ETH zürich

Introduction to Scientific Computation
Lecture 3
Fall 2020

Sorting, Fourier transform, FFT

Mikhail Usvyatsov October 7, 2020

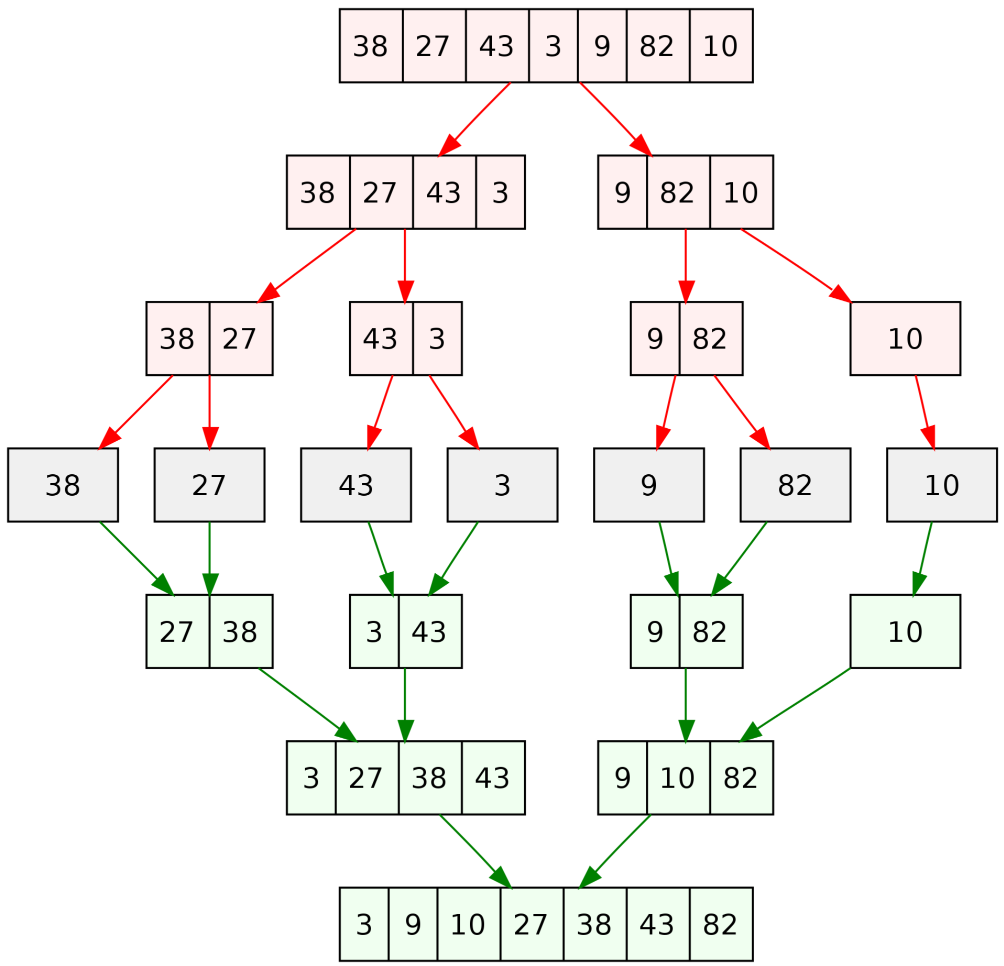
Sorting

6 5 3 1 8 7 2 4

Divide and Conquer

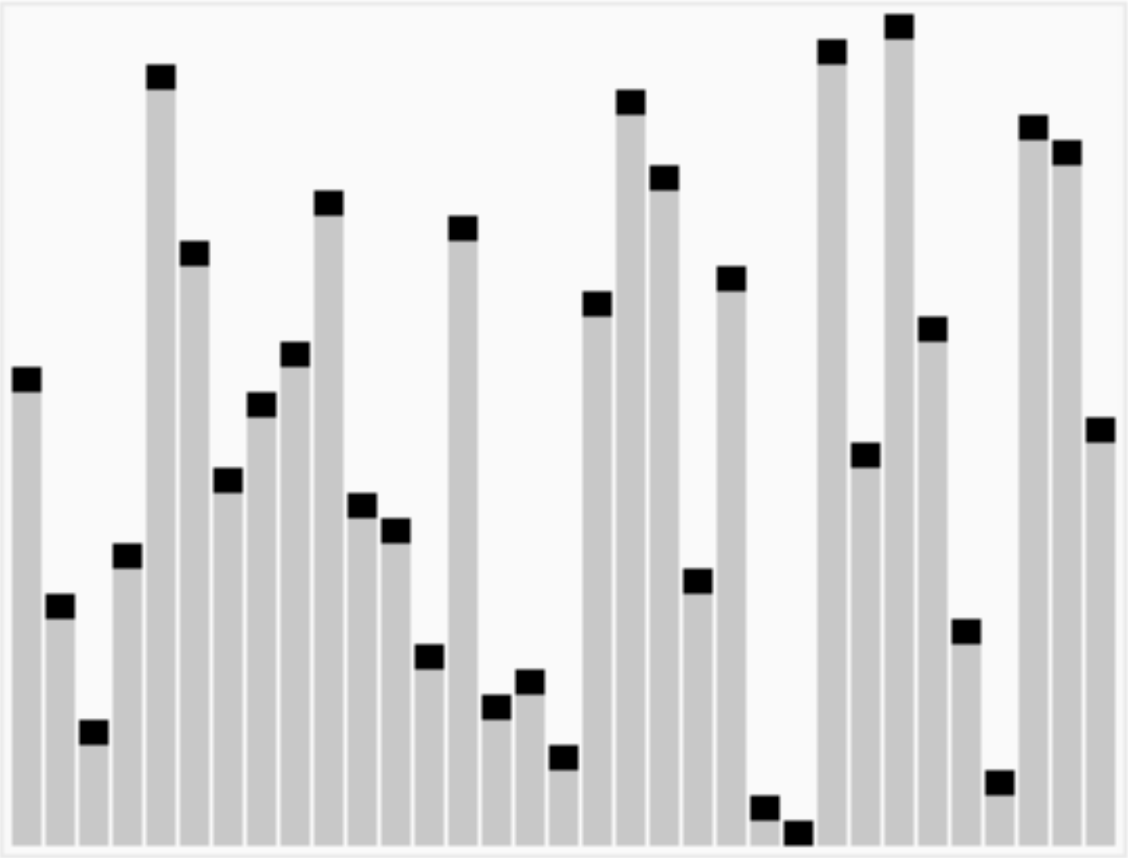
Divide: break problems into several similar to the original problems but of smaller size

