Limit

Limit of a function: When x approaches a constant quantity a from either side, if there exists a definite finite number l, towards which $f(x)$ approaches such that the numerical difference of $f(x)$ and l can be made as small as we please by taking x sufficiently close to a, then l is defined as the limit of $f(x)$ as x tends to a. This is symbolically written as $\lim\limits_{x\to a} f(x)=l$.

P.1: A function $f(x)$ is defined as follows: $f(x)=\begin{cases} 2x-3\ , & 0\le x\le 2 \\ x^2-3\ , & 2<x\le 4 \end{cases}$ **, Does $\lim\limits_{x\to 2} f(x)$ exist?**

Solution:

L.H.L = $\lim\limits_{x\to 2^-} f(x)$

$= \lim\limits_{x\to 2^-} 2x-3$

$=2.2-3$

$=4-3$

$=1$

R.H.L = $\lim\limits_{x\to 2^+} f(x)$

$= \lim\limits_{x\to 2^+} x^2-3$

$=2^2-3$

$=4-3$

$=1$

Since L.H.L.=R.H.L, then $\lim\limits_{x\to 2} f(x)$ exist

Continuity

Continuity of a function: Geometrically, If the graph of a function $y=f(x)$ is a continuous curve we naturally call the function a continuous one. It means there should not be any sudden change in the value of the function i.e. a small change in the value of x should produce a small change in the value of y and so the graph of the function should be a continuous curve without any break in it.

Problem 1: Determine whether the function $f(x)=\begin{cases}2x+3\ , & x\le 4\\ 7+\dfrac{16}{x}\ , & x>4\end{cases}$ **is continuous at** $x=4$.

Solution:

L.H.L = $\lim_{x\to 4^-} f(x)$

$= \lim_{x\to 4^-}(2x+3)$

$=11$

R.H.L = $\lim_{x\to 4^+} f(x)$

$= \lim_{x\to 4^+}\left(7+\dfrac{16}{x}\right)$

$=11$

and $f(4)=2.4+3=11$

Since $L.H.L=R.H.L=f(4),\ f(x)$ is continuous at $x=4$.

Problem-2: Test the continuity of $f(x)$ **at** $x=0$, **where** $f(x)=\begin{cases}\dfrac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}+1}, & when\,x\neq 0\\ 0\ , & when\,x=0\end{cases}$.

Solution:

L.H.L = $\lim_{x\to 0^-} f(x)$

$= \lim_{x\to 0^-}\dfrac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}+1}$

$= \lim_{x\to 0^-}\dfrac{1}{1+e^{-\frac{1}{x}}}$

$=\dfrac{1}{1+e^{-\frac{1}{0}}}$

$=\dfrac{1}{1+e^{-\infty}}$

$=\dfrac{1}{1+0}$

$=1$

R.H.L = $\lim_{x\to 0^+} f(x)$

$= \lim_{x\to 0^+}\dfrac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}+1}$

$= \lim_{x\to 0^+}\dfrac{1}{1+e^{-\frac{1}{x}}}$

$=\dfrac{1}{1+e^{-\frac{1}{0}}}$

$=\dfrac{1}{1+e^{-\infty}}$

$=\dfrac{1}{1+0}$

$=1$

and $f(0)=0$

Since L.H.L. = R.H.L. ≠ $f(0)$, the given function is discontinuous at $x=0$.

Problem-3: Test the continuity of $f(x)$ **at** $x=0$ **where** $f(x)=\begin{cases}x^2\sin\dfrac{1}{x}, & when\,x\neq 0\\ 0\ , & when\,x=0\end{cases}$.

Solution:

L.H.L = $\lim_{x\to 0^-} f(x)$

$= \lim_{x\to 0^-} x^2 \sin\frac{1}{x}$

$= 0\times[\text{a finite number between } -1 \text{ and } 1]$

$= 0$

R.H.L = $\lim_{x\to 0^+} f(x)$

$= \lim_{x\to 0^+} x^2 \sin\frac{1}{x}$

$= 0$

and $f(0) = 0$

Since L.H.L. = R.H.L. = $f(0)$, the given function is continuous at $x=0$.

Problem 4: Test the continuity of $f(x)$ at $x=0$, where $f(x)=\begin{cases} \dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, & when\, x\neq 0 \\ 0 \quad , & when\, x=0 \end{cases}$ **.**

Solution:

L.H.L = $\lim_{x\to 0^-} f(x)$

$= \lim_{x\to 0^-} \dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$

$= \lim_{x\to 0^-} \dfrac{1-e^{-\frac{1}{x}}}{1+e^{-\frac{1}{x}}}$

$= \dfrac{1-0}{1+0}$

$= 1$

R.H.L = $\lim_{x\to 0^+} f(x)$

$= \lim_{x\to 0^+} \dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$

$= \lim_{x\to 0^+} \dfrac{1-e^{-\frac{1}{x}}}{1+e^{-\frac{1}{x}}}$

$= \dfrac{1-0}{1+0}$

$= 1$

and $f(0) = 0$

Since L.H.L. = R.H.L. $\neq$ $f(0)$, the given function is discontinuous at $x=0$.

Problem-5: Find the value for the constant k, that will make the function $f(x)=\begin{cases} 7x-2 \ , & x\leq 1 \\ kx^2 \ , & x>1 \end{cases}$ **continuous at** $x=1$.

