations of G and $\mathbb{C}[G]$-modules. Morphisms and sub-objects (sub-representations and submodules) are also in correspondence.

Proof. It suffices to verify 1), as identifications in 2) and 3) are induced by 1).

Suppose that V is a representation of G, Then V has a structure of $\mathbb{C}[G]$-module, whose addition is the same as in the vector space, and scalar multiplication is given by

$$\left(\sum_{g \in G} (a_g \cdot \alpha(g)) \right) \cdot v = \sum_{g \in G} (a_g \cdot g(v))$$

where the sums are finite. Conversely, if M is a $\mathbb{C}[G]$-module, then it has a vector space structure via considering the action $\mathbb{C} \hookrightarrow \mathbb{C}[G]$ which acts on M; and the linear map is given by $(g, -)$, where $(g, x) \mapsto \alpha(g) \cdot x$ as specified by the $\mathbb{C}[G]$ module. □

Remark 2.7. In general, for a representation over a field $\mathbb{F}$ of G, it can be identified with $\mathbb{F}[G]$.

3 Examples of Representations

The following gives some common examples of representations:

1) Suppose that G acts on a set X. Let V be the free $\mathbb{C}$-vector space associated to X, with basis $\{\alpha(u) \mid u \in X\}$ in bijection with X. Define $G \xrightarrow{\rho} \mathrm{GL}(V), g \mapsto \rho_g$, with $\rho_g(\alpha(u)) = \alpha(gu)$. This is the permutation representation associated with X where action of elements in the group corresponds to a permutation of the set. This is essentially just the group action, as the representation is completely fixed via specifying its behavior on elements in X (i.e. with coefficient 1).

2) Example 1) applied to the action of G on itself, $G \times G \to G$, $(g,h) \mapsto (gh)$ induces a representation $\mathbb{C}[G]$. This is the regular representation of G. Viewed under the context of Proposition 2.6, this is the standard left $\mathbb{C}[G]$-module structure of itself (rings are left-modules over itself).

3) Direct sum of representations. If $\rho_V : G \to \mathrm{GL}(V)$ and $\rho_W : G \to \mathrm{GL}(W)$ are representations of G, then we can get a representation $\rho : G \to \mathrm{GL}(V \oplus W)$, given by

$$\rho_g = (\rho_g^V, \rho_g^W) : G \times (V \oplus W) \to (V \oplus W), \quad (g,(v,w)) \mapsto (gv, gw)$$

Under the context of Proposition 2.6, this corresponds to the direct sum of modules.

4) Tensor product of representations. Suppose that we have $\rho : G \to \mathrm{GL}(V)$ and $\rho' : G \to \mathrm{GL}(V')$ two representations of G. Then we can have

$$\widetilde{\rho} = \rho \otimes \rho' : G \to \mathrm{GL}(V \otimes_{\mathbb{C}} V'), \quad g \mapsto (\rho_g \otimes \rho'_g)$$

This is indeed a group homomorphism, as tensor product of maps behave functorially. That is, it commutes with composition of maps by the universal property of tensor product:

$$(f \otimes g) \circ (f' \otimes g') = (f \circ f') \otimes (g \circ g')$$

4 Irreducible Representations

Similar to the introduction of simple groups in group theory, we would like to have some simple objects in terms of representation, s.t. for any representation it can be decomposed into the "combination" of these simple objects, and understanding them provides understanding of the whole object.

Consider the simplest case of representation of G, where it is 1-dimensional, and is given by $G \to \mathbb{C}^* \simeq (\mathbb{C} \to \mathbb{C})$ as a group homomorphism. The composition of $\mathbb{C}^*$ is the multiplication, as here $\mathbb{C}$ is considered as the 1-by-1 complex matrix (1-dimensional linear map). Since $\mathbb{C}^*$ is commutative, this is the same as the representation $\bar{\rho} : G^{\mathrm{ab}} \to \mathbb{C}^*$. By Remark 1.6, two representations are isomorphic if and only if they are conjugate; and since $\mathbb{C}^*$ is commutative, this implies that two representations are isomorphic if and only if they are identical.

