



Introduction to Scientific Computation

Lecture 3

Fall 2023

Fourier transform, FFT, SVD

Convolution

$$c(t) = \int_{-\infty}^{\infty} a(t - \tau)h(\tau)d\tau$$

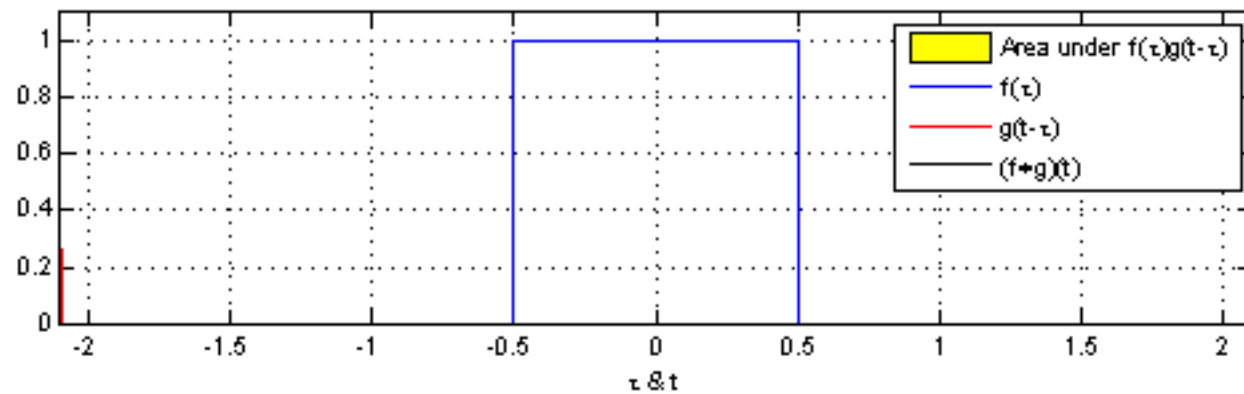
Convolution

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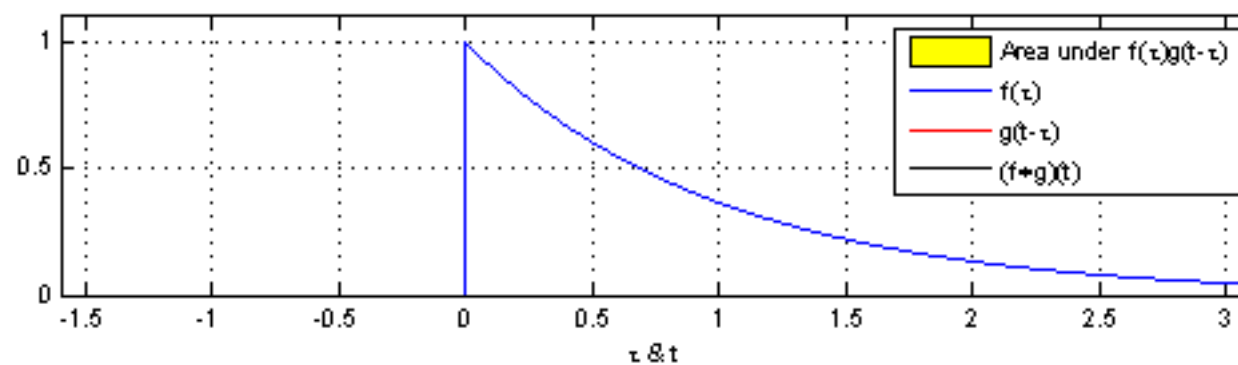
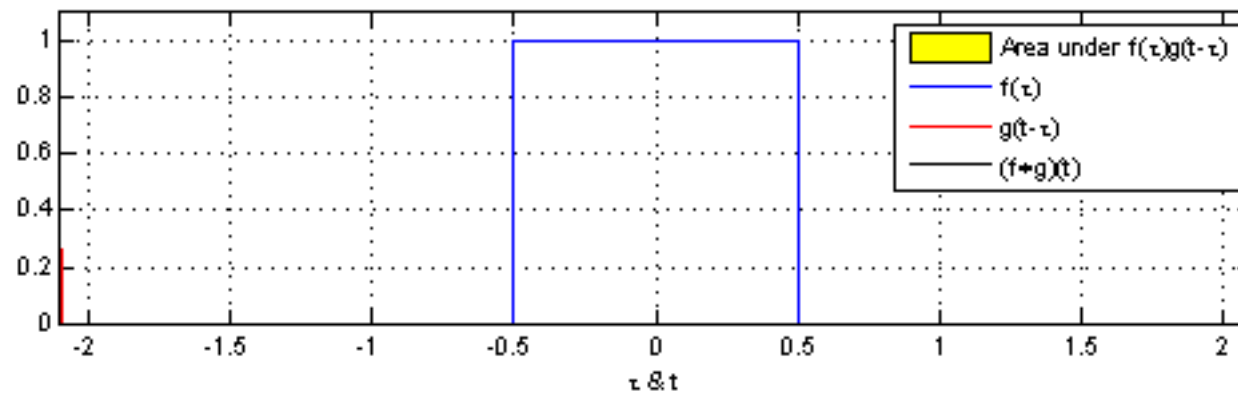
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Convolution

