

\* Set :- A set is any well defined collection of distinct or distinguishable elements by a well defined collection means that there exists a rule with the help of which we should be able to say whether any given object belongs to or not to the collection under some specific rule.

The set of students in a class.

The set of months in a year.

$$A = \{1, 2, 3\}$$

Generally capital letters A, B, C, X, Y, Z etc are used to denote sets and its elements by lowercase letters a, b, c, x, y, z etc.

\* Elements of a set :-

If  $A = \{1, 2, 3\}$  the objects in a set are called elements or members.

The distinct elements means no element is repeated.

The Distinguishable means that given any object or element is either in the set or not in the set. so the elements of a set must be distinct and distinguishable.

The symbol  $\in$  (epsilon) is used to indicate "belongs to" if 1 is the element of set A then symbolically we write

$$1 \in A$$

The symbol  $\notin$  is used to indicate "does not belongs to" if 4 is not an element of set A then symbolically we write -

$$4 \notin A$$

\* Symbols :-

$$N = \text{Natural Numbers} = \{1, 2, 3, \dots\}$$

$$I \text{ or } Z = \text{set of Integers} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

$$Z^+ = \text{set of Positive Integers} = \{0, 1, 2, \dots\}$$

$$Z^- = \text{set of Negative Integers} = \{\dots, -2, -1\}$$

$$R = \text{set of Real Numbers.}$$

$$Q = \text{set of Rational Numbers (where } \frac{p}{q}, q \neq 0)$$

$$C = \text{set of Complex Numbers.}$$

Finite Set :- A set is finite if it contains finite no. of elements.

$$n(A) = 3$$

$$A = \{1, 2, 3\}$$

$$O(A) \text{ or } n(A) \text{ for } |A|.$$

The number of distinct elements in set A is called cardinal no. of set.

The cardinality of set (or) cardinal no. of set :-

Types of sets \*

$C = \{x : x \text{ is a vowel in English alphabet}\}$

$= \{x : x \text{ is an even positive integer}\}$   
or

$= \{x : x \text{ is an even natural number}\}$   
or

$B = \{x : x = 2n \text{ } \forall n \in \mathbb{N}\}$

$A = \{x : x \in \mathbb{N} \text{ } \& \text{ } x > 4\}$

$A = \{1, 2, 3\} \quad B = \{2, 4, 6, 8, \dots\} \quad C = \{a, e, i, o, u\}$

E.g. Let consider the set

Set builder form :- In this method the elements of a set are uniquely characterized by stating a property which is satisfied by all its elements.

$$N = \{1, 2, 3, \dots\}$$

The set of natural numbers.

$$V = \{a, e, i, o, u\}$$

E.g. The set of vowels in English alphabet can be written as.

Set builder form :- In this method the elements of a set are described by listing all the elements in braces.

• Finite form :- In this method the elements of a set are represented by two forms:-

• Set builder form.

• Roster or Enumeration or Tabular form.

Set and represented by two forms:-

• Set Representation \*

The set of rivers in India.

- Infinite Set :- A set which is not finite is called infinite set.  
e.g. set of Natural Numbers.
- Singleton Set :- A set which contains only 1 element is called singleton set.  
e.g.:  $A = \{x : 4 < x < 5 \text{ } \& \text{ } x \in \mathbb{N}\}$   
 $A = \{\emptyset\}$ .
- Null set (or) Empty (or) Void set :- A set which contains no or zero elements is called null set.  
This set is denoted by  $\emptyset$  (or)  $\{\}$ .  
Cardinality of null set is always zero.
- Equality of sets (or) Equal sets :- Two sets A and B are said to be equal if every element of set A is an element of B and also every element of set B is an element of A. The equality of two sets A and B are denoted by -  
$$A = B$$
  
symbolically,  
$$A = B \text{ if and only if } x \in A \iff x \in B$$
- Equivalence set :- If the elements of 1 set can be put in one to one mapping with the elements of another set then two sets are called equivalence set.  
It is denoted by  $\sim$  (or)  $\equiv$  (or)  $\cong$ .  
If the elements of sets are same then it is known as equivalent set
- \*. Subset :- Let A and B be two non-empty sets. The set A is subset of B if and only if every element of set A is an element of set B.  
Subset is denoted by " $\subseteq$ ".  
Symbolically,  $A \subseteq B$  iff ~~and~~  $x \in A \implies x \in B$ .

e.g.,  $A = \{1, 2\}$

$B = \{1, 2, 3\}$ .

Here,  $A \subseteq B$

A is contained in B.

\* Superset :- It is denoted by " $\supseteq$ "

ii) If A is subset of B then, B is superset of A.  
 $B \supseteq A$ .

\* Properties of Subset :-

- $\emptyset$  is the subset of every set.
  - If  $A \subseteq B \wedge B \subseteq A$ , then  $A = B$ .
  - If  $A \subseteq B \wedge B \subseteq C$  then  $A \subseteq C$ . ( $A$  is also subset of  $C$ ).

\* Power set :- The set of all possible subsets of set A is called Power set of set A. Denoted by "P(A)".

$$\frac{e.g.}{\cdot \cdot \cdot} \quad A = \{1, 2\}$$

$$P(\infty) = \{\{\emptyset\}, \{\{1\}\}, \{\{2\}\}, \{\{1,2\}\}\}$$

$$A = \{1, 2, 3\}$$

$$P(A) = \{ \emptyset, \{1\}, \{2\}, \{3\}; \{1,2\}, \{1,3\}; \{2,3\}, \{1,2,3\} \}.$$

NOTE :- If a set has ' $n$ ' elements then ' $2^n$ ' subsets of that set are possible.

\* Proof :- Let Set A has n elements.

So, no. of subsets taking one element =  $m_1$

No. of subsets taking two elements =  ${}^n C_2$

No. of subsets taking three element =  $nC_3$ .

No. of subsets taking "n" element =  $n^n$ .

We know that  $\phi$  is subset of every set.

So. Total no. of subjects

$$= 1 + n_{c_1} + n_{c_2} + n_{c_3} + \dots + n_{c_n}$$

$$= n_{c_0} + n_{c_1} + n_{c_2} + n_{c_3} + \dots + n_{c_r}$$

$$= (1+1)^n$$

$$= (2)^n.$$

Prob - Find  $P(A)$  for the following sets.

$$1) \quad A = \{a\}. \quad P(A) = \{\emptyset, \{a\}\}$$

$$2) \quad A = \{ \Phi, \{\Phi\} \}, P(A) = \{ \emptyset, \{\emptyset\}, \{\Phi\}, \{\{\Phi\}\} \}$$

$$3) \quad A = \{\phi\}, \quad P(A) = \{\phi, \{\phi\}\}$$

$$4) \quad A = \{\{a\}\} \quad P(A) = \quad \{\emptyset, \{\{a\}\}\}$$

5)  $D = \phi(D)$

- Union :- Let A and B be two sets non-empty sets then union of A and B is the set of all elements which are either in A or in B or in both A and B and the union of A and B is denoted by 'A ∪ B'. It is also known as Joint or logical sum of A and B. Symbolically,

$$A \cup B = \{ x : x \in A \text{ or } x \in B \}.$$

e.g.  $A = \{1, 2, 3\}$        $B = \{1, 2, 4, 5\}$ .

$$A \cup B = \{1, 2, 3, 4, 5\}.$$

### ② Properties of Union of Set :-

- ★ Commutative Property :- Union of sets is commutative i.e.  $A \cup B = B \cup A$ .

Proof :- Let  $x \in (A \cup B)$ ,

$$\begin{aligned} &\Rightarrow x \in A \text{ or } x \in B \\ &\Rightarrow x \in B \text{ or } x \in A \\ &\Rightarrow x \in (B \cup A) \end{aligned}$$

from eqn ① & ②

$$\boxed{A \cup B = B \cup A}$$

$$\text{So, } (A \cup B) \subseteq (B \cup A) \quad \text{--- ①}$$

Let  $x \in (B \cup A)$

$$\begin{aligned} &\Rightarrow x \in B \text{ or } x \in A \\ &\Rightarrow x \in A \text{ or } x \in B \\ &\Rightarrow x \in (A \cup B) \end{aligned}$$

$$\text{So, } (B \cup A) \subseteq (A \cup B) \quad \text{--- ②}$$

- ★ Associative Property :- Union of sets is associative i.e.  $A \cup (B \cup C) = (A \cup B) \cup C$ .

Proof :- Let  $x \in (A \cup (B \cup C))$

$$\begin{aligned} &\Rightarrow x \in A \text{ or } x \in (B \cup C) \\ &\Rightarrow x \in A \text{ or } (x \in B \text{ or } x \in C) \\ &\Rightarrow (x \in A \text{ or } x \in B) \text{ or } x \in C \\ &\Rightarrow x \in (A \cup B) \text{ or } x \in C \\ &\Rightarrow x \in ((A \cup B) \cup C) \end{aligned}$$

$$\text{So, } A \cup (B \cup C) \subseteq (A \cup B) \cup C \quad \text{--- ①}$$

Let  $x \in ((A \cup B) \cup C)$

$$\begin{aligned} &\Rightarrow x \in (A \cup B) \text{ or } x \in C \\ &\Rightarrow (x \in A \text{ or } x \in B) \text{ or } x \in C \\ &\Rightarrow x \in A \text{ or } (x \in B \text{ or } x \in C) \\ &\Rightarrow x \in A \text{ or } x \in B \cup C \\ &\Rightarrow x \in A \cup (B \cup C) \end{aligned}$$

$$\text{So, } (A \cup B) \cup C \subseteq A \cup (B \cup C) \quad \text{--- (2)}$$

From eqn (1) & (2),

$$A \cup (B \cup C) = (A \cup B) \cup C.$$

\* Idempotent Property :- Union of sets is idempotent.  
i.e.  $A \cup A = A$ .

Proof :- Let,  $x \in A \cup A$

$$\Rightarrow x \in A \text{ or } x \in A$$

$$\Rightarrow x \in A.$$

$$\text{So, } (A \cup A) \subseteq A \quad \text{--- (1)}$$

$$\text{Let } x \in A.$$

$$\Rightarrow x \in A \text{ or } x \in A.$$

$$\Rightarrow x \in (A \cup A)$$

$$\text{So, } A \subseteq (A \cup A) \quad \text{--- (2)}$$

$$\text{From eqn (1) & (2)} \quad A \cup A = A$$

\* Let A and B be two non-empty sets.

$$A = \{1, 2, 3\}$$

$$B = \{1, 2, 4, 5\}$$

$$A \cup B = \{1, 2, 3, 4, 5\}.$$

$$\Rightarrow A \subseteq (A \cup B)$$

$$\Rightarrow B \subseteq (A \cup B)$$

$$A \cup \emptyset = A$$

$$A \cup U = U$$

\* If  $A \subseteq B$  then  $A \cup B = B$ .

$$\text{Let } x \in A \cup B$$

$$\Rightarrow x \in A \text{ or } x \in B.$$

$$\Rightarrow x \in B \text{ or } x \in B \quad (\because A \subseteq B \text{ then } x \in A \Rightarrow x \in B)$$

$$\Rightarrow x \in B.$$

$$\text{So, } A \cup B \subseteq B \quad \text{--- (1)}$$

$$\text{We know that, } B \subseteq A \cup B. \quad \text{--- (2)}$$

From (1) & (2)

$$A \cup B = B$$

$$\text{e.g. } A = \{1, 2\}$$

$$B = \{1, 2, 3\}$$

.....

is the set of elements which belongs to both A and B.  
 (common to both A and B).  
 The intersection of A and B is denoted by "A ∩ B."  
 Symbolically,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

e.g. A = {1, 2, 3}

B = {1, 3, 5}.

A ∩ B = {1, 3}.

### ① Properties of Intersection of sets :-

⇒ Commutative Property :- Intersection of sets is commutative.  
 i.e. A ∩ B = B ∩ A.

Proof :- Let  $x \in A \cap B$

$$\Rightarrow x \in A \text{ and } x \in B.$$

$$\Rightarrow x \in B \text{ and } x \in A.$$

$$\Rightarrow x \in (B \cap A)$$

$$\text{So, } A \cap B \subseteq B \cap A. \quad \text{--- (1)}$$

Let  $x \in B \cap A$

$$\Rightarrow x \in B \text{ and } x \in A$$

$$\Rightarrow x \in A \text{ and } x \in B$$

$$\Rightarrow x \in (A \cap B)$$

$$\text{So, } B \cap A \subseteq A \cap B. \quad \text{--- (2)}$$

From eqn (1) & (2) -

$$A \cap B = B \cap A$$

# Associative Property :- Intersection of sets is associative.  
 i.e.  $A \cap (B \cap C) = (A \cap B) \cap C$ .

Proof :- Let  $x \in A \cap (B \cap C)$

$$\Rightarrow x \in A \text{ and } x \in B \cap C$$

$$\Rightarrow x \in A \text{ and } (x \in B \text{ and } x \in C)$$

$$\Rightarrow x \in A \cap B \text{ and } x \in C.$$

$$\Rightarrow x \in (A \cap B) \cap C$$

$$\text{So, } A \cap (B \cap C) \subseteq (A \cap B) \cap C. \quad \text{--- (1)}$$

Let  $x \in (A \cap B) \cap C$ .

$$\Rightarrow x \in (A \cap B) \text{ and } x \in C$$

$$\Rightarrow (x \in A \text{ and } x \in B) \text{ and } x \in C$$

$$\Rightarrow x \in A \text{ and } x \in (B \cap C)$$

$$\Rightarrow x \in A \cap (B \cap C)$$

$$\text{So, } (A \cap B) \cap C \subseteq A \cap (B \cap C). \quad \text{--- (2)}$$

From eqn ① & ② -

$$A \cap (B \cap C) = (A \cap B) \cap C$$

7. Commutative Property :- Intersection of sets is Idempotent  
i.e.  $A \cap A = A$ .

Proof :- Let  $x \in A \cap A$

$$\Rightarrow x \in A \text{ and } x \in A.$$

$$\Rightarrow x \in A.$$

$$\text{So, } A \cap A \subseteq A \quad \text{--- ①}$$

Let  $x \in A$ .

$$\Rightarrow x \in A \text{ and } x \in A.$$

$$\Rightarrow x \in (A \cap A).$$

$$\text{So, } A \subseteq A \cap A \quad \text{--- ②}$$

From eqn ① & ②

$$A \cap A = A$$

$$A = \{1, 2, 3\}$$

$$B = \{1, 3, 5\}.$$

$$A \cap B = \{1, 3\}.$$

$$\Rightarrow A \cap B \subseteq A$$

$$\Rightarrow A \cap B \subseteq B.$$

$$A \cap \emptyset = \emptyset$$

$$A \cap U = A.$$

NOTE: A is  $\subseteq$  A  $\cap$  B.

• Compliment of sets :- Let U be the universal set  
and A be any subset of universal set the compliment of A is a set containing elements of universal set which do not belong to A.  
The compliment of A is denoted by  $A'$  (or)  $A^c$  (or)  $\bar{A}$ .

E.g.:  $U = \{1, 2, 3, 4, 5, 6\}$

$$A = \{2, 4, 6\}.$$

$$A' = \{1, 3, 5\}.$$

Symbolically,

$$A' = \{x \in U \text{ and } x \notin A\}.$$

NOTE:  $x \in A^c$  (or)  $x \notin A$  are same.

those elements which belongs to A but does not belongs to B.  
It is denoted by 'A - B' (or) 'A / B'.  
Symbolically,

$$A - B = \{ x : x \in A \text{ and } x \notin B \}$$

e.g.  $A = \{ 1, 2, 3 \}$   
 $B = \{ 2, 3, 4, 5 \}$ .  
 $A - B = \{ 1 \}$   
 $B - A = \{ 4, 5 \}$ .

NOTE :-  $A - B = A \cap B^c$   
 $B - A = B \cap A^c$

$A - B \neq B - A$

\* Symmetric Difference of sets :- Let A and B be two sets of A and B is a set containing all the elements that belong to A or B but not both.

It is denoted by 'A  $\oplus$  B'.

e.g.  $A = \{ 1, 2, 3, 4, 5 \}$   
 $B = \{ 4, 5, 6 \}$   
 $A \oplus B = \{ 1, 2, 3, 6 \}$ .

$$A \oplus B = (A \cup B) - (A \cap B)$$

$$A \cup B = \{ 1, 2, 3, 4, 5, 6 \}$$
  
 $A \cap B = \{ 4, 5 \}$

$$A \oplus B = (A \cup B) - (A \cap B) = \{ 1, 2, 3, 6 \}.$$

\* Disjoint sets :- Let A and B be two sets then, if there is no common element b/w A and B then, they are said to be disjoint sets.

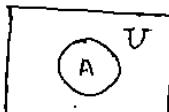
e.g.  $A = \{ 1, 2, 3 \}$   
 $B = \{ 4, 5, 6 \}$ .

If  $A \cap B = \emptyset$  Then A and B are disjoint sets.

\* Proper Subset :- Let A and B be two non-empty sets and  $A \subset B$  if there is at least one element in B which does not belong to A then, A is called proper subset of B and it is denoted by ' $\subset$ ' ( $A \subset B$ )

\* Venn Diagram :- Venn Diagram is a pictorial representation of sets which are used to show relationships among sets.  
The universal set is represented by interior of a rectangle and its subsets are represented by circular areas within the rectangle.

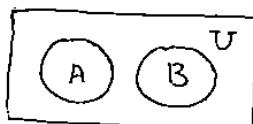
#  $A \subseteq U$



#  $A \subseteq B$



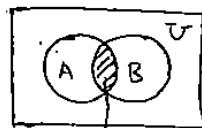
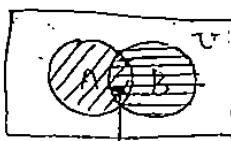
# Disjoint sets



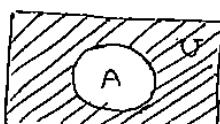
#  $A \cap B \neq \emptyset$

$$A \cup B$$

$$A \cap B$$

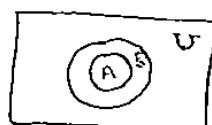


#  $A \subset C$



#  $A - B$

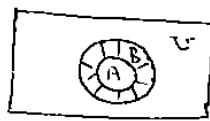
$$A \subseteq B$$



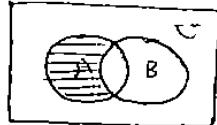
No shaded area means  
So,  $A - B = \emptyset$ .

#  $B - A$

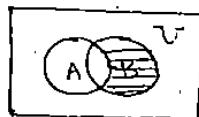
$$A \subseteq B$$



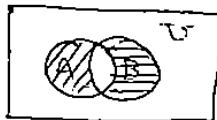
#  $A - B$



$B - A$



#  $A \oplus B$



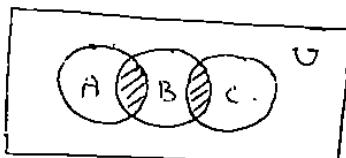
NOTE:

$$A \oplus B = (A - B) \cup (B - A)$$

Q. Draw a Venn diagram of sets A, B, & C where-

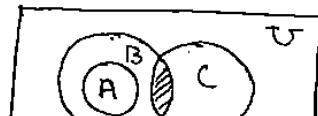
A and B have common elements ; B and C have common elements but A and C are disjoint.

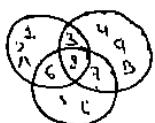
Soln



Q.  $A \subseteq B$ , set A and C are disjoint but B and C have elements in common.

Soln.





- 1)  $A - (B \cap C)$   
 2)  $(A \cup B) - C$   
 3)  $B \oplus C$   
 4)  $A - (B - C)$

- 5)  $A - (B \cup C)$   
 6)  $A \cap (B \oplus C)$

- 1)  $A - (B \cap C) = \{1, 2, 3, 6, 9\}$   
 2)  $(A \cup B) - C = \{1, 2, 3, 4, 9\}$   
 3)  $(B \oplus C) = \{6, 3, 4, 5, 9\}$   
 4)  $A - (B - C) = \{1, 2, 6, 8\}$   
 5)  $A - (B \oplus C) = \{1, 2, 8\}$ .  
 6)  $A \cap (B \oplus C) = \{6, 3\}$ .

$$A = \{1, 2, 6, 3, 8\}$$

$$B = \{3, 4, 5, 8, 9\}$$

$$C = \{5, 6, 7, 8\}$$

$$B \cap C = \{5, 8\}$$

$$A \cup B = \{1, 2, 3, 4, 6\}$$

$$B \oplus C = (B \cup C) - (B \cap C)$$

$$B - C = \{3, 4, 9\}$$

## \* Algebra of Sets :-

- Idempotent Law :- 1)  $A \cup A = A$   
2)  $A \cap A = A$ .
- Commutative Law :- 1)  $A \cup B = B \cup A$   
2)  $A \cap B = B \cap A$ .
- Associative Law :- 1)  $A \cup (B \cup C) = (A \cup B) \cup C$ .  
2)  $A \cap (B \cap C) = (A \cap B) \cap C$ .
- Identity Law :- 1)  $A \cup \phi = A$   
2)  $A \cap U = A$ .
- Bound Law :- 1)  $A \cup U = U$  (Universal set).  
2)  $A \cap \phi = \phi$ . (Empty set).
- Compliment Law :- 1)  $U^c = \phi$   
2)  $\phi^c = U$   
3)  $A \cup U^c = U$   
4)  $A \cap A^c = \phi$ .
- Involution Law :-  $(A^c)^c = A$ .
- Distributive Law :-

Q Prove that union of sets is distributive over intersect of sets and vice-versa.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Prove :-  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ .

Proof :- Let  $x \in (A \cup (B \cap C))$

$$\Rightarrow x \in A \text{ or } x \in (B \cap C)$$

$$\Rightarrow x \in A \text{ or } x \in B \text{ and } x \in C.$$

$$\begin{aligned}
 &\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C) \\
 &\Rightarrow (x \in A \cup B) \text{ and } (x \in A \cup C) \\
 &\Rightarrow x \in (A \cup B) \cap (A \cup C) \quad \boxed{\text{①}} \\
 \text{So, } A \cup (B \cap C) &\subseteq (A \cup B) \cap (A \cup C) \quad \boxed{\text{②}}
 \end{aligned}$$

Let,  $x \in (A \cup B) \cap (A \cup C)$

$$\begin{aligned}
 &\Rightarrow x \in (A \cup B) \text{ and } x \in (A \cup C) \\
 &\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C) \\
 &\Rightarrow x \in A \text{ or } x \in B \text{ and } x \in C \\
 &\Rightarrow x \in A \text{ or } x \in B \cap C \\
 &\Rightarrow x \in A \cup (B \cap C) \\
 \text{So, } (A \cup B) \cap (A \cup C) &\subseteq A \cup (B \cap C) \quad \boxed{\text{③}}
 \end{aligned}$$

From eqn ① & ③

$$\Rightarrow A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Proof  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof: Let  $x \in A \cap (B \cup C)$

$$\begin{aligned}
 &\Rightarrow x \in A \text{ and } x \in (B \cup C) \\
 &\Rightarrow x \in A \text{ and } x \in B \text{ or } x \in C \\
 &\Rightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\
 &\Rightarrow x \in (A \cap B) \text{ or } x \in (A \cap C) \\
 &\Rightarrow x \in (A \cap B) \cup (A \cap C) \\
 \text{So, } A \cap (B \cup C) &\subseteq (A \cap B) \cup (A \cap C) \quad \boxed{\text{①}}
 \end{aligned}$$

Let,  $(A \cap B) \cup (A \cap C) \ni x$ .

$$\begin{aligned}
 &\Rightarrow x \in (A \cap B) \text{ or } x \in (A \cap C) \\
 &\Rightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\
 &\Rightarrow x \in A \text{ and } (x \in B \text{ or } x \in C) \\
 &\Rightarrow x \in A \text{ and } x \in (B \cup C) \\
 &\Rightarrow x \in A \cap (B \cup C) \\
 \text{So, } (A \cap B) \cup (A \cap C) &\subseteq A \cap (B \cup C) \quad \boxed{\text{②}}
 \end{aligned}$$

From eqn ① & ②

$$\Rightarrow A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$\neg(A \cup B) = A \cap B^c$$

$$\Rightarrow (A \cap B)^c = A^c \cup B^c$$

Prove :-  $(A \cup B)^c = A^c \cap B^c$

Proof :- Let  $x \in (A \cup B)^c$ .  $\{ \begin{array}{l} x \in A^c \\ x \notin A \end{array} \}$

$$\Rightarrow x \notin (A \cup B)$$

$$\Rightarrow x \notin A \text{ and } x \notin B.$$

$$\Rightarrow x \in A^c \text{ and } x \in B^c.$$

$$\Rightarrow x \in (A^c \cap B^c)$$

$$\text{So, } (A \cup B)^c \subseteq (A^c \cap B^c) \quad \text{--- ①}$$

Let,  $x \in A^c \cap B^c$

$$\Rightarrow x \in A^c \text{ and } x \in B^c$$

$$\Rightarrow x \notin A \text{ and } x \notin B$$

$$\Rightarrow x \notin (A \cup B)$$

$$\Rightarrow x \in (A \cup B)^c$$

$$\text{So, } (A^c \cap B^c) \subseteq (A \cup B)^c \quad \text{--- ②}$$

From eqn ① & ② -

$$\Rightarrow (A \cup B)^c = (A^c \cap B^c)$$

Prove :-  $(A \cap B)^c = A^c \cup B^c$

Proof :- Let  $x \in (A \cap B)^c$

$$\Rightarrow x \notin (A \cap B)$$

$$\Rightarrow x \notin A \text{ and } x \notin B$$

$$\Rightarrow x \in A^c \text{ or } x \in B^c$$

$$\Rightarrow x \in A^c \cup B^c$$

$$\text{So, } (A \cap B)^c \subseteq (A^c \cup B^c) \quad \text{--- ①}$$

Let, ~~also~~  $x \in A^c \cup B^c$

$$\Rightarrow x \in A^c \text{ or } x \in B^c$$

$$\Rightarrow x \notin A \text{ or } x \notin B$$

$$\Rightarrow x \notin (A \cap B)$$

$$\Rightarrow x \in (A \cap B)^c$$

$$\text{So, } (A^c \cup B^c) \subseteq (A \cap B)^c \quad \text{--- ②}$$

From eqn ① & ② -

$$\Rightarrow (A \cap B)^c = A^c \cup B^c$$

Q Show that  $(A - B) - C = A - (B \cup C)$ .

$$\text{Let } x \in (A - B) - C$$

$$\Rightarrow x \in (A - B) \text{ and } x \notin C.$$

$$\Rightarrow (x \in A \text{ and } x \notin B) \text{ and } x \notin C.$$

$$\Rightarrow x \in A \text{ and } (x \notin B \text{ and } x \notin C).$$

$$\Rightarrow x \in A \text{ and } (x \notin (B \cup C))$$

$$\Rightarrow x \in A - (B \cup C)$$

$$\text{So, } (A - B) - C \subseteq A - (B \cup C) \quad \text{--- (1)}$$

$$\text{Let, } x \in A - (B \cup C)$$

$$\Rightarrow x \in A \text{ and } x \notin (B \cup C)$$

$$\Rightarrow x \in A \text{ and } (x \notin B \text{ and } x \notin C)$$

$$\Rightarrow (x \in A \text{ and } x \notin B) \text{ and } x \notin C$$

$$\Rightarrow x \in A - B \text{ and } x \notin C$$

$$\Rightarrow x \in (A - B) - C.$$

$$\text{So, } A - (B \cup C) \subseteq (A - B) - C. \quad \text{--- (2)}$$

From eqn (1) & (2)

$$\Rightarrow (A - B) - C = A - (B \cup C)$$

Q Prove that  $(A - B) \cap (B - A) = \emptyset$ .

$$\text{Let, } x \in (A - B) \cap (B - A)$$

$$\Rightarrow x \in (A - B) \text{ and } x \in (B - A)$$

$$\Rightarrow (x \in A \text{ and } x \notin B) \text{ and } (x \in B \text{ and } x \notin A)$$

$$\Rightarrow (x \in A \text{ and } x \notin A) \text{ and } (x \in B \text{ and } x \notin B)$$

$$\Rightarrow x \in \emptyset \text{ and } x \in \emptyset$$

$$\Rightarrow x \in \emptyset.$$

$$\text{So, } (A - B) \cap (B - A) \subseteq \emptyset. \quad \text{--- (1)}$$

Let,  $x \in \emptyset$  We know that  $\emptyset$  is subset of every set.

$$\emptyset \subseteq (A - B) \cap (B - A) \quad \text{--- (2)}$$

From (1) & (2)

$$\Rightarrow (A - B) \cap (B - A) = \emptyset.$$

Q Prove that  $A - (A \cap B) = (A - B)$

$$\text{Let, } x \in A - (A \cap B)$$

$$\Rightarrow x \in A \text{ and } x \notin (A \cap B)$$

$$\Rightarrow x \in A \text{ and } (x \notin A \text{ or } x \notin B)$$

$$\Rightarrow (x \in A \text{ and } x \notin A) \text{ or } (x \in A \text{ and } x \notin B)$$

$$\Rightarrow x \in \emptyset \text{ or } x \in (A - B)$$

$$\Rightarrow x \in \emptyset \cup (A - B)$$

$$\Rightarrow x \in (A - B)$$

$$\text{So, } A - (A \cap B) \subseteq (A - B) \quad \text{--- (1)}$$

$$\begin{aligned}
&\Rightarrow x \in A \text{ and } x \notin B \\
&\Rightarrow x \in \phi \cup (A - B) \\
&\Rightarrow x \in \phi \text{ or } x \in (A - B) \\
&\Rightarrow x \in \phi \text{ or } x \in A \text{ and } x \notin B \\
&\Rightarrow (x \in \phi \text{ and } x \notin A) \text{ or } (x \in A \text{ and } x \notin B) \\
&\Rightarrow x \in \phi \text{ and } (x \notin A \text{ or } x \notin B) \\
&\Rightarrow x \in A \text{ and } x \notin (A \cap B) \\
&\Rightarrow x \in A - (A \cap B) \\
\text{So, } (A - B) &\subseteq A - (A \cap B) \quad \text{--- (2)}
\end{aligned}$$

$$\Rightarrow [A - (A \cap B) = A - B] \text{ From (1) \& (2).}$$

Q If A and B are two sets then  $(A \cap B) \cup (A \cap \bar{B})$  and  $A \cap (A \cup B)$  are equal to ?

$$(A \cap B) \cup (A \cap \bar{B})$$

$$\Rightarrow A \cap (B \cup \bar{B}) \quad [\text{By Distributive Law}]$$

$$\Rightarrow A \cap U \quad [\text{By complement law}]$$

$$\Rightarrow A \quad [\text{By Identity Law}]$$

$$A \cap (A \cup B)$$

$$\Rightarrow (A \cap A) \cup (A \cap B) \quad [\text{Distributive Law}]$$

$$\Rightarrow \emptyset \cup (A \cap B) \quad [\text{By complement law}]$$

$$\Rightarrow A \cap B \quad [\text{By Identity Law}].$$

Q If A and B are two subsets of universal set then prove the following -

$$1) A - B = B - A \text{ if \& only if } A = B.$$

$$2) A - B = A \text{ iff } A \cap B = \emptyset.$$

$$3) (i) \text{ If } A = B \text{ then } A - B = B - A.$$

$$\text{Let } x \in A - B$$

$$\Rightarrow x \in A \text{ and } x \notin B.$$

$$\Rightarrow x \in B \text{ and } x \notin B.$$

$$\Rightarrow x \in (B - A)$$

$$\text{So, } (A - B) \subseteq B - A \quad \text{--- (1)}$$

$$\text{Let } x \in (B - A)$$

$$\Rightarrow x \in B \text{ and } x \notin A.$$

$$\Rightarrow x \in A \text{ and } x \notin B.$$

$$\Rightarrow x \in (A - B)$$

$$\text{So, } (B - A) \subseteq (A - B) \quad \text{--- (2)}$$

From (1) \& (2).

$$\Rightarrow [A - B = B - A]$$

(1) If  $(A - B) = (B - A)$  then  $A = B$ .

$$x \in (B - A)$$

$\Rightarrow x \in B$  and  $x \notin A$ . — ①

$$x \in (A - B)$$

$\Rightarrow x \in A$  and  $x \notin B$ . — ②

From ① & ② can be equal only when  $A = B$ .

(2) (i) If  $A \cap B \neq \emptyset$  then if  $A - B = A$  then  $A \cap B = \emptyset$

$$x \in A - B$$

$\Rightarrow x \in A$  and  $x \notin B$ .

$$\text{Let, } A \cap B \neq \emptyset$$

$$\Rightarrow x \in (A \cap B)$$

$$\Rightarrow x \in (A - B) \cap B. \quad [A - B = A]$$

$$\Rightarrow x \in A - B \text{ and } x \in B.$$

$$\Rightarrow (x \in A \text{ and } x \notin B) \text{ and } x \in B.$$

$$\Rightarrow x \in A \text{ and } (x \notin B \text{ and } x \in B)$$

$$\Rightarrow x \in A \text{ and } x \notin \emptyset$$

$$\Rightarrow x \in A \cap \emptyset$$

$$\Rightarrow x \in \emptyset.$$

So,  $A \cap B \subseteq \emptyset$ . — ① But this contradicts our assumption  $A \cap B \neq \emptyset$ .

But, we know that  $\emptyset$  is subset of every set. — ②

$$\emptyset \subseteq A \cap B. \quad \text{— ②}$$

From eqn ① & ②

$$\boxed{A \cap B = \emptyset}$$

(ii) If  $A \cap B = \emptyset$  then  $A - B \subseteq A$ .

$$\text{Let, } x \in A - B.$$

$$\Rightarrow x \in A \text{ and } x \notin B.$$

$$\Rightarrow x \in A \quad (\because A \cap B = \emptyset).$$

$$\text{So, } A - B \subseteq A \quad \text{— ①}$$

$$\text{Let, } x \in A$$

$$\Rightarrow x \in A \text{ and } x \notin B. \quad [\because A \cap B = \emptyset].$$

$$\Rightarrow x \in (A - B)$$

$$\text{So, } A \subseteq A - B \quad \text{— ②}$$

From ① & ②

$$\boxed{A - B = A}$$

(OR)

If  $(A \cap B) = \emptyset$  then  $A - B = A$ .

Let,  $x \in (A - B)$   
 $\Rightarrow x \in (A - B) \cup \emptyset$   
 $\Rightarrow x \in (A - B) \text{ or } x \in (A \cap B)$   
 $\Rightarrow (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \in B)$   
 $\Rightarrow (x \in A \text{ and } x \in B^c) \text{ or } (x \in A \text{ and } x \in B)$   
 $\Rightarrow x \in A \text{ and } (x \in B^c \text{ or } x \in B)$   
 $\Rightarrow x \in A \text{ and } x \in (B \cup B^c)$   
 $\Rightarrow x \in A \text{ and } x \in U$   
 $\Rightarrow x \in (A \cap U)$   
 $\Rightarrow x \in A$   
 $\text{So, } A - B \subseteq A \quad \text{--- ①}$

## \* Cartesian Product :-

$$A \times B = \{ (x, y) : x \in A \text{ and } y \in B \}.$$

e.g. ordered pair :-  $A = \{1, 2, 3\}$   
 $B = \{a, b\}$ .

$$A \times B = \{ (1, a), (1, b), (2, a), (2, b), (3, a), (3, b) \}.$$

$$B \times A = \{ (a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3) \}$$

Ordered Pair :- It is a pair of objects formed by using the two components in specified order  
 In the ordered pair  $(x, y)$ ,  $x$  is the first component and  $y$  is the second component.

Consider two sets  $A$  and  $B$ , then cartesian product of  $A$  and  $B$  is denoted by ' $A \times B$ ' and it is the set of all possible ordered pairs  $(x, y)$  with  $x \in A$  and  $y \in B$ .

Symbolically,

$$A \times B = \{ (x, y) : x \in A \text{ and } y \in B \}.$$

NOTE:  $A \times B \neq B \times A$ .

\* Multiset :- Multisets are sets where an element appears more than ones.

$$\text{e.g. } A = \{1, 1, 1, 2, 2, 3\}.$$

Multiset  $A$  can also be written as-

$$A = \{3 \cdot 1, 2 \cdot 2, 1 \cdot 3\}.$$

Multiplicity :- The multiplicity of an element in a multiset is defined to be the no. of times an element appears in the multiset.

3 is the multiplicity of 1.

2 " " " " 2.

1 " " " " 3.

NOTE: It is a special type of multiset in which the multiplicity of every element is one or zero.

\* Cardinality of Multiset :- Cardinality of Multiset is equal to the cardinality of corresponding set.

(OR)

Cardinality of Multiset is defined as the cardinality of its corresponding set. Assuming that all the elements are distinct in the multiset.

So cardinality of multiset A = cardinality of corresponding set A.

e.g.  $A = \{1, 1, 1, 2, 2, 3\}$

$A = \{1, 2, 3\}$ .

cardinality of Multiset A = 3. { , , , , }

\* Operations in Multisets :-

Let,  $A = \{3.a, 2.b, 1.c\}$ .

$B = \{2.a, 1.b, 1.d\}$ .

Union of multisets (U) :-  $(A \cup B) = \{3.a, 2.b, 1.c, 1.d\}$

Intersection of Multisets (N) :-  $A \cap B = \{2.a, 1.b\}$

Difference (-) :-  $A - B = \{1.a, 1.b, 1.c\}$

Sum (+) :-  $B - A = \{1.d\}$ .

$A + B = \{5.a, 3.b, 1.c, 1.d\}$ .

Union :-  $A \cup B$  is the multiset where the multiplicity of an element is maximum of its multiplicities in A and B.

e.g.  $A \cup B = \{3.a, 2.b, 1.c, 1.d\}$ .

Intersection :-  $A \cap B$  is the multiset where the multiplicity of a common element is minimum of its multiplicities in A and B.

e.g.  $A \cap B = \{2.a, 1.b\}$ .

Difference :-  $A - B$  is the multiset where the multiplicity of an element is equal to multiplicity of element in A minus multiplicity of element in B, if the difference is positive but it is equal to zero if the difference is zero and negative.

e.g.  $A - B = \{1.a, 1.b, 1.c\}$

$B - A = \{1.d\}$ .

Sum :-  $A + B$  is the multiset where multiplicity of an element is equal to the sum of multiplicities of an element in both multisets A and B.

e.g.  $A + B = \{5.a, 3.b, 1.c, 1.d\}$ .

$$Q = \{ 3 \cdot a, 3 \cdot b, 2 \cdot d \}$$

Find 1)  $P \cup Q$ , 2)  $P \cap Q$ , 3)  $P - Q$ , 4)  $Q - P$ , 5)  $P + Q$

$$\Rightarrow P \cup Q = \{ 4 \cdot a, 3 \cdot b, 1 \cdot c, 2 \cdot d \}$$

$$\Rightarrow P \cap Q = \{ 3 \cdot a, 3 \cdot b \}$$

$$\Rightarrow P - Q = \{ 1 \cdot a, 1 \cdot c \}$$

$$\Rightarrow Q - P = \{ 2 \cdot d \}$$

$$\Rightarrow P + Q = \{ 7 \cdot a, 6 \cdot b, 1 \cdot c, 2 \cdot d \}$$

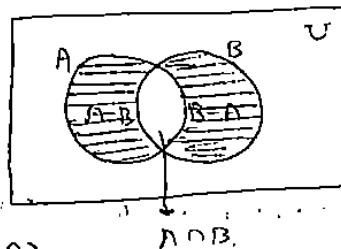
\* Set Inclusion Exclusion Principle :-  
(OR)

Counting Principle :-

$$n(A) = n(A - B) + n(A \cap B) \quad \text{--- (1)}$$

$$n(B) = n(B - A) + n(A \cap B) \quad \text{--- (2)}$$

$$n(A \cup B) = n(A - B) + n(B - A) + n(A \cap B) \quad \text{--- (3)}$$



Put the values of  $n(A - B)$  &  $n(B - A)$  in eqn (3).

from eqn (1) & (2) in eqn (3) -

$$n(A \cup B) = n(A) + n(A \cap B) + n(B) - n(A \cap B) + n(A \cap B)$$

$$\boxed{n(A \cup B) = n(A) + n(B) - n(A \cap B)}$$

Corollary :-

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

Proof: Let  $B \cup C = D$ .

$$\begin{aligned} n(A \cup D) &= n(A) + n(D) - n(A \cap D) \\ &= n(A) + n(B \cup C) - n[A \cap (B \cup C)] \\ &= n(A) + n(B) + n(C) - n(B \cap C) - n[(A \cap B) \cup (B \cap C)] \\ &= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B) - n(B \cap C) \\ &\quad + n[(A \cap B) \cap (B \cap C)] \\ &= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B) - n(A \cap C) \\ &\quad + n(A \cap B \cap C). \end{aligned}$$

Q 40 lecturers were interviewed for a job, 25 were mathematicians, 28 were physicist and 7 were neither. How many lecturers were mathematicians and physicists?

→ Let, M be the set of mathematicians and P be the set of physicists.

Given :-  $n(M) = 25$

$n(P) = 28$

Find :-  $n(M \cap P) = ?$

Given :-  $n(M \cup P) = 40 - 7 = 33$

$$n(M \cup P) = n(M) + n(P) - n(M \cap P) \quad \begin{matrix} \text{By set incl.} \\ \text{ex. principle} \end{matrix}$$

$$n(M \cap P) = n(M) + n(P) - n(M \cup P)$$

$$= 25 + 28 - 33$$

$$= 53 - 33$$

$$\boxed{n(M \cap P) = 20}$$

So, 20 lecturers were both mathematicians and physicists.

Q In a survey of 600 TV viewers given the following information.

1) 385 watch cricket matches.

2) 295 watch hockey matches.

3) 215 watch football matches.

4) 145 watch cricket & football matches both.

5) 170 watch cricket & hockey matches both.

6) 150 watch hockey & football matches both.

7) 150 does not watch any of the three games.

Find 1) How many people watch all three kind of matches.

2) How many people watch exactly one sport.

→ Let, C be the set of Cricket viewers.

H be the set of Hockey viewers.

F be the set of Football viewers.

Given :-  $n(C) = 385$

$n(H) = 295$

$n(F) = 215$

$n(C \cap F) = 145$

$n(C \cap H) = 170$

$n(H \cap F) = 150$

$$n(C \cup H \cup F) = 600 - 150 = 450$$

$$\text{Now, } n(C \cup H \cup F) = n(C) + n(H) + n(F) - n(C \cap H) - n(C \cap F) \\ - n(H \cap F) + n(C \cap H \cap F)$$

$$450 = 385 + 295 + 215 - 170 - 145 - 150 + n(C \cap H \cap F)$$

$$450 = 895 - 465 + n(C \cap H \cap F)$$

$$450 = 430 + n(C \cap H \cap F)$$

$$\boxed{n(C \cap H \cap F) = 20}$$

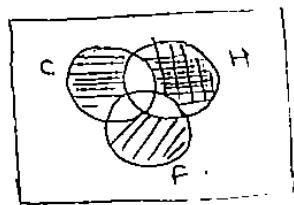
(By set incl. ex. principle).

2) Only cricket viewers.

$$n(C) = n(C \cap H) + n(C \cap F) - n(C \cap H \cap F)$$

$$= 385 - 170 + 20 - 145$$

$$= 90.$$



Only Hockey viewers.

$$n(H) = n(C \cap H) + n(C \cap F) - n(C \cap H \cap F)$$

$$= 295 - 170 + 20 - 145$$

$$= 55.$$

Only football viewers.

$$n(F) = n(C \cap F) + n(H \cap F) - n(C \cap H \cap F)$$

$$= 215 - 145 + 20 - 150$$

$$= -60.$$

No. of viewers exactly watch =  $n(\text{exactly Cricket}) + n(\text{exactly Hockey}) + n(\text{exactly Football})$

$$= 90 - 5 - 60$$

$$= 25$$

Q. Among first 500 integers find.

1) The no. of integers which are not divisible by 2 nor by 3 nor by 5.

2) The no. of integers which are exactly divisible by one of them.

→ Let ; A be the set of integers divisible by 2.  
B be the set of integers divisible by 3.  
C be the set of integers divisible by 5.

$$n(A) = \left\lfloor \frac{500}{2} \right\rfloor = 250.$$

$$n(B) = \left\lfloor \frac{500}{3} \right\rfloor = 166.$$

$$n(C) = \left\lfloor \frac{500}{5} \right\rfloor = 100.$$

$$n(A \cap B) = \left\lfloor \frac{500}{2 \times 3} \right\rfloor = 83$$

$$n(B \cap C) = \left\lfloor \frac{500}{3 \times 5} \right\rfloor = 33.$$

$$n(A \cap C) = \left\lfloor \frac{500}{2 \times 5} \right\rfloor = 50.$$

$$n(A \cap B \cap C) = \left\lfloor \frac{500}{2 \times 3 \times 5} \right\rfloor = 16.$$

1)  $n(A' \cap B' \cap C') = ?$

$$n(A' \cap B' \cap C') = n(A \cup B \cup C)^c$$

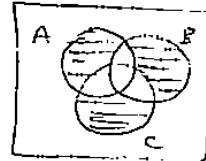
$$= 75 - n(A \cup B \cup C)$$

$$\begin{aligned}
 n(A \cup B \cup C) &= n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) \\
 &\quad + n(A \cap B \cap C) \\
 &= 250 + 166 + 100 - 83 - 33 - 50 + 16 \\
 &= 366.
 \end{aligned}$$

$$\begin{aligned}
 n(A' \cap B' \cap C') &= n(\text{U}) - n(A \cup B \cup C) \\
 &= 500 - 366 \\
 &= 134
 \end{aligned}$$

(2) only divisible by 2.

$$\begin{aligned}
 n(F) &= n(A \cap B) - n(A \cap C) + n(A \cap B \cap C) \\
 &= 250 - 83 - 33 + 16 \\
 &= 150
 \end{aligned}$$



only divisible by 3.

$$\begin{aligned}
 n(B) &= n(A \cap B) - n(B \cap C) + n(A \cap B \cap C) \\
 &= 166 - 83 - 50 + 16 \\
 &= 49.
 \end{aligned}$$

only divisible by 5

$$\begin{aligned}
 n(C) &= n(A \cap C) - n(B \cap C) + n(A \cap B \cap C) \\
 &= 100 - 33 - 50 + 16 \\
 &= 33
 \end{aligned}$$

No. of integers exactly divisible by one of them

$$\begin{aligned}
 &= n(\text{exactly divisible by 2}) + n(3) + n(5) \\
 &= 150 + 49 + 33 \\
 &= 232
 \end{aligned}$$

Q 75 children went to a circus where they attended a magic show, comedy show and animal show. 20 of them attended all three shows and 55 attended atleast two shows. Each show cost 5 Rs. The total money collected 700 Rs. find the no. of children who did not attend any of three shows.

→ Let, M be the set of Magic show.

C be the set of Comedy Show.

A be the set of Animal Show.

Total money collected = ₹ 700.

Cost per show = ₹ 5

Total no. of shows attended =  $\frac{700}{5} = 140$

$$n(M \cap C \cap A) = 20.$$

No. of children attended atleast two shows = 55.

⇒ No. of children attended exactly two shows + No. of children attended all shows = 55.

$$\Rightarrow \text{No. of children attended Exactly 2 shows} + 20 = 55.$$

$$\Rightarrow \text{No. of children attended exactly 2 shows} = 35.$$

$$\begin{aligned}
 \text{No. of children attended exactly 2 shows} &= 140 - (35 + 20 + 10) \\
 &= 140 - (70 + 60) \\
 &= 140 - 130 \\
 &= 10
 \end{aligned}$$

$$\begin{aligned}
 \text{No. of children did not attend any of 3 shows} &= 75 - (35 + 20 + 10) \\
 &= 75 - (65) \\
 &= 10
 \end{aligned}$$

## \* Countable and Uncountable Sets :-

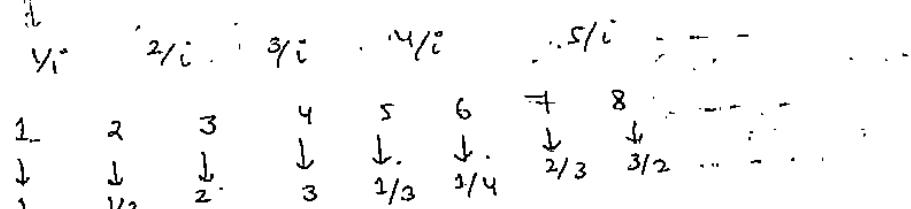
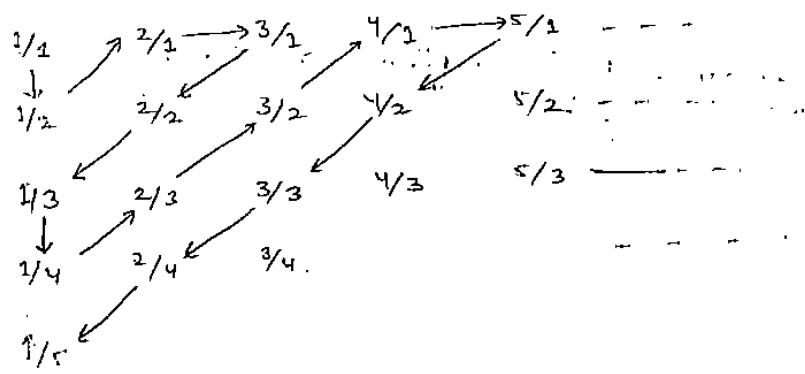
**Countable Infinite Set :-** An infinite set, A is said to be countably infinite or denumerable if it is equivalent to set 'N' (set of Natural No.'s) i.e. if there exist a one to one mapping  $f : N \rightarrow A$  eq set of Irrational no.'s

**Countable set :-** A set A is called countable when it is either

- Empty or Null set i.e.  $A = \emptyset$ .
- Finite set.
- Countable Infinite set.

Natural numbers : 1, 2, 3, 4, 5, 6, 7, 8, ...  
Even numbers : 2, 4, 6, 8, 10, 12, 14, 16, ...

Q Show that the set of +ve rational no.'s is countable  
 $p+q=2, p+q=3, p+q=4, \dots$



Q  $A = \{n^2 : n \in \mathbb{N}\}$ . Cardinality.

$$A = \{1^2, 2^2, 3^2, 4^2, \dots\}$$

$n(A)$  = This set is countable infinite.

$$\begin{matrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1^2 & 2^2 & 3^2 & 4^2 \end{matrix}$$

\* **Relation :-** Let A and B be two non-empty sets then R is a relation from A to B if  $R \subseteq A \times B$ . and it is the set of ordered pairs  $(a, b)$  where  $a \in A$  and  $b \in B$ . It is denoted by  $a R b$  or  $(a, b) \in R$ . and it is read as "a is related to b". If there is no relation b/w a and b then  $a \not R b$  or  $(a, b) \notin R$ . and it is read as "a is not related to b". Symbolically,

$$R = \{ (a, b) : a \in A \text{ & } b \in B \text{ and } a R b \text{ or } (a, b) \in R \}.$$

Q. e.g.  $A = \{1, 2, 3, 4\}$ ,  $B = \{1, 2\}$ . A relation R is defined from A to B i.e.  $a R b$  iff  $a \times b = \text{even no.}$  Find relation R.

$$\rightarrow R \subseteq A \times B.$$

$$A = \{1, 2, 3, 4\}, B = \{1, 2\}.$$

$$A \times B = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2), (4, 1), (4, 2)\}.$$

$$R = \{(1, 2), (2, 1), (2, 2), (3, 2), (4, 1), (4, 2)\}.$$

Q.  $A = \{2, 4, 6, 8\}$ ,  $B = \{1, 2, 3\}$ . Find the relation R such that  $a R b$  iff  $a \geq b$ .

$$\rightarrow A = \{2, 4, 6, 8\}, B = \{1, 2, 3\}.$$

$$A \times B = \{(2, 1), (2, 2), (2, 3), (4, 1), (4, 2), (4, 3), (6, 1), (6, 2), (6, 3), (8, 1), (8, 2), (8, 3)\}$$

$$R = \{(2, 1), (2, 2), (4, 1), (4, 2), (4, 3), (6, 1), (6, 2), (8, 1), (6, 3), (8, 2), (8, 3)\}.$$

Q.  $A = \{1, 2, 3, 4\}$  A relation R is defined on set A such that  $a R b$  iff a is divisible by b.

$$\rightarrow A = \{1, 2, 3, 4\}$$

$$A = \{1, 2, 3, 4\}.$$

$$A \times A = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}.$$

$$R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 3), (4, 1), (4, 2), (4, 4)\}.$$

### \* Domain and Range of a Relation :-

Domain of a Relation :- It is the set consisting of all first elements of ordered pairs of relation R. It is denoted by  $D(R)$  or  $\text{Dom}(R)$ .

