Value-Based Reinforcement Learning

Shusen Wang

Value Functions

Action-Value Function $Q(s, a)$

Definition: Discounted return (aka cumulative discounted future reward).

- $R_t = r_t + \gamma\, r_{t+1} + \gamma^2\, r_{t+2} + \gamma^3\, r_{t+3} + \cdots$ (to infinity.)

Definition: Action-value function.

- $Q_\pi(s, a) = \mathbb{E}\left[R_t | s, a, \pi\right]$.

  - Taken w.r.t. the randomness in the state transition $p(s'|s, a)$.
  - The state transition $(s_t, a_t) \mapsto s_{t+1}$ is random.

Action-Value Function $Q(s, a)$

Definition: Discounted return (aka cumulative discounted future reward).

- $R_t = r_t + \gamma\, r_{t+1} + \gamma^2\, r_{t+2} + \gamma^3\, r_{t+3} + \cdots$ (to infinity.)

Definition: Action-value function.

- $Q_\pi(s, a) = \mathbb{E}\left[R_t | s, a, \pi\right]$.

Definition: Optimal action-value function.

- $Q^\star(s, a) = \max_\pi Q_\pi(s, a)$.

Deep Q-Network (DQN)

Approximate the Q Function

Goal: Win the game (≈ maximize the total reward.)

Question: If we know $Q^\star(s, a)$, what is the best action?

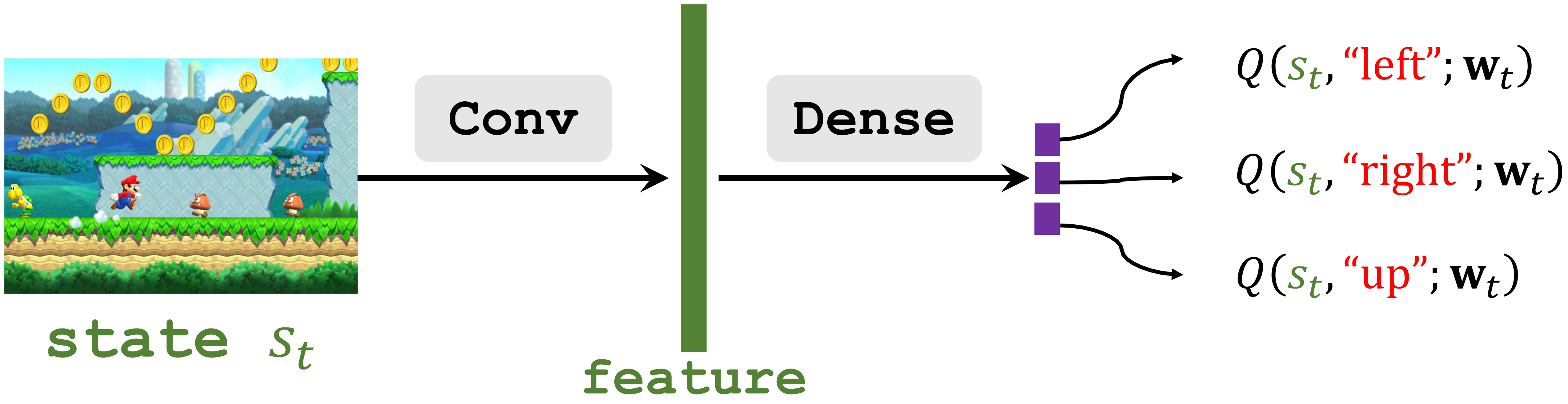
- Obviously, the best action is $a^\star = \underset{a}{\operatorname{argmax}}\, Q(s, a)$.

Challenge: We do not know $Q^\star(s, a)$.

- **Solution:** Use a deep neural network to approximate $Q^\star(s, a)$.
- Let $Q(s, a; \mathbf{w})$ be a neural network parameterized by $\mathbf{w}$.
- The inputs are state and action; the output is the approximate $Q^\star$.

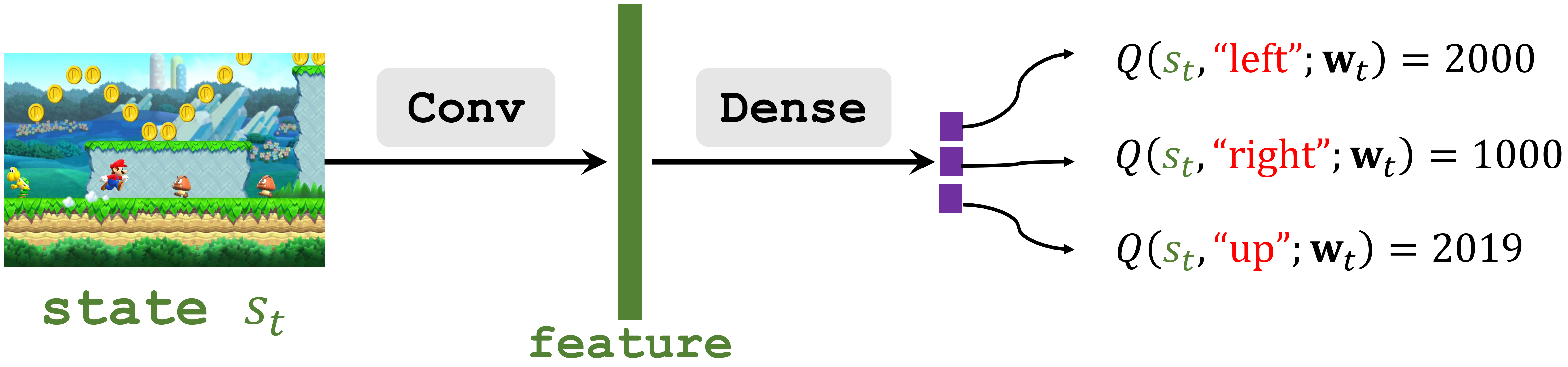
Deep Q Network (DQN)

- Input shape: size of the screenshot.
- Output shape: dimension of action space.



