Predicate Logic

Predicate Logic : The proposition Logic is not sufficient to expression the meaning of statements in mathematics and in natural Language. Predicate Logic is more powerful as compare to propositional Logic & it can be used to express the meaning of wide range of Stmts in mathematics and computer Science.

Predicate :

"x is greater than 5"

This stmt is consist of two part one is variable x which is subject of the stmt and another one is " is greater than 5" which specifies the property of variable x. This statement can be denoted as $P(x)$.

In predicate logic the domain must be specified.

Q⇒ $P(x)$: "$x = x+1$"

Domain — "Real No"

find the value of Universal Quantifier and Existential Quantifier for this stmt.

$\forall x\, P(x)$ — false Let $x=1$, $1 \neq 1+1$

$\exists x\, P(x)$ — false.

Q⇒ $P(x)$: "$x < x+1$"

Domain : all set of real numbers.

$\forall x\, P(x)$ — True

$\exists x\, P(x)$ — True.

Q⇒ $P(x, y)$: "$x = y+1$" find, $P(3, 2)$ — True

$P(4, 5)$ — False

Qualifiers :

when variables in a funⁿ are assigned values then resulting stmt becomes proposition with certain truth value. The Quantification is a way to create proposition from a given function. It expresses the extent to which a given predicate is true over a range of elements.

Quantifiers are of two types :

1) <u>Universal Quantifier</u> : $(\forall)$ {for all}

The Universal Quantifier say that for given predicate it is true for every element under consideration.

2.) <u>Existential Quantifier</u> : $(\exists)$ {there exist}

Existential Quantifier says that there exist at least one element under consideration for which given predicate is true and area of logic that deals Quantifiers and predicate is called predicate Calculus.

* If specified domain is empty in that case universal Quantifier is true bcoz there does Universal Quantifier is true bcoz there does not exist single case for which its value can be falsed and existential Quantifier have false truth value bcoz there does not exist single case for which its value can be true. *

$\forall x P(x) = P(x_1) \wedge P(x_2) \wedge P(x_3) \ldots\ldots \wedge P(x_n)$

True when all cases are true

False when single case is false.

$\exists x P(x) = P(x_1) \vee P(x_2) \vee P(x_3) \ldots\ldots \vee P(x_n)$

False when all cases are false

True when single case is true

Q⇒ Find the counter example if possible to following universally quantified statement where domain for all variables is consist of all Integer.

domain : all Integer

a) $\forall x\ (x^2 \geq x)$

It is true, no counter example is there.

b) $\forall x\ (x > 0 \vee x < 0)$

It is false, counter example $x = 0$

c) $\forall x\ (x = 1)$

It is false, Counter example $x = 0, 2, 3, \ldots$

d) $\forall x\ (x^2 \neq 2)$

It is true, No counter example

e) $\forall x\ (x^2 \neq x)$

It is false, counter example, $x = 0, 1$

f) $\forall x\ (|x| > 0)$

It is false, counter example, $x = 0$

Negating Quantifiers:

1) Negation of Universal Quantifier :

Stmt : "All Indians are honest"

$P(x)$: "x is honest"

domain : Indians

$\forall x\, P(x)$ — All Indians are honest

negation of $\forall x\, P(x) = \neg \forall x\, P(x)$

↓

All Indians are not honest

$\underbrace{\text{Some}}_{\exists}$ $\underbrace{\text{Indians}}_{x}$ (or) $\underbrace{\text{are not honest}}_{\neg P(x)}$.

$\neg P(x)$: "x is not honest."

Some Indians are not honest

$$\downarrow$$

$$\exists x\, \neg P(x)$$

that means:

$$\boxed{\neg \forall x\, P(x) \;=\; \exists x\, \neg P(x)}$$

2.) <u>Negation</u> of <u>Existential</u> <u>Quantifier</u>:

<u>Stmt</u>: "Some Indians are honest"

$P(x)$: "x is honest"

domain : Indians.

