1

Consider an $n \times n$ grid graph with a sink attached to every node on the boundary (and double connection between the corner nodes and the sink). Let L_0 be the Laplacian of this graph. Take the principal minor with respect to the diagonal entry corresponding to the sink node. Call this matrix Δ. In general, this matrix looks like,

$$\Delta = \begin{pmatrix} 4 & -1 & -1 & \dots & 0 \\ -1 & 4 & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & -1 & -1 & 4 \end{pmatrix}$$

Now, replace the first row with the first row of same sized identity matrix to obtain the matrix Δ',

$$\Delta' = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 4 & 0 & \dots & 0 \\ -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & -1 & -1 & 4 \end{pmatrix}$$

Consider the following two vectors,

$$V = \begin{pmatrix} 1 \\ v_2 \\ \vdots \\ v_{n^2-1} \\ v_{n^2} \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

We are trying to solve for the system

$$\Delta' V = B$$

$$V = \Delta'^{-1}.B$$

In particular, we are interested in finding the bottommost entry of V, v_{n^2} which is clearly,

$$v_{n^2} = \frac{det(\Delta'(1, n^2))}{det(\Delta')}$$

Here, $\Delta'(1, n^2)$ is the minor of Δ' obtained by deleting the first row and last column. Since the first row is all thats different between Δ and Δ', this is the same as the minor $\Delta(1, n^2)$. Also, as there is only one non-zero entry in the first row of Δ', its determinant is the same as the determinant of the minor with respect to this entry times the value of the entry. So, $det(\Delta') = det(\Delta'(1,1)) = det(\Delta(1,1))$. The last equality follows from the fact that Δ' and Δ are different in one row only (which gets out of the way while taking minors). So we have,

$$v_{n^2} = \frac{det(\Delta(1, n^2))}{det(\Delta(1,1))}$$

We need a lower bound on this value. Which means we need k_n and k_d such that,

$$k_n \leq det(\Delta(1, n^2))$$

and,

$$k_d \geq det(\Delta(1,1))$$

By the interlacing property of eigenvalues of principal minors of symmetric matrices, we know that eigenvalues of $\Delta(1,1)$ are interlaced between those of Δ. So,

$$det(\Delta(1,1)) \leq \frac{det(\Delta)}{\lambda_{min}} = k_d$$

where λ_is are the eigenvalues of the matrix Δ.

We now need a similar interlacing result for the eigenvalues of the minor $\Delta(1, n^2)$ to be able to say something about k_n.

2

Consider the dual of above stated problem. That is, we have an $n \times n$ grid of one ohm resistances. A one ampere current source-sink pair is connected to the neighbours of a corner vertex and we want estimates on the lower bound on the current flowing through the opposite corner vertex. If the combinatorial laplacian of a grid ghraph is denoted by L, the potentials that appear at each of the nodes because of the current flowing by vector v and let i be the vector containing *net currents* flowing in at any node. The equations of Kirchoff's laws at each node can be succinctly written as,

$$Lv = i$$

For now, we will refer to a node in the grid by its cartesian coordinates in its canonical embedding in euclidean plane. In our case, the vector i has all entries 0 except for a 1 at position corresponding to node $(0,1)$ and -1 for node $(1,0)$. We need to estimate the potential difference between nodes $(n-2, n-1)$ and $(n-1, n-2)$ given the values of A and i. Denote by $L(\epsilon)$ the matrix $L + \epsilon I$. Note that for small values of ϵ, unlike L, $L(\epsilon)$ is invertible. Call its inverse $G(\epsilon)$. If we denote the row of $G(\epsilon)$ corresponding to node $(n-2, n-1)$ by $G(\epsilon)_{(n-2,n-1)}$, then

$$V_{(n-2,n-1)} = \lim_{\epsilon \to 0} (G(\epsilon)_{(n-2,n-1)}.i) \quad (1)$$
$$V_{(n-1,n-2)} = \lim_{\epsilon \to 0} (G(\epsilon)_{(n-1,n-2)}.i) \quad (2)$$

