



Introduction to Scientific Computation

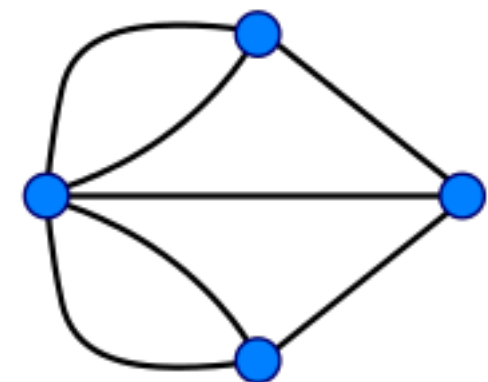
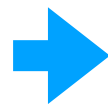
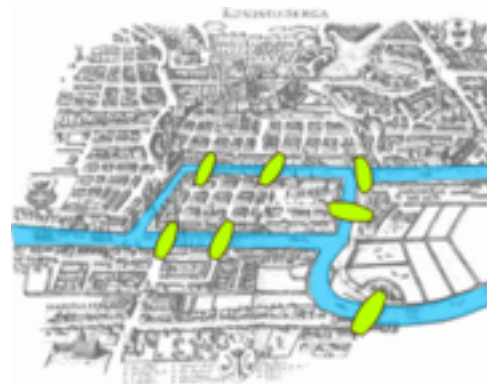
Lecture 5

Fall 2021

Graphs, graph algorithms

Graph

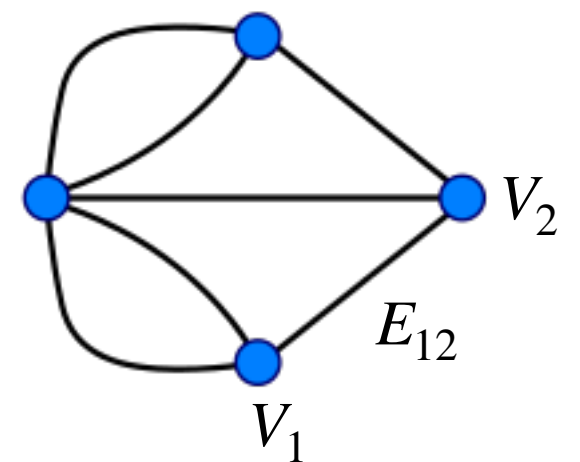
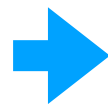
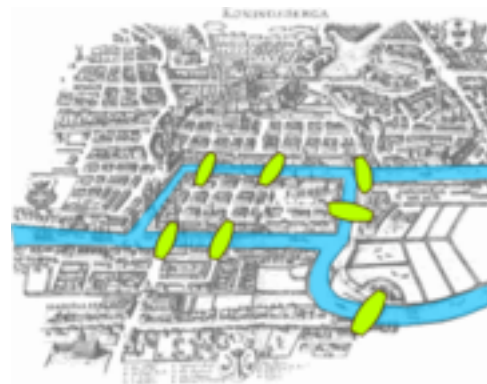
Seven Bridges of Königsberg



Leonhard Euler, 1736

Graph

$$G :< V, E >$$



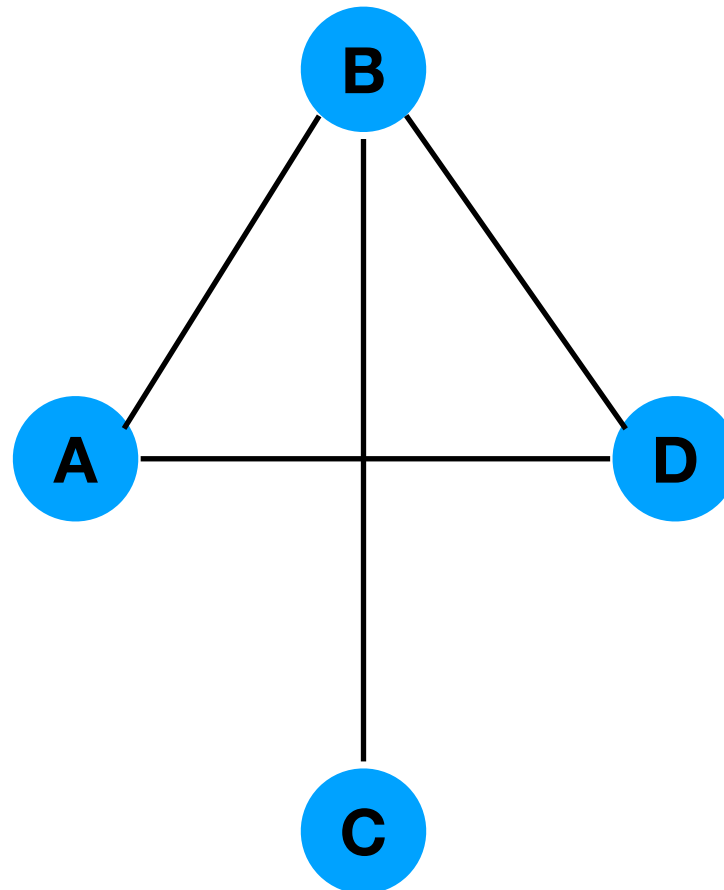
V node in the graph

E edge in the graph, the connection between two nodes

Applications

- Road networks
- Electronic circuits
- Telecommunication networks
- Social networks
- Any relationships ...

Types

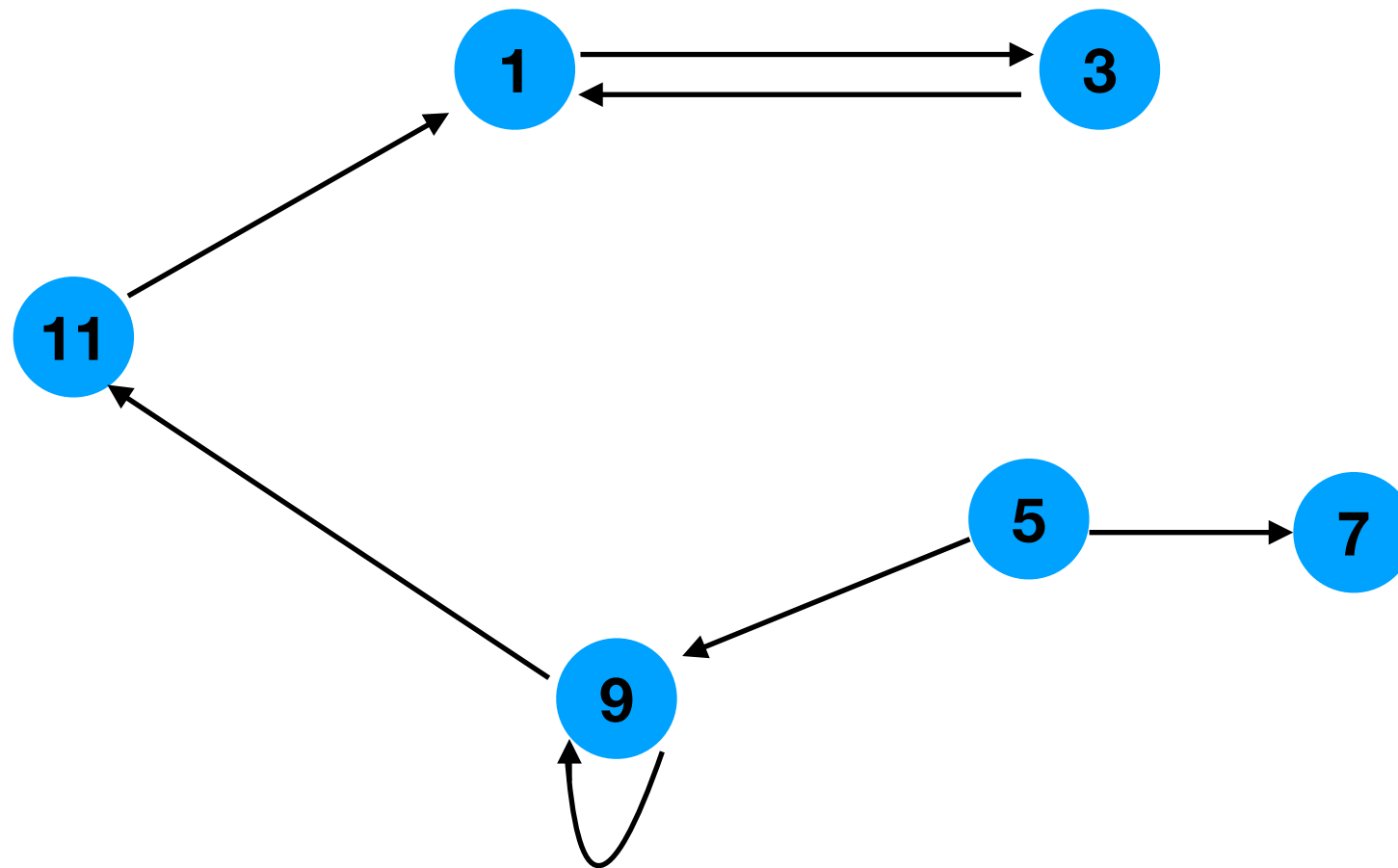


Undirected G

$$V = \{A, B, C, D\}$$

$$E = \{AB, BD, AD, BC\}$$

Types

Directed G

$$V = \{1, 3, 5, 7, 9, 11\}$$

$$E = \{(1, 3), (3, 1), (5, 7), (5, 9), (5, 7), (5, 9), (9, 9), (9, 11), (11, 1)\}$$

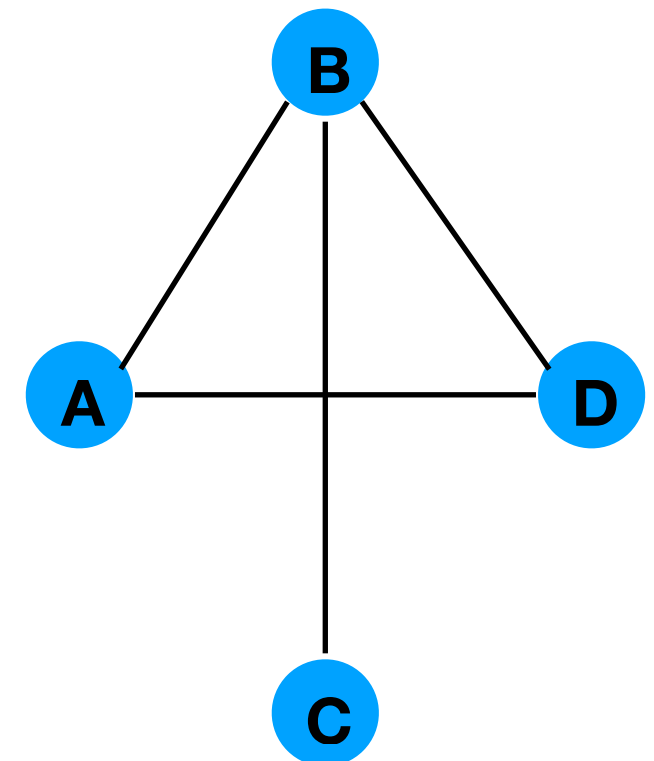
Definitions

Adjacency - two vertices are called **adjacent** if they are connected by edge

Path - the sequence of vertices which connects two nodes in a graph

Complete graph - every vertex is connected to every other vertex

Wighted graph - graph with the values assigned to edges



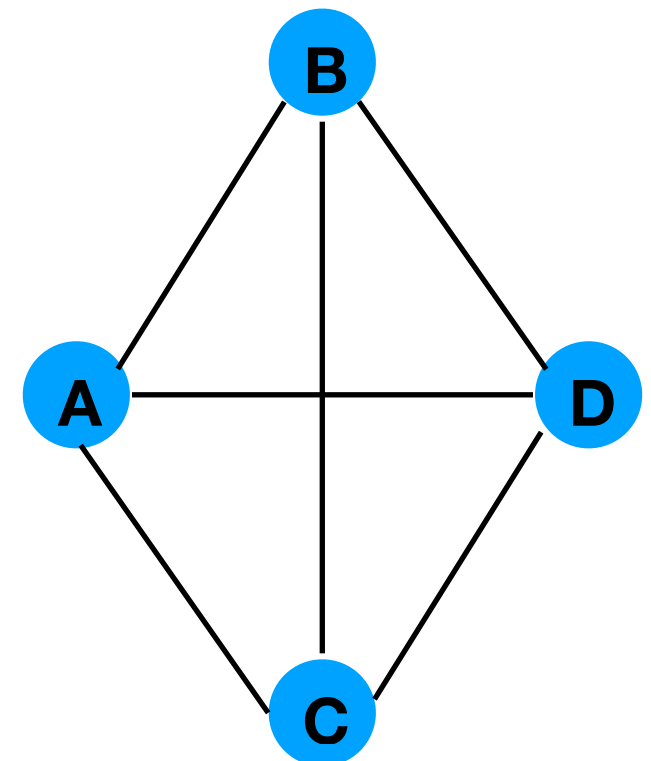
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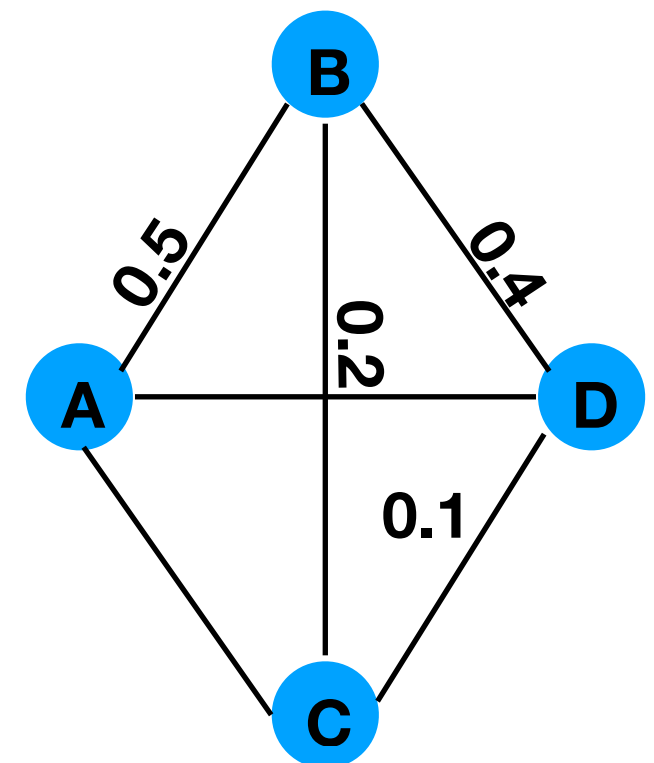
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Definitions

Quiz: how many edges exist in a complete graph ?

