Limit and Continuity

LIMITS (AN INTUITIVE APPROACH)

THE TANGENT LINE PROBLEM Given a function *f* and a point $P(x_0, y_0)$ on its graph, find an equation of the line that is tangent to the graph at *P* (Figure 1)..

THE AREA PROBLEM Given a function f , find the area between the graph of f and an interval [a, b] on the x-axis (Figure 2).

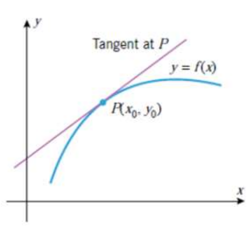

Figure 1

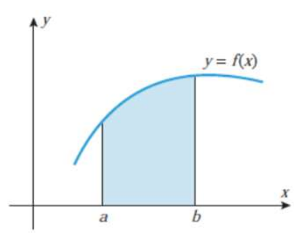

Figure 2

TANGENT LINES AND LIMITS

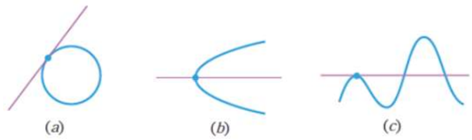

Figure 1

Figure 2

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AREAS AND LIMITS

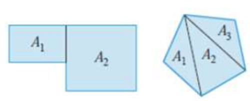

Figure 1

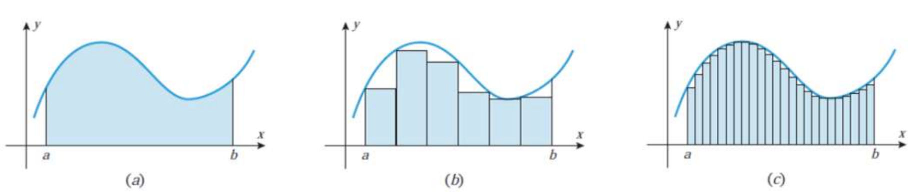

Figure 2

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LIMITS

The most basic use of limits is to describe how a function behaves as the independent variable approaches a given value.

Example 1: Examine the behavior of the function $f(x) = x^2 - x + 1$ for x-values closer and closer to 2.

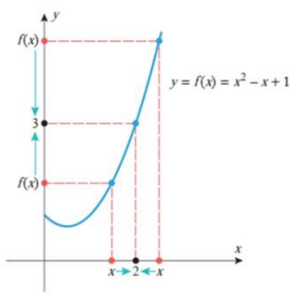

Figure 1

From Figure 1 and Table 1: The values of $f(x)$ get closer and closer to 3 as values of x are selected closer and closer to 2 on either the left or the right side of 2.

We describe this by saying that the "limit of $x^2 - x + 1$ is 3 as x approaches 2 from either side," and we write

$$\lim_{x \to 2} (x^2 - x + 1) = 3$$

Table 1

x	1.0	1.5	1.9	1.95	1.99	1.995	1.999	2	2.001	2.005	2.01	2.05	2.1	2.5	3.0
$f(x)$	1.000000	1.750000	2.710000	2.852500	2.970100	2.985025	2.997001		3.003001	3.015025	3.030100	3.152500	3.310000	4.750000	7.000000

Left side → ← Right side

LIMITS (cont…)

LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x \to a} f(x) = L$$

which is read "the limit of $f(x)$ as x approaches a is L" or "$f(x)$ approaches L as x approaches a." The above expression can also be written as

$$f(x) \to L \text{ as } x \to a$$

LIMITS (cont…)

Example 2 Use numerical evidence to make a conjecture about the value of

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}.$$

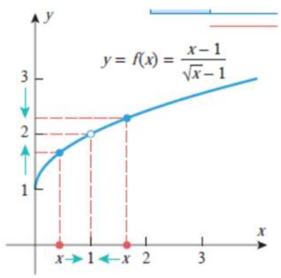

Figure 1

Solution. Although the function

$$f(x)=\frac{x-1}{\sqrt{x}-1}$$

is undefined at $x = 1$, this has no bearing on the limit.