Deep Q Network (DQN)

- Input shape: size of the screenshot.
- Output shape: dimension of action space.



Question: Based on the predictions, what should be the action?

Apply DQN to Play Game

$s_t \longrightarrow a_t$

$$a_t = \mathrm{argmax}_a\, Q(s_t, a; \mathbf{w})$$

Apply DQN to Play Game

$$s_{t+1} \sim p(\cdot \mid s_t, a_t)$$

$$\begin{array}{ccccc} s_t & \longrightarrow & a_t & \longrightarrow & s_{t+1} \\ & & & \searrow & r_t \end{array}$$

$$a_t = \mathrm{argmax}_a \, Q(s_t, a; \mathbf{w})$$

Apply DQN to Play Game

$$s_{t+1} \sim p(\cdot \mid s_t, a_t)$$

$$\begin{tikzcd} s_t \arrow[r] & a_t \arrow[r] \arrow[dr] & s_{t+1} \arrow[r] & a_{t+1} \\ & & r_t & \end{tikzcd}$$

$$a_t = \text{argmax}_a \, Q(s_t, a; \mathbf{w})$$

$$a_{t+1} = \text{argmax}_a \, Q(s_{t+1}, a; \mathbf{w})$$

Apply DQN to Play Game

$s_{t+1} \sim p(\cdot \mid s_t, a_t)$

$s_{t+2} \sim p(\cdot \mid s_{t+1}, a_{t+1})$

$$s_t \rightarrow a_t \rightarrow s_{t+1} \rightarrow a_{t+1} \rightarrow s_{t+2}$$

$$a_t \rightarrow r_t \qquad a_{t+1} \rightarrow r_{t+1}$$

$a_t = \text{argmax}_a \, Q(s_t, a; \mathbf{w})$

$a_{t+1} = \text{argmax}_a \, Q(s_{t+1}, a; \mathbf{w})$

Apply DQN to Play Game

$$s_{t+1} \sim p(\cdot \mid s_t, a_t) \qquad s_{t+2} \sim p(\cdot \mid s_{t+1}, a_{t+1})$$

$$\begin{tikzcd}
s_t \arrow[r] & a_t \arrow[r] \arrow[dr] & s_{t+1} \arrow[r] & a_{t+1} \arrow[r] \arrow[dr] & s_{t+2} \arrow[r] & a_{t+2} \\
 & & r_t & & r_{t+1} &
\end{tikzcd}$$

$$a_t = \text{argmax}_a \, Q(s_t, a; \mathbf{w})$$

$$a_{t+1} = \text{argmax}_a \, Q(s_{t+1}, a; \mathbf{w})$$

$$a_{t+2} = \text{argmax}_a \, Q(s_{t+2}, a; \mathbf{w})$$

Apply DQN to Play Game

$$s_{t+1} \sim p(\cdot \mid s_t, a_t) \qquad s_{t+2} \sim p(\cdot \mid s_{t+1}, a_{t+1}) \qquad s_{t+3} \sim p(\cdot \mid s_{t+2}, a_{t+2})$$

$$s_t \rightarrow a_t \rightarrow s_{t+1} \rightarrow a_{t+1} \rightarrow s_{t+2} \rightarrow a_{t+2} \rightarrow s_{t+3} \quad \cdots$$

$$a_t \rightarrow r_t \qquad a_{t+1} \rightarrow r_{t+1} \qquad a_{t+2} \rightarrow r_{t+2}$$

$$a_t = \text{argmax}_a\, Q(s_t, a; \mathbf{w})$$

$$a_{t+2} = \text{argmax}_a\, Q(s_{t+2}, a; \mathbf{w})$$

$$a_{t+1} = \text{argmax}_a\, Q(s_{t+1}, a; \mathbf{w})$$

Temporal Difference (TD) Learning

Temporal Difference (TD) Learning

- I want to drive from NYC to Atlanta.
- Model $Q(\mathbf{w})$ estimate the time cost, e.g., 1000 minutes.

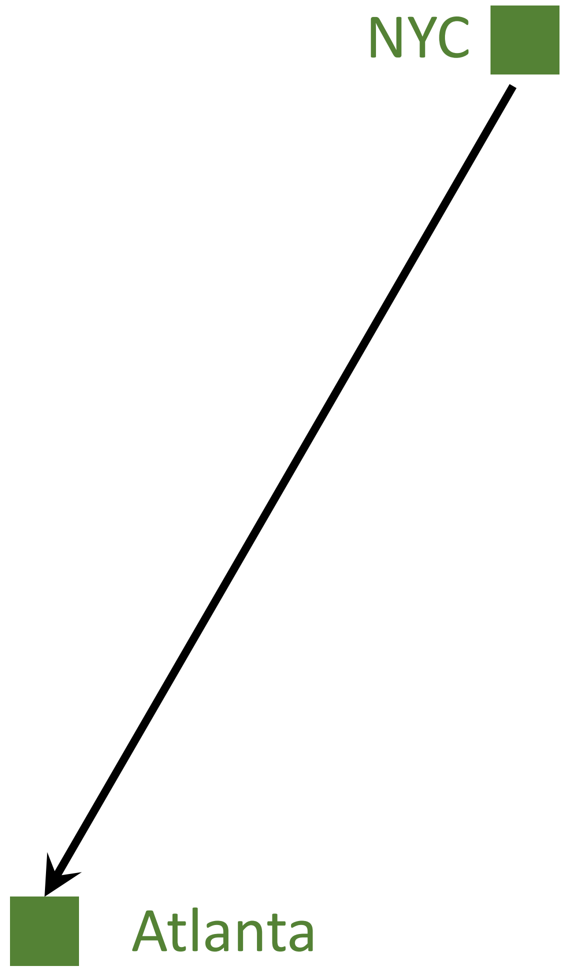
Question: How do I update the model?

