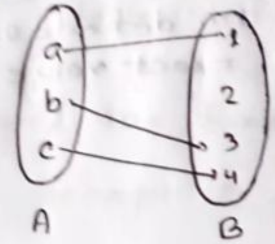
class-2

Function

1)

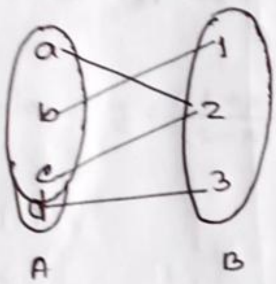


→ function (unique output)

→ 1:1 function

→ not on-to function (cause domain এ একটি element থেকে যায়)

ii)

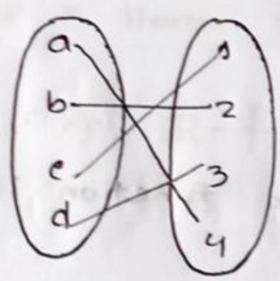


→ function

→ many-one function (cause not 1.1 f.)

→ on-to function

iii)

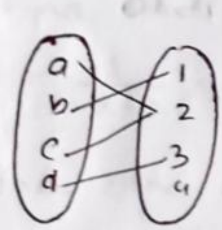


→ one-one + on-to function

↓

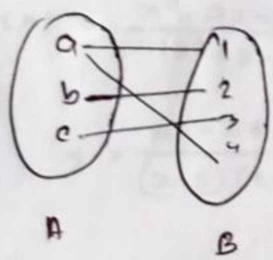
→ bijective function

iv)



→ many-one funct.

v)



→ not function, its relation

Domain and Range :

Domain → for the inputs the function is defined

Range → output values of domain

$\frac{7}{0}$ → undefined → cause there is no definite or exact value

M-1

Find the domain and range for the function

$f(x) = x^2+1$ where $x = \{0,1,2,3\}$

Sol^n Let, $y = f(x) = x^2+1$, here

when $x=0 \to y=1$

$x=1 \to y=2$

$x=2 \to y=5$

$x=3 \to y=10$

here for all the values of x the function is defined.

The set of output values are = 1, 2, 5, 10

$\therefore$ Range = $\{1,2,5,10\}$

2. Find the domain & Range of the function $f(x) = \frac{1+x}{5-x}$.

Sol^n

Let, $y = f(x) = \frac{1+x}{5-x}$

Here the function $f(x)$ is defined for all values of x except 5. So, the domain of function -

$$D_f = \mathbb{R} - \{5\}$$

(ans)

Again, $y = \frac{1+x}{5-x} \Rightarrow 5y - xy = 1 + x \Rightarrow 5y - 1 = x + xy = x(1+y)$

$$\therefore x = \frac{5y-1}{1+y}$$

$\therefore$ ~~R_f~~ = Here x is defined for all values except -1.

$\therefore R_f = \mathbb{R} - \{-1\}$ (ans)

$\mathbb{R}$ $\mathbb{N}$

3. Find the domain and range of the function $f(x) = \frac{1}{x-3}$.

4. Find the domain and range of the function

$$y = f(x) = \frac{x^2 - 3x + 2}{x^2 + x - 6}$$

Sol^n

$$y = \frac{x^2 - 3x + 2}{(x-2)(x+3)}$$

Here the function is defined for all the values of x except $2, -3$,

$$\therefore D_f = \mathbb{R} - \{-3, 2\}$$

$$y = \frac{x^2 - 3x + 2}{x^2 + x - 6}$$

$x^2y + xy - 6y = x^2 - 3x + 2$

$x^2y - x^2 + xy + 3x = 2 + 6y$

$\therefore x^2(y-1) + x(y+3) - (2+6y) = 0$

$$x = \frac{-(y+3) \pm \sqrt{(y+3)^2 + 4(y-1)(2+6y)}}{2 \times (y-1)}$$

$$= \frac{-(y+3) \pm \sqrt{25y^2 - 10y + 1}}{2(y-1)}$$

$$= \frac{-(y+3) \pm \sqrt{(5y-1)^2}}{2(y-1)}$$

$$= \frac{-y+3 \pm (5y-1)}{2(y-1)}$$

here, x is defined for all the values except 1

$\therefore R_f = \mathbb{R} - \{1\}$ (Ans)

$x \cdot y = 1$

$x = y$

$x^2 = xy$

$\Rightarrow x^2 - y^2 = xy - y^2$

$\Rightarrow (x-y)(x+y) = (x-y)y$

$x + y = y$

$2 = 1$

$(y-1)(2+6y)$

$2y + 6y^2 - 2 - 6y$

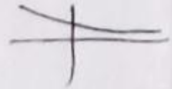
$6y^2 - 4y - 2$

$y^2 + 6y + 9$

$+ 24y^2 - 16y - 8$

* $\frac{5x + \sqrt{3x+1}}{x-1}$

* $R_f = \{x : x \neq 1 \text{ and } x \geq 1/3\}$



21-05.2023

$x^2+y^2=a^2$, $y^2 x = x$ $\longrightarrow$ they are not function. [they are relation]

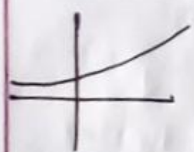
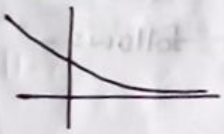
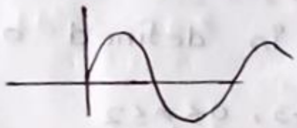
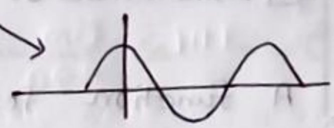
$\Rightarrow y = \pm\sqrt{a^2-x^2}$ $\quad y = \pm\sqrt{x}$ $\rightarrow$ they don't have unique outputs.

