Wavelet analysis of signals with gaps

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A recently introduced algorithm [Frick *et al.*, Astrophys. J. **483**, 426 (1997)] of spectral analysis of data with gaps via a modified continuous wavelet transform is developed and studied. This algorithm is based on a family of functions called ″gapped wavelets″ which fulfill the admissibility condition on the gapped support. The wavelet family is characterized by an additional parameter which should be calculated for every scale and position. Three theorems concerning the properties of gapped wavelet transform are formulated and proved. They affirm the global stability of the algorithm as well as its stability in both limits of large and small scales. These results are illustrated by some numerical examples, which show that the algorithm really attenuates the artifacts coming from gaps (and/or boundaries), and is particularly efficient at small and large scales. © *1998 American Institute of Physics.* [S0022-2488(98)01208-0]

I. INTRODUCTION

Finding periodicities in observed quantities has been a most important enterprise for a very long time. There are, however, many fields for which the data come with inevitable gaps due to the existence of time (or space) intervals at which measurements cannot be made. A prominent example is astronomy, where the gaps are caused by seasonal windows of observation, cloudy skies, telescope maintenance, etc. An illustrative example can be seen in Fig. 1.

The causes of the interruptions are in general not related to the quantities under study, and their effects are felt in several ways. The most direct one is, of course, the loss of useful information. A more insidious effect is the possible contamination of data with spurious periodicities arising from the spectral properties of the set of gaps.

A straightforward ''filling of the holes,'' either by interpolation or by arbitrary extension can have quite undesirable properties. A typical result of interpolation will be the smoothing of the signal, leading to the loss of information in high frequencies.

Fourier transform techniques have been adapted to such a situation: Welsh estimators[1] are perhaps the most commonly used. An extension of Fourier transform to an uneven data set is known in astronomy as the Lomb-Scargle periodogram.[2,3] The idea of this technique is to correct the basic functions $\cos(\omega t)$ and $\sin(\omega t)$ by a phase shift and a mean value subtraction to preserve their normalization conditions on a given set of observations.

In some instances, it is desirable to dispose of time-scale transforms. To this end, Foster[4] recently introduced an analogous algorithm for wavelet transform on an irregular data set. He considers the Morlet wavelets and proposes to reorthogonalize the three basic functions (the real and the imaginary parts of Morlet wavelet and a constant) by rotating the matrix of their scalar products.

We study here another technique for analyzing signals with gaps, introduced in Refs. 5,6 in connection with the spectral study of stellar chromospheric activity. Two examples of data records taken from Ref. 7 are given in Fig. 1. They show that the whole time of observations includes only few periods of main oscillation and that the total length of the gaps is comparable with the length of the record. The main idea of the proposed procedure is to consider the wavelet transform not as

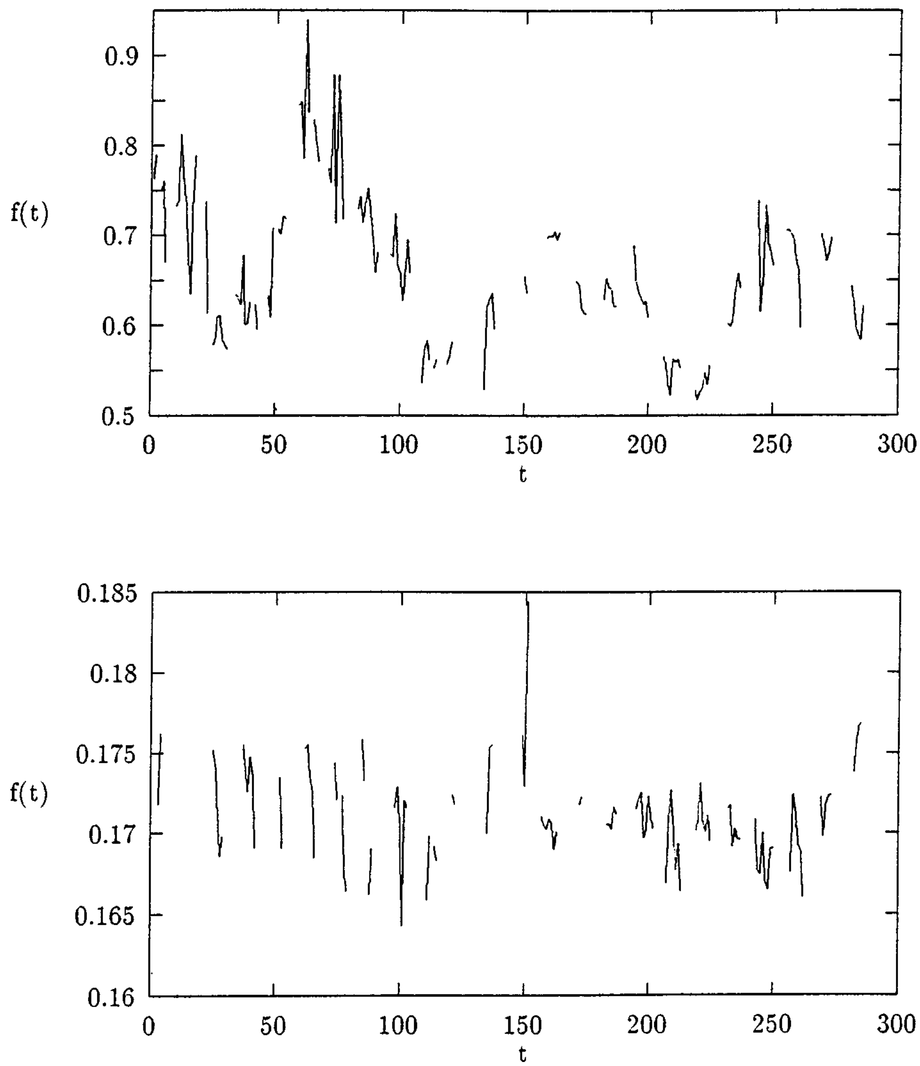

FIG. 1. Two typical examples of data with gaps which concern the stellar chromospheric activity of stars near the lower main sequence at the Mount Wilson Observatory. The figures are taken from Ref. 7.

a convolution of a signal with gaps and a given analyzing wavelet, but as a convolution of a signal and of a wavelet with gaps, thus transferring the gap problem from the unknown signal function to the known wavelet function. Punching holes in a wavelet will of course destroy its useful properties, and one tries to correct the broken wavelet by making it at least admissible.

The algorithm under consideration has been called in Refs. 5 and 6 ''adaptive wavelets'' to emphasize that the analyzing functions are modified by the gaps in the data. This name does not appear to be fortunate because of the extensive use of the word ''adaptive'' in wavelet analysis. This is why we shall write ''the gapped wavelet method'' or ''wavelets with gaps.''

The paper is organized as follows. In Sec. II the definition of wavelets with gaps is given. Some theorems concerning their properties are formulated. The possible developments of the method are also discussed. The properties of wavelet transform with gapped wavelets are illustrated by some examples in Sec. III. The proofs of theorems are given in Sec. IV.

II. DEFINITIONS AND MAIN STATEMENTS

For the sake of simplicity, we limit our study to the one-dimensional setting. The interested reader can verify that all the foregoing statements extend straightforwardly to the n-dimensional case.

Let $\Omega\subset\mathbb{R}$ be an open set. We want to develop a wavelet analysis for functions supported on Ω, which depends as little as possible on Ω. It turns out that the naive approach consisting in performing a standard wavelet analysis of f is not satisfactory. See Sec. III. The reason is the following. Let ψ be the mother wavelet, and denote as usual

$$\psi_{a,b}(x)=\frac{1}{a}\psi\left(\frac{x-b}{a}\right). \tag{1}$$

It is well known that ψ must satisfy the so-called *admissibility condition*, which essentially amounts to demand that the first moment of ψ vanishes. Obviously, when this condition is fulfilled, so it is for all the wavelets $\psi_{a,b}$.

