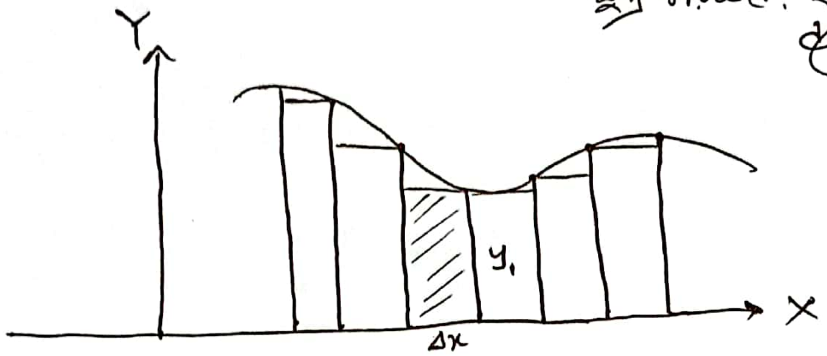
* Differential calculus focuses on rates of change such as slopes of tangent lines and velocities, integral calculus deals with total size or value, such as lengths, area and volumes.

ক্ষুদ্র পরিবর্তন হলে
differentiation করা)



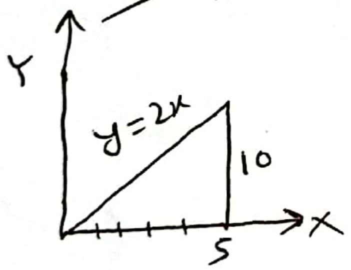
$\int$ Newton
Leibniz

$$\text{Area} = \int y\,dx$$

আয়তক্ষেত্র $= \Delta x \times y$
$=$ দৈর্ঘ্য $\times$ প্রস্থ
$= m \times m = m^2$

$$\lim_{\Delta x \to 0} y\,\Delta x$$

⊛ যোগজের ক্ষুদ্রাংশকে একটি সমাকলনের হিসাবে গণ্য করা যায়।



⊛ $A = \frac{1}{2} \times$ ভূমি $\times$ উচ্চতা
$= \frac{1}{2} \times 5 \times 10 = 25$

⊛ $$A = \int_0^5 y\,dx = \int_0^5 2x\,dx = 2\left[\frac{x^2}{2}\right]_0^5 = \left[x^2\right]_0^5 = 5^2 - 0 = 25.$$

⊛ Integration as a process of summation, which is used to calculate area, volumes and their generalizations. Integration started as a method to solve problems in mathematics such as finding the area under a curve. Today integration is used in a wide variety of scientific fields.

Integral Calculus

★ Integrate with respect to the variable indicated.

★ $\int \sin^2 ax\,dx = \frac{1}{2}\int 2\sin^2 ax\,dx = \frac{1}{2}\int (1-\cos 2ax)\,dx$

$= \frac{1}{2}\left(x - \frac{\sin 2ax}{2a}\right) + c$

★ $\int \cos^3 bx\,dx = \frac{1}{4}\int 4\cos^3 bx\,dx = \frac{1}{4}\int (\cos 3bx + 3\cos bx)\,dx$

$= \frac{1}{4}\left(\frac{\sin 3bx}{3b} + \frac{3\sin bx}{b}\right) + c$

★ $\int \sin^4 x\,dx = \int (\sin^2 x)^2\,dx = \frac{1}{4}\int (1-\cos 2x)^2\,dx$

$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx = \frac{1}{4}\int dx - \frac{1}{2}\int \cos 2x\,dx + \frac{1}{8}\int 2\cos^2 2x\,dx$

$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}\int (1+\cos 4x)\,dx$

$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{x}{8} + \frac{1}{32}\sin 4x + c$

$= \frac{3x}{4} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c.$

★ $\frac{dx}{\sqrt{x}+\sqrt{1+x}} = \int \frac{(\sqrt{x}-\sqrt{1+x})\,dx}{(\sqrt{x}+\sqrt{1+x})(\sqrt{x}-\sqrt{1+x})} = \int \frac{\sqrt{x}-\sqrt{1+x}}{x-1-x}\,dx$

$= -\int (\sqrt{x} - \sqrt{1+x})\,dx = -\frac{2}{3}x^{3/2} + \frac{2}{3}(1+x)^{3/2} + c.$

★ $\int \frac{dx}{1+\sin x} = \int \frac{(1-\sin x)}{1-\sin^2 x} = \frac{(1-\sin x)}{\cos^2 x}\,dx$

$= \int (\sec^2 x - \tan x \sec x)\,dx = \int \sec^2 x\,dx - \int \tan x \sec x\,dx$

$= \tan x - \sec x + c.$

$\star \quad \dfrac{(1-2x^2)^2}{x\sqrt[3]{x}}dx = \displaystyle\int \dfrac{1-4x^2+4x^4}{x\sqrt[3]{x}}\,dx$

$= \displaystyle\int \dfrac{1}{x \cdot x^{1/3}}\,dx - 4\int \dfrac{x^2}{x \cdot x^{1/3}}\,dx + 4\int \dfrac{x^4}{x \cdot x^{1/3}}\,dx$

$= \displaystyle\int x^{-4/3}\,dx - 4\int x^{2/3}\,dx + 4\int x^{8/3}\,dx$

$= -3x^{-1/3} - \dfrac{12}{5}x^{5/3} + {}^{12}/_{11}\, x^{11/3} + c.$

$\star \displaystyle\int \sqrt{(1+\sin 2x)}\,dx = \int \sqrt{\sin^2 x+\cos^2 x + 2\sin x\cos x}\,dx$

$= = \displaystyle\int \sqrt{(\sin x+\cos x)^2}\,dx = \pm\int (\sin x+\cos x)\,dx = \pm(\sin x - \cos x)+c$

$\star \displaystyle\int \cos^4 x\,dx = \dfrac{3x}{8} + \dfrac{\sin 2x}{4} + \dfrac{\sin 4x}{32} + c$

$\star\star \displaystyle\int \dfrac{dx}{e^x+e^{-x}} = \int \dfrac{e^x\,dx}{e^x(e^x+e^{-x})} = \int \dfrac{e^x\,dx}{e^{2x}+1}$

numerator e^x

Let, $e^x = z \Rightarrow e^x dx = dz$

$I = \displaystyle\int \dfrac{dz}{z^2+1} = \tan^{-1} z + c = \tan^{-1} e^x + c.$

⊛ $\int \frac{dx}{x^2\sqrt{1+x^2}}$

Let, $x = \tan\theta$

$\therefore dx = \sec^2\theta\, d\theta$

$\therefore \cot\theta = 1/x$

$= \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta \sqrt{1+\tan^2\theta}}$

$= \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta \sec\theta}$

$= \int \frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta} d\theta$

$= -\int \operatorname{cosec}\theta \cdot \cot\theta\, d\theta$

$= -\operatorname{cosec}\theta + c$

$= -\sqrt{\operatorname{cosec}^2\theta} + c$

$= -\sqrt{1+\cot^2\theta} + c$

$= -\sqrt{1+(1/x)^2} + c$

$= -\sqrt{1+1/x^2} + c$ [Ans]

⊛ $\int \frac{dx}{\sqrt{1-x^2}\sqrt{\sin^{-1}x}}$

Let, $x = \sin\theta \quad \therefore \theta = \sin^{-1}x$

$dx = \cos\theta\, d\theta$

$\therefore \int \frac{\cos\theta\, d\theta}{\sqrt{\cos^2\theta}\sqrt{\theta}}$

$= \frac{d\theta}{\sqrt{\theta}} = 2\sqrt{\theta} + c$

$= 2\sqrt{\sin^{-1}x} + c$ [Ans]

Alternative by (substitution)

$\sin^{-1}x = z$

$\Rightarrow dz = \frac{dx}{\sqrt{1-x^2}}$

$\therefore I = \int \frac{dz}{\sqrt{z}}$

$= 2\sqrt{\sin^{-1}x} + c$

⊛ $\int \frac{dx}{x\sqrt{x^2-1}}$

$= \int \frac{\sec\theta \tan\theta\, d\theta}{\sec\theta\sqrt{\sec^2\theta - 1}}$

$= \int \frac{\tan\theta\, d\theta}{\tan\theta}$

$= \int d\theta = \theta + c = \sec^{-1}x + c.$ [Ans]

Integration By Substitution

$\star \int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

Let, $z = \sqrt{x}$

$\Rightarrow \frac{dz}{dx} = \frac{1}{2\sqrt{x}}$

$\Rightarrow dz = \frac{dx}{2\sqrt{x}}$

$\Rightarrow 2dz = \frac{dx}{\sqrt{x}}$

$= \int 2\cos z \, dz$

$= 2\int \cos z \, dz = 2\sin z + c$

$= 2\sin\sqrt{x} + c$

$\star \int \frac{(x+1)(x+\ln x)^2}{x}\,dx$

Let, $x + \ln x = z$

$\Rightarrow (1 + 1/x)\,dx = dz$

$\therefore \frac{1+x}{x}\,dx = dz$

$= \int z^2\,dz = \frac{z^3}{3} + c$

$= \frac{1}{3}(x + \ln x)^3 + c$

$\star \int \frac{\cos^3 x}{\sqrt[3]{\sin x}}\,dx = \int \frac{\cos x \cos^2 x}{\sqrt[3]{\sin x}}\,dx$

$= \int \frac{\cos x\,(1-\sin^2 x)}{\sqrt[3]{\sin x}}\,dx$

Let, $\sin x = z$

$\therefore \cos x\,dx = dz$

$= \int \frac{(1-z^2)\,dz}{\sqrt[3]{z}} = \int \frac{(1-z^2)}{z^{1/3}}\,dz$

$= \int z^{-1/3}\,dz - \int z^{2-1/3}\,dz = \int z^{-1/3}\,dz - z^{5/3}\,dz$

$= \frac{z^{-1/3+1}}{1/3+1} - \frac{z^{5/3+1}}{5/3+1} + c = \frac{3}{2} z^{2/3} - 3/8\, z^{8/3} + c$

$= 3/2\,(\sin x)^{2/3} - 3/8\,(\sin x)^{8/3} + c$ (Ans).

$\star \displaystyle\int \frac{\tan^2 x \sec^2 x}{1+\tan^6 x}\,dx$

Let $\tan^3 x = z$

$\therefore 3\tan^2 x \sec^2 x \, dx = dz$

$\therefore \tan^2 x \sec^2 x = \dfrac{dz}{3}$

$= \displaystyle\int \frac{\tan^2 x \sec^2 x}{1+(\tan^3 x)^2}\,dx$

$= \dfrac{1}{3}\displaystyle\int \frac{dz}{1+z^2} = \frac{1}{3}\tan^{-1} z + c$

$= \dfrac{1}{3}\tan^{-1}(\tan^3 x) + c$

O/C, $z = \tan x$

$dz = \sec^2 x\,dx$

$\displaystyle\int \frac{z^2\,dz}{(1+z^6)}$ $\quad \left| \begin{array}{l} p = z^3 \\ dp = 3z^2 dz \end{array} \right.$

$= \displaystyle\int \frac{dp}{3(1+p^2)}$

$= \dfrac{1}{3}\tan^{-1} p = \dfrac{1}{3}\tan^{-1} z^3 = \dfrac{1}{3}\tan^{-1}(\tan^3 x) + c$

$\star \displaystyle\int \frac{x^2 \tan^{-1} x}{1+x^2}\,dx$

Let $\tan^{-1} x = z$

$\therefore x = \tan z$

$\therefore \dfrac{dx}{1+x^2} = dz$

$= \displaystyle\int (\tan z)^2 z \, dz$

$= \displaystyle\int z \tan^2 z \, dz$

$= \displaystyle\int z(\sec^2 z - 1)\,dz$

$= \displaystyle\int z \sec^2 z \, dz - \int z \, dz$

$= z\displaystyle\int \sec^2 z \, dz - \int \left\{ \frac{dz}{dz} \int \sec^2 z \, dz \right\} dz - \int z \, dz$

$= z \tan z - \displaystyle\int \tan z \, dz - \frac{z^2}{2} + c$

$= x \tan^{-1} x - \ln \sec z - \dfrac{z^2}{2} + c$

$= x \tan^{-1} x - \ln \sec(\tan^{-1} x) - \dfrac{(\tan^{-1} x)^2}{2} + c.$

★ $\int \frac{e^x(1+x)\,dx}{\cos^2(xe^x)}$

Let $xe^x = z$

$\Rightarrow (xe^x + e^x)\,dx = dz$

$\therefore e^x(1+x)\,dx = dz$

$= \int \frac{dz}{\cos^2 z}$

$= \int \sec^2 z\,dz = \tan z + c = \tan(xe^x) + c$ (Ans)

★ $\int \frac{(e^x - 1)\,dx}{e^x + 1}$

$= \int \frac{(e^x+1) - 2}{e^x+1}\,dx = \int dx - 2\int \frac{dx}{e^x + 1}$

$e^x + 1 = z$

$\therefore e^x = z - 1$

$\Rightarrow e^x dx = dz$

$\Rightarrow (z-1)\,dx = dz$

$\therefore dx = \frac{dz}{z-1}$

$= x - 2\int \frac{dz}{z(z-1)}$

$= x - 2\int \frac{dz}{z-1} + 2\int \frac{dz}{z}$

$= x - 2\int \ln(z-1) + 2\ln z + c$

$= x - 2\ln e^x + 2\ln(e^x+1) + c$

$= x - 2x + 2\ln(e^x+1) + c = -x + 2\ln(e^x+1) + c$

⊛ $\int \frac{(x+1)\ (x+\ln x)^2\, dx}{x}$

Let, $z = x + \ln x$

$\Rightarrow (1 + 1/x)\, dx = dz$

$\therefore \frac{1+x}{x}\, dx = dz$.

$= \int z^2 dz = \frac{z^3}{3} + c$

$= \frac{1}{3}(x + \ln x)^3 + c$

[Ans]

⊛ $\int \frac{\cos x\, dx}{a + b^2 \sin^2 x}$

$z = b \sin z$

$\therefore b \cos x\, dx = dz$

$\therefore \cos x\, dx = \frac{dx}{b}$.

$= \frac{1}{b} \int \frac{dz}{a + z^2}$

$= \frac{1}{b} \int \frac{dz}{z^2 + (\sqrt{a})^2}$

$= \frac{1}{b} \frac{1}{\sqrt{a}} \tan^{-1} \frac{z}{\sqrt{a}} + c$

$= \frac{1}{b\sqrt{a}} \tan^{-1} \frac{b \sin x}{\sqrt{a}} + c$

$= \frac{\sqrt{a}}{ab} \tan^{-1} \frac{b \sin x}{\sqrt{a}} + c$

[Ans]

Type (Some standard Forms)

F: $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$.