$y^2 = x$, Let us calculate for each value of 'x' we find the value of y.

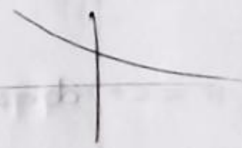
x	1	4
y	± 1	± 2

Special equation $\longrightarrow$ Must learn their graphs :-

$\boxed{y = e^x}$, $\boxed{y = e^{-x}}$, $\boxed{y = \sin x}$, $\boxed{y = \cos x}$

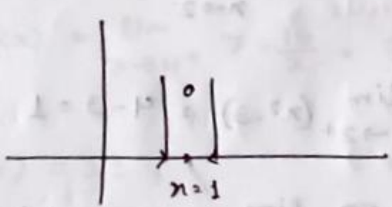
$\boxed{y = 2^x}$, Logarithm function is inverse funtion of e^x funtion?



, $\boxed{y = a^x}$, $\boxed{y = |x|}$

$2^2 = 4 \Rightarrow 2 = \log_2 4$

$\longrightarrow f(x) = \dfrac{3x+2}{x-1}$



$x \longrightarrow 1^-$

$x \longrightarrow 1^+$

$\lim\limits_{x \to 1} f(x)$

imp* **Limit:**

Limit of a function: When x approches to a constant quantity 'a' from either side, if there exists a definite finite number 'l', which f(x) approches such that the numerical differrence, of f(x) and 'l' can be made as small as we please by taking x sufficiently close to 'a' then 'l' is defined as the limit of f(x) as 'x' tends to 'a'. This is symbolically written as $\lim_{x \to c} f(x) = l$

Existence of Limit:

1. A function f(x) is defined as follows -

$$f(x) = \begin{cases} 2x-3, & 0 \le x \le 2 \\ x^2-3, & 2 \le x \le 4 \end{cases}$$

does $\lim_{x \to 2} f(x)$ exist?

Sol^n LHL $= \lim_{x \to 2^-} f(x) = \cancel{2x-3} \lim_{x \to 2^-} (2x-3) = 2 \times 2 - 3 = 1$

RH.L $= \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2-3) = 4-3 = 1$

Since, L.H.L = R.H.L, so $\lim_{x \to 2} f(x)$ exist. And

$\lim_{x \to 2} f(x) = 1$

ans

Continuty and Discontinuty : → Learn their defination

Continuty:

A function $y = f(x)$ is said to be continuous at the point $x = a$ if
① $f(a)$ is defined
② $\lim_{x \to a} f(x)$ exists
③ $\lim_{x \to a} f(x) = f(a)$

2. Determine whether the function, ~~f(x)~~

$$f(x) = \begin{cases} 2x+3 & , x \leq 4 \\ 7 + \frac{16}{x} & , x > 4 \end{cases}$$ is continuous at $x = 4$.

Soln

~~$f(a) = 2 \times 4 + 3 = 11$~~

At, $x = 4$ function $f(x)$ is defined. $f(a) = 2x+3 = 2 \times 4 + 3 = 11$

LHL. $= \lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} 2x+3 = 2 \times 4 + 3 = 11$

RHL $= \lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} 7 + \frac{16}{x} = 7 + \frac{16}{4} = 11$

So, $\lim_{x \to 4} f(x) = 11$,

$\therefore f(4) = \lim_{x \to 4} f(x) = 11$

(ans)

H.W

① Test the continuity at $x=0$ where,

$$f(x) \begin{cases} \dfrac{e^{1/x}}{e^{1/x}+1} & , \text{ when } x \neq 0 \\ 0 & , \text{ when } x = 0 \end{cases}$$

② Test the continuity of $f(x)$ at $x=0$ where

$$f(x) = \begin{cases} x^2 \sin(1/x) & , \text{ where } x \neq 0 \\ 0 & , \text{ where } x = 0 \end{cases}$$

Answer

1.

$$LHL = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^{1/x}}{e^{1/x}+1} = \lim_{x\to 0^-} \frac{e^{\infty}}{e^{\infty}+1} = \frac{\infty}{\infty}$$

L.Hopital

applying L'hopita Rule,

$$= \lim_{x\to 0^-} \frac{e^{1/x}(-1/x^2)}{e^{1/x}(-1/x^2)} = 1$$

RHL

$$= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^{1/x}}{e^{1/x}+1} \left(\frac{\infty}{\infty}\right)$$

applying L.hopital rule,

$$= \lim_{x\to 0^+} \frac{e^{1/x}.(-1/x^2)}{e^{1/x}(-1/x^2)} = 1$$

$f(x) = 0$

$\therefore$ L.H.L $=$ R.H.L $\neq f(x)$

$\therefore$ It is discontinuous. function.

2. We know,

$$-1 \le \sin(\tfrac{1}{x}) \le 1$$

$$-x^2 \le x^2 \sin(1/x) \le \cancel{x}^2$$

$$\lim_{x \to 0^\circ} -x^2 \le \lim_{x\to 0} x^2 \sin(\tfrac{1}{x}) \le \lim_{x \to 0} x^2$$

$$0 \le \lim_{x \to 0} x^2 \sin(\tfrac{1}{x}) \le 0$$ (sandwich theoram)

$\therefore \lim_{x \to 0} x^2 \sin(1/x) = 0 \longrightarrow$ Limit exists at $x = 0$

$f(x) = f(0) = 0$

$\therefore$ $f(x)$ is continuous at $x = 0$.

