PHYSICS FOR ENGINEERS

VOLUME ONE

(Waves & Oscillations, Properties of Matter, Heat & Thermodynamics and Optics)

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Revised and enlarged edition of
Outlines of physics (vol. 1)

CHAPTER I

SIMPLE HARMONIC MOTION

Periodic motion - Simple Harmonic Motion - Phase and epoch of a particle executing simple harmonic motion - Time period and frequency of a body executing SHM - Energy of a body executing SHM - Average value of kinetic and potential energies of a harmonic oscillator - Some examples of simple harmonic motion - Relation between simple harmonic motion and uniform circular motion - Solved problems - Exercises.

1.1 Periodic motion

A motion which repeats itself over and over again after a regular interval of time is referred to as a *periodic motion*. The time required for each repetition is called *time-period*. The motion of moon about the earth, the oscillation of a pendulum, the motion of a mass suspended from a coil spring are examples of periodic motion.

When the particle, undergoing periodic motion, covers the same path back and forth about a mean position, it is said to be executing an *oscillatory or vibratory* motion. The oscillatory motion is, therefore, a *to and fro* (forward and backward) motion. One complete to and fro motion is called an *oscillation or vibration or a cycle*. Further, the oscillatory motion is not only periodic but also *bounded, i.e.*, the displacement of the particle on either side of its mean position remains confined within a well-defined limit. The number of complete oscillations or cycles in unit time is called the *frequency* of vibration.

Of all the trigonometrical ratios, the sines and cosines alone are periodic as well as bounded. Thus the displacement of a particle executing an oscillatory motion is usually expressed in terms of sines or cosines or combination of both. This, coupled with the fact that this type of motion is generally associated with musical instruments, oscillatory motion is also referred to as *harmonic* motion.

Simple Harmonic Motion

Let a particle oscillate along a straight line within some fixed limits.

The magnitude and direction of its displacement y will be changed periodically and along with displacement the velocity and acceleration of the particle will also be changed periodically.

Fig. 1.1 shows a particle oscillating between y_1 and y_2 with the point O as its equilibrium or mean position. When the particle is at the equilibrium position its potential energy is minimum but its kinetic energy is maximum. When the particle is at its maximum displacement, its potential energy is maximum but its kinetic energy is minimum.

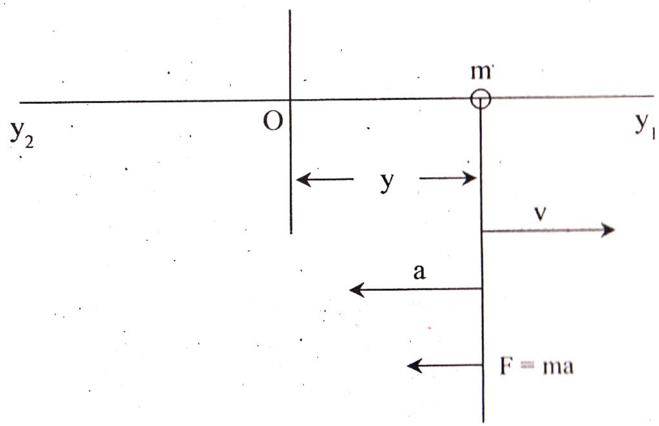

Fig. 1.1

Fig. 1.2 shows the relationship of the potential energy of the oscillating particle with its displacement. For a displacement y of the particle, the potential energy is represented by $U(y)$. It can be observed that $U(y) = 0$ at the equilibrium or mean position. A restoring force $F = -\frac{dU(y)}{dy}$ acts upon the particle at every point of its displacement, except at the mean position (where it is zero), tending to bring it back to its mean or equilibrium position. Or, in other words, if the displacement is on the right side of the mean position the force acts towards the left side and *vice versa*.

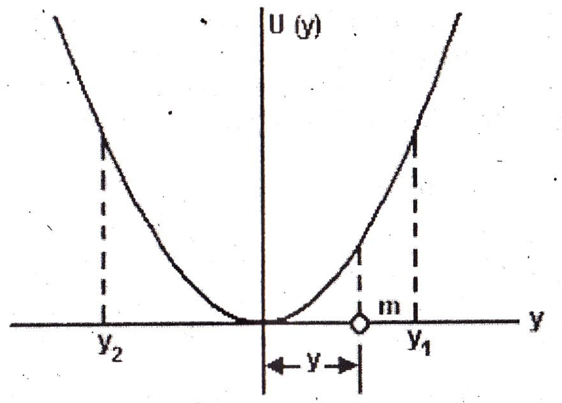

Fig. 1.2

The potential energy of an oscillating particle is $U(y) = \frac{1}{2}ky^2$; then the force acting on the particle is

$$F(y) = -\frac{dU(y)}{dy} = -ky \qquad (1.1)$$

where k is a *positive constant* called the *force constant*

Eqn. (1.1) indicates that a restoring force acts on the particle tending to bring it back to its mean or equilibrium position.

The motion executed by such an oscillating particle is called harmonic oscillation and the particle itself a harmonic oscillator. In case the limits (y_1 and y_2) are equally spaced about the equilibrium position, it is called simple harmonic motion (S. H. M) and the particle a simple harmonic oscillator.

Thus a clear-cut definition to simple harmonic motion can be given a follows :

Whenever a force acting on a particle, and hence the acceleration of the particle, is proportional to its displacement from its equilibrium position or any other fixed point in its path, but is always directed in a direction opposite to the direction of the displacement (i.e., directed towards the mean position) and if the maximum displacement of the particle is the same on either side of the mean position, the particle is said to execute a simple harmonic notion.

In case the displacement is not the same on either side of the mean position, the motion is still harmonic but not simple harmonic. However, it can be shown that even in this case the oscillation can still be regarded as simple harmonic provided the displacement is small.

1.2 Differential Equation of Simple Harmonic Motion

If F be the force acting on a particle executing simple harmonic motion and y its displacement from its mean or equilibrium position, then $F = -ky$. Again, according to Newton's laws of motion, $F = ma'$ where m is the mass of the particle and a' its acceleration.

Substituting $-ky$ for F and $\frac{d^2y}{dt^2}$ for a', we can write (for $F = ma'$)

$$-ky = m\frac{d^2y}{dt^2}$$

$$\text{or,}\quad \frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \qquad (1.2)$$

Equation (1.2) is called the differential equation of motion of a body executing simple harmonic motion, because the solution of eqn. (1.2) gives us the nature of variation of displacement with time. Thus the correct nature of the motion of the particle can be known,

Rearranging eqn. (1.2), we can write,

$$\frac{d^2y}{dt^2} = -\frac{k}{m}y = -w^2y \qquad (1.3)$$

$$= -\mu y \qquad (1.4)$$

where $w = \sqrt{\frac{k}{m}}$ is the angular velocity of the particle and μ is a constant equal to w^2. Since $\frac{d^2y}{dt^2} = -\mu$ when $y = 1$, μ may be defined as the *acceleration per unit displacement of the particle.*

To obtain a general solution of the differential equation of simple harmonic motion, let us multiply both sides of eqn. (1.3) by $2\frac{dy}{dt}$ when we get

$$2.\frac{dy}{dt}.\frac{d^2y}{dt^2} = -w^2y.2.\frac{dy}{dt}$$

$$\text{or,}\quad 2\frac{dy}{dt}.\frac{d^2y}{dt^2} = -w^2.2.\frac{dy}{dt}.y$$

Integrating with respect to time, we have

$$\left(\frac{dy}{dt}\right)^2 = -w^2y^2 + C \qquad (1.5)$$

where C is a constant of integration. C can be evaluated by applying boundary conditions. To evaluate C we recall that velocity and acceleration of a simple harmonic motion are oppositely directed. We further recall that the velocity is zero (or K.E. is zero) at maximum displacement (or amplitude).

$$\text{or,}\quad \frac{dy}{dt} = 0 \text{ when } y = a \text{ (amplitude).}$$

Substituting these values in eqn. (1.5)

$$0 = -w^2a^2 + C\,;\ \text{or, } C = w^2a^2$$

Substituting this value of C in eqn. (1.5), we have

$$\left(\frac{dy}{dt}\right)^2 = -w^2y^2 + w^2a^2 = w^2(a^2 - y^2) \qquad (1.6)$$

17

$$\text{or,}\quad \frac{dy}{dt} = \pm w\sqrt{a^2 - y^2}$$

$$= \pm\sqrt{\frac{k}{m}}\sqrt{a^2 - y^2} \qquad (1.7)$$

Eqn. (1.7) can be rearranged as

$$\frac{dy}{\sqrt{a^2 - y^2}} = w.dt$$

Integrating again with respect to time, we have

$$\sin^{-1}\frac{y}{a} = wt + \phi$$

or, $y = a \sin (wt + \phi)$ (1.8)

Equation (1.8) gives the displacement of the particle at an instant t in terms of its amplitude a and its total phase $(wt + \phi)$ and is the general solution of the differential equation of simple harmonic motion.

Expanding eqn. (1.8), we have

$$y = a \sin wt \cos \phi + a \cos wt \sin \phi$$

$$= A \sin wt + B \cos wt$$

where $A = a \cos \phi$ and $B = a \sin\phi$

In special cases where either A or B may be zero, the displacement may be written as either

$$y = A \sin wt$$

or, $y = B \cos wt$

Hence the most general form of the differential equation is

$$y = A \sin wt + B \cos wt \quad (1.9)$$

which is a combination of both the sine and the cosine terms.

Velocity and acceleration of a body executing SHM

The displacement of a particle executing simple harmonic motion is given by

$$y = a \sin (wt + \phi)$$

(i) Hence, the velocity of the particle at any instant of time t is

$$\frac{dy}{dt} = wa \cos (wt + \phi)$$

$$= \pm wa \sqrt{1 - \sin^2 (wt + \phi)}$$

Now $\sin (wt + \phi) = \frac{y}{a}$.

$$\therefore \frac{dy}{dt} = \pm wa \sqrt{1 - \frac{y^2}{a^2}}$$

$$= \pm wa \sqrt{\frac{a^2 - y^2}{a^2}}$$

$$= \pm w \cancel{a} \sqrt{a^2 - y^2}$$

$$= \pm \sqrt{\frac{k}{m}} \sqrt{a^2 - y^2} \quad (1.10)$$

(ii) The acceleration of the particle at any time t is given by

$$\frac{d^2y}{dt^2} = -w^2 a \sin (wt + \phi)$$

$$\cancel{=} w^2 . a . \frac{y}{a} = -w^2 . y$$

$$= -\frac{k}{m} . y. \quad (1.11)$$

From eqns. (1.10) and (1.11) it can be seen that *maximum value of velocity, i.e.,* $v_{max} = \pm w.a$ or $\pm a \sqrt{\frac{k}{m}}$ and occurs when $y = 0$ *i.e.*, when the particle is passing through its mean position.

Similarly, the *maximum value of acceleration* occurs when y is maximum *i.e.*, the particle is at the position of one of its extreme displacements.

1.3 Phase and epoch of a particle executing simple harmonic motion

Going back to eqn. (1.8), we see that the total phase of a particle executing simple harmonic motion is made up of the *phase angle wt* and ϕ which is called the *initial phase* or *phase constant* or the *epoch* of the particle, usually denoted by the letter e. This initial phase or epoch is a direct consequence of the fact that we start to count time, not from the instant when the particle is in some standard position, like its mean position or one of its extreme position, but from the instant when it is anywhere else in between. The following cases will then arise :

(i) if we start counting time when the particle is in its mean position, *i.e.*, when $y = 0$ at $t = 0$, we have $\phi = 0$. Eqn. (1.8), therefore, reduces to

$y = a \sin wt$

(ii) If the counting of time starts *when the particle is in one of its extreme positions, i.e.*, when y = a at t = 0, we have from eqn (1.8),

$a = a \sin(0 + \phi)$

or, $\sin\phi = \frac{a}{a} = 1$; or, $\phi = \frac{\pi}{2}$.

Eqn. (1.9), therefore, becomes

$$y = a \sin\left(wt + \frac{\pi}{2}\right) = a \cos wt.$$

x

As can be seen, both $y = a \sin wt$ and $y = a \cos wt$ represent the equations of simple harmonic motion with the difference lying only in the initial position of the vibrating particle. Either of these equations can, therefore, be taken as the general equation of simple harmonic motion. In fact, the general solution of the differential equation of simple harmonic motion is of the form

$$y = A \sin wt + B \cos wt$$

which is a combination of both the sine and cosine terms.

