

Stochastic Gradient Variance Reduction by Solving a Filtering Problem

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Abstract—Deep neural networks (DNN) are typically optimized using stochastic gradient descent (SGD). However, the estimation of the gradient using stochastic samples tends to be noisy and unreliable, resulting in large gradient variance and bad convergence. In this paper, we propose Filter Gradient Decent (FGD), an efficient stochastic optimization algorithm that make consistent estimation of the local gradient by solving an adaptive filtering problem with different design of filters. Our method reduces estimation variance in stochastic gradient descent by incorporating the historic state of the optimization. It aims to improve noisy gradient direction as well as accelerate the convergence of learning. We demonstrate the effectiveness of the proposed Filter Gradient Decent under various optimization tasks where it is shown to achieve superior and robust performance. To the best of our knowledge, we are the first to make analogy between gradient estimation between filtering problem in signal processing.

I. INTRODUCTION

Among common techniques to solve the optimization problem, stochastic gradient decent (SGD) method has been a widespread practice in machine learning. It aims to minimize the empirical risk by taking the gradient computed from the randomly sampled data iteratively. Considering training a parametric model $f(\cdot; \theta)$ parameterized by $\theta \in \mathbb{R}^d$ over a finite set of training data $\mathcal{D} = \{(\mathbf{x}_i, y_i)\}_{i=1}^N$. At iteration t , a subset of the dataset $\mathcal{D}_t \subseteq \mathcal{D}$ is randomly sampled. Denote $J(\theta_t; \cdot)$ as the objective function and $\mathbf{g}_t = \nabla_{\theta} J(\theta_t; \mathcal{D}_t)$ as the observed gradient, we minimize the objective by updating the weight θ by

$$\theta_{t+1} = \theta_t - \eta \mathbf{g}_t \quad (1)$$

The learning rate η determines the stepsize heading to the minimum.

A critical problem of SGD is that it is highly “greedy”. On the one hand, the sampled data points \mathcal{D}_t might not well represent the overall data statistics, e.g. class-biased and noise-corrupted, which leads to undesirable error for the gradient estimation. Meanwhile, the stochastic gradient only considers local information for the error surface, therefore might spend much time bouncing around ravines with poorly conditioned Hessian matrix. Consequently, the estimated gradient has a large variance [1], leading to slower convergence and worse performance. This phenomenon is most severe for the non-smooth and non-convex optimization problems, like training deep neural networks. Existing problems in stochastic gradient

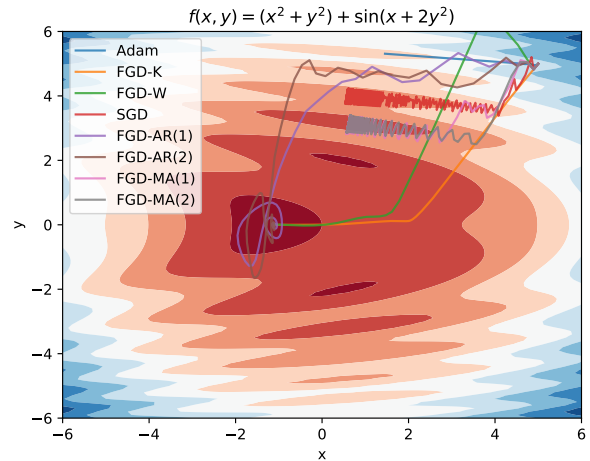


Fig. 1. Illustration of Filter Gradient Decent (FGD) for a non-convex optimization problem. Using common filtering techniques to “denoise” in gradient estimation, FGD is able to reduce the variance of the gradient as well as to get rid of the local minimum in SGD.

descent forces us to think about how to alleviate the randomness in gradient calculation and solve the problem of poor convergence.

In the theory of stochastic processes, the filtering problem is a model for a state estimation problems to establish a “best estimate” for the true value of some system from an incomplete, potentially noisy set of observations on that system. This is closely related to our problem: *Given observations $\{\mathbf{g}_i\}$ for $0 \leq i \leq t$, what is the best estimate $\hat{\mathbf{g}}_{t|t}$ of the true state \mathbf{g}_t of the system based on those observations?*

Inspired by the success experience of applying filters in signal denoising and reconstruction, we introduce a general framework **Filter Gradient Decent (FGD)**, a remedy to the noisy gradient with the filtering to reduce variance and make consistent estimation of gradient in stochastic optimization. It takes the stochastic gradient as a special signal and solving the filtering problem or denoising problem to estimate the true gradient direction under the noisy observations. As a simple illustration in Figure 1, FGD is able to reduce the variance of the gradient as well as to get rid of the local minimum in SGD. This general framework can be applied to any filtering

techniques.

Our contributions are summarized as follows:

- We propose Filter Gradient Decent (FGD), which incorporates signal filtering into stochastic gradient descent to reduce the noise and randomness of gradient estimation.
- We show that our framework can be applied to any filters. In this paper, we compare four kinds of filters including (1) Moving-average (MA) filter (2) Autoregressive (AR) filter (3) Kalman filter (4) Wavelet filter.
- Experiments verify that Filter Gradient Decent is effective in gradient variance reduction and finding the global minimum for non-convex optimization problem.
- We train deep neural networks with Filter Gradient Decent under various datasets, which demonstrates superior performance and fast convergence.