F: $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{x-a}{x+a}$

$$\int \frac{dx}{4x^2+9}$$

$$= \frac{1}{4}\int \frac{dx}{x^2+9/4}$$

$$= \frac{1}{4}\int \frac{dx}{x^2+(3/2)^2}$$

$$= \frac{1}{4}\,\frac{2}{3}\tan^{-1}\frac{x}{3/2}+c$$

$$= \frac{1}{6}\tan^{-1}\frac{2x}{3}+c.$$

$$\int \frac{dx}{9x^2-4}$$

$$= \frac{1}{9}\int \frac{dx}{x^2-(2/3)^2}$$

$$= \frac{1}{9}\cdot\frac{1}{2}\cdot\frac{3}{2}\ln\frac{x-2/3}{x+2/3}+c$$

$$= \frac{1}{12}\ln\frac{3x-2}{3x+2}+c.$$

$$\int \frac{dx}{20-5x^2}$$

$$= \frac{1}{5}\int \frac{dx}{20/5-x^2}$$

$$= \frac{1}{5}\int \frac{dx}{2^2-x^2}$$

$$= \frac{1}{5}\cdot\frac{1}{2\cdot 2}\ln\frac{2+x}{2-x}+c.$$

$$= \frac{1}{20}\ln\frac{2+x}{2-x}+c.$$

Note:

- এইটি টার্ম এবং তারা ফর্ম
- টার্ম ভ্যারিয়েবল নম্বর 1
- টার্ম ভ্যারিয়েবল ধরব 2

$$\star \int \frac{dx}{2x^2+x+1} = \frac{1}{2}\int \frac{dx}{x^2+x/2+1/2}$$

$$= \frac{1}{2}\int \frac{dx}{x^2+2.x.1/4+(1/4)^2+(1/2-1/16)}$$

$$= \frac{1}{2}\int \frac{dx}{(x+1/4)^2+\left(\frac{\sqrt{7}}{4}\right)^2}$$

$$= \frac{1}{2}\cdot\frac{4}{\sqrt{7}}\tan^{-1}\frac{4x+1}{\sqrt{7}}+c$$

$$= \frac{2}{\sqrt{7}}\tan^{-1}\frac{4x+1}{\sqrt{7}}+c$$

$$\star \int \frac{dx}{x^2-x+1}$$

$$= \int \frac{dx}{\left\{x^2-2.x.\frac{1}{2}+\left(\frac{1}{2}\right)^2\right\}+(1-1/4)}$$

$$= \int \frac{dx}{(x-1/2)^2+(\sqrt{3}/2)^2}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}+c$$

$$\star \int \frac{dx}{3x^2-2x+4}$$

$$= \frac{1}{3}\int \frac{dx}{x^2-2x/3+4/3}$$

$$= \frac{1}{3}\int \frac{dx}{x^2-2.x.\frac{1}{3}+(1/3)^2+4/3-1/9}$$

$$= \frac{1}{3}\int \frac{dx}{(x-1/3)^2+(\sqrt{11}/3)^2}$$

$$= \frac{1}{\sqrt{11}}\tan^{-1}\frac{3x-1}{\sqrt{11}}+c$$

$$\star \int \frac{dx}{x^2+7x-18}$$

$$= \int \frac{dx}{x^2+2.x.7/2+(7/2)^2-18-\frac{49}{4}}$$

$$= 1/11\ \ln\frac{x-2}{x+9}+c$$

⊛ $\int \frac{\cos x \, dx}{5+7\cos^2 x}$

Put, $\sin x = z$

$\therefore \cos x \, dx = dz$

$= \int \frac{\cos x \, dx}{5+7(1-\sin^2 x)}$

$= \int \frac{\cos x \, dx}{12-7\sin^2 x}$

$= \int \frac{dz}{12-7z^2}$

$= -\frac{1}{7}\int \frac{dz}{z^2 - 12/7}$

$= -\frac{1}{7}\int \frac{dz}{z^2 - \left(\frac{\sqrt{12}}{\sqrt{7}}\right)^2}$

$= \frac{1}{7}\int \frac{dz}{\left(\frac{\sqrt{12}}{\sqrt{7}}\right)^2 - (z)^2}$

$= \frac{1}{7} \frac{1}{2\frac{\sqrt{12}}{\sqrt{7}}} \ln \frac{\sqrt{12}/\sqrt{7} + z}{\sqrt{12}/\sqrt{7} - z} + c$

$= \frac{1}{2}$

⊛ $\int \frac{dx}{4\cos^2 x + 9\sin^2 x}$

$= \int \frac{dx}{\cos^2 x(4+9\tan^2 x)}$

$= \int \frac{\sec^2 x}{4+9\tan^2 x} dx$

$= \frac{1}{9}\int \frac{\sec^2 x}{4/9 + \tan^2 x} dx$

Let, $\tan x = z$

$\therefore \sec^2 x \, dx = dz$

$= \frac{1}{9}\int \frac{dz}{(2/3)^2 + z^2}$

$= \frac{1}{9} \frac{1}{2/3} \tan^{-1} \frac{z}{2/3} + c$

$= \frac{1}{6} \tan^{-1}\left(\frac{3\tan x}{2}\right) + c$ (Ans) type $\sqrt{\ }$

⊛ $$\int \frac{dx}{(x-3)\sqrt{x+1}}$$

Let, $x+1 = z^2$

$\therefore dx = 2z\,dz$

$$= 2\int \frac{z\,dz}{(z^2-1-3)z}$$

$$= 2\int \frac{dz}{z^2-2^2}$$

$$= 2\left[\frac{1}{4}\ln\frac{z-2}{z+2}\right]+c$$

⊛ $$\int \frac{dx}{(x-1)\sqrt{x+2}}$$

Let, $x+2 = z^2$

$\therefore dx = 2z\,dz$

Ans: $\frac{1}{\sqrt{3}} \ln \frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}$.

⊛ $$\int \frac{dx}{(1-x)\sqrt{1+x}}$$

Let,

$1+x = z^2$

$\Rightarrow dx = 2z\,dz$

$\therefore x = z^2-1$.

$\therefore z = \sqrt{1+x}$

$$= 2\int \frac{z\,dz}{(1-z^2+1)z^2}$$

$$= 2\int \frac{dz}{(\sqrt{2})^2-z^2}$$

$$= 2\left[\frac{1}{2\sqrt{2}}\ln\frac{\sqrt{2}+z}{\sqrt{2}-z}\right]+c$$

$$= 2\left[\frac{1}{2\sqrt{2}}\ln\frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{2}-\sqrt{1+x}}\right]+c$$

$$= \frac{1}{\sqrt{2}}\ln\frac{\sqrt{2}+\sqrt{1+x}}{\sqrt{2}-\sqrt{1+x}}+c.$$

Let, $x-1 = z^2$

$\therefore x = z^2+1$

$\therefore dx = 2z\,dz$

$\therefore z = \sqrt{x-1}$.

⊛ $$\int \frac{dx}{(x-3)\sqrt{x-1}}$$

$$= 2\int \frac{z\,dz}{(x^2-2)z^2}$$

$$= 2\int \frac{dx}{z^2-(\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}}\ln\frac{\sqrt{x+1}-\sqrt{2}}{\sqrt{x-1}+\sqrt{2}}+c$$

Special types of Integration

যদি কোন ইন্টিগ্রেল $\int \frac{dx}{a+b\cos x}$, $\int \frac{dx}{a+b\sin x}$ এবং

$\int \frac{dx}{a\cos x + b\sin x + c}$ এর যে কোন একআকারে থাকে, তবে

এই ইন্টিগ্রেলে $\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$ এবং

$\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$ দ্বারা বসিয়ে প্রতিস্থাপন করতে হবে।

অতঃপর $\tan x/2 = z$ ধরে পুনরায় প্রতিস্থাপন করে তারপর ইন্টিগ্রেশন করতে হবে।

$$\int \frac{dx}{5+4\cos x}$$

$$= \int \frac{dx}{5+4\cos x}$$

$$= \int \frac{dx}{5(\cos^2 x/2 + \sin^2 x/2) + 4(\cos^2 x/2 - \sin^2 x/2)}$$

$$= \int \frac{dx}{9\cos^2 x/2 + \sin^2 x/2}$$

Let $\tan\frac{x}{2} = z$

$\therefore \sec^2\frac{x}{2}\,dx = 2dz$

$$= \int \frac{\sec^2 x/2\, dx}{9+\tan^2 x/2}$$

$$= 2\int \frac{dz}{3^2+z^2} = 2\cdot\frac{1}{3}\tan^{-1}\frac{z}{3} + c = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right) + c$$

$$\int \frac{d\theta}{3+2\cos\theta}$$

Let $\tan\frac{\theta}{2} = z$

$\therefore \sec^2\frac{\theta}{2}\, d\theta = 2dz$

$$= \int \frac{d\theta}{3+2\,\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}}$$

$$= \int \frac{(1+\tan^2\theta/2)\, d\theta}{3(1+\tan^2\theta/2)+2(1-\tan^2\theta/2)}$$

$$= \int \frac{(\sec^2\theta/2)\, d\theta}{5+\tan^2\theta/2} = 2\int \frac{dz}{5+z^2}$$

$$= 2\int \frac{dz}{(\sqrt{5})^2+z^2} = 2\cdot\frac{1}{\sqrt{5}}\tan^{-1}\frac{z}{\sqrt{5}}+c$$

$$= \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{\tan\theta/2}{\sqrt{5}}\right)+c$$

Let $\tan x/2 = z$

$$\int \frac{dx}{5+3\cos\theta} = \frac{1}{2}\int \frac{\sec^2 x/2}{\tan^2 x/2+4} = \int \frac{dz}{z^2+2^2}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan x/2\right)+c$$

★ $\int \frac{2\,dx}{1-2\cos x}$

$= 2\int \frac{dx}{1-2\,\frac{1-\tan^2 x/2}{1+\tan^2 x/2}}$

$= 2\int \frac{(1+\tan^2 x/2)\,dx}{1+\tan^2 x/2 - 2 + 2\tan^2 x/2}$

$= 2\int \frac{\sec^2 x/2\,dx}{3\tan^2 x/2 - 1}$

Let, $\tan\frac{x}{2} = z$

$\sec^2 x/2\,dx = 2\,dz$

$= 4\int \frac{dz}{3z^2-1}$

$= 4/3 \int \frac{dz}{z^2 - (1/\sqrt{3})^2}$

$= \frac{4}{3}\,\frac{\sqrt{3}}{2}\,\ln \frac{z - 1/\sqrt{3}}{z + 1/\sqrt{3}} + c$

$= \frac{2}{\sqrt{3}}\,\ln \frac{\sqrt{3}\tan x/2 - 1}{\sqrt{3}\tan x/2 + 1} + c$

$$\circledast \int \frac{dx}{1+2\sin x}$$

$$= \int \frac{dx}{\sin^2 x/2 + \cos^2 x/2 + 2\cdot 2 \sin x/2 \cdot \cos x/2}$$

$$= \int \frac{dx}{\cos^2 x/2 \left(\tan^2 x/2 + 1 + 4 \tan x/2\right)}$$

$$= \int \frac{\sec^2 x/2 \, dx}{\left(\tan^2 x/2 + 4 \tan x/2 + 1\right)}$$

$$= 2\int \frac{dz}{z^2+4z+1}$$

Let $\tan x/2 = z$

$\therefore \sec^2 x/2 \, dx = 2dz$

$$= 2\int \frac{dz}{(z^2+2\cdot 2z+2^2)+1-4}$$

$$= 2\int \frac{dz}{(z+2)^2-(\sqrt{3})^2}$$

$$= 2\,\frac{1}{2\sqrt{3}} \ln \frac{z+2-\sqrt{3}}{z+2+\sqrt{3}} + c$$

$$= \frac{1}{\sqrt{3}} \ln \frac{\tan x/2 + 2 - \sqrt{3}}{\tan x/2 + 2 + \sqrt{3}} + c$$

★ $\int \frac{dx}{4+5\sin x}$

$= \int \frac{dx}{4(\cos^2 x/2 + \sin^2 x/2) + 5\cdot 2\sin x/2 \cdot \cos x/2}$

$= \int \frac{dx}{\cos^2 x/2 \{4 + 4\tan^2 x/2 + 10\tan^2 x/2)}$

$= \int \frac{\sec^2 x/2 \, dx}{4\tan^2 x/2 + 10\tan x/2 + 4}$

Let, $\tan x/2 = z$

$\therefore \sec^2 x/2 \, dx = 2dz$

$= \int \frac{2\ dz}{4z^2+10z+4}$

$= \frac{1}{2}\int \frac{dz}{z^2 + 5/2\, z + 1}$

$= \frac{1}{2}\int \frac{dz}{(z)^2 + 2\cdot 5/4\, z + (5/4)^2 + (1 - 25/16)}$

$= \frac{1}{2}\int \frac{dz}{(z+5/4)^2 - (3/4)^2} = \frac{1}{2}\cdot 4/6 \ln \frac{z+5/4-3/4}{z+5/4+3/4} + c$

$= \frac{1}{3}\ln \frac{4\tan x/2 + 2}{4\tan x/2 + 8} = \frac{1}{3}\ln \frac{2\tan x/2 + 1}{2\tan x/2 + 4} + c.$

Special Integration

★ $$I = \int \frac{px+q}{\sqrt{ax^2+bx+c}}$$

Since $d(ax^2+bx+c) = (2ax+b)\,dx$ then above integral can be written as.

$$I = k\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}\,dx + \int \frac{dx}{\sqrt{ax^2+bx+c}}$$