Analyzing a given f supported in Ω means computing the coefficients $\langle f,\psi_{a,b}\rangle$. Hence, even if f is flat on Ω (for example, $f=\chi_\Omega$), this analysis will give significant coefficients for those values of b and a such that $\int_\Omega\psi_{a,b}\neq 0$. This is in contradiction with the very nature of wavelet transform. This is why it is desirable to modify the wavelets so as to enforce an adapted admissibility condition.

A. Gapped wavelets

We start by choosing a mother wavelet ψ satisfying the admissibility condition. We introduce next a non-negative envelope function ϕ such that $|\psi|\leqslant\phi$. Throughout the paper we assume ψ and ϕ to be rapidly decreasing at infinity, as well as their derivative. We define $\phi_{a,b}$ as in (1).

1. The family $\theta_{a,b}$

If $a>0$ and $b\in\mathbb{R}$ we define

$$\theta_{a,b}=\psi_{a,b}\chi_\Omega-C(a,b)\phi_{a,b}\chi_\Omega, \tag{2}$$

where

$$C(a,b)=\begin{cases}\dfrac{\int_\Omega\psi_{a,b}}{\int_\Omega\phi_{a,b}} & \text{if} \quad \displaystyle\int_\Omega\phi_{a,b}\neq 0,\\ 0 & \text{otherwise.}\end{cases} \tag{3}$$

This ensures that

$$\int_{\mathbb{R}^n}\theta_{a,b}=0. \tag{4}$$

Let us note that as $|\psi|\leqslant\phi$, we have

$$|C(a,b)|\leqslant 1 \tag{5}$$

for all a,b. We emphasize that the analyzing function $\theta_{a.b}$ changes for every scale a and every position b if only a gap falls in its vicinity. Thus, strictly speaking, the family $\theta_{a,b}$ is not a family of wavelets because its self-similarity is broken.

2. The family $\theta^{(1)}_{a,b}$

This construction can be generalized by requiring the modified wavelets to have vanishing moments of higher orders. This can be achieved by introducing, for a given integer k, the functions

$$\theta^{(k)}_{a,b}(t)=\psi_{a,b}(t)\chi_\Omega(t)-\left(C_0(a,b)+C_1(a,b)\left(\frac{t-b}{a}\right)+\ldots.+C_k(a,b)\left(\frac{t-b}{a}\right)^k\right)\phi_{a,b}(t)\chi_\Omega(t). \tag{6}$$

The constants C_j are readily obtained from the corresponding conditions

$$0 \leqslant j \leqslant k, \quad \int_\Omega \theta_{a,b}^{(k)}(t) \left(\frac{t-b}{a} \right)^j dt = 0. \tag{7}$$

For example, for $k=1$ one finds

$$C_0(a,b) = \frac{\int_\Omega \psi_{a,b} \int_\Omega ((t-b)/a)^2 \phi_{a,b} - \int_\Omega ((t-b)/a) \psi_{a,b} \int_\Omega ((t-b)/a) \phi_{a,b}}{\int_\Omega \phi_{a,b} \int_\Omega ((t-b)/a)^2 \phi_{a,b} - (\int_\Omega ((t-b)/a) \phi_{a,b})^2}, \tag{8}$$

$$C_1(a,b) = \frac{\int_\Omega \phi_{a,b} \int_\Omega ((t-b)/a) \psi_{a,b} - \int_\Omega \psi_{a,b} \int_\Omega ((t-b)/a) \phi_{a,b}}{\int_\Omega \phi_{a,b} \int_\Omega ((t-b)/a)^2 \phi_{a,b} - (\int_\Omega ((t-b)/a) \phi_{a,b})^2}. \tag{9}$$

Let us note that C_0 and C_1 are no longer necessarily bounded by 1. In numerical calculations, described below, we checked the values of these coefficients. They really can become larger than 1, but this only happens for large values of a, such that the (effective) support of the wavelet is at least of the same size as Ω. However, in such a case, the physical meaning of our analysis becomes doubtful.

B. Gapped wavelet transform: Definition and properties

In this subsection, we only consider the case $k=0$.

Definition 1: The gapped wavelet transform is the linear map which associates to any $f \in L^2(\Omega)$ *the set of scalars* $\langle f, \theta_{a,b} \rangle$, $b \in \mathbb{R}$, $a > 0$.

As explained before, we may view f as the restriction to Ω of some square-integrable function globally defined on $\mathbb{R}$. Since only the values of f on Ω are relevant, there is no harm in not distinguishing between the various extensions of f in $\mathbb{R}$. For simplicity we will denote all of them by the same letter f. Therefore we have by construction

$$\langle f, \theta_{a,b} \rangle = \langle f \chi_\Omega, \theta_{a,b} \rangle. \tag{10}$$

Our first result states that this map is continuous from $L^2(\Omega)$ to $L^2(\mathbb{R} \times \mathbb{R}^+, (db\ da)/a)$, as for the usual wavelet transform. This is a stability result, in which the most remarkable point is that the stability constant A_0 (see the theorem below) does not depend on Ω. In other words, whatever the shape of Ω, the stability of the gapped wavelet transform is uniformly controlled.

Theorem 2: *There exists a constant* A_0, *independent of* Ω, *such that*

$$\int_0^{+\infty} \int_{\mathbb{R}} |\langle f, \theta_{a,b} \rangle|^2 \frac{da\ db}{a} \leqslant A_0 \int_\Omega |f|^2. \tag{11}$$

When Ω is of finite measure, we may assume that $m_\Omega f$, the mean value of f over Ω, vanishes, since $\int_\Omega \theta_{a,b} = 0$. Therefore, inequality (11) improves in

$$\int_0^{+\infty} \int_{\mathbb{R}} |\langle f, \theta_{a,b} \rangle|^2 \frac{da\ db}{a} \leqslant A_0 \int_\Omega |f - m_\Omega f|^2. \tag{12}$$

We now examine to what extent our gapped wavelet transform carries out some time-scale information. Let $(S_a)_{a>0}$ be a Littlewood–Paley approximation of identity, defined by a function Φ such that $\hat{\Phi}(\omega) = 1$ if $|\omega| \leqslant 1$, $\hat{\Phi}(\omega) = 0$ if $|\omega| \geqslant 2$. Denoting by Φ_a the function $(1/a)\, \Phi(./a)$ we write

$$S_a f = f * \Phi_a. \tag{13}$$

By construction, $S_a f$ is a smoothed version of f, at scale a. Since $\hat{\Phi}_a(\omega) = 1$ when $|\omega| \leqslant (1/a)$ and

$$\widehat{S_a f}(\omega) = \hat{f}(\omega) \hat{\Phi}_a(\omega), \tag{14}$$

the spectral content of f in the interval $|\omega| \leqslant 1/a$ is preserved under the action of S_a. Also, since $\hat{\Phi}_a(\omega) = 0$ when $|\omega| \geqslant 2/a$, the spectral content of f in the domain $|\omega| \geqslant 2/a$ is totally erased.

Here of course, the constants 1 and 2 which we use when defining $\hat{\Phi}$ are arbitrary, and may be replaced by any other choice without changing the foregoing statements.

The next theorem describes how the coefficients $\langle f, \theta_{a,b} \rangle$, for fixed $a > 0$, attenuate the low-frequency content of f.

Theorem 3: *There exists a constant $A_1 > 0$, independent of Ω, such that*

$$\int_{\mathbb{R}^n} |\langle S_{a'} f, \theta_{a,b} \rangle|^2 db \leqslant A_1 \left(\frac{a}{a'} \right)^2 \int_{\mathbb{R}} |S_{a'} f|^2, \tag{15}$$

for all $a' \geqslant a > 0$.

This means that $\langle f, \theta_{a,b} \rangle$ filters in f the frequencies lower in modulus than $1/a$. The exponent 2 in the term $(a/a')^2$ is directly linked to the vanishing of the first moment of the gapped wavelets. When using the functions $\theta_{a,b}^{(k)}$ for $k \geqslant 1$, one gets a term $(a/a')^{2k+2}$, provided the constants $C_i(a,b)$, $0 \leqslant i \leqslant k$, in (6) are uniformly controlled. This fact may be proved by a straightforward adaptation of the proof of the theorem (see Sec. IV below).