(iii) If, on the other hand, we start counting time from an instant t', before the particle has passed through its mean position, we have y = 0 at t = t'.

Then we have

$$0 = a \sin(wt' + \phi)$$

epoch

or, $(wt' + \phi) = 0$ whence $\phi = -wt' = -e$, say. Therefore, the expression for simple harmonic motion becomes

$$y = a \sin(wt - e)$$

(iv) Similarly, if the counting of time is started from an instant t', after the particle has passed through its mean position, we have

$$y = a \sin(wt + e)$$

1.4 Time period, frequency and angular frequency of a body executing SHM

we know displacement y=

If the time t in the relation $y = a \sin(wt + \phi)$ is increased by $2\pi/w$, then we have

$$y = a \sin\left[w\left(t + \frac{2\pi}{w}\right) + \phi\right]$$

of t is increased by $\frac{2\pi}{\omega}$

or, $y = a \sin(wt + 2\pi + \phi)$

$= a \sin(wt + \phi)$

the same as before. This indicates that the particle executing simple harmonic motion *repeats its motion after every $2\pi/w$ seconds*. In other words, the *time period of the particle* is given by

$$T = \frac{2\pi}{w} = 2\pi\sqrt{\frac{1}{w^2}} = 2\pi\sqrt{\frac{1}{\mu}}$$

or, $$T = 2\pi\sqrt{\frac{1}{\text{acceleration per unit displacement}}}$$

$$= 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

Since $w^2 = \frac{k}{m}$, the time period may also be expressed as

$$T = 2\pi\sqrt{\frac{m}{k}} \qquad (1.12)$$

The number of oscillations (or vibrations) made by the particle per second (unit time) is called its *frequency of oscillation* or, simply, its *frequency*, usually denoted by the letter n.

Thus, frequency is the reciprocal of the time period.

or, $$n = \frac{1}{T} = \frac{w}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

From the above relation we also have $w = 2\pi n = \frac{2\pi}{T}$. w is also referred to as the angular frequency of the particle. It is the angle described per second and has the unit radians per second, the same as angular velocity, also designated w. Angular frequency is very closely related to the angular velocity of a circular motion which is associated with simple harmonic motion. The use of w instead of n to describe the frequency of simple harmonic motion simplifies these expressions by avoiding factors of 2π, which usually accompany n in such expressions.

It may be noted that the angular frequency and thus the time period $T = \frac{2\pi}{w}$ are determined only by the force constant and the mass. They do not depend either on the amplitude a or the initial phase ϕ. This means that the oscillations of a simple harmonic oscillator are isochronous, i.e., they take the same time irrespective of the values of a and ϕ. This is an important property of simple harmonic motion where the frequency and time period are independent of the amplitude and epoch./ phase

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Example 1.1. *A spring, hung vertically, is found to be stretched by 0.02 m from its equilibrium position when a force of 4N acts on it. Then a 2 kg body is attached to the end of the spring and is pulled 0.04 m from its equilibrium position along the vertical line. The body is then released and it executes simple harmonic motion.*

(i) what is the force constant of the spring ?

A force of 4N on the spring produces a displacement of 0.02 m. Hence.

from F = ky we have $k = \frac{F}{y} = \frac{4N}{0.02m}$

$= \mathbf{200\ N\ per\ m.}$

(ii) what is the force executed by the spring on the 2 kg body just before it is released ?

The spring is stretched 0.04 m. Hence the force executed by the spring is

$$F = -ky = -(200\ N.m^{-1})\ (0.04\ m)$$

$$= \mathbf{-8\ N.}$$

The minus sign indicates that the force is directed opposite to the displacement.

(iii) what is the period and frequency of oscillation after release ?

$$T = 2\pi.\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2\ kg}{200\ N.m^{-1}}}$$

$$= \frac{\pi}{5}\ sec = \mathbf{0.628\ sec.}$$

$$n = \frac{1}{T} = \frac{1}{0.628} = \mathbf{159\ Hz.}$$

$$w = 2\pi n = (2)\ (3.14)\ (1.59)$$

$$= \mathbf{10\ sec^{-1}}.$$

Also,

$$w = 2\pi n = 2\pi.\frac{1}{T} = \frac{1}{2\pi} = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{k}{m}} = \sqrt{\frac{200\ N.m^{-1}}{2\ kg}} = 10\ sec.^{-1}$$

(iv) what is the amplitude of motion ?

Amplitude is the initial displacement to the body, *i.e,* **0.04 m.**

(v) what is the maximum velocity of the oscillating body ?

The maximum velocity occurs when the body passes through the position of equilibrium, *i.e.*, y = 0.

Hence, from

$$v = \pm\ w\sqrt{a^2 - y^2}, \text{ we get}$$

$$v_{max} = \pm\ w.a$$

$$\therefore\ v_{max} = \left(10\ sec^{-1}\right)\ (0.04\ m)$$

$$= \pm\ 0.4\ m\ sec^{-1}.$$

(vii) what is the mechanical (total) energy of the oscillating system ?

Total energy = P.E + K.E

$$= \frac{1}{2}ka^2 = \frac{1}{2}(200)\ (0.04)^2$$

$$= \mathbf{0.16\ joules.}$$

Also, total energy

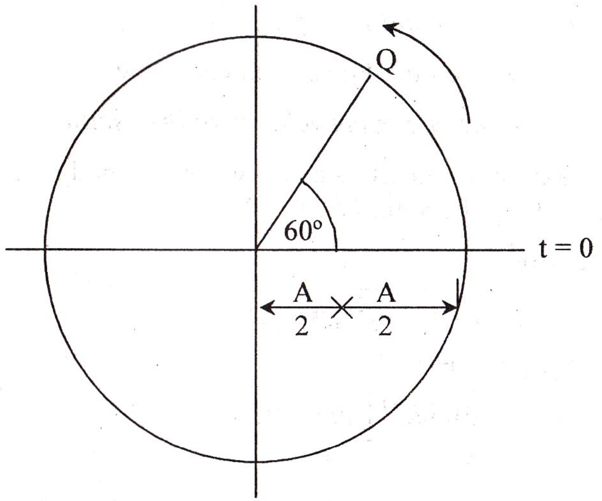
$$\frac{1}{2}ka^2 = 2\pi 2.m.a^2.n^2 \quad \text{(Art. 1.5)}$$

$$= 2.\,(3.14)^2.\,2.\,(0.04)^2.\,(1.59)^2$$

$$= \mathbf{0.16\ joules.}$$

(ix) how much time is required for the body to move half-way in to the centre from its initial position ?

The motion is neither one of constant velocity nor one of constant acceleration. The simplest method is to make use of the reference circle. While the body moves half-way in, the reference point revolves through an angle of 60°. Since the reference point moves with constant angular speed and in this example makes one complete revolution in $\frac{\pi}{5}$ sec, the time to rotate through 60° is



$$\frac{1}{6}.\frac{\pi}{5}\text{sec} = \frac{\pi}{30}\text{sec} = 0.105 \text{ sec.}$$

The time can also be computed directly from the relation,

$$y = a \sin wt$$

$$\frac{a}{2} = a \sin \ (10 \text{ sec}^{-1}) \,.\, t$$

$$\sin (10 \text{ sec}^{-1}) \,.\, t = \frac{1}{2}$$

$$10 \text{ sec}^{-1} \,.\, t = \sin^{-1}\frac{1}{2} = \frac{\pi}{3}$$

$$\therefore \ t = \frac{\pi}{3 \times 10} \ \textbf{sec}$$

(x) what is the displacement of the body as a function of time ?

The general equation for displacement of a body executing simple harmonic motion is

$$y = a \sin (wt + \phi)$$

The value of w, as already obtained, is

$$w = \frac{2\pi}{T} = 10 \text{ radian/}sec.$$

$$\therefore \ y = a \ \sin \ (10t + \phi)$$

At t = 0, y = a = 0.04 m, so that at that instant

$$y = 0.04 \sin \phi = 0.04$$

$$\therefore \ \sin \phi = 1$$

$$\text{or,} \ \phi = \sin^{-1} 1 = \frac{\pi}{2} \text{ radian.}$$

Therefore, with a = 0.04 m, w = 10 rad./sec and $\phi = \frac{\pi}{2}$ radian, we get

$$y = 0.04 \sin (10\,t + \frac{\pi}{2})$$

$$= \mathbf{0.04 \sin 10t.}$$

Example 1.2. *A body is vibrating with simple harmonic motion of amplitude 15 cm and frequency 4 Hz. Compute (a) the maximum values of the acceleration and velocity and (b) the acceleration and velocity when the displacement is 9 cm.*

Soln. (a) $v_{max} = w.a$

$a = 15$ cm

$n = 4$ Hz

$\therefore \quad w = 2\pi n = 2 \times 3.14 \times 4$

$= 25.12$ rad/sec.

$\therefore \quad v_{max} = 25.12 \times 15 = \mathbf{376.8\ cm/sec.}$

$(accln)_{max} = -w^2.a$

$= -(25.12)^2 \times 15$

$= \mathbf{-9470\ cm/sec^2.}$

(b) when $y = 9$ cm

$v = w.\sqrt{a^2 - y^2} = 25.12 \sqrt{(15)^2 - 9^2}$

$= \mathbf{300\ cm/sec}$

accln. $= -w^2.y = -(25.12)^2 \times 9$

$= \mathbf{-5680\ cm/sec^2.}$

Example 1.3. *A particle executes linear harmonic motion about the point x = 0. At t = 0, it has displacement y = 0.37 cm and zero velocity. If the frequency of the motion is 0.25/sec, determine (a) the period, (b) the amplitude, (c) the maximum speed and (d) the maximum acceleration.*

Soln.

(a) $T = \frac{1}{n} = \frac{1}{0.25} = \mathbf{4\ sec.}$

(b) $y = a \sin (wt + \delta)$

at $t = 0$, $y = 0.37$ cm.

$\therefore a \sin \delta = 0.37$ (i)

Again $v = \frac{dy}{dt} = wa \cos (wt + \delta)$

at $t = 0$, $v = 0$

$\therefore wa \cos \delta = 0$

or, $a \cos \delta = 0$ (ii)

from (i) and (ii),

$a^2 = (0.37)^2$

or, $a = \pm$ **0.37 cm.**

(c) $v = w.\sqrt{a^2 - y^2}$

v is maximum when $y = 0$.

$\therefore \quad v_{max} = w.a = a \times 2\pi n$

$= 0.37 \times 2 \times 3.14 \times 0.25$

$= \mathbf{0.5809\ cm/sec.}$

(d) $(accln)_{max} = -w^2.a$

$= -(2\pi n)^2 . a$

$= -(2 \times 3.14 \times 0.25)^2 \times 0.37$

$= \mathbf{-0.912013\ cm/sec^2.}$

Example 1.4. *The displacement of an oscillating particle at any instant t is given by*

$y = a \cos wt + b \sin wt.$

Show that it is executing a simple harmonic motion.

If a = 5 cm, b = 12 cm and w = 4 radian/sec, calculate (i) the amplitude, (ii) the time period, (iii) the maximum velocity and (iv) the maximum acceleration of the particle.

Soln.

$y = a \cos wt + b \sin wt$

or, $\frac{dy}{dt} = -aw \sin wt + bw \cos wt$

or, $\frac{d^2y}{dt^2} = -w^2.a \cos wt - w^2 b \sin wt$

$= -w^2 (a \cos wt + b \sin wt)$

$= -w^2 y$

or, $$\frac{d^2y}{dt^2} + w^2y = 0$$

Hence the motion is simple harmonic.

(i) Let $a = A \sin \alpha$ and $b = A \cos \alpha$.

Then

$$y = A \sin\alpha \cos wt + A \cos \alpha \sin wt$$
$$= A \sin (wt + \alpha)$$

This represents a simple harmonic motion with amplitude A.

