Recurrence Relation

example : ① Fibonacci Seq^n :

$\langle 0, 1, 1, 2, 3, 5, 8 \ldots \rangle$

$a_n = a_{n-1} + a_{n-2}$ $\quad n \geq 2$

$a_2 = a_1 + a_0$

$a_0 = 0$

$a_1 = 1$

~~3·2~~ ~~3·2·~~ ~~3·2·2~~

② Geometric Progression

$(3, 6, 12, 24, 48 \ldots)$ Let's find recurrence Relation.

$a_n = 2a_{n-1}$

$a_0 = 3$

$a_n = 3(2^n)$

suppose we need to calculate a_{667} then

$$\boxed{\left.\begin{aligned} a_n &= d\,a_{n-1} \\ a_0 &= K \end{aligned}\right\}} \quad \boxed{a_n = K(d^n) \quad n \geq 0}$$

③ ~~a_0 a_1 a_2 a_3 a_4~~

$0, 2, 6, 12, 20, 30, 42 \ldots$ }

$a_0 \quad a_1$

$a_1 - a_0 = 2$

$+ \, a_2 - a_1 = 4$

$+ \, a_3 - a_2 = 12 - 6 = 6$

$\vdots$

$+ \, a_n - a_{n-1} = 2n$

$a_n - a_0 = 2 + 4 + 8 + \ldots 2n$

$= 2(1 + 2 + \ldots + n)$

$= 2 \sum^{n} i$

$= 2\left(\frac{n(n+1)}{2}\right) = n(n+1)$

$a_n - a_0 = n(n+1)$

$\boxed{a_n = n(n+1)}$

$$a_6 = 42$$
$$a_6 = 6(6+1)$$
$$= 6 \times 7$$
$$= 42$$

if $\boxed{a_n - a_{n-1} = k}$ (some no.)

$$a_0 = c \qquad n \geq 1$$

$$a_n = a_0 + \sum_{i=1}^{n} k$$

$$a_n - a_{n-1} = 2n$$
$$a_0 = 0$$
$$n \geq 1$$

$$a_n = a_0 + \sum_{i=1}^{n} 2n$$
$$= 0 + 2\sum_{i=1}^{n} n$$
$$= 2\left(\frac{n(n+1)}{2}\right)$$
$$= n(n+1)$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$S = 1 + 2 + 3 + 4 + \cdots + n$$
$$+ \quad S = n(n-1) + (n-2) + \cdots + 1$$
$$2S = (n+1) + (n+1) + (n+1) + \cdots \quad n+1$$
$$S = \frac{n(n+1)}{2} \Rightarrow \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

efinition

Σa_n is a

Definition: A recurrence relation of the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the seq^n namely $a_0, a_1, a_2 \ldots a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is non-negative integer.

A recurrence relation are also called difference equation

Eg let $a_n = a_{n-1} - a_{n-2}$

$n = 2, 3 \ldots$

$a_0 = 3$. $a_1 = 5$.

What is a_2 & a_3 ?

if $n = 2$

$a_2 = a_1 - a_0$

$a_2 = 5 - 3$

$a_2 = 2$

$n = 3$

$a_3 = a_2 - a_1$

$= 2 - 5$

$a_3 = -3$

Linear Recurrence Relation with Constant Coefficients

A recurrence relation of the form

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + C_3 a_{r-3} + \ldots + C_k a_{r-k} = f(r) \quad \text{---} (1)$$

C_0, C_1, C_2, C_3, C_k are constant coefficient

$a_r, a_{r-1}, a_{r-2}, \ldots a_{r-k} \rightarrow$ kth order R.R.

$C_0 \neq 0$ & $C_k \neq 0$

Second order RR,

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} = f(r)$$

Third oder RR.

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + C_3 a_{r-3} = f(r)$$

Solu^n of eq^n

$$\boxed{a_r = a_r^{(h)} + a_r^{(P)}}$$

Linear R.R depends on $f(r)$

where $a_r^{(h)} \rightarrow$ Homogeneous Solu^n.