Figure 1 and Table 1 shows that as x-values approaching 1 from the left side and from the right side, in both cases the corresponding values of $f(x)$ appear to get closer and closer to 2.

Hence we conjecture that

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}=2$$

Table 1

x	0.99	0.999	0.9999	0.99999		1.00001	1.0001	1.001	1.01
$f(x)$	1.994987	1.999500	1.999950	1.999995		2.000005	2.000050	2.000500	2.004988

Left side → ← Right side

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LIMITS (cont…)

Example 3 Use numerical evidence to make a conjecture about the value of

$$\lim_{x\to 0}\frac{\sin x}{x}.$$

Table 1

x (RADIANS)	$y=\frac{\sin x}{x}$
±1.0	0.84147
±0.9	0.87036
±0.8	0.89670
±0.7	0.92031
±0.6	0.94107
±0.5	0.95885
±0.4	0.97355
±0.3	0.98507
±0.2	0.99335
±0.1	0.99833
±0.01	0.99998

Solution. Although the function

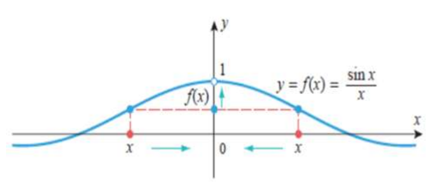
$$f(x)=\frac{\sin x}{x}$$

is undefined at $x = 0$, this has no bearing on the limit.

Figure 1 and Table 1 indicates that as x-values approaching 0 from the left side and from the right side, in both cases the corresponding values of $f(x)$ appear to get closer and closer to 1.

Hence we conjecture that

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$



As x approaches 0 from the left or right, $f(x)$ approaches 1.

Figure 1

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ONE-SIDED LIMITS

For example, consider the function

$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

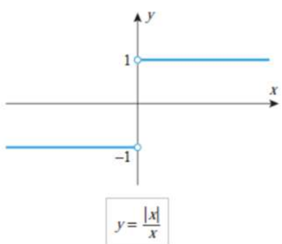

Figure 1

From Figure 1:

As x approaches 0 from the *right*, the values of $f(x)$ approach a limit of 1 [in fact, the values of $f(x)$ are exactly 1 for all such x].

$$\lim_{x \to 0^+} \frac{|x|}{x} = 1$$

Similarly, as x approaches 0 from the *left*, the values of $f(x)$ approach a limit of −1.

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1$$

Note:

The superscript "+" indicates a limit from the right and the superscript "−" indicates a limit from the left.

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ONE-SIDED LIMITS (cont…)

ONE-SIDED LIMITS (AN INFORMAL VIEW) If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than a), then we write

$$\lim_{x \to a^+} f(x) = L \qquad (1)$$

and if the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write

$$\lim_{x \to a^-} f(x) = L \qquad (2)$$

Expression (1) is read "the limit of $f(x)$ as x approaches a from the right is L" or "$f(x)$ approaches L as x approaches a from the right."

Similarly, expression (2) is read "the limit of $f(x)$ as x approaches a from the left is L" or "$f(x)$ approaches L as x approaches a from the left."

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THE RELATIONSHIP BETWEEN ONE-SIDED ANDTWO-SIDED LIMITS

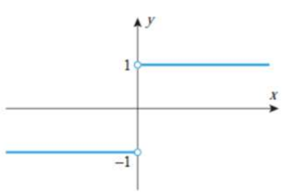
THE RELATIONSHIP BETWEEN ONE-SIDED ANDTWO-SIDED LIMITS The two sided limit of a function *f(x)* exists at *a* if and only if both of the one-sided limits exist at *a* and have the same value; that is,

$$\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L.$$

Example 1: Explain why $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Solution: Consider the function

$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$



$y = \frac{|x|}{x}$

Figure

From Figure: As x approaches 0 from the *right*, the values of *f(x)* approach a limit of 1 [in fact, the values of *f(x)* are exactly 1 for all such x].

$$\lim_{x\to 0^+} \frac{|x|}{x} = 1$$

Similarly, as x approaches 0 from the *left*, the values of *f(x)* approach a limit of −1.

