Differentiability

$$\mathrm{Lf}'(\mathrm{a}) = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}$$

$$\mathrm{Rf}'(\mathrm{a}) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

When $\mathrm{Lf}'(\mathrm{a}) = \mathrm{Rf}'(\mathrm{a})$ and finite, then the given function is differentiable at $x=a$

Problem-1: Show that $f(x)=\begin{cases} x^2+1, & x\le 1 \\ 2x, & x>1 \end{cases}$ **is continuous and differentiable at** $x=1$.

Solution:

Continuity Test:

L.H.L. $= \lim_{x\to 1^-} f(x)$

$= \lim_{x\to 1^-} (x^2+1)$

$= (1)^2+1$

$= 2$

L.H.L. $= \lim_{x\to 1^+} f(x)$

$= \lim_{x\to 1^+} 2x$

$= 2.1$

$= 2$

and $f(1) = (1)^2 + 1 = 2$

Since $L.H.L = R.H.L = f(1)$, $f(x)$ is continuous at $x=1$.

Differentiability Test:

$\mathrm{Lf}'(1) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h}$

$= \lim_{h\to 0} \frac{\{(1-h)^2+1\}-\{(1)^2+1\}}{-h}$

$= \lim_{h\to 0} \frac{h^2-2h+1+1-2}{-h}$

$= \lim_{h\to 0} \frac{h^2-2h}{-h}$

$= \lim_{h\to 0} \frac{-h(2-h)}{-h}$

$= \lim_{h\to 0} (2-h)$

$= 2$

$\because f(x) = x^2+1$

$\therefore f(1-h) = (1-h)^2+1$

$\therefore f(1) = (1)^2+1$

$$\mathrm{Rf}'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h\to 0} \frac{2(1+h)-2}{h}$$
$$= \lim_{h\to 0} \frac{2h}{h}$$
$$= 2$$

$\because f(x) = 2x$
$\therefore \ f(1+h) = 2(1+h)$
$\therefore \ f(1) = 2.1 = 2$

Since $\mathrm{Lf}'(1) = \mathrm{Rf}'(1)$ the given function $f(x)$ is differentiable at $x=1$.

Problem-2: Show that $f(x) = \begin{cases} x^2+2, & x \le 1 \\ x+2, & x > 1 \end{cases}$ **is continuous but not differentiable at** $x=1$.

Solution:

Continuity Test:

L.H.L. $= \lim_{x\to 1^-} f(x)$
$= \lim_{x\to 1^-} (x^2+2)$
$= (1)^2 + 2$
$= 3$

R.H.L. $= \lim_{x\to 1^+} f(x)$
$= \lim_{x\to 1^+} (x+2)$
$= 1+2$
$= 3$

and $f(1) = (1)^2 + 2 = 3$

Since $L.H.L = R.H.L = f(1)$, $f(x)$ is continuous at $x=1$.

Differentiability Test:

$$\mathrm{Lf}'(1) = \lim_{h\to 0} \frac{f(1-h) - f(1)}{-h}$$
$$= \lim_{h\to 0} \frac{\{(1-h)^2+2\} - \{(1)^2+2\}}{-h}$$
$$= \lim_{h\to 0} \frac{h^2 - 2h + 1 + 2 - 3}{-h}$$
$$= \lim_{h\to 0} \frac{h^2 - 2h}{-h}$$
$$= \lim_{h\to 0} \frac{-h(2-h)}{-h}$$
$$= \lim_{h\to 0} (2-h)$$
$$= 2$$

$\because f(x) = x^2 + 2$
$\therefore \ f(1-h) = (1-h)^2 + 2$
$\therefore \ f(1) = (1)^2 + 2$

$$\mathrm{Rf}'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h\to 0} \frac{h+3-3}{h}$$

$\because f(x) = x+2$
$\therefore \ f(1+h) = 1+h+2 = h+3$
$\therefore \ f(1) = 1+2 = 3$

$$=\lim_{h\to 0}\frac{h}{h}$$
$$=1$$

Since $Lf'(1)\neq Rf'(1)$ the given function $f(x)$ is not differentiable at $x=1$.

