Value-Based Reinforcement Learning

Shusen Wang

Action-Value Functions

Discounted Return

Definition: Discounted return (aka cumulative discounted future reward).

- $R_t = r_t + \gamma \, r_{t+1} + \gamma^2 \, r_{t+2} + \gamma^3 \, r_{t+3} + \cdots$ (to infinity.)

- The rewards depend on actions $a_t, a_{t+1}, a_{t+2}, a_{t+3}, \cdots$ and states $s_{t+1}, s_{t+2}, s_{t+3}, \cdots$
- Observing s_t, the future rewards are random.
- Actions are randomly sampled: $a_t \sim \pi(\cdot \mid s_t)$. (Policy function.)
- States are randomly sampled: $s_{t+1} \sim p(\cdot \mid s_t, a_t)$. (State transition.)

Action-Value Function $Q(s, a)$

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Definition: Action-value function for policy π.

- $Q_\pi(s_t, a_t) = \mathbb{E}\,[R_t | s_t, a_t, \pi]$.

  - Taken w.r.t. actions $a_{t+1}, a_{t+2}, a_{t+3}, \cdots$ and states $s_{t+1}, s_{t+2}, s_{t+3}, \cdots$
  - Actions are randomly sampled: $a_t \sim \pi(\cdot \,| s_t)$. (Policy function.)
  - States are randomly sampled: $s_{t+1} \sim p(\cdot \,| s_t, a_t)$. (State transition.)

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Definition: Action-value function for policy π.

- $Q_\pi(s_t, a_t) = \mathbb{E}\left[R_t | s_t, a_t, \pi\right]$.

Definition: Optimal action-value function.

- $Q^\star(s_t, a_t) = \max_\pi Q_\pi(s_t, a_t)$.

Deep Q-Network (DQN)

Approximate the Q Function

Goal: Win the game (≈ maximize the discounted return.)

Question: If we know $Q^\star(s, a)$, what is the best action?

- Obviously, the best action is $a^\star = \underset{a}{\operatorname{argmax}}\, Q^\star(s, a)$.

Approximate the Q Function

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Question: If we know $Q^\star(s, a)$, what is the best action?

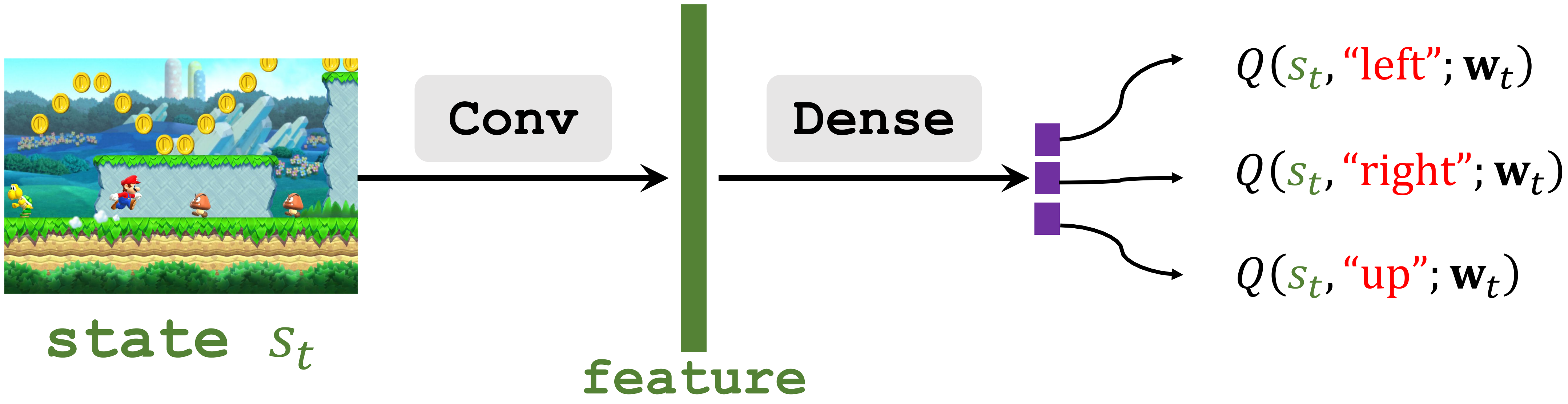
- Obviously, the best action is $a^\star = \underset{a}{\operatorname{argmax}}\, Q^\star(s, a)$.

Challenge: We do not know $Q^\star(s, a)$.

- **Solution:** Use a neural network to approximate $Q^\star(s, a)$.
- Let $Q(s, a; \mathbf{w})$ be a neural network parameterized by $\mathbf{w}$.
- The inputs are state and action; the output is the approximate $Q^\star$.

Deep Q Network (DQN)

- Input shape: size of the screenshot.
- Output shape: dimension of action space.



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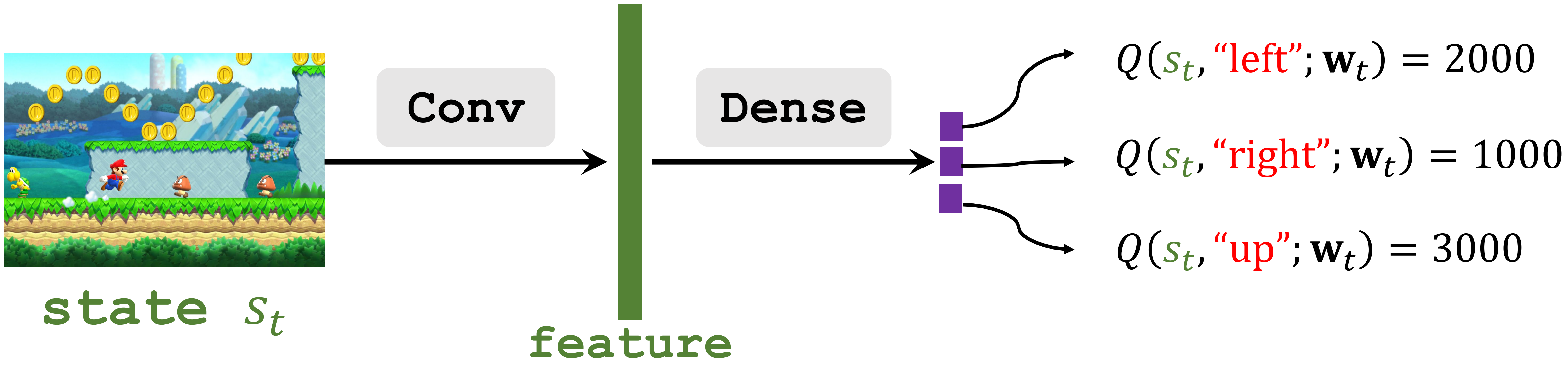

Question: Based on the predictions, what should be the action?