$\exists x\, P(x)$: Some Indians are honest

negation of $\exists x\, P(x)$ is $-\ \neg[\exists x\, P(x)]$

$$\downarrow$$

Some Indians are not honest

(or)

$\overbrace{\text{All Indians}}^{\forall x}$ $\overbrace{\text{are not honest}}^{\neg P(x)}$.

$$\downarrow$$

$\neg P(x)$ – "x is not honest" $\qquad \forall x\, \neg P(x)$

that means:

$$\boxed{\neg \exists x\, P(x) \;=\; \forall x\, \neg P(x)}$$

<u>Q</u>⇒ find out negation of following statements:

i) All politicans are honest

ii) Some Americans eat cheese Burger

iii) $\forall x\, (x+1 > x)$

iv) $\forall x\, (x < 2)$

i) All politicians are honest

$P(x)$: "x is honest"

domain : Politicians

Predicate form :- $\forall x\, P(x)$

Negation :- $\neg \forall x\, P(x)$

and $\neg \forall x\, P(x) = \exists x \neg P(x)$

Some politicians are not honest.

ii) Some Americans eat cheese Burger.

$P(x)$: "x eat cheese Burger"

domain : Americans

Predicate form : $\exists x\, P(x)$

Negation : $\neg \exists x\, P(x)$

and we know, $\neg \exists x\, P(x) = \forall x \neg P(x)$

All Americans does not eat cheese Burger.

iii) $\forall x\, (x+1 > x)$

$$\neg \forall x\, (x+1 > x) = \exists x \neg P(x)$$
$$= \exists x \neg (x+1 > x)$$
$$= \exists x\, (x+1 \leq x)$$ Ans

iv) $\forall x\, (x < 2)$

$$\neg \forall x\, (x < 2) = \exists x \neg P(x)$$
$$= \exists x \neg (x < 2)$$
$$= \exists x\, (x \geq 2)$$ Ans

Representation of sentences into Predicate symbolic form :

Q⇒ Convert these statement into Logical form :

i) Every student of this class has studied Data structure.

$P(x)$: "x has studied Data structure"

domain : student of class

$\forall x\, P(x)$ Ans

ii) Some student of this class has visited Dipu chauhan

$P(x)$: " x has visited dipu chauhan"

domain : student of class

$\exists x\, P(x)$ Ans

iii) Each student of this class has visited either Kanpur Centre or Bus stop.

" x has visted either Kanpur Centre or Bus Stop".

It is compound stmt - Convert into simple stmt.

$P(x)$: x has visited Kanpur Centre

$Q(x)$: x has visited Bus stop.

domain : student of class

$\forall x\, (P(x) \lor Q(x))$ Ans

Examples: $P(x, y)$ = " $x + y = 5$ "

$P(x, y, z)$ = " $x + y = z$ "

Q⇒ Convert these stmt into logical stmt where domain is all animals.

1) All lions are cat.
2) Some lions do not drink coffee.
3) Some cats do not drink coffee.

Sol^n⇒ $P(x)$: " x is lion"

$Q(x)$: " x is cat"

$R(x)$: " x drink coffee".

Predicates

1) $\forall x\, (P(x) \rightarrow Q(x))$

2) $\exists x\, (P(x) \land \neg R(x))$

3) $\exists x\, (Q(x) \land \neg R(x))$

Q⇒ Convert these stmts into Logical stmts.
Domain : All creatures.
1) " All humming birds are richly coloured "
2) " No large bird live on honey "
3) " Birds that donot live on honey are dull in Colour".

Soln ⇒ Predicates are :
$P(x)$: " x is humming Bird"
$Q(x)$: " x is richly coloured"
$R(x)$: " x is large".
$S(x)$: " x live on honey"

1) Logical stmt : $\forall x\ (P(x) \rightarrow Q(x))$
2.) Logical stmt : $\forall x\ \neg (R(x) \wedge S(x))$
$$\parallel$$
$$\neg \exists x\ (R(x) \wedge S(x))$$
3) Logical stmt : $\exists x\ (\neg S(x) \rightarrow \neg Q(x))$
4) Logical stmt : $\exists x\ (P(x) \wedge \neg R(x))$

Q⇒ $Q(x)$: " word x contain letter 'a' "
find the truth value of $Q(x)$.
i) $Q(x)$: " class " True (bcoz word 'class' contain letter 'a')
ii) $Q(x)$: " Red ". False (bcoz word 'Red' not contain letter 'a')

Q⇒ Suppose $Q(x, y)$ denotes.
$Q(x, y)$: x is capital of y
and then find out the truth values of
i) Q (Lucknow , UP) — True
ii) Q (Punjab, India) — False.

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Q⇒ Let Predicate $P(x)$ denotes.