Symbolically,

$$D(R) = \{a : a \in A \text{ and } a R b\}.$$

NOTE:  $D(R) \subseteq A$ .

$\cap$  second elements of ordered pairs of  
relation  $R$ . It is denoted by  $R(R)$  or  $\text{Ran}(R)$ .  
Symbolically,

$$R(R) = \{ b : b \in B \text{ and } a R b \}.$$

NOTE:  $R(R) \subseteq B$ .

e.g.  $A = \{1, 2, 3, 4\}$ ,  $B = \{1, 2\}$ .

Domain  $\rightarrow D(R) = \{1, 2, 3, 4\}$

Range  $\rightarrow R(R) = \{1, 2\}$ .

$$A = \{2, 4, 6, 8\}, B = \{1, 2, 3\}.$$

Domain  $\rightarrow D(R) = \{2, 4, 6, 8\}$

Range  $\rightarrow R(R) = \{1, 2, 3\}$

$$A = \{1, 2, 3, 4\}, B = \{1, 2, 3, 4\}$$

Domain  $\rightarrow D(R) = \{1, 2, 3, 4\}$

Range  $\rightarrow R(R) = \{1, 2, 3, 4\}$ .

Q. Let  $A = \{a, b, c\}$ ,  $B = \{a, b, c, d\}$ ,  $R$  and  $S$  are two relations defined from  $A$  to  $B$  such that  $R = \{(a, b), (b, c), (c, a)\}$  and  $S = \{(a, a), (a, b), (b, d)\}$ . Find domain and range ( $R \cup S$ ), ( $R \cap S$ ) and ( $R - S$ ).

$$R \cup S = \{(a, a), (a, b), (b, c), (c, a), (a, d), (b, d)\}$$

$$R \cap S = \{(a, a)\}$$

$$R - S = \{(a, b), (b, c), (c, a)\}.$$

$$D(R \cup S) = \{a, b, c, d\} \\ = \{a, b, c\}$$

$$R(R \cup S) = \{a, b, c, d\}.$$

$$D(R \cap S) = \{a\}.$$

$$R(R \cap S) = \{a\}$$

$$D(R - S) = \{a, b, c\}$$

$$R(R - S) = \{b, c, d\}.$$

## \* Types of Relation :-

- Universal relation :- A relation  $R$  is called universal relation from  $A$  to  $B$  if  $R = A \times B$ .

<sup>For one set</sup>  
A relation  $R$  is called universal relation defined on

Set  $A$ . If  $R = A \times A$ .

e.g.  $A = \{1, 2\}$

$$\text{R} = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

- Identity Relation (OR) Diagonal Relation :- A relation  $R$  is called identity relation on set  $A$  if every element of set is related to itself only.

It is denoted by ' $I_A$ ' or ' $\Delta_A$ ' or ' $\Delta$ '.

$$I_A = \{(a, a) : \forall a \in A \text{ and } aRa\}.$$

e.g.  $A = \{1, 2, 3\}$

$$I_A = \{(1, 1), (2, 2), (3, 3)\}.$$

- Inverse Relation :- A relation defined from  $B$  to  $A$  is called Inverse relation if a relation  $R$  is defined from  $A$  to  $B$ . It is denoted by ' $R^{-1}$ '. Symbolically,

$$R^{-1} = \{(b, a) : b \in B, a \in A \text{ and } aRb\}.$$

e.g.  $R = \{(1, 1), (1, 2), (1, 3), (3, 2)\}$

$$R^{-1} = \{(1, 1), (2, 1), (3, 1), (2, 3)\}.$$

- Empty Relation :- (OR) NULL (OR) VOID Relation :- A relation  $R$  is called empty relation if  $R = \emptyset$ .

e.g.  $A = \{1, 2, 3\}$ .

and.  $a R b$  iff  $a+b > 6$ .

Here,  $R = \emptyset$ .

- Compliment of Relation :- If a relation is defined from  $A$  to  $B$  then compliment of  $R$  is set of ordered pairs such that

$$R^c = \{(a, b) : (a, b) \notin R\}.$$

$$R^c = (A \times A) - R. \quad [ \text{Defined on one set} ]$$

$$R^c = (A \times B) - R. \quad [ \text{Defined on two sets} ].$$

It is denoted by  $R^c$  or  $R'$ .

e.g.  $A = \{1, 2, 3\}$

$$B = \{4, 5\}$$

$$R = \{(1, 4), (1, 5), (3, 4), (3, 5)\}.$$

$$A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}.$$

$$R^c = (A \times B) - R.$$

$$R^c = \{(2, 4), (2, 5)\}.$$

$$R: A \rightarrow B$$

$$R = \{ (a, b) : a R b \}$$

$$S: B \rightarrow C$$

$$S = \{ (b, c) : b S c \}$$

Let,  $R$  be the relation from  $A$  to  $B$  and  $S$  be the relation from  $B$  to  $C$  then composition of relation  $R$  and  $S$  is a relation consisting of ordered pairs  $(a, c)$  where  $a \in A$  and  $c \in C$  provided that there exists some  $b \in B$  such that  $a R b$  and  $b S c$ . It is denoted by ' $R \circ S$ '.

Symbolically,

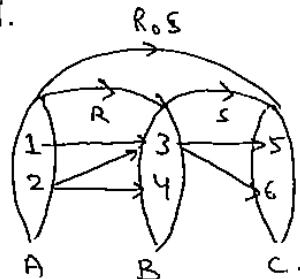
$$R \circ S = \{ (a, c) : \exists b \in B, a R b \text{ and } b S c \}.$$

Eg.  $A = \{1, 2\}, B = \{3, 4\}, C = \{5, 6\}$ .

$$R = \{(1, 3), (2, 3), (2, 4)\}$$

$$S = \{(3, 5), (3, 6)\}$$

$$R \circ S = \{(1, 5), (1, 6), (2, 5), (2, 6)\}.$$



- Q.  $R$  and  $S$  relations are defined on set  $A = \{1, 2, 3, 4\}$  and  
 $R = \{(1, 2), (2, 2), (2, 3), (2, 4)\}, (3, 2), (4, 3)\}$ .  
 $S = \{(2, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 2)\}$ . find  $R \circ S$  &  $S \circ R$
- $R \circ S = \{(1, 2), (1, 3), (2, 2), (2, 3), (2, 4), (3, 3), (3, 2), (4, 3), (4, 4)\}$ .
- $S \circ R = \{(2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (3, 2), (4, 2), (4, 3), (4, 4)\}$ .

Q. Prove that inverse of  $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$ .

→ Let,  $(a, b) \in (R \cup S)^{-1}$

$$\Rightarrow (b, a) \in (R \cup S)$$

$$\Rightarrow (b, a) \in R \text{ or } (b, a) \in S$$

$$\Rightarrow (a, b) \in R^{-1} \text{ or } (a, b) \in S^{-1}$$

$$\Rightarrow (a, b) \in (R^{-1} \cup S^{-1})$$
 ①

$$\text{So, } (R \cup S)^{-1} \subseteq (R^{-1} \cup S^{-1})$$

Let,  $(a, b) \in (R^{-1} \cup S^{-1})$

$$\Rightarrow (a, b) \in R^{-1} \text{ or } (a, b) \in S^{-1}$$

$$\Rightarrow (b, a) \in R \text{ or } (b, a) \in S$$

$$\Rightarrow (b, a) \in (R \cup S)$$

$$\Rightarrow (a, b) \in (R \cup S)^{-1}$$

$$\text{So, } (R^{-1} \cup S^{-1}) \subseteq (R \cup S)^{-1}$$
 ②

From eqn ① & ②.

$$\boxed{(R \cup S)^{-1} = R^{-1} \cup S^{-1}}$$

Q Prove that  $(R \cap S)^{-1} = (R^{-1} \cap S^{-1})$ .

Let  $(a, b) \in (R \cap S)^{-1}$

$\Rightarrow (b, a) \in (R \cap S)$

$\Rightarrow (b, a) \in R$  and  $(b, a) \in S$ .

$\Rightarrow (a, b) \in R^{-1}$  and  $(a, b) \in S^{-1}$

$\Rightarrow (a, b) \in R^{-1} \cap S^{-1}$ .

So,  $(R \cap S)^{-1} \subseteq R^{-1} \cap S^{-1}$ . — ①

Let,  $(a, b) \in (R^{-1} \cap S^{-1})$

$\Rightarrow (a, b) \in R^{-1}$  and  $(a, b) \in S^{-1}$

$\Rightarrow (b, a) \in R$  and  $(b, a) \in S$ .

$\Rightarrow (b, a) \in (R \cap S)$

$\Rightarrow (a, b) \in (R \cap S)^{-1}$

So,  $(R^{-1} \cap S^{-1}) \subseteq (R \cap S)^{-1}$  — ②

From eqn ① & ②

$$(R \cap S)^{-1} = (R^{-1} \cap S^{-1})$$

Q If  $R \subseteq S$  then,  $R^{-1} \subseteq S^{-1}$ .

Given,  $R \subseteq S$ .

Let,  $(a, b) \in R^{-1}$ ,

$\Rightarrow (b, a) \in R$ .

$\Rightarrow (b, a) \in S$  ( $\because R \subseteq S$ )

$\Rightarrow (a, b) \in S^{-1}$

So,  $R^{-1} \subseteq S^{-1}$ .

Q Prove that composition of relation follows the associative law.

Proof :-  $(R \circ S) \circ T = R \circ (S \circ T)$ .

Let,  $R$  be a relation from  $A$  to  $B$ ,  $S$  be a relation from  $B$  to  $C$  and  $T$  be a relation from  $C$  to  $D$ .

Let  $(a, d) \in (R \circ S) \circ T$ .

$\Rightarrow$  (such that)  $\exists c \in C$ ,  $(a, c) \in (R \circ S)$  and  $(c, d) \in T$ . — ①

Now,  $(a, c) \in R \circ S$ .

$\Rightarrow \exists b \in B$ ,  $(a, b) \in R$  and  $(b, c) \in S$  — ②

From eqn ① & ②,

$(b, c) \in S$  and  $(c, d) \in T$ .

$\Rightarrow (b, d) \in (S \circ T)$  — ③.

From eqn ② & ③,

$(a, b) \in R$  and  $(b, d) \in (S \circ T)$

$\Rightarrow (a, d) \in R \circ (S \circ T)$  — ④

So,  $(R \circ S) \circ T \subseteq R \circ (S \circ T)$  — ⑤

Conversely, let,  $(a, c) \in \text{no } \cup \cup$

$\Rightarrow \exists b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in (S \circ T) \text{ --- (6)}$

Now,  $(b, c) \in S \circ T$ .

$\Rightarrow \exists c \in C, (b, c) \in S \text{ and } (c, d) \in T \text{ --- (7)}$

From eqn (6) & (7)

$(a, b) \in R \text{ and } (b, c) \in S$

$\Rightarrow (a, c) \in R \circ S \text{ --- (8)}$

From eqn (7) & (8)

$(a, c) \in R \circ S \text{ and } (c, d) \in T$

$\Rightarrow (a, d) \in (R \circ S) \circ T \text{ --- (9)}$

So,  $R \circ (S \circ T) \subseteq (R \circ S) \circ T \text{ --- (10)}$

From eqn (5) and (10)

$$[(R \circ S) \circ T] = R \circ (S \circ T)$$

Q. Prove that  $(R \circ S)^{-1} = (S^{-1} \circ R^{-1})$

→ Let,  $R$  be a relation from  $A$  to  $B$ .  
and  $S$  be a relation from  $B$  to  $C$ .

Let,  $(c, a) \in (R \circ S)^{-1}$

$\Rightarrow (a, c) \in (R \circ S)$

$\Rightarrow \exists b \in B, (a, b) \in R \text{ and } (b, c) \in S \text{ --- (1)}$

From eqn (1)

$(a, b) \in R$

$\Rightarrow (b, a) \in R^{-1} \text{ --- (2)}$

From eqn (1)

$(b, c) \in S$

$\Rightarrow (c, b) \in S^{-1} \text{ --- (3)}$

From eqn (3) and (2)

$(c, b) \in S^{-1} \text{ and } (b, a) \in R^{-1}$

$\Rightarrow (c, a) \in S^{-1} \circ R^{-1} \text{ --- (4)}$

So,  $(R \circ S)^{-1} \subseteq (S^{-1} \circ R^{-1}) \text{ --- (5)}$

Conversely, Let,  $(c, a) \in (S^{-1} \circ R^{-1})$

$\Rightarrow \exists b \in B, (c, b) \in S^{-1} \text{ and } (b, a) \in R^{-1} \text{ --- (6)}$

From eqn (6)

$(c, b) \in S^{-1}$

$\Rightarrow (b, c) \in S \text{ --- (7)}$

From eqn (6)

$(b, a) \in R^{-1}$

$$\Rightarrow (a, b) \in R \quad \text{--- (8)}$$

From eqn (7) & (8)

$$(a, b) \in R \text{ and } (b, c) \in S$$

$$\Rightarrow (a, c) \in R \circ S.$$

$$\Rightarrow (c, a) \in (R \circ S)^{-1} \quad \text{--- (9)}$$

$$So, S^{-1} \circ R^{-1} \subseteq (R \circ S)^{-1} \quad \text{--- (10)}$$

From eqn (8) and (10).

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}.$$

Q. Let S, R and T are relations on set  $A = \{0, 1, 2, 3\}$  defined by  $R = \{(a, b) : a+b=3\}$ ,  $S = \{(a, b) : 3 \text{ is divisor of } a+b\}$  and  $T = \{(a, b) : \max(a, b) = 3\}$ . Find.

- i)  $R^2$ . ii)  $R \circ S$  iii)  $R \circ T$  iv)  $S \circ T$ , v)  $(R \circ S) \circ T$

$$A = \{0, 1, 2, 3\}$$

$$[R^n = R^{n-1} \circ R]$$

$$R = \{(0, 3), (3, 0), (1, 2), (2, 1)\}$$

$$S = \{(0, 3), (3, 0), (1, 2), (2, 1)\}, (3, 3)\}$$

$$T = \{(0, 3), (3, 0), (1, 3), (3, 1), (2, 3), (3, 2), (3, 3)\}$$

i)  $R^2$  ( $R \circ R$ ) =  $\{(0, 0), (3, 3), (1, 1), (2, 2)\}$

ii)  $R \circ S$  =  $\{(0, 0), (0, 3), (3, 3), (1, 1), (2, 2)\}$

iii)  $R \circ T$  =  $\{(0, 0), (0, 1), (0, 2), (0, 3), (3, 3), (1, 3), (2, 3)\}$

iv)  $S \circ T$  =  $\{(0, 0), (0, 1), (0, 2), (0, 3), (3, 3), (1, 3), (2, 3), (3, 0), (3, 1), (3, 2)\}$

v)  $(R \circ S) \circ T$  =  $\{(0, 3), (0, 0), (3, 0), (3, 3), (1, 3), (2, 3), (3, 2), (3, 1), (0, 1), (0, 2)\}$ .

## \* Properties of Relation :-

Reflexive Relation :- A binary relation  $\circ R$  on set A is said to be reflexive relation if every element of set A is related to itself. i.e.  $\forall a \in A, aRa$  or  $(a, a) \in R$ .

e.g.  $A = \{1, 2, 3\}$ .

$$R = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\}.$$

R is reflexive relation

since,  $(1, 1) \in R$ .

$(2, 2) \in R$

$(3, 3) \in R$ .

$|A| = n$  (cardinality of set A)

Total binary relations =  $2^{n^2}$

Total reflexive relations =  $2^{n(n-1)}$

Total symmetric relations =  $2^{n(n+1)/2}$

Note :- Every identity relation is a reflexive relation but every reflexive relation is not identity relation.

Irreflexive Relation :- A relation R is defined on set A is said to be irreflexive if there is no element in A which is related to itself.

Eg.  $A = \{1, 2, 3\}$ .  
 $R = \{(1,2), (2,1), (2,3)\}$ ,  $R$  is irreflexive relation  
 since,  $(1,1) \notin R$ .  
 $(2,2) \notin R$ .  
 $(3,3) \notin R$ .

- Non-Reflexive Relation :- A relation  $R$  is defined on set  $A$  is said to be non-reflexive relation if it is neither reflexive nor irreflexive i.e. some elements are related to itself but there exist atleast one element which is not related to itself.

Eg.  $A = \{1, 2, 3\}$ .  
 $R = \{(1,1), (2,2), (1,3), (2,3), (3,1)\}$ .  
 $R$  is Non-Reflexive  
 since.  $(3,3) \notin R$ .

- Symmetric Relation :- A relation  $R$  is defined on set  $A$  is said to be symmetric relation if whenever  $a R b$  then  $b R a$ .

i.e.  $(a,b) \in R \Rightarrow (b,a) \in R$ .

Eg.  $A = \{1, 2, 3\}$ .  
 $R = \{(1,1), (1,2), (2,1), (2,3), (3,2)\}$ .  
 $R$  is symmetric relation.

since,  
 $(1,1) \in R$ .  
 $(1,2) \in R \Rightarrow (2,1) \in R$ .  
 $(2,3) \in R \Rightarrow (3,2) \in R$ .

Therefore, it is symmetric relation.

NOTE: A relation  $R$  is symmetric if  $R = R'$ .

- Asymmetric Relation :- A relation  $R$  on set  $A$  is said to be asymmetric relation if  $(a,b) \in R \Rightarrow (b,a) \notin R$ .

i.e.  $(a,b) \in R \Rightarrow (b,a) \notin R$ .

$\left\{ \begin{array}{l} p \rightarrow q \\ \text{if then} \end{array} \right.$

Eg.  $A = \{1, 2, 3\}$

$R = \{(1,2), (3,2)\}$ .

$R$  is asymmetric relation.

since,  $(1,2) \in R \Rightarrow (2,1) \notin R$ .  
 $(2,3) \in R \Rightarrow (3,2) \notin R$ .

- Anti-Symmetric Relation :- A relation  $R$  is defined on set  $A$  is said to be anti-symmetric relation if  $(a,b) \in R$  and  $(b,a) \in R \Rightarrow a = b$ .

Eg. contrapositive definition of Anti-Symmetric Relation :-

If  $\forall (a,b) \in R$  and  $(b,a) \notin R$ .  
 $\Rightarrow a \neq b$ .

E.g.  $A = \{1, 2, 3\}$ .  
 $R = \{(1,1), (1,2)\}$ .

$R$  is antisymmetric relation.  
Since,  $(1,1) \in R$ .

$R$  is contra positive anti-symmetric relation.  
Since,  $(1,2) \in R$   
 $(2,1) \notin R$ .  
and also,  $1 \neq 2$ .

{ Divide in always  
a anti-symmetric  
relation }

\* Transitive Relation :- A relation  $R$  is defined on set  $A$  is said to be transitive relation if

$$[(a,b) \in R \text{ and } (b,c) \in R \Rightarrow (a,c) \in R.]$$

E.g.  $A = \{1, 2, 3\}$ .

$$R = \{(1,2), (2,3), (1,3)\}.$$

$R$  is transitive relation

Since,  $(1,2) \in R$  and  $(2,3) \in R$ .  
 $\therefore \Rightarrow (1,3) \in R$ .

\* Equivalence Relation :- A relation  $R$  is defined on set  $A$  is said to be equivalence relation if it is reflexive, symmetric and transitive. i.e. equivalence relation satisfies following properties.

- 1)  $(a,a) \in R$ ,  $\forall a \in A$ . (Reflexive)
- 2)  $(a,b) \in R \Rightarrow (b,a) \in R$ . (symmetric)
- 3)  $(a,b) \in R$  and  $(b,c) \in R \Rightarrow (a,c) \in R$ . (transitive).

Q Determine whether the relation  $R$  on set  $A = \{-3, -2, -1, 0, 1, 2, 3\}$  is reflexive, symmetric, irreflexive, anti-symmetric, asymmetric, and transitive. where  $(a,b) \in R$  iff i)  $a=1$ , ii)  $a=2b$ , iii)  $a+b = \text{even}$ .

$$\rightarrow A = \{-3, -2, -1, 0, 1, 2, 3\}.$$

$$R_1 = \{(1, -3), (1, -2), (1, -1), (1, 0), (1, 1), (1, 2), (1, 3)\}.$$

1) Every element of  $A$  is not related to itself therefore, it is not reflexive relation.

$$\therefore (2,2) \notin R, (3,3) \notin R \text{ etc.}$$

2)  $(1, -3) \in R \Rightarrow (-3, 1) \notin R$ . Therefore, it is not symmetric relation.

3) Since  $(1,1) \notin R$  - Therefore it is not irreflexive relation.

4)  $(1,1) \in R$  and  $1=1$ .  $\therefore (1, -3) \in R \Rightarrow (-3, 1) \notin R$ .  
 $1 \neq -3$ .

Therefore, it is anti-symmetric relation.

5)  $(1,1) \in R$  - Therefore it is not anti-symmetric relation.

6)  $(1,1) \in R, (1, -1) \in R$  etc. Therefore, it is transitive relation.

- $R_2 = \{(x, -x) : x \in \{0, 1, 2, 3\}\}$
- 1) Every element is not related to itself. Therefore, it is not a reflexive relation.
  - (-2, -2) and (2, 2)  $\in R$  so, not a reflexive relation.
  - 2)  $(-2, -1) \in R \Rightarrow (1, -2) \notin R$ . Therefore, it is not symmetric relation.
  - 3)  $(0, 0) \in R$ . Therefore, it is not irreflexive relation.
  - 4)  $(0, 0) \in R, (2, 1) \in R, (1, 2) \notin R \Rightarrow 1 \neq 2$ . so, it is antisymmetric relation.
  - 5) Since,  $(0, 0) \notin R$ . so, it is not asymmetric relation.
  - 6) Not transitive relation.

[ $a+b = \text{even}$ ]

$$R_3 = \{(0, 1), (1, 2), (2, 3), (-1, -1), (-2, -2), (-3, -3), (-1, -3), (1, 3), (0, 2), (0, -2), (1, -1), (2, -2), (3, -3), (-3, -1), (-2, 2), (2, 0), (-2, 0), (-1, 1), (-3, 3), (3, 1), (1, -3), (-3, 1), (-1, 3), (3, -1)\}.$$

- 1) It is not reflexive.
- 2) symmetric relation.
- 3) Not reflexive relation.
- 4) Not antisymmetric.
- 5) Not Asymmetric
- 6) Transitive relation.

Q Prove that the relation  $R$  "  $(a-b)$  is divisible by 6" &  $a, b \in \mathbb{Z}^+$  is an equivalence relation.

$\rightarrow$  Congruent Relation :-      }      Equivalence relation  
 $a \equiv b \pmod m$ .      } satisfies when reflexive  
 $\Rightarrow a-b$  divisible by  $m$ .      } symmetric and transitive  
relation satisfies.

- 1) 0 is divisible by 6.  
 $\Rightarrow (a-a)$  is divisible by 6,  $\forall a \in \mathbb{Z}^+$ .  $\Rightarrow (0, a) \in R$ .  
 So, it follows reflexive relation.
- 2) Let  $(a, b) \in R$ .  
 $\Rightarrow (a-b)$  is divisible by 6.  
 $\Rightarrow -(a-b)$  is divisible by 6.  
 $\Rightarrow (b-a)$  is divisible by 6.  
 $\Rightarrow (b, a) \in R$ .

It follows symmetric property.

- 3) Let.  $(a, b) \in R$  and  $(b, c) \in R$ .  
 $\Rightarrow (a-b)$  is divisible by 6 and  $(b-c)$  is divisible by 6.  
 $\Rightarrow (a-b) + (b-c)$  is divisible by 6.  
 $\Rightarrow (a-c)$  is divisible by 6.  
 $\Rightarrow (a, c) \in R$ .

So, It follows transitive relation. So,  $R$  is an equivalence relation.

Q State whether following relations are equivalent or not.  
on the set of lines in a plane.

i)  $x \parallel y$

ii)  $x \perp y$ .

For equivalence relation reflexive, symmetric and transitive relation must satisfy.

i)  $x \parallel y$ .

Reflexive:- Every line is parallel to itself. i.e.  $x \parallel x$ .  
So, it is reflexive.

Symmetric:  $x \parallel y \Rightarrow y \parallel x$ . It follows symmetric relation.

Transitive:  $x \parallel y$  and  $y \parallel z \Rightarrow x \parallel z$ . It follows transitive relation. Hence, it is an equivalence relation.

ii)  $x \perp y$ .

Reflexive:- No line is  $\perp$  to itself. so, it does not follow reflexive property.

Hence, it is not an equivalence relation.

i) Show that  $R = \{(a,b) : a \equiv b \pmod{m}\}$  is an equivalence relation on  $\mathbb{Z}$ .

ii) Show also if  $x_1 \equiv y_1$  and  $x_2 \equiv y_2$  then  $(x_1 + x_2) \equiv (y_1 + y_2)$ .

iii)  $x_1 \equiv y_1 \Rightarrow (x_1 - y_1)$  is divisible by  $m$  ①

and  $x_2 \equiv y_2 \Rightarrow (x_2 - y_2)$  is divisible by  $m$  ②

Add eqn ① & ② -

$\Rightarrow ((x_1 - y_1) + (x_2 - y_2))$  is divisible by  $m$ .

$\Rightarrow (x_1 + x_2) - (y_1 + y_2)$  is divisible by  $m$ .

$\Rightarrow (x_1 + x_2) \equiv (y_1 + y_2) \pmod{m}$ .

$\therefore (x_1 + x_2) \equiv (y_1 + y_2)$ .

- Show whether relation on collection of sets.
- For equivalence relation reflexive, symmetric and transitive relation must satisfy.

For Reflexive Property - If  $(a,a) \in R$ .  
then,  $a \subseteq a \Rightarrow a \subseteq a$ .  
So, it is reflexive.

For symmetric property - If  $(a,b) \in R$ , then  $a \subseteq b$ .  
then,  $b \subseteq a \text{ i.e. } (b,a) \in R$ .

It does not follow symmetric property.

so, it is not an equivalence relation.

Also, it is not an equivalence relation on collection of sets.

- Q Check whether  $R = \{(a,b) : a \geq b\}$  on set of real numbers is an equivalence relation or not.

$$\rightarrow R = \{(a,b) : a \geq b\}.$$

For Reflexive Property :-

If  $(a,a) \in R$ , i.e.,  $a \geq a$ .  
 $\Rightarrow a \geq a$ .  
So, it is reflexive.

For symmetric property :-

If  $(a,b) \in R \Rightarrow a \geq b$ .  
but,  $b \neq a \Rightarrow (b,a) \notin R$ .

So, it is not a symmetric property.

Therefore, it is not an equivalence relation.

- Q Prove that if  $R$  is an equivalence relation then  $R^{-1}$  is also an equivalence on set  $A$ .

→ Given  $R$  is an equivalence relation.

i) For Reflexive Property :- If  $R$  is reflexive relation.

then  $(a,a) \in R$ .  
 $\Rightarrow (a,a) \in R^{-1}$

$R^{-1}$  is an eq. reflexive relation.

ii) For symmetric Property :- If  $R$  is symmetric relation.

then  $(a,b) \in R$  and  $(b,a) \in R$ .

then  $(b,a) \in R^{-1}$  and  $(a,b) \in R^{-1}$ .

So,  $R^{-1}$  is a symmetric relation.

- 3) for transitive property :- since  $R$  is transitive relation.  
then  $(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R$ .

so, then  $(b,a) \in R^{-1}$  and  $(c,b) \in R^{-1} \Rightarrow (c,a) \in R^{-1}$

so,  $R^{-1}$  is a transitive relation.

∴ Therefore,  $R^{-1}$  is an equivalence relation.

- Q If  $R$  and  $S$  are two equivalence relations then  $R \cap S$  is also an equivalence relation.

→ Given  $R$  and  $S$  is an equivalence relation.

- 1) for Reflexive relation :- If  $(a,a) \in R$ ,

and  $(a,a) \in S$ .

then,  $(a,a) \in R \cap S$ .

so, it is reflexive.

- 2) for symmetric property :-  $(a,b) \in R \& (b,a) \in R$ .  
 $(a,b) \in S \& (b,a) \in S$

For  $(R \cap S)$  :-  $(a,b) \in R \cap S \Leftrightarrow (b,a) \in R \cap S$ .

so, it is symmetric relation.

- 3) for transitive property :-  $(a,b) \in R \& (b,c) \in R \& (a,c) \in R$ .  
 $\Rightarrow (a,b) \in S \& (b,c) \in S \& (a,c) \in S$ .

then,  $(a,b) \in R \cap S \& (b,c) \in R \cap S \Rightarrow (a,c) \in R \cap S$ .

so, it is transitive relation.

Therefore, it is an equivalence relation.

- \* Equivalence Class :- Let  $R$  be an equivalence relation defined on set  $A$  the set of all elements that are related to an element  $a \in A$  is called equivalence class of that element  $a$ .

If  $a \in A$

then,  $[a] = \{b : aRb\}$ .

It is represented by  $[a]_R$  or  $[a]$ .

e.g.  $A = \{1, 2, 3\}$ .

$$R = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}.$$

$$\{1\} = \{1, 2\},$$

$$\{2\} = \{1, 2\},$$

$$\{3\} = \{3\}.$$

Properties :-

- $a \in [a]_R \quad \forall a \in A$ .
- $[a] = [b] \text{ iff } (a,b) \in R$ .
- $[a] \neq [b] \text{ or } [a] \cap [b] = \emptyset$ .

elements of set  $A$  under equivalence relation.

$R$  is called quotient set of  $A$  by  $R$ :  
It is denoted by  $[A/R]$  ( $A$  is induced by  $R$ ).

$$A = \{1, 2, 3\}.$$

$$R = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}.$$

$$[1] = \{1, 2\}.$$

$$[2] = \{1, 2\}$$

$$[3] = \{3\}$$

$$A/R = \{[1]_R, [2]_R, [3]_R\}.$$

\* **Partition :-** The partition of  $A$  set is a collection of mutually disjoint, non-empty subsets of  $A$  whose union is  $A$ .

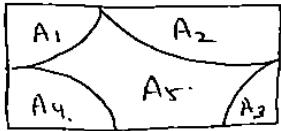
Partition so,  $P$  is called partition of  $A$  if -

- $P$  contains non-empty subset of  $A$  i.e.  $A_1, A_2, A_3, \dots, A_n$  where,  $A_i \subseteq A$  for  $i = \{1, 2, 3, \dots, n\}$ .
- Union of all subsets under partition  $P$  gives set  $A$  i.e.  $A_1 \cup A_2 \cup A_3 \cup A_4 \cup \dots \cup A_n = A$ .

$$\text{Q2} \quad \bigcup_{i=1}^n A_i = A.$$

- The elements of partition  $P$  are pairwise disjoint i.e.  $[A_i \cap A_j = \emptyset]$  or  $[A_i = A_j]$ .

**Theorem :-** Every equivalence relation on a set generates a unique partition of set. The blocks of this partition form equivalence classes.



Q Let  $R$  be defined on set  $A = \{1, 2, 3, 4\}$  as  $R = \{(a,b) : a-b \in 1\}$ . Find all the equivalence classes.

$$\rightarrow A = \{1, 2, 3, 4\}.$$

$$R = \{(1,1), (1,3), (2,2), (2,4), (3,1), (3,3), (4,2), (4,4)\}.$$

$$[1] = \{1, 3\}.$$

$$[2] = \{2, 4\}.$$

$$[3] = \{1, 3\}.$$

$$[4] = \{2, 4\}.$$

Q. Let  $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$  and let  $R$  be the relation ~~of~~  $\subseteq$   $A \times A$  defined by  $(a,b) R (c,d)$  iff  $a+d = b+c$ .

- Prove that  $R$  is an equivalence relation.
- Find  $[2, 5]_R$ .

(1) (i) Reflexive :-  $(a+b) = (a+b)$   
 $\Rightarrow (a,b) R (a,b)$ .  
 So, it is reflexive.

(ii) Symmetric :- Let  $(a,b) R (c,d)$ .  
 $\Rightarrow a+d = b+c$ .  
 $\Rightarrow b+c = a+d$ .  
 $\Rightarrow c+b = d+a$ .  
 $\Rightarrow (c,d) R (a,b)$ .  
 So, it is symmetric.

(iii) Transitive :- Let,  $(a,b) R (c,d)$ .  
 $\Rightarrow a+d = b+c \quad \dots \text{--- } ①$ .  
 and,  $(c,d) R (e,f)$ .  
 $\Rightarrow (c+f) = (d+e) \quad \dots \text{--- } ②$ .  
 Adding eqn ① & ②.  
 $a+d+e+f = c+d+b+e$ .  
 $a+f = b+e$ .  
 $\Rightarrow (a,b) R (e,f)$

So, it is transitive.

$\Rightarrow$  It is an equivalence relation.

(2)  $[2,5]_R = ?$

$(2,5) R (x,y)$

$$\Rightarrow 2+y = x+5$$

$$\Rightarrow y = x+3.$$

$$x=1, y=4$$

$$x=2, y=5$$

$$x=3, y=6$$

$$x=4, y=7$$

$$x=5, y=8$$

$$x=6, y=9$$

$$[2,5]_R = \{(1,4), (2,5), (3,6), (4,7), (5,8), (6,9)\}$$

### \* Closure of Relation :-

Reflexive closure :- Let  $R$  be a relation on set  $A$  then  
 $R \cup I_A$  is called reflexive closure of  $R$ .

$$I_A = \{(a,a) : \forall a \in A\}$$

$$R_F = R \cup I_A$$

Eg. Let  $A = \{1, 2, 3\}$   
 $R = \{(1,1), (1,2), (2,3)\}$

$$R_F = \{(1,1), (1,2), (2,3), (2,2), (3,3)\}$$

$R \cup R^{-1}$  is called symmetric closure of  $R$ .  
 where,  $R^{-1}$  is the inverse of relation  $R$ .

$$R_S = R \cup R^{-1}$$

e.g.  $A = \{1, 2, 3\}$ .

$$R = \{(1,1), (1,2), (2,3)\}.$$

$$R^{-1} = \{(1,1), (2,1), (3,2)\}.$$

$$R_S = \{(1,1), (1,2), (2,3), (2,1), (3,2)\}.$$

Transitive closure :- let  $R$  be a relation on set  $A$  then transitive closure  $R^*$  or  $R^+$  is equals to

$$R^* = R \cup R^2 \cup R^3 \cup R^4 \dots \cup R^n.$$

$$\text{or } R^* = \bigcup_{i=1}^n R^i.$$

where, ' $n$ ' is the number of elements in the set.

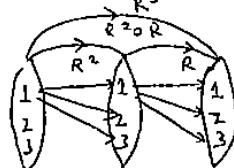
e.g.  $A = \{1, 2, 3\}$ .

$$R = \{(1,1), (1,2), (2,3)\}.$$

$$R_0 R = \{(1,2), (1,3), (1,1)\}.$$

$$R^2 \circ R = \{(1,1), (1,2), (1,3)\}.$$

$$R^3 = \{(1,1), (1,2), (1,3)\}.$$



## \* Pictorial Representation of Relation :-

A relation can be expressed in following ways-

- Relation as a Matrix.
- Relation as a Directed Graph (Di-graph).
- Relation as an Arrow Diagram.
- Relation as a Table.

### • Relation as a Matrix :-

Let,  $A = \{a_1, a_2, \dots, a_i, \dots, a_m\}$ .  $\leftrightarrow$   
 $B = \{b_1, b_2, \dots, b_j, \dots, b_n\}$ .

are finite sets & contain  $m$  and  $n$  elements respectively.

Let,  $R$  be a relation from set  $A$  to set  $B$  then the relation matrix of  $R$  is a  $m \times n$  matrix defined by -

$$M_R = [m_{ij}] = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin R \end{cases}$$

e.g.  $A = \{1, 2, 3\}$ .

$$R = \{(1,1), (1,2), (2,3), (3,3), (2,1)\}.$$

$$M_R = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}$$

This matrix is also called zero one matrix or Boolean matrix or adjacency matrix of relation  $R$ .

\*(1) If  $m_{ii} = 1$  then relation is reflexive i.e. main diagonal of matrix contain 1.

\*(2) If  $m_{ij} = m_{ji}$  then relation is symmetric.

$$\underline{\text{Q.E.D.}} \quad [M_R = M_R^T] \quad (T = \text{Transpose of Matrix}).$$

Q Let  $N = \{1, 2, 3, \dots\}$  and the relation  $R$  is defined in  $N \times N$  as follows.  $(a, b) R (c, d)$  iff  $ad = bc$ . Then show that  $R$  is an equivalence relation or not.

$\rightarrow$  1) Reflexive :-  $ab = ab \cdot a$ .

$$\Rightarrow (a, b) R (a, b)$$

So, it is reflexive.

2) Symmetric :- Let,  $(a, b) R (c, d)$

$$ad = bc.$$

$$bc = ad.$$

$$cb = da. \text{ But } da = cb$$

$$\Rightarrow (c, d) R (a, b).$$

So, it is symmetric.

3) Transitive : Let  $(a, b) R (c, d)$

$$ad = bc. \quad \text{--- (1)}$$

$$\& (c, d) R (e, f)$$

$$cf = de. \quad \text{--- (2)}$$

Multiply eqn (1) & (2)

$$ad \cdot cf = bc \cdot de$$

$$af = be.$$

$$af = ef.$$

$$(a, b) R (e, f)$$

So, it is transitive.

Therefore, it is an equivalence relation.

\*(3) A relation  $R$  is transitive if

$$[M_R^2 + M_R = M_R]$$

$$\begin{bmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & T \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = M_{R \wedge S}$$

$$\begin{bmatrix} T & T & T \\ T & T & T \\ T & T & T \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = M_{R \wedge S}$$

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \& \quad M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Find  $M_{R \wedge S}$ ,  $M_{R \oplus S}$ ,  $M_{R \ominus S}$ .

following matrices.

Set  $A = \{a, b, c\}$  how two relation  $R$ 's is represented by

Example of a relation :-  $M_R^{-1} = M_T^{-1}$

$$\begin{bmatrix} T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

if

$$\begin{bmatrix} T & T & T \\ T & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \wedge \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

eg

Inverse of Relation :-  $M_{R \wedge S}^{-1}$

by programming & meet of corresponding elements of two matrices can be obtained

Effect :- If it is denoted by  $\wedge$ ,  $\forall$ ,  
union of two Matrices:-  $M_{R \wedge S} = M_R \vee M_S$

join of two matrices can be obtained by performing join of corresponding elements of matrices.

$$\begin{array}{rcl} 0 \vee 0 & = & 0 \\ 0 \vee 1 & = & 1 \\ 1 \vee 0 & = & 1 \\ 1 \vee 1 & = & 1 \end{array}$$

Join :- It is denoted by  $\wedge$ ,  $\forall$ .

Meet ( $\wedge$ )  $\Leftrightarrow$  AND.

Join ( $\vee$ )  $\Leftrightarrow$  OR

thus we have two of relation union :-

$$M_{R \oplus S} = M_{R \cup S} - M_{R \cap S} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

{Non-zero element }  
formula :-

$$= \begin{bmatrix} 1+0+0 & 1+0+0 & 1+0+1 \\ 0+0+0 & 0+0+0 & 0+0+1 \\ 1+1+0 & 1+1+0 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

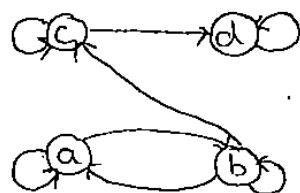
$$M_{R \ominus S} = M_{R \cup S} - M_{R \cap S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

x(R)   
 Non empty - S.   
 Non empty - S.

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

\* Relation as a Directed Graph (Di-Graph) :-

e.g.  $A = \{a, b, c, d\}$   
 $R = \{(a,a), (a,b), (b,a), (b,b), (b,c), (c,c), (c,d), (d,d)\}$



\* Partial order relation (POR) :- A relation R defined on set A is said to

be POR if it is -

- Reflexive  $\Rightarrow [(a,a) \in R, \forall a \in A]$
- Antisymmetric  $\Rightarrow [(a,b) \in R \text{ and } (b,a) \in R \Rightarrow a=b]$
- Transitive  $\Rightarrow [(a,b) \in R \text{ and } (b,c) \in R \Rightarrow (a,c) \in R]$

Any POR relation can be represented as ' $\leq$ '

\* Poset :- The set A along with POR is called Poset or Partially ordered set.

It is represented by  $(A, \leq)$ .

e.g. The relation " $\leq$ " is a POR on the set of integers 'Z'.  
 $(Z, \leq)$ .

- 1) Reflexive -  $a \leq a, \forall a \in Z$ .
- 2) Antisymmetric -  $a \leq b \text{ and } b \leq a \Rightarrow a=b$ .
- 3) Transitive -  $a \leq b \text{ and } b \leq c \Rightarrow a \leq c$ .

$\leq$  is a POR

$\geq$  is a POR

$\subset$  is a POR

defined on set A is said to be total order relation if and only if it is -

- Partial order relation.
- Either  $(a,b) \in R$  or  $(b,a) \in R$ , &  $a, b \in A$  : (It is called comparability).

e.g.  $R = \{(a,b) : a \leq b\}$  over the set of integers is a T.O.R

Reflexive,  $\rightarrow a \leq a \in R$ .

Antisymmetric  $\rightarrow a \leq b$  and  $b \leq a \Rightarrow a = b$ .

Transitive  $\rightarrow a \leq b$  and  $b \leq c \Rightarrow a \leq c$ .

It is POR.

\* Function :- Let X and Y be two non-empty sets. A function f from X to Y is a rule that assigns to each element  $x \in X$ , a unique element  $y \in Y$ . It is denoted by  $f: X \rightarrow Y$ .

function is also called Mapping, transformation or correspondence.

If  $x \in X$ ,  $y \in Y$  and  $f: X \rightarrow Y$  then we write  $f(x) = y$ .

If  $[X]$  has  $[m]$  elements and  $[Y]$  has  $[n]$  elements.

$$f: X \rightarrow Y \quad n^m \quad (\text{No. of functions})$$

$$f: Y \rightarrow X \quad m^n$$

$$f: X \rightarrow X \quad m^m$$

$(X \rightarrow Y)$

Note: Total no. of relations from one set to another set  $\gg$  Total no. of relations from one set to one set (same set)  $(X \rightarrow X)$

No. of relations  $>$  No. of functions. (Always).

$$2^{mn} > n^m$$

\* Domain and Codomain of a function :-

If  $f: X \rightarrow Y$

X should be domain & Y should be codomain

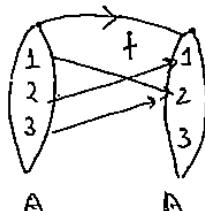
$$A = \{1, 2, 3\}$$

$$f: A \rightarrow A$$

$$f(1) = 2$$

$$f(2) = 1$$

$$f(3) = 2$$



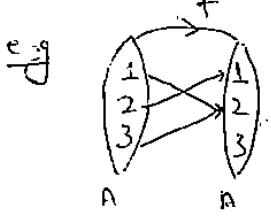
$$\text{Domain} = A = \{1, 2, 3\}$$

$$\text{Co-domain} = A = \{1, 2, 3\}$$

$y$  is image of  $x$ .

$x$  is preimage of  $y$ .

\* Range of a function :- Range is subset of Codomain.  
 $\text{Range} \subseteq \text{Codomain}$



$$\text{Codomain} = \{1, 2, 3\} = A.$$

$$\text{Range} = \{1, 2\}.$$

y is image of x.

x is preimage of y.

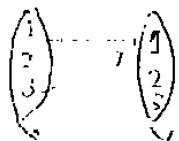
- Q. Let  $A = \{1, 2, 3, 4\}$ ,  $B = \{a, b, c, d\}$ , &  $f: A \rightarrow B$  such that  $f = \{f(1, a), f(2, c), f(3, d), \cancel{f(4, a)}\}$ . Find domain, codomain and range.

$$\text{Domain} = A = \{1, 2, 3, 4\}.$$

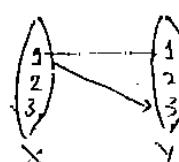
$$\text{Codomain} = B = \{a, b, c, d\}.$$

$$\text{Range} = \{a, c, d\}.$$

NOTE:



Not a function.  
(No element should  
be left in X)

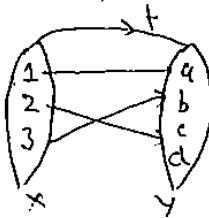


Not a function  
(No two images  
of single element  
is not possible)

### \* Types of Function :-

- One to One (Injective) Function :- Let  $f: X \rightarrow Y$  then  $f$  is called one to one function if for distinct elements of  $X$  there are distinct images in  $Y$ . i.e.  $f$  is one to one iff  $\boxed{f(x_1) = f(x_2) \Rightarrow x_1 = x_2}$ ,  $\forall x_1, x_2 \in X$ .

e.g.



$$\{(1, a), (2, b), (3, c)\}$$

- e.g. Let  $f(x) = 5x + 1$ . on set of integer.

$$\text{Let } f(x_1) = f(x_2).$$

$$\Rightarrow 5x_1 + 1 = 5x_2 + 1$$

$$\Rightarrow 5x_1 = 5x_2$$

$$\Rightarrow \boxed{x_1 = x_2}.$$

- Onto (surjective) Function :- Let  $f: X \rightarrow Y$  then  $f$  is called onto function if

for every element  $y \in Y$  there is an element  $x \in X$  such that  $\boxed{f(x) = y}$ . (or) function  $f$  is onto if  $\boxed{\text{codomain} = \text{range}}$ .

- $\Rightarrow f(x) = 5x+1$  over the set of real numbers.  
 $f: \mathbb{R} \rightarrow \mathbb{R}$
- $\Rightarrow f(x) = 5x+1$
- $\Rightarrow y = 5x+1$
- $\Rightarrow x = \frac{y-1}{5}$  &  $y \in \mathbb{R} \Rightarrow x \in \mathbb{R}$ . (there exist  $\exists x \in \mathbb{R}$ )
- $f(x) = 5x+1$  over the set of integer no's  $f: \mathbb{Z} \rightarrow \mathbb{Z}$
- $\Rightarrow f(x) = 5x+1$
- $\Rightarrow y = 5x+1$
- $\Rightarrow x = \frac{y-1}{5}$  &  $y \in \mathbb{Z} \Rightarrow x \notin \mathbb{Z}$ . (there does not exist  $\exists x \in \mathbb{Z}$ ).

- One to one onto (Bijective) function :- A function which reaches one to one and onto both is called bijective function.

Let  $f(x) = 5x+1$  on set of real no's.

$$\begin{aligned}\Rightarrow \text{Let } f(x_1) &= f(x_2) \\ \Rightarrow 5x_1 + 1 &= 5x_2 + 1 \\ \Rightarrow 5x_1 &= 5x_2 \\ \Rightarrow x_1 &= x_2\end{aligned}$$

Let,  $f(x) = 5x+1$ .

$$\begin{aligned}\Rightarrow y &= 5x+1 \\ x &= \frac{y-1}{5} \quad \& y \in \mathbb{R} \Rightarrow \text{there exist } \exists x \in \mathbb{R}\end{aligned}$$

- Into function :- A function which is not onto is called into function.

- Identity function :- Let  $f(x) : x \rightarrow x$  function  $f$  is called identity function if  $[f(x) = x] \forall x \in X$ .

If image of every element is element itself, then it is called Identity function.

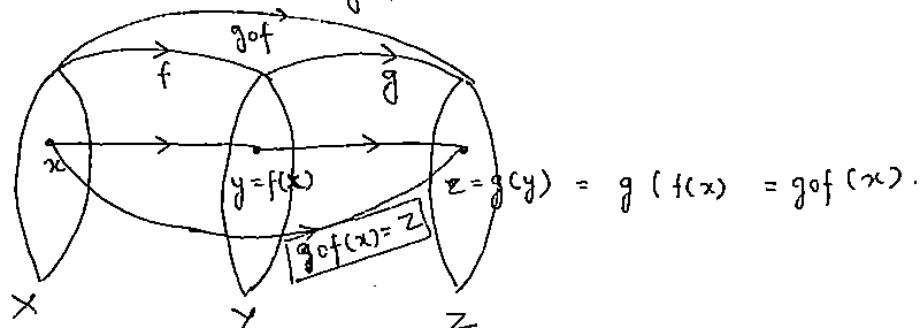
It is denoted by  $I_x$ .

- Composition of function :-

$$f : x \rightarrow y. \quad f(x) = y$$

$$g : y \rightarrow z. \quad g(y) = z$$

$$gof : x \rightarrow z. \quad gof(x) = z$$



- Composition of function is not commutative.  
i.e.  $[fog \neq gof]$
- Composition of function is associative.  
i.e.  $[(h \circ g) \circ f = h \circ (g \circ f)]$

Proof Let,  $f(x) = y$  let,  $f: x \rightarrow y$   
 $g(y) = z$ .  $g: y \rightarrow z$   
 $h(z) = w$   $h: z \rightarrow w$ .

$$\begin{array}{ll}
 \text{LHS} & (h \circ g) \circ f \\
 \Rightarrow & (h \circ g) \circ f(x) \\
 \Rightarrow & (h \circ g) f(x) \\
 \Rightarrow & (h \circ g)y \\
 \Rightarrow & h(g(y)) \\
 \Rightarrow & h(z) \\
 \Rightarrow & w \\
 \text{RHS} & h \circ (g \circ f) \\
 \Rightarrow & h \circ (g \circ f)(x) \\
 \Rightarrow & h \circ (g(f(x))) \\
 \Rightarrow & h \circ (g(y)) \\
 \Rightarrow & h(z) \\
 \Rightarrow & w
 \end{array}$$

LHS = RHS. (Hence Proved). Function is associative.

Q.  $f(x) = 3x + 2$ . on the set of real no.'s. find  $gof$  &  
 $g(x) = 4x + 8$  fog. [gof]

$$\begin{array}{ll}
 \text{fog}(x) = f(g(x)) & \text{gof}(x) = g(f(x)) \\
 = 3(g(x)) + 2 & = 4(f(x)) + 8 \\
 = 3(4x + 8) + 2 & = 4(3x + 2) + 8 \\
 = 12x + 24 + 2 & = 12x + 8 + 8 \\
 \boxed{\text{fog}(x) = 12x + 26.} & \boxed{\text{gof}(x) = 12x + 16.}
 \end{array}$$

Q. Let  $h, g$  &  $f$  be functions from  $N$  to  $N$ . such that  
 $f(x) = x+1$ ,  $g(x) = 2x$  &  $h(x) = \begin{cases} 0, & \text{if } x \text{ is even} \\ 1, & \text{if } x \text{ is odd} \end{cases}$ .