$a_r^{(P)} \rightarrow$ Particular solu^n.

Linear RR — HRR, NHRR

① Homogenous Recurrence Relation

If $f(r) = 0$ then eq^n(1) is called Homogeneous R.R.

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + C_3 a_{r-3} + \dots + C_k a_{r-k} = 0$$

② Non Homogeneous RR

If $f(r) \neq 0$, then eq^n(1) is called Non-homogenous recurrence Relation.

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + C_3 a_{r-3} + \dots + C_k a_{r-k} = f(r)$$

Homogenous Linear Recurrence Relation with Constant coefficients

Suppose the second oder H.Li.R.R. is

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} = 0$$

The Characteristics Eq^n (Auxillary Eq^n) is

$$C_0 m^2 + C_1 m + C_2 = 0$$

Case I : If the roots of A.E. are ~~eq~~ real & unequal

$m_1 \neq m_2$ e.g. 2 & 3

The general solution is

$$\boxed{a_r = C_1 m_1^r + C_2 m_2^r}$$

Case II : If the roots of A.E. are Real and Equal

$m_1 = m_2 = m$

The general solun is

$$\boxed{a_r = [C_1 + r C_2] m^r}$$

Case III : If the roots of A.E are in complex no.

$m = \alpha \pm i\beta$

∴ The general solun is

$$a_r = (C_1 \cos r\theta + C_2 \sin r\theta) R^r$$

where $R = \sqrt{\alpha^2 + \beta^2}$

$\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

Q1 Solve the R.R

$$a_r + 5a_{r-1} + 6a_{r-2} = 0$$

Solⁿ Given recurrence relation is

$$a_r + 5a_{r-1} + 6a_{r-2} = 0 \quad —(1)$$

This is second order R.R.

The characteristic eqⁿ is

$$m^2 + 5m + 6 = 0 \rightarrow m^2 + 2m + 3m + 6 = 0$$

$$(m+2)(m+3) = 0$$

Case I

$$m = -2, -3$$

The general solⁿ is

$$\boxed{a_r = c_1(-2)^r + c_2(-3)^r}$$ Ans

Q2 Solve the R.R.

$$a_r - 7a_{r-1} + 10a_{r-2} = 0 \quad —(1)$$

given $a_0 = 0 \quad a_1 = 3$

Solu'

Order eqⁿ

$$m^2 - 7m + 10 = 0$$

$$(m-2)(m-5) = 0$$

$$m = 2, 5.$$

$$a_r = c_1(2)^r + c_2(5)^r \quad —(2)$$

Putting in eqⁿ(2) $a_0 = 0 \quad a_r = 0$ & $r = 0$

$$c_1(2)^0 + c_2(5)^0 = 0$$

$$c_1 + c_2 = 0 \quad —(3)$$

Again putting in eqⁿ —(2) $a_1 = 3$ i.e. $a_r = 3$ & $r = 1$

$$c_1(2)^1 + c_2(5)^1 = 3$$

$2(c_1)^1 + c_2(5)^1 = 3$

$2c_1 + 5c_2 = 3$ —(4)

Solvg eqn (3) & (4)

we get

$c_1 = -1$ $c_2 = 1$

∴ The reqn general solun

$a_r = 5^r - 2^r$ Ans

Q) Solve the recurrence Relation

$a_r - 3a_{r-1} + 3a_{r-2} - a_{r-3} = 0$ —(1)

given that $a_0 = 1$ ✓ $a_1 = -2$ ✓ $a_2 = -1$

This is third order recurrence relation

$a^3 \; 3a^2$.

The characteristic eqn

$m^3 - 3m^2 + 3m - 1 = 0$

$(m-1)^3 = 0$

$m = 1, 1, 1$

$1 =$

∴ The general solution is

$a_r = [c_1 + c_2 r + c_3 r^2] m^r$

$a_r = (c_1 + c_2 r + c_3 r^2)(1)^r$ —(2)

Putting in eqn (2)

$a_0 = 1$ i.e. $a_r = 1$ & $r = 0$ >

$c_1 = 1$

Puttg eqn $a_1 = -2$ i.e $a_r = -2$ & $r = 1$

in (2) $c_1 + c_2 + c_3 = -2$ —(3)

Putting in eqn - 2.