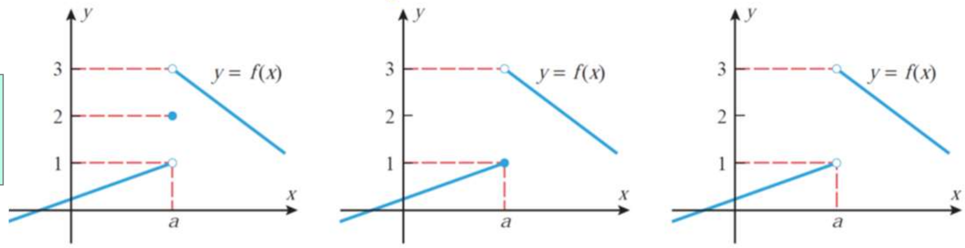
$$\lim_{x\to 0^-} \frac{|x|}{x} = -1$$

$$\therefore \lim_{x\to 0^+} \frac{|x|}{x} \neq \lim_{x\to 0^-} \frac{|x|}{x}$$

Thus, $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Example 2: For the functions in Figure, find the one-sided and two-sided limits at $x = a$ if they exist.

Figure



Solution:

From Figure: The functions in all three figures have the same one-sided limits as $x \to a$, since the functions are identical, except at $x = a$. These limits are:

$$\lim_{x \to a^+} f(x) = 3$$

$$and \qquad \lim_{x \to a^-} f(x) = 1$$

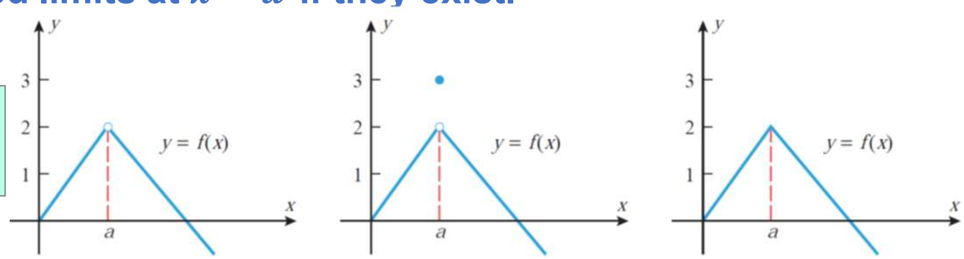
$$\therefore \lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$$

In all three cases: the one-sided limits are not equal.

Therefore, two-sided limit does not exist as $x \to a$.

Example 3: For the functions in Figure, find the one-sided and two-sided limits at $x = a$ if they exist.

Figure



Solution:

From Figure: The value of f at $x = a$ has no bearing on the limits as $x \to a$. These limits are:

$$\lim_{x \to a^+} f(x) = 2$$

$$and \qquad \lim_{x \to a^-} f(x) = 2$$

$$\therefore \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = 2$$

Since in all three cases the one-sided limits are equal, the two-sided limit exists as $x \to a$.

$$\therefore \lim_{x \to a} f(x) = 2$$

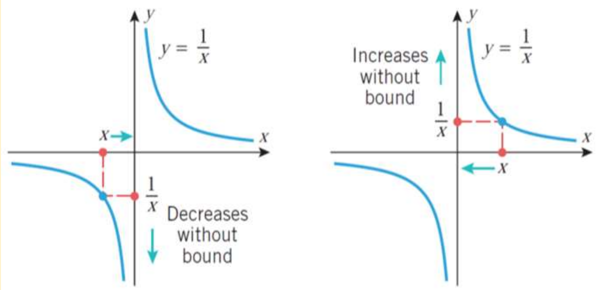
INFINITE LIMITS

Sometimes one-sided or two-sided limits fail to exist because the values of the function increase or decrease without bound.