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Convolution

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Autocorrelation

$$c(t) = \sum_{\tau=-\infty}^{\infty} a(\tau)a(t - \tau)$$

Convolution

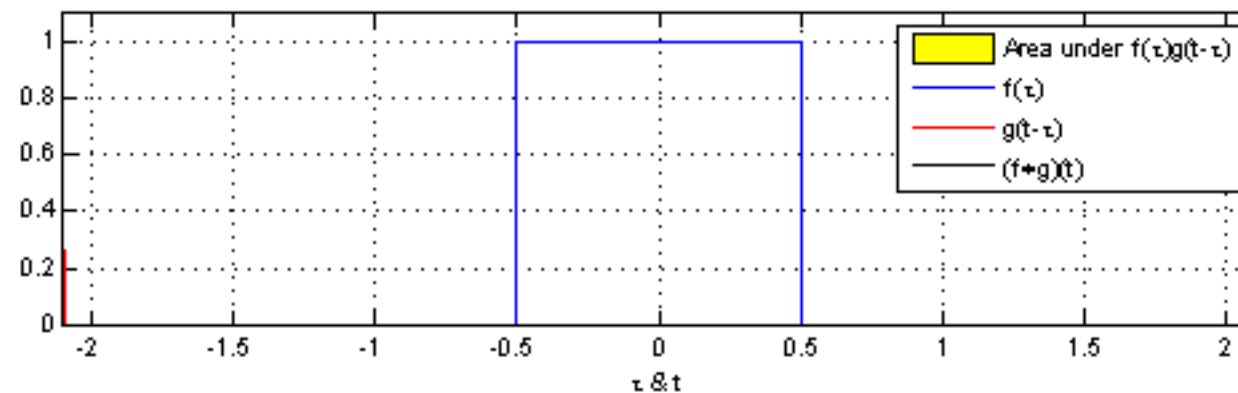
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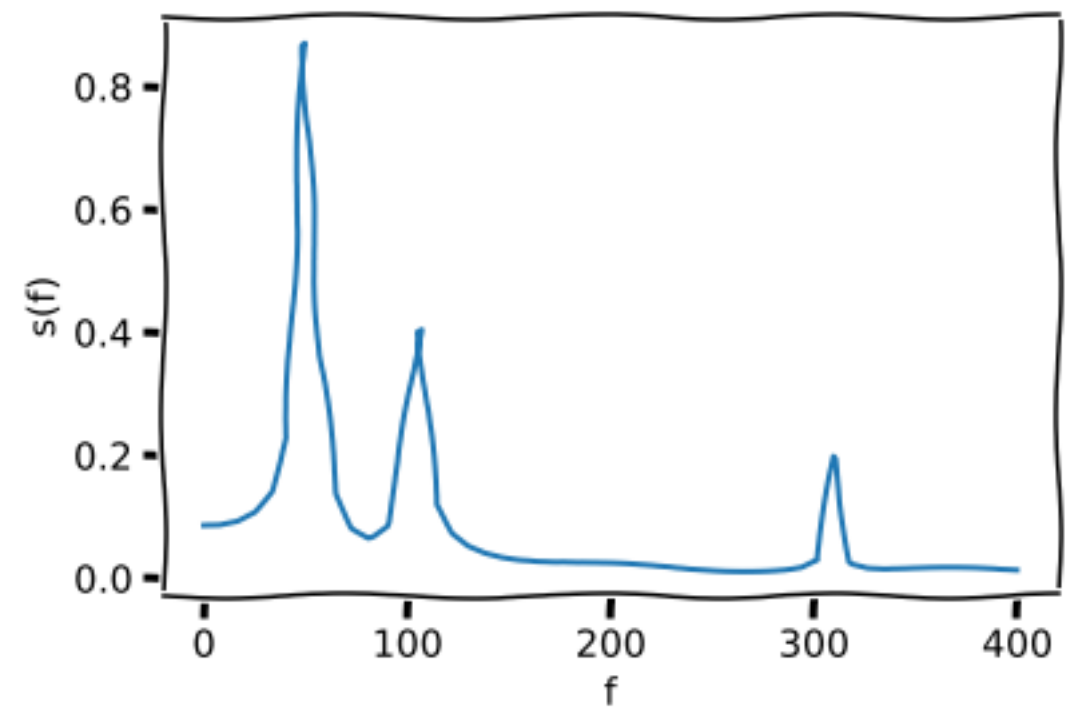
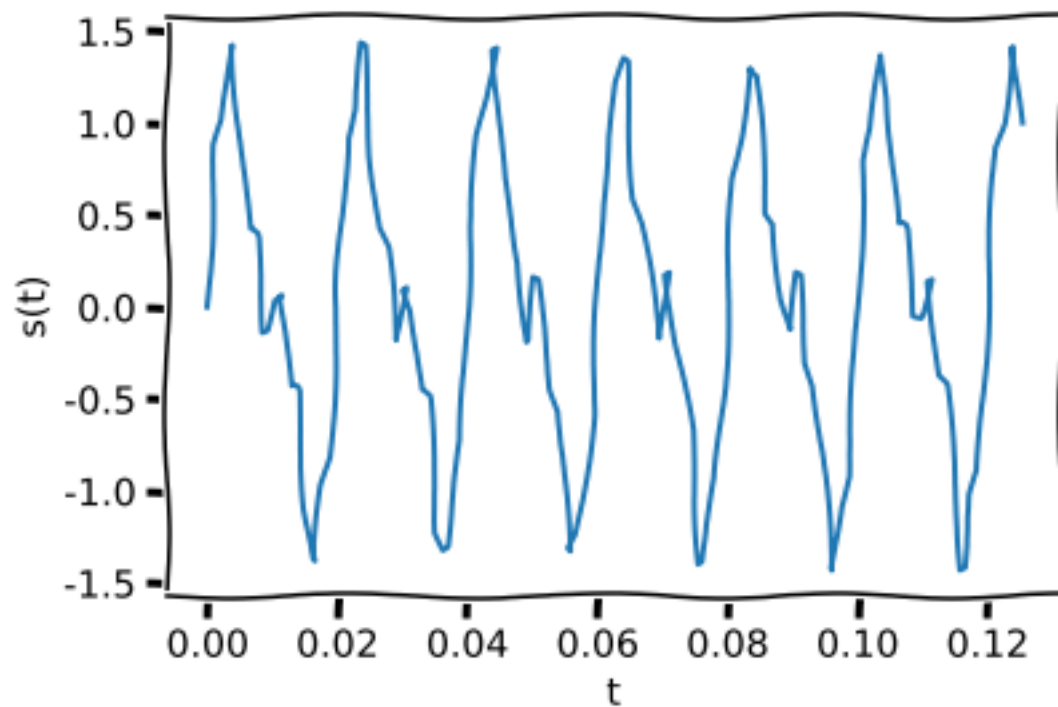
Fourier transform