II. BACKGROUND

A. Stochastic Gradient Descent

Stochastic gradient descent is a way to minimize an objective function by updating the parameters in the opposite direction of the gradient. Vanilla stochastic gradient descent, however, does not guarantee good convergence, either suffering from the large variance of the gradient or trapped in suboptimal local minima. Momentum [2] helps accelerate SGD by adding a ratio γ of the update vector of the past time step to the current update vector. Adagrad [3] uses a different learning rate for every parameter θ_i at every time step t . Adadelta [4] and RMSprop [5] are two extensions of Adagrad that seeks to restrict its aggressive learning rate. Adaptive Moment Estimation (Adam)[6] incorporates adaptive learning rates with a decaying average of past gradients similar to momentum. Later variants like AdaMax[6] and Nadam[7] incrementally extends the framework of Adam.

B. Variance Reduction in Gradient Descent

SGD uses the noisy gradient estimated from random data samples. This method introduces large variance to gradient estimation, which is harmful to the model in terms of convergence. Apparently, averaging is the off-the-shelf yet simple solution. Averaged Stochastic Gradient Descent (ASGD) [8] is the simplest method of variance reduction by keeping an average of some set of the weight vectors. Stochastic Averaged Gradient (SAG) [9] builds on the ASGD method by keeping a moving average of n past gradients for updates. [1] builds a variance reduction technique, which makes use of control variates to augment the noisy gradient and thereby reduce its variance. [10] proposes to control the variance of the reparameterization gradient estimator in Monte Carlo variational inference (MCVI).

C. Filter

In signal processing, a filter is a device or process that removes some unwanted components or features from a signal. We introduce four types of filters used in this paper and their basic concepts.

Moving-average Filter. In time series analysis, the moving-average (MA) model specifies that the output variable depends linearly on the current and various past values of a stochastic term. $MA(q)$ refers to the moving average model of order q , where the output y_t is a weighted linear combination of x_t and q past input $\{x_{t-1}, \dots, x_{t-p}\}$

$$y_t = x_t + \sum_{i=1}^q a_i x_{t-i} \quad (2)$$

Autoregressive Filter. An autoregressive (AR) model is a type of random process that the output variable depends linearly on its own previous values and on a stochastic term. $AR(p)$ indicates an autoregressive model of order p , where the output y_t is a weighted linear combination of input x_t and p past output value $\{y_{t-1}, \dots, y_{t-p}\}$

$$y_t = x_t + \sum_{i=1}^p b_i y_{t-i} \quad (3)$$

Kalman Filter. The Kalman filter [11] addresses the general problem of trying to estimate the state $\mathbf{x} \in \mathbb{R}^n$ of a discrete-time controlled process that is governed by the linear stochastic difference equation with a measurement $\mathbf{y} \in \mathbb{R}^m$

$$\mathbf{x}_t = \mathbf{F}_{t-1} \mathbf{x}_{t-1} + \mathbf{I}_{t-1} \mathbf{u}_t + \mathbf{w}_{t-1}, \quad \mathbf{y}_t = \mathbf{C}_{t-1} \mathbf{x}_t + \mathbf{v}_t \quad (4)$$

$\mathbf{F}_t \in \mathbb{R}^{n \times n}$ is the state transition matrix which relates the state to the previous step, $\mathbf{I}_t \in \mathbb{R}^{n \times k}$ is the control matrix that relates the optional control input $\mathbf{u} \in \mathbb{R}^k$ to current state, and $\mathbf{C}_t \in \mathbb{R}^{m \times n}$ is measurement matrix. The random variables $\mathbf{w}_t \sim \mathcal{N}(0, Q)$ and $\mathbf{v}_t \sim \mathcal{N}(0, R)$ represent the process and measurement noise respectively. They are assumed to be independent of each other, white, and with normal probability distributions.

The Kalman filter solves the problem in two steps: the prediction step, where the next state of the system is predicted given the previous measurements, and the update step, where the current state of the system is estimated given the measurement at that time step. The steps translate to equations as follows

- Prediction Step

$$\hat{\mathbf{x}}_{t|t-1} = \mathbf{F}_{t-1} \mathbf{x}_{t-1} + \mathbf{I}_t \mathbf{u}_t \quad (5)$$

$$\hat{\mathbf{P}}_{t|t-1} = \mathbf{F}_{t-1} \mathbf{P}_{t-1|t-1} \mathbf{F}_{t-1}^T + Q \quad (6)$$

- Update Step

$$\hat{\mathbf{y}}_t = \mathbf{y}_t - \mathbf{C} \hat{\mathbf{x}}_{t|t-1} \quad (7)$$

$$\mathbf{K}_t = \hat{\mathbf{P}}_{t|t-1} \mathbf{C}^T (\mathbf{C} \hat{\mathbf{P}}_{t|t-1} \mathbf{C}^T + R)^{-1} \quad (8)$$

$$\hat{\mathbf{x}}_{t|t} = \hat{\mathbf{x}}_{t|t-1} + \mathbf{K}_t \hat{\mathbf{y}}_t \quad (9)$$

$$\mathbf{P}_{t|t} = (\mathbf{I} - \mathbf{K}_t \mathbf{C}) \hat{\mathbf{P}}_{t|t-1} \quad (10)$$

Wavelet Filter. A wavelet is a mathematical function used to divide a given function or continuous-time signal into different scale components. A wavelet transform is the representation of a function by wavelets. By shifting and scaling a mother

wavelet function $\Psi(t)$, any function $f(t)$ can be decomposed into coefficients with multi-resolution wavelet function.