For example, consider the behavior of $f(x) = 1/x$ for values of x near 0.
From Figure and Table:

$$\lim_{x\to 0^+} \frac{1}{x} = +\infty$$

and $$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$



Figure

Note: The symbols $+\infty$ and $-\infty$ here are not real numbers; they simply describe particular ways in which the limits fail to exist. Do not make the mistake of manipulating these symbols using rules of algebra. For example, it is incorrect to write $(+\infty) - (+\infty) = 0$.

x	−1	−0.1	−0.01	−0.001	−0.0001	0	0.0001	0.001	0.01	0.1	1
$\frac{1}{x}$	−1	−10	−100	−1000	−10,000		10,000	1000	100	10	1

Left side → ← Right side

Table

INFINITE LIMITS (Cont…)

INFINITE LIMITS (AN INFORMAL VIEW) The expressions

$$\lim_{x\to a^-} f(x) = +\infty \quad \text{and} \quad \lim_{x\to a^+} f(x) = +\infty$$

denote that $f(x)$ increases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x\to a} f(x) = +\infty$$

Similarly, the expressions

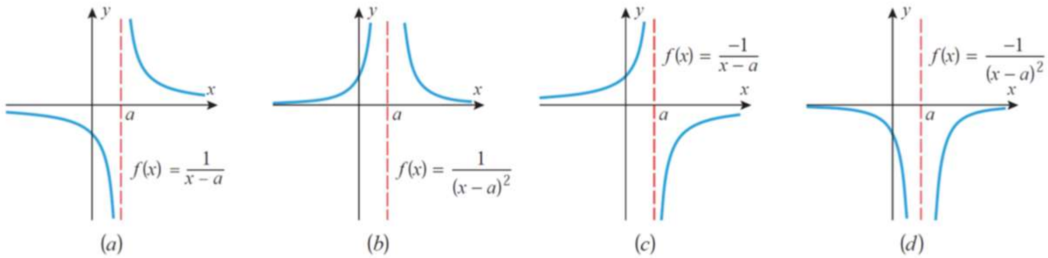
$$\lim_{x\to a^-} f(x) = -\infty \quad \text{and} \quad \lim_{x\to a^+} f(x) = -\infty$$

denote that $f(x)$ decreases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x\to a} f(x) = -\infty$$

Example: For the functions in Figure, describe the limits at $x = a$ in appropriate limit notation.

Figure



Solution (a): In Figure (a), the function increases without bound as x approaches a from the right and decreases without bound as x approaches a from the left. Thus,

$$\lim_{x\to a^+}\frac{1}{x-a} = +\infty \qquad \text{and} \qquad \lim_{x\to a^-}\frac{1}{x-a} = -\infty$$

Solution (b): In Figure (b), the function increases without bound as x approaches a from both the left and right. Thus,

$$\lim_{x\to a^+}\frac{1}{(x-a)^2} = +\infty \qquad \text{and} \qquad \lim_{x\to a^-}\frac{1}{(x-a)^2} = +\infty$$

$$\therefore \lim_{x\to a^+}\frac{1}{(x-a)^2} = \lim_{x\to a^-}\frac{1}{(x-a)^2} = +\infty$$

$$\therefore \lim_{x\to a}\frac{1}{(x-a)^2} = +\infty$$

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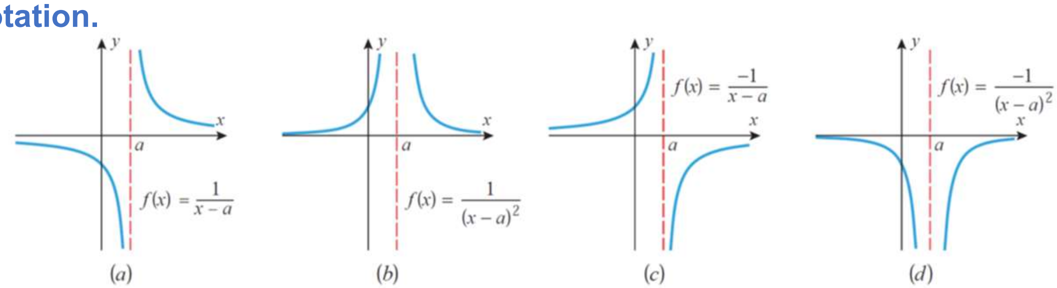
Example: For the functions in Figure, describe the limits at $x = a$ in appropriate limit notation.