$\therefore \quad A^2 \sin^2 \alpha + A^2 \cos^2 \alpha = a^2 + b^2$

or, $A = \sqrt{a^2 + b^2}$

a = 5 cm

b = 12 cm

$\therefore \quad A = \sqrt{5^2 + 12^2}$

= 13 cm

(ii) $T = \frac{2\pi}{w} = \frac{2\pi}{4} = \frac{\pi}{2}$ sec.

(iii) $v_{max} = w.A = 13 \times 4$

= 52 cm/sec

(iv) $(accln)_{max} = -w^2.A = -(4)^2 \times 13$

$= \mathbf{-208\ cm/sec^2}$.

Example 1.5. *The positions of a particle executing simple harmonic motion along the x-axis are x = A and x = B at time t an 2t respectively. Show that its period of oscillation is given by*

$\therefore \quad T = (2\pi t) / \cos^{-1} (B/2A)$

Soln.

$$A = a \sin wt$$
$$B = a \sin (w.\ 2t)$$
$$= a \sin 2wt$$
$$= a\ 2 \sin wt \cos wt$$

$$\therefore \quad \frac{A}{B} = \frac{a \sin wt}{a\ 2 \sin wt \cos wt}$$
$$= \frac{1}{2 \cos wt}$$

or, $\cos wt = \frac{B}{2A}$

or, $wt = \cos^{-1}\left(\frac{B}{2A}\right)$

or, $w = \frac{1}{t} \cos^{-1}\left(\frac{B}{2A}\right)$

or, $\frac{2\pi}{T} = \frac{\cos^{-1}\left(\frac{B}{2A}\right)}{t}$

$$\therefore \quad T = \frac{2\pi t}{\cos^{-1}\left(\frac{B}{2A}\right)}$$

Example 1.6. *For a particle executing simple harmonic motion the displacement is 8 cm at the instant the velocity is 6 cm/sec and the displacement is 6 cm at the instant the velocity is 8 cm/sec. Calculate (i) amplitude, (ii) frequency and (iii) time period.*

Soln.

Velocity of a particle executing simple harmonic motion,

$$v = \frac{dy}{dt} = w\sqrt{a^2 - y^2}$$

Now v = 6 cm/sec when y = 8 cm.

$$\therefore \quad 6 = w\sqrt{a^2 - 64} \qquad \text{(i)}$$

Again v = 8 cm/sec when y = 6 cm

$$\therefore \quad 8 = w\sqrt{a^2 - 64} \qquad \text{(ii)}$$

Dividing (ii) by (i) and squaring

$$\frac{64}{36} = \frac{a^2 - 36}{a^2 - 64}$$

or, a = **10 cm.**

Substituting a = 10 cm in eqn. (i)

$$\therefore \quad 6 = w\sqrt{100 - 64}$$

or, w = **1 rad/sec.**

Hence frequency,

$$n = \frac{w}{2\pi} = \frac{1}{2\pi} \text{ Hz.}$$

time period

$$T = \frac{1}{n} = \mathbf{2\pi \text{ seconds.}}$$

Example 1.7. *A simple harmonic motion is represented by*

$$y = 10 \sin(10t - \frac{\pi}{6})$$

where y is measured in metres, t in seconds and the phase angle in radians. Calculate (i) the frequency, (ii) the time period, (iii) the ***maximum*** *displacement, (iv) the maximum, velocity and (v) the* ***maximum*** *acceleration and (vi) displacement, velocity and acceleration at time t = 0 and t = 1 second.*

Soln.

Here $y = 10 \sin (10t - \frac{\pi}{6})$ (1)

Comparing with the displacement equation

$y = a \sin (wt + \delta)$ (2)

we get, (i) $w = 2\pi n = 10$

or, $n = \frac{10}{2\pi}$ = **1.6 Hz.**

(ii) time period, $T = \frac{1}{n} = \frac{2\pi}{10}$ = **0.63 sec.**

(iii) maximum displacement (amplitude)
a = **10 m.**

(iv) maximum velocity,

$v_{max} = w.a = 10 \times 10$ = **100 m/sec.**

(v) $(\text{accln.})_{max} = -w^2a = -(10)2 \times 10$

= **– 1000 m/sec².**

minus sign shows that the acceleration is directed towards the mean position.

(vi) From eqn. (1)

(a) at t = 0

$$y = 10 \sin(-\frac{\pi}{6}) = \mathbf{-5\ m.}$$

velocity, $\frac{dy}{dt} = a\, w \cos \delta$

$= 10 \times 10 \cos(-\frac{\pi}{6})$

$= 100 \times 0.866$ = **86.6 m/sec.**

Acceleration, $\frac{d^2y}{dt^2} = -aw^2 \sin \delta$

$= -10 \times 10^2 \times \sin(-\frac{\pi}{6})$

$= -10 \times 100 \times 0.5$

= **– 500 m/sec².**

(b) From eqn. (1), at t = 1,

displacement

$$y = 10 \sin(10 - \frac{\pi}{6})$$

$$= 10 \sin \left(\frac{60 - 3.142}{6}\right)$$

$$= 10 \sin \left(\frac{56.858}{6}\right)$$

$= 10 \sin (3\pi)$ approximately

$= 10 \sin \pi$

$= \mathbf{0}$

velocity, $\frac{dy}{dt} = aw \cos \left(10 - \frac{\pi}{6}\right)$

$= aw \cos (\pi)$ approximately

$= 10 \times 10 \times (-1)$

$= -100$ m/sec.

Accln., $\frac{d^2y}{dt^2} = -a w^2 \sin \left(10 - \frac{\pi}{6}\right)$

$= -aw^2 \sin (\pi)$ approximately.

$= \mathbf{0.}$

Example 1.8. *A particle performs simple harmonic motion given by the equation*

$y = 20 \sin (wt + \alpha)$

If the time period is 30 seconds and the particle has a displacement of 10 cm at $t = 0$, find (i) epoch, (ii) the phase angle at $t = 5$ seconds and (iii) the phase difference between two positions of the particle 15 seconds apart.

Soln.

Here

$y = 20 \sin (wt + \alpha)$

$T = 30$ secs.

$\therefore \quad w = \frac{2\pi}{T} = \frac{2\pi}{30} = \frac{\pi}{15}$ rad/sec.

(i) at $t = 0$, $y = 10$ cm.

$\therefore \quad 10 = 20 \sin \left(\frac{\pi}{15} \times 0 + \alpha\right)$

or, $\sin \alpha = \frac{10}{20} = 0.5$

or, $\alpha = \frac{\pi}{6}$ **radian.**

(ii) at $t = 5$ sec,

the phase angle $= (wt + \alpha)$

$= \left(\frac{\pi}{15} \times 5 + \frac{\pi}{6}\right)$

$= \frac{\pi}{2}$

(iii) at $t = 0$ the phase angle

$\theta_1 = \left(\frac{\pi}{15} \times 0 + \frac{\pi}{6}\right) = \frac{\pi}{6}$ radian.

at $t = 15$ sec, the phase angle

$\theta_2 = \left(\frac{\pi}{15} \times 15 + \frac{\pi}{6}\right)$

$= \frac{7\pi}{6}$ radian.

$\therefore$ the phase difference,

$\theta_2 - \theta_1 = \frac{7\pi}{6} - \frac{\pi}{6} = \pi$ **radian.**

Example 1.9. *A body describing SHM has a maximum acceleration of 8π m/s² and a maximum speed of 1.6 m/s. Find the period T and the amplitude a.*

Soln :

$a_{max} = w^2 a$ (ignoring the minus sign)

$$= (2\pi n)^2 a = \left(\frac{2\pi}{T}\right)^2 a = \frac{4\pi^2}{T^2} a = 8\pi \text{ m/s}^2.$$

again $v_{max} = wa = \left(\frac{2\pi}{T}\right) a = 1.6$ m/s.

$$\frac{a_{max}}{v_{max}} = \frac{2\pi}{T} = \frac{8\pi}{1.6}; \quad \text{or,} \quad T = 0.4 \text{ s.}$$

Then $\frac{2\pi a}{T} = \frac{2\pi a}{0.4} = 1.6$; or, $a = \frac{0.4 \times 1.6}{2\pi}$

$$= \mathbf{0.102\ m}$$

Example 1.10. *A ball moves in a circular path of 0.15 m diameter with a constant angular speed of 20 rev/min. Its shadow performs simple-harmonic motion on the wall behind it. Find the acceleration and speed of the shadow (i) at a turning point of the motion (ii) at the equilibrium position, and (iii) at a point 6 cm from the equilibrium position.*

Soln :

Amplitude is the radius of the circle.

$$\therefore \quad a = \frac{0.15}{2} \text{ m} = \mathbf{0.075\ m}$$

The period and frequency of the shadow are the same as that of the circular motion.

So

$$20 \text{ rev/min } \frac{20}{60} \text{ rev./sec.}$$

$$w = (2\pi) \times \frac{20}{60} \text{ rad/s} = \frac{2\pi}{3} \text{ rad/sec.}$$

From the general formulas for SHM, we have

(i) at the turning point $v = 0$ and $a = a_{max}$.

$$\therefore \quad a_{max} = w^2 a = \left(\frac{4\pi^2}{9}\right)(0.075) = \mathbf{0.329\ m/sec^2}$$

(ii) at the equilibrium position,

$a = 0$ and $v = v_{max}$

$$\therefore \quad v_{max} = wa = \left(\frac{2\pi}{3}\right)(0.075) = \mathbf{0.157\ m/sec.}$$

(iii) at y = 0.06 m (6 cm)

$$v = w.\sqrt{a^2 - y^2} = \frac{2\pi}{3}\sqrt{(0.075)^2 - (0.06)^2}$$

$$= \frac{2\pi}{3}(0.045) = \mathbf{0.0942\ m/sec.}$$

$$a = w^2 y = \left(\frac{4\pi^2}{9}\right)(0.06) = \mathbf{0.263\ m/sec^2}.$$

1.5 Energy of a body executing S.H.M.

The acceleration of a particle and hence the force acting on the particle executing simple harmonic motion is, as we know, directed towards its mean or equilibrium position, *i.e.*, opposite to the direction in which the displacement y increases. Work is, therefore, done during the displacement of the particle. Hence the particle possesses *potential energy (U)*. As the particle also possesses velocity, it possesses *kinetic energy (K)* too. Thus the mechanical energy E of a particle executing simple harmonic motion is partly kinetic and partly potential. *If no non-conservative forces, such as the force of friction act on the particle, the sum of its kinetic energy and potential energy remains constant*. or,

$$E = K + U = \text{constant.}$$

As the displacement increase, the potential energy increases and the kinetic energy decreases and *vice versa*. But the total energy $E (= K + U)$ is conserved.

Let the displacement of a particle executing simple harmonic motion at any instant be y. If the mass of the particle be m and its velocity at that instant be v, then its kinetic energy is $\frac{1}{2}mv^2$. The potential energy of the particle at the same instant is the amount of work that must be done in overcoming the force through a displacement y and is given by the relation $\int_0^y F.\,dy$ where F is the force required to maintain the displacement and dy is a small displacement.

Now the displacement is given by the relation $y = a \sin(wt + \phi)$.

Hence the acceleration $\frac{d^2y}{dt^2}$ is given by

$$\frac{d^2y}{dt^2} = -aw^2 \sin(wt + \phi) = -w^2.y.$$

Then, force F = mass x acceleration

$$= m.(-w^2y) = -mw^2.y.$$

Then the potential energy of the particle is

$$\text{P.E.} = \int_0^y F.\,dy = \int_0^y mw^2.\,y.\,dy$$

$$= \frac{1}{2}\,m.\,w^2.\,y$$

$$= \frac{1}{2}\,m.\,w^2.\,a^2 \sin^2(wt + \phi)$$

$$= \frac{1}{2}k.\,a^2 \sin^2(wt + \phi)\ (\because\ w^2 = \frac{k}{m}) \qquad (1.14)$$

ignoring the minus sign in the expression for F, which simply shows that the direction of the force and displacement are opposite to each other.