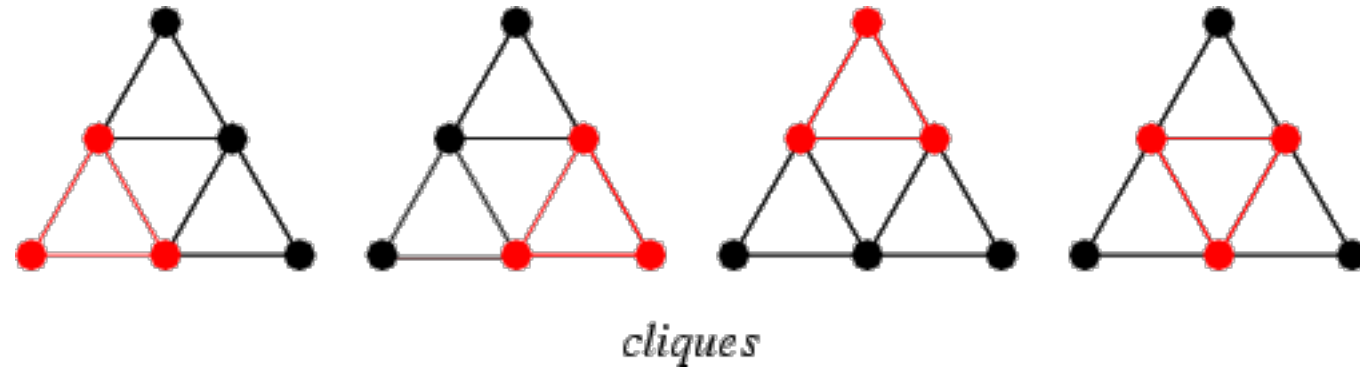
Definitions

Quiz: how many edges exist in a complete graph ?

Answer:
$$\frac{N^2 - N}{2}$$

Definitions

Clique - complete subgraph



Euler trail (path) - the path in a finite graph which visits every edge exactly once

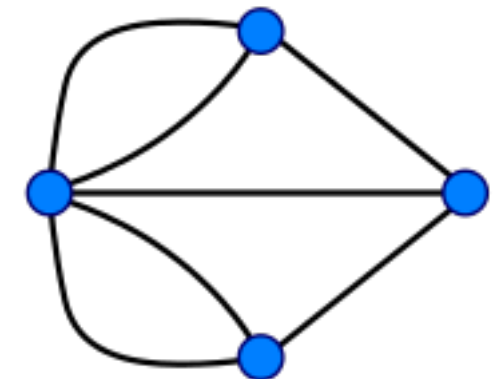
Degree $d(V)$ - number of edges incident to V

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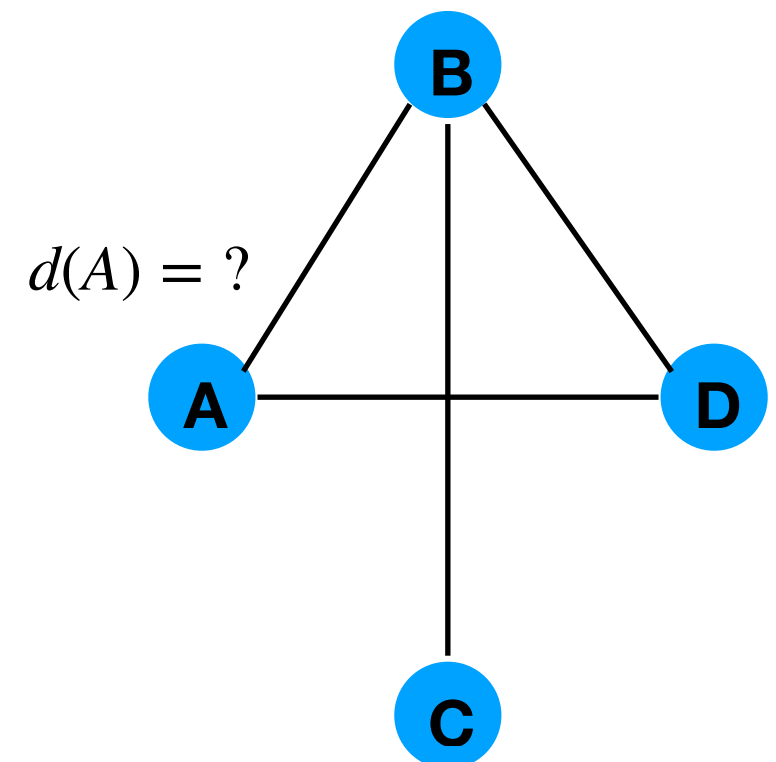


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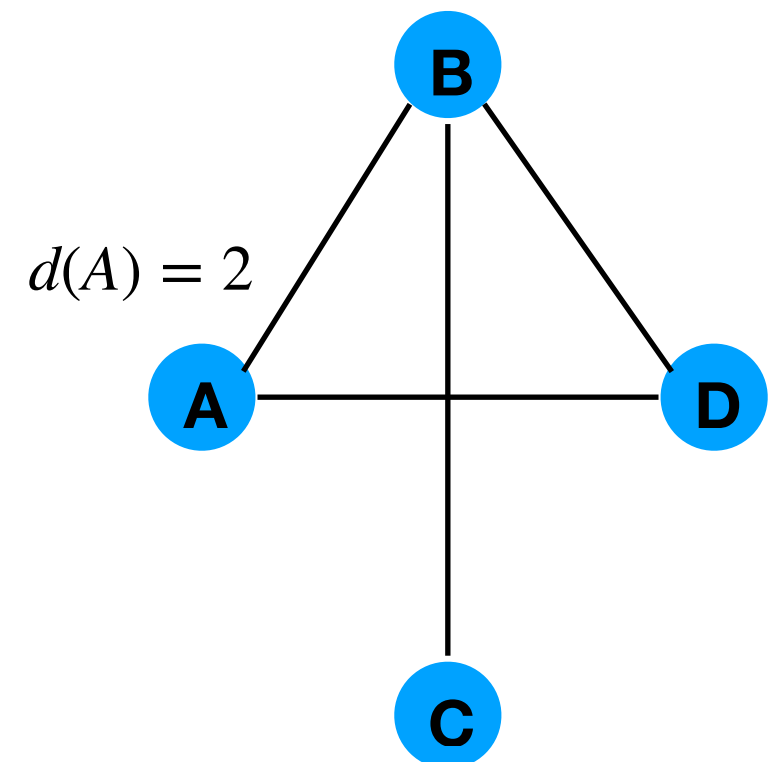


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Definitions

Handshaking lemma

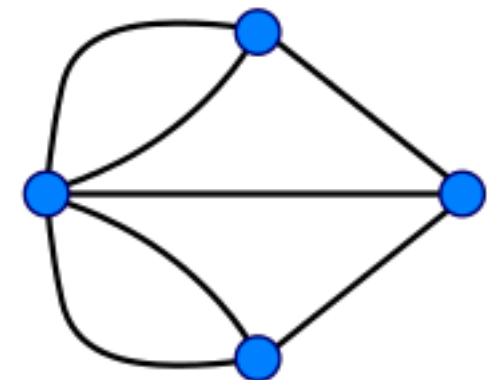
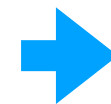
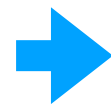
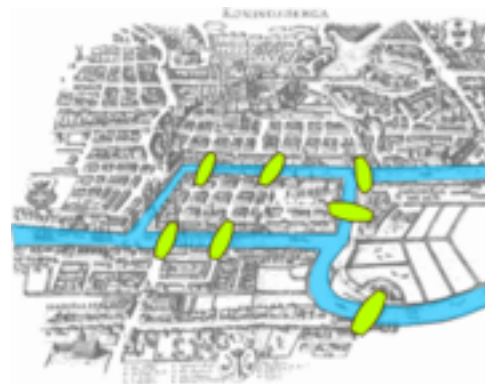
$$G = \langle V, E \rangle$$

$$\sum_{u \in V} d(u) = 2 |E|$$

in a party of people some of whom shake hands, an even number of people must have shaken an odd number of other people's hands.

Graph

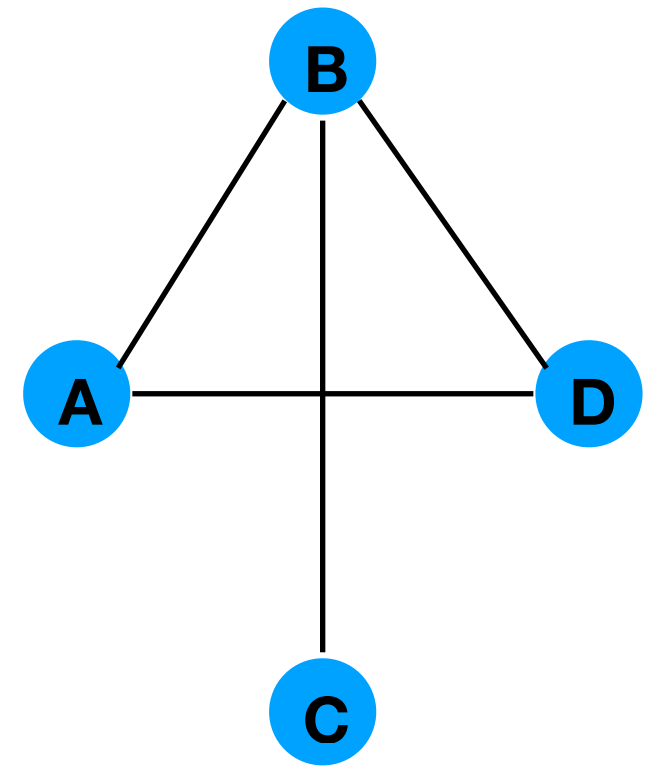
Back to Seven Bridges of Königsberg



An undirected graph has an Euler path if and only if exactly zero or two vertices have odd degree, and all of its vertices with nonzero degree belong to a single connected component.

Leonhard Euler, 1736

CS representation



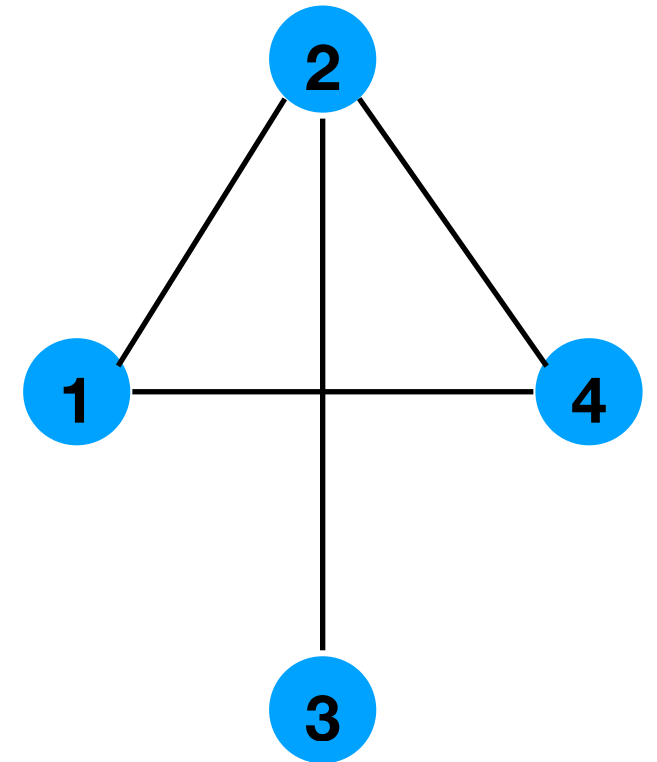
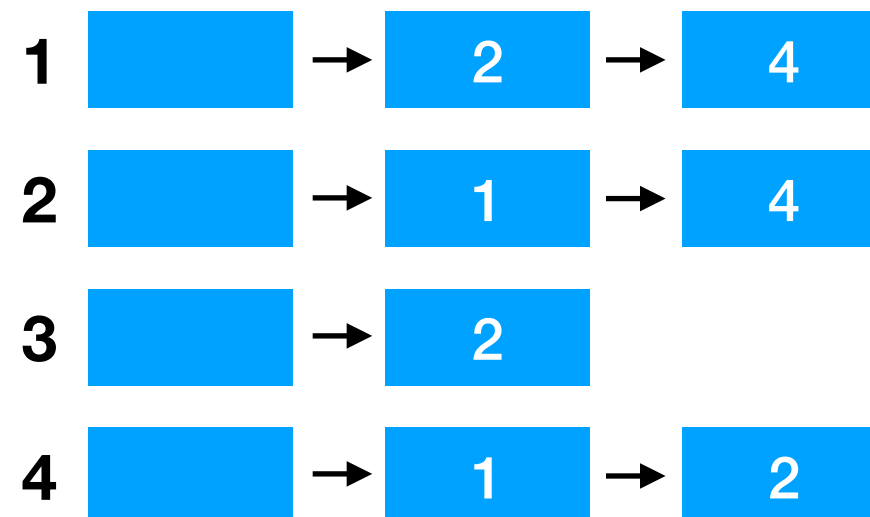
CS representation