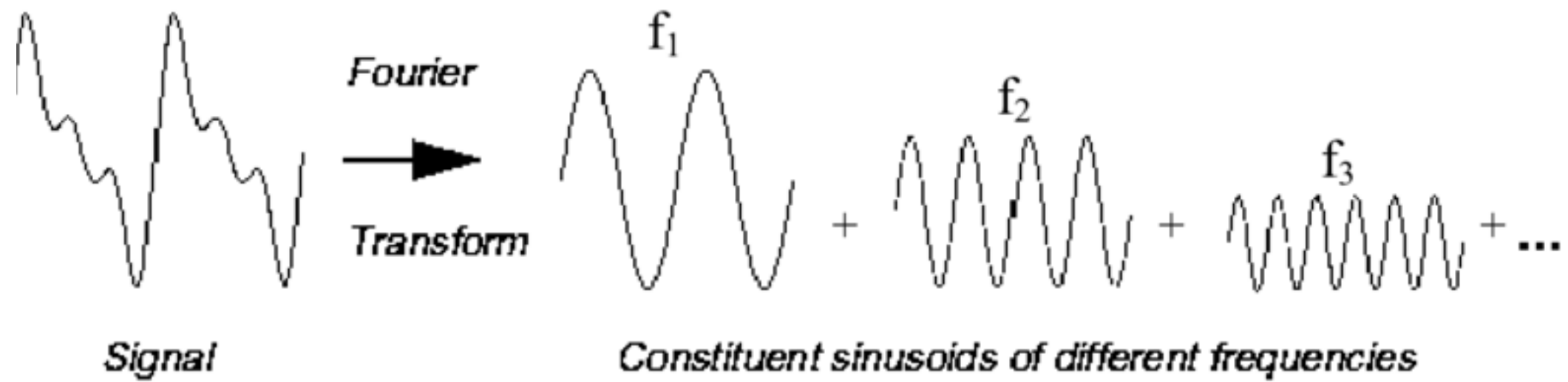
$$S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi ft} dt$$

Fourier transform

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi ft} df$$

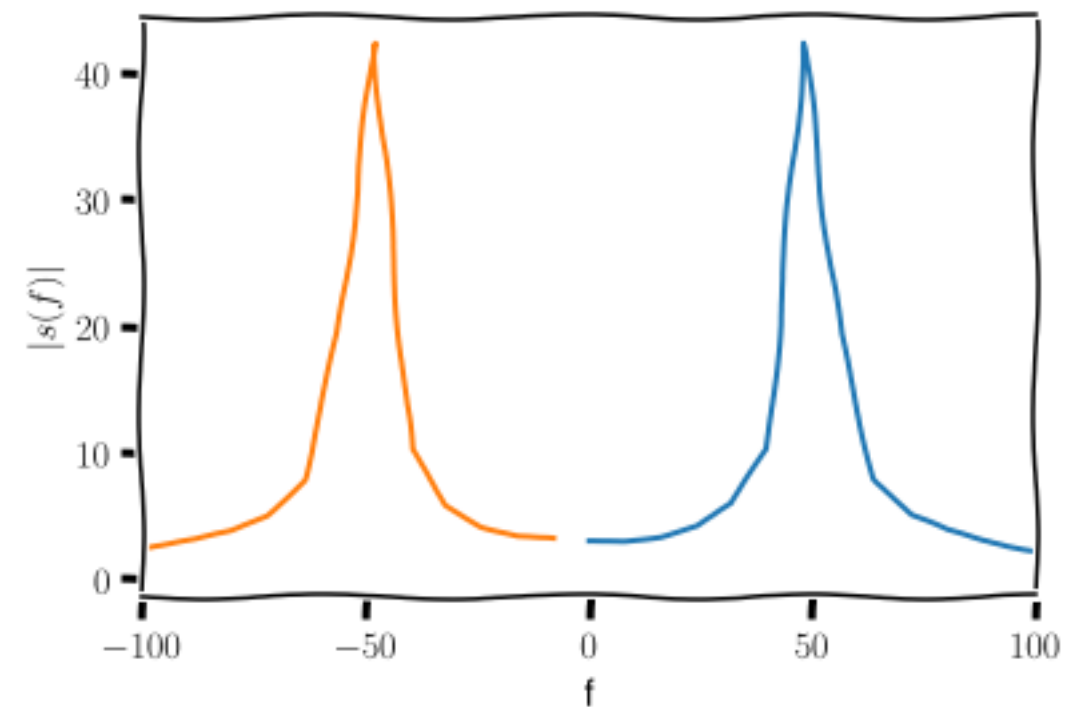
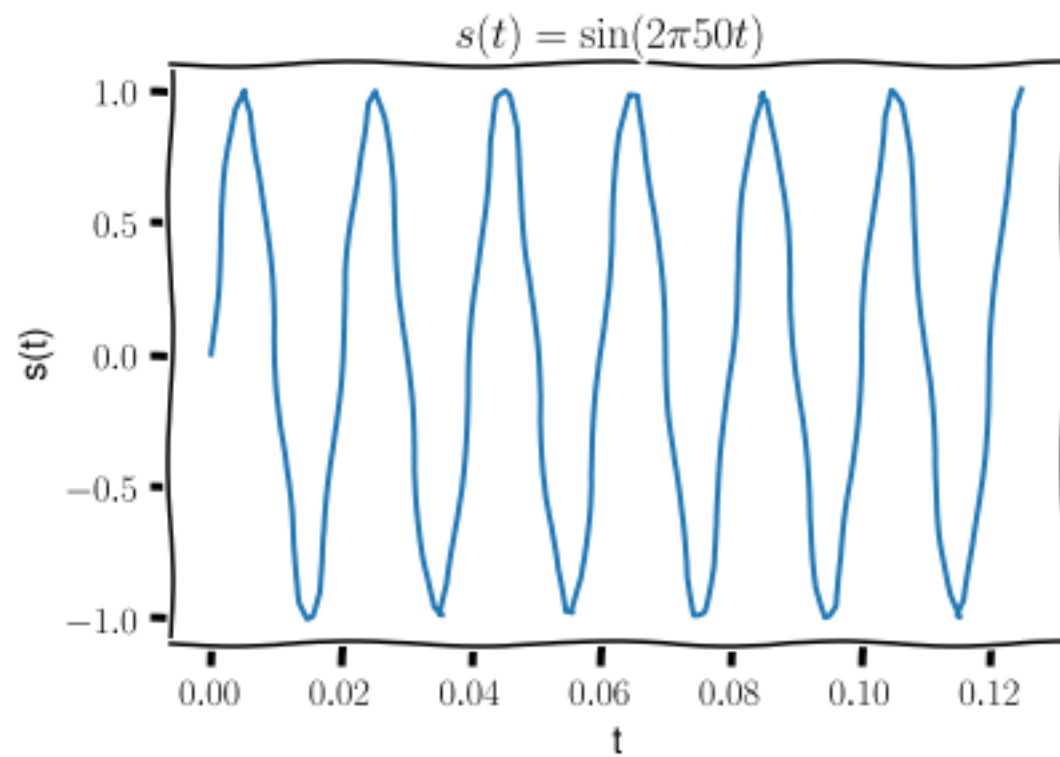
Inverse Fourier transform