$P(x)$: "x spends more than 5 hours every week day in class".

where domain of $P(x)$ is : all students

Now express the following logical stmts in form of stmt.

a) $\exists x\, P(x)$ b) $\forall x\, P(x)$ c) $\exists x\, \neg P(x)$ d) $\forall x\, \neg P(x)$

a) $\exists x\, P(x)$

Some students spends more than 5 hours every week day in class.

b) $\forall x\, P(x)$

All students spend more than 5 hours every week day in class.

c) $\exists x\, \neg P(x)$

Some student do not spends more than 5 hours every week day in class.

or

No students spend more than 5 hours every week day in class. — $\neg \forall x\, P(x)$

d) $\forall x\, \neg P(x)$

All student do not spend more than 5 hours every week day in class.

or

Some student do not spend more than 5 hours every week day in class — $\neg \exists x\, P(x)$

Q⇒ Convert the following logical stmt in statements:

$C(x)$: "x is Comedian"

$f(x)$: "x is funny"

a) $\forall x\, (C(x) \rightarrow P(x))$

b) $\forall x\,(C(x) \wedge F(x))$

c) $\exists x\,(C(x) \rightarrow F(x))$

d) $\exists x\,(C(x) \wedge F(x))$

a) $\forall x\,(C(x) \rightarrow F(x))$

"All comedians are funny"

b) $\forall x\,(C(x) \wedge F(x))$

"All people are comedian as well as funny"

c) $\exists x\,(C(x) \rightarrow F(x))$

"Some Comedians are funny"

(or)

"Some people who are comedian are funny."

d) $\exists x\,(C(x) \wedge F(x))$

"Some people are comedian as well as funny".

Q⇒ $P(x)$: "x can speak Russian"

$Q(x)$: "x Knowns C++".

where domain for x consist of all students. Convert the following expression into logical stmt.

1.) "There is a student in your class who can speak Russian & who Knowns C++."

2.) "There is a student in class who can speak Russian but does not Know C++."

3.) "Every student in your class either speak Russian or Knowns C++."

4) "No student at your school can speak Russian or Knowns C++."

1) $\exists x\,(P(x) \wedge Q(x))$

2.) $\exists x\,(P(x) \wedge \neg Q(x))$

3.) $\forall x\,(P(x) \vee Q(x))$

4.) $\forall x \neg (P(x) \vee Q(x))$

(or) $\neg \exists x\,(P(x) \vee Q(x))$

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Rules of Inference for Predicate Logic :

There are 13 rules of Inferences — 9 Rules (in Proposition)
+ 4 Rules.

1.) Universal Specification (US) :

$$\frac{\forall x\, P(x)}{\therefore\ P(a)}$$

where 'a' is an arbitary variable in domain

2.) Universal Generalization (UG) :

$$\frac{P(a)}{\therefore\ \forall x\, P(x)}$$

where 'a' is an arbitary variable in domain.

3.) Existential Specification (ES) :

$$\frac{\exists x\, P(x)}{\therefore\ P(a)}$$

where 'a' is a specified variable from domain.

4.) Existential Generalization (EG) :

$$\frac{P(a)}{\therefore\ \exists x\, P(x)}$$

where 'a' is a specified variable from domain.

Q ⇒ Prove that with the help of Inference Rule

$$\left.\begin{array}{l}\forall x\,(H(x) \rightarrow M(x)) \\ \exists x\, H(x)\end{array}\right\}\ \text{Hypothesis}$$

$$\overline{\therefore\ \exists x\, M(x)} \longrightarrow \text{Conclusion}$$

1) $\exists x\, H(x)$ — Hypothesis

2) $\forall x\,(H(x) \rightarrow M(x))$ — Hypothesis

3) $\therefore\ H(y)$ ES in 1

4) $H(y) \rightarrow M(y)$ US in 2

5) $\therefore\ M(y)$ apply Modus ponens in 3) & 4)

$\therefore\ \exists x\, M(x)$ EG in 5

Q⇒ $\exists x\,(P(x) \wedge Q(x))$ — Hypothesis

$\therefore \exists x\, P(x) \wedge \exists x\, Q(x)$ — Conclusion

Prove:

1) $\exists x\,(P(x) \wedge Q(x))$ — Hypothesis

2) $\therefore P(a) \wedge Q(a)$ ES in ①

3) $\therefore P(a)$ By simplification in ②

4) $\therefore Q(a)$ By simplification in ②

5) $\exists x\, P(x)$ EG in ③

6) $\exists x\, Q(x)$ EG in ④

7) $\therefore \exists x\, P(x) \wedge \exists x\, Q(x)$ By conjunction in ⑤ & ⑥

Q⇒ Show that the Hypothesis "A student in this class has not read the book." "Everyone in this class pass the Exam has the exam". Conclusion: "Someone who pass the Exam has not read the book.