$$\Psi_{a,b}(t) = \frac{1}{\sqrt{b}} \Psi\left(\frac{t-a}{b}\right), \gamma_{a,b} = \int f(t) \Psi_{a,b}(t) dt \quad (11)$$

where a and b are translation and scaling coefficients, $\gamma_{a,b}$ is the wavelet coefficients correspond to wavelet $\Psi_{a,b}(t)$. A discrete wavelet transform (DWT) is any wavelet transform for which the wavelets are discretely sampled. There are three basic steps to filtering signals using wavelets. (1) Decompose the signal using the DWT.

$$\Psi_{j,k}(t) = \frac{1}{\sqrt{2^j}} \Psi\left(\frac{t-k2^j}{2^j}\right), \gamma_{j,k} = \sum_t f(t) \Psi_{j,k}(t) \quad (12)$$

(2) Filter the signal in the wavelet space using thresholding operation T .

$$\gamma'_{j,k} = T(\gamma_{j,k}) \quad (13)$$

(3) Invert the filtered signal to reconstruct the filtered signal using the inverse DWT.

$$f'(t) = \sum_i \sum_k \gamma'_{j,k} \Psi_{j,k}(t) \quad (14)$$

The filtering of signals using wavelets is based on the idea that as the DWT decomposes the signal into details and approximation parts. At some scale the details contain mostly the insignificant noise and can be removed or zeroed out using thresholding without affecting the signal.

III. GRADIENT ESTIMATION WITH A FILTERING PROBLEM

In this paper, we make a bold analogy between gradient and high-dimensional signal and reliably estimate gradient by solving a filtering problem. Given the current and past noisy gradient observation $\{g_i\}_{i=0}^t$, we estimate the true state of the current gradient $\hat{g}_{t|t}$ using filter F

$$\hat{g}_{t|t} = F(g_t, g_{t-1}, \dots, g_0) \quad (15)$$

The estimated gradient then is utilized to update the model weight θ via typical gradient decent. Algorithm 1 shows the overall framework of Filter Gradient Decent.

Algorithm 1: Filter Gradient Decent

Result: Network Parameter θ

Initialize the Network Parameter with θ_0 and Filter F with parameter Γ ;

for $t = 0, \dots, T-1$ **do**

 Compute Gradient:

$$g_t = \nabla_{\theta} J(\theta_t; \mathcal{D}_t);$$

 Filtering Step:

$$\hat{g}_{t|t} = F(g_t, g_{t-1}, \dots, g_0);$$

 Gradient Decent:

$$\theta_{t+1} = \theta_t - \eta \hat{g}_{t|t};$$

end

The different filter selection models the various parametric dependencies between the true state and the gradient computed from randomly sampled data. It works as a low-pass filter that removes the high frequency components (randomness and

variance) in the gradient observations and enhance the low frequency components (consistent direction of the gradient). Gradient estimation with filter gets rid of the drawbacks of local gradient and data randomness by considering the historical state of the gradient, thus become more consistent and robust. We select four most well-known filters (1) MA filter (2) AR filter (3) Kalman filter (4) Wavelet Filter as F and would like to study how different filters affect the gradient estimation.

A. Solution 1: Moving-average for Gradient Decent

When assuming that the observed and true gradients follows a moving-average model, we can approximate the true gradient $\hat{g}_{t|t}$ using an $MA(q)$ model

$$\hat{g}_{t|t} = g_t + \sum_{i=1}^q a_i g_{t-i}$$

g_{t-i} is the past stochastic gradient and a_i is the weight for historical gradient at $t-i$ iteration. We call this method Filter Gradient Decent with Moving-average filter (FGD-MA). It keep a weighted averaging of q past gradients observation.

B. Solution 2: Autoregressive for Gradient Decent

When the autoregressive model best describe the relation between the observed and true gradients follows, we may also estimate the true gradient $\hat{g}_{t|t}$ using an $AR(p)$ model

$$\hat{g}_{t|t} = g_t + \sum_{i=1}^p b_i \hat{g}_{t-i|t-i}$$

The b_i is the weight corresponds to historical gradient estimation $\hat{g}_{t-i|t-i}$. We call this model Filter Gradient Decent with Autoregressive filter (FGD-AR). Interestingly, FGD-AR with order 1 turns out to be the momentum method that commonly used in stochastic gradient decent

$$v_t = g_t + \gamma v_{t-1}$$

This shows that momentum method is one special case for FGD-AR. FGD-AR extends the current momentum term to allow longer time dependency for gradient estimation.

C. Solution 3: Kalman filter for Gradient Decent

To reduce the variance of the gradient, we adopt a Kalman filter to estimate the true state of gradient upon a filtering problem. We assume the gradient follows a linear scholastic model

$$\hat{g}_t = \mathbf{F}_{t-1} \hat{g}_{t-1} + \mathbf{w}_{t-1}, \quad g_t = \mathbf{C}_{t-1} \hat{g}_t + \mathbf{v}_{t-1} \quad (16)$$

where g_t is the observed gradient and \hat{g}_t refers to real gradient. We adopt the Kalman filter to give the minimum mean-square estimate of real gradient $\hat{g}_{t|t}$ given previous state $\hat{g}_{t-1|t-1}$ and observation g_t at time t .

$$\hat{g}_{t|t} = KF(g_t, \hat{g}_{t-1|t-1}) \quad (17)$$

By assuming that the gradient for each parameter is independent, we need to maintain a Kalman filter for each parameter.