Temporal Difference (TD) Learning

- I want to drive from NYC to Atlanta.
- Model $Q(\mathbf{w})$ estimate the time cost, e.g., 1000 minutes.

Question: How do I update the model?

- Make a prediction: $q = Q(\mathbf{w})$, e.g., $q = 1000$.
- Finish the trip and get the target y, e.g., $y = 860$.
- Loss: $L = \frac{1}{2}(q - y)^2$.
- Gradient: $\frac{\partial L}{\partial \mathbf{w}} = \frac{\partial q}{\partial \mathbf{w}} \cdot \frac{\partial L}{\partial x} = (q - y) \cdot \frac{\partial Q(\mathbf{w})}{\partial \mathbf{w}}$.
- Gradient descent: $\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \cdot \frac{\partial L}{\partial \mathbf{w}} \Big|_{\mathbf{w}=\mathbf{w}_t}$.

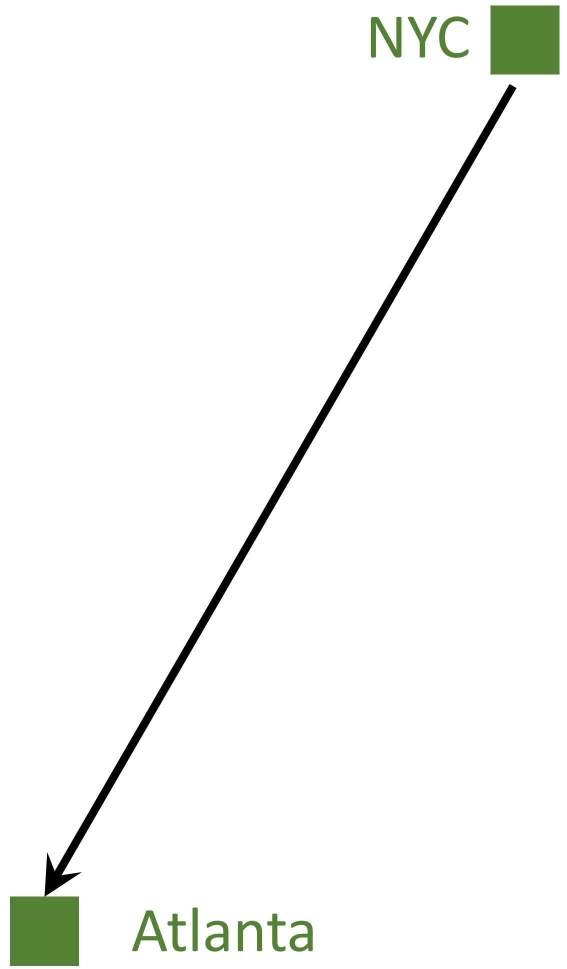


Temporal Difference (TD) Learning

- I want to drive from NYC to Atlanta.
- Model $Q(\mathbf{w})$ estimate the time cost, e.g., 1000 minutes.

Question: How do I update the model?

- Can I update the model before finishing the trip?

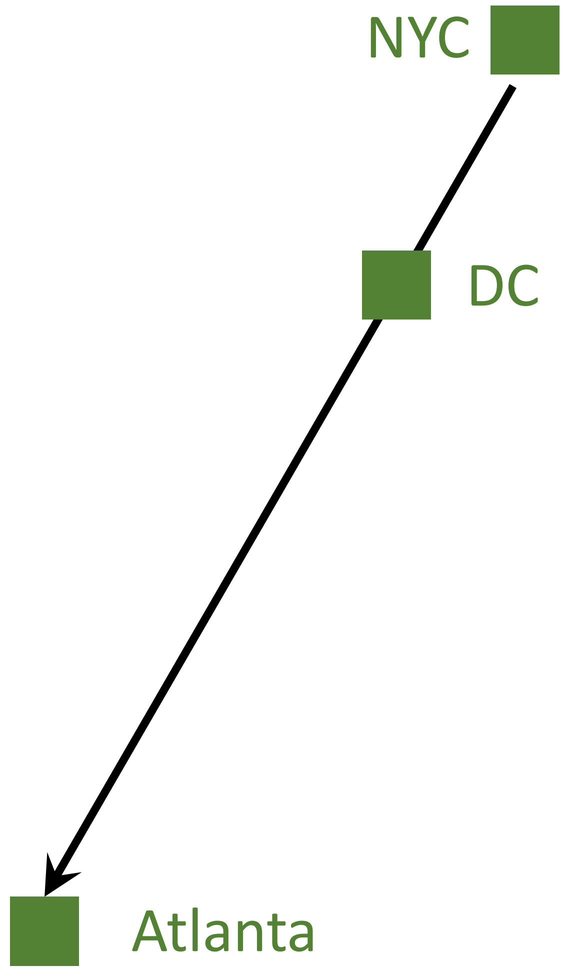


Temporal Difference (TD) Learning

- I want to drive from NYC to Atlanta (via DC).
- Model $Q(\mathbf{w})$ estimate the time cost, e.g., 1000 minutes.