Maths ↓

1) Limits
ii) Continuity
iii) ...

24.05.23

Squeezing — Theorem

g, f, h

$$g(x) \leq f(x) \leq h(x)$$

(E) → Let f, g and h be functions satisfying $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing the number c, with the possible exception that the inequalities need not hold at $x = c$.

If g and h have the same limit say,

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

Then, $\lim_{x \to c} f(x) = L$

M-1 Find the limit of $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$ by using squeezing theorem.

Soln Let, $f(x) = x \sin(1/x)$

We know the value of $\sin(1/x)$ varies from -1 to 1.

So, we can write, $-1 \leq \sin(1/x) \leq 1$

if $x \neq 0$ then,

$$-|x| \leq |x| \sin(\tfrac{1}{x}) \leq |x| \quad -①$$

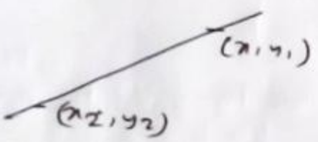
$\quad g(x) \qquad\qquad f(x) \qquad\qquad h(x)$

Since, $|x| \to 0$ as $x \to 0$.

The inequalities in ① and the squeezing theorem imply that, $\lim_{x \to 0} x \sin(\frac{1}{x}) = 0$.

(ans)

absolute value $\lvert x \rvert$
$\lvert x \rvert = \begin{cases} x ; & x > 0 \\ -x , & x < 0 \\ 0 ; & x = 0 \end{cases}$



$m = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{\Delta y}{\Delta x}$ rate of chang / slope with respect of x

Viva

$\frac{dy}{dx} \to$ derivative of y in respect of x

$\to$ [illegible] হার: rate of change

CT (E) Defination: of differentiability:-

A function 'f' is to be differentiable at ~~x→~~ x_0 if the limit, $f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists.

* Left hand Derivative:

$$Lf'(x) = \lim_{h\to 0} \frac{f(x_0-h)-f(x_0)}{-h}$$

Righ hand derivative:

$$Rf'(x) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

M-2 Show that $f(x) = \begin{cases} x^2+1, & x \leq 1 \\ 2x, & x > 2 \end{cases}$ is continuous and differentiable at $x=1$.

Ans.

2nd part:

$f(x) = x^2+1$

$f(1-h) = (1-h)^2+1$

$$\text{L.H.D}: Lf'(1) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{(1-h)^2+1-(1^2+1)}{-h} = \lim_{h\to 0} \frac{1-2h+h^2+1-2}{-h}$$

$$= \lim_{h\to 0} \frac{h^2-2h}{-h} = \lim_{h\to 0} (-h+2) = 0+2 = 2$$

$$Rf'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$ $f(x) = 2x$, $f(1+h) = 2(1+h)$

$$= \lim_{h\to 0} \frac{2(1+h) - 2}{h}$$

$$= \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} 2 = 2$$

Since, $Lf'(1) = Rf'(1)$

H.W

1. $f(x) = \begin{cases} x^2+2 , & x \le 1 \\ 2+x , & x > 2 \end{cases}$ is continuous

and differentiable at $x = 1$

2. $f(x) = \begin{cases} x^2+2 , & x \le 1 \\ x+2 , & x > 1 \end{cases}$ is continuous but

not differentiable at $x = 1$.

Soln $Lf'(a) = \lim_{h\to 0} \frac{f(a-h) - f(a)}{-h}$

$$= \lim_{h\to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{(1-h)^2 + 2 - 3}{-h}$$

$$= \lim_{h\to 0} \frac{\cancel{1} - 2h + h^2 \cancel{-1}}{-h} = \lim_{h\to 0} \frac{-h(-2+h)}{-h}$$

$$= 2$$

$$Rf'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$ $f(a+h) = 1+2+h$

$$= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{1+2+h-(1+2)}{h}$$

$$= 1$$

$$\therefore Lf'(a) \ne Rf'(a)$$

(E) If a function is differentiable - will it be continuous too?

Theorem:

If the f(x) function $f(x)$ is differential at $x = a$ then the function is continuous at that point.

But the inverse may not be true.

Proof:

Since the function $f(x)$ is differentiable at that point $x = a$. So from the defination of differentiability we have,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$ exist and finite.

Now,

$$\lim_{h\to 0} \frac{[f(a+h) - f(a)]}{h} = \lim_{h\to 0} \left[\frac{f(a+h) - f(a)}{h} \times h\right]$$

$$= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \times \lim_{h\to 0} h$$

$$= f'(a) \times 0 = 0$$

$$\lim_{h\to 0} [f(a+h)-f(a)] = 0$$

$$\Rightarrow \lim_{h\to 0} f(a+h) - \lim_{h\to 0} f(a) = 0$$

$$\Rightarrow \lim_{h\to 0} f(a+h) - f(a) = 0$$

$$\Rightarrow \lim_{h\to 0} f(a+h) = f(a)$$

Hence, the function is differentiable.

But,

$$f(x) = |x|$$

At $x=0$, $f(0) = 0$

$$LHL = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} |x| = |0| = 0$$

$$RHL = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} |x| = |0| = 0$$

$\therefore$ LHL = RHL

Also, the function is continuous.