$a_1 = -1$ i.e $a_r = -1$ & $r = 2$ - (4)

$c_1 + 2c_2 + 4c_3 = -1$

~~Solving eqn - 3 & 4~~

Solving eqn 3 & 4 we get

$c_2 = -11/2$ and $c_3 = 5/2$

∴ The required general eqn is

$a_r = 1 - \frac{11}{2} r + \frac{5}{2} r^2$ Ans

$a_r + 2a_{r-1} + 2a_{r-2} = 0$

$a_0 = 0$ & $a_1 = -1$

<u>Solution</u> Given recurrence relation is

$a_r + 2a_{r-1} + 2a_{r-2} = 0$ — 1

and given $a_0 = 0$ $a_1 = -1$

This is second order R.R.

The characteristic eqn. is

$m^2 + 2m + 2 = 0$ $[ax^2+bx+c]=0$

$m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)}$

$m = \dfrac{-2 \pm 2i}{2} = +1 \pm 1$

Here $\alpha = -1$ and $\beta = 1$

$R = \sqrt{\alpha^2 + \beta^2}$

$= \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

and $\Rightarrow \theta = \tan^{-1}\left(\dfrac{\beta}{\alpha}\right) = \tan^{-1}\dfrac{(-1)}{(1)}$

$= \tan^{-1}(1)$

Formula $\theta = \pi - \tan^{-1}(1)$

$= \pi - \dfrac{\pi}{4}$

$= \dfrac{3}{4}\pi$

The general eqn: $a_r = [c_1 \cos r\theta + c_2 \sin r\theta] R^r$

$a_r = [c_1 \cos \frac{3\pi}{4} r + c_2 \sin \frac{3\pi}{4} r](\sqrt{2})^r$ — 2

Putting in eq^n (2)

$a_0 = 0$ i.e $a_r = 0$ $r = 0$

$$[C_1(1) + C_2(0)](\sqrt{2})^0 = 0$$

$C_1 = 0$

Putting in eq^n (2)

$a_1 = -1$ $a_r = -1$ & $r = 1$

$$\left(C_1 \cos\left(\frac{3\pi}{4}\right) + C_2 \sin\left(\frac{3\pi}{4}\right)\right)(\sqrt{2})^1 = -1$$

$$= \left[C_1\left(-\frac{1}{\sqrt{2}}\right) + C_2\left(\frac{1}{\sqrt{2}}\right)\right](\sqrt{2})^1 = -1$$

$\because C_1 = 0$

$C_2 = -1$

Putting the value in eq^n (2) we get

$$a_r = \left[0 + (-1)\sin\left(\frac{3\pi}{4}r\right)\right](\sqrt{2})^r$$

$$a_r = -\sin\left(\frac{3\pi}{4}r\right)(\sqrt{2})^r$$ Ans

Solve the

i.e) a

solut

Solve the following recurrence relations.

i) $a_{r+1} - 1.5 a_r = 0$, $r > 0$.

Solution

$$a_r = A\alpha^r$$

$$a_{r+1} = A\alpha^{r+1} \quad \text{--- (1)}$$

$$A\alpha^{r+1} - 1.5\alpha^r = 0$$

$$A\alpha^r(\alpha - 1.5) = 0$$

But $A\alpha^r \neq 0$

$\alpha = 1.5$ then solution of homogeneous Recurrence Relation is given by

$$a_r = A_1\alpha_1^r$$

$$\boxed{a_r = A_1(1.5)^r}$$

ii) $a_r = 5a_{r-1} + 6a_{r-2}$, $r \geq 2$,

$a_0 = a_1 = 3$.