> **Corollary 4.1.** If $F : V \to W$ is a morphism of G-representations, then $\ker f \subseteq V$ and $\operatorname{im} f \subseteq W$ are sub-representations. This can be seen via using Proposition 2.6 to identify representations with $\mathbb{C}[G]$-modules, and see that the kernel and image of a morphism of R-modules are both submodules.

The following gives some tools for properly define the concept of "simple" objects in terms of representations, and decompose complex objects to those simple ones. From now on, we will consider only G being finite groups, and all representations are finite-dimensional.

> **Parenthesis 4.2.** Let V be a vector space, and $W \subseteq V$ a linear subspace. Then giving the followings are equivalent:
>
> 1) A vector subspace $W' \subseteq V$ s.t. $V = W \oplus W'$ which is the internal direct sum, i.e. every element in V can be uniquely decomposed into the sum of an element in W and an element in W'; and the two vector subspaces W and W' are linearly independent, i.e. $W \cap W' = \{0\}$.

2) A linear map $p : V \to V$ s.t. $p^2 = p$, and $\operatorname{im} p = W$.

Proof. Consider implication in two directions:

- *1)* $\implies$ *2).* Given $V = W \oplus W'$, we know that for all $v \in V$, there exists unique $w \in W$ and $w' \in W'$ s.t. $v = w + w'$. Define $p : V \to W$ s.t. $p(v) = w \in W$ with w the same as in the decomposition above. It is then clear that $p^2 = p$.
- *2)* $\implies$ *1).* Define $W = p(V)$, and $W' = \ker p$. Check: $W \cap W' = \{0\}$, as $v \in W' \implies p(v) = 0$; and $v \in W \implies p(v) = v$, which gives $W \cap W' = \{0\}$; and the decomposition can be seen by $v \mapsto (p(v), v - p(v))$.

It is further clear that these two transforms are inverse to each other, which proves the assertion. □

The general result above is also true for representations:

Theorem 4.3. Let G be a finite group. Let V be a finite-dimensional $\mathbb{C}$-representation of G, and $W \subseteq V$ a sub-representation. Then there exists another sub-representation W' of V s.t. $V = W \oplus W'$.

Proof. First show that we have similar identifications as in the scenario for vector spaces: for V a representation of G, and $W \subseteq V$ a sub-representation, then

$$\text{Giving } W' \text{ sub-repr. s.t. } V = W \oplus W' \quad \iff \quad \text{Giving } p : V \to V \text{ morphism of repr. s.t. } p^2 = p \text{, and } \operatorname{im} p = W$$

$\Rightarrow$: For W' being a sub-representation, it is in particular a vector subspace of V. Then by Parenthesis 4.2 we have $p : V \to V$ linear map which satisfies the desired conditions. Further verify that this is a morphism of representation: for all $v \in V, g \in G$ we have

$$g(p(v)) = g(p(w + w')) = g(p(w)) = g(w) = p(g(w))$$

since $g(w) \in W$ as W is a sub-representation of V.

$\Leftarrow$: The decomposition is clear from the result when considering V, W, W' as vector spaces; and the fact that W' is a sub-representation results from the result that $W' = \ker p$ and kernel of morphism of representations is still a representation (Corollary 4.1).

Approach the proof via providing the construction in RHS. For $W \subseteq V$ being a sub-representation, by Parenthesis 4.2 we have a linear map $p : V \to V$ s.t. $p^2 = p$ and $\operatorname{im} p = W$. Now seek using p to construct a morphism of representations: Define

$$\tilde{p} : V \to V, \quad v \mapsto \sum_{g \in G} g^{-1} p(gv)$$

Verify that this is a morphism of representations. For all $v \in V$ and $h \in G$, we have

$$\begin{aligned}
\tilde{p}(hv) &= \sum_{g \in G} g^{-1} p(ghv) = h \left(\sum_{g \in G} h^{-1} g^{-1} p(ghv) \right) \\
&= h \left(\sum_{g' \in G} (g')^{-1} p(g'v) \right) && \text{(Use substitution } g' = gh\text{)} \\
&= h(\tilde{p}(v))
\end{aligned}$$

Check that $\tilde{p}$ also satisfies the other two conditions, i.e. $\tilde{p}^2 = \tilde{p}$, and $\operatorname{im} \tilde{p} = W$. Notice $p(\tilde{h}v) = h\left(\sum_{g' \in G} (g')^{-1} p(g'v)\right)$, where $p(g'v) \in W$; and linear combination of elements in W still remains in W. Further W is a sub-representation gives the fact that W is invariant under g-actions. Let $W' = \ker \tilde{p}$ finishes the proof. □

Remark 4.4. In general linear maps between vector spaces, or even endomorphisms on a specific vector space, are not morphisms of representations, as ρ_g as a linear map in general does not commute with p (which is required by the definition of morphism of representations).