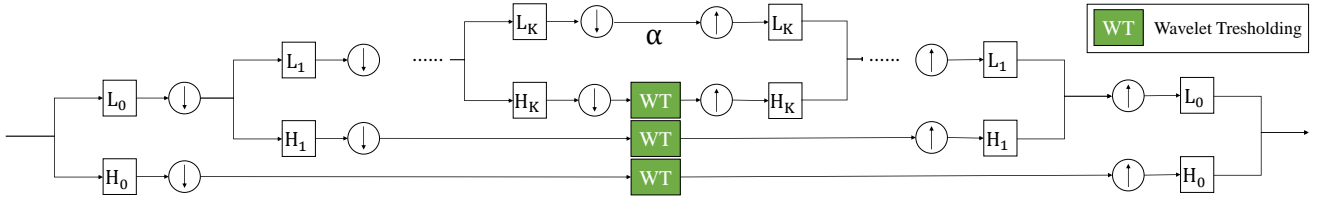


Fig. 2. Pipeline for Wavelet Gradient Filtering for Gradient Decent. (1) Forward DWT to get multi-resolution representation of the gradients (2) Thresholding and amplification on wavelet domain (3) Inverse DWT to spatial domain.

To reduce the computational burden, we simplified our method to a scalar Kalman filter: $\mathbf{F}_t = \gamma I$, $\mathbf{C}_t = cI$, $Q = \sigma_Q^2 I$, $R = \sigma_R^2 I$, $\mathbf{P}_0 = p_0 I$, where I is the identity matrix. This design cancels the time-consuming matrix multiplication, with only a scalar Kalman filter per parameter.

D. Solution 4: Wavelet Gradient Decent

Wavelet Filtering is widely adopted in signal processing to remove certain frequency component of the signal. In order to better eliminate gradient noise, we consider the multi-resolution wavelet filter that soft-threshold highpass coefficients at each level. We also intend to boost the low-frequency component of the gradient by amplify the lowpass coefficient by a hyper-parameter of α . Figure 2 explain the overall pipeline for our proposed Filter Gradient Decent with Wavelet filter (FGD-W). The procedure can be summarized into three steps

- *Forward DWT* We first apply K -level wavelet transformation for L historical gradients to get multi-resolution representation of the signal

$$(\mathbf{L}_K, \mathbf{H}_K, \dots, \mathbf{H}_0) = DWT(\mathbf{g}_t, \dots, \mathbf{g}_{t-L}) \quad (18)$$

\mathbf{L}_i and \mathbf{H}_i are the lowpass and highpass coefficient vector at level i . Particularly, we assume independence for different elements of the gradient and maintain a 1D DWT for each parameter.

- *Filter in wavelet domain* Soft-thresholding is performed on all highpass components \mathbf{H}_i at each level with operation T and amplify the lowpass band by a coefficient of $\alpha \geq 1$

$$\mathbf{H}'_i = T(\mathbf{H}_i), \quad \mathbf{L}'_K = \alpha \mathbf{L}_K \quad (19)$$

where \mathbf{H}'_i and \mathbf{L}'_K is the filtered highpass and low pass coefficients in wavelets domain. For simplicity, we adopt the same threshold on all levels.

- *Inverse DWT* The filtered coefficients in wavelet domain are converted back to spatial domain using inverse DWT

$$\mathbf{g}_{t|t}, \dots, \mathbf{g}_{t-L|t} = IDWT(\mathbf{L}'_K, \mathbf{H}'_K, \dots, \mathbf{H}'_0)$$

IV. EXPERIMENT

To empirically evaluate the proposed method, we first show a numerical problem for non-convex optimization. We also investigated its performance in optimizing different popular machine learning models, including multilayer fully connected

neural networks and deep convolutional neural networks. Using various models and datasets, we demonstrate FGD with various filter designs can efficiently solve the numerical optimization problems and practical deep learning problems.

A. Study 1: Non-convex optimization

We compare the scenario of optimizing a well designed non-convex problem using (1) SGD (2) Adam [6] (3) FGD-MA with order 1 and 2 (4) FGD-AR with order 1 (vanilla momentum) and 2 (5) FGD-K and (6) FGD-W. In this experiment, we intend to minimize a non-convex function

$$f(x, y) = (x^2 + y^2) + 5 \sin(x + y^2) \quad (20)$$

Experiment Setting For each experiment, we select a learning rate from $\eta = \{0.01, 0.03\}$ to minimize this function. The initial point is fixed to $(x, y) = (5, 5)$. For Adam, we use the default setting that $\beta_1 = 0.9, \beta_2 = 0.99$. For FGD-MA and FGD-AR, we select $a_1 = b_1 = 0.9$ for order 1 model and $a_1 = b_1 = 0.8, a_2 = b_2 = 0.1$ for order 2. For FGD-K, we select $\gamma = 2, c = 1, \sigma_Q^2 = 0.01, \sigma_R = 2$. For FGD-W, we use Daubechies 4 (db4) wavelet and $K = 3$ and $L = 8$. We apply soft-thresholding on each highpass band with threshold of 0.2 and select $\alpha = 5$. Each Algorithm runs for $T = 500$ iterations.

Result Figure 3(a) visualizes the function value surface. We also demonstrate the trajectory of the optimization process for different methods in Figure 3(b) and 3(c). Figure 3(d) and 3(e) is the plot of the function value along with the number of iterations. Under both learning rates, SGD, FGD-MA(1) and FGD-MA(2) was finally trapped in a local minimum. In comparison, the momentum method (FGD-AR(1)) and FGD-AR(2) succeeded in finding the optimal solution when the learning rate is properly set (0.01 in this experiment) but still fails when the learning rate increases to 0.03. Surprisingly, Adam, as the most popular method, did not improve much and performed extremely poor on both setting, mainly because the non-smoothing property of the optimization problem. On the other hand, FGD-K and FGD-W is able to find the global solution constantly. This shows that, compared with vanilla momentum method that accelerate the gradient blindly, FGD-K and FGD-W can robustly solve the non-convex optimization problem regardless of hyper-parameter setting.

