

MATH 594 - Representation of Finite Groups

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1 Complex Representation

The motivation of introducing the representation of G is to have a linearized version of group action on sets. Recall that we have the correspondence between action of G on a set X and group homomorphism $G \rightarrow S_x$ where S_x is the group of bijective maps on S , with the operation defined as composition. Explicitly, this is given by

$$\varphi : G \times X \rightarrow X \quad \rightsquigarrow \quad G \rightarrow S_x, g \mapsto \varphi(g, -) \in (X \rightarrow X)$$

We now give the formal definition on vector spaces:

Definition 1.1 (Representation). A **(complex) representation** of a group G is a vector space V over \mathbb{C} , together with a group homomorphism

$$\rho : G \rightarrow \text{GL}(V) := \{\varphi : V \rightarrow V \mid \varphi \text{ } \mathbb{C}\text{-linear isomorphism}\}$$

Equivalently, a representation of G is a vector space over \mathbb{C} with an action of G on V $\rho : G \times V \rightarrow V$ s.t. for all $g \in G$, the induced map $\varphi(g, -)$ is \mathbb{C} -linear.

Notation. The map $\rho(g, -) : V \rightarrow V$ is often abbreviated as ρ_g . The representation is denoted by V or ρ , with V emphasizing the vector space structure.

Definition 1.2 (Dimension of Repr.). The **dimension** of the representation is $\dim_{\mathbb{C}} V$, with the same notation as above.

For most of the time, we will only consider the representation of finite groups on finite-dimensional vector spaces.

Remark 1.3. In general, one can consider representations over other fields than \mathbb{C} . The reasons why \mathbb{C} is chosen are the followings:

- 1) If G is finite, then $|G| \in \mathbb{C}$ is always invertible.
- 2) \mathbb{C} is algebraically closed. The implications include, for example, every linear map has an eigenvalue.

These specialties will often appear in subsequent proofs.

Definition 1.4 (Morphism of Repr.). Given two representations of G , V and W , a **morphism of representations** (or simply **G -morphism**) is a linear map $f : V \rightarrow W$ s.t. $f(gv) = g(fv)$ for all $g \in G, v \in V$. This is an **isomorphism** if f is further bijective.

Remark 1.5. Following from the definitions we have the immediate results:

- 1) If $V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3$ are morphisms of representation, then so is $g \circ f$ since $g(f(hv)) = g(hf(v)) = h(g(f(v)))$ for all $h \in G, v \in V$. This gives the morphisms of objects, i.e. representations of G give a category.
- 2) If $f : V \rightarrow W$ is an isomorphism of representations, then so is f^{-1} (simply by writing the equation for definition in the inverse order).
- 3) If V and W are representations of G , then $\{f : V \rightarrow W \mid f \text{ } G\text{-morphism}\} \subseteq \text{Hom}_{\mathbb{C}}(V, W)$ gives a \mathbb{C} -vector subspace.

This is clear as by the fact that f is linear, V as a representation is closed under addition and scalar multiplication.

Remark 1.6. Given a finite-dimensional representation $\rho : G \rightarrow \text{GL}(V)$, choosing a basis $\{e_1, \dots, e_n\}$ of V gives us an isomorphism $V \simeq \mathbb{C}^n$, i.e. we have the description of representations in matrices

$$\rho : G \rightarrow \text{GL}(V) \simeq \text{GL}_n(\mathbb{C}), \quad g \mapsto \rho_g = (a_{ij}(g))$$

This implies that two representations are isomorphic if and only if there exists some matrix $A \in \text{GL}_n(\mathbb{C})$ s.t. $(a_{ij}(g)) = A(b_{ij}(g))$. In particular, applying the result twice gives that (with identification of representations and its matrix form) $\rho_g = A\rho'_g A^{-1}$, i.e. conjugate representations are isomorphic. Such morphisms of representations (A) are equivariant.

Definition 1.7 (Sub-representation). Given a representation V of G , a **sub-representation** of V is a vector space $W \subseteq V$ s.t. $gv \in W$ for all $v \in W, g \in G$.

Remark 1.8. In particular, for W a sub-representation of V , it is itself a representation with the map ρ' being $\rho(-)|_W$. The inclusion $W \hookrightarrow V, x \mapsto X$ is a morphism of representation.

2 Interpretation via the Group Algebra

Similar to the case of group action where we interpreted the structure of group action by the group homomorphism $G \rightarrow S_x$, we would like to have some equivalence to structures that are more explicit, and easier to analyze. This introduces the following definitions:

Definition 2.1 (Group Algebra). Let G be a group. Then the **group algebra over \mathbb{C}** , denoted $\mathbb{C}[G]$ is a vector space with a basis $\{\alpha(g) \mid g \in G\}$ in bijection with elements in G (formally). Endow it with a multiplication $\alpha(g) \cdot \alpha(g) = \alpha(gh)$ compatible with the group structure gives the desired ring structure.

Remark 2.2. Verifying the ring axioms, we have the fact that the identity in $\mathbb{C}[G]$ to be $\alpha(e)$. This is in fact a \mathbb{C} -algebra, with the associated morphism given by $\mathbb{C} \rightarrow \mathbb{C}[G]$. Since the image of it are scalars, it is clearly in the center of the group.

Notice that G is not necessarily a finite group. Therefore the vector space can be infinite-dimensional, which we have imposed the requirement that every element should be a finite sum of linear combination of basis. In the following deduction, denote \sum' to be the finite sum.

