A PHILOSOPHICAL ESSAY

ON

PROBABILITIES.

BY

PIERRE SIMON, MARQUIS DE LAPLACE.

TRANSLATED FROM THE SIXTH FRENCH EDITION

BY

FREDERICK WILSON TRUSCOTT, PH.D. (HARV.),
Professor of Germanic Languages in the West Virginia University,

AND

FREDERICK LINCOLN EMORY, M.E. (WOR. POLY. INST.),
Professor of Mechanics and Applied Mathematics in the West Virginia University; Mem. Amer. Soc. Mech. Eng.

[redacted]

JOHN WILEY & SONS.
LONDON: CHAPMAN & HALL, LIMITED.
1902.

E. H.

ROBERT DRUMMOND, PRINTER, NEW YORK

TABLE OF CONTENTS.

PART I.

A PHILOSOPHICAL ESSAY ON PROBABILITIES.

PART II.

APPLICATION OF THE CALCULUS OF PROBABILITIES.

A PHILOSOPHICAL ESSAY ON PROBABILITIES.

CHAPTER I.

INTRODUCTION.

THIS philosophical essay is the development of a lecture on probabilities which I delivered in 1795 to the normal schools whither I had been called, by a decree of the national convention, as professor of mathematics with Lagrange. I have recently published upon the same subject a work entitled *The Analytical Theory of Probabilities*. I present here without the aid of analysis the principles and general results of this theory, applying them to the most important questions of life, which are indeed for the most part only problems of probability. Strictly speaking it may even be said that nearly all our knowledge is problematical; and in the small number of things which we are able to know with certainty, even in the mathematical sciences themselves, the principal means for ascertaining truth —induction and analogy—are based on probabilities;

so that the entire system of human knowledge is connected with the theory set forth in this essay. Doubtless it will be seen here with interest that in considering, even in the eternal principles of reason, justice, and humanity, only the favorable chances which are constantly attached to them, there is a great advantage in following these principles and serious inconvenience in departing from them: their chances, like those favorable to lotteries, always end by prevailing in the midst of the vacillations of hazard. I hope that the reflections given in this essay may merit the attention of philosophers and direct it to a subject so worthy of engaging their minds.

CHAPTER II.

CONCERNING PROBABILITY.

ALL events, even those which on account of their insignificance do not seem to follow the great laws of nature, are a result of it just as necessarily as the revolutions of the sun. In ignorance of the ties which unite such events to the entire system of the universe, they have been made to depend upon final causes or upon hazard, according as they occur and are repeated with regularity, or appear without regard to order; but these imaginary causes have gradually receded with the widening bounds of knowledge and disappear entirely before sound philosophy, which sees in them only the expression of our ignorance of the true causes.

Present events are connected with preceding ones by a tie based upon the evident principle that a thing cannot occur without a cause which produces it. This axiom, known by the name of *the principle of sufficient reason*, extends even to actions which are considered indifferent; the freest will is unable without a determinative motive to give them birth; if we assume two positions with exactly similar circumstances and find that the will is active in the one and inactive in the other, we say that its choice is an effect without a cause. It is then, says Leibnitz, the blind chance of the Epicureans. The contrary opinion is an illusion of the mind, which, losing sight of the evasive reasons of the choice of the will in indifferent things, believes that choice is determined of itself and without motives.

We ought then to regard the present state of the universe as the effect of its anterior state and as the cause of the one which is to follow. Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it—an intelligence sufficiently vast to submit these data to analysis—it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes. The human mind offers, in the perfection which it has been able to give to astronomy, a feeble idea of this intelligence. Its discoveries in mechanics and geometry, added to that of universal gravity, have enabled it to comprehend in the same analytical expressions the past and future states of the system of the world. Applying the same method to some other objects of its knowledge, it has succeeded in referring to general laws observed phenomena and in foreseeing those which given circumstances ought to produce. All these efforts in the search for truth tend to lead it back continually to the vast intelligence which we have just mentioned, but from which it will always remain infinitely removed. This tendency, peculiar to the human race, is that which renders it superior to animals; and their progress in this respect distinguishes nations and ages and constitutes their true glory.

Let us recall that formerly, and at no remote epoch, an unusual rain or an extreme drought, a comet having in train a very long tail, the eclipses, the aurora borealis, and in general all the unusual phenomena were regarded as so many signs of celestial wrath. Heaven was invoked in order to avert their baneful influence. No one prayed to have the planets and the sun arrested in their courses: observation had soon made apparent the futility of such prayers. But as these phenomena, occurring and disappearing at long intervals, seemed to oppose the order of nature, it was supposed that Heaven, irritated by the crimes of the earth, had created them to announce its vengeance. Thus the long tail of the comet of 1456 spread terror through Europe, already thrown into consternation by the rapid successes of the Turks, who had just overthrown the Lower Empire. This star after four revolutions has excited among us a very different interest. The knowledge of the laws of the system of the world acquired in the interval had dissipated the fears begotten by the ignorance of the true relationship of man to the universe; and Halley, having recognized the identity of this comet with those of the years 1531, 1607, and 1682, announced its next return for the end of the year 1758 or the beginning of the year 1759. The learned world awaited with impatience this return which was to confirm one of the greatest discoveries that have been made in the sciences, and fulfil the prediction of Seneca when he said, in speaking of the revolutions of those stars which fall from an enormous height: "The day will come when, by study pursued through several ages, the things now concealed will appear with evidence; and posterity will be astonished that truths so clear had escaped us." Clairaut then undertook to submit to analysis the perturbations which the comet had experienced by the action of the two great planets, Jupiter and Saturn; after immense calculations he fixed its next passage at the perihelion toward the beginning of April, 1759, which was actually verified by observation. The regularity which astronomy shows us in the movements of the comets doubtless exists also in all phenomena.

The curve described by a simple molecule of air or vapor is regulated in a manner just as certain as the planetary orbits; the only difference between them is that which comes from our ignorance.

Probability is relative, in part to this ignorance, in part to our knowledge. We know that of three or a greater number of events a single one ought to occur; but nothing induces us to believe that one of them will occur rather than the others. In this state of indecision it is impossible for us to announce their occurrence with certainty. It is, however, probable that one of these events, chosen at will, will not occur because we see several cases equally possible which exclude its occurrence, while only a single one favors it.

The theory of chance consists in reducing all the events of the same kind to a certain number of cases equally possible, that is to say, to such as we may be equally undecided about in regard to their existence, and in determining the number of cases favorable to the event whose probability is sought. The ratio of this number to that of all the cases possible is the measure of this probability, which is thus simply a fraction whose numerator is the number of favorable cases and whose denominator is the number of all the cases possible.

The preceding notion of probability supposes that, in increasing in the same ratio the number of favorable cases and that of all the cases possible, the probability remains the same. In order to convince ourselves let us take two urns, A and B, the first containing four white and two black balls, and the second containing only two white balls and one black one. We may imagine the two black balls of the first urn attached by a thread which breaks at the moment when one of them is seized in order to be drawn out, and the four white balls thus forming two similar systems. All the chances which will favor the seizure of one of the balls of the black system will lead to a black ball. If we conceive now that the threads which unite the balls do not break at all, it is clear that the number of possible chances will not change any more than that of the chances favorable to the extraction of the black balls; but two balls will be drawn from the urn at the same time; the probability of drawing a black ball from the urn A will then be the same as at first. But then we have obviously the case of urn B with the single difference that the three balls of this last urn would be replaced by three systems of two balls invariably connected.

When all the cases are favorable to an event the probability changes to certainty and its expression becomes equal to unity. Upon this condition, certainty and probability are comparable, although there may be an essential difference between the two states of the mind when a truth is rigorously demonstrated to it, or when it still perceives a small source of error.

In things which are only probable the difference of the data, which each man has in regard to them, is one of the principal causes of the diversity of opinions which prevail in regard to the same objects. Let us suppose, for example, that we have three urns, A, B, C, one of which contains only black balls while the two others contain only white balls; a ball is to be drawn from the urn C and the probability is demanded that this ball will be black. If we do not know which of the three urns contains black balls only, so that there is no reason to believe that it is C rather than B or A, these three hypotheses will appear equally possible, and since a black ball can be drawn only in the first hypothesis, the probability of drawing it is equal to one third. If it is known that the urn A contains white balls only, the indecision then extends only to the urns B and C, and the probability that the ball drawn from the urn C will be black is one half. Finally this probability changes to certainty if we are assured that the urns A and B contain white balls only.

It is thus that an incident related to a numerous assembly finds various degrees of credence, according to the extent of knowledge of the auditors. If the man who reports it is fully convinced of it and if, by his position and character, he inspires great confidence, his statement, however extraordinary it may be, will have for the auditors who lack information the same degree of probability as an ordinary statement made by the same man, and they will have entire faith in it. But if some one of them knows that the same incident is rejected by other equally trustworthy men, he will be in doubt and the incident will be discredited by the enlightened auditors, who will reject it whether it be in regard to facts well averred or the immutable laws of nature.

It is to the influence of the opinion of those whom the multitude judges best informed and to whom it has been accustomed to give its confidence in regard to the most important matters of life that the propagation of those errors is due which in times of ignorance have covered the face of the earth. Magic and astrology offer us two great examples. These errors inculcated in infancy, adopted without examination, and having for a basis only universal credence, have maintained themselves during a very long time; but at last the progress of science has destroyed them in the minds of enlightened men, whose opinion consequently has caused them to disappear even among the common people, through the power of imitation and habit which had so generally spread them abroad. This power, the richest resource of the moral world, establishes and conserves in a whole nation ideas entirely contrary to those which it upholds elsewhere with the same authority. What indulgence ought we not then to have for opinions different from ours, when this difference often depends only upon the various points of view where circumstances have placed us! Let us enlighten those whom we judge insufficiently instructed; but first let us examine critically our own opinions and weigh with impartiality their respective probabilities.

The difference of opinions depends, however, upon the manner in which the influence of known data is determined. The theory of probabilities holds to considerations so delicate that it is not surprising that with the same data two persons arrive at different results, especially in very complicated questions. Let us examine now the general principles of this theory.

CHAPTER III.

THE GENERAL PRINCIPLES OF THE CALCULUS OF PROBABILITIES.

First Principle.—The first of these principles is the definition itself of probability, which, as has been seen, is the ratio of the number of favorable cases to that of all the cases possible.

Second Principle.—But that supposes the various cases equally possible. If they are not so, we will determine first their respective possibilities, whose exact appreciation is one of the most delicate points of the theory of chance. Then the probability will be the sum of the possibilities of each favorable case. Let us illustrate this principle by an example.

Let us suppose that we throw into the air a large and very thin coin whose two large opposite faces, which we will call heads and tails, are perfectly similar. Let us find the probability of throwing heads at least one time in two throws. It is clear that four equally possible cases may arise, namely, heads at the first and at the second throw; heads at the first throw and tails at the second; tails at the first throw and heads at the second; finally, tails at both throws. The first three cases are favorable to the event whose probability is sought; consequently this probability is equal to $\frac{3}{4}$; so that it is a bet of three to one that heads will be thrown at least once in two throws.

We can count at this game only three different cases, namely, heads at the first throw, which dispenses with throwing a second time; tails at the first throw and heads at the second; finally, tails at the first and at the second throw. This would reduce the probability to $\frac{2}{3}$ if we should consider with d'Alembert these three cases as equally possible. But it is apparent that the probability of throwing heads at the first throw is $\frac{1}{2}$, while that of the two other cases is $\frac{1}{4}$, the first case being a simple event which corresponds to two events combined: heads at the first and at the second throw, and heads at the first throw, tails at the second. If we then, conforming to the second principle, add the possibility $\frac{1}{2}$ of heads at the first throw to the possibility $\frac{1}{4}$ of tails at the first throw and heads at the second, we shall have $\frac{3}{4}$ for the probability sought, which agrees with what is found in the supposition when we play the two throws. This supposition does not change at all the chance of that one who bets on this event; it simply serves to reduce the various cases to the cases equally possible.