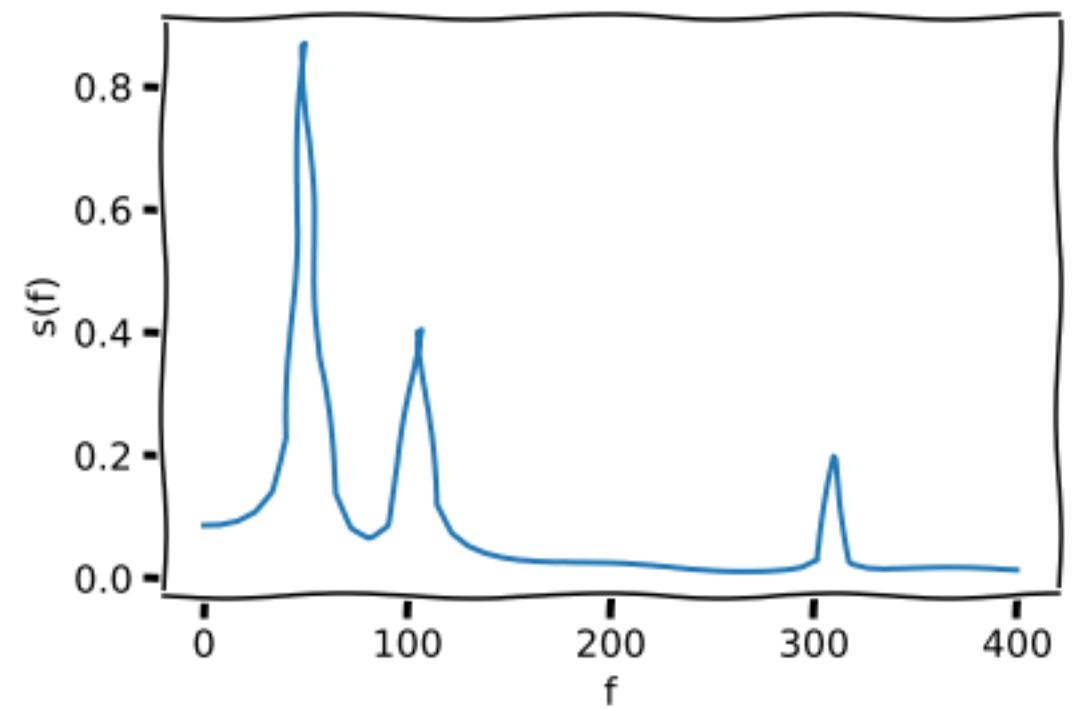
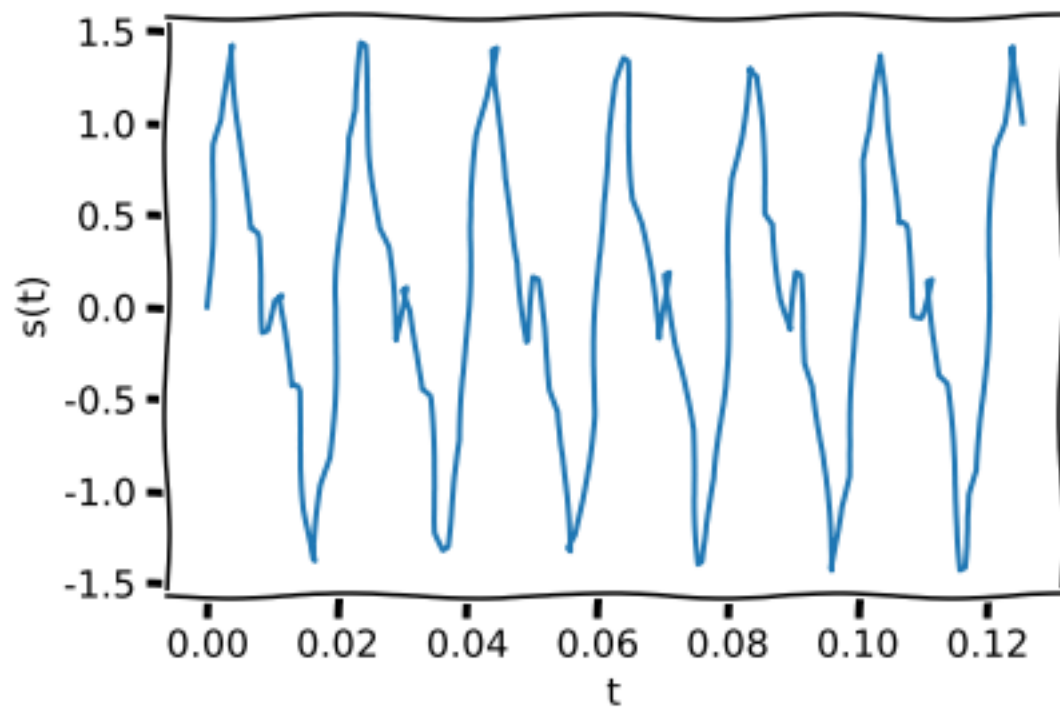


$$S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi ft} dt \quad 1/s, \text{ Hz}$$

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi ft} df \quad s$$



How much 100Hz is in $s(t)$?