Solution

we have

$$a_r = 5a_{r-1} + 6a_{r-2}$$

So

$$a_r - 5a_{r-1} + 6a_{r-2} = 0$$

Put $a_r = A\alpha^r$, $a_{r-1} = A\alpha^{r-1}$, $a_{r-2} = A\alpha^{r-2}$ $\Big\}$ eqⁿ – (1)

put values

$$A\alpha^r - 5A\alpha^{r-1} + 6A\alpha^{r-2} = 0$$

But $A\alpha^r \neq 0$

$\Rightarrow \alpha^2 - 5\alpha + 6 = 0 \rightarrow$ characteristic eqⁿ.

Solve we get $\alpha = 2, 3$

homogeneous recurrence Relation is given by

$$a_r = A_1(\alpha_1)^r + A_2(\alpha_2)^r$$

$$a_r = A_1(2)^r + A_2(3)^r$$

Put $r = 0, 1$ and used $a_0 = a_1 = 3$,

$$a_0 = A_1 + A_2 \Rightarrow 3 = A_1 + A_2$$

$$a_1 = 2A_1 + 3A_2 \Rightarrow 3 = 2A_1 + 3A_2$$

Solve eq^n - we get

$$A_1 = 6 \qquad A_2 = -3$$

The req solution is $a_r = 6(2)^r - 3(3^r)$

Linear Recurrence Relation

Non-Homogenous Linear Recurrence Relation

- → Concept of N.H.RR.
- → Solu^n of NHRR
- → Method to finding particular solution.
- → Case 1 & case 2 when $f(r) = b^r$

NHRR with constant coefficients.

Suppose the k^{th} oder Non-Homogeneous linear recurrence relation with constant coefficients is

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + C_3 a_{r-3} + \cdots C_k a_{r-k} = f(r) \quad (1)$$

The solu^n of eq^n-(1)

$$a_r = a_{(r)}^{(h)} + a_r^{(p)}$$

$a_r^{(h)}$ = Homogenous sol^n

$a_r^{(p)}$ = Particular Solution.

Solution of H.L.R.R. when $f(r) = 0$

→ ~~for~~ see same as Homogeneous Solu^n of LRR.

* Method of finding the Particular Solution when $f(r) \neq 0$

To find PS of RR, we will consider a trial solution on the basis nature of $f(r)$

Case I When $f(r) = b^r$ and b is not a root of characteristic eq^n

Suppose the trial solution is $\boxed{a_r = A \cdot b^r}$

a_P

Q Solve the recurrence relation

$$a_r - 5a_{r-1} + 6a_{r-2} = 5^r \quad \text{—— (1)}$$

Soln

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0$$

$$m = 2, 3$$

The Homogeneous Solution is

$$a_r^{(h)} = C_1 (2)^r + C_2 (3)^r$$

Since $f(r) = 5^r$ and 5 is not root of Homogeneous Solution.

then we will assume the particular solution is

$$a_r^{(P)} = A(5)^r \quad \text{—— (2)}$$ [Put value of $r, r-1, r-2$]

Putting the value of a_r in eq^n — (1) we get

$$A \cdot 5^r - 5(A \cdot 5^{r-1}) + 6[A \cdot 5^{r-2}] = 5^r$$

$$A 5^r \left[1 - 1 + \frac{6}{25}\right] = 5^r$$

$$A = \frac{25}{6}$$

Putting in eqn (2) we get

$$a_r^{(P)} = \frac{25}{6}.(5)^r = \frac{1}{6} 5^{r+2}$$

The total solution eqn (1) is given by

$$\boxed{a_r = a_r^{(h)} + a_r^{(P)}}$$

$$a_r^{(t)} = C_1(2)^r + C_2(3)^r + \frac{1}{6} 5^{r+2}$$ Ans

Q2 Solve the recurrence relation

$$a_r - 7a_{r-1} + 10a_{r-2} = 3^r$$

given that $a_0 = 0$ and $a_1 = 1$

Solution

$$a_r - 7a_{r-1} + 10a_{r-2} = 3^r$$

the characteristic eqn

$$m^2 - 7m + 10 = 0$$

$$m = 2, 5$$

The homogeneous solution is

$$a_r^{(h)} = C_1(2)^r + C_2(5)^r$$

Since $f(r) = 3^r$ and 3 is not a root of H.S. then we will assume particular solution.