Problem-3: Show that $f(x)=\begin{cases}x^2+x+1, & x\le 1\\ 3x, & x>1\end{cases}$ **is continuous at** $x=1$. **Determine whether** $f(x)$ **is differentiable at** $x=1$, **if so find the value of the derivative there.**

Solution:

Continuity Test:

L.H.L. $=\lim_{x\to 1^-} f(x)$
$$=\lim_{x\to 1^-}\left(x^2+x+1\right)$$
$$=(1)^2+1+1$$
$$=3$$

R.H.L. $=\lim_{x\to 1^+} f(x)$
$$=\lim_{x\to 1^+}3x$$
$$=3.1$$
$$=3$$

and $f(1)=(1)^2+1+1=3$

Since $L.H.L=R.H.L=f(1)$, $f(x)$ is continuous at $x=1$.

Differentiability Test:

$$Lf'(1)=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}$$
$$=\lim_{h\to 0}\frac{\left(h^2-3h+3\right)-3}{-h}$$
$$=\lim_{h\to 0}\frac{h^2-3h}{-h}$$
$$=\lim_{h\to 0}\frac{-h(3-h)}{-h}$$
$$=\lim_{h\to 0}(3-h)$$
$$=3$$

$\because f(x)=x^2+x+1$

$\therefore\ f(1-h)=(1-h)^2+(1-h)+1$
$$=1-2h+h^2+1-h+1$$
$$=h^2-3h+3$$

$\therefore\ f(1)=(1)^2+1+1=3$

$$Rf'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$
$$=\lim_{h\to 0}\frac{3+3h-3}{h}$$
$$=\lim_{h\to 0}\frac{3h}{h}$$
$$=3$$

$\because f(x)=3x$

$\therefore\ f(1+h)=3(1+h)=3+3h$

$\therefore\ f(1)=3.1=3$

Since $Lf'(1)=Rf'(1)$ the given function $f(x)$ is differentiable at $x=1$ and the derivative of $f(x)$ is 3 i.e., $f'(x)=3$.

Problem-4: Show that $f(x)=\sqrt[3]{x}$ is continuous at $x=0$ but not differentiable at $x=0$.

Solution:

Continuity Test:

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\sqrt[3]{x}=\lim_{x\to 0} x^{\frac{1}{3}}=0$$

and $f(0)=0$

Since $\lim_{x\to 0} f(x)=f(0)$, $f(x)$ is continuous at $x=0$.

Differentiability Test:

$$\mathrm{Lf}'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$=\lim_{h\to 0}\frac{(-h)^{\frac{1}{3}}-0}{-h}$$

$$=\lim_{h\to 0}\frac{(-h)^{\frac{1}{3}}}{(-h)}$$

$$=\lim_{h\to 0}\frac{1}{(-h)^{\frac{1}{3}}} \text{ which does not exist.}$$

$\because f(x)=\sqrt[3]{x}=x^{\frac{1}{3}}$

$\therefore f(0-h)=(0-h)^{\frac{1}{3}}=(-h)^{\frac{1}{3}}$

$\therefore f(0)=(0)^{\frac{1}{3}}=0$

$$\mathrm{Rf}'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{h^{\frac{1}{3}}-0}{h}$$

$$=\lim_{h\to 0}\frac{h^{\frac{1}{3}}}{h}$$

$$=\lim_{h\to 0}\frac{1}{h^{\frac{2}{3}}}$$

$$=\infty$$

$\because f(x)=\sqrt[3]{x}=x^{\frac{1}{3}}$

$\therefore f(0+h)=(0+h)^{\frac{1}{3}}=h^{\frac{1}{3}}$

$\therefore f(0)=(0)^{\frac{1}{3}}=0$

Therefore, the given function $f(x)$ is not differentiable at $x=0$.

Problem-5: Test the differentiability of the function $f(x)=\begin{cases} x\tan^{-1}\left(\frac{1}{x}\right), & x\neq 0 \\ 0, & x=0 \end{cases}$ **at** $x=0$.

Solution:

Continuity Test:

L.H.L. = $\lim\limits_{x\to 0^-} f(x)$

$$= \lim_{x\to 0^-} x\tan^{-1}\left(\frac{1}{x}\right)$$

$$=0.\tan^{-1}(\infty)=0.\frac{\pi}{2}$$

$$=0$$

R.H.L. = $\lim\limits_{x\to 0^+} f(x)$

$$= \lim_{x\to 0^+} x\tan^{-1}\left(\frac{1}{x}\right)$$

$$=0.\tan^{-1}(\infty)=0.\frac{\pi}{2}$$

$$=0$$

and $f(0)=0$

Since $L.H.L=R.H.L=f(0)$, $f(x)$ is continuous at $x=0$.