Discrete Fourier transform

\int



Σ

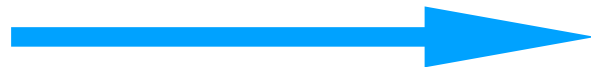
Discrete Fourier transform

\int



Σ

$$S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi ft} dt$$



$$S[f] = \sum_{i=0}^{N-1} s[i] e^{\frac{-j2\pi fi}{N}}$$

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi ft} df$$



$$s[t] = \sum_{j=0}^{M-1} S[j] e^{\frac{-j2\pi jt}{M}}$$

2D Fourier transform

$$S(f_1, f_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(t_1, t_2) e^{-j2\pi(f_1 t_1 + f_2 t_2)} dt_1 dt_2$$

2D Fourier transform

$$s(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(f_1, f_2) e^{j2\pi(f_1 t_1 + f_2 t_2)} df_1 df_2$$

Inverse 2D Fourier transform

Discrete 2D Fourier transform

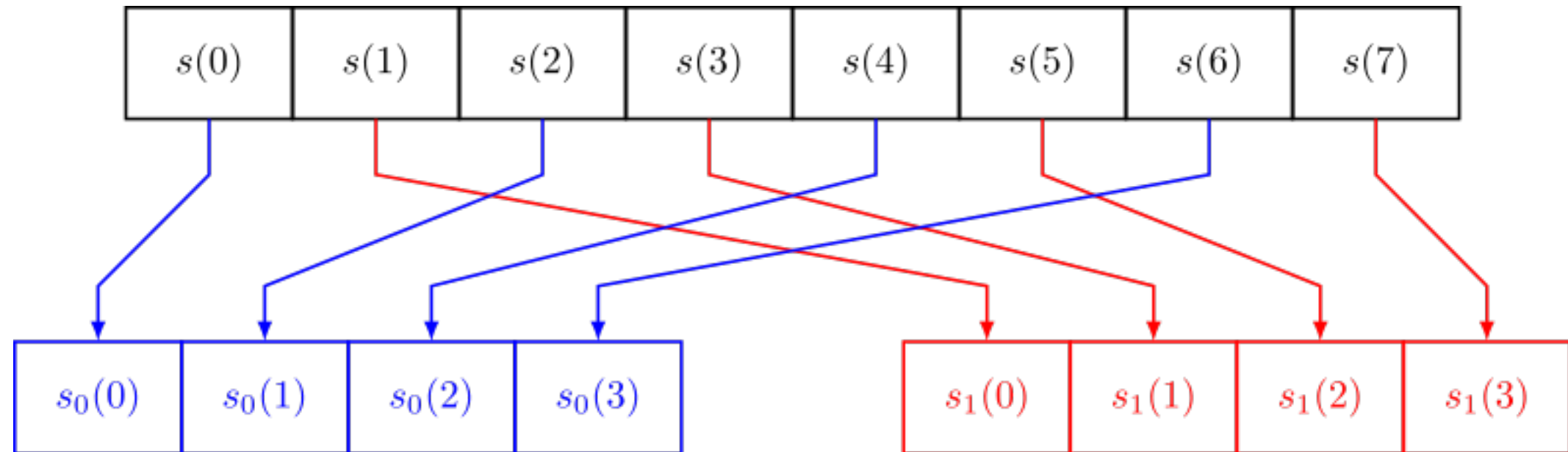
$$S[f_1, f_2] = \sum_{t_1=0}^{N-1} \sum_{t_2=0}^{N-1} s[t_1, t_2] e^{-j2\pi(f_1 t_1 + f_2 t_2)}$$

2D Fourier transform

$$s[t_1, t_2] = \sum_{f_1=0}^{M-1} \sum_{f_2=0}^{M-1} S[f_1, f_2] e^{j2\pi(f_1 t_1 + f_2 t_2)}$$