Third Principle.—One of the most important points of the theory of probabilities and that which lends the most to illusions is the manner in which these probabilities increase or diminish by their mutual combination. If the events are independent of one another, the probability of their combined existence is the product of their respective probabilities. Thus the probability of throwing one ace with a single die is $\frac{1}{6}$; that of throwing two aces in throwing two dice at the same time is $\frac{1}{36}$. Each face of the one being able to combine with the six faces of the other, there are in fact thirty-six equally possible cases, among which one single case gives two aces. Generally the probability that a simple event in the same circumstances will occur consecutively a given number of times is equal to the probability of this simple event raised to the power indicated by this number. Having thus the successive powers of a fraction less than unity diminishing without ceasing, an event which depends upon a series of very great probabilities may become extremely improbable. Suppose then an incident be transmitted to us by twenty witnesses in such manner that the first has transmitted it to the second, the second to the third, and so on. Suppose again that the probability of each testimony be equal to the fraction $\frac{9}{10}$; that of the incident resulting from the testimonies will be less than $\frac{1}{8}$. We cannot better compare this diminution of the probability than with the extinction of the light of objects by the interposition of several pieces of glass. A relatively small number of pieces suffices to take away the view of an object that a single piece allows us to perceive in a distinct manner. The historians do not appear to have paid sufficient attention to this degradation of the probability of events when seen across a great number of successive generations; many historical events reputed as certain would be at least doubtful if they were submitted to this test.

In the purely mathematical sciences the most distant consequences participate in the certainty of the principle from which they are derived. In the applications of analysis to physics the results have all the certainty of facts or experiences. But in the moral sciences, where each inference is deduced from that which precedes it only in a probable manner, however probable these deductions may be, the chance of error increases with their number and ultimately surpasses the chance of truth in the consequences very remote from the principle.

Fourth Principle.—When two events depend upon each other, the probability of the compound event is the product of the probability of the first event and the probability that, this event having occurred, the second will occur. Thus in the preceding case of the three urns A, B, C, of which two contain only white balls and one contains only black balls, the probability of drawing a white ball from the urn C is $\frac{2}{3}$, since of the three urns only two contain balls of that color. But when a white ball has been drawn from the urn C, the indecision relative to that one of the urns which contain only black balls extends only to the urns A and B; the probability of drawing a white ball from the urn B is $\frac{1}{2}$; the product of $\frac{2}{3}$ by $\frac{1}{2}$, or $\frac{1}{3}$, is then the probability of drawing two white balls at one time from the urns B and C.

We see by this example the influence of past events upon the probability of future events. For the probability of drawing a white ball from the urn B, which primarily is $\frac{2}{3}$, becomes $\frac{1}{2}$ when a white ball has been drawn from the urn C; it would change to certainty if a black ball had been drawn from the same urn. We will determine this influence by means of the following principle, which is a corollary of the preceding one.

Fifth Principle.—If we calculate *à priori* the probability of the occurred event and the probability of an event composed of that one and a second one which is expected, the second probability divided by the first will be the probability of the event expected, drawn from the observed event.

Here is presented the question raised by some philosophers touching the influence of the past upon the probability of the future. Let us suppose at the play of heads and tails that heads has occurred oftener than tails. By this alone we shall be led to believe that in the constitution of the coin there is a secret cause which favors it. Thus in the conduct of life constant happiness is a proof of competency which should induce us to employ preferably happy persons. But if by the unreliability of circumstances we are constantly brought back to a state of absolute indecision, if, for example, we change the coin at each throw at the play of heads and tails, the past can shed no light upon the future and it would be absurd to take account of it.

Sixth Principle.—Each of the causes to which an observed event may be attributed is indicated with just as much likelihood as there is probability that the event will take place, supposing the event to be constant. The probability of the existence of any one of these causes is then a fraction whose numerator is the probability of the event resulting from this cause and whose denominator is the sum of the similar probabilities relative to all the causes; if these various causes, considered *à priori*, are unequally probable, it is necessary, in place of the probability of the event resulting from each cause, to employ the product of this probability by the possibility of the cause itself. This is the fundamental principle of this branch of the analysis of chances which consists in passing from events to causes.

This principle gives the reason why we attribute regular events to a particular cause. Some philosophers have thought that these events are less possible than others and that at the play of heads and tails, for example, the combination in which heads occurs twenty successive times is less easy in its nature than those where heads and tails are mixed in an irregular manner. But this opinion supposes that past events have an influence on the possibility of future events, which is not at all admissible. The regular combinations occur more rarely only because they are less numerous. If we seek a cause wherever we perceive symmetry, it is not that we regard a symmetrical event as less possible than the others, but, since this event ought to be the effect of a regular cause or that of chance, the first of these suppositions is more probable than the second. On a table we see letters arranged in this order, *Constantinople*, and we judge that this arrangement is not the result of chance, not because it is less possible than the others, for if this word were not employed in any language we should not suspect it came from any particular cause, but this word being in use among us, it is incomparably more probable that some person has thus arranged the aforesaid letters than that this arrangement is due to chance.

This is the place to define the word *extraordinary*. We arrange in our thought all possible events in various classes; and we regard as *extraordinary* those classes which include a very small number. Thus at the play of heads and tails the occurrence of heads a hundred successive times appears to us extraordinary because of the almost infinite number of combinations which may occur in a hundred throws; and if we divide the combinations into regular series containing an order easy to comprehend, and into irregular series, the latter are incomparably more numerous. The drawing of a white ball from an urn which among a million balls contains only one of this color, the others being black, would appear to us likewise extraordinary, because we form only two classes of events relative to the two colors. But the drawing of the number 475813, for example, from an urn that contains a million numbers seems to us an ordinary event; because, comparing individually the numbers with one another without dividing them into classes, we have no reason to believe that one of them will appear sooner than the others.

From what precedes, we ought generally to conclude that the more extraordinary the event, the greater the need of its being supported by strong proofs. For those who attest it, being able to deceive or to have been deceived, these two causes are as much more probable as the reality of the event is less. We shall see this particularly when we come to speak of the probability of testimony.

Seventh Principle.—The probability of a future event is the sum of the products of the probability of each cause, drawn from the event observed, by the probability that, this cause existing, the future event will occur. The following example will illustrate this principle.

Let us imagine an urn which contains only two balls, each of which may be either white or black. One of these balls is drawn and is put back into the urn before proceeding to a new draw. Suppose that in the first two draws white balls have been drawn; the probability of again drawing a white ball at the third draw is required.

Only two hypotheses can be made here: either one of the balls is white and the other black, or both are white. In the first hypothesis the probability of the event observed is $\frac{1}{4}$; it is unity or certainty in the second. Thus in regarding these hypotheses as so many causes, we shall have for the sixth principle $\frac{1}{5}$ and $\frac{4}{5}$ for their respective probabilities. But if the first hypothesis occurs, the probability of drawing a white ball at the third draw is $\frac{1}{2}$; it is equal to certainty in the second hypothesis; multiplying then the last probabilities by those of the corresponding hypotheses, the sum of the products, or $\frac{9}{10}$, will be the probability of drawing a white ball at the third draw.

When the probability of a single event is unknown we may suppose it equal to any value from zero to unity. The probability of each of these hypotheses, drawn from the event observed, is, by the sixth principle, a fraction whose numerator is the probability of the event in this hypothesis and whose denominator is the sum of the similar probabilities relative to all the hypotheses. Thus the probability that the possibility of the event is comprised within given limits is the sum of the fractions comprised within these limits. Now if we multiply each fraction by the probability of the future event, determined in the corresponding hypothesis, the sum of the products relative to all the hypotheses will be, by the seventh principle, the probability of the future event drawn from the event observed. Thus we find that an event having occurred successively any number of times, the probability that it will happen again the next time is equal to this number increased by unity divided by the same number, increased by two units. Placing the most ancient epoch of history at five thousand years ago, or at 182623 days, and the sun having risen constantly in the interval at each revolution of twenty-four hours, it is a bet of 1826214 to one that it will rise again to-morrow. But this number is incomparably greater for him who, recognizing in the totality of phenomena the principal regulator of days and seasons, sees that nothing at the present moment can arrest the course of it.

Buffon in his *Political Arithmetic* calculates differently the preceding probability. He supposes that it differs from unity only by a fraction whose numerator is unity and whose denominator is the number 2 raised to a power equal to the number of days which have elapsed since the epoch. But the true manner of relating past events with the probability of causes and of future events was unknown to this illustrious writer.

CHAPTER IV.

CONCERNING HOPE.

THE probability of events serves to determine the hope or the fear of persons interested in their existence. The word *hope* has various acceptations; it expresses generally the advantage of that one who expects a certain benefit in suppositions which are only probable. This advantage in the theory of chance is a product of the sum hoped for by the probability of obtaining it; it is the partial sum which ought to result when we do not wish to run the risks of the event in supposing that the division is made proportional to the probabilities. This division is the only equitable one when all strange circumstances are eliminated; because an equal degree of probability gives an equal right to the sum hoped for. We will call this advantage *mathematical hope*.

Eighth Principle.—When the advantage depends on several events it is obtained by taking the sum of the products of the probability of each event by the benefit attached to its occurrence.

Let us apply this principle to some examples. Let us suppose that at the play of heads and tails Paul receives two francs if he throws heads at the first throw and five francs if he throws it only at the second. Multiplying two francs by the probability $\frac{1}{2}$ of the first case, and five francs by the probability $\frac{1}{4}$ of the second case, the sum of the products, or two and a quarter francs, will be Paul's advantage. It is the sum which he ought to give in advance to that one who has given him this advantage; for, in order to maintain the equality of the play, the throw ought to be equal to the advantage which it procures.

If Paul receives two francs by throwing heads at the first and five francs by throwing it at the second throw, whether he has thrown it or not at the first, the probability of throwing heads at the second throw being $\frac{1}{2}$, multiplying two francs and five francs by $\frac{1}{2}$ the sum of these products will give three and one half francs for Paul's advantage and consequently for his stake at the game.

Ninth Principle.—In a series of probable events of which the ones produce a benefit and the others a loss, we shall have the advantage which results from it by making a sum of the products of the probability of each favorable event by the benefit which it procures, and subtracting from this sum that of the products of the probability of each unfavorable event by the loss which is attached to it. If the second sum is greater than the first, the benefit becomes a loss and hope is changed to fear.

Consequently we ought always in the conduct of life to make the product of the benefit hoped for, by its probability, at least equal to the similar product relative to the loss. But it is necessary, in order to attain this, to appreciate exactly the advantages, the losses, and their respective probabilities. For this a great accuracy of mind, a delicate judgment, and a great experience in affairs is necessary; it is necessary to know how to guard one's self against prejudices, illusions of fear or hope, and erroneous ideas, ideas of fortune and happiness, with which the majority of people feed their self-love.

The application of the preceding principles to the following question has greatly exercised the geometricians. Paul plays at heads and tails with the condition of receiving two francs if he throws heads at the first throw, four francs if he throws it only at the second throw, eight francs if he throws it only at the third, and so on. His stake at the play ought to be, according to the eighth principle, equal to the number of throws, so that if the game continues to infinity the stake ought to be infinite. However, no reasonable man would wish to risk at this game even a small sum, for example five francs. Whence comes this difference between the result of calculation and the indication of common sense? We soon recognize that it amounts to this: that the moral advantage which a benefit procures for us is not proportional to this benefit and that it depends upon a thousand circumstances, often very difficult to define, but of which the most general and most important is that of fortune.

Indeed it is apparent that one franc has much greater value for him who possesses only a hundred than for a millionaire. We ought then to distinguish in the hoped-for benefit its absolute from its relative value.

But the latter is regulated by the motives which make it desirable, whereas the first is independent of them. The general principle for appreciating this relative value cannot be given, but here is one proposed by Daniel Bernoulli which will serve in many cases.

Tenth Principle.—The relative value of an infinitely small sum is equal to its absolute value divided by the total benefit of the person interested. This supposes that every one has a certain benefit whose value can never be estimated as zero. Indeed even that one who possesses nothing always gives to the product of his labor and to his hopes a value at least equal to that which is absolutely necessary to sustain him.