Conquer: solve small problems



(c) Wikipedia

Quicksort



(c) Wikipedia

Quicksort

```
def quicksort(array, left, right):
    pivot = # select the pivot somehow
    if left < right:
        pivot_idx = partition(array, left, right, pivot)
        quicksort(array, left, pivot_idx)
        quicksort(array, pivot_idx + 1, right)
```

Insert quiz here

Here was quiz

Convolution

$$c(t) = \int_{-\infty}^{\infty} a(t-\tau)h(\tau)d\tau$$

Convolution

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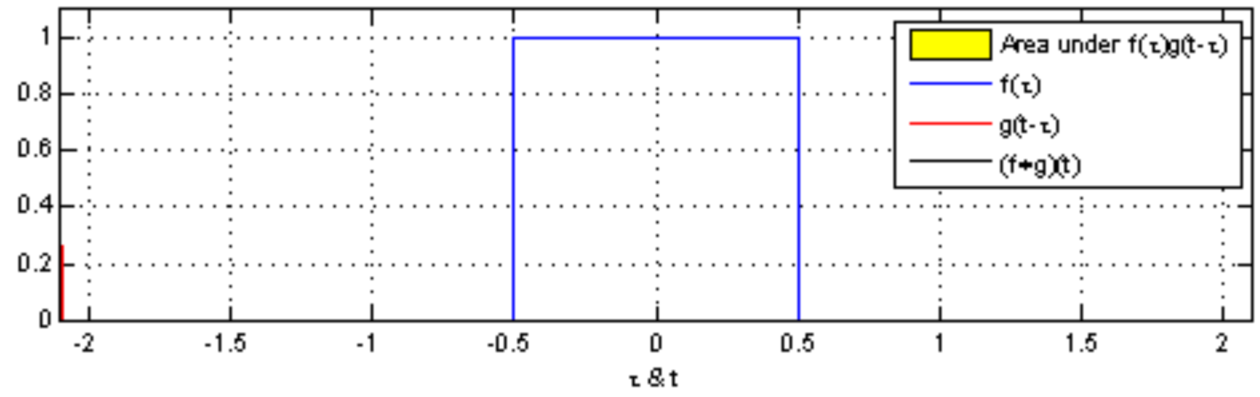

(c) Wikipedia

Convolution

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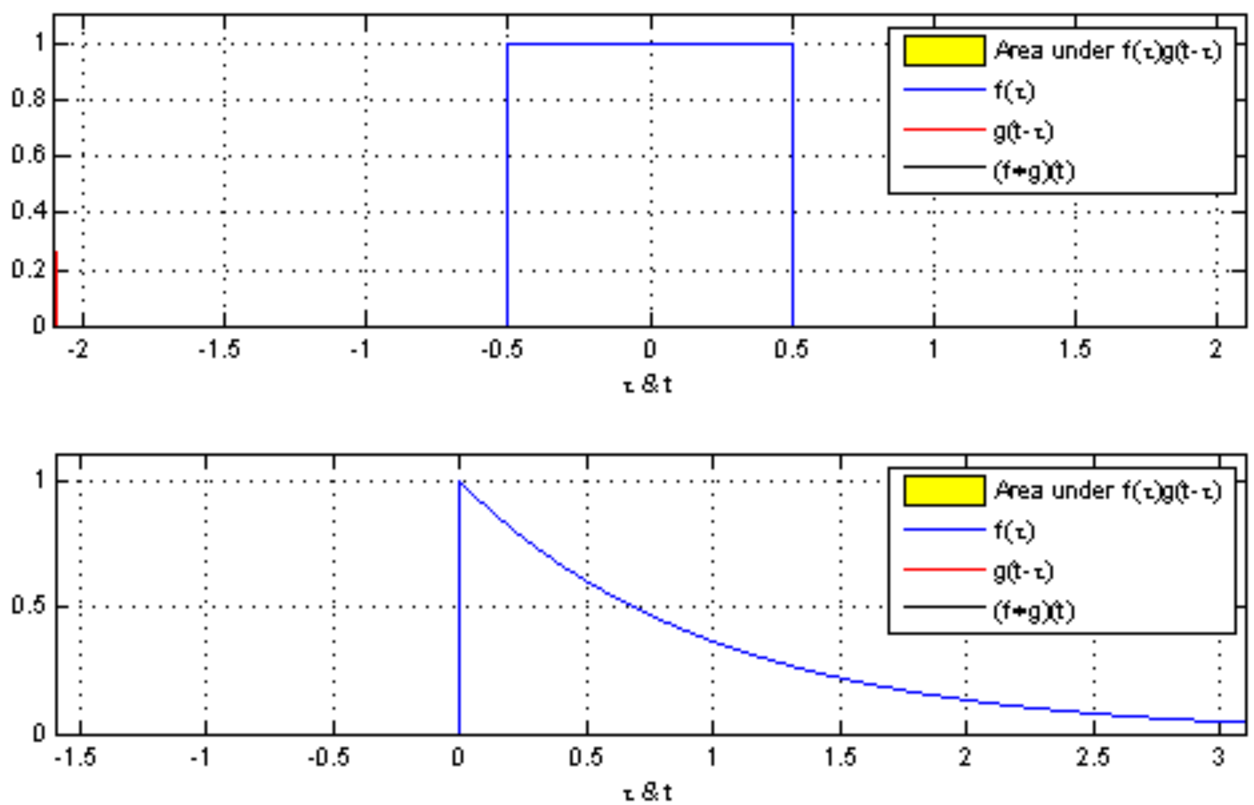