Now the kinetic energy of the particle is given by

$$\text{K.E.} = \frac{1}{2}\,mv^2$$

$$= \frac{1}{2}\,m\,(\frac{dy}{dt})^2$$

$$= \frac{1}{2}\,m.[\,wa\cos(wt + \phi)]^2$$

$$= \frac{1}{2}\,m.\,a^2.\,w^2\cos^2(wt + \phi)$$

$$= \frac{1}{2}\,k.\,a^2\cos^2(wt + \phi) \qquad (1.15)$$

It can be seen from eqns. (1.14) and (1.15) that both the potential and kinetic energies have a maximum value of $\frac{1}{2}$ ka^2. or $\frac{1}{2}$ m $(wa)^2$. During the motion the potential energy as well as the kinetic energy vary between zero to this maximum value.

The total energy which is just the sum of the kinetic and potential energy is

$$E = K + U = \frac{1}{2}\,ka^2\ \sin^2(wt + \phi) + \frac{1}{2}\,ka^2\ \cos^2(wt + \phi)$$

$$= \frac{1}{2}\,ka^2. \qquad (1.16)$$

We see that the total energy, as expected, is constant and has the value $\frac{1}{2}$ ka^2. Thus, *the total energy of the system is the same as the maximum value of any one of the two forms of energy.* At the maximum displacement the kinetic energy is zero, but the potential energy has the value $\frac{1}{2}$ka^2. At the position of equilibrium the potential energy is zero but the kinetic energy has the value $\frac{1}{2}$ ka^2. At any other position, the kinetic and potential energies each contribute energy whose sum is always $\frac{1}{2}$ ka^2.

Fig. 1.3 shows the kinetic, potential and total energies as a function of time while in Fig. 1.4, the same energies are plotted as a function of displacement from the equilibrium position. *The total energy of a particle executing simple harmonic motion is proportional to the square of the amplitude of the motion.*

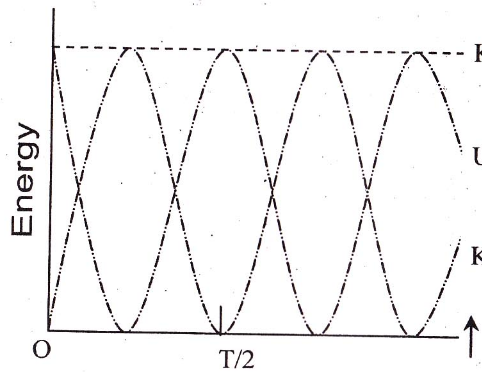


Fig. 1.3

Now

$$\frac{1}{2}ka^2 = \frac{1}{2}m\,w^2a^2$$

$$= \frac{1}{2}m\,.\left(\frac{2\pi}{T}\right)^2 a^2 = \frac{2\pi^2 ma^2}{T^2}$$

But, since $\frac{1}{T}$ = n, the frequency of oscillation, the total energy of the system can also be expressed as

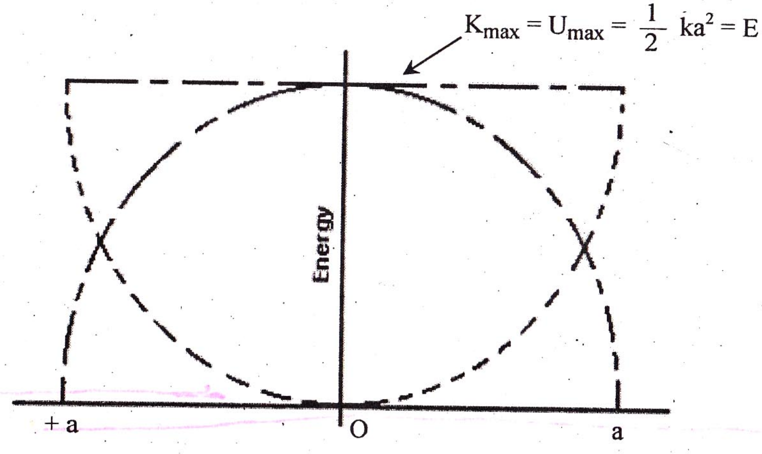


Fig. 1.4

$$E = \frac{1}{2}ka^2 = 2\pi^2 m\,.\,a^2\,.\,n^2 \qquad (1.17)$$

Thus, *total energy*

= maximum value of potential energy

= maximum value of kinetic energy

$$= \frac{1}{2}ka^2 = \frac{1}{2}mw^2a^2 = 2\pi^2n^2a^2m.$$

The total energy in Fig. 1.3 is represented by the upper horizontal line parallel to the time axis and touching the two curves at points representing the maximum values of kinetic and potential energies respectively. The upper horizontal line in Fig. 1.4 which passes through the two points of maximum potential energy corresponding to the two points of maximum displacement +a and -a on either side of the mean position represents the total energy. This line is also parallel to the displacement axis. Since the line representing total energy in either figure is a horizontal line, being parallel to either the time axis or the displacement axis, it follows that *total energy of the particle executing simple harmonic motion remains constant throughout and is independent of both time and displacement.*

1.6 Average value of kinetic and potential energies of a harmonic oscillator

The potential energy (P.E.) of the particle at a displacement y is given by

$$= \frac{1}{2}m\,w^2y^2$$

$$= \frac{1}{2}m\,.\,w^2a^2\sin^2(wt + \phi)$$

So the average P.E. of the particle over a complete cycle or a whole time period T

$$= \frac{1}{T}\int_0^T \frac{1}{2}m\,w^2a^2\sin^2(wt+\phi)\,dt$$

$$= \frac{1}{T} \cdot \frac{m w^2 a^2}{4} \int_O^T 2 \sin^2 (wt + \phi)\, dt$$

$$= \frac{m w^2 a^2}{4T} \int_O^T [1 - \cos 2 (wt + \phi)]\, dt$$

$$= \frac{m w^2 a^2}{4T} [\int_O^T dt - \int_O^T \cos 2 (wt + \phi)]\, dt$$

The average value of both a sine and a cosine function for a complete cycle or a whole time period T is zero. We, therefore, have

average P.E. of the particle

$$= \frac{1}{4T} mw^2a^2 [\, t \,]_0^T - 0$$

$$= \frac{1}{4T} mw^2a^2T$$

$$= \frac{1}{4} mw^2a^2$$

$$= \frac{1}{4} ka^2 \qquad [\because mw^2 = K] \qquad (1.18)$$

The kinetic energy (K.E.) of the particle at displacement y is given by

$$= \frac{1}{2} m \left(\frac{dy}{dt}\right)^2$$

$$= \frac{1}{2} m \left[\frac{d}{dt} a \sin (wt + \phi)\right]^2$$

$$= \frac{1}{2} m\, w^2a^2 \cos^2 (wt + \phi)$$

The average K.E. of the particle over a complete cycle or a whole time period T, as in the case of P.E., is given by

$$\frac{1}{T} \int_0^T \frac{1}{2} m\, w^2\, a^2 \cos^2 (wt + \phi)\, dt$$

$$= \frac{m w^2 a^2}{4T} \int_0^T 2 \cos^2 (wt + \phi)\, dt$$

$$= \frac{m w^2 a^2}{4T} \int_0^T [1 + \cos 2 (wt + \phi)]\, dt$$

$$= \frac{m w^2 a^2}{4T} \left[\int_0^T dt + \int_0^T \cos 2 (wt + \phi)\, dt\right]$$

Again, the average value of a sine or cosine function over a complete cycle or a whole time period is zero. Hence

average K.E. of the particle

$$= \frac{m w^2 a^2}{4T} [\, t \,]_0^T$$

$$= \frac{m w^2 a^2}{4T} . T$$

$$= \frac{1}{4} m w^2 a^2 = \frac{1}{4} ka^2. \qquad (1.19)$$

Thus,

average value of P.E. of the particle

= average value of K.E. of the particle

$$= \frac{1}{4} m w^2 a^2 = \frac{1}{4} ka^2$$

= half the total energy.

Example 1.11. *(i) What is the mechanical (total) energy of the oscillating system of Example 1.1 ?*

Soln.

Total energy = P.E. + K.E. $\frac{1}{4} ka^2$

Now k = 200N/m and a = 0.04m [Ex. 1.1 (i) and (iv)]

$\therefore$ Total energy $= \frac{1}{2}(200)(0.04)^2 = \mathbf{0.16\ joules}$

Also total energy,

$$\frac{1}{4} ka^2 = 2\pi^2 m\, a^2 n^2$$

From Ex. 1.1

a = 0.04 m

m = 2 Kg

n = 159 Hz

$$= 2\,(3.14)^2\,(2)\,(0.04)^2\,(159)$$

$$= \mathbf{0.16\ joules}$$

(ii) Compute the velocity, the acceleration and the kinetic and potential energies of the body when it has moved in half-way from its initial position toward the centre of motion.

$$v = \pm w \sqrt{a^2 - y^2}$$

At half-way,

$$y = \frac{a}{2} = \frac{0.04}{2}\,m = 0.02m$$

$$w = 10\ sec^{-1}$$

$$= \pm (10\ sec^{-1})\sqrt{(0.04m)^2 - (0.02m)^2}$$

$$= \pm \frac{2\sqrt{3}}{10}\ m/sec^{-1}$$

$$= \pm\ \mathbf{0.346\ m\ sec^{-1}}$$

$$\text{acceleration} = -w^2 y = -(10\ sec^{-1}) . (0.02\ m)$$

$$= \mathbf{-2.0\ m\ sec^{-2}}$$

$$K.E. = \frac{1}{2} mv^2 = \frac{1}{2}(2)(0.346)^2 \simeq \mathbf{0.12\ joules}$$

$$P.E. = \frac{1}{2} ky^2 = \frac{1}{2}(200)(0.02)^2 \simeq \mathbf{0.04\ joules}$$

$\therefore$ Total energy = P.E. + K.E. = (0.04 + 0.12) joules

$$= \mathbf{0.16\ joules.}$$

Note that total energy is constant.

Example 1.12. *A block whose mass is 680 gm is at rest on a table top and is fastened to an anchored horizontal spring. The spring constant of the spring is 65 N/m. There is negligible friction between the block and the table top. The block is pulled a distance x = 11 cm from its equilibrium position at x = 0 and released from rest at t = 0.*

(i) What force does the spring exert on the block just before the block is released ?

from Hooke's law

$$F = -ky = -(65\ N/m)(0.11\ m)$$

$$= \mathbf{7.2\ N}$$

minus sign simply indicates that force and displacement are oppositely directed.

(ii) What are the angular frequency, the frequency, and the period of the resulting oscillation ?

$$\text{angular frequency, } w = \sqrt{\frac{k}{m}} = \sqrt{\frac{65\ N/m}{0.68\ Kg}} = 9.78\ rad/s.$$

$$\text{frequency, } n = \frac{w}{2\pi} = \frac{9.78\ N/m}{2 \times 3.14} \simeq \mathbf{1.56\ Hz}$$

$$\text{and the period, } T = \frac{1}{n} = \frac{1}{1.56\ Hz} = \mathbf{0.64\ s}$$

(iii) What is the amplitude of the oscillation ?

since the block is released from rest 11 cm from its equilibrium position, its kinetic energy will be zero whenever it is again 11 cm from that position.

Hence its maximum displacement, *i.e.*, amplitude is zero.

or, a = 11 cm = 0.11 m.

(iv) What are the maximum velocity and acceleration of the oscillating block ?

$$v_{max} = wa = (9.78\ rad/s)(0.11\ m) = \mathbf{1.1\ m/s.}$$

$$a_{max} = -w^2 a = (9.78\ rad/s)^2 (0.11\ m) = \mathbf{-11\ m/s^2.}$$

(v) What is the phase constant φ for the motion ?

At t = 0 *i.e.*, the moment of release, the displacement of the block has its maximum value. Hence the equation for the displacement is

$$y = a\cos(wt + \varphi) \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

and the expression for velocity

$$\frac{dy}{dt} = -wa\sin(wt + \varphi) \quad \ldots\ldots\ldots\ldots \quad (2)$$

At $t = 0$ $y = a$. So from eqn. (1) we have

$$1 = \cos\varphi \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

Again at $t = 0$, $\frac{dy}{dt} = 0$. So from eqn (2) we get

$$0 = \sin\varphi \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

The smallest angle that satisfies both these requirements is $\varphi = 0$.