If we apply analysis to the principle just propounded, we obtain the following rule: Let us designate by unity the part of the fortune of an individual, independent of his expectations. If we determine the different values that this fortune may have by virtue of these expectations and their probabilities, the product of these values raised respectively to the powers indicated by their probabilities will be the physical fortune which would procure for the individual the same moral advantage which he receives from the part of his fortune taken as unity and from his expectations; by subtracting unity from the product, the difference will be the increase of the physical fortune due to expectations: we will call this increase *moral hope*. It is easy to see that it coincides with mathematical hope when the fortune taken as unity becomes infinite in reference to the variations which it receives from the expectations. But when these variations are an appreciable part of this unity the two hopes may differ very materially among themselves.

This rule conduces to results conformable to the indications of common sense which can by this means be appreciated with some exactitude. Thus in the preceding question it is found that if the fortune of Paul is two hundred francs, he ought not reasonably to stake more than nine francs. The same rule leads us again to distribute the danger over several parts of a benefit expected rather than to expose the entire benefit to this danger. It results similarly that at the fairest game the loss is always greater than the gain. Let us suppose, for example, that a player having a fortune of one hundred francs risks fifty at the play of heads and tails; his fortune after his stake at the play will be reduced to eighty-seven francs, that is to say, this last sum would procure for the player the same moral advantage as the state of his fortune after the stake. The play is then disadvantageous even in the case where the stake is equal to the product of the sum hoped for, by its probability. We can judge by this of the immorality of games in which the sum hoped for is below this product. They subsist only by false reasonings and by the cupidity which they excite and which, leading the people to sacrifice their necessaries to chimerical hopes whose improbability they are not in condition to appreciate, are the source of an infinity of evils.

The disadvantage of games of chance, the advantage of not exposing to the same danger the whole benefit that is expected, and all the similar results indicated by common sense, subsist, whatever may be the function of the physical fortune which for each individual expresses his moral fortune. It is enough that the proportion of the increase of this function to the increase of the physical fortune diminishes in the measure that the latter increases.

CHAPTER VIII.

CONCERNING THE LAWS OF PROBABILITY WHICH RESULT FROM THE INDEFINITE MULTIPLICATION OF EVENTS.

AMID the variable and unknown causes which we comprehend under the name of *chance*, and which render uncertain and irregular the march of events, we see appearing, in the measure that they multiply, a striking regularity which seems to hold to a design and which has been considered as a proof of Providence. But in reflecting upon this we soon recognize that this regularity is only the development of the respective possibilities of simple events which ought to present themselves more often when they are more probable. Let us imagine, for example, an urn which contains white balls and black balls; and let us suppose that each time a ball is drawn it is put back into the urn before proceeding to a new draw. The ratio of the number of the white balls drawn to the number of black balls drawn will be most often very irregular in the first drawings; but the variable causes of this irregularity produce effects alternately favorable and unfavorable to the regular march of events which destroy each other mutually in the totality of a great number of draws, allowing us to perceive more and more the ratio of white balls to the black balls contained in the urn, or the respective possibilities of drawing a white ball or black ball at each draw. From this results the following theorem.

The probability that the ratio of the number of white balls drawn to the total number of balls drawn does not deviate beyond a given interval from the ratio of the number of white balls to the total number of balls contained in the urn, approaches indefinitely to certainty by the indefinite multiplication of events, however small this interval.

This theorem indicated by common sense was difficult to demonstrate by analysis. Accordingly the illustrious geometrician Jacques Bernouilli, who first has occupied himself with it, attaches great importance to the demonstrations he has given. The calculus of discriminant functions applied to this matter not only demonstrates with facility this theorem, but still more it gives the probability that the ratio of the events observed deviates only in certain limits from the true ratio of their respective possibilities.

One may draw from the preceding theorem this consequence which ought to be regarded as a general law, namely, that the ratios of the acts of nature are very nearly constant when these acts are considered in great number. Thus in spite of the variety of years the sum of the productions during a considerable number of years is sensibly the same; so that man by useful foresight is able to provide against the irregularity of the seasons by spreading out equally over all the seasons the goods which nature distributes in an unequal manner. I do not except from the above law results due to moral causes. The ratio of annual births to the population, and that of marriages to births, show only small variations; at Paris the number of annual births is almost the same, and I have heard it said at the post-office in ordinary seasons the number of letters thrown aside on account of defective addresses changes little each year; this has likewise been observed at London.

It follows again from this theorem that in a series of events indefinitely prolonged the action of regular and constant causes ought to prevail in the long run over that of irregular causes. It is this which renders the gains of the lotteries just as certain as the products of agriculture; the chances which they reserve assure them a benefit in the totality of a great number of throws. Thus favorable and numerous chances being constantly attached to the observation of the eternal principles of reason, of justice, and of humanity which establish and maintain societies, there is a great advantage in conforming to these principles and of grave inconvenience in departing from them. If one consult histories and his own experience, one will see all the facts come to the aid of this result of calculus. Consider the happy effects of institutions founded upon reason and the natural rights of man among the peoples who have known how to establish and preserve them. Consider again the advantages which good faith has procured for the governments who have made it the basis of their conduct and how they have been indemnified for the sacrifices which a scrupulous exactitude in keeping their engagements has cost them. What immense credit at home! What preponderance abroad! On the contrary, look into what an abyss of misfortunes nations have often been precipitated by the ambition and the perfidy of their chiefs. Every time that a great power intoxicated by the love of conquest aspires to universal domination the sentiment of independence produces among the menaced nations a coalition of which it becomes almost always the victim. Similarly in the midst of the variable causes which extend or restrain the divers states, the natural limits acting as constant causes ought to end by prevailing. It is important then to the stability as well as to the happiness of empires not to extend them beyond those limits into which they are led again without cessation by the action of the causes; just as the waters of the seas raised by violent tempests fall again into their basins by the force of gravity. It is again a result of the calculus of probabilities confirmed by numerous and melancholy experiences. History treated from the point of view of the influence of constant causes would unite to the interest of curiosity that of offering to man most useful lessons. Sometimes we attribute the inevitable results of these causes to the accidental circumstances which have produced their action. It is, for example, against the nature of things that one people should ever be governed by another when a vast sea or a great distance separates them. It may be affirmed that in the long run this constant cause, joining itself without ceasing to the variable causes which act in the same way and which the course of time develops, will end by finding them sufficiently strong to give to a subjugated people its natural independence or to unite it to a powerful state which may be contiguous.

In a great number of cases, and these are the most important of the analysis of hazards, the possibilities of simple events are unknown and we are forced to search in past events for the indices which can guide us in our conjectures about the causes upon which they depend. In applying the analysis of discriminant functions to the principle elucidated above on the probability of the causes drawn from the events observed, we are led to the following theorem.

When a simple event or one composed of several simple events, as, for instance, in a game, has been repeated a great number of times the possibilities of the simple events which render most probable that which has been observed are those that observation indicates with the greatest probability; in the measure that the observed event is repeated this probability increases and would end by amounting to certainty if the numbers of repetitions should become infinite.

There are two kinds of approximations: the one is relative to the limits taken on all sides of the possibilities which give to the past the greatest probability; the other approximation is related to the probability that these possibilities fall within these limits. The repetition of the compound event increases more and more this probability, the limits remaining the same; it reduces more and more the interval of these limits, the probability remaining the same; in infinity this interval becomes zero and the probability changes to certainty.

If we apply this theorem to the ratio of the births of boys to that of girls observed in the different countries of Europe, we find that this ratio, which is everywhere about equal to that of 22 to 21, indicates with an extreme probability a greater facility in the birth of boys. Considering further that it is the same at Naples and at St. Petersburg, we shall see that in this regard the influence of climate is without effect. We might then suspect, contrary to the common belief, that this predominance of masculine births exists even in the Orient. I have consequently invited the French scholars sent to Egypt to occupy themselves with this interesting question; but the difficulty in obtaining exact information about the births has not permitted them to solve it. Happily, M. de Humboldt has not neglected this matter among the innumerable new things which he has observed and collected in America with so much sagacity, constancy, and courage. He has found in the tropics the same ratio of the births as we observe in Paris; this ought to make us regard the greater number of masculine births as a general law of the human race. The laws which the different kinds of animals follow in this regard seem to me worthy of the attention of naturalists.

The fact that the ratio of births of boys to that of girls differs very little from unity even in the great number of the births observed in a place would offer in this regard a result contrary to the general law, without which we should be right in concluding that this law did not exist. In order to arrive at this result it is necessary to employ great numbers and to be sure that it is indicated by great probability. Buffon cites, for example, in his *Political Arithmetic* several communities of Bourgogne where the births of girls have surpassed those of boys. Among these communities that of Carcelle-le-Grignon presents in 2009 births during five years 1026 girls and 983 boys. Although these numbers are considerable, they indicate, however, only a greater possibility in the births of girls with a probability of $\frac{9}{10}$, and this probability, smaller than that of not throwing heads four times in succession in the game of heads and tails, is not sufficient to investigate the cause for this anomaly, which, according to all probability, would disappear if one should follow during a century the births in this community.

The registers of births, which are kept with care in order to assure the condition of the citizens, may serve in determining the population of a great empire without recurring to the enumeration of its inhabitants—a laborious operation and one difficult to make with exactitude. But for this it is necessary to know the ratio of the population to the annual births. The most precise means of obtaining it consists, first, in choosing in the empire districts distributed in an almost equal manner over its whole surface, so as to render the general result independent of local circumstances; second, in enumerating with care for a given epoch the inhabitants of several communities in each of these districts; third, by determining from the statement of the births during several years which precede and follow this epoch the mean number corresponding to the annual births. This number, divided by that of the inhabitants, will give the ratio of the annual births to the population in a manner more and more accurate as the enumeration becomes more considerable. The government, convinced of the utility of a similar enumeration, has decided at my request to order its execution. In thirty districts spread out equally over the whole of France, communities have been chosen which would be able to furnish the most exact information. Their enumerations have given 2037615 individuals as the total number of their inhabitants on the 23d of September, 1802. The statement of the births in these communities during the years 1800, 1801, and 1802 have given:

Births.	Marriages.	Deaths.
110312 boys	46037	103659 men
105287 girls		99443 women

The ratio of the population to annual births is then $28\frac{352845}{1000000}$; it is greater than had been estimated up to this time. Multiplying the number of annual births in France by this ratio, we shall have the population of this kingdom. But what is the probability that the population thus determined will not deviate from the true population beyond a given limit? Resolving this problem and applying to its solution the preceding data, I have found that, the number of annual births in France being supposed to be 1000000, which brings the population to 28352845 inhabitants, it is a bet of almost 300000 against 1 that the error of this result is not half a million.

The ratio of the births of boys to that of girls which the preceding statement offers is that of 22 to 21; and the marriages are to the births as 3 is to 4.

At Paris the baptisms of children of both sexes vary a little from the ratio of 22 to 21. Since 1745, the epoch in which one has commenced to distinguish the sexes upon the birth-registers, up to the end of 1784, there have been baptized in this capital 393386 boys and 377555 girls. The ratio of the two numbers is almost that of 25 to 24; it appears then at Paris that a particular cause approximates an equality of baptisms of the two sexes. If we apply to this matter the calculus of probabilities, we find that it is a bet of 238 to 1 in favor of the existence of this cause, which is sufficient to authorize the investigation. Upon reflection it has appeared to me that the difference observed holds to this, that the parents in the country and the provinces, finding some advantage in keeping the boys at home, have sent to the Hospital for Foundlings in Paris fewer of them relative to the number of girls according to the ratio of births of the two sexes. This is proved by the statement of the registers of this hospital. From the beginning of 1745 to the end of 1809 there were entered 163499 boys and 159405 girls. The first of these numbers exceeds only by $\frac{1}{38}$ the second, which it ought to have surpassed at least by $\frac{1}{24}$. This confirms the existence of the assigned cause, namely, that the ratio of births of boys to those of girls is at Paris that of 22 to 21, no attention having been paid to foundlings.