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Convolution

$$c(t) = \sum_{\tau=-\infty}^{\infty} a(t-\tau)h(\tau)$$

Convolution

$$c(t) = \sum_{\tau=-\infty}^{\infty} a(\tau)h(t-\tau)$$

Autocorrelation

$$c(t) = \sum_{\tau=-\infty}^{\infty} a(\tau)a(t-\tau)$$

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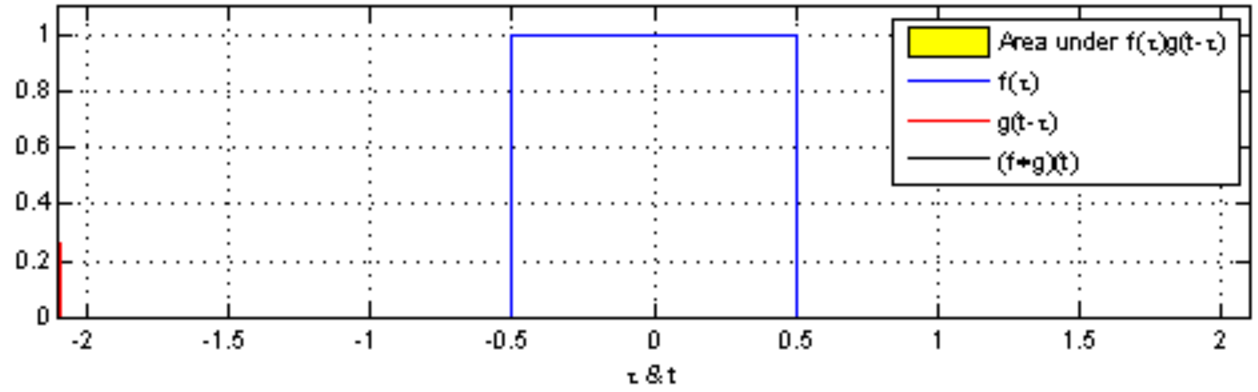
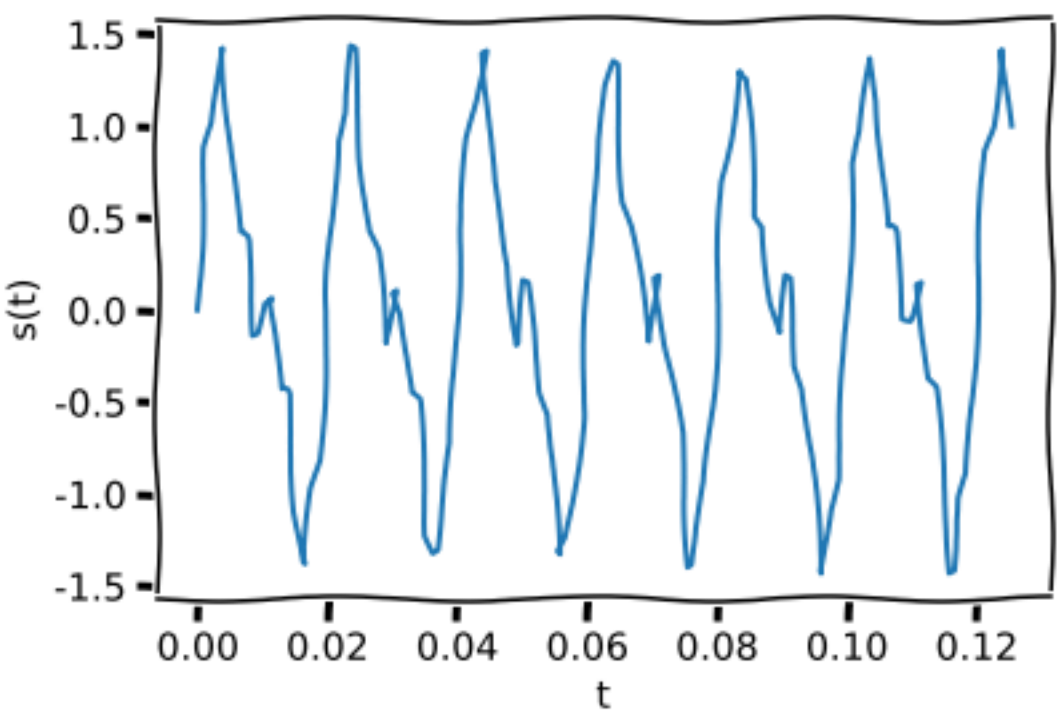


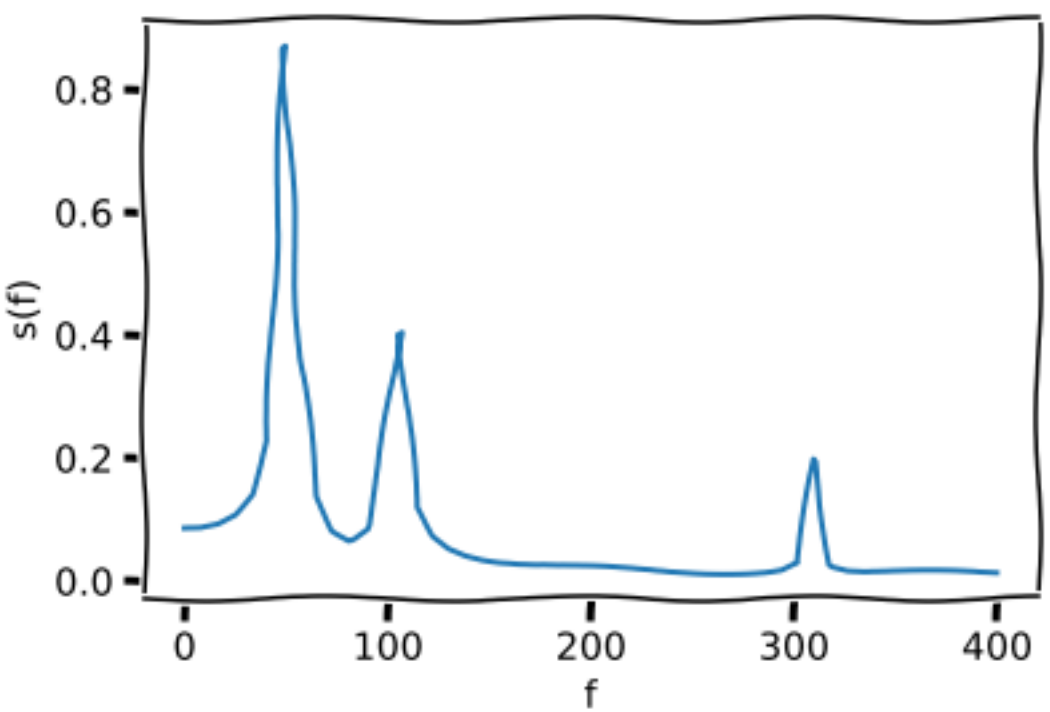
(c) Wikipedia

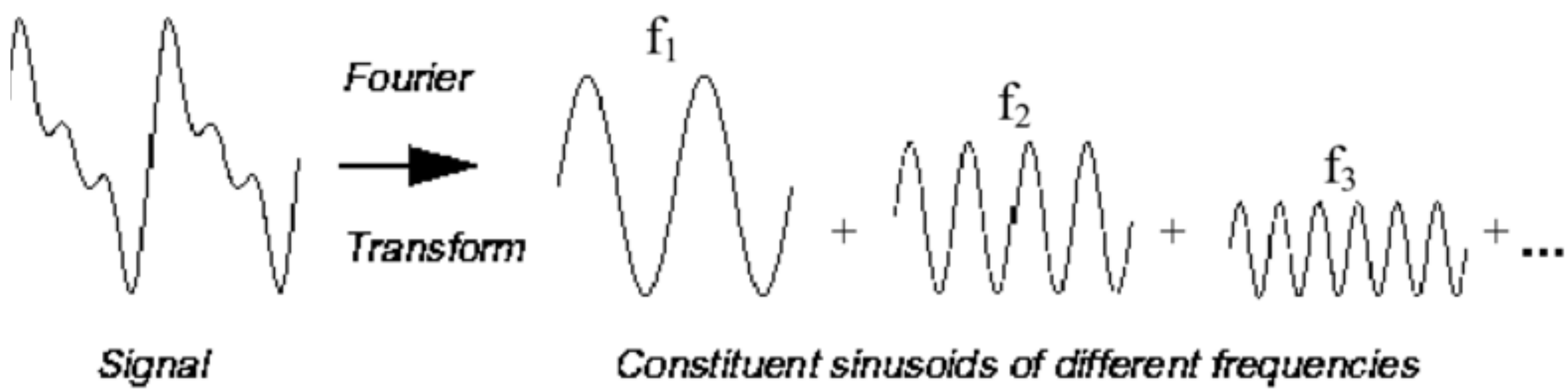
Fourier transform

$$S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi ft} dt$$ Fourier transform