NOW,

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{f(-h)-0}{-h} = \lim_{h\to 0} \frac{|-h|-0}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1$$

$$Rf'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)-0}{h} = \lim_{h\to 0} \frac{|h|-0}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

$$Lf'(0) \neq Rf'(0)$$

Hence, the function is not differentiable.

Limits - def, exists, find out
Function - def, types, domain, range, find out

Co...

C.T

Wednesday (30.05.23)

সবটুকু যা পড়ানো হবে।

Week-5

04.06.2023

Q. Test the continuity and differentiability of

$$f(x) = \begin{cases} 1, & \text{when } x<0 \\ 1+\sin x, & \text{when } 0 \le x < \pi/2 \\ 2+(x-\pi/2)^2, & \text{when } x \ge \pi/2 \end{cases}$$

at the point $x = \pi/2$

Sol^n

continuity test.

$$f(\pi/2) = 2+(\pi/2-\pi/2)^2 = 2$$

$$LHL = \lim_{x\to \frac{\pi}{2}^-} 1+\sin \pi/2 = 2$$

$$RHL = \lim_{x\to \frac{\pi}{2}^+} 2+(\pi/2-\pi/2)^2 = 2 \text{ ; continuous}$$

∴ differentiability test,

$\pi/2 - h$

$$Lf'(x) = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}$$

$f(\pi/2-h)$

$= 1+\sin(\pi/2-h)$

$$= \lim_{h\to 0} \frac{f(\pi/2-h)-f(\pi/2)}{-h}$$

$$= \lim_{h\to 0} \frac{1+\sin(\pi/2-h)-(1+\sin\pi/2)}{-h}$$

$$= \lim_{h\to 0} \frac{1+\cos h-2}{-h} = \lim_{h\to 0} \frac{\cos h-1}{-h}$$

$$= \lim_{h\to 0} (1-\cos h)$$

$$= 1-\cos 0 = \lim_{h\to 0}\frac{1-\cos h}{h} = \lim_{h\to 0}\frac{2\sin^2 h/2}{h} = \lim_{h\to 0}\frac{\sin^2 h/2}{h/2}$$

$$= \lim_{h\to 0}\left(\frac{\sin h/2}{h/2}\right)^2 \times \frac{h}{2}$$

$$= h/2 = 0$$

$$Rf'(x) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$$= \lim_{h\to 0}\frac{f(\pi/2+h)-f(\pi/2)}{h}$$

$$= \lim_{h\to 0}\frac{2+(\pi/2+h-\pi/2)^2-\{2+(\pi/2-\pi/2)^2\}}{h}$$

$$= \lim_{h\to 0}\frac{2+h^2-2}{h} = 0$$

$$\therefore Lf'(x) = Rf'(x)$$

(ans)

Q. $\frac{d}{dx}\ y = (x^2 \ln x^2)$

$$\Rightarrow \frac{dy}{dx} = x^2 \cdot \frac{2x}{x^2} + \ln x^2 \cdot 2x$$

$$= 2x(1+\ln x^2)$$

Q. $y = x^{\cos^{-1}x}$

$$\Rightarrow \ln y = \cos^{-1}x \cdot \ln x$$

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \cos^{-1}x \cdot \frac{1}{x} + \ln x\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$\frac{dy}{dx} = x^{\cos^{-1}x}\left(\frac{\cos^{-1}x}{x} - \frac{\ln x}{\sqrt{1-x^2}}\right)$$

ans

$x^2 = y \rightarrow$ explicit function

$x^2 - y = 0 \rightarrow$ implicit "

* $x^2 + 2xy + 6y^2 + 5x - 6y = 0$

$\Rightarrow x^2 + 2xy + 5x = 6y - 6y^2$

$\Rightarrow 2x + \cancel{2x\frac{dy}{dx}} + 2(x\frac{dy}{dx} + y) + 5 = 6\frac{dy}{dx} -$

$\frac{dy}{dx} = -\frac{2x + 2y + 5}{2x + 12y - 6}$

$y = 2t^2$, $x = 2t$

$\frac{dy}{dx} = \frac{\frac{d}{dt}(2t^2)}{\frac{d}{dt}(2t)} = \frac{4t}{2} = 2t$

differentiate $\tan^{-1}\frac{2x}{1-x^2}$ with respect to $\sin^{-1}\frac{2x}{1+x^2}$

$\frac{dy}{d(\sin^{-1}\frac{2x}{1+x^2})}$ $\tan^{-1}\frac{2x}{1-x^2}$ a b

$= \frac{\frac{da}{dx}(2\tan^{-1}x)}{\frac{db}{dx}(2\tan^{-1}x)} = \frac{d}{dx}1 =$

$y = \tan^{-1}\frac{2x}{1+x^2}$

$\frac{dy}{dx} = 2\tan^{-1}x$

$\frac{dy}{dt} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{2\tan^{-1}x}{2\tan^{-1}x} = 1$

Application of differentiation

If the area of a circle incress at a ~~unfortunate~~ uniform rate, then show that the rate of increase of the circumference (s) will vary inversly as the radius.

area of circle $= A = \pi r^2$

according to the ques. $\rightarrow \frac{dA}{dt} = k$

$\Rightarrow \frac{d}{dt}(\pi r^2) = \frac{dA}{dt}$

$\Rightarrow 2\pi r\frac{dr}{dt} = k$

$\Rightarrow \frac{dr}{dt} = \frac{k}{2\pi r}$

Now,

$S = 2\pi r \Rightarrow \frac{ds}{dt} = 2\pi\frac{dr}{dt} = 2\pi \times \frac{k}{2\pi r} = \frac{k}{r}$

$\therefore \frac{ds}{dt} \propto \frac{k}{r}$ [showed]

Homework:

The volume of a spherical ballon is increaseing at the rate of 100 cc/sec. find the rate of change of its surface at the instant when its radius is 16 cm.