The preceding results suppose that we may compare the births to the drawings of balls from an urn which contains an infinite number of white balls and black balls so mixed that at each draw the chances of drawing ought to be the same for each ball; but it is possible that the variations of the same seasons in different years may have some influence upon the annual ratio of the births of boys to those of girls. The Bureau of Longitudes of France publishes each year in its annual the tables of the annual movement of the population of the kingdom. The tables already published commence in 1817; in that year and in the five following years there were born 2962361 boys and 2781997 girls, which gives about $\frac{16}{15}$ for the ratio of the births of boys to that of girls. The ratios of each year vary little from this mean result; the smallest ratio is that of 1822, where it was only $\frac{17}{16}$; the greatest is of the year 1817, when it was $\frac{15}{14}$. These ratios vary appreciably from the ratio of $\frac{22}{21}$ found above. Applying to this deviation the analysis of probabilities in the hypothesis of the comparison of births to the drawings of balls from an urn, we find that it would be scarcely probable. It appears, then, to indicate that this hypothesis, although closely approximated, is not rigorously exact. In the number of births which we have just stated there are of natural children 200494 boys and 190698 girls. The ratio of masculine and feminine births was then in this regard $\frac{20}{19}$, smaller than the mean ratio of $\frac{16}{15}$. This result is in the same sense as that of the births of foundlings; and it seems to prove that in the class of natural children the births of the two sexes approach more nearly equality than in the class of legitimate children. The difference of the climates from the north to the south of France does not appear to influence appreciably the ratio of the births of boys and girls. The thirty most southern districts have given $\frac{16}{15}$ for this ratio, the same as that of entire France.

The constancy of the superiority of the births of boys over girls at Paris and at London since they have been observed has appeared to some scholars to be a proof of Providence, without which they have thought that the irregular causes which disturb without ceasing the course of events ought several times to have rendered the annual births of girls superior to those of boys.

But this proof is a new example of the abuse which has been so often made of final causes which always disappear on a searching examination of the questions when we have the necessary data to solve them. The constancy in question is a result of regular causes which give the superiority to the births of boys and which extend it to the anomalies due to hazard when the number of annual births is considerable. The investigation of the probability that this constancy will maintain itself for a long time belongs to that branch of the analysis of hazards which passes from past events to the probability of future events; and taking as a basis the births observed from 1745 to 1784, it is a bet of almost 4 against 1 that at Paris the annual births of boys will constantly surpass for a century the births of girls; there is then no reason to be astonished that this has taken place for a half-century.

Let us take another example of the development of constant ratios which events present in the measure that they are multiplied. Let us imagine a series of urns arranged circularly, and each containing a very great number of white balls and black balls; the ratio of white balls to the black in the urns being originally very different and such, for example, that one of these urns contains only white balls, while another contains only black balls. If one draws a ball from the first urn in order to put it into the second, and, after having shaken the second urn in order to mix well the new ball with the others, one draws a ball to put it into the third urn, and so on to the last urn, from which is drawn a ball to put into the first, and if this series is recommenced continually, the analysis of probability shows us that the ratios of the white balls to the black in these urns will end by being the same and equal to the ratio of the sum of all the white balls to the sum of all the black balls contained in the urns. Thus by this regular mode of change the primitive irregularity of these ratios disappears eventually in order to make room for the most simple order. Now if among these urns one intercalate new ones in which the ratio of the sum of the white balls to the sum of the black balls which they contain differs from the preceding, continuing indefinitely in the totality of the urns the drawings which we have just indicated, the simple order established in the old urns will be at first disturbed, and the ratios of the white balls to the black balls will become irregular; but little by little this irregularity will disappear in order to make room for a new order, which will finally be that of the equality of the ratios of the white balis to the black balls contained in the urns. We may apply these results to all the combinations of nature in which the constant forces by which their elements are animated establish regular modes of action, suited to bring about in the very heart of chaos systems governed by admirable laws.

The phenomena which seem the most dependent upon hazard present, then, when multiplied a tendency to approach without ceasing fixed ratios, in such a manner that if we conceive on all sides of each of these ratios an interval as small as desired, the probability that the mean result of the observations falls within this interval will end by differing from certainty only by a quantity greater than an assignable magnitude. Thus by the calculations of probabilities applied to a great number of observations we may recognize the existence of these ratios. But before seeking the causes it is necessary, in order not to be led into vain speculations, to assure ourselves that they are indicated by a probability which does not permit us to regard them as anomalies due to hazard. The theory of discriminant functions gives a very simple expression for this probability, which is obtained by integrating the product of the differential of the quantity of which the result deduced from a great number of observations varies from the truth by a constant less than unity, dependent upon the nature of the problem, and raised to a power whose exponent is the ratio of the square of this variation to the number of observations. The integral taken between the limits given and divided by the same integral, applied to a positive and negative infinity, will express the probability that the variation from the truth is comprised between these limits. Such is the general law of the probability of results indicated by a great number of observations.

CHAPTER XI.

CONCERNING THE PROBABILITIES OF TESTIMONIES.

THE majority of our opinions being founded on the probability of proofs it is indeed important to submit it to calculus. Things it is true often become impossible by the difficulty of appreciating the veracity of witnesses and by the great number of circumstances which accompany the deeds they attest; but one is able in several cases to resolve the problems which have much analogy with the questions which are proposed and whose solutions may be regarded as suitable approximations to guide and to defend us againt the errors and the dangers of false reasoning to which we are exposed. An approximation of this kind, when it is well made, is always preferable to the most specious reasonings. Let us try then to give some general rules for obtaining it.

A single number has been drawn from an urn which contains a thousand of them. A witness to this drawing announces that number 79 is drawn; one asks the probability of drawing this number. Let us suppose that experience has made known that this witness deceives one time in ten, so that the probability of his testimony is $\frac{1}{10}$. Here the event observed is the witness attesting that number 79 is drawn. This event may result from the two following hypotheses, namely: that the witness utters the truth or that he deceives. Following the principle that has been expounded on the probability of causes drawn from events observed it is necessary first to determine *à priori* the probability of the event in each hypothesis. In the first, the probability that the witness will announce number 79 is the probability itself of the drawing of this number, that is to say, $\frac{1}{1000}$. It is necessary to multiply it by the probability $\frac{9}{10}$ of the veracity of the witness; one will have then $\frac{9}{10000}$ for the probability of the event observed in this hypothesis. If the witness deceives, number 79 is not drawn, and the probability of this case is $\frac{999}{1000}$. But to announce the drawing of this number the witness has to choose it among the 999 numbers not drawn; and as he is supposed to have no motive of preference for the ones rather than the others, the probability that he will choose number 79 is $\frac{1}{999}$; multiplying, then, this probability by the preceding one, we shall have $\frac{1}{1000}$ for the probability that the witness will announce number 79 in the second hypothesis. It is necessary again to multiply this probability by $\frac{1}{10}$ of the hypothesis itself, which gives $\frac{1}{10000}$ for the probability of the event relative to this hypothesis. Now if we form a fraction whose numerator is the probability relative to the first hypothesis, and whose denominator is the sum of the probabilities relative to the two hypotheses, we shall have, by the sixth principle, the probability of the first hypothesis, and this probability will be $\frac{9}{10}$; that is to say, the veracity itself of the witness. This is likewise the probability of the drawing of number 79. The probability of the falsehood of the witness and of the failure of drawing this number is $\frac{1}{10}$.

If the witness, wishing to deceive, has some interest in choosing number 79 among the numbers not drawn, —if he judges, for example, that having placed upon this number a considerable stake, the announcement of its drawing will increase his credit, the probability that he will choose this number will no longer be as at first, $\frac{1}{999}$, it will then be $\frac{1}{2}$, $\frac{1}{3}$, etc., according to the interest that he will have in announcing its drawing. Supposing it to be $\frac{1}{9}$, it will be necessary to multiply by this fraction the probability $\frac{999}{1000}$ in order to get in the hypothesis of the falsehood the probability of the event observed, which it is necessary still to multiply by $\frac{1}{10}$, which gives $\frac{111}{10000}$ for the probability of the event in the second hypothesis. Then the probability of the first hypothesis, or of the drawing of number 79, is reduced by the preceding rule to $\frac{9}{120}$. It is then very much decreased by the consideration of the interest which the witness may have in announcing the drawing of number 79. In truth this same interest increases the probability $\frac{9}{10}$ that the witness will speak the truth if number 79 is drawn. But this probability cannot exceed unity or $\frac{10}{10}$; thus the probability of the drawing of number 79 will not surpass $\frac{10}{121}$. Common sense tells us that this interest ought to inspire distrust, but calculus appreciates the influence of it.

The probability *à priori* of the number announced by the witness is unity divided by the number of the numbers in the urn; it is changed by virtue of the proof into the veracity itself of the witness; it may then be decreased by the proof. If, for example, the urn contains only two numbers, which gives $\frac{1}{2}$ for the probability *à priori* of the drawing of number 1, and if the veracity of a witness who announces it is $\frac{4}{10}$, this drawing becomes less probable. Indeed it is apparent, since the witness has then more inclination towards a falsehood than towards the truth, that his testimony ought to decrease the probability of the fact attested every time that this probability equals or surpasses $\frac{1}{2}$. But if there are three numbers in the urn the probability *à priori* of the drawing of number 1 is increased by the affirmation of a witness whose veracity surpasses $\frac{1}{3}$.

Suppose now that the urn contains 999 black balls and one white ball, and that one ball having been drawn a witness of the drawing announces that this ball is white. The probability of the event observed, determined *à priori* in the first hypothesis, will be here, as in the preceding question, equal to $\frac{9}{10000}$. But in the hypothesis where the witness deceives, the white ball is not drawn and the probability of this case is $\frac{999}{1000}$. It is necessary to multiply it by the probability $\frac{1}{10}$ of the falsehood, which gives $\frac{999}{10000}$ for the probability of the event observed relative to the second hypothesis. This probability was only $\frac{1}{10000}$ in the preceding question; this great difference results from this—that a black ball having been drawn the witness who wishes to deceive has no choice at all to make among the 999 balls not drawn in order to announce the drawing of a white ball. Now if one forms two fractions whose numerators are the probabilities relative to each hypothesis, and whose common denominator is the sum of these probabilities, one will have $\frac{9}{1008}$ for the probability of the first hypothesis and of the drawing of a white ball, and $\frac{999}{1008}$ for the probability of the second hypothesis and of the drawing of a black ball. This last probability strongly approaches certainty; it would approach it much nearer and would become $\frac{999999}{1000008}$ if the urn contained a million balls of which one was white, the drawing of a white ball becoming then much more extraordinary. We see thus how the probability of the falsehood increases in the measure that the deed becomes more extraordinary.

We have supposed up to this time that the witness was not mistaken at all; but if one admits, however, the chance of his error the extraordinary incident becomes more improbable. Then in place of the two hypotheses one will have the four following ones, namely: that of the witness not deceiving and not being mistaken at all; that of the witness not deceiving at all and being mistaken; the hypothesis of the witness deceiving and not being mistaken at all; finally, that of the witness deceiving and being mistaken. Determining *à priori* in each of these hypotheses the probability of the event observed, we find by the sixth principle the probability that the fact attested is false equal to a fraction whose numerator is the number of black balls in the urn multiplied by the sum of the probabilities that the witness does not deceive at all and is mistaken, or that he deceives and is not mistaken, and whose denominator is this numerator augmented by the sum of the probabilities that the witness does not deceive at all and is not mistaken at all, or that he deceives and is mistaken at the same time. We see by this that if the number of black balls in the urn is very great, which renders the drawing of the white ball extraordinary, the probability that the fact attested is not true approaches most nearly to certainty.

Applying this conclusion to all extraordinary deeds it results from it that the probability of the error or of the falsehood of the witness becomes as much greater as the fact attested is more extraordinary. Some authors have advanced the contrary on this basis that the view of an extraordinary fact being perfectly similar to that of an ordinary fact the same motives ought to lead us to give the witness the same credence when he affirms the one or the other of these facts. Simple common sense rejects such a strange assertion; but the calculus of probabilities, while confirming the findings of common sense, appreciates the greatest improbability of testimonies in regard to extraordinary facts.