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi ft} df$$ Inverse Fourier transform





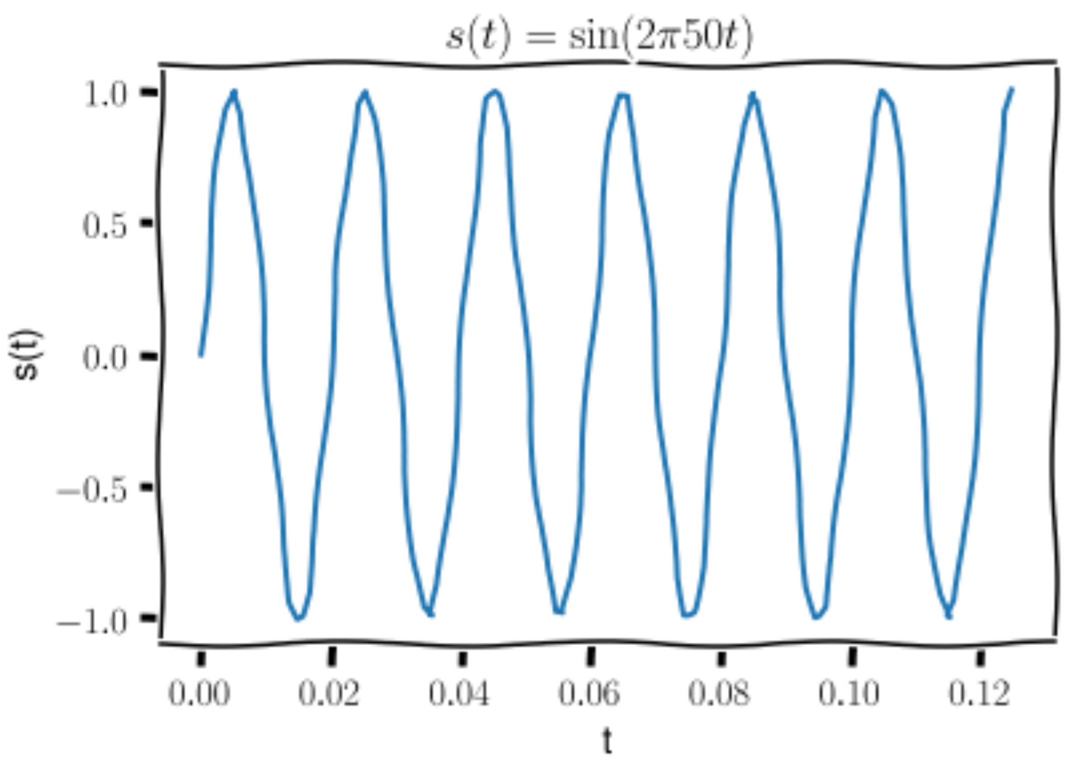

Fourier
Transform
f_1
f_2
f_3
+
+
+ ...
Signal
Constituent sinusoids of different frequencies

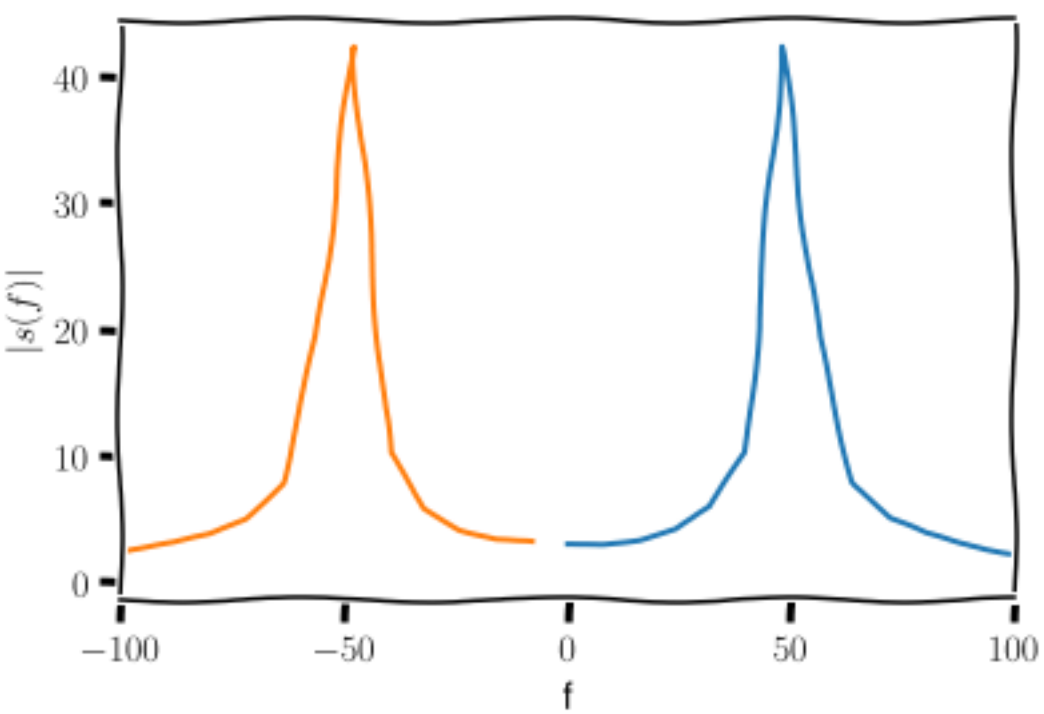
$$S(f) = \int_{-\infty}^{\infty} s(t)e^{-j2\pi ft}dt$$