04.5.23

Succecive differentiation:-

1. $y = x^n$, $y_n = ?$

$\Rightarrow y_1 = n x^{n-1}$

$y_2 = n(n-1) x^{n-2}$

$y_3 = n(n-1)(n-2) x^{n-3}$

$\vdots$

$y_n = n(n-1)(n-2) \dots 3.2.1\, x^{n-n} = n!$

$* \; y_{n+1} = 0$

(2) If $y = e^{a\sin^{-1}x}$ then show that $(1-x^2)y_2 - xy_1 - a^2y = 0$

Soln

$y_1 = e^{a\sin^{-1}x} . a . \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\, y_1 = a e^{a\sin^{-1}x} = ay$

$\Rightarrow (1-x^2) y_1^2 = a^2 y^2$

DS

$\Rightarrow (1-x^2) 2y_1 y_2 + y_1^2 . (0-2x) = a^2 . 2y . y_1$

$\Rightarrow (1-x^2) \cancel{2}\cancel{y_1} y_2 - \cancel{2} x y_1^{\cancel{2}} = a^2 \cancel{2} y \cancel{y_1}$

$\Rightarrow (1-x^2) y_2 - xy_1 - a^2 y = 0$ [showed]

③ If $y=(ax+b)^n$ and $n \in N$ then show that $y_n = n!\,a^n$

Soln

$y = (ax+b)^n$

$\Rightarrow y_1 = na\,(ax+b)^{n-1}$

$\Rightarrow y_2 = n(n-1).a^2\,(ax+b)^{n-2}$

$\Rightarrow y_3 = n(n-1)(n-2).a^3.(ax+b)^{n-3}$

$\vdots$

$\Rightarrow y_n = n(n-1)(n-2)\ldots\ 3.2.1\ a^n\,(ax+b)^{n-n}$

$\boxed{y_n = n!\,a^n}$

④ If ~~y=(ax+b)~~ $y = \sin(ax+b)$, then show that $y_n = a^n \sin(n.\frac{\pi}{2}+ax+b)$

Soln $y = \sin(ax+b)$

$y_1 = \cos(ax+b).a = a.\sin(1.\frac{\pi}{2}+ax+b)$

$\cancel{y_2 = -\sin(ax+b)a^2 = a^2\,\sin(2.\frac{\pi}{2}+ax+b)}$

$y_2 = a^2\cos(\frac{\pi}{2}+ax+b) = a^2(\sin\frac{\pi}{2}+\frac{\pi}{2}+ax+b)$

$y_2 = a^2\sin(2.\frac{\pi}{2}+ax+b)$

$\boxed{y_n = a^n\sin(n\frac{\pi}{2}+ax+b)}$

⑤ If $y = \cos(ax+b)$ then show than $y_n = a^n\cos(n\frac{\pi}{2}+ax+b)$

Soln

$y = \cos(ax+b)$

$y_1 = -a.\sin(\frac{\pi}{2}+ax+b) = a\cos(\frac{\pi}{2}+ax+b)$

$y_2 = -a^2\sin(\frac{\pi}{2}+ax+b) = a^2\cos(\frac{\pi}{2}+\frac{\pi}{2}+ax+b)$

$= a^2\cos(2.\frac{\pi}{2}+ax+b)$

$\vdots$

$\boxed{y_n = a^n.\cos(n.\frac{\pi}{2}+ax+b)}$

Tough ↓ ∴

⊞ Libnitz's Theorem

⊞ If each of the function $u=u(x)$ and $v=v(x)$ are differentiable in times then,

* $\boxed{(uv)_n = u_n v + {}^nc_1.u_{n-1}.v_1 + {}^nc_2.u_{n-2}.v_2 + \ldots + \ldots + {}^nc_r.u_{n-r}.v_r + \ldots + u.v_n}$ *

Here, u = first function

v = second function

M-1 Show that $\frac{d^{n+1}}{dx^{n+1}}(x^n \ln x) = \frac{n!}{x}$

Sol^n Let, $y = x^n . \ln x$

$y_1 = x^n . \frac{1}{x} + \ln x . n x^{n-1} = \frac{x^n}{x} + x n x^{n-1} . \ln x$

$\Rightarrow y_1 x = x^n + n x^n \ln x = x^n + ny$

$\Rightarrow (xy_1)_n$ or $\frac{d^n}{dx^n}(xy_1) = \frac{d^n}{dx^n}(x^n) + \frac{d^n}{dx^n}(ny)$

1st derivative

$\Rightarrow y_{n+1} x + y_n . 1 . {}^nc_1 + {}^nc_2 \cancel{y_{n-1} . 0} = n! + ny_n$

$\Rightarrow y_{n+1} . x + y_n . n = n! + ny_n$

$\Rightarrow y_{n+1} . x = n!$

$\Rightarrow y_{n+1} = \frac{n!}{x} \Rightarrow \frac{d^{n+1} y}{dx^{n+1}} = \frac{n!}{x}$