Question: How do I update the model?

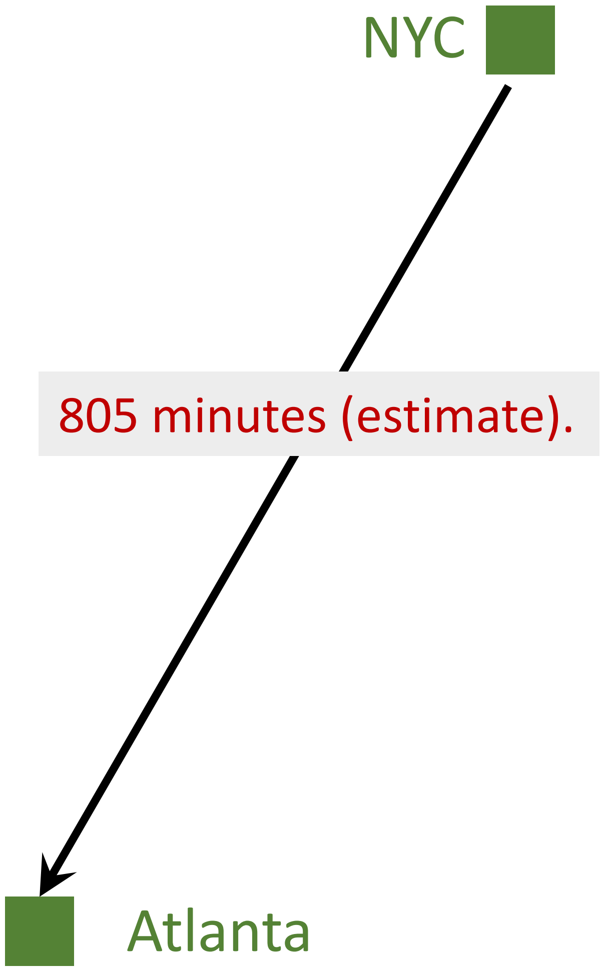
- Can I update the model before finishing the trip?
- Can I get a better $\mathbf{w}$ as soon as I arrived DC?



Temporal Difference (TD) Learning

- Model's estimate:

  NYC to Atlanta: 805 minutes (estimate).



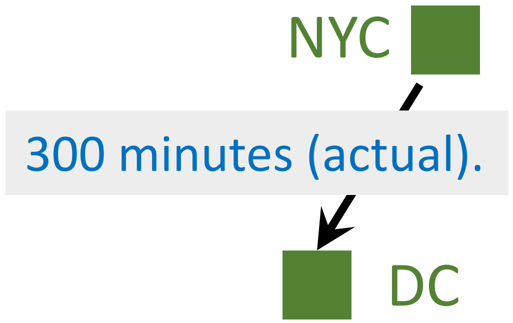
Temporal Difference (TD) Learning

- Model's estimate:

  NYC to Atlanta: 805 minutes (estimate).

- I arrived at DC; actual time cost:

  NYC to DC: 300 minutes (actual).



Temporal Difference (TD) Learning

- Model's estimate:

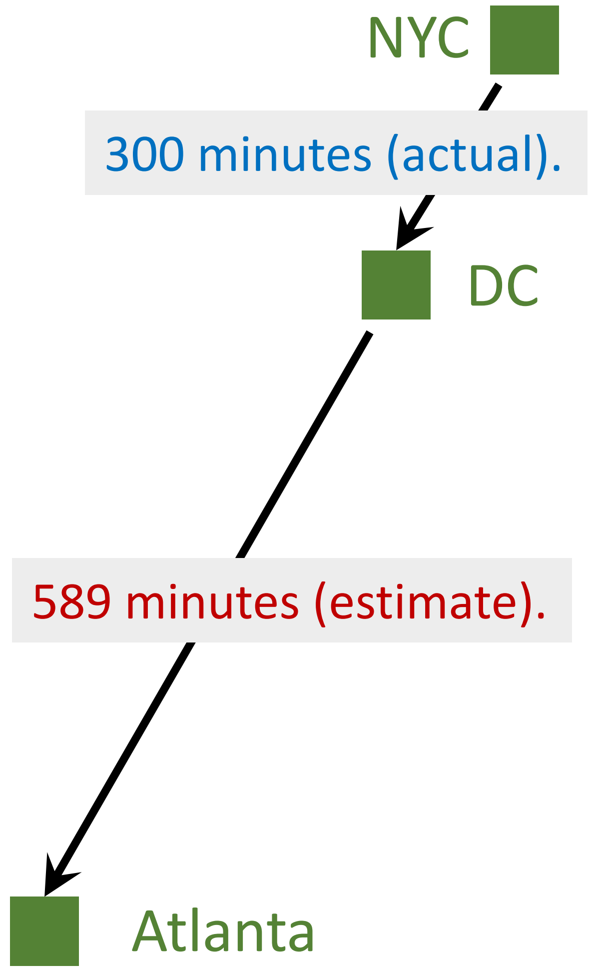
  NYC to Atlanta: 805 minutes (estimate).

- I arrived at DC; actual time cost:

  NYC to DC: 300 minutes (actual).