1/s, Hz

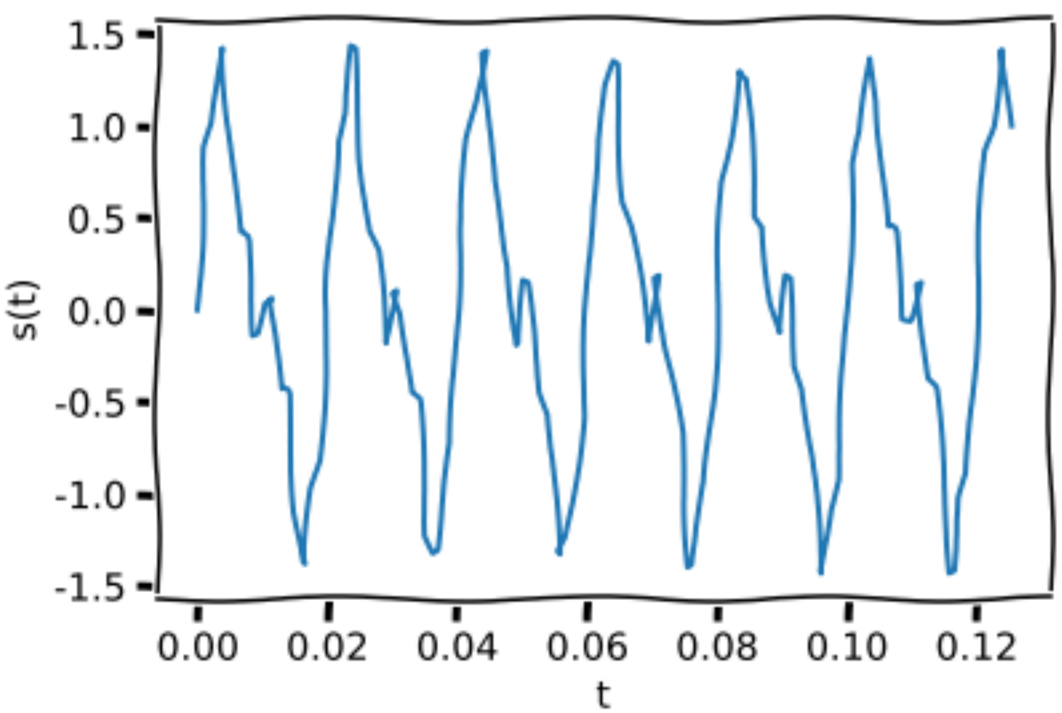
$$s(t) = \int_{-\infty}^{\infty} S(f)e^{j2\pi ft}df$$

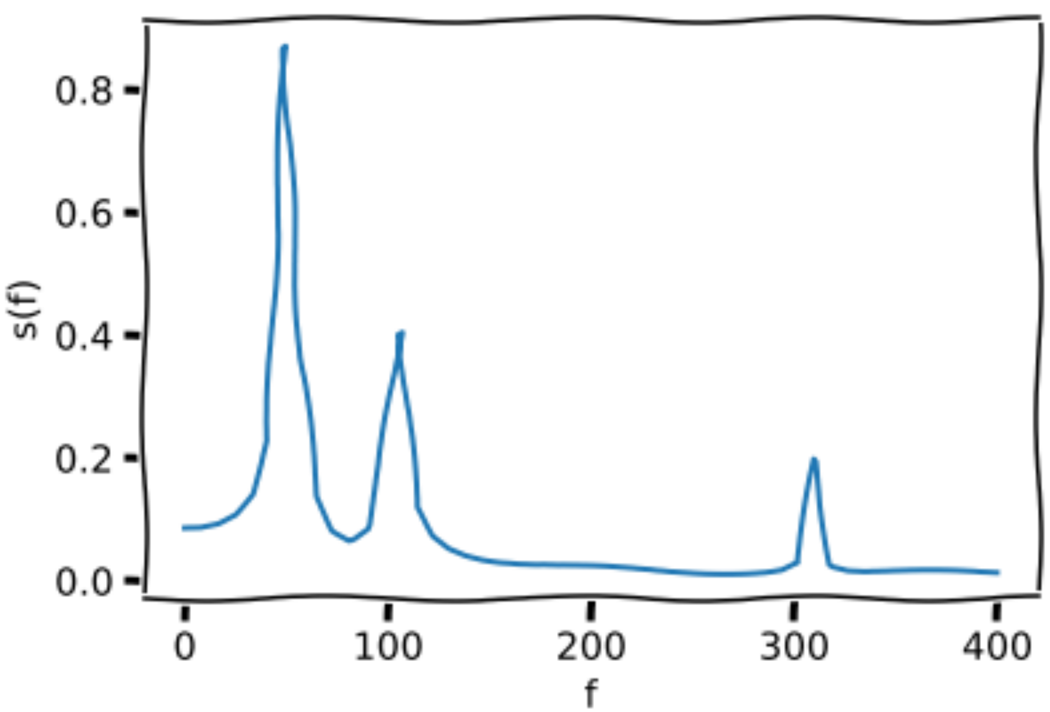
s





How much 100Hz is in $s(t)$?





Discrete Fourier transform

$$\int \longrightarrow \sum$$

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$$S(f) = \int_{-\infty}^{\infty} s(t) e^{-j2\pi f t} dt \longrightarrow S[f] = \sum_{i=0}^{N-1} s[i] e^{\frac{-j2\pi f i}{N}}$$

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{j2\pi f t} df \longrightarrow s[t] = \sum_{j=0}^{M-1} S[j] e^{\frac{-j2\pi j t}{M}}$$

2D Fourier transform

$$S(f_1, f_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s(t_1, t_2) e^{-j2\pi(f_1 t_1 + f_2 t_2)} dt_1 dt_2$$ 2D Fourier transform

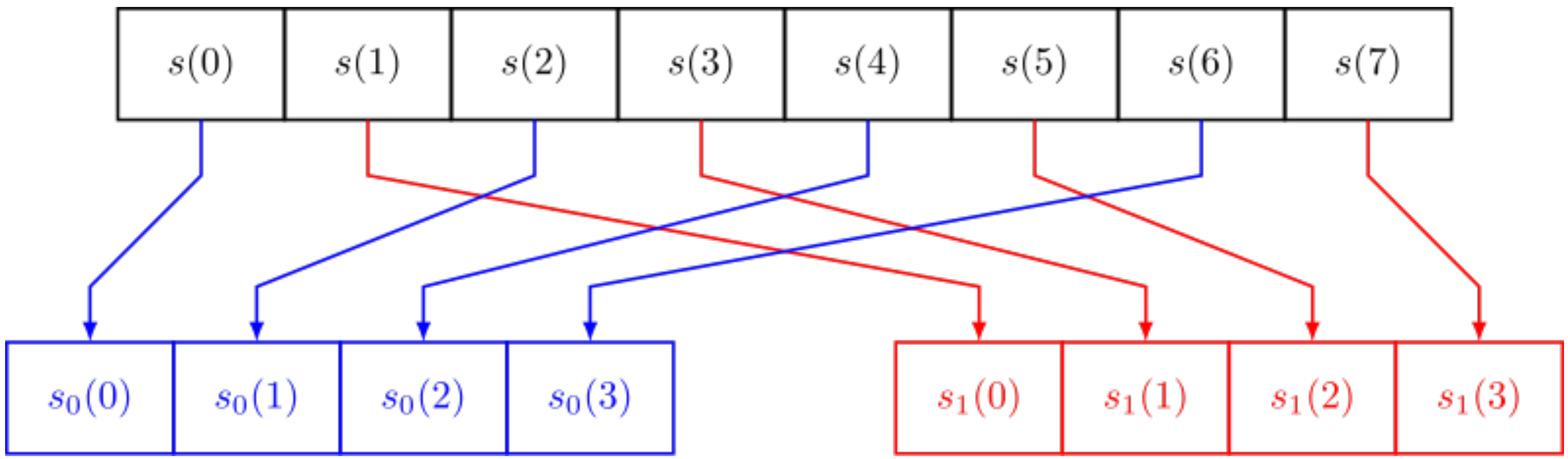
$$s(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(f_1, f_2) e^{j2\pi(f_1 t_1 + f_2 t_2)} df_1 df_2$$ Inverse 2D Fourier transform