Remark 4.5. It is also vital that we require that G is finite; and the field is of characteristic zero. Otherwise the "averaging" process where we divide the sum of all possible representations by the order of G is not valid; and in general Theorem 4.3 is *not* true for representations over positive-characteristic fields, or for infinite groups.

A good example given in the homeworks is the followings: take $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in K^2 = V$ for K some field, and the representation $\rho : G \to \mathrm{GL}(V)$. Such projection p (as in the proof) does not exist for the following cases:

- $K = \mathbb{C}$, $G = \mathbb{Z}$, and $\rho(1) = A$.
- $K = \mathbb{Z}/2\mathbb{Z}$, $G = \mathbb{Z}/2\mathbb{Z}$, and $\rho(\bar{1}) = A$.

Remark 4.6. By Proposition 2.6 we have the identification between G-representations on V and $\mathbb{C}[G]$-module structure on V. This implies that for G finite, $W \subseteq V$ being a $\mathbb{C}[G]$-module implies that there exists another $\mathbb{C}[G]$-submodule W' s.t. $V = W \oplus W'$.

Definition 4.7 (Irreducible). A representation V of G is **irreducible** if

- $V \neq \{0\}$.
- For every representation $W \subseteq V$, either $W = \{0\}$ or $W = V$.

Corollary 4.8. By applying iteratively Theorem 4.3 for V any representation of a finite group we have the irreducible decomposition $V = W_1 \oplus \cdots \oplus W_r$ for W_is irreducible decompositions.

Remark 4.9. In general irreducible representations do not have to be degree-1. The counterexamples provided in Remark 4.5 are good counterexamples.

Remark 4.10. The $W_i \subseteq V$ as sub-representations of G are not necessarily unique. For example let V being the trivial representation with degree greater than 1; then any decomposition of V into 1-dimensional subspaces gives the irreducible decompositions, and they are not unique as they can be any linearly-independent subspaces.

However they are unique up to isomorphisms, i.e. given any irreducible representation W, and $V = \bigoplus_{i \in I} W_i$, $\#\{i \mid W_i \simeq W\}$ and $\sum_{W_i \simeq W} W_i$ are independent of the decomposition. That is, the "space" that can be represented by W (up to isomorphism) is fixed in for a given V.

The following lemma gives important foundation for computing morphisms between irreducible representations:

Lemma 4.11 (Schur). Suppose that V and W are irreducible representations of G; and $f : V \to W$ morphism of representations. Then

- If $V \not\simeq W$, then $f = 0$.
- If $V \simeq W$, then $f = \lambda \cdot \mathrm{Id}$ for some $\lambda \in \mathbb{C}$.

Proof. Use the result from Corollary 4.1, that kernel and image of morphism of representations are also representations. Consider $\ker f$ and $\operatorname{im} f$. Since V is irreducible, either $\ker f = V$ (where $f = 0$) or $\ker f = \{0\}$. Similarly either $\operatorname{im} f = \{0\}$ or $\operatorname{im} f = W$.

Since a morphism of representations is in particular a linear map, $\dim \ker f + \dim \operatorname{im} f = \dim V$; and in particular we cannot have both $\operatorname{im} f = \{0\}$ and $\ker f = \{0\}$. Therefore if $\ker f = \{0\}$, $\operatorname{im} f = W$, which implies that f is an isomorphism. Since $\mathbb{C}$ is algebraically closed, we know that f (as a linear map) has an eigenvalue λ, i.e. $f - \lambda \cdot \mathrm{Id}$ is not injective. But by the fact that W is irreducible, $f - \lambda \cdot \mathrm{Id} = 0$, i.e. $f = \lambda \cdot \mathrm{Id}$. □

Now we seek to prove the first assertion in Remark 4.10. First we need to introduce the structure of representations on the linear maps $\mathrm{Hom}_{\mathbb{C}}(V, W)$.