The situation is more subtle regarding the damping of the high-frequency content of f, because the geometry of Ω must be taken into consideration. This is the reason why we assume a mild regularity property on the boundary $\partial\Omega$ of Ω, quantified by an exponent $\alpha > 0$, and defined as follows (for the sake of completeness, the definition is given in arbitrary dimension n).

Definition 4: Ω verifies the property (P_α), where $\alpha \in]0,1]$, when there exists a constant A_Ω, such that for all $b \in \mathbb{R}^n$ and for all $\rho, \varepsilon > 0$, $\varepsilon \leqslant \rho$ one has

$$\min \left\{ \begin{array}{l} |\{y \in B(b,\rho) \cap \Omega; d(y,\partial\Omega) \leqslant \varepsilon\}| \\ |\{y \in B(b,\rho) \cap \Omega^c; d(y,\partial\Omega) \leqslant \varepsilon\}| \end{array} \right\} \leqslant A_\Omega \left(\frac{\varepsilon}{\rho} \right)^\alpha |B(b,\rho)|. \tag{16}$$

Here $B(b,\rho)$ is the Euclidean ball of center b and radius ρ, while $|E|$ stands for the Lebesgue measure of E.

Such a property means that, at each scale ρ, the local density in Ω or in Ω^c of the strip around $\partial\Omega$ of width ε is uniformly controlled by $(\varepsilon/\rho)^\alpha$.

For example, if Ω is a finite union of open intervals in dimension 1, or if Ω is a regular domain (at least Lipschitz) in several dimensions, then (P_1) is true. Also, most classical examples of sets with fractal boundary, such as von Koch curve, do fulfill (P_α) for some $\alpha > 0$.

Let us go back to our gapped wavelet transform, for an open set Ω fulfilling (P_α). The next theorem states that α describes the damping of the high-frequency content of f in the coefficients $\langle f, \theta_{a,b} \rangle$ for fixed $a > 0$.

Theorem 5: *For any $\alpha \in]0,1]$, there exists a constant A_2, independent of Ω, such that if $\partial\Omega$ verifies the property (P_α)*

$$\int_{\mathbb{R}} |\langle f - S_{a'} f, \theta_{a,b} \rangle|^2 \, db \leqslant A_2 (1 + A_\Omega) \left(\frac{a'}{a} \right)^\alpha \int_{\mathbb{R}} |f - S_{a'} f|^2, \tag{17}$$

for all $a \geqslant a' > 0$.

The proof will show that this result does not rely on the cancellation properties of the gapped wavelets and thus is also valid for standard wavelet transforms.

III. ILLUSTRATIONS

The three theorems formulated above show that no dramatical changes (which lead to the growth of the spectral energy) are expected in the spectra of the signal calculated using the gapped wavelets $\theta_{a,b}$ and that the density of spectral energy which reflects the artifacts coming from the boundaries and gaps should be attenuated (at different rates) in both high and low frequencies.

No exact results have been obtained concerning the functions $\theta_{a,b}^{(1)}$, but in numerical simulations we used them as well as the functions $\theta_{a,b}$. We denote by $T0$ the results of the transform using the functions $\theta_{a,b}$, which are only admissible (the zeroth order moment vanishes), and as

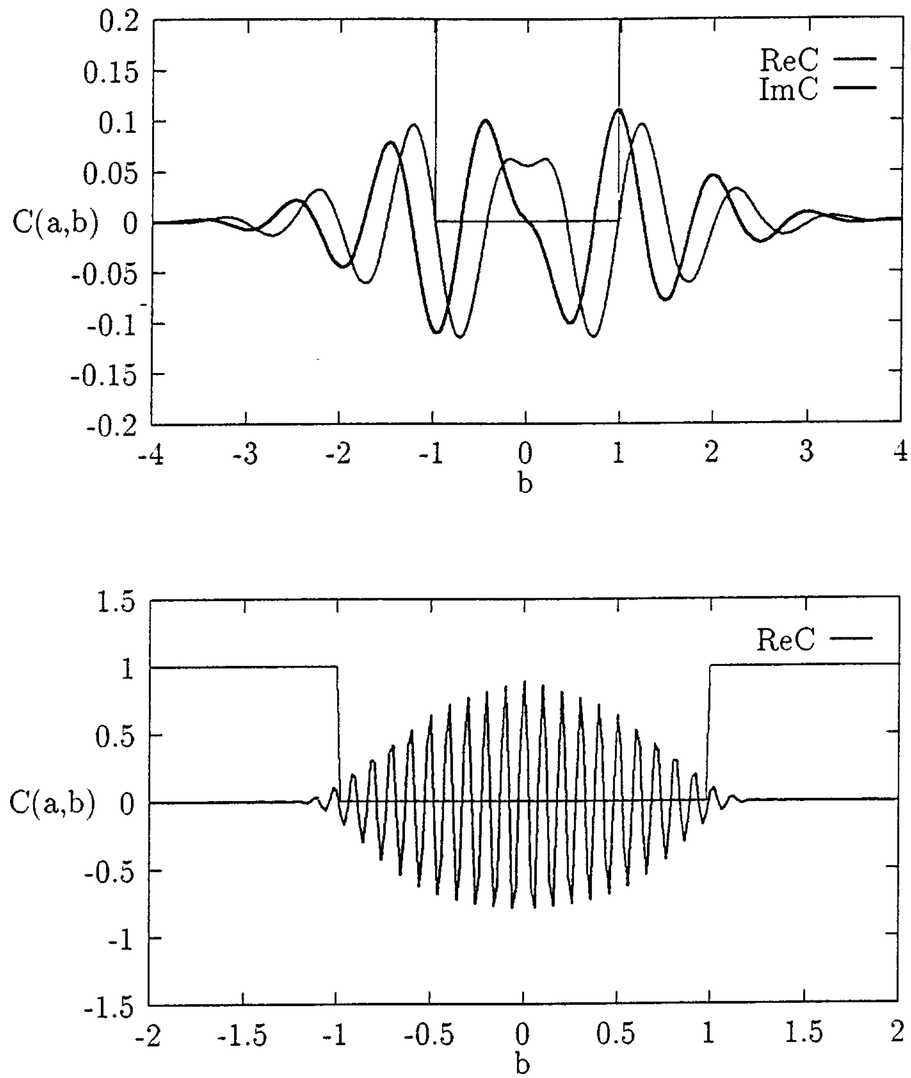

FIG. 2. Variations of parameter $C(a,b)$ in the vicinity of a gap $[-1,1]$ (the characteristic function is shown by the solid line). The real and the imaginary parts of $C(a,b)$ for $a=1$ are shown in the upper panel; the real part of $C(a,b)$ for $a=0.1$ is presented in the lower panel.

$T1$ the results of the transform using the functions $\theta_{a,b}^{(1)}$ for which the first order moment vanishes as well as the zeroth. We compare both with the standard wavelet transform with Morlet wavelet (denoted in the figures as M). Let us emphasize that in the latter case (taken as the reference) we also do not use any interpolation of signals in the gap, and we do not try to eliminate the mean value, the trend, etc.

In the examples to be discussed below, ψ is the Morlet wavelet

$$\psi(t)=e^{i\omega t}e^{-t^2/2} \tag{18}$$

with appropriate ω.