Inverse 2D Fourier transform

Fast Fourier transform

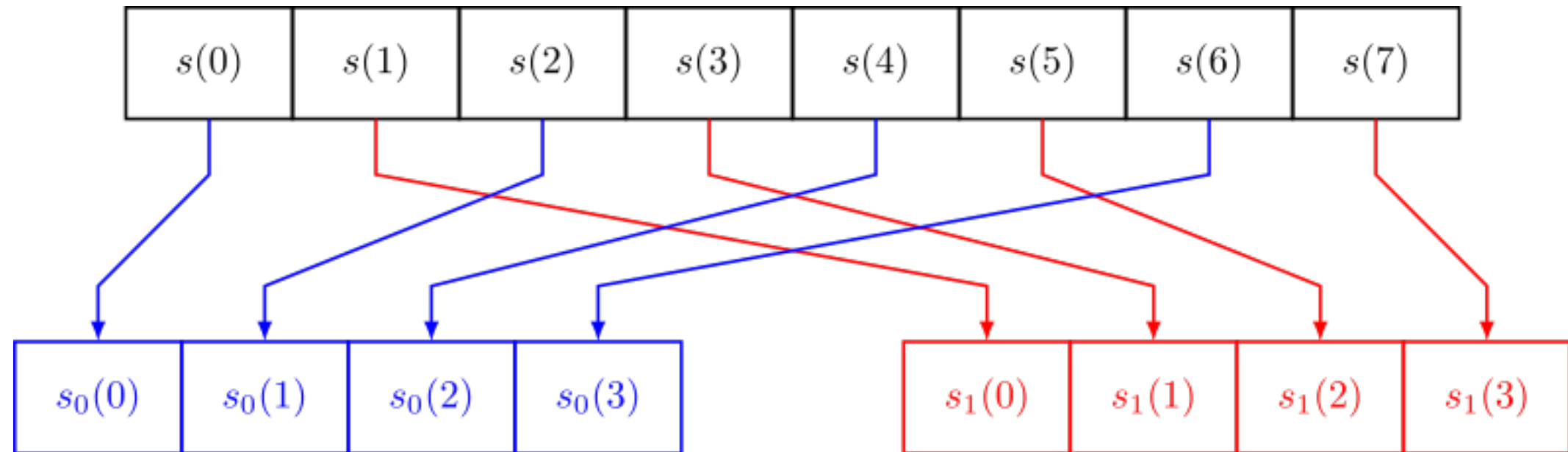


$$S[f] = \sum_{i=0}^{N-1} s[i] e^{\frac{-j2\pi fi}{N}} = \sum_{m=0}^{\frac{N}{2}-1} s[2m] W_N^{2mf} + \sum_{m=0}^{\frac{N}{2}-1} s[2m+1] W_N^{(2m+1)f}$$

$$W_N^f = \exp\left(-j \frac{2\pi}{N} f\right)$$

Fast Fourier transform

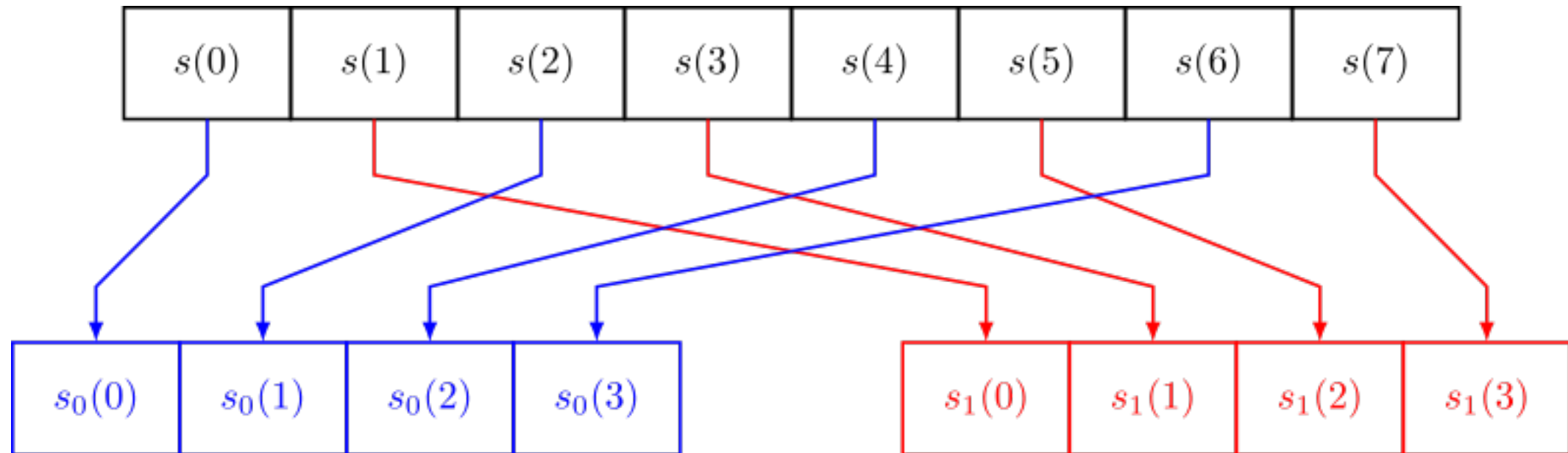
$$W_N^{2mf} = \exp \left(-j \frac{2\pi}{N} 2mf \right) = \exp \left(-j \frac{2\pi}{\frac{N}{2}} mf \right) = W_{\frac{N}{2}}^{mf}$$



$$S[f] = \sum_{i=0}^{N-1} s[i] e^{\frac{-j2\pi fi}{N}} = \sum_{m=0}^{\frac{N}{2}-1} s[2m] W_N^{2mf} + \sum_{m=0}^{\frac{N}{2}-1} s[2m+1] W_N^{(2m+1)f}$$

Fast Fourier transform

$$W_N^{2mf} = \exp \left(-j \frac{2\pi}{N} 2mf \right) = \exp \left(-j \frac{2\pi}{\frac{N}{2}} mf \right) = W_{\frac{N}{2}}^{mf}$$