(Any angle that is an integral multiple of 2π also satisfies the requirement)

(vi) What is the mechanical energy of the oscillating system?

$$E = \frac{1}{2}ka^2 = \left(\frac{1}{2}\right)(65\ \text{N/m})(0.11\ \text{m})^2$$

$$= \mathbf{0.393\ J.}$$

(vii) What are the potential and kinetic energies of the oscillator when $y = \frac{dy}{dt}$?

$$\text{P.E.} = \frac{1}{2}ky^2 = \frac{1}{2}k\left(\frac{a}{2}\right)^2 = \frac{1}{4}\left(\frac{1}{2}ka^2\right)$$

$$= \left(\frac{1}{4}\right)(0.393\ \text{J}) = 0.098\ \text{J}.$$

Since the total energy remains constant,

$$\text{K. E.} = \text{E} - \text{P. E.} = 0.393\ \text{J} - 0.098\ \text{J}$$

$$\simeq 0.30\ \text{J}.$$

Thus when the oscillator is half-way to its end point, 25% of the energy is potential and 75% is Kinetic.

Example 1.13. *A 0.2 kg mass suspended from a spring describes a simple harmonic motion with a period T of 3 s and amplitude R of 10 cm. At t = 0 the mass passed upward through the equilibrium position. (i) Find the force constant k of the spring (ii) Find the displacement, velocity and acceleration of the mass when t = 1 s. (iii) Also show that at t = 1, the sum of the potential and kinetic energy is equal to* $\frac{1}{2}$ *ka ?*

Soln.

(i) $w = \sqrt{\frac{k}{m}}$ or, $w^2 = \frac{k}{m}$

$$\therefore \quad (2\pi n)^2 = \frac{k}{m} \; ; \; \left(\frac{2\pi}{T}\right)^2 = \frac{k}{m}$$

$$\frac{4\pi^2}{T^2} = \frac{k}{m}; \quad \text{or, } k = \frac{4\pi^2}{T^2} \times m = \frac{4\pi^2}{3^2} \times (0.2)$$

$$= \mathbf{0.88\ N/m.}$$

(ii) Let us consider the upward direction as the positive direction. Then from

$y = a\sin wt$, we have

at $t = 0$, $y = 0$ and $v > 0$.

At $t = 1$ s,

$$y = 0.10\sin\left[\left(\frac{2\pi}{3}\right)(1)\right] \qquad w = \frac{2\pi}{T}$$

$$= 0.10\sin 120^\circ$$

$$= \mathbf{0.0866\ m.}$$

$$v = \frac{dy}{dt} = wa\cos wt$$

$$= \left(\frac{2\pi}{T}\right)(a)\cos\left[\left(\frac{2\pi}{T}\right)(1)\right]$$

$$= \left(\frac{2\pi}{3}\right)(0.10)\cos 120°$$

$$= -0.105 \text{ m/s}.$$

$$\text{acc ln.} = \frac{d^2y}{dt^2} = -w^2 a \sin wt$$

$$= -\left(\frac{2\pi}{T}\right)^2 (a) \sin\left[\left(\frac{2\pi}{T}\right)(1)\right]$$

$$= -\left(\frac{2\pi}{3}\right)^2 (0.10) \sin 120°$$

$$= \mathbf{-0.38 \ m/s^2}.$$

(iii) E = P. E. + K. E.

$$\frac{1}{2}ka^2 = \frac{1}{2}ky^2 + \frac{1}{2}mv^2$$

$$\left(\frac{1}{2}\right)(0.88)(0.1)^2 = \left(\frac{1}{2}\right)(0.88)(0.0866)^2 + \left(\frac{1}{2}\right)(0.2)(0.105)^2$$

$$4.4 \times 10^{-3} \text{ J} = (3.3 \times 10^{-3} + 1.1 \times 10^{-3}) \text{ J}$$

$$= \mathbf{4.4 \times 10^{-3} \ J}.$$

1.7 Some examples of simple harmonic motion

Some important examples of simple harmonic motion will be examined below.

(i) Motion of a body suspended from a coil spring

Fig. 1.5 shows a coil spring whose upper end is fixed to a rigid support. A mass m is attached to its free end. Let l be the no-load length of the spring as shown in (a). When the load m is attached to the spring, it hangs in equilibrium with the spring extended by an amount Δl as in (b). Under this condition the upward force F exerted by the spring is equal to the weight of the body, *mg*. If the spring obey's Hooke's law (Art. 8.6), then the force on the body is given by

$$F = -k.\Delta l$$

where k is the force constant of the spring and is referred to as the *spring constant*. Again, the minus sing simply indicates that the force and displacement are oppositely directed. Since F = m g, ignoring the minus sign, we have

$$k\Delta l = mg\,; \quad \text{or, } k = \frac{mg}{\Delta l} \qquad (1.20)$$

Thus spring constant may be defined as the tension (mg) per unit displacement (Δl).

If the body is now displaced from its equilibrium position and released, it will oscillate along the vertical direction. Suppose the body is at a distance y above its equilibrium position as in (c) Then the extension of the spring is (Δl – y); the upward force it exerts on

= K(Δl - y)

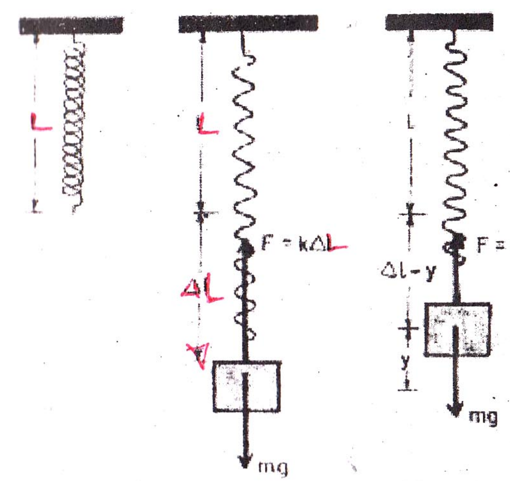


Fig. 1.5

the body is k (Δl – y) and the downward force acting on the body is mg. Hence the resultant force F on the body is

$$F = k(\Delta l - y) - mg = -ky \qquad (\because k\Delta l = mg)$$

The resultant force is, therefore, proportional to the displacement of the body from its equilibrium position and is oppositely directed.

∴ F = -ky

According to Newton's second law of motion, F = ma

$$F = m\frac{d^2y}{dt^2} = -ky$$

$$\text{or,}\quad m\frac{d^2y}{dt^2} + ky = 0$$

$$\text{or,}\quad m\frac{d^2y}{dt^2} + \left(\frac{k}{m}\right).\, y = 0 \qquad (1.21)$$

or, $m\frac{d^2y}{dt^2} + \omega^2 y = 0$ $\quad \omega^2 = \frac{k}{m}$

The equation is similar to the differential equation of simple harmonic motion

$$\frac{d^2y}{dt^2} + w^2y = 0$$

Hence the motion of a mass suspended from a coil spring is simple harmonic.

Comparing eqns. (1.2) and (1.21)

$$w^2 = \frac{k}{m}\ ;\ \text{or,}\ w = \sqrt{\frac{k}{m}}$$

The time period of oscillation is

$$T = \frac{2\pi}{w} = 2\pi\sqrt{\frac{m}{k}}$$

Thus the body oscillates simple harmonically with angular frequency $w = \sqrt{k/m}$ or a time period $T = 2\pi\sqrt{m/k}$. Measuring the time period, the spring constant k can be determined. The constant k depends on the shear modulus (n) of the wire, its radius (r), the radius of the coil (R) and the number of turns (N) in the coil and is given by $n = \frac{4NR^3k}{r^4}$.

Alternately, k can also be determined from the relation $k = mg/\Delta l$ where Δl is the increase in length of the spring when a small mass m is attached to its free end (eqn. 1.20). It is to be noted that $mg/\Delta l$ is constant for a given spring provided it obeys Hooke's law.

(ii) The simple pendulum GA

An important example of periodic motion is that of a simple pendulum. *A simple pendulum consists of a point mass suspended by an inextensible weightless string in a uniform gravitational field.* When pulled to one side of its equilibrium position and released, the bob of the pendulum vibrates about the position of equilibrium. We would like to investigate whether the motion is simple harmonic.

Let the mass of the bob be m and the length of the string be l. The path of the bob is not a straight line, but the arc of a circle of radius l, l being the length of the supporting string. The co-ordinate s refers to distances measured along this arc. The necessary condition for the motion to be simple harmonic is that the *restoring force F shall be directly proportional to the co-ordinate s and oppositely directed i.e.,*

$F = -ks$ where k is the force constant.

Fig. 1.6 shows the forces on the bob at an instant when its coordinate is s. The weight is resolved into components along the radius and along the tangent to the circle. The restoring force F is

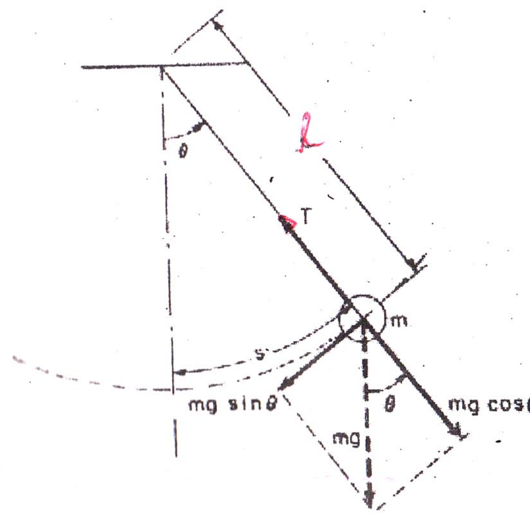


Fig. 1.6

$$F = -mg\sin\theta \qquad (1.22)$$

The restoring force is, therefore, proportional not to θ but to sin θ. Hence the motion is not simple harmonic. However, if the angle θ is small, $\sin\theta$ is very nearly equal to θ. For example, when $\theta = 0.1$ radian (about 6°), $\sin\theta = 0.0998$, a difference of only 0.2 per cent. With this approximation eqn. (1.22) becomes

$$F = -mg\,\theta$$

$$\text{But } \theta = \frac{s}{l}$$

$$\therefore \quad F = -mg\frac{s}{l} = -\frac{mg}{l}\cdot s \qquad (1.23)$$

Thus, for small displacements, the restoring force is proportional to s, and the constant mg/l represents the force constant k.

Now $F = m.a$ where a is the tangential acceleration $\frac{d^2s}{dt^2}$. Eqn. (1.23) then becomes

$$m.\frac{d^2s}{dt^2} = -\frac{mg}{l}.s$$

$$\text{or,} \quad \frac{d^2s}{dt^2} = -\frac{g}{l}.s$$

$$\text{or,} \quad \frac{d^2s}{dt^2} + \frac{g}{l}.s = 0 \qquad (1.24)$$

Eqn. (1.24) is similar to the equation $\frac{d^2y}{dt^2} + w^2y = 0$, the familiar form of the differential equation of simple harmonic motion.

Thus, provided the amplitude of oscillation is small, the motion of a simple pendulum is simple harmonic. The angular frequency of oscillation is $w = \sqrt{\frac{g}{l}}$. The time period of oscillation of the simple pendulum when its amplitude is small is, therefore,

$$T = \frac{2\pi}{w} = 2\pi\sqrt{\frac{l}{g}} \qquad (1.25)$$

It may be noted that the above expressions do not contain the mass

of the particle; this is because *the restoring force which is a component of the particle's weight is proportional to m*. Hence mass appears on both sides of $F = ma$ and is cancelled out. For small oscillations the period of a pendulum for a given value of g is determined entirely by its length.

It should, however, be emphasized that the motion of pendulum is only *approximately* simple harmonic, and when the amplitude is not small, the departures from simple harmonic motion can be substantial. *What then is a small amplitude ?*

It can be shown that the general equation for the time of swing, when the maximum angular displacement is θ, is

$$T = 2\pi\sqrt{\frac{l}{g}}\left[1 + \frac{2}{2^2}\sin^2\frac{\theta}{2} + \frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\sin^4\frac{\theta}{2} + \ldots\ldots\right]$$

Thus the period of oscillation may be computed to any desired degree of precision by taking enough terms in the infinite series. When θ = 15° on either side of the central position, the true period differs from the approximate period as given by eqn. (1.25), by less than 0.5 per cent. Fig. 1.7 shows a plot of θ and $\sin\theta$. As can be seen, even for a rather large angle $\theta = \frac{\pi}{6}$ (= 0.523) radian which corresponds to 30°, $\sin\theta = \sin 30° = 0.500$; the difference between θ and $\sin\theta$ is only 4.6 per cent.