Apply DQN to Play Game

$s_t \longrightarrow a_t$

$a_t = \text{argmax}_a \, Q(s_t, a; \mathbf{w})$

Apply DQN to Play Game

$$s_{t+1} \sim p(\cdot \mid s_t, a_t)$$

$$\begin{tikzcd} s_t \arrow[r] & a_t \arrow[r] \arrow[dr] & s_{t+1} \\ & & r_t \end{tikzcd}$$

$$a_t = \mathrm{argmax}_a \, Q(s_t, a; \mathbf{w})$$

Apply DQN to Play Game

$$s_{t+1} \sim p(\cdot \mid s_t, a_t)$$

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Apply DQN to Play Game

$s_{t+1} \sim p(\cdot \mid s_t, a_t)$

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$$
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Apply DQN to Play Game

$$s_{t+1} \sim p(\cdot \mid s_t, a_t) \qquad s_{t+2} \sim p(\cdot \mid s_{t+1}, a_{t+1})$$

$$\begin{tikzcd}
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$$a_t = \mathrm{argmax}_a\, Q(s_t, a; \mathbf{w})$$

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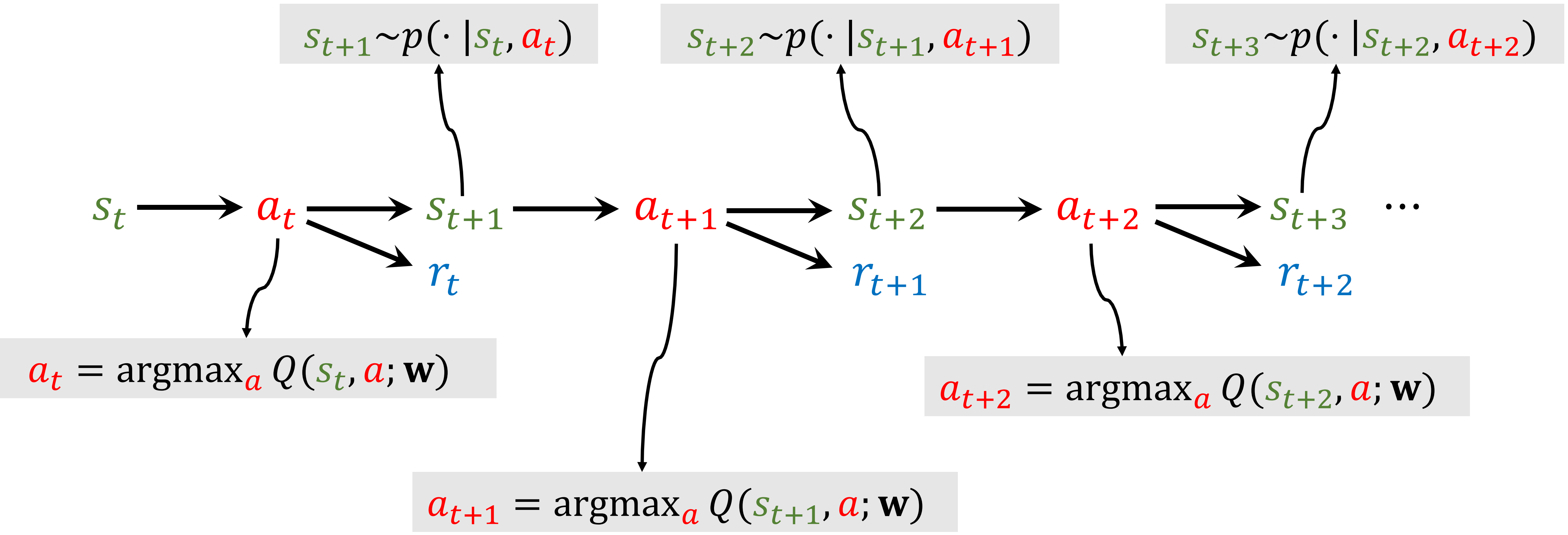

How to train DQN?

Naïve approach:

1. Play a game to the end (totally T steps.)
2. Make predictions using DQN at every step. Get

$$Q(s_1, a_1; \mathbf{w}),\ Q(s_2, a_2; \mathbf{w}),\ \cdots,\ Q(s_T, a_T; \mathbf{w}).$$

3. Observe rewards: $r_1, r_2, \cdots, r_T$.
4. Compute returns: $R_1, R_2, \cdots, R_T$.

- $R_t = r_t + \gamma \cdot r_{t+1} + \cdots + \gamma^{T-t} \cdot r_T$.
- R_t is unknown until the game ends.

How to train DQN?

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3. Observe rewards: $r_1, r_2, \cdots, r_T$.
4. Compute returns: $R_1, R_2, \cdots, R_T$.
5. $\text{loss} = \sum_{t=1}^{T} [Q(s_t, a_t; \mathbf{w}) - R_t]^2$.
6. Use $\frac{\partial \, \text{loss}}{\partial \, \mathbf{w}}$ to update $\mathbf{w}$.

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Naïve approach:

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$$Q(s_1, a_1; \mathbf{w}),\ Q(s_2, a_2; \mathbf{w}),\ \cdots,\ Q(s_T, a_T; \mathbf{w}).$$
3. Observe rewards: $r_1, r_2, \cdots, r_T$.
4. Compute returns: $R_1, R_2, \cdots, R_T$.
5. Loss: $L = \sum_{t=1}^{T} [Q(s_t, a_t; \mathbf{w}) - R_t]^2$.
6. Use $\frac{\partial L}{\partial \mathbf{w}}$ to update $\mathbf{w}$.

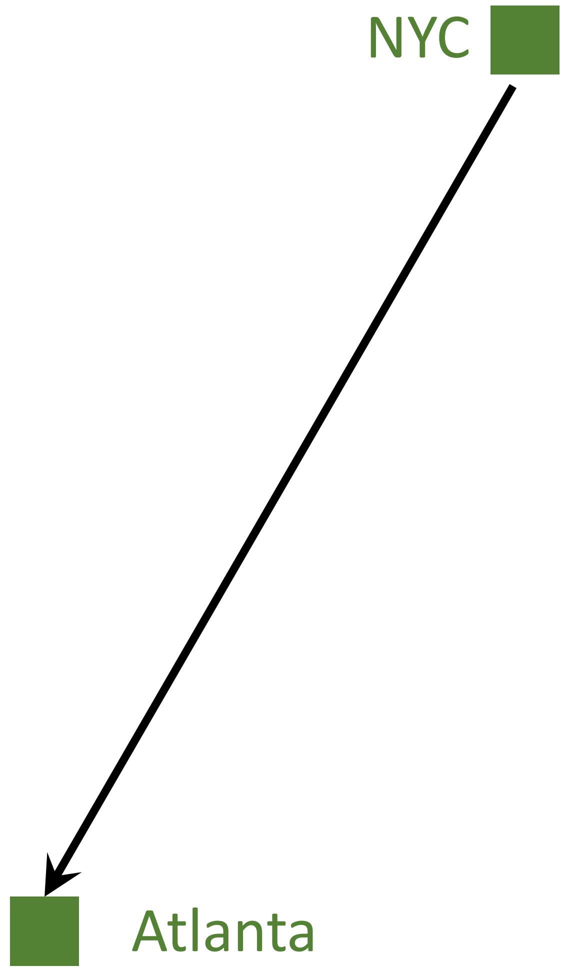
Problem: What if the game takes long to end or does not end at all?