We have seen that $\mathrm{Hom}_{\mathbb{C}}(V, W)$ obtains a vector space structure with addition and scalar multiplication given by the corresponding operation on the output of the map in W. Now suppose that V and W are both G-representations. Then there exists a natural G-representation structure in $\mathrm{Hom}_{\mathbb{C}}(V, W)$, given by

$$(g\varphi)(v) := g\left(\varphi(g^{-1}(v))\right)$$

It is clear that this is linear. Check that this is a group homomorphism:

$$((g_1g_2)\varphi)(v) = (g_1g_2)(\varphi(g_2^{-1}g_1^{-1}(v))) = g_1\left(g_2(\varphi(g_2^{-1}(g_1^{-1}(v))))\right) = (g_1(g_2\varphi))(v)$$

Remark 4.12. For V any G-representation, define $V^G := \{v \in V \mid gv = v \ \forall g \in G\}$ which is the largest trivial subrepresentation of G. Then $\mathrm{Hom}_{\mathbb{C}}(V, W)$ can be identified with $\{\varphi : V \to W \mid \varphi \text{ morphism of repr.}\}$. This follows directly from the fact that $g\varphi(g^{-1}-) = \varphi(-)$ implies that $g^{-1}\varphi(-) = \varphi(g^{-1}-)$ which is exactly the definition of morphism of representations.

Corollary 4.13 (Result 1 in Remark 4.10). If V is a G-representation with irreducible decompositions $V = W_1 \oplus \cdots \oplus W_r$. Then $\#\{i \mid W_i \simeq W\} = \dim_{\mathbb{C}}\left(\mathrm{Hom}_{\mathbb{C}}(V, W)^G\right)$

Proof. By the structure of representations in V, we have the isomorphism of representations:

$$\mathrm{Hom}_{\mathbb{C}}(W, V)^G \simeq \mathrm{Hom}_{\mathbb{C}}(W, W_1) \oplus \cdots \oplus \mathrm{Hom}_{\mathbb{C}}(W, W_r)$$

Schur (Lemma 4.11) gives

$$\dim_{\mathbb{C}}(W, W_i)^G = \begin{cases} 0, & W \not\simeq W_i \\ 1, & W \simeq W_i \end{cases}$$

Since we are in the context of vector spaces, we have

$$\dim_{\mathbb{C}}(W, V) = \sum_{i=1}^{r} \dim \operatorname{Hom}_{\mathbb{C}}(W, W_i)$$

and summing up the dimensions gives the desired result. $\square$

5 Character Theory

Definition 5.1 (Character). Fix a representation $\rho : G \to \mathrm{GL}(V)$, with $g \mapsto \rho_g$, the **character** of ρ is a function

$$\chi_\rho : G \to \mathbb{C}, \quad g \mapsto \mathrm{Tr}(\rho_g)$$

Remark 5.2. The definition of character is invariant w.r.t. choice of basis as the trace of a linear map is independent of the choice of basis.

Notation. $\chi(\rho) \in \mathbb{C}$ is often abbreviated to be χ_ρ as is in the notation of representation.

The following gives some immediate properties of trace function:

1) If $\rho \simeq \rho'$, then $\chi_\rho = \chi_{\rho'}$. Recall that by Remark 1.6, $\rho \simeq \rho'$ if and only if there exists some linear map A s.t. for all $g \in G$. $\rho_g = A \circ \rho'_g \circ A^{-1}$, i.e. they are similar. Then the equality in character results directly from the fact that similar matrices have identical trace.

2) An immediate corollary to 1) is the fact that each χ_ρ is a class function, i.e. takes constant value on the conjugacy class of any $g \in G$. This can be seen via taking $A = \rho_g$, and apply the fact that ρ is a group homomorphism, i.e. $\rho_{g^{-1}} = \rho_g^{-1}$.

3) $\chi_\rho(e) = \mathrm{Tr}(\mathrm{Id}) = \dim_{\mathbb{C}} V$.