$\therefore \frac{d^{n+1} y}{dx^{n+1}}(x^n . \ln x) = \frac{n!}{x}$

[showed]

Exam-G এরকম আসবে,

M-2 If $y = \tan^{-1} x$ then show that,

(i) $(1+x^2) y_{n+1} + 2nxy_n + n(n-1) y_{n-1} = 0$

(ii) $(1+x^2) y_{n+2} + 2(n+1) x y_{n+1} + n(n+1) y_n = 0$

Sol^n

(i)

Given that,

$y = \tan^{-1} x$

$\Rightarrow y_1 = \frac{1}{1+x^2} \Rightarrow (1+x^2) y_1 = 1$

[applying Libnitz's theorem]

$\Rightarrow (1+x^2) y_1 = 1 \Rightarrow \frac{d^n}{dx^n}(1+x^2) y_1 = \frac{d^n}{dx^n}(1)$

$\Rightarrow y_{n+1}(1+x^2) + {}^nc_1 . y_n . 2x + {}^nc_2 y_{n-1} . 2 + \cancel{{}^nc_3 . y_{n-2} . 0} = 0$

$\Rightarrow (1+x^2) y_{n+1} + n 2x . y_n + \frac{n(n-1)}{2!} . 2 . y_{n-1} = 0$

$\Rightarrow (1+x^2) y_{n+1} + 2nx . y_n + n(n-1) y_{n-1} = 0$

[showed]

(11)

Given that,

$$y = \tan^{-1} x \qquad (1+x)^{-1}$$

*** cross multification must

$\Rightarrow y_1 = \dfrac{1}{1+x^2} \Rightarrow \cancel{y_2 = -1(1+x^2)^{-2}.2x}$

$\Rightarrow y_2 = \dfrac{-2x}{(1+x^2)^2} \Rightarrow (1+x^2)\, y_1 = 1$

$\Rightarrow (1+x^2)\, y_2 + y_1 2x = 0$

$\Rightarrow (1+x^2)\, y_2 + 2x y_1 = 0$

$\Rightarrow \dfrac{d^n}{dx^n}(1+x^2)\, y_2 + \dfrac{d^n}{dx^n}(2x y_1) = 0$

$\Rightarrow y_{2+n} \cdot (1+x^2) + {}^nc_1 \cdot y_{n+1} \cdot 2x + {}^nc_2 \cdot y_n \cdot 2 + 0 + 2\,(y_{1+n} \cdot x + y_n \cdot 1 \cdot {}^nc_1) = 0$

$\Rightarrow y_{n+2}(1+x^2) + 2x.n.y_{n+1} + n(n-1)\, y_n + 2x\, y_{n+1} + 2n\, y_n = 0$

$\Rightarrow y_{n+2}(1+x^2) + 2x\, y_{n+1}(n+1) + n y_n (n-1+2) = 0$

$\therefore (1+x^2)\, y_{n+2} + 2x y_{n+1}(n+1) + n y_n (n+1) = 0$

[showed]

05-06-2023

M-1

$\square$ $(1+x^2)\, y_1$ এর n derivative

Find out $\dfrac{d^n}{dx^n}(1+x^2)\, y_1$

$\Rightarrow \dfrac{d^n}{dx^n}(1+x^2)\, y_1 = y_{1+n}(1+x^2) + {}^nc_1 \cdot y_n \cdot 2x + {}^nc_2 \cdot y_{n-1} \cdot 2 + 0$

$= y_{n+1}(1+x^2) + n2x\, y_n + n(n-1)\, y_{n-1}$

(ans)

M-2

If $\ln y = \tan^{-1}x$ then show that,

$(1+x^2)\, y_{n+2} + (2nx + 2x - 1)\, y_{n+1} + n(n+1)\, y_n = 0$

Soln

$\ln y = \tan^{-1} x$

$\Rightarrow y = e^{\tan^{-1}x} \Rightarrow y_1 = e^{\tan^{-1}x} \cdot \dfrac{1}{1+x^2} = \dfrac{e^{\tan^{-1}x}}{1+x^2}$

$\Rightarrow (1+x^2)\, y_1 = e^{\tan^{-1}x} = y$

$\Rightarrow (1+x^2)\, y_2 + y_1 2x = y_1 \Rightarrow (1+x^2)\, y_2 + \cancel{2x(y_1+1)}\; y_1(2x-1) = 0$

By applying Libnitz theorem,

$\dfrac{d^n}{dx^n}\{(1+x^2)\, y_2\} + \dfrac{d^n}{dx^n}\{y_1(2x-1)\} = 0$

$\Rightarrow y_{n+2}(1+x^2) + y_{n+1} 2x\, {}^nc_1 + {}^nc_2 \cdot y_n \cdot 2 + y_{n+1}(2x-1) + {}^nc_1 \cdot y_n \cdot 2 = 0$

$\Rightarrow y_{n+2}(1+x^2) + 2xn.y_{n+1} + n(n-1).y_n + y_{n+1}(2x-1) + n2y_n = 0$

$\Rightarrow y_{n+2}(1+x^2) + y_{n+1}(2nx+2x-1) + y_n(2n+n-1)n = 0$

$\therefore y_{n+2}(1+x^2) + y_{n+1}(2xn+2x-1) + n y_n (n+1) = 0$

[showed]