Find  $fof$ ,  $gof$ ,  $fog$ ,  $goh$ ,  $hog$ ,  $(fog)oh$ ,  $go(fog)oh$ .

$$\begin{aligned}
 \rightarrow fof(x) &= f(f(x)) \\
 &= f(x+1) + 1 \\
 &= x+1+1
 \end{aligned}$$

$$fof(x) = x+2$$

$$\begin{aligned}
 gof(x) &= g(f(x)) \\
 &= 2(f(x)) \\
 &= 2(x+1) = 2x+2.
 \end{aligned}$$

$$\begin{aligned}
 fog(x) &= f(g(x)) \\
 &= g(x)+1 = 2x+1.
 \end{aligned}$$

$$j_{uv} = \frac{1}{2} \ln \frac{1}{\sin \theta}$$

$\Rightarrow$  if  $x$  is even.

$$\begin{cases} g_{0h} = 0 & \text{if } x \text{ is odd,} \\ g_{0h} = 2 \times 1 = 2. \end{cases}$$

$$h \circ g(x) = h(g(x))$$

$$h \circ g(x) = 0$$

if  $x$  is even.

$$(f \circ g) \circ h \cdot (x) = \begin{cases} f(f \circ g)(h(x)) & \text{if } x \text{ is even} \\ (f \circ g)(h(x)) & \text{if } x \text{ is odd} \end{cases}$$

$$= \begin{cases} (f \circ g)^0 & \text{if } x \text{ is even} \\ (f \circ g)^1 & \text{if } x \text{ is odd.} \end{cases}$$

$$= \begin{cases} f \circ (g(0)) & \text{if } x \text{ is even} \\ f \circ (g(1)) & \text{if } x \text{ is odd.} \end{cases}$$

$$= \begin{cases} f^{(0)} & \text{if } x \text{ is zero.} \\ f^{(2)} & \text{if } x \text{ is 2.} \end{cases}$$

$$= \begin{cases} 1 & \\ 3 & \end{cases}$$

$$g_0 \circ (f \circ g) \circ h = \begin{cases} g_0(f \circ g)(h(x)) & \text{if } x \text{ is even} \\ g_0(f \circ g)(h(x)) & \text{if } x \text{ is odd.} \end{cases}$$

$$= \begin{cases} \cancel{g_0 \circ f} \circ g_0 \circ f^{(0)} & \text{if } x \text{ is even} \\ g_0 \circ f^{(1)} & \text{if } x \text{ is odd.} \end{cases}$$

$$= \begin{cases} g_0 \circ f^{(0)} & \text{if } x \text{ is even} \\ g_0 \circ f^{(2)} & \text{if } x \text{ is odd.} \end{cases}$$

$$= \begin{cases} g^{(1)} & \text{if } x \text{ is even} \\ g^{(3)} & \text{if } x \text{ is odd.} \end{cases}$$

$$= \begin{cases} 2 & \\ 6 & \end{cases}$$

$$g \circ f(x) = x^2 + 2$$

$$h + (x^2 - 2) =$$

$$(x^2 + 2) g = (x^2 + 2)$$

$$f \circ g(x) = x^2 + 8x + 14$$

$$= x^2 + 16 + 8x - 2$$

$$= (x+4)^2 - 2$$

$$f \circ g(x) = f(g(x))$$

These functions are injective, surjective + bijective.

Q Let  $f: R \rightarrow R$  &  $g: R \rightarrow R$  where  $R$  is the set of real numbers. Find  $f \circ g$  &  $g \circ f$  where  $f(x) = x^2 - 2$  &  $g(x) = x + 4$ . State whether  $f \circ g$  &  $g \circ f$  are injective, surjective + bijective.

$$h_I = (h \circ g \circ f) \Leftarrow$$

$$h = (h \circ g \circ f)$$

$$(x) f =$$

$$(h \circ g \circ f) = (h \circ g \circ f) \quad \text{as}$$

$$h \leftarrow h : g \circ f$$

$$x_I = (x) f \circ g \Leftarrow$$

$$x = (x) f \circ g$$

$$(h \circ g \circ f) =$$

$$(h \circ g \circ f) = (x) f \circ g \quad \text{now, } g \circ f$$

$$x \leftarrow x : f \circ g$$

$$x = h \circ g \circ f \quad \text{as } x \in X \text{ then } x \neq y \in Y \text{ such that } g(y) = x$$

$$h \circ g \circ f : y \rightarrow y \quad \text{as } y \in Y \text{ then } y \neq z \in Y \text{ such that } f(z) = y$$

*Proof:* Function  $f$  &  $g$  should be bijective functions.

$$x_I = f \circ g \circ f \quad \text{as } f \circ g \circ f = I_Y = g \circ f \circ f$$

Inverse of  $f \circ g \circ f$  is  $I_R$ . If  $x \leftarrow y$  &  $y \leftarrow z$  then  $g$  is called

$$h_{I_R} = (h)_{I_R} \circ f \Leftarrow$$

$$h_{I_R} = x \Leftarrow$$

$$h_{I_R} = x \Leftarrow$$

$$x = h_{I_R} \Leftarrow \frac{x}{x} = R \Leftarrow$$

$$\frac{x}{x} = (x) f \quad \text{if}$$

$$x = (h)_{I_R} \circ f \quad \text{as } h = (x) f \quad \text{i.e. } I_R$$

Inverse of  $f$  denoted by  $f^{-1}$

domain of  $f$  such that  $f(x) = y$

If  $f$  is the function that assigns an element  $y \in Y$  to  $x \in X$  such that  $f(x) = y$  then  $f$  is a unique

function  $f$  is a bijective function.

From  $x$  to  $y$  the same

domain  $X$  such that  $f(x) = y$  for some  $x \in X$  such that  $f(x) = y$

element  $y \in Y$  such that  $f(x) = y$

unique function

so, if it is also not bijective  
so, it is not onto.

∴ if  $y \in \mathbb{Z}$  therefore there is no such  $x$ .

$$\Leftrightarrow x_2 = y - 2$$

$$\Leftrightarrow y = x_2 + 2$$

$$\text{let, } g_{0f} = x^2 + 2$$

for Bijective :-

If is not onto to one function

$$x_1 \neq x_2 \Leftrightarrow x_1 = tx_2$$

$$\Leftrightarrow x_1 - x_2 = x_2 - x_1$$

$$\Leftrightarrow x_1 - x_2 = x_2^2 + 2$$

$$\text{let, } g_{0f}(x_1) = g_{0f}(x_2)$$

for Non-Bijective

If is also not bijective function.

so, it is not onto function

$$\Leftrightarrow y + 2 < 0$$

$$\text{so, } y < -2$$

$$\Leftrightarrow (x+4)^2 > 0$$

$$\Leftrightarrow (x+4)^2 = y + 2$$

$$\Leftrightarrow y = (x+4)^2 - 2$$

$$\Leftrightarrow y = x^2 + 8x + 14 + 2 - 2$$

$$\Leftrightarrow y = x^2 + 8x + 14$$

for Bijective

If is not onto to one function

$$x_1 \neq x_2 \Leftrightarrow$$

$$\Rightarrow 20 = +20$$

$$\Rightarrow 2(a+8) = -10(-10+8)$$

$$\text{let, } x_1 = 2, x_2 = 8 - 10$$

$$\Leftrightarrow x_1(x_1+8) = x_2(x_2+8)$$

$$\Leftrightarrow x_1^2 + 8x_1 + 16 = x_2^2 + 8x_2 + 16$$

$$\text{let, } f_{0g}(x_1) = f_{0g}(x_2)$$

- Q)  $f: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ ,  $f(x) = x^4$ .  
 $f: \mathbb{Z} \rightarrow \mathbb{Z}$ ,  $f(x) = x + x^2$ .  
 $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = \sin x$ .  
 $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = |x|$

Whether these functions are injective or not.

1) Let,  $f(x_1) = f(x_2)$

$$x_1^4 = x_2^4$$

$$x_1^4 = \pm x_2$$

but,  $f: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$

$$\Rightarrow \boxed{x_1 = x_2} \text{ for } \mathbb{Z}^+ \text{ set it is one to one.}$$

2) Let,  $f(x_1) = f(x_2)$

$$\Rightarrow x_1 + x_1^2 = x_2 + x_2^2$$

$$\Rightarrow x_1(1+x_1) = x_2(1+x_2)$$

Let,  $x_1 = 1$  &  $x_2 = -2$ .

$$\Rightarrow 1(1+1) = -2(1-2)$$

$$\Rightarrow 2 = 2$$

$$\Rightarrow \boxed{x_1 \neq x_2} \text{ so, it is not one to one.}$$

3) Let,  $f(x_1) = f(x_2)$

$$\Rightarrow \sin x_1 = \sin x_2$$

Let,  $x_1 = 0$ ,  $x_2 = \pi$ .

$$\Rightarrow \sin 0 = \sin \pi$$

$$\Rightarrow 0 = 0$$

$$\Rightarrow \boxed{x_1 \neq x_2} \text{ so, it is not one to one.}$$

4) Let,  $f(x_1) = f(x_2)$

$$\Rightarrow |x_1| = |x_2|$$

$$\Rightarrow \boxed{x_1 = \pm x_2} \text{ so, it is not one to one.}$$

Q) State whether following functions are onto or not.

1)  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = e^x$ .

2)  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = \sin x$ .

3)  $f: \mathbb{Z} \rightarrow \mathbb{Z}$ ,  $f(x) = x^2 + 4x - 51$ .

1)  $f(x) = e^x$ .

$$\Rightarrow y = e^x$$

$$\Rightarrow \boxed{y = \log x}$$

$\therefore \log$  is not defined for  $-ve$  no's. So, for  $-ve$  value of  $y$   $x$  does not exist. So, it is not onto.

$$2) y = \frac{x+1}{x-1}$$

$$\boxed{x = \sin^{-1}y}$$

For  $-1 \leq y \leq 1$   $\sin^{-1}y$  does not have any value.

So, it is not onto.

$$3) y = x^2 + 14x - 51.$$

$$y = x^2 + 14x - 51 + 49 - 49.$$

$$y = (x+7)^2 - 100.$$

$$(x+7)^2 = y+100.$$

$$\Rightarrow (x+7)^2 \geq 0.$$

$$\text{for } y < -100.$$

$$\Rightarrow y+100 < 0.$$

so, it is not onto.

\* Theorem :- If  $f : x \rightarrow y$  &  $g : y \rightarrow z$  are invertible functions  
then  $gof : x \rightarrow z$  is also invertible function and

$$\boxed{(gof)^{-1} = f^{-1} \circ g^{-1}}$$

Proof Since  $f$  and  $g$  are invertible therefore they are bijective functions. so,  $gof : x \rightarrow z$  is bijective. Hence,  $gof$  is also invertible.

$f : x \rightarrow y$  &  $y \in Y$  there exist  $\exists x \in X$  such that  $f(x) = y \Rightarrow f^{-1}(y) = x$   
 $g : y \rightarrow z$  &  $z \in Z$  there exist  $\exists y \in Y$  such that  $g(y) = z \Rightarrow g^{-1}(z) = y$

Since,  $gof : x \rightarrow z$  is bijective and invertible.

$$\Rightarrow \boxed{gof(x) = z}.$$

$$\Rightarrow \boxed{(gof)^{-1}(z) = x}$$

(OR)

$$\begin{aligned} gof(x) &= g(f(x)) \\ &= g(y) \end{aligned}$$

$$gof(x) = z$$

$$\Rightarrow \boxed{(gof)^{-1}(z) = x}. \quad \text{--- ①}$$

$$\begin{aligned} f^{-1} \circ g^{-1}(z) &= f^{-1}(g^{-1}(z)) \\ &= f^{-1}(y) \end{aligned}$$

$$\Rightarrow \boxed{f^{-1} \circ g^{-1}(z) = x.} \quad \text{--- ②} \quad \text{from eqn ① \& ②.}$$

$$\therefore \boxed{(gof)^{-1}(z) = (f^{-1} \circ g^{-1})(z)}$$

Q) show that the function  $f(x) = x^5$  and  $g(x) = x^{1/5}$  are inverse of each other over the set of real numbers.

$$\Rightarrow fog(x) = \boxed{f(g(x))}$$

$$fog(x) = (x^{\frac{1}{5}})^6$$

$$\Rightarrow \boxed{fog(x) = x} \Rightarrow \boxed{fog = Ix}$$

$$gof(x) = g(f(x))$$

$$= (x^5)^{\frac{1}{2}}$$

$$\Rightarrow \boxed{gof(x) = x} \Rightarrow \boxed{gof = Ix}$$

Since,  $fog = gof = Ix$

Therefore, they are inverse of each other.

Q Let,  $x = \{1, 2, 3\}$  a function  $f: x \rightarrow x$  is defined as  $f = \{(1, 2), (2, 1), (3, 3)\}$ , find  $f^2, f^3, f^{-1}$ .

$$f^{-1} = \{(2, 1), (1, 2), (3, 3)\}.$$

$$f(1) = 2$$

$$f(2) = 1$$

$$f(3) = 3$$

$$f^2 = fog \Rightarrow fog(1) = f(f(1)) = f(2) = 1.$$

$$fog(2) = f(f(2)) = f(1) = 2.$$

$$fog(3) = f(f(3)) = f(3) = 3.$$

$$f^2 = fog = \{(1, 2), (2, 1), (3, 3)\}.$$

$$f^3 = f^2 \circ f \Rightarrow f^2 \circ f(1) = f^2(f(1)) = f^2(2) = 2.$$

$$\Rightarrow f^2 \circ f(2) = f^2(f(2)) = f^2(1) = 1.$$

$$\Rightarrow f^2 \circ f(3) = f^2(f(3)) = f^2(3) = 3.$$

$$f^3 = f^2 \circ f = \{(1, 2), (2, 1), (3, 3)\}.$$

\* Permutation Function :- Let, A be a finite set the set of all bijective functions from A to A is called the set of permutation functions from A to A. If set A contains n elements then there are  $n!$  different permutations on A.

The matrix form to represent a permutation function on a finite set is to list all the elements of the set across the top row and the images of each element exactly below it.

$$\text{Consider } A = \{x_1, x_2, x_3, \dots, x_n\}.$$

Let, 'P' be a permutation on set A. then

$$P = \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ P(x_1) & P(x_2) & P(x_3) & \dots & P(x_n) \end{pmatrix}$$

where,  $P(x_1), P(x_2), P(x_3), \dots, P(x_n) \in A$ . ( $P(x_1), P(x_2), \dots, P(x_n)$  are images.)

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad P = \{P_1, P_2\}.$$

Inverse of permutation :-

$$A = \{1, 2, 3, 4\}.$$

$$(OR) \quad P(1)=2, \quad P(2)=3, \quad P(3)=1, \quad P(4)=4$$

$$P^{-1}(2)=1, \quad P^{-1}(3)=2, \quad P^{-1}(1)=3, \quad P^{-1}(4)=4.$$

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 2 & 3 & 1 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} 4 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

Composition or Product of Permutation :-

Let  $P_1$  &  $P_2$  be two permutations then composition of  $P_1$  &  $P_2$  denoted by  $P_1 \circ P_2$  is another permutation defined by

$$[P_1 \circ P_2](x) = P_1(P_2(x))$$

$$\text{eg } A = \{a, b, c\}.$$

$$P_1 = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}, \quad P_2 = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$$

$$P_1 \circ P_2 = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$$

$$P_1 \circ P_2 = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}$$

$$P_1 \circ P_2(a) = P_1(P_2(a)) = P_1(b) = b.$$

$$P_1 \circ P_2(b) = P_1(P_2(b)) = P_1(c) = a.$$

$$P_1 \circ P_2(c) = P_1(P_2(c)) = P_1(a) = c.$$

$$P_2 \circ P_1 = \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}$$

$$P_2 \circ P_1 = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}$$

$$Q \quad A = \{5, 6, 7\}. \quad P_1 = \begin{pmatrix} 5 & 6 & 7 \\ 6 & 5 & 7 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 5 & 6 & 7 \\ 7 & 5 & 6 \end{pmatrix}$$

$$P_1 \circ P_2 = \begin{pmatrix} 5 & 6 & 7 \\ 7 & 6 & 5 \end{pmatrix} \quad \text{find } P_1^{-1} \circ P_2^{-1}, \quad P_1 \circ P_2.$$

$$P_1 \circ P_2 = \begin{pmatrix} 5 & 6 & 7 \\ 7 & 5 & 6 \end{pmatrix}$$

$$P_1(5) = 6, \quad P_1(6) = 5, \quad P_1(7) = 4 \\ P_1^{-1}(6) = 5, \quad P_1^{-1}(5) = 6, \quad P_1^{-1}(4) = 7.$$

$$P_1^{-1} = \begin{pmatrix} 5 & 6 & 7 \\ 6 & 5 & 4 \end{pmatrix}$$

$$P_2^{-1} = \begin{pmatrix} 5 & 6 & 7 \\ 6 & 7 & 5 \end{pmatrix}$$

$$P_1^{-1} \circ P_2 = \begin{pmatrix} 5 & 6 & 7 \\ 7 & 6 & 5 \end{pmatrix}$$

$$P_2^{-1} \circ P_1^{-1} = \begin{pmatrix} 5 & 6 & 7 \\ 6 & 5 & 7 \\ 7 & 6 & 5 \end{pmatrix}$$

$$P_2^{-1} \circ P_1^{-1} = \begin{pmatrix} 5 & 6 & 7 \\ 7 & 6 & 5 \end{pmatrix}$$

\* **Mathematical Induction** :- mathematical induction is a technique of proving a proposition over the positive integers.

A formal statement of principle of mathematical induction can be stated as follows.

Let,  $P(n)$  be ~~a~~ a statement over the +ve integers then

- Inductive Base : show that  $P(1)$  is true for  $n \geq 1$ .  
 $P(n_0)$  is true for  $n \geq n_0$ .

- Inductive Hypothesis : let  $P(k)$  is true for  $n=k$ . (arbitrary  $k$ )

- Inductive Step : verify  $P(k+1)$  is true for  $n=k+1$  using Inductive Hypothesis.

Q Show that sum of square of first  $n$  +ve integers is

$$\frac{n(n+1)(2n+1)}{6}$$

$$\rightarrow \text{Let, } P(n) : 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

- 1) RHS for  $n=1$ ,  $P(1)$ , LHS =  $1^2 = 1$

$$\text{Inductive Base :-} \quad \text{RHS} = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1(2)(3)}{6} = \frac{k!}{k}$$

$$\therefore \text{RHS} = 1.$$

$$\text{LHS} = \text{RHS.}$$

$P(1)$  is verified.

Let  $P(k) = 1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$  is true.

### 3) Inductive Step :-

Now,  $R(k \rightarrow n)$  for  $n = k+1$

$$\begin{aligned}
 LHS &= 1^2 + 2^2 + 3^2 + \dots + (k+1)^2 \\
 &= 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 \\
 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \{ \text{By using step 2} \} \\
 &= (k+1) \left[ \frac{k(2k+1) + 6(k+1)}{6} \right] \\
 &= \left( \frac{k+1}{6} \right) [ 2k^2 + k + 6k + 6 ] \\
 &= \left( \frac{k+1}{6} \right) [ 2k^2 + 7k + 6 ] \\
 &= \left( \frac{k+1}{6} \right) [ 2k(k+2) + 3(k+2) ] \\
 &= \frac{(k+1)(2k+3)(k+2)}{6}.
 \end{aligned}$$

$\therefore P(k+1)$  is true. Proved.

① Prove by Mathematical Induction  $1 \cdot 1! + 2 \cdot 2! + \dots + n \cdot n! = (n+1)! - 1$

→ Let,  $P(n) : 1 \cdot (1!) + 2 \cdot (2!) + \dots + n \cdot (n!) = (n+1)! - 1$ .

### 1) Inductive Base:-

For,  $n=1$ ,  $LHS = 1 \cdot (1!) = 1 \cdot 1 = 1$

$RHS = (n+1)! - 1 = 2! - 1 = 2 - 1 = 1$ .

$LHS = RHS$ .

Hence,  $P(1)$  is true.

### 2) Inductive Hypothesis:-

Let,  $P(k) = k^2 + k^2 + \dots + k \cdot (k!) = (k+1)! - 1$  is true.

### 3) Inductive Step:-

$P(k+1) : 1 \cdot (1!) + 2 \cdot (2!) + \dots + (k+1) \cdot (k+1)! = (k+2)! - 1$ .

Now, For  $n = k+1$ .

$$\begin{aligned}
 LHS &= 1 \cdot (1!) + 2 \cdot (2!) + \dots + (k+1) \cdot (k+1)! \\
 &= 1 \cdot (1!) + 2 \cdot (2!) + \dots + k \cdot (k!) + (k+1) \cdot (k+1)! \\
 &= (k+1)! - 1 + (k+1) \cdot (k+1)! \\
 &= (k+1)! [ 1 + k+1 ] - 1 \\
 &= (k+1)! (k+2) - 1 \\
 &= (k+2)! - 1 \\
 &= RHS.
 \end{aligned}$$

$\therefore P(k+1)$  is true. Hence,  $(P(n))$  is true for natural no.'s.

$$B = 11.11k^{+2} + 12^2 \cdot 12k^{+2} + 33.33k^{+2} + 44.44 \cdot 12k^{+2}$$

3) Inductive Step :-

a) Implement the HyPath algorithm.

• 132. 13. 13. 13. 13. 13. 13. 13. 13.

$$\log n = 1, \quad P(1) = 11^8 + 12^3 = 3659, \quad 133 \times 25$$

### 4) Industrial Bar :-

(Q) Prove that  $\int_{n+2}^{n+1} \lambda^m dx$  is divisible by 133

$P(k+1)$  is true.

$$= \frac{RHS}{10^{k+2} - 9(k+1) - 10}$$

$$\frac{t^2}{(t+1)^2} = \left(\frac{t}{t+1}\right)^2$$

٧٢

$$\frac{10k+1 - 9k - 10 + 9 \cdot 10k + 1}{3} = \frac{2 + 89k}{3}$$

$$= 10^{k+1} - 10^k \cdot 10 = 10^{k+1}$$

$$b = -bb + \frac{ab - b}{1+b}$$

$$\sum_{k=1}^{\infty} \frac{(1+k)^2}{3^k} = \frac{1}{3} + \frac{4}{9} + \frac{9}{27} + \dots$$

Step :- For  $n = k+1$ .

$$3 + \frac{3}{3 + \frac{3}{3 + \dots}}} = 3$$

(2) its future

$$RHS = \frac{10^2 - 9 - 10}{2t} = \frac{1}{16t - 1}$$

- Square

$$\begin{array}{r} \text{---} \\ - 333 \\ + 333 \\ \hline \text{---} \end{array} \quad : 3 + 33 + 333 = 369$$

$$\begin{aligned}
 &= 11 \cdot 11^{2k+2} + 133 \cdot 12^{2k+1} + 11 \cdot 12^{2k+1} \\
 &= 11(11^{2k+2} + 12^{2k+1}) + 133 \cdot 12^{2k+1} \\
 &= 11 \cdot 133 \cdot m + 133 \cdot 12^{2k+1}, \quad \{ \therefore m \in \mathbb{Z}^+ \} \\
 &= (11m + 12^{2k+1}) \cdot 133. \\
 &= 133 \cdot s \quad \{ \therefore s \in \mathbb{Z}^+ \}.
 \end{aligned}$$

So,  $P(k+1)$  is divisible by 133.

(Q1) Prove that  $8^m - 3^m$  is divisible by 5.

(Q-2) Prove that  $n^5 - n$  is divisible by 5.

(Q-3) Prove that  $n^3 + 2n$  is divisible by 3.

→  $P(m)$ :  $8^m - 3^m$ , is divisible by 5.

1) Inductive Base:-

$$\text{for } m=1, P(1) \text{ is } 8^1 - 3^1 = 5$$

∴  $P(1)$  is divisible by 5.

2) Inductive Hypothesis:-

Let,  $P(k)$ :  $8^k - 3^k$  is divisible by 5.

3) Inductive Step:-

$P(k+1)$ :  $8^{k+1} - 3^{k+1}$  is divisible by 5.

$$= 8 \cdot 8^k - 3 \cdot 3^{k+1}.$$

$$= (5+3) 8^k - 3 \cdot 3^k.$$

$$= 5 \cdot 8^k + 3 \cdot 8^k - 3 \cdot 3^k.$$

$$= 5 \cdot 8^k + 3(8^k - 3^k).$$

$$= 5 \cdot 8^k + 3 \cdot 5 \cdot s. \quad \{ \therefore s \in \mathbb{Z}^+ \}$$

$$= 5(8^k + 3s).$$

$$= 5 \cdot m. \quad \{ \therefore m \in \mathbb{Z}^+ \}.$$

∴  $P(k+1)$  is true i.e. it is divisible by 5.

(2).  $P(n)$ :  $n^5 - n$  is divisible by 5.

1) Inductive Base:-

$$\text{for } n=1, P(1) = (1)^5 - 1 = 0.$$

∴  $P(1)$  is divisible by 5.

2) Inductive Hypothesis:-

$P(k)$ :  $k^5 - k$  is divisible by 5.

3) Inductive Step:-

$P(k+1) = (k+1)^5 - (k+1)$  is divisible by 5.

$$= (s_{c_0} + s_{c_1}k + s_{c_2}k^2 + s_{c_3}k^3 + s_{c_4}k^4 + s_{c_5}k^5) - (k+1)$$

$$= (1 + 5k + 10k^2 + 10k^3 + 5k^4 + k^5) - k - 1.$$

$$= (k^5 - k) + (5k^4 + 10k^3 + 10k^2 + 5k).$$

$$\begin{aligned}
 &= 5m + 5(k^4 + 2k^3 + 2k^2 + k) \\
 &= 5m + 5p \\
 &= 5(m+p) \\
 &= 5x
 \end{aligned}$$

$\therefore m+p \in \mathbb{Z}^+$   
 $\therefore x \in \mathbb{Z}^+$

$\therefore P(k+1)$  is divisible by 5.  $\therefore x \in \mathbb{Z}^+$

3)  $P(n): n^3 + 2n$  is divisible by 3.

1) Inductive Base:-

$$\text{For, } n=1, P(1) : 1^3 + 2(1) = 3$$

$\therefore P(1)$  is divisible by 3.

2) Inductive Hypothesis:-

Let,  $P(k): k^3 + 2k$  is divisible by 3.

3) Inductive Step:-

$P(k+1) : (k+1)^3 + 2(k+1)$  is divisible by 3.

$$\begin{aligned}
 &= (k+1)(k+1)^2 + 2(k+1) \\
 &= k^3 + 1 + 3k^2 + 3 + 2k + 2 \\
 &= (k^3 + 2k) + 3k^2 + 6 \\
 &= 3 \cdot m + 3(k^2 + 2) \\
 &= 3 \cdot m + 3 \cdot s \\
 &= 3(m+s)
 \end{aligned}$$

$\therefore P(k+1)$  is divisible by 3.

Q Using MI show that  $2^n < n!$  for  $n \geq 4$ .

$\rightarrow P(n): 2^n < n!$

1) Inductive Base:-

$$\begin{aligned}
 \text{For, } n=4, P(4) : 2^4 < 4! \\
 16 < 24.
 \end{aligned}$$

$\therefore P(4)$  is true.

2) Inductive Hypothesis :-

Let,  $P(k): 2^k < k!$  is true for  $k \geq 4$ .

3) Inductive Step:-

$P(k+1) : 2^{k+1} < (k+1)!$  is true for  $(k+1) \geq 4$ .

Using Inductive Hypothesis-

$$2^k < k!$$

$$\Rightarrow 2^k \cdot 2 < 2(k!)$$

$$\Rightarrow 2^{k+1} < 2(k!) \quad \text{--- (1)}$$

Again,  $k \geq 4$

$$\Rightarrow (k+1) \geq 5$$

$$\Rightarrow (k+1) > 2$$

$$\Rightarrow (k+1)(k!) > 2(k!).$$

(Ans)  $\therefore$  true.  $\square$

From eqn ① & ②.

$$\Rightarrow 2^{k+1} < (k+1)!$$

$\therefore P(k+1)$  is true.

Q Prove that  $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} > \sqrt{n}$  for  $n \geq 2$ .

$$\rightarrow P(n) : \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

1) Inductive Base:-

$$\text{For } n=2, P(2) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1.707\ldots$$

$$\sqrt{2} = 1.414.$$

$$\Rightarrow 1.707 > 1.414.$$

$\therefore P(2)$  is true.

2) Inductive Hypothesis:-

$P(k) : \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{k}} > \sqrt{k}$  is true for  $k \geq 2$

3) Inductive Step:-

$P(k+1) : \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$  is true for  $k+1 \geq 2$

Using Inductive Hypothesis -

$$\Rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} > \sqrt{k}$$

$$\Rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$$

$$\Rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \frac{\sqrt{k} \cdot \sqrt{k+1} + 1}{\sqrt{k+1}}$$

$$\Rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \frac{\sqrt{k} \cdot \sqrt{k+1}}{\sqrt{k+1}} \quad \therefore \sqrt{k+1} >$$

$$\Rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \frac{k+1}{\sqrt{k+1}}$$

$$\Rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$$

$\therefore P(k+1)$  is true.

Q If a set has  $n$  elements then prove that no. of possible relations on set A is greater than no. of possible functions on set A.  $[2^{n^2} > n^n]$ .

$$\rightarrow P(n) : 2^{n^2} > n^n.$$

1) Inductive Base:-

$$\text{For } n=1, 2^{1^2} > 1^1 \Rightarrow 2 > 1.$$

$\therefore P(1)$  is true.

$$y = 2n+1, \quad n_1, n_2 \in \mathbb{Z} \\ x = 2n_1+1$$

Now, let  $p$  is true. i.e.  $x+y$  are odd integers.

$P \rightarrow q$ : If  $x+y$  are two odd integers than  $(x+y)$  is even.

$$q: (x+y) \text{ is even.}$$

Let,  $P$ :  $x$  and  $y$  are odd integers.

Using method of direct proof shows that sum of 2 odd integers is even.

Therefore,  $P \rightarrow q$  is true.

So,  $n^2$  is even. i.e.  $q$  is true.

$$\Leftrightarrow n^2 = 2 \cdot m \quad (\text{where } m = 2k^2 \in \mathbb{Z})$$

$$\Leftrightarrow n^2 = 2 \cdot 2k^2$$

$$\Leftrightarrow n^2 = 4k^2$$

$$\Leftrightarrow n^2 = (2k)^2$$

$$\Leftrightarrow n = 2k, \text{ where } k \in \mathbb{Z}$$

Now, let  $p$  is true i.e.  $n$  is even.

$P \rightarrow q$ : If  $n$  is even, then  $n^2$  is even.

$$q: n^2 \text{ is even}$$

Let  $P$ :  $n$  is even.

is shown by using method of direct proof if  $n$  is even then  $n^2$  is even.

$P \rightarrow q$  is true by using some suitable and theorems. This will show that  $q$  is true by using some suitable and theorems ( $P \rightarrow q$ ). We can prove

• Direct Proof: In this method we will assume that hypothesis

$\Rightarrow$   $\leftarrow$  Induction

## \* Methods of Proofs :-

$$P(k+1) \Rightarrow : 2^{(k+1)^2} < (k+1)^{k+1}$$

3) Inductive step:-

$$P(k) \Rightarrow 2^{k^2} < k^k$$

2) Inductive Hypothesis:-

$$= 2(m_1 + n_2 + 1) \\ = 2(k+1), \quad k \in \mathbb{Z}, \text{ where, } k = (m_1 + n_2) - 2$$

$\Rightarrow$ ,  $x+y$  is even. i.e.  $q$  is true.  
Therefore,  $p \rightarrow q$  is true.

Indirect Proof :-

Proof By contrapositive :- In this method we can prove  $p \rightarrow q$  is true by showing  $\neg q \rightarrow \neg p$  is true.

Q Prove that if  $n$  is integer and  $3n+2$  is even then  $n$  is even.

Let,  $p$ :  $3n+2$  is even,  $n \in \mathbb{Z}$ .  
 $q$ :  $n$  is even

$p \rightarrow q$ : If  $3n+2$  is even where  $n \in \mathbb{Z}$  then  $n$  is even.

$\neg p$ :  $3n+2$  is odd,  $n \in \mathbb{Z}$ .

$\neg q$ :  $n$  is odd.

$\neg p \rightarrow \neg q$ : If  $n$  is odd then  $(3n+2)$  is odd where  $n \in \mathbb{Z}$ .

Now, let,  $\neg q$  is true i.e.  $n$  is odd.

$$n = 2k+1, \quad k \in \mathbb{Z}$$

$$\text{So, } 3n+2 = 3(2k+1)+2 \\ = 6k+5 \\ = 6k+4+1 \\ = 2(3k+2)+1 \\ = 2m+1 \quad \text{where } m = 3k+2 \in \mathbb{Z}$$

$\Rightarrow 3n+2$  is odd i.e.  $\neg p$  is  $\neg q$  true.

Hence,  $\neg q \rightarrow \neg p$  is true.

Therefore,  $p \rightarrow q$  is true.

Proof By contradiction :- In this method we assume that statement is false i.e. if  $p$  is statement

then assume that  $\neg p$  is true. then by using some rules & theorems we will show that given statement  $p$  is true as well as false i.e. we will reach at a contradiction. Therefore,  $p$  must be true.

Proof that  $\sqrt{3}$  is irrational.

Let,  $\sqrt{3}$  is rational.

$$\Rightarrow \sqrt{3} = \frac{p}{q}, \text{ where, } p \& q \text{ are prime no's and } q \neq 0. \quad \text{--- (1)}$$

By squaring both sides-

$$\Rightarrow 3 = \frac{p^2}{q^2}$$

$$\Rightarrow p^2 = 3q^2 \quad \text{--- (2)}$$

$\Rightarrow 3$  is a factor of ' $p^2$ '.

$\Rightarrow 3$  is a factor of ' $p$ '.

Let,  $p = 3n, n \in \mathbb{Z}$ .

Put,  $p = 3n$  in eqn (1).

$$\begin{aligned}
 &\Rightarrow (3n)^2 = 3q^2 \\
 &\Rightarrow 9n^2 = 3q^2 \\
 &\Rightarrow q^2 = 3n^2 \\
 &\Rightarrow 3 \text{ is a factor of } q^2 \\
 &\Rightarrow 3 \text{ is a factor of } q.
 \end{aligned}$$

$\therefore 3$  is a common factor of  $p$  &  $q$ , both which means  $p$  and  $q$  are not prime, which leads to a contradiction.  
i.e.  $p$  and  $q$  are not prime no.  $\therefore$   $\sqrt{3}$  is not a rational number.  
i.e.  $\sqrt{3}$  is irrational.

Q Prove that  $5 - 3\sqrt{2}$  is irrational.

$\rightarrow$  Let,  $5 - 3\sqrt{2}$  is rational.  $\therefore$  ①

$$\Rightarrow 5 - 3\sqrt{2} = \frac{p}{q}, \text{ where, } p \text{ & } q \text{ are prime} \& q \neq 0.$$

$$\Rightarrow 3\sqrt{2} = 5 - \frac{p}{q}$$

$$\Rightarrow \sqrt{2} = \frac{5q - p}{3q}$$

$$\Rightarrow \sqrt{2} = \frac{s}{t}, \text{ where, } s = 5q - p \& t = 3q \neq 0.$$

On squaring both sides-

$$\Rightarrow 2 = \frac{s^2}{t^2}$$

$$\Rightarrow 2s^2 = t^2. \quad \boxed{2}$$

$$\Rightarrow s^2 = 2t^2$$

$\Rightarrow 2$  is factor of  $s^2$ .

$\Rightarrow 2$  is factor of  $s$ .

Let,  $s = 2n \dots n \in \mathbb{Z}$ .

Put in eqn ② -

$$\Rightarrow 2(2n)^2 = 2s^2$$

$$\Rightarrow 4n^2 = 2s^2$$

$$\Rightarrow 2n^2 = s^2$$

$$\Rightarrow s^2 = 2n^2$$

$\Rightarrow 2$  is factor of  $s^2$

$\Rightarrow 2$  is factor of  $s$ .

$\therefore 2$  is a common factor of  $s$  &  $t$  both which means  $s$  &  $t$  are not prime which leads to a contradiction.  
i.e.  $s$  &  $t$  are not prime no.  $\therefore \sqrt{2}$  is not a rational no.

i.e.  $5 - 3\sqrt{2}$  is irrational.

11) Q. By counter example :- In this method we will give an example for which given statement is false. This example is called 'counter example' and negation of that statement is true.

→ an counter example.

Let  $f$  &  $g$  be two functions such that  $f(x) = x^2$   
 $g(x) = 2x+1$ .

$$fog = f(g(x)) = f(2x+1) \\ = (2x+1)^2 = 4x^2 + 4x + 1.$$

$$gof = g(f(x)) = g(x^2) \\ = 2x^2 + 1.$$

So, composition of function is not commutative.

① Find a counter example to a statement that every +ve integer can be written as sum of square of 3 integers.

$$\rightarrow 1 = 1^2 + 0^2 + 0^2$$

$$2 = 1^2 + 1^2 + 0^2$$

$$3 = 1^2 + 1^2 + 1^2$$

$$4 = 2^2 + 0^2 + 0^2$$

$$5 = 2^2 + 0^2 + 1^2$$

$$6 = 2^2 + 1^2 + 1^2$$

$$7 = ?$$

⇒ 7 can not be written as sum of square of 3 integers so, 7 is the counter example to that statement and statement is a

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UNIT-3Partial order Set, Lattice and Boolean Algebra.

$$\begin{cases} a \leq b & \{a \text{ is immediate} \\ & \text{order successor} \\ a \lessdot b & \text{to } b\} \end{cases}$$

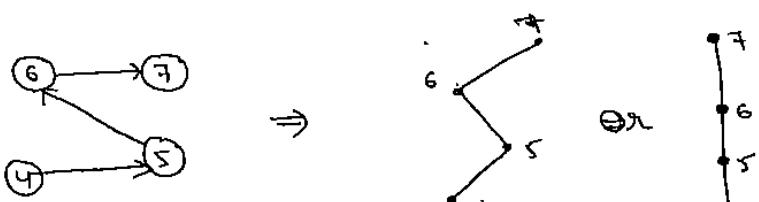
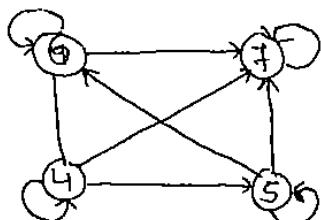
- \* Hasse Diagram of Poset :- Let  $(A, \leq)$  is a poset and  $(a, b) \in R$ . i.e.  $a \leq b$ . It means 'a' is immediate predecessor of 'b' or 'b' is immediate successor of 'a'. 'b' is also called cover of 'a'. It also means 'b' succeeds 'a' or 'a' precedes 'b'. If  $a \leq b$  but no element of set A lies b/w  $a$  &  $b$  then it is denoted by  $[a \lessdot b]$ . Strictly succeeds 'b' that means there is no element in between 'a' & 'b'.

Hasse Diagram of Poset  $(A, \leq)$  is a directed graph whose vertices are elements of  $A$  and there is directed path from  $a$  to  $b$ . whenever  $a \lessdot b$  [a strictly precedes b]. In Hasse Diagram we will place 'b' higher to 'a' and draw a line b/w 'a' and 'b' to indicate succession.

As 'POR' is reflexive as well as transitive so, edges showing these properties will be removed.

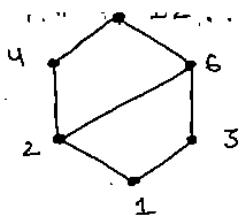
e.g. Consider  $A = \{4, 5, 6, 7\}$ .

Let  $R$  be a relation ' $\leq$ ' on  $A$  the directed graph of  $R$  is shown below. (Less than equal to)



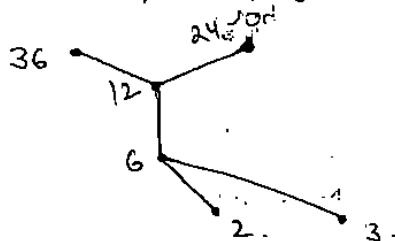
Q Let  $A = \{1, 2, 3, 4, 6, 12\}$ . Draw the Hasse Diagram of  $(A, |)$ .

$\Rightarrow (A, |) \Rightarrow a/b \quad \{a \text{ divides } b\}$   
 'a is divisible by b' . 'b' is not divisible by 'a'.

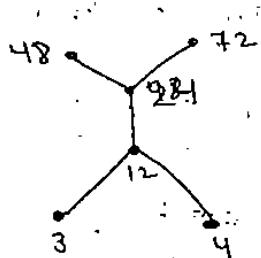


Q.  $A = \{2, 3, 6, 12, 24, 36\}$ ,  $(A, |)$ .

$\Rightarrow (A, |) \Rightarrow a/b \rightarrow a \mid b$

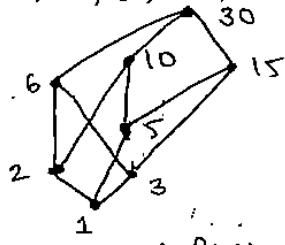


Q.  $A = \{3, 4, 12, 24, 48, 72\}$ ,  $(A, |)$ .



Q. If  $D_m$  is a set of factors of  $m$ . Draw the Hasse Di of Poset  $(D_m, |)$ , where  $m = 30$ .

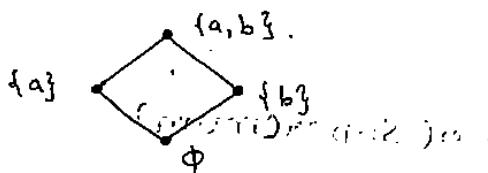
$\Rightarrow D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$ .



Q. Draw Hasse dia. of  $(P(A), \subseteq)$ ,  $A = \{a, b\}$ .

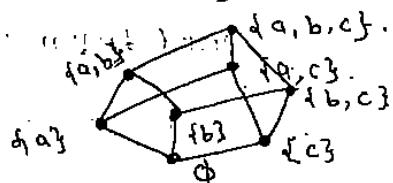
$$P(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

$$\Rightarrow (\{\emptyset, \{a\}, \{b\}, \{a, b\}\}, \subseteq).$$



Q.  $A = \{a, b, c\}$ ,  $(P(A), \subseteq)$ .

$$P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$



## \* Maximal and Minimal Elements :-

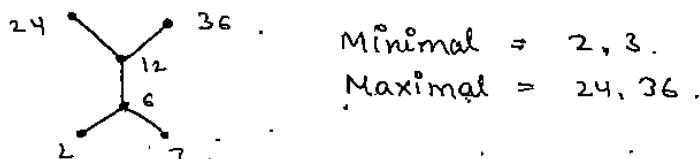
Minimal Element :- An element is called minimal element if no other element strictly precedes it.

In Hasse Diagram an element is minimal element if no edge enters to this element from below.

Maximal Element :- An element is called maximal element if no other element strictly succeeds it.

In Hasse Diagram an element is maximal element if no edge leads to this element from above.

There is more than one maximal and minimal elements.

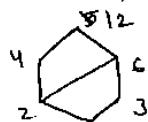


## \* First Element (or) Least Element and Last Element (or)

### Greatest Element :-

- First Element (or) Least Element :- 'm' is first element if 'm' precedes all the elements i.e.  $\{m \leq b, \forall b \in A\}$
- Last element (or) Greatest element :- 'n' is last element if 'n' succeeds all the elements i.e.  $\{a \leq n, \forall a \in A\}$

Eg.



First element = 1

Last element = 12

There is always only one least and one last element.

## \* Upper Bound and Lower Bound :-

- Upper Bound :- If  $x$  be a subset of poset  $(A, \leq)$ , then  $M$  element ( $M \in A$ ) is called upper bound of subset  $x$  if  $M$  succeeds every element of  $A$ :  
i.e.  $\{M : \forall x \in x, x \leq M, M \in A\}$

- Lower Bound :- If  $x$  be a subset of  $(A, \leq)$ , then  $N \in A$  is called lower bound of subset  $x$  if  $N$  precedes every element of  $A$ .  
i.e.  $\{N : \forall x \in x, N \leq x, N \in A\}$

- Least Upper Bound (or) (Supremum) [lub]  
If an upper bound of  $x$  precedes any other upper bound of  $x$  then it is called 'lub' or 'supremum'.  
It is denoted as 'lub( $x$ ) or sup( $x$ )'.

- Greatest Lower Bound (glb) (or) (Infimum)  
If an lower bound of  $x$  succeeds any other lower bound of  $x$  then it is called 'glb' and infimum.

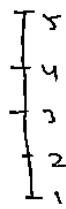
Eg

$$\text{UB of } \{3, 4\} = \{4, 5\}.$$

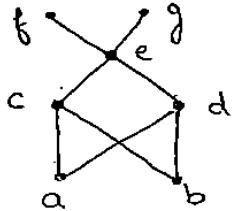
$$\text{lub} = 4.$$

$$\text{LB of } \{3, 4\} = \{1, 2, 3\}.$$

$$\text{glb} = 3.$$



Q.



$$A = \{c, d, e\}$$

Find upper bound and lower bound of A.

$$\text{UB of } c = \{c, e, f, g\}$$

$$\text{UB of } d = \{d, e, f, g\}$$

$$\text{UB of } e = \{e, f, g\}$$

$$\Rightarrow \text{UB of } A = \{e, f, g\}$$

$$\Rightarrow \text{lub of } A = \emptyset$$

Minimal element = a, b.

Maximal element = f, g.

Least and last element does not exist.

$$\text{LB of } c = \{c, a, b\}$$

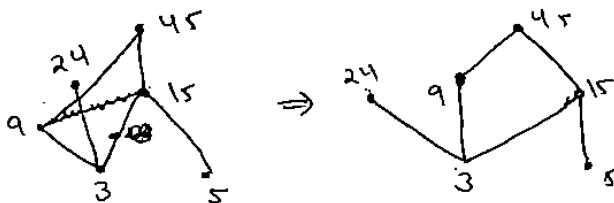
$$\text{LB of } d = \{d, a, b\}$$

$$\text{LB of } e = \{c, d, a, b, e\}$$

$$\Rightarrow \text{LB of } A = \{a, b\}$$

$$\Rightarrow \text{glb of } A \text{ does not exist.}$$

Q Draw the Hasse Diagram of Poset  $\mathcal{E} = \{(3, 5, 9, 15, 24, 45)\}$ , Find LB, UB, glb, lub of  $\{3, 5\}$ .



Minimal element = 3, 5

Maximal element = 24, 45

Least and last element does not exist.

$$\text{UB of } \{3, 5\} = \{9, 24, 15, 45\}$$

$$\text{UB of } 5 = \{15, 45\}$$

$$\text{UB of } \{3, 5\} = \{15, 45\}$$

$$\text{lub of } \{3, 5\} = 15$$

$$\text{LB of } 3 = 3$$

$$\text{LB of } 5 = 5$$

$$\text{LB of } \{3, 5\} = \emptyset \Rightarrow \text{glb of } \{3, 5\} = \emptyset$$

$$\text{LB of } 15 = 15, 5, 3$$

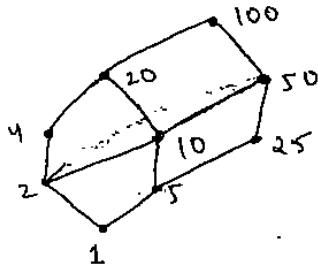
$$\text{LB of } 45 = 45, 15, 5, 3$$

$$\text{LB of } \{15, 45\} = 3, 5, 15$$

$$\text{glb of } \{15, 45\} = 15$$

Q Draw a Hasse Dia. of  $(D_{100}, \sqsubseteq)$ . Find glb of  $\{10, 20\}$ , lub of  $\{10, 20\}$ , glb of  $\{5, 10, 20, 25\}$  & lub of  $\{5, 10, 20, 25\}$ .  
 $D_{100} = \text{Set of factors of } 100.$

$$A_{\text{Base}} = (\{1, 2, 4, 5, 10, 20, 25, 50, 100\}, \sqsubseteq).$$



$$\text{UB of } \{10, 20\} = \{20, 50, 100\}.$$

$$\text{lub of } \{10, 20\} = 20 \text{ & } 20.$$

$$\text{LB of } \{10, 20\} = \{1, 2, 4, 5, 10\}.$$

$$\text{glb of } \{10, 20\} = 10.$$

$$\text{glb of } \{5, 10, 20, 25\} = \{1, 5\}$$

$$\text{glb of } \{5, 10, 20, 25\} = 5.$$

$$\text{UB of } \{5, 10, 20, 25\} = 100,$$

$$\text{lub of } \{5, 10, 20, 25\} = 100.$$

$$\text{UB of } \{10\} = \{10, 20, 50, 100\}.$$

$$\text{UB of } \{20\} = \{100, 20, 50, 100\},$$

$$\text{UB of } \{10\} = \{5, 2, 1, 10\}.$$

$$\text{LB of } \{20\} = \{5, 10, 20, 100\},$$

$$\text{LB of } \{5\} = \{1, 5\}.$$

$$\text{LB of } \{25\} = \{5, 1, 25\}.$$

$$\text{UB of } \{5\} = \{5, 10, 20, 50, 100\}$$

$$\text{UB of } \{25\} = \{50, 100\}$$

NOTE :- For  $(P_m, \sqsubseteq)$  where,  $m$  is factors of  $m$ .

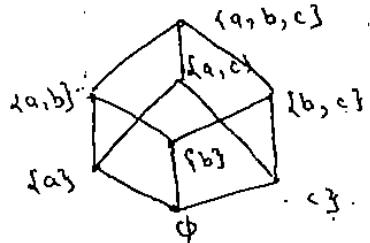
$$\text{glb of } \{a, b\} : \text{HCF of } \{a, b\}.$$

$$\text{lub of } \{a, b\} : \text{LCM of } \{a, b\}.$$

Q A =  $\{a, b, c\}$ ,  $(P(A), \sqsubseteq)$ , find Max., Min., greatest, least.

$$\Rightarrow P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}.$$

$$\Rightarrow (\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}, \sqsubseteq).$$



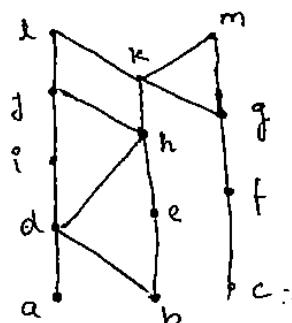
$$\text{Maximal} = \{a, b, c\}$$

$$\text{Minimal} = \{\emptyset\}.$$

$$\text{Least} = \{\emptyset\}$$

$$\text{Greatest} = \{a, b, c\}.$$

Q



Find min, max, greatest & least element.

$$\text{lub of } \{a, b, c\}$$

$$\text{glb of } \{f, g, h\}.$$

Minimal element =  $a, b, c$ .

Maximal Element =  $d, m$ .

Least and greatest element does not exist.

UB of  $\{a\} = \emptyset, a, d, e, f, i, h, k, l, m$ .

UB of  $\{b\} = b, d, e, f, h, j, k, l, m$ .

UB of  $\{c\} = c, f, g, m, k, l$ .

UB of  $\{a, b, c\} = d, k, m$ .

Lub glb of  $\{a, b, c\} = k$ .

LB of  $\{f\} = e, f$ .

LB of  $\{g\} = g, f, e$ .

LB of  $\{h\} = h, e, b, d, a$ .

LB of  $\{f, g, h\}$  does not exist.

and Hence, glb of  $\{f, g, h\}$  also does not exist.

\* Lattice :- A poset lattice is a poset  $(A, \leq)$  in which every subset  $(a, b)$  consisting of two elements has least upper bound (lub) and greatest lower bound (glb).

$\text{lub}\{a, b\} = a \vee b \rightarrow$  Join of  $a \& b$ . [V - join]

$\text{glb}\{a, b\} = a \wedge b \rightarrow$  Meet of  $a \& b$ . [ $\wedge$  - meet]

Thus, Lattice is a poset with two binary operations join & meet satisfying following properties.

Commutative Property :-  
•  $a \vee b = b \vee a$ .  
•  $a \wedge b = b \wedge a$ .

Associative Property :-  
•  $a \vee (b \vee c) = (a \vee b) \vee c$ .  
•  $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ .

Absorption law :-  
•  $a \vee (a \wedge b) = a$ .  
•  $a \wedge (a \vee b) = a$ .