The following results are also quite useful but are not so immediate, so we formalize them as propositions:

Proposition 5.3. Let ρ be a representation of a finite group G. Then $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$.

Proof. Since we have $V \simeq \mathbb{C}^n$, we have the identification of ρ as $\rho : G \to \mathrm{GL}(\mathbb{C}^n)$. Gram-Schmidt gives a basis in which the matrix representation of ρ_g is upper-triangular, i.e. in the form of $\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ with λ_is the eigenvalues. It is clear then that the inverse must also be in the form of $\begin{pmatrix} \lambda_1^{-1} & & * \\ & \ddots & \\ & & \lambda_n^{-1} \end{pmatrix}$. This gives

$$\mathrm{Tr}(\rho_g) = \sum_{i=1}^{n} \lambda_i, \quad \mathrm{Tr}(\rho_{g^{-1}}) = \sum_{i=1}^{n} \lambda_1^{-1}$$

But since we have the requirement that ρ is a group homomorphism, for $|G| = m$ we have the constraint

$$\begin{pmatrix} \lambda_1^{-1} & & * \\ & \ddots & \\ & & \lambda_n^{-1} \end{pmatrix}^n = \mathrm{Id} \implies \lambda_i^m = 1 \ \forall i$$

In particular we have $|\lambda_i| = 1$, which gives $\lambda_i^{-1} = \overline{\lambda_i}$. Summing together gives the desired result. $\square$

Proposition 5.4. The characters of direct sum of representations, and tensor product of representations have the following relations:

- For $\rho = \rho_1 \oplus \rho_2$, $\chi_\rho = \chi_{\rho_1} + \chi_{\rho_2}$.
- For $\rho = \rho' \otimes \rho''$, $\chi_\rho = \chi_{\rho'} \cdot \chi_{\rho''}$.

Proof. Choose an appropriate basis for the vector space, and express the representations in that correspondingly.

Let V_1 and V_2 be the corresponding vector spaces of ρ_1 and ρ_2. Then given bases in both V_1 and V_2, we have a basis of $V = V_1 \oplus V_2$, and the corresponding matrix representation of ρ: $\left(\begin{array}{c|c} (\rho_1)_g & \\ \hline & (\rho_2)_g \end{array}\right)$ which is block diagonal. The trace corresponding to the character can then be computed via

$$\mathrm{Tr}(\rho) = \mathrm{Tr}((\rho_1)_g) + \mathrm{Tr}((\rho_2)_g)$$

For the case with tensor product, consider the representations to be $\rho' : G \to \mathrm{GL}(V_1)$ and $\rho'' : G \to \mathrm{GL}(V_2)$. Choose bases $e_1, \ldots, e_n$ for V_1, and $f_1, \ldots, f_m$ for V_2. Let $(a_{ij})_{n \times n}$ and $(b_{kl})_{m \times m}$ be the corresponding matrix representations for ρ' and ρ''. Then $V_1 \otimes V_2$ has a basis $e_i \otimes f_j$. Compute the matrix representation for ρ:

$$\begin{aligned}\rho_g(e_j \otimes f_\ell) &= \rho'_g(e_j) \otimes \rho''_g(f_\ell) \\ &= \left(\sum_{i=1}^{n} a_{ij} e_i\right) \otimes \left(\sum_{k=1}^{m} b_{k\ell} f_k\right) \\ &= \sum_{i,k} a_{ij} b_{k\ell} \left(e_i \otimes f_k\right)\end{aligned}$$

Then

$$\mathrm{Tr}(\rho_g) = \sum_{i=j,k=\ell} a_{ij} b_{k\ell} (e_i \otimes f_k) = \sum_{i,k} a_{ii} b_{kk} (e_i \otimes f_k) = \left(\sum_i a_{ii}\right)\left(\sum_k b_{kk}\right) = \mathrm{Tr}(\rho'_g) \cdot \mathrm{Tr}(\rho''_g)$$

$\square$

The character of a representation gives information on its properties, e.g. checking whether a given set of sub-representations gives a decomposition, or checking whether a representation is irreducible, etc. To utilize this concept we need some extra tools on the object:

Notation. Define $\mathrm{Func}(G) := \{f : G \to \mathbb{C}\}$. This gives a $\mathbb{C}$-vector space of dim $|G|$. Explicitly, for $G = \{g_1, \ldots, g_m\}$ we have the isomorphism $\mathrm{Func}(G) \simeq \mathbb{C}$, $f \mapsto (f(g_1), \ldots, f(g_m))$.