M-3

If $y=(x+\sqrt{x^2+1})^m$ then show that,

$$(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$$

Soln

Given that,

$$y=(x+\sqrt{x^2+1})^m$$

$$\Rightarrow y_1 = m(x+\sqrt{x^2+1})^{m-1}\cdot\left(1+\frac{0+\cancel{2}x}{\cancel{2}\sqrt{1+x^2}}\right)$$

$$= m(x+\sqrt{x^2+1})^{m-1}\,\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}$$

$$= \frac{m(x+\sqrt{x^2+1})^m}{\sqrt{1+x^2}}$$

$$\Rightarrow y_1\sqrt{1+x^2}=my$$

$$\Rightarrow y_1\frac{0+\cancel{2}x}{\cancel{2}\sqrt{1+x^2}}+\sqrt{1+x^2}\cdot y_2 = \cancel{m}y_1\sqrt{1+x^2}$$

$$\Rightarrow 2xy_1+(1+x^2)y_2 = my_1\sqrt{1+x^2}$$

$$\Rightarrow y_2(1+x^2)+y_1(x-m)=0$$

Libnitz,

$$\frac{d^n}{dx^n}\{y_2(1+x^2)\}+\frac{d^n}{dx^n}\{y_1(x-m)\}=0$$

$$\Rightarrow y_{n+2}$$

$$\Rightarrow y_1^2(1+x^2)=m^2y^2$$

$$\Rightarrow y_1^2\,\cancel{2}x+(1+x^2)\,\cancel{2}y_1y_2=m^2\,\cancel{2}yy_1$$

$$\Rightarrow y_1x+(1+x^2)y_2=m^2y$$

$$\Rightarrow y_2(1+x^2)+xy_1-m^2y=0$$

$$\frac{d^n}{dx^n}\{y_2(1+x^2)\}+\frac{d^n}{dx^n}(xy_1)-\frac{d^n}{dx^n}(m^2y)=0$$

$$\Rightarrow y_{2+n}(1+x^2)+y_{n+1}\cdot 2x\cdot {}^nC_1+{}^nC_2\cdot 2\cdot y_n+y_{n+1}\cdot x+y_n\cdot 1\cdot {}^nC_1-y_nm^2=0$$

$$\Rightarrow (1+x^2)y_{n+2}+n2xy_{n+1}+n(n-1)y_n+xy_{n+1}+y_n\cdot n-y_nm^2=0$$

$$\Rightarrow y_{n+2}(1+x^2)+xy_{n+1}(2n+1)+y_n(n^2-n+n-m^2)=0$$

$$y_{n+2}(1+x^2)+xy_{n+1}(2n+1)+y_n(n^2-m^2)=0$$

[Showed]

Home-work:

1. If $y=\sin(m\sin^{-1}x)$ then ~~find~~ show that, $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(m^2-n^2)y_n=0$
2. If $y=e^{a\sin^{-1}x}$ then show that, $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0$

11-06-2023

H.W

M-1

1. If $y=(x^2-1)^n$ then show that,
$(x^2-1)y_{n+2}+2xy_{n+1}-n(n+1)y_n=0$

2. $y=\tan^{-1}x \longrightarrow$ show: $(1+x^2)y_{n+2}+(2nx+2x-1)y_{n+1}+n(n+1)y_n=0$

3. $y=\cos\{\ln(1+x)\} \longrightarrow$ show: $(1+x^2)y_{n+2}+(2n+1)(1+x)y_{n+1}+(n^2+1)y_n=0$

Rolle's Theorem

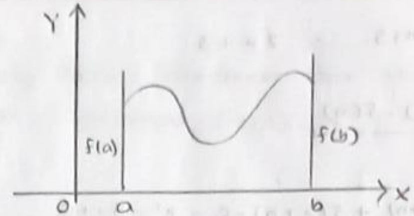
If a function $y=f(x)$ is –

1) continuous in the closed interval $[a,b]$

11) differentiable in the open interval (a,b)

111) $f(a)=f(b)$

Then there exists atleast a value of $c\{x\}$ between a and b; such that first derivative of c; $[f'(c)]=0$



কোনো function দেওয়া থাকবে তাদের জন্য Rolle's theorem varify করতে দিলে হবে → exam এ এই question আসবে,

M-1

Verify Rolle's theorem for the function $f(x)=x^2+5x-6$ at in the interval of $(-6,1)$

Soln

Givens that, $f(x)=x^2+5x-6$
$f(-6)=0$
$f(1)=0$

$$Lf'(x)=\lim_{h\to0}\frac{f(x-h)-f(x)}{-h}$$

$$=\lim_{h\to0}\frac{(x-h)^2+5(x-h)-\cancel{6}-x^2-5x+\cancel{6}}{-h}$$

$$=\lim_{h\to0}\frac{\cancel{x^2}-2hx+h^2+\cancel{5x}-5h-\cancel{x^2}-\cancel{5x}}{-h}$$

$$=\lim_{h\to0}\frac{h^2-2hx-5h}{-h}$$

$$= \lim_{h\to 0} -h+2x+5 = 2x+5$$

$$R f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2+5(x+h)-6-x^2-5x+6}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2+\cancel{5x}+5h-\cancel{6}-\cancel{x^2}-\cancel{5x}+\cancel{6}}{h}$$

$$= \lim_{h\to 0} \frac{2xh+h^2+5h}{h}$$

$$= \lim_{h\to 0} 2x+h+5 = 2x+5$$

$$\therefore Lf'(x) = Rf'(x)$$

Since $Lf'(x) = Rf'(x)$, the the function is differentiable at the close interval $(-6,1)$.