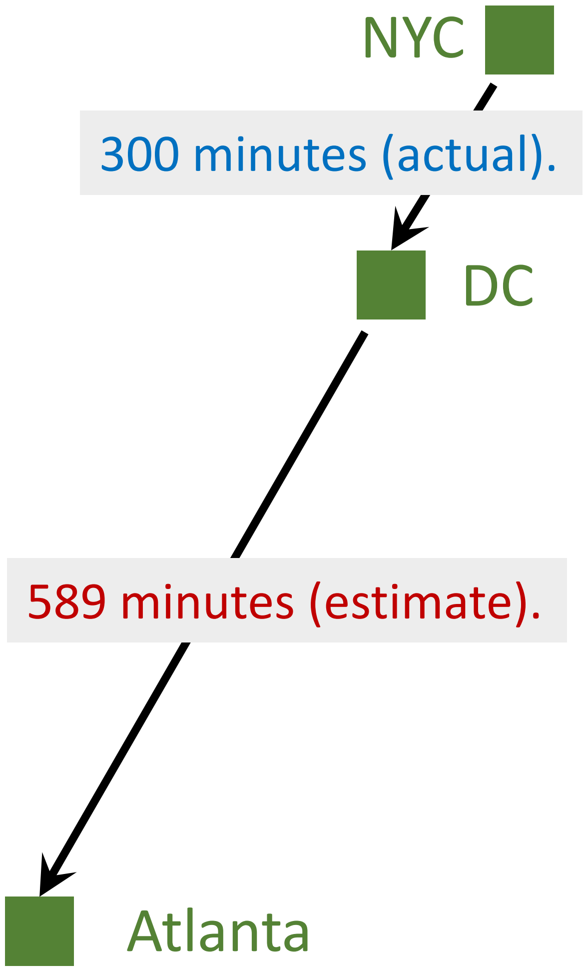
- Model now updates its estimate:

  DC to Atlanta: 589 minutes (estimate).



Temporal Difference (TD) Learning

- Model's estimate: $Q(\mathbf{w}) = 805$ minutes .
- Updated estimate: $300 + 589 = \underbrace{889}_{\text{TD target.}}$ minutes.

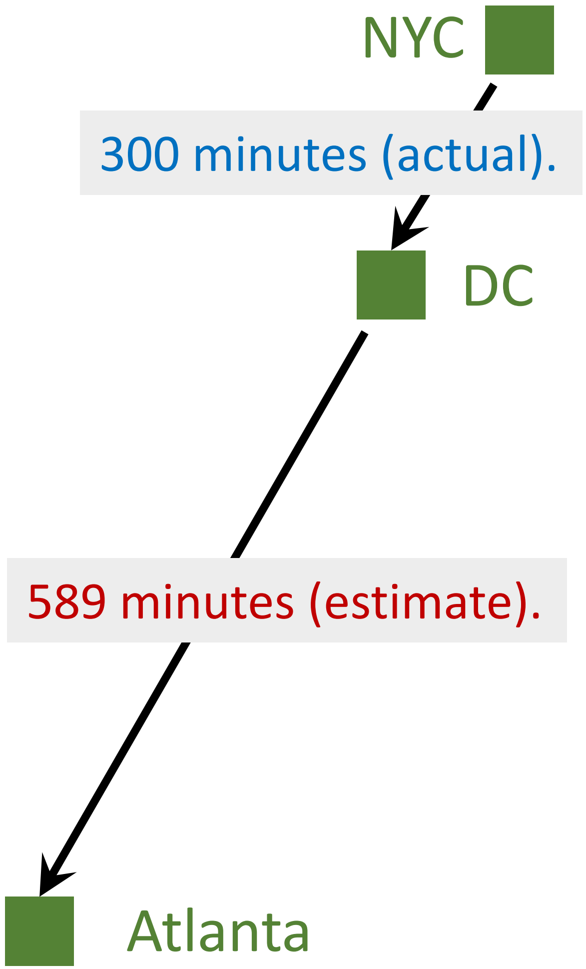


Temporal Difference (TD) Learning

- Model's estimate: $Q(\mathbf{w}) = 805$ minutes .
- Updated estimate: $300 + 589 = 889$ minutes.

TD target.

- TD target $y = 889$ is a more reliable estimate than 805.

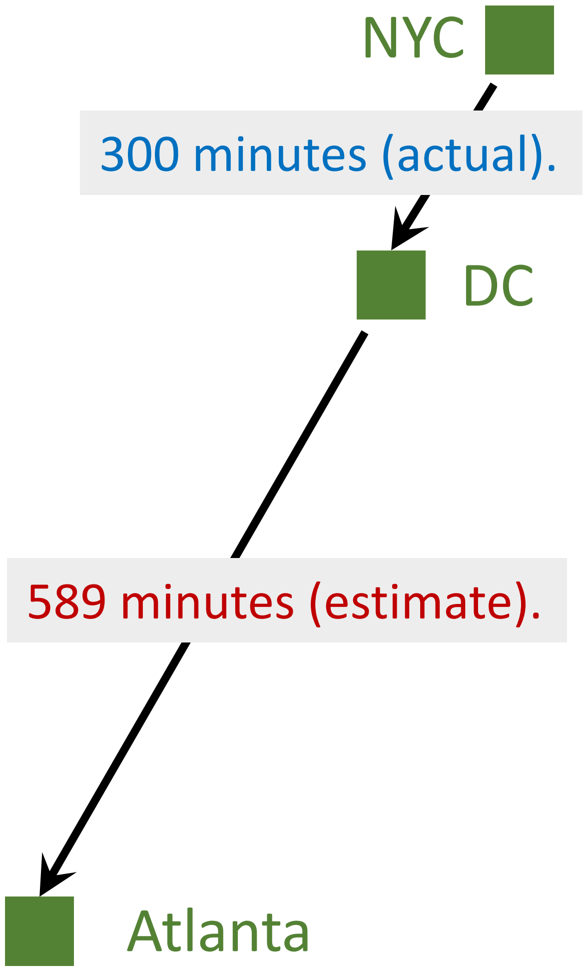


Temporal Difference (TD) Learning

- Model's estimate: $Q(\mathbf{w}) = 805$ minutes .
- Updated estimate: $300 + 589 = 889$ minutes.