$$G = \langle V, E \rangle$$

Adjacency matrix

	1	2	3	4
1	0	1	0	1
2	1	0	1	1
3	0	1	0	0
4	1	1	0	0

Adjacency list



CS representation

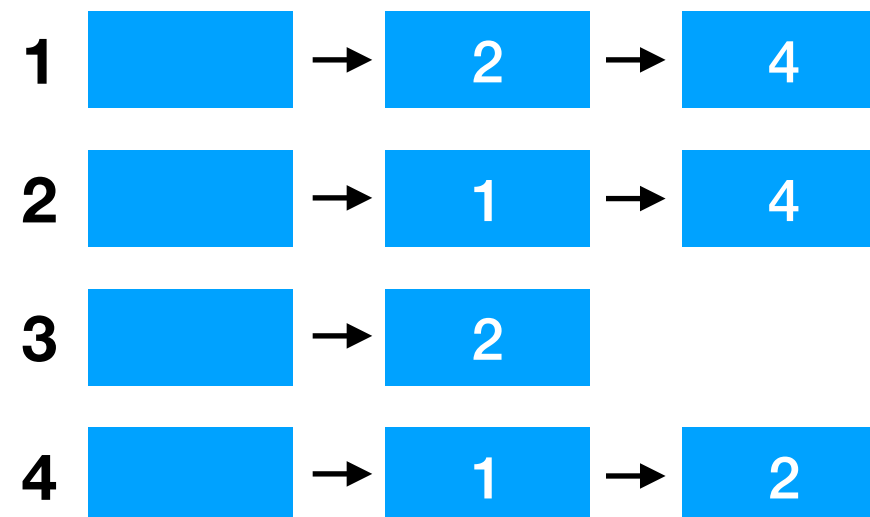
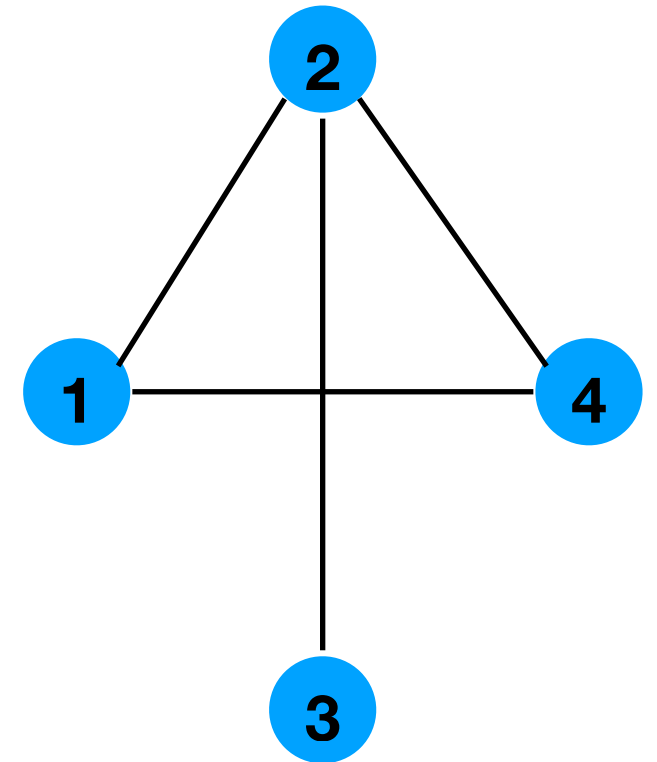
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Directed: N^2 Undirected: N^2

Adjacency list

Directed: $N + M$ Undirected: $N + 2M$ 

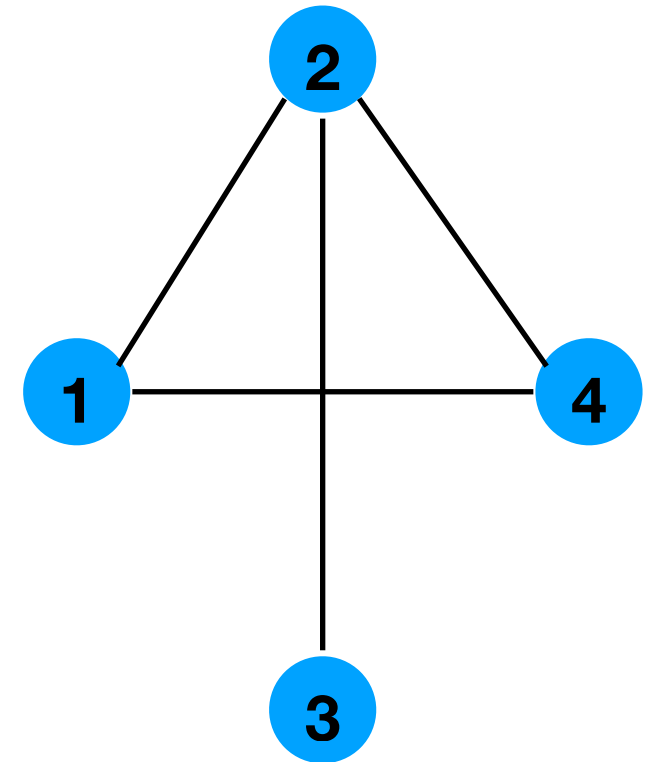
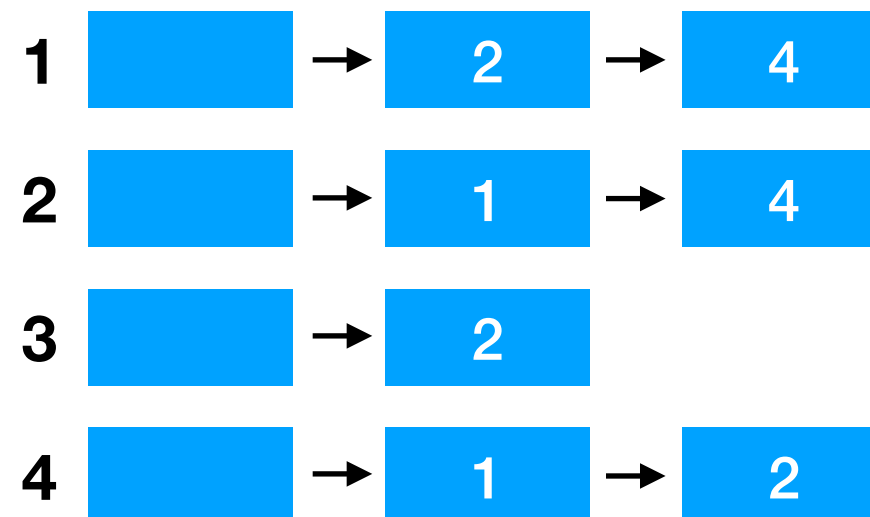
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Quiz: What is better to test if an edge is in the graph ?

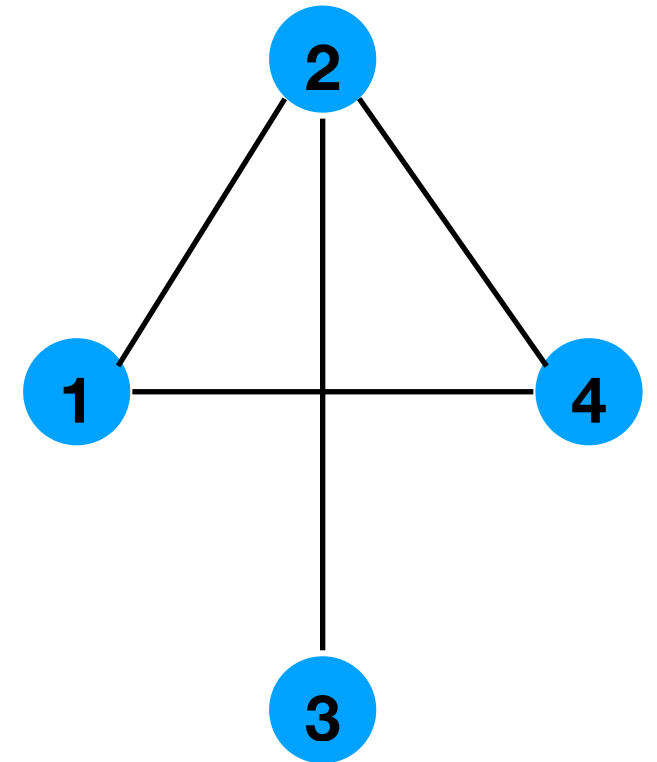
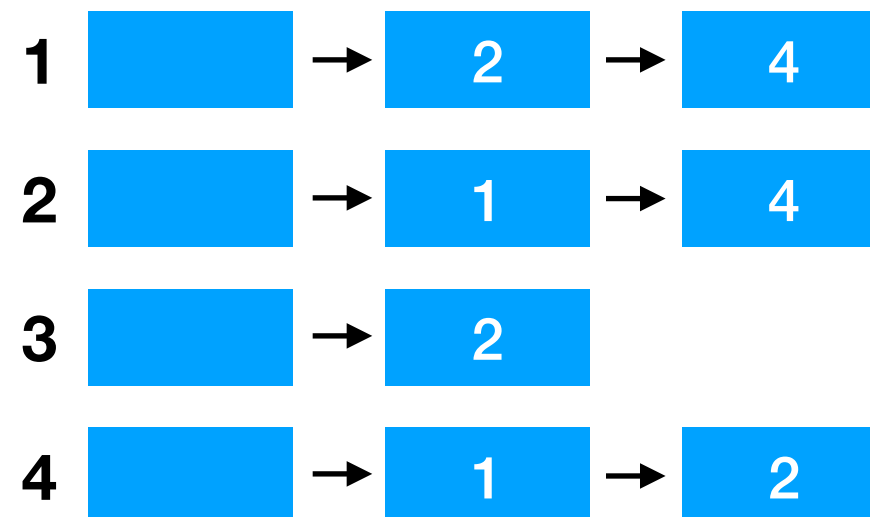
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Answer: matrix

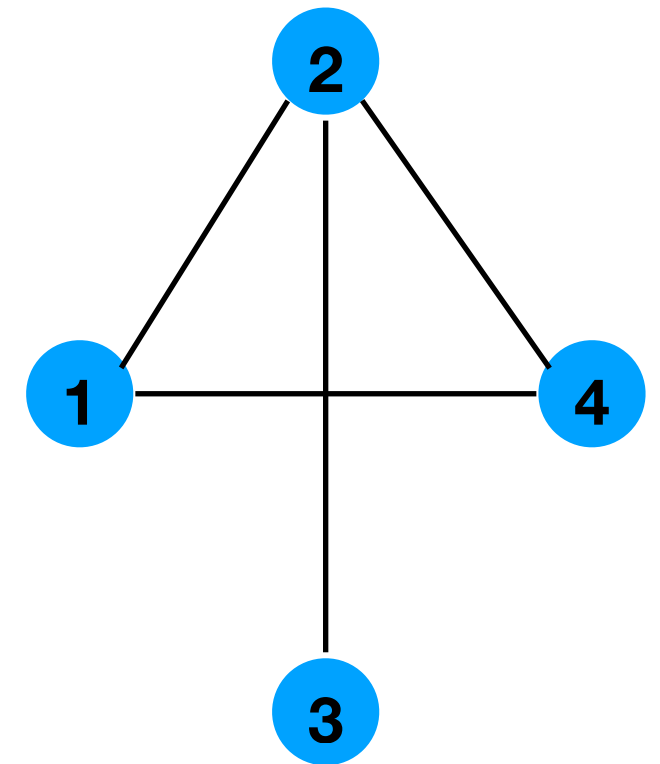
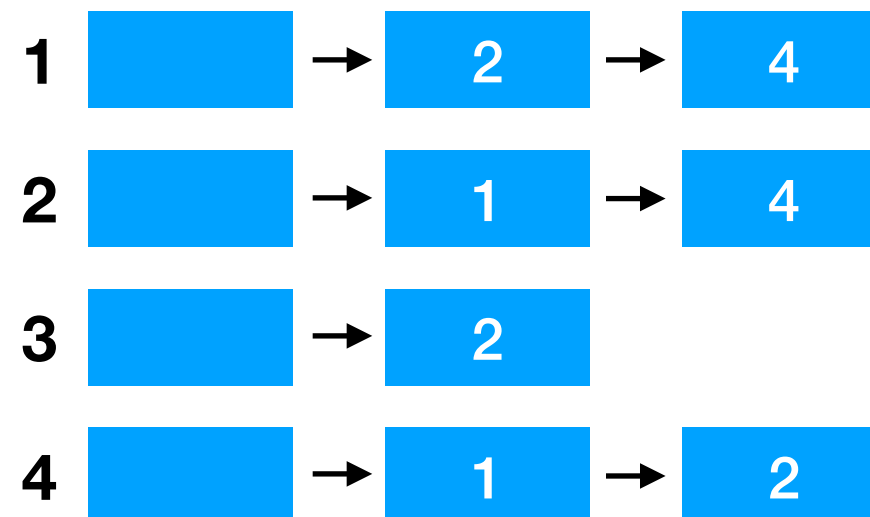
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Adjacency list



Quiz: What is faster to find the degree of vertex ?

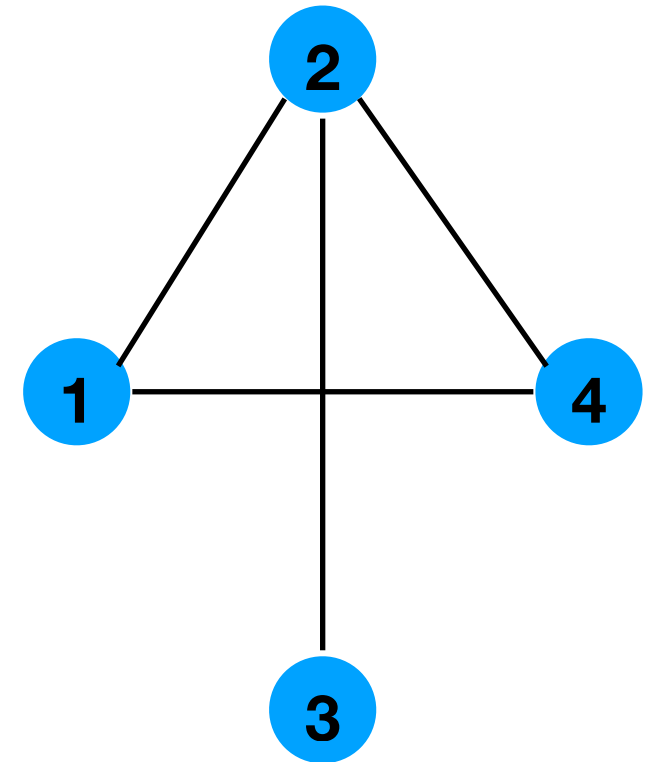
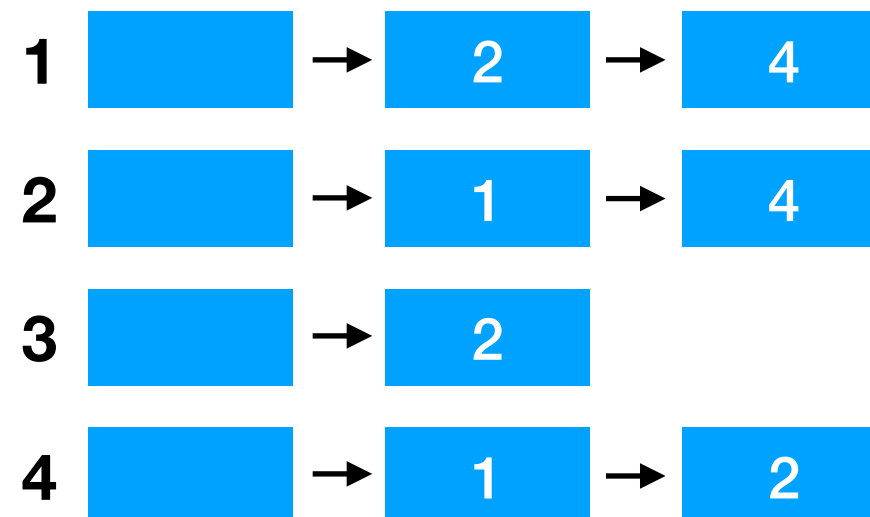
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Adjacency list



Quiz: What is better to test if an edge is in the graph ?