These authors insist and suppose two witnesses equally worthy of belief, of whom the first attests that he saw an individual dead fifteen days ago whom the second witness affirms to have seen yesterday full of life. The one or the other of these facts offers no improbability. The reservation of the individual is a result of their combination; but the testimonies do not bring us at all directly to this result, although the credence which is due these testimonies ought not to be decreased by the fact that the result of their combination is extraordinary.

But if the conclusion which results from the combination of the testimonies was impossible one of them would be necessarily false; but an impossible conclusion is the limit of extraordinary conclusions, as error is the limit of improbable conclusions; the value of the testimonies which becomes zero in the case of an impossible conclusion ought then to be very much decreased in that of an extraordinary conclusion. This is indeed confirmed by the calculus of probabilities.

In order to make it plain let us consider two urns, A and B, of which the first contains a million white balls and the second a million black balls. One draws from one of these urns a ball, which he puts back into the other urn, from which one then draws a ball. Two witnesses, the one of the first drawing, the other of the second, attest that the ball which they have seen drawn is white without indicating the urn from which it has been drawn. Each testimony taken alone is not improbable; and it is easy to see that the probability of the fact attested is the veracity itself of the witness. But it follows from the combination of the testimonies that a white ball has been extracted from the urn A at the first draw, and that then placed in the urn B it has reappeared at the second draw, which is very extraordinary; for this second urn, containing then one white ball among a million black balls, the probability of drawing the white ball is $\frac{1}{1000001}$. In order to determine the diminution which results in the probability of the thing announced by the two witnesses we shall notice that the event observed is here the affirmation by each of them that the ball which he has seen extracted is white. Let us represent by $\frac{9}{10}$ the probability that he announces the truth, which can occur in the present case when the witness does not deceive and is not mistaken at all, and when he deceives and is mistaken at the same time. One may form the four following hypotheses:

1st. The first and second witness speak the truth. Then a white ball has at first been drawn from the urn A, and the probability of this event is $\frac{1}{2}$, since the ball drawn at the first draw may have been drawn either from the one or the other urn. Consequently the ball drawn, placed in the urn B, has reappeared at the second draw; the probability of this event is $\frac{1}{1000001}$, the probability of the fact announced is then $\frac{1}{2000002}$. Multiplying it by the product of the probabilities $\frac{9}{10}$ and $\frac{9}{10}$ that the witnesses speak the truth one will have $\frac{81}{200000200}$ for the probability of the event observed in this first hypothesis.

2d. The first witness speaks the truth and the second does not, whether he deceives and is not mistaken or he does not deceive and is mistaken. Then a white ball has been drawn from the urn A at the first draw, and the probability of this event is $\frac{1}{2}$. Then this ball having been placed in the urn B a black ball has been drawn from it: the probability of such drawing is $\frac{1000000}{1000001}$; one has then $\frac{1000000}{2000002}$ for the probability of the compound event. Multiplying it by the product of the two probabilities $\frac{9}{10}$ and $\frac{1}{10}$ that the first witness speaks the truth and that the second does not, one will have $\frac{9000000}{200000200}$ for the probability for the event observed in the second hypothesis.

3d. The first witness does not speak the truth and the second announces it. Then a black ball has been drawn from the urn B at the first drawing, and after having been placed in the urn A a white ball has been drawn from this urn. The probability of the first of these events is $\frac{1}{2}$ and that of the second is $\frac{1000000}{1000001}$; the probability of the compound event is then $\frac{1000000}{2000002}$. Multiplying it by the product of the probabilities $\frac{1}{10}$ and $\frac{9}{10}$ that the first witness does not speak the truth and that the second announces it, one will have $\frac{9000000}{200000200}$ for the probability of the event observed relative to this hypothesis.

4th. Finally, neither of the witnesses speaks the truth. Then a black ball has been drawn from the urn B at the first draw; then having been placed in the urn A it has reappeared at the second drawing: the probability of this compound event is $\frac{1}{2000002}$. Multiplying it by the product of the probabilities $\frac{1}{10}$ and $\frac{1}{10}$ that each witness does not speak the truth one will have $\frac{1}{200000200}$ for the probability of the event observed in this hypothesis.

Now in order to obtain the probability of the thing announced by the two witnesses, namely, that a white ball has been drawn at each draw, it is necessary to divide the probability corresponding to the first hypothesis by the sum of the probabilities relative to the four hypotheses; and then one has for this probability $\frac{81}{18000082}$, an extremely small fraction.

If the two witnesses affirm the first, that a white ball has been drawn from one of the two urns A and B; the second that a white ball has been likewise drawn from one of the two urns A′ and B′, quite similar to the first ones, the probability of the thing announced by the two witnesses will be the product of the probabilities of their testimonies, or $\frac{81}{100}$; it will then be at least a hundred and eighty thousand times greater than the preceding one. One sees by this how much, in the first case, the reappearance at the second draw of the white ball drawn at the first draw, the extraordinary conclusion of the two testimonies decreases the value of it.

We would give no credence to the testimony of a man who should attest to us that in throwing a hundred dice into the air they had all fallen on the same face. If we had ourselves been spectators of this event we should believe our own eyes only after having carefully examined all the circumstances, and after having brought in the testimonies of other eyes in order to be quite sure that there had been neither hallucination nor deception. But after this examination we should not hesitate to admit it in spite of its extreme improbability; and no one would be tempted, in order to explain it, to recur to a denial of the laws of vision. We ought to conclude from it that the probability of the constancy of the laws of nature is for us greater than this, that the event in question has not taken place at all—a probability greater than that of the majority of historical facts which we regard as incontestable. One may judge by this the immense weight of testimonies necessary to admit a suspension of natural laws, and how improper it would be to apply to this case the ordinary rules of criticism. All those who without offering this immensity of testimonies support this when making recitals of events contrary to those laws, decrease rather than augment the belief which they wish to inspire; for then those recitals render very probable the error or the falsehood of their authors.

But that which diminishes the belief of educated men increases often that of the uneducated, always greedy for the wonderful.

There are things so extraordinary that nothing can balance their improbability. But this, by the effect of a dominant opinion, can be weakened to the point of appearing inferior to the probability of the testimonies; and when this opinion changes an absurd statement admitted unanimously in the century which has given it birth offers to the following centuries only a new proof of the extreme influence of the general opinion upon the more enlightened minds. Two great men of the century of Louis XIV.—Racine and Pascal—are striking examples of this. It is painful to see with what complaisance Racine, this admirable painter of the human heart and the most perfect poet that has ever lived, reports as miraculous the recovery of Mlle. Perrier, a niece of Pascal and a day pupil at the monastery of Port-Royal; it is painful to read the reasons by which Pascal seeks to prove that this miracle should be necessary to religion in order to justify the doctrine of the monks of this abbey, at that time persecuted by the Jesuits. The young Perrier had been afflicted for three years and a half by a lachrymal fistula; she touched her afflicted eye with a relic which was pretended to be one of the thorns of the crown of the Saviour and she had faith in instant recovery. Some days afterward the physicians and the surgeons attest the recovery, and they declare that nature and the remedies have had no part in it. This event, which took place in 1656, made a great sensation, and "all Paris rushed," says Racine, "to Port-Royal. The crowd increased from day to day, and God himself seemed to take pleasure in authorizing the devotion of the people by the number of miracles which were performed in this church." At this time miracles and sorcery did not yet appear improbable, and one did not hesitate at all to attribute to them the singularities of nature which could not be explained otherwise.

This manner of viewing extraordinary results is found in the most remarkable works of the century of Louis XIV.; even in the Essay on the Human Understanding by the philosopher Locke, who says, in speaking of the degree of assent: "Though the common experience and the ordinary course of things have justly a mighty influence on the minds of men, to make them give or refuse credit to anything proposed to their belief; yet there is one case, wherein the strangeness of the fact lessens not the assent to a fair testimony of it. For where such supernatural events are suitable to ends aimed at by him who has the power to change the course of nature, there, under such circumstances, they may be the fitter to procure belief, by how much the more they are beyond or contrary to ordinary observation." The true principles of the probability of testimonies having been thus misunderstood by philosophers to whom reason is principally indebted for its progress, I have thought it necessary to present at length the results of calculus upon this important subject.

There comes up naturally at this point the discussion of a famous argument of Pascal, that Craig, an English mathematician, has produced under a geometric form. Witnesses declare that they have it from Divinity that in conforming to a certain thing one will enjoy not one or two but an infinity of happy lives. However feeble the probability of the proofs may be, provided that it be not infinitely small, it is clear that the advantage of those who conform to the prescribed thing is infinite since it is the product of this probability and an infinite good; one ought not to hesitate then to procure for oneself this advantage.

This argument is based upon the infinite number of happy lives promised in the name of the Divinity by the witnesses; it is necessary then to prescribe them, precisely because they exaggerate their promises beyond all limits, a consequence which is repugnant to good sense. Also calculus teaches us that this exaggeration itself enfeebles the probability of their testimony to the point of rendering it infinitely small or zero. Indeed this case is similar to that of a witness who should announce the drawing of the highest number from an urn filled with a great number of numbers, one of which has been drawn and who would have a great interest in announcing the drawing of this number. One has already seen how much this interest enfeebles his testimony. In evaluating only at $\frac{1}{2}$ the probability that if the witness deceives he will choose the largest number, calculus gives the probability of his announcement as smaller than a fraction whose numerator is unity and whose denominator is unity plus the half of the product of the number of the numbers by the probability of falsehood considered *à priori* or independently of the announcement. In order to compare this case to that of the argument of Pascal it is sufficient to represent by the numbers in the urn all the possible numbers of happy lives which the number of these numbers renders infinite; and to observe that if the witnesses deceive they have the greatest interest, in order to accredit their falsehood, in promising an eternity of happiness. The expression of the probability of their testimony becomes then infinitely small. Multiplying it by the infinite number of happy lives promised, infinity would disappear from the product which expresses the advantage resultant from this promise which destroys the argument of Pascal.

Let us consider now the probability of the totality of several testimonies upon an established fact. In order to fix our ideas let us suppose that the fact be the drawing of a number from an urn which contains a hundred of them, and of which one single number has been drawn. Two witnesses of this drawing announce that number 2 has been drawn, and one asks for the resultant probability of the totality of these testimonies. One may form these two hypotheses: the witnesses speak the truth; the witnesses deceive. In the first hypothesis the number 2 is drawn and the probability of this event is $\frac{1}{100}$. It is necessary to multiply it by the product of the veracities of the witnesses, veracities which we will suppose to be $\frac{9}{10}$ and $\frac{7}{10}$: one will have then $\frac{63}{10000}$ for the probability of the event observed in this hypothesis. In the second, the number 2 is not drawn and the probability of this event is $\frac{99}{100}$. But the agreement of the witnesses requires then that in seeking to deceive they both choose the number 2 from the 99 numbers not drawn: the probability of this choice if the witnesses do not have a secret agreement is the product of the fraction $\frac{1}{99}$ by itself; it becomes necessary then to multiply these two probabilities together, and by the product of the probabilities $\frac{1}{10}$ and $\frac{3}{10}$ that the witnesses deceive; one will have thus $\frac{1}{330000}$ for the probability of the event observed in the second hypothesis. Now one will have the probability of the fact attested or of the drawing of number 2 in dividing the probability relative to the first hypothesis by the sum of the probabilities relative to the two hypotheses; this probability will be then $\frac{2079}{2080}$, and the probability of the failure to draw this number and of the falsehood of the witnesses will be $\frac{1}{2080}$.

If the urn should contain only the numbers 1 and 2 one would find in the same manner $\frac{21}{22}$ for the probability of the drawing of number 2, and consequently $\frac{1}{22}$ for the probability of the falsehood of the witnesses, a probability at least ninety-four times larger than the preceding one. One sees by this how much the probability of the falsehood of the witnesses diminishes when the fact which they attest is less probable in itself. Indeed one conceives that then the accord of the witnesses, when they deceive, becomes more difficult, at least when they do not have a secret agreement, which we do not suppose here at all.