① $$I = \int \frac{(x+1)\,dx}{\sqrt{x^2-x+1}} = \int \frac{\frac{1}{2}(2x-1)+\frac{3}{2}}{\sqrt{x^2-x+1}}\,dx$$

$$= \frac{1}{2}\int \frac{(2x-1)\,dx}{\sqrt{x^2-x+1}} + \frac{3}{2}\int \frac{dx}{\sqrt{x^2-x+1}} \quad \Bigg|\ \because \int \frac{dx}{\sqrt{x}} = 2\sqrt{x}$$

$$= \frac{1}{2} \times 2\sqrt{x^2-x+1} + \frac{3}{2}\int \frac{dx}{\sqrt{\{x^2-(1/2)^2\} + 3/4}}$$

$$= \sqrt{x^2-x+1} + \frac{3}{2}\sinh^{-1}\frac{x-\frac{1}{2}}{\sqrt{\frac{3}{2}}} + c$$

$$= \sqrt{x^2-x+1} + \frac{3}{2}\sinh^{-1}\frac{2x-1}{\sqrt{3}} + c.$$

$$\circledast \quad I = \int \frac{dx}{(ax+b)\sqrt{cx+d}} \qquad \text{Put, } 4x+3 = z^2$$

$$\int \frac{dx}{(2x+1)\sqrt{4x+3}} \qquad \Rightarrow x = \tfrac{1}{4}(z^2-3)$$

$$dx = \frac{1}{2} z\,dz$$

$$I = \tfrac{1}{2}\int \frac{z\,dz}{\left(\frac{2z^2-2}{4}+1\right)}$$

$$= \int \frac{dz}{z^2-1} = \frac{1}{2}\ln\frac{z-1}{z+1} + c$$

$$= \frac{1}{2}\ln\frac{\sqrt{4x+3}-13}{\sqrt{4x+3}+1} + c \qquad \text{(Ans)}$$

$$\circledast \quad I = \int \frac{dx}{(ax^2+b)\sqrt{cx^2-d}}$$

$$I = \int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$$

Put, $x = 1/z$, $dx = -dz/z^2$

$$I = -\frac{dz/z^2}{(1+1/z^2)\sqrt{1-1/z^2}} = -\frac{z\,dz}{(z^2+1)\sqrt{z^2-1}}$$

Put, $z^2-1 = u^2$ or, $z\,dz = u\,du$

$$I = \int \frac{u\,du}{(u^2+2)u} = -\int \frac{du}{u^2+2} = -\int \frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{\sqrt{2}} + c$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1} \frac{\sqrt{z^2-1}}{\sqrt{2}} + c$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1} \frac{\sqrt{1/x^2-1}}{\sqrt{2}} + c$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1} \frac{\sqrt{1-x^2}}{x\sqrt{2}} + c \quad \text{(Ans)}.$$

Integration By Parts

* $\int uv\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$

$$= u\int v\,dx - \int u'v_1\,dx \qquad u' = \frac{du}{dx}\ ;\ v' = \frac{dv}{dx}.$$

u = first function, v = 2nd function.

* $\int e^x\left[f(x) + f'(x)\right]dx = e^x f(x) + c$

In order to identify 'u' from the given product of two functions, we use the following series,

LIATE

L = Logarithmic function

I = Inverse

A = Algebric

T = Trigonometric

E = Exponential

$\star \int \ln x \, dx = \ln x \int dx - \int \left\{ \frac{d}{dx}(\ln x) \int dx \right\} dx$

$= x \ln x - \int \frac{1}{x} \times x \, dx$

$= x \ln x - \int dx = x \ln x - x + c$

$\star \int x \ln x \, dx$

$= \ln x \int x \, dx - \int \left\{ \frac{d}{dx}(\ln x) \int x \, dx \right\} dx$

$= \frac{x^2}{2} \ln x - \int \frac{1}{x} \frac{x^2}{2} dx$

$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx$

$= \frac{x^2}{2} \ln x - \frac{1}{4} x^2 + c$

$\star \int x \sin x \, dx$

$= x \int \sin x \, dx - \int \left\{ \frac{dx}{dx} \int \sin x \, dx \right\} dx$

$= -x \cos x + \int \cos x \, dx$

$= \sin x - x \cos x + c$

$\star \int x \tan^{-1} x \, dx$

$= \tan^{-1} x \int x \, dx - \int \left\{ \frac{d}{dx}(\tan^{-1} x) \int x \, dx \right\} dx$

$= \frac{x^2}{2} \tan^{-1} x - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2} dx$

$= x^2/2 \tan^{-1} x - \frac{1}{2} \int \frac{(1+x^2) - 1}{(1+x^2)} dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int dx + \frac{1}{2} \int \frac{dx}{1+x^2}$

$= x^2/2 \tan^{-1} x - x/2 + 1/2 \tan^{-1} x + c$

$= \frac{1}{2} \tan^{-1} x \, (x^2+1) - x/2 + c.$

$$\int x^2 \sin^2 x \, dx$$

$$= \frac{1}{2}\int x^2 \, 2\sin^2 x \, dx$$

$$= \frac{1}{2}\int x^2 (1-\cos 2x)\, dx$$

$$= \frac{1}{2}\int x^2 dx - \frac{1}{2}\int x^2 \cos 2x \, dx$$

$$= \frac{x^3}{3\times 2} - \frac{1}{2}\left[x^2 \int \cos 2x \, dx - \int \left\{\frac{d}{dx}(x^2)\int \cos 2x \, dx\right]\right.$$

$$= \frac{x^3}{6} - \frac{1}{2}\left[x^2 \frac{\sin 2x}{2} - \int 2x \frac{\sin 2x}{2} dx\right.$$

$$= \frac{x^3}{6} - \frac{1}{2}\left[\frac{x^2 \sin 2x}{2} - \int x \sin 2x \, dx\right]$$

$$= \frac{x^3}{6} - \frac{x^2 \sin 2x}{4} - \frac{1}{2}\left[x\int \sin 2x \, dx - \int\left\{\frac{d}{dx}(x)\int \sin 2x \, dx\right]\right.$$

$$= \frac{x^3}{6} - \frac{x^2 \sin 2x}{4} - \frac{1}{2}\left[x \frac{\cos 2x}{2} - \int \frac{\cos 2x}{2} dx\right.$$

$$= \frac{x^3}{6} - \frac{x^2 \sin 2x}{4} - \frac{x \cos 2x}{4} + \int \frac{\cos 2x}{4} dx$$

$$= \frac{x^3}{6} - \frac{x^2 \sin 2x}{4} - \frac{x \cos 2x}{4} + \frac{\sin 2x}{2\times 4} + C$$

$$= x^3/6 - \frac{x^2}{4}\sin 2x - x/4 . \cos 2x + 1/8 \sin 2x + C.$$

$\boxed{\text{Ans}}$

* $\int x^2 e^x dx = x^2 \int e^x dx - \int \left\{ \frac{d}{dx}(x^2) \int e^x dx \right\} dx$

$= x^2 e^x - 2 \int x e^x dx$

$= x^2 e^x - 2 \left[x \int e^x dx - \int e^x dx \right]$

$= x^2 e^x - 2 \, (x e^x - e^x)$

$= x^2 e^x - 2x e^x + 2e^x + c$ (Ans)

(*) $\int x^2 (\ln x)^2 dx$

$= (\ln x)^2 \int x^2 dx - \int \left\{ \frac{d}{dx} (\ln x)^2 \cdot \int x^2 dx \right\} dx$

$= \frac{x^3}{3} (\ln x)^2 - \int 2 \ln x \cdot \frac{1}{x} \cdot \frac{x^3}{3} dx$

$= \frac{x^3}{3} (\ln x)^2 - \frac{2}{3} \int \ln x \cdot x^2 dx$

$= \frac{x^3}{3} (\ln x)^2 - \frac{2}{3} \left[\ln x \cdot \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3} dx \right]$

$= \frac{x^3}{3} (\ln x)^2 - \frac{2x^3}{9} \left[\ln x + \frac{2}{9} \int x^2 dx \right]$

$= \frac{x^3}{3} (\ln x)^2 - \frac{2x^3}{9} \ln x + \frac{2x^3}{27} + c$

$= \frac{x^3}{3} (\ln x)^2 - \frac{2}{9} x^3 \ln x + \frac{2}{27} x^3 + c$.

$$\int \sqrt{x}(\ln x)^2 dx$$

$$= \int x^{1/2} (\ln x)^2 dx$$

$$= (\ln x)^2 \int x^{1/2} dx - \int \left\{ \frac{d}{dx}(\ln x)^2 \int x^{1/2} dx \right\} dx$$

$$= \frac{2}{3}(\ln x)^2 \, x^{3/2} - \int 2 \ln x \cdot \tfrac{1}{x} \, \tfrac{2}{3} \, x^{3/2} dx$$

$$= \frac{2}{3} x^{3/2} (\ln x)^2 - \tfrac{4}{3} \int \sqrt{x} \, \ln x \, dx$$

$$= \frac{2}{3} x^{3/2} (\ln x)^2 - \frac{4}{3} \left[\ln x \int \sqrt{x} \, dx - \int \left\{ \frac{d}{dx}(\ln x) \int \sqrt{x} \, dx \right\} dx \right]$$

$$= \frac{2}{3} x^{3/2} (\ln x)^2 - \frac{8}{9} x^{3/2} \ln x + \tfrac{4}{3} \left[\int \tfrac{1}{x} \cdot \tfrac{2}{3} \, x^{3/2} dx \right]$$

$$= \tfrac{2}{3} \, x^{3/2} (\ln x)^2 - \tfrac{8}{9} \, x^{3/2} \ln x + \tfrac{8}{9} \int x^{1/2} dx$$

$$= \frac{2}{3} \, x^{3/2} (\ln x)^2 - \tfrac{8}{9} \, x^{3/2} \ln x + \frac{16}{27} \, x^{3/2} + c$$

$$= x^{3/2} \left[\tfrac{2}{3} (\ln x)^2 - \tfrac{8}{9} \ln x + \frac{16}{27} \right] + c$$

★ $I = \int \frac{x e^x}{(1+x)^2} dx$

$= \int e^x \frac{x \, dx}{(1+x)^2} = \int e^x \frac{(1+x)-1}{(1+x)^2} dx$

Put, $\frac{1}{1+x} = f(x)$

$\therefore \frac{dx}{(1+x)^2} = f'(x)$

$= \int e^x \left\{ \frac{1}{1+x} dx - \frac{1}{(1+x)^2} \right\} dx$

$= \int e^x \{ f(x) + f'(x)] dx$

$= e^x f(x) + c = e^x/(1+x) + c$

★ $I = \int e^x \frac{(1+x)}{(2+x)^2} dx = \int e^x \frac{(2+x)-1}{(2+x)^2} dx$

$= \int e^x \left\{ \frac{1}{2+x} - \frac{1}{(2+x)^2} \right\} dx$

$f(x) = \frac{1}{2+x}$

$\therefore f'(x) = -\frac{1}{(2+x)^2}$

$= \int e^x \{ f(x) + f'(x) \} dx$

$= e^x f(x) + c = e^x/(2+x) + c$

★ $\int \frac{1 + x \ln x}{x} e^x dx$

$= \int e^x \left(\frac{1}{x} + \ln x \right) dx$

$= \int e^x \{ f'(x) + f(x)] dx$

$= e^x f(x) + c = e^x \ln x + c.$

$$\star\ I = \int e^x \cos x\,dx$$

$$= e^x \sin x - \int e^x \sin x\,dx$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx + c$$

$$\Rightarrow I + I = e^x \sin x + e^x \cos x + c$$

$$\Rightarrow 2I = e^x \sin x + e^x \cos x + c$$

$$\therefore I = \frac{1}{2}\left(e^x \sin x + e^x \cos x\right) + c.$$

$$\star\ \int e^x (\tan x - \ln \cos x)\,dx$$

$$= \int e^x \{ f'(x) + f(x) \}\,dx$$

$$= e^x f(x) + c = -e^x \ln \cos x + c$$

$f(x) = -\ln \cos x$

$f'(x) = \tan x.$

* $\int x^2 \cos x \, dx$

$= x^2 \int \cos x \, dx - \int \{ \frac{d}{dx}(x^2) \int \cos x \, dx \} \, dx$

$= x^2 \sin x - \int 2x \sin x \, dx$

$= x^2 \sin x - 2 \int x \sin x \, dx$

$= x^2 \sin x - 2 [x \int \sin x \, dx - \int \{ \frac{dx}{dx} \int \sin x \, dx \} \, dx]$

$= x^2 \sin x - 2 [- x \cos x + \int \cos x \, dx]$

$= x^2 \sin x + 2x \cos x - 2 \sin x + c$

$= \sin x (x^2 - 2) + 2x \cos x + c$ [Ans]

Integration By Partial Fraction

1. $$\frac{ax^2+bx+c}{(x+p)(x+q)(x+r)} = \frac{A}{x+p} + \frac{B}{x+q} + \frac{C}{x+r}$$

2. $$\frac{ax^2+bx+c}{(x+p)(qx^2+r)} = \frac{A}{x+p} + \frac{Bx+C}{qx^2+r}$$

3. $$\frac{ax^2+bx+c}{(x+p)(x+q)^2} = \frac{A}{x+p} + \frac{B}{x+q} + \frac{C}{(x+q)^2}$$

4. $$\frac{ax^2+bx+c}{(x+p)(x+q)} = A + \frac{B}{x+p} + \frac{C}{x+q}$$

* $\int \frac{dx}{x(x+1)(x-1)}$

Let, $\frac{1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$

Multiplication by $x(x+1)(x-1)$ the both sides

$1 = A(x+1)(x-1) + Bx(x-1) + C(x+1)x$

$x=1$ in eq (i)

$1 = 0+0+2C \quad \therefore C = \frac{1}{2}$

$\frac{1}{x(0+1)(0-1)} + \frac{1}{1(2)(x-1)} + \frac{1}{-1\times-2(x+1)}$

$= -\frac{1}{x} + \frac{1}{2(x-1)} + \frac{1}{2(x+1)}$

Again, $x=-1$ in eqn (i)

$1 = 0+2B+0 \quad \therefore B = \frac{1}{2}$

$x=0$ in the eq (i)

$1 = -A+0+0 \quad \therefore A = -1$

Now, $\frac{1}{x(x+1)(x-1)} = -\frac{1}{x} + \frac{1}{2}\frac{1}{(x+1)} + \frac{1}{2}\frac{1}{(x-1)}$

$\therefore I = \int \frac{dx}{x(x+1)(x-1)} = -\int \frac{dx}{x} + \frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{dx}{x-1}$

$= -\ln x + \frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(x-1) + c$

$= \frac{1}{2}\{\ln(x+1) + \ln(x-1)\} - \ln x + c$

$= \frac{1}{2}\ln(x+1)(x-1) - \ln x + c = \frac{1}{2}\{\ln(x+1)(x-1) - 2\ln x\} + c$