Answer: list

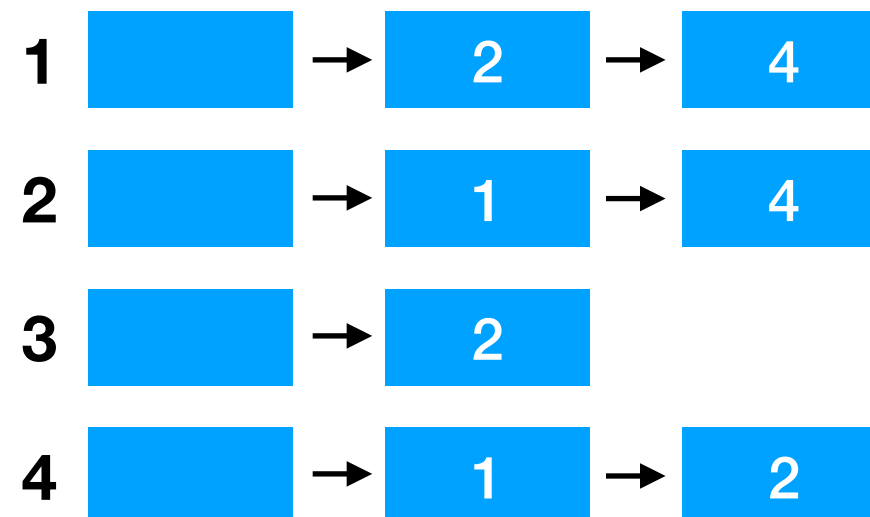
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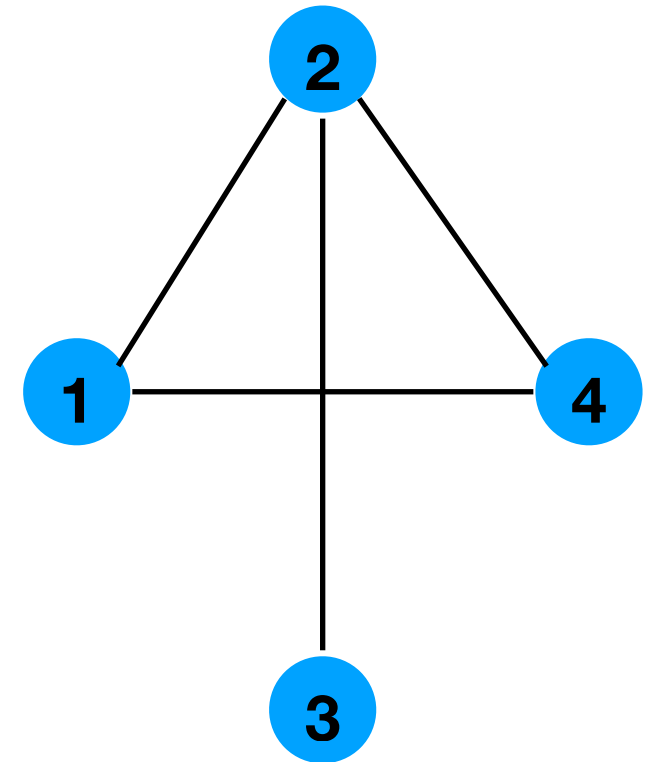
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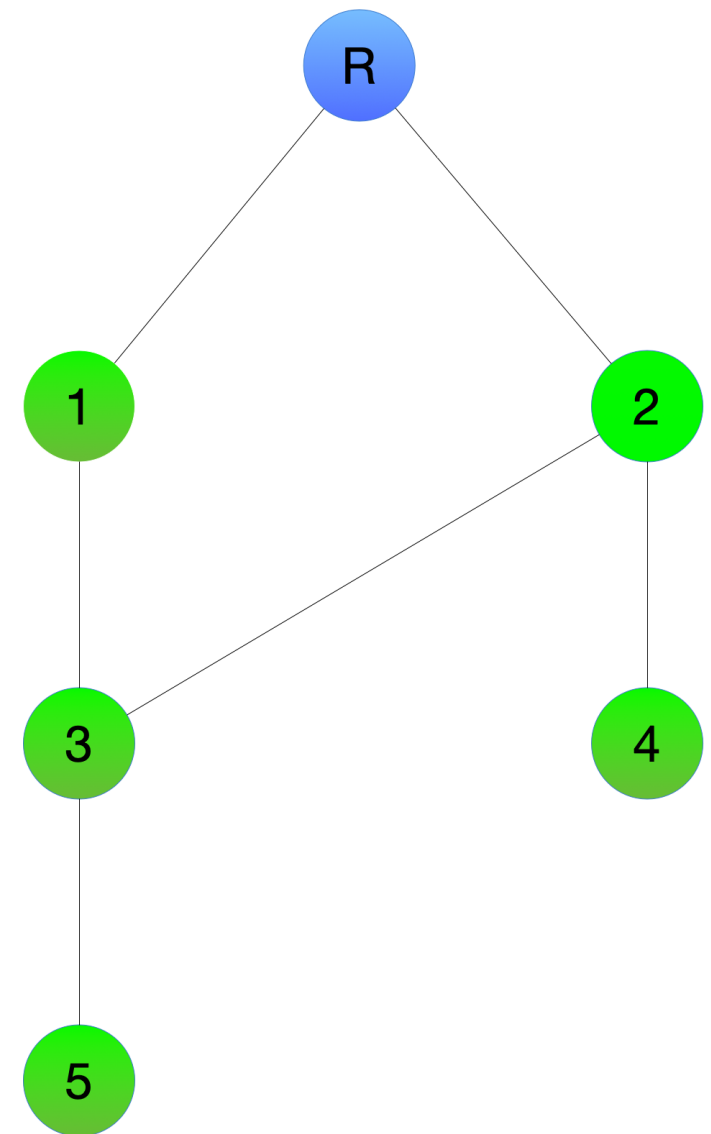


$$M + N \textbf{ VS } N^2$$



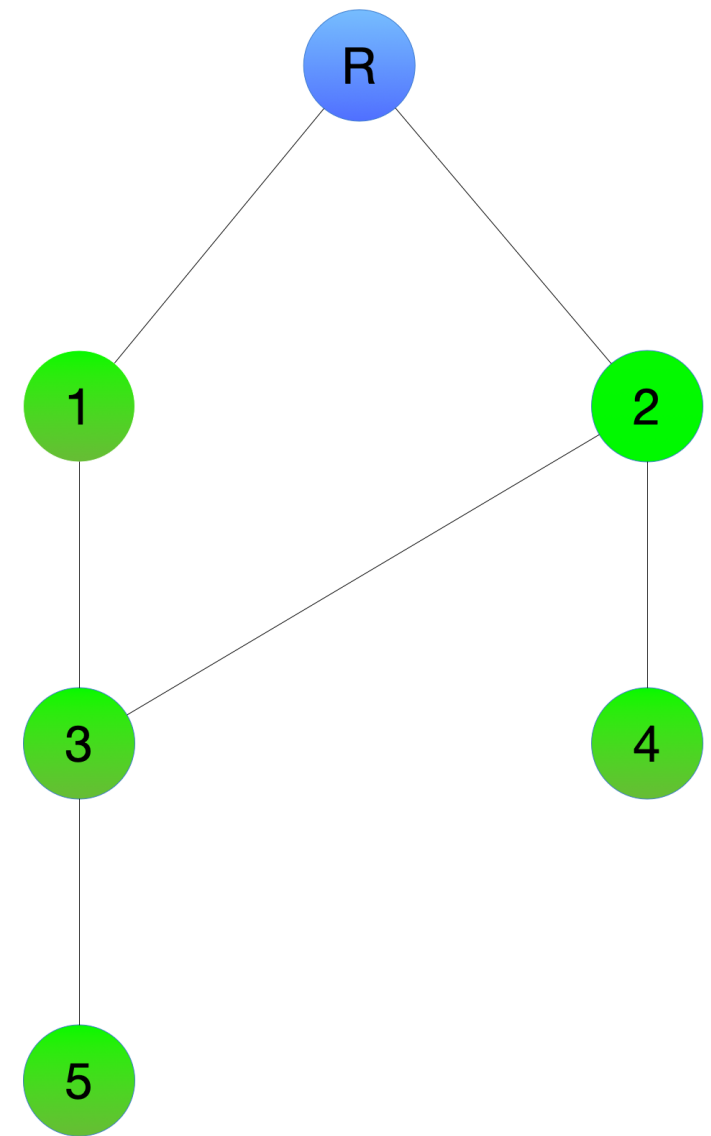
Graph traversals

- Breadth-First Search (BFS)
- Depth-First Search (DFS)



Breadth-First Search (BFS)

Main Idea: mark each vertex when we **first** visit it & monitor completely unexplored items.



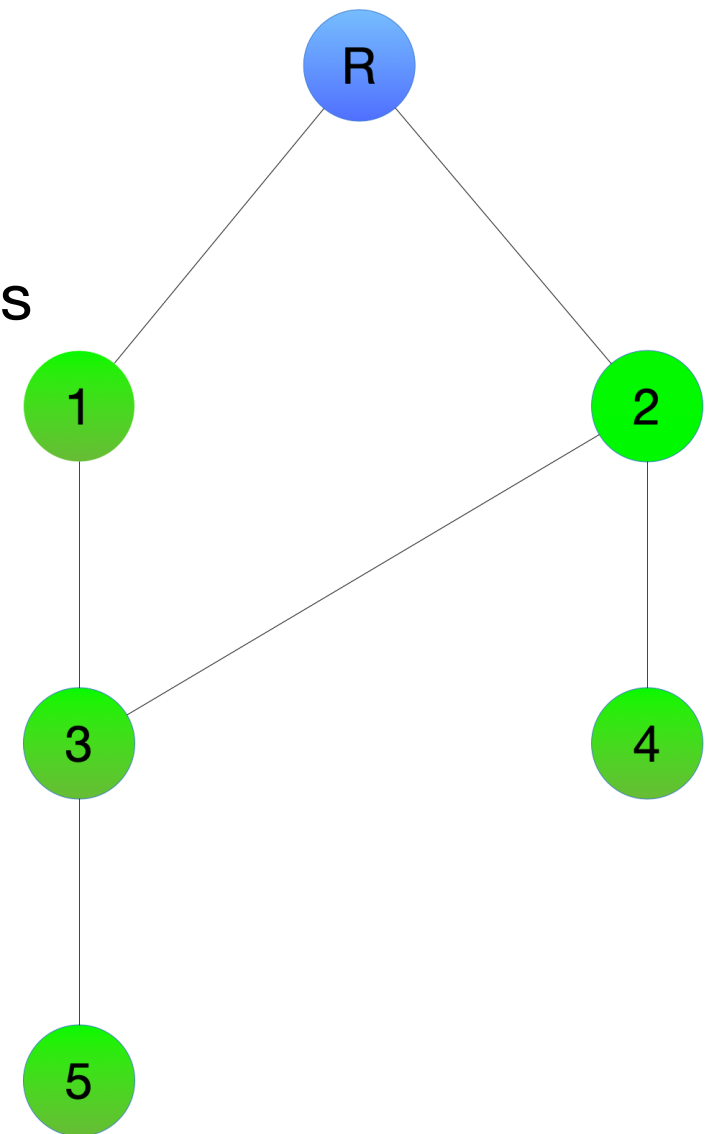
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Each vertex can exist in one of three states:

- Undiscovered - the node has never been visited
- Discovered - the node has been found, but not all its edges are visited
- Processed - all adjacent edges of the node are visited

Container to be used - queue.



Breadth-First Search (BFS)

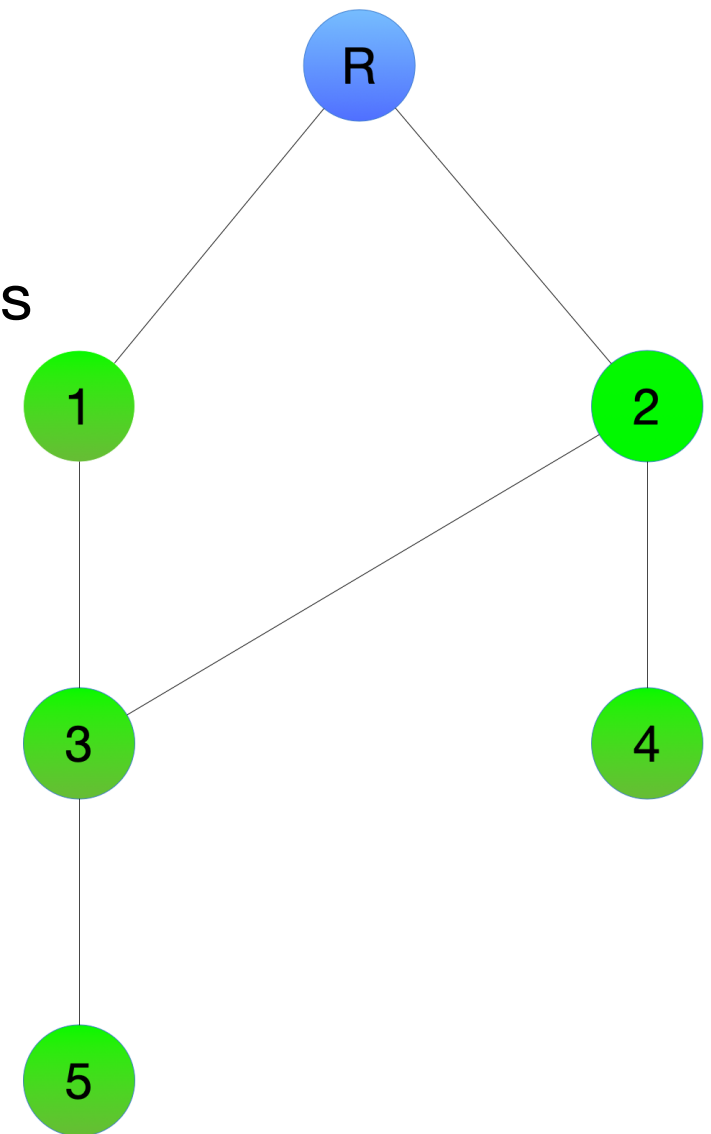
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Quiz: What is the complexity of BFS ?