As the function is differentiable, the function must be continuous.

So, all the three conditions of Rolle's theorem are satisfied. So by Rolle's theorem,

$f'(c) = 0 \Rightarrow 2c+5 = 0 \Rightarrow c = -5/2 \in (-6,2)$

Hence, the Rolle's theorem is verified.

M-2

Verify Rolle's theorem for the function $f(x) = x^{2/3}$ in the interval $(-1,1)$.

M-3

Verify Rolle's theorem .... $f(x) = 2x^3 + x^2 - 4x - 2$ in the interval $[-\sqrt{2}, \sqrt{2}]$

MEAN VALUE THEOREM

If the function $y = f(x)$ is continuous in the closed interval $[a,b]$ and differential in the open interval (a,b), then there exists a value of c between a and b of x such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

M-1 Verify main. v. th. for the function $f(x) = 3+2x-x^2$ in the interval $(0,1)$

Sol^n

Given, $f(x) = -x^2+2x+3$

$$Lf'(x) = \lim_{h\to 0} \frac{f(x-h) - f(x)}{-h}$$

$$= \lim_{h\to 0} \frac{-(x-h)^2+2(x-h)+3+x^2-2x-3}{-h}$$

$$= \lim_{h\to 0} \frac{-\cancel{x^2}-h^2+2hx+\cancel{2x}-2h+\cancel{x^2}-\cancel{2x}-\cancel{3}+\cancel{3}}{-h}$$

$$= \lim_{h\to 0} \frac{-h^2+2hx-2h}{-h} = \lim_{h\to 0} \frac{-h(h-2x+2)}{-h}$$

$$= -2x+2$$

$$Rf'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{-(x+h)^2+2(x+h)+\cancel{3}+x^2-2x-\cancel{3}}{h}$$

$$= \lim_{h\to 0} \frac{-\cancel{x^2}-h^2-2xh+\cancel{2x}+2h+\cancel{x^2}-\cancel{2x}}{h}$$

$$= \lim_{h\to 0} \frac{-h^2-2xh+2h}{h} = \lim_{h\to 0} \frac{h(-h-2x+2)}{h}$$

$$= -2x+2$$

Since, $Lf'(x) = Rf'(x)$

~~Now~~ So, the funtion is differentiable for $f(x)$, Again, Since the " " ", hence the function must be continuous.

We can say that the function satisfies all the condition of m. v. th.

By m. v. th we have, $f'(c) = \frac{f(b)-f(a)}{b-a}$; $a=0$, $b=1$

$f(1) = 3+2-1 = 4$

$f(0) = 3$

So, $f'(c) = \frac{4-3}{1-0} = 1 \in [0,1]$

(ans)

M-2

Verify m.v.th for the function $f(x) = (x-1)(x-2)(x-3)$ in the interval $(0, 4)$.

Sol^n

Given, $f(x) = (x-1)(x-2)(x-3)$

$$= (x^2 - 3x + 2)(x-3) = x^3 - 3x^2 + 2x - 3x^2 + 9x - 6$$

$$= x^3 - 6x^2 + 11x - 6$$

$$Lf'(x) = \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h}$$

$$= \lim_{h \to 0} \frac{(x-h)^3 - 6(x-h)^2 + 11(x-h) - \cancel{6} - x^3 + 6x^2 - 11x + \cancel{6}}{-h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} - 3x^2h + 3xh^2 + h^3 - \cancel{6x^2} - 6h^2 + 12xh + \cancel{11x} - 11h - \cancel{x^3} + \cancel{6x^2} - \cancel{11x}}{-h}$$

$$= \lim_{h \to 0} \frac{-3x^2h + 3xh^2 + h^3 - 6h^2 + 12xh - 11h}{-h}$$

$$= \lim_{h \to 0} \frac{-h(3x^2 - 3xh - h^2 + 6h - 12x + 11)}{-h}$$

$$= 3x^2 - 12x + 11$$

MACLAURIN THEOREM

If $f(x)$, $f'(x)$, $f''(x)$ ---- $f^n(x)$ exist at the point '0' $x=0$' then Maclaurin polynomial will be,

$$P_n(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^n(0)$$

M-1

Expand $(1-x)^{-1}$ or $\frac{1}{1-x}$ by Maclaurin's theorem.

Soln

Given that,

$$f(x) = \frac{1}{1-x} \quad ; f(0) = 1$$

$$f'(x) = -\frac{1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2} \Rightarrow f'(0) = 1 = 1!$$

$$f''(x) = \frac{2}{(1-x)^3} \Rightarrow f''(0) = 2 = 2!$$

$$f''(x) = \frac{6}{(1-x)^4} \Rightarrow f'''(0) = 6 = 3!$$

$\dots$

$\dots$

$$f^n(x) = \frac{n!}{(1-x)^n} \Rightarrow f^n(0) = n!$$

$$P_n(x) = f(0) + f'(0)x + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^n(0)$$

$$= 1 + x.1 + 2!\frac{x^2}{2!} + \frac{x^3}{3!} \times 3! + \dots + \frac{x^n}{n!} \times n!$$

$$= 1 + x + x^2 + x^3 + \dots + x^n$$

(ans)

A.W **M-2**

Expand e^x at the point $x=0$ by Maclaurin's theorem.

M-3 $e^x.\sin x$ at the point $x=0$ by ...

সাবধান : only $x=0$ point - এ থেকে expand করা যাবে।