$$a_r^{(P)} = A.(3)^r \quad (2)$$

Putting the value of a_r in eqn (1) we get

$$A.3^r - 7(A.3^{r-1}) + 10(A.3^{r-2}) = 3^r$$

$$A \cdot 3^r\left(1 - \frac{7}{3} + \frac{10}{9}\right) = 3^r$$

$$A = -\frac{9}{2}$$

Putting in eq^n — (2) we get

$$a_r^{(P)} = -\frac{9}{2}(3)^r = -\frac{1}{2} \cdot 3^{r+2}$$

The total solution of eq^n (1) is given by

$$a_r = a_r^{(h)} + a_r^{(P)}$$

$$a_r = C_1(2)^r + C_2(5)^r - \frac{1}{2} 3^{r+2} \quad \text{— (3)}$$

Condition (1)

$a_0 = 0$ $\quad a_r = 0$ & $r = 0$ in eq^n

we get

$$C_1(2)^0 + C_2(5)^0 - \frac{1}{2} 3^2 = 0$$

$$C_1 + C_2 = \frac{9}{2} \quad \text{— (4)}$$

Again putting $a_1 = 1$ $a_r = 1$ and $r = 1$ in eq^n — (3) we get

$$C_1(2) + C_2(5) - \frac{1}{2} \cdot 3^3 = 1$$

$$2C_1 + 5C_2 = \frac{29}{2} \quad \text{— (5)}$$

Solving eq^n — (4) & (5)

$$C_1 = \frac{8}{3} \quad \& \quad C_2 = \frac{11}{6}$$

Putting in eq^n - (3) we get

$$a_r = \frac{8}{3}(2)^r + \frac{11}{6}(5)^r - \frac{1}{2}3^{r+2}$$

Ans

Case(2) $f(r) = b^r$ & b is a root of Characteristic eq^n.

(1) If multiplicity of b is One i.e b is root of multiplicity 1, then suppose the trial solution is

$$a_r = A \cdot r b^r$$

(2) If multiplicity of b is two, b is a root of multiplicity 2, then suppose the trial solution is

$$a_r = A r^2 b^r$$

3. If multiplicity of b is three i.e b is root of multiplicity 3, then suppose the trial solu^n is

$$a_r = A r^3 b^r$$

Problem

$a_r - 3a_{r-1} + 2a_{r-2} = 2^r$ ——— (1)

Solution

$$m^2 - 3m + 2 = 0$$
$$(m-1)(m-2) = 0$$
$$m = 1, 2$$

The H.S.

$$a_r^{(h)} = C_1(1)^r + C_2(2)^r$$

Since $f(r) = 2^r$ & 2 is a characteristic root of multiplicity is 1

then we will assume the particular solution is

$$a_r^{(p)} = Ar.2^r \quad \text{——— (2)}$$

$$Ar.2^r - 3A[r-1]2^{r-1} + 2[A[r-2]2^{r-2}] = 2^r$$

By Putting the value of a_r in eq^n -(1)

$$\Rightarrow \cancel{Ar\,2^r} - \cancel{\frac{3}{2}Ar.2^r} + \frac{3}{2}A2^r + \cancel{\frac{1}{2}Ar.2^r} - A2^r = 2^r$$

$$\frac{1}{2}A.2 = 2^r$$

$$A = 2$$

Putting in eq^n (2) we get

$$a_r^{(p)} = 2r(2)^r = r.2^{r+1}$$

The total solution of eq^n (1) is given by

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$a_r = C_2(1)^r + C_2(2)^r + r.2^{r+1}$$

Q $a_r - 4a_{r-1} + 4a_{r-2} = 2^r$ — (1)

Solⁿ The characteristic eqⁿ

$$m^2 - 4m + 4 = 0$$

$$m = 2, 2$$ no matchg'