Differentiability Test:

$$\text{Lf}'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$=\lim_{h\to 0}\frac{-h\tan^{-1}\left(\frac{1}{-h}\right)}{-h}$$

$$=\lim_{h\to 0}\tan^{-1}\left(-\frac{1}{h}\right)$$

$$=\tan^{-1}(-\infty)$$

$$=-\frac{\pi}{2}$$

$\because f(x)=x\tan^{-1}\left(\frac{1}{x}\right)$

$\therefore f(0-h)=(0-h)\tan^{-1}\left(\frac{1}{0-h}\right)$

$=-h\tan^{-1}\left(\frac{1}{-h}\right)$

$\therefore f(0)=(0)\tan^{-1}\left(\frac{1}{0}\right)=0.\tan^{-1}(\infty)=0.\frac{\pi}{2}=0$

$$\text{Rf}'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{h\tan^{-1}\left(\frac{1}{h}\right)}{h}$$

$$=\lim_{h\to 0}\tan^{-1}\left(\frac{1}{h}\right)$$

$$=\tan^{-1}(\infty)$$

$$=\frac{\pi}{2}$$

$\because f(x)=x\tan^{-1}\left(\frac{1}{x}\right)$

$\therefore f(0+h)=(0+h)\tan^{-1}\left(\frac{1}{0+h}\right)$

$=h\tan^{-1}\left(\frac{1}{h}\right)$

$\therefore f(0)=(0)\tan^{-1}\left(\frac{1}{0}\right)=0.\tan^{-1}(\infty)=0.\frac{\pi}{2}=0$

Since $\text{Lf}'(0)\neq \text{Rf}'(0)$ the given function $f(x)$ is not differentiable at $x=0$.

Problem-6: Discuss the continuity and differentiability of the following function $f(x)=\begin{cases}\sqrt{|x|}, & x\ge 0\\ -\sqrt{|x|}, & x<0\end{cases}$ **at** $x=0$.

Solution:

Continuity Test:

L.H.L. = $\lim\limits_{x\to 0^-} f(x)$

$= \lim\limits_{x\to 0^-}\left(-\sqrt{|x|}\right)$

$=0$

R.H.L. = $\lim\limits_{x\to 0^+} f(x)$

$= \lim\limits_{x\to 0^-}\left(\sqrt{|x|}\right)$

$=0$

and $f(0)=\sqrt{|0|}=0$

Since $L.H.L = R.H.L = f(0)$, $f(x)$ is continuous at $x=0$.

Differentiability Test:

$\text{Lf}'(0)=\lim\limits_{h\to 0}\dfrac{f(0-h)-f(0)}{-h}$

$=\lim\limits_{h\to 0}\dfrac{-\sqrt{h}-0}{-h}$

$=\lim\limits_{h\to 0}\dfrac{-\sqrt{h}}{-h}$

$=\lim\limits_{h\to 0}\dfrac{1}{h^{\frac{1}{2}}}$

$=\infty$

$\because f(x)=-\sqrt{|x|}$

$\therefore\ f(0-h)=-\sqrt{|0-h|}=-\sqrt{h}$

$\therefore\ f(0)=-\sqrt{|0|}=0$

$\text{Rf}'(0)=\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$

$=\lim\limits_{h\to 0}\dfrac{\sqrt{h}-0}{h}$

$=\lim\limits_{h\to 0}\dfrac{\sqrt{h}}{h}$

$=\lim\limits_{h\to 0}\dfrac{1}{h^{\frac{1}{2}}}$

$=\infty$

$\because f(x)=\sqrt{|x|}$

$\therefore\ f(0+h)=\sqrt{|0+h|}=-\sqrt{h}$

$\therefore\ f(0)=\sqrt{|0|}=0$

Since $\text{Lf}'(0)=\text{Rf}'(0)$ but not finite, the given function $f(x)$ is not differentiable at $x=0$.