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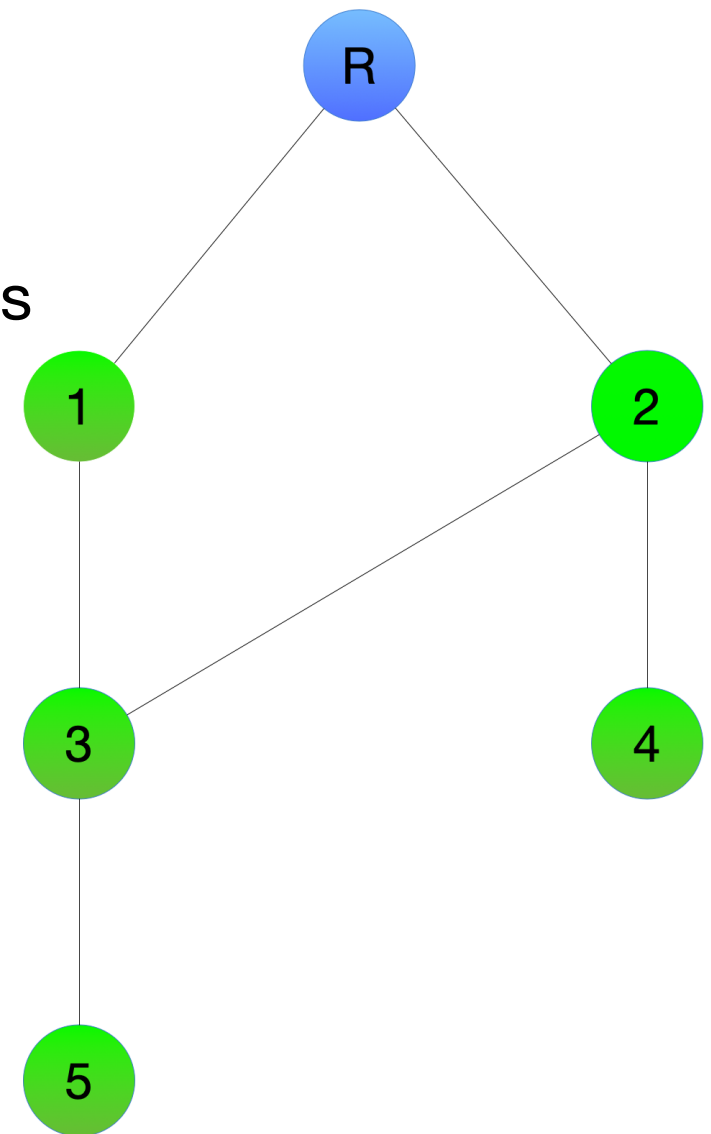
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Quiz: What is the complexity of BFS ?

Answer: $O(V + E)$



Depth-First Search (DFS)

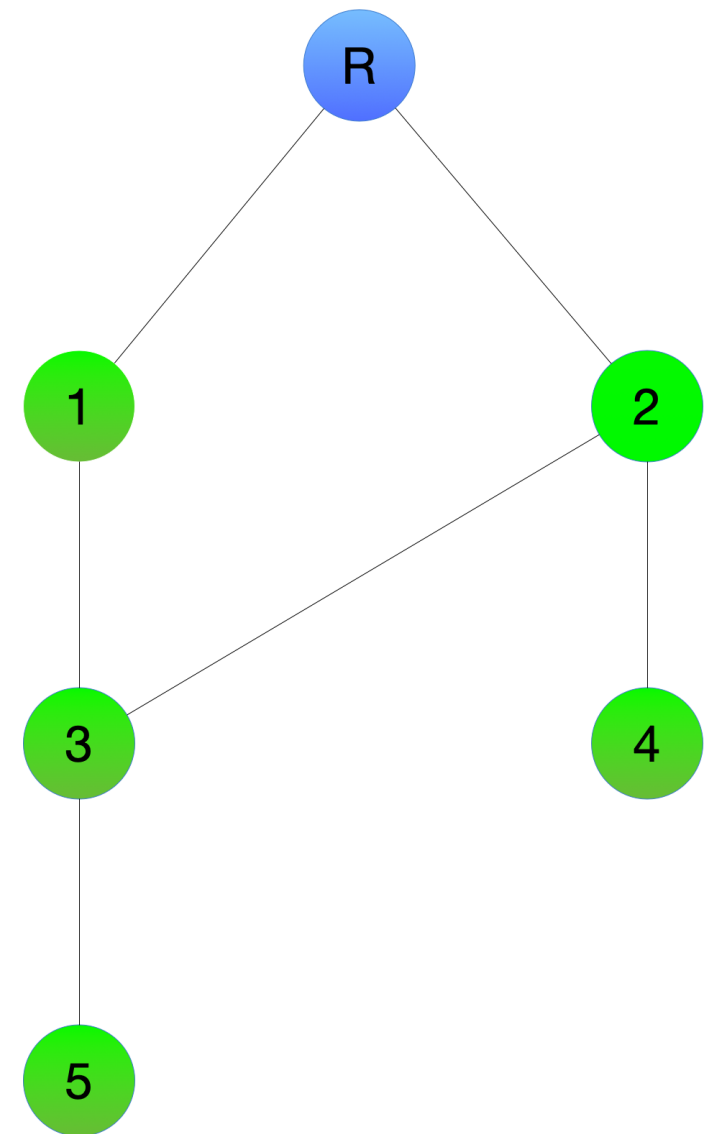
Main Idea: start DFS from every next node until there are nodes which are not visited.

Each vertex can exist in one of two states:

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Container to be used - stack (recursion).

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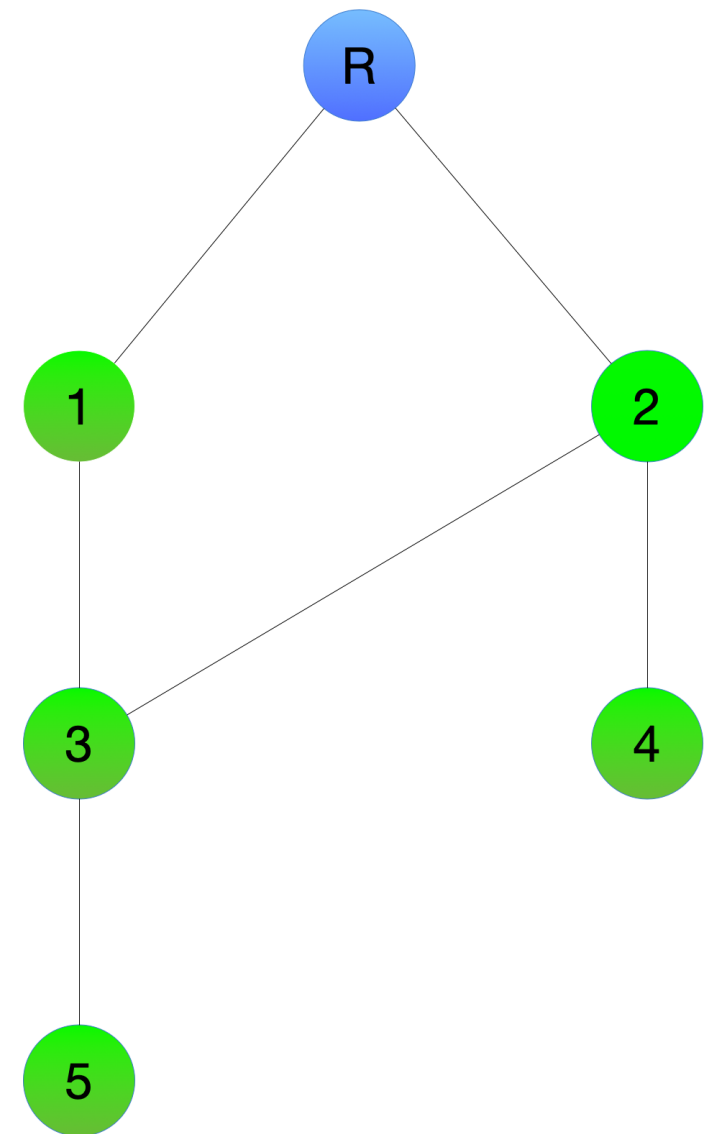
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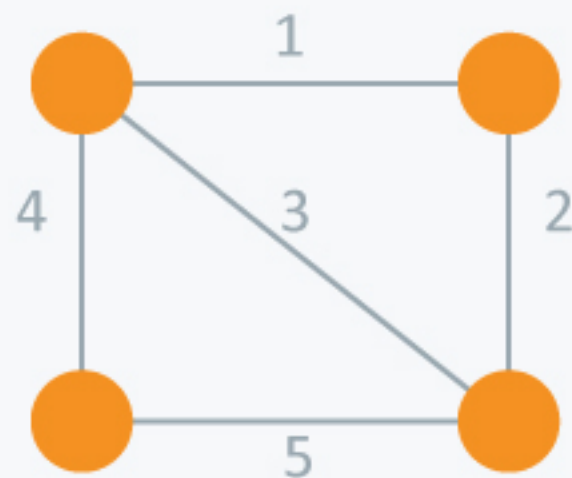
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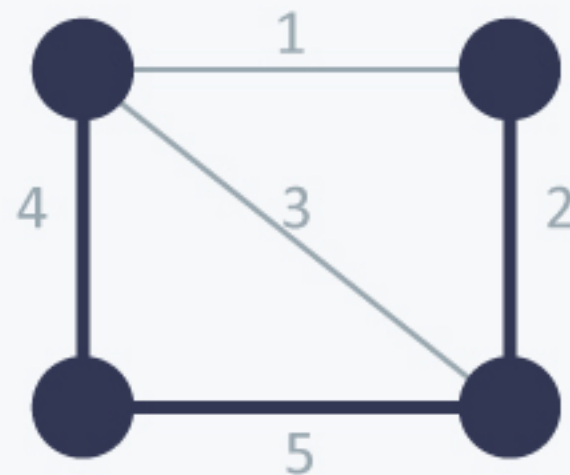
Minimum Spanning Tree

Spanning tree: for graph $G = (V, E)$ is a subset of edges from E , forming a tree connecting all vertices of V .

For edge-weighted graph we are usually interested in minimum value spanning tree

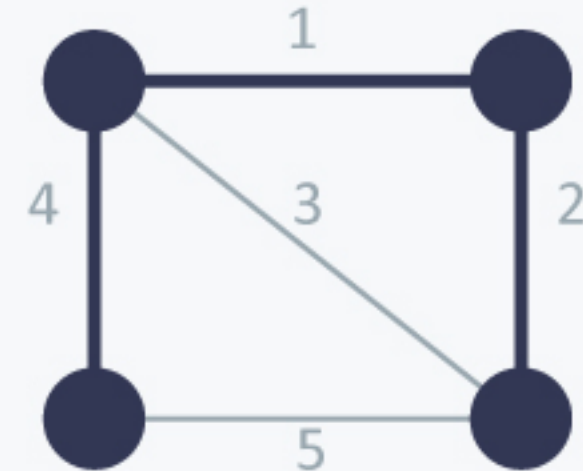


Undirected
Graph



Spanning
Tree

Cost = 11(=4+5+2)



Minimum Spanning
Tree

Cost = 7(=4+1+2)

Minimum Spanning Tree

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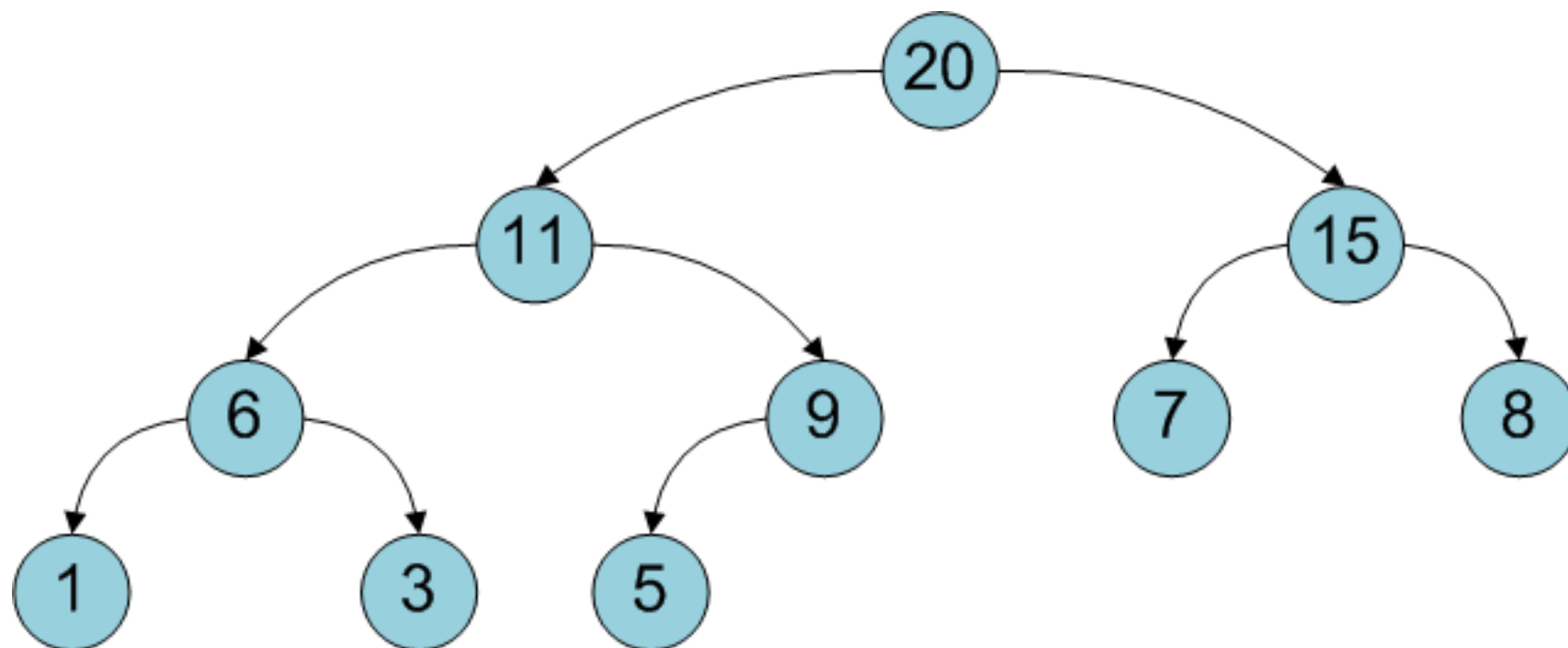
Two main algorithms:

- Prim
 - Kruskal
-
- Minimum spanning tree minimises total length (weight) over all possible spanning trees.
 - There could be more than one minimum spanning tree in the graph.