Figure 3(f), 3(h), 3(g) and 3(i) illustrates the variance of the gradient for each parameter. SGD bounces around ravines, resulting in noisy gradients and large variance on both x and y . Moving-average model introduces large variance to the

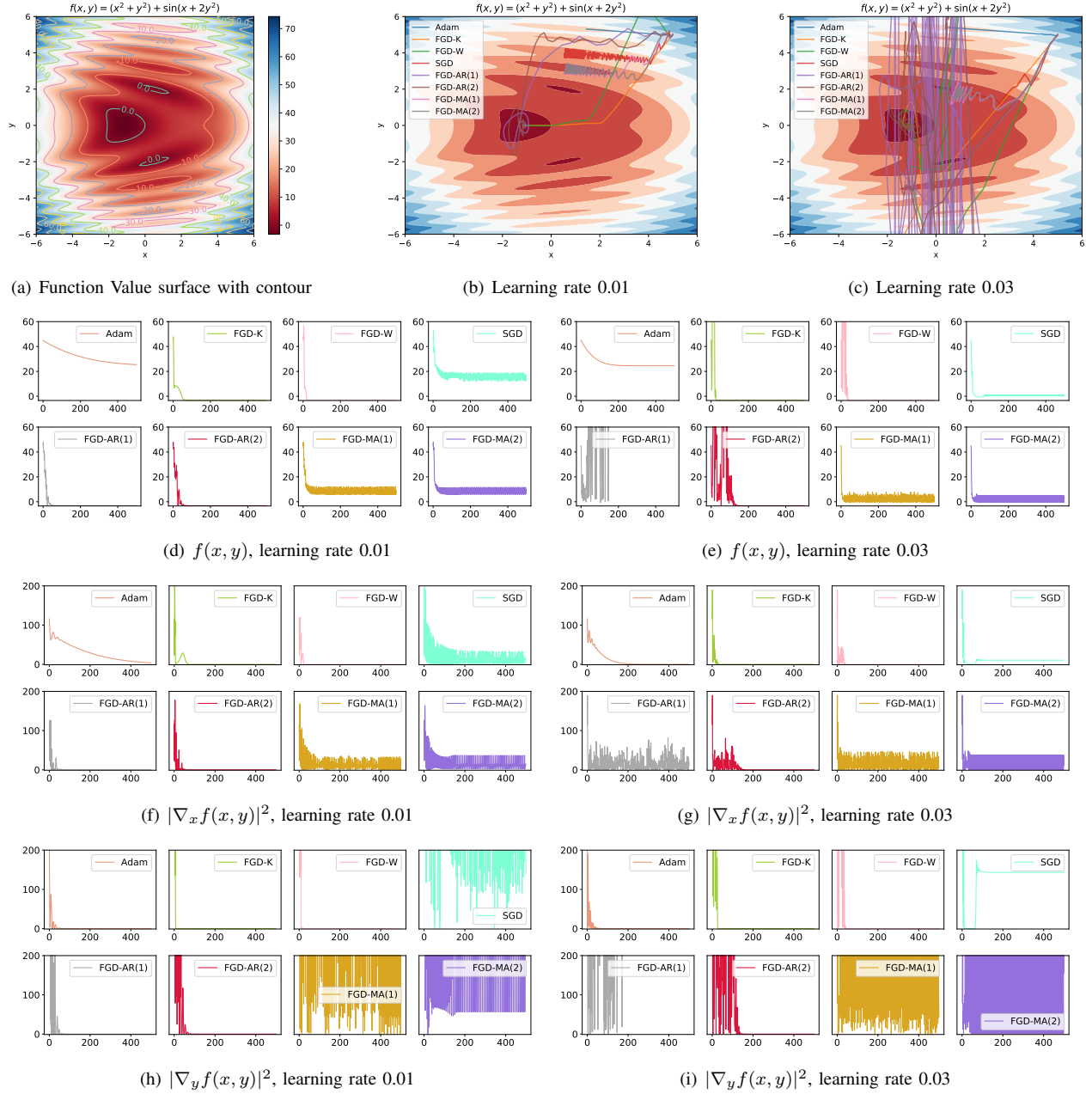


Fig. 3. Plot of Value surface of $f(x, y)$ with contour, the function value, gradient variance for non-convex function minimization

gradient by blindly adding the past gradient to the current update. FGD-AR(1) (momentum) and FGD-AR(2) relieve the variance problem when learning rate is small, but fails when large learning rate is used. Adam succeed in suppressing the gradient variance by keep a running average of second order momentum, but nearly stop learning from 200th iteration. It is clear that FGD-K and FGD-W are able to reduce the variance of the gradient compared with the momentum-based methods and vanilla SGD. This experiment show promising results that FGD not only reduce the variance in gradient decent, but also help to get rid of the local minimum in non-convex optimization problem. It encourages us to apply FGD to other

optimization problem, like training neural networks.

B. Study 2: Multi-layer perceptron

We train a multi-layer perceptron (MLP) on MNIST [12] dataset using (1) SGD (2) Adam [6] (3) FGD-MA with order 1 and 2 (4) FGD-AR with order 1 (vanilla momentum) and 2 (5) FGD-K and (6) FGD-W. We compare their (1) performance, measured by test accuracy (2) convergence rate, measured by training loss.