Soln: Convert the stmt in Predicate Logic:

$P(x)$: "x is in this class"

$Q(x)$: "x has read the book"

$R(x)$: "x has passed the exam"

domain : student

Logical Form:

$$\left.\begin{array}{l} ①\ \exists x\,(P(x) \wedge \neg Q(x)) \\ ②\ \forall x\,(P(x) \rightarrow R(x)) \end{array}\right\} \text{Hypothesis}$$

③ $\exists x\,(R(x) \wedge \neg Q(x))$ — conclusion

Inference form:

① $\exists x\,(P(x) \wedge \neg Q(x))$

② $\forall x\,(P(x) \rightarrow R(x))$

$\therefore \exists x\,(R(x) \wedge \neg Q(x))$

Prove with the help of Inference Rule:

① $\exists x (P(x) \wedge \neg Q(x))$ ⟶ Hypothesis

② $\therefore P(y) \wedge \neg Q(y)$ — ES in ①

③ $\therefore P(y)$ — By Simplification in ②

④ $\therefore \neg Q(y)$ — By simplification in ②

⑤ $\forall x (P(x) \rightarrow R(x))$ ⟶ Hypothesis

⑥ $P(y) \rightarrow R(y)$ ⟶ US in ⑤

⑦ $\therefore R(y)$ — Modus ponens in ③ and ⑥

⑧ $R(y) \wedge \neg Q(y)$ — By Conjunction in ⑦ and ④

⑨ $\therefore \exists x (R(x) \wedge \neg Q(x))$ — EG in ⑧

Hence, Conclusion is $\exists x (R(x) \wedge \neg Q(x))$ proved.

Q⇒ Show that the Hypothesis

" Every one in discrete class has taken a course in Computer Science."

" Ram is a student in this class."

Conclusion: " Ram has taken a course in computer Science"

Soln ⇒ Predicate statements are:

$P(x)$: " x is in discrete class"

$Q(x)$: " x has taken a course in CS".

Logical form:

① $\forall x (P(x) \rightarrow Q(x))$ } Hypothesis

② $P(\text{Ram})$

③ $Q(\text{Ram})$ — conclusion

Inference form:

$$\begin{array}{l} \forall x (P(x) \rightarrow Q(x)) \\ \underline{P(\text{Ram})} \\ \therefore Q(\text{Ram}) \end{array}$$

Prove: ① $\forall x\,(P(x) \rightarrow Q(x))$ — Hypothesis

② $\therefore P(a) \rightarrow Q(a)$ By US in ①

Let $a =$ Ram

② $\therefore P(Ram) \rightarrow Q(Ram)$ By US in ①

③ $P(Ram)$ — Hypothesis

$\therefore Q(Ram)$ — By Modus ponens.

Hence, conclusion is $Q(Ram)$ proved.

Nested Quantifier :

* when we use more than one quantifier in a single predicate statement then it is known as Nested Quantifier.

① $\forall x \forall y\, P(x, y)$

② $\forall x \exists y\, P(x, y)$

③ $\exists x \forall y\, P(x, y)$

④ $\exists x \exists y\, P(x, y)$

Example: $\forall x \forall y\, P(x, y)$

$P(x, y)$: " $x + y = 5$ "

Domain : all Integers

$x = 1$	$x = 2$	$x = 3$	----	$x = n$
then $y = 1$	then $y = 1$	then $y = 1$		then $y = 1$
$y = 2$	$y = 2$	$y = 2$		$y = 2$
$y = 3$	$y = 3$	$y = 3$		$y = 3$
⋮	⋮	⋮		⋮
$y = n$	$y = n$	$y = n$		$y = n$

Q⇒ $P(x, y)$: " $x + y = 5$ "

domain : all Integers

find the truth value for the following :

1) $\forall x \forall y\, P(x, y)$ False

2) $\forall x \exists y\, P(x, y)$ True $\{(x = 1, y = 4), (x = 7, y = -2)\}$

3) $\exists x \forall y\, P(x, y)$ False

4) $\exists x \exists y\, P(x, y)$ True $\{(x = 1, y = 4)\}$

Note:

Order matters for two different Quantifiers

$$\left.\begin{array}{l}\forall x \forall y\, P(x,y) = \forall y \forall x\, P(x,y) \\ \exists x \exists y\, P(x,y) = \exists y \exists x\, P(x,y)\end{array}\right\}$$

but,

$$\left.\begin{array}{l}\forall x \exists y\, P(x,y) \neq \exists x \forall y\, P(x,y) \\ \& \quad \exists x \forall y\, P(x,y) \neq \forall x \exists y\, P(x,y)\end{array}\right\}$$

Q ⇒ Translate the statement into sentence and find the truth value of statement.

$$\forall x \forall y\, (((x>0) \wedge (y<0)) \rightarrow (xy<0))$$

Soln : Statement:

For all x and for all y if x is positive and y is negative then their multiplication is negative.