We show in Fig. 2 how the parameter $C(a,b)$ varies when the function $\theta_{a,b}$ passes through an isolated gap $[-1,1]$. In the upper panel of Fig. 2 is presented the graph for $a=1$. Both real and imaginary parts are shown. Notice that the modulus of C does not surpass the value 0.15. In the lower panel of the same figure we present the variations of the real part of $C(a,b)$ for $a=0.1$ (the imaginary one is not shown due to the high density of lines). Now, the maxima of C approach to one in the central part of the gap. Obviously, these large values of $C(a,b)$ correspond to strong

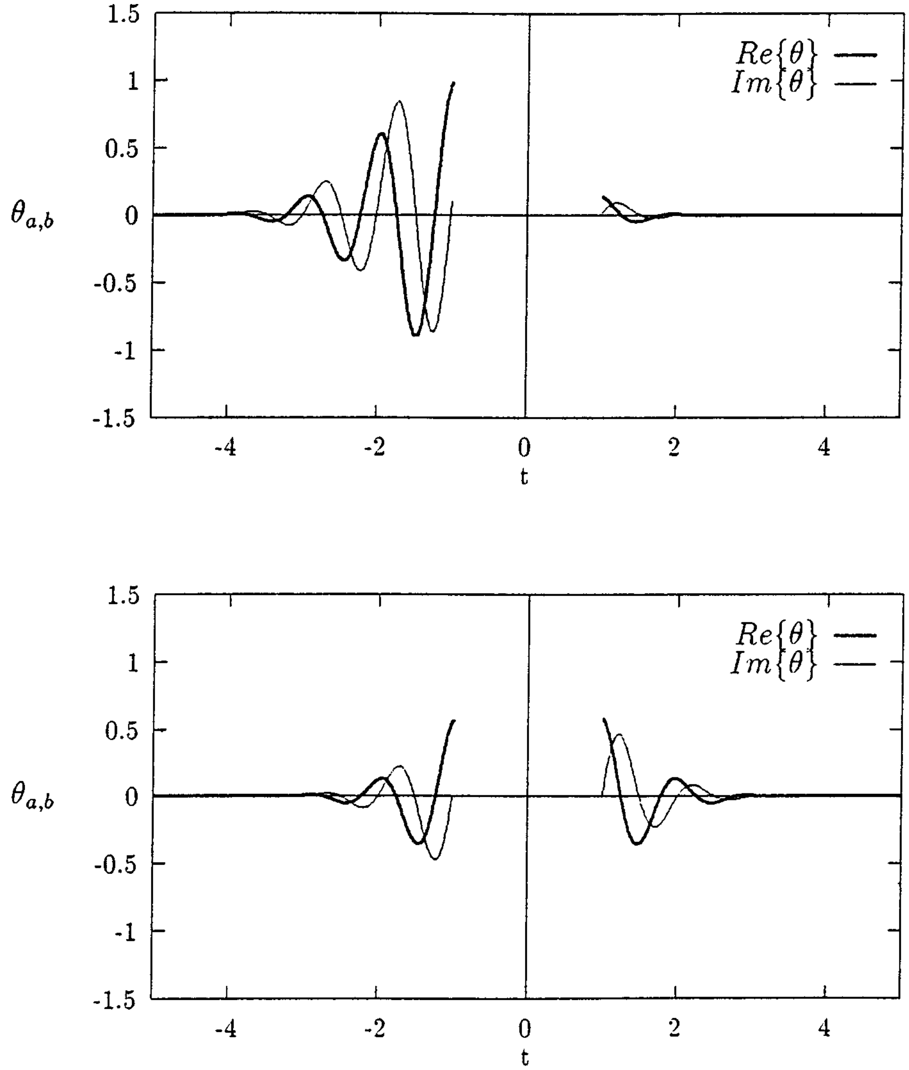

FIG. 3. Gapped wavelet $\theta_{a,b}(t)$ for $a=1$ when going through the gap $[-1,1]$: $b=-1$ (upper panel), $b=0$ (lower panel).

deformations of the analyzing wavelet, provided by the fact that only a far part of exponentially decreasing tail of the wavelet reaches the boundary of the gap. But this also implies that these strongly deformed functions do not give a considerable contribution.

Two examples of gapped wavelets $\theta_{a,b}(t)$ are shown on Fig. 3. We consider the same isolated gap $[-1,1]$ and the scale $a=1$. The functions $\theta_{a,b}(t)$ are plotted for two positions: $b=-1$, where the center of the wavelet is on the boundary of the gaps (upper panel of Fig. 3) and $b=0$, where the wavelet is centered in the gap. One can see that the analyzing wavelet does not undergo drastic changes in the domain lying away from the gap. The discontinuities of the gapped wavelet are on the boundary of the gap and do not introduce additional contributions in the wavelet transform.

We start by analyzing a function with a simple (gaussian) spectrum. The function $f(t)=1+t+\exp(-t^2/2)\cos(2\pi t)$ is a superposition of a Morlet wavelet and a linear trend. The function is given only in a single interval (window) $[-0.6,2.6]$ (Fig. 4.). The true spectrum of this function (the mean value of $|W(a,b)|^2$ for given a) is also shown on the same figure. One can see that the standard algorithm M gives a spectrum that increases with scale a, without any well pronounced peak at $a=1$. The algorithm $T0$ clearly displays the top of the Gaussian bump, but gives the wrong maximum at large scales. Algorithm $T1$ efficiently suppress the wrong power in the large scales, and also slightly improves the small scales part of the spectrum.

In Fig. 5 we show the wavelet spectrum for a harmonic signal superposed with a linear function and known on an interval with regular gaps. The basic spectrum M displays two wrong

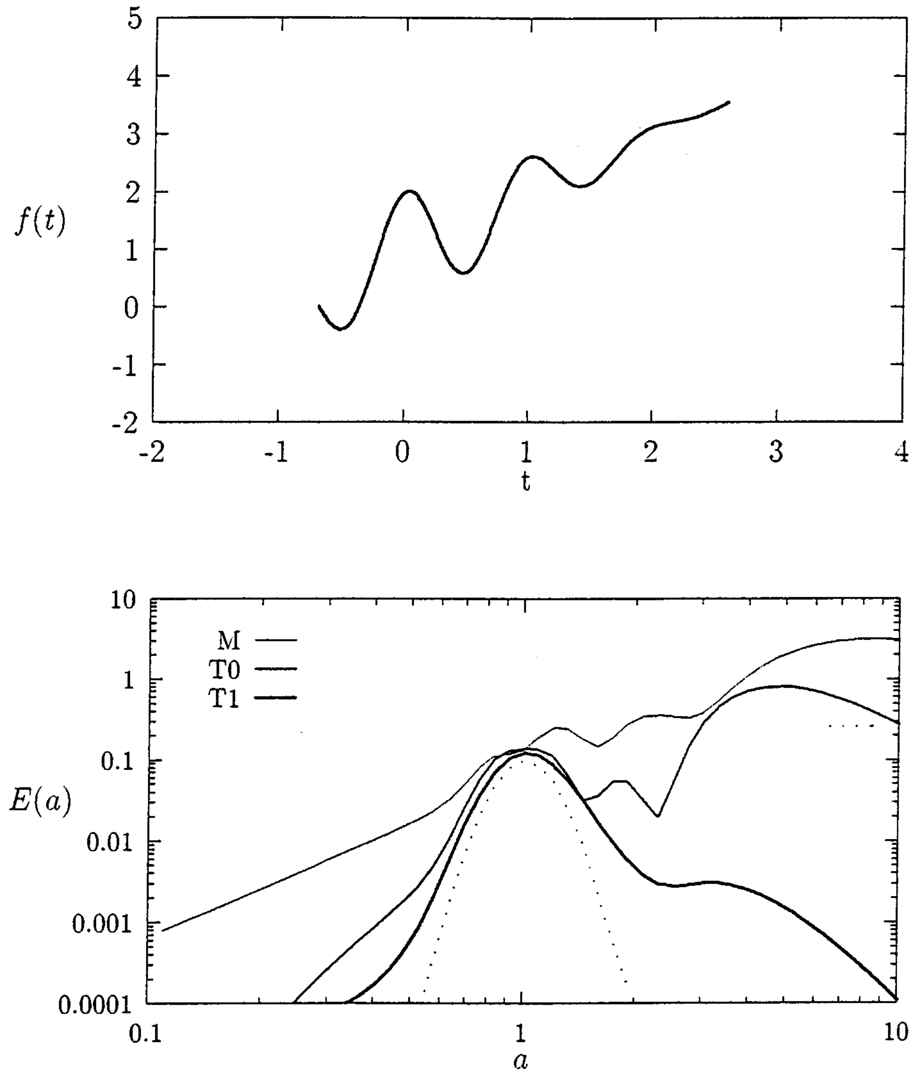

FIG. 4. The function $f(t)=1+t+\exp(-t^2/2)\cos(2\pi t)$ given on the interval $[-0.6,2.6]$ and its wavelet spectra M (line), $T0$(line) and $T1$ (line). The ''ideal'' spectrum is shown by a dotted line.

maxima with orders of magnitude comparable with the main one: in large scales and in a scale smaller than the main one. As indicated by the theorems proved above, the wrong maxima are suppressed in both large and small scales. The transform $T1$ is again essentially more effective in large scales. Let us note that the standard algorithm displays the main maximum at $a=0.74$, $T0$ gives $a=0.76$ and $T1$ shows the maximum at $a=0.79$ (the true value is $a=0.8$ and is marked in the figure by an arrow). This indicates that the gapped wavelet technique not only suppresses the low and high noise frequencies, but is also better at estimating the frequency of the signal.