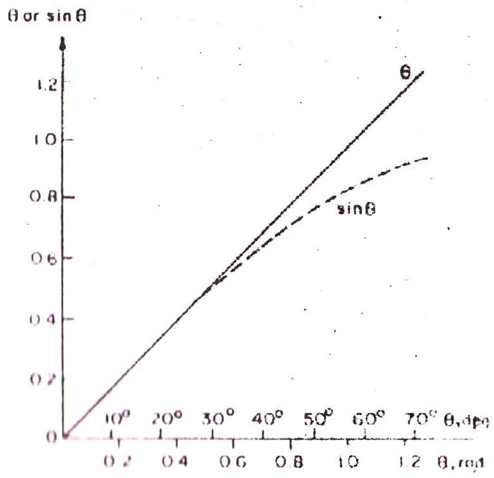


Fig. 1.7

The utility of the pendulum as a time keeper is based on the fact that the period is practically independent of the amplitude. Thus, as a clock runs down and the amplitude of the swings becomes slightly smaller, the clock will keep very nearly correct time. However, for accurate time keeping, the amplitude of the swing must be kept constant despite the frictional losses that affect all mechanical systems. Even so small a change in amplitude as from 5° to 4° would cause a pendulum clock to run fast by 0.25 minute per day, an unacceptable amount even for household time-keeping. To keep the amplitude constant in a pendulum clock, energy is automatically supplied in small increments from a weight or a spring by an escapement mechanism to compensate for friction losses. The pendulum clock with escapement was invented by Christiaan Huygens.

The simple pendulum was used for early determinations of the value of acceleration due to gravity because both the period and the length are easily measured. Direct measurement by observation of free fall, in contract, is difficult because the time of fall over reasonable distance is too short for easy measurement. From eqn. (1.25), we have, in terms of l and T, $g = \frac{4\pi^2 l}{T}$.

(iii) The Torsional osciliator (Torsion Pendulum)

Fig. 1.8 shows a disk suspended by a wire or a shaft. One end of the wire is securely fixed to a solid support or a clamp while the other end is fixed to the centre of mass of the disk. With the disk in equilibrium, a radial line is drawn from its centre O to a point P on its rim. If the disk is rotated in a horizontal plane so that the reference line OP moves to a position OQ, the wire will be twisted. The twisted wire will exert a restoring torque on the disk tending to return the reference line to its equilibrium position. Thus if the disk is given a small twist and released, it will execute angular oscillations about its position of equilibrium. The device is then called a *torsion pendulum*, with torsion referring to the twisting.

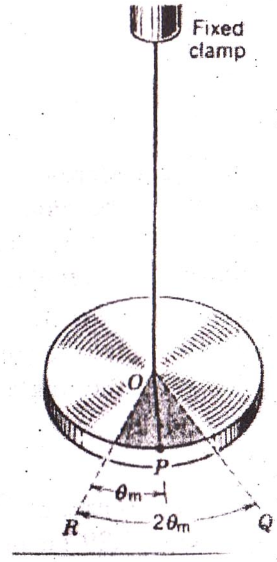


Fig. 1.8

For small twists the restoring torque is found to be proportional to the angular displacement (Hooke's law), so that

$$\tau = -k\theta \tag{1.26}$$

where k is a constant that depends on the properties of the wire and is called the *torsional constant*. The minus sign shows that the torque is directed opposite to the angular displacement θ. Eqn. 1.26 is a condition of *angular simple harmonic motion*. The equation of motion for such a system is based on the angular form of Newton's second law,

$$\tau = I\,\alpha = I\,\frac{d^2\theta}{dt^2}$$

where I is the rotational inertia of the oscillating disk.

Combining eqns. 1.26 and 1.27 we obtain

$$-K\theta = I\frac{d^2\theta}{dt^2}$$

$$\text{or,}\quad \frac{d^2\theta}{dt^2} = -\left(\frac{\kappa}{I}\right)\theta$$

$$\text{or,}\quad \frac{d^2\theta}{dt^2} + \left(\frac{\kappa}{I}\right)\theta = 0 \qquad (1.28)$$

Eqn. 1.28 is similar to the differential equation of simple harmonic motion (eqn. 1.2). Hence the angular oscillations of the torsion pendulum is simple harmonic. We can simply substitute angular displacement θ for linear displacement y, rotational inertia I for mass m, and torsional constant κ for force constant k. Making these substitutions, it can be shown that the solution of eqn. 1.28 to be a simple harmonic oscillation in the angle coordinate θ and is given by

$$\theta = \theta_m \sin(wt + \varphi) \qquad (1.29)$$

Here θ_m is the maximum angular displacement, that is, the amplitude of the angular oscillation. w in eqn. 1.29 means angular frequency, not angular velocity. In eqn. 1.29 $w \neq d\theta/dt$.

In Fig. 1.8 the disk oscillates about the equilibrium position $\theta = 0$, the total angular range being $2\theta_m$ (from OQ to OR). By analogy with eqn. 1.12, the period of oscillation is given by

$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$

If κ is known and T is measured, the rotational inertia I about the axis of rotation of any oscillating rigid body can be determined. If I is known and T is measured, the torsional constant κ of any sample of wire can be determined.

Example 1.14. *A particle performs simple harmonic motion given by the equation.*

$$y = 20\sin(wt + \alpha)$$

If the time period is 30 seconds and the particle has a displacement of 10 cm at t = 0, find (i) epoch, (ii) the phase angle at t = 5 seconds and (iii) the phase difference between two positions of the particle 15 seconds apart.

Soln.

Here

$$y = 20\sin(wt + \alpha)$$

$$T = 30 \text{ secs.}$$

$$\therefore\quad w = \frac{2\pi}{T} = \frac{2\pi}{30} = \frac{\pi}{15}\ rad/sec.$$

(i) at t = 0, y = 10 cm.

$$\therefore\quad 10 = 20\sin\left(\frac{\pi}{15} \times 0 + \alpha\right)$$

$$\text{or,}\quad \sin\alpha = \frac{10}{20} = 0.5$$

$$\text{or,}\quad \alpha = \frac{\pi}{6}\ \textbf{radian.}$$

(ii) at t = 5 sec,

the phase angle = $(wt + \alpha)$

$$= \left(\frac{\pi}{15} \times 5 + \frac{\pi}{6}\right)$$

$$= \frac{\pi}{2}$$

(iii) at t = 0 the phase angle

$$\theta_1 = \left(\frac{\pi}{15} \times 0 + \frac{\pi}{6}\right) = \frac{\pi}{6}\ \text{radian.}$$

at $t = 15$ *sec*, the phase angle

$$\theta_2 = \left(\frac{\pi}{15} \times 15 + \frac{\pi}{6}\right)$$

$$= \frac{7\pi}{6}\ \text{radian.}$$

$\therefore$ the phase difference,

$$\theta_2 - \theta_1 = \frac{7\pi}{6} - \frac{\pi}{6} = \pi\ \textbf{radian.}$$

Example. 1.15. *A vertically suspended spring of negligible mass and force constant R is stretched by an amount l when a body of mass m is hung on it. The body is pulled by hand an additional distance y (positive direction downward) and then released.*

(a) Show that the motion of the body is governed by $a' = -\dfrac{k}{m}y$, *so that the body executes harmonic motion about its equilibrium position, and*

(b) show that the period of this motion is the same as that of a simple pendulum of length l.

Soln.

When the mass m is attached to the spring it hangs in equilibrium with the spring extended by an amount l. From Hooke's law, the upward force exerted by the spring on the body is

$F = kl$ where k is the force constant of the spring.

Under the condition of equilibrium we therefore have $mg = kl$ where mg is the downward force on the spring.

When the spring is stretched by an additional length y and released, the net force acting on the spring is

$$mg - k(l + y)$$

According to Newton's second law of motion,

$$F = ma' = mg - k(l + y)$$

$$= mg - kl - ky = -ky \qquad (mg = kl)$$

$$\therefore \quad a' = -\frac{k}{m}y.$$

(b) The period of oscillation of the spring is given by

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/l}} \qquad (\text{from } mg = kl)$$

$$= 2\pi\sqrt{\frac{l}{g}}$$

The above expression is the period of oscillation of a simple pendulum of length l placed in uniform gravitational field g.

Example 1.16. *A thin rod of length L = 12.4 cm and mass m = 135 g is suspended at its midpoint from a long wire. Its period* T_a *of angular simple harmonic motion is measured to be 2.53 s. An irregular object X is then hung from the same wire and its period* T_b *of the angular SHM is found to be 4.76 s.*

(i) What is the rotational inertia of object X about its suspension axis?

The rotational inertia of the thin rod about a perpendicular axis through its midpoint is given by $\frac{1}{12}$ m L^2. Thus we have

$$I_a = \frac{1}{12}mL^2 = \left(\frac{1}{12}\right)(0.135\ \text{kg})(0.124\ \text{m})^2$$

$$= 1.73 \times 10^{-4}\ \text{kg.m}^2$$

Now

$$T_a = 2\pi\sqrt{\frac{I_a}{\kappa}} \quad \text{and} \quad T_b = 2\pi\sqrt{\frac{I_b}{\kappa}}$$

The torsion constant κ, which is a property of the wire, is the same for both the objects. Only the periods and rotational inertias are different.

$$\frac{T_b}{T_a} = \sqrt{\frac{I_b}{I_a}} \quad \text{or,} \quad \frac{T_b^2}{T_a^2} = \frac{I_b}{I_a}$$

$$\text{or,} \quad I_b = \frac{T_b^2}{T_a^2}.I_a = \frac{(4.76\ \text{s})^2}{(2.53\ \text{s})^2}\left(1.73 \times 10^{-4}\ \text{kg.m}^2\right)$$

$$= \mathbf{6.12 \times 10^{-4}\ kg.m^2}$$

(ii) If both the objects are fastened together and hung from the same wire, what would be the period of oscillation ?

$$T_a = 2\pi\sqrt{\frac{I_a}{\kappa}} \quad \text{and} \quad T_C = 2\pi\sqrt{\frac{I_C}{\kappa}}$$

$$\text{or,}\quad = \frac{T_C}{T_a} = \sqrt{\frac{I_C}{I_a}} \qquad \therefore\quad T_c = T_a\sqrt{\frac{I_c}{I_a}} = T_a\sqrt{\frac{I_a + I_b}{I_a}}$$

$$\text{or,}\quad T_C = T_a\sqrt{\frac{I_C}{I_a}} \qquad = Ta\sqrt{1+\frac{I_b}{I_a}} = (2.53)\sqrt{1+\frac{6.12\times10^{-4}\,Kg.m^2}{1.73\times10^{-4}\,Kg.m^2}}$$

$$= 5.37\ s.$$

Now $I_c = I_a + I_b$

(iv) LC circuit

Just as harmonic oscillations in mechanical system, so also we come across harmonic oscillations in electrical systems. Under suitable conditions, charge, current or voltage can execute simple harmonic motion. As an example let us consider the case of an *LC* circuit.

In an LC circuit, a capacitor of capacitance C and an inductance coil of inductance L (of negligible ohmic resistance) are connected with a battery through a Morse key K in the manner as shown in Fig. 1.9. When the key is pressed, the capacitor gets directly connected to the battery and thus gets charged. On being released, the key gets disconnected to the inductance coil. The capacitance thus discharges itself through the inductance coil.

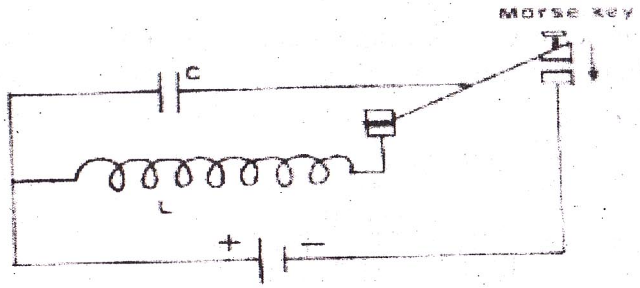


Fig. 1.9

An inductive circuit, *i.e.*, a circuit containing an inductance coil, opposes both growth and decay of current in the circuit. Thus as the capacitor discharges itself through the inductance coil and the current in the latter grows, the magnetic flux due to the current increases. This increasing magnetic flux induces an electro-motive

force (*emf*) in the circuit which opposes the growth of current in it. Thus, if *i* be the instantaneous value of the current in the coil (or the circuit) at any given instant, then the opposing *emf* set up across the coil is $-L\frac{di}{dt}$ where $\frac{di}{dt}$ is the rate of change of current through the coil or the circuit.