Discrete 2D Fourier transform

$$S[f_1, f_2] = \sum_{t_1=0}^{N-1} \sum_{t_2=0}^{N-1} s[t_1, t_2] e^{-j2\pi(f_1 t_1 + f_2 t_2)}$$ 2D Fourier transform

$$s[t_1, t_2] = \sum_{f_1=0}^{M-1} \sum_{f_2=0}^{M-1} S[f_1, f_2] e^{j2\pi(f_1 t_1 + f_2 t_2)}$$ Inverse 2D Fourier transform

Fast Fourier transform



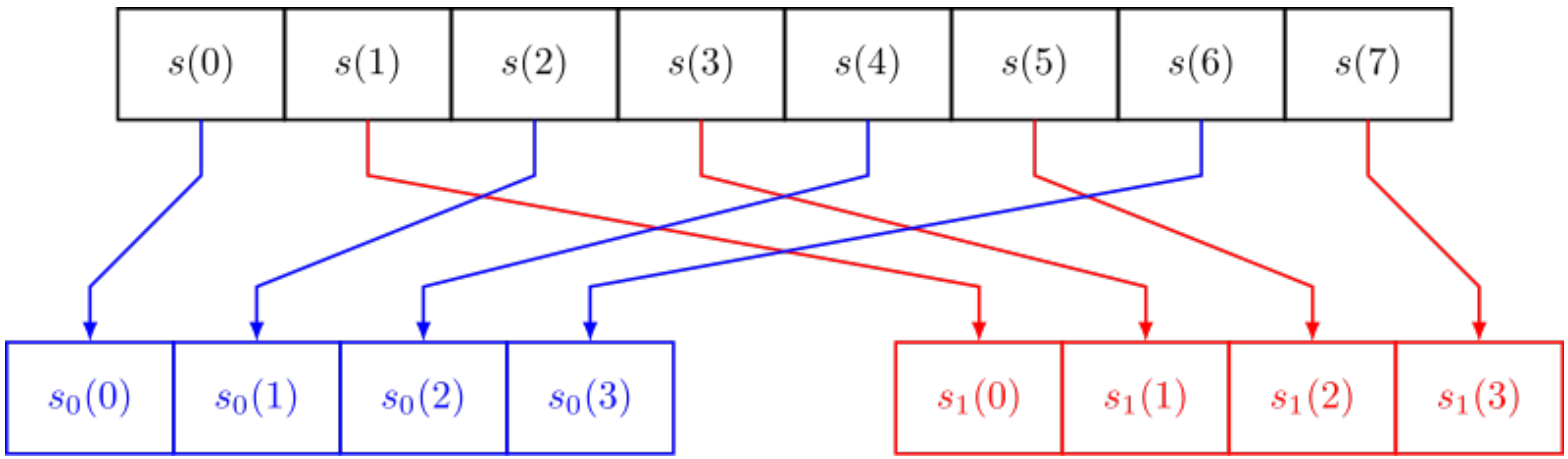
$$S[f] = \sum_{i=0}^{N-1} s[i] e^{\frac{-j2\pi fi}{N}} = \sum_{m=0}^{\frac{N}{2}-1} s[2m] W_N^{2mf} + \sum_{m=0}^{\frac{N}{2}-1} s[2m+1] W_N^{(2m+1)f}$$

$$W_N^f = \exp\left(-j\frac{2\pi}{N} f\right)$$

(c) http://ru.dsplib.org/

Fast Fourier transform

$$W_N^{2mf} = \exp\left(-j\frac{2\pi}{N}2mf\right) = \exp\left(-j\frac{2\pi}{\frac{N}{2}}mf\right) = W_{\frac{N}{2}}^{mf}$$



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Fast Fourier transform

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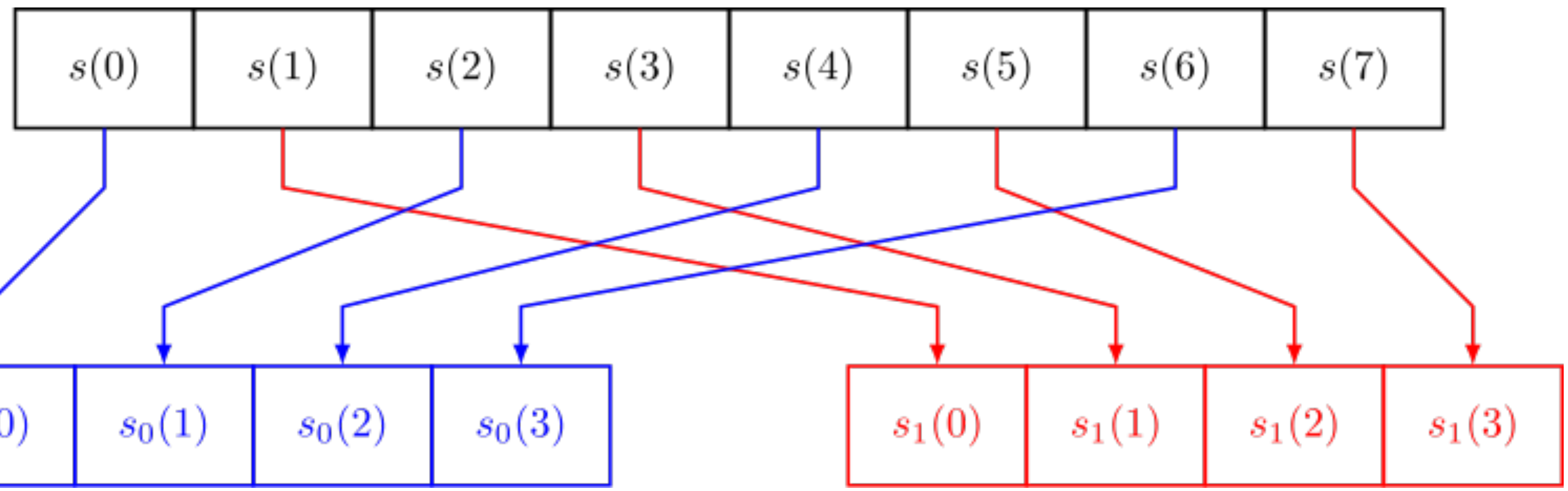

