Logical Implication:

A stmt 'A' is said to tautologically imply a stmt 'B' iff –

$A \rightarrow B$ is a tautology

represented by –

$A \Rightarrow B$

Implication Formulae (or) Rules of Inference:

Inference Rule	Tautology	Name
1) $\dfrac{P \quad P \rightarrow Q}{\therefore Q}$	$[P \wedge (P \rightarrow Q)] \rightarrow Q$ or $[P \wedge (P \rightarrow Q)] \Rightarrow Q$	Modus Ponens
2) $\dfrac{\neg Q \quad P \rightarrow Q}{\therefore \neg P}$	$[\neg Q \wedge (P \rightarrow Q)] \rightarrow \neg P$	Modus Tollens
3) $\dfrac{P \rightarrow Q \quad Q \rightarrow R}{\therefore P \rightarrow R}$	$[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R)$	Hypothetical Syllogism
4) $\dfrac{P \vee Q \quad \neg P}{\therefore Q}$	$[(P \vee Q) \wedge \neg P] \rightarrow Q$	Disjunctive Syllogism
5) $\dfrac{P}{\therefore P \vee Q}$ or $\dfrac{Q}{\therefore P \vee Q}$	$P \rightarrow (P \vee Q)$ $Q \rightarrow (P \vee Q)$	Addition
6) a) $\dfrac{P \wedge Q}{\therefore P}$	$(P \wedge Q) \rightarrow P$	Simplification

b) $\dfrac{P \wedge Q}{\therefore Q}$ $\qquad [P \wedge Q] \rightarrow Q$ $\qquad$ Simplification

7) ~~a)~~ $\dfrac{\begin{matrix} P \\ Q \end{matrix}}{\therefore P \wedge Q}$ $\qquad [P \wedge Q] \rightarrow (P \wedge Q)$ $\qquad$ Conjunction

~~b)~~

8) $\dfrac{\begin{matrix} P \vee Q \\ \neg P \vee R \end{matrix}}{\therefore Q \vee R}$ $\qquad [(P \vee Q) \wedge [\neg P \vee R)] \rightarrow (Q \vee R)$ $\qquad$ Resolution

9) $\dfrac{\begin{matrix} P \vee Q \\ P \rightarrow R \\ Q \rightarrow R \end{matrix}}{\therefore R}$ $\qquad [(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R)] \rightarrow R$ $\qquad$ Dilemma

<u>Q</u>⇒ Prove the following Implications using logical Equivalence:

i) $(P \wedge Q) \Rightarrow (P \rightarrow Q)$

ii) $P \rightarrow Q \Rightarrow P \rightarrow (P \wedge Q)$

iii) $(P \rightarrow Q) \rightarrow Q \Rightarrow (P \vee Q)$

i) $(P \wedge Q) \Rightarrow (P \rightarrow Q)$ $\quad$ or $\quad$ $(P \wedge Q) \rightarrow (P \rightarrow Q)$ is a tautology

$\Rightarrow [(P \wedge Q) \rightarrow (P \rightarrow Q)]$ $\qquad$ Using Conditional stmt

$\Rightarrow [(P \wedge Q) \rightarrow (\neg P \vee Q)]$ $\qquad$ Using Conditional stmt

$\Rightarrow [\neg(P \wedge Q) \vee (\neg P \vee Q)]$ $\qquad$ Using De-Morgan Law

$\Rightarrow [\neg P \vee \neg Q \vee \neg P \vee Q]$ $\qquad$ Commutative Law

$\Rightarrow [\underbrace{\neg P \vee \neg P}_{\text{Idempotent}} \vee \underbrace{\neg Q \vee Q}_{\text{Negation}}]$ $\qquad$ Using Idempotent & Negation Law.

$\Rightarrow (\neg P \vee T)$ $\qquad$ Using Domination Law

$\Rightarrow T$ (true)

Hence, given stmt is tautology

ii) $(P \to Q) \to (P \to (P \wedge Q))$ is a tautology.