The next example (Fig. 6) presents a signal with two frequencies in a linear background. In this case the standard technique produces an additional peak in intermediate scales. The gapped algorithm eliminates it. The large scale artifact is again better suppressed by the wavelets $T1$.

The last example (Fig. 7) presents a signal with two harmonic components without any background but with regular gaps. The results show that there is an unavoidable artifact (the left peak, which remains on the same magnitude in all three spectra). At the same time one sees that the standard techniques absolutely lost the high frequency, which is drowned in noise, in spite of being well observed by the naked eye. In the gapped wavelet method spectra this frequency gives a pronounced peak.

Surely, there are a lot of possibilities to improve the results of wavelet transform of signals

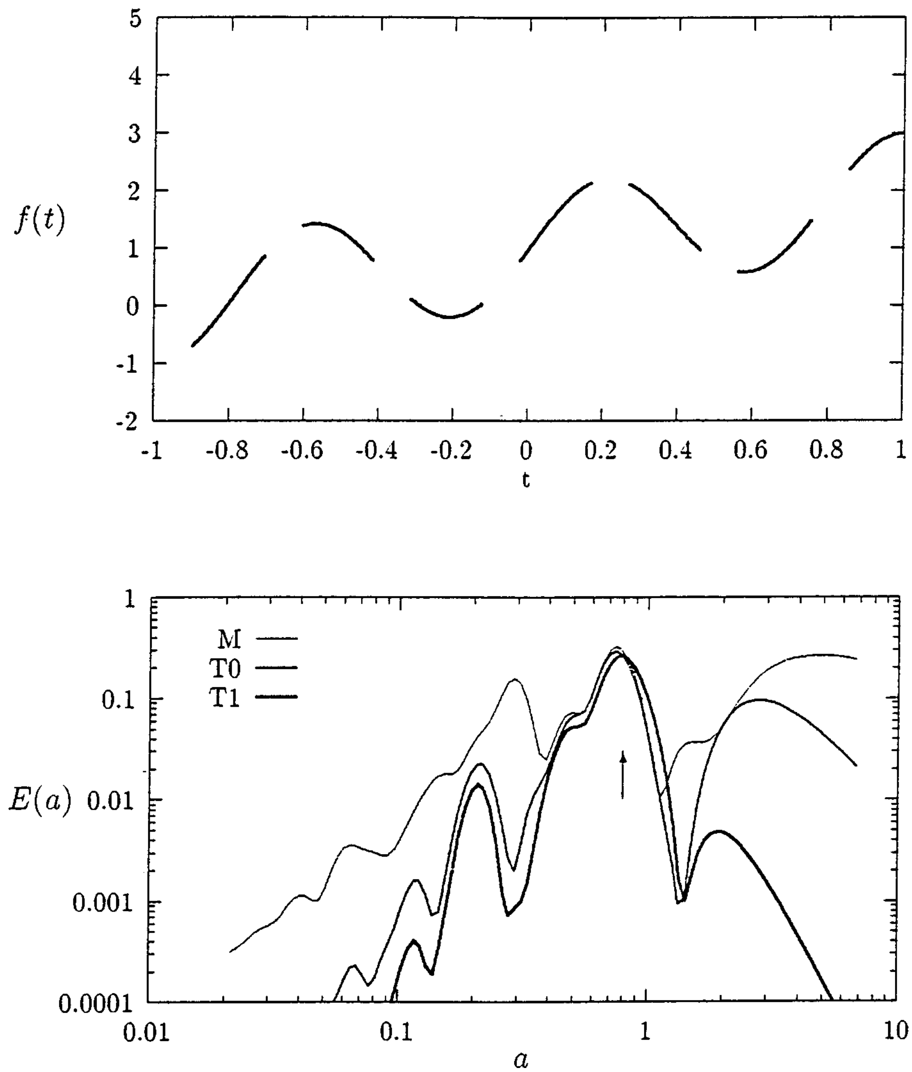

FIG. 5. The analyzed signal $f(t)=1+t+\sin((4\pi/5)\,t)$ with gaps (above) and its spectra (below), obtained by standard Morlet wavelet (M) and gapped wavelets θ ($T0$) and $\theta^{(1)}$ ($T1$). The true frequency is marked by the arrow.

with gaps, especially in a concrete situation when the expected character of the signal is known *a priori*.

Our examples illustrate the proved properties of gapped wavelets $\theta_{a,b}$ and show that the wavelets $\theta^{(1)}_{a,b}$ provide further improvement of spectral resolution of the wavelet transform.

IV. PROOFS

In what follows, the letter A denotes a constant, which may change from line to line.

A. Proof of Theorem 2

This proof is based on classical estimates in the theory of singular integral operators. One can show the following result:

Lemma 6: The matrix $(\langle \theta_{a,b},\theta_{a',b'}\rangle)$, where $a,a'>0,\ b,b'\in\mathbb{R}$, *defines a continuous operator in* $L^2(\mathbb{R}^2_+,db\ da/a)$.

In this statement, the word ''matrix'' is used by analogy with the standard case where the parameters a,b belong to a discrete, or even finite, set.