Lattice is denoted by  $(L, \vee, \wedge)$ .  
e.g. 1)  $(P(A), \cup, \cap)$  is a lattice under  $\subseteq$ , where  $A = \{a, b\}$ .

$$\begin{aligned} \{a\} \cup \{b\} &= \{a\} \cup \{b\} = \{a, b\}. & \text{lub } \{a, b\} &= \{a, b\}. \\ \{a\} \cap \{b\} &= \{a\} \cap \{b\} = \emptyset. & \text{glb } \{a, b\} &= \emptyset. \end{aligned}$$

2)  $(D_m, \text{lcm}, \text{gcd})$  is a lattice under ' $/$ ' where,

$$\text{lub}(a, b) = \text{lcm}(a, b)$$

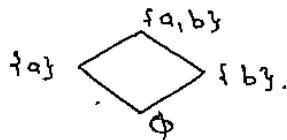
$$\text{glb}(a, b) = \text{gcd}(a, b)$$

where,  $a, b \in D_m$ .

e.g.  $A = \{a, b\}$ ,  $P(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ .

$\cup$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\emptyset$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\{a\}$	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b\}$
$\{b\}$	$\{b\}$	$\{b\}$	$\{a, b\}$	$\{a, b\}$
$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$

[Join].



$$\text{lub} = \{a, b\}$$

$\wedge$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\{a\}$	$\{a\}$	$\{a\}$
$\{b\}$	$\emptyset$	$\{b\}$	$\{b\}$	$\{b\}$
$\{a, b\}$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$

[Meet]

$$\text{glb} = \emptyset.$$

Join and Meet exist for every pair so it is a lattice.

But lub or glb may not exist. So, lattice doesn't



Here, 2 and 3 are not related by each other.  
 $\Rightarrow$  glb & lub does not exist.  
So, it is not a lattice.



It is a lattice. Bcoz each element are related to each other. glb & lub exist.

### Properties :-

(1) Commutative :-

(2) Associative :-

(3) Absorption :-

(4) Idempotent law :-  $\rightarrow a \vee a = a$   
 $\rightarrow a \wedge a = a$ .

(5) Lattice law :-  $\rightarrow a \vee b = b$ . if  $a \leq b$   
 $\rightarrow a \wedge b = a$ . if  $a \leq b$ .

(6) Distributive law or Property :-  
 $a, b, c \in L$  (lattice)  $\therefore$  if  $a \leq b$ ,

$$\rightarrow a \vee c \leq b \vee c.$$

$$\rightarrow a \wedge c \leq b \wedge c.$$

(7) Distributive Inequality :-

$$\rightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$$

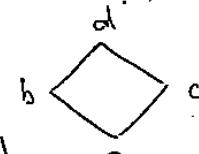
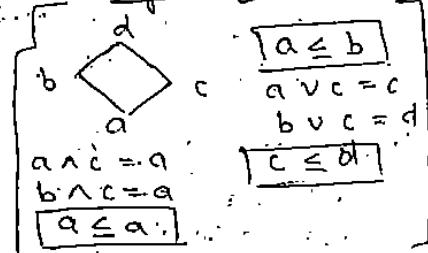
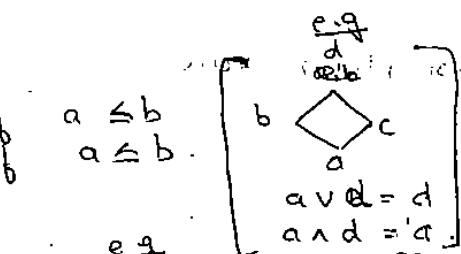
$$\rightarrow a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$$

e.g.  $a \vee (b \wedge c) = a \vee a = a.$  }  $a \leq a$

$$(a \vee b) \wedge (a \vee c) = b \wedge c = a$$

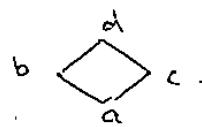
$$a \wedge (b \vee c) = a \wedge a = a$$

$$(a \wedge b) \vee (a \wedge c) = a \wedge a = a$$



$\Rightarrow$  If  $a \leq b$  and  $b \leq c$   
then,  $a \leq b \vee c$ ,  $a \leq b \wedge c$ .

e.g.  $a \leq b$ ,  $a \leq c$ .  
 $b \vee c = d$ .  
 $a \leq d$ ,  $a \leq a$ .

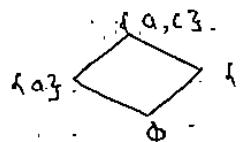


$\Rightarrow$  If  $b \leq a$  and  $c \leq a$ .  
then,  $b \vee c \leq a$ ,  $b \wedge c \leq a$ .

\* Sub-Lattice :- Let, L be a lattice. ( $L, \leq$ ) a non-empty subset 'S' of 'L' is called sublattice of L  
[If  $a \vee b \in S$  &  $a \wedge b \in S$  whenever  $a, b \in S$ .]

- Every sublattice is a lattice.
- Every singleton set is sublattice of 'L'
- If n divides m, then  $D_n$  is sub-lattice of  $D_m$ .
- e.g. If 25 divides 100, then  $D_{25}$  is sub-lattice of  $D_{100}$ .

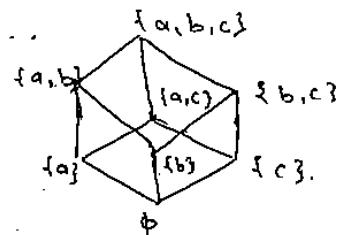
e.g.  $A = \{a, b, c\} \subset (PCA) \Rightarrow \leq$



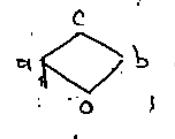
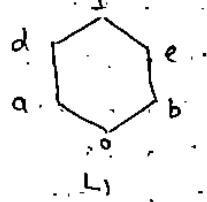
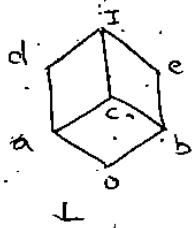
$$\{a\} \vee \{c\} = \{a, c\} \in A$$

$$\{a\} \wedge \{c\} = \emptyset$$

so, it is a sublattice.



e.g.



Find  $L_1$  and  $L_2$  are sublattice of  $L$ .

V	o	a	b	c
o	o	ab	bc	
a	a	a	bc	ac
b	b	b	b	bc
c	c	c	c	c

$L_2$  is a sublattice.

$$a \vee b = c \in L_2$$

V	o	a	b	d	e	I
o	o	ab	ab	ad, ae	ae, I	
a	a, o	a, b		ad, ae	ae, I	ad, ae, I
b	b	b	b	I	I	I
d	d			d	I	I
e	e				I	I
I	I	I	I	I	I	I

$L_1$  is not a sublattice  
 $a \vee b = c \notin L_1$

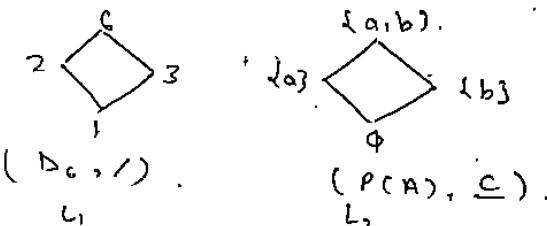
## \* Isomorphic Lattice :-

For Isomorphic lattice we check the no. of elements in both lattice are equal or not.  
If not equal, so, it is not a isomorphic lattice.

$$f : L_1 \rightarrow L_2$$

$$\begin{aligned} f(a \vee b) &= f(a) \vee f(b), \quad a, b \in L_1 \\ f(a \wedge b) &= f(a) \wedge f(b) \end{aligned}$$

e.g.



Let,  $f : L_1 \rightarrow L_2$  be a bijective function.

such that,  $f(1) = \phi$ ,  $f(2) = \{a\}$ ,  $f(3) = \{b\}$ ,  $f(4) = \{a, b\}$

$$f(1 \vee 2) = f(2) = \{a\} \Rightarrow f(1) \vee f(2) = \phi \vee \{a\} = \{a\}$$

$$f(1 \vee 3) = f(3) = \{b\} \Rightarrow f(1) \vee f(3) = \phi \vee \{b\} = \{b\}$$

$$f(1 \vee 4) = f(4) = \{a, b\} \Rightarrow f(1) \vee f(4) = \phi \vee \{a, b\} = \{a, b\}$$

$$f(2 \vee 3) = f(3) = \{b\} \Rightarrow f(2) \vee f(3) = \{a\} \vee \{b\} = \{a, b\}$$

$$f(2 \vee 4) = f(4) = \{a, b\} \Rightarrow f(2) \vee f(4) = \{a\} \vee \{a, b\} = \{a, b\}$$

$$f(3 \vee 4) = f(4) = \{a, b\} \Rightarrow f(3) \vee f(4) = \{b\} \vee \{a, b\} = \{a, b\}$$

$f(\text{LHS}) \sim \text{RHS}$

LHS = RHS

$$f(1 \wedge 2) = f(1) = \phi \Rightarrow f(1) \wedge f(2) = \phi \wedge \{a\} = \phi$$

$$f(1 \wedge 3) = f(1) = \phi \Rightarrow f(1) \wedge f(3) = \phi \wedge \{b\} = \phi$$

$$f(1 \wedge 4) = f(1) = \phi \Rightarrow f(1) \wedge f(4) = \phi \wedge \{a, b\} = \phi$$

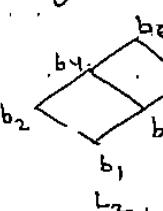
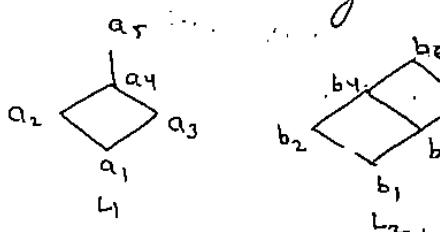
$$f(2 \wedge 3) = f(1) = \phi \Rightarrow f(2) \wedge f(3) = \{a\} \wedge \{b\} = \phi$$

$$f(2 \wedge 4) = f(1) = \phi \Rightarrow f(2) \wedge f(4) = \{a\} \wedge \{a, b\} = \phi$$

$$f(3 \wedge 4) = f(1) = \phi \Rightarrow f(3) \wedge f(4) = \{b\} \wedge \{a, b\} = \phi$$

LHS = RHS

Q Consider the lattice given below are they Isomorphic.



Let,  $f : L_1 \rightarrow L_2$  be a bijective function.

such that,  $f(a_1) = b_1$ ,  $f(a_2) = b_2$ ,  $f(a_3) = b_3$ ,  $f(a_4) = b_4$ ,  $f(a_5) = b_6$ .

Here, no. of elements of two lattice i.e.  $L_1$  and  $L_2$  are not equal. So, they are not isomorphic.

\* **Bounded Lattice** :- A lattice  $L$  is said to be bounded if it has a greatest or last ( $I$  or  $\top$ ) element and least or first ( $0$  or  $\bot$ ) element.

In such lattice we have :-

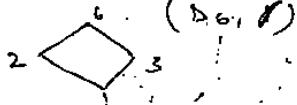
$$\begin{cases} a \vee I = I \\ a \wedge I = a \end{cases}$$

$$\begin{cases} a \vee 0 = a \\ a \wedge 0 = 0 \end{cases} \quad \forall a \in L, 0 \leq a \leq I$$

E.g.  $(D_m, \leq)$  is always bounded.

$$0 = 1$$

$$I = m$$



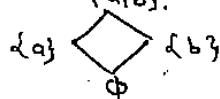
$$0 = 1$$

$$I = 6$$

$(P(A), \subseteq)$  is always bounded.

$$0 = \emptyset$$

$$I = A$$



NOTE:  $(Z, \leq)$  and  $(N, \leq)$  is not a bounded lattice

$(Z, \geq)$  and  $(N, \geq)$  is not a bounded lattice

bcoz no. of integers and Natural No.'s are not defined.

\* **Complemented Lattice** :- (Lattice must be bounded in complemented lattice).

Let,  $L$  be a bounded lattice with first element  $0$  and last element  $I$ .

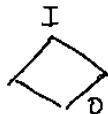
Let,  $a \in L$ . an element,  $a' \in L$  is called complement of 'a' if

$a \vee a' = I$
$a \wedge a' = 0$

A lattice  $L$  is said to be complemented lattice if there exist complement of each element.

NOTE

$I \vee 0 = I$
$I \wedge 0 = 0$



$I$  and  $0$  are complements of each other.

i.e.

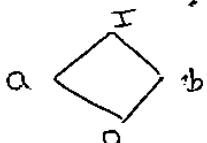
$I' = 0$
$0' = I$

(2) Complement of an element in a lattice is not unique.

(3) Complement is symmetric.

$a' = b$ then $b' = a$ .
--------------------------

Q



$$a \vee b = I$$

$$a \wedge b = 0$$

$$I' = 0$$

$$0 = I$$

so,  $a' = b$ .

and also  $b' = a$ . bcoz it is symmetric



Q. Find complements.

$$I^1 = 0$$

$$0^1 = I.$$

$$0 \vee 0^1 = I$$

$$0 \wedge 0^1 = 0$$

$$1 \vee 2 = I$$

$$1 \wedge 2 = 0$$

$$1 \vee 3 = I$$

$$1 \wedge 3 = 0$$

$$\therefore 1^1 = 2, 3 \Rightarrow 2 \vee 3 = I$$

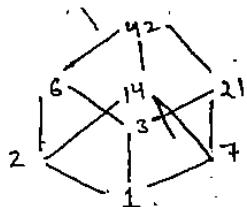
$$2 \wedge 3 = 0$$

$$\Rightarrow 2^1 = 1, 3$$

$$3^1 = 1, 2.$$

Q. ( $D_{42}, /$ ). Find complement of each element.

Factors of  $42 = \{1, 2, 3, 6, 7, 14, 21, 42\}$ .



$$I = 42$$

$$0 = 1.$$

$$So, 1^1 = 42.$$

$$42^1 = 1.$$

$$2 \vee 21 = 42$$

$$2 \wedge 21 = 1.$$

$$2^1 = 21$$

$$21^1 = 2.$$

$$3 \vee 14 = 42$$

$$3 \wedge 14 = 1$$

$$3^1 = 14$$

$$14^1 = 3.$$

$$6 \vee 7 = 42$$

$$6 \wedge 7 = 1$$

$$6^1 = 7$$

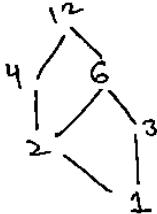
$$7^1 = 6.$$

In this only there  
element would be the  
complement if there  
is no relation b/w  
these 2 elements.

(OR)  
Mole. Join & meet b/w  
and first & last  
at place where join and  
Meet are same

Q. Is  $D_{12}$  a complemented lattice.

Factors of  $12 = \{1, 2, 3, 4, 6, 12\}$ .



$$I = 12$$

$$0 = 1.$$

$$So, 12^1 = 1$$

$$1^1 = 12$$

$$2 \vee 3 = 6.$$

$$2 \wedge 3 = 1.$$

Complement of 2 does not exist. So,  $D_{12}$  is not a complemented lattice.

Q. In the following lattice how many complements does the element 'e' have. Give all.

$$I = a$$

$$0 = f$$

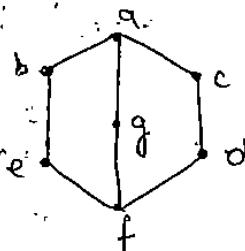
$$e \vee c = a$$

$$e \wedge c = f$$

$$e \vee d = a$$

$$e \wedge d = f$$

$$e$$



$$e \vee g = a$$

$$e \wedge g = f$$

$$e^l = c, d, g$$

Distributive Lattice :- A lattice  $L$  is said to be distributive lattice if for  $a, b, c \in L$ , following properties are satisfied.

- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$  {These are distributive}
- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ . {properties}

otherwise lattice  $L$  is non-distributive.

e.g.  $(P(A), \subseteq)$  is a poset  $\Rightarrow (P(A), \cup, \cap)$ .

The power set  $P(A)$  of set  $A$  under two operations union and intersection is a distributive lattice. Since,

$$\begin{aligned} A \vee (B \wedge C) &= (A \cup B) \cap (A \cup C) & \vee \rightarrow \cup \\ A \wedge (B \vee C) &= (A \cap B) \cup (A \cap C) & \wedge \rightarrow \cap \end{aligned}$$

for,  $A, B, C \in P(A)$ .

- Q If  $L$  be a bounded distributive lattice then prove that  
 If a complement exists it is unique.  
 $\rightarrow$  If let,  $a_1$  &  $a_2$   $\in L$  are complements of  $L$ .  
 Let,  $a \in L$ .

$$\begin{aligned} a \vee a_1 &= I \\ a \wedge a_1 &= O \end{aligned} \quad \left. \begin{array}{l} \text{--- ①} \\ \text{--- ②} \end{array} \right.$$

$$\begin{aligned} a \vee a_2 &= I \\ a \wedge a_2 &= O \end{aligned} \quad \left. \begin{array}{l} \text{--- ③} \\ \text{--- ④} \end{array} \right.$$

$$\begin{aligned} a_1 &= a_1 \vee O \\ a_1 &= a_1 \vee (a \wedge a_2) \quad \text{From eqn ②} \\ a_1 &= (a_1 \vee a) \wedge (a_1 \vee a_2) \quad \left. \begin{array}{l} \text{From eqn ①} \\ \text{By distributive lattice} \end{array} \right. \\ a_1 &= I \wedge (a_1 \vee a_2) \\ a_1 &= (a_1 \vee a_2) \quad \text{--- ⑤} \end{aligned}$$

Again,  $a_2 = a_2 \vee O$ .

$$\begin{aligned} a_2 &= a_2 \vee (a \wedge a_1) \quad \text{From eqn ①} \\ a_2 &= (a_2 \vee a) \wedge (a_2 \wedge a_1) \quad \text{[By distributive lattice]} \\ a_2 &= I \wedge (a_2 \vee a_1) \quad \text{From eqn ②} \end{aligned}$$

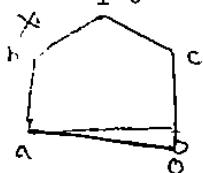
$$a_2 = (a_2 \vee a_1) \quad \text{--- ⑥}$$

From eqn ⑤ & ⑥

$$\Rightarrow a_1 = a_2$$

NOTE :- A lattice  $L$  is Non-Distributive if and only if it contains a sub-lattice i.e. isomorphic to one of the two lattices given below-

Pentagon.

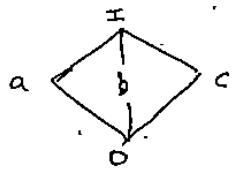


$$a \vee (b \wedge c) = a \vee b = a$$

$$(a \vee b) \wedge (a \vee c) = b \wedge c = b$$

$$\therefore a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$$

Diamond.



$$a \vee (b \wedge c) = a \vee c = a$$

$$(a \vee b) \wedge (a \vee c) = b \wedge c = b$$

$$\therefore a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$$

Therefore, lattices isomorphic to these two lattices are also non-distributive.

\* Complete Lattice :- A lattice  $L$  is called complete lattice if each of its non-empty subset has a glb and lub.

Let  $A$  be the class of all subsets of some universal set  $U$  and a relation ' $\leq$ ' is defined as

$$x \leq y \Rightarrow x \subseteq y$$

such that  $x \vee y = x \cup y$  &

$$x \wedge y = x \cap y$$

Every subset of  $A$  has a glb and a lub. So,  $A$  is complete lattice.

\* Modular Lattice :- A lattice  $L$  is called Modular if

$$a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$$

\*  $a, b, c \in L$

NOTE :-  $\vee \rightarrow +$ ,  $\wedge \rightarrow \cdot$

$$\text{eg. } a_1 \diamond a_2 \diamond a_3 \quad a_1 \leq I. \\ a_1 \vee (a_2 \wedge I) \quad | \quad (a_1 \vee a_2) \wedge I \\ = a_1 \vee a_2 \quad | \quad = I \wedge I \\ = I. \quad | \quad = I.$$

\* Boolean Algebra :-

(De Morgan's Law)

$$(a \cdot b) + (a' + b') = 1$$

$$(a \cdot b) \cdot (a' + b') = 0$$

$$(a \cdot b)' = a' + b'$$

$$(a' + b)' = a \cdot b$$

# Boolean Algebra Properties

# SOP & POS form

# MAX and MIN term

# Logic Gates and its Implementations

# K-Map. 3 & 4 variables with Don't Care condition

# Simplification of Boolean Expressions

# De Morgan's Law

From PLD

# Prepositional Logic & Predicate Calculus

- \* **Preposition :-** Preposition is a declarative statement which is either true or false but not both. It is usually denoted by lower case letters (p, q, r, ...). These are also called boolean variables, logic variables or propositional variables.

- e.g.
1. Narendra Modi is the Prime Minister of India.
  2. New Delhi is the capital of India.
  3. Paris is in Australia.
  4.  $2+2=5$
  5.  $x=2$  is a solution of  $x^2=4$ .
  6. Come Here.
  7. Shut the door.
  8. Where are you going?

First five are prepositions in which 1, 2 and 5 are true and 3 and 4 are false.  
6, 7, 8 are not prepositions.

- \* **Compound Preposition :-** A compound preposition is formed by composition of two or more prepositions called sub-prepositions or components.

NOTE: The preposition which can not be broken into sub-prepositions is called primitive preposition.

sky is blue and clouds are white.

e.g. sky is blue and clouds are white.  
This preposition contains two sub-prepositions  
 $\rightarrow$  sky is blue.  
 $\rightarrow$  clouds are white.

- \* **Fundamental Connectives :-**

Connective word	Name of connective	Symbol.
NOT	Negation	$\sim$ or ' $\perp$ ' or $\neg$
OR	Disjunction	$\vee$
AND	Conjunction	$\wedge$
If - then	Implication (conditional)	$\rightarrow$
If and only if	Bi-conditional	$\leftrightarrow$ , $\iff$

## \* Truth Table :-

(A T T F)

- NOT (Negation).

P	$\neg P$
T	F
F	T

- Disjunction. (OR)

P	q	$P \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- Conjunction. (AND)

P	q	$P \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- Implication (If - Then).

P	q	$P \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

In this it is only true when condition is true & then condition is false.

$P \rightarrow q \rightarrow \neg P \text{ is true}$   
True      False

- Biconditional (If and only if).

P	q	$P \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

It is only true when both p and q is true then it is true & both p and q are false then it is true.

- ◎ Consider the following statements.

p : It is raining today.

q : It is cloudy today.

r : It is windy today.

Write following statements in symbolic form.

- 1) It is either raining or cloudy today.

$p \vee q$ .

- 2) Neither it is raining nor windy today.

$\neg p \wedge \neg r$ .

P	Q	R	$P \leftarrow Q$	$(P \rightarrow Q) \vee R$	$P \leftarrow Q \rightarrow R$	AVB
T	T	F	T	T	T	F
T	F	F	F	T	T	F
F	T	F	F	T	T	F
F	F	F	F	F	F	F

Q) Construct the truth table for  $((P \rightarrow Q) \vee R) \vee ((P \rightarrow Q) \rightarrow R)$ .

P	Q	R	$up$	$upq$	$p \vee q$	$upqr$	$u(pqr)$
T	T	T	T	T	T	T	F
T	T	F	F	T	T	T	F
T	F	T	F	F	F	T	F
T	F	F	F	F	F	F	F
F	T	T	F	F	F	T	T
F	T	F	F	F	F	F	T
F	F	T	F	F	F	F	T
F	F	F	F	F	F	F	F

Q) Construct the truth table for  $u(pqr) \vee (up \wedge q)$ .

I should not go to market and if it will rain then I shall go to market.

If it will not rain tomorrow then I shall go to market.

Otherwise it will rain tomorrow and I shall go to market.

up and upq.

Q) If  $P \equiv$  It will rain tomorrow then I shall go to market.

Result meaning of following propositions.

Q: I shall go to market.

Following proposition.

It is windy else it is better sailing and cloudy.

$\neg (P \wedge Q) \vee (P \wedge \neg Q)$

It is false that, it is raining or cloudy but not windy.

$(\neg P \vee \neg Q) \wedge up$

It is cloudy or windy today but not raining.

$\neg (P \wedge Q) \wedge \neg R$

D. D.

**Logical Equivalent :-** If two propositions  $P(p, q, r, s, t)$  and  $Q(p, q, r, s, t, \dots)$  where  $p, q, r, \dots$  are propositional variables have always identical truth values. Then,  $P$  and  $Q$  are logical equivalent. It is denoted as  $P \equiv Q$ .

$$\text{eq. } \bullet \quad p \rightarrow q \equiv (\neg p \vee q)$$

(supra)

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الطبعة الأولى

P		q		P → q		¬ P ∨ q		P ↔ q		¬ P ↔ q		P → q → P		¬ P → q → P		¬ P → q → q → P		¬ P → q → q → q → P	
P	q	P	q	P	q	P	q	P	q	P	q	P	q	P	q	P	q	P	q
T	T	F	F	T	F	T	F	T	T	T	F	T	F	T	F	T	F	T	F
T	F	F	T	F	T	F	T	F	F	F	T	F	T	F	T	F	F	T	F
F	T	T	F	F	T	T	F	F	F	F	T	F	T	F	T	F	F	T	F
F	F	F	F	T	F	F	T	F	F	F	T	F	T	F	T	F	F	T	F
P		q		¬ P		P → q		¬ P → q		P → q → P		¬ P → q → P		¬ P → q → q → P		¬ P → q → q → q → P		¬ P → q → q → q → q → P	
T	T	F	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F
T	F	T	F	F	T	F	T	F	F	T	F	T	F	T	F	T	F	T	F
F	T	F	T	F	T	T	F	F	F	T	F	T	F	T	F	T	F	T	F
F	F	F	F	T	F	F	T	F	F	F	T	F	T	F	T	F	F	T	F

Q Establish following equivalence using truth Table.  
 $(p \vee q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$ .

$$P \leftrightarrow q \equiv (P \vee q) \rightarrow (\neg P \wedge q)$$

$p \vee q$	$p \wedge q$	$\neg(p \vee q) \rightarrow (\neg p \wedge \neg q)$
T	T	F
F	F	T
T	F	F
F	F	T

Let  $P \otimes Q \otimes R$  be one proposition

Reema is not well.

Began mining the major examination

Keema Mela 11

5 : Recma passes the course .  
Express each of the following prepositions as English  
equivalents.

$$1) (p \rightarrow \neg r) \vee (\neg q \rightarrow \neg r).$$

2b Reema is not well ~~or~~ then she does not pass the course  
or if she misses the major examination then she does not pass

2)  $(q \rightarrow (\neg r))$   
If Reema misses the major examination then she does not pass the course.

3)  $\neg q \leftrightarrow r$ .  
Reema does not miss the major examination if and only if she passes the course.

Q Write the following in symbolic form.

If either Ram takes Maths or Shyam takes science then Hari will take Biology.

$$(p \vee q) \rightarrow r$$

Let, p : Ram takes Maths

q : Shyam takes Science

r : Hari will take Biology.

\* Derived Connectives :- (NAND, NOR, X-OR).

$$\text{NAND} \quad p \uparrow q \equiv \neg(p \wedge q)$$

$$\text{NOR} \quad p \downarrow q \equiv \neg(p \vee q)$$

$$\text{X-OR} \quad p \oplus q \equiv \neg(p \leftrightarrow q)$$

P	q	$p \uparrow q$	$p \wedge q$	$\neg(p \wedge q)$	$p \downarrow q$	$p \vee q$	$\neg(p \vee q)$
T	T	F	T	F			
T	F	T	F	T			
F	T	T	F	T			
F	F	T	F	T			

P	q	$p \oplus q$	$p \leftrightarrow q$	$\neg(p \leftrightarrow q)$
T	T	F	T	F
T	F	T	F	T
F	T	T	F	T
F	F	F	T	F

\* Well-formed Formula :- In propositional logic a statement which is collected of variables or statements represented in well-defined form using parenthesis according to priority of operations is known as well-formed formula.

To generate well-formed formula recursively following rules are used -

→ Negation has priority over  $\wedge$  &  $\vee$ .  $((\neg p) \wedge q) \vee r$

- An atomic statement p is well-formed formula.
- If p is well formed formula then  $\neg p$  is also well formed formula.
- If  $(p \wedge q)$  are well formed formula then  $(p \wedge q)$ ,  $(p \vee q)$ ,  $(p \rightarrow q)$  &  $(p \leftrightarrow q)$  are also well formed formula.
- A statement consist of variables, parenthesis & connectives is recursively a well formed formula if & only if it can be obtained by applying above 3 rules.

Q Which of the following are well-formed formula. If no, explain why.

- 1)  $(P \rightarrow (P \vee Q))$  Well-formed.
- 2)  $(P \wedge Q) \rightarrow (Q \wedge P)$  Not well formed because conjunction is used with two variables.
- 3)  $((P \leftrightarrow Q) \rightarrow R)$  well formed.
- 4)  $((P \wedge Q) \leftrightarrow (Q \rightarrow R) \rightarrow R)$  Not well formed. because in bi-conditional statement is ~~is~~ not in parenthesis. So, not well formed bi-conditional statements.

### \* (Converse, Inverse and Contrapositive of Conditional Statement :-

Let,  $P \rightarrow Q$  is a statement then-

Converse :-  $Q \rightarrow P$

Inverse :-  $\neg P \rightarrow \neg Q$

Contrapositive :-  $\neg Q \rightarrow \neg P$ .

P	Q	$\neg P$	$\neg Q$	$\neg P \rightarrow \neg Q$	$Q \rightarrow P$	$\neg Q \rightarrow \neg P$	$P \rightarrow Q$
T	T	F	F	T	T	F	T
T	F	F	T	T	T	F	F
F	T	T	F	F	F	T	T
F	F	T	T	T	T	T	T

Inverse                  Converse                  Contrapositive

Q Let  $P \rightarrow Q$  : If A,B,C,D is a square then it is a rectangle. Write down converse, inverse & contrapositive of this statement.

P : ABCD is a square.  $\neg P$  : ABCD is not a square.

Q : ABCD is a rectangle.  $\neg Q$  : ABCD is not a rectangle.

Converse :-  $Q \rightarrow P \Rightarrow$  If ABCD is a rectangle then ABCD is a square.

Inverse :-  $\neg P \rightarrow \neg Q \Rightarrow$  If ABCD is not a square then it is not a rectangle.

Contrapositive :-  $\neg Q \rightarrow \neg P \Rightarrow$  If ABCD is not a rectangle then it is not a square.

Q If  $x=2$  or  $x=-2$  then  $x^2=4$ .

$$\begin{array}{ll} P : x=2 & (P \vee Q) \rightarrow R \\ Q : x=-2 & \neg P : x \neq 2 \\ R : x^2=4 & \neg Q : x \neq -2 \\ & \neg R : x^2 \neq 4. \end{array}$$

Converse :-  $R \rightarrow (P \vee Q) \Rightarrow$  If  $x^2=4$  then  $x=2$  or  $x=-2$ .

Inverse :-  $\neg (P \vee Q) \rightarrow \neg R \Rightarrow (\neg P \wedge \neg Q) \rightarrow \neg R$   
If neither  $x=2$  nor  $x=-2$  then  $x^2 \neq 4$ .

Contrapositive :-  $\neg R \rightarrow \neg (P \vee Q)$

If  $x^2 \neq 4$  then neither  $x=2$  nor  $x=-2$ .

then  $x+4 < 6$ . write the statement  $s$  & its converse.

$$\neg q : x < 2$$

$$\neg p : x+4 \leq 6$$

$\neg q \Rightarrow p$  i.e. If  $x < 2$  then  $x+4 < 6$ .

$$p : x+4 \neq 6, q : x \neq 2$$

$s$  : If  $x+4 \neq 6$  then  $x \neq 2$ .

( $x < = \geq$ )  
( $x > = \leq$ )

Converse :- If  $x \neq 2$  then  $x+4 \neq 6$ .

## \* Algebra of Propositions :-

• Idempotent Law :-  $\begin{cases} p \vee p \equiv p \\ p \wedge p \equiv p \end{cases}$

• Commutative Law :-  $\begin{cases} p \vee q \equiv q \vee p \\ p \wedge q \equiv q \wedge p \end{cases}$

• Associative Law :-  $\begin{cases} (p \vee q) \vee r \equiv p \vee (q \vee r) \\ (p \wedge q) \wedge r \equiv p \wedge (q \wedge r) \end{cases}$

• Absorption Law :-  $\begin{cases} p \vee (p \wedge q) \equiv p \\ p \wedge (p \vee q) \equiv p \end{cases}$

• Distributive Law :-  $\begin{cases} p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \\ p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \end{cases}$

• Involution Law :-  $\neg(\neg p) \equiv p$

• Identity Law :-  $\begin{aligned} p \vee T &\equiv T \\ \Rightarrow p \wedge T &\equiv p \\ \Rightarrow p \vee F &\equiv p \\ p \wedge F &\equiv F \end{aligned}$

• Complement Law :-  $\begin{aligned} \neg T &\equiv F, \quad p \vee \neg p \equiv T \\ \neg F &\equiv T, \quad p \wedge \neg p \equiv F \end{aligned}$

• DeMorgan's Law :-  $\begin{aligned} \neg(p \vee q) &\equiv \neg p \wedge \neg q \\ \therefore \neg(p \wedge q) &\equiv \neg p \vee \neg q. \end{aligned}$

Q Using Algebra of Propositions. show that,

$$(p \leftrightarrow q) = (p \vee q) \rightarrow (p \wedge q).$$

→

$$\text{LHS } p \leftrightarrow q$$

$$\Rightarrow (p \rightarrow q) \wedge (q \rightarrow p)$$

$$\Rightarrow (\neg p \vee q) \wedge (\neg q \vee p)$$

$$\Rightarrow ((\neg p \vee q) \wedge \neg q) \vee ((\neg p \vee q) \wedge p) \quad \{ \text{Distributive Law} \}$$

$$\Rightarrow ((\neg q \wedge \neg p) \vee (\neg q \wedge q)) \vee ((p \wedge \neg p) \vee (p \wedge q)) \quad \{ \text{Distributive Law} \}$$

$$\Rightarrow (((\neg p \wedge \neg q) \vee f) \vee (\neg q \wedge q)) \quad \{ \text{Using Complement Law} \}$$

$$\Rightarrow ((\neg p \vee \neg q) \vee (q \wedge p)) \quad \{ \text{Identity Law} \}$$

$$\Rightarrow (\neg(p \wedge q)) \vee (q \wedge p) \quad \{ \text{DeMorgan's Law} \}$$

$$\Rightarrow \neg(p \wedge q) \vee (p \wedge q)$$

$$\Rightarrow (p \vee q) \rightarrow (p \wedge q)$$

Q  $(p \wedge q) \vee (p \wedge \neg q) \equiv p$

LHS  $\equiv (p \wedge q) \vee (p \wedge \neg q)$   
 $\equiv p \wedge (q \vee \neg q)$  [Distributive Law]  
 $\equiv p \wedge T$  [Complement Law]  
 $\equiv p$  [Identity Law]

Q  $(\neg p \rightarrow (\neg p \rightarrow (\neg p \wedge q))) \equiv p \vee q$

LHS  $\equiv (\neg p \rightarrow (\neg (\neg p) \vee (\neg p \wedge q)))$   $[p \rightarrow q \equiv \neg p \vee q]$   
 $\equiv (\neg p \rightarrow (p \vee (\neg p \wedge q)))$  [Involution Law]  
 $\equiv (\neg p \rightarrow ((p \vee \neg p) \wedge (p \vee q)))$  [Distributive Law]  
 $\equiv (\neg p \rightarrow (T \wedge (p \vee q)))$  [Complement Law]  
 $\equiv (\neg p \rightarrow (p \vee q))$  [Identity Law]  
 $\equiv \neg(\neg p) \Leftrightarrow (p \vee q)$   $[p \rightarrow q \equiv \neg p \vee q]$   
 $\equiv p \vee (p \vee q)$  [Involution Law]  
 $\equiv (p \vee p) \vee q$  [Associative Law]  
 $\equiv p \vee q$  [Idempotent Law]

Q  $\neg(p \vee q) \vee (\neg p \wedge q) \equiv \neg p$

LHS  $\equiv \neg(p \vee q) \vee (\neg p \wedge q)$   
 $\equiv (\neg p \wedge \neg q) \vee (\neg p \wedge q)$  [DeMorgan's Law]  
 $\equiv \neg p \wedge \{\neg q \vee q\}$  [Distributive Law]  
 $\equiv \neg p \wedge T$  [Complement Law]  
 $\equiv p$  [Identity Law]

\* Negation of Compound Statement :-

Q If  $\det(A) = 0$  then A is a singular matrix.

$$\neg(p) \equiv \neg p$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \rightarrow q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q \quad (\text{or}) \quad \neg p \leftrightarrow q$$

Q- If  $|A| = 0$  then A is a singular matrix.

$$p : |A| \neq 0$$

q : A is a singular matrix.

$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$

$|A| = 0$  and A is not a singular Matrix.

i)  $\Delta ABC$  is a isosceles  $\Delta$ , if & only if 2 sides are equal  
 $\Delta ABC$  is not isosceles if & only if two sides of  $\Delta$  are not equal.

ii) No. is divisible by 3 if & only if sum of digit of no. is divisible by 3.

$\sim(p \leftrightarrow q) \equiv \sim p \leftrightarrow q \equiv p \leftrightarrow \sim q$ .  
 No. is divisible by 3 if and only if sum of digit of no. is not divisible by 3.

iii) If tomorrow is holiday then we will go either to market or cinema.

p : Tommorow is Holiday.

q : we will go to market

r : we will go to cinema.

$$\sim(p \rightarrow (q \vee r))$$

$$\equiv p \wedge (\sim q \wedge \sim r)$$

$$\equiv p \wedge \sim q \wedge \sim r$$

Tommorow is Holiday and we will neither go to market and cinema.

\* Tautology, contradiction and Contingency :-

• Tautology :- A compound proposition that is always true for all possible truth values of its propositional variables is called tautology. i.e. it contains only 'T' in last column of its truth table.  
 e.g. The doctor is either male or female.

• Contradiction :- A compound proposition that is always false for all possible truth values of its propositional variables is called contradiction. i.e. it contains only 'F' in last column of its truth table.  
 e.g.  $x$  is even and  $x$  is odd.

It is also called 'falsity'.

• Contingency :- A proposition that is neither tautology nor contradiction is called contingency.  
 i.e. the last column contains both T and F. in its truth table. It is also called "satisfiable".  
 Satisfiable :- It contains atleast one 'T'.

(3) Determine which contradiction and contingency follow in tautology.

- (1)  $(C \rightarrow A) \rightarrow (A \rightarrow B)$  . *Assumption*

(2)  $C \rightarrow A$  . *From 1*

(3)  $A \rightarrow B$  . *From 1*

(4)  $(A \rightarrow B) \wedge (B \rightarrow C)$  . *From 2,3*

(5)  $(A \rightarrow B) \wedge (B \rightarrow C) \wedge ((A \wedge B) \rightarrow C)$  . *From 4*

(6)  $((A \wedge B) \rightarrow C) \rightarrow ((A \wedge B) \rightarrow C)$  . *Tautology*

$p$	$q$	$\neg p$	$\neg q$	$\neg p \rightarrow q$	$\neg q \rightarrow p$	$(\neg p \rightarrow q) \rightarrow (\neg q \rightarrow p)$
T	T	F	T	T	T	T
T	F	F	T	T	F	T
F	T	T	F	F	T	F
F	F	F	F	T	F	T

P	T	F	F	T
F	T	F	F	T
F	T	F	F	T
F	T	F	F	T
T	T	F	F	T
T	T	F	F	T
F	F	T	T	T
T	F	T	F	T

Q Find a formula A that uses the variables p, q & r such that A is a contradiction.

$$\neg(p \vee (q \wedge r)) \leftrightarrow ((p \vee q) \wedge (p \vee r))$$

\* Logical Implication ( $\Rightarrow$ ) :- A proposition P is said to logically imply a proposition Q if implication  $P \rightarrow Q$  is a tautology and it is denoted by  $P \Rightarrow Q$ .

& Show that  $(p \vee q) \wedge \neg q \rightarrow p$  is a logical implication

P	q	$\neg q$	$p \vee q$	$(p \vee q) \wedge \neg q$	$(p \vee q) \wedge \neg q \rightarrow p$
T	T	F	T	F	T
T	F	T	T	T	T
F	T	F	T	F	T
F	F	T	F	F	

So, this implication is a logical implication and we can write  $(p \vee q) \wedge \neg q \Rightarrow p$

\* Dual :- The dual of any compound proposition can be obtained by replacing  $\vee$  and  $\wedge$  by  $\wedge$  and  $\vee$  resp and T and F by F and T respectively.

e.g. •  $p \vee q$

$$\text{Dual} \equiv p \wedge q$$

•  $(p \vee q) \wedge \neg q$

$$\text{Dual} \equiv (p \wedge q) \vee \neg q$$

•  $p \Leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$$\equiv (\neg p \vee q) \wedge (\neg q \vee p)$$

$$\text{Dual} \equiv (\neg p \wedge q) \vee (\neg q \wedge p)$$

\* Theorem of Duality :- If  $P(p, q, r, \dots) \equiv Q(p, q, r, \dots)$

$$\text{then, } P^*(p, q, r, \dots) \equiv Q^*(p, q, r, \dots)$$

where,  $P^*$  and  $Q^*$  are dual of  $P$  &  $Q$  respectively.

Q Show that following logical expressions are equivalent by using duality theorem without using truth table

$$(p \vee q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$$

$$\sim (p \vee q) \vee r \equiv (\sim p \vee r) \wedge (\sim q \vee r).$$

Dual of above statement -

$$\text{Dual} \equiv \sim (p \wedge q) \wedge r \equiv (\sim p \wedge r) \vee (\sim q \wedge r).$$

$$\begin{aligned}\text{Taking LHS.} &\equiv \sim (p \wedge q) \wedge r \\ &\equiv (\sim p \vee \sim q) \wedge r. \quad (\text{DeMorgan's Law}) \\ &\equiv (\sim p \wedge r) \vee (\sim q \wedge r) \quad (\text{Distributive Law}) \\ &\equiv \text{RHS.}\end{aligned}$$

Since, dual are equal.

$$\text{So, } \boxed{(p \vee q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)}$$

- Q Given that the value of  $p \rightarrow q$  is false. Determine the value of  $(\bar{p} \vee \bar{q}) \rightarrow \bar{q}$ .

$$\begin{aligned}\rightarrow p \rightarrow q, \\ &T \quad F. \\ &\equiv (\bar{p} \vee \bar{q}) \rightarrow \bar{q} \\ &\equiv (\bar{T} \vee \bar{F}) \rightarrow F. \\ &\equiv (F \vee T) \rightarrow F. \\ &\equiv T \rightarrow F. \\ &\equiv F\end{aligned}$$

- Q Show that the truth values of the following formulae are independent of their components.

$$\begin{aligned}1) \quad (p \rightarrow q) &\Leftrightarrow (\sim p \vee q) \\ &\equiv ((p \rightarrow q) \rightarrow (\sim p \vee q)) \wedge ((\sim p \vee q) \rightarrow (p \rightarrow q)) \\ &\equiv (\sim (p \rightarrow q) \vee (\sim p \vee q)) \wedge (\sim (\sim p \vee q) \vee (p \rightarrow q)) \\ &\equiv (\sim (\sim p \vee q) \vee (\sim p \vee q)) \wedge (\sim (\sim p \vee q) \vee (\sim p \vee q)) \\ &\equiv T \wedge T \\ &\equiv T \quad (\text{Tautology}).\end{aligned}$$

$$\begin{aligned}2) \quad (p \wedge (p \rightarrow q)) \rightarrow q \\ &\equiv (\sim (p \wedge (p \rightarrow q))) \vee q \\ &\equiv (\sim (p \wedge (\sim p \vee q))) \vee q \\ &\equiv (\sim (\sim (p \wedge p) \vee (p \wedge q))) \vee q \\ &\equiv (\sim (F \vee (p \wedge q))) \vee q \\ &\equiv (\sim (F)) \vee q \\ &\equiv T \vee q \\ &\equiv T.\end{aligned}$$

$$\begin{aligned}3) \quad (p \rightarrow q) \wedge (q \rightarrow r) &\rightarrow (p \rightarrow r) \\ &\equiv \sim ((p \rightarrow q) \wedge (q \rightarrow r)) \vee (p \rightarrow r) \\ &\equiv \sim ((\sim p \vee q) \wedge (\sim q \vee r)) \vee (\sim p \vee r) \\ &\equiv ((\sim p \wedge \sim q) \vee (\sim q \wedge r)) \vee (\sim p \vee r)\end{aligned}$$

$$\begin{aligned}
 & \rightarrow ((P \vee Q) \wedge (T) \wedge (P \vee \neg R) \wedge (\neg Q \vee \neg R)) \vee (\neg P \vee R) \\
 & \rightarrow ((P \vee Q) \wedge (P \vee \neg R) \wedge (\neg Q \vee \neg R)) \vee (\neg P \vee R) \\
 & \rightarrow ((P \vee Q \vee \neg P) \vee (P \vee \neg R)) \wedge ((\neg P \vee \neg R \vee \neg P) \vee (\neg P \vee R)) \wedge \\
 & \quad ((\neg Q \vee \neg R \vee \neg P) \vee (\neg Q \vee R)) \\
 & \rightarrow ((T \vee Q) \vee (P \vee \neg R)) \wedge ((T \vee \neg R) \vee (\neg P)) \wedge ((\neg Q \vee \neg R \vee P) \vee (\neg Q)) \\
 & \rightarrow (T \vee (P \vee \neg R)) \wedge (T \vee \neg R) \wedge (\neg Q \vee \neg R \vee P) \vee T
 \end{aligned}$$

\* **Argument :-** An argument is an assertion that a given set of preposition  $P_1, P_2, \dots, P_n$  give another preposition  $\theta$ . Here,  $P_1, P_2, \dots, P_n$  are called Premises and  $\theta$  is called conclusion or consequent such argument is denoted by :

$$\frac{\begin{array}{l} P_1 \\ P_2 \\ P_3 \\ | \\ \vdots \\ P_n \end{array}}{\therefore \theta} \text{ or } P_1, P_2, P_3, \dots, P_n \vdash \theta.$$

\* **Valid Argument :-** An argument  $P_1, P_2, P_3, \dots, P_n \vdash \theta$  is said to be valid if  $\theta$  is true whenever premises  $P_1, P_2, \dots, P_n$  are true.

eg ①  $P \rightarrow Q, P \vdash Q$

P		Q		$P \rightarrow Q$
T	F	T	F	
T	T	T	T	T
T	F	T	F	F
F	F	T	F	F
F	F	T	T	T

critical row.

A valid argument.

Since Conclusion is true  
in critical row.

②  $P \rightarrow Q, Q \vdash P$

P		Q		$P \rightarrow Q$
T	F	T	F	
T	T	T	T	T
T	F	T	F	F
F	T	T	T	T
F	F	T	F	F

critical rows.

It is not a valid argument.

Since, 2nd critical row. premises  
are true but Conclusion is false.

A critical row  
is a row in w  
premises are  
all true.  
 If in all crit  
rows premi  
are true &  
conclusion is  
then it is  
argument.

NOTE :- We can say that  $P_1, P_2, \dots, P_n \vdash \theta$  is an argument if and only if the premises  $P_1, P_2, \dots, P_n$  are true as well as conclusion  $\theta$  is true i.e.

$P_1 \wedge P_2 \wedge \dots \wedge P_n \rightarrow \theta$  is a tautology.

For any preposition  $P(p, q, r, \dots)$  and  $\theta(p, q, r, \dots)$  The following three statements are equivalent -

- The argument  $P(p, q, r, \dots) \vdash \theta(p, q, r, \dots)$  is valid.
- $P \rightarrow \theta$  is a tautology.
- $P \Rightarrow \theta$

Q Test the validity of following arguments.

- $p \vee q, q \vee r, \neg r \vdash p$ .
- $p \leftrightarrow q, \neg p \vee q, \neg r \vdash \neg q$ .
- $p \rightarrow \neg q, r \rightarrow p, q \vdash \neg r$ .

1)  $p \vee q, q \vee r, \neg s \vdash p$ .

			$\boxed{p}$	$\boxed{p}$	$\boxed{p}$
$\neg s$	$\neg r$	$\neg p$	$p \vee q$	$q \vee r$	
T	F	T	T	T	
T	T	F	T	T	
T	F	F	T	T	
F	F	T	T	T	
T	F	F	T	F	
F	F	T	F	T	
F	F	F	F	F	

Since, conclusion  $c$  is true in 2<sup>nd</sup> critical row & false in 6<sup>th</sup> critical row.  
So, this is not a valid argument.

2)  $p \rightarrow q, \neg p \vee q, \neg r \vdash \neg q$ .

			$\boxed{p}$	$\boxed{p}$	$\boxed{p}$	$\boxed{\neg p}$	$\boxed{\neg p}$	$\boxed{\neg p}$
$\neg r$	$\neg q$	$\neg p \vee q$	$\neg r$	$\neg p \vee q$	$\neg r$	$\neg p \vee q$	$\neg r$	$\neg p \vee q$
T	T	F	F	F	T	T	T	
T	F	F	F	T	T	T	T	
T	T	F	T	F	T	F	T	
F	F	F	T	T	F	F	T	
T	T	F	T	F	T	F	T	
F	F	T	F	T	F	T	T	
F	F	T	T	F	T	F	T	

Since, conclusion  $c$  is true in 8<sup>th</sup> row & false in 2<sup>nd</sup> row.  
So, this is not a valid argument.

3)  $p \rightarrow \neg q, \neg r \rightarrow p, q \vdash \neg r$ .

			$\boxed{p}$	$\boxed{p}$	$\boxed{p}$	$\boxed{\neg r}$	$\boxed{\neg r}$	$\boxed{\neg r}$
$\neg r$	$\neg q$	$\neg r \rightarrow p$	$\neg r$	$\neg r \rightarrow p$	$\neg r$	$\neg r \rightarrow p$	$\neg r$	$\neg r \rightarrow p$
T	T	F	F	F	T	T	T	
T	F	F	F	T	F	T	T	
T	F	T	T	F	T	T	T	
F	T	T	T	T	T	T	T	
T	T	F	F	F	T	F		
F	F	T	F	T	T	F		
F	F	T	T	F	T	T		

Since conclusion  $c$  is true in 6<sup>th</sup> row. So, it is a valid argument.

## \* Rules of Inference :-

- Modus Ponens (Law of Detachment) :-

$$\begin{array}{c} p \rightarrow q \text{ is true} \\ \hline p \text{ is true} \\ \therefore q \text{ is true} \end{array}$$

- Modus Tollens :-

$$\begin{array}{c} p \rightarrow q \text{ is true} \\ \neg q \text{ is true} \\ \hline \neg p \text{ is true} \end{array}$$

- Hypothetical Syllogism :-

$$\begin{array}{c} p \rightarrow q \text{ is true} \\ q \rightarrow r \text{ is true} \\ \hline \therefore p \rightarrow r \text{ is true} \end{array}$$

$$\begin{array}{c} \text{---} \\ \text{P} \vee q \quad \text{is true} \\ \text{---} \\ \therefore p \quad \text{is true.} \end{array} \qquad \begin{array}{c} \text{---} \\ \text{P} \vee q \quad \text{is true} \\ \text{---} \\ \therefore q \quad \text{is true.} \end{array}$$

• Addition :-

$$\begin{array}{c} \text{P} \quad \text{is true} \\ \text{---} \\ \therefore p \vee q \quad \text{is true.} \end{array}$$

• Simplification :-

$$\begin{array}{c} \text{P} \wedge q \quad \text{is true} \\ \text{---} \\ \therefore p \quad \text{is true} \end{array}$$

• Conjunction :-

$$\begin{array}{c} \text{P} \quad \text{is true} \\ \text{---} \\ \text{q} \quad \text{is true} \\ \text{---} \\ \therefore p \wedge q \quad \text{is true.} \end{array}$$

• Absorption law :-

$$\begin{array}{c} \text{P} \rightarrow q \quad \text{is true} \\ \text{---} \\ \therefore p \rightarrow (p \wedge q) \quad \text{is true.} \end{array}$$

• Constructive Dilemma :-

$$\begin{array}{c} (p \rightarrow q) \wedge (r \rightarrow s) \quad \text{is true} \\ \text{---} \\ \text{p} \vee r \quad \text{is true.} \\ \text{---} \\ \therefore q \vee s \quad \text{is true.} \end{array}$$

• Destructive Dilemma :-

$$\begin{array}{c} (p \rightarrow q) \wedge (r \rightarrow s) \quad \text{is true} \\ \text{---} \\ \text{u} q \vee u s \quad \text{is true} \\ \text{---} \\ \therefore u p \vee u r. \quad \text{is true.} \end{array}$$

Q Prove the argument  $p, q \vdash (p \vee r) \wedge q$  is a valid argument without using Truth table.