For all $f \in [0, \frac{N}{2} - 1]$

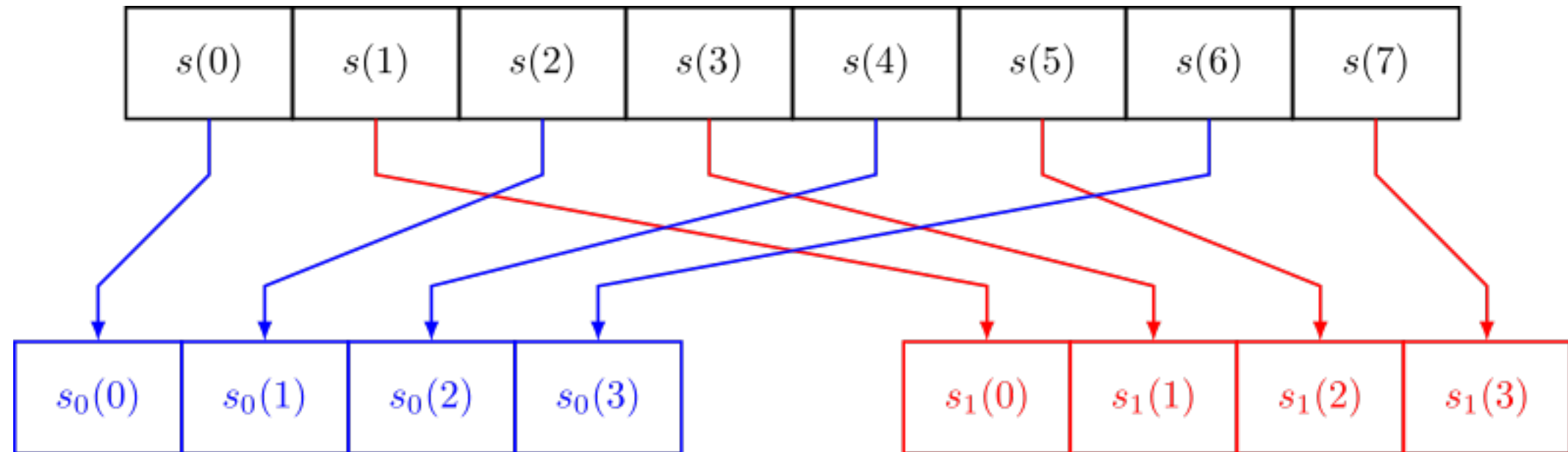
$$S[f] = \sum_{i=0}^{N-1} s[i] e^{-j \frac{2\pi f i}{N}} = \sum_{m=0}^{\frac{N}{2}-1} s[2m] W_N^{2mf} + \sum_{m=0}^{\frac{N}{2}-1} s[2m+1] W_N^{(2m+1)f}$$

$$S[f] = \sum_{m=0}^{\frac{N}{2}-1} s[2m] W_N^{2mf} + W_N^f \sum_{m=0}^{\frac{N}{2}-1} s[2m+1] W_N^{2mf} = \sum_{m=0}^{\frac{N}{2}-1} s[2m] W_{\frac{N}{2}}^{mf} + W_N^f \sum_{m=0}^{\frac{N}{2}-1} s[2m+1] W_{\frac{N}{2}}^{mf}$$

Fast Fourier transform

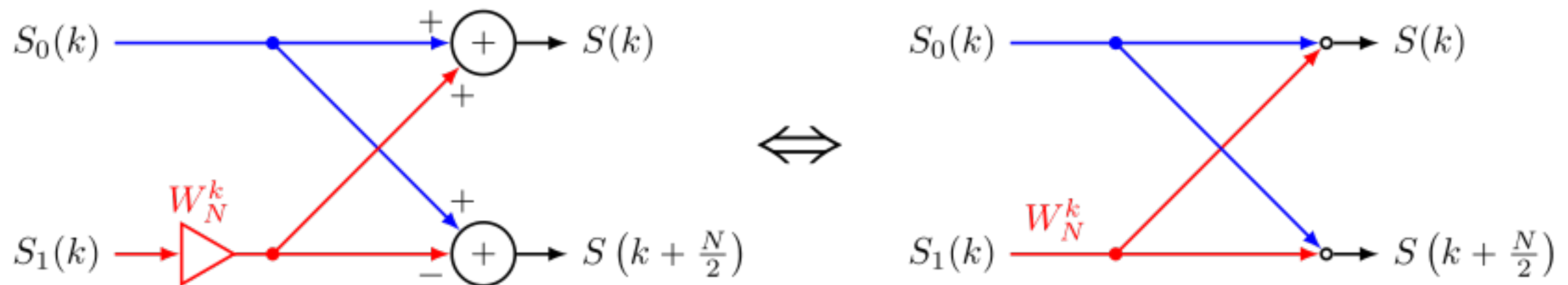
$$W_N^{2m(f + \frac{N}{2})} = W_N^{2mf} W_N^{mN} = W_{\frac{N}{2}}^{mf}$$

$$W_N^{(2m+1)(f + \frac{N}{2})} = W_N^{2mf} W_N^{mN} W_N^f W_N^{\frac{N}{2}} = -W_N^f W_{\frac{N}{2}}^{mf}$$