$= \frac{1}{2}\ln\frac{x^2-1}{x^2} + c$

$$I = \int \frac{x^2}{(x+1)^2(x+2)}\,dx \quad \text{—(i)}$$

$$\frac{x^2}{(x+1)^2(x+2)} = \frac{A}{(x+1)^2} + \frac{B}{(x+1)} + \frac{C}{x+2} \quad \text{—(ii)}$$

$$x^2 = A(x+2) + B(x+1)(x+2) + C(x+1)^2 \quad \text{—(iii)}$$

multiplying both sides by $(x+1)^2(x+2)$

Now, substituting $x = -1$ and $x = -2$ successively we get,

$A = 1$; $C = 4$

Hence, equating the co-efficient of x on both sides

(iii) we get $A + 3B + 2C = 0 \Rightarrow B = -9/3 = -3$

Thus (ii) gives,

$$\frac{x^2}{(x+1)^2(x+2)} = \frac{1}{(x+1)^2} + \frac{-3}{x+1} + \frac{4}{x+2}$$

Hence (i) gives

$$I = \int \frac{dx}{(x+1)^2} - \int \frac{3\,dx}{x+1} + \int \frac{4\,dx}{x+2}$$

$$= \frac{(x+1)^{-2+1}}{(-2+1)} - 3\ln(x+1) + 4\ln(x+2) + C$$

$$\therefore I = -\frac{1}{x+1} - 3\ln(x+1) + 4\ln(x+2) + C$$

(Ans)

$$\int \frac{x e^x}{(1+x)^2} dx = \int e^x \frac{x}{(1+x)^2} dx$$

Now, $\dfrac{x}{(1+x)} = \dfrac{A}{(1+x)} + \dfrac{B}{(1+x)^2}$

$\Rightarrow x = A(1+x) + B$ ——— (1)

$x = -1$ gives, $B = -1$ also,

$A + B = 0 \Rightarrow A = -B = 1$

Hence, $\dfrac{x}{(1+x)^2} = \dfrac{1}{1+x} - \dfrac{1}{(1+x)^2}$

Thus the integral becomes,

$$\int e^x \left[\frac{1}{1+x} - \frac{1}{(1+x)^2}\right] dx$$

$$= \int e^x \left[\frac{1}{(1+x)} + \frac{d}{dx}\left(\frac{1}{1+x}\right)\right] dx$$

$$= e^x \frac{1}{1+x} + c \quad \left[\because e^x \int (f(x) + f'(x))\, dx = e^x f(x) + c\right]$$

$$\int \frac{x^2}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

$x = -1$, $x = -2$, $x = 0$

$A = 1$, $B = 0$, $C = -4$

Now, $\dfrac{x^2}{(x+1)(x+2)^2} = \dfrac{1}{x+1} + 0 - \dfrac{4}{(x+2)^2} = \int \dfrac{dx}{x+1} - 4\int \dfrac{dx}{(x+2)^2}$

$= \ln(x+1) + \dfrac{4}{x+2} + c$ [Ans]

$$\int \frac{dx}{2e^{2x}+3e^{x}+1}$$

Let, $e^{x}=z$

$\therefore e^{x}dx = dz$

$\therefore dx = \frac{dz}{z}$

$$= \int \frac{dz}{z(2z^2+2z+z+1)}$$

$$= \int \frac{dz}{z(2z^2+2z+z+1)} = \int \frac{dz}{z\{2z(z+1)+1(z+1)\}}$$

$$= \int \frac{dz}{z(z+1)(2z+1)}$$

Let, $\frac{1}{z(z+1)(2z+1)} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{2z+1}$

Multiplying both sides by $z(z+1)(2z+1)$

then, we get

$\therefore 1 = A(z+1)(2z+1) + Bz(2z+1) + Cz(z+1)$ ---- (i)

Putting $z=0$ in (i)

$1 = A \quad \therefore A=1$

Putting $z=-1$ in (i)

$1 = 0 - B(-1) + 0$

$\therefore B = 1$

Putting $z = -\frac{1}{2}$ in (i)

$1 = -\frac{C}{2}\left(-\frac{1}{2}+1\right)$

$\therefore C = -4$

Now, $\int \frac{1}{z(z+1)(2z+1)} = \int \frac{dz}{z} + \int \frac{dz}{z+1} - 4\int \frac{dz}{2z+1}$

$= \ln z + \ln(z+1) - 4\cdot\frac{1}{2}\ln(2z+1) + C$

$= \ln e^x + \ln(e^x+1) - 2\ln(2e^x+1) + C$

$= x + \ln(e^x+1) - 2\ln(2e^x+1) + C$

(★) $\int \frac{3x^2+5x}{(x-1)(x+1)^2}\,dx$

Let, $\frac{3x^2+5x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$\therefore 3x^2+5x = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$ ——— (1)

Puttin $x=1$ in (1)

$8 = 4A$

$\therefore A = 2$

Puttin $x=-1$ in (1)

$3-5 = -2C$

$\therefore C = 1$

Puttn $x=0$ in (1)

$A-B-C=0$ or, $2-B-1=0$

$\therefore B=1$

Now $\frac{3x^2+5x}{(x-1)(x+1)^2} = \frac{2}{x-1} + \frac{1}{x+1} + \frac{1}{(x+1)^2}$

$\therefore \int \frac{3x^2+5x}{(x-1)(x+1)^2}\,dx = 2\int \frac{dx}{x-1} + \int \frac{dx}{x+1} + \int \frac{dx}{(x+1)^2}$

$= 2\ln(x-1) + 1\ln(x+1) - \frac{1}{x+1} + c$

$= \ln\left\{(x-1)^2(x+1)\right\} - \frac{1}{x+1} + c.$

$\int \frac{dx}{x^2+x-42} = -\frac{1}{13}\int \frac{dx}{x+7} + \frac{1}{13}\int \frac{dx}{x-6}$

$= \frac{1}{13}\ln \frac{x-6}{x+7} + c.$

Integration By Reduction

The process which is used to reduce the power of integrals successively is known as "integration by successive reduction."

⊛ Establish a reduction formula for $\int \sin^n x \, dx$ and evaluate $\int_0^{\pi/6} \sin^4 3x \, dx$

$$I_n = \int \sin^n x \, dx = \int \sin^{n-1} x \sin x \, dx$$

$$= \sin^{n-1} x \int \sin x \, dx - \int \left\{ \frac{d}{dx}\left(\sin^{n-1} x\right) \int \sin x \, dx \right\} dx$$

$$= \sin^{n-1} x \left(-\cos x\right) - (n-1) \int \sin^{n-2} x \cos x \left(-\cos x\right) dx$$

$$= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \left(1 - \sin^2 x\right) dx$$

$$= -\cos x \sin^{n-1} x + (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx$$

$$= -\cos x \cdot \sin^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow (1 + n - 1) I_n = -\cos x \cdot \sin^{n-1} x + (n-1) I_{n-2}$$

$$\Rightarrow n I_n = -\cos x \cdot \sin^{n-1} x + (n-1) I_{n-2}$$

$$\Rightarrow I_n = \frac{-\cos x \cdot \sin^{n-1} x}{n} + \frac{n-1}{n} I_{n-1} \quad \cdots (1)$$

that means $\int \sin^n x\,dx = \frac{-\cos x \cdot \sin^{n-1} x}{n} + \frac{(n-1)}{n}\int \sin^{n-2} x\,dx \quad \ldots(2)$

which is the required reduction formula.

2nd Part: $U_4 = \int_0^{\pi/6} \sin^4 3x\,dx$. $\quad 3x = z$.

$\Rightarrow 3\,dx = dz$

If $x = 0$ then, $z = 0$

$x = \pi/6$ then, $z = 3(\pi/6) = \pi/2$

$\therefore U_4 = \frac{1}{3}\int_0^{\pi/2} \sin^4 z\,dz$

$= \frac{1}{3}\int_0^{\pi/2} \sin^4 x\,dx \quad \because \int_a^b f(x)dx = \int_a^b f(z)dz$

taking limit 0 to $\pi/2$, Put, $n = 4, 2$ respectively.

$\int_0^{\pi/2} \sin^4 x\,dx = -\left[\frac{\cos x \sin^3 x}{4}\right]_0^{\pi/2} + \frac{3}{4}\int_0^{\pi/2} \sin^2 x\,dx$

$\Rightarrow \int_0^{\pi/2} \sin^4 x\,dx = 0 + \frac{3}{4}\int_0^{\pi/2} \sin^2 x\,dx \quad \ldots(3)$

$\Rightarrow$ Now, $\int_0^{\pi/2} \sin^2 x\,dx = -\left[\frac{\cos x \sin x}{2}\right]_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2} \sin^0 x\,dx$

$= 0 + \frac{1}{2}\int_0^{\pi/2} dx = \frac{1}{2}[x]_0^{\pi/2} = \frac{1}{2}(\pi/2 - 0)$

$\int_0^{\pi/2} \sin^2 x\,dx = \pi/4 \quad \ldots(4)$

From (3) & (4) $\Rightarrow$ $\int_0^{\pi/2} \sin^4 x\,dx = \frac{3}{4}\cdot\frac{\pi}{4}$

$\Rightarrow \int_0^{\pi/2} \sin^4 x\,dx = 3\pi/16$.

$U_4 = \frac{1}{3}\int_0^{\pi/2} \sin^4 x\,dx \Rightarrow U_4 = \frac{1}{3}\cdot 3\pi/16$ by (5)

$\therefore \int_0^{\pi/6} \sin^4 3x\,dx = \pi/16$ (Ans).

$\circledast$ $\int \cos^n x \, dx$

$$I_n = \int \cos^{n-1} x \cos x \, dx$$

$$= \cos^{n-1} x \int \cos x \, dx - \int \left\{ \frac{d}{dx}(\cos^{n-1} x) \int \cos x \, dx \right\} dx$$

$$= \cos^{n-1} x \sin x - (n-1) \int \cos^{n-2} x (-\sin x) \sin x \, dx$$

$$= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) \, dx$$

$$= \sin x \cos^{n-1} x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$$

$$= \sin x \cos^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$$

$$\Rightarrow (1+n-1) I_n = \sin x \cos^{n-1} x + (n-1) I_{n-2}$$

$$I_n = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}$$

$$\Rightarrow \int \cos^n x \, dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

(Ans)

$\circledast$ $\int x^m \sin nx \, dx$

$$I_m = \int x^m \sin nx \, dx$$

$$= x^m \int \sin nx - \int \left[\frac{d}{dx}(x^m) \int \sin nx \, dx \right] dx$$

$$= -x^m \frac{\cos nx}{n} + \int m x^{m-1} \frac{\cos nx}{n} \, dx$$

$$= -\frac{x^m \cos nx}{n} + \frac{m}{n}\int x^{m-1} \cos nx\, dx$$

$$= \frac{-x^m \cos nx}{n} + \frac{m}{n}\left[x^{m-1}\int \cos nx\, dx - \int \left\{\frac{d}{dx} x^{m-1} \int \cos nx\, dx\right] dx\right.$$

$$= \frac{-x^m \cos nx}{n} + \frac{m}{n}\left[\frac{x^{m-1}}{n} \sin nx - \frac{m-1}{n}\int x^{m-2} \sin nx\, dx\right]$$

$$= \frac{-x^m \cos nx}{n} + \frac{m}{n^2} x^{m-1} \sin nx - \frac{m(m-1)}{n^2}\int x^{m-2} \sin nx\, dx$$

$$= \frac{-x^m \cos nx}{n} + \frac{m \cdot x^{m-1} \sin nx}{n^2} - \frac{m(m-1)}{n^2} I_{m-2}$$

$$\Rightarrow I_m + \frac{m(m-1)}{n^2} I_{m-2} = \frac{-x^m \cos nx}{n} + \frac{m \cdot x^{m-1} \cdot \sin nx}{n^2}$$

which is required reduction formula (Ans)

(*) If $I_{m,n} = \int \cos^m x \cos nx \, dx$ then show that

$$I_{m,n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

and, evaluate $\int_0^{\pi/2} \cos^3 x \cos 2x \, dx$

Ans: $I_{m,n} = \int \cos^m x \cos nx \, dx$

$$= \cos^m x \int \cos nx \, dx - \int \left\{ \frac{d}{dx}(\cos^m x) \int \cos nx \, dx \right\} dx$$

$$= \frac{\cos^m x \sin nx}{n} - m \int \cos^{m-1} x (-\sin x) \frac{\sin nx}{n} dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \sin nx \sin x \, dx \quad \ldots (1)$$

Now, $\cos(n-1)x = \cos(nx - x)$

$\Rightarrow \cos(n-1)x = \cos nx \cdot \cos x + \sin nx \sin x$

$\Rightarrow \cos(n-1)x - \cos nx \cos x = \sin nx \sin x \quad$ —— (2)

From (1) & (2) $\Rightarrow$

$$I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} \int \cos^{m-1} x \left\{ \cos(n-1)x - \cos nx \cos x \right\} dx$$

$$\Rightarrow I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1}x \cos(n-1)x\,dx - \frac{m}{n}\int \cos^m x \cos nx\,dx$$

$$\Rightarrow I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} I_{m-1,n-1} - \frac{m}{n} I_{m,n}$$

$$\Rightarrow \left(1+\frac{m}{n}\right) I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow \left(\frac{m+n}{n}\right) I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$\therefore I_{m,n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1} \quad \text{--- (3)}$$

$$\int \cos^m x \cos nx\,dx = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n}\int \cos^{m-1}x \cos(n-1)x\,dx \quad \text{--- (4)}$$

take $0 \to \pi/2$, Put, $m=3,\ n=2$ successively
$m=2,\ n=1$

$$\int_0^{\pi/2} \cos^3 x \cos 2x\,dx = \left[\frac{\cos^3 x \sin 2x}{3+2}\right]_0^{\pi/2} + \frac{3}{3+2}\int_0^{\pi/2} \cos^2 x \cdot \cos x\,dx$$

$$\Rightarrow \int_0^{\pi/2} \cos^3 x \cos 2x \, dx = 0 + \frac{3}{5} \int_0^{\pi/2} \cos^2 x \cos x \, dx \quad \text{--- (5)}$$

And, $$\int_0^{\pi/2} \cos^2 x \cos x \, dx = \left[\frac{\cos^2 x \sin x}{2+1} \right]_0^{\pi/2} + \frac{2}{2+1} \int_0^{\pi/2} \cos x \, dx$$

$$\Rightarrow \int_0^{\pi/2} \cos^2 x \cos x \, dx = 0 + \frac{2}{3} \int_0^{\pi/2} \cos x \, dx = \frac{2}{3} \left[\sin x \right]_0^{\pi/2}$$

$$\Rightarrow \int_0^{\pi/2} \cos^2 x \cos x \, dx = \frac{2}{3} \left[\sin \frac{\pi}{2} - \sin 0 \right]$$

$$= \frac{2}{3} (1 - 0)$$

$$\therefore \int_0^{\pi/2} \cos^2 x \cos x \, dx = \frac{2}{3} \quad \text{--- (6)}$$

Now, (5) & (6)

$$\int_0^{\pi/2} \cos^3 x \cos 2x \, dx = \frac{3}{5} \cdot \frac{2}{3}$$

$$\therefore \int_0^{\pi/2} \cos^3 x \cos 2x \, dx = \frac{2}{5}.$$

If $I_{m,n} = \int \cos^m x \sin nx \, dx$ then,

$$I_{m,n} = \frac{-\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}.$$ Also.

evaluate: $\int_0^{\pi/2} \cos^5 x \sin 3x \, dx$. Ans: $(1/3)$.

