13.LIMITS AND DERIVATIVES
SYNOPSIS

- Limit of a function at a point is the common value of the left and right hand limits , if they coincide.

- For functions f and g the following holds :
  1. $\lim_{x\to a}\left[f(x)\pm g(x)\right]=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$
  2. $\lim_{x\to a}\left[f(x).g(x)\right]=\lim_{x\to a} f(x).\lim_{x\to a} g(x)$
  3. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$

- Following are some of the standard limits:
  1. $\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$
  2. $\lim_{x\to 0}\frac{\sin x}{x}=1$
  3. $\lim_{x\to 0}\frac{1-\cos x}{x}=0$
  4. $\lim_{x\to 0}\frac{e^x-1}{x}=1$
  5. $\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a$
  6. $\lim_{x\to 0}\frac{\log(1+x)}{x}=1$

- The derivative of a function f at any point x is defined by $f'(x)=\frac{df(x)}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

- For the functions u and v following holds:
  1. $(u\pm v)'=u'\pm v'$
  2. $(uv)'=u'v+v'u$ (product rule)
  3. $\left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2}$ (Quotient rule)
- Some standard derivatives :

  $\frac{d}{dx}(x^n)=nx^{n-1}$

  $\frac{d}{dx}(\sin x)=\cos x$

  $\frac{d}{dx}(\cos x)=-\sin x$

LIMITS AND DERIVATIVES

SECTION A (1 MARK)

MCQ:

1. $\lim_{x\to 0} \frac{sinx}{x\,(1+\cos x)}$ is equal to

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) -1

2. $\lim_{x\to \frac{\pi}{2}} \frac{1-sinx}{cosx}$ is equal to

(a) 0 (b) -1 (c) 1 (d) does not exist

3. $\lim_{x\to 0} \frac{|x|}{x}$

(a) 1 (b) -1 (c) 0 (d) does not exist

4. $\lim_{n\to\infty} \frac{1+2+3+\ .....+n}{n^2}, n \in N$ is equal to

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

5. If $f(x) = x\ sinx$, then $f'(\frac{\pi}{2})$ is equal to

(a) 0 (b) 1 (c) -1 (d) $\frac{1}{2}$

6. $\lim_{x\to \pi} \frac{sinx}{x-\pi}$ is

(a) 1 (b) 2 (c) -1 (d) -2

7. $\lim_{x\to 0} \frac{x^2 cosx}{1-cosx}$ is

(a) 2 (b) $\frac{3}{2}$ (c) $-\frac{3}{2}$ (d) 1

8. $\lim_{x\to 0} \frac{(1+x)^n-1}{x}$ is

(a) n (b) 1 (c) – n (d) 0

9. $\lim_{x\to 1} \frac{x^m-1}{x^n-1}$ is

(a) 1 (b) $\frac{m}{n}$ (c) $-\frac{m}{n}$ (d) $\frac{m^2}{n^2}$

10. $\lim_{x\to 0} \frac{1-cos4\theta}{1-cos6\theta}$ is

(a) $\frac{4}{9}$ (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) -1

11. $\lim_{x\to 0} \frac{cosec\ x-\cot x}{x}$ is

(a) $-\frac{1}{2}$ (b) 1 (c) $\frac{1}{2}$ (d) 1

12. $\lim_{x\to 0} \frac{\sin x}{\sqrt{x+1}-\sqrt{1-x}}$ is

(a) 2 (b) 0 (c) 1 (d) -1

13. $\lim_{x\to\frac{\pi}{4}} \frac{\sec^2 x-2}{tanx-1}$ is

(a) 3 (b) 1 (c) 0 (d) $\sqrt{2}$

14. $\lim_{x\to 1} \frac{(\sqrt{x}-1)(2x-3)}{2x^2+x-3}$ is

(a) $\frac{1}{10}$ (b) $-\frac{1}{10}$ (c) 1 (d) None of these

Fill in the blanks:

15. If $f(x) = \frac{tanx}{x-\pi}$, then $\lim_{x\to\pi} f(x) =$ _____

16. $\lim_{x\to 0}\left(\sin mx \cot\frac{x}{\sqrt{3}}\right) = 2$, then m = _____

17. $\lim_{x\to 3^+} \frac{x}{[x]}$ = _____

18. If $y = \sqrt{x} + \frac{1}{\sqrt{x}}$, then $\frac{dy}{dx}$ at x = 1 is _____

19. If $f(x) = \frac{x-4}{2\sqrt{x}}$, then $f'(1)$ is _____

VSA:

20. Evaluate: $\lim_{x\to -1} \frac{x^3+1}{x^2-1}$

21. Evaluate: $\lim_{x\to 0} \frac{\sin 3x}{x}$

22. Find the derivative of $1/x^2$.

23. Evaluate $\lim_{x\to 0} \frac{\sin x^\circ}{x}$

24. Find the derivative of $f(x) = x^2 - 5x + 9$ at x = 3.

Section-B (2 Marks)

25. Let $f(x) = \begin{cases} cosx & if\ x \geq 0 \\ x + k & if\ x < 0 \end{cases}$, find the value of constant k , given that $\lim_{x\to 0} f(x)\ exists$

26. Find the positive value of n so that $\lim_{x\to 3} \frac{x^n-3^n}{x-1}$ = 108.

27. If y = $\frac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}}$, find $\frac{dy}{dx}$.

28. Find $\lim_{x\to 1}[x-1]$ where $[.]$ is greatest integer function.

29. Evaluate $\lim_{x\to 2} \frac{x^{10}-1024}{x^5-32}$.

Section C(4/6 marks)

30. Differentiate with respect to x, from first principles.

a. $\sin x$ b. $\tan x$ c. $\sec x$ d. $x\cos x$

e. $cos^2 x$ f. $cos\, x^2$ g. $\frac{1}{\sqrt{x}}$ h. $\tan 2x$

i. $\sin\sqrt{x}$ j. $x^3 - 1$ k. $\sqrt{sin\, x}$ l. $sin^3 x$

m. $\cot(2x+3)$ n. $\sec 3x$ o. $x^2 \cos x$ p. $\frac{\sin x}{x}$

q. $\frac{x^2+3}{x}$

31. Evaluate

a. $\lim_{x\to 0} \frac{\tan 2x - \sin 2x}{x^3}$ b. $\lim_{x\to 0} \frac{1-\cos 5x}{1-\cos 6x}$ c. $\lim_{x\to 0} \frac{1-\cos x}{\sin^2 x}$

d. $\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1-2x}}{\sin x}$ e. $\lim_{x\to \frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1}$ f. $\lim_{x\to \frac{\pi}{6}} \frac{\sqrt{3}\sin x - \cos x}{x - \frac{\pi}{6}}$

g. $\lim_{x\to 2} \frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}$ h. $\lim_{x\to a} \frac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}$ i. $\lim_{x\to \frac{\pi}{4}} \frac{\tan^3 x - \tan x}{\cos\left(\frac{\pi}{4}+x\right)}$

j. $\lim_{x\to \frac{\pi}{2}} \frac{1+\cos 2x}{(\pi-2x)^2}$ k. $\lim_{x\to \frac{\pi}{2}} \frac{1-\sin^3 x}{\cos^2 x}$ l. $\lim_{x\to 0} \frac{1-\cos x\sqrt{\cos 2x}}{x^2}$

m. $\lim_{x\to \frac{\pi}{4}} \frac{1-\tan^2 x}{\cos x - \sin x}$

32. If $y = \sec x + \tan x$, show that $\cos x \frac{dy}{dx} = y$.

33. If $y = \operatorname{cosec} x + \cot x$, show that $\sin x \frac{dy}{dx} + y = 0$.

34. If $y = a\sin x + b\cos x$, show that $y^2 + \left(\frac{dy}{dx}\right)^2 = a^2 + b^2$.

35. If $y = x \sin x$, prove that $\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \cot x$

36. If $f(x) = \begin{cases} x + 1/2, & x > 1/2 \\ 0, & x = 1/2 \\ 2x, & x < 1/2 \end{cases}$ find $\lim_{x\to 1/2} f(x)$

37. For what value of p does $\lim_{x\to 1} f(x)$ exists where $f(x) = \begin{cases} 2px + 3, & \text{if } x < 1 \\ 1 - px^2, & \text{if } x > 1 \end{cases}$