Figure



Solution (c): In Figure (c), the function decreases without bound as x approaches a from the right and increases without bound as x approaches a from the left. Thus,

$$\lim_{x\to a^+}\frac{-1}{x-a} = -\infty \qquad \text{and} \qquad \lim_{x\to a^-}\frac{-1}{x-a} = +\infty$$

Solution (d): In Figure (d), the function decreases without bound as x approaches a from both the left and right. Thus,

$$\lim_{x\to a^+}\frac{-1}{(x-a)^2} = -\infty \qquad \text{and} \qquad \lim_{x\to a^-}\frac{1}{(x-a)^2} = -\infty$$

$$\therefore \lim_{x\to a^+}\frac{-1}{(x-a)^2} = \lim_{x\to a^-}\frac{-1}{(x-a)^2} = -\infty$$

$$\therefore \lim_{x\to a}\frac{1}{(x-a)^2} = -\infty$$

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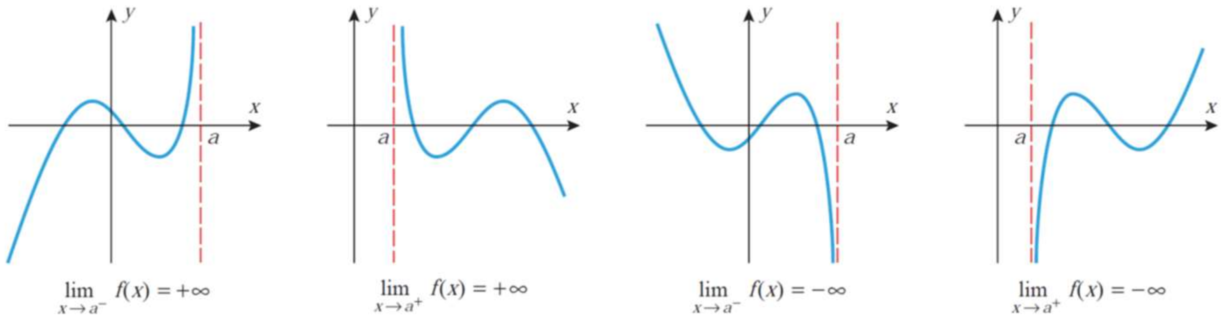
VERTICAL ASYMPTOTES



Figure

In each case the graph of $y = f(x)$ either rises or falls without bound, squeezing closer and closer to the vertical line $x = a$ as x approaches a from the side indicated in the limit.

The line $x = a$ is called a ***vertical asymptote*** of the curve $y = f(x)$ (from the Greek word *asymptotos*, meaning nonintersecting”).

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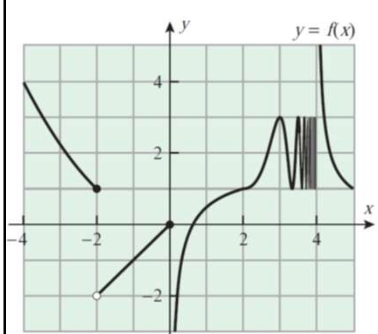
Example: For the functions in Figure, find

(a) $\lim_{x \to -2^-} f(x)$ (b) $\lim_{x \to -2^+} f(x)$ (c) $f(-2)$ (d) $\lim_{x \to 0^-} f(x)$ (e) $\lim_{x \to 0^+} f(x)$ (f) $f(0)$

(g) $\lim_{x \to 4^-} f(x)$ (h) $\lim_{x \to 4^+} f(x)$ (i) the vertical asymptotes of the graph of f



Figure

Solution (a), (b), (c):

(a) $\lim_{x \to -2^-} f(x) = 1$ (b) $\lim_{x \to -2^+} f(x) = -2$ (c) $f(-2) = 1$

Solution (d), (e), (f):