$\Rightarrow ((\neg P \vee Q) \to (P \to (P \wedge Q))$ Conditional Stmt

$\Rightarrow ((\neg P \vee Q) \to (\neg P \vee (P \wedge Q))$ Conditional Stmt

$\Rightarrow ((\neg P \vee Q) \to ((\underbrace{\neg P \vee P}_{T}) \wedge (\neg P \vee Q))$ Distributive Law

$\Rightarrow (\neg P \vee Q) \to (T \wedge (\neg P \vee Q))$ Negation Law

$\Rightarrow (\neg P \vee Q) \to (\neg P \vee Q)$ Identity Law

$\Rightarrow \neg(\underbrace{\neg P \vee Q}_{A}) \vee (\underbrace{\neg P \vee Q}_{A})$ Conditional stmt

Negation Law

$\Rightarrow$ True

Hence, given statement is tautology

iii) $(P \to Q) \to Q \Rightarrow (P \vee Q)$ is tautology.

$= [(\neg P \vee Q) \to Q \to (P \vee Q)]$ {Using Conditional stmt]

$= [(\neg(\neg P \vee Q) \vee Q] \to (P \vee Q)]$ Using Conditional stmt

$= (((\neg(\neg P) \wedge \neg Q) \vee Q) \to (P \vee Q)]$ Using de-Morgan's Law

$= ((P \wedge \neg Q) \vee Q) \to (P \vee Q)]$ Using double Negation

$= [(P \wedge \underbrace{\neg Q \vee Q}_{T}) \to (P \vee Q)]$

Using Negation Law

$= [(P \wedge T) \to (P \vee Q)]$ Identity Law

$= [P \to (P \vee Q)]$ Using Conditional Law

$= \neg P \vee (P \vee Q)$ Using Associative Law

$= (\underbrace{\neg P \vee P}_{T}) \vee Q$ Negation Law

$= T \vee Q$ (Domination Law)

$= T$ (true)

Hence, Stmt is tautology

Q ⇒ $(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \equiv \neg P \vee Q$

Taking L.H.S

$(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q))$

$= ((P \vee Q) \wedge \neg P) \wedge Q$

$= ((P \wedge \neg P) \vee (Q \wedge \neg P)) \wedge Q$

$= (F \vee (\neg P \wedge Q)) \wedge Q$

$= (\neg P \wedge Q) \wedge Q$

$= \neg P \wedge (Q \wedge Q)$

$= \neg P \wedge Q$.

Theory of Inference of Propositional Logic :

The main function of Logic is to provide rules of Inference of principles of reasoning. The theory associated with such rules is known as principles of reasoning. Because it is concerned with the inferring of a conclusion from certain Premises or Hypothesis.

When a conclusion is derived from a set of premises by using the expected rules of reasoning then such a process of derivation is called detection or a formal proof.

Argument :

An argument is a sequence of stmts. All stmts except final stmt (conclusion) are called premises (Assumption or Hypothesis)

The final stmt is called conclusion.

Let S_1, S_2, S_3, S_4 are some stmts

These stmts are called premises or Hypothesis $\left\{ \begin{array}{l} S1 \\ S2 \\ S3 \\ S4 \end{array} \right.$

C : conclusion

Example :

G : If I get up early in the morning.

P : I will go to see the picture.

1) $G \rightarrow P, G$
   $\therefore P$

2) $G \rightarrow P, G \Rightarrow P$

3) $G \wedge (G \rightarrow P) \Rightarrow P$

4) $(G \wedge (G \rightarrow P)) \rightarrow P$ is a tautology.

An argument is called logically valid iff the Conjunction of the premises implies the conclusion. That is. if the premises are all T, the conclusion must also be true.

Q⇒ show that Hypothesis : " It is not sunny afternoon and it is colder than yesterday," "we will go swimming only if it is sunny". " If we do not go swimming then we will take a holiday trip." " If we take a holiday trip the we will be home by sunset." The above premises let to the conclusion : " we will be home by sunset."

Compound Proposition : It is not sunny this afternoon and it is colder than yesterday.

p : It is sunny this afternoon

q : It is colder than yesterday.

Scanned with CamScanner

Different Proposition are :

P : It is sunny this afternoon

Q : It is colder than yesterday

R : we will go swimming.

S : we will take a holiday trip

T : we will be home by sunset.

In Inference form :

$$\begin{array}{l} \neg P \wedge Q \\ R \rightarrow P \\ \neg R \rightarrow S \\ S \rightarrow T \\ \hline \therefore T \end{array}$$

Prove of Inference form with the help of Inference formula

Prove:

$$\frac{\neg P \wedge Q}{\therefore \neg P}$$ (Hypothesis) (By simplification)

$$\frac{R \rightarrow P}{\therefore \neg R}$$ (Hypothesis) (By modus tollens)

$$\frac{\neg R \rightarrow S}{\therefore S}$$ (Hypothesis) (By Modus Ponens)

$$\frac{S \rightarrow T}{\therefore T}$$ (Hypothesis) (By modus ponens).