Solution:

L.H.L = $\lim_{x\to 1^-} f(x)$

$= \lim_{x\to 1^-} (7x-2)$

$= 5$

R.H.L = $\lim_{x\to 1^+} f(x)$

$= \lim_{x\to 1^+} kx^2$

$= k$

and $f(1) = 7.1-2 = 5$

Since $f(x)$ is continuous at $x=1$, for $L.H.L = R.H.L = f(1)$

So, $k = 5 = 5$.
Therefore, $k = 5$.

Problem-6: Find the value for the constant k, **that will make the function** $f(x) = \begin{cases} kx^2 \ , \ x \le 2 \\ 2x + k \ , \ x > 2 \end{cases}$ **continuous at** $x = 2$.

Solution:

L.H.L = $\lim_{x \to 2^-} f(x)$
$= \lim_{x \to 2^-} kx^2$
$= 4k$

R.H.L = $\lim_{x \to 2^+} f(x)$
$= \lim_{x \to 2^+} (2x + k)$
$= 4 + k$

and $f(2) = k.(2)^2 = 4k$
Since $f(x)$ is continuous at $x = 2$,
So, $L.H.L = R.H.L = f(2)$
$\Rightarrow 4k = 4 + k = 4k$
$i.e.,\ 4k = 4 + k$
$\Rightarrow 3k = 4$
$\therefore k = \frac{4}{3}$
Therefore, $k = \frac{4}{3}$.

Problem 7: Find a nonzero value for the constant k **that makes** $f(x) = \begin{cases} \frac{\tan kx}{x}, x < 0 \\ 3x + k^2, x \ge 0 \end{cases}$ **continuous at** $x = 0$.

Solution:

L.H.L = $\lim_{x \to 0^-} f(x)$
$= \lim_{x \to 0^-} \frac{\tan kx}{x}$
$= \lim_{x \to 0^-} \frac{\tan kx}{kx} \cdot k$
$= \lim_{kx \to 0^-} \frac{\tan kx}{k\,x} . k$
$= 1 . k$
$= k$

R.H.L = $\lim_{x \to 0^+} f(x)$
$= \lim_{x \to 0^+} 3x + k^2$
$= 3(0) + k^2$
$= k^2$

and $f(0) = 3.0 + k^2$
$= k^2$

Since the given function is continuous at $x = 0$; L.H.L. = R.H.L. = $f(0)$

which gives $k^2 = k$

$$k(k-1)=0$$
$$k=0,1$$

Therefore the nonzero value of the constant k is 1.

Problem-8: Determine the value of a, b, c for which the following function

$$f(x)=\begin{cases}\dfrac{\sin\left((a+1)x\right)+\sin x}{x}, & x<0\\ c, & x=0\\ \dfrac{\left(x+bx^2\right)^{\frac{1}{2}}-x^{\frac{1}{2}}}{bx^{\frac{3}{2}}}, & x>0\end{cases}$$ **is continuous at** $x=0$.

Solution:

L.H.L = $\lim\limits_{x\to 0^-} f(x)$

$$=\lim_{x\to 0^-}\frac{\sin\left((a+1)x\right)+\sin x}{x}$$

$$=\lim_{x\to 0^-}\frac{\sin\left((a+1)x\right)}{x}+\lim_{x\to 0^-}\frac{\sin x}{x}$$

$$=\lim_{(a+1)x\to 0^-}\frac{\sin\left((a+1)x\right)}{(a+1)x}(a+1)+1$$

$$=(a+1).1+1$$

R.H.L = $\lim\limits_{x\to 0^+} f(x)$

$$=\lim_{x\to 0^+}\frac{\left(x+bx^2\right)^{\frac{1}{2}}-x^{\frac{1}{2}}}{bx^{\frac{3}{2}}}$$

$$=\lim_{x\to 0^+}\frac{\{x(1+bx)\}^{\frac{1}{2}}-x^{\frac{1}{2}}}{bx^{\frac{3}{2}}}$$

$$=\lim_{x\to 0^+}\frac{x^{\frac{1}{2}}(1+bx)^{\frac{1}{2}}-x^{\frac{1}{2}}}{bx^{\frac{3}{2}}}$$

$$=\lim_{x\to 0^+}\frac{x^{\frac{1}{2}}\left[1+\frac{1}{2}(bx)+\frac{1}{2}\left(\frac{1}{2}-1\right)\frac{(bx)^2}{2!}+\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\frac{(bx)^3}{3!}+\cdots\cdots\right]-x^{\frac{1}{2}}}{bx^{\frac{3}{2}}}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{2}bx^{\frac{3}{2}} + \frac{1}{2}\left(\frac{1}{2}-1\right)\frac{(bx)^2}{2!}x^{\frac{1}{2}} + \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\frac{(bx)^3}{3!}x^{\frac{1}{2}} + \cdots\cdots\cdots}{bx^{\frac{3}{2}}}$$

$\therefore R.H.L. = \frac{1}{2}$

and $f(0)=c$

Since $f(x)$ is continuous at $x=0$,

So, $L.H.L = R.H.L = f(0)$

$\Rightarrow a+2 = \frac{1}{2} = c$

i.e., $c = \frac{1}{2}$ and $a+2 = \frac{1}{2} \Rightarrow a = -\frac{3}{2}$

If $b=0$ the function is undefined.

So, $\mathrm{b} \neq 0$.