If Q be the charge on the capacitor at the instant considered, then the voltage across it tending to drive the current through the circuit (or the coil) is $\frac{Q}{C}$. Since there is no external *emf* in the circuit (the battery being cut off), the net *emf* in the circuit is

$$\frac{Q}{C} + L\frac{di}{dt} = 0 \tag{1.27}$$

But current is the rate of flow of charge ;

hence $$i = \frac{dQ}{dt}$$

Eqn. (1.27) thus takes the form

$$\frac{Q}{C} + L\frac{d^2Q}{dt^2} = 0$$

$$\text{or,}\quad \frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0$$

$$\text{or,}\quad \frac{d^2Q}{dt^2} + w^2Q = 0 \tag{1.28}$$

where $w^2 = \frac{1}{LC}$

or, $\frac{1}{\sqrt{LC}}$ *is a constant t.*

Eqn. (1.28) is similar to eqn. (1.2), the differential equation of simple harmonic motion with *y* replaced by *Q*, mass (*m*) by inductance *L* and the force constant by the reciprocal of the capacitance ($\frac{1}{C}$).

Thus the charge on the capacitor oscillates simple harmonically with time, *i.e.*, the discharge of the capacitor is oscillatory in nature, its time period T is given by

$$T = \frac{2\pi}{w} = 2\pi\sqrt{LC}$$

and, therefore, its frequency

$$n = \frac{1}{T} = \frac{1}{2\pi\sqrt{LC}} \qquad (1.29)$$

The solution of eqn. (1.28) is given by

$$Q = Q_o \sin(wt + \phi) \qquad (1.30)$$

where Q_o is the maximum value or amplitude of the charge and $w = \frac{1}{\sqrt{LC}}$, the angular frequency (of the variation of charge) and ϕ, the phase constant which depends, as usual, on the initial conditions.

The charge in the circuit, therefore, oscillates between $+Q_o$ and $-Q_o$ with a frequency $n = \frac{1}{2\pi\sqrt{LC}}$

Differentiating eqn. (1.29) with respect to time, we have

$$i = \frac{dQ}{dt} = Q_0\, w \cos(wt + \phi)$$

where $i = \frac{dQ}{dt}$ is the instantaneous value of the current, $Q_0 w$ is the maximum value or the amplitude of current when

$$\cos(wt + \phi) = 1.$$

Denoting this maximum value by i_0, we get

$$i = i_0 \cos(wt + \phi) \qquad (1.30)$$

Thus current in the circuit is also oscillatory in character and has the same frequency as the charge, *viz.* $n = \frac{1}{2\pi\sqrt{LC}}$.

1.8 Simple harmonic motion and uniform circular motion

Let us consider a particle P moving along the circumference of a circle with a constant angular speed w radians per seconds. Q is the perpendicular projection of P on the horizontal diameter *i.e.*,

along the x-axis while R is the perpendicular projection of P on the vertical diameter *i.e.*, along the y-axis. Let us consider the moment when P crosses the position X at time $t = 0$ (Fig. 1.10). At this instant the displacement of P from the centre O measured along the horizontal direction is equal to radius a of the circle while that measured along the vertical direction is zero. When it arrives at P after, say, an interval of time t, its respective displacements along the x-and y-axes are $a\cos wt$ and $a\sin wt$. On arrival at Y, the horizontal displacement vanishes but the vertical displacement attains its maximum value namely a. In the second quadrant of the

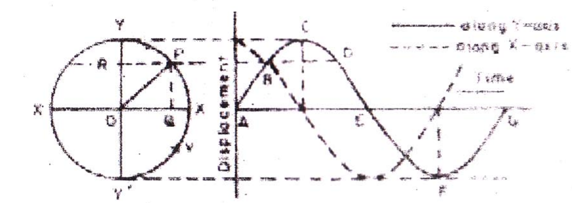


Fig. 1.10

circle, the vertical displacement gradually decreases, becoming zero as the particle arrives at X'. But, at the same time, the horizontal displacement increases attaining the maximum value a at X'. However, the maximum value a attained this time is in the negative x-direction. Thus as the particle completes one half-cycle from X to X', the displacement measured along the vertical diameter (y-axis), starts from zero, rises to a maximum ($= a$) and then falls back to zero. In the next half - cycle (from X' to X) the vertical displacement again increases (though in the negative direction), becomes maximum ($= -a$) when the particle reaches Y' and starts decreasing till it becomes zero when the particle arrives back at X. At each instant, the displacement in the y-direction as measured from the central point O is given by

$$y = a \sin wt \qquad (1.31)$$

where t is measured from the instant when the particle crosses the point X. If y is plotted against t, the sine curve given in Fig. 1.10 is obtained.

Similarly, the displacement along the x-axis at each instant is given by

$$x = a \cos wt \qquad (1.32)$$

The dotted line in Fig. (1.10) represents a plot of x vs. t. As can be seen, the cosine curve can be obtained by displacing the sine curve to the left by an interval $T/4$ where T is the time required by the particle to complete one revolution along the circumference of the circle.

If the particle is already ahead of X when we start counting time, *i.e.*, if the radius OP has already made an angle δ with the x-axis at $t = 0$, then the angle between OP and the x-axis at any subsequent time t is $(wt + \delta)$.

Eqns. (1.31) and (1.32) will then assume the forms

$$y = a \sin (wt + \delta) \qquad (1.33)$$

and $x = a \cos (wt + \delta)$ see Art. (1.3) (1.34)

Thus as P moves along the circumference of the circle, its projections along x - and y - directions execute a to and fro (vibratory) motion along the two axes about the central point O. Moreover, by differentiating eqn. (1.31) twice with respect to time we find that.

$$\frac{dy}{dt} = aw \cos (wt + \delta)$$

and $$\frac{d^2y}{dt^2} = -aw^2 \sin (wt + \delta) = -w^2 y$$

or, acceleration $= -w^2 y$.

similarly the acceleration along the x-axis is given by $-w^2x$.

Thus the acceleration of the projection whether along the x - or along the y - axis is directly proportional to the instantaneous displacement of the projection and is oppositely directed to this displacement. Thus motion of the projection either along the x- or along the y- axis satisfies the necessary condition of simple harmonic motion.

Thus the motion of the projection of uniform circular motion on a diameter of the circle in which the circular motion occurs is simple harmonic.

Conversely, the uniform motion of a particle along the circumference of a circle can be described as a combination of two simple harmonic motions. It is the combination of those simple harmonic motions which occur along perpendicular lines and have the same amplitude and

frequency but differ in phase by 90°. *When one component is at the maximum displacement, the other component is at the equilibrium point. Combining eqns. (1.33) and (1.34) we at once obtain the relation*

$$r = \sqrt{x^2 + y^2} = a$$

The point P is called the reference point and the circle on which it moves is called the reference circle. The angular frequency w of simple harmonic motion is the same as the angular speed of the reference point. The frequency of simple harmonic motion is the number of revolutions per unit time of the reference point. Hence, $n = \dfrac{w}{2\pi}$ or, $w = 2\pi n$. The time for a complete revolution of the reference point is the same as the period T of the simple harmonic motion. Hence, $T = \dfrac{2\pi}{w}$ or, $w = \dfrac{2\pi}{T}$. $(wt + \delta)$ in eqn. (1.33) or (1.34) is called the phase of the simple harmonic motion while δ is called the epoch or initial phase of the motion. The amplitude of the simple harmonic motion is the same as the radius of the reference circle.

Two-body oscillations

Many objects on the microscopic level, such as molecules, atoms, nuclei, execute oscillations that are approximately simple harmonic. One example is a diatomic molecule, in which the two atoms are bound together with a force. If displaced a small distance from its equilibrium position, the molecule will oscillate about the equilibrium position. Let us suppose that the molecule can be represented by two particles of masses m_1 and m_2 connected by a spring of force constant k, as shown in Fig. 1.11.

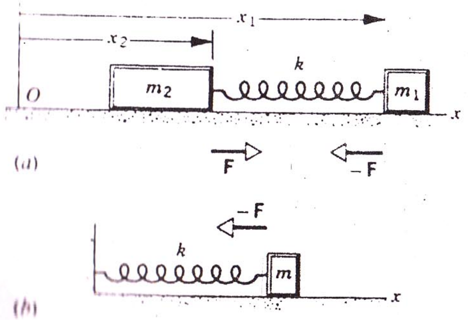


Fig. 1.11

The motion of the system can be described in terms of the separate motions of the two particles, which are located relative to the origin O by the two coordinates x_1 and x_2, as shown in Fig. 1.11 a. As we shall see, this leads to a different and often more useful description, which is given in terms of the *relative* separation and velocity of the two particles. In effect, the two coordinates x_1 and x_2 are replaced with two other coordinates: the relative separation x_1-x_2 and the location x_{cm} of the centre of mass. But, in the absence of external forces, the centre of mass moves at constant velocity and as such, its motion is of no real interest in studying the oscillation of the system. So the system can be analyzed in terms of the relative coordinate alone.

The relative separation $x_1 - x_2$ gives the length of the spring at any time. If the unstretched length of the spring is L, then the change in length of the spring is given by $x = (x_1 - x_2) - L$. The magnitude of the force that the spring exerts on *each particle* is $F = Kx$. As can be seen in Fig. 1.11 a, if the spring exerts a force $-\boldsymbol{F}$ on m_1, then it exerts a force $+\boldsymbol{F}$ on m_2.

Taking the force component along the x axis, let us apply Newton's second law separately to the two particles,

$$m_1 \frac{d^2x_1}{dt^2} = -kx$$

$$m_2 \frac{d^2x_2}{dt^2} = +kx$$

Multiplying the first of these equations by m_2 and the second by m_1, we get

$$m_1m_2 \frac{d^2x_1}{dt^2} = -m_2kx$$

$$m_1m_2 \frac{d^2x_2}{dt^2} = -m_1kx$$

Subtracting,

$$m_1m_2 \frac{d^2x_1}{dt^2} - m_1m_2 \frac{d^2x_2}{dt^2} = -m_2kx - m_1kx$$

$$= -kx\,(m_1 - m_2)$$

or,
$$\frac{m_1m_2}{m_1+m_2} \frac{d^2}{dt^2}(x_1 - x_2) = -kx \qquad (1.35)$$

The quantity $m_1m_2/(m_1 + m_2)$ has the dimension of mass and is known as the *reduced mass m* of the system.

The unstretched length L of the spring is a constant. Hence the derivatives of $(x_1 - x_2)$ are the same as the derivatives of x.

$$\frac{d}{dt}(x_1 - x_2) = \frac{d}{dt}(x + L) = \frac{dx}{dt}$$

Eqn. 1.35 therefore becomes

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

or,
$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad (1.36)$$

Eqn. 1.36 is identical to the differential equation of simple harmonic motion of a single oscillating mass. Hence, from the standpoint of oscillation, the system of Fig. 1.11 a can be replaced by a single particle, as represented by Fig. 1.11, whose mass is given by the reduced mass of the system. In particular, the frequency of oscillation of the system is given by

$$n = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

where m in the expression is the reduced mass of the system. It may be noted that the reduced mass is always smaller than either of the masses of the system. If one of the mass is very much smaller than the other, then the reduced mass is roughly equal to the smaller mass. If the masses are equal, then m is half as large as either mass.

Example 1.16. *A body of mass 5 kg is suspended by a spring, which stretches 0.1 m when the body is attached. It is then displaced downward an additional 0.05 m and released. Find the amplitude, period and frequency of the resulting simple harmonic motion.*

Soln.

Since the initial position is 0.05 m from equilibrium and there is no initial velocity, the amplitude.

a = 0.05 m.