Conclusion

Hence, conclusion is T i.e we will be home by sunset

Proved.

Q⇒ Show that the hypothesis

1.) If you set me a message then I will stop writing the Program

2.) "If you do not sent me message then I will go to sleep early."

3) "If I will go to sleep early then I will breakup feeling with fresh."

Conclusion : "If I do not stop writing the program then I will break up feeling with fresh".

With the help of three Hypothesis prove that conclusion is true with the help of Inference formula.

Sol ⇒ Prepositions are :

P : You sent me a message.

Q : I will stop writing the program.

R : I will go to sleep early

S : I will break up feeling with fresh.

Logical stmt of hypothesis with the help of connectives

1) $P \rightarrow Q$

2.) $\neg P \rightarrow R$

3.) $R \rightarrow S$

Conclusion : $\neg Q \rightarrow S$

Inference formula form :

$$\begin{array}{c} P \rightarrow Q \\ \neg P \rightarrow R \\ R \rightarrow S \\ \hline \therefore \neg Q \rightarrow S \\ \hline \end{array}$$

Proving : $P \rightarrow Q$ (Hypothesis)

$\neg Q \rightarrow \neg P$ (Contrapositive)

$\neg P \rightarrow R$ (Hypothesis)

$\therefore \neg Q \rightarrow R$ (By Hypothetical Syllogism)

$R \rightarrow S$ (Hypothesis)

Scanned with CamScanner

$\therefore \neg Q \rightarrow R$

$$\begin{array}{l} R \rightarrow S \\ \hline \therefore \neg Q \rightarrow S \\ \hline \end{array}$$ (By Hypothetical Syllogism)

Conclusion

Hence, conclusion is $\neg Q \rightarrow S$. proved

Q⇒ Prove that R is valid Inference from premises

Conclusion : R

Hypothesis : $P \rightarrow Q$, $Q \rightarrow R$, P

Inference formula form is :

$$\begin{array}{l} P \rightarrow Q \\ Q \rightarrow R \\ P \\ \hline \therefore R \end{array}$$

Prove:

$$\begin{array}{ll} 1)\ P \rightarrow Q & \text{(Hypothesis)} \\ 2)\ Q \rightarrow R & \text{(Hypothesis)} \\ \hline \therefore P \rightarrow R & \text{(By Hypothetic syllogism)} \\ P & \text{(Hypothesis)} \\ \hline \therefore R & \text{(By Modus ponens)} \\ \hline \end{array}$$

Hence, conclusion is R. Hence R is valid Inference proved.

P and T Rule (Rules of Inference):

There are two rules of Inference Known as P & T rule :

1) Rule P : A premises may be introduced at any point in the derivation and any no. of times.

2) Rule T : A formula 'S' may be introduced in the derivation if 'S' is tautologically implied

by any one or more of the preceeding formulas in the derivation.

S1
S2
S3
S4

Using S2 & S4, we form new stmt S5 which is a valid stmt.

C : Conclusion

Q⇒ Test the validity of the following arguments

Stmt 1: If my brother stands first in the class ~~or I~~ then I give him a watch.

Stmt 2: Either he stood first in the class or I was out of station

Stmt 3: I did not give my brother watch this time, therefore I was out of station.

Sol ⇒ Step 1:

B : My brother stands first in the class.
W : I give him a watch
S : I was out of station.

Step 2:

Premises
1) $B \rightarrow W$
2) $B \vee S$
3) $\neg W$ (stmt after therefore)

Conclusion
4) S

By Using P & T Rule we proof the conclusion :

i) $B \rightarrow W$ (Rule P)
ii) $\neg W$ (Rule P)
iii) $\neg B$ (Rule T, by combining step i) & ii) Modus tollens)
iv) $B \vee S$ (Rule P)
v) S (Rule T, iii) & iv) Disjunction Syllogism)

Therefore, conclusion is valid.