Lemma 6 implies Theorem 2. Indeed, if we define an operator T by

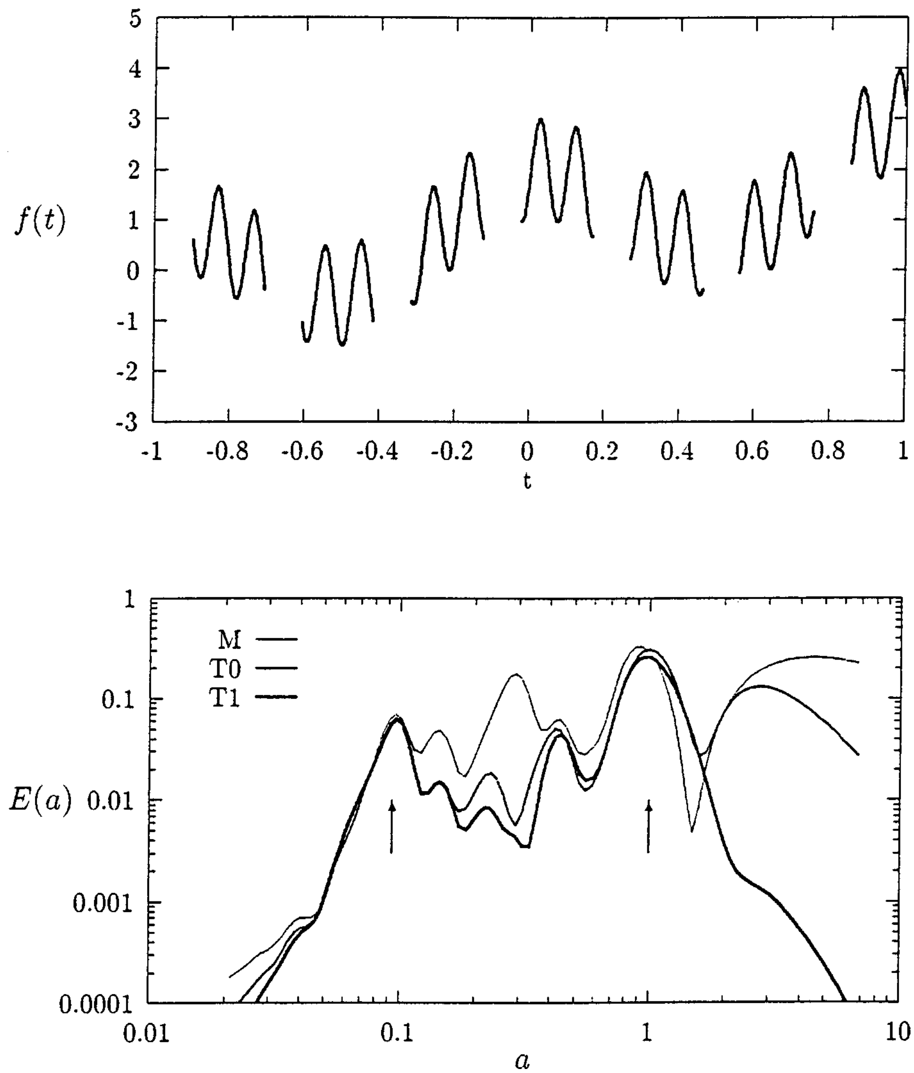

FIG. 6. The analyzed signal $f(t)=t+c+\cos(2\pi t)+\sin(21\pi t)$ with gaps and its spectra. Both frequencies are marked in the scales by arrows.

$$Tf=\int\int\langle f,\theta_{a,b}\rangle\theta_{a,b}\frac{da\ db}{a}\tag{19}$$

then we have

$$\langle Tf,f\rangle=\int\int|\langle f,\theta_{a,b}\rangle|^2\frac{da\ db}{a}\tag{20}$$

and

$$\langle Tf,Tf\rangle=\int\int\int\int\langle f,\theta_{a,b}\rangle\langle\theta_{a,b},\theta_{a',b'}\rangle\langle f,\theta_{a',b'}\rangle\frac{da\ db}{a}\frac{da'db'}{a'}.\tag{21}$$

Applying Lemma 6 to the r.h.s. of (21), we get

$$\langle Tf,Tf\rangle\leqslant A_0\langle Tf,f\rangle\tag{22}$$

which implies continuity of T with $\|T\|\leqslant A_0$. Here A_0 is the norm of the operator defined by $(\langle\theta_{a,b},\theta_{a',b'}\rangle)$. We have to show that it can be bounded uniformly in Ω.

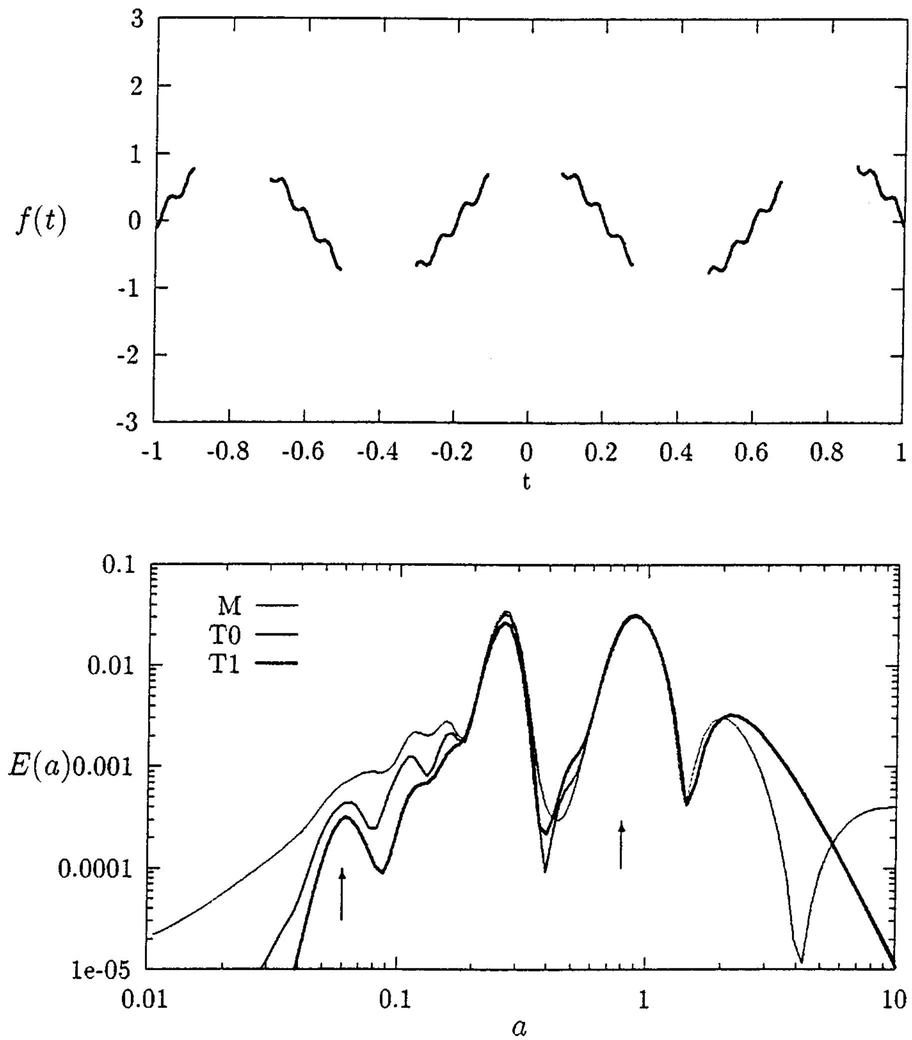

FIG. 7. The analyzed signal $f(t) = \cos(5\pi t/2) + 0.1\ \sin(33\pi t)$ with gaps and its spectra.

To do so, we shall show that there exists a constant A, independent on Ω, such that

$$\int \int |\langle \theta_{a,b}, \theta_{a',b'} \rangle| \frac{da'\ db'}{a'} \leqslant A \tag{23}$$

for all $a>0$, $b \in \mathbb{R}$. This inequality implies that the matrix $(\langle \theta_{a,b}, \theta_{a',b'} \rangle)$ defines a continuous operator on $L^{\infty}(\mathbb{R}_+^2, da\ db/a)$. By symmetry one gets the continuity of the same operator on $L^1(\mathbb{R}_+^2, da\ db/a)$ and, by interpolation, on $L^2(\mathbb{R}_+^2, da\ db/a)$. In addition, the corresponding operator norms are controlled by the constant A of (23), which does not depend on Ω.

The proof of (23) depends on the following estimates. By assumption, there exists A such that the wavelet ψ and the function ϕ satisfy the estimates

$$|\psi(x)| + \phi(x) \leqslant A(1+|x|)^{-3}, \tag{24}$$

$$|\psi(x+h) - \psi(x)| + |\phi(x+h) - \phi(x)| \leqslant A|h|(1+|x|)^{-3}, \tag{25}$$

if $x, h \in \mathbb{R}$ and $|h| \leqslant |x|/2$. We set

$$\omega(x) = (1+|x|)^{-3}. \tag{26}$$

Lemma 7: There exists a constant A, independent of Ω, *such that*

$$|\langle \theta_{a,b}, \theta_{a',b'} \rangle| \leqslant A \frac{a'}{a^2} \omega\left(\frac{b'-b}{a}\right) \tag{27}$$

for all $a \geqslant a' > 0$ *and* $b, b' \in \mathbb{R}$.