38. Find the derivatives of:

a. $\frac{x\sin x + 2x^2}{\cos x}$ b. $\frac{\sin x - x\cos x}{x\sin x + \cos x}$ c. $\frac{3+5\cos x}{5+3\cos x}$

d. $\frac{3}{x^2 + \sin x - \cos x}$ e. $\frac{1-\tan x}{1+\tan x}$ f. $x^3 \sin x + 2x\cos x$

g. $(x + \sec x)(x - \tan x)$ h. $\frac{3x-4}{x+1}$ i. $x^2 \sin x$

j. $\frac{x+\cos x}{\tan x}$ k. $\sqrt{\frac{1+\sin x}{1-\sin x}}$

39. a. $\lim_{x\to 2}\frac{e^x-e^2}{x-2}$ b. $\lim_{x\to 0}\frac{e^x-e^{-x}}{x}$ c. $\lim_{x\to 0}\frac{e^x+e^{-x}-2}{x^2}$ d. $\lim_{x\to 0}\frac{3^{2x}-2^{3x}}{x}$

e. $\lim_{x\to 0}\frac{3^{2x}-1}{2^{3x}-1}$ f. $\lim_{x\to 0}\frac{e^x-x-1}{x}$ g. $\lim_{x\to 0}\frac{e^{bx}-e^{ax}}{x}$

h. $\lim_{x\to 0}\frac{a^x-b^x}{x}$ i. $\lim_{x\to 0}\frac{e^{\tan x}-1}{x}$ j. $\lim_{x\to 0}\frac{e^{3+x}-\sin x-e^3}{x}$

SCORING KEY

1.	b	2.	a	3.	d	4.	c	5.	b
6.	c	7.	a	8.	a	9.	b	10.	a
11.	c	12.	c	13.	d	14.	b		
15.	1								
16.	$m=\frac{2\sqrt{3}}{3}$								
17.	1								
18.	0								
19.	$\frac{5}{4}$								
20.	$-\frac{3}{2}$								
21.	3								
22.	$\frac{-2}{x^3}$								
23.	$\frac{\pi}{180}$								
24.	1								
25.	$k=1$								
26.	4								
27.	$\frac{-4x}{(x^2-1)^2}$								
28.	Limit does not exist								
29.	64								
30.	a. cos x						b. $\sec^2 x$		
	c. $\sec x\tan x$						d. cosx – xsinx		
	e. –sin2x						f. $-2x\sin x^2$		
	g. $\frac{-1}{2x^{\frac{3}{2}}}$						h. $2\sec^2 2x$		
	i. $\frac{\cos\sqrt{x}}{2\sqrt{x}}$						j. $3x^2$		
	k. $\frac{\cos x}{2\sqrt{\sin x}}$						l. $3\sin^2 x\cos x$		
	m. $-2\cos ec^2(2x+3)$						n. 3sec3x tan3x		

	o. $2x\cos x - x^2 \sin x$	p. $\dfrac{x\cos x - \sin x}{x^2}$
	q. $\dfrac{x^2 - 3}{x^2}$	
31.	a. 4	b. $\dfrac{25}{36}$
	c. $\dfrac{1}{2}$	d. 2
	e. 2	f. 2
	g. 8	h. $\dfrac{5}{3}(a+2)^{2/3}$
	i. -4	j. 1/2
	k. 3/2	l. 3/2
	m. $2\sqrt{2}$	
36.	1	
37.	-2/3	
38.	a. $\dfrac{\cos x \sin x + x + 4x\cos x + 2x^2 \sin x}{\cos^2 x}$	
	b. $\dfrac{x^2}{(x\sin x + \cos x)^2}$	
	c. $\dfrac{-16\sin x}{(5+3\cos x)^2}$	
	d. $\dfrac{-3(2x + \cos x = \sin x)}{(x^2 + \sin x - \cos x)^2}$	
	e. $\dfrac{-2}{(\cos x + \sin x)^2}$	
	f. $x(3x-2)\sin x + (x^3+2)\cos x$	
	g. $(1+\sec x \tan x)(x - \tan x) + (x + \sec x)(1 - \sec^2 x)$	
	h. $\dfrac{7}{(x+1)^2}$	
	i. $x^2 \cos x + 2x \sin x$	
	j. $\dfrac{\tan x(1-\sin x) - (x+\cos x)\sec^2 x}{\tan^2 x}$	
	k. $\dfrac{1}{1-\sin x}$	

39.	a. e^2	b. 2	c. 1	d. $\log\dfrac{9}{8}$	e. $\dfrac{\log 9}{\log 8}$	f. 0	g. b-a
	h. loga- logb		i. 1	j. $e^3 - 1$			