Definition 5.5 (Inner Product of Character)**.** Let $f, g \in \mathrm{Func}(G)$. Define

$$\langle f, g \rangle := \frac{1}{|G|} \sum_{x \in G} f(x) \overline{g(x)}$$

Remark 5.6. The inner product gives a Hermitian product on $\mathrm{Func}(G)$. Recall that Hermitian product needs to satisfy $\mathbb{C}$-linearity in the first entry, conjugate $\mathbb{C}$-linear in the second entry, and positive-definite, i.e. for all $f \neq 0, \langle f, f\rangle > 0$.

The following is the main theorem for character of representations:

Theorem 5.7. If ρ and ρ' are non-isomorphic irreducible representations, then $\langle \chi_\rho, \chi_{\rho'}\rangle = 0$, and $\langle \chi_\rho, \chi_\rho\rangle = 1$.

The tools we have for such situations are Schur's result (Lemma 4.11), and the averaging process which makes a linear map into a morphism of representations used in Theorem 4.3. The general strategy is to notice that we have application of g both in the domain and image in the "averaged" representation, and $\chi(g^{-1}) = \overline{\chi(g)}$ gives the conjugacy and thus the desired form of the inner product.

Notation. In the proof we use $(a_{ij})_{m\times n}$ to denote the matrices, with the indices dropped if apparent, and a_{ij} (without the parenthesis) to denote the elements in the matrix.

Proof of Theorem 5.7. Assume that $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ and $\rho' : G \to \mathrm{GL}_m(\mathbb{C})$ are the representations. Denote the matrix representations for ρ_g, ρ'_g and $\rho'_{g^{-1}} = (\rho'_g)^{-1}$ to be $(a_{ij})_{n\times n}$, $(b_{ij})_{m\times m}$ and $(c_{ij})_{m\times m}$, correspondingly. Consider a linear map $\varphi : V \to V'$ with matrix representation $(\varphi_{ij})_{m\times n}$, and the induced morphism of representations $\psi : V \to V'$ given by

$$\psi(v) := \sum_{g\in G} g^{-1}(\varphi(gv))$$

For proof of this being indeed a representation check Proof of Theorem 4.3. Now consider the matrix representation of $\psi(v)$:

- Consider $\rho \not\simeq \rho'$. Then by Schur's Lemma (Lemma 4.11), we have $\psi = 0$. In particular, every entry of matrix representation of ψ is 0, which gives

  $$\psi_{ik} = \sum_{g\in G}\left(\sum_{j,\ell} c_{ij}(g)\varphi_{j\ell}a_{\ell k}\right)$$

  Since this holds for all $\varphi : V \to V'$, decomposing φ into the basis given by each entry in the m-by-n matrix i.e. with

  $$\varphi_{j\ell} = \begin{cases} 1, & j = j',\ \ell = \ell' \\ 0, & \text{otherwise} \end{cases}$$

  we have for all i, j', ℓ', k, $\sum_{g\in G}(\sum_{i,k} c_{ij'}a_{\ell' k}(g)) = 0$. In particular, we can take $i = j$ and $\ell = k$, and summing up along the diagonal entries, which gives

  $$0 = \sum_{g\in G}\left(\sum_{i=1}^{n} c_{ii}\right)\left(\sum_{k=1}^{n} a_{kk}\right) = \sum_{g\in G}\chi_{\rho'}(g^{-1})\cdot\chi_\rho(g) = \sum_{g\in G}\overline{\chi_{\rho'}(g)}\cdot\chi_\rho(g) = \langle\chi_\rho, \chi_{\rho'}\rangle$$

- Now consider the case where $\rho = \rho'$, i.e. $V \simeq V'$. Then Schur (Lemma 4.11) gives that there exists $\lambda_{j\ell}$ s.t. for $\varphi_{j'\ell'}$ as defined in the previous case

  $$\psi_{j'\ell',ik} = \sum_{g\in G} c_{ij'}(g)a_{\ell' k}(g) = \lambda_{j',\ell'}\delta_{i,k}$$