Establish a reduction formula for $\int x^n e^{ax}\,dx$ and apply it to evaluate $\int x^2 e^{ax}\,dx$.

$$I_n = \int x^n e^{ax}\,dx = x^n \frac{e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\,dx \quad \text{--- (i)}$$

$$I_n = e^{ax}\frac{x^n}{a} - \left(\frac{n}{a}\right) I_{n-1}$$

which is the required reduction formula.

Put $n = 2, 1$ successively in (i)

$$I_2 = \int x^2 e^{ax}\,dx = x^2 e^{ax/a} - \frac{2}{a} I_1 \quad \text{--- (ii)}$$

$$I_1 = \int x e^{ax}\,dx = x e^{ax/a} - \int \left(e^{ax}/a\right) dx$$

$$= x e^{ax/a} - e^{ax/a^2} \quad \text{--- (iii)}$$

Now from (ii) & (iii)

$$I_2 = \int x^2 e^{ax}\,dx = x^2 e^{ax/a} - 2x e^{ax/a} + 2 e^{ax/a^3} + c$$

$$= \left(e^{ax/a^3}\right)\left(a^2x^2 - 2ax + 2\right) + c \quad \text{(Ans).}$$

$$\Rightarrow \int_0^{\pi/2} \sin^7 x\,dx = \frac{(7-1)(7-3)(7-5)}{7\cdot 5\cdot 3\cdot 1}\cdot 1 \quad \text{(by Wallis Formula)}$$

$$= \frac{6\cdot 4\cdot 2}{7\cdot 5\cdot 3\cdot 1}\cdot 1$$

⊛ Integrate $\sin^7 x$

Let, $I = \int \sin^7 x \, dx = \int (1-\cos^2 x)^3 \sin x \, dx$

Put, $\cos x = z$, $\sin x \, dx = -dz$

$\therefore I = -\int (1-z^2)^3 dz = -\int (1-3z^2+3z^4-z^6) dz$

$= -z + z^3 - \frac{3}{5} z^5 + \frac{1}{7} z^7 + C$

$= -\cos x + \cos^3 x - \frac{3}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C.$

When the power of $\sin x$ or $\cos x$ is positive odd integer it is unnessary to follow the reduction formula. The integral is easily obtained by the method.

⊛ Evaluate $\int \sin^6 x \, dx$

$I = \int \sin^6 x \, dx = -\frac{1}{6} \sin^5 x \cos x + \frac{5}{6} \int \sin^4 x \, dx$

$= -\frac{1}{6} \sin^5 x \cos x + \frac{5}{6} \left\{ -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 x \, dx \right\}$

$= -\frac{1}{6} \sin^5 x \cos x - \frac{5}{24} \sin^5 x \cos^4 x + \frac{5}{8} \int \sin^2 x \, dx$

$= -\frac{1}{6} \sin^5 x \cos x - \frac{5}{24} \sin^3 x \cos x - \frac{5}{16} \sin x \cos x$

$+ \frac{5}{16} x + C$

(Ans)

⊛ $\int \sin^m x \cdot \cos^n x \, dx$

Let, $I_{m,n} = \int \sin^m x \cdot \cos^n x \, dx$

$= \int \cos^{n-1} x \cos x \cdot \sin^m x \, dx$

$= \cos^{n-1} x \int \sin^m x \cdot \cos x \, dx - \int \left[\frac{d}{dx}(\cos^{n-1} x) \int \sin^m x \cdot \cos x \, dx \right] dx$

$= \cos^{n-1} x \int \sin^m x \cdot d(\sin x) - \int \left[(n-1) \cos^{n-2} x \, (-\sin x) \int \sin^m x \cdot d(\sin x) \right]$

$\left(\because \frac{d}{dx}(\sin x) = \cos x \;\; \therefore d(\sin x) = \cos x \, dx \right)$

$= \cos^{n-1} x \cdot \frac{\sin^{m+1} x}{m+1} + (n-1) \int \cos^{n-2} x \cdot \frac{\sin^{m+1} x}{m+1} \sin x \, dx$

$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \sin^{m+2} x \, dx$

$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \cdot \sin^m x \sin^2 x \, dx$

$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \sin^m x (1 - \cos^2 x) \, dx$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\left[\int \cos^{n-2} x \sin^m x - \int \cos^n x \sin^m x \, dx\right]$$

$$= \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2} x \sin^m x \, dx$$

$$- \left(\frac{n-1}{m+1}\right) I_{m,n}$$

$$\left(\text{where } I_{m,n} = \int \cos^n x \sin^m x \, dx\right]$$

$$\Rightarrow \left(1 + \frac{n-1}{m+1}\right) I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2} x \sin^m x \, dx$$

$$\Rightarrow \frac{m+n}{m+1} I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+1} + \frac{n-1}{m+1}\int \cos^{n-2} x \sin^m x \, dx$$

$$\Rightarrow I_{m,n} = \frac{1}{m+n} \cos^{n-1} x \sin^{m+1} x + \frac{n-1}{m+n}\int \cos^{n-2} x \sin^m x \, dx \quad \text{——— (ii)}$$

which is the required reduction formula.

Walli's formula

(n be a positive integer)

Statement: If n is positive integer, show that

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{when } (n \text{ is even})$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1 \quad (\text{when } n \text{ is odd})$$

Improper Integrals

(*) If the range of integration $[a,b]$ is not finite or if $f(x)$ is not defined or not bounded at one or more points of $[a,b]$ then the integral of $f(x)$ over this range is called improper integral.

Example:

$$\int_1^{\infty} \frac{1}{x^3}\,dx,\quad \int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}\,dx,\quad \int_{-\infty}^{\infty} x e^{-x^2}\,dx,\quad \int_0^3 \frac{1}{\sqrt{3-x}}\,dx$$

(*) The improper integral of f over the interval $[a, +\infty)$ is defined to be

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to +\infty} \int_a^b f(x)\,dx$$

In this case where the limit exists, the improper integral is said to be convergent and the limit is defined to be the value of the integral. In this case, where the limit does not exist, the improper integral is said to diverge and it is not assigned a value.

★ The improper integral of f over the integral $(-\infty, b]$ is defined to be $\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)\,dx$

The integral is said to converge if the limit exists and diverge if it does not.

★ The improper integral of a function f over the integerval $(-\infty, \infty)$ is defined as,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$$

where c is any real number and $c \in (-\infty, \infty)$. The integral is said to converge if both terms converge.

★ If f is continuous on the interval $[a, b]$ except for an infite discontinuity at a (the left-hand endpoint) then the improper integral of f over $[a, b]$ is defined as

$$\int_{a}^{b} f(x)\,dx = \lim_{k \to a^{+}} \int_{k}^{b} f(x)\,dx.$$

★ If f is continuous on the interval $[a, b]$ except for an infinite discontinuity at a point c in $[a, b]$ then the improper integral of f over $[a, b]$ is defined as

$$\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx$$

The improper integral is said to converge if both terms converge and diverge if both terms diverge.

যদি নির্দিষ্ট ইন্টিগ্রাল $\int_a^b f(x)\,dx$ এর (i) নিম্নসীমা a ঊর্ধ্বসীমা b সসীম হয় এবং

(ii) $a \le x \le b$ ব্যবধিতে $f(x)$ সংজ্ঞায়িত অবিচ্ছিন্ন হয় তবে $\int_a^b f(x)\,dx$ কে প্রকৃত (Proper) ইন্টিগ্রাল বলা হয়। প্রত্যেক প্রকৃত ইন্টিগ্রালই অভিসারী। (Convergent).

যদি উভয়ের যেকোনো একটি অথবা উভয় অসীম হয়, তবে $\int_a^b f(x)\,dx$ কে অপ্রকৃত ইন্টিগ্রাল বলা হয়।

যদি ইন্টিগ্রালের সীমা সসীম হয় অথচ $f(x)$ সংজ্ঞায়িত ব্যবধি এবং ভিন্ন ব্যবধিতে $f(x)$ সংজ্ঞায়িত অবিচ্ছিন্ন অথবা ব্যবধি সীমা বিন্দু ও ব্যবধির এক বা একাধিক বিন্দুতে $f(x)$ অসংজ্ঞায়িত বিচ্ছিন্ন হয়, তবে ঐ ধরনের ইন্টিগ্রালকে অপ্রকৃত ইন্টিগ্রাল বলা হয়।

অপ্রকৃত ইন্টিগ্রাল প্রধানত দুই প্রকার।

Improper integral of the first kind.

Improper integral of the 2nd kind.

Improper integral of the first kind:

এই প্রকারে: ইন্টিগ্রাল সীমাগুলির একটি ইন্টিগ্রাল একটি ইন্টিগ্রেশন সীমা (অসীম) হয় কিন্তু ব্যবধানে $f(x)$ সংজ্ঞায়িত অবিচ্ছিন্ন হয় অর্থাৎ $f(x)$ সংজ্ঞায়িত হলে থাকে। তবে ইন্টিগ্রাল সংজ্ঞায়িত

1. $$\int_a^{\infty} f(x)\,dx = \lim_{\epsilon \to 0} \int_a^{1/\epsilon} f(x)\,dx.$$

2. $$\int_{-\infty}^{b} f(x)\,dx = \lim_{\epsilon \to 0} \int_{-1/\epsilon}^{b} f(x)\,dx$$

3. $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

$$= \lim_{\epsilon \to 0} \int_{-1/\epsilon}^{a} f(x)\,dx + \lim_{\epsilon \to 0} \int_a^{1/\epsilon} f(x)\,dx.$$

যদি ডানপাশের সীমাগুলি বিদ্যমান হয় তবে convergent (অভিসারী) অন্যথায় অপসারী (divergent) হবে।

দ্বিতীয় প্রকার অপ্রকৃত ইন্টিগ্রাল:

এই প্রকার ইন্টিগ্রাল হলো যেখানে সসীম সীমা বিশিষ্ট ব্যবধির এক বা একাধিক বিন্দুতে $f(x)$ অসন্তত-বিচ্ছিন্ন হয়, অর্থাৎ $f(x)$ এর মান অসীম হয়। ইহার বিভিন্ন সংজ্ঞাসমূহ:

(i) যদি $[a,b]$ ব্যবধিতে a বিন্দুতে $f(x)$ অসন্তত-বিচ্ছিন্ন হয় এবং a বিন্দুতে $f(x)$ এর মান অসীম হয়, তবে

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0} \int_{a+\epsilon}^{b} f(x)\,dx$$

(ii) যদি $[a,b]$ ব্যবধিতে b বিন্দুতে $f(x)$ অসন্তত-বিচ্ছিন্ন হয় এবং b বিন্দুতে $f(x)$ এর মান অসীম হয় তবে,

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0} \int_{a}^{b-\epsilon} f(x)\,dx$$

(iii) যদি $a<c<b$ ব্যবধিতে c বিন্দুতে $f(x)$ অসন্তত-বিচ্ছিন্ন হয়, এবং c বিন্দুতে $f(x)$ এর মান অসীম হয় তবে

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{\epsilon\to 0} \int_a^{c-\epsilon} f(x)\,dx + \lim_{\epsilon_1\to 0} \int_{c+\epsilon_1}^{b} f(x)\,dx.$$

- Evaluate $\int_1^\infty \frac{dx}{x^3}$ using the definition of improper integral. Also, find out its convergent.

OR, Show whether the following integrals converge or diverge: (i) $\int_1^\infty \frac{dx}{x^3}$ (ii) $\int_1^\infty \frac{dx}{x}$.

Ans: Using the definition of improper integral of a function over some interval $[a, +\infty)$, we can write the given function in the following limiting form.

$$\int_1^\infty \frac{dx}{x^3} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^3} = \lim_{b\to+\infty} \left[\frac{1}{-2x^2}\right]_1^b$$

$$= \frac{1}{-2}\left[\lim_{b\to+\infty} \frac{1}{b^2} - 1\right]$$

$$= \frac{1}{-2}[0-1] = \frac{1}{2}.$$

In this case, the improper integral is said to converge.

$$\int_1^\infty \frac{dx}{x}.$$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty} \left[\ln|x|\right]_1^b.$$

$$= \lim_{b\to\infty} \left[\ln|b| - \ln|1|\right) = \lim_{b\to\infty} \ln|b| = \infty$$

hence has no value, divergent.

Evaluate the improper integral

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to-\infty} \int_a^0 \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty} \left[\tan^{-1}(x)\right]_a^0$$

$$= \lim_{a\to-\infty} \left[\tan^{-1}(0) - \tan^{-1}(a)\right]$$

$$= \lim_{a\to-\infty} \left[-\tan^{-1}(a)\right] = -\left(-\pi/2\right) = \pi/2$$

$\therefore$ Convergent.