- 1)  $p$  (Given Premise)
- 2)  $q$  (Given Premise)
- 3)  $p \vee r$  (Addition of 1)
- 4)  $(p \vee r) \wedge q$  (Conjunction using 2 & 3)

This is valid argument.

Q  $p \vee (q \rightarrow p), u p \wedge r \vdash u q.$

- 1)  $p \vee (q \rightarrow p)$  (Given Premise).
- 2)  $u p \wedge r$  (Given Premise)
- 3)  $u p$  (Simplification of 2).
- 4)  $q \rightarrow p$  (Dijunctive Syllogism of 1 & 3).
- 5)  $u q$  (Modus Tollens of 3 & 4).

This is valid argument.

Q  $p \rightarrow q, r \rightarrow p, q \vdash \neg r \dots$

- 1)  $p \rightarrow q$  (Given Premise)
- 2)  $r \rightarrow p$  (Given Premise)
- 3)  $q$  (Given Premise)
- 4)  $\neg p$  (Modus Tollens 1 & 3)
- 5)  $\neg r$  (Modus Tollens using 2 & 4)

This is valid argument.

Q Check the validity of the arguments.

- D If I tried hard & If I tooted hard & I have talent  
then I will become an engineer. If I become an engineer  
Let,  $p$ : I tried hard.  
 $q$ : I have talent.  
 $r$ : I will become an engineer.

then I will be happy. Therefore, If will not be happy then  
I didn't try hard or I don't have talent.  
 $s$ : I will be happy.

- 1)  $(p \wedge q) \rightarrow r$  (Given Premise)
  - 2)  $r \rightarrow s$  (Given Premise)
- $\therefore \neg s \rightarrow (\neg p \vee \neg q)$

- 1)  $(p \wedge q) \rightarrow r$  (Given Premise)
- 2)  $s \rightarrow r$  (Given Premise)
- 3)  $(p \wedge q) \rightarrow s$  (Hypothetical Syllogism using 1 & 2)
- 4)  $\neg s \rightarrow (\neg(p \wedge q))$  (Contrapositive) of 3)
- 5)  $\neg s \rightarrow (\neg p \vee \neg q)$  (De Morgan's Law of 4).

- 2) If Rohit is married, he is sad. If he is sad then  
he does not watch TV. If he watches TV therefore  
Rohit is unmarried.

→  $p$ : Rohit is married.

$q$ : Rohit is sad.

$r$ : Rohit watches TV.

$$\begin{array}{c} p \rightarrow q, \\ q \rightarrow \neg r \\ \hline \therefore \neg p. \end{array}$$

- 1)  $p \rightarrow q$  (Given Premise)
- 2)  $q \rightarrow \neg r$  (Given Premise)
- 3)  $r$  (Given Premise)
- 4)  $p \rightarrow \neg r$ . (Hypothetical Syllogism of 1 & 2)
- 5)  $\neg p$  (Modus Tollens using 3 & 4).

Q Show that  $s$  is a valid conclusion from the given premises.

$$\neg p \wedge q, \quad \neg r \rightarrow p, \quad \neg s \rightarrow r \quad \text{and} \quad s \rightarrow t$$

- 1)  $\neg p \wedge q$  (Given Premise)
- 2)  $\neg r \rightarrow p$  (Given Premise)
- 3)  $\neg s \rightarrow r$  (Given Premise)
- 4)  $s \rightarrow t$  (Given Premise)
- 5)  $\neg p$  (Simplification using 1)
- 6)  $\neg r$  (Modus Tollen using 2 & 5).
- 7)  $\neg s \rightarrow t$  (Hypothetical Syllogism using 3 & 4).
- 8)  $t$ . (Modus Ponens using 6 & 7)

Q Show that  $s$  is a valid conclusion from given premise  
 $p \rightarrow q, \quad q \rightarrow r, \quad \neg r \rightarrow (s \wedge t), \quad p$ .

- 1)  $p \rightarrow q$  (Given Premise)
- 2)  $q \rightarrow r$  (Given Premise)
- 3)  $\neg r \rightarrow (s \wedge t)$  (Given Premise)
- 4)  $p$  (Given Premise)
- 5)  $p \rightarrow r$  (Hypothetical Syllogism using ① & ②)
- 6)  $\neg r$  (Modus Ponens using 4 & 5).
- 7)  $p \rightarrow (s \wedge t)$  (Hypothetical Syllogism using ③ & ⑤)
- 8)  $s \wedge t$  (Modus Ponens using 3 & 6).
- 9)  $s$ . (Simplification of 8)

Q Translate the following in symbolic form and test the validity of the argument.

If 6 is even then 2 does not divide seven either. So if 6 is not prime or 2 divides 7 but 5 is prime. Therefore, 6 is

$\rightarrow$  P: 6 is even.

q: 2 does not divide seven.

r: 5 is prime.

$$P \rightarrow \neg q$$

$$\neg r \vee q$$

$$\underline{r}$$

$$\therefore \neg p$$

now.

or

$$[(P \rightarrow q) \equiv (\neg p \vee q)]$$

- 1)  $P \rightarrow \neg q$  (Given Premise)
- 2)  $\neg r \vee q$  (Given Premise)
- 3)  $r$  (Given Premise)
- 4)  $q$  (Disjunctive Syllogism using 2 & 3)
- 5)  $\neg p$ . (Modus Tollens using 1 & 4)

Tom bought a washing machine or a TV. If he bought a TV then he likes to watch movie. He does not like movies therefore Tom bought a washing machine.

- P: Tom bought washing machine  
 q: Tom bought a TV  
 r: Tom likes to watch movie.

$$\begin{array}{c} p \vee q \\ q \rightarrow r \\ \hline \therefore p \end{array}$$

- 1)  $p \vee q$  (Given Premise)
- 2)  $q \rightarrow r$  (Given Premise)
- 3)  $\neg r$  (Given Premise)
- 4)  $\neg q$  (Modus Tollens 2 & 3)
- 5)  $p$  (Disjunctive Syllogism of 1 & 4)

### \* Predicate Calculus : (First order Logic).

1) Functional logic calculus is generalisation and Propositional logic is particularisation.

Functional logic is generalisation of propositional logic.

$\forall$  Universal Quantifier,  $\exists$  Existential Quantifier.

First order Logic is the extension of propositional logic by generalising and quantifying the propositions over given universe of discourse.

In general in first order logic every individual has a property 'P'. It is also called first order predicate calculus.

Predicate Calculus :-

Consider the following two statements

p: sparrow can fly.

q: Parrot can fly.

These two propositions can be replaced by a single proposition i.e. "x can fly".

In this sentence 'x' is subject and 'can fly' is predicate.

This single sentence generalise two propositions or more so this type of generalisation is known as predicate calculus.

'x' is called predicate variable & sentence "x can fly" is represented by  $P(x)$  i.e.

$P(x) : x \text{ can fly.}$

Universal discourse ( $\forall P$ ) - it is the set of all possible values that can be substituted in place of predicate variable.

e.g. Let,  $P(x)$ :  $x$  can fly.

$\forall D$  = set of all birds which can fly.

\* Quantifiers :- There are two types of quantifiers.

- Universal Quantifier

- Existential Quantifier.

Universal Quantifier :-  $P(x)$  be a propositional function defined on set A.

consider the expression  $\forall (\forall x \in A) P(x)$  or  $\forall x P(x)$ .

Here, the symbol " $\forall$ " is read as "for all" or "for every" and it is called Universal Quantifier then, the above statement is read as

for every  $x$  belongs to A,  $P(x)$  is true.

Existential Quantifier :- Let,  $P(x)$  be a propositional function defined on set A

consider the expression  $\exists (\exists x \in A) P(x)$  or  $\exists x P(x)$ .

Here, the symbol " $\exists$ " is read as "for some" or "for at least one" or "There exists" and it is called Existential Quantifier then the above statement is read as

for some  $x$  belongs to A,  $P(x)$  is true.

NOTE: The value of  $x$  for which  $P(x)$  is false is called counter example.

Q. Express following in symbolic form using quantifiers.

1) Some chalk boards are green.

2) Every student has a laptop.

3) All chairs has four legs.

4) Panjabad is an industrial city.

5) Some integers are prime and some are composite.

1)  $P(x)$ :  $x$  is a chalk board.

$Q(x)$ :  $x$  is green.

$$\exists x (P(x) \wedge Q(x)).$$

{ for some }

{ use  $\wedge$  for and }

{ for every }

→ (complement)

2)  $P(x)$ :  $x$  is a student.

$Q(x)$ :  $x$  has a laptop.

$$\forall x (P(x) \rightarrow Q(x)).$$

3)  $P(x)$ :  $x$  is a chair.

$Q(x)$ :  $x$  has four legs.

$$\forall x (P(x) \rightarrow Q(x)).$$

1)  $P(x)$  :  $x$  is Faridabad city.  
 $Q(x)$  :  $x$  is an Industrial city.  
 $\exists x (P(x) \wedge Q(x))$ .

2)  $P(x)$  :  $x$  is an integer.  
 $Q(x)$  :  $x$  is prime.  
 $R(x)$  :  $x$  is composite.

$\exists x \{ (P(x) \wedge Q(x)) \wedge \exists x (P(x) \wedge R(x)) \}$

3) Consider the function  $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$   
Determine the truth value of following proposition where the UD is set of real numbers.

- 1)  $\forall x (x^2 \geq 0)$
- 2)  $\forall x (|x| > 0)$ .

$$\rightarrow UD = \mathbb{R}.$$

- 1)  $\forall x (x^2 \geq 0)$  This is true because the square of every real number is non-negative or positive.
- 2)  $\forall x (|x| > 0)$  This is false because  $x=0$ ,  $|0|=0$ .  $0 > 0$  which is not greater.

### \* Variant of Quantified Conditional Statement :-

- 1) Converse :  $\forall x (Q(x) \rightarrow P(x))$
- 2) Inverse :  $\forall x (\neg P(x) \rightarrow \neg Q(x))$ .
- 3) Contrapositive :  $\forall x (\neg Q(x) \rightarrow \neg \neg P(x))$ .

### \* Multiple Quantifiers :- A proposition having more than one variable can be quantified using multiple quantifiers.

Propositions containing both Universal and Existential quantifiers can not be exchanged without altering the meaning of the proposition.

Ex.  $P(x, y)$  :  $x$  loves  $y$ .

$\exists x \forall y P(x, y)$  → There is a person who loves everyone in the world.

$\forall y \exists x P(x, y)$  → Everyone in the world is loved by at least one person.

NOTE:  $\forall x \forall y$  is same as  $\forall y \forall x$ .

$\exists x \exists y$  is same as  $\exists y \exists x$ .

Ex.  $x$  is father of mother of  $y$ .

$UD = \text{All the persons.}$

$P(x) : x$  is a person.

$F(x, y) : x$  is father of  $y$ .

$M(x, y) : x$  is mother of  $y$ .

$x$  is father of  $z$   
 $z$  is mother of  $y$ .

$$\exists z (P(z) \wedge F(x, z) \wedge M(z, y)).$$

### \* Negation of Quantified Preposition :-

$$\neg (\forall x P(x)) \equiv \exists x \neg P(x)$$

$$\neg (\exists x P(x)) \equiv \forall x \neg P(x)$$

Q Write the following preposition using quantifier then negate at each quantified preposition.

1) some students don't leave live in hostel.

$P(x) : x$  is a student.

$Q(x) : x$  don't live in hostel.

$$\exists x (P(x) \wedge Q(x)).$$

$$\Rightarrow \neg (\exists x (P(x) \wedge Q(x)))$$

$$\forall x \neg (P(x) \wedge Q(x))$$

$$\equiv \forall x (\neg P(x) \vee \neg Q(x))$$

$$\equiv \forall x (P(x) \rightarrow \neg Q(x)). \quad \text{All students live in hostel}$$

2) Every bird can fly.

$\Theta(x) : x$  can fly.

$B(x) : x$  is a bird.

$$\forall x (B(x) \rightarrow \Theta(x))$$

$$\Rightarrow \neg (\forall x (B(x) \rightarrow \Theta(x)))$$

$$\equiv \exists x \neg (\neg B(x) \vee \Theta(x))$$

$$\equiv \exists x (B(x) \wedge \neg \Theta(x)) \quad \text{some birds cannot fly.}$$

3) All boys can run faster than all girls.

$P(x) : x$  is a boy.

$R(x, y) : x$  can run faster than  $y$ .

$G(y) : y$  is a girl.

$$\forall x \forall y (P(x) \rightarrow R(x, y)).$$

$$\Rightarrow \neg (\forall x \forall y (P(x) \rightarrow R(x, y)))$$

$$\equiv \exists x \exists y (P(x) \wedge \neg R(x, y)).$$

Some boys can not run faster than some girls.

Q Use quantifier to say that  $\sqrt{3}$  is irrational.

$$\neg (\exists x (x^2 = 3)).$$

QD = set of rational no.'s.

## \* Rules of Inference :-

### • Universal Specification :-

$$\frac{\forall x \ P(x)}{P(a)} \text{ is true}$$

$\therefore P(a)$  where,  $a$  is particular member of UD.

### • Universal Generalization :- If $P(a)$ is true for all ' $a$ ' in UD

$$\frac{P(a)}{\forall x \ P(x)}$$

$\therefore \forall x \ P(x)$  is true.

### • Existential Specification :-

$$\frac{\exists x \ P(x)}{P(a)}$$

$\therefore P(a)$  is true.

### • Existential Generalization :-

$$\frac{P(a)}{\exists x \ P(x)}$$

$\therefore \exists x \ P(x)$  is true.

### • Universal Modus Ponens :-

$$\frac{\forall x \ (P(x) \rightarrow Q(x))}{\forall x \ (P(x) \rightarrow Q(x))} \text{ is true}$$

$$\frac{P(a)}{P(a)} \text{ is true}$$

$$\therefore Q(a) \text{ is true}$$

### • Universal Modus Tollens :-

$$\frac{\forall x \ (P(x) \rightarrow Q(x))}{\forall x \ (P(x) \rightarrow Q(x))} \text{ is true}$$

$$\frac{\neg Q(a)}{\neg Q(a)} \text{ is true}$$

$$\therefore \neg P(a) \text{ is true}$$

Q If a number is odd then its square is odd.  $k$  is a particular no. that is odd. Therefore,  $k^2$  is odd.

$\rightarrow P(x) : x$  is odd.

$(S(x) : x^2$  is odd).

$$\frac{\forall x \ (P(x) \rightarrow Q(x))}{\forall x \ (P(x) \rightarrow Q(x))}$$

$P(k)$  where,  $k \in \text{UD}$ .

$$\therefore Q(k)$$

Above argument is valid by Universal Modus Ponens Rule.

Q Leopards are dangerous animals; there are leopards.

Therefore, there are dangerous animals.

$\rightarrow P(x) : x$  is a leopard.

$Q(x) : x$  is dangerous animals.

$$\frac{\forall x \ (P(x) \rightarrow Q(x))}{\forall x \ (P(x) \rightarrow Q(x))}$$

$$\frac{\exists x \ P(x)}{\exists x \ Q(x)}$$

$$1) \quad \frac{\forall x \ (P(x) \rightarrow Q(x))}{\forall x \ (P(x) \rightarrow Q(x))} \text{ (Given Premise)}$$

$$2) \quad \frac{\exists x \ P(x)}{\exists x \ P(x)} \text{ (Given Premise)}$$

(Universal Existential Specification)

(using 2).

(Universal Modus Ponens of 1 & 3)

(Existential Generalization of 4).

$$3) \quad \frac{P(a)}{P(a)}$$

$$4) \quad \frac{Q(a)}{Q(a)}$$

$$5) \quad \frac{\exists x \ Q(x)}{\exists x \ Q(x)}$$

Therefore, all human being are mortal.

$P(x)$  :  $x$  is an animal.

$Q(x)$  :  $x$  is mortal.

$R(x)$  :  $x$  is human being.

$\forall x (P(x) \rightarrow Q(x))$ .

$\forall x (R(x) \rightarrow P(x))$ .

$\forall x (R(x) \rightarrow Q(x))$ .

$$1) \forall x (P(x) \rightarrow Q(x)) \quad (\text{Given Premise})$$

$$2) \forall x (R(x) \rightarrow P(x)) \quad (\text{Given Premise})$$

$$3) P(x) \rightarrow Q(x) \quad (\text{Universal Specification of } 1)$$

$$4) R(x) \rightarrow P(x) \quad (\text{Universal Specification of } 2)$$

$$5) R(x) \rightarrow Q(x) \quad (\text{Hypothetical Syllogism of } 3 \& 4)$$

$$6) \forall x (R(x) \rightarrow Q(x)) \quad (\text{Universal Generalization of } 5)$$

(e) Let,  $M(x)$  :  $x$  is mammal.

$A(x)$  :  $x$  is an animal.

$W(x)$  :  $x$  is warm blooded.

Translate following into a formula -

1) Every mammal is warm blooded.

Translate into English sentence.

$$1) \exists x (M(x) \rightarrow W(x))$$

2) Some animals are not warm blooded.

(f)  $g(x, y, z) : x + y = z$   
Determine the truth value of following prepositions on

statements.

$$1) \forall x \forall y \exists z g(x, y, z) \quad \text{True} \quad (\text{For all } x \text{ & } y \text{ only then will } z \text{ exist such that } x + y = z).$$

$$2) \exists z \forall x \forall y \quad \text{False.} \quad (\text{For some } z \text{ there is no } x \text{ & } y).$$

(g) Show that  $\neg P(a, b)$  follows logically from.

$$\neg x \neg y (P(x, y) \rightarrow W(x, y)) \quad (\text{Given Premise})$$

$$1) \neg x \neg y (P(x, y) \rightarrow W(x, y)) \quad (\text{Given Premise})$$

$$2) \neg W(a, b) \quad (\text{Given Premise})$$

$$3) \neg \neg P(a, b) \rightarrow \neg W(a, b) \quad (\text{Universal Specification of } 1)$$

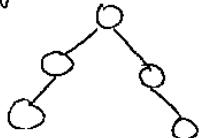
$$4) \neg P(a, b) \quad (\text{Modus Tollens of } 2 \& 3).$$

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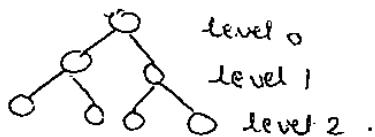
UNIT-5Graph Theory and Recurrence Relation

## \* Properties of Tree :-

- A tree having 'n' nodes have  $n-1$  branches.



- In a tree at 'l' level there will be  $2^l$  nodes



$$n = 2^0 + 2^1 + 2^2 + 2^3 + \dots + 2^h$$

$$n = 2^{h+1} - 1$$

$$\Rightarrow n+1 = 2^{h+1}$$

Taking log on both sides

$$\Rightarrow \log_2(n+1) = h+1$$

$$\Rightarrow h = \log_2(n+1) - 1$$

\* Recurrence Relations :- For a sequence of numbers or numeric function ( $a_0, a_1, a_2, a_3, \dots, a_n, \dots$ ) an equation relating  $a_n$ , for any  $n$ , to one or more of the  $a_i$ 's,  $i < n$  is called recurrence relations.

Recurrence Relation are also called difference equation because they can be written in terms of the difference b/w the consecutive terms of a sequence.

$$\text{eg } a_n = 4 \cdot a_{n-1}, n \geq 1.$$

$$a_n = a_{n-1} - 2a_{n-2}, n \geq 2.$$

## \* Order of Recurrence Relation :- The difference b/w the highest and lowest

subscripts of  $a_n$  or  $f(x)$  or  $y_n$  or  $a_n$  is called the order of recurrence relation.

$$1) a_n = 4 \cdot a_{n-1}, n \geq 1. \quad n - (n-1) = 1.$$

$$2) a_n = a_{n-2} - 2a_{n-2}, n \geq 2, \quad n - (n-2) = 2$$

$$3) a_n = 2a_{n-1} - a_{n-2}, n \geq 2, \quad n - (n-2) = 2$$

$$4) f(x+2) + 8f(x+1) + 2f(x) = 5. \quad x+2 - x = 2.$$

\* Degree of Recurrence Relation :- The highest power of  $a_n$  or  $f(x)$  or  $y_n$  is called degree of recurrence relation.

e.g.  $a_{n+2} + a_{n+1} - a_n = 0$ .

Degree = 2.

Q. Find the first four terms of  $a_k = k(a_{k-1})^2$ ,  $\forall k \geq 1$ ,  $a_0 = 1$ .

$$\rightarrow a_k = k(a_{k-1})^2$$

Put,  $k=1$ .

$$a_1 = 1(a_0)^2$$

$$a_1 = 1(1)^2$$

$$\boxed{a_1 = 1}$$

Put,  $k=2$ .

$$a_2 = 2(a_1)^2$$

$$= 2 \cdot (1)^2$$

$$\boxed{a_2 = 2}$$

Put,  $k=3$ .

$$a_3 = 3(a_2)^2$$

$$= 3 \cdot (2)^2$$

$$\boxed{a_3 = 12}$$

### \* Linear Recurrence Relation with Constant Coefficients

If the degree of the recurrence relation is 1 then the recurrence relation is called linear recurrence relation.

Generally,

$$\{c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} = f(n)\}$$

Where,  $c_i$  ( $0 \leq i \leq k$ ) is constant & ( $0 \leq i \leq k$ )  
and  $f(n)$  is a function which satisfies the equation is called linear recurrence relation.

If  $f(n) = 0$  then recurrence relation is linear Homogeneous recurrence relation.

& if  $f(n) \neq 0$  then recurrence relation is called non-homogeneous recurrence relation.

e.g.  $a_n = 5a_{n-1}$ . Linear Homogeneous.

$$a_n = a_{n-1} \cdot a_{n-2}$$
 Non Linear

$$a_n = a_{n-1} = 5$$
 Linear Non-Homogeneous.

### \* Solution of Recurrence Relation :- An explicit formula which satisfy the recurrence relation with initial condition is called solution to the recurrence relation.

There are three methods -

- Iteration Method.
- Characteristic Roots Method.
- Using Generating Function.

## Iteration Method :-

Q.  $a_k = 2 \cdot a_{k-1}$ ,  $a_0 = 1$ .

$$a_k = 2 \cdot 2 \cdot a_{k-2}$$

$$a_k = 2^2 a_{k-2}$$

$$a_k = 2^2 \cdot 2 \cdot a_{k-3}$$

$$a_k = 2^3 \cdot a_{k-3}$$

⋮

$$a_k = 2^{k-1} \cdot a_{k-(k-1)}$$

$$a_k = 2^k \cdot a_0$$

$$a_k = 2^k \cdot (1)$$

$$a_k = 2^k$$

G.P

$$\frac{a \cdot (2^{n-1})}{2^n - 1}$$

Q.  $a_n = a_{n-1} + n$ ,  $a_1 = 4$ .

$$a_n = a_{n-2} + n + (n-1)$$

$$a_n = a_{n-2} + (n-1) + n$$

$$a_n = a_{n-3} + (n-2) + (n-3) + n$$

⋮

$$a_n = a_{n-k} + (n-2) + (n-3) + \dots + (n-(k-1))$$

$$a_n = a_{n-k} + (n-(k-1)) + (n-(k-2)) + \dots + (n-2) + (n-1) + n$$

Replace  $k = n-1$

$$a_n = a_{n-(n-1)} + (n-(n-1-1)) + (n-(n-1-2)) + \dots + (n-2) + (n-1) + n$$

$$= a_1 + 2 + 3 + \dots + (n-2) + (n-1) + n$$

$$= 4 + 2 + 3 + \dots + (n-2) + (n-1) + n$$

$$= 3 + 1 + 2 + 3 + \dots + (n-2) + (n-1) + n$$

$$a_n = 3 + \frac{n(n-1)}{2}$$

Q.  $a_n = n \cdot a_{n-1}$ ,  $a_0 = 5$ .

$$a_n = n(n-1) a_{n-2}$$

$$a_n = n(n-1)(n-2) a_{n-3}$$

$$a_n = n(n-1)(n-2) \dots (n-(n-1)) \text{ out of } a_{n-n}$$

$$a_n = n(n-1)(n-2) \dots \cancel{(n-1)} \cdot 1 \cdot a_0$$

$$a_n = n! \cdot 5$$

$$a_n = 5n!$$

Q.  $a_n = 2a_{n-1} + 1$ ,  $a_1 = 7$ .

$$a_n = 2(2a_{n-2} + 1) + 1$$

( $n-k=1$ )

$$a_n = 2(2(2a_{n-3} + 1) + 1) + 1$$

( $k = n-1$ )

$$a_n = 2(2(2a_{n-3} + 1) + 1) + 1$$

$$a_n = 2^3 a_{n-3} + 2^2 + 2 + 1$$

⋮

$$a_n = 2^k a_{n-k} + 2^{k-1} + 2^{k-2} + 2^{k-3} + \dots + 2^{n-k} + 2 + 1$$

Replace,  $k = n-1$

$$\begin{aligned}
 a_n &= 2^{n+1} a_{n-1} + 2^{n+1} + 2^n + \dots + 2^2 + 2 + 1 \\
 &= 2^{n+1} a_1 + 2^{n+2} + 2^{n+4} + \dots + 2^2 + 2 + 1 \\
 &= 7 \cdot 2^{n+1} + 2^{n+2} + \dots + 2^2 + 2 + 1 \\
 &= 7 \cdot 2^{n+1} + 1 \frac{(2^{n+1}-1)}{2-1} \\
 &= 7 \cdot 2^{n+1} + 2^{n+1} - 1 \\
 &= 2^{n+1} (7+1) - 1 \\
 &= 8 \cdot 2^{n+1} - 1 \\
 &= 2^3 \cdot 2^{n+1} - 1 = 2^{3+n+1} - 1 \\
 \boxed{a_n = 2^{n+2} - 1}.
 \end{aligned}$$

### \* Methods of characteristics of Roots :-

Solution of Linear Homogeneous Recurrence Relation

When  $f(n) = 0$ .

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} = 0 \quad \text{--- (1)}$$

1) Put,  $a_n = \alpha^n$  in eqn (1)

$$\alpha^n + c_1 \alpha^{n-1} + c_2 \alpha^{n-2} + \dots + c_k \alpha^{n-k} = 0 \quad \text{--- (2)}$$

2) Divide eqn (2) by  $\alpha^{n-k}$ .

$$\frac{\alpha^n}{\alpha^{n-k}} + c_1 \frac{\alpha^{n-1}}{\alpha^{n-k}} + c_2 \frac{\alpha^{n-2}}{\alpha^{n-k}} + \dots + c_k \frac{\alpha^{n-k}}{\alpha^{n-k}} = 0$$

$$\Rightarrow [\alpha^k + c_1 \alpha^{k-1} + c_2 \alpha^{k-2} + \dots + c_k] = 0 \quad \text{--- (3)}$$

Equation (3) is called characteristic eqn and this eqn has k no. of roots and these roots are characteristic roots.

Let,  $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_k$  be the characteristic roots.

1) If roots are real.

⇒ If roots are distinct.

$$a_n = b_1 \alpha_1^n + b_2 \alpha_2^n + b_3 \alpha_3^n + \dots + b_k \alpha_k^n$$

⇒ If some roots are equal.

Let,  $\alpha_1 = \alpha_2 = \alpha_3 = \alpha, \alpha_4, \alpha_5, \dots, \alpha_k$

$$a_n = (b_1 + b_2 n + b_3 n^2) \alpha^n + b_4 \alpha_4^n + \dots + b_k \alpha_k^n$$

2) If roots are Imaginary.

⇒ If roots are distinct.

$$a_n = b_1 (\alpha_1 + i\alpha_2)^n + b_2 (\alpha_1 - i\alpha_2)^n$$

⇒ If roots are equal. (If roots are repeated then the complete pair is repeated in where, b is constant)

$$a_n = (b_1 + b_2 n) (\alpha_1 + i\alpha_2)^n + (b_3 + b_4 n) (\alpha_3 - i\alpha_2)^n$$

$$a_n - 3a_{n-1} + 2a_{n-2} = 0, \quad (1) \quad a_0 = 1, \quad a_1 = 4.$$

Put,  $a_n = \alpha^n$  in eqn (1)

$$\alpha^n - 3\alpha^{n-1} + 2\alpha^{n-2} = 0. \quad (2)$$

Divide eqn (2) by  $\alpha^{n-2}$

$$\frac{\alpha^n}{\alpha^{n-2}} - 3 \frac{\alpha^{n-1}}{\alpha^{n-2}} + 2 \frac{\alpha^{n-2}}{\alpha^{n-2}} = 0$$

$$\alpha^2 - 3\alpha + 2 = 0. \quad (3)$$

$$\alpha^2 - 2\alpha - \alpha + 2 = 0$$

$$\alpha(\alpha-2) + -1(\alpha-2) = 0$$

$$(\alpha-2)(\alpha-1) = 0$$

$\alpha = 2, 1$  ( Roots are real & distinct)

Then soln

$$a_n = b_1 \cdot 1^n + b_2 \cdot 2^n$$

$$\boxed{a_n = b_1 + b_2 2^n} \quad (4)$$

Now, Put,  $n=0$  in eqn (4)

$$a_0 = b_1 + b_2 2^0$$

$$b_1 + b_2 = 1. \quad (5)$$

Now, Put,  $n=1$  in eqn (4)

$$a_1 = b_1 + 2b_2$$

$$b_1 + 2b_2 = 4 \quad (6)$$

$$\begin{array}{r} b_1 + b_2 = 1 \\ + b_1 + 2b_2 = 4 \\ \hline + b_2 = 3 \end{array}$$

$$\boxed{b_2 = 3}$$

$$\boxed{b_1 = -2}$$

Now Put,  $b_1$  &  $b_2$  in eqn (4)

$$a_n = -2 + 3 \cdot 2^n$$

$$\Rightarrow \boxed{a_n = 3 \cdot 2^n - 2}$$

$$a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0 \quad (1) \quad a_0 = 1, \quad a_1 = 4, \quad a_2 = 8.$$

Put  $a_n = \alpha^n$  in eqn (1)

$$\alpha^n - 7\alpha^{n-1} + 16\alpha^{n-2} - 12\alpha^{n-3} = 0 \quad (2)$$

Divide eqn (2) by  $\alpha^{n-3}$

$$\alpha^3 - 7\alpha^2 + 16\alpha - 12 = 0 \quad (3)$$

$$\alpha^2(\alpha-2) - 5\alpha(\alpha-2) + 6(\alpha-2) = 0$$

$$\Rightarrow (\alpha-2)(\alpha^2 - 5\alpha + 6) = 0$$

$$\Rightarrow (\alpha-2)(\alpha-2)(\alpha-3) = 0$$

$a = 2, 2, 3 \dots$ , i.e., 2 roots are equal & 1 unequal

Then soln

$$a_n = b_1 \cdot 3^n + (b_2 + b_3 n) \cdot 2^n \quad (4)$$

Now Put,  $n = 0$  in eqn (4)

$$a_0 = b_1 + b_2.$$

$$b_1 + b_2 = 1. \quad \text{---} \quad (5)$$

Now, Put,  $n = 1$  in eq<sup>n</sup> (4)

$$a_1 = 3b_1 + 2(b_2 + b_3)$$

$$3b_1 + 2b_2 + 2b_3 = 4. \quad \text{--- (6)}$$

Now, Put,  $n = 2$  in eq<sup>n</sup> (4).

$$a_2 = 8b_1 + 4b_2 + 8b_3$$

$$9b_1 + 4b_2 + 8b_3 = 8 \quad - \textcircled{4}$$

By Solving eq'n ⑤, ⑥ & ⑦

$$b_1 = -4, \quad b_2 = 5, \quad b_3 = 3$$

$$a_n = -4 \cdot 3^n + (5 + 3n) \cdot 2^n$$

$$f_n = f_{n-1} + f_{n-2} \rightarrow \textcircled{1} \quad n \geq 2, \quad f_0 = f_1 = 1.$$

Put,  $\phi_n = x^n$ . in eqn ①

$$d^n - d^{n-1} - d^{n-2} = 0 \quad \dots \quad (2)$$

Divide eqn (2) by  $a^{n-2}$

$$\alpha^2 - \alpha - 1 = 0 \quad \text{--- (3)}$$

$$\alpha = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}$$

Then, so I

$$q_n = b_1 \left( \frac{1+\sqrt{5}}{2} \right)^n + b_2 \left( \frac{1-\sqrt{5}}{2} \right)^n. \quad (4)$$

Put,  $n = 0$  in eq<sup>n</sup> (4)

$$f_0 = b_1 + b_2$$

$$b_1 + b_2 = 1 \quad \text{---} \quad (5)$$

Put,  $n=1$  in eqn 4

$$f_1 \quad \hat{=} \quad \left( \frac{1+\sqrt{5}}{2} \right) b_1 + \left( \frac{1-\sqrt{5}}{2} \right) b_2.$$

$$\frac{1+\sqrt{5}}{2} b_1 + \frac{(1-\sqrt{5})}{2} b_2 = 1. \quad -\textcircled{6}$$

from eqn. (5) & (6)

$$\begin{aligned} & \left(\frac{1+\sqrt{5}}{2}\right) b_1 + \left(\frac{1-\sqrt{5}}{2}\right) b_2 = 0 \quad \left(\frac{1+\sqrt{5}}{2}\right), \\ & + \left(\frac{1+\sqrt{5}}{2}\right) b_1 + \left(\frac{1-\sqrt{5}}{2}\right) b_2 = 1 \\ \hline & \left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) b_2 = \left(\frac{1+\sqrt{5}}{2}\right) - 1. \end{aligned}$$

$$\begin{aligned} \sqrt{5} b_2 &= \frac{-\sqrt{5}-1}{2} \\ b_2 &= \frac{-1}{2\sqrt{5}} \\ b_1 &= \frac{-\sqrt{5}}{2\sqrt{5}} \end{aligned}$$

$$\Rightarrow f_n = -\frac{1}{2\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n + \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n$$

$$\Rightarrow f_n = \left( \frac{1+\sqrt{5}}{2\sqrt{5}} \right)^n - \left( \frac{1-\sqrt{5}}{2\sqrt{5}} \right)^n.$$

$$\Rightarrow f_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1+\sqrt{5}}{2} \right)^{n+1} - \left( \frac{1-\sqrt{5}}{2} \right)^{n+1} \right]$$

Q

$$\alpha^{n+4} + 8\alpha^{n+2} + 16\alpha^n = 0 \quad \dots \textcircled{1}$$

Put,  $\alpha^n = x^n$  in eqn \textcircled{1}

$$x^{n+4} + 8x^{n+2} + 16x^n = 0 \quad \dots \textcircled{2}$$

Divide eqn \textcircled{2} by  $x^n$ .

$$x^4 + 8x^2 + 16 = 0 \quad \dots \textcircled{3}$$

$$x^4 + 4x^2 + 4x^2 + 16 = 0$$

$$x^2(x^2 + 4) + 4(x^2 + 4) = 0$$

$$(x^2 + 4)(x^2 + 4) = 0$$

$$x = \pm 2i, \pm 2i$$

Then sol<sup>n</sup>

$$a_n = (b_1 + b_2 i)(2i)^n + (b_3 + b_4 i)(-2i)^n$$

Q

$$\text{Solve } a_{n+1} - 1.5a_n = 0, \forall n \geq 0$$

Put,  $a_n = x^n$  in eqn \textcircled{1}

$$x^{n+1} - 1.5x^n = 0. \quad \dots \textcircled{2}$$

Divide eqn \textcircled{2} by  $x^n$ .

$$x - 1.5 = 0$$

$$x = 1.5$$

Then sol<sup>n</sup>

$$a_n = b_1 (1.5)^n$$

Put,  $a_n = \alpha^n$  in eqn ①

$$\alpha^n - 2\alpha^{n-1} + \alpha^{n-2} = 0 \quad \text{--- ②}$$

Divide eqn ② by  $\alpha^{n-2}$ .

$$\alpha^2 - 2\alpha + 1 = 0 \quad \text{--- ③}$$

$$\alpha^2 - \alpha - \alpha + 1 = 0$$

$$\alpha(\alpha-1) - 1(\alpha-1) = 0$$

$$(\alpha-1)(\alpha-1) = 0$$

$$\boxed{\alpha = 1, 1}$$

Then soln.

$$\boxed{a_n = (b_1 + b_2 n)}$$

Q.  $a_n = 5a_{n-1} + 6a_{n-2} \quad \text{--- ①} \quad n \geq 0, a_0 = a_1 = 3.$

Put,  $a_n = \alpha^n$  in eqn ①

$$\alpha^n - 5\alpha^{n-1} - 6\alpha^{n-2} = 0 \quad \text{--- ②}$$

Divide eqn ② by  $\alpha^{n-2}$ .

$$\alpha^2 - 5\alpha - 6 = 0 \quad \text{--- ③}$$

$$\alpha^2 - 6\alpha - 5\alpha - 6 = 0$$

$$\alpha(\alpha-6) + 5(\alpha-6) = 0$$

$$(\alpha+1)(\alpha-6) = 0$$

$$\alpha = -1, 6$$

Then soln.

$$\boxed{a_n = b_1(-1)^n + b_2(6)^n} \quad \text{--- ④}$$

Put,  $\alpha = n=0$

$$a_0 = b_1(-1)^0 + b_2(6)^0$$

$$b_1 + b_2 = 3 \quad \text{--- ⑤}$$

Put,  $n=1$ .

$$a_1 = b_1(-1)^1 + b_2(6)^1$$

$$-b_1 + 6b_2 = 3 \quad \text{--- ⑥}$$

on solving eqn ⑤ & ⑥.

$$\cancel{b_1 + b_2 = 3}$$

$$\cancel{-b_1 + 6b_2 = 3}$$

$$\underline{+b_2 = 6}$$

$$\boxed{b_2 = \frac{6}{7}}$$

$$\boxed{b_1 = \frac{15}{7}}$$

$$\Rightarrow \boxed{a_n = \frac{15}{7}(-1)^n + \left(\frac{6}{7}\right)(6)^n}$$

- Solution of Non-Homogeneous Recurrence Relation:  
 $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} = f(n) \quad \text{--- (1)}$
- 1) Put,  $f(n)=0$  and find  $a_n^{(h)}$  (Homogeneous solution)
  - 2) Find particular solution or trial solution  $a_n^{(P)}$  with the help of  $f(n)$ .
  - 3) Determine coefficients of  $a_n^{(P)}$  by putting it into eqn (1)
  - 4) complete solution  $a_n = a_n^{(h)} + a_n^{(P)}$
  - 5) Determine coefficients with the help of initial condition if given.

S.N	$f(n)$	Trial function (or) Particular solution ( $a_n^{(P)}$ )
1.	$A$ (constant)	$A_0$
2.	$c^n$ [ $c$ is not a characteristic root]	$A_0 c^n$
3.	$c^n$ [ $c$ is a characteristic root with multiplicity s]	$A_0 n^s c^n$
4.	$P(n)$ [Polynomial of degree m]	$A_0 + A_1 n + A_2 n^2 + \dots + A_m n^m$
5.	$c^n \cdot P(n)$ [ $c$ is not a characteristic root]	$[A_0 + A_1 n + A_2 n^2 + \dots + A_m n^m] c^n$
6.	$c^n \cdot P(n)$ [ $c$ is a characteristic root with multiplicity s]	$[A_0 + A_1 n + A_2 n^2 + \dots + A_m n^m] n^s c^n$

Q.  $y_k - y_{k-1} - 6y_{k-2} = -30 \quad \text{--- (1)} \quad y_0 = 20, y_1 = -5$   
 For Homogeneous soln make RHS of eqn (1) = 0 i.e.  
 $f(n) = 0$ .

Now consider the eqn -

$$y_k - y_{k-1} - 6y_{k-2} = 0 \quad \text{--- (2)}$$

Put,  $y_k = \alpha^k$  in eqn (2)

$$\alpha^k - \alpha^{k-1} - 6\alpha^{k-2} = 0$$

Divide eqn  $\alpha^{k-2}$   $\otimes$

$$\alpha^2 - \alpha - 6 = 0 \quad \text{--- (3)}$$

$$\Rightarrow \alpha = -2, 3$$

$$y_k^{(h)} = b_1(-2)^k + b_2(3)^k \quad \text{--- (4)}$$

Now,  $f(k) = -30$  i.e. constant.

So, particular soln.

$$y_k^{(P)} = A \quad \text{--- (5)}$$

Now, substitute eqn (5) in eqn (1).

$$A - A - 6A = -30$$

$$A = 5$$

$$\text{so, } y_k^{(P)} = 5 \quad \text{--- (6)}$$

Complete soln will be -

$$y_k = y_k^{(n)} + y_k^{(p)}$$

$$y_k = b_1(-2)^k + b_2 3^k + 5 \quad \text{--- (1)}$$

Put,  $k=0$  in eqn (1) &  $k=1$

$$y_0 = b_1 + b_2 + 5$$

$$20 = b_1 + b_2 + 5$$

$$\Rightarrow b_1 + b_2 = 15$$

$$y_1 = -2b_1 + 3b_2 + 5$$

$$-5 = -2b_1 + 3b_2 + 5$$

$$\Rightarrow 2b_1 - 3b_2 = 10$$

$$2b_1 + 2b_2 = 30$$

$$+ 2b_1 - 3b_2 = 10$$

$$\underline{\underline{5b_2 = 20}}$$

$$b_2 = 4$$

$$b_1 = 15 - 4$$

$$b_1 = 11$$

Sol<sup>n</sup> will be -

$$y_k = (11)(-2)^k + 4 \cdot 3^k + 5$$

$$Q \quad a_n - 6a_{n-1} + 8a_{n-2} = 3^n \quad \text{--- (1)} \quad a_0 = 3, a_1 = 9$$

$$\alpha^n - 6\alpha^{n-1} + 8\alpha^{n-2} = 0$$

$$\alpha^2 - 6\alpha + 8 = 0$$

$$\alpha^2 - 4\alpha - 2\alpha + 8 = 0$$

$$\alpha(\alpha-4) - 2(\alpha-4) = 0$$

$$(\alpha-4)(\alpha-2) = 0$$

$$\alpha = 4, 2$$

$$a_n^{(h)} = b_1(2)^n + b_2(4)^n$$

$$f(n) = 3^n$$

Particular sol<sup>n</sup> will be -

$$a_n^{(p)} = A 3^n$$

Put in eqn (1)

$$A \cdot 3^n - 6 \cdot A \cdot 3^{n-1} + 8 \cdot A \cdot 3^{n-2} = 3^n$$

$$A \cdot 3^n \left(1 - \frac{6}{3} + \frac{8}{9}\right) = 3^n$$

$$A \left(\frac{9-18+8}{9}\right) = 1$$

$$\Rightarrow A = -9$$

$$a_n^{(p)} = -9 \cdot 3^n$$

Complete sol<sup>n</sup>.

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = b_1 \cdot 2^n + b_2 \cdot 4^n - 9 \cdot 3^n \quad \text{--- (2)}$$

Put,  $n=0$  &  $n=1$  in eqn (2)

$$a_0 = b_1 + b_2 - 9$$

$$b_1 + b_2 - 9 = 3$$

$$a_n(p) = \frac{q}{2} + \frac{5}{2}a_1 + \frac{1}{2}a_2$$

$$A_0 = \frac{q}{2}$$

$$\Rightarrow 2A_0 - \frac{q}{2} - \frac{1}{2} = 1$$

$$2A_0 - 3A_1 - A_2 = 1$$

$$A_0 + 2A_1 + 4A_2 - 5A_0 - 5A_1 - 5A_2 + 6A_0 = 1$$

Comparing Coefficients of  $a_1$

$$A_1 = \frac{5}{2}$$

$$\Rightarrow 2A_1 - 6A_2 = 2$$

$$A_1 + 4A_2 - 5A_1 - 10A_2 + 6A_2 = 2$$

Comparing Coefficients of  $a_2$

$$A_2 = \frac{1}{2}$$

$$A_2 - 5A_2 + 6A_2 = 1$$

Comparing Coefficients of  $a_0$  in both sides

$$+ 6(A_0 + A_1 a_1 + A_2 a_2) = a_2 + 2a_1 + 1 \\ \Rightarrow A_0 + A_1(a_1 + 2) + A_2(a_2 + 4 + 4a_1) - 5(A_0 + A_1(a_1 + 1) + A_2(a_1 + 2a_1))$$

$$6(A_0 + A_1 a_1 + A_2 a_2) = a_2 + 2a_1 + 1 \\ A_0 + A_1(a_1 + 2) + A_2(a_2 + 4 + 4a_1) - 5(A_0 + A_1(a_1 + 1) + A_2(a_1 + 2a_1)) +$$

$$a_n(p) = A_0 + A_1 a_1 + A_2 a_2$$

$$f(n) = (a_1 + 1)^2 = a_1^2 + 2a_1 + 1$$

$$a_n(l_n) = b_1 a_1 + b_2 3a_1$$

Solving

$$\alpha = 3$$

$$(\alpha - 3)(\alpha - 2) = 0$$

$$\alpha(\alpha - 3) - 2(\alpha - 2) = 0$$

$$\alpha^2 - 3\alpha - 2\alpha + 6 = 0$$

$$\alpha^2 - 5\alpha + 6 = 0$$

$$\alpha^{a_1+2} - 5\alpha^{a_1+1} + 6\alpha^{a_1} = 0$$

$$a_{a_1+2} - 5a_{a_1+1} + 6a_{a_1} = (a_1 + 1)^2$$

$$a_n = T-a_n + 5-a_n - a_1 a_n$$

Solving

$$-b_1 = T$$

$$b_2 = 5$$

$$+b_2 = 7.5$$

$$-b_1 + 2b_2 = 17$$

$$b_1 + b_2 = 12$$

$$b_1 + 2b_2 = 17$$

$$2b_1 + 4b_2 = 34$$

$$T = 3b_1 + 4b_2 - 27$$

$$a_1 = 2b_1 + 4b_2 - 27$$

$$b_1 + b_2 = 12$$

$$a_{\alpha} = [A_0 \alpha^2 + A_1 \alpha^2 - A_0 \alpha - A_1 \alpha] \Rightarrow$$

$$(A_0 \alpha + A_1 \alpha) \alpha - \frac{1}{4} [A_0 \alpha^2 + A_1 \alpha^2 + A_0 - 2A_1 \alpha] + \frac{1}{16} (A_0 \alpha - 2A_0)$$

$$\text{Put in eqn ①} \quad (A_0 \alpha + A_1 \alpha) \alpha - \frac{1}{4} (A_0 \alpha^2 + A_1 \alpha^2) \alpha + \frac{1}{16} (A_0 \alpha^2 + A_1 \alpha^2)^2$$

$$a_{\alpha}(p) = (A_0 + A_1 \alpha) \cdot \alpha \cdot \alpha$$

Partial fraction  $A_0 \alpha$

$$a_{\alpha} = b_1(3)\alpha + b_2(4)\alpha$$

so in  $\alpha$

$$\alpha = 3, 4.$$

$$\alpha^2 - 7\alpha + 12 = 0$$

$$\textcircled{1} \quad a_{\alpha} - 7a_{\alpha-1} + 12a_{\alpha-2} = \alpha \cdot \alpha$$

$$a_{\alpha} = b_1 \cdot 3\alpha + b_2 \cdot 4\alpha + 10 \cdot a(5)\alpha$$

$$a_{\alpha} = a_{\alpha}(p) + a_{\alpha}(r)$$

$$\text{complete } A_0 \alpha$$

$$a_{\alpha}(p) = +\frac{10}{10} \alpha(5)\alpha \Leftrightarrow$$

$$A_0 = +\frac{10}{10}$$

$$A_0 = +\frac{10}{10} \alpha$$

$$+ \frac{10}{10} A_0 = 50$$

$$\frac{25}{25} A_0 \alpha - \frac{45}{25} A_0 \alpha + \frac{20}{25} A_0 \alpha - \frac{40}{25} A_0 \alpha = 2$$

$$A_0 \alpha - \frac{9}{5} A_0 \alpha + \frac{9}{5} A_0 \alpha + \frac{20}{25} A_0 \alpha - \frac{40}{25} A_0 \alpha = 2$$

$$A_0 \alpha - \frac{9}{5} A_0 \alpha (5\alpha-1) + \frac{20}{25} A_0 \alpha (5\alpha-2) = 2 \cdot 5\alpha$$

$$A_0 \cdot 5\alpha - 9 A_0 (5\alpha-1) + 20 A_0 (5\alpha-2) = 2 \cdot 5\alpha$$

$$2(5\alpha) = 9 \cdot 2(5\alpha-1) - 20 \cdot 2(5\alpha-2) = -2(5\alpha)$$

Put in eqn ①

$$a_{\alpha}(p) = A_0 \alpha (5)\alpha \quad \text{partial fraction } A_0 \alpha$$

$$a_{\alpha} = b_1(5)\alpha + b_2(5)\alpha$$

$$\alpha = 5, 5$$

$$0 = (5-5)(5-5)$$

$$\alpha(5-5) - 5(5-5) = 0$$

$$\alpha^2 - 4\alpha - 5\alpha + 25 = 0$$

$$\alpha^2 - 9\alpha + 25 = 0$$

$$\alpha \quad a_{\alpha} - 9a_{\alpha-1} + 20a_{\alpha-2} = 2$$

$$! \Rightarrow A_0 - \frac{7}{4}A_0 + \frac{14}{4}A_1 + \frac{3}{4}A_0 - \frac{12}{4}A_1 = 1.$$

~~$\frac{A_0}{4}$~~   $\frac{1}{2}A_1 = 1$

$\Rightarrow A_1 = 2$

$$\Rightarrow \frac{7}{4}A_0 - \frac{7}{4}A_1 - \frac{6}{4}A_0 + \frac{12}{4}A_1 = 0.$$

$$\Rightarrow \frac{1}{4}A_0 + \frac{5}{4}A_1 = 0$$

$$\frac{1}{4}A_0 = -\frac{5}{4}A_1$$

$$A_0 = -10$$

Soln will be

$$a_n^{(P)} = (2n-10)n \cdot 4^n$$

complete soln

$$a_n = a_n^{(P)} + a_n^{(h)}$$

$$a_n = b_1(3)^n + b_2(4)^n + (2n-10)n \cdot 4^n$$

Q.  $s(k) - 9s(k-1) + 8s(k-2) = 9k+1 \rightarrow ① s(0) = s(1) = 1$

$$\alpha^2 - 9\alpha + 8 = 0$$

$$\alpha = 1, 8$$

$$s(k) = b_1(1)^k + b_2(8)^k$$

$$s(k)^{(h)} = b_1 + b_2(8)^k \quad ②$$

Particular soln -

$$s(k)^{(P)} = (A_0 + A_1 k)k$$

Put in eqn ① -

$$(A_0k + A_1k^2) - 9(A_0(k-1) + A_1(k-1)^2) + 8(A_0(k-2) + A_1(k-2)^2) = 9k+1$$

$$A_0k + A_1k^2 - 9A_0k + 9A_0 + 9A_1k^2 - 18A_1k + 8A_0k - 16A_0 + 8A_1k^2 - 32A_1k = 9k+1$$

Comparing coefficients of  $k$ .