For all $\quad f \in [0, \frac{N}{2} - 1]$

$$S[f] = \sum_{i=0}^{N-1} s[i]e^{\frac{-j2\pi fi}{N}} = \sum_{m=0}^{\frac{N}{2}-1} s[2m]W_N^{2mf} + \sum_{m=0}^{\frac{N}{2}-1} s[2m+1]W_N^{(2m+1)f}$$

$$S[f] = \sum_{m=0}^{\frac{N}{2}-1} s[2m]W_N^{2mf} + W_N^f \sum_{m=0}^{\frac{N}{2}-1} s[2m+1]W_N^{2mf} = \sum_{m=0}^{\frac{N}{2}-1} s[2m]W_{\frac{N}{2}}^{mf} + W_N^f \sum_{m=0}^{\frac{N}{2}-1} s[2m+1]W_{\frac{N}{2}}^{mf}$$

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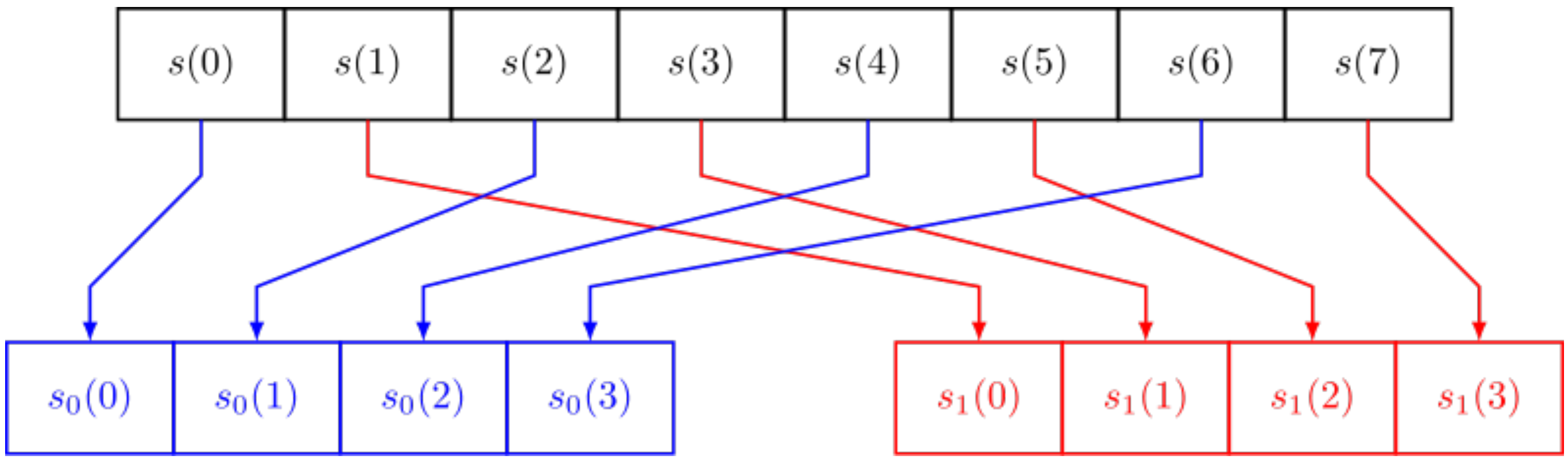
Fast Fourier transform

$$W_N^{2m\left(f+\frac{N}{2}\right)} = W_N^{2mf} W_N^{mN} = W_{\frac{N}{2}}^{mf}$$

$$W_N^{(2m+1)\left(f+\frac{N}{2}\right)} = W_N^{2mf} W_N^{mN} W_N^{f} W_N^{\frac{N}{2}} = -\, W_N^{f} W_{\frac{N}{2}}^{mf}$$



$$S[f+\frac{N}{2}] = \sum_{m=0}^{\frac{N}{2}-1} s[2m] W_N^{2m\left(f+\frac{N}{2}\right)} + \sum_{m=0}^{\frac{N}{2}-1} s[2m+1] W_N^{(2m+1)\left(f+\frac{N}{2}\right)} =$$

$$= \sum_{m=0}^{\frac{N}{2}-1} s[2m] W_{\frac{N}{2}}^{mf} - W_N^{f} \sum_{m=0}^{\frac{N}{2}-1} s[2m+1] W_{\frac{N}{2}}^{mf}$$

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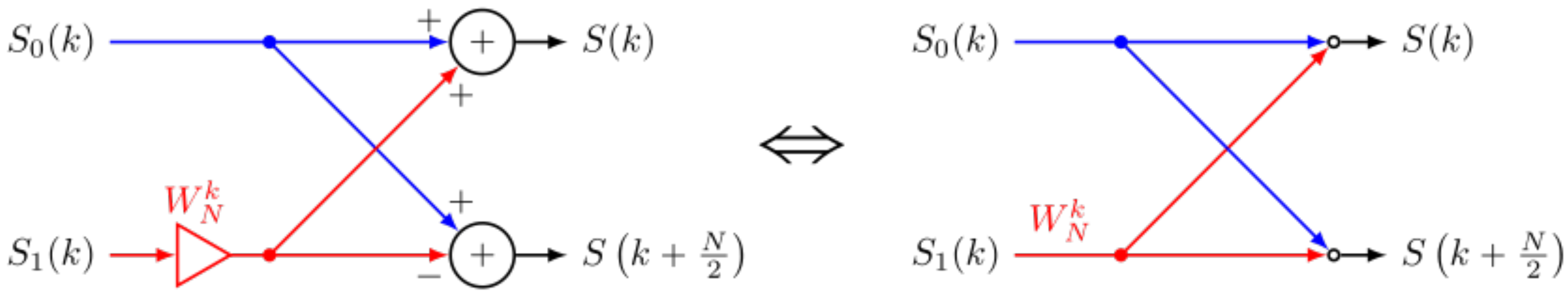
Fast Fourier transform



$$S[f] = \sum_{m=0}^{\frac{N}{2}-1} s[2m] W_{\frac{N}{2}}^{mf} + W_N^f \sum_{m=0}^{\frac{N}{2}-1} s[2m+1] W_{\frac{N}{2}}^{mf}$$

$$S[f+\frac{N}{2}] = \sum_{m=0}^{\frac{N}{2}-1} s[2m] W_{\frac{N}{2}}^{mf} - W_N^f \sum_{m=0}^{\frac{N}{2}-1} s[2m+1] W_{\frac{N}{2}}^{mf}$$