Experiment setting We establish a 3-layer neural network with 784 input units, 10 hidden units, and 10 output units. All training images are augmented with zero-padding of 2

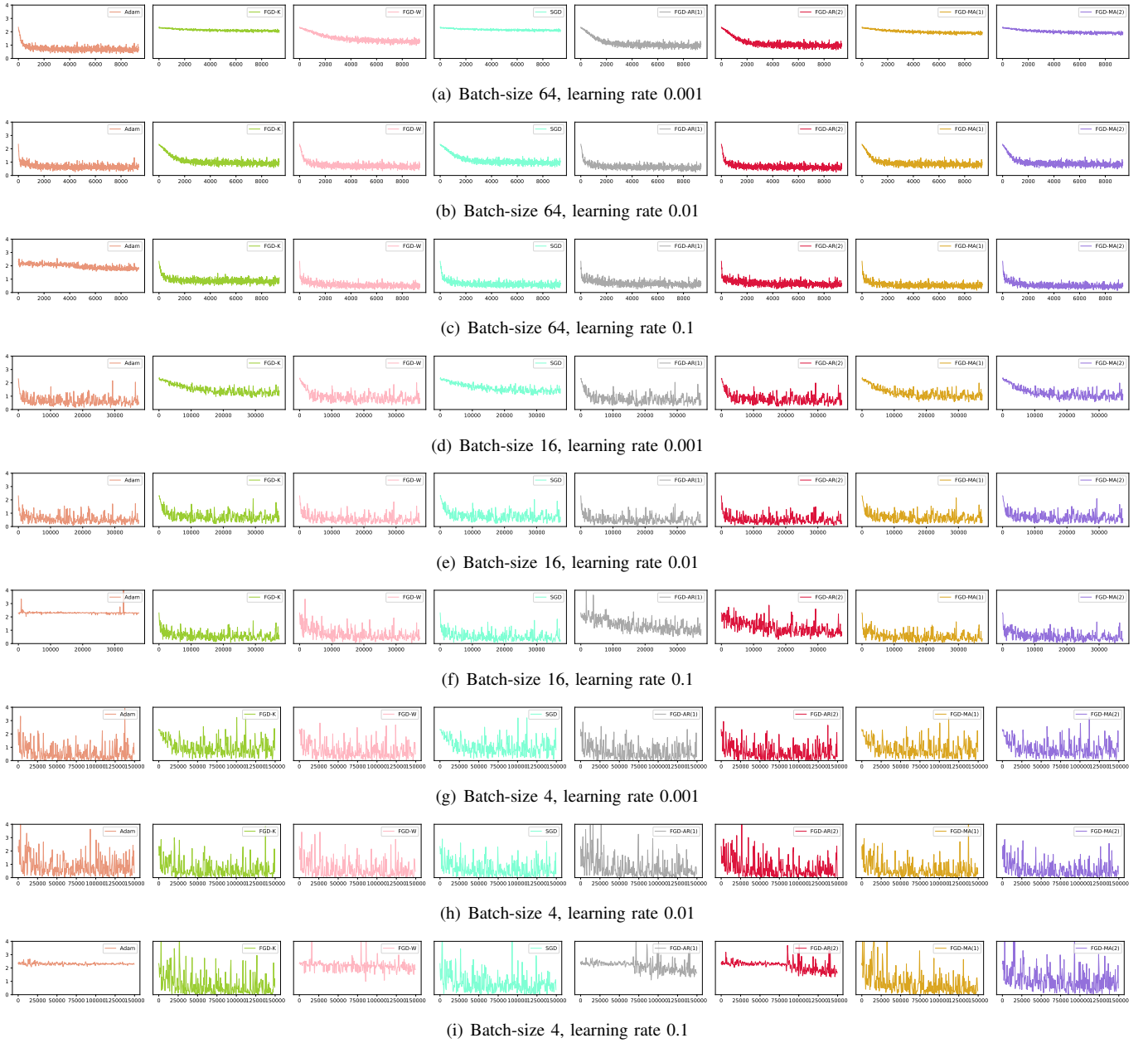


Fig. 4. Training loss on MNIST with MLP(x-axis: iteration; y-axis: loss)

pixels and random cropped to 28×28 . Each image is flattened as a 1-dimension vector. We set the initial learning rate $\eta = \{0.1, 0.01, 0.001\}$ and mini-batch size $batchsize = \{4, 16, 64\}$. The learning rate is multiplied by 0.7 every epoch. For Adam, we use the default setting that $\beta_1 = 0.9, \beta_2 = 0.99$. For FGD-MA and FGD-AR, we select $a_1 = b_1 = 0.9$ for order 1 model and $a_1 = b_1 = 0.8, a_2 = b_2 = 0.1$ for order 2. For FGS-K, we select $\gamma = 5, c = 1, \sigma_Q^2 = 0.01, \sigma_R = 2$. For FGD-W, we use Daubechies 4 (db4) wavelet and $K = 3$ and $L = 16$. We apply soft-thresholding on each highpass band with threshold of 0.2 and select $\alpha = 5$. We train the network for 10 epoch. The network is trained with a cross-entropy loss.