(*) Evaluate $\int_{-\infty}^{\infty} f(x)\,dx$ where $f(x) = \frac{1}{1+x^2}$

Using the definition of improper integral over $(-\infty,\infty)$ with $c=0$ as $0\in(-\infty,\infty)$, we get

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx + \lim_{b\to+\infty} \int_{0}^{b} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to-\infty} \left[\tan^{-1}x\right]_a^0 + \lim_{b\to+\infty} \left[\tan^{-1}x\right]_0^b$$

$$= \pi/2 + \pi/2 = \pi$$

Since the integral converges in both terms hence the integral is convergent and its value is π.

$$\circledast \int_0^{\infty} \frac{x\,dx}{x^4+1} = \lim_{\epsilon \to 0} \int_0^{1/\epsilon} \frac{x\,dx}{(x^2)^2+1}$$

$$= \lim_{\epsilon \to 0} \frac{1}{2} \int_0^{1/\epsilon} \frac{2x}{(x^2)^2+1}\,dx$$

$$= \lim_{\epsilon \to 0} \frac{1}{2} \int_0^{1/\epsilon} \frac{d(x^2)}{(x^2)^2+1} = \lim_{\epsilon \to 0} \frac{1}{2} \left(\tan^{-1} x^2\right)_0^{1/\epsilon}$$

$$= \lim_{\epsilon \to 0} \frac{1}{2}\left[\tan^{-1} \frac{1}{\epsilon^2} - 0\right] = \frac{1}{2}\left[\tan^{-1} \infty\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}.$$ (Convergent)

$$\circledast \int_{-\infty}^{3} \frac{dx}{\sqrt{7-x}}$$

$$= \int_{-\infty}^{3} \frac{dx}{\sqrt{7-x}} = \lim_{\epsilon \to 0} \int_{-1/\epsilon}^{3} \frac{dx}{\sqrt{7-x}} = -\lim_{\epsilon \to 0} \int_{-1/\epsilon}^{3} \frac{-dx}{\sqrt{7-x}}$$

$$= -\lim_{\epsilon \to 0} \left[2\sqrt{7-x}\right]_{-1/\epsilon}^{3} = -2\lim_{\epsilon \to 0}\left[\sqrt{4} - \sqrt{7+1/\epsilon}\right]$$

$$= -2\left[2 - \sqrt{7+\infty}\right) = 2\left[-2+\infty\right) = \infty$$

$\therefore \int_{-\infty}^{3} \frac{dx}{\sqrt{7-x}}$ ও এর কোনো মান নাই। (has no value)
(divergent)

$$\int_{-\infty}^{\infty} \frac{x\,dx}{x^4+1}$$

দেওয়া আছে ইন্টিগ্রেশনের নিম্নসীমা এবং উর্ধ্বসীমা অসীম কাজেই সমাধান.

$$\int_{-\infty}^{\infty} \frac{x\,dx}{x^4+1} = \int_{-\infty}^{a} \frac{x\,dx}{x^4+1} + \int_{a}^{\infty} \frac{x\,dx}{x^4+1}$$

$$= \lim_{\epsilon\to 0} \frac{1}{2}\int_{-1/\epsilon}^{a} \frac{2x}{(x^2)^2+1}\,dx + \lim_{\epsilon\to 0} \frac{1}{2}\int_{0}^{1/\epsilon} \frac{2x\,dx}{(x^2)^2+1}$$

$$= \lim_{\epsilon\to 0} \frac{1}{2}\int_{-1/\epsilon}^{a} \frac{d(x^2)}{(x^2)^2+1} + \lim_{\epsilon\to 0} \frac{1}{2}\int_{0}^{1/\epsilon} \frac{d(x^2)}{(x^2)^2+1}$$

$$= \lim_{\epsilon\to 0} \frac{1}{2}\left(\tan^{-1}(x^2)\right]_{-1/\epsilon}^{a} + \lim_{\epsilon\to 0} \frac{1}{2}\left[\tan^{-1}(x^2)\right]_{a}^{1/\epsilon}$$

$$= \lim_{\epsilon\to 0} \frac{1}{2}\left(\tan^{-1}(a^2) - \tan^{-1}(1/\epsilon^2)\right] +$$

$$\lim_{\epsilon\to 0} \frac{1}{2}\left[\tan^{-1}(1/\epsilon^2) - \tan^{-1}(a^2)\right]$$

$$= \lim_{\epsilon\to 0} \frac{1}{2}\left[\tan^{-1}(a^2) - \tan^{-1}(1/\epsilon^2) + \tan^{-1}(1/\epsilon^2) - \tan^{-1}(a^2)\right]$$

$$= \lim_{\epsilon\to 0} (0) = 0. \quad \text{(Ans)}$$

(Convergent)

$\circledast$ $\displaystyle\int_0^\infty \frac{x\,dx}{e^{x^2}}$ (উচ্চসীমা ∞)

$$\int_0^\infty x e^{-x^2} dx = \lim_{\epsilon\to 0} \int_0^{1/\epsilon} x e^{-x^2} dx = \lim_{\epsilon\to 0} \frac{1}{2}\int_0^{1/\epsilon} 2x e^{-x^2} dx$$

$$= \lim_{\epsilon\to 0} \frac{1}{2}\int_0^{1/\epsilon} e^{-x^2} d(x^2) = \lim_{\epsilon\to 0} \frac{1}{2}\left[-e^{-x^2}\right]_0^{1/\epsilon}$$

$$= \lim_{\epsilon\to 0} \frac{1}{2}\left(-e^{-1/\epsilon^2} + 1\right) = \frac{1}{2}\left(-e^{-\infty} + 1\right)$$

$$= \frac{1}{2}(-0+1) = \frac{1}{2}.$$

$\circledast$ $\displaystyle\int_0^\infty x e^{-x} dx$

$$= \int_0^\infty x e^{-x} dx = \lim_{\epsilon\to 0} \int_0^{1/\epsilon} x e^{-x} dx$$

$$= \lim_{\epsilon\to 0} \left[x(-e^{-x}) - (1)(e^{-x})\right]_0^{1/\epsilon}$$

$$= -\lim_{\epsilon\to 0} \left[x e^{-x} + e^{-x}\right]_0^{1/\epsilon}$$

$$= -\lim_{\epsilon\to 0} \left[\frac{1}{\epsilon} e^{-1/\epsilon} + e^{-1/\epsilon} - 0 - 1\right]$$

$$= -\lim_{\epsilon \to 0} \frac{1/\epsilon}{e^{1/\epsilon}} - \lim_{\epsilon \to 0} e^{-1/\epsilon} + 1$$

$$= -\lim_{\epsilon \to 0} \frac{1/\epsilon}{e^{1/\epsilon}} - 0 + 1 \qquad (e^{-\infty} = 0)$$

$$= -\lim_{\epsilon \to 0} \frac{-1/\epsilon^2}{(-1/\epsilon^2)} + 1$$

$$= -\lim_{\epsilon \to 0} e^{-1/\epsilon} + 1 = -0 + 1 = 1 \quad \text{(Ans)}$$

(*) Verify whether the given integral is improper and hence evaluate it.

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

Ans:

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^2}} \text{ ——— } (1) \quad \text{as } x \to 1$$

$I \to \infty$ hence the given integral (1) is an improper integral.

Now using the definition of improper integral we can write,

$$I = \int_0^1 \frac{dx}{\sqrt{1-x}} \quad \cancel{\epsilon \to 0}\ \lim_{k \to 1^-}$$

$$= \lim_{\epsilon \to 0} \int_0^{1+\epsilon} \frac{dx}{\sqrt{1-x}} = \lim_{\epsilon \to 0} \int_0^{1+\epsilon} (1-x)^{-1/2}\,dx$$

$$= \lim_{\epsilon \to 0} \left[\frac{(1-x)^{1-1/2}}{1-1/2} \right]_0^{1+\epsilon}$$

$$\frac{2-1}{2} = 1/2$$

$$= +2 \lim_{\epsilon \to 0} \left[\sqrt{1-x} \right]_0^{1+\epsilon}$$

$$1 - 1/2 \qquad \frac{2-1}{2} = 1/2$$

$$= 2 \lim_{\epsilon \to 0} \left[\sqrt{1-1} - 1 \right]$$

$$= 2(0-1) = -2$$

OR

$$\lim_{k \to 1^-} \int_0^k \frac{dx}{\sqrt{1-x}} = \lim_{k \to 1^-} \int_0^k (1-x)^{-1/2}\,dx = \lim_{k \to 1^-} \left[\frac{(1-x)^{1-1/2}}{1-1/2} \right]_0^k$$

$$= 2 \lim_{k \to 1^-} \left[\sqrt{1-k} - 1 \right] = -2.$$

$$\int_1^3 \frac{x\,dx}{(x^2-1)^{2/3}} \qquad f(x) = \frac{x}{(x^2-1)^{2/3}}$$

$$\therefore f(1) = 1/0 = \infty$$

যেহেতু ব্যবধির $x = 1$ বিন্দুতে $f(x)$ অসংজ্ঞায়িত সেহেতু $x=1$ একটি অসীমতা বিন্দু, সুতরাং

$$\int_1^3 \frac{x\,dx}{(x^2-1)^{2/3}} = \lim_{\epsilon\to 0} \frac{1}{2}\int_{1+\epsilon}^{3} \frac{2x\,dx}{(x^2-1)^{2/3}}$$

$$= \lim_{\epsilon\to 0} \frac{1}{2}\int_{1+\epsilon}^{3} \frac{d(x^2-1)}{(x^2-1)^{2/3}}$$

$$= \lim_{\epsilon\to 0} \frac{1}{2}\left[\frac{(x^2-1)^{-2/3+1}}{1/3}\right]_{1+\epsilon}^{3}$$

$$= \lim_{\epsilon\to 0} \frac{3}{2}\left[(x^2-1)^{1/3}\right]_{1+\epsilon}^{3}$$

$$= \lim_{\epsilon\to 0} \frac{3}{2}\left[(2^3)^{1/3} - \{(1+\epsilon)^2 - 1\}^{1/3}\right]$$

$$= \lim_{\epsilon\to 0} \frac{3}{2}\left(2-(2\epsilon+\epsilon^2)^{1/3}\right] = \frac{3}{2}(2-0) = 3$$

Ans : 3

$$\int_0^1 \frac{1}{x}\,dx \qquad \begin{matrix} 0 \to 1 \\ \text{ও } 0+ \end{matrix}$$

$$= \lim_{a\to 0^+} \int_a^1 \frac{1}{x}\,dx = \lim_{a\to 0^+} \Big[\ln|x|\Big]_a^1$$

$$= \lim_{a\to 0^+} \Big[\ln|1| - \ln|a|\Big]$$

$$= \lim_{a\to 0^+} \Big[-\ln(a)\Big] = -(-\infty) = \infty.$$

(divergent)

$$\int_0^4 \frac{1}{\sqrt[3]{4-x}}\,dx \qquad \begin{matrix} 0 \to 4. \\ 4^- \end{matrix}$$

$$= \lim_{b\to 4^-} \int_0^b (4-x)^{-1/3}\,dx$$

$$= \lim_{b\to 4^-} \Big[-\tfrac{3}{2}(4-x)^{2/3}\Big]_0^b$$

$$= \lim_{b\to 4^-} \Big[-\tfrac{3}{2}(4-b)^{2/3} - \big(-\tfrac{3}{2}(4-0)^{2/3}\Big]$$

$$= \tfrac{3}{2}(4)^{2/3} = \tfrac{3}{2}(16)^{1/3} = \tfrac{3}{2}\times 2\times 2^{1/3}$$

$$= 3.2^{1/3}$$

(Convergent)

(*) $\displaystyle\int_0^6 \frac{x}{x-2}\,dx$ except for 2 in $(0,6)$

$$= \int_0^2 \frac{x}{x-2}\,dx + \int_2^6 \frac{x}{x-2}\,dx$$

$$= \lim_{b\to 2^-} \int_0^b \frac{x}{x-2}\,dx = \lim_{b\to 2^-} \int_0^b \frac{x-2+2}{x-2}\,dx$$

$$= \lim_{b\to 2^-} \left[x+2\ln(x-2)\right]_0^b = \lim_{b\to 2^-}\left[\left(b+2\ln|b-2|\right) - \left(0+2\ln|0-2|\right)\right]$$

$$= -\infty$$

(divergent)

same as $\displaystyle\int_2^6 \frac{x}{x-2}\,dx$

(4) $\displaystyle\int_1^4 \frac{dx}{(x-2)^{2/3}}$

The integrand $\to \infty$ as $x \to 2$ and $2 \in (1,4)$

Hence by appropriate definition of improper integral with infinite discontinuity at some point $c \in (a,b)$.

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}}$$

$$= \lim_{k \to 2^-} \int_1^k \frac{dx}{(x-2)^{2/3}} + \lim_{k \to 2^+} \int_k^4 \frac{dx}{(x-2)^{2/3}}$$

$$= \lim_{k \to 2^-} \left[3(k-2)^{1/3} - 3(1-2)^{1/3} \right] +$$

$$\lim_{k \to 2^+} \left[3(4-2)^{1/3} - 3(k-2)^{1/3} \right]$$

$= 3 + 3\sqrt[3]{2}$ and the integral converges as each term convergent

Definite Integrals

If $F(x)$ is an integral of $f(x)$ then we write

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

Fundamental Theorem of Integral Calculus:

If $f(x)$ is integrable in (a, b), $a < b$ and if there exists a function $F'(x) = f(x)$ in (a, b) then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

☆ $\int_0^1 x^3 \sqrt{1+3x^4}\, dx$

Let, $1+3x^4 = z$

$\therefore 12x^3\, dx = dz$

$\Rightarrow x^3\, dx = \frac{1}{12}\, dz$

when, $x=1,\ z=4$

$x=0,\ z=1$

$= \int_1^4 \frac{1}{12}\sqrt{z}\, dz$

$= \int_1^4 \frac{1}{12} z^{1/2} dz$

$= \frac{1}{12}\left[\frac{z^{3/2}}{3/2}\right]_1^4$

$= \frac{1}{12} \times \frac{2}{3}\left(4^{3/2} - 1^{3/2}\right)$

$= \frac{1}{18}\left(2^3 - 1\right)$

$= 7/18$ Ans: $7/18$

☆ $\int_0^1 \frac{dx}{(1+x^2)^{3/2}}$

Put,

$x = \tan\theta$

$\therefore dx = \sec^2\theta\, d\theta$

$x=1,\ \theta = \pi/4$

$x=0,\ \theta=0$

$= \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)^{3/2}}$

$= \int_0^{\pi/4} \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}}\, d\theta$