$$A_0 - 9A_0 - 18A_1 + 8A_0 - 32A_1 = 9$$

$$-14A_1 = 9$$

$$A_1 = -\frac{9}{14}$$

Comparing constant -

$$9A_0 + 9A_1 - 16A_0 + 32A_1 = 1$$

$$-7A_0 + 23A_1 = 1$$

$$-7A_0 = -23 \times \frac{9}{14} - 1$$

$$A_0 = -\frac{221}{98}$$

Put,  $k=0$  in eq ②

$$S(0) = b_1 + b_2$$

$$1 = b_1 + b_2$$

Put,  $k=1$  in eq ②

$$S(1) = b_1 + 8b_2$$

$$b_1 + 8b_2 = 1.$$

$$\begin{array}{r} b_1 + b_2 = 1 \\ + b_1 + 8b_2 = 1 \\ \hline -7b_2 = 0 \end{array}$$

complete soln

$$S(k) = S(k^P) + S(k^M)$$

$$S(k) = b_1 + b_2 (8)^k + (A_0 + A_1 k) k$$

$$= b_1 + b_2 (8)^k + \left(-\frac{221}{98} + \frac{9}{14} k\right) k$$

$$S(k) = b_1 + b_2 (8)^k - \frac{221}{98} k - \frac{9}{14} k^2 \quad \leftarrow \textcircled{3}$$

Put  $k=0$  in eq ③

$$S(0) = b_1 + b_2$$

$$b_1 + b_2 = 1.$$

Put,  $k=1$  in eq ③

$$S(1) = b_1 + 8b_2 - \frac{221}{98} - \frac{9}{14}$$

$$1 = b_1 + 8b_2 - \frac{284}{98}$$

$$b_1 + 8b_2 = 1 + \frac{284}{98} = \frac{98+284}{98} = \frac{382}{98}$$

$$b_1 + 8b_2 = \frac{191}{98}$$

$$\begin{array}{r} b_1 + b_2 = 1 \\ + b_1 + 8b_2 = \frac{191}{98} \\ \hline -7b_2 = \frac{49-191}{98} \end{array}$$

$$b_2 = \frac{142}{49 \times 7} \Rightarrow b_2 = \frac{142}{343}$$

$$b_2 = 1 - \frac{142}{343} \Rightarrow b_2 = \frac{201}{343}$$

$$S(k) = \frac{201}{343} + \frac{142}{343} (8)^k - \frac{221}{98} k - \frac{9}{14} k^2$$

$$y_{n+2} - y_{n+1} - y_n = n^2 \quad \text{--- (1)}$$

$$\alpha^2 - \alpha - 1 = 0$$

$$\alpha = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$$

$$\frac{1 \pm \sqrt{5}}{2}$$

$$y_n^{(h)} = b_1 \left( \frac{1+\sqrt{5}}{2} \right)^n + b_2 \left( \frac{1-\sqrt{5}}{2} \right)^n$$

Particular sol<sup>n</sup> will be -

$$y_n^{(p)} = (A_0 + A_1 n + A_2 n^2)$$

Put this in eq<sup>n</sup> (1)

$$[A_0 + A_1(n+2) + A_2(n+2)^2] - [A_0 + A_1(n+1) + A_2(n+1)^2] -$$

$$[A_0 + A_1 n + A_2 n^2] = n^2$$

$$A_0 + A_1 n + 2A_1 + A_2 n^2 + 4A_2 + 4A_2 n - A_0 - A_1 n - A_1 - A_2 n^2 - A_2$$

$$- 2A_2 n - A_0 - A_1 n - A_2 n^2 = n^2$$

Compare the coefficients of  $n^2$  -

$$\cancel{A_2} - A_2 - A_2 = 1$$

$$\Rightarrow \boxed{A_2 = -1}$$

Compare the coefficients of  $n$ .

$$\cancel{A_1} + 4A_2 - A_1 - 2A_2 - \cancel{A_1} = 0$$

$$-A_1 + 2 = 0$$

$$-A_1 = +2$$

$$\Rightarrow \boxed{A_1 = -2}$$

Compare constants -

$$A_0 + 2A_1 + 4A_2 - A_0 - A_1 - A_2 - \cancel{A_0} = 0$$

$$A_1 + 3A_2 - A_0 = 0$$

$$-2 - 3 - A_0 = 0$$

$$\Rightarrow \boxed{A_0 = -5}$$

$$y_n^{(p)} = (-5 - 2n - n^2)$$

complete sol<sup>n</sup>

$$y_n = y_n^{(h)} + y_n^{(p)}$$

$$y_n = b_1 \left( \frac{1+\sqrt{5}}{2} \right)^n + b_2 \left( \frac{1-\sqrt{5}}{2} \right)^n - 5 - 2n - n^2$$

$$a_n - 6a_{n-1} + 8a_{n-2} = n \cdot 4^n \quad \text{--- (1)} \quad a_0 = 8, a_1 = 22$$

$$\alpha^2 - 6\alpha + 8 = 0$$

$$\alpha^2 - 4\alpha - 2\alpha + 8 = 0$$

$$\alpha(\alpha-4) - 2(\alpha-4) = 0$$

$$(\alpha-2)(\alpha-4) = 0$$

$$\alpha = 2, 4$$

$$y_n^{(h)} = b_1 2^n + b_2 4^n$$

Particular sol<sup>n</sup> -

$$a_n^{(p)} = (A_0 + A_1 n) n \cdot 4^n$$

$$(A_0 + A_1 \pi) \pi \cdot 4^\pi = 6(A_0(\pi-1) + A_1(\pi-1)^2)4^\pi + 8(A_0(\pi-2) + A_1(\pi-1)^2)4^\pi + 8(A_0\pi \cdot 4^\pi + A_1\pi^2 \cdot 4^\pi) = 6A_0\pi \cdot 4^\pi + 6A_04^\pi - 6A_1\pi^2 \cdot 4^\pi - 6A_14^\pi + 12A_1\pi^4 + 8[A_0\pi - 2A_0 + A_1\pi^2 + 4A_1 - 4A_1\pi]4^\pi = \pi 4^\pi.$$

$$A_0\pi + A_1\pi^2 - 6A_0\pi + 6A_0 - 6A_1\pi^2 - 6A_1 + 12A_1\pi + 8A_0\pi - 16A_0 + 8A_1\pi^2 + 32A_1 - 32A_1\pi = \pi.$$

Compare coefficients of  $\pi$ :

$$2(A_0\pi + A_1\pi^2) - 3(A_0(\pi-1) + A_1(\pi^2 - 2\pi + 1)) + (A_0(\pi-2) + A_1(\pi^2 - 4\pi + 4)) = 2\pi.$$

$$\cancel{A_0} - 3\cancel{A_0} + \cancel{A_1} + \cancel{A_0} - 4A_1 = 2 \\ 2A_1 = 2 \\ \Rightarrow A_1 = 1$$

$$3A_0 - 3A_1 - 2A_0 + 4A_1 = 0.$$

$$A_0 + A_1 = 0$$

$$\Rightarrow A_0 = -1$$

$$a_n^{(EP)} = (\pi-1) \cdot \pi \cdot 4^\pi.$$

complete soln

$$a_n = a_n^{(H)} + a_n^{(P)} \\ a_n = b_1 \cdot 2^\pi + b_2 \cdot 4^\pi + (\pi-1) \cdot \pi \cdot 4^\pi. \quad (2)$$

Put,  $\pi=0$  &  $\pi=1$  in eqn (2)

$$a_0 = b_1 + b_2.$$

$$\Rightarrow b_1 + b_2 = 8. \quad \leftarrow$$

$$a_1 = 2b_1 + 4b_2$$

$$\Rightarrow b_1 + 2b_2 = 11.$$

$$\boxed{b_1 = 5}$$

$$\begin{array}{r} \cancel{b_1 + b_2 = 8} \\ + \cancel{b_1 + 2b_2 = 11} \\ \hline \cancel{+ b_2 = 7} \\ \boxed{b_2 = 3} \end{array}$$

Now, soln -

$$a_n = 5 \cdot 2^\pi + 3 \cdot 4^\pi + (\pi-1) \pi \cdot 4^\pi$$

\* Generating Function (G.P.) :- The generating function for  $n$ th sequence  $a_0, a_1, a_2, a_3, a_4, \dots, a_k, \dots$  of real numbers is infinite series given by

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_kx^k + \dots = \sum_{k=0}^{\infty} a_k \cdot x^k$$

Here, 'x' is considered as just a symbol called indeterminate and it is not a variable which is replaced

by numbers belonging to same domain.

Q. Find the generating function of the given sequence.

1, 2, 4, 8, -----

→ Here,  $a_0 = 1, a_1 = 2, a_2 = 4, a_3 = 8$

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

$$G(x) = 1 + 2x + 4x^2 + 8x^3 + \dots$$

$$G(x) = 1 + 2x + 2^2 \cdot x^2 + 2^3 \cdot x^3 + \dots$$

$$G(x) = \boxed{\frac{1}{1-2x}} = \sum_{n=0}^{\infty} 2^n \cdot x^n$$

$$a_n, n$$

$$T_n = a_{n-1}$$

$$S_n = \frac{n(1-a_n)}{1-x}$$

$$S_\infty = \frac{1}{1-x}$$

Q

1, -1, 1, -1, -----

$a_0 = 1, a_1 = -1, a_2 = 1, a_3 = -1$

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

$$G(x) = 1 - x + x^2 - x^3 + \dots$$

$$G(x) = \boxed{\frac{1}{1+x}} = \sum_{n=0}^{\infty} (-x)^n$$

Q

1, 1, 0, 1, 1, 1, -----

Here,  $a_0 = 1, a_1 = 1, a_2 = 0, a_3 = 1, a_4 = 1, a_5 = 1$

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$$

$$G(x) = (1 + x + x^3 + x^4 + x^5 + \dots + x^2) - x^2$$

$$G(x) = \boxed{\frac{1}{1-x} - x^2}$$

Q

0, 0, 1, 1, 1, -----

Here,  $a_0 = 0, a_1 = 0, a_2 = 1, a_3 = 1, a_4 = 1$

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$$

$$G(x) = 0 + 0 + x^2 + x^3 + x^4 + \dots$$

$$G(x) = \boxed{\frac{x^2}{1-x}}$$

Q. Find the generating function of the sequence whose  $n$ th term is  $a_n = n$ .

Here,  $a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 3, \dots$

$$G(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

$$G(x) = 0 + x + 2x^2 + 3x^3 + 4x^4 + \dots$$

$$\frac{x}{(1-x)} G(x) = \frac{x}{1-x} + \frac{x^2}{1-x} + \frac{x^3}{1-x} + \frac{x^4}{1-x} + \dots$$

$$(1-x) G(x) = x + x^2 + x^3 + x^4 + \dots$$

$$\Rightarrow G(x) = \boxed{\frac{x}{(1-x)^2}}$$

Q. Determine the generating function.

Here, it is given  
A.P.G.P; therefore  
we multiply the  
common ratio of  
A.P & shifts the  
zero term.

$$\begin{aligned}
 & a(x) = \frac{1-3x+2x^2}{2-3x} \\
 & a(x) [1-3x+2x^2] = 2-3x \\
 & a(x) [1-3x+2x^2] - 2-3x + 6x = 0 \\
 & a(x) [1-3x+2x^2] - a_0 - a_1x + 3xa_0 = 0 \\
 & a(x) - a_0 - a_1x - 3x [a(x) - a_0] + 2x^2 (a(x)) = 0 \\
 & a(x) - a_0 - a_1x - 3x \cdot \frac{a(x) - a_0}{x} + 2x^2 a(x) = 0 \\
 & a(x) - a_0 - a_1x - 3 \cdot \frac{a(x) - a_0}{x} + 2 a(x) = 0 \\
 & + 2(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots) = 0 \\
 & \Rightarrow x^2(a_2 + a_3x + a_4x^2 + \dots) - 3x(a_1 + a_2x + a_3x^2 + \dots) + \\
 & 2(a_0 + a_1x + a_2x^2 + \dots) - 3(a_0 + a_1x + a_2x^2 + \dots) = 0 \\
 & \Rightarrow (a_2 + a_3x + a_4x^2 + \dots) - 3(a_1 + a_2x + a_3x^2 + \dots) = 0 \\
 & \text{Now, Expand all terms.} \\
 & \text{Multiplying eqn ① by } x^2 \text{ and summing from } x=0 \text{ to } \infty
 \end{aligned}$$

\* Solution of recurrence relation using G.F.

$$a(x) = \frac{(1+x)^2}{(3-x)^2} = \frac{1+2x+x^2}{9-6x+x^2}$$

$$\begin{aligned}
 (1-\frac{x}{3}) a(x) &= 1 - \frac{1}{3}x + \frac{1}{9}x^2 + \frac{1}{27}x^3 + \dots \\
 \frac{d}{dx} a(x) &= \frac{1}{3} - \frac{1}{3}x + \frac{2}{9}x^2 + \frac{2}{27}x^3 + \dots \\
 a(x) &= 1 + \frac{1}{3}x + \frac{3}{9}x^2 + \frac{4}{27}x^3 + \dots \\
 a(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \dots
 \end{aligned}$$

Hence,  $a_0 = 1$ ,  $a_1 = \frac{1}{3}$ ,  $a_2 = \frac{1}{9}$ ,  $a_3 = \frac{4}{27}$ .  
 $1, \frac{1}{3}, \frac{3}{9}, \frac{4}{27}, \dots = \frac{1}{(x+1)}, \dots$

Find the simple expansion for the G.F. if the following

$$a(x) = \frac{1+2x}{1-2x} = \sum_{x=0}^{\infty} (-2x)^n x^n$$

$$\begin{aligned}
 a(x) &= 1 + 2x + 2^2x^2 + 2^3x^3 + 2^4x^4 + \dots \\
 a(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots \\
 \text{Hence, } a_0 &= 1, a_1 = -2, a_2 = 2^2, a_3 = -2^3, a_4 = 2^4, \dots
 \end{aligned}$$

$\{ -2^n \text{ if } n \text{ is odd}, 0 \text{ if } n \text{ is even} \}$

$$\Rightarrow u(x) = \frac{x-2x}{(1-2x)(1-x)} = \frac{A}{1-2x} + \frac{B}{1-x}$$

$$u(x) = \frac{1}{1-2x} + \frac{1}{1-x}$$

$$u(x) = \sum_{n=0}^{\infty} (2x)^n + \sum_{n=0}^{\infty} (x)^n$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (2x)^n + \sum_{n=0}^{\infty} (x)^n$$

$$a_n = 2^n + 1.$$

$$\text{S } a_{n+2} - 2a_{n+1} + a_n = 2^n. \quad \text{--- (1)} \quad a_0 = 2, a_1 = 1.$$

Let,  $G(x) = \sum_{n=0}^{\infty} a_n x^n$  is the gf of this function.

Multiply eqn (1) by  $x^n$  & summing  $\forall n=0$  to  $n=\infty$

$$\sum_{n=0}^{\infty} a_{n+2} x^n - 2 \sum_{n=0}^{\infty} a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2^n x^n$$

$$(a_2 + a_3 x + a_4 x^2 + \dots) - 2(a_1 + a_2 x + a_3 x^2 + \dots) \\ + (a_0 + a_1 x + a_2 x^2 + \dots) = 1 + 2x + 2^2 x^2 + \dots$$

$$\frac{x^2}{x^2} (a_2 + a_3 x^2 + a_4 x^4 + \dots) + \frac{2x}{x} (a_1 + a_2 x + a_3 x^2 + \dots) \\ + (a_0 + a_1 x + a_2 x^2 + \dots) = 1 + 2x + 2^2 x^2 + \dots$$

$$\frac{(G(x) - a_0 - a_1 x)}{x^2} + \frac{2x}{x} (G(x) - a_0) + G(x) = 1 + 2x + 2^2 x^2 + \dots$$

$$G(x) - a_0 - a_1 x + 2x(G(x) - a_0) + x^2 G(x) = x^2 (1 + 2x + 2^2 x^2 + \dots)$$

$$\Rightarrow G(x) [1 + 2x + x^2] - a_0 - a_1 x + 2a_0 x = x^2 + 2x^3 + 2^2 x^4 + \dots$$

$$\Rightarrow G(x) [1 - 2x + x^2] - 2 - x + 4x = x^2 [1 + 2x + 2^2 x^2 + \dots]$$

$$\Rightarrow G(x) [x^2 - 2x + 1] = x^2 [1 + 2x + 2^2 x^2 + \dots] + 2 + 3x.$$

$$\Rightarrow G(x) (x^2 - 2x + 1) x^2 + x^3 + 4x^4 + \dots = \frac{x^2}{1-2x} + 2 + 3x.$$

$$\Rightarrow G(x) = \frac{4x^2 - 4x + 2}{(1-2x)(x^2 - 2x + 1)}$$

$$\Rightarrow G(x) = \frac{4x^2 - 4x + 2}{(1-2x)(1-x)^2}$$

$$\Rightarrow G(x) = \frac{A}{(1-x)^2} + \frac{B}{(1-x)} + \frac{C}{(1-2x)}$$

$$4x^2 - 4x + 2 = A(1-2x) + B(2x^2 - 3x + 1) + C(x^2 - 2x + 1)$$

$$B = 3, \quad A = -2, \quad C = 1.$$

$$\Rightarrow G(x) = \frac{-2}{(1-x)^2} + \frac{3}{(1-x)} + \frac{1}{(1-2x)}$$

$1-2x = 0 \Rightarrow x = \frac{1}{2}$   
 $1-x = 0 \Rightarrow x = 1$   
 equal x values  
 used check for  
 particular values  
 $x=1$  value  
 R.H.S.  
 $1-x = 1$   
 $(2-3)x^2$   
 $(1-2x) - 1$

$$\Rightarrow G(x) = -2 \left( 1 + 2x + \frac{2 \cdot 3}{2!} x^2 + \frac{2 \cdot 3 \cdot 4}{3!} x^3 + \dots \right) + 3 \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} 2^n \cdot x^n$$

$$\Rightarrow G(x) = -2 \left( 1 + 2x + 3x^2 + 4x^3 + \dots \right) + 3 \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} 2^n \cdot x^n$$

$$\Rightarrow G(x) = -2 \sum_{n=0}^{\infty} (n+1) \cdot x^n + 3 \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} 2^n \cdot x^n$$

$$\Rightarrow a_n = -2n - 2 + 3 + 2^n$$

$$\Rightarrow a_n = 2^n - 2n + 1.$$

Q  $f(k) = f(k-1) + f(k-2) \quad \dots \quad \text{Eqn } ① \quad f(0) = f(1) = 1.$   
 Let,  $G(x) = \sum_{k=0}^{\infty} f(k) \cdot x^k$  is the g.f. of this func.

Multiply eqn ① by  $x^k$  & summing  $k=0$  to  $k=\infty$ .

$$\sum_{k=2}^{\infty} f(k+2) x^k - \sum_{k=2}^{\infty} f(k-1) x^k - \sum_{k=2}^{\infty} f(k-2) x^k = 0.$$

$$(f(2)x^2 + f(3)x^3 + \dots) - (f(1)x^2 + f(2)x^3 + \dots) - (f(0)x^2 + f(1)$$

$$+ \dots) = 0.$$

$$(G(x) - f(0) - f(1)x) - x(G(x) - f(0)) - x^2 G(x) = 0.$$

$$G(x)[1 - x - x^2] - 0 = 1 - x + x^2 = 0.$$

$$G(x)(1 - x - x^2) = 1.$$

$$G(x) = \frac{1}{1 - x - x^2}$$

$$G(x) = \frac{1}{(x^2 + x - 1)}$$

$$G(x) = \frac{1}{\left(x - \frac{-1+\sqrt{5}}{2}\right)\left(x - \frac{-1-\sqrt{5}}{2}\right)}$$

$$G(x) = \frac{A}{\left(x - \frac{-1+\sqrt{5}}{2}\right)} + \frac{B}{\left(x - \frac{-1-\sqrt{5}}{2}\right)}$$

$$A = \frac{1}{\frac{-1+\sqrt{5}}{2} - \left(\frac{-1-\sqrt{5}}{2}\right)} = \frac{1}{-1 + \sqrt{5}} = \frac{1}{\sqrt{5}}$$

$$A = \frac{1}{\sqrt{5}}, \quad B = \frac{1}{\sqrt{5}}$$

$$G(x) = \frac{\frac{1}{\sqrt{5}}}{\left(x - \left(\frac{-1+\sqrt{5}}{2}\right)\right)} + \frac{\frac{1}{\sqrt{5}}}{\left(x + \frac{1+\sqrt{5}}{2}\right)}$$

$$G(x) = \frac{1}{\sqrt{5}} \left( \frac{\sqrt{5}-1}{2} - x \right) + \frac{1}{\sqrt{5}} \left( \frac{\sqrt{5}+1}{2} + x \right)$$

$$G(x) = \frac{2}{\sqrt{5}(\sqrt{5}-1)} \left( 1 - \frac{2x}{\sqrt{5}-1} \right) + \frac{2}{\sqrt{5}(\sqrt{5}+1)} \left( 1 + \frac{2x}{\sqrt{5}+1} \right)$$

$$\sum_{k=0}^{\infty} F(k) x^k = \frac{1}{\sqrt{5}} \left[ \frac{2}{\sqrt{5}-1} \sum_{k=0}^{\infty} \left( \frac{2x}{\sqrt{5}-1} \right)^k + \frac{2}{\sqrt{5}+1} \sum_{k=0}^{\infty} \left( \frac{-2x}{\sqrt{5}+1} \right)^k \right]$$

Compare the coefficients of  $x^k$ .

$$F(k) = \frac{1}{\sqrt{5}} \left[ \frac{2}{\sqrt{5}-1} \left( \frac{2}{\sqrt{5}-1} \right)^k + \frac{2}{\sqrt{5}+1} \left( \frac{-2}{\sqrt{5}+1} \right)^k (-1)^k \right]$$

$$F(k) = \frac{1}{\sqrt{5}} \left[ \left( \frac{2}{\sqrt{5}-1} \right)^{k+1} + \left( \frac{2}{\sqrt{5}+1} \right)^{k+1} (-1)^{k+1} \right].$$

$$\textcircled{1} \quad a_0 - 2a_{n-1} - 3a_{n-2} = 0, \quad \text{for } n \geq 2, \quad a_0 = 3, \quad a_1 = 1$$

Let,  $G(x) = \sum_{n=0}^{\infty} a_n x^n$  is the GF of the func.

Multiply the eqn- $\textcircled{1}$  by  $x^n$  & summing  $n=2$  to  $n=\infty$ .

$$\sum_{n=2}^{\infty} a_n x^n - 2 \sum_{n=2}^{\infty} a_{n-1} x^n - 3 \sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

$$(a_2 x^2 + a_3 x^3 + \dots) - 2(a_1 x^2 + a_2 x^3 + \dots) - 3(a_0 x^2 + a_1 x^3 + \dots) = 0.$$

$$(G(x) - a_0 x^2) - 2x(G(x) - a_0) - 3x^2 G(x) = 0.$$

$$G(x)[1 - 2x - 3x^2] - a_0 - a_1 x + 2x a_0 = 0.$$

$$G(x)[1 - 2x - 3x^2] = 3 - x + 6x = 0.$$

$$G(x)[-3x^2 - 2x + 1] = 3 - 5x.$$

$$G(x) = \frac{3 - 5x}{-3x^2 - 2x + 1} = \frac{3 - 5x}{3x^2 + 2x - 1}$$

$$G(x) = \frac{5x - 3}{(3x-1)(x+1)}$$

$$\textcircled{2} \quad \frac{5x-3}{3x-1} = Ax + B + \frac{B}{x+1}$$

$$\textcircled{3} \quad \frac{5x-3}{3x-1} = Ax + B + \frac{B}{x+1} = Ax + A + \frac{B}{x+1} = Ax + A + Bx - B.$$

$$\begin{array}{rcl} A + 3B & = & 5 \\ A - B & = & -3 \\ \hline 4B & = & 8 \end{array}$$

$$A - B = -3.$$

$$\begin{array}{l} A = -1 + \frac{3}{4} \\ A = -\frac{1}{4} \end{array}$$

$$A = \frac{1}{4}, \quad B = \frac{2}{3}$$

$$\begin{array}{rcl} A + 3B & = & 5 \\ A - B & = & -3 \\ \hline 4B & = & 8 \end{array} \quad A - B = -3.$$

$$G(x) = \frac{\frac{1}{4} + \frac{2}{3}x}{3x-1}$$

$$\boxed{B = \frac{2}{3}}, \quad A = -\frac{1}{4} + \frac{2}{3} \Rightarrow \boxed{A = -\frac{1}{4}}$$

$$G(x) = \frac{1}{1-3x} + \frac{2}{x+1}$$

$$G(x) = \sum_{k=0}^{\infty} (3x)^k + 2 \sum_{k=0}^{\infty} (-x)^k$$

$n = N$  . of Pigeons  
 $m = N$  . of Pigeon holes  
 $\lceil x \rceil \Rightarrow$  Smallest integer zero than or equals to  $x$   
 $\lfloor x \rfloor \Rightarrow$  Greatest integer zero than or equals to  $x$

Pigeon Hole Principle \*

$$a_n = \dots - \frac{q}{2} (a)_n + \frac{t}{2} (-s)_n$$

$$a_n = \frac{q}{2} \frac{t}{2} (2x)_n + \frac{t}{2} (5x)_n$$

$$(1+2x) \frac{t}{2} - \frac{q}{2} (5x+1) = (x)$$

$$B = \frac{t}{2}$$

$$B = -\frac{t}{2}$$

$$\frac{t}{2} = 1 + \frac{q}{2}$$

$$+ qA + 5B = 5$$

$$A + B = 1.$$

$$5A - 2B = \pm$$

$$5x+1 = 5Ax + A + B - 2Bx$$

$$a(x) = \frac{A}{5x+1} + \frac{B}{1-2x}$$

$$\frac{(5x+1)(2x-1)}{5x+1} = \frac{(5x+1)(2x-1)}{(5x+1)} = a(x)$$

$$\frac{10x^2 - 3x - 1}{5x+1} = a(x)$$

$$\frac{10x^2 + 3x - 1}{5x+1} = a(x)$$

$$5x(2x-1) + 1$$

$$a(x) [ -10x^2 + 3x + 1 ] = 1 - 4x - 3x = 0$$

$$a(x) [ 1 + 3x - 10x^2 ] - a_0 - a_1x - 3a_2x = 0$$

$$a(x) - a_0 - a_1x + 3x(a(x) - a_0) - 10x^2(a(x)) = 0$$

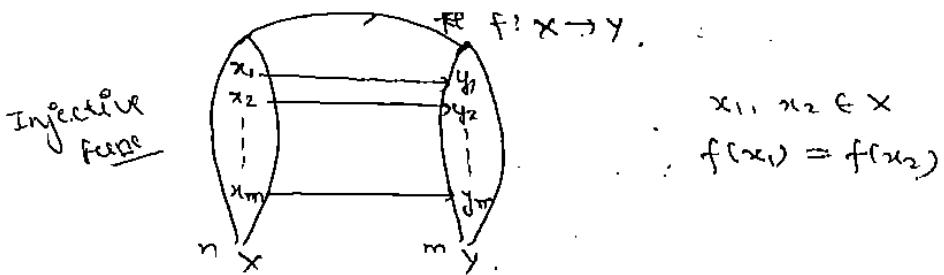
$$(a_2x^2 + a_3x^3 + \dots) + 3(a_1x^2 + a_2x^3 + \dots) - 10(a_0x^2 + a_1x^3 + \dots) = 0$$

$$a_n x^n + 3 \sum_{n=2}^{\infty} a_{n-1} x^n - 10 \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

Multiplying the eqn ① by  $x^n$  & summing n=2 to n=∞

$$\text{L.H.S., } a(x) = \sum_{n=0}^{\infty} a_n x^n \text{ as the C.F. of the func.}$$

$$a_n + 3a_{n-1} - 10a_{n-2} = 0. \quad \text{④} \quad n \geq 2, a_0 = 1, a_1 = 1$$



According to pigeon hole principle, if  $n$  pigeons are assigned to  $m$  pigeon holes and  $n > m$ . Then, there is atleast one pigeon hole that contains two or more pigeons.

Proof:- Let,  $f: X \rightarrow Y$  where,  $X$  &  $Y$  are finite sets and cardinality of  $X$  is ' $n$ ' & cardinality of  $Y$  is ' $m$ '. &  $n > m$ . Then there exist atleast 2 elements  $x_1$  &  $x_2$  such that  $f(x_1) = f(x_2)$

Proof:- Let,  $X = x_1, x_2, x_3, \dots, x_m$ .

Suppose,  $f$  is injective. Then  $f(x_1), f(x_2), \dots, f(x_m)$  are distinct elements in  $Y$ . So,

As  $f$  is one to one.  $n \leq m$

But, this is a contradiction to our assumption  $n > m$ . Therefore,  $f$  is not injective. and there must be atleast two distinct elements  $x_1$  &  $x_2 \in X$  such that

$$f(x_1) = f(x_2)$$

\* Find the minimum number of elements to be selected from the set  $A = \{3, 4, 5, 6, 7, 8, 9\}$  such that sum of any two elements is 12.

→ The possible sets or cases selecting two elements from set  $A$  such that sum of them is 12 will be as follows-

$$\{3, 9\}, \{4, 8\}, \{5, 7\}, \{6, 6\}$$

Now, we can observe that each element of  $A$  belongs to 1 and only 1 of these sets.

Thus, minimum no. of digits to be selected from set  $A$  should be greater than number of sets. So, minimum no. of digits selected is 5.

\* Extended or Generalise Pigeon Hole Principle :-

If ' $n$ ' pigeons are accommodated in ' $m$ ' pigeon holes. and  $n > m$ . then one of the pigeon hole must contained atleast  $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ .

Minimum no. of cars painted with same colour.

$$\text{No. of cars} = 100$$

$$\text{No. of colours} = 9.$$

Let, cars be the pigeons and colours be the pigeon holes.

$$\text{so, } n = 100$$

$$m = 9.$$

So, Acc. to Pigeon hole principle,

The minimum no. of cars painted with same colour -

$$\left\lfloor \frac{n-1}{m} \right\rfloor + 1 = \left\lfloor \frac{100-1}{9} \right\rfloor + 1 = \left\lfloor \frac{99}{9} \right\rfloor + 1 = 12.$$

- Q How many people must you have to guarantee that atleast 9 of them will have birthdays in the same day of the week  
→  $n = ?$   
 $m = \text{Days of the week} = 7.$

$$\Rightarrow \left\lfloor \frac{n-1}{m} \right\rfloor + 1 = 9.$$

$$\Rightarrow \left\lfloor \frac{n-1}{7} \right\rfloor + 1 = 9. \Rightarrow \frac{n-1}{7} + \frac{1}{7} = 9.$$

$$\Rightarrow n-1 = 9 \times 7 - 1$$

$$\Rightarrow n = 54.$$

- Q What is the minimum no. of students required in a class to be sure that atleast 5 of them will receive same grades if there are four grades A, B, C, D.

$$\rightarrow n = ?$$

$$m = 4 \quad (\text{No. of grades}).$$

$$\Rightarrow \left\lfloor \frac{n-1}{m} \right\rfloor + 1 = 5 \Rightarrow \frac{n-1}{4} + 1 = 5.$$

$$\Rightarrow n-1+4 = 20.$$

$$\Rightarrow n = 20 - 3.$$

$$\Rightarrow \boxed{n=17}$$

- Q In how many ways 6 numbers can be chosen from 1 to 15 so that all the choices have same sum.

$$\binom{15}{6} = 5005 \text{ ways}.$$

$$1+2+3+4+5+6 = 21 \quad \text{to} \quad 10+11+12+13+14+15 = 75$$

$$\text{Total sum} = 75 - 21 + 1 = 55.$$

$$\text{No. of pigeons (n)} = 5005.$$

$$\text{No. of pigeon holes (m)} = 55.$$

$$\left\lfloor \frac{n-1}{m} \right\rfloor + 1 = \left\lfloor \frac{5005-1}{55} \right\rfloor + 1 = \left\lfloor \frac{5004}{55} \right\rfloor + 1 = \boxed{90.98} + 1$$

$$= 91.$$

Atleast ways to select 6 no. = 91

Exactly ways to select 6 no. =  $91 - 1 = 90$

Yes, they saying the same thing:

T	T	T	T	T	T	T
T	F	T	F	T	F	T
F	T	F	F	T	F	T
T	F	F	F	T	F	T
F	T	F	F	T	F	T
P	q	q	p	q	p	q

$$q \leftarrow p \\ p \leftarrow q$$

q : Food is cheap.

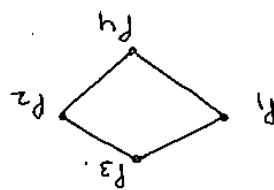
p : Food is good.

Saying the same thing.

Again that says "good food is not good", & the other has a saying that says "cheap food is not cheap"; one of the two statements next to each other must have a saying that says "good food is not cheap".

V	$P_1$	$P_2$	$P_3$	$P_4$	$P_1$	$P_2$	$P_3$	$P_4$
✓	$P_1$	$P_2$	$P_3$	$P_4$	$P_1$	$P_2$	$P_3$	$P_4$
	$P_1$	$P_2$	$P_3$	$P_4$	$P_3$	$P_4$	$P_1$	$P_2$
	$P_2$	$P_3$	$P_4$	$P_1$	$P_2$	$P_3$	$P_4$	$P_1$
	$P_3$	$P_4$	$P_1$	$P_2$	$P_4$	$P_1$	$P_2$	$P_3$
	$P_4$	$P_1$	$P_2$	$P_3$	$P_4$	$P_1$	$P_2$	$P_3$

2)



1)

1) Draw the Hasse diagram of  $(P, \leq)$ .

2) Is this poset a lattice.

A poset on  $P$  is a ordered such that  $R \subseteq P^2$  or  $P_i \leq P_j$  whenever  $i, j = 1, 2, 3, 4$ . If & only if all the elements of  $P$  are subsets of elements of  $P$ .

1)  $(a), (b), (c), (d), (e)$

$P_4 = \{a, b, c, d, e\}$

$P_3 = \{a, b\}, \{c, d, e\}$

$P_2 = \{a, b\}, \{c, d, e\}$

$P_1 = \{a, b\}, \{c, d, e\}$

such that  $P = \{P_1, P_2, P_3, P_4\}$ .

## Algebraic Structure

\* n-ary operation: Let  $X$  be a non-empty set, a mapping function  $f: X^n \rightarrow X$  is called an  $n$ -ary operation on set  $X$  where,  $n$  is called order of operation.  
 For  $n=1$ , it is unary operation.  
 For  $n=2$ , it is binary operation.  
 For  $n=3$ , it is ternary operation.

For  $n=n$  it is  $n$ -ary operation.

\* Algebraic System or Structure: A system consisting of a non-empty set and 1 or more an  $n$ -ary operations on the set is called algebraic system or structure.

An algebraic system is represented as -

$$(S, f_1, f_2, \dots)$$

Where,  $S$  = Non-empty sets.

$f_1$  &  $f_2$  are  $n$ -ary operations on  $S$ .

e.g.  $(R, +)$ ,  $(Z, +)$ ,  $(Z, \cdot)$ ,  $(R, \leq)$

\* Properties of Algebraic System:

Let,  $(S, *, +)$  be an algebraic system where  $*$  and  $+$  are binary operations on set  $S$ .

$S$  is closed under operation  $*$ .

Inverse Property:  $a * b \in S, \forall a, b \in S$ .

Associative Property:  $(a * b) * c = a * (b * c) \quad \forall a, b, c \in S$

Identity element:  $\exists e \in S \quad a * e = a \quad (\text{Right identity})$   
 $e * a = a \quad (\text{Left identity})$

Inverse element:  $\exists a^{-1} \in S \quad a^{-1}$  is called inverse of  $a$  under the operation  $*$ .  
 $a * a^{-1} = e$ .  
 $a^{-1} * a = e \quad \forall a \in S$ .

Commutative property:  $a * b = b * a, \forall a, b \in S$ .

Where,  $e$  is called identity element with respect to operation  $*$ .

Cancellation Law:  $a * b = a * c \Rightarrow b = c$ . (Right cancellation)

$b * a = c * a \Rightarrow b = c$ . (Left cancellation)

1. Distributive laws :-

$$a * (b+c) = (a*b) + (a*c) \quad (\text{Right Distribution})$$

$$(b+c)*a = (b*a) + (c*a) \quad (\text{Left Distribution})$$

2. Important Property :- An element  $a \in S$  is called idempotent element w.r.t operation  $*$ , if  $a * a = a$ .

Q. The binary operation  $*$  on  $\mathbb{Q}$  (set of Rational No.'s) defined by  $a * b = a+b+2ab$ .

Determine whether, i) Associative, ii) Commutative  
Also find Identity & Inverse Element.

→ Let  $a, b, c \in \mathbb{Q}$ .

$$\text{L.H.S.} : (a * b) * c = a * (b * c)$$

$$\text{LHS} \quad (a * b) * c$$

$$\Rightarrow (a+b+2ab) * c$$

$$\Rightarrow a+b+2ab+c+2(a+b+2ab)c$$

$$\Rightarrow a+b+c+2ab+2ac+2bc+4abc$$

$$\text{RHS} \quad (a * (b * c))$$

$$\Rightarrow a * (b+c+2bc)$$

$$\Rightarrow a+b+c+2bc+2(a+b+2bc)$$

$$\Rightarrow a+b+c+2ab+2bc+2ac+4bca$$

Here, LHS = RHS.

So, Associative property follows.

$$\text{Commutative: } a * b = b * a.$$

$$\text{LHS} \quad a * b$$

$$\Rightarrow a+b+2ab$$

$$\text{RHS} \quad b * a$$

$$\Rightarrow b+a+2ba$$

Here, LHS = RHS.

So, Commutative property follows.

Let,  $e \in \mathbb{Q}$  be the identity element w.r.t  $*$ .

$$\forall a \in \mathbb{Q}, \quad a * e = a$$

$$\Rightarrow a+e+2ae = a$$

$$\Rightarrow e(1+2a) = 0$$

$$\Rightarrow \boxed{e=0} \quad (\because 1+2a \neq 0)$$

Let,  $a^{-1} \in \mathbb{Q}$  be the inverse of  $a \in \mathbb{Q}$  w.r.t  $*$

$$a * a^{-1} = e = 0$$

$$\Rightarrow a+a^{-1}+2aa^{-1}=0$$

$$\Rightarrow a+a^{-1}(1+2a)=0$$

1+2a.

$l \times 1 + 2a + 0 \times l$ .

\* Semigroup :- Let,  $S$  be a non-empty set and ' $*$ ' be a binary operation on ' $S$ '. Then  $(S, *)$  is called a semigroup if it follows -

- Closure Property.
- Associative Property.

\* Monoid :- Let,  $S$  be a non-empty set and ' $*$ ' be a binary operation on ' $S$ ', then  $(S, *)$  is called a monoid if it follows -

- Closure Property.
- Associative Property.
- Identity Property.

Q Is it a semigroup?  $(N, +)$  and  $(N, \otimes)$ .

Closure Property -  $2+3 = 5 \in N$ . &  $2, 3 \in S$ .

$$\begin{aligned} \text{Associative Property} - & (2+3)+4 = 2+(3+4) \\ & \Rightarrow 5+4 = 2+7 \\ & \Rightarrow 9 \text{ (LHS)} \quad 9 \text{ (RHS)} \end{aligned}$$

LHS = RHS

Remainder.

\* Addition Modulo.  $a +_{\text{mod}} b$  e.g.  $2+3 = \frac{4}{4} = 0$

\* Multiplication Modulo.  $a *_{\text{mod}} b$ . firstly add then multiply

Q Let,  $S = \{0, 1, 2, 3, 4, 5\}$  &  $*$  be the operation addition modulo 6. Show that  $(S, *)$  is a semigroup.

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

For Associative -

$$\begin{aligned} 1+_6(3+_6 5) &= 1+_6 2 = 3 \\ (1+_6 3) +_6 5 &= 4 +_6 5 = 3 \end{aligned}$$

From the above table closure & associative property follows.

Q Let,  $*$  be operation on set  $S = \mathbb{Q} \times \mathbb{Q}$ . Given by  $(a, b) * (x, y) = (ax, ay+b)$  show that whether  $(S, *)$  is a semigroup. also find identity & inverse elements on set  $S$  if exists.

Given,  $(a, b) * (x, y) = (ax, ay+b)$

Closure Property -  $(a, b) \& (x, y) \in \mathbb{Q} \times \mathbb{Q}$ .  
 $a, b, x, y \in \mathbb{Q}$ .

$$\begin{aligned}\Rightarrow ax &\in \mathbb{Q}, \\ \Rightarrow ay &\in \mathbb{Q}, \\ \Rightarrow ay+b &\in \mathbb{Q}, \\ \Rightarrow (ax, ay+b) &\in \mathbb{Q} \times \mathbb{Q}.\end{aligned}$$

It follows closure property.

### 2) Associative Property :-

Let,  $(a, b), (x, y) \in \mathbb{Q} \times \mathbb{Q}$ .

$$*((a, b) * (x, y)) * (u, v) = (a, b) * ((x, y) * (u, v)).$$

$$\text{LHS} = ((a, b) * (x, y)) * (u, v)$$

$$\Rightarrow (ax, ay+b) * (u, v)$$

$$\Rightarrow (axu, axv + ay + b)$$

$$\text{RHS} \quad (a, b) * ((x, y) * (u, v))$$

$$\Rightarrow (a, b) * (xu, xv + y)$$

$$\Rightarrow (axu, axv + ay + b)$$

$$\text{LHS} = \text{RHS}$$

Therefore, Associative Property follows.

### 3) Identity element -

Let,  $(e_1, e_2) \in S$ .

$$\begin{aligned}\text{For, } (a, b) \in S, \quad (a, b) * (e_1, e_2) &= (a, b) \\ \Rightarrow (ae_1 + ae_2 + b) &= (a, b) \\ \Rightarrow ae_1 = a &\Rightarrow e_1 = 1 \\ \Rightarrow ae_2 + b &= b \\ \Rightarrow e_2 = 0 &\end{aligned}$$

$$\text{So, } (e_1, e_2) \Rightarrow (1, 0) \in S.$$

### 4) Inverse element -

Let,  $(c, d) \in S$  be the inverse of  $(a, b)$

$$(a, b) * (c, d) = (1, 0)$$

$$(ac + ad + b) = (1, 0)$$

$$\Rightarrow ac = 1 \Rightarrow c = \frac{1}{a}.$$

$$\Rightarrow ad + b = 0 \Rightarrow d = -\frac{b}{a}.$$

$$\left(\frac{1}{a}, -\frac{b}{a}\right).$$

$$\left\{ \begin{array}{l} \text{e.g. } \left(\frac{1}{2}, \frac{3}{5}\right) \\ \left(2, -\frac{3/5}{1/2}\right) \\ \left(2, -\frac{6}{5}\right). \end{array} \right\}$$

**Group :-** Let  $\ast$  be a non-empty set with binary operation on  $\ast$ , then  $(\ast, \ast)$  is called a group if it follows -

- Closure Property
- Associative Property.
- Existence of Identity element.
- Existence of Inverse element.

Let  $(G, \ast)$  be a binary algebraic structure

e.g.  $(Z, +)$ ,  $(R, +)$  are groups.

\* **Abelian Group :-** A group  $(G, \ast)$  is called an abelian group or commutative group if binary operation  $\ast$  is commutative i.e.

$$a \ast b = b \ast a, \forall a, b \in G.$$

e.g.  $(Z, +)$  is an abelian group.

Q Prove that  $G = \{1, w, w^2\}$  is a group w.r.t multiplication where,  $1, w$  &  $w^2$  are cube roots of unity.

$\times$	1	w	$w^2$
1	1	w	$w^2$
w	w	$w^2$	1
$w^2$	$w^2$	1	w

$$w^3 = 1$$

$$w^4 = w^3 \cdot w = w.$$

1) Closure Property :- From the table closure property satisfied.

2) Associative Property :-  $(1 \times w) \times w^2 = 1 \times (w \times w^2)$

$$\begin{array}{ll} \text{LHS} & w \times w^2 \\ & \Rightarrow w^3 \\ & \Rightarrow 1. \end{array} \quad \begin{array}{ll} \text{RHS} & 1 \times w^3 \\ & \Rightarrow 1 \times 1. \\ & \Rightarrow 1. \end{array}$$

3) Existence of Identity Element :-

$1 \in G$  such that

$$1 \times 1 = 1.$$

$$w \times 1 = w$$

$$w^2 \times 1 = w^2$$

∴ So, 1 is the identity element.

4) Existence of Inverse Element :-

$$1 \times 1 = 1$$

$$\therefore 1^{-1} = 1$$

$$w \times w^2 = 1$$

$$\therefore w^{-1} = w^2$$

$$w^2 \times w = 1$$

$$\therefore (w^2)^{-1} = w.$$

∴ Inverse exist of each element. So,

so, it is a group.

Q Show that  $G = \{1, 2, 3, 4, 5, 6\}$  is an abelian group under the operation multiplication modulo 7.

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

- 1) Closure Property :- From the above table closure property satisfied.
- 2) Associative Property :- From the above table associative property satisfied.
- 3) Existence of Identity element :-

$1 \in G$  is the identity element.

$$1 \times_7 1 = 1$$

$$2 \times_7 1 = 2$$

$$3 \times_7 1 = 3$$

$$4 \times_7 1 = 4$$

$$5 \times_7 1 = 5$$

$$6 \times_7 1 = 6$$

So, 1 is the identity element.

- 4) Existence of Inverse element :-

$$1 \times_7 1 = 1 \Rightarrow 1^{-1} = 1$$

$$2 \times_7 4 = 1 \Rightarrow 2^{-1} = 4$$

$$3 \times_7 5 = 1 \Rightarrow 3^{-1} = 5$$

$$4 \times_7 2 = 1 \Rightarrow 4^{-1} = 2$$

$$5 \times_7 3 = 1 \Rightarrow 5^{-1} = 3$$

$$6 \times_7 6 = 1 \Rightarrow 6^{-1} = 6$$

- 5) commutative Property :- From the composition table it is clear that  $\times_7$  is commutative. i.e.  $a * b = b * a$ ,  $\forall a, b \in G$ .

- Q Is  $G = \{1, 2, 3, 4\}$  a group under addition modulo 5

$+_5$	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

Here, 0  $\notin G$ . so, it does not follow closure property.

so, it is not a group.

$\Rightarrow (\mathbb{Z}_m, +_m)$  is always a group.

$$\begin{array}{r} - \\ - \\ - \\ - \end{array}$$

3) Existance of Identity element :-

2) Additive property :- From the above table, additive property is satisfied.

1) Closure property :- From the above table closure property is satisfied.

-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

←

Shows under the operation multiplication.

commutative. i.e.  $a * b = b * a$ ;  $a, b \in G$ .

5) Commutative property :- From the composition table

$$\begin{aligned} a * b &= b & \Leftrightarrow a = b \\ t * s &= t & \Leftrightarrow s = t \\ s * t &= t & \Leftrightarrow s = t \\ t * 1 &= t & \Leftrightarrow t = 1 \end{aligned}$$

4) Existence of inverse element :-

$$\begin{aligned} b * 1 &= b \\ t * 1 &= t \\ 3 * 3 &= 1 \\ 1 * 1 &= t \end{aligned}$$

4) Existance of identity element :-

3) Existence of identity element :-

2) Additive property :- From the above table, additive property is satisfied.

1) Closure property :- From the above table closure property is satisfied.

b	t	3	1	t	b	3	t	t	b
t	b	3	1	t	t	b	3	1	t
3	t	b	1	t	3	t	b	1	t
1	t	3	t	b	1	t	3	b	t
t	b	3	t	1	t	b	1	3	t

1)

#### 4) Existence of Inverse Element :-

$$\begin{aligned} 1 \times 1 &= 1 \Rightarrow 1^{-1} = 1 \\ -1 \times -1 &= 1 \Rightarrow (-1)^{-1} = -1 \\ i \times -i &= 1 \Rightarrow i^{-1} = -i \\ -i \times i &= 1 \Rightarrow (-i)^{-1} = i \end{aligned}$$

5) Commutative Property :- From the composition table  
It is clear that  $\times$  is  
commutative i.e.  $a * b = b * a$ ,  $\forall a, b \in G$ .

#### \* Properties of a Group :-

- The Identity element of a group  $(G, *)$  is unique.

$\Rightarrow$  Let,  $e_1$  &  $e_2 \in G$  are identity elements of  $(G, *)$ .

$$e_1 * e_2 = e_2 * e_1 = e_2 \quad \text{--- (1)}$$

$$e_2 * e_1 = e_1 * e_2 = e_1 \quad \text{--- (2)}$$

From (1) & (2)

$$\boxed{e_1 = e_2}$$

- The Inverse of every element of  $(G, *)$  is unique.

$\Rightarrow$  Let, if possible  $b$  &  $c$  are inverse of  $a \in G$ , and  
 $e \in G$  is the identity element.

$$b * a = a * b = e \quad \text{--- (1)}$$

$$c * a = a * c = e \quad \text{--- (2)}$$

$$\text{Now, } b = e * b$$

$$b = ((*)^a) * b$$

$$b = c * (a * b)$$

$$b = c * e$$

$$\boxed{b = c}$$

$$\boxed{a * a^{-1} = e}$$

- $(a * b)^{-1} = b^{-1} * a^{-1}$  for  $a, b \in G$ .

$$\Rightarrow (a * b) * (b^{-1} * a^{-1}) \Rightarrow (b^{-1} * a^{-1}) * (a * b)$$

$$\Rightarrow a * (b * b^{-1}) * a^{-1} \Rightarrow b^{-1} * (a^{-1} * a) * b$$

$$\Rightarrow a * e * a^{-1} \Rightarrow b^{-1} * e * b$$

$$\Rightarrow a * a^{-1} \Rightarrow b^{-1} * b$$

$$\Rightarrow e \Rightarrow e$$

- Cancellation Laws are true in a group.

$$a * b = a * c \Rightarrow b = c \quad (\text{Left cancellation})$$

$$b * a = c * a \Rightarrow b = c \quad (\text{Right cancellation})$$

$$i) a * b = a * c$$

$$\Rightarrow a^{-1} * (a * b) = a^{-1} * (a * c)$$

$$\Rightarrow (a^{-1} * a) * b = (a^{-1} * a) * c$$

$$\Rightarrow e * b = e * c$$

$$\Rightarrow \boxed{b = c}$$

$$2) b * a = c * a$$

$$(b * a) * a^{-1} = (c * a) * a^{-1}$$

$$\Rightarrow b * (a * a^{-1}) = c * (a * a^{-1})$$

$$\Rightarrow b * e = c * e$$

$$\Rightarrow \boxed{b = c}$$

•  $(G, *)$  can not have an idempotent element except the Identity element  $e$ .

$\Rightarrow$  Let if possible  $a \in G$  be an idempotent element of  $(G, *)$  rather than identity element 'e'.

$$a * a = a \quad \text{--- (1)}$$

$$\text{Now, } e = a * a^{-1}$$

$$e = (a * a) * a^{-1} \quad (\text{From 1})$$

$$e = a * (a * a^{-1}) \quad (\text{Associative})$$

$$e = a * e$$

$$\boxed{e = a}$$

$$\left\{ \begin{array}{l} e * e = e \\ e * e = e \\ e * e = e \\ e * e = e \end{array} \right.$$

Q Prove that inverse of the inverse of an element of a group is equal to element itself.

$\rightarrow$  We need to prove that,  $(a^{-1})^{-1} = a$ ,  $\forall a \in G$ .