The homo solⁿ

$$a_r^{(h)} = (C_1 + C_2 r)(2)^r$$

The

$$a_r^{(p)} = A r^2 . 2^r$$ — (2)

Putty the value of a_r in eqⁿ - (1)

$$A r^2 2^r - 4\left[A (r-1)^2 2^{r-1}\right] + 4\left[A(r-2)^2 2^{r-2}\right] = 2^r$$

Solve

$$A = \frac{1}{2}$$

Putty in eqⁿ - (2)

$$a_r^{(p)} = \frac{1}{2} r^2 (2)^r$$

$$= r^2 2^{r-1}$$

Total solⁿ

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$a_r = (C_1 + C_2 r)(2)^r + r^2 2^{r-1}$$

Case III

Suppose the H. L R. R.

$$C_0 a_r - - - - - - C_k a_{r-k} = f(r) \quad (1)$$

The solu

$$a_r = a_r^{(h)} + a_r^{(p)} \longrightarrow f(r) \neq 0$$

we consider $f(r) = 0$

Case III

When $f(r)$ is a polynomial

1. If $f(r)$ is Ist degree, the Trial solu^n

$$a_r^{(p)} = A_0 + A_1 r$$

2. If $f(r)$ is II degree

$$a_r^{(p)} = A_0 + A_1 r + A_2 r^2$$

3. If $f(r)$ is III degree

$$a_r^{(p)} = A_0 + A_1 r + A_2 r^2 + A_3 r^3$$

Q $a_r - 4a_{r-1} + 3a_{r-2} = 5^r$

Solution $a_r = C_1(1)^r + C_2(3)^r + \frac{5^r}{8}$

Q $a_r - 2a_{r-1} + a_{r-2} = 2^r$

given $a_0 = 2$ $a_1 = 1$

Solution $a_r = 1 - 2r + 2^r$

Solve the recurrence relation

$$a_r - 5a_{r-1} + 6a_{r-2} = 2 + r \quad \text{--- (1)}$$

given $a_0 = 1 \quad a_1 = 1$

Solu^n

$$m = 2, 3.$$

The Homo solution

$$a_r^{(h)} = C_1 (2)^r + C_2 (3)^r.$$

Since $f(r) = 2 + r$ is a polynomial of degree one then.

$$a_r^{(p)} = A_0 + A_1 r \quad \text{--- (2)}$$

Put in eq^n (1)

$$[A_0 + A_1 r] - 5[A_0 + A_1(r-1)] + 6[A_0 + A_1(r-2)] = 2 + r$$

$$\Rightarrow 2A_0 \cancel{+} -7A_1 + 2A_1 r = 2 + r$$

equating the coefficient of r^0 + r^1 on both sides we get

$$2A_0 - 7A_1 = 2 \quad \text{--- (3)}$$

$$2A_1 = 1, \quad A_1 = \frac{1}{2}$$

Putting the value of A_1 in eq^n (3)

$$A_0 = \frac{11}{4}$$

Putting the value in eqn (2) we get

$a_r^{(P)} = \frac{11}{4} + \frac{1}{2}r$

Total solun

$a_r = c_1(2)^r + c_2(3)^r + \frac{11}{4} + \frac{1}{2}r$ — (4)

Putting $a_0 = 1$ i.e $r = 0$, $a_r = 1$ in eqn 4

$c_1(2)^0 + c_2(3)^0 + \frac{11}{4} + 0 = 1$

$c_1 + c_2 = \frac{7}{4}$ — (5)

Putting $a_1 = 1$ i.e $a_r = 1$ & $r = 1$ in eqn 4

$2c_1 + 3c_2 = \frac{9}{4}$ — (6)

Solve 5 & 6

$c_1 = -3 \quad c_2 = \frac{5}{4}$

Putting in eqn (4)

$a_r = -3(2)^r + \frac{5}{4}(3)^r + \frac{11}{4} + \frac{1}{2}r$

Ans