Solutions to questions 4(c) and 7

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4(c). **Corrected Problem Statement:** Let α be a formula in negative normal form with v and v' two valuations. Suppose that $pos(v, \alpha) \subseteq pos(v', \alpha)$. Show that if $v \models \alpha$ then $v' \models \alpha$.

Proof. We shall prove this using induction on structure of α

Base cases:

- $\alpha = p$: $pos(v, p) \subseteq pos(v', p)$. Now if $v \models p$, then $pos(v, p) = \{p\}$ and hence $pos(v', p)$ has to be $\{p\}$ which means that $v' \models p$
- $\alpha = \neg p$: Proof similar to the previous case

Inductive cases:

- $\alpha = \phi \vee \psi$: It is easy to see that,

  $$pos(v, \phi \vee \psi) = pos(v, \phi) \cup pos(v, \psi) \tag{1}$$

  and similarly

  $$pos(v', \phi \vee \psi) = pos(v', \phi) \cup pos(v', \psi) \tag{2}$$

  Now, if $v \models \phi \vee \psi$, we know,

  $$v \models \phi \; or \; v \models \psi \tag{3}$$

  From (1) and (2), we get:

  $$pos(v, \phi) \cup pos(v, \psi) \subseteq pos(v', \phi) \cup pos(v', \psi) \tag{4}$$

  Now we claim that:

  $$pos(v, \phi) \subseteq pos(v', \phi) \tag{5}$$

  and

  $$pos(v, \psi) \subseteq pos(v', \psi) \tag{6}$$

  Let us try to prove equation (5) by contradiction. Suppose (5) does not hold, then, there exists a literal l in $pos(v, \phi)$ such that $l \notin pos(v', \phi)$. Since the literal l is in $pos(v, \phi)$, it must appear in the formula ϕ. But then by (4), l must be in $pos(v', \psi)$. This means that $v' \models l$. But since l appears in ϕ and is satisfied by v', it must be an element of $pos(v', \phi)$, which contradicts our assumption. Hence, (5) holds. By a similar argument one can prove (6).

  Now using (3), (5), (6) and the induction hypothesis, we get,

  $$v' \models \phi \; or \; v' \models \psi \tag{7}$$

  Thus $v' \models \phi \vee \psi$

- $\alpha = \phi \wedge \psi$: The same argument as in the previous case, should work.

Hence, proved! □

7. We'll prove that the following are equivalent:

1. α is valid in the field of sets sense.
2. α is valid in the usual sense.

Proof. ($1 \implies 2$) Suppose that α is valid in the field of sets sense. Now, suppose X is some set and $\mathcal{F} = \{\phi, X\}$. Let v be any valuation in $PV \to \{\top, \bot\}$. We construct $u : PV \to \mathcal{F}$, such that $u(p) = X$ if $v(p) = \top$ and $u(p) = \phi$ otherwise, for all $p \in PV$. If we show that $v \models \alpha$ iff $u \models \alpha$, we will be done since we know that u indeed satisfies α (because α is valid in the field of sets sense).

So, we show that $u \models \alpha$ if and only if $v \models \alpha$. The proof is by induction on the structure of α.

1. If α is a propositional variable, then the claim holds by construction of u.
2. If $\alpha = \varphi \vee \psi$, then $\hat{u}(\alpha) = \hat{u}(\varphi) \cup \hat{u}(\psi)$. Suppose $v \models \varphi \vee \psi$. Then, without loss of generality, say $v \models \varphi$. But by induction hypothesis, this means $u \models \varphi$ and hence $\hat{u}(\varphi) = X = \hat{u}(\varphi) \cup \hat{u}(\psi)$, and thus $u \models \varphi \vee \psi$. The converse direction is also similar.
3. (The proof for $\alpha = \varphi \wedge \psi$ is similar. Please try this.)
4. If $\alpha = \neg\varphi$, then $v \models \neg\varphi$ if and only if $v \not\models \varphi$ if and only if (by induction hypothesis) $u \not\models \varphi$ if and only if $\hat{u}(\varphi) = \phi$ if and only if $\hat{u}(\neg\varphi) = X$ if and only if $u \models \neg\varphi$.

($2 \implies 1$)

We prove the contrapositive of the statement. Suppose that $\mathcal{F}$ is a field of sets over some set X along with a valuation u such that $u \not\models \alpha$. We will show that there is a valuation v (in the usual sense) such that $v \not\models \alpha$.

By semantics, our assumption means that there is an $x \in X$ such that $x \notin \hat{u}(\alpha)$. Construct $v : PV \to \{\top, \bot\}$ such that $v(p) = \top$ if $x \in u(p)$ and $v(p) = \bot$ otherwise.

Now, show by induction on the structure of formulae that for any well formed formula φ, $\hat{v}(\varphi) = \top$ if and only if $x \in \hat{u}(\varphi)$. But this means that $v \not\models \varphi$ because of the choice of x. $\square$