(d) $\lim_{x \to 0^-} f(x) = 0$ (e) $\lim_{x \to 0^+} f(x) = -\infty$ (f) $f(0) = 0$

Solution (g), (h):

(g) $\lim_{x \to 4^-} f(x)$ does not exists due to oscillation (h) $\lim_{x \to 4^+} f(x) = +\infty$

Solution (i):

(i) The $y-$axis and the line $x = 4$ are vertical asymptotes of the graph of f

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HOMEWORK

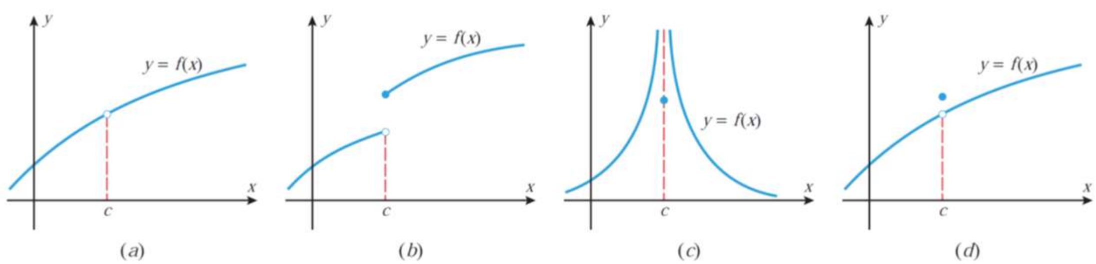
Exercises 1.1: (1-10)

Continuity

The graph of a function can be described as a "continuous curve" if it has no breaks or holes.

From Figure, we see that the graph of a function has a break or hole if any of the following conditions occur:

- The function f is undefined at c (Figure a).
- The limit of $f(x)$ does not exist as x approaches c (Figures b, c).
- The value of the function and the value of the limit at c are different (Figure d).



Figure

Continuity

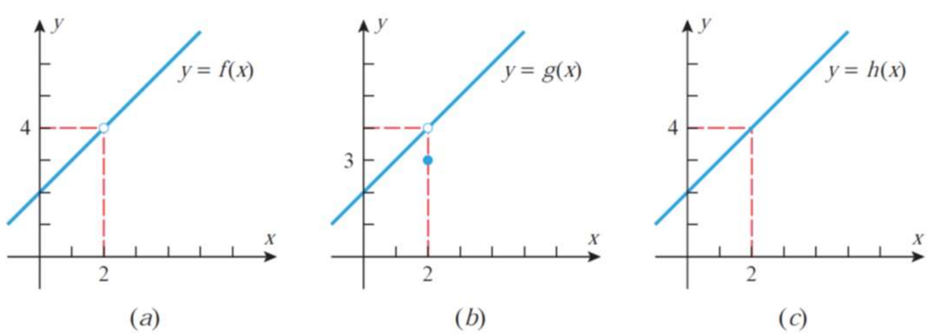
CONTINUITY A function f is said to be continuous at $x = c$ provided the following conditions are satisfied:

1) $f(c)$ is defined.
2) $\lim_{x \to c} f(x)$ exists.
3) $\lim_{x \to c} f(x) = f(c)$.

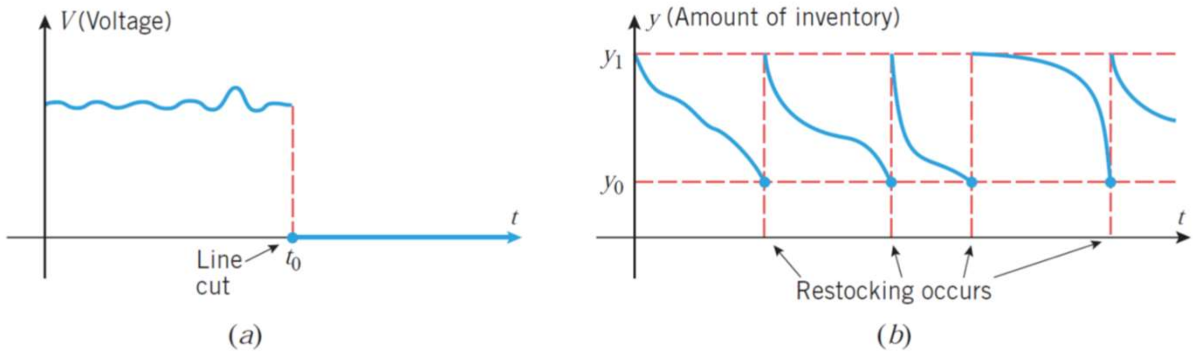
In other words:
A function f is said to be continuous at $x = c$ if $f(c)$exists and