Temporal Difference (TD) Learning

Example

- I want to drive from NYC to Atlanta.
- Model $Q(\mathbf{w})$ estimates the time cost, e.g., 1000 minutes.

Question: How do I update the model?

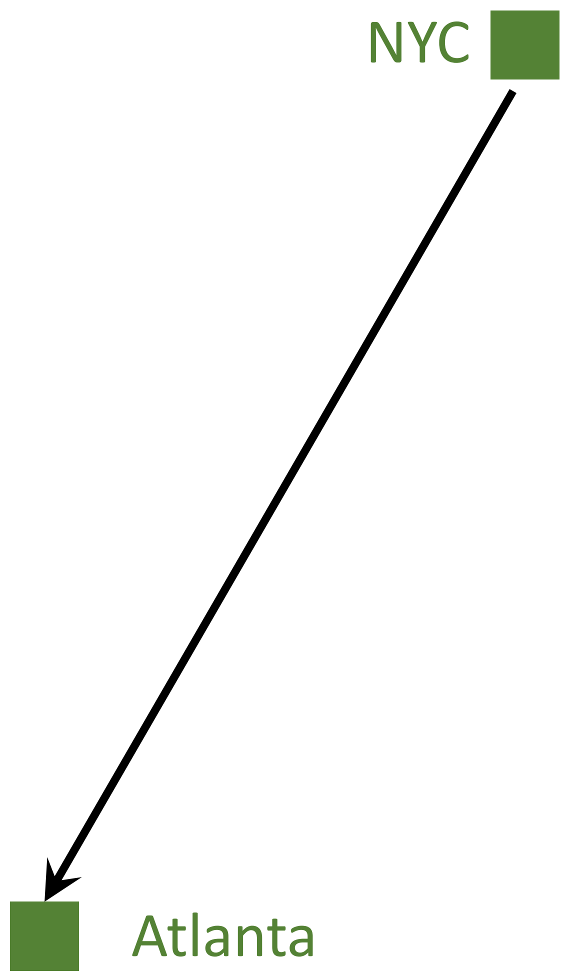


Example

- I want to drive from NYC to Atlanta.
- Model $Q(\mathbf{w})$ estimates the time cost, e.g., 1000 minutes.

Question: How do I update the model?

- Make a prediction: $q = Q(\mathbf{w})$, e.g., $q = 1000$.
- Finish the trip and get the target y, e.g., $y = 860$.
- Loss: $L = \frac{1}{2}(q - y)^2$.
- Gradient: $\frac{\partial\, L}{\partial\, \mathbf{w}} = \frac{\partial\, q}{\partial\, \mathbf{w}} \cdot \frac{\partial\, L}{\partial\, x} = (q - y) \cdot \frac{\partial\, Q(\mathbf{w})}{\partial\, \mathbf{w}}$.
- Gradient descent: $\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \cdot \frac{\partial\, L}{\partial\, \mathbf{w}} \Big|_{\mathbf{w}=\mathbf{w}_t}$.

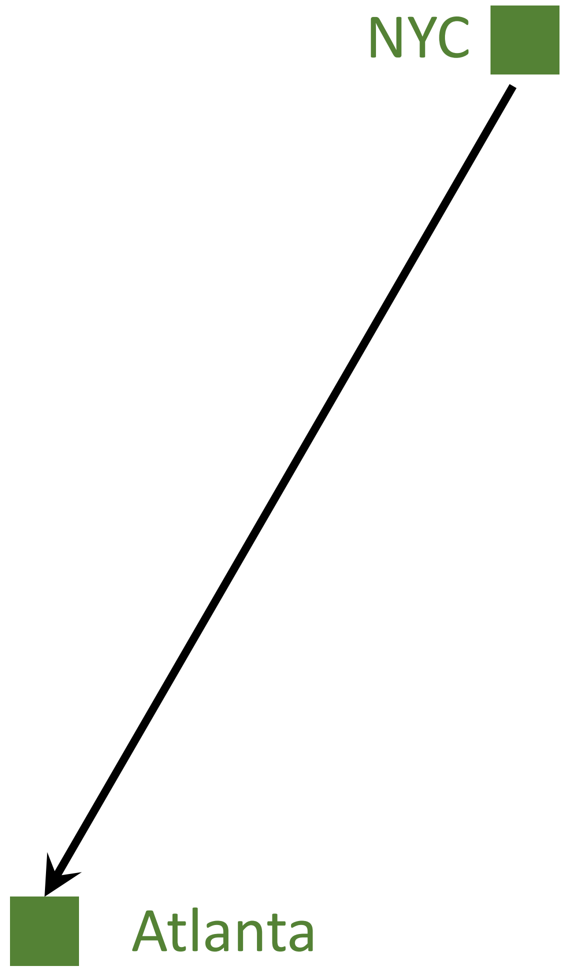


Example

- I want to drive from NYC to Atlanta.
- Model $Q(\mathbf{w})$ estimates the time cost, e.g., 1000 minutes.

Question: How do I update the model?

- Can I update the model before finishing the trip?

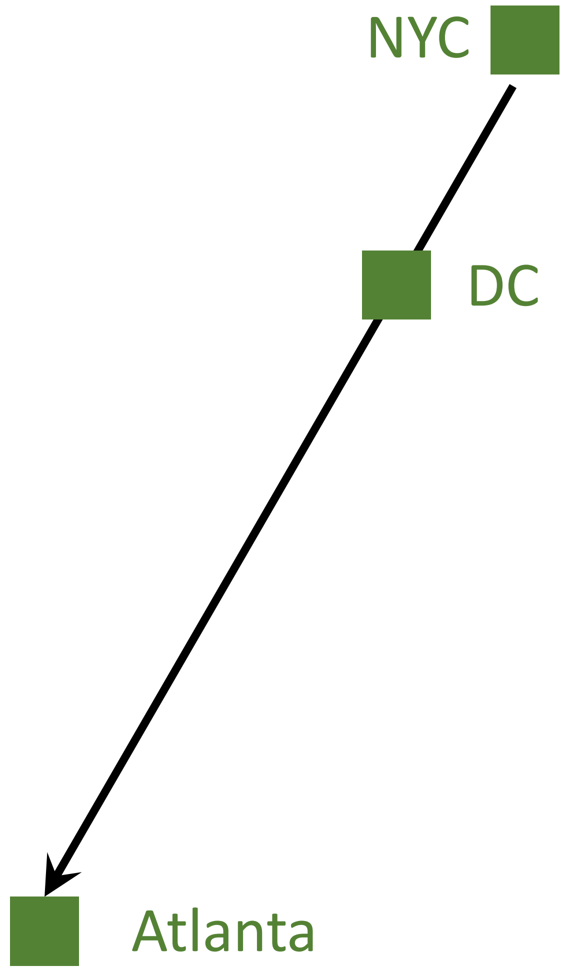


Example

- I want to drive from NYC to Atlanta (via DC).
- Model $Q(\mathbf{w})$ estimates the time cost, e.g., 1000 minutes.