Result Figure 4 compares the training loss under different learning rate and batch size setting on MNIST classification task. **First**, convergence is slower when learning rate is small and batch size is large. In such circumstances, filtering techniques accelerate the training by enhance the low frequency part of the gradient. For example, when $lr = 0.001$ and $batchsize = 64$, variants of FGD show faster convergence compared with vanilla SGD. **Second**, a large learning rate and small batch size result in a sharply jittering curves, which demonstrates large randomness in training. For example, Adam and FGD-AR(1) (momentum) fails under the case with $lr = 0.1$ and $batchsize = 4$. It again shows that momentum method is very sensitive to the hyper-parameter setting. In such

TABLE I
TEST ACCURACY ON MNIST WITH MLP WITH DIFFERENT HYPER-PARAMETER SETTING

Setting	Adam	FGD-K	FGD-W	SGD	FGD-AR(1)	FGD-AR(2)	FGD-MA(1)	FGD-MA(2)
Batch 4 lr 0.001	91.55	85.41	90.16	86.79	90.60	90.93	88.96	88.92
Batch 16 lr 0.001	89.95	80.59	87.58	77.90	88.85	88.87	84.38	84.38
Batch 64 lr 0.001	88.55	36.79	81.02	35.86	85.89	85.88	50.63	50.61
Batch 4 lr 0.01	85.55	91.41	91.48	90.41	91.18	90.42	91.34	90.73
Batch 16 lr 0.01	90.71	89.09	91.46	88.71	91.66	91.61	89.68	89.51
Batch 64 lr 0.01	87.86	86.22	89.26	85.96	90.22	90.16	89.41	89.41
Batch 4 lr 0.1	11.52	91.29	20.53	88.07	26.52	26.16	91.48	85.94
Batch 16 lr 0.1	12.03	90.64	89.54	91.77	71.81	79.36	92.13	92.07
Batch 64 lr 0.1	30.31	81.80	90.94	90.18	89.67	89.14	91.15	91.23
Mean	65.34	81.47	81.33	81.74	80.71	81.39	85.46	84.76

TABLE II
TEST ACCURACY ON CIFAR-10 WITH RESNET-18

Method	Top1 test
Adam [6]	92.50
FGD-AR(1)	92.77
SGD	93.33
FGD-K	93.54
FGD-M	93.42
FGD-AR(2)	93.60
FGD-MA(1)	93.23
FGD-MA(2)	93.76

case, FGD-K and FGD-MA successfully produces smoother curves and reduce the noise and randomness in the training. In comparison, FGD is more robust to learning rate changes compared to momentum based method.

Table I shows the test accuracy given various parameter setup on MNIST. **First**, FGD outperforms vanilla SGD when learning with small learning rate and large batch size. For example, FGD-W, FGD-MA, FGD-AR(1)&(2) and FGD-MA(1)&(2) outperform SGD when $lr = 0.01$ and $lr = 0.001$ for all batch setting. FGD-W and FGD-AR even improves SGD by over 130% for case where $lr = 0.001$ and $batchsize = 64$. It reveals that FGD is able to accelerate the learning and achieve better convergence when the step-size is small. **Another observation** from the table is that momentum based model (FGD-AR(1) and Adam) is very sensitive to the hyper-parameter setting. Test accuracy with Adam dramatically drop to around 20% for situations with a large learning rate (0.1). In comparison, FGD-MA and FGD-K are robust to all kinds of hyper-parameter setting. This result further provide evidences that blind acceleration with momentum method is not universally optimal. On the other hand, gradient estimation with filters is able to adaptively find a robust estimation of the true gradient regardless of parameter setup. **In addition**, FGD-MA with order 1 and order 2 makes the best performance in average. Compared with traditional momentum method, it provides a cheap solution to training neural network with great potential.

C. Study 3: Convolutional neural networks

We train a well-designed convolutional neural network (ResNet-18 [13]) on CIFAR10 [14] dataset using (1) SGD (2) Adam [6] (3) FGD-MA with order 1 and 2 (4) FGD-AR

with order 1 (vanilla momentum) and 2 (5) FGD-K and (6) FGD-W. We compare their (1) convergence rate, measured by the test error curve, and (2) performance, measured by best test accuracy.

Experiment setting All training images are augmented with random horizontal flip, zero-padding of 4 pixel and random cropped to 32×32 . We set the initial learning rate of $\eta = 0.1$, mini-batch size $batchsize = 64$ and train the network for 200 epoch. The learning rate is multiplied by 0.1 at the 80 and 160 epoch. For Adam, we use the default setting that $\beta_1 = 0.9, \beta_2 = 0.99$. For FGD-MA and FGD-AR, we select $a_1 = b_1 = 0.9$ for order 1 model and $a_1 = b_1 = 0.8, a_2 = b_2 = 0.1$ for order 2. For FGS-K, we select $\gamma = 5, c = 1, \sigma_Q^2 = 0.01, \sigma_R = 2$. For FGD-W, we use Daubechies 4 (db4) wavelet and $K = 3$ and $L = 16$. We apply soft-thresholding on each highpass band with threshold of 0.2 and select $\alpha = 5$. The network is trained with a cross-entropy loss.

Result Table II list the test accuracy for different optimization methods on CIFAR-10 dataset. **First**, resembling with results in Study 2, momentum-based method (Adam & FGD-AR(1)) may not be optimal even compared with vanilla SGD. It again shows that blindly accelerating the gradient decent by using an AR(1) model does not capture the real dependency the noisy gradient observations and the true state. **Second**, FGD variants outperform SGD and momentum method experiment. These results reemphasize a major merit of the FGD method – it is able to help the the network to get rid of the global minimum while less sensitive to hyper-parameter selection. **Third**, we can make the observation that FGD is not sensitive to the filter selection. FGD achieves similar test accuracy regardless of the which filter is applied. This result gives supporting evidence that FGD is a generally efficient framework in optimization. **Impressively**, among all variants, FGD-MA(2) achieves the best test performance of 93.76% for ResNet-18 on CIFAR-10 dataset. This result reemphasizes the huge potential for FGD-MA method in training neural network.