Q⇒ $\underbrace{P \vee Q,\ Q \rightarrow R,\ P \rightarrow M,\ \neg M}_{\text{Premises}}$ $\underbrace{C : R}_{\text{Conclusion}}$

1) $P \rightarrow M$ (Rule P)
2) $\neg M$ (Rule P)
3) $\neg P$ (Rule T, (1) & (2), Modus tollens)
4) $P \vee Q$ (Rule P)
5) Q (Rule T), (3) & (4), Disjunction Syllogism)
6) $Q \rightarrow R$ (Rule P)
7) R (Rule T, (5) & (6), Modus Ponens)

Rule CP : (Conditional Proof (or) Conditional Premise)

If we have a conclusion in the form of the conditional statement and a set of premises where conditional statement is "$P \rightarrow Q$" and "premises R" then hypothesis of conditional stmt is taken as "additional Premise" and we have to derive "conclusion part" of conditional stmt.

Q⇒ Show that $R \rightarrow S$ (Conclusion) can be derived from premises $\underbrace{P \rightarrow (Q \rightarrow S),\ \neg R \vee P,\ Q}_{\text{Premises}}$ with the help of conditional proof:

Inference formula form:

$P \rightarrow (Q \rightarrow S)$

$\neg R \vee P$

Q

R

$\therefore S$

Prove:

i) $\neg R \vee P$ (Hypothesis)

ii) R (Hypothesis)

---

iii) $\therefore P$ (By Disjunction syllogism)

iv) $P \rightarrow (Q \rightarrow S)$ Hypothesis

---

v) $\therefore Q \rightarrow S$ (By modus ponens)

vi) Q (Hypothesis)

$\therefore S$ (By modus ponens)

Hence, conclusion is S.

Q⇒ In previous question:

Premises are: $P \rightarrow Q$, $\neg P \rightarrow R$, $R \rightarrow S$

Conclusion: $\neg Q \rightarrow S$ — conclusion

Hypothesis

Inference form is:

$P \rightarrow Q$

$\neg P \rightarrow R$

$R \rightarrow S$

$\neg Q$

$\therefore S$

Prove: 1) $P \rightarrow Q$ (Hypothesis)

2) $\neg Q$ (Hypothesis)

---

3) $\therefore \neg P$ (modus tollens)

4) $\neg P \rightarrow R$ (Hypothesis)

---

5) $\therefore R$ (By modus ponens)

6) $R \rightarrow S$ (Hypothesis)

$\therefore S$ (By modus ponens)

Hence, conclusion is S proved

Fallacies :

Some fallacies arise in incorrect arguments these fallacies resemble rules of Inference but are actually based on Contigencies instead of tautologies.

If the argument with premises $P \rightarrow Q$ and Q with Conclusion P is treated as valid argument then this type of reasoning is Called Fallacy of Affirming Conclusion.

and if an argument with $P \rightarrow Q$ and $\neg P$ with Conclusion $\neg R$ is treated as Valid argument then this type of reasoning is Called Fallacy of Denying Hypothesis.

1) $$\begin{array}{c} P \rightarrow Q \\ P \\ \hline \therefore Q \end{array} \qquad \begin{array}{c} P \rightarrow Q \\ Q \\ \hline \therefore P \end{array}$$ → It is fallacy of Affirming Conclusion.

2) $$\begin{array}{c} P \rightarrow Q \\ \neg Q \\ \hline \therefore \neg P \end{array} \qquad \begin{array}{c} P \rightarrow Q \\ \neg P \\ \hline \therefore \neg Q \end{array}$$ → Case of Denying Hypothesis

Q⇒ "If it rains then we will play foot ball"
we will play football
Therefore it rains".
Verify it is Valid or Not.

P : It rains

Q : we will play football

Stmts with Connectives

1) $P \rightarrow Q$

2) Q

Inference formula form

$$\begin{array}{l} P \rightarrow Q \\ \underline{Q} \\ \therefore P \end{array}$$

According to first Inference Rule it is not valid.

Functionally Complete Set of Connectives:

Any set of connectives in which every formula can be expressed in terms of an equivalent formula containing the connective from this set is called functionally complete set of connectives.

NAND and NOR are known as Universal Connectives bcoz with the help of these two connective we can derive any expression.