Truth ~~Sta~~ Value :

Truth value of $\forall x \forall y\, (((x>0) \wedge (y<0)) \rightarrow (xy<0))$ is True bcoz whenever x is positive and y is negative then product of ~~to~~ x and y is negative for all value of x and y.

Table

Statement	True	False
1) $\forall x P(x)$	$P(x)$ is true for all x	There is x for which $P(x)$ is false
2) $\exists x P(x)$	There is x for which $P(x)$ is true	$P(x)$ is false for all x

Negation

$\neg \exists x P(x) \equiv \forall x \neg P(x)$

3) $\neg \exists x P(x)$	for each x, $P(x)$ is false	There is x for which $P(x)$ is true

$\neg \forall x P(x) \equiv \exists x \neg P(x)$

4) $\neg \forall x P(x)$	There is x, for which $P(x)$ is false	$P(x)$ is true for all x.

5) $\forall x \forall y\, P(x,y) \equiv \forall y \forall x\, P(x,y)$	When $P(x,y)$ is true for each pair of (x,y)	There is a pair (x,y) for which $P(x,y)$ is false.
6) $\exists x \forall y\, P(x,y)$	There is x for which $P(x,y)$ is true for all value of y.	for each x, $P(x,y)$ is false, for all y.
7) $\forall x \exists y\, P(x,y)$	for each x, there is y for which $P(x,y)$ is true	For each y, there is x which is false for $P(x,y)$
8) $\exists x \exists y\, P(x,y) \equiv \exists y \exists x\, P(x,y)$	There is a pair of (x,y) for which $P(x,y)$ is true	For each pair of (x,y), $P(x,y)$ is false.

Negation of Nested Quantifier:

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

Q ⇒ Negate the following statement:

1) $\forall x \forall y\, P(x,y)$

$\neg(\forall x \forall y\, P(x,y))$

$\exists x\, \neg(\forall y\, P(x,y))$

$\exists x \exists y\, \neg P(x,y)$ Ans

2) $\forall x \exists y\, P(x,y)$

$\neg(\forall x \exists y\, P(x,y))$

$\exists x\, \neg(\exists y\, P(x,y))$

$\exists x \forall y\, \neg P(x,y)$ Ans

3) $\exists x \exists y\, P(x,y)$

$\neg(\exists x \exists y\, P(x,y))$

$\forall x\, \neg(\exists y\, P(x,y))$

$\forall x \forall y\, \neg P(x,y)$ Ans

4) $\exists x \forall y\, P(x,y)$

$\neg(\exists x \forall y\, P(x,y))$

$\forall x\, \neg(\forall y\, P(x,y))$

$\forall x \exists y\, \neg P(x,y)$ Ans

Q⇒ $\forall x \exists y \forall z\ P(x,y,z)$

$\neg(\forall x \exists y \forall z\ P(x,y,z))$

$\exists x \neg(\exists y \forall z\ P(x,y,z))$

$\exists x \forall y \neg(\forall z\ P(x,y,z))$

$\exists x \forall y \exists z\ \neg P(x,y,z)$ Ans

Q⇒ Translate the following stmt into sentence.

$C(x)$: " x has computer".

$F(x,y)$: " x and y are friends".

domain for x, y is all students.

$\forall x\ (C(x) \vee \exists y\ (C(y) \wedge F(x,y)))$

Stmt : All student have computer at school or in School either student has computer or they have friend who have computer.

Excercise

Q⇒ Express given stmt using quantifier then form negation of stmt and express the negation in simple English sentence.

a) Some old dogs can learn new tricks.

b) No Rabbit knowns Calculus.

c) Every Bird can fly

d) No one can keep secret.

e) There is no dog that can talk

f) There is no one in this class who knows french & Russian.

Q⇒ Translate the following into Logical stmt :

" If a person is female and is parent then this person is someone's mother."