Figure 5 demonstrates the test error VS epoch through the training process. The bottom-left and bottom-right plots provide two zoom-in views for the training process. The bottom-left plot is in the middle of training and the bottom-right plot visualizes the test error at the end of the training. **First**, in bottom-left plot, we may observe that the variance

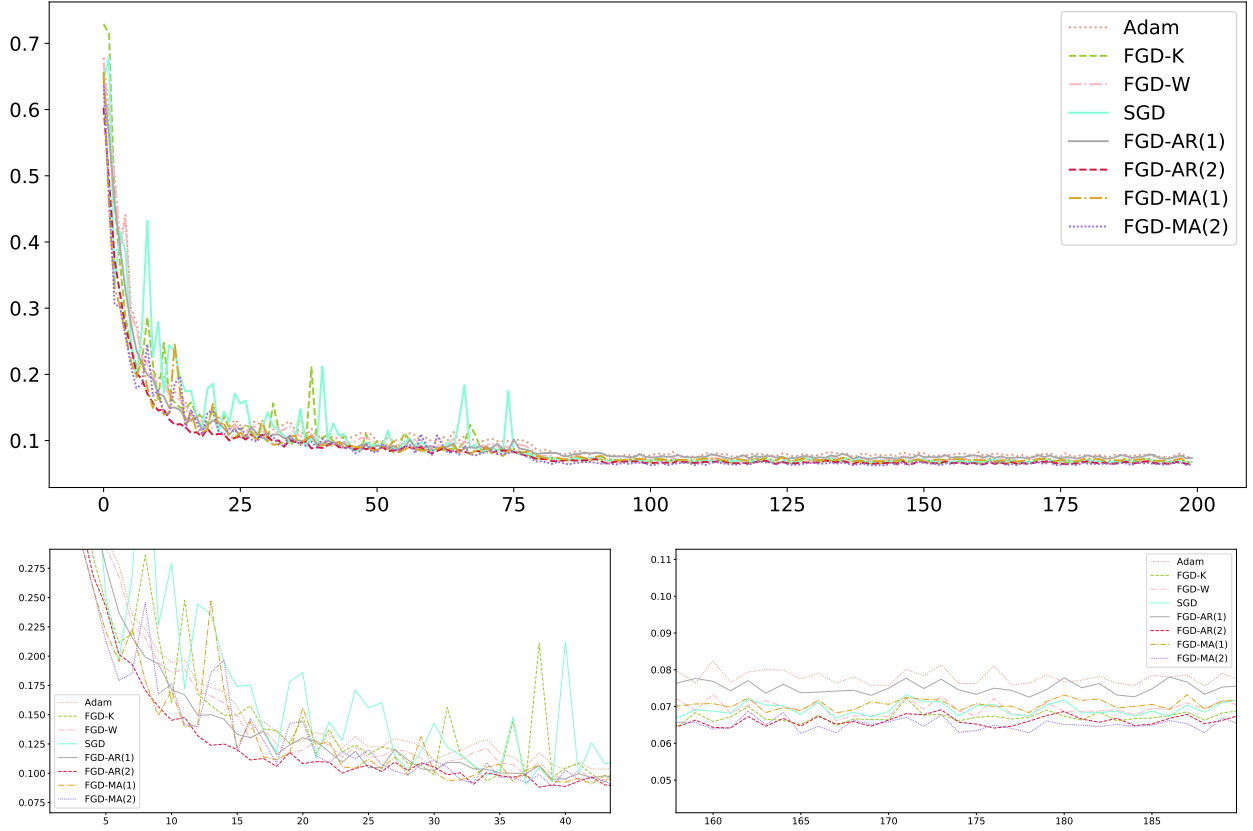


Fig. 5. Test error curve On CIFAR-10 with ResNet-18, with two detail zoom in (x-axis: epoch; y-axis: test error)

in the error curve is successfully reduced by the different filters compared with vanilla SGD (lightblue solid). The curve SGD is highly non-smooth, with tremendous ups and downs. Filtering in gradients, especially for FGD-AR(2) and FGD-K, succeed in reducing the variance of the gradient, resulting in smoother curves. **Second**, in bottom-right plot, we see that FGD results in better convergence in training neural networks. Compared with Adam (orange dotted) and FGD-AR(1) (gray solid), FGDs generally achieve lower test error. For example, FGD-MA(2) (purple dotted) and FGD-AR(2) (crimson dashed) achieves the lowest test error. This observation illustrates that FGD is efficient in training neural network.

V. CONCLUSION

In this paper, we propose Filter Gradient Decent, which reduces the variance in gradient estimation by solving a filtering problem. It is a general framework that applies filter techniques to stochastic gradient descent to estimate the true state of the gradient from noisy observations. FGD not only greatly reduces the noise in the gradient estimation, but also helps to get rid of the local optima problem in the optimization. A series of comprehensive experiments show that FGD is able to achieve superior performance in solving non-convex optimization problems and training neural networks.

FUTURE WORK

Filtering in signal processing is a well-studied subjects. A large number of researchers have provided mathematical proofs that different filtering techniques can efficiently eliminate different noise models. In future works, I would like to provide some theoretical provements that Filter Gradient Decent is mathematically explainable in reduce the gradient variance.

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