Now consider $\varphi = \mathrm{Id}$, i.e. $\varphi_{ij} = \delta_{i,j}$. Taking $i = j$ and $k = \ell$ we have

$$\sum_{g \in G} \left(\sum_{i,k} c_{ii}(g) a_{kk}(g) \right) = \sum_{i,k} \lambda_{i,k} \delta_{i,k} = \sum_{q} \lambda_{q,q}$$

But $\lambda_{i,i} = 1$ for all i, which gives

$$\begin{aligned} |G| = \sum_{k} \lambda_{k,k} &= \sum_{g \in G} \left(\sum_{i,k} c_{ii}(g) a_{kk}(g) \right) \\ &= \sum_{g \in G} \left(\sum_{i} c_{ii}(g) \right) \left(\sum_{k} a_{kk}(g) \right) \\ &= \sum_{g \in G} \chi_\rho(g) \cdot \chi_\rho(g^{-1}) = \sum_{g \in G} \chi_\rho(g) \cdot \overline{\chi_\rho(g)} \\ &= \sum_{g \in G} \langle \chi_\rho, \chi_\rho \rangle \end{aligned}$$

where dividing both sides by $|G|$ gives the desired result.

□

Corollary 5.8. From the theorem we have the following immediate results:

1) The number of isomorphism classes of irreducible representations is bounded above by $\dim_{\mathbb{C}} \mathrm{Func}(G) = |G|$. Notice that if $\rho_1, \dots, \rho_r$ are pairwise non-isomorphic irreducible representations, then by Theorem 5.7 $\langle \chi_i, \chi_j \rangle = 0$ for all $i \neq j$, i.e. they are orthonormal, and in particular linearly independent. But $\dim_{\mathbb{C}} \mathrm{Func}(G) \leq |G|$, so there are at most $|G|$ of them.

2) Let V be any representation of G, and W an irreducible representation of G, then for irreducible decomposition of V: $V = \bigoplus_{i=1}^{r} W_i$, $\#\{i \mid W_i \simeq W\} = \langle \chi_V, \chi_W \rangle$.

   Proof. To see this, by Proposition 5.4 we have $\chi_V = \sum_{i=1}^{r} \chi_{W_i}$. Then

   $$\langle \chi_V, \chi_W \rangle = \left\langle \sum_{i=1}^{r} \chi_{W_i}, \chi_W \right\rangle = \sum_{i=1}^{r} \langle \chi_{W_i}, \chi_W \rangle$$

   where since both W_i and W are irreducible, we have $\langle \chi_{W_i}, \chi_W \rangle = 1$ if and only if $W \simeq W_i$; and 0 otherwise. □

3) A representation V of G is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$.

   Proof. The "only if" part is prove in the theorem above. To see the "if" part, consider the decomposition of V:

   $$V \simeq W_1^{\oplus a_1} \oplus \cdots \oplus W_r^{\oplus a_r}, \quad \chi_V = \sum_{i=1}^{r} a_i \chi_{W_i}$$

   Then use the formula given in the theorem, we have $\langle \chi_V, \chi_V \rangle = \sum_{i=1}^{r} a_i^2$, which is 1 if and only if $r = 1$ and $a_1 = 1$. But this is exactly paraphrasing of the V being irreducible. □

4) For every representation V, we have $V \simeq W$ if and only if $\chi_V = \chi_W$.

Proof. The "only if" part results from Remark 1.6, where representations are isomorphic if and only if they are conjugate; but conjugate matrices have the same trace, i.e. the representations have the same character.

To see the "if" part, since $\chi_V = \chi_W$, in particular for every irreducible G-representation W' we have $\langle \chi_V, \chi_{W'} \rangle = \langle \chi_W, \chi_{W'} \rangle$. Testing this through all non-isomorphic irreducible G-representations gives the irreducible decomposition of V and W, which are the same as the inner products which gives the powers (a_is as in part 3) are the same. This implies that they are actually isomorphic. □

6 Counting Irreducible G-Representations