Question: How do I update the model?

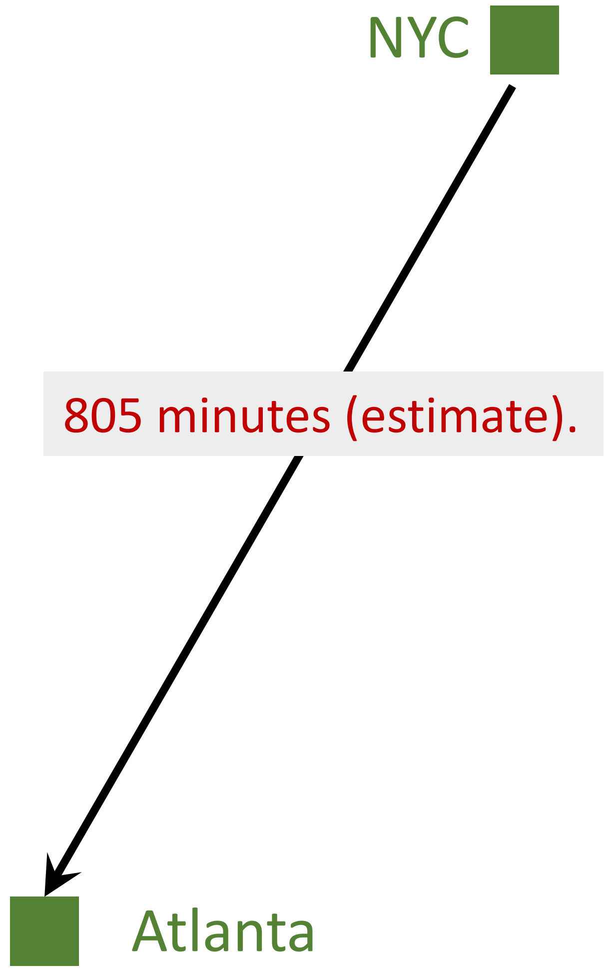
- Can I update the model before finishing the trip?
- Can I get a better $\mathbf{w}$ as soon as I arrived DC?



Temporal Difference (TD) Learning

- Model's estimate:

  NYC to Atlanta: 805 minutes (estimate).



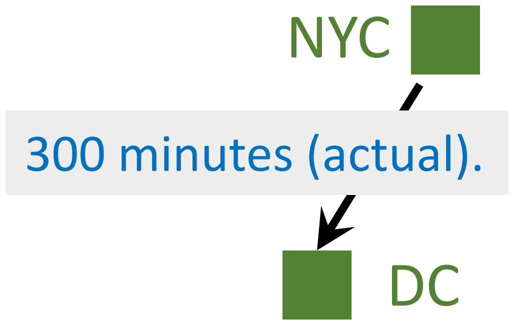
Temporal Difference (TD) Learning

- Model's estimate:

  NYC to Atlanta: 805 minutes (estimate).

- I arrived at DC; actual time cost:

  NYC to DC: 300 minutes (actual).



Temporal Difference (TD) Learning

- Model's estimate:

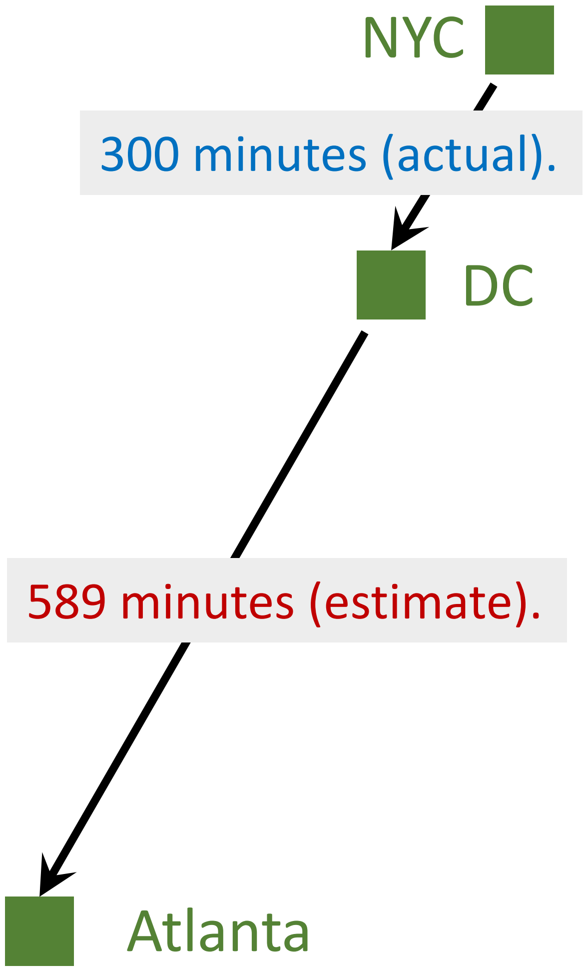
  NYC to Atlanta: 805 minutes (estimate).

- I arrived at DC; actual time cost:

  NYC to DC: 300 minutes (actual).

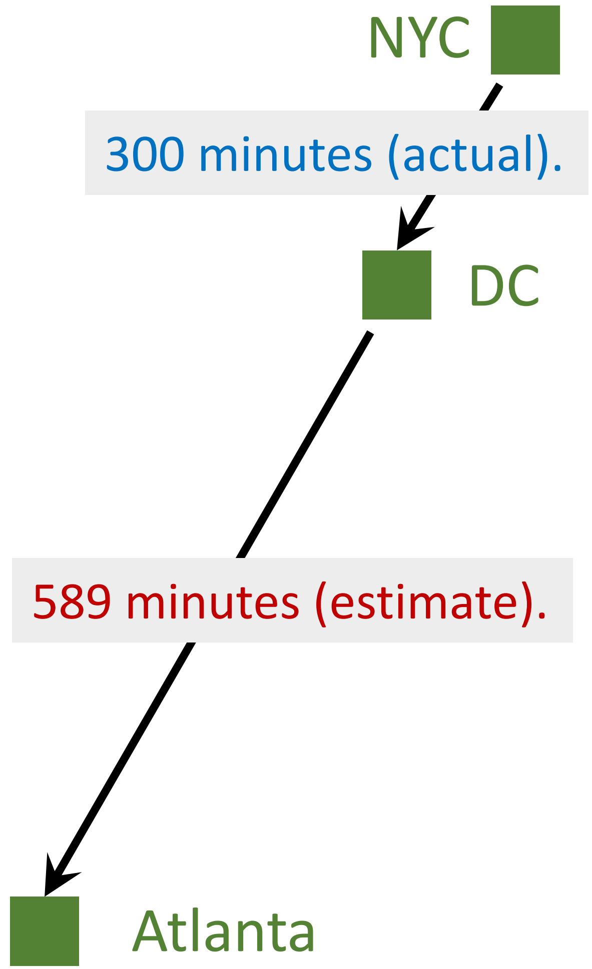
- Model now updates its estimate:

  DC to Atlanta: 589 minutes (estimate).