Since,  $a^{-1}$  is inverse of  $a$ .

$$\text{then, } a * a^{-1} = a^{-1} * a = e \quad \text{--- (1)}$$

Now, Inverse of  $a^{-1}$  is  $(a^{-1})^{-1}$ .

$$a^{-1} * (a^{-1})^{-1} = e$$

$$\Rightarrow a * (a^{-1} * (a^{-1})^{-1}) = a * e$$

$$\Rightarrow (a * a^{-1}) * (a^{-1})^{-1} = a$$

$$\Rightarrow e * (a^{-1})^{-1} = a$$

$$\Rightarrow \boxed{(a^{-1})^{-1} = a}$$

Q If  $a$  &  $b$  are any two elements of a group  $G$ , then  $(a * b)^2 = a^2 * b^2$  if & only if  $G$  is abelian.

$$(a * b)^2 = a^2 * b^2$$

$\rightarrow$  1) If  $G$  is abelian. i.e.,  $a * b = b * a$ ,

$$\text{Now, } (a * b)^2 = (a * b) * (a * b) \quad [\text{Associative}]$$

$$= a * (b * a) * b \quad (\text{from given cond})$$

$$= a * (a * b) * b \quad (\text{Associative})$$

$$= (a * a) * (b * b)$$

$$\boxed{(a * b)^2 = a^2 * b^2}$$

$$\begin{aligned}
 2) \text{ Let, } (a * b)^2 &= (a^2 * b^2) \\
 \Rightarrow (a * b) * (a * b) &= (a * a) * (b * b) \\
 \Rightarrow e * (b * a) * b &= e * a * (a * b) * b \\
 \Rightarrow (b * a) * b &= (a * b) * b \quad \{ \text{Left cancellation} \} \\
 \Rightarrow (b * a) &= (a * b) \quad \{ \text{Right cancellation} \}
 \end{aligned}$$

Hence,  $G$  is abelian.

\* **finite Group** :- A group  $(G, *)$  be a finite group if no. of elements in  $G$  is finite.

e.g.  $G = \{0, 1, 2, 3, 4, 5\}$ , under  $+_{\mathbb{Z}_6}$  is a finite group.

\* **Infinite Group** :- A group  $(G, *)$  is called infinite group if no. of elements in  $G$  is infinite.

e.g.  $\mathbb{C} \setminus \{z, +\}$  is infinite.

\* **Order of Group** :- Order of group  $G$  is the no. of elements in group  $G$ , it is denoted by  $|G|$  or  $O(G)$ .

Note: A group of order 1 i.e.  $O(G_1) = 1$  has only the identity element.

\* **Order of an Element** :- If the element  $a \in G$ , then the least positive integer  $n$  for which  $[a^n = e]$ . w.r.t. to binary operation '\*' where  $e$  is an identity element is called order of element  $a$ . and it is denoted by  $O(a)$ .

Q Consider  $G = \{1, 2, 4, 7, 8, 11, 13, 14\}$ ; under  $\times_{15}$ . Find the order of all the elements.

→ Here,  $e = 1$ .

$$O(1) \Rightarrow 1^n = e \Rightarrow 1^n = 1.$$

$$O(1) \Rightarrow 1^1 = 1 \Rightarrow O(1) = 1.$$

$$O(2) \Rightarrow 2^4 = 1 \Rightarrow O(2) = 4. \quad \text{as } 2^4 = 2 \times 2 \times 2 \times 2 = 16 \equiv 1 \pmod{15}$$

$$O(4) \Rightarrow 4^2 = 1 \Rightarrow O(4) = 2. \quad 16 \equiv 1 \pmod{15} \Rightarrow O(4) = 2.$$

$$O(7) \Rightarrow 7^4 = 1 \Rightarrow O(7) = 4. \quad 2401 \equiv 1 \pmod{15} \Rightarrow O(7) = 4.$$

$$O(8) \Rightarrow 8^4 = 1 \Rightarrow O(8) = 4.$$

$$O(11) \Rightarrow 11^2 = 1 \Rightarrow O(11) = 2.$$

$$O(13) \Rightarrow 13^4 = 1 \Rightarrow O(13) = 4.$$

$$O(14) \Rightarrow 14^2 = 1 \Rightarrow O(14) = 2.$$

Q  $G = \{a, a^2, a^3, a^4, a^5, a^6\} = e\}$ . Find order of every element?

→ Here,  $a^6 = e$ ,  $O(G) = 6$

$$O(a) \Rightarrow a^6 = e \Rightarrow O(a) \geq 6.$$

$$a(a^2)^3 = e \Rightarrow O(a^2) = 3.$$

$$O(a^0) \Rightarrow (a^0)^n = e \Rightarrow O(a^0) = n.$$

$$O(a^4) \Rightarrow (a^4)^3 = a^{12} \Rightarrow (a^6)^2 = (e)^2 = e \Rightarrow O(a^4) = 3.$$

$$O(a^5) \Rightarrow (a^5)^6 = a^{30} \Rightarrow (a^6)^5 = (e)^5 = e \Rightarrow O(a^5) = 6.$$

$$O(a^6) \Rightarrow (a^6)^1 = e \Rightarrow O(a^6) = 1.$$

Q.  $G = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ . Is a group under  $+_{10}$ . Find the order of every element.

→ Here,  $e = 0$ .  $O(G) = 10$ .

$$O(0) \Rightarrow 0^1 = 0 \Rightarrow O(0) = 1. [0 \equiv 0 \pmod{10}]$$

$$O(1) \Rightarrow 1^{10} = \underbrace{1+1+1+\dots+1}_{10 \text{ times}} = 10 \equiv 0 \pmod{10} \Rightarrow O(1) = 10.$$

$$O(2) \Rightarrow 2^5 = 2+2+2+2+2 = 10 \equiv 0 \pmod{10} \Rightarrow O(2) = 5.$$

$$O(3) \Rightarrow 3^{10} = \underbrace{3+3+\dots+3}_{10 \text{ times}} = 30 \equiv 0 \pmod{10} \Rightarrow O(3) = 10.$$

$$O(4) \Rightarrow 4^5 = 4+4+4+4+4 = 20 \equiv 0 \pmod{10} \Rightarrow O(4) = 5.$$

$$O(5) \Rightarrow 5^2 = 5+5 = 10 \equiv 0 \pmod{10} \Rightarrow O(5) = 2.$$

$$O(6) \Rightarrow 6^5 = 6+6+6+6+6 = 30 \equiv 0 \pmod{10} \Rightarrow O(6) = 5.$$

$$O(7) \Rightarrow 7^{10} = \underbrace{7+7+\dots+7}_{10 \text{ times}} = 70 \equiv 0 \pmod{10} \Rightarrow O(7) = 10.$$

$$O(8) \Rightarrow 8^5 = 8+8+8+8+8 = 40 \equiv 0 \pmod{10} \Rightarrow O(8) = 5.$$

$$O(9) \Rightarrow 9^{10} = \underbrace{9+9+\dots+9}_{10 \text{ times}} = 90 \equiv 0 \pmod{10} \Rightarrow O(9) = 10.$$

### Properties of Order of Element :-

- If a group  $G$  is finite then order of every element of the group is less than or equal to order of group.

$$[O(a) \leq O(G), \forall a \in G]$$

- If order of an element  $a \in G$  is  $n$  then,

$$ak = e \text{ iff } n \text{ is divisor of } k.$$

Proof:  $n$  is order of  $a$ . So,  $a^n = e$ .

Let,  $k \in \mathbb{Z}^+$   
for some  $p, q \in \mathbb{Z}^+$

$$k = n \cdot p + q$$

$$0 \leq q < n$$

$$\begin{aligned} \text{Now, } a^k &= a^{np+q} = a^{np} \cdot a^q \\ &= (a^n)^p \cdot a^q \\ &= e^p \cdot a^q \\ &= e \cdot a^q \end{aligned}$$

$$\Rightarrow [a^k = a^q] = e.$$

It is not possible unless  $q = 0$ .

$$k = np$$

So,  $n$  is the divisor of  $k$ .

⇒ Let,  $n$  is divisor of  $k$ . then  $\exists p \in \mathbb{Z}^+$ .

$$k = np.$$

$$\text{Now, } a^k = a^{np} = (a^n)^p = (e)^p = e. \\ \text{So, } \boxed{a^k = e}.$$

\* Cyclic Group :- A group  $(G, *)$  is said to be cyclic if there exist an element  $a \in G$  such that every element  $x \in G$  can be expressed as  $x = a^n$ .

$$x = a^n \quad \exists \text{ integer } n. [n \in \mathbb{Z}]$$

In this case cyclic group  $G$  is said to be generated by ' $a$ '. So, ' $a$ ' is called Generator of ' $G$ '. ' $G$ ' is also denoted by  $\langle a \rangle$ .

E.g. If  $G = \{1, -1, i, -i\}$   
then,  $(G, *)$  is a cyclic group.

$$(i)^1 = i$$

$$(i)^2 = -1$$

$$(i)^3 = i^2 \cdot i = -i$$

$$i^4 = i^2 \cdot i^2 = 1$$

So,  $i$  is the generator of that group.

For this cyclic group ' $-i$ ' is also a generator.

Properties :-

- A cyclic group is abelian.

$(G, *)$  &  $\langle a \rangle$  is generator.

Let,  $x, y \in G$ .

$$x = a^m, y = a^n, m, n \in \mathbb{Z}$$

$$\begin{aligned} \text{Now, } x * y &= a^m * a^n \\ &= a^{m+n} \\ &= a^{n+m} \\ &= a^n * a^m \end{aligned}$$

$$\boxed{x * y = y * x}$$

- If ' $a$ ' is generator of cyclic group so, ' $a^{-1}$ ' is also generator of cyclic group.

$$x = a^n, \quad n \in \mathbb{Z}$$

$$\boxed{x = (a^{-1})^{-n}} \quad , -n \in \mathbb{Z}, \therefore$$

- If  $(G, *)$  is a finite cyclic group generated by an element  $a \in G$  &  $a$  is of order ' $n$ ' then  $a^n = e$ .

$$a^n = e \quad \text{so that } G = \{a, a^2, a^3, \dots, a^{n-1}\} = \{e\}$$

- If  $(G, *)$  is a finite cyclic group of order ' $n$ '  $[O(G) = n]$  and generated by ' $a$ ' then  $a^m$  is also generator of  $(G, *)$  if & only if  $m \& n$  are relative primes.

So, total no. of generators are 1, 2, 3, 4.

So, 4 and 3 are also generators.

4 is the inverse of 1 & 3 is the inverse of 2.

$$2^5 \equiv 10 \equiv 0 \pmod{5}$$

$$2^4 \equiv 8 \equiv 3 \pmod{5}$$

$$2^3 \equiv 6 \equiv 1 \pmod{5}$$

$$2^2 \equiv 4 \equiv 4 \pmod{5}$$

$$2^1 \equiv 2 \equiv 2 \pmod{5}$$

$$1^5 = 1+1+1+1+1 = 5 \equiv 0 \pmod{5}$$

$$1^4 = 1+1+1+1 = 4 \equiv 4 \pmod{5}$$

$$1^3 = 1+1+1 = 3 \equiv 3 \pmod{5}$$

$$1^2 = 1+1 = 2 \equiv 2 \pmod{5}$$

$$1^1 = 1 \equiv 1 \pmod{5}$$

Q. If the identity element of  $\mathbb{Z}_5$  under  $+$ .

	4	0	1	2	3
3	4	0	1	2	
2	3	4	0	1	
1	2	3	4	0	
0	0	1	2	3	4

$$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}.$$

Q. Show that  $(\mathbb{Z}_5, +)$  is a cyclic group. What are its gen-

There are total 2 generators i.e. 2 & 5.

Since, 5 is the inverse of 2, so 5 is the generator.

Since, 2 is the inverse of 5, so 2 is the generator.

$$\begin{aligned} 2^6 &= 64 \equiv 1 \pmod{9} \\ 2^5 &= 32 \equiv 5 \pmod{9} \\ 2^4 &= 16 \equiv 7 \pmod{9} \\ 2^3 &= 8 = 8 \pmod{9} \\ 2^2 &= 4 = 4 \pmod{9} \\ 2^1 &= 2 = 2 \pmod{9} \end{aligned}$$

1 is the identity element.

	8	4	5	1	2	3
7	4	1	2	7	8	9
6	5	8	4	2	1	5
5	2	4	3	7	5	4
4	1	2	4	5	3	8

$$G = \{1, 2, 4, 5, 7, 8\}, X_9\}$$

many generators are there and what are they?

Show that  $G = \{1, 2, 4, 5, 7, 8\}$  under  $\times_9$  is cyclic. How

Q. Is the generator of  $G$ ?

$$(m)^3 = 1$$

$$(m)^2 = m^2$$

$$(m) = m$$

\* Permutation Group :- A bijective function from  $A$  to  $A$  where  $A$  is non-empty is called a permutation of  $A$ .

E.g.  $A = \{1, 2\}$

$$P_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad P_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

\* Equality of Permutations :- Let,  $P_1$  and  $P_2$  are two permutations on set  $A$ ,  $P_1$  and  $P_2$  are said to be equal, if their corresponding images are equal.

$$P_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 3 & 2 & 1 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$

\* Identity Permutation :- If the image of element is same as element itself then permutation is said to be identity permutation. It is denoted by ' $I$  or  $I_A$ '.

E.g. If  $A = \{1, 2, 3\} \Rightarrow I_A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

\* Inverse Permutation :- (since permutation is a bijective function)

Each permutation ' $P$ ' is invertible since it is bijective & it is denoted by  $P^{-1}$

$$P = \begin{pmatrix} a & b & c & d \\ b & c & d & a \end{pmatrix} \Rightarrow P^{-1} = \begin{pmatrix} a & b & c & d \\ d & a & b & c \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} b & c & d & a \\ a & b & c & d \end{pmatrix}$$

\* Product of Permutation :-

(The no. of elements is known as degree of permutation)

The product of permutations  $P_1$  &  $P_2$  of same degree is denoted by  $P_1 \circ P_2$ .

$$A = \{1, 2, 3, 4\}$$

$$P_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \quad P_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 2 \end{pmatrix}$$

$$P_1 \circ P_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

\* Permutation Group :- A set  $G$  of all permutations on a non-empty set  $A$  under the binary operation '\*' (product of permutation) is called permutation group. If  $A$  has ' $n$ ' elements then the given permutation group formed by  $A$  is also called symmetric group of degree ' $n$ '. Denoted by ' $S_n$ ' and ' $S_n$ ' has  $n!$  elements.

\* Cyclic Permutation :- Let,  $A = \{x_1, x_2, \dots, x_n\}$  and let,  $x_1, x_2, \dots, x_k \in A$  and

Permutation ' $P$ ':  $A \rightarrow A$  defined by

$$\begin{pmatrix} f & e & b & d \\ a & b & c & d \\ a & e & b & d \\ a & b & c & d \end{pmatrix} =$$

$$\begin{pmatrix} f & e & b & d \\ a & e & b & d \\ a & b & c & d \\ a & b & c & d \end{pmatrix}$$

$$\begin{pmatrix} f & b & c & d \\ a & b & c & d \\ a & b & c & d \\ a & b & c & d \end{pmatrix}$$

$$\text{Hence can write } P = (a, f) \circ (b, c) = (a, f) \circ (c, b, e, d)$$

[It is even permutation].

$$P = \begin{pmatrix} f & e & b & d \\ a & b & c & d \\ a & b & c & d \\ a & b & c & d \end{pmatrix}$$

Q. Expression given permutation as product of cycles where -

Product of even & odd permutation is odd.

Product of two even permutations is even.

NOTE: A permutation can't be both even or odd.

of odd no. of transpositions.

\* Even permutation :- A permutation can be expressed as a product of even no. of transpositions. If it can be expressed as a product of odd permutations. A permutation can be expressed as a product of even no. of even no. of transpositions.

\* Even permutation can be solved as a product of even permutations.

NOTE: If there is a cycle of length  $n$ , then there are  $(n-1)$  fix points.

E.g.  $(a, b) \cdot (1, 2)$  has  $2$  fix points.

↳ Transpositions :- A cycle of length  $2$  is called a transposition.

e.g.  $P_1 = (b \ d \ e) \cdot P_2 = (a \ c \ f)$  are disjoint cycles if  $a \in$  such that  $a \in$  both cycles.

\* Disjoint cycles :- Two cycles  $P_1$  and  $P_2$  for those  $\in$  the same permutation can be calculated the same way.

This permutation would be called a disjoint cycles. It is not necessary that  $P_1$  and  $P_2$  have a common length  $k$ .

$$P(+k) = k,$$

$$P(+k-1) = k$$

$$P = (\underbrace{b \ d \ e}, c)$$

$$(f \ e \ c \ d \ b \ a) = P$$

$$P(+3) = 74$$

$$P(+2) = 73$$

$$P(+1) = 72$$

\* Q Determine whether the given permutations are even or odd.

$$P_1 = \begin{pmatrix} a & b & c & d & e & f \\ c & f & b & d & a & \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 4 & 5 & 2 & 1 \end{pmatrix}$$

$$P_1 = (a, e) \circ (b, f, c) = (a, e) \circ (b, f) \circ (b, c) \quad [\text{odd per}]$$

$$P_2 = (1, 6) \circ (2, 3, 4, 5) = (1, 6) \circ (2, 3), (2, 4), (2, 5). \quad [\text{Even per}]$$

Q A = (1, 2, 3, 4, 5) - find the product (1, 3)  $\circ$  (2, 4, 5)  $\circ$  (2, 3)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 4 & 5 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 5 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix} \quad \dots \quad .$$

\* Subgroup :- If  $(G, *)$  is a group and  $H$  "subset of  $G$ " then  $(H, *)$  is said to be subgroup of  $(G, *)$ .  
i)  $(H, *)$  is also a group itself. i.e.  $H$  must be following properties - satisfy

1) Closure Property :-  $\forall a, b \in H$   
 $a * b \in H$ .

2) Identity Element :-  $\exists e \in H$ ,  $a * e = a$ ,  $\forall a \in H$ .

3) Inverse Element :-  $\exists a^{-1} \in H$ ,  $a * a^{-1} = e$ ,  $\forall a \in H$ .

One Step Subgroup Test :-

The necessary & sufficient condition for a non-empty subset ' $H$ ' of  $(G, *)$  to be subgroup of ' $G$ ' is  
 $[a * b^{-1} \in H ; \forall a, b \in H]$

Proof :-

1) Necessary Condition :- Let  $(H, *)$  be a subgroup. and  
 $a, b \in H \Rightarrow b^{-1} \in H$ .  
 $\Rightarrow a * b^{-1} \in H$  (closure property).

2) Sufficient Condition :- If  $a * b^{-1} \in H$ ,  $\forall a, b \in H$ .  
Put,  $b = a$   
 $\Rightarrow a * a^{-1} \in H$ .  
 $\Rightarrow e \in H$  (e is identity element)  
∴ Identity Element exists in  $H$ .  $(G, *)$ ,  $H \subseteq G$  &  $e \in G$ ,  $e \in H$

Remark: 1) Identity of  $H$  will be same as Identity of  $G$ .

2)  $(G, *)$  &  $(e, *)$  are trivial subgroups of  $G$ .

3) Other than trivial subgroups all subgroups are called

$\Rightarrow a * b \in H$ .

∴ closure property exists.

so,  $H$  is a subgroup.

The intersection of any 2 sub-groups of a group  $(G, *)$  is again a subgroup of  $(G, *)$ .

Let,  $H_1$  &  $H_2$  be two subgroups of a group  $(G, *)$ .  
Since, identity element exist or belong to both subgroups  $H_1$  &  $H_2$ .

Therefore,  $(H_1 \cap H_2, *)$  is  $H_1 \cap H_2 \neq 0$

Let,  $a, b \in H_1 \cap H_2$

$\Rightarrow a, b \in H_1 \text{ & } a, b \in H_2$ .

$\Rightarrow (a \in H_1 \text{ and } a \in H_2) \text{ and } (b \in H_1 \text{ and } b \in H_2)$  — (1)

Since,  $H_1$  &  $H_2$  are subgroups:

$b^{-1} \in H_1 \text{ and } b^{-1} \in H_2$  — (2)

From eqn ① & ②.

$a * b^{-1} \in H_1 \text{ and } a * b^{-1} \in H_2$

$\Rightarrow a * b^{-1} \in (H_1 \cap H_2)$

By one step sub-group test 2 subgroup is a group of  $(G, *)$ .

Note:- Every subgroup of a cyclic group is cyclic.

Q Prove the following & given as counter example. If  $(H_1, *)$  &  $(H_2, *)$  are both subgroups of the group  $(G, *)$ . Then  $(H_1 \cup H_2, *)$  is also a subgroup of  $(G, *)$ .

→ Let,  $(\mathbb{Z}, +)$  The union of two subgroups may or may not be a subgroup. The counter example is given below-

Let  $H_1 = \{ \dots, -6, -3, 0, 3, 6, \dots \}$

Let,  $H_2 = \{ \dots, -10, -5, 0, 5, 10, \dots \}$

$\Rightarrow H_1 \cup H_2 = \{ \dots, \pm 10, \pm 6, \pm 5, \pm 3, 0 \}$ .

Since,  $6+5=11 \notin H_1 \cup H_2$ .

∴  $H_1 \cup H_2$  is not a subgroup.

\* Coset :- Let,  $H$  be a subgroup of group  $G$ . And let  $a \in G$ . Then-

$Ha = \{ ah, h \in H \}$  is called Right coset generated by  $H$  &  $a$ . also the set

$aH = \{ ab, h \in H \}$  is called Left coset generated by  $a$  &  $H$ .

The element  $a$  is called representative of  $aH$  or  $Ha$ .

Eg-  $(z_9 + a)$ .

$$a(z_9) = 9$$

$$\begin{cases} aH = 11 \text{ & } 11 \\ a + H = 11 \text{ & } 11 \end{cases}$$

$$z_9 = \{ 0, 1, 2, 3, 4, 5, 6, 7, 8 \}.$$



If  $G$  is a group and  $H$  is a subgroup of  $G$ , then the no. of distinct right or left cosets of  $H$  in  $G$  is called the Index of  $H$  in  $G$ . It is denoted by  $[G:H]$  or  $f_G(H)$  [Index of  $H$  in  $G$ ].

Eg.  $H = \{1, 11\}$

$$[G:H] = f_4(H) = 4.$$

**Lagrange's Theorem :-** Order of subgroup divides order of  $G$ .

If  $G$  is a finite group and  $H$  is a subgroup of  $G$ , then order of subgroup divides the order of group. i.e.  $[G:H]$  divides  $|G|$ . Moreover the number of distinct left or right cosets of  $H$  in  $G$  equals to  $\frac{|G|}{|H|}$ .

Proof :- Let,  $H$  be a subgroup of order 'm' of a group  $G$ .

of order 'n'. i.e.  $|G|=n$  &  $|H|=m$ .

Let,  $H = \{h_1, h_2, h_3, \dots, h_m\} \subset G = \{a_1, a_2, \dots, a_n\}$  be a left coset

Let,  $a \in G$  then  $aH = \{ah_1, ah_2, ah_3, \dots, ah_m\}$  is a left coset

containing  $m$  distinct elements.

since,  $ahi = ah_j \Rightarrow h_i = h_j$  By cancellation law of  $G$ .

Thus, every left coset of  $H$  in  $G$  has  $m$  distinct elements.

Since,  $G$  is a finite group then the no. of distinct left cosets

will also be finite. Let it be ' $k$ '. Then the union of

these ' $k$ ' distinct left cosets of ' $H$ ' in ' $G$ ' =  $G$ . i.e.

if  $a_1H, a_2H, a_3H, \dots, a_kH$  are distinct left cosets of  $H$ .

then,  $G = a_1H \cup a_2H \cup a_3H \dots \cup a_kH$

$$\Rightarrow |G| = |a_1H| + |a_2H| + |a_3H| + \dots + |a_kH|. \quad (k \text{ times})$$

$$\Rightarrow pm = m + m + m + \dots + m,$$

$$\Rightarrow pm = m \cdot k.$$

$$\Rightarrow k = \frac{n}{m}.$$

$$\therefore k = \frac{|G|}{|H|}, \quad \text{where, } k \in \mathbb{Z}^+.$$

Q Find all the subgroups of -

$$(Z_{12}, +_{12})$$

$$(Z_5, +_5)$$

$$i) Z_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}.$$

Possible order of subgroups = 1, 2, 3, 4, 6, 12.

$$2^1 = 2 \quad \{0, 2, 4, 6, 8, 10\} \Rightarrow |G| = 6.$$

$$2^2 = 4$$

$$2^3 = 8$$

$$2^4 = 16$$

$$2^5 = 32$$

$$2^6 = 64$$

$$\Rightarrow (0, +_{12})$$

$$\dots (Z_{12}, +_{12})$$



normal subgroup of  $G$ . if  $\forall$   
 $H = aH$ , i.e.  $a \in G$ , i.e. the right coset & left coset  
of  $H$  is generated by 'a' are same.

Remark: • Clearly every subgroup 'H' of an abelian group 'G'  
a normal subgroup of 'G' for  $a \in G$  and  $h \in H$   
 $ah = ha$   
• since a cyclic group is abelian therefore every subgroup  
of a cyclic group is normal.

Theorem: A subgroup  $H$  of a group  $G$  is normal if & only if

$$g^{-1}hg \in H \quad \text{for } \forall h \in H, \forall g \in G.$$

Q. Show that such  $H = \{(1, b) : b \in R\}$  is a normal subgroup  
of  $G = \{(a, b) : a \neq 0, b \in R\}$  under the composition '\*' defined  
by  $(a, b) * (c, d) = (ac, bc+d)$ .

$\rightarrow$  Let  $(e_1, e_2) \in G$  be the identity element.

$$\forall (a, b) \in G.$$

$$\Rightarrow (a, b) * (e_1, e_2) = (a, b).$$

$$\Rightarrow (ae_1, be_1 + e_2) = (a, b).$$

$$\Rightarrow ae_1 = a \quad \& \quad be_1 + e_2 = b.$$

$$\Rightarrow e_1 = 1. \quad \& \quad b + e_2 = b \\ \Rightarrow e_2 = 0.$$

So,  $(1, 0)$  is the identity element.

Let,  $(c, d)$  be the inverse of  $(a, b) \in G$ .

$$\Rightarrow (a, b) * (c, d) = (1, 0).$$

$$\Rightarrow (ac, bc+d) = (1, 0)$$

$$\Rightarrow ac = 1 \quad \& \quad bc+d = 0.$$

$$\Rightarrow c = \frac{1}{a}. \quad \& \quad d = -bc. \quad \Rightarrow d = -b \cdot a^{-1}.$$

$$\Rightarrow c = a^{-1}$$

$(\frac{1}{a}, -\frac{b}{a})$  be the inverse of  $(a, b)$

Let,  $(a, b) \in G$  &  $(1, b) \in H$  and,  $(a, b)^{-1} = (\frac{1}{a}, -\frac{b}{a})$ .

Now,  $(a, b)^{-1} * (1, b) * (a, b)$

$$\cdot ((\frac{1}{a}, -\frac{b}{a}) * (1, b)) * (a, b).$$

$$\Rightarrow (\frac{1}{a}, -\frac{b}{a} + b) * (a, b) \Rightarrow (\frac{1}{a}, -\frac{b+ab}{a}) * (a, b)$$

$$\Rightarrow (\frac{1}{a} * a, -\frac{b+ba}{a} * a + b)$$

$$\Rightarrow (1, -b+ba+b)$$

$$\Rightarrow (1, ab), \quad ab \in R.$$

$$\Rightarrow (1, ab) \in H.$$

\* Q If  $N$  is a Normal subgroup of  $G$ , then show that

$$\Rightarrow N a N b = N a b$$

$$2) a N b N = a b N, \quad a, b \in G.$$

$$\rightarrow 1) N a N b = N(aN)b.$$

$$= N(Na)b$$

$$= NN ab.$$

$$\boxed{N a N b = N a b.}$$

$(aN = Na)$  (since  $N$  is  
Normal subgroup).

$(NN = N)$ .

$$2) a N b N = a(Nb)N$$

$$= a(bN)N$$

$$= ab NN$$

$$\boxed{aN b N = ab N.}$$

$(aNb = bN)$  (since  $N$  is  
Normal subgroup)

$(NN = N)$ .

\* Homomorphism :- Let  $(G_1, *)$  and  $(G_2, *)$  be two groups then a function or mapping

$f : G_1 \rightarrow G_2$  is called a Homomorphism if

$$\boxed{|f(a \cdot b)| = f(a) * f(b)| \quad \forall a, b \in G_1|}$$

Thus, 'f' is Homomorphism from  $G_1 \rightarrow G_2$  then 'f'  
preserves the composition of  $G_1$  &  $G_2$  i.e. image of composition

$$|f(a \cdot b)| = f(a) * f(b)| \quad \text{= composition of images.}$$

The group  $G_2$  is said to be Homomorphic image of  
group  $G_1$ .

\* Isomorphism :- Let,  $(G_1, *)$  and  $(G_2, *)$  be two groups  
then a function 'f' :  $G_1 \rightarrow G_2$  is an

Isomorphism if

- $f$  is homomorphism.  
i.e.  $|f(a \cdot b)| = f(a) * f(b)| \quad \forall a, b \in G_1|$
- $f$  is Bijective (one to one & onto).

\* Automorphism :- An isomorphism of a group 'G' onto itself  
is called an automorphism of 'G'.

\* Isomorphic Groups :- A group  $G_1$  is said to be isomorphic  
to group  $G_2$  if there exist an isomor-  
phism of  $G_1$  on to  $G_2$  and it is denoted by  $\boxed{G_1 \cong G_2}$

Q Let,  $G_1 = (Z, +)$  and  $G_2 = G_1$ , and, let func.  $f : G_1 \rightarrow G_2$  is  
defined as  $f(x) = 2x$ ,  $x \in Z$ . Prove that 'f' is  
homomorphism.  $|f(a \cdot b)| = f(a) * f(b)$ .

$$\rightarrow G_1 = (Z, +), \quad G_2 = G_1 = (Z, +)$$

$$f(x) = 2x, \quad \forall x \in Z$$

We have to show that

$$f : G_1 \rightarrow G_2$$

$$\text{such that, } f(a+b) = f(a) + f(b)$$

$$\text{or } f(x+y) = f(x) + f(y), \quad \forall x, y \in Z.$$

$$\text{Let } f(x) = y \in \mathbb{Z}_n, \text{ then } x = a^y \in \mathbb{Z}_n$$

$$x = a^y \Leftrightarrow a^x = y \in \mathbb{Z}_n$$

$$\text{Let } f(x) = y \in \mathbb{Z}_n, x \in \mathbb{Z}_n, y \in \mathbb{Z}_n$$

$$\text{Let } f(x_1) = y_1 \in \mathbb{Z}_n, f(x_2) = y_2 \in \mathbb{Z}_n, \text{ then } f(x_1 + x_2) = f(x_1) + f(x_2)$$

$$f(x_1 + x_2) = a^{x_1 + x_2} = a^{x_1} \cdot a^{x_2} = f(x_1) \cdot f(x_2)$$

$$f(x+y) = a^{x+y} = a^x \cdot a^y = f(x) \cdot f(y) \quad (1)$$

(2)  $f$  is bijective.

(3)  $f$  is homomorphism.

$$f(x) = a^x, \forall x \in \mathbb{Z}_n$$

$$f : \mathbb{Z}_n \hookrightarrow \mathbb{Z}_n$$

Show that a finite cyclic group of odd order  $n$  is isomorphic to  $\mathbb{Z}_n$ . The group of integers mod  $n$ .

$$f(-x) = -[f(x)] \quad \leftarrow$$

$$f(-x) = (-x)f \cdot (x)f \quad \leftarrow$$

$$f(-x) = (-x \cdot x)f \quad \leftarrow$$

$$f(-x) = x^{-1} \cdot x = e_1. \quad \leftarrow$$

Let  $x \in \mathbb{G}_1$ , then  $f(x) \in \mathbb{G}_2$ .

$$f(x^{-1}) = [f(x)]^{-1} \quad \leftarrow$$

$$e_2 = f(e_1). \quad \leftarrow$$

$$f(x) \cdot e_2 = f(x) \cdot f(e_1). \quad \leftarrow$$

$$f(x) \cdot e_2 = f(x \cdot e_1). \quad (e_1 \text{ is identity of } \mathbb{G}_1)$$

$$f(x) \cdot e_2 = f(x). \quad (e_2 \text{ is identity of } \mathbb{G}_2)$$

$$\text{Let } x \in \mathbb{G}_1 \text{ then } f(x) \in \mathbb{G}_2. \quad \leftarrow$$

$$f(e_1) = e_2. \quad \leftarrow$$

$$f(-x) = [f(x)]^{-1}, \forall x \in \mathbb{G}_1.$$

$$f(e_1) = e_2 \quad \text{where, } e_1, e_2 \text{ are identity elements of } \mathbb{G}_1 \text{ and } \mathbb{G}_2 \text{ respectively.}$$

THEOREM 3: Let  $f: \mathbb{G}_1 \rightarrow \mathbb{G}_2$  is a homomorphism then-

Theorems on HOmomorphism:-

$$(h+x)f = h + (x)f$$

$$= a^{h+x} = a^h + a^x$$

$$(h+x)f = f(h+x)$$

Q Show that any infinite cyclic group is isomorphic to  $(\mathbb{Z}, +)$ .

Let,  $G = \langle a \rangle = \{a^n : n \in \mathbb{Z}\}$ .

$$f : \mathbb{Z} \rightarrow G,$$

$$f(x) = a^x, \forall x \in \mathbb{Z}.$$

i)  $f$  is homomorphism.

ii)  $f$  is bijective

$$\begin{aligned} f(x+y) &= f(x) + f(y) \\ &= a^x \cdot a^y \end{aligned}$$

$$\boxed{f(x+y) = f(x) \cdot f(y)}$$

so, it is homomorphism

iii) Let,  $f(x_1) = f(x_2), x_1, x_2 \in \mathbb{Z}$

$$\Rightarrow a^{x_1} = a^{x_2}$$

$$\Rightarrow \boxed{x_1 = x_2}$$

so, it is one to one

iv) Let,  $f(x) = y$

$$a^x = y$$

$$\boxed{x = \log_a y}.$$

so, it is onto.

v)

$$g \in G$$

$$\Rightarrow a^x \in G \quad g = a^x, x \in \mathbb{Z}$$

$$\Rightarrow \boxed{g \in (\mathbb{Z})} \quad (\text{It is bijective})$$

so, it is isomorphic to  $(\mathbb{Z}, +)$ .

\* **Ring** :- An algebraic system  $(R, +, \cdot)$  where 'R' is a non-empty set and '+' and ' $\cdot$ ' are any two binary operations in called a ring if following conditions are satisfied.

•  $(R, +)$  is abelian group.

•  $(R, \cdot)$  is a semigroup.

$$\text{i.e. } a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \forall a, b, c \in R.$$

• operation ' $\cdot$ ' is distributed over '+'.  
 $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$

$$(b+c) \cdot a = (b \cdot a) + (c \cdot a) \quad \forall a, b, c \in R.$$

e.g. Set of Integers  $(\mathbb{Z})$   
 Set of Real No.  $(\mathbb{R})$  under addition and multiplication  
 collect is a ring. (with Unity 1).

Remark: • Ring  $(R)$  is commutative iff  $a \cdot b = b \cdot a$   
 $\forall a, b \in R$ . i.e.  $(R, \cdot)$  is commutative

- An unit or identity element of ring are (if it exists) is an element of semigroup  $(R, \cdot)$ .
- If  $(R, \cdot)$  is monoid then the ring  $(R, +, \cdot)$  is called a ring with identity or unit.
- Additive identity of  $(R, +)$  is called zero element, or zero identity of ring  $R$ .

$a \cdot b = 0$  then  $a$  &  $b$  are divisors of 0 or zero divisor

show that  $(\mathbb{Z}_5, +_5, \times_5)$  is a commutative ring with unity

$$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$$

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

1)  $(\mathbb{Z}_5, +_5)$  is abelian:-

1) Closure Property: From above table it is satisfied

then Ring

1)  $(\mathbb{Z}_5, +_5)$  is abelian.

2)  $(\mathbb{Z}_5, \times_5)$  is semigroup.

3) ' $\times_5$ ' is distributed over ' $+_5$ '

2) Associative Property: From above table it is satisfied

3) Existence of Identity element :- 0 is the identity element

4) Inverse element:-

$$0+50 = 0 \Rightarrow 0^{-1} = 0$$

$$1+54 = 0 \Rightarrow 1^{-1} = 4$$

$$2+53 = 0 \Rightarrow 2^{-1} = 3$$

$$3+52 = 0 \Rightarrow 3^{-1} = 2$$

$$4+51 = 0 \Rightarrow 4^{-1} = 1$$

5) Commutative Property:- from above table, it is satisfied

$\Rightarrow (\mathbb{Z}_5, +_5)$  is abelian.

6)  $(\mathbb{Z}_5, \times_5)$  is semigroup:-

1) Closure Property: from above table, it is satisfied.

2) Associative Property: from above table, it is satisfied.

$\Rightarrow (\mathbb{Z}_5, \times_5)$  is semigroup.

7) ' $\times_5$ ' is distributed over ' $+_5$ '.

$$(1 \times_5 (2+53)) = ((1 \times_5 2) +_5 (1 \times_5 3)) \Rightarrow LHS = RHS$$

$$\Rightarrow 1 \times_5 0 = 2+53$$

$$\Rightarrow 0 = 0 \Rightarrow LHS = RHS$$

$$(2+53) \times_5 1 = (2 \times_5 1) +_5 (3 \times_5 1)$$

$$\Rightarrow 0 \times_5 1 = 2+53$$

$$\Rightarrow 0 = 0 \Rightarrow LHS = RHS$$

so, it is a ring.

4) Ring is commutative.  $(\mathbb{Z}_5, \times_5)$ . [  $a \times_5 b = b \times_5 a$  ].

$$1 \times_5 2 = 2 \times_5 1 \Rightarrow 2 = 2$$

From above table commutative property follows.

5)  $(\mathbb{Z}_5, \times_5)$  Identity element

$i \in \mathbb{Z}_5$  is a identity element

$$0 \times_5 i = 0$$

$$1 \times_5 i = 1$$

$$2 \times_5 i = 2$$

$$3 \times_5 i = 3$$

$$4 \times_5 i = 4$$

$\therefore 1$  is identity element of  $(\mathbb{Z}_5, \times_5)$ .

6)  $(\mathbb{Z}_5, \times_5)$  is a monoid.

1) Closure Property: From table, it is satisfied

2) Associative Property: From table, it is satisfied.

3) Identity Element: - From above it is clear  $1$  is identity element.

4) It is monoid.

5)  $(\mathbb{Z}_5, +_5)$  is called zero element

6) It is a commutative ring with unity.

Q Show that the system  $(E, +, \cdot)$  of Even integers is a commutative ring without unity w.r.t usual addition and multiplication.

→  $E$  = Set of Even Integers.

1)  $(E, +)$  is an abelian.

$$a+b \in E, \forall a, b \in E$$

2) Closure:

3) Associativity:  $\forall a, b, c \in E$ .

$$a + (b+c) = (a+b) + c$$

4) Existence of Identity element:

$0 \in E$ . is an identity element

$$\text{Since, } a+0 = a, \forall a \in E$$

5) Existence of Inverse Element:

$$a^{-1} = -a$$

$$\text{Since, } a + (-a) = 0, \forall a \in E$$

6) Commutative Property:

$$a+b = b+a, \forall a, b \in E$$

7)  $(E, \cdot)$  is a semigroup:

1) Closure Property:  $a \cdot b \in E, \forall a, b \in E$

2) Associative Property:  $\forall a, b, c \in E$ .

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

3)  $\cdot$  is distributed over  $+$ .

$$a \cdot (b+c) = (a \cdot b) + (a \cdot c)$$

$$\forall a, b, c \in E$$

4) Commutative Ring  $(E, \cdot)$

$$\forall a, b \in E$$

$1 \notin E$  so, Identity element does not exist.

so. it is a commutative ring without unity.

$$\text{Q} (a+b)^2 = a^2 + 2ab + b^2.$$

If  $R$  is a commutative ring then show that  $\exists$

$$\Rightarrow (a+b)^2 = (a+b) \cdot (a+b)$$

$$= (a+b) \cdot a + (a+b) \cdot b$$

$$= a \cdot a + b \cdot a + a \cdot b + b \cdot b$$

$$= a \cdot a + a \cdot b + a \cdot b + b \cdot b$$

$$\boxed{(a+b)^2 = a^2 + 2ab + b^2}$$

(Right Distribution)

(Left Distribution)

( $R$  is commutative.  
( $a \cdot b = b \cdot a$ ).

\* Properties of Ring :-

- The additive identity or zero element of a ring  $(R, +, \cdot)$  is unique.

Proof Let, if possible  $0$  and  $0'$  be additive identity of ring  $R$ .

Let  $0$  is additive identity and  $0' \in R$ .

$$\Rightarrow 0'+0 = 0+0' = 0' \quad \text{--- (1)}$$

Let  $0'$  is additive identity and  $0 \in R$ .

$$\Rightarrow 0+0' = 0'+0 = 0 \quad \text{--- (2)}$$

From eqn (1) & (2) -

$$\boxed{0 = 0'}$$

- The additive inverse of every element of the ring is unique.

Proof Let,  $b$  &  $c$  are additive inverse of  $a \in R$ .  
since  $0 \in R$  is additive identity.

$$\text{Then, } a+b = b+a = 0 \quad \text{--- (1)}$$

$$a+c = c+a = 0 \quad \text{--- (2)}$$

Now,

$b = b+0$		from eqn (2)	from eqn (1) [Associative Law]
$b = b+(a+c)$	$b = (b+a)+c$		

$\leftarrow$

$$\boxed{b = a+c}$$

- Cancellation Laws of Addition.

Proof If  $a, b, c \in R$ .

$$\text{i)} a+b = a+c \Rightarrow b = c \quad \begin{matrix} [\text{Left cancellation}] \\ [\text{Right cancellation}] \end{matrix}$$

$$\text{ii)} b+a = c+a \Rightarrow b = c$$

$\Rightarrow$  Let,  $a+b = a+c$  {where  $-a$  is inverse of  $a \in R$ }

$$\Rightarrow (-a) + (a+b) = (-a) + a+c \quad \begin{matrix} \\ \end{matrix}$$

$$\Rightarrow (-a+a) + b = (-a+a) + c$$

$$\Rightarrow 0+b = 0+c$$

$$\Rightarrow \boxed{b = c}$$

$$\begin{aligned} \Rightarrow b+a &= c+a \\ \Rightarrow b+a+(-a) &= c+a+(-a) \\ \Rightarrow b+(a-a) &= c+(a-a) \\ \Rightarrow b+0 &= c+0 \\ \Rightarrow b &= c \end{aligned}$$

- If  $(R, +, \cdot)$  is a ring and  $a \in R$  then  

$$a \cdot 0 = 0 \cdot a = 0$$
 where, 0 is the additive identity or zero element of  $R$ .

Proof: Consider,  $a \cdot 0 + a \cdot a = a \cdot (0+a)$  (Distributive law)  
 $= a \cdot a$ . ( $\because 0+a=a$ )  
 $a \cdot 0 + a \cdot a = 0 + a \cdot a$  [By Right Cancellation Law of Addition]  

$$a \cdot 0 = 0$$

Consider,  $a \cdot a + 0 \cdot a = (a+0) \cdot a$ . (Distributive)  
 $a \cdot a + 0 \cdot a = a \cdot a$ . ( $\because 0+a=0$ )  
 $a \cdot a + 0 \cdot a = a \cdot a + 0$  [By Left Cancellation]  

$$0 \cdot a = 0$$

- 1) If  $(R, +, \cdot)$  is a ring then  $[-(-a)] = a$ .

Proof:- 0  $\in R$  is additive identity of Ring  $R$

$$\text{So, } a+a^{-1} = 0 \quad \forall a \in R$$

$$\Rightarrow a^{-1} = -a$$

i.e.  $(-a)$  is additive inverse of  $a \in R$ .

$$\text{So, } a+(-a) = (-a)+a = 0$$

$$\text{Now, } (-a)^{-1} = [-(-a)] = a$$

Since, additive inverse is unique.

$$2) [a \cdot (-b) = (-a) \cdot b = -(a \cdot b)], \forall a, b \in R$$

Proof: If 0  $\in R$  is zero element

$$a \cdot 0 = 0 \quad \forall a \in R$$

$$\Rightarrow a \cdot (b+(-b)) = 0$$

$$\Rightarrow a \cdot b + a \cdot (-b) = 0 \quad \text{[Distributive]}$$

We know that  $-a$  is additive inverse of  $a \in R$  and

it is unique.

$$\text{So, } [-(a \cdot b)] = a \cdot (-b)$$

$$0 \cdot b = 0 \quad \forall a \in R$$

$$\Rightarrow 0 \cdot (b+(-b)) \cdot b = 0$$

$$\Rightarrow 0 \cdot b + 0 \cdot (-b) \cdot b = 0 \quad \text{[Distributive]}$$

$$\Rightarrow 0 \cdot b + (-a) \cdot b = 0$$

$$\text{So, } [-(a \cdot b)] = (-a) \cdot b$$

$$3) [(-a) \cdot (-b) = a \cdot b], \forall a, b \in R$$

Proof: Consider  $(-a) \cdot (-b) = -a \cdot (-b)$   
 $= -[-(a \cdot b)]$

Proof :- consider,  $a \cdot (b+c) = a \cdot b + a \cdot c$  [distributive]

$$\begin{aligned} a \cdot (b+c) &= a \cdot b + a \cdot c \\ &= a \cdot b - a \cdot c \end{aligned}$$

Proof  $(a-b) \cdot c = a \cdot c - b \cdot c$

consider,  $(a-b) \cdot c = (a + (-b)) \cdot c$   
 $= a \cdot c + (-b) \cdot c$   
 $= a \cdot c - b \cdot c$

\* Boolean Ring :- An element  $a \in R$  is said to be idempotent if  $a^2 = a$ .  
A ring  $R$  is called a Boolean Ring if all its elements are idempotent.  
i.e.  $\forall a \in R : a^2 = a$ .

\* Integral Domain :- A ring is called integral domain if:

- It is commutative.
- It has unique element.
- It is without zero divisors

[ $a \cdot b = b \cdot a$   
 $\exists a \in R$  Ring with unity]  
 $[a \cdot b = 0 \Rightarrow a, b = 0]$

e.g.  $(C, +, \cdot)$ ,  $(\mathbb{Q}, +, \cdot)$ ,  $(\mathbb{R}, +, \cdot)$  are integral domains.

\* Field :- A ring  $R$  with at least two elements is called a field if it has following property -

- Ring  $R$  is commutative.
- $R$  has unity or unit element.
- $R$  is such that each non-zero element has multiplicative inverse.

e.g. Set of Real Numbers in a field, set of complex No. is field.

Set of complex  $(x+iy) + (e_1+ie_2) = (x+iy)$

Remark:  
Every field is an integral domain

$$(x+e_1) + i(y+e_2) = x+iy$$

$$\Rightarrow x+e_1 = x \quad \& \quad y+e_2 = y$$

$$\Rightarrow e_1 = 0 \quad \& \quad e_2 = 0$$

$$\Rightarrow (x+iy) + (0+0i) \rightarrow (0+0i)$$

Identity element of  $(x+iy)$   $\rightarrow (0+0i)$

$$(x+iy) + (a+ib) = 0+0i$$

$$\Rightarrow a = -x \quad \& \quad b = -y$$

$$(x+iy)^{-1} = -(x+iy)$$

Identity element of multiplicative inverse =  $(1+0i)$

$$w = -a^2 - b^2$$

$$\Rightarrow (x+iy) \cdot (a+ib) = (1+0i)$$

$$\Rightarrow (x+iy) \cdot a + (x+iy) \cdot ib = (1+0i)$$

$$\Rightarrow x \cdot a + i \cdot y \cdot a + ix \cdot b - iy \cdot b = (1+0i)$$

$$\Rightarrow (x \cdot a - y \cdot b) + i(y \cdot a + x \cdot b) = (1+0i)$$

$$\Rightarrow x \cdot a - y \cdot b = 1 \quad \& \quad y \cdot a + x \cdot b = 0$$

$$\Rightarrow xb = -ay$$

$$\Rightarrow b = -\frac{ay}{x}$$

$$b = -\frac{x}{(x^2+y^2)}$$

$$b = -\frac{y}{x^2+y^2}$$

$$\Rightarrow x^2a + y^2a = x$$

$$\Rightarrow a \cdot \frac{x}{x^2+y^2}$$

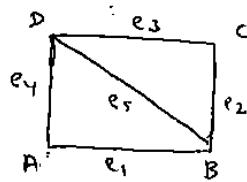
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## GRAPH THEORY

$$V(G) = \{A, B, C, D\}$$

$$E(G) = \{e_1, e_2, e_3, e_4, e_5\}$$

$$E(G) = \{(A, B), (B, C), (C, D), (D, A), (B, D)\}.$$



The degree of self loop is 2.

Degree of vertex :  $\deg(v_i)$  or  $d(v_i)$ . [No. of edges associated with it].

$d(A) = 2$  vertex A is even.

$d(B) = 3$  vertex B is odd.

$d(C) = 2$  vertex C is even.

$d(D) = 3$ . vertex D is odd.

'e' Edge is incident on A & B

A & B are adjacent vertices.

A tree is always a graph.  
but a graph is not always a tree

$$\sum \deg(v_i) = 10.$$

$$\sum \deg(v_i) = 2e = \text{even} \quad \boxed{\text{Handshaking Theorem}}$$

$\Rightarrow$  sum of degrees of vertices is also even.

$\Rightarrow e = \text{Distinct edges of graph.}$

Handshaking Theorem Improved :-

The no. of odd vertices in a graph is also even.

\* Nil vertex :- vertex having zero degree.

\* Half vertex :- vertex having degree 1.

\* Hand Shaking Theorem :- Theorem

$$\sum_{i=1}^n \deg(v_i) = 2e.$$

Where, e is No. of Distinct edges.

Statement: The sum of degrees of all vertices in a graph is always even. and it is equal to  $2 \times$  Total No. of edges in the graph.

\* Theorem : The number of vertices of odd degree in an undirected graph is always even.

Proof Let. G(V, E) be a undirected graph then by

Handshaking Theorem

$$\sum_{i=1}^n \deg(v_i) = 2e = \text{even} \quad \text{--- } ①$$

Let,  $V_1$  and  $V_2$  be the set of even and odd vertices

$$\sum \deg(v_1) + \sum \deg(v_2) = \sum \deg(v).$$

= even

[from eqn ①]

$$\Rightarrow \text{even} + \sum \deg(v_2) = \text{even}.$$

$$\Rightarrow \sum \deg(v_2) = \text{even} - \text{even}$$

$$\Rightarrow \boxed{\sum \deg(v_2) = \text{even.}}$$

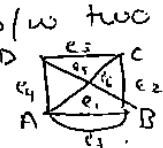
$\left\{ \begin{array}{l} \text{subtraction of} \\ \text{even no. is} \\ \text{always even} \end{array} \right.$

Since sum of odd degree vertices is even. Then, no. of odd vertices of degree will be even.

\* **Simple Graph** :- A graph which has neither self loop nor parallel edges.

Parallel edges. There is more than 1 edge b/w two vertices.

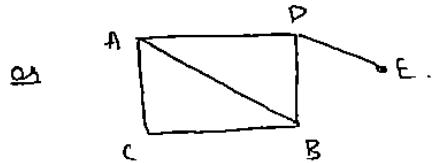
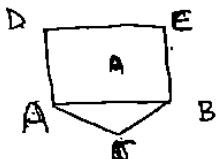
e.g.  $e_1$  &  $e_7$  are parallel edges.



\* **Multi Graph** :- A graph which contains some parallel edges or self-loop is called multi-graph

- Q Draw a graph with vertices A, B, C, D, E such that  $d(A) = 3$ , B is an odd vertex and D and E are adjacent vertices.

→



- Q Does there exist a simple graph with 5 vertices of given degrees. If yes, draw such graph.