Example: $\{\neg, \vee\}$, $\{\neg, \wedge\}$, $\{\neg, \rightarrow\}$, $\{\uparrow\}$, $\{\downarrow\}$

Replacing of $\neg$ by $\uparrow$:

$\neg P$

$\equiv \neg(P \wedge P)$

$\equiv P \uparrow P$

Replacing of $\rightarrow$ by $\uparrow$:

$P \rightarrow Q$

$\equiv \neg P \vee Q$

$\equiv \neg\neg(\neg P \vee Q)$

$\equiv \neg(P \wedge \neg Q)$

$\equiv P \uparrow \neg Q$

$\equiv P \uparrow (Q \uparrow Q)$

* A functionally complete set of connectives does not contain a connective which can be expressed in terms of other connectives.

Dual :

Two formula A and A^* are called duals of each other if either one can be obtained from the other by replacing disjunction by conjunction, conjunction by disjunction, True by false and false by true.

i) $\wedge$ with $\vee$

ii) $\vee$ with $\wedge$

iii) T by F

iv) F by T

Example : $A = P \vee Q \wedge R$

$A^* = P \wedge Q \vee R$ (dual of P)

$A = P \vee Q \wedge T$

$A^* = P \wedge Q \vee F$ (dual of B).

Principle of Duality :

The principle of duality states that if two statements are equal then their duals are also equal.

(demorgan's Law)

Ex: $\neg(P \vee Q) = \neg P \wedge \neg Q$

L.H.S $= \neg(P \vee Q)$

$=$ dual of $\neg(P \vee Q)$

$= \neg(P \wedge Q)$

R.H.S $= \neg P \wedge \neg Q$

$=$ dual of $\neg P \wedge \neg Q$

$= \neg P \vee \neg Q = \neg(P \wedge Q)$

Hence, dual of $\neg(P \vee Q)$ and dual of $\neg P \wedge \neg Q$ are equal.

Consistency of Premises:

Inconsistency means A and $\neg A$ both are true like it is raining & it is not raining both are ~~true~~ not possible bcoz they contradict each other.

Ex: 1) If John misses many classes through illness then he fails high school.

2) If John fails high school then he is uneducated

3) If John reads alot of books then he is not uneducated

4) John misses many classes through illness and reads alot of books.

Show that the above premises are inconsistence.

Soln → I : John misses many classes through illness

H : He fails high school

U : He is uneducated

B : John reads a lot of books.

P1 : $I \rightarrow H$

P2 : $H \rightarrow U$

P3 : $B \rightarrow \neg U$

P4 : $I \wedge B$.

1) $I \rightarrow H$ (Rule P)

2) $H \rightarrow U$ (Rule P)

3) $I \rightarrow U$ (Rule T)

4) $B \rightarrow \neg U$ (Rule P)

5) $U \rightarrow \neg B$

6) $I \rightarrow \neg B$

7) $\neg I \vee \neg B$

8) $\neg(I \wedge B)$

9) $I \wedge B$

From step 8 and 9 we are getting contradiction therefore the premises are actually Inconsistent

Exercise

Q1⇒ $\neg(P \wedge T) \wedge (F \vee \neg P) \equiv F$ Verify the duality of this stm

Q2⇒ Use CP Rule to derive the Conclusion:

a) $P \to (Q \to S), \neg R \vee P, Q$ C : $R \to S$

b) $P \to Q, \neg P \to R, R \to S$ C : $\neg Q \to S$

Q3⇒ Use P & T Rule to derive:

a) $R \to E, M \to R, M \vee I, \neg I$ C : E

b) $J \wedge W \to P, P \to H, \neg H$ C : $\neg J \vee \neg W$

c) $\neg P \vee Q, \neg Q \vee R, R \to S$ C : $P \to S$

d) $P \leftrightarrow Q, P \wedge \neg S, Q \to S, \neg P \to R$ C : R

Q4⇒ Prove using Logical formulae

a) $\neg(P \wedge Q) \to (\neg P \vee (\neg P \vee Q)) \equiv \neg P \vee Q$.

b) $(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \equiv \neg P \vee Q$.

Q5⇒ Show that the following argument are valid

A) · If Today is tuesday, I have a test in Mathematics or Economics.
If My Economic professor is sick then I will not have a test in Economics.
Today is tuesday and my Economics professor is sick therefore I have a test in Mathematics.

B) If I get the job and work hard then I will get promoted
If I get promoted then I will be happy.
I am not happy therefore I will not get job or I will not work hard.

Q6⇒ $P \to Q, Q \to R, Q \to \neg R, P$. Prove that there four premises are Inconsistence.