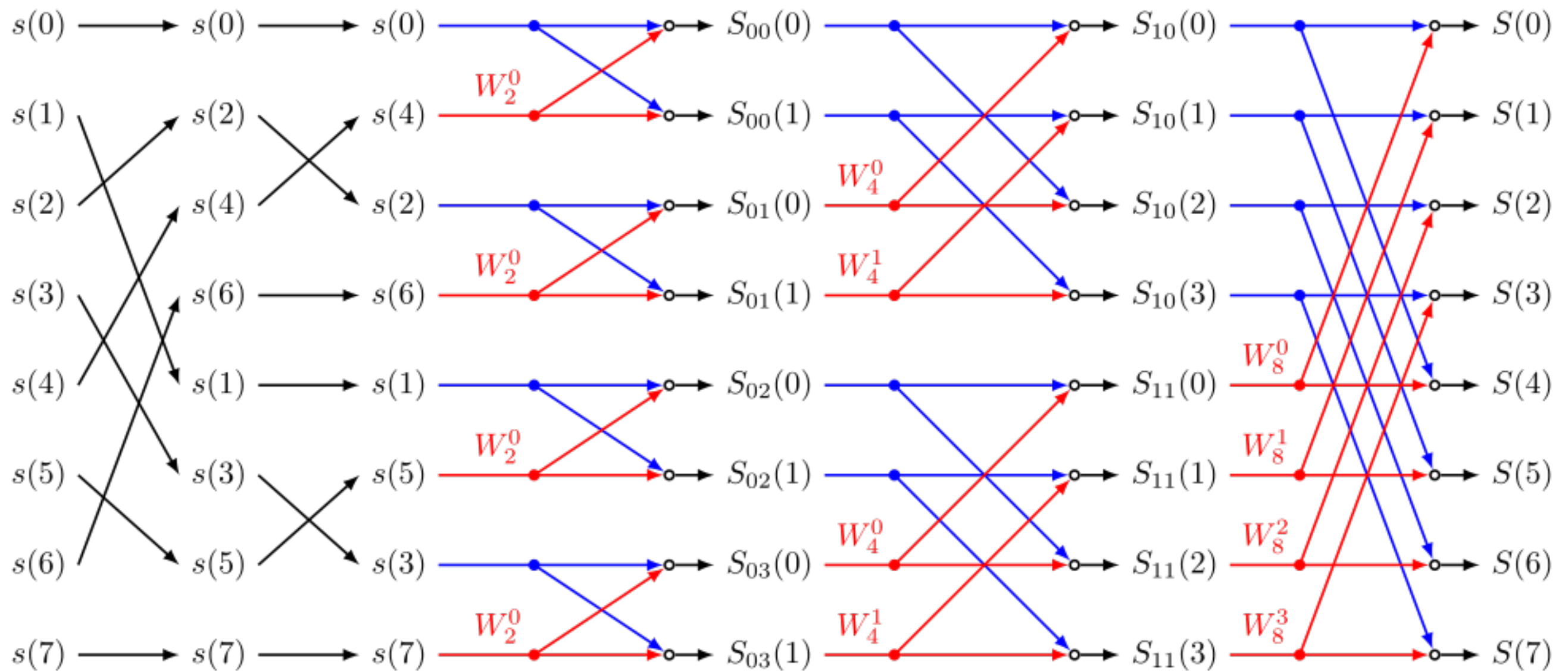
$$\begin{aligned} S[f + \frac{N}{2}] &= \sum_{m=0}^{\frac{N}{2}-1} s[2m] W_N^{2m(f + \frac{N}{2})} + \sum_{m=0}^{\frac{N}{2}-1} s[2m + 1] W_N^{(2m+1)(f + \frac{N}{2})} = \\ &= \sum_{m=0}^{\frac{N}{2}-1} s[2m] W_{\frac{N}{2}}^{mf} - W_N^f \sum_{m=0}^{\frac{N}{2}-1} s[2m + 1] W_{\frac{N}{2}}^{mf} \end{aligned}$$

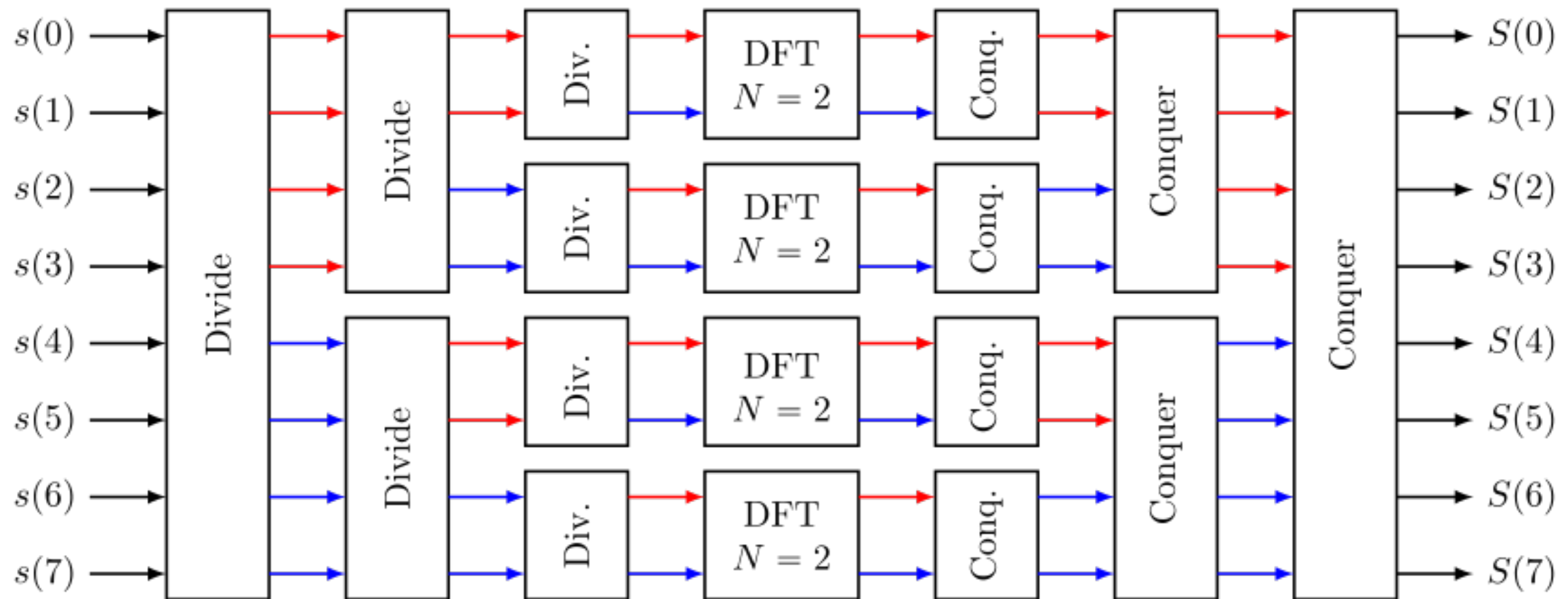
Fast Fourier transform



$$S[f] = \sum_{m=0}^{\frac{N}{2}-1} s[2m] W_{\frac{N}{2}}^{mf} + W_N^f \sum_{m=0}^{\frac{N}{2}-1} s[2m+1] W_{\frac{N}{2}}^{mf}$$

$$S[f + \frac{N}{2}] = \sum_{m=0}^{\frac{N}{2}-1} s[2m] W_{\frac{N}{2}}^{mf} - W_N^f \sum_{m=0}^{\frac{N}{2}-1} s[2m+1] W_{\frac{N}{2}}^{mf}$$





$O(N \log N)$ Operations

Singular Value Decomposition (SVD)