- 1) 1, 2, 3, 4, 5  
By handshaking theorem,

$\sum \deg(v) = 15$ . which is odd

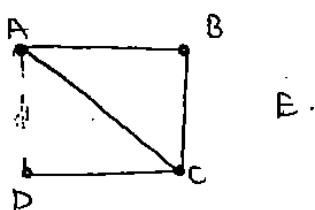
But, By HST the sum of degrees of all vertices in a graph is even.

- 2) 1, 2, 3, 4, 4

$\sum \deg(v) = 1+2+3+4+4 = 14$ .  
Here, Two vertices of degree 4 exists. Each remaining 3 vertices will be of atleast degree 2. But the given sequence of degree contains 1. So, no such graph is possible of given sequence.

- 3) 0, 1, 2, 2, 3.

$$\sum \deg(v) = 1+2+2+3+0 = 8.$$



Q Show that the maximum no. of edges in a simple graph with  $n$  vertices is  $\frac{n(n-1)}{2}$ .

$$\Rightarrow \text{i.e. } e_{\max} = \frac{n(n-1)}{2}$$

$G(V, E)$ .

$$\pi \sum_{i=1}^n \deg(v_i) = 2e.$$

$$\Rightarrow e = \frac{1}{2} \sum_{i=1}^n \deg(v_i). \quad \text{--- (1)}$$

If there are ' $n$ ' vertices then ' $n$ ' can be adjacent to maximum  $(n-1)$  vertices.

So, No. of maximum edges associated with one vertex =  $(n-1)$

So, No. of Total maximum edges of graph =  $n(n-1)$

$$\text{So, } \sum_{i=1}^n \deg(v_i) = n(n-1) \rightarrow \text{Put in eqn (1).}$$

$$\Rightarrow [e_{\max} = \frac{1}{2} n(n-1)].$$

Q Show that if all the vertices of an undirected graph are each of odd degree ' $k$ '. Show that no. of edges of the graph is a 'multiple of  $k$ '.

$\rightarrow$  since, each vertex is of an odd degree ' $k$ '.

so, vertices must be in even number say it is  $2n$ ,  
 $n \in \mathbb{Z}^+$ .

If No. of edges =  $e$ , then acc. to Handshaking Theorem -

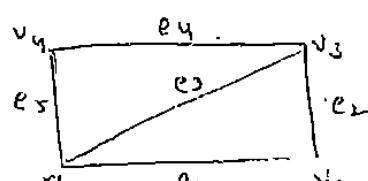
$$\sum_{i=1}^{2n} \deg(v_i) = 2e.$$

$$\Rightarrow 2n \times k = 2e.$$

$$\Rightarrow e = nk.$$

### \* Terminologies :-

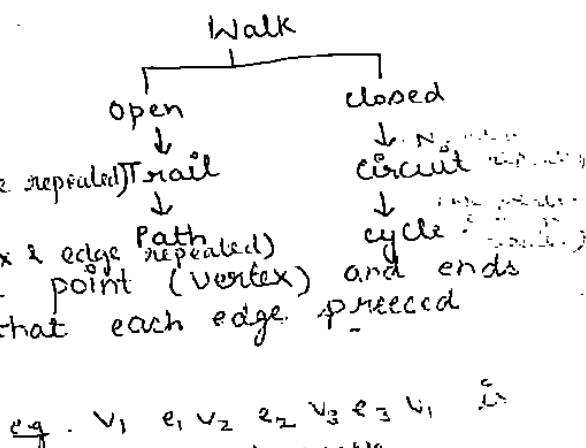
• Walk :- It starts from a point such that each edge precedes the next vertex.



There are two types of walk :-

Open walk :-  $v_1 \neq v_n$ .

Closed walk :-  $v_1 = v_n$ .



e.g.  $v_1 e_1 v_2 e_2 v_3 e_3 v_1$  is a closed walk.  
 $v_1 e_1 v_2 e_2 v_3 e_3$  is an open walk.

- Trail :- Trail is an open walk with no edge repeated.
- Path :- Path is an open walk with neither vertex nor

repeated.

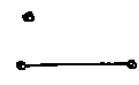
- Cycle :- cycle is a closed walk with neither edge nor vertex repeated.

### \* Some Special types of Graph :-

- Complete Graph :- A simple graph is called complete graph in which exactly one edge present between every pair of vertices. It is denoted by  $K_n$ , where 'n' is no. of vertices.

$$nC_2 = \frac{n(n-1)}{2}$$

e.g.  $K_1$



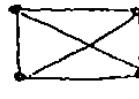
$K_2$



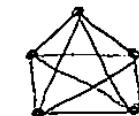
$K_3$



$K_4$

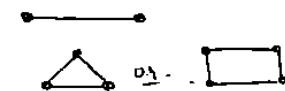


$K_5$

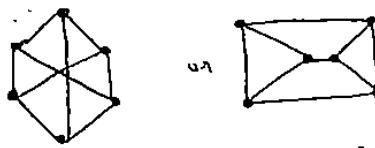


- n - Regular Graph :- It is a graph where degree of every vertex is same. If degree is 'n' then it is called n- regular graph.

e.g. 1-regular graph  
2-regular graph  
3-regular graph  
3-regular graph



•



- Null graph (or) void (or) Empty graph :- The graph with isolated vertices called Null graph. A null graph with 'k' vertices is denoted by  $N_k$ .

e.g.  $v_1$   $v_2$   $v_3$   $v_4$

- Bipartite Graph :- A graph  $G(V, E)$  is said to be bipartite if vertex set 'V' can be partition into two disjoint subsets  $V_1$  and  $V_2$  such that each edge in 'E' connects a vertex of  $V_1$  to a vertex of  $V_2$ . So that no edge in 'E' connects other two vertices in  $V_1$  or two vertices in  $V_2$ . So,  $(V_1, V_2)$  is called bipartite of  $G$ .

If  $V_1$  has 'm' vertices and  $V_2$  has 'n' vertices then complete bipartite graph is denoted by  $K_{m,n}$ .

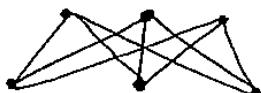
NOTE: A bipartite graph has no self loop.

e.g

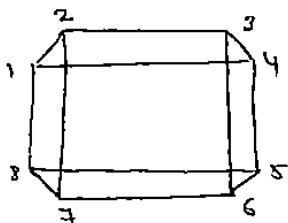
$K_{2,3}$



$K_{3,3}$



Q



$$V_1 = \{1, 3, 5, 7\}$$

$$V_2 = \{2, 4, 6, 8\}.$$

Here, vertex 1 is connected to 2, 4, 3 not connected to 6.

vertex 3 is connected to 2, 4, 6 not connected to 8.

vertex 5 is connected to 4, 6, 3 not connected to 2

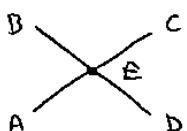
vertex 7 is connected to 2, 6, 3 not connected to 4

so, it is a bipartite graph.

since, there is no edge b/w 1 & 6, 3 & 8, 5 & 2, 4 & 7

so, it is not a complete bipartite graph.

Q



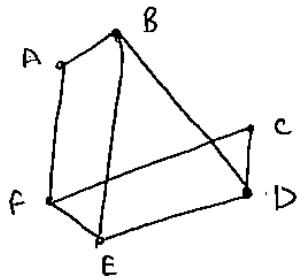
$$V_1 = \{A, B, C, D\}$$

$$V_2 = \{E\}.$$

Here, vertices A, B, C, D are not connected to each other they all are connected to E.

so, it is not a complete bipartite graph  $K_{4,4}$ .

Q



It is not a bipartite graph.

Because two disjoint sets are obtained but vertices of set  $V_2$  are connected to each other.

$$V_1 = \{B, C\}$$

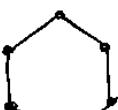
$$V_2 = \{A, D, E, F\}.$$

- Sub Graph :- A graph  $H(V', E')$  is called sub-graph of  $G(V, E)$  if  $V' \subseteq V$  and  $E' \subseteq E$ .  
or  $G(V, E)$  is called super-graph of  $H(V', E')$ .

e.g



$\Rightarrow$



Subgraph

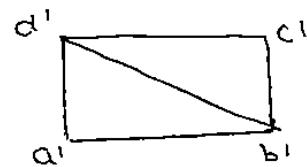
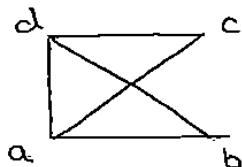
isomorphic to graph  $G_2 (V_2, E_2)$

if there exists a function  $f : V_1 \rightarrow V_2$  such that

- $f$  is bijective. (one to one, onto).
- for all  $\{a, b\} \in E_1 \quad \{f(a), f(b)\} \in E_2$
- $a, b \in V_1$

such a correspondence is called isomorphism.

e.g.



$$f(a) = b'$$

$$f(b) = a'$$

$$f(c) = c'$$

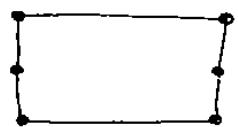
$$f(d) = d'$$

Image of  $v_i$  is that which has same order of degrees. e.g.  $f(a)$  is  $b'$  and we take any one in that means that we take  $b'$ .

- Homomorphic Graph :- Two graphs  $G(v, E)$  and  $G^*(v^*, E^*)$  are said to be Homomorphic if  $G^*$  can be obtained from  $G$  by dividing edges of  $G$  by additional edges.



$G$ .

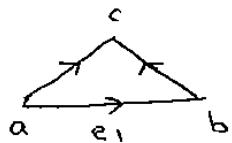


This is a homomorphic graph of  $G$ .



This is also a homomorphic graph.

### \* Directed Graph :-



$e_1$  is incident from  $a$ .

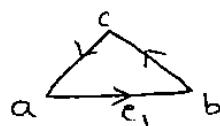
$e_1$  is incident to  $b$ .

$a$  is adjacent to  $b$ .

$b$  is adjacent from  $a$ .

$a$  is the initial vertex.

$b$  is the terminal vertex.



Indegree

$\text{Indeg}(v)$

$d^-(v)$

Outdegree

$\text{outdeg}(v)$

$d^+(v)$

$$d^-(a) = 1, d^-(c) = 1$$

$$d^+(a) = 1, d^+(c) = 1$$

$$d^-(b) = 1$$

$$d^+(b) = 1$$

### Total Degree :-

$$\deg(v) = d(v) = d^-(v) + d^+(v)$$

$$d(a) = 2$$

$$d(b) = 2$$

$$d(c) = 2$$

- SCIRCE :- Vertex whose indegree is zero.

$$d^-(v) = 0 \quad \text{source.}$$

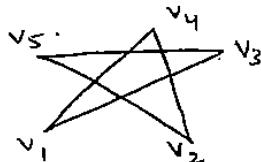
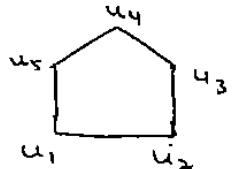
- SINK :- Vertex whose outdegree is zero.

$$d^+(v) = 0 \quad \text{sink.}$$

- Self Loop :- Indegree = 1  
outdegree = 1  
Total degree = 2

NOTE :- Sum of Indegree = sum of outdegree = e.  
 $\sum d^-(v_i) = \sum d^+(v_i) = e$  (No. of edges).

Q.



$f: u_1 \rightarrow v_1, u_2 \rightarrow v_2, \dots$   
 There are equal no. of edges so, it is bijective.

order of degree of all vertices of graph  $G_1$  &  $G_2$  are 2 and are same.

$$f(u_1) = v_1, \quad (u_1, u_2) \in E_1 \Rightarrow \{f(u_1), f(u_2)\}$$

$$f(u_2) = v_3, \quad = (v_1, v_3) \in E_2$$

$$f(u_3) = v_5, \quad (u_1, u_5) \in E_1 \Rightarrow \{f(u_1), f(u_5)\}$$

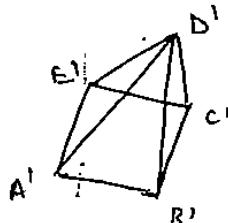
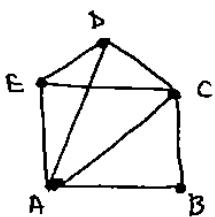
$$f(u_4) = v_2, \quad = (v_1, v_4) \in E_2$$

$$f(u_5) = v_4, \quad (u_2, u_3) \in E_1 \Rightarrow \{f(u_2), f(u_3)\}$$

$$= (v_3, v_5) \in E_2.$$

$$(u_3, u_4) \in E_1 \Rightarrow \{f(u_3), f(u_4)\} = (v_5, v_2) \in E_2.$$

Q.



$$d(A) = 4, \quad d(B) = 2,$$

$$d(C) = 4, \quad d(D) = 3.$$

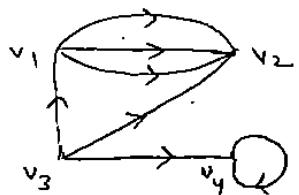
$$d(E) = 3.$$

$$d(A') = 3, \quad d(B') = 3, \quad d(C') = 3$$

$$d(D') = 4, \quad d(E') = 3.$$

The degree of Graph  $G_1$  &  $G_2$  are not same.

Therefore, it is not isomorphic.



$$\begin{array}{ll}
 d^-(v_1) = 1 & d^+(v_1) = 3 \\
 d^-(v_2) = 3 & d^+(v_2) = 0 \\
 d^-(v_3) = 0 & d^+(v_3) = 3 \\
 d^-(v_4) = 2 & d^+(v_4) = 1
 \end{array}$$

Total degree :-

$$\begin{aligned}
 \sum d^-(v_i) + \sum d^+(v_i) \\
 7 + 7 = 14
 \end{aligned}$$

$$\begin{aligned}
 d(v_1) &= d^-(v_1) + d^+(v_1) = 1+3 = 4 \\
 d(v_2) &= d^-(v_2) + d^+(v_2) = 4+0 = 4 \\
 d(v_3) &= d^-(v_3) + d^+(v_3) = 0+3 = 3 \\
 d(v_4) &= d^-(v_4) + d^+(v_4) = 2+1 = 3
 \end{aligned}$$

\* Pendant vertex in directed graph :- vertex whose total degree is 1 (by adding indegree and outdegree).

- Indegree may be zero then outdegree is 1.
- Outdegree may be zero then Indegree is 1.

But both Indegree & outdegree at same time not equal to 1.

### \* Representation of Graph :-

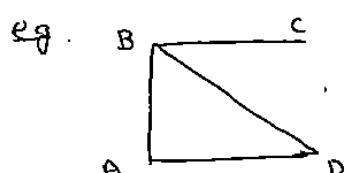
• Undirected graph:- There are two methods to represent undirected graph.

- ① Adjacency (vertex - vertex adjacency matrix)
- ② Incident (vertex - edge incident matrix).

#### ① Adjacency Matrix :-

$$A_{ij} = [a_{ij}]$$

where,  $a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise.} \end{cases}$



$$A = \begin{bmatrix} A & B & C & D \\ B & 0 & 1 & 0 & 1 \\ C & 1 & 0 & 1 & 1 \\ D & 0 & 1 & 0 & 0 \\ A & 1 & 1 & 0 & 0 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

If a graph has M vertices then the order of adjacency matrix will be  $m \times n$ . and adjacency matrix is represented as -

$$A_{ij} = a_{ij} \quad \text{where, } a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise.} \end{cases}$$

#### Properties :-

- # Adjacency Matrix is always symmetric.
- # If we add rows of matrix then we can get degree of corresponding vertex.



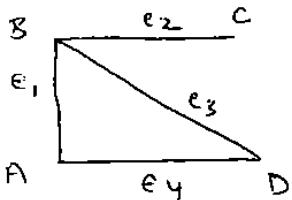
If we add all non-zero elements of matrix then we get  $2e$ . where,  $e$  is edges of graph.  
No. of

## • Incidence Matrix :-

$$Bij = [bij]$$

where,

$$bij = \begin{cases} 1, & \text{if } ej \text{ is incident on } vi \\ 0, & \text{otherwise} \end{cases}$$

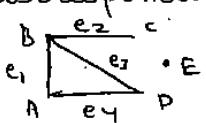


$$I = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ A & 1 & 0 & 0 & 1 \\ B & 1 & 1 & 1 & 0 \\ C & 0 & 1 & 0 & 0 \\ D & 0 & 0 & 1 & 1 \end{bmatrix}$$

# If we add row of matrix then we can get degree of matrix.

# On every column there is only or exactly two one's because edge is incident on two vertices.

# If a vertex is isolated then it has all zeroes in its corresponding row/ column.



$$A = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ A & 1 & 0 & 0 & 1 \\ B & 1 & 1 & 1 & 0 \\ C & 0 & 1 & 0 & 0 \\ D & 0 & 0 & 1 & 1 \\ E & 0 & 0 & 0 & 0 \end{bmatrix}$$

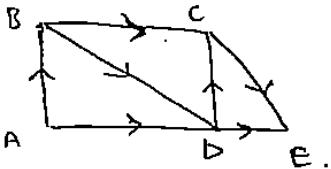
## • Directed Graph:-

### • Adjacency Matrix :-

$$Aij = [a_{ij}]$$

where,

$$a_{ij} = \begin{cases} 1, & \text{if there is an edge -dir from } v_i \text{ to } v_j \\ 0, & \text{otherwise} \end{cases}$$



$$A = \begin{bmatrix} A & B & C & D & E \\ A & 0 & 1 & 0 & 1 & 0 \\ B & 0 & 0 & 1 & 1 & 0 \\ C & 0 & 0 & 0 & 0 & 1 \\ D & 0 & 0 & 1 & 0 & 1 \\ E & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# It is not symmetric.

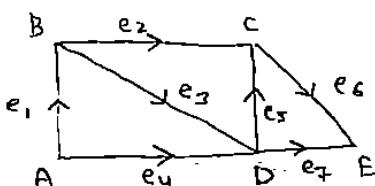
# The sum of elements of row gives the outdegree of the corresponding vertex.

# The sum of elements of column gives the indegree of the corresponding vertex.

$\forall$  Incidence Matrix :-

$$B_{ij} = [b_{ij}]$$

where,  $b_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is initial vertex of edge } e_j \\ -1, & \text{if } v_i \text{ is terminal vertex of edge } e_j \\ 0, & \text{otherwise.} \end{cases}$

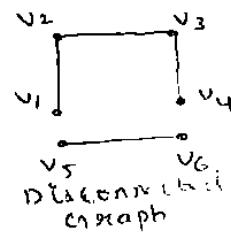
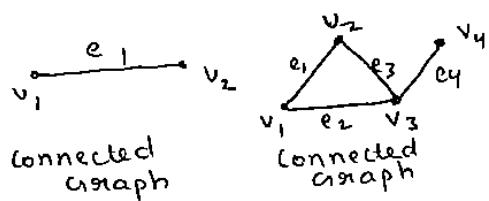
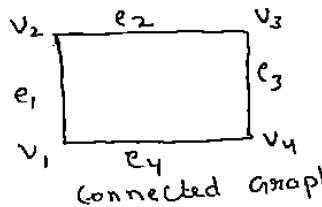


$$A \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ \hline A & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ B & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ C & 0 & -1 & 0 & 0 & -1 & 1 & 0 \\ D & 0 & 0 & -1 & -1 & 1 & 0 & 1 \\ E & 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$\forall$  Distance and Diameter :- Consider a connected graph.

\* Connected Graph (or) Connectivity in Undirected Graph :-

An undirected graph is said to be connected if there exist a path b/w every pair of distinct vertices. A graph which is not connected is called disconnected.



\* Every disconnected graph contains two or more connected graphs. and each of the connected graph (subgraph) is called component of disconnected graph.

\* Theorem :- A graph 'G' without parallel edges or self loop with 'n' nodes and 'k' components can have almost  $\frac{(n-k)(n-k+1)}{2}$  edges.

$\forall$  Proof Let, No. of vertices in  $i^{th}$  component be  $n_i$

$$n_1 + n_2 + n_3 + \dots + n_k = n$$

$$\Rightarrow \sum_{i=1}^k n_i = n$$

$$\Rightarrow \sum_{i=1}^k n_i - k = n - k$$

$\dots n - k$

and the same and

$$\Leftrightarrow \sum_{k=1}^n (n_k^{\phi}-1)^2 + 2 \sum_{\substack{1 \leq i < j \leq n \\ i \neq j}} (n_i^{\phi}-1)(n_j^{\phi}-1) = n^2 + k^2 - 2nk.$$

$$\frac{1}{x^2} + \frac{1}{x^2} + \frac{1}{x^2} = x + kx + \frac{1}{x^2}$$

$$\sum_{k=1}^{\infty} \left( (-1)^{k-1} \frac{1}{k} + (-1)^k \frac{1}{k} \right) = -\ln 2 + \frac{\pi^2}{12}$$

$$\begin{aligned}
 & \sum_{i=1}^k n_i - \overbrace{(1+1+1+\dots)}^{k \text{ terms}} = n - k \\
 \Rightarrow & \sum_{i=1}^k n_i - \sum_{i=1}^k 1 = n - k \\
 \Rightarrow & \sum_{i=1}^k (n_i - 1) = n - k \\
 \Rightarrow & \left[ \sum_{i=1}^k (n_i - 1) \right]^2 = (n - k)^2
 \end{aligned}$$

leave the one 'digit'.

$$\Rightarrow \sum_{i=1}^k (n_i - 1)^2 \leq n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^k n_i^2 + \sum_{i=1}^k 1 - 2 \sum_{i=1}^k n_i \leq n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^k n_i^2 + k - 2n \leq n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^k n_i^2 \leq n^2 + k^2 - 2nk - k + 2n$$

We know that the maximum no. of edges in the  $i^{th}$  component

$$= \frac{1}{2} n_i (n_i - 1)$$

so, Total no. of edges of the graph

$$= \frac{1}{2} \sum_{i=1}^k n_i (n_i - 1)$$

$$= \frac{1}{2} \sum_{i=1}^k (n_i^2 - n_i)$$

$$= \frac{1}{2} \sum_{i=1}^k n_i^2 - \frac{1}{2} \sum_{i=1}^k n_i$$

$$= \frac{1}{2} [ n^2 + k^2 - 2nk + 2n - k - n ]$$

$$= \frac{1}{2} [ n^2 + k^2 - 2nk + n - k ]$$

$$= \frac{1}{2} [ (n-k)^2 + (n-k) ]$$

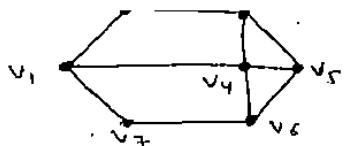
$$= \boxed{\frac{1}{2} (n-k)(n-k+1)}$$

\* Distance and Diameter :- Consider a connected graph 'G' the distance

b/w vertices  $u$  &  $v$  denoted by  $d(u,v)$  is minimum

No. of edges b/w vertices  $u$  and  $v$ .

The diameter of graph 'G' denoted by 'dia(G)' or ' $\Delta(G)$ ' or ' $\Delta$ ' is maximum distance b/w any two vertices.



$$\begin{aligned} d(v_1, v_2) &= 1 \\ d(v_1, v_3) &= 2 \\ d(v_1, v_4) &= 1 \\ d(v_1, v_5) &= 2 \\ d(v_1, v_7) &= 1 \end{aligned}$$

$$\begin{aligned} d(v_2, v_3) &= 1 \\ d(v_2, v_4) &= 2 \\ d(v_2, v_5) &= 2 \\ d(v_2, v_6) &= 3 \\ d(v_2, v_7) &= 2 \end{aligned}$$

$$\begin{aligned} d(v_3, v_4) &= 1 \\ d(v_3, v_5) &= 1 \\ d(v_3, v_6) &= 2 \\ d(v_3, v_7) &= 3 \end{aligned}$$

$$\begin{aligned} d(v_4, v_5) &= 1 \\ d(v_4, v_6) &= 1 \\ d(v_4, v_7) &= 2 \end{aligned}$$

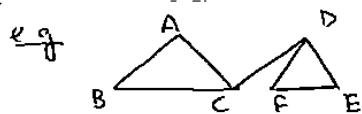
$$\begin{aligned} d(v_5, v_6) &= 1 \\ d(v_5, v_7) &= 2 \end{aligned}$$

$$\boxed{\Delta = 3},$$

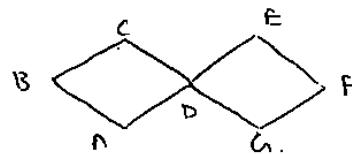
$$\boxed{\Delta = \max d(u, v)}.$$

### \* Cut-Points and Bridge :-

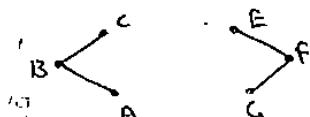
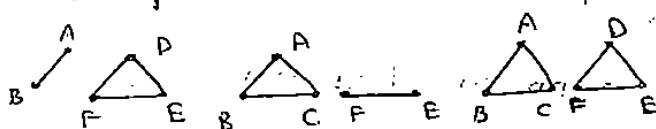
Let 'G' be a connected graph. A vertex  $v_i$  in 'G' is called a cut-point if  $G - v_i$  is disconnected. And edge 'e' of 'G' is called a Bridge if ' $G - e$ ' is disconnected.



e.g. C and D are cut points.  
{C,D} edge is Bridge.

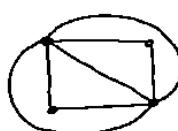


D is a cut point.  
There is not Bridge.



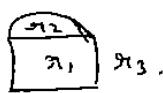
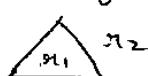
\* Planar Graph :- A graph or a multigraph which can be drawn in the plane so that its edges don't cross each other is said to be planar.

e.g.



⇒ The planar graph divides the plane into two or more parts or regions where one of the parts will be infinite.

e.g.



or has only a graph which has one region which is infinite & it is planar.

\* **Euler's Theorem :-** Consider any connected planar graph 'G(V, E)' with 'R' regions.

'V' vertices & 'E' edges then-

$$[V + R - E = 2]$$

Proof By Mathematical Induction

1) Induction Based :  $E=1$

$$\begin{array}{l} \text{Graph 1: } V_1, V_2, V_3 \text{ with edge } e_1 \\ V_1 + R_1 - E_1 = 2 \\ \Rightarrow 2 + 1 - 1 = 2 \end{array} \quad \begin{array}{l} \text{Graph 2: } V_1, V_2, V_3 \text{ with edge } e_2 \\ V_1 + R_2 - E_2 = 2 \\ \Rightarrow 1 + 2 - 1 = 2 \end{array}$$

2) For  $k$  it is true.

$$[V + R - E = 2] \text{ is true.} \quad (\text{on } V + R - k = 2)$$

3) For  $(k+1)$  ~~it~~. (Inductive Hypothesis).

Case 1  $V = V_1 + 1$  &  $E = E_1 + 1$ .

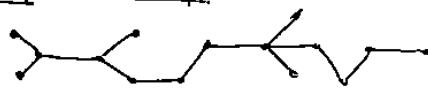
$$V + R - E = 2$$

$$V_1 + X + R_1 - E_1 - X = 2$$

$$[V_1 + R_1 - E_1 = 2]$$

It is true.

for open graph



Case 2 For closed graph



$$R = R_1 + 1$$

$$E = E_1 + 1$$

$$V + R - E = 2$$

$$V + R_1 + X - E_1 - X = 2$$

$$[V + R_1 - E_1 = 2]$$

① If a connected planar graph 'G' has  $E$  edges &  $R$  regions then,  $R \leq \frac{2}{3} E$

② If a connected planar graph 'G' has  $V$  vertices &  $E$  edges then,  $3V - E \geq 6$

③ A complete graph  $K_n$  is planar iff  $n < 5$

④ A complete bipartite graph  $: m, n$  is planar iff  $m < 3$  &  $n < 3$

Q. A planar graph has 30 vertices each of degree 3. Determine the no. of regions into which this planar graph can be split.

$$\rightarrow V = 30, d(V_i) = 3, \text{ if } \sum d(V_i) = 3V$$

$$d(V_i) = 3$$

$$\therefore \sum d(V_i) = 2e \quad \text{or} \quad \text{sum of degrees} = 3V$$

$$30 \times 3 = 2e$$

$$\therefore e = 45$$

$$\text{or} \quad \text{sum of degrees} = 3V \quad = 3 \times 30$$

$$\text{By Hand Shaking formula} \quad \sum d(V_i) = 2e$$

$$90 = 2e \Rightarrow e = 45$$

$$V - E + R = 2 \Rightarrow 30 - 45 + R = 2$$

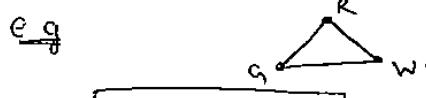
$$\therefore R = 17$$

$\Rightarrow$  If  $E$  edges is bounded by  $k$  edges then prove that  
 $E = \frac{k(n-2)}{(k-2)}$ .  $\Rightarrow$  Solution next page

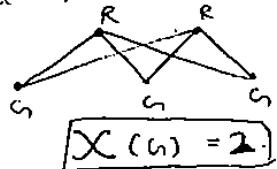
\* Non-Planar Graph :- A graph ' $G$ ' is said to be a planar graph if it can not be drawn in a plane so that no edges cross.

\* Graph Colouring :- Suppose that  $G(V, E)$  with no multiple edges, a vertex coloring of  $G$  is an assignment of colours to vertices of  $G$  such that adjacent vertices have different colours.  
A graph ' $G$ ' is  $m$ -colourable if there exist a colouring ' $G$ ' which uses  $m$  colours.  
Colouring the vertices such a way such that no two adjacent vertices have same colour is called proper colouring. otherwise it is called improper colouring.

\* Chromatic Number of Graph :- The minimum number of colours needed to produce a proper colouring of a graph ' $G$ ' is called chromatic no. of ' $G$ ' and it is denoted by  $[X(G)]$ .



$$[X(G) = 3]$$



$$[X(G) = 2]$$

\* Algorithm for Graph Colouring :- (Welch and Powell)  
Welch and Powell gave algorithm for the colouring of graph ' $G$ '.

Step 1 : Order the vertices of the graph ' $G$ ' acc. to decreasing degree.

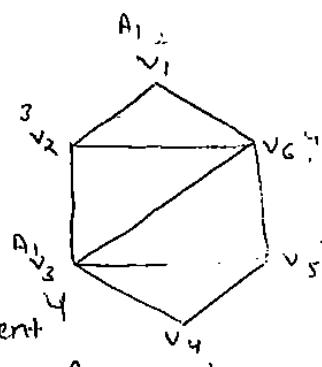
Step 2 : Assign first colour  $A_1$  to the first vertex and then assign  $A_1$  to each vertex which is not adjacent to previous vertex which was assigned colour  $A_1$  but in sequential order.

Step 3 : Repeat step 2 with second colour  $A_2$  and subsequent vertices.

Step 4 : Repeat step 3 with a third colour  $A_3$  then fourth colour  $A_4$  and so on until all vertices are coloured.

e.g.

Vertices	$v_3$	$v_6$	$v_2$	$v_5$	$v_1$	$v_4$
Degrees	4	4	3	3	2	2



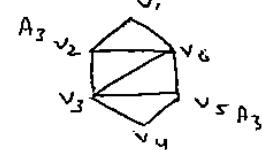
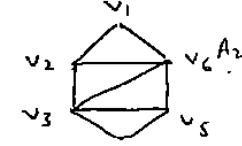
- ① Now Assign colour  $A_1$  to the vertex  $v_3$ . Now, the vertex  $v_1$  is not adjacent to ~~edge~~ vertex  $v_3$ . So we assign colour  $A_1$  also to vertex  $v_1$ .

Q) Now assign second colour  $A_2$  to vertex  $v_6$ . and Now  $v_4$  is not adjacent to vertex  $v_4$ . so we assign colour  $A_2$  to vertex  $v_4$ .

3) Now assign third colour  $A_3$  to vertex  $v_2$ . and Now  $v_5$  is not adjacent to vertex  $v_5$ . so we assign colour  $A_3$  to vertex  $v_5$ .

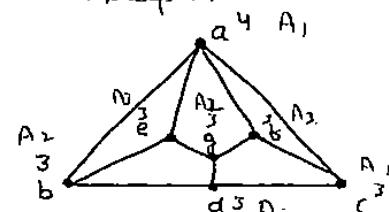
Now all the vertices are coloured  
The chromatic no. of this graph is

$$X(a) = 3$$



Q) find the chromatic no. of this graph.

vertices	a	b	c	d	e	f	g
degree	4	3	3	3	3	3	3



1) Now assign colour  $A_1$  to the vertex 'a'. Now, 'd' is not adjacent to vertex 'a'. so, we assign colour  $A_1$  to vertex 'd'.

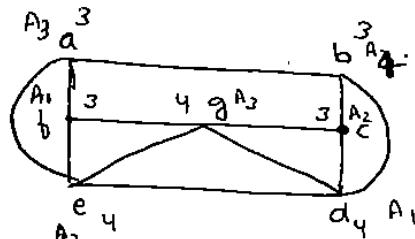
2) Now assign second colour  $A_2$  to the vertex 'b'. Now vertex 'g' and 'c' are not adjacent to vertex 'b'. so we assign colour  $A_2$  to vertex 'g' and 'c'.

3) We assign third colour  $A_3$  to vertex 'e'. Now vertex 'f' are not adjacent to vertex e. so, we assign colour  $A_3$  to vertex 'f'.

### \* Bipartite

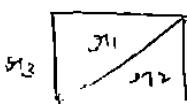
vertices	d	e	g	a	b	c	f
degree	4	4	4	3	3	3	3

$$\{ X(a) = 4 \}$$



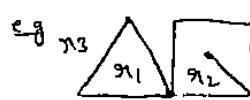
\* Degree of Region R in a planar graph:- The no. of edges traversed in a closed walk of the region such that every vertex of the region is traversed is called degree of the region.

e.g.



$$\begin{aligned} d(r_1) &= 3 \\ d(r_2) &= 3 \\ d(r_3) &= 4 \end{aligned}$$

$$\begin{aligned} 10 &= 2e \\ 10 &= 2 \times 5 \end{aligned}$$



$$\begin{aligned} d(r_1) &= 3 \\ d(r_2) &= 3 \\ d(r_3) &= 3 \end{aligned}$$

$$\begin{aligned} 16 &= 2e \\ 16 &= 2 \times 8 \end{aligned}$$

$$\sum \{ \deg(r) = 2e \}$$

$r = \text{regions.}$

If such a no. of edges let be ' $k$ ' then that region is said to be bounded by  $k$  regions. edges.

n. no. of outer edges in a closed closed walk of the graph is said to be degree of infinite region.

$$E = \frac{k(n-2)}{(k-2)}$$

$$kR = 2E$$

$$\Rightarrow R = \frac{2E}{k}$$

$$V + R - E = 2$$

$$\Rightarrow n + \frac{2E}{k} - E = 2$$

$$\Rightarrow nk + 2E - Ek = 2k$$

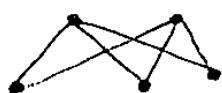
$$\Rightarrow E(2-k) = k(2-n)$$

$$\Rightarrow E = \frac{k(2-n)}{(2-k)} \Rightarrow E = \frac{k(n-2)}{(k-2)}$$

In "planar graph" region colouring is also done  
 $\pi_1 \rightarrow A_1$  colour  
 $\pi_2 \rightarrow A_2$  colour  
 $\pi_1, \pi_3 \rightarrow A_1$  colour  
 $\pi_2 \rightarrow A_2$  colour  
adjacent region doesn't have same colour

- \* Bichromatic No. of Graph :- If a graph 'G' requires 2 colo for colouring then it is called bichromatic graph.

e.g.



Every bipartite graph is bichromatic.

- Note:
- chromatic no. of  $G_i$  will always be less than equal to no. of vertices in Graph ' $G_i$ '. ( $\leq n$ ).
  - If chromatic no. of a subgraph of ' $G_i$ ' is 'm' then  $\chi(G_i) \geq m$
  - Chromatic no. of complete graph ' $K_n$ ' no. of vertices in  $K_n$
  - The chromatic no. of a Null graph is "1"

### \* Euler Graph :-

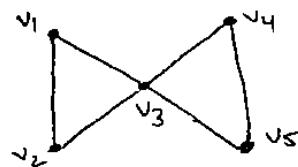
Euler Path :- A path of graph ' $G$ ' which includes each edge of ' $G$ ' exactly once is called Eulerian path

Eulerian Circuit :- A circuit of graph ' $G$ ' which includes each edge of ' $G$ ' exactly once.

Eulerian Graph :- A graph containing Eulerian circuit is called Eulerian Graph. bridge vertex

e.g.

$v_3 v_1 v_2 v_3 v_4 v_5 v_3$



- \* Theorem :- A connected graph contain an Euler circuit if and only if each of its vertices of even degree.

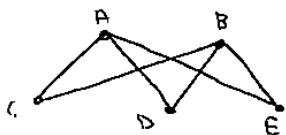
- \* Theorem :- A connected graph contain an Euler path if it has only exactly two vertices of odd degree.

Q



Ans :- There is no Euler path & Euler circuit because each vertex is of order 3 which is odd and also there are more than two vertices of odd degree.

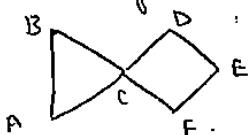
Q



There is Euler path because A and B vertex is of order 3 and there is only two vertex which has odd degree.

$$A \rightarrow C \rightarrow B \rightarrow E \rightarrow A \rightarrow D \rightarrow B$$

Q



There is no Euler circuit because two vertices A & B is of order 3, i.e. odd.

There is no Euler path because there is no vertex which is of odd degree.

There is Euler circuit.

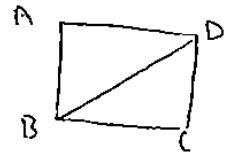
$$C-A-B-C-D-E-F \text{ so, it is Euler graph.}$$

## \* Hamiltonian Graph :-

Hamiltonian Graph is named after Irish mathematician Sir William Hamilton.

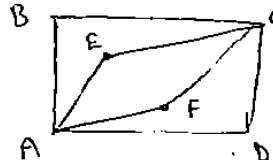
A Hamiltonian circuit in a graph 'G' is a closed path that visit every vertex in 'G' exactly once except the end vertices.

A graph 'G' is called Hamiltonian graph, if it contains a hamiltonian circuit.



$$A-B-C-D-A$$

Hamiltonian graph.



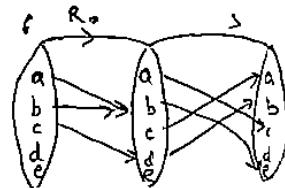
Not a Hamiltonian graph.

Hamiltonian Path :- The path obtained by removing any one edge from a hamiltonian circuit is called Hamiltonian path.

Hamiltonian path is subgraph of Hamiltonian circuit but converse is not true.

The length of a Hamiltonian path in a connected graph of 'n' vertices is  $(n-1)$  if it exists.

- Q  $R = \{(a,b), (c,d), (b,b)\}$   $S = \{(d,b), (b,c), (c,a), (a,c)\}$ .
- $A = \{a, b, c, d\}$
- Compute  $R \circ S$  &  $S \circ R$ .
- $R \circ S = \{(a, c), (b, c), (c, b)\}$
- $S \circ R = \{(d, b), (c, b), (a, d)\}$ .
- $(R \circ S) \circ R = \{(c, b)\}$
- $(S \circ R) \circ S = \{(d, e), (e, e), (a, b)\}$ .



- Q  $R = \{(a, 3a) : a \in \mathbb{Z}^+\}$ .  $S = \{(a, a+1) : a \in \mathbb{Z}^+\}$ .
- $R \circ S = \{(a, 3a+1) : a \in \mathbb{Z}^+\}$
- $R \circ R = \{(a, 9a)\}$
- $R \circ R \circ R = \{(a, 27a)\}$ .
- $R \circ S \circ R = \{(a, 9a+3)\}$

Show that if set A has 3 elements then we can have  $2^6$  symmetric relations on A.

Total symmetric relations.  $= 2^{n(n+1)/2}$

- Q  $A = \{1, 2\}$ . Find  $P(A)$
- $P(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ .
- Q Define multiset.
- Q Prove  $n^3 + n$  by MI is divisible by 3 by Eq
- Q  $A = \{1, 2, 3, 4\}$ . i)  $R = \{(1, 3), (3, 1), (1, 1), (1, 2), (3, 3), (4, 4)\}$ . ii)  $R = A \times A$ . Equivalence
- Show it is equivalence or not.
- $\Rightarrow$  i) Reflexive:  $(a, a) \in R, \forall a \in A$ .
- ii) Symmetric:  $(a, b) \in R \Rightarrow (b, a) \in R, \forall (a, b) \in A$ .
- iii) Transitive:  $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R, \forall a, b, c \in A$ .

i) It is not reflexive.

Q How many 4 digit no. we make when repetition is allowed.

$\frac{4}{2} \times \frac{4}{3} \times \frac{4}{2} \times \frac{4}{1} = 256$ . i.e. we find sol.

Q: DNF  $(x+y)(x'+y')$  (Disjunctive Normal Form)

CNF (Conjunctive Normal Form) AND (Pos) form.

$$\Rightarrow xx' + xy' + yx' + yy'$$

$$\Rightarrow xy' + x'y$$

i) Not all books have bibliographies. Quantifier  
 $P(x)$ : All books  $x$  in a book having bibliography

$$\neg (\forall x: P(x))$$

$$Q) Y = ((AB)^T + A^T + AB)^T$$

$$A^T B^T C^T D^T + A^T B^T C^T D + A^T B^T C D + A^T B^T C D^T = A^T B^T$$

Q)  $A = \{1, 2, 3, 4\}$  &  $a R b$  iff  $a$  is a factor of  $b$ .

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}.$$

$$A = \{1, 2, 3, 4, 6\}.$$

$$R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (6,6)\}.$$

$R(3) =$  The set of those elements which are related to 3 in  $R$ .  
 $= \{(x \in A | (3, x) \in R)\} = \{3, 6\}$ .

$$R(6) = \{6\}.$$

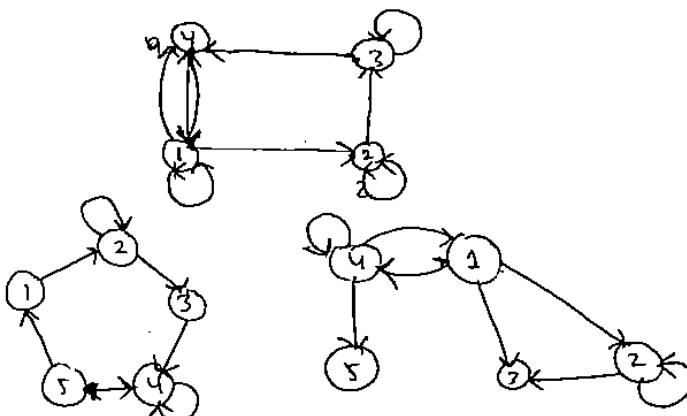
$$\begin{aligned} R(2,4,6) &= R(2) \cup R(4) \cup R(6) \\ &= \{2, 4, 6\} \cup \{4\} \cup \{6\} \\ &= \{2, 4, 6\}. \end{aligned}$$

Q)  $A = \{1, 2, 3, 4\}$ , find relation  $R$  & draw the digraph represented by relation  $R$ .

$$R = \{(1,1), (1,2), (1,4), (2,2), (2,3), (3,3), (3,4), (4,1)\}.$$

$$(directed graph)$$

$$MR = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R = \{(1,2), (2,2), (2,3), (3,4), (4,1), (4,4), (5,4)\}, \quad R = \{(1,2), (1,3), (1,4), (2,2), (2,3), (4,1), (4,4), (4,5)\}.$$

$$Q) A = \{1, 2, 3, 4\}.$$

$$R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}.$$

1) It is reflexive  $(a, a) \in R$ . (Every element related to itself).

Since,  $(1,1) \in R, (2,2) \in R, (3,3) \in R, (4,4) \in R$ .

2) Not irreflexive.

3) It is symmetric. Since,  $(1,1) \in R, (1,2) \in R \Rightarrow (2,1) \in R$   
 $(2,2) \in R, (3,4) \in R \Rightarrow (4,3) \in R$ .  
 $(4,4) \in R, (3,3) \in R$ .

$\text{exp}_\theta \phi_\theta \beta = \phi_{\text{exp}_\theta \beta}$  shows that  $\phi_\theta$  is a  $C^\infty$  function of  $\theta$ .

$$g_{\text{down}}(h) = \Delta \left( \begin{pmatrix} 1,2 \\ 1,2 \end{pmatrix}, \begin{pmatrix} 2,1 \\ 2,1 \end{pmatrix}, \begin{pmatrix} 3,4 \\ 3,4 \end{pmatrix}, \begin{pmatrix} 4,5 \\ 4,5 \end{pmatrix}, \begin{pmatrix} 5,1 \\ 5,1 \end{pmatrix} \right).$$

$$\begin{aligned} S_1 &= \{1, 3, 5, 7, 9\} \\ S_2 &= \{2, 4, 6, 8, 10\} \\ S_3 &= \{1, 2, 3, 4, 5\} \end{aligned}$$

∴  $S = \{1, 2, 3, 4, 5\}$  and the four figures are obtained from  $x$ .

$$A \cup B = \{1, 2, 3, 4, 5, 7, 9\}, \quad A \cap B = \{1, 3, 5\}.$$

Also  $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$  & the odd elements of  $U$  are  $\{1, 3, 5, 7, 9\}$ . What is the intersection of all odd integers in  $U$ ,  $A' = \{2, 4, 6, 8, 10\}$ . What is the intersection of all odd integers in  $U$ ,  $B' = \{1, 3, 5, 7, 9\}$ .

Even integers in  $U$  are  $\{2, 4, 6, 8, 10\}$ . The intersection of all even integers in  $U$  is  $\emptyset$ .

Find the union & intersection of  
 $\{0, 000, 010, 101, 111\}$  and  
 $\{0, 000, 010, 101, 111\}$

Not on equivalence relation  $\{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}$ . The best strategy for the set  $\{1, 2, 3, 4, 5\}$  is to always choose  $i$  such that  $i + j = 6$ . We can do this by  $i = 1, 2, 3, 4, 5$  &  $j = 5, 4, 3, 2, 1$ . This can be done in  $1111100000$  &  $1010101010000000$  ways. And the winner & elimination of sets.

If it is not a formalistic approach.

$$|z - h| > |h - x|$$

3) Transitivity :- Let  $(x_1, R, y_1), (y_1, R, z_1) \in R$   $\Rightarrow (x_1, R, z_1) \in R$ .

$$y \ni (x_0, h) \Leftarrow \begin{aligned} & 1 > (x-h) \\ & 1 > (x-h) - 1 \\ & -1 > h - x_1 \end{aligned}$$

$\text{symmetric}$ :  $\text{let}(x,y) \in R$

$$1) \text{ Reflexivity} \rightarrow L(x_1 x_1) \in R \rightarrow (x_1 x_1) \in R$$

$\Rightarrow$  Note Anti-symmetric rule  
Let  $R$  be a relation on the set of real No.'s such that  
 $R = \{ (x, y) : |x-y| \leq 1 \}$ . Is it an equivalence relation?

(ii) Transitive note  $(3,1) \in R$  &  $(1,2) \in R \Rightarrow (3,2) \in R$ .

33 Note symmetric ER  $\Leftrightarrow$  (3,1) ER  
 32 If  $\Delta = \{x_1, x_2, x_3\}$  then  $(x_1, x_2) \in \Delta$   $\Leftrightarrow$   $(x_2, x_1) \in \Delta$

33 If  $\exists x \forall y$   $\neg p(x,y)$  is true then  $\neg p(a,b)$  is true  
 $\neg p(a,b) \Leftrightarrow p(a,b) \rightarrow \perp$

34  $\neg p(a,b) \Leftrightarrow \neg p(a,b) \rightarrow \perp$

4) Note scatterplot (2,2) & R.

$$R = \{ (1,3), (1,2), (3,1), (1,2), (3,3) \}$$

• Transitive Properties

$$\begin{array}{ll}
 1) & f \circ g(1) = f[g(1)] = f(3) = 4 \\
 & f \circ g(2) = f[g(2)] = f(5) = 3 \\
 & f \circ g(3) = f[g(3)] = f(1) = 2 \\
 & f \circ g(4) = f[g(4)] = f(2) = 1 \\
 & f \circ g(5) = f[g(5)] = f(4) = 5
 \end{array}
 \quad
 \begin{array}{l}
 f \circ g = \{(1,4), (2,3), (3,2), \\
 \quad (4,1), (5,5)\}
 \end{array}$$
  

$$\begin{array}{ll}
 & g \circ f(1) = g[f(1)] = g(2) = 5 \\
 & g \circ f(2) = g[f(2)] = g(1) = 3 \\
 & g \circ f(3) = g[f(3)] = g(4) = 2 \\
 & g \circ f(4) = g[f(4)] = g(5) = 4 \\
 & g \circ f(5) = g[f(5)] = g(3) = 1
 \end{array}
 \quad
 \begin{array}{l}
 g \circ f = \{(1,5), (2,3), (3,2), \\
 \quad (4,4), (5,1)\}
 \end{array}$$

$$\Rightarrow \boxed{f \circ g \neq g \circ f}$$

2)  $f$  and  $g$  are bijective functions therefore inverse of  $f^{-1} \circ g$  exists but  $h$  is not bijective fun. therefore inverse of  $h$  does not exist.

$$h(1) = h(2) \Rightarrow 1 \neq 2$$

$$\begin{array}{l}
 3) f^{-1} = \{(2,1), (7,2), (4,3), (5,4), (3,5)\} = \{(1,2), (2,1), (3,5), \\
 \quad g^{-1} = \{(3,1), (5,2), (1,3), (2,4), (4,5)\} = \{(4,3), (5,4)\} \\
 \quad = \{(1,3), (2,4), (3,1), (4,5), (5,2)\} \neq
 \end{array}$$

$$\begin{array}{ll}
 4) (f \circ g)^{-1} = \{(4,1), (3,2), (2,3), (1,4), (5,5)\} = \{(1,4), (2,3), (3,2), (4,1), \\
 \quad (5,5)\} \\
 g^{-1} \circ f^{-1}(1) = 4 \quad - g^{-1} \circ f^{-1} = \{(1,4), (2,3), (3,2), (4,1), (5,5)\} \\
 g^{-1} \circ f^{-1}(2) = 3 \\
 g^{-1} \circ f^{-1}(3) = 2 \\
 g^{-1} \circ f^{-1}(4) = 1 \\
 g^{-1} \circ f^{-1}(5) = 5
 \end{array}
 \Rightarrow \boxed{(f \circ g)^{-1} = g^{-1} \circ f^{-1}} \text{ ful}$$

Ques 10 M  
 Consider the set  $\mathbb{Z}$  of integers defined arb. by  $b = a^x$  for some +ve integer ' $x$ '. Show that ' $\leq$ ' is a partial order on  $\mathbb{Z}$ .  
For POF

1) Reflexive: -  $b = a^1$  for  $\forall a \in \mathbb{Z}^+$   
 $\Rightarrow aR a, \forall a \in \mathbb{Z}$

2) Antisymmetric: - Let  $aRb \& bRa$   
 $\Rightarrow b = a^x \& a = b^y ; x, y \in \mathbb{Z}^+$ .

$$\begin{aligned}
 a &= (a^x)^y \\
 \Rightarrow a &= a^{xy}
 \end{aligned}$$

Three conditions  $\Rightarrow 1) x=1, 2) a=1, 3) a=-1$   
 $x=1 \& y=1 \Rightarrow b=a \& a=b$

$$\left. \begin{array}{l}
 2) a=1, \\
 \Rightarrow b=1^y=1=a \\
 \Rightarrow b=a
 \end{array} \right\} \Rightarrow b=(-1)^{x=-1}=a$$

$$\boxed{b \neq 1}$$

$$a \neq b \Rightarrow$$

$$(-1) \neq b$$

$$\Rightarrow \boxed{b \neq 1}$$

3) Transitive: -  $aRb \Rightarrow b=a^x$   
 $bRc \Rightarrow c=b^y$   
 $\Rightarrow c=a^{xy} \Rightarrow c=a^p, p \in \mathbb{Z}^+$ ,  
aRc