MVE155 - Statistial Inference
Assignment 2 - Parameter Estimation/Bootstrap

Adam Wirehed
wirehed@student.chalmers.se

February 21, 2020

1 Exercise A

The data given in the file "gamma-arrivals.txt" is illustrated in a histogram in figure 1. The distribution of the data looks to be similar to an exponential distribution. Which the gamma distribution is also able to resemble if its shape paramater α is close to one. From this simple figure I would say that the gamma distribution is a plausible model.

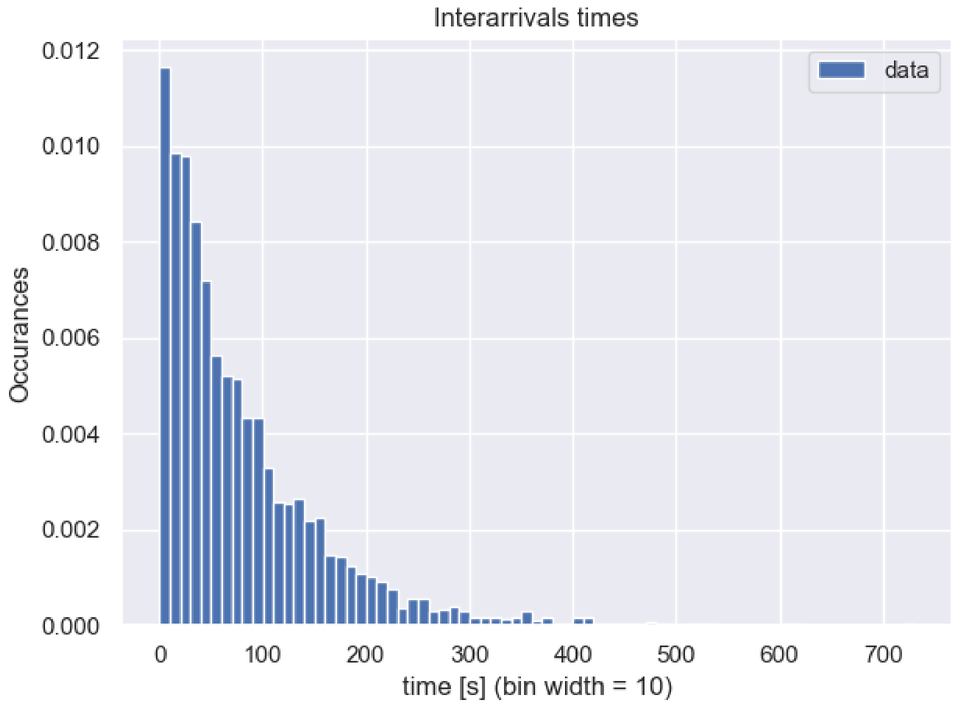

Figure 1: Histogram of interarrival times of photons

2 Exercise B

To estimate the parameters of the gamma distribution using method of moments we use the following equations:

$$\bar{x} = \frac{x_1 + ... + x_n}{n}$$

$$\bar{x^2} = \frac{x_1^2 + ... + x_n^2}{n}$$

$$\alpha = E(x)\lambda$$

$$E(x^2) = \sigma^2 + \mu^2 = \frac{\alpha(1+\alpha)}{\lambda^2}$$

We can re-write the last two equations using sample mean of the data:

$$E(x) = \bar{x}, E(x^2) = \bar{x^2}$$

$$\bar{\alpha} = \frac{\bar{x}^2}{\bar{x^2} - \bar{x}^2}$$

$$\bar{\lambda} = \frac{\bar{\alpha}}{\bar{x}}$$

From the said data, we then get parameter estimations using MME (method of moments) to be the values presented in table 1.

When estimating the gamma parameters using maximum likelihood, we need to maximize the likelihood function (1), by solving (2) numerically (derivative of (1) set to 0).

$$L(\alpha, \lambda) = \frac{\lambda^{n\alpha}}{\Gamma^n(\alpha)}(t_2)^{\alpha-1}e^{-\lambda(t_1)} \tag{1}$$

$$\frac{\Gamma'(\hat{\alpha})}{\Gamma(\hat{\alpha})} - ln(\gamma) - ln(t_2) = 0 \tag{2}$$

$$t_1 = \sum_{i=1}^{n} x_j$$

$$t_2 = \prod_{j=1}^{n} x_j$$

Instead of implementing all of this in Python, the library *Scipy* does parameter estimation using MLE (maximum likelihood) with the method *"distribution".fit*. Here I use the input argument *floc=0* to fixate the location parameter to 0. The parameter estimations computed are presented in table 1.