Temporal Difference (TD) Learning

- Model's estimate: $Q(\mathbf{w}) = 805$ minutes .
- Updated estimate: $\underbrace{300 + 589 = 889}_{\text{TD target.}}$ minutes.

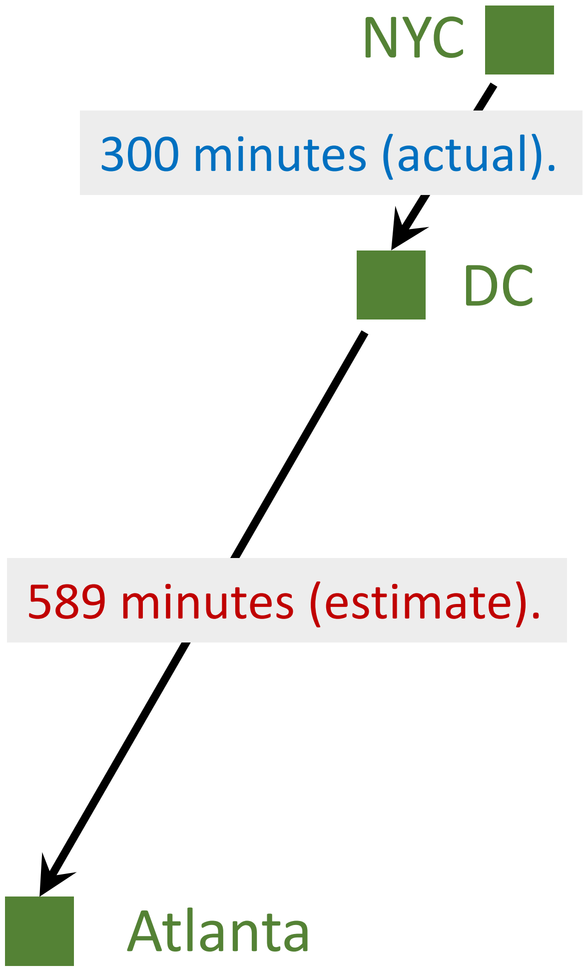


Temporal Difference (TD) Learning

- Model's estimate: $Q(\mathbf{w}) = 805$ minutes .
- Updated estimate: $300 + 589 = 889$ minutes.

TD target.

- TD target $\mathrm{y} = 889$ is a more reliable estimate than 805.



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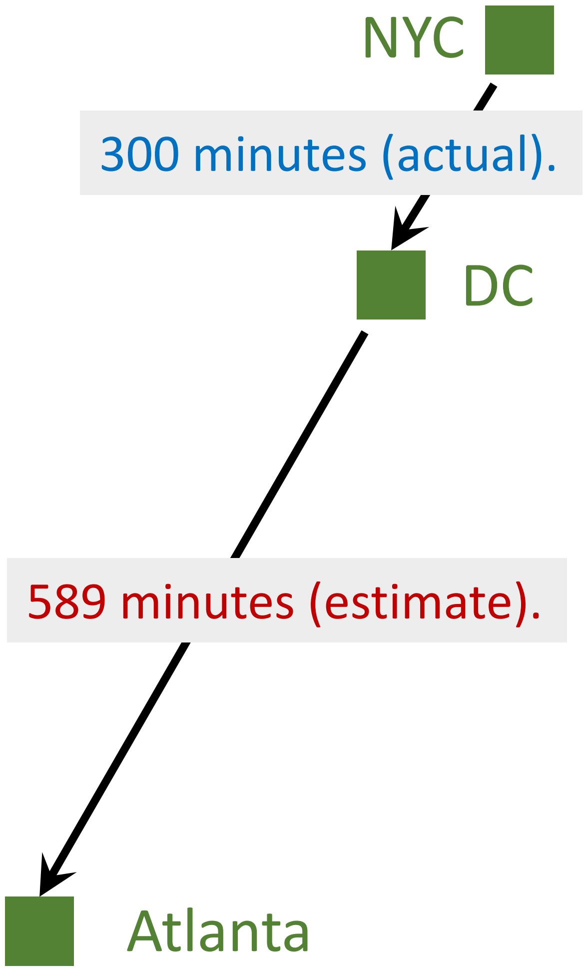
- TD target $y = 889$ is a more reliable estimate than 805.
- Loss: $L = \frac{1}{2}\underbrace{(Q(\mathbf{w}) - y)}_{\text{TD error}}^2$.



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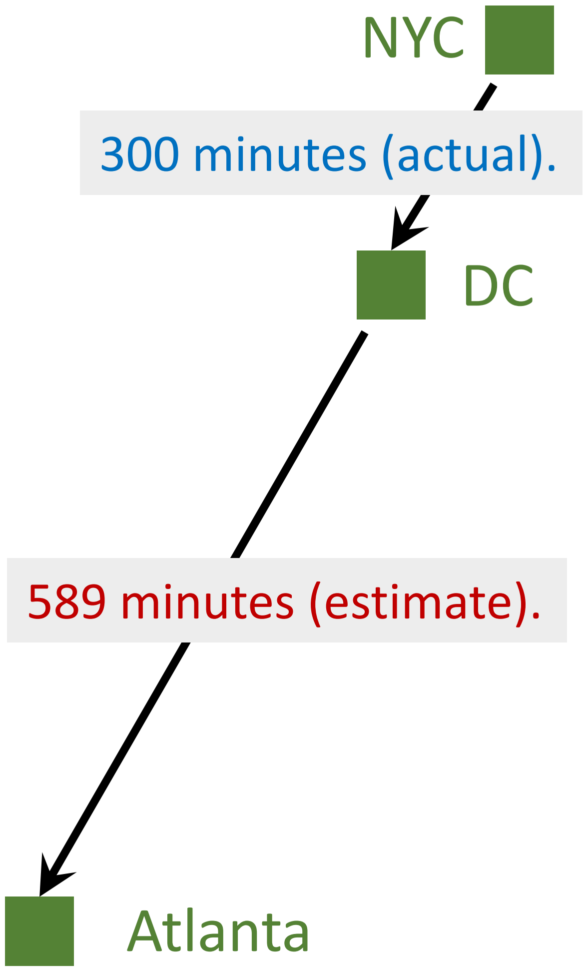
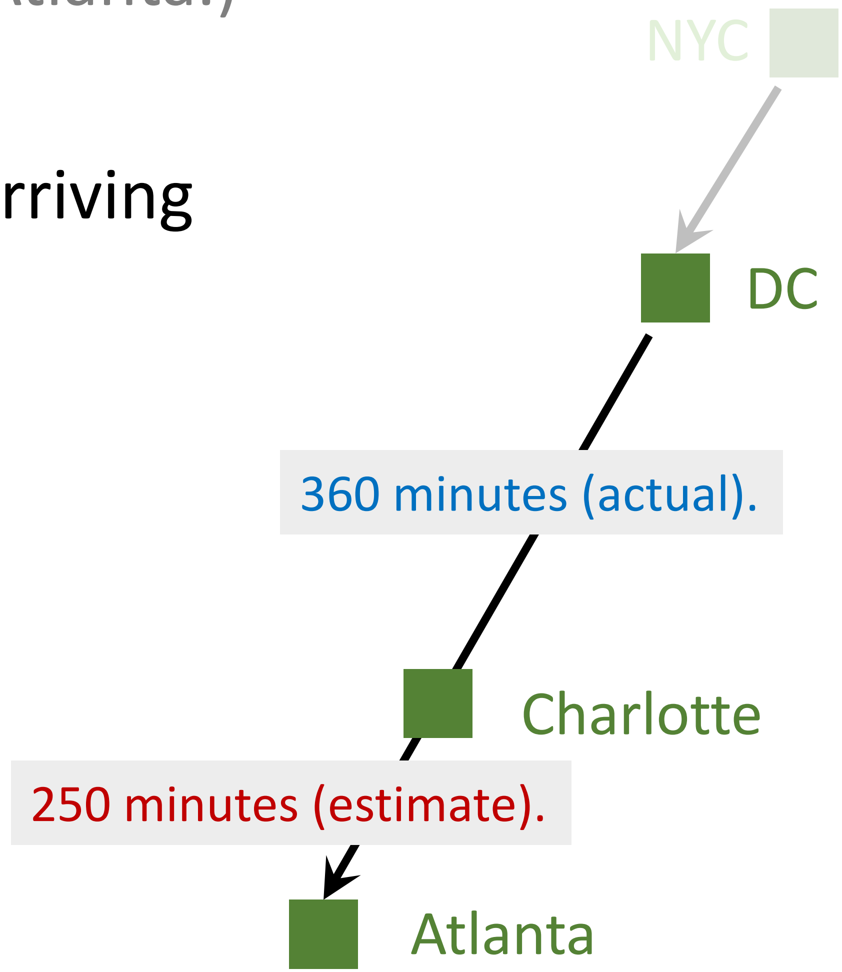
- TD target $y = 889$ is a more reliable estimate than 805.
- Loss: $L = \frac{1}{2}\left(Q(\mathbf{w}) - y\right)^2$.
- Gradient: $\frac{\partial\, L}{\partial\, \mathbf{w}} = (805 - 889) \cdot \frac{\partial\, Q(\mathbf{w})}{\partial\, \mathbf{w}}$.
- Gradient descent: $\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \cdot \frac{\partial\, L}{\partial\, \mathbf{w}} \Big|_{\mathbf{w}=\mathbf{w}_t}$.