TD target.

- TD target $y = 889$ is a more reliable estimate than 805.
- Loss: $L = \frac{1}{2}\left(Q(\mathbf{w}) - y\right)^2$.
- Gradient: $\frac{\partial L}{\partial \mathbf{w}} = (805 - 889) \cdot \frac{\partial Q(\mathbf{w})}{\partial \mathbf{w}}$.
- Gradient descent: $\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \cdot \frac{\partial L}{\partial \mathbf{w}} \Big|_{\mathbf{w}=\mathbf{w}_t}$.



TD Learning for DQN

Apply DQN to Play Game

$s_{t+1} \sim p(\cdot \mid s_t, a_t)$

$s_{t+2} \sim p(\cdot \mid s_{t+1}, a_{t+1})$

$s_{t+3} \sim p(\cdot \mid s_{t+2}, a_{t+2})$

$$s_t \longrightarrow a_t \longrightarrow s_{t+1} \longrightarrow a_{t+1} \longrightarrow s_{t+2} \longrightarrow a_{t+2} \longrightarrow s_{t+3} \quad \cdots$$

$r_t \qquad r_{t+1} \qquad r_{t+2}$

$a_t = \text{argmax}_a \, Q(s_t, a; \mathbf{w})$

$a_{t+1} = \text{argmax}_a \, Q(s_{t+1}, a; \mathbf{w})$

$a_{t+2} = \text{argmax}_a \, Q(s_{t+2}, a; \mathbf{w})$

Apply DQN to Play Game

$$
\begin{tikzcd}
s_t \arrow[r] & a_t \arrow[r] \arrow[dr] & s_{t+1} \\
 & & r_t
\end{tikzcd}
$$