If $a \geqslant a'$

$$\langle \theta_{a,b}, \theta_{a',b'} \rangle = \int [\psi_{a,b}(x) - C(a,b)\phi_{a,b}(x)] \overline{\theta_{a',b'}(x)} \tag{28}$$

as $G^2 = G$. Since the first moment of $\theta_{a',b'}$ is zero, one can write

$$\langle \theta_{a,b}, \theta_{a',b'} \rangle = \int [\psi_{a,b}(x) - \psi_{a,b}(b')] \overline{\theta_{a',b'}(x)} dx - C(a,b) \int [\phi_{a,b}(x) - \phi_{a,b}(b')] \overline{\theta_{a',b'}(x)} dx. \tag{29}$$

The first term on the r.h.s. is controlled by

$$A \int \frac{|x-b'|}{a^2} \left[\omega\left(\frac{x-b}{a}\right) + \omega\left(\frac{b'-b}{a}\right) \right] \frac{1}{a'} \omega\left(\frac{x-b'}{a'}\right) dx \tag{30}$$

by (5), (24), and (25). This integral can be evaluated as

$$A \frac{a'}{a^2} \omega\left(\frac{b-b'}{a}\right). \tag{31}$$

The second term on the r.h.s. of (29) is controlled in the same way. This completes the proof of Lemma 7.

One then deduces (23) by writing

$$\begin{aligned} &\int \int |\langle \theta_{a,b}, \theta_{a',b'} \rangle| \frac{db'\,da'}{a'} \\ &\leqslant A \int \int_{a' \geqslant a} \frac{a}{a'^2} \omega\left(\frac{b-b'}{a'}\right) \frac{db'\,da'}{a'} + A \int \int_{a' \leqslant a} \frac{a'}{a^2} \omega\left(\frac{b-b'}{a}\right) \frac{db'\ da'}{a'} \\ &\leqslant A \int_a^{+\infty} \frac{a}{a'} \frac{da'}{a'} + A \int_0^a \frac{a'}{a} \frac{da'}{a'} = 2A. \end{aligned} \tag{32}$$

Thus inequality (23) is proved and the proof of theorem 2 is completed.

B. Proof of Theorem 3

Theorem 3 is a consequence of estimates which are close analogs of the preceding ones. Denoting by $\tilde{\Phi}$ the function

$$\tilde{\Phi}(x) = 2\Phi(2x) \tag{33}$$

one has by construction $\tilde{\Phi} * \Phi = \Phi$. From this one gets the identity

$$\tilde{S}_a S_a = S_a, \tag{34}$$

where $\tilde{S}_a$ is the convolution operator with $\tilde{\Phi}_a$. Thus, if $0 < a \leqslant a'$, then

$$\langle S_{a'} f, \theta_{a,b} \rangle = \langle S_{a'} f, \tilde{S}_{a'} \theta_{a,b} \rangle. \tag{35}$$

Inequality (15) is a simple consequence of

Lemma 8: If $0<a\leqslant a'$ *then*

$$|\tilde{S}_{a'}\theta_{a,b}(x)|\leqslant A\frac{a}{a'^2}\omega\left(\frac{x-b}{a'}\right), \tag{36}$$

where A is independent of Ω.

We leave it to the reader to verify that (35) and (36) imply (15) and only explain why (36) is true. We write

$$\tilde{S}_{a'}\theta_{a,b}(x)=\int\tilde{\Phi}_{a'}(x-y)\theta_{a,b}(y)dy=\int[\tilde{\Phi}_{a'}(x-y)-\tilde{\Phi}_{a'}(x-b)]\theta_{a,b}(y)dy. \tag{37}$$

Thus one gets as in the proof of (27)

$$|\tilde{S}_{a'}\theta_{a,b}(x)|\leqslant A\int\frac{|y-b|}{a'^2}\left[\omega\left(\frac{x-y}{a'}\right)+\omega\left(\frac{x-b}{a'}\right)\right]\frac{1}{a}\omega\left(\frac{y-b}{a}\right)dy\leqslant A\frac{a}{a'^2}\omega\left(\frac{x-b}{a'}\right). \tag{38}$$

C. Proof of Theorem 5

Let Ψ be an even C^∞ function, such that Supp $\Psi\subset\{\omega\in\mathbb{R};1/4\leqslant|\omega|\leqslant1\}$, and

$$\int_0^{+\infty}\hat{\Psi}(\tau\omega)\frac{d\tau}{\tau}=1 \tag{39}$$

for all $\omega\neq0$.

Then, if Δ_τ is the convolution operator by Ψ_τ, where $\Psi_\tau(t)=(1/\tau)\Psi(t/\tau)$, the above identity means that for any $f\in L^2(\mathbb{R})$ one has

$$f=\int_0^{+\infty}\Delta_\tau f\frac{d\tau}{\tau}. \tag{40}$$

By construction, one has

$$\Delta_\tau(I-S_{a'})=0 \tag{41}$$

if $\tau\geqslant a'$. It follows that

$$(I-S_{a'})f=\int_0^{a'}\Delta_\tau(I-S_{a'})f\frac{d\tau}{\tau}. \tag{42}$$

This identity enables us to deduce Theorem 5 from the inequality

$$\int_{\mathbb{R}}|\langle\Delta_\tau f,\theta_{a,b}\rangle|^2db\leqslant A(1+A_\Omega)\left(\frac{\tau}{a}\right)^\alpha\int_{\mathbb{R}}|\Delta_\tau f|^2, \tag{43}$$

where $0<\tau\leqslant a$.

Indeed, if $a'\leqslant a$ one applies (42) and then the Minkowski inequality and (43). That gives

$$\left(\int_{\mathbb{R}}|\langle f-S_{a'}f,\theta_{a,b}\rangle|^2db\right)^{1/2}\leqslant A(1+A_\Omega)^{1/2}\int_0^{a'}\left(\frac{\tau}{a}\right)^{\alpha/2}\left(\int_{\mathbb{R}}|\Delta_\tau(I-S_{a'})f|^2\right)^{1/2}\frac{d\tau}{\tau}. \tag{44}$$

Using the Cauchy–Schwarz inequality one gets

$$\int_{\mathbb{R}}|\langle f-S_{a'}f,\theta_{a,b}\rangle|^2db\leqslant A(1+A_\Omega)\left(\int_0^{a'}\left(\frac{\tau}{a}\right)^\alpha\frac{d\tau}{\tau}\right)\left(\int_0^{+\infty}\|\Delta_\tau(I-S_{a'})f\|_2^2\frac{d\tau}{\tau}\right). \tag{45}$$

To deduce inequality (17) one computes

$$\int_0^{+\infty} \|\Delta_\tau (I-S_{a'})f\|_2^2 \frac{d\tau}{\tau} = A\|(I-S_{a'})f\|_2^2, \tag{46}$$

and

$$\int_0^{a'} \left(\frac{\tau}{a}\right)^\alpha \frac{d\tau}{\tau} = \frac{1}{\alpha}\left(\frac{a'}{a}\right)^\alpha. \tag{47}$$

It remains to prove (43), by using the geometric hypothesis (P_α). One starts by taking a function $\tilde{\Psi}$, even and C^∞, such that $\hat{\tilde{\Psi}}(\omega)=1$ if $\omega \in \operatorname{Supp} \hat{\Psi}$, with

$$\operatorname{Supp} \widehat{\tilde{\Psi}} \subset \left\{\omega \in \mathbb{R}^n; \tfrac{1}{8} \leqslant |\omega| \leqslant 2\right\}. \tag{48}$$

Denoting as usual by $\tilde{\Delta}_\tau$ the convolution operator by $\tilde{\Psi}_\tau$ one gets

$$\tilde{\Delta}_\tau \Delta_\tau = \Delta_\tau, \tag{49}$$

from which one obtains

$$\langle \Delta_\tau f, \theta_{a,b}\rangle = \langle \Delta_\tau f, \tilde{\Delta}_\tau \theta_{a,b}\rangle. \tag{50}$$

Lemma 9: There exists a constant A, independent of Ω, *such that, if* $\tau \leqslant a$, *then for any* $x \in \mathbb{R}$

$$\int_{\mathbb{R}} |\tilde{\Delta}_\tau \theta_{a,b}(x)|\, db \leqslant A, \tag{51}$$

and for any $b \in \mathbb{R}$

$$\int_{\mathbb{R}} |\tilde{\Delta}_\tau \theta_{a,b}(x)|\, dx \leqslant A(1+A_\Omega)\left(\frac{\tau}{a}\right)^\alpha. \tag{52}$$

Inequality (51) means that the operator associating to any function g the function

$$b \mapsto \langle g, \tilde{\Delta}_\tau \theta_{a,b}\rangle \tag{53}$$

(where a and τ are two fixed parameters with $\tau \leqslant a$), is uniformly bounded on $L^1(\mathbb{R})$. Inequality (52) means that the same operator is bounded on $L^\infty(\mathbb{R})$ and that its norm is controlled by

$$A(1+A_\Omega)\left(\frac{\tau}{a}\right)^\alpha. \tag{54}$$

Inequality (43) follows by interpolation.