$= \int_0^{\pi/4} \frac{1}{\sec\theta}\, d\theta$

$= \int_0^{\pi/4} \cos\theta\, d\theta$

$= [\sin\theta]_0^{\pi/4}$

$= \sin \pi/4 - \sin 0^\circ$

$= \frac{1}{\sqrt{2}} - 0 = \frac{1}{\sqrt{2}}$

Ans: $\frac{1}{\sqrt{2}}$

$$\int_0^1 \frac{\sqrt{\tan^{-1}x}}{1+x^2}\,dx = \int_0^{\pi/4} \sqrt{z}\,dz$$

Let,

$\tan^{-1}x = z$

$\therefore \frac{1}{1+x^2}dx = dz$

when, $x = 1,\ z = \pi/4$

$x = 0,\ z = 0$

$$= \frac{2}{3}\left[z^{3/2}\right]_0^{\pi/4}$$

$$= \frac{2}{3}\left(\pi/4\right)^{3/2} - 0$$

$$= \frac{2}{3}\,\frac{\pi^{3/2}}{4^{3/2}} = \frac{2}{3}\,\frac{\pi^{3/2}}{2^3} = \frac{2}{3}\cdot\frac{2}{8}\,\pi^{3/2} = \frac{2}{24}\,\pi^{3/2} \text{ (Ans)}.$$

$$\int_0^{\infty} \frac{x^2\,dx}{(1+x^2)^2}$$

$$= \int_{\pi/4}^{\pi/2} \frac{\tan^2\theta\,\sec^2\theta}{(1+\tan^2\theta)^2}\,d\theta$$

Let, $x = \tan\theta$

$\therefore dx = \sec^2\theta\,d\theta$

$$= \int_{\pi/4}^{\pi/4} \frac{\tan^2\theta\,\sec^2\theta\,d\theta}{\sec^4\theta}$$

when, $x = \infty,\ \theta = \pi/2$

$x = 1,\ \theta = \pi/4$

$$= \int_{\pi/4}^{\pi/4} \frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/4} \frac{\sin^2\theta}{\cos^2\theta}\,\cos^2\theta\,d\theta.$$

$$= \frac{1}{2}\int_{\pi/4}^{\pi/4} 2\sin^2\theta \, d\theta = \frac{1}{2}\int_{\pi/4}^{\pi/2} (1-\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\Big[\theta - \frac{\sin 2\theta}{2}\Big]_{\pi/4}^{\pi/2}$$

$$= \frac{1}{2}\Big[\pi/2 - \frac{1}{2}\sin 2\,\pi/2 - \pi/4 + 1/2 \sin 2\,\pi/4\Big]$$

$$= 1/2\Big[\pi/2 - 1/2 \sin\pi - \pi/4 + 1/2 \sin \pi/2\Big]$$

$$= \pi + 1/4 = 1/8(\pi+2) \text{ (Ans)}$$

$$\int_{-a}^{a} x e^{-x^2} dx$$

$$= \int x e^{-x^2} dx$$

Let, $x^2 = z$

$\therefore 2x\,dx = dz$

$\therefore x\,dx = \frac{1}{2} dz$

$$= \int \frac{1}{2} dz\, e^{-z}$$

$$= \frac{1}{2}\int e^{-z} dz$$

$$= \frac{1}{2} \frac{e^{-z}}{-1}$$

$$= -1/2\, e^{-z} = -\frac{1}{2} e^{-x^2}$$

$(a)^2 = a^2$

$(-a)^2 = a^2$

$x^2 = z$

Now, taking the limits we get

$$\int_{-a}^{a} x e^{-x^2} dx = \Big[-\frac{1}{2} e^{-x^2}\Big]_{-a}^{a} = -\frac{1}{2} e^{-a^2} + \ldots$$

$$= 0$$

Ans: 0

(*) $\int_0^1 \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Put, $x = \sin\theta$

$\therefore dx = \cos\theta \, d\theta$

$x \to 0, \theta \to 0$

$x \to 1, \theta \to \pi/2$

Let, $I = \int_0^1 \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

$= \int_0^{\pi/2} \frac{\cos\theta \, d\theta}{(1+\sin^2\theta)\sqrt{1-\sin^2\theta}}$

$= \int_0^{\pi/2} \frac{1}{1+\sin^2\theta} d\theta$

$= \int_0^{\pi/2} \frac{1}{\cos^2\theta(\sec^2\theta + \tan^2\theta)} d\theta$

$= \int_0^{\pi/2} \frac{\sec^2\theta}{\sec^2\theta + \tan^2\theta} d\theta = \int_0^{\pi/2} \frac{\sec^2\theta}{(1+\tan^2\theta) + \tan^2\theta}$

$= \int_0^{\pi/2} \frac{\sec^2\theta \, d\theta}{1+2\tan^2\theta}$

Put, $\tan\theta = t$

$\Rightarrow \sec^2\theta \, d\theta = dt$

θ	0	$\pi/2$
t	0	∞

$= \int_0^{\infty} \frac{dt}{1+2t^2} = \frac{1}{2}\int_0^{\infty} \frac{dt}{\frac{1}{2}+t^2}$

$= \frac{1}{2}\int_0^{\infty} \frac{dt}{(\frac{1}{\sqrt{2}})^2 + (t)^2}$

$= \frac{1}{2}\left[\frac{1}{\sqrt{2}} \tan^{-1} \frac{1}{\sqrt{2}}\right]_0^{\infty}$

$= \frac{1}{2}\left[\sqrt{2} \tan^{-1} \sqrt{2}t\right]_0^{\infty}$

$= \frac{1}{2}\sqrt{2} \tan^{-1} \sqrt{2}(\infty) - 0 = \frac{\sqrt{2}}{2} \tan^{-1} \infty$

$= \frac{\sqrt{2}}{2} \cdot \pi/2 = \frac{\pi}{2\sqrt{2}}$

Ans: $\pi/2\sqrt{2}$

Properties of Definite Integrals

1. $\int_a^b f(x)\,dx = \int_a^b f(z)\,dz$

2. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

4. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ Lower limit $= 0$

5. $\int_a^{na} f(x)\,dx = n\int_0^a f(x)\,dx$ if $f(a+x) = f(x)$

$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$ if $f(2a-x) = f(x)$

$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$

$\int_0^a f(x)\,dx = 2\int_0^{a/2} f(x)\,dx$ if $f(a-x) = f(x)$

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx \quad \ldots (1)$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2 - x)} + \sqrt{\cos(\pi/2 - x)}}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}\, dx}{\sqrt{\cos x} + \sqrt{\sin x}} \quad \ldots (2)$$

(1) + (2) $\Rightarrow$

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\, dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \pi/2 \quad \therefore I = \pi/4$$

(Ans)

$$\boxed{\therefore \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} = \pi/4}$$

★ $\displaystyle\int_0^{\pi/2} \frac{d\theta}{1+\tan\theta}$

$$I = \int_0^{\pi/2} \frac{d\theta}{1+\tan\theta}$$

$$= \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta+\cos\theta} \quad \text{--- (i)}$$

$$= \int_0^{\pi/2} \frac{\cos(\pi/2-\theta)}{\sin(\pi/2-\theta)+\cos(\pi/2-\theta)}$$

$$I = \int_0^{\pi/2} \frac{\sin\theta}{\sin\theta+\cos\theta}\,d\theta \quad \text{--- (ii)}$$

$$\therefore I + I = \int_0^{\pi/2} \frac{\sin\theta+\cos\theta}{\sin\theta+\cos\theta}\,d\theta$$

$$\Rightarrow 2I = \int_0^{\pi/2} d\theta = [\theta]_0^{\pi/2} = \pi/2$$

$$\therefore I = \pi/4.$$

OR

$$I = \int_0^{\pi/2} \frac{d\theta}{1+\tan\theta} \quad \text{--- (i)}$$

$$= \int_0^{\pi/2} \frac{d\theta}{1+\tan(\pi/2-\theta)}$$

$$= \int_0^{\pi/2} \frac{d\theta}{1+\cot\theta}$$

$$I = \int_0^{\pi/2} \frac{\tan\theta}{1+\tan\theta}\,d\theta \quad \text{--- (ii)}$$

$$\therefore 2I = \int_0^{\pi/2} \frac{1+\tan\theta}{1+\tan\theta}\,d\theta \quad \text{(i+ii)}$$

$$\Rightarrow 2I = \int_0^{\pi/2} d\theta$$

$$= [\theta]_0^{\pi/2}$$

$$= \pi/2$$

$$\therefore I = \pi/4.$$

Prove that, $\displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x - \cos x} = \pi/4$

$$I = \int_0^{\pi/2} \frac{\sin x \, dx}{\sin x - \cos x} \quad \text{---} (i)$$

Do $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\cot x} = \pi/4$.

$$= \int_0^{\pi/2} \frac{\sin(\pi/2 - x)}{\sin(\pi/2 - x) - \cos(\pi/2 - x)} \, dx$$

$$I = \int_0^{\pi/2} \frac{\cos x \, dx}{\cos x - \sin x} = \int_0^{\pi/2} \frac{-\cos x}{\sin x - \cos x} \quad \text{---} (ii)$$

$$\therefore 2I = \int_0^{\pi/2} \frac{\sin x \, dx}{\sin x - \cos x} - \int_0^{\pi/2} \frac{\cos x \, dx}{\sin x - \cos x}$$

$$= \int_0^{\pi/2} \frac{\sin x - \cos x}{\sin x - \cos x} \, dx$$

$$\therefore 2I = \int_0^{\pi/2} dx$$

$$= [x]_0^{\pi/2} = \pi/2 \qquad \therefore I = \pi/4 \quad \text{(Ans)}$$

$$\star \int_0^{\pi} \frac{x}{1+\sin x}\,dx$$

$$\underline{\text{change}}$$

$$\int_0^{\pi} \frac{x}{1+\sin x} \longleftrightarrow \int_0^{\pi} \frac{(\pi-x)}{1+\sin x}$$

$$I = \int_0^{\pi} \frac{x\,dx}{1+\sin x} \quad \ldots (i)$$

$$= \int_0^{\pi} \frac{(\pi-x)\,dx}{1+\sin(\pi-x)}$$

$$I = \int_0^{\pi} \frac{(\pi-x)}{1+\sin x} \quad \ldots (2)$$

$\underline{(1)+(ii)}$

$$2I = \int_0^{\pi} \frac{x+\pi-x}{1+\sin x}\,dx$$

$$= \pi \int_0^{\pi} \frac{dx}{1+\sin x}$$

[illegible] $f(a-x) = f(x)$

$$\frac{dx}{1+\sin(\pi-x)}$$

$$\Rightarrow 2I = 2\pi \int_0^{\pi/2} \frac{dx}{1+\sin x}$$

$$\left(\frac{dx}{1+\sin x} \longleftrightarrow \frac{dx}{1+\sin x} \right)$$

(No change)

$$I = \pi \int_0^{\pi/2} \frac{dx}{1 + \frac{2\tan x/2}{1+\tan^2 x/2}}$$

$$= \pi \int_0^{\pi/2} \frac{1+\tan^2 x/2}{1+\tan^2 x/2 + 2\tan x/2}\,dx$$

$$I = \pi \int_0^{\pi/2} \frac{1+\tan^2 x/2}{1+\tan^2 x/2 + 2\tan x/2}\, dx$$

$$= \pi \int_0^{\pi/2} \frac{\sec^2 x/2\, dx}{(1+\tan x/2)^2}$$

Let, $1+\tan x/2 = z$

$\Rightarrow \sec^2 x/2\, dx = 2dz$

$\therefore x = 0$, $z = 1+0 = z$

$x = \pi/2$, $z = 1+\tan \pi/4 = 2$

$$I = 2\pi \int_1^2 \frac{dz}{z^2}$$

$$= -2\pi \left[\frac{1}{z}\right]_1^2$$

$$= -2\pi\left(\tfrac{1}{2}-1\right)$$

$$= \pi$$

$\therefore I = \pi$

(Ans)

$$\int_0^{\pi} \frac{x\,dx}{1+\cos^2 x} \quad \text{———(I)}$$

$$I = \int_0^{\pi} \frac{x\,dx}{1+\cos^2 x} = \int_0^{\pi} \frac{(\pi - x)\,dx}{1+\{\cos(\pi - x)\}^2} = \int_0^{\pi} \frac{(\pi - x)\,dx}{1+\cos^2 x} \quad \text{———(II)}$$

(1) + (2) $\Rightarrow$

$$2I = \int_0^{\pi} \frac{(x+\pi-x)\,dx}{1+\cos^2 x} = \pi \int_0^{\pi} \frac{dx}{1+\cos^2 x}$$

(same)

$$\Rightarrow 2I = 2\pi \int_0^{\pi/2} \frac{dx}{1+\cos^2 x}$$

$\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$

$f(2a-x) = f(x)$

OR, $\int_0^{a} f(x)dx = 2\int_0^{a/2} f(x)dx$ if $f(a-x) = f(x)$

$$\Rightarrow 2I = 2\pi \int_0^{\pi/2} \frac{\sec^2 x\,dx}{\sec^2 x + 1}$$

$$= 2\pi \int_0^{\pi/2} \frac{\sec^2 x\,dx}{\tan^2 x + 2}$$

[divided by $\cos^2 x$]

Let, $\tan x = z \Rightarrow \sec^2 x\,dx = dz$

when, $x = 0$ then $z = \tan 0 = 0$

$x = \pi/2$ then $z = \tan \pi/2 = \infty$

$$\therefore I = \pi \int_0^{\infty} \frac{dz}{z^2 + (\sqrt{2})^2} = \frac{\pi}{\sqrt{2}} \left[\tan^{-1} \frac{z}{\sqrt{2}}\right]_0^{\infty}$$

$$= \pi/\sqrt{2}\,[\tan^{-1}\infty - \tan^{-1} 0] = \pi/\sqrt{2}\,(\pi/2 - 0)$$

$$\therefore \int_0^{\pi} \frac{x\,dx}{1+\cos^2 x} = \pi^2/2\sqrt{2}$$

(Ans)