In the preceding case where the urn contained only two numbers the *à priori* probability of the fact attested is $\frac{1}{2}$, the resultant probability of the testimonies is the product of the veracities of the witnesses divided by this product added to that of the respective probabilities of their falsehood.

It now remains for us to consider the influence of time upon the probability of facts transmitted by a traditional chain of witnesses. It is clear that this probability ought to diminish in proportion as the chain is prolonged. If the fact has no probability itself, such as the drawing of a number from an urn which contains an infinity of them, that which it acquires by the testimonies decreases according to the continued product of the veracity of the witnesses. If the fact has a probability in itself; if, for example, this fact is the drawing of the number 2 from an urn which contains an infinity of them, and of which it is certain that one has drawn a single number; that which the traditional chain adds to this probability decreases, following a continued product of which the first factor is the ratio of the number of numbers in the urn less one to the same number, and of which each other factor is the veracity of each witness diminished by the ratio of the probability of his falsehood to the number of the numbers in the urn less one; so that the limit of the probability of the fact is that of this fact considered *à priori*, or independently of the testimonies, a probability equal to unity divided by the number of the numbers in the urn.

The action of time enfeebles then, without ceasing, the probability of historical facts just as it changes the most durable monuments. One can indeed diminish it by multiplying and conserving the testimonies and the monuments which support them. Printing offers for this purpose a great means, unfortunately unknown to the ancients. In spite of the infinite advantages which it procures the physical and moral revolutions by which the surface of this globe will always be agitated will end, in conjunction with the inevitable effect of time, by rendering doubtful after thousands of years the historical facts regarded to-day as the most certain.

Craig has tried to submit to calculus the gradual enfeebling of the proofs of the Christian religion; supposing that the world ought to end at the epoch when it will cease to be probable, he finds that this ought to take place 1454 years after the time when he writes. But his analysis is as faulty as his hypothesis upon the duration of the moon is bizarre.

CHAPTER XII.

CONCERNING THE SELECTIONS AND THE DECISIONS OF ASSEMBLIES.

THE probability of the decisions of an assembly depends upon the plurality of votes, the intelligence and the impartiality of the members who compose it. So many passions and particular interests so often add their influence that it is impossible to submit this probability to calculus. There are, however, some general results dictated by simple common sense and confirmed by calculus. If, for example, the assembly is poorly informed about the subject submitted to its decision, if this subject requires delicate considerations, or if the truth on this point is contrary to established prejudices, so that it would be a bet of more than one against one that each voter will err; then the decision of the majority will be probably wrong, and the fear of it will be the better based as the assembly is more numerous. It is important then, in public affairs, that assemblies should have to pass upon subjects within reach of the greatest number; it is important for them that information be generally diffused and that good works founded upon reason and experience should enlighten those who are called to decide the lot of their fellows or to govern them, and should forewarn them against false ideas and the prejudices of ignorance. Scholars have had frequent occasion to remark that first conceptions often deceive and that the truth is not always probable.

It is difficult to understand and to define the desire of an assembly in the midst of a variety of opinions of its members. Let us attempt to give some rules in regard to this matter by considering the two most ordinary cases: the election among several candidates, and that among several propositions relative to the same subject.

When an assembly has to choose among several candidates who present themselves for one or for several places of the same kind, that which appears simplest is to have each voter write upon a ticket the names of all the candidates according to the order of merit that he attributes to them. Supposing that he classifies them in good faith, the inspection of these tickets will give the results of the elections in such a manner that the candidates may be compared among themselves; so that new elections can give nothing more in this regard. It is a question now to conclude the order of preference which the tickets establish among the candidates. Let us imagine that one gives to each voter an urn which contains an infinity of balls by means of which he is able to shade all the degrees of merit of the candidates; let us conceive again that he draws from his urn a number of balls proportional to the merit of each candidate, and let us suppose this number written upon a ticket at the side of the name of the candidate. It is clear that by making a sum of all the numbers relative to each candidate upon each ticket, that one of all the candidates who shall have the largest sum will be the candidate whom the assembly prefers; and that in general the order of preference of the candidates will be that of the sums relative to each of them. But the tickets do not mark at all the number of balls which each voter gives to the candidates; they indicate solely that the first has more of them than the second, the second more than the third, and so on. In supposing then at first upon a given ticket a certain number of balls all the combinations of the inferior numbers which fulfil the preceding conditions are equally admissible; and one will have the number of balls relative to each candidate by making a sum of all the numbers which each combination gives him and dividing it by the entire number of combinations. A very simple analysis shows that the numbers which must be written upon each ticket at the side of the last name, of the one before the last, etc., are proportional to the terms of the arithmetical progression 1, 2, 3, etc. Writing then thus upon each ticket the terms of this progression, and adding the terms relative to each candidate upon these tickets, the divers sums will indicate by their magnitude the order of their preference which ought to be established among the candidates. Such is the mode of election which The Theory of Probabilities indicates. Without doubt it would be better if each voter should write upon his ticket the names of the candidates in the order of merit which he attributes to them. But particular interests and many strange considerations of merit would affect this order and place sometimes in the last rank the candidate most formidable to that one whom one prefers, which gives too great an advantage to the candidates of mediocre merit. Likewise experience has caused the abandonment of this mode of election in the societies which had adopted it.

The election by the absolute majority of the suffrages unites to the certainty of not admitting any one of the candidates whom this majority rejects, the advantage of expressing most often the desire of the assembly. It always coincides with the preceding mode when there are only two candidates. Indeed it exposes an assembly to the inconvenience of rendering elections interminable. But experience has shown that this inconvenience is nil, and that the general desire to put an end to elections soon unites the majority of the suffrages upon one of the candidates.

The choice among several propositions relative to the same object ought to be subjected, seemingly, to the same rules as the election among several candidates. But there exists between the two cases this difference, namely, that the merit of a candidate does not exclude that of his competitors; but if it is necessary to choose among propositions which are contrary, the truth of the one excludes the truth of the others. Let us see how one ought then to view this question.

Let us give to each voter an urn which contains an infinite number of balls, and let us suppose that he distributes them upon the divers propositions according to the respective probabilities which he attributes to them. It is clear that the total number of balls expressing certainty, and the voter being by the hypothesis assured that one of the propositions ought to be true, he will distribute this number at length upon the propositions. The problem is reduced then to this, namely, to determine the combinations in which the balls will be distributed in such a manner that there may be more of them upon the first proposition of the ticket than upon the second, more upon the second than upon the third, etc.; to make the sums of all the numbers of balls relative to each proposition in the divers combinations, and to divide this sum by the number of combinations; the quotients will be the numbers of balls that one ought to attribute to the propositions upon a certain ticket. One finds by analysis that in going from the last proposition these quotients are among themselves as the following quantities: first, unity divided by the number of propositions; second, the preceding quantity, augmented by unity, divided by the number of propositions less one; third, this second quantity, augmented by unity, divided by the number of propositions less two, and so on for the others. One will write then upon each ticket these quantities at the side of the corresponding propositions, and adding the relative quantities to each proposition upon the divers tickets the sums will indicate by their magnitude the order of preference which the assembly gives to these propositions.

Let us speak a word about the manner of renewing assemblies which should change in totality in a definite number of years. Ought the renewal to be made at one time, or is it advantageous to divide it among these years? According to the last method the assembly would be formed under the influence of the divers opinions dominant during the time of its renewal; the opinion which obtained then would be probably the mean of all these opinions. The assembly would receive thus at the time the same advantage that is given to it by the extension of the elections of its members to all parts of the territory which it represents. Now if one considers what experience has only too clearly taught, namely, that elections are always directed in the greatest degree by dominant opinions, one will feel how useful it is to temper these opinions, the ones by the others, by means of a partial renewal.

CHAPTER XIII.

CONCERNING THE PROBABILITY OF THE JUDGMENTS OF TRIBUNALS.

ANALYSIS confirms what simple common sense teaches us, namely, the correctness of judgments is as much more probable as the judges are more numerous and more enlightened. It is important then that tribunals of appeal should fulfil these two conditions. The tribunals of the first instance standing in closer relation to those amenable offer to the higher tribunal the advantage of a first judgment already probable, and with which the latter often agree, be it in compromising or in desisting from their claims. But if the uncertainty of the matter in litigation and its importance determine a litigant to have recourse to the tribunal of appeals, he ought to find in a greater probability of obtaining an equitable judgment greater security for his fortune and the compensation for the trouble and expense which a new procedure entails. It is this which had no place in the institution of the reciprocal appeal of the tribunals of the district, an institution thereby very prejudicial to the interest of the citizens. It would be perhaps proper and conformable to the calculus of probabilities to demand a majority of at least two votes in a tribunal of appeal in order to invalidate the sentence of the lower tribunal. One would obtain this result if the tribunal of appeal being composed of an even number of judges the sentence should stand in the case of the equality of votes.

I shall consider particularly the judgments in criminal matters.

In order to condemn an accused it is necessary without doubt that the judges should have the strongest proofs of his offence. But a moral proof is never more than a probability; and experience has only too clearly shown the errors of which criminal judgments, even those which appear to be the most just, are still susceptible. The impossibility of amending these errors is the strongest argument of the philosophers who have wished to proscribe the penalty of death. We should then be obliged to abstain from judging if it were necessary for us to await mathematical evidence. But the judgment is required by the danger which would result from the impunity of the crime. This judgment reduces itself, if I am not mistaken, to the solution of the following question: Has the proof of the offence of the accused the high degree of probability necessary so that the citizens would have less reason to doubt the errors of the tribunals, if he is innocent and condemned, than they would have to fear his new crimes and those of the unfortunate ones who would be emboldened by the example of his impunity if he were guilty and acquitted? The solution of this question depends upon several elements very difficult to ascertain. Such is the eminence of danger which would threaten society if the criminal accused should remain unpunished. Sometimes this danger is so great that the magistrate sees himself constrained to waive forms wisely established for the protection of innocence. But that which renders almost always this question insoluble is the impossibility of appreciating exactly the probability of the offence and of fixing that which is necessary for the condemnation of the accused. Each judge in this respect is forced to rely upon his own judgment. He forms his opinion by comparing the divers testimonies and the circumstances by which the offence is accompanied, to the results of his reflections and his experiences, and in this respect a long habitude of interrogating and judging accused persons gives great advantage in ascertaining the truth in the midst of indices often contradictory.

The preceding question depends again upon the care taken in the investigation of the offence; for one demands naturally much stronger proofs for imposing the death penalty than for inflicting a detention of some months. It is a reason for proportioning the care to the offence, great care taken with an unimportant case inevitably clearing many guilty ones. A law which gives to the judges power of moderating the care in the case of attenuating circumstances is then conformable at the same time to principles of humanity towards the culprit, and to the interest of society. The product of the probability of the offence by its gravity being the measure of the danger to which the acquittal of the accused can expose society, one would think that the care taken ought to depend upon this probability. This is done indirectly in the tribunals where one retains for some time the accused against whom there are very strong proofs, but insufficient to condemn him; in the hope of acquiring new light one does not place him immediately in the midst of his fellow citizens, who would not see him again without great alarm. But the arbitrariness of this measure and the abuse which one can make of it have caused its rejection in the countries where one attaches the greatest price to individual liberty.

Now what is the probability that the decision of a tribunal which can condemn only by a given majority will be just, that is to say, conform to the true solution of the question proposed above? This important problem well solved will give the means of comparing among themselves the different tribunals. The majority of a single vote in a numerous tribunal indicates that the affair in question is very doubtful; the condemnation of the accused would be then contrary to the principles of humanity, protectors of innocence. The unanimity of the judges would give very strong probability of a just decision; but in abstaining from it too many guilty ones would be acquitted. It is necessary, then, either to limit the number of judges, if one wishes that they should be unanimous, or increase the majority necessary for a condemnation, when the tribunal becomes more numerous. I shall attempt to apply calculus to this subject, being persuaded that it is always the best guide when one bases it upon the data which common sense suggests to us.