A force of (5 kg) (9.8 m/sec^2) produces a displacement of 0.1 m. Hence from F = ky,

the force constant $k = \frac{F}{y} = \frac{mg}{y}$

$$= \frac{(5\,kg)(9.8 m/sec^2)}{0.1 m}$$

= 490 N/m.

the time period,

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{5\,kg}{490\,N/m}}$$

= 0.635 sec.

the frequency

$$n = \frac{1}{T} = \frac{1}{0.635} = \mathbf{1.57\ Hz.}$$

Example 1.17. *The scale of a spring balance reading from 0 to 10 kg is 0.25 m long. A body suspended from the balance is found to oscillate vertically with a frequency of 10/π hertz. Calculate the mass of the body attached to the spring.*

Soln.

Clearly, a mass of 10 kg suspended from the spring balance extends the spring by 0.25 m. Hence from F = ky, the force constant

$$k = \frac{F}{y} = \frac{10 \times 9.8}{0.25}\ N/m.$$

From

$$T = 2\pi\sqrt{\frac{m}{k}} \text{ we have}$$

$$T = \frac{1}{n} = 2\pi\sqrt{\frac{m}{k}} \qquad k = \frac{10 \times 9.8}{0.25}\ N/m.$$

$$or, \frac{1}{n^2} = 4\pi^2 . \frac{m}{k} \qquad n = \frac{10}{\pi}\ Hz.$$

$$or,\ m = \frac{k}{4\pi^2 . n^2} \qquad m = ?$$

$$\therefore \quad m = \frac{10 \times 9.8}{0.25 \times 4 \times (3.14)^2 \times \frac{100}{(3.14)^2}}$$

$$= \frac{98}{100}$$

$$= 0.98\ kg.$$

Example 1.18. *A body of mass 0.5 kg is suspended from a spring of negligible mass and it stretches the spring by 0.07 m. For a displacement of 0.03 m it has a downward velocity of 0.4 m/sec. Calculate (i) the time period, (ii) the frequency and (iii) the amplitude of vibration of the spring.*

Soln.

From F = ky, the force constant of the spring

$$k = \frac{F}{y} = \frac{0.5 \times 9.8}{0.07}$$

(i) the time period,

$$T = 2\pi\sqrt{\frac{m}{k}} \qquad m = 0.5\ kg, \quad k = \frac{0.5 \times 9.8}{0.07}$$

$$\therefore \quad T = 2 \times 3.14 \times \sqrt{\frac{0.5 \times 0.07}{0.5 \times 9.8}} = \mathbf{0.5307\ sec.}$$

(ii) the frequency,

$$n = \frac{1}{T} = \frac{1}{0.5309} = \mathbf{1.8843\ Hz.}$$

the angular frequency, $w = 2\pi n$

$= 2 \times 3.14 \times 1.8843$

$= \mathbf{11.833\ rad/sec.}$

(iii) For the amplitude, we have

$$v = w\sqrt{a^2 - y^2}$$

or, $0.4 = 11.383\sqrt{a^2 - (0.03)^2}$

or, $(0.4)^2 = (11.383)^2\left[a^2 - (0.03)^2\right]$

or, $\mathbf{a = 0.04526\ m.}$

Example 1.19. *A simple pendulum of length 100 cm has an energy equal to 2 x 10^6 ergs when its amplitude is 4 cm. Calculate its energy when (i) its length is doubled and (ii) its amplitude is doubled.*

Soln.

The energy of a simple pendulum

= its maximum P.E.

= its maximum K.E.

$$= \frac{1}{2}\ mw^2a^2$$

Now $w = \frac{2\pi}{T}$ where T is the time period of the pendulum $= 2\pi.\sqrt{\frac{l}{g}}$

or, $w = 2\pi.\frac{1}{2\pi}\sqrt{\frac{g}{l}} = \sqrt{\frac{g}{l}}$

Initially $l = 100$ cm. $\therefore w = \sqrt{g/100}.$

So, initially with a = 4 cm, the energy of the pendulum

$$E = \frac{1}{2}\ m.\left(\sqrt{g/100}\right)^2.\ (4)^2 = 8mg/100$$

$= 2 \times 10^6$ ergs.

(i) If the length of the pendulum is doubled, $l = 200$ cm. and $w = \sqrt{g/200}$. Hence energy of the pendulum

$$E' = \frac{1}{2}.m.\left(\sqrt{g/200}\right)^2.(4)^2$$

$= \mathbf{8mg/200}$

$$\therefore \frac{E'}{E} = \frac{8\ mg}{200} / \frac{8\ mg}{100} = \frac{1}{2}$$

or, $E' = \frac{1}{2}\ .E = \frac{1}{2} \times 2 \times 10^6$ ergs

$= \mathbf{10^6\ ergs.}$

(ii) If the amplitude is doubled, a = 8 cm, its length remaining 100 cm. Hence energy of the pendulum

$$E'' = \frac{1}{2}\ .\ m\ .\left(\sqrt{g/100}\right)^2.\ (8)^2$$

$= 32\ mg / 100$

$$\therefore \frac{E''}{E} = \frac{32\ mg}{100} / \frac{8\ mg}{100} = 4$$

or, $E'' = 4.E = 4 \times 2 \times 10^6$ ergs

$= 8 \times 10^6$ ergs.

Example 1.20. *A small body of mass 0.10 kg is undergoing simple harmonic motion of amplitude 1.0 metre and period 0.20 sec. (i) what is the maximum value of the force acting on it? (ii) if the oscillations are produced by a spring, what is the force constant of the spring?*

Soln.

(ii) From $T = 2\pi\sqrt{\frac{m}{k}}$, we have

$$T^2 = 4\pi^2\ .\ \frac{m}{k}$$

$T = 0.20$ sec

$m = 0.10$ kg.

$\therefore \quad (0.2)^2 = 4 \times (3.14)^2 \times \dfrac{0.10}{k}$

or, $k = \dfrac{4 \times 9.86 \times 0.10}{0.04}$

$= \dfrac{3.944}{0.04} \simeq$ **99 nt/m.**

force constant of the spring

$\simeq$ **99 nt/m.**

(i) force acting on the body is given by

$F = ky.$

Obviously F is maximum when y is maximum. The maximum value of y is its amplitude *i.e.*, 1.0 metre.

$\therefore$ maximum force $= k.y = 99 \times 1$

$=$ **99 nt.**

Example 1.21. *An oscillating mass-spring system has a mechanical (total) energy of 1.0 joule, amplitude of 0.10 metre and maximum speed of 1.0 m/sec. Find (i) the force constant of the spring, (ii) the mass, and (iii) the frequency of oscillation.*

(iii) $v_{max} = w.a$

or, $1.0 = w \times 0.1$

or, $w = \dfrac{1.0}{0.1} = 10$

Now $w = 2\pi n$ where n is the frequency of oscillation

$\therefore \quad 10 = 2 \times 3.14 \times n$

or, $n = \dfrac{10}{6.28}$

$=$ **1.6 cycle/sec.**

(ii) Mechanical (or total energy),

$2\pi^2 n^2 a^2 m = 1.0$ joule.

$2 \times 9.8 \times (1.6)^2 \times (0.1)^2 \times m$

$=$ **1.0 joule.**

or, $m = \dfrac{1.0}{0.5}$ kg $=$ **2 kg.**

(i) $T = \dfrac{1}{n} = \dfrac{1}{1.6} = 0.625$ sec.

But $T = 2\pi \sqrt{\dfrac{m}{k}}$

or, $T^2 = 4\pi^2 \dfrac{m}{k}$

or, $(0.625)^2 = \dfrac{4 \times 9.8 \times 2.0}{k}$

or, $k = \dfrac{78.4}{0.30} \simeq$ **200 nt/m.**

EXERCISES

[1] Define simple harmonic motion and discuss its characteristics.

[2] Establish the differential equation of simple harmonic motion and solve it to obtain an expression for the displacement of a particle executing simple harmonic motion.

[3] Show that for a body vibrating simple harmonically the time period is given by

$$T = 2\pi \sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

[4] Establish the differential equation of simple harmonic motion and show that the time period is equal to 2π times the square root of displacement per unit acceleration.

[5] Define simple harmonic motion. Prove that the motion of a simple pendulum is simple harmonic. Hence obtain an expression for the time period of a simple pendulum.

[6] Solve the differential equation $\dfrac{d^2y}{dt^2} + w^2 y = 0$ to obtain the expression $y = a \sin(wt + \phi)$ for the displacement of a particle executing simple harmonic motion.

[7] Explain simple harmonic motion. Obtain expressions for the frequency, amplitude, velocity and acceleration of a body executing simple harmonic motion.

[8] Show that the motion of a body suspended from a coil spring is simple harmonic.

[9] Show that a simple harmonic motion may be expressed as either a sine or a cosine function, there being only a difference of initial phase in the two cases.

[10] Show that for a body executing simple harmonic motion, mechanical energy remains conserved and that its energy is, on an average, half kinetic and half potential in form. At what particular displacement is this exactly so? What is the ratio between its kinetic and potential energies at a displacement equal to half its amplitude.

[11] Calculate the average kinetic energy and the total energy of a body executing simple harmonic motion. Show that the principle of conservation of energy is obeyed by a harmonic oscillator.

[12] Show that the motion of a simple pendulum is simple harmonic. Obtain an expression for the frequency of oscillations of the simple pendulum. Is the motion of a simple pendulum really simple harmonic?

[13] Suppose we have a block of unknown mass and a spring of unknown force constant. Show how we can predict the period of oscillation of this block-spring system simply by measuring the extension of the spring produced by attaching the block to it.

[14] What is a torsion pendulum? Show that for small angular displacement the oscillation of a torsion pendulum is simple harmonic. Obtain an expression for the period of oscillation.

[15] Show that the motion of the projection of uniform circular motion on a diameter of the circle in which the circular motion occurs is simple harmonic.

[16] Show that the uniform motion of a particle along the circumference of a circle can be described as a combination of two simple harmonic motions which occur along perpendicular lines and have the same amplitude and frequency but differ in phase by 90°.

[17] Show that the charge on the capacitor in an LC circuit oscillates simple harmonically and hence obtain an expression for the frequency of oscillations.

[18] A mass suspended on a vertical spring oscillates with a period of 0.5 sec. When the mass is attached to the spring and allowed to hang at rest, by how much is the spring stretched? **[6.205 cm]**.

[19] A body is vibrating with simple harmonic motion of amplitude 15 cm and frequency 4.0 cycles per second. Calculate (i) the maximum values of its velocity and acceleration, (ii) the acceleration and speed when the body is 9.0 cm from the equilibrium position, and (iii) the time required for the body to move from the equilibrium position to a point 12 cm away. **[(i) 3.8 m/sec; 94 m/sec^2. (ii) 57 m/sec^2; 3.0 m/sec. (iii) 3.7 x 10^{-2} sec.]**

[20] The general equation of simple harmonic motion, $y = A \sin (wt + \theta_0)$

can be written in the equivalent form $y = B \sin wt + C \cos wt$.

Find the expressions for the amplitude B and C in terms of the amplitude A and the initial phase angle θ_0. **[B = A cos θ_0 and C = A sin θ_0]**

[21] A particle executes simple harmonic motion given by the equation

$$y = 12 \sin\left(\frac{2\pi t}{10} + \frac{\pi}{4}\right)$$

Calculate (i) amplitude (ii) frequency (iii) epoch (iv) displacement at t = 1.25 sec (v) velocity at t = 2.5 sec and (vi) acceleration at t = 5 sec.

[(i) 12 umits (ii) 0.1 Hz (iii) $\frac{\pi}{4}$ (iv) 12 units (v) –5.552 units (vi) 3.35 units]

[22] (a) What is the frequency of a simple pendulum 2.0 metre long ? (b) Assuming small amplitudes, what would its frequency be in an elevator accelerating upward at a rate of 2.0 m/sec^2 ? (c) What would its frequency be in free fall ? **[(a) 0.35 cycles/sec (b) 0.39 cycles per sec (c) zero]**

[23] (a) When the displacement is one-half the amplitude A, what fraction of the total energy is kinetic and what fraction is potential in simple harmonic motion? (b) At what displacement is the energy half kinetic and half potential?

[(a) $\frac{3}{4}$, $\frac{3}{4}$ (b) A.$\sqrt{2}$].