Problem-7: Test the continuity and differentiability of $f(x)=\begin{cases} 1, \text{ when } x<0 \\ 1+\sin x \quad , \text{ when } 0\le x<\frac{\pi}{2} \\ 2+\left(x-\frac{\pi}{2}\right)^2, \text{ when } x\ge\frac{\pi}{2} \end{cases}$

at $x=\frac{\pi}{2}$ **or** $x=0$**.**

Solution: For $x=\frac{\pi}{2}$**:**

Continuity Test: Here $f\left(\frac{\pi}{2}\right)=2+\left(\frac{\pi}{2}-\frac{\pi}{2}\right)^2=2$

L.H.L. = $\lim\limits_{x\to\frac{\pi}{2}^-} f(x)$

$= \lim\limits_{x\to\frac{\pi}{2}^-} \{1+\sin(x)\}$

$=1+\sin\left(\frac{\pi}{2}\right)$

$=2$

R.H.L = $\lim\limits_{x\to\frac{\pi}{2}^+} f(x)$

$= \lim\limits_{x\to\frac{\pi}{2}^+} 2+\left(x-\frac{\pi}{2}\right)^2$

$=2+\left(\frac{\pi}{2}-\frac{\pi}{2}\right)^2$

$=2$

Since L.H.L.= R. H. L. = $f\left(\frac{\pi}{2}\right)=2$, the given function is continuous at $x=\frac{\pi}{2}$.

Differentiability Test:

$Lf'\left(\frac{\pi}{2}\right)=\lim\limits_{h\to0}\frac{f\left(\frac{\pi}{2}-h\right)-f\left(\frac{\pi}{2}\right)}{-h}$

$=\lim\limits_{h\to0}\frac{1}{-h}\{\cos h-1\}$

$=\lim\limits_{h\to0}\frac{1-\cos h}{h}$

$=\lim\limits_{h\to0}\frac{2\sin^2\frac{h}{2}}{h}$

$=\lim\limits_{\frac{h}{2}\to0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\quad\lim\limits_{\frac{h}{2}\to0}\sin\frac{h}{2}$

$=1\times0$

$=0$

$Rf'\left(\frac{\pi}{2}\right)=\lim\limits_{h\to0}\frac{f\left(\frac{\pi}{2}+h\right)-f\left(\frac{\pi}{2}\right)}{h}$

$=\lim\limits_{h\to0}\frac{2+\left(\frac{\pi}{2}+h-\frac{\pi}{2}\right)^2-2}{h}$

$=\lim\limits_{h\to0}\frac{h^2}{h}$

$=\lim\limits_{h\to0} h$

$=0$

Since $\mathrm{Lf}'(\mathrm{a})$ = $\mathrm{Rf}'(\mathrm{a})$ = 0, the given function is differentiable at $x=\dfrac{\pi}{2}$.

For $x=0$:

Continuity Test:

Here $f(0)=1+\sin(0)=1+0=1$

L.H.L. = $\lim\limits_{x\to 0^-} f(x)$

$= \lim\limits_{x\to 0^-} 1$

$=1$

R.H.L .= $\lim\limits_{x\to 0^+} f(x)$

$= \lim\limits_{x\to 0^+} \{1+\sin(x)\}$

$=1+\sin(0)$

$=1$

Since L.H.L.= R. H. L. $= f(0)$, the given function is continuous at $x=0$.

Differentiability Test:

$$\mathrm{Lf}'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$=\lim_{h\to 0}\frac{1-1}{-h}$$

$$=\lim_{h\to 0}\frac{0}{-h}$$

$$=\lim_{h\to 0} 0$$

$$=0$$

$$\mathrm{Rf}'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{1+\sin h-1}{h}$$

$$=\lim_{h\to 0}\frac{\sin h}{h}$$

$$=1$$

Since $\mathrm{Lf}'(\mathrm{a})\neq \mathrm{Rf}'(\mathrm{a})$, the given function is not differentiable at $x=0$.

Problem-8: Test the continuity and differentiability of $f(x)=\begin{cases} x^2+1, & \text{when } x<0 \\ x \quad , & \text{when } 0\le x\le 1 \\ \dfrac{1}{x}, & \text{when } x>1 \end{cases}$ **at** $x=0$ **or** $x=1$.

Problem-9: Test the continuity and differentiability of $g(x)=\begin{cases} \ln x; & \text{when } 0<x\le 1 \\ 0; & \text{when } 1<x\le 2 \\ 1+x^2; & \text{when } x>2 \end{cases}$ **at** $x=2$

Solution: For $x=2$**:**

Continuity Test:

Here $f(2)=0$**.**

L.H.L. = $\lim\limits_{x\to 2^-} g(x)$

$= \lim\limits_{x\to 2^-} 0$

$=0$

R.H.L = $\lim\limits_{x\to 2^+} g(x)$

$= \lim\limits_{x\to 2^+} 1+x^2$

$=1+2^2$

$=5$

Since $L.H.S \neq R.H.S \neq f(2)$, the given function is not continuous at $x=2$**.**

Hence, the given function is not differentiable at $x=2$**.**