The probability that the opinion of each judge is just enters as the principal element into this calculation. If in a tribunal of a thousand and one judges, five hundred and one are of one opinion, and five hundred are of the contrary opinion, it is apparent that the probability of the opinion of each judge surpasses very little $\frac{1}{2}$; for supposing it obviously very large a single vote of difference would be an improbable event. But if the judges are unanimous, this indicates in the proofs that degree of strength which entails conviction; the probability of the opinion of each judge is then very near unity or certainty, provided that the passions or the ordinary prejudices do not affect at the same time all the judges. Outside of these cases the ratio of the votes for or against the accused ought alone to determine this probability. I suppose thus that it can vary from $\frac{1}{2}$ to unity, but that it cannot be below $\frac{1}{2}$. If that were not the case the decision of the tribunal would be as insignificant as chance; it has value only in so far as the opinion of the judge has a greater tendency to truth than to error. It is thus by the ratio of the numbers of votes favorable, and contrary to the accused, that I determine the probability of this opinion.

These data suffice to ascertain the general expression of the probability that the decision of a tribunal judging by a known majority is just. In the tribunals where of eight judges five votes would be necessary for the condemnation of an accused, the probability of the error to be feared in the justice of the decision would surpass $\frac{1}{4}$. If the tribunal should be reduced to six members who are able to condemn only by a plurality of four votes, the probability of the error to be feared would be below $\frac{1}{4}$. There would be then for the accused an advantage in this reduction of the tribunal. In both cases the majority required is the same and is equal to two. Thus the majority remaining constant, the probability of error increases with the number of judges; this is general whatever may be the majority required, provided that it remains the same. Taking, then, for the rule the arithmetical ratio, the accused finds himself in a position less and less advantageous in the measure that the tribunal becomes more numerous. One might believe that in a tribunal where one might demand a majority of twelve votes, whatever the number of the judges was, the votes of the minority, neutralizing an equal number of votes of the majority, the twelve remaining votes would represent the unanimity of a jury of twelve members, required in England for the condemnation of an accused; but one would be greatly mistaken. Common sense shows that there is a difference between the decision of a tribunal of two hundred and twelve judges, of which one hundred and twelve condemn the accused, while one hundred acquit him, and that of a tribunal of twelve judges unanimous for condemnation. In the first case the hundred votes favorable to the accused warrant in thinking that the proofs are far from attaining the degree of strength which entails conviction; in the second case, the unanimity of the judges leads to the belief that they have attained this degree. But simple common sense does not suffice at all to appreciate the extreme difference of the probability of error in the two cases. It is necessary then to recur to calculus, and one finds nearly one fifth for the probability of error in the first case, and only $\frac{1}{8192}$ for this probability in the second case, a probability which is not one thousandth of the first. It is a confirmation of the principle that the arithmetical ratio is unfavorable to the accused when the number of judges increases. On the contrary, if one takes for a rule the geometrical ratio, the probability of the error of the decision diminishes when the number of judges increases. For example, in the tribunals which can condemn only by a plurality of two thirds of the votes, the probability of the error to be feared is nearly one fourth if the number of the judges is six; it is below $\frac{1}{7}$ if this number is increased to twelve. Thus one ought to be governed neither by the arithmetical ratio nor by the geometrical ratio if one wishes that the probability of error should never be above nor below a given fraction.

But what fraction ought to be determined upon? It is here that the arbitrariness begins and the tribunals offer in this regard the greatest variety. In the special tribunals where five of the eight votes suffice for the condemnation of the accused, the probability of the error to be feared in regard to justice of the judgment is $\frac{65}{256}$, or more than $\frac{1}{4}$. The magnitude of this fraction is dreadful; but that which ought to reassure us a little is the consideration that most frequently the judge who acquits an accused does not regard him as innocent; he pronounces solely that it is not attained by proofs sufficient for condemnation. One is especially reassured by the pity which nature has placed in the heart of man and which disposes the mind to see only with reluctance a culprit in the accused submitted to his judgment. This sentiment, more active in those who have not the habitude of criminal judgments, compensates for the inconveniences attached to the inexperience of the jurors. In a jury of twelve members, if the plurality demanded for the condemnation is eight of twelve votes, the probability of the error to be feared $\frac{1093}{8192}$, or a little more than one eighth, it is almost $\frac{1}{22}$ if this plurality consists of nine votes. In the case of unanimity the probability of the error to be feared is $\frac{1}{8192}$, that is to say, more than a thousand times less than in our juries. This supposes that the unanimity results only from proofs favorable or contrary to the accused; but motives that are entirely strange, ought oftentimes to concur in producing it, when it is imposed upon the jury as a necessary condition of its judgment. Then its decisions depending upon the temperament, the character, the habits of the jurors, and the circumstances in which they are placed, they are sometimes contrary to the decisions which the majority of the jury would have made if they had listened only to the proofs; this seems to me to be a great fault of this manner of judging.

The probability of the decision is too feeble in our juries, and I think that in order to give a sufficient guarantee to innocence, one ought to demand at least a plurality of nine votes in twelve.

CHAPTER XIV.

CONCERNING TABLES OF MORTALITY, AND OF MEAN DURATIONS OF LIFE, OF MARRIAGES, AND OF ASSOCIATIONS.

THE manner of preparing tables of mortality is very simple. One takes in the civil registers a great number of individuals whose birth and death are indicated. One determines how many of these individuals have died in the first year of their age, how many in the second year, and so on. It is concluded from these the number of individuals living at the commencement of each year, and this number is written in the table at the side of that which indicates the year. Thus one writes at the side of zero the number of births; at the side of the year 1 the number of infants who have attained one year; at the side of the year 2 the number of infants who have attained two years, and so on for the rest. But since in the first two years of life the mortality is very great, it is necessary for the sake of greater exactitude to indicate in this first age the number of survivors at the end of each half year.

If we divide the sum of the years of the life of all the individuals inscribed in a table of mortality by the number of these individuals we shall have the mean duration of life which corresponds to this table. For this, we will multiply by a half year the number of deaths in the first year, a number equal to the difference of the numbers of individuals inscribed at the side of the years 0 and 1. Their mortality being distributed over the entire year the mean duration of their life is only a half year. We will multiply by a year and a half the number of deaths in the second year; by two years and a half the number of deaths in the third year; and so on. The sum of these products divided by the number of births will be the mean duration of life. It is easy to conclude from this that we will obtain this duration, by making the sum of the numbers inscribed in the table at the side of each year, dividing it by the number of births and subtracting one half from the quotient, the year being taken as unity. The mean duration of life that remains, starting from any age, is determined in the same manner, working upon the number of individuals who have arrived at this age, as has just been done with the number of births. But it is not at the moment of birth that the mean duration of life is the greatest; it is when one has escaped the dangers of infancy and it is then about forty-three years. The probability of arriving at a certain age, starting from a given age is equal to the ratio of the two numbers of individuals indicated in the table at these two ages.

The precision of these results demands that for the formation of tables we should employ a very great number of births. Analysis gives then very simple formulæ for appreciating the probability that the numbers indicated in these tables will vary from the truth only within narrow limits. We see by these formulæ that the interval of the limits diminishes and that the probability increases in proportion as we take into consideration more births; so that the tables would represent exactly the true law of mortality if the number of births employed were infinite.

A table of mortality is then a table of the probability of human life. The ratio of the individuals inscribed at the side of each year to the number of births is the probability that a new birth will attain this year. As we estimate the value of hope by making a sum of the products of each benefit hoped for, by the probability of obtaining it, so we can equally evaluate the mean duration of life by adding the products of each year by half the sum of the probabilities of attaining the commencement and the end of it, which leads to the result found above. But this manner of viewing the mean duration of life has the advantage of showing that in a stationary population, that is to say, such that the number of births equals that of deaths, the mean duration of life is the ratio itself of the population to the annual births; for the population being supposed stationary, the number of individuals of an age comprised between two consecutive years of the table is equal to the number of annual births, multiplied by half the sum of the probabilities of attaining these years; the sum of all these products will be then the entire population. Now it is easy to see that this sum, divided by the number of annual births, coincides with the mean duration of life as we have just defined it.

It is easy by means of a table of mortality to form the corresponding table of the population supposed to be stationary. For this we take the arithmetical means of the numbers of the table of mortality corresponding to the ages zero and one year, one and two years, two and three years, etc. The sum of all these means is the entire population; it is written at the side of the age zero. There is subtracted from this sum the first mean and the remainder is the number of individuals of one year and upwards; it is written at the side of the year 1. There is subtracted from this first remainder the second mean; this second remainder is the number of individuals of two years and upwards; it is written at the side of the year 2, and so on.

So many variable causes influence mortality that the tables which represent it ought to be changed according to place and time. The divers states of life offer in this regard appreciable differences relative to the fatigues and the dangers inseparable from each state and of which it is indispensable to keep account in the calculations founded upon the duration of life. But these differences have not been sufficiently observed. Some day they will be and then will be known what sacrifice of life each profession demands and one will profit by this knowledge to diminish the dangers.

The greater or less salubrity of the soil ~~sun~~, its elevation, its temperature, the customs of the inhabitants, and the operations of governments have a considerable influence upon mortality. But it is always necessary to precede the investigation of the cause of the differences observed by that of the probability with which this cause is indicated. Thus the ratio of the population to annual births, which one has seen raised in France to twenty-eight and one third, is not equal to twenty-five in the ancient duchy of Milan. These ratios, both established upon a great number of births, do not permit of calling into question the existence among the Milanese of a special cause of mortality, which it is of moment for the government of our country to investigate and remove.

The ratio of the population to the births would increase again if we could diminish and remove certain dangerous and widely spread maladies. This has happily been done for the smallpox, at first by the inoculation of this disease, then in a manner much more advantageous, by the inoculation of vaccine, the inestimable discovery of Jenner, who has thereby become one of the greatest benefactors of humanity.

The smallpox has this in particular, namely, that the same individual is not twice affected by it, or at least such cases are so rare that they may be abstracted from the calculation. This malady, from which few escaped before the discovery of vaccine, is often fatal and causes the death of one seventh of those whom it attacks. Sometimes it is mild, and experience has taught that it can be given this latter character by inoculating it upon healthy persons, prepared for it by a proper diet and in a favorable season. Then the ratio of the individuals who die to the inoculated ones is not one three hundredth. This great advantage of inoculation, joined to those of not altering the appearance and of preserving from the grievous consequences which the natural smallpox often brings, caused it to be adopted by a great number of persons. The practice was strongly recommended, but it was strongly combated, as is nearly always the case in things subject to inconvenience. In the midst of this dispute Daniel Bernoulli proposed to submit to the calculus of probabilities the influence of inoculation upon the mean duration of life. Since precise data of the mortality produced by the smallpox at the various ages of life were lacking, he supposed that the danger of having this malady and that of dying of it are the same at every age. By means of these suppositions he succeeded by a delicate analysis in converting an ordinary table of mortality into that which would be used if smallpox did not exist, or if it caused the death of only a very small number of those affected, and he concludes from it that inoculation would augment by three years at least the mean duration of life, which appeared to him beyond doubt the advantage of this operation. D'Alembert attacked the analysis of Bernoulli: at first in regard to the uncertainty of his two hypotheses, then in regard to its insufficiency in this, that no comparison was made of the immediate danger, although very small, of dying of inoculation, to the very great but very remote danger of succumbing to natural smallpox. This consideration, which disappears when one considers a great number of individuals, is for this reason immaterial for governments and the advantages of inoculation for them still remain; but it is of great weight for the father of a family who must fear, in having his children inoculated, to see that one perish whom he holds most dear and to be the cause of it. Many parents were restrained by this fear, which the discovery of vaccine has happily dissipated. By one of those mysteries which nature offers to us so frequently, vaccine is a preventive of smallpox just as certain as variolar virus, and there is no danger at all; it does not expose to any malady and demands only very little care. Therefore the practice of it has spread quickly; and to render it universal it remains only to overcome the natural inertia of the people, against which it is necessary to strive continually, even when it is a question of their dearest interests.