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c).$$

Example 1: For the functions in Figure, determine whether the functions are continuous at $x = 2$..



Continuity in Application



P.1: A function $f(x)$ is defined as follows: $f(x)=\begin{cases} 2x-3, & 0\le x\le 2 \\ x^2-3, & 2<x\le 4 \end{cases}$ **, Does $\lim_{x\to 2} f(x)$ exist?**

Solution

$$\text{L.H.L.} = \lim_{x\to 2^-} f(x)$$
$$= \lim_{x\to 2^-} 2x-3$$
$$= 2.2-3$$
$$= 1$$

$$\text{R.H.L.} = \lim_{x\to 2^+} f(x)$$
$$= \lim_{x\to 2^+} x^2-3$$
$$= 2^2-3$$
$$= 1$$

Since $\text{L.H.L} = \text{R.H.L}$, then $\lim_{x\to 2} f(x)$ exists

Problem 1: Determine whether the function $f(x)=\begin{cases}2x+3\ , & x\le 4\\ 7+\dfrac{16}{x}\ , & x>4\end{cases}$ **is continuous at** $x=4$.

Solution

$$\text{L.H.L.}=\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-}(2x+3) = 11$$

$$\text{R.H.L.}=\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+}\left(7+\frac{16}{x}\right) = 11$$

$$f(4)=2.4+3=11$$

Since $\text{L.H.L}=\text{R.H.L}=f(4)$, $f(x)$ is continuous at $x=4$

Problem 4: Test the continuity of $f(x)$ **at** $x=0$, **where** $f(x)=\begin{cases}\dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, & when\, x\neq 0\\ 0\ , & when\, x=0\end{cases}$.

Solution

$$\text{L.H.L.}=\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} = \lim_{x\to 0^-}\frac{1-e^{-\frac{1}{x}}}{1+e^{-\frac{1}{x}}} = \frac{1-0}{1+0} = 1$$

$$\text{R.H.L.}=\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} = \lim_{x\to 0^+}\frac{1-e^{-\frac{1}{x}}}{1+e^{-\frac{1}{x}}} = \frac{1-0}{1+0} = 1$$

$$\therefore f(0)=0$$

Since $\text{L.H.L}=\text{R.H.L}\neq f(0)$, $f(x)$ is discontinuous at $x=0$

Problem 7: Find a nonzero value for the constant k that makes $f(x)=\begin{cases}\dfrac{\tan kx}{x}, x<0\\ 3x+k^2, x\geq 0\end{cases}$ **continuous at** $x=0$.

Solution

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x)$$
$$= \lim_{x\to 0^-} \frac{\tan kx}{x}$$
$$= \lim_{x\to 0^-} \frac{\tan kx}{kx}\cdot k$$
$$= \lim_{kx\to 0^-} \frac{\tan kx}{k\,x}\,.\,k$$
$$=1\,.\,k$$
$$=k$$

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x)$$
$$= \lim_{x\to 0^+} 3x+k^2$$
$$=3(0)+k^2$$
$$=k^2$$

$$\therefore f(0)=3.0+k^2$$
$$=k^2$$

Since the given function is continuou at $x=0$; $\text{L.H.L}=\text{R.H.L}=f(0)$

$$\therefore k=k^2=k^2$$
$$i.e.,\ k=k^2$$
$$k(k-1)=0$$
$$k=0,1$$

Therefore the nonzero value of the constant k is 1

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