Proposition 2.3. The group algebra is well-defined.

Proof. This is clear for the cases where G is finite. Consider the case where G is infinite. Then by definition of the group algebra, for all $u, v \in \mathbb{C}[G]$, we have their decomposition into elements in the basis:

$$u = \sum'_{g \in G} a_g \alpha(g), \quad v = \sum'_{g \in G} b_g \alpha(g)$$

Multiplying these two terms together gives

$$u \cdot v = \sum_{g \in G} \left(\sum_{g_1 g_2 = g} (a_{g_1} b_{g_2}) \right) \alpha(g)$$

Furthermore there are only finitely many such a_g s and b_g s being nonzero, implying that there are only finitely many nonzero such products. \square

Notation. If G is abelian, and the correspondence of elements in G and in $\mathbb{C}[G]$ is written additively. Instead of $\alpha(g)$ one usually writes χ^g (with the convention that $\chi^g \cdot \chi^h = \chi^{g+h}$).

Remark 2.4. $\mathbb{C}[G]$ is a commutative ring if and only if G is an abelian group. “Only if” is clear as if $\mathbb{C}[G]$ is commutative implies for all $g, h \in G$, they commute. “If” results from the fact that for every element in $x \in \mathbb{C}[G]$ there exists a scalar λ s.t. $\lambda x = \alpha(g)$ for some $g \in G$ as \mathbb{C} is a field.

Example 2.5. If $G = (\mathbb{Z}, +)$, identifying $x \leftrightarrow \chi^x$ for $x \in \mathbb{Z}$, we have $\mathbb{C}[G] \simeq \bigoplus_{m \in \mathbb{Z}} \mathbb{C} \chi^m \simeq S^{-1} \mathbb{C}[x]$ for $S = \langle x \rangle = \{1, x, x^2, \dots\}$. These are the Laurent Polynomials.

If $G = (\mathbb{Z}/n\mathbb{Z}, +)$, we have the identification $x^n = 1$, giving $\mathbb{C}[G] \simeq \mathbb{C}[x]/(x^n - 1)$.

Proposition 2.6. We have the identification between representations of G and $\mathbb{C}[G]$ -modules. Morphisms and sub-objects (sub-representations and submodules) are also in correspondence.

Proof. It suffices to verify 1), as identifications in 2) and 3) are induced by 1).

Suppose that V is a representation of G , Then V has a structure of $\mathbb{C}[G]$ -module, whose addition is the same as in the vector space, and scalar multiplication is given by

$$\left(\sum'_{g \in G} (a_g \cdot \alpha(g)) \right) \cdot v = \sum'_{g \in G} (a_g \cdot \alpha(gv))$$

Conversely, if M is a $\mathbb{C}[G]$ -module, then it has a vector space structure via considering the action $\mathbb{C} \hookrightarrow \mathbb{C}[G]$ which acts on M ; and the linear map is given by $(g, -)$, where $(g, x) \mapsto \alpha(g) \cdot x$ as specified by the $\mathbb{C}[G]$ module. \square

Remark 2.7. In general, for a representation over a field \mathbb{F} of G , it can be identified with $\mathbb{F}[G]$.

3 Examples of Representations

The following gives some common examples of representations:

- 1) Suppose that G acts on a set X . Let V be the free \mathbb{C} -vector space associated to X , with basis $\{\alpha(u) \mid u \in X\}$ in bijection with X . Define $G \xrightarrow{\rho} \text{GL}(V)$, $g \mapsto \rho_g$, with $\rho_g(\alpha(u)) = \alpha(gu)$. This is the permutation representation associated with X . This is essentially just the group action, as the representation is completely fixed via specifying its behavior on elements in X (i.e. with coefficient 1).

- 2) Example 1) applied to the action of G on itself, $G \times G \rightarrow G$, $(g, h) \mapsto (gh)$ induces a representation $\mathbb{C}[G]$. This is the regular representation of G . Viewed under the context of Proposition 2.6, this is the standard left $\mathbb{C}[G]$ -module structure of itself (rings are left-modules over itself).
- 3) Direct sum of representations. If $\rho_V : G \rightarrow \text{GL}(V)$ and $\rho_W : G \rightarrow \text{GL}(W)$ are representations of G , then we can get a representation $\rho : G \rightarrow \text{GL}(V \oplus W)$, given by

$$\rho_g = (\rho_g^V, \rho_g^W) : G \times (V \oplus W) \rightarrow (V \oplus W), \quad (g, v \oplus w) \mapsto (gv, gw)$$

Under the context of Proposition 2.6, this corresponds to the direct sum of modules.

- 4) Tensor product of representations. Suppose that we have $\rho : G \rightarrow \text{GL}(V)$ and $\rho' : G \rightarrow \text{GL}(V')$ two representations of G . Then we can have

$$\tilde{\rho} = \rho \otimes \rho' : G \rightarrow \text{GL}(V \otimes_{\mathbb{C}} V'), \quad g \mapsto (\rho_g \otimes \rho'_g)$$

This is indeed a group homomorphism, as tensor product of maps behave functorially. That is, it commutes with composition of maps by the universal property of tensor product:

$$(f \otimes g) \circ (f' \otimes g') = (f \circ f') \otimes (g \circ g')$$

4 Irreducible Representations

5 Character Theory