Now, let us prove (51). It is enough to write

$$\tilde{\Delta}_\tau \theta_{a,b}(x) = \int_{\mathbb{R}} \tilde{\Psi}_\tau(x-y)\, \theta_{a,b}(y)\, dy, \tag{55}$$

and to note that by (5) for any y

$$\int_{\mathbb{R}} |\theta_{a,b}(y)|\, db \leqslant \|\psi\|_1 + \|\phi\|_1. \tag{56}$$

This yields (51).

The proof of (52) is more elaborate. We write

$$\tilde{\Delta}_\tau \theta_{a,b}(x) = \int_{\mathbb{R}} \tilde{\Psi}_\tau(x-y) G(y)[\psi_{a,b}(y) - C(a,b)\phi_{a,b}(y)]dy. \tag{57}$$

Let us start with

$$\int_{\mathbb{R}} \tilde{\Psi}_\tau(x-y) G(y) \phi_{a,b}(y) dy = \int_{\mathbb{R}} \tilde{\Psi}_\tau(x-y) G(y)[\phi_{a,b}(y) - \phi_{a,b}(x)]dy + \tilde{\Delta}_\tau G(x)\phi_{a,b}(x). \tag{58}$$

The first term on the right is controlled by

$$A\frac{\tau}{a^2}\omega\left(\frac{x-b}{a}\right), \tag{59}$$

as we have seen several times. Its integral with respect to x is dominated by $A\tau/a$.

The hypothesis (P_α) is used for the second term.

Lemma 10: There exists an absolute constant A such that

$$\left| \int_{\mathbb{R}} \tilde{\Delta}_\tau G(x) \phi_{a,b}(x) dx \right| \leqslant A A_\Omega \left(\frac{\tau}{a}\right)^\alpha, \tag{60}$$

for all $\tau \leqslant a,\ b \in \mathbb{R}$.

Admit this result for the moment. Then one has

$$\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \tilde{\Psi}_\tau(x-y) G(y) \phi_{a,b}(y) dy \right| dx \leqslant A(1+A_\Omega)\left(\frac{\tau}{a}\right)^\alpha. \tag{61}$$

By the same argument, one obtains

$$\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \tilde{\Psi}_\tau(x-y) G(y) \psi_{a,b}(y) dy \right| dx \leqslant A(1+A_\Omega)\left(\frac{\tau}{a}\right)^\alpha. \tag{62}$$

Lemma 9 is thus completely proved using (58) and (5) once more.

It remains to prove Lemma 10. There exist two rapidly decreasing sequences of numbers $(c_k)_{k\in\mathbb{Z}}$ and $(d_l)_{l\in\mathbb{Z}}$ and two sequences of functions $(\phi_k)_{k\in\mathbb{Z}}$ and $(\tilde{\Psi}_l)_{l\in\mathbb{Z}}$ satisfying the following properties:

$$\begin{cases} \phi(x) = \sum_{k\in\mathbb{Z}} c_k \phi_k(x), & \tilde{\Psi}(x) = \sum_{l\in\mathbb{Z}} d_l \tilde{\Psi}_l(x), \\ \text{Supp } \phi_k \subset [k-1, k+1], & \text{Supp } \tilde{\Psi}_l \subset [l-1, l+1], \\ \|\phi_k\|_\infty = \|\tilde{\Psi}_l\|_\infty = 1, & \\ \int_{\mathbb{R}} \tilde{\Psi}_l = 0. & \end{cases} \tag{63}$$

One constructs these objects with the help of a partition of the identity which is used to decompose ϕ on one hand, and the primitive of $\tilde{\Psi}$ vanishing at infinity on the other hand. This gives directly the c_k, ϕ_k, and by differentiation the d_l, $\tilde{\Psi}_l$. The sequences (c_k) and (d_l) are rapidly decreasing because so are ϕ and the primitive of $\tilde{\Psi}$, since $\int \tilde{\Psi} = 0$. Verifying the other properties in (63) is straightforward.

One can now write

$$\tilde{\Delta}_\tau G(x)\phi_{a,b}(x)=\sum_k \sum_l c_k d_l \int_{\mathbb{R}} \frac{1}{\tau}\tilde{\Psi}_l\left(\frac{x-y}{\tau}\right) G(y)\,dy\,\frac{1}{a}\phi_k\left(\frac{x-b}{a}\right). \tag{64}$$

Let us examine the integral

$$I_{kl}=\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\frac{1}{\tau}\tilde{\Psi}_l\left(\frac{x-y}{\tau}\right)G(y)\,dy\right|\frac{1}{a}\phi_k\left(\frac{x-b}{a}\right)dx, \tag{65}$$

(we may suppose ϕ_k non-negative.) By support considerations, we may assume that $y\in\Omega$, $(x-y)/\tau\in[k-1,k+1]$ and $(x-b)/a\in[l-1,l+1]$. Moreover, since $\int\tilde{\Psi}_l=0$, the integral w.r.t. y does not vanish only when Supp $\tilde{\Psi}_l((x-)/\tau$ intersects Ω^c, which implies $d(y,\partial\Omega)\leqslant 2\tau$. We notice that

$$\left\{(x,y);\frac{x-b}{a}\in[l-1,l+1],\frac{x-y}{\tau}\in[k-1,k+1],y\in\Omega,d(y,\partial\Omega)\leqslant 2\tau\right\}\subset\mathscr{E}_{kl}, \tag{66}$$

where

$$\mathscr{E}_{kl}=\{(x,y);x\in[y+(k-1)\tau,y+(k+1)\tau],$$
$$y\in\Omega\cap[b-k\tau+a(l-2),b-k\tau+a(l+2)],d(y,\partial\Omega)\leqslant 2\tau\}. \tag{67}$$

We thus have, using (63)

$$I_{kl}\leqslant\frac{1}{\tau a}|\mathscr{E}_{kl}|\leqslant\frac{2}{a}|\mathscr{E}'_{kl}|, \tag{68}$$

where

$$\mathscr{E}'_{kl}=\{y\in\Omega\cap[b-k\tau+a(l-2),b-k\tau+a(l+2)],d(y,\partial\Omega)\leqslant 2\tau\}. \tag{69}$$

Using once more $\int\tilde{\Psi}_l=0$, we replace G by $1-G$ in I_{kl} without changing anything, which amounts to replacing Ω by Ω^c. Hence we also have

$$I_{kl}\leqslant\frac{2}{a}|\mathscr{E}''_{kl}|, \tag{70}$$

where

$$\mathscr{E}''_{kl}=\{y\in\Omega^c\cap[b-k\tau+a(l-2),b-k\tau+a(l+2)],d(y,\partial\Omega)\leqslant 2\tau\}. \tag{71}$$

By (P_α), we have

$$I_{kl}\leqslant 8A_\Omega\left(\frac{\tau}{a}\right)^\alpha. \tag{72}$$

Summing over $k,l\in\mathbb{Z}$ achieves the proof of Lemma 10, and the theorem follows.

V. FURTHER CONSIDERATIONS

The gapped wavelet transform is a simple alternative to the well-known construction of wavelet bases on an interval or a domain (e.g., Ref. 8), which could also be used to define spectral estimators (see Ref. 9).

It should be noted that the computation of the coefficients $\langle f,\theta_{a,b}\rangle$ basically involves standard wavelet and scaling function coefficients of the signal and of the characteristic function of its support. It is therefore possible in principle to use a pyramidal algorithm, following the ideas of Refs. 10,11.

Finally, let us point out that the existence of a reconstruction formula similar to the usual one is an open problem.

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