Algorithms are greedy

Heap

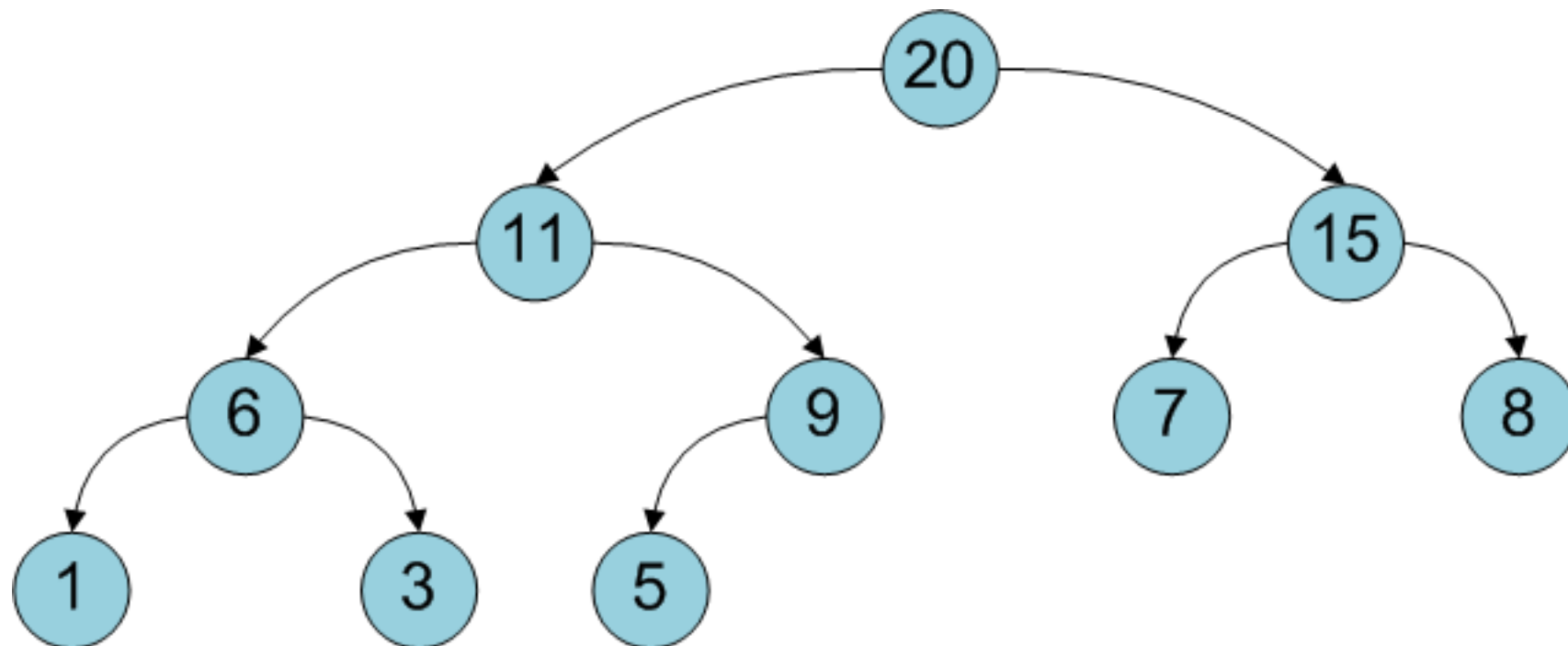
Def: is a binary tree with the property value of each vertex is less than the value of its ancestors



Heap

Def: is a binary tree with the property value of each vertex is greater than the value of its ancestors

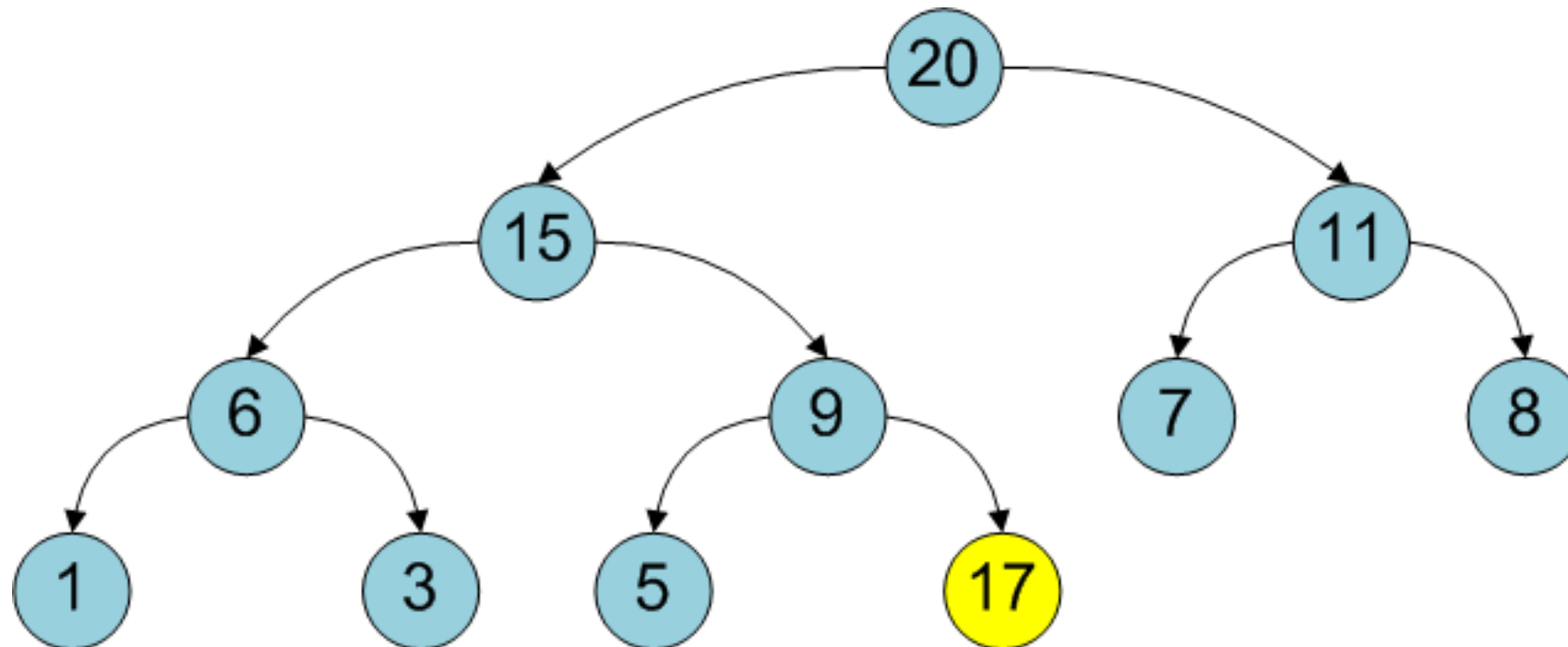
One can store entire data structure in an array. Root element is in index 0, left child of vertex i would have an index $2 * i + 1$, while the right one would have an index $2 * i + 2$



Heap

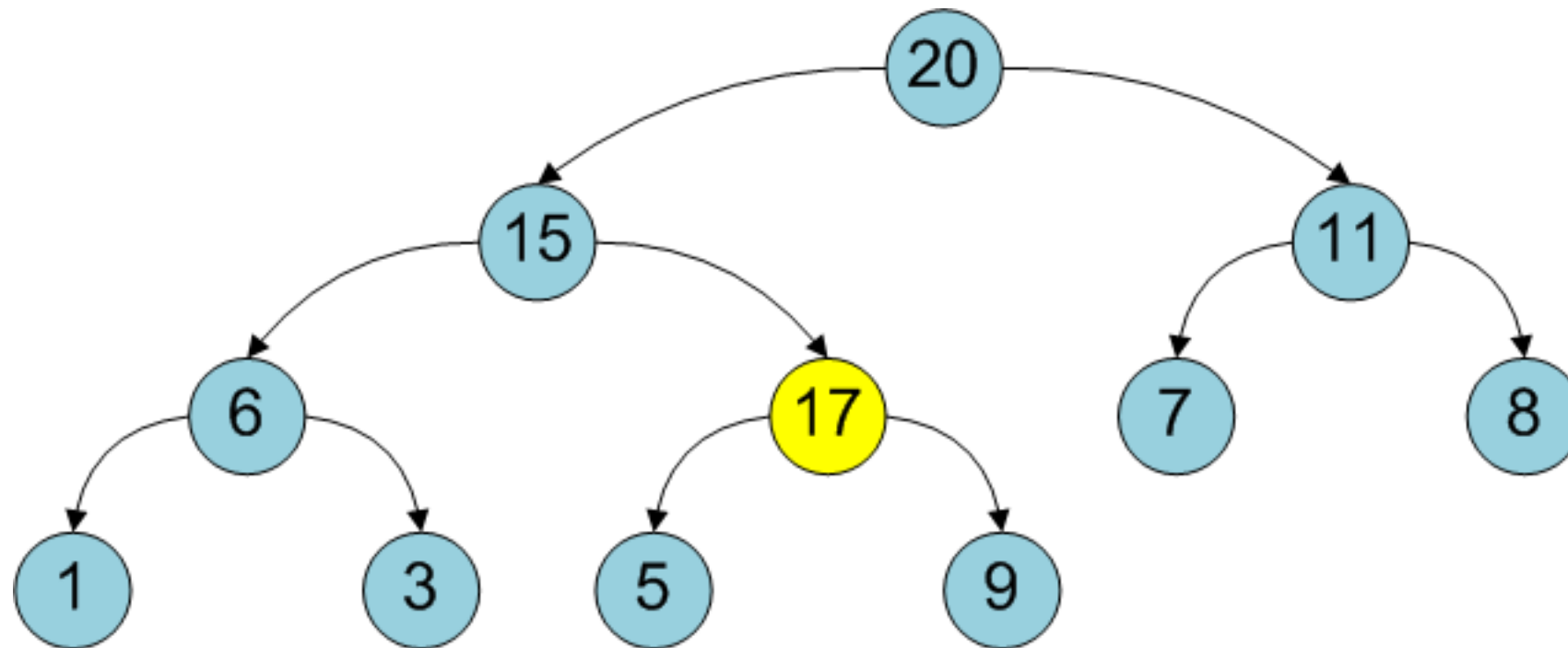
Push

Add element to the end of the array. Then fix the property of the heap.