Temporal Difference (TD) Learning

- Model's estimate: $Q(\mathbf{w}) = 589$ minutes (DC to Atlanta.)

- Continue driving. Update the model again upon arriving at Charlotte.

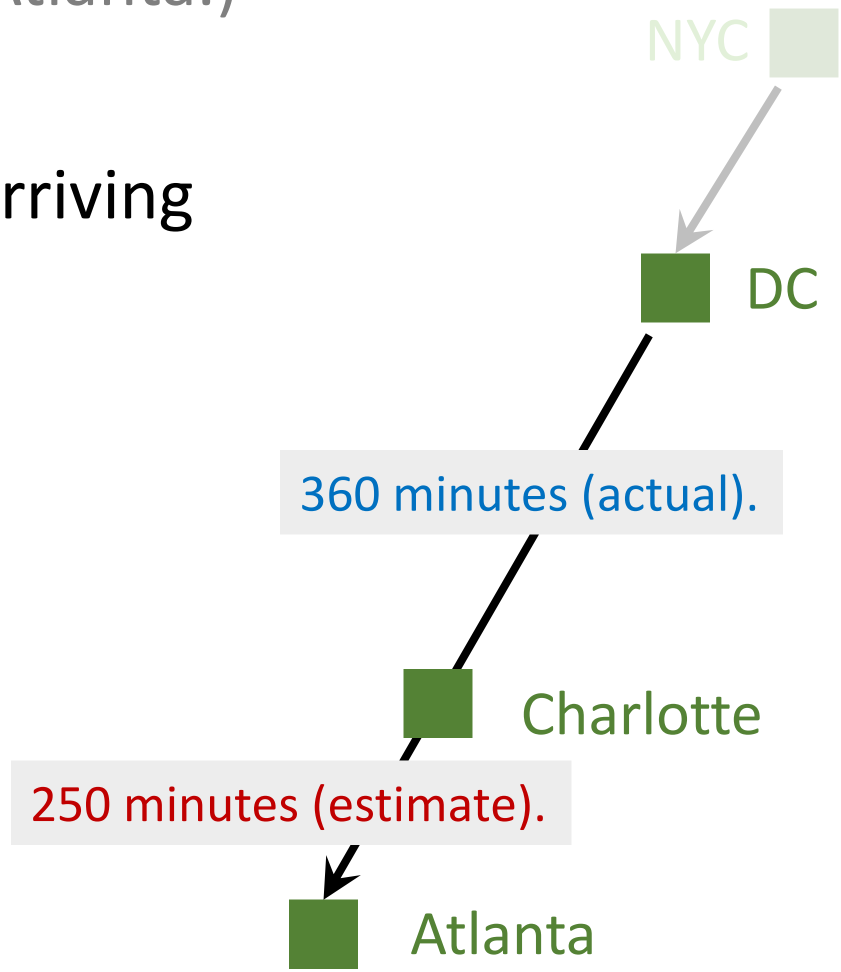


Temporal Difference (TD) Learning

- Model's estimate: $Q(\mathbf{w}) = 589$ minutes (DC to Atlanta.)

- Continue driving. Update the model again upon arriving at Charlotte.
- I arrive at Charlotte, I have the new TD target:

$$\underbrace{360}_{\text{Actual time}} + \underbrace{250}_{\text{Updated estimate}} = \underbrace{610}_{\text{TD target.}} \text{ minutes.}$$