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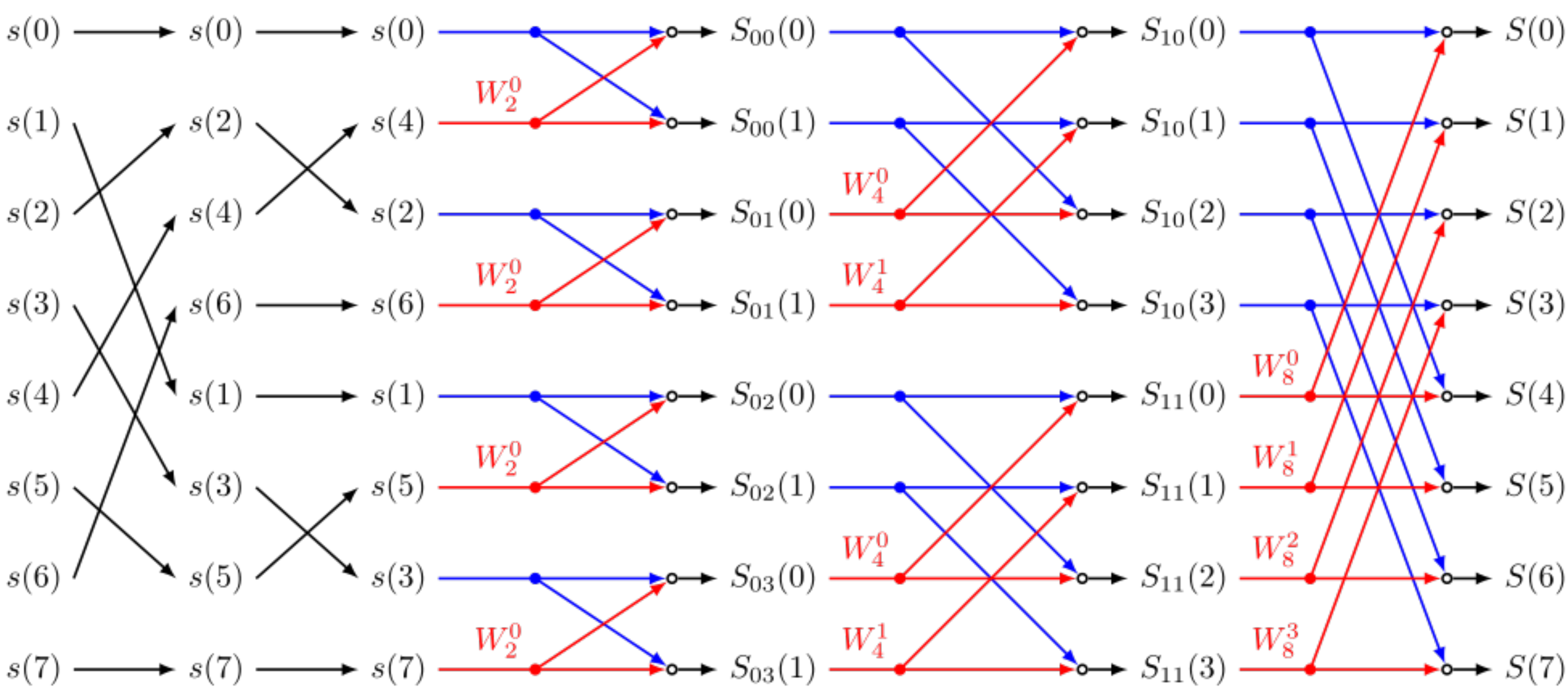

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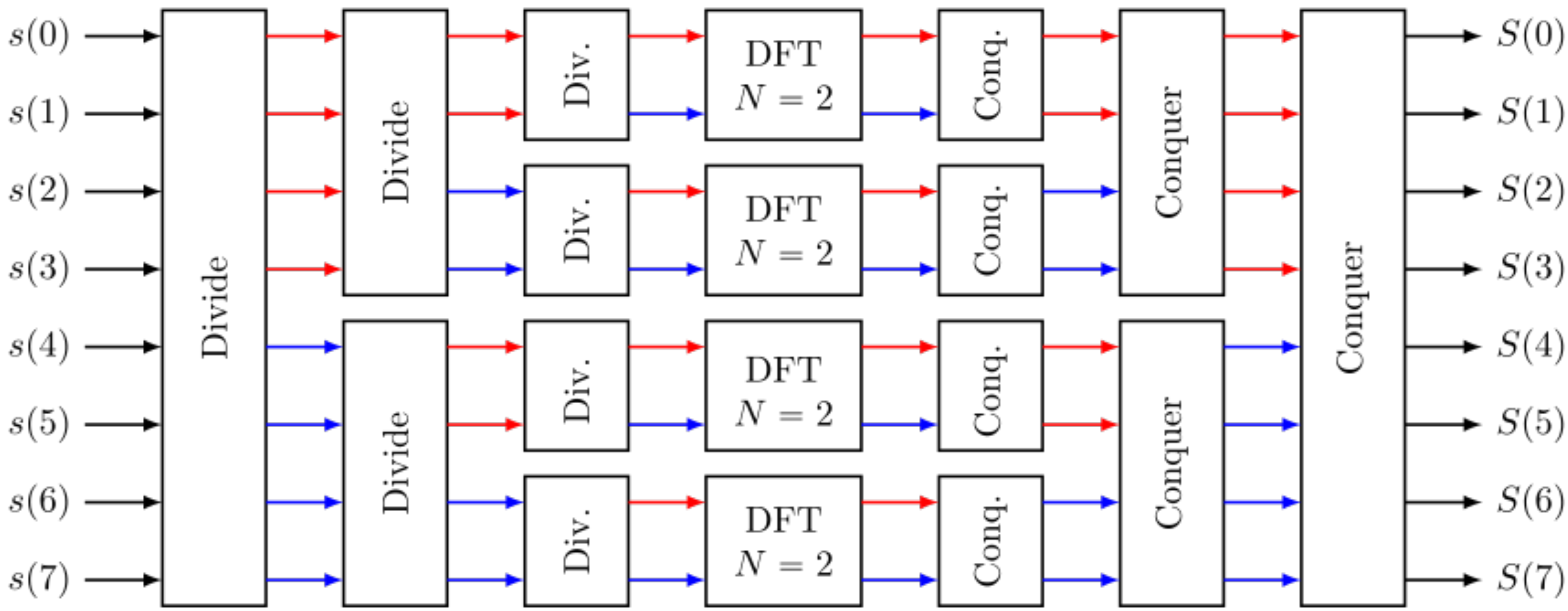

$O(N \log N)$ Operations