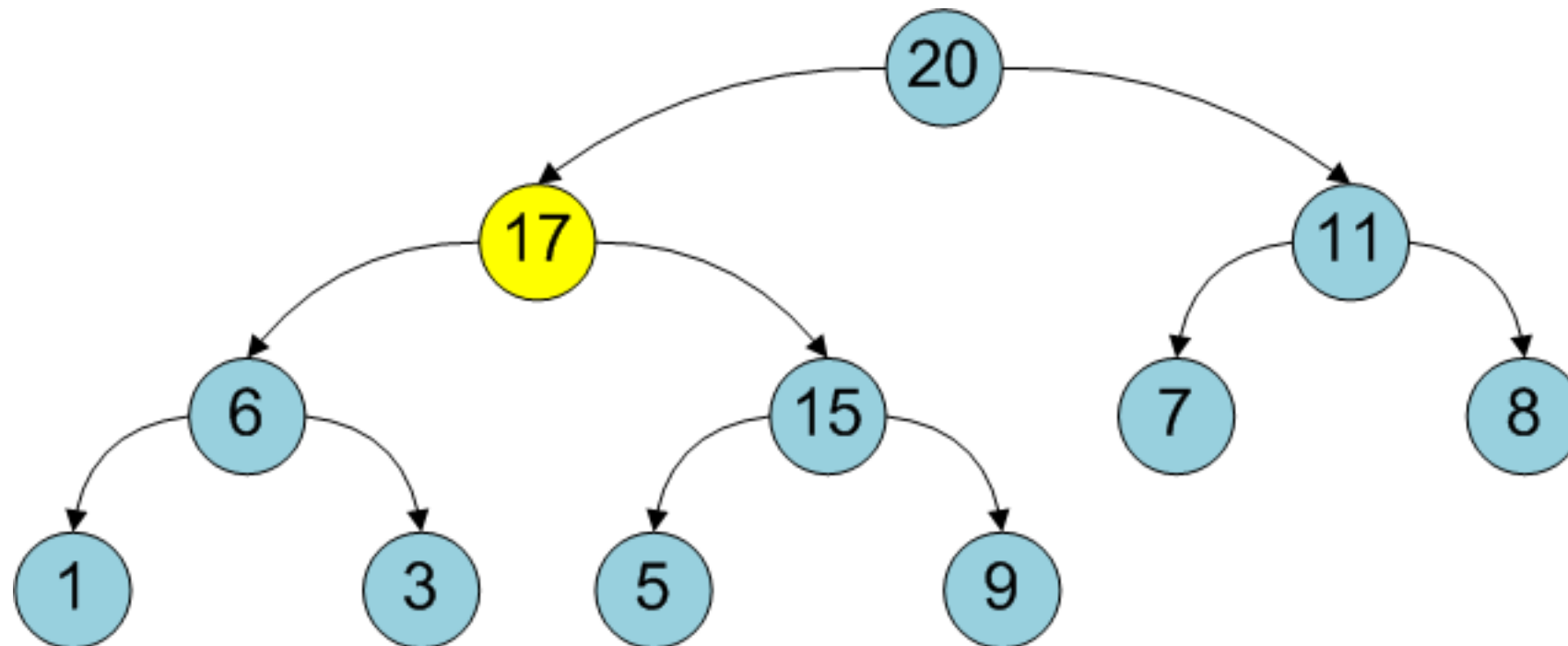
Heap

Push



Heap

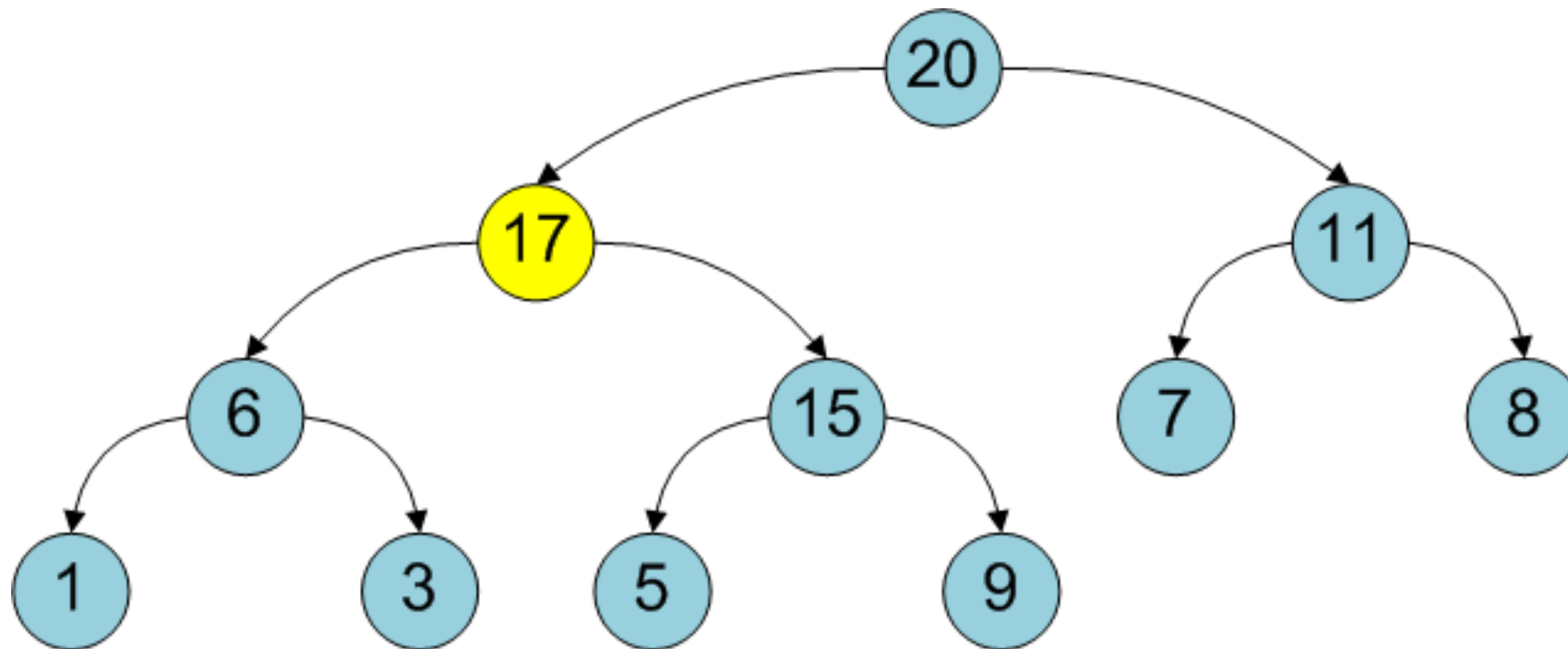
Push



Quiz: What is the complexity of adding an element in the heap ?

Heap

Push

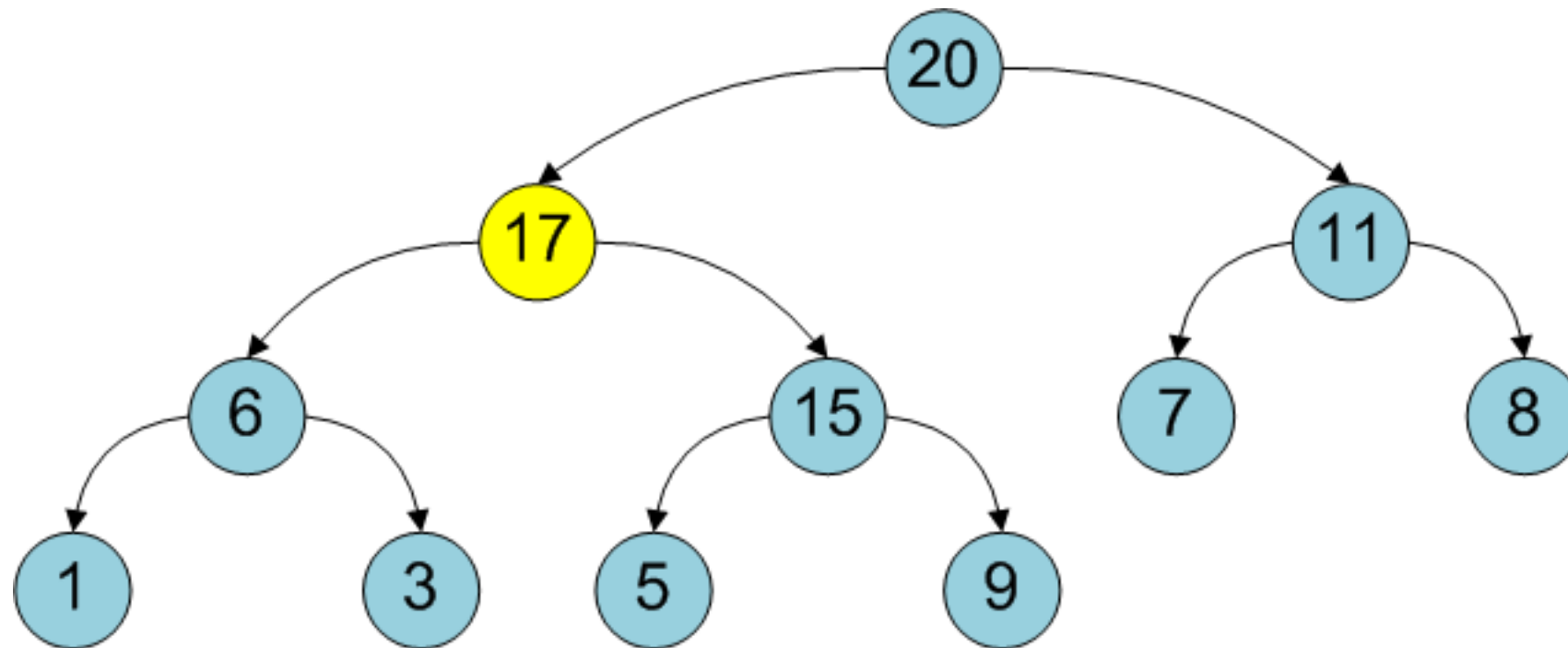


Quiz: What is the complexity of adding an element in the heap ?

Answer: $O(\log(N))$

Heap

Push

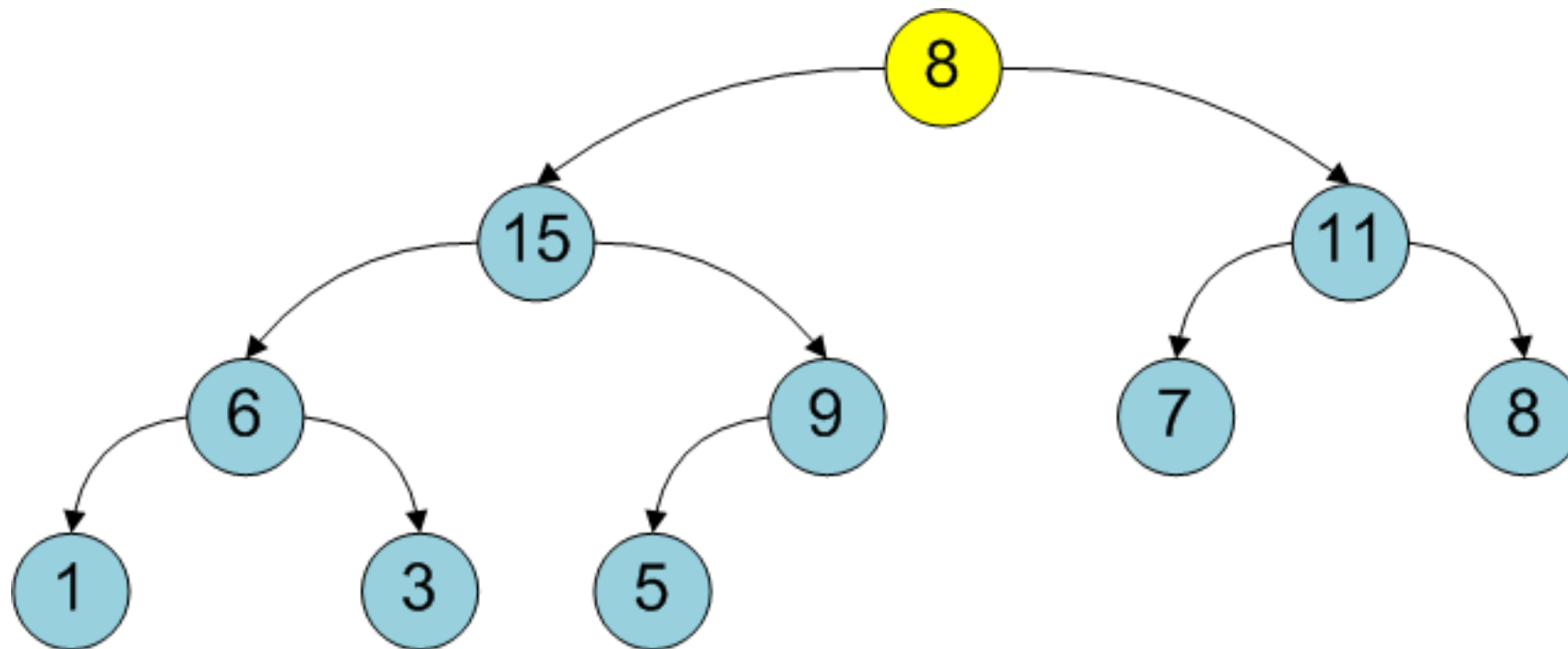


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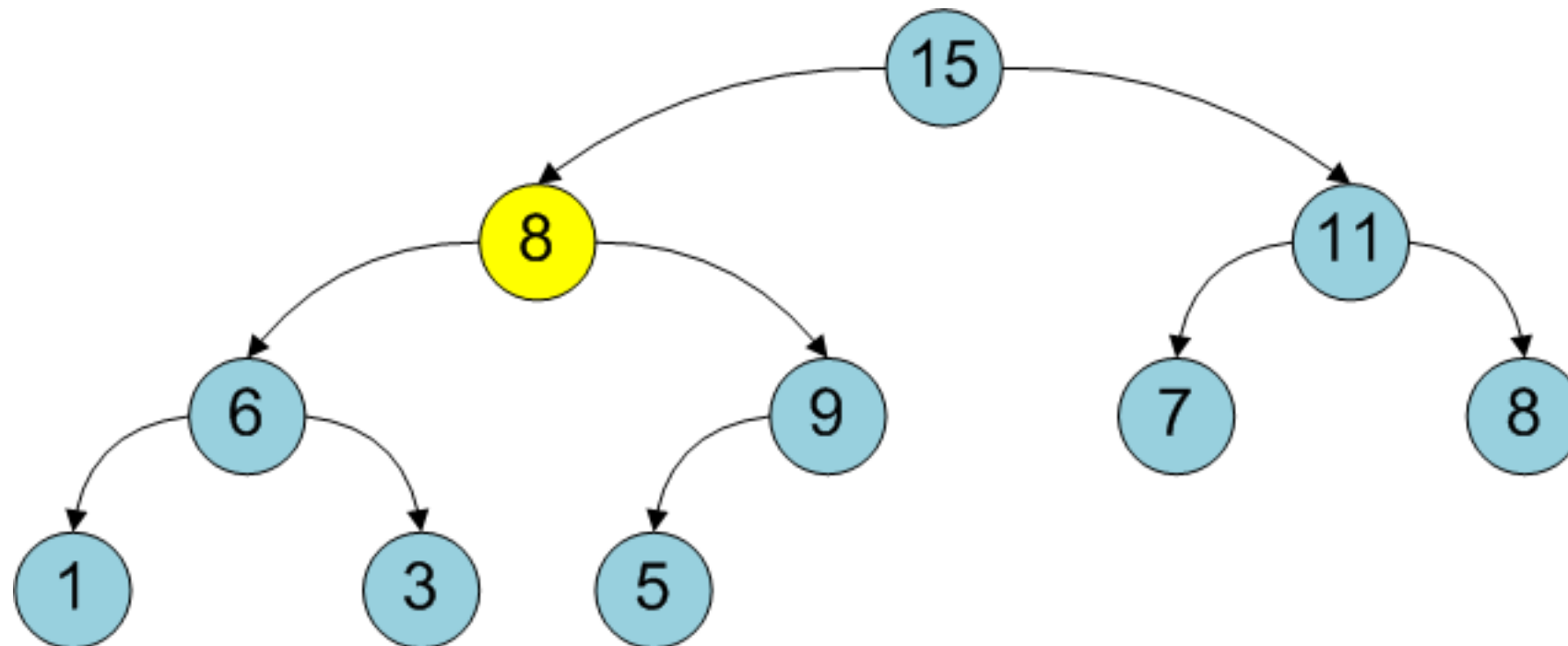
Heap