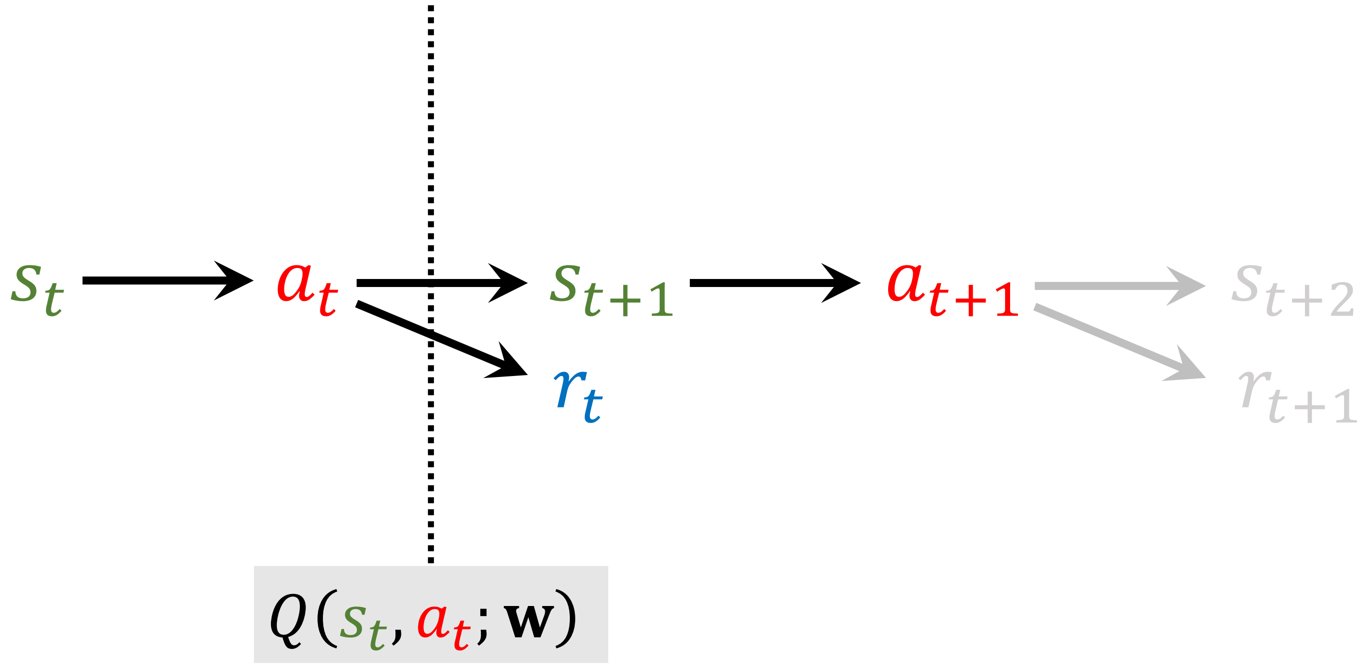
Apply DQN to Play Game

$$s_t \longrightarrow a_t \longrightarrow s_{t+1}$$

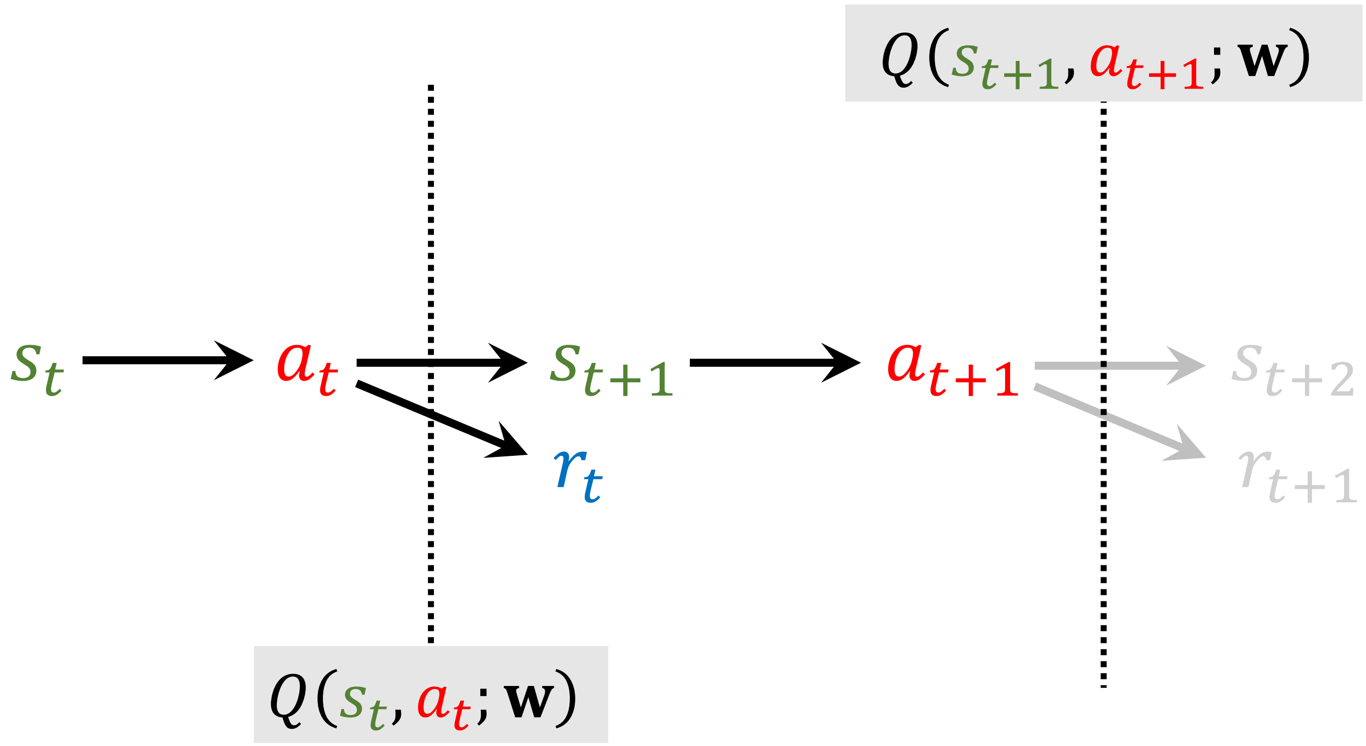
$$r_t$$

$$Q(s_t, a_t; \mathbf{w})$$

Apply DQN to Play Game



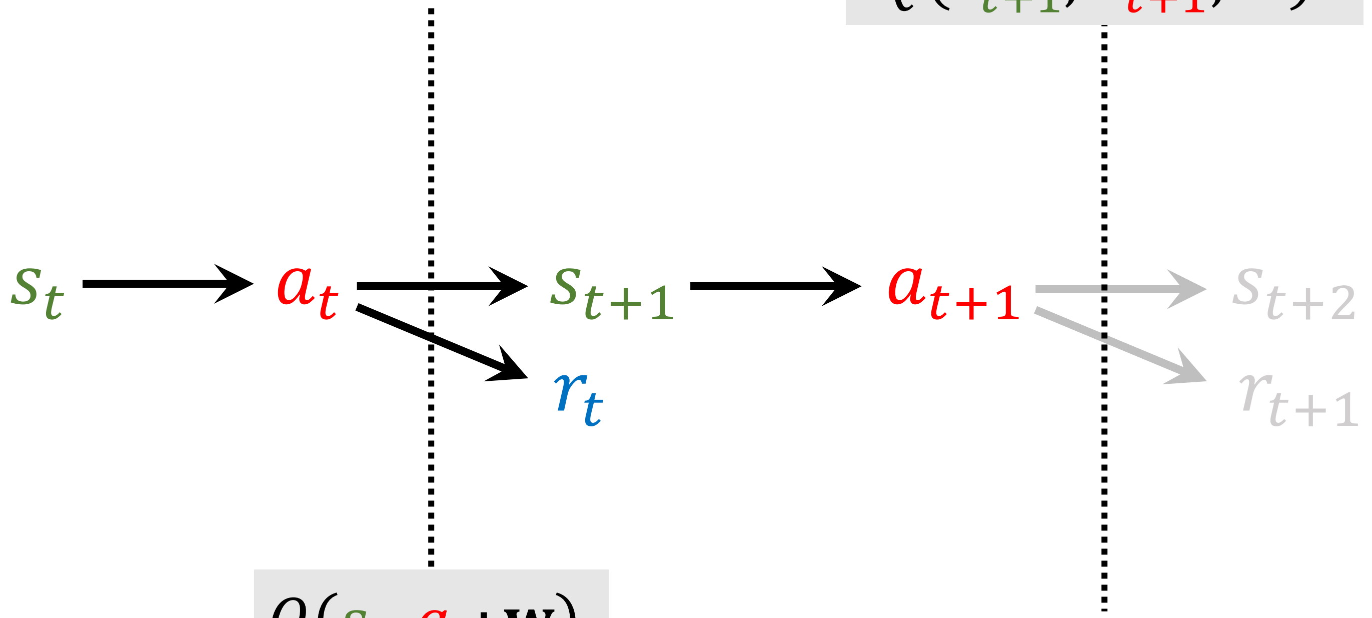
Apply DQN to Play Game



Train DQN using TD learning

If it is accurate estimate, then $Q(s_{t+1}, a_{t+1}; \mathbf{w}) = \mathbb{E}[r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \cdots]$