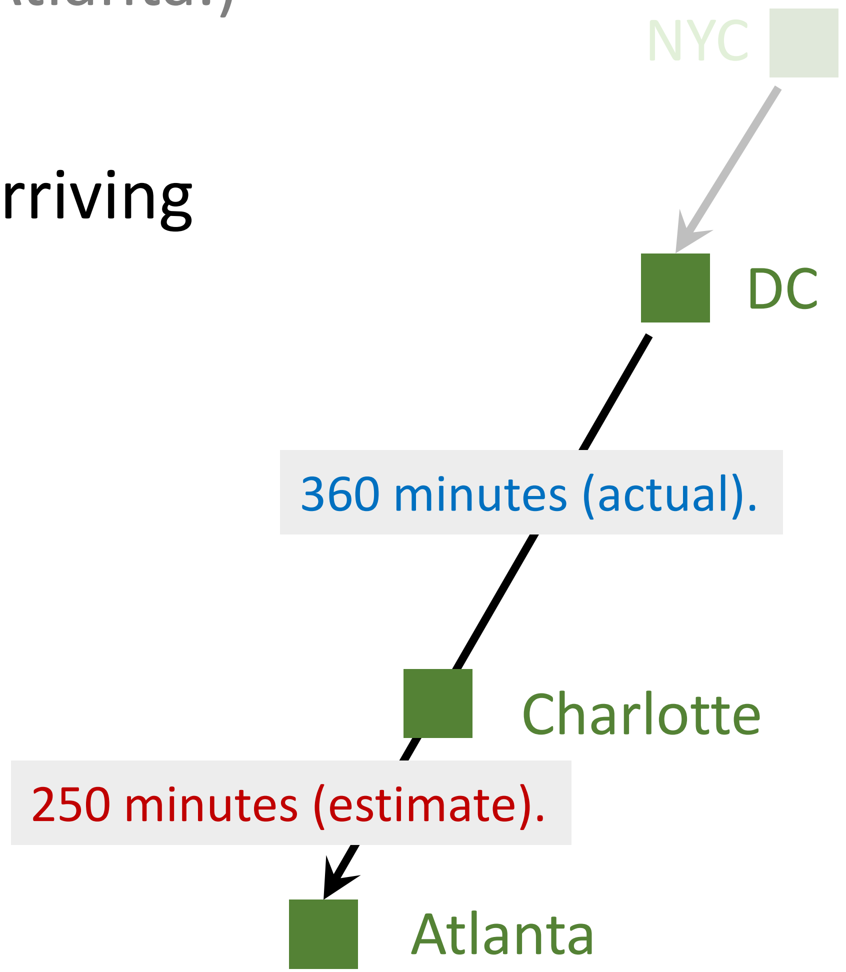
Temporal Difference (TD) Learning

- Model's estimate: $Q(\mathbf{w}) = 589$ minutes (DC to Atlanta.)

- Continue driving. Update the model again upon arriving at Charlotte.
- I arrive at Charlotte, I have the new TD target:

$$360 + 250 = 610 \text{ minutes.}$$

- Loss: $L = \frac{1}{2}(Q(\mathbf{w}) - 610)^2$.
- Update the model again using gradient descent.



TD Learning for DQN

Apply DQN to Play Game

$$
\begin{tikzcd}
s_t \arrow[r] & a_t \arrow[r] \arrow[dr] & s_{t+1} \\
 & & r_t
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$$

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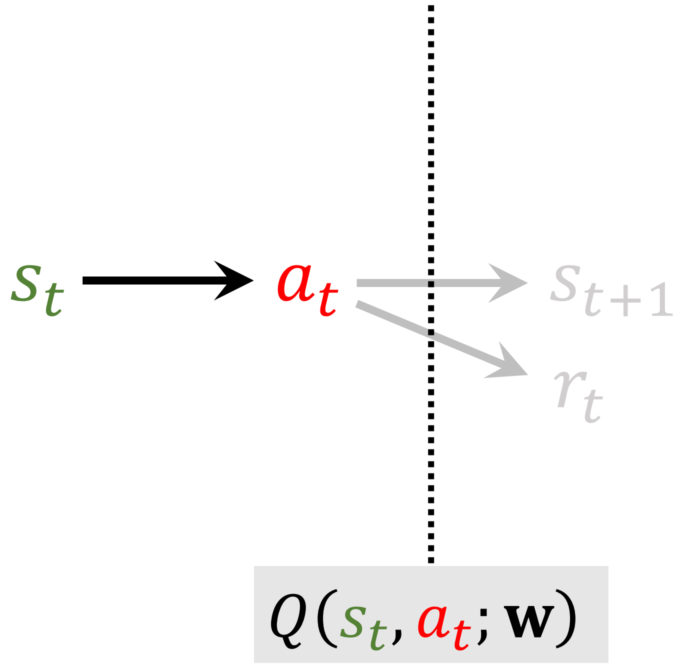


Apply DQN to Play Game

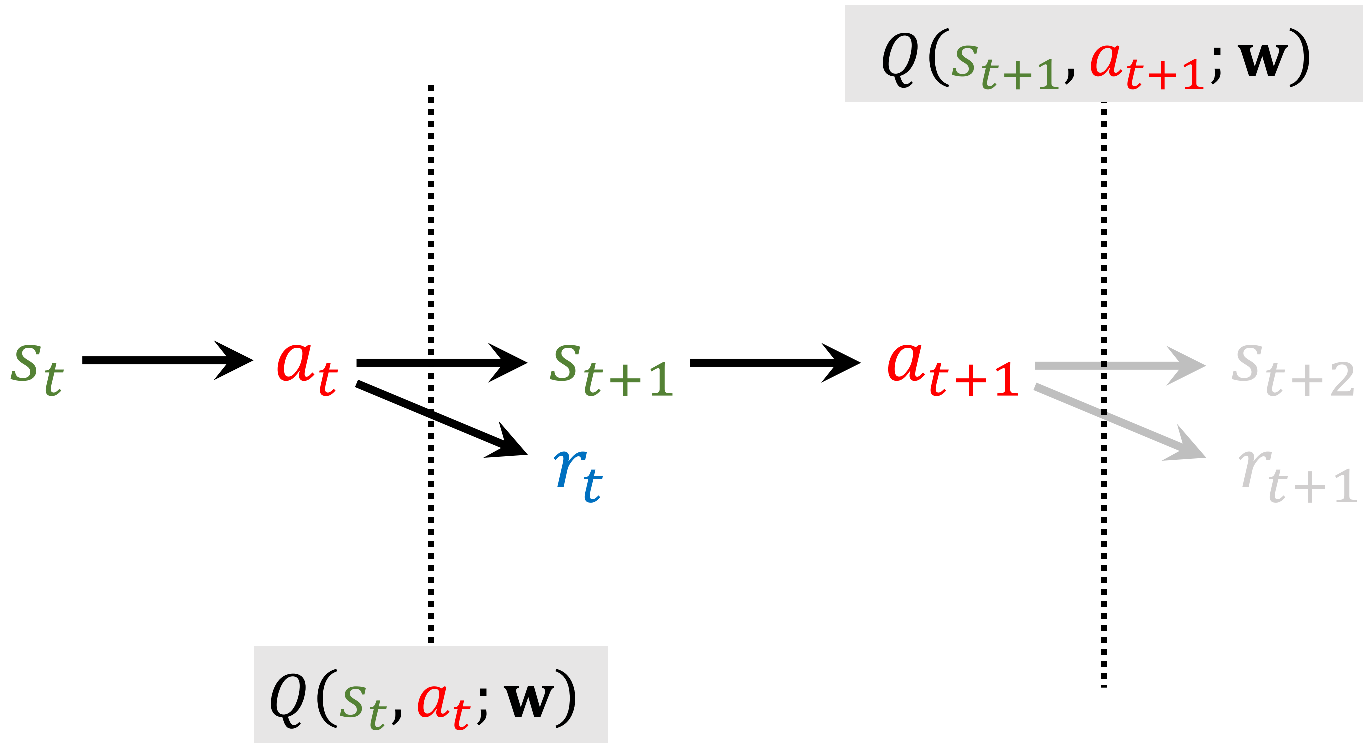
$$s_t \longrightarrow a_t \longrightarrow s_{t+1} \longrightarrow a_{t+1} \longrightarrow s_{t+2}$$