Heapify



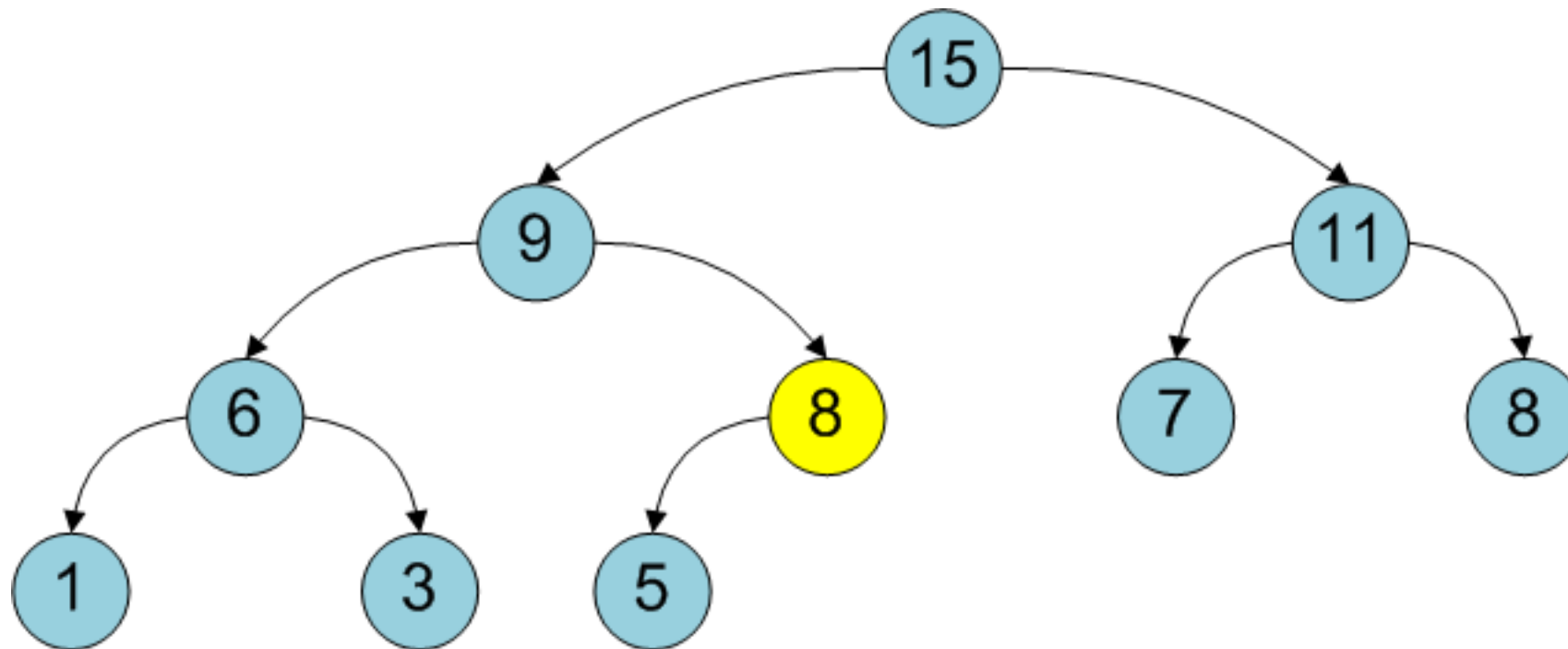
Heap

Heapify



Heap

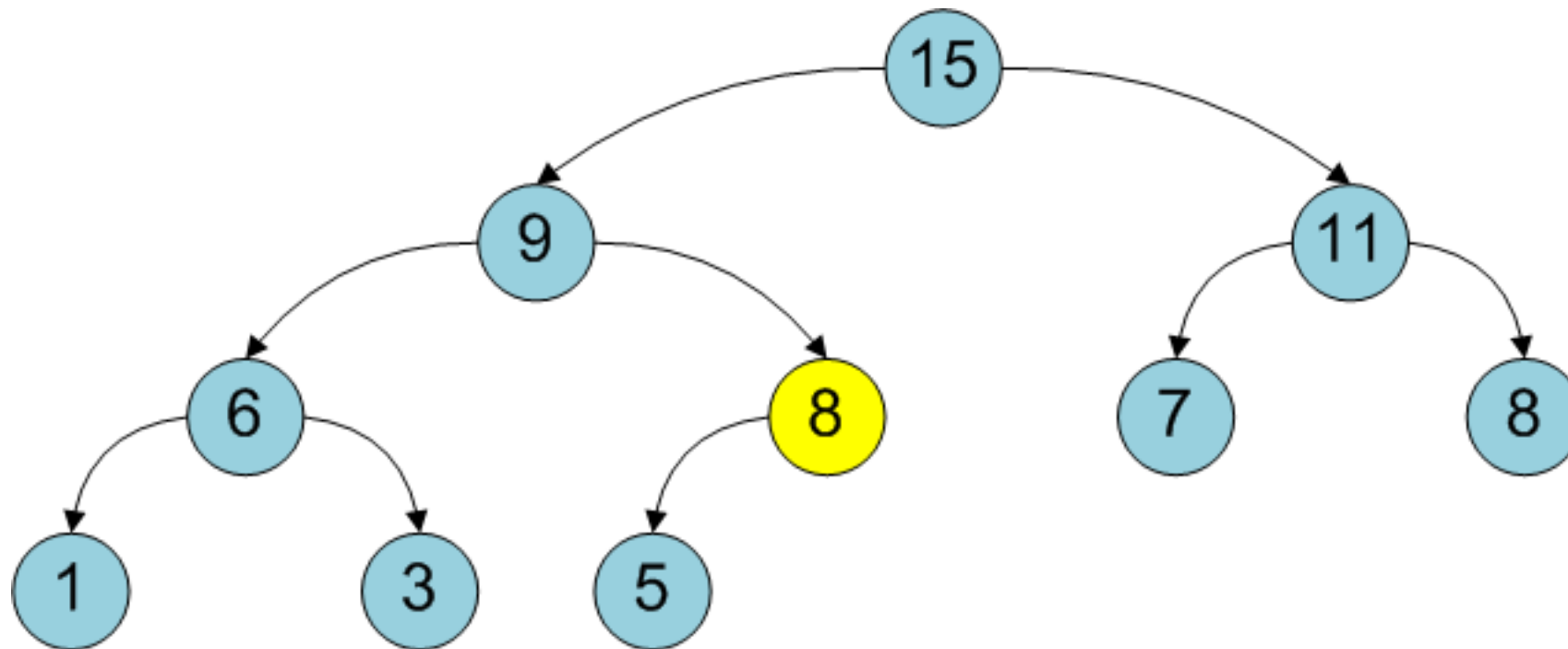
Heapify



Quiz: What is the complexity of heaping operation ?

Heap

Heapify



Quiz: What is the complexity of heaping operation ?

Answer: $O(\log(N))$

Heap

Pop

1. Take root out
2. Put the last heap element on the place of the root
3. Heapify

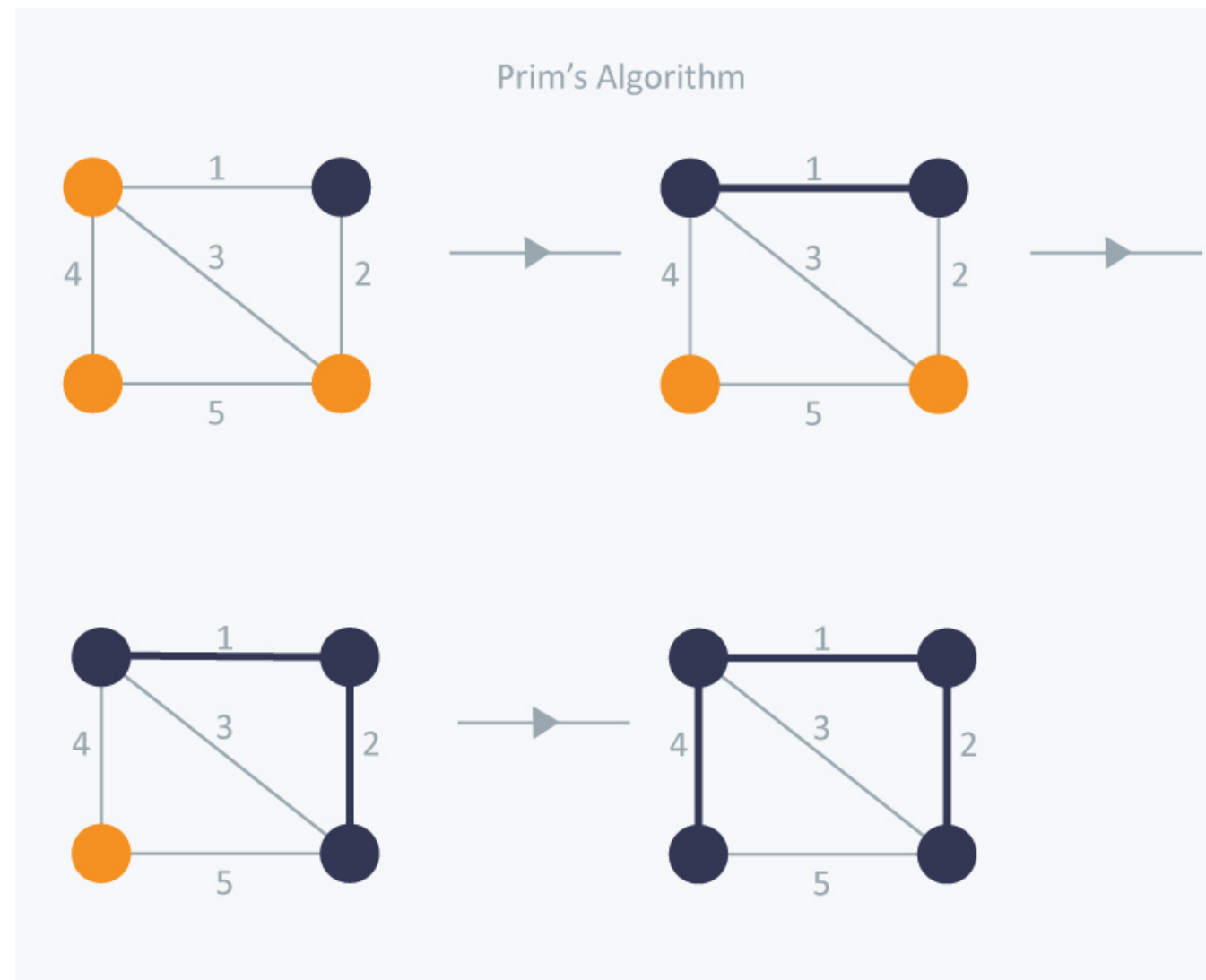
Heap

Applications

- Heap sort
- Priority queue

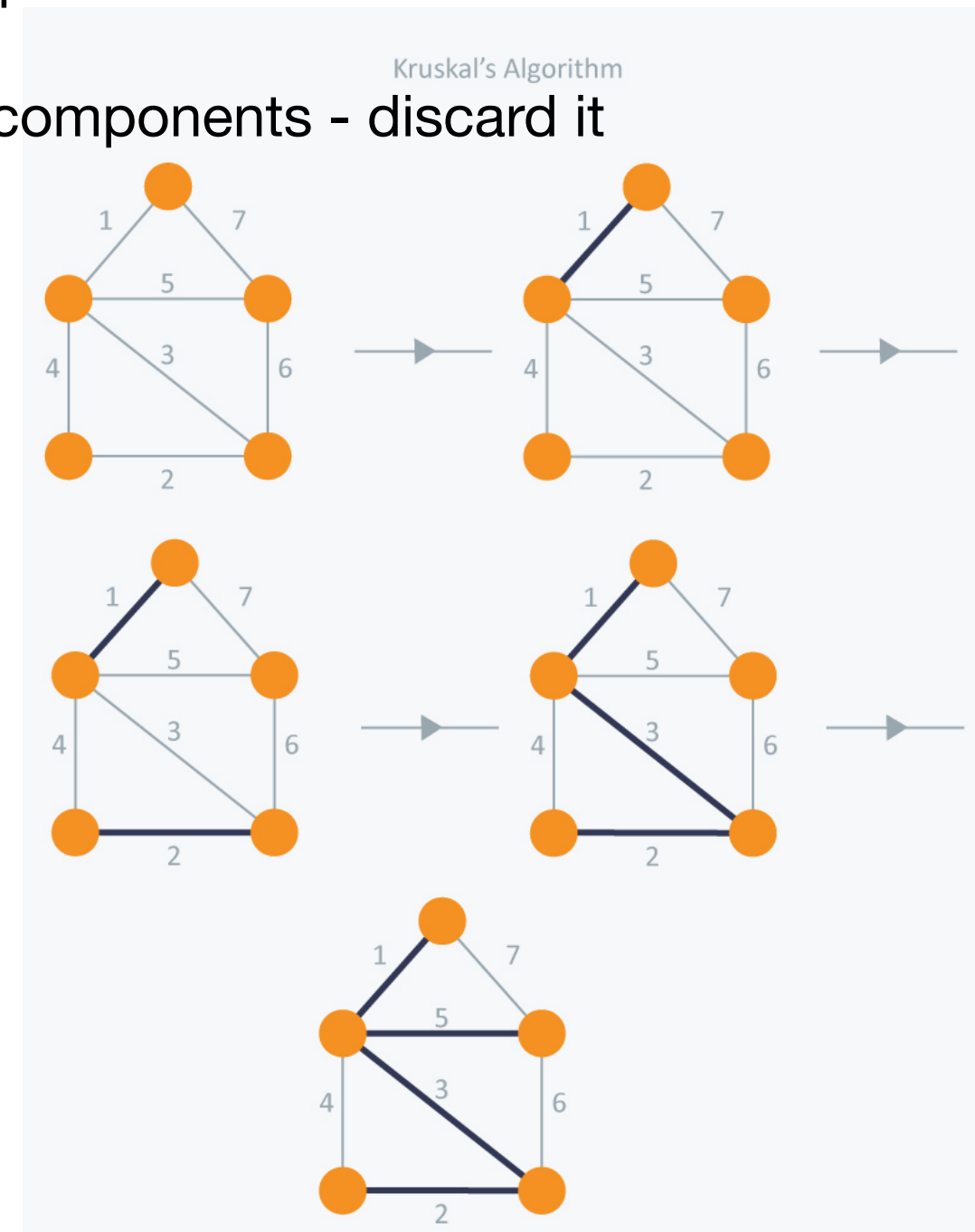
Prim algorithm

1. Select random node to start from
2. While non-tree nodes remaining:
 1. Select an edge of minimum weight between a tree and non-tree vertex
 2. Add selected edge and vertex to the tree



Kruskal algorithm

1. Initialise every node to be a single connected component
2. Consider the lightest edge
 1. If adjacent nodes are in the same connected components - discard it
 2. Otherwise, add edge, merge components



Prim vs Kruskal

- Prim **$O(VE)$** , if we don't keep track of cheapest edge
 - Prim **$O(VV)$** , if we do, but use simple data structure
 - Prim **$O(E + V \log V)$** , if use priority queue
 - Kruskal **$O(E \log E)$**
-
- Prim **is better on dense graphs**
 - Kruskal **is better on sparse graphs**

Shortest paths

Path is a sequence of edges connecting two vertices

- For unweighted graphs can be found with BFS
- Same for the graphs with equal weights

Dijkstra algorithm find shortest path between start and end vertices.

- $O(N^2)$
- Greedy
- On each step selects the cheapest to add edge
- Almost Prim

Dijkstra algorithm

What is different from Prim:

Instead of considering only the weight of edge consider the length of the path from the start node