$Q(s_{t+1}, a_{t+1}; \mathbf{w})$



$Q(s_t, a_t; \mathbf{w})$

If it is accurate estimate, then $Q(s_t, a_t; \mathbf{w}) = \mathbb{E}[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \cdots]$

Train DQN using TD learning

If it is accurate estimate, then $Q(s_{t+1}, a_{t+1}; \mathbf{w}) = \mathbb{E}[r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \cdots]$

If DQN is accurate estimate, then

$$
\begin{aligned}
Q(s_t, a_t; \mathbf{w}) &= r_t + \gamma \cdot Q(s_{t+1}, a_{t+1}; \mathbf{w}) \\
&= r_t + \gamma \cdot \max_a Q(s_{t+1}, a; \mathbf{w})
\end{aligned}
$$

If it is accurate estimate, then $Q(s_t, a_t; \mathbf{w}) = \mathbb{E}[r_t + \gamma r_{t+1} + \gamma^2 r_{t+2} + \cdots]$

Train DQN using TD learning

If DQN is accurate estimate, then

$$
\begin{aligned}
Q(s_t, a_t; \mathbf{w}) &= r_t + \gamma \cdot Q(s_{t+1}, a_{t+1}; \mathbf{w}) \\
&= r_t + \gamma \cdot \max_{a} Q(s_{t+1}, a; \mathbf{w})
\end{aligned}
$$

Train DQN using TD learning

If DQN is accurate estimate, then

$$\underbrace{Q(s_t, a_t; \mathbf{w})}_{\text{Old estimate (less reliable)}} = r_t + \gamma \cdot Q(s_{t+1}, a_{t+1}; \mathbf{w})$$

$$= \underbrace{r_t + \gamma \cdot \max_{a} Q(s_{t+1}, a; \mathbf{w})}_{\text{TD target (more reliable estimate of the value)}}$$

Train DQN using TD learning

If DQN is accurate estimate, then

$$Q(s_t, a_t; \mathbf{w}) = r_t + \gamma \cdot Q(s_{t+1}, a_{t+1}; \mathbf{w})$$
$$= r_t + \gamma \cdot \max_a Q(s_{t+1}, a; \mathbf{w})$$

- TD target: $y_t = r_t + \gamma \cdot \max_a Q(s_{t+1}, a; \mathbf{w}_t)$.
- Loss: $L_t = \frac{1}{2}[Q(s_t, a_t; \mathbf{w}) - y_t]^2$.
- Gradient descent: $\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \cdot \frac{\partial L_t}{\partial \mathbf{w}} \Big|_{\mathbf{w}=\mathbf{w}_t}$

Summary

Action-Value Function Approximation

Definition: Optimal action-value function.

- $Q^{\star}(s, a) = \max_{\pi} \mathbb{E}\left[R_t | s, a, \pi\right]$.

DQN: Approximate $Q^{\star}(s, a)$ using a neural network.

- $Q(s, a; \mathbf{w})$ is a neural network parameterized by $\mathbf{w}$.
- Input: observed state s (e.g., a screenshot of game.)
- Output: a vector, each entry of which corresponds to an action a.

Temporal Difference (TD) Learning

Algorithm: One iteration of TD learning.

1. Observe state s_t and action a_t.
2. Predict the value: $q_t = Q(s_t, a_t; \mathbf{w}_t)$.
3. Differentiate the value network: $\mathbf{d}_t = \frac{\partial\, Q(s_t, a_t; \mathbf{w})}{\partial\, \mathbf{w}} \Big|_{\mathbf{w}=\mathbf{w}_t}$.

Temporal Difference (TD) Learning

Algorithm: One iteration of TD learning.

1. Observe state s_t and action a_t.
2. Predict the value: $q_t = Q(s_t, a_t; \mathbf{w}_t)$.
3. Differentiate the value network: $\mathbf{d}_t = \frac{\partial\, Q(s_t, a_t; \mathbf{w})}{\partial\, \mathbf{w}} \Big|_{\mathbf{w}=\mathbf{w}_t}$.
4. Environment provides new state s_{t+1} and reward r_t.
5. Compute TD target: $y_t = r_t + \gamma \cdot \max_a Q(s_{t+1}, a; \mathbf{w}_t)$.

Temporal Difference (TD) Learning

Algorithm: One iteration of TD learning.

1. Observe state s_t and action a_t.
2. Predict the value: $q_t = Q(s_t, a_t; \mathbf{w}_t)$.
3. Differentiate the value network: $\mathbf{d}_t = \frac{\partial\, Q(s_t, a_t; \mathbf{w})}{\partial\, \mathbf{w}} \Big|_{\mathbf{w}=\mathbf{w}_t}$.
4. Environment provides new state s_{t+1} and reward r_t.
5. Compute TD target: $y_t = r_t + \gamma \cdot \max_a Q(s_{t+1}, a; \mathbf{w}_t)$.
6. Gradient descent: $\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \cdot (q_t - y_t) \cdot \mathbf{d}_t$.