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$$

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

If $x = 0 \Rightarrow 0 = \tan\theta$
$\Rightarrow \theta = 0^\circ$

If $x = 1 \Rightarrow$
$1 = \tan\theta$
$\Rightarrow \theta = \pi/4$

$$\therefore I = \int_0^{\pi/4} \frac{\ln(1+\tan\theta)\cdot\sec^2\theta\, d\theta}{1+\tan^2\theta}$$

$$= \int_0^{\pi/4} \frac{\ln(1+\tan\theta)\,\sec^2\theta}{\sec^2\theta}$$

$$= \int_0^{\pi/4} \ln(1+\tan\theta)\,d\theta \quad \cdots (1)$$

$$\Rightarrow I = \int_0^{\pi/4} \ln\{1+\tan(\pi/4-\theta)\}\,d\theta$$

$$= \int_0^{\pi/4} \ln\left\{1+\frac{\tan\pi/4-\tan\theta}{1+\tan\pi/4\cdot\tan\theta}\right\}d\theta$$

$$= \int_0^{\pi/4} \ln\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta = \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan\theta}\right)$$

$$= \int_0^{\pi/4} \{\ln 2 - \ln(1+\tan\theta)\}\,d\theta$$

$$= \ln 2 \int_0^{\pi/4} d\theta - \int_0^{\pi/4} \ln(1+\tan\theta)\,d\theta$$

$$\Rightarrow I = \ln 2\,[\theta]_0^{\pi/4} - I \quad \Rightarrow 2I = \ln 2\,(\pi/$$

$$\Rightarrow 2I = \pi/4 \ln 2$$

$$\therefore I = \pi/8 \ln 2$$

(A)

$$I = \int_0^\infty \frac{dx}{(1+x)(1+x^2)} = \frac{\pi}{4}.$$

Put $x = \tan\theta$

$\therefore dx = \sec^2\theta \, d\theta$

x	0	∞
θ	0	$\pi/2$

$$= \int_0^{\pi/2} \frac{\sec^2\theta \, d\theta}{(1+\tan\theta)(1+\tan^2\theta)}$$

$$= \int_0^{\pi/2} \frac{d\theta}{1+\tan\theta}$$

$$I = \int_0^{\pi/2} \frac{\cos\theta \, d\theta}{\sin\theta + \cos\theta} \quad \text{--- (i)}$$

$$= \int_0^{\pi/2} \frac{\cos(\pi/2 - \theta) \, d\theta}{\sin(\pi/2-\theta) + \cos(\pi/2 - \theta)}$$

$$I = \int_0^{\pi/2} \frac{\sin\theta}{\sin\theta + \cos\theta} \, d\theta \quad \text{--- (ii)}$$

$\therefore$ or, $$2I = \int_0^{\pi/2} \frac{\sin\theta + \cos\theta}{\sin\theta + \cos\theta} \, d\theta$$

$$\therefore 2I = \int_0^{\pi/2} d\theta = [\theta]_0^{\pi/2} = \pi/2$$

$$\therefore I = \pi/4 \quad \text{(Ans)}$$

$$\star \quad A = \int_1^2 \int_0^1 \int_{-1}^1 (x^2+y^2+z^2)\,dx\,dy\,dz$$

$$= \int_1^2 \int_0^1 \left[x^3/3 + xy^2 + xz^2 \right]_{-1}^{1} dy\,dz$$

$$= \int_1^2 \int_0^1 \left\{ \left(\frac{1}{3} + y^2 + z^2\right) - \left(-1/3 - y^2 - z^2\right) \right\} dy\,dz$$

$$= \int_1^2 \int_0^1 \left(2/3 + 2y^2 + 2z^2 \right) dy\,dz$$

$$= \int_1^2 \left[2/3\, y + 2y^3/3 + 2yz^2 \right]_0^1 dz$$

$$= \int_1^2 \left(2/3 + 2/3 + 2z^2 - 0 \right) dz = \int_1^2 \left(\frac{4}{3} + 2z^2 \right) dz$$

$$= \left[\frac{4z}{3} + \frac{2z^3}{3} \right]_1^2 = 8/3 + \frac{16}{3} - \frac{4}{3} - 2/3$$

$$= 8 - 2 = 6.$$

Ans 06

$$\int_0^1 \int_0^{1-x} \int_0^{1-y^2} z \, dz \, dy \, dx$$

$$A = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_0^{1-y^2} dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2}(1-y^2)^2 \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} (1-2y^2+y^4) \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 \left(y - \frac{2y^3}{3} + \frac{y^5}{5}\right]_{y=0}^{1-x} dx$$

$$= \frac{1}{2} \int_0^1 \left((1-x) - \frac{2}{3}(1-x)^3 + \frac{1}{5}(1-x)^5\right] dx$$

$$= \frac{1}{2} \left[\frac{(1-x)^2}{2(-1)} - \frac{2}{3} \frac{(1-x)^4}{4(-1)} + \frac{1}{5} \frac{(1-x)^6}{6(-1)} \right]_0^1$$

$$= \frac{1}{2} \left[-\frac{1}{2}(1-x)^2 + \frac{1}{6}(1-x)^4 - \frac{1}{30}(1-x)^6 \right]_0^1$$

$$= \frac{1}{2} \left[(0+0+0) - \left(-\frac{1}{2} + \frac{1}{6}\right) - \frac{1}{30} \right]$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{30}\right) = \frac{11}{60} \qquad \text{Ans: } \frac{11}{60}$$

Gamma and Beta Function

Gamma function :

The integral $\int_0^\infty x^{n-1} e^{-x} dx$ defined for positive value of n $(n>0)$ is known as gamma function and denoted by $\Gamma(n)$.

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx.$$

Beta function :

The integral $\int_0^1 x^{m-1} (1-x)^{n-1} dx$ and defined for positive values for m, n and $m>0$, $n>0$ is known as Beta function and is denoted by $B(m,n)$.

$$\therefore B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

⊛ Show that, $\Gamma(1) = 1$.

We know, $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx, \; n>0$

$$\Gamma(1) = \int_0^\infty x^{1-1} e^{-x} dx = \int_0^\infty e^{-x} dx = \left[e^{-x}\right]_0^\infty$$

$$= -(e^{-\infty} - e^{0}) = -(0-1) = 1 \qquad \therefore \Gamma(1) = 1$$

⊛ Prove that, $\beta(m,n) = \beta(n,m)$.

We know, $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx, \quad m>0, n>0$

Let, $x = 1-y \quad \therefore dx = -dy$

$x = 0 \Rightarrow 0 = 1-y \Rightarrow y = 1$

$x = 1 \Rightarrow 1 = 1-y \Rightarrow y = 0$

$$\therefore \beta(m,n) = -\int_1^0 (1-y)^{m-1} \{1-(1-y)\}^{n-1} dy$$

$$\Rightarrow \beta(m,n) = \int_0^1 (1-y)^{m-1} y^{n-1} dy$$

$$= \int_0^1 y^{n-1} (1-y)^{m-1} dy \qquad \because \int_a^b f(x)\,dx = \int_a^b f(z)\,dz$$

$$= \int_0^1 x^{n-1} (1-x)^{m-1} dx$$

$$= \beta(n,m) \qquad \therefore \beta(m,n) = \beta(n,m)$$

(Proved)

✪ Prove that, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

We know, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, $m>0, n>0$

$m = n = \frac{1}{2}$

$$\beta(\tfrac{1}{2},\tfrac{1}{2}) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}\,dx$$

$$\Rightarrow \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})} = \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\,dx$$

$$\Rightarrow \frac{\{\Gamma(\frac{1}{2})\}^2}{\Gamma(1)} = \int_0^1 \frac{dx}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}$$

Let, $x = \sin^2\theta \Rightarrow dx = 2\sin\theta.\cos\theta\,d\theta$

If $x = 0 \Rightarrow 0 = \sin^2\theta \Rightarrow \theta = 0$

$x = 1 \Rightarrow 1 = \sin^2\theta \Rightarrow \theta = \pi/2$

$$\therefore \{\Gamma(\tfrac{1}{2})\}^2 = \int_0^{\pi/2} \frac{2\sin\theta.\cos\theta\,d\theta}{\sin\theta.\cos\theta} = 2\int_0^{\pi/2} d\theta$$

$$\Rightarrow \{\Gamma(\tfrac{1}{2})\}^2 = 2[\theta]_0^{\pi/2} = 2(\pi/2 - 0) = \pi$$

$\therefore \Gamma(\frac{1}{2}) = \sqrt{\pi}$ (Proved)

Relation between Gamma & Beta function

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$$

$$\Gamma(n+1) = n! = n\,\Gamma(n) \quad ; \quad \Gamma(1/2) = \sqrt{\pi}$$

★ $\Gamma(n+1) = n\Gamma(n) = n!$

We know, $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx,\ n>0$ —— (1)

Put, $n = n+1$ in (1)

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx$$

$$\Rightarrow \Gamma(n+1) = \left[x^n(-e^{-x})\right]_0^\infty + n\int_0^\infty x^{n-1} e^{-x} dx$$

$$\Rightarrow \Gamma(n+1) = -\left[x^n e^{-x}\right]_0^\infty + n\Gamma(n) \quad [\text{by}(1)]$$

$$\Rightarrow \Gamma(n+1) = 0 + n\Gamma(n) \quad \because e^{-\infty} = 0$$

$$\therefore \Gamma(n+1) = n\Gamma(n) \quad (\text{Proved}) \text{ —— } (2)$$

Put, $n = n-1,\ n-2$

$$\Gamma(n) = (n-1)\Gamma(n-1) \text{ —— } (3)$$

and, $\Gamma(n-1) = (n-2)\Gamma(n-2)$ —— (4)

by (2), (3) and (4)

$$\Gamma(n+1) = n(n-1)(n-2)\Gamma(n-2)$$

Similarly,

$$\Gamma(n+1) = n(n-1)(n-2)\cdots 3.2.1\ \Gamma(1)$$

$$\Rightarrow \Gamma(n+1) = n(n-1)(n-2)\cdots 3.2.1$$

$$\because \Gamma(1) = 1$$

$$\therefore \Gamma(n+1) = n! \quad (\text{Proved}).$$

$\circledast$ $\int_0^1 x^6 \sqrt{1-x^2}\, dx$

Let, $x = \sin\theta$
$\Rightarrow dx = \cos\theta\, d\theta$
$x = 0 \Rightarrow 0 = \sin\theta \Rightarrow \theta = 0$
$x = 1 \Rightarrow 1 = \sin\theta \Rightarrow \theta = \pi/2$

$$I = \int_0^1 x^6 \sqrt{1-x^2}\, dx$$

$$= \int_0^{\pi/2} \sin^6\theta \sqrt{\cos^2\theta}\, \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^6\theta \cos^2\theta\, d\theta$$

$$= \frac{\Gamma(7/2)\,\Gamma(3/2)}{2\Gamma(5)} = \frac{5/2 \cdot 3/2 \cdot 1/2 \cdot \Gamma(1/2)\ 1/2\, \Gamma(1/2)}{2 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= \frac{5/2 \cdot 3/2 \cdot 1/2 \cdot \sqrt{\pi}\ 1/2 \cdot \sqrt{\pi}}{2 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{5\pi}{256} \quad \text{(Ans)}$$

$\circledast$ $\int_0^1 x^2 \sqrt{1-x^2}\, dx$

Put, $x = \sin\theta$
$\Rightarrow dx = \cos\theta\, d\theta$

$$= \int_0^{\pi/2} \sin^2\theta \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta \cos^2\theta\, d\theta$$

$$= \frac{\Gamma(3/2)\,\Gamma(3/2)}{2\,\Gamma\left(\frac{2+2+2}{2}\right)} = \frac{1/2\sqrt{\pi}\ 1/2\sqrt{\pi}}{2\,\Gamma 3} = \frac{\pi}{2\times2\times2\times2\,\Gamma 1}$$

$$= \frac{\pi}{8\times2} = \pi/16.$$

Ans $\pi/16$

(*) $\int_0^1 x^3 (1-x^2)^{5/2}\, dx$

Put $x = \sin\theta$

$dx = \cos\theta\, d\theta$

x	0	1
θ	0	$\pi/2$

$$= \int_0^{\pi/2} \sin^3\theta\, (1-\sin^2\theta)^{5/2} \cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^3\theta\, \cos^6\theta\, d\theta$$

$$= \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{6+1}{2}\right)}{2\Gamma\left(\frac{3+6+2}{2}\right)} = \frac{\Gamma 2\, \Gamma 7/2}{2\Gamma 11/2} = \frac{\Gamma(7/2)}{2 \cdot 9/2 \cdot 7/2\, \Gamma 7/2}$$

$$= 2/63 \qquad \text{Ans: } 2/63$$

(*) $\int_0^\infty \frac{x^3\, dx}{(1+x^2)^{9/2}}$

Put $x = \tan\theta$

$\Rightarrow dx = \sec^2\theta\, d\theta$

$x = 0,\ 0 = \tan\theta \Rightarrow \theta = 0$

$x = \infty,\ \infty = \tan\theta \Rightarrow \theta = \pi/2$

$$\therefore I = \int_0^{\pi/2} \frac{\tan^3\theta \cdot \sec^2\theta}{(\sec^2\theta)^{9/2}}\, d\theta$$

$$= \int_0^{\pi/2} \frac{\tan^3\theta\, \sec^2\theta}{\sec^9\theta}\, d\theta$$

$$= \int_0^{\pi/2} \tan^3\theta \cdot \cos^7\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{\sin^3\theta}{\cos^3\theta} \cos^7\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^3\theta \cdot \cos^4\theta\, d\theta$$

$$\frac{\Gamma(2)\, \Gamma(5/2)}{2\Gamma(9/2)}$$

$$= \frac{\Gamma(5/2)}{2 \cdot 7/2 \cdot 5/2 \cdot \Gamma(5/2)}$$

$$= 2/35$$

Ans: $\frac{2}{35}$.

$\int_0^1 x^2 (1-x^2)^{5/2} = \frac{5\pi}{256}$

$\left| \Gamma(1/2) = \sqrt{\pi} \right|$

$\int_0^1 \frac{x^6\, dx}{\sqrt{1-x^2}} = \frac{5\pi}{32}$

$I = \int_0^1 \frac{x^6\, dx}{\sqrt{1-x^2}}$

$x = \sin\theta$

$\therefore dx = \cos\theta\, d\theta$

x	0	1
θ	0	$\pi/2$

$= \int_0^{\pi/2} \frac{\sin^6\theta \cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}}$

$= \int_0^{\pi/2} \sin^6\theta\, d\theta = \frac{\Gamma\frac{6+1}{2}\, \Gamma 1/2}{2\Gamma\frac{6+2}{2}} = \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{2\cdot3\cdot2\cdot1}$

$= \frac{5\cdot3\pi}{2\cdot2\cdot2\cdot2\cdot2\cdot3\cdot1} = \frac{5\pi}{32}$.

Prove that $\beta\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{2\pi}{\sqrt{3}}$

$\beta(1/3, 2/3) = \frac{\Gamma 1/3\, \Gamma 2/3}{\Gamma(1/3 + 2/3)} = \frac{\Gamma(1/3)\, \Gamma(2/3)}{\Gamma(1)}$

$= \Gamma(1/3)\, \Gamma(1-1/3) = \frac{\pi}{\sin\frac{1}{3}\pi} = \frac{2\pi}{\sqrt{3}}$

We know: $\Gamma(n)\, \Gamma(1-n) = \frac{\pi}{\sin n\pi}$