$$r_t \qquad\qquad r_{t+1}$$

$$Q(s_t, a_t; \mathbf{w})$$

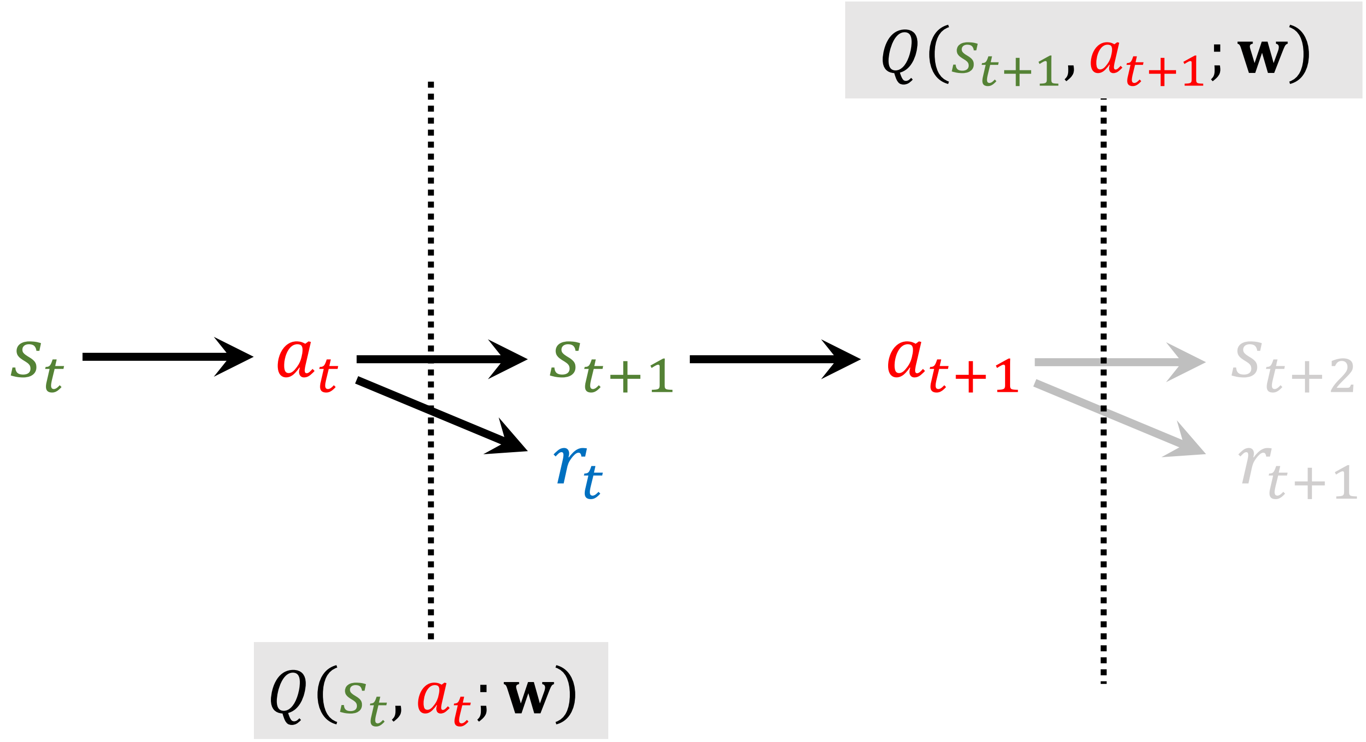
$$a_{t+1} = \mathrm{argmax}_a \, Q(s_{t+1}, a; \mathbf{w})$$

Apply DQN to Play Game



Train DQN using TD learning

If it is accurate estimate, then $Q(s_{t+1}, a_{t+1}; \mathbf{w}) = \mathbb{E}[\underbrace{r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \cdots}_{\text{Discounted return: } R_{t+1}}]$



If it is accurate estimate, then $Q(s_t, a_t; \mathbf{w}) = \mathbb{E}[\underbrace{r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \cdots}_{\text{Discounted return: } R_t}]$

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Fact: $R_t = r_t + \gamma \cdot R_{t+1}.$

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$$
\begin{aligned}
Q(s_t, a_t; \mathbf{w}) &= r_t + \gamma \cdot Q(s_{t+1}, a_{t+1}; \mathbf{w}) \\
&= r_t + \gamma \cdot \max_a Q(s_{t+1}, a; \mathbf{w})
\end{aligned}
$$

If it is accurate estimate, then $Q(s_t, a_t; \mathbf{w}) = \mathbb{E}[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \cdots]$

Train DQN using TD learning

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$$Q(s_t, a_t; \mathbf{w}) = r_t + \gamma \cdot \max_a Q(s_{t+1}, a; \mathbf{w})$$

Train DQN using TD learning

If DQN is accurate estimate, then

$$\underbrace{Q(s_t, a_t; \mathbf{w})}_{\text{Old estimate (less reliable)}} = \underbrace{r_t + \gamma \cdot \max_a Q(s_{t+1}, a; \mathbf{w})}_{\text{TD target (more reliable estimate of the value)}}$$

Train DQN using TD learning

If DQN is accurate estimate, then

$$Q(s_t, a_t; \mathbf{w}) = r_t + \gamma \cdot \max_a Q(s_{t+1}, a; \mathbf{w})$$

- TD target: $y_t = r_t + \gamma \cdot \max_a Q(s_{t+1}, a; \mathbf{w}_t)$.
- Loss: $L_t = \frac{1}{2}[Q(s_t, a_t; \mathbf{w}) - y_t]^2$.
- Gradient descent: $\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \cdot \frac{\partial L_t}{\partial \mathbf{w}} \Big|_{\mathbf{w}=\mathbf{w}_t}$

Summary

Action-Value Function Approximation

Definition: Optimal action-value function.

- $Q^{\star}(s, a) = \max_{\pi} \mathbb{E}\left[R_t | s, a, \pi\right].$

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DQN: Approximate $Q^{\star}(s, a)$ using a neural network (DQN).

- $Q(s, a; \mathbf{w})$ is a neural network parameterized by $\mathbf{w}$.
- Input: observed state s (e.g., a screenshot of game.)
- Output: a vector, each entry of which corresponds to an action a.

Temporal Difference (TD) Learning

Algorithm: One iteration of TD learning.

1. Observe state s_t and action a_t.
2. Predict the value: $q_t = Q(s_t, a_t; \mathbf{w}_t)$.
3. Differentiate the value network: $\mathbf{d}_t = \frac{\partial\, Q(s_t, a_t; \mathbf{w})}{\partial\, \mathbf{w}} \Big|_{\mathbf{w}=\mathbf{w}_t}$.

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4. Environment provides new state s_{t+1} and reward r_t.
5. Compute TD target: $y_t = r_t + \gamma \cdot \max_{a} Q(s_{t+1}, a; \mathbf{w}_t)$.

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5. Compute TD target: $y_t = r_t + \gamma \cdot \max_a Q(s_{t+1}, a; \mathbf{w}_t)$.
6. Gradient descent: $\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \cdot (q_t - y_t) \cdot \mathbf{d}_t$.