Table 1: Computed parameter estimations for Gamma distribution

Param. / Method	α	λ
MME	1.01235	0.01266
MLE	1.02633	0.01284

There is a very slight difference in value between the different estimate methods. Two gamma distribution based on the values computed is plotted together with the original data in figure 2. The fit does look reasonable and the shape parameter *alpha* is close to 1 which was expected because of the resemblance to the exponential distribution.

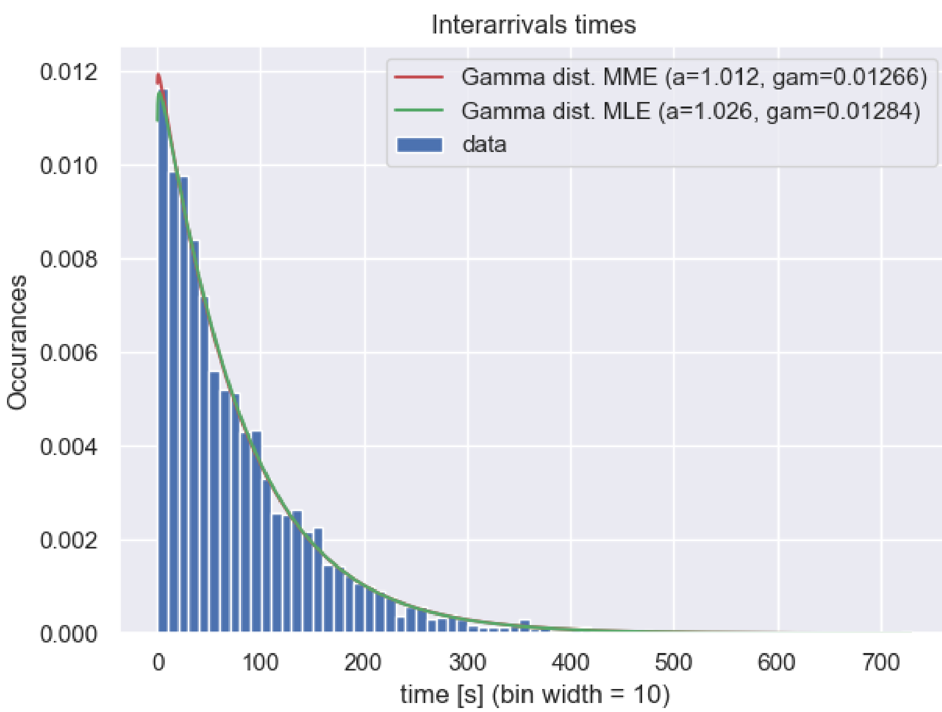

Figure 2: Histogram of interarrival times of photons together with two gamma distributions (red curve behind the green)

3 Exercise C

When using bootstrap to estimate the parameters we use the values computed in the previous exercise (table 1). These parameters are used to "create" two gamma distributions to sample from (bootstrap sampling):

$$\text{Gamma dist. from MME: } Gam(\bar{\alpha} = 1.01235, \bar{\lambda} = 0.01266)$$

$$\text{Gamma dist. from MLE: } Gam(\hat{\alpha} = 1.02633, \hat{\lambda} = 0.01284)$$

For each of these samples (B=100 for this task) we compute the parameters using MME or MLE like in exercise B.

The mean of all the samples for both α and λ (for both MME and MLE) is computed with:

$$\bar{\alpha}(\text{mean for the samples}) = \frac{1}{B}(\alpha_{s,1} + ... + \alpha_{s,B}), \quad \text{where } \alpha_{s,i} \text{ is the computed alpha for sample i}$$

The standard error is then computed by:

$$s_\alpha = \sqrt{\frac{1}{B}\sum_{j=1}^{B}(\alpha_{s,i} - \bar{\alpha})^2}$$

The different values for the standard error of α and λ using MME and MLE are presented in table 2. Here we can see that the standard error for α using MLE is almost half the size of MME.

Table 2: Computed parameter estimations for Gamma distribution

Method \ Param.	s_α	s_λ
MME	0.03368	0.00046
MLE	0.01918	0.00033

4 Exercise D

To form a 95% approximately confidence interval we simple compute it as:

$$I_\alpha = [\bar{\alpha} - 1.96 * s_\alpha, \bar{\alpha} + 1.96 * s_\alpha]$$

Where $\bar{\alpha}$ and the standard error used is the values computed in exercise C. The different confidence intervals computed are presented in table 3. The intervals follows the standard error as expected. So the interval for α is smaller for MLE than MME.

Table 3: Computed confidence intervals (95%) for gamma parameters (MME and MLE)

Param. / Method	I_α	I_λ
MME	[0.94633, 1.07836]	[0.01176, 0.01354]
MLE	[0.98874, 1.06391]	[0.01219, 0.01348]