The simplest means of calculating the advantage which the extinction of a malady would produce consists in determining by observation the number of individuals of a given age who die of it each year and subtracting this number from the number of deaths at the same age. The ratio of the difference to the total number of individuals of the given age would be the probability of dying in the year at this age if the malady did not exist. Making, then, a sum of these probabilities from birth up to any given age, and subtracting this sum from unity, the remainder will be the probability of living to that age corresponding to the extinction of the malady. The series of these probabilities will be the table of mortality relative to this hypothesis, and we may conclude from it, by what precedes, the mean duration of life. It is thus that Duvilard has found that the increase of the mean duration of life, due to inoculation with vaccine, is three years at the least. An increase so considerable would produce a very great increase in the population if the latter, for other reasons, were not restrained by the relative diminution of subsistences.

It is principally by the lack of subsistences that the progressive march of the population is arrested. In all kinds of animals and vegetables, nature tends without ceasing to augment the number of individuals until they are on a level of the means of subsistence. In the human race moral causes have a great influence upon the population. If easy clearings of the forest can furnish an abundant nourishment for new generations, the certainty of being able to support a numerous family encourages marriages and renders them more productive. Upon the same soil the population and the births ought to increase at the same time simultaneously in geometric progression. But when clearings become more difficult and more rare then the increase of population diminishes; it approaches continually the variable state of subsistences, making oscillations about it just as a pendulum whose periodicity is retarded by changing the point of suspension, oscillates about this point by virtue of its own weight. It is difficult to evaluate the *maximum* increase of the population; it appears after observations that in favorable circumstances the population of the human race would be doubled every fifteen years. We estimate that in North America the period of this doubling is twenty-two years. In this state of things, the population, births, marriages, mortality, all increase according to the same geometric progression of which we have the constant ratio of consecutive terms by the observation of annual births at two epochs.

By means of a table of mortality representing the probabilities of human life, we may determine the duration of marriages. Supposing in order to simplify the matter that the mortality is the same for the two sexes, we shall obtain the probability that the marriage will subsist one year, or two, or three, etc., by forming a series of fractions whose common denominator is the product of the two numbers of the table corresponding to the ages of the consorts, and whose numerators are the successive products of the numbers corresponding to these ages augmented by one, by two, by three, etc., years. The sum of these fractions augmented by one half will be the mean duration of marriage, the year being taken as unity. It is easy to extend the same rule to the mean duration of an association formed of three or of a greater number of individuals.

CHAPTER XVII.

CONCERNING THE VARIOUS MEANS OF APPROACHING CERTAINTY.

INDUCTION, analogy, hypotheses founded upon facts and rectified continually by new observations, a happy tact given by nature and strengthened by numerous comparisons of its indications with experience, such are the principal means for arriving at truth.

If one considers a series of objects of the same nature one perceives among them and in their changes ratios which manifest themselves more and more in proportion as the series is prolonged, and which, extending and generalizing continually, lead finally to the principle from which they were derived. But these ratios are enveloped by so many strange circumstances that it requires great sagacity to disentangle them and to recur to this principle: it is in this that the true genius of sciences consists. Analysis and natural philosophy owe their most important discoveries to this fruitful means, which is called *induction*. Newton was indebted to it for his theorem of the binomial and the principle of universal gravity. It is difficult to appreciate the probability of the results of induction, which is based upon this that the simplest ratios are the most common; this is verified in the formulæ of analysis and is found again in natural phenomena, in crystallization, and in chemical combinations. This simplicity of ratios will not appear astonishing if we consider that all the effects of nature are only mathematical results of a small number of immutable laws.

Yet induction, in leading to the discovery of the general principles of the sciences, does not suffice to establish them absolutely. It is always necessary to confirm them by demonstrations or by decisive experiences; for the history of the sciences shows us that induction has sometimes led to inexact results. I shall cite, for example, a theorem of Fermat in regard to primary numbers. This great geometrician, who had meditated profoundly upon this theorem, sought a formula which, containing only primary numbers, gave directly a primary number greater than any other number assignable. Induction led him to think that two, raised to a power which was itself a power of two, formed with unity a primary number. Thus, two raised to the square plus one, forms the primary number five; two raised to the second power of two, or sixteen, forms with one the primary number seventeen. He found that this was still true for the eighth and the sixteenth power of two augmented by unity; and this induction, based upon several arithmetical considerations, caused him to regard this result as general. However, he avowed that he had not demonstrated it. Indeed, Euler recognized that this does not hold for the thirty-second power of two, which, augmented by unity, gives 4,294,967,297, a number divisible by 641.

We judge by induction that if various events, movements, for example, appear constantly and have been long connected by a simple ratio, they will continue to be subjected to it; and we conclude from this, by the theory of probabilities, that this ratio is due, not to hazard, but to a regular cause. Thus the equality of the movements of the rotation and the revolution of the moon; that of the movements of the nodes of the orbit and of the lunar equator, and the coincidence of these nodes; the singular ratio of the movements of the first three satellites of Jupiter, according to which the mean longitude of the first satellite, less three times that of the second, plus two times that of the third, is equal to two right angles; the equality of the interval of the tides to that of the passage of the moon to the meridian; the return of the greatest tides with the syzygies, and of the smallest with the quadratures; all these things, which have been maintained since they were first observed, indicate with an extreme probability, the existence of constant causes which geometricians have happily succeeded in attaching to the law of universal gravity, and the knowledge of which renders certain the perpetuity of these ratios.

The chancellor Bacon, the eloquent promoter of the true philosophical method, has made a very strange misuse of induction in order to prove the immobility of the earth. He reasons thus in the *Novum Organum*, his finest work: "The movement of the stars from the orient to the occident increases in swiftness, in proportion to their distance from the earth. This movement is swiftest with the stars; it slackens a little with Saturn, a little more with Jupiter, and so on to the moon and the highest comets. It is still perceptible in the atmosphere, especially between the tropics, on account of the great circles which the molecules of the air describe there; finally, it is almost inappreciable with the ocean; it is then nil for the earth." But this induction proves only that Saturn, and the stars which are inferior to it, have their own movements, contrary to the real or apparent movement which sweeps the whole celestial sphere from the orient to the occident, and that these movements appear slower with the more remote stars, which is conformable to the laws of optics. Bacon ought to have been struck by the inconceivable swiftness which the stars require in order to accomplish their diurnal revolution, if the earth is immovable, and by the extreme simplicity with which its rotation explains how bodies so distant, the ones from the others, as the stars, the sun, the planets, and the moon, all seem subjected to this revolution. As to the ocean and to the atmosphere, he ought not to compare their movement with that of the stars which are detached from the earth; but since the air and the sea make part of the terrestrial globe, they ought to participate in its movement or in its repose. It is singular that Bacon, carried to great prospects by his genius, was not won over by the majestic idea which the Copernican system of the universe offers. He was able, however, to find in favor of that system, strong analogies in the discoveries of Galileo, which were continued by him. He has given for the search after truth the precept, but not the example. But by insisting, with all the force of reason and of eloquence, upon the necessity of abandoning the insignificant subtilities of the school, in order to apply oneself to observations and to experiences, and by indicating the true method of ascending to the general causes of phenomena, this great philosopher contributed to the immense strides which the human mind made in the grand century in which he terminated his career.

Analogy is based upon the probability, that similar things have causes of the same kind and produce the same effects. This probability increase as the similitude becomes more perfect. Thus we judge without doubt that beings provided with the same organs, doing the same things, experience the same sensations, and are moved by the same desires. The probability that the animals which resemble us have sensations analogous to ours, although a little inferior to that which is relative to individuals of our species, is still exceedingly great; and it has required all the influence of religious prejudices to make us think with some philosophers that animals are mere automatons. The probability of the existence of feeling decreases in the same proportion as the similitude of the organs with ours diminishes, but it is always very great, even with insects. In seeing those of the same species execute very complicated things exactly in the same manner from generation to generation, and without having learned them, one is led to believe that they act by a kind of affinity analogous to that which brings together the molecules of crystals, but which, together with the sensation attached to all animal organization, produces, with the regularity of chemical combinations, combinations that are much more singular; one might, perhaps, name this mingling of elective affinities and sensations *animal affinity.* Although there exists a great analogy between the organization of plants and that of animals, it does not seem to me sufficient to extend to vegetables the sense of feeling; but nothing authorizes us in denying it to them.

Since the sun brings forth, bythe beneficent action of its light and of its heat, the animals and plants which cover the earth, we judge by analogy that it produces similar effects upon the other planets; for it is not natural to think that the cause whose activity we see developed in so many ways should be sterile upon so great a planet as Jupiter, which, like the terrestrial globe, has its days, its nights, and its years, and upon which observations indicate changes which suppose very active forces. Yet this would be giving too great an extension to analogy to conclude from it the similitude of the inhabitants of the planets and of the earth. Man, made for the temperature which he enjoys, and for the element which he breathes, would not be able, according to all appearance, to live upon the other planets. But ought there not to be an infinity of organization relative to the various constitutions of the globes of this universe? If the single difference of the elements and of the climates make so much variety in terrestrial productions, how much greater the difference ought to be among those of the various planets and of their satellites! The most active imagination can form no idea of it; but their existence is very probable.

We are led by a strong analogy to regard the stars as so many suns endowed, like ours, with an attractive power proportional to the mass and reciprocal to the square of the distances; for this power being demonstrated for all the bodies of the solar system, and for their smallest molecules, it appears to appertain to all matter. Already the movements of the small stars, which have been called *double*, on account of their conjunction, appear to indicate it; a century at most of precise observations, by verifying their movements of revolution, the ones about the others, will place beyond doubt their reciprocal attractions.

The analogy which leads us to make each star the centre of a planetary system is far less strong than the preceding one; but it acquires probability by the hypothesis which has been proposed in regard to the formation of the stars and of the sun; for in this hypothesis each star, having been like the sun, primitively environed by a vast atmosphere, it is natural to attribute to this atmosphere the same effects as to the solar atmosphere, and to suppose that it has produced, in condensing, planets and satellites.

A great number of discoveries in the sciences is due to analogy. I shall cite as one of the most remarkable, the discovery of atmospheric electricity, to which one has been led by the analogy of electric phenomena with the effects of thunder.

The surest method which can guide us in the search for truth, consists in rising by induction from phenomena to laws and from laws to forces. Laws are the ratios which connect particular phenomena together: when they have shown the general principle of the forces from which they are derived, one verifies it either by direct experiences, when this is possible, or by examination if it agrees with known phenomena; and if by a rigorous analysis we see them proceed from this principle, even in their small details, and if, moreover, they are quite varied and very numerous, then science acquires the highest degree of certainty and of perfection that it is able to attain. Such, astronomy has become by the discovery of universal gravity. The history of the sciences shows that the slow and laborious path of induction has not always been that of inventors. The imagination, impatient to arrive at the causes, takes pleasure in creating hypotheses, and often it changes the facts in order to adapt them to its work; then the hypotheses are dangerous. But when one regards them only as the means of connecting the phenomena in order to discover the laws; when, by refusing to attribute them to a reality, one rectifies them continually by new observations, they are able to lead to the veritable causes, or at least put us in a position to conclude from the phenomena observed those which given circumstances ought to produce.

If we should try all the hypotheses which can be formed in regard to the cause of phenomena we should arrive, by a process of exclusion, at the true one. This means has been employed with success; sometimes we have arrived at several hypotheses which explain equally well all the facts known, and among which scholars are divided, until decisive observations have made known the true one. Then it is interesting, for the history of the human mind, to return to these hypotheses, to see how they succeed in explaining a great number of facts, and to investigate the changes which they ought to undergo in order to agree with the history of nature. It is thus that the system oi Ptolemy, which is only the realization of celestial appearances, is transformed into the hypothesis of the movement of the planets about the sun, by rendering equal and parallel to the solar orbit the circles and the epicycles which he causes to be described annually, and the magnitude of which he leaves undetermined. It suffices, then, in order to change this hypothesis into the true system of the world, to transport the apparent movement of the sun in a sense contrary to the earth.

It is almost always impossible to submit to calculus the probability of the results obtained by these various means; this is true likewise for historical facts. But the totality of the phenomena explained, or of the testimonies, is sometimes such that without being able to appreciate the probability we cannot reasonably permit ourselves any doubt in regard to them. In the other cases it is prudent to admit them only with great reserve.