Knowing the value i explicitly, we can write

$$V_{(i,j)} = \lim_{\epsilon \to 0} (G(\epsilon)_{(i,j)}(0,1) - G(\epsilon)_{(i,j)}(1,0))$$

where $G(\epsilon)_{(i,j)}(x, y)$ is the entry in the row of the node (i, j) and column of node (x, y). The formula for $G(\epsilon)$ is,

$$G(\epsilon) = \Psi \Lambda(\epsilon)^{-1} \Psi^{\dagger}$$

where Ψ is the unitary matrix containing the eigenvectors of L (and consequently of $L(\epsilon)$ and $G(\epsilon)$) as its columns, $\Psi^{\dagger}$ is its hermitian and $\Lambda(\epsilon)$ is the diagonal matrix containing the eigenvalues of $L(\epsilon)$.Which means,

$$G(\epsilon)_{(p,q)}(i,j) = \frac{1}{n^2 \epsilon} + \sum_{\substack{0 \leq a,b \leq n-1 \\ (a,b) \neq (0,0)}} \frac{\psi_{(a,b),(p,q)} \psi^{\dagger}_{(a,b),(i,j)}}{\lambda_{(a,b)} + \epsilon}$$

where $\psi_{(a,b),(p,q)}$ is the entry of (p, q) node in the $(a, b)^{th}$ eigenvector. This leads us to,

$$G(\epsilon)_{(p,q)}(i_1, j_1) - G(\epsilon)_{(p,q)}(i_2, j_2) = \sum_{\substack{0 \leq a,b \leq n-1 \\ (a,b) \neq (0,0)}} \frac{\psi_{(a,b),(p,q)} (\psi_{(a,b),(i_1,j_1)} - \psi_{(a,b),(i_2,j_2)})^{\dagger}}{\lambda_{(a,b)} + \epsilon}$$

or in particular,

$$V_{(n-2,n-1)} = \lim_{\epsilon \to 0} \left(\sum_{\substack{0 \leq a,b \leq n-1 \\ (a,b) \neq (0,0)}} \frac{\psi_{(a,b),(n-2,n-1)} (\psi_{(a,b),(0,1)} - \psi_{(a,b),(1,0)})^{\dagger}}{\lambda_{(a,b)} + \epsilon} \right) \tag{3}$$

$$V_{(n-2,n-1)} = \sum_{\substack{0 \leq a,b \leq n-1 \\ (a,b) \neq (0,0)}} \frac{\psi_{(a,b),(n-2,n-1)} (\psi_{(a,b),(0,1)} - \psi_{(a,b),(1,0)})^{\dagger}}{\lambda_{(a,b)}} \tag{4}$$

The potential difference we want takes the form,

$$V_{(n-2,n-1)} - V_{(n-1,n-2)} = \sum_{\substack{0 \leq a,b \leq n-1 \\ (a,b) \neq (0,0)}} \frac{(\psi_{(a,b),(n-2,n-1)} - \psi_{(a,b),(n-1,n-2)}) (\psi_{(a,b),(0,1)} - \psi_{(a,b),(1,0)})^{\dagger}}{\lambda_{(a,b)}} \tag{5}$$

We will now consider the general form of the eigenvalues and eigenvectors of a grid graph. The eigenvalues indexed by ordered pair (i, j) with $0 \leq i, j \leq n - 1$ are,

$$\lambda_{i,j} = 4 - 2\cos(\frac{i\pi}{n}) - 2\cos(\frac{j\pi}{n}) \tag{6}$$

and the corresponding eigenvectors are,

$$\psi_{0,0} = \frac{1}{n}[111\ldots1] \tag{7}$$

$$\psi_{(i,j),(x,y)} = \frac{2}{n} \cos\frac{(x+1/2)i\pi}{n} \cos\frac{(y+1/2)j\pi}{n} \tag{8}$$

Note that,

$$\begin{aligned}
\psi_{(a,b),(0,1)} - \psi_{(a,b),(1,0)} &= \frac{2}{n}\{\cos\frac{a\pi}{2n}\cos\frac{3b\pi}{2n} - \cos\frac{3a\pi}{2n}\cos\frac{b\pi}{2n}\} \\
&= -\frac{2}{n}\{\cos\frac{b\pi}{2n}\cos\frac{3a\pi}{2n} - \cos\frac{3b\pi}{2n}\cos\frac{a\pi}{2n}\} \\
&= -\{\psi_{(b,a),(0,1)} - \psi_{(b,a),(1,0)}\} \qquad (9)
\end{aligned}$$

which implies that in the expressions for $V_{(n-2,n-1)}$ and $V_{(n-1,n-2)}$, we can pair up the summands corresponding to pair (a,b) and (b,a) and ignore the symmetric terms of type (a,a) to get,

$$V = \sum_{\substack{0\le a<b\le n-1\\ a\neq b}} \frac{(\psi_{(a,b),(n-2,n-1)} - \psi_{(b,a),(n-2,n-1)} - \psi_{(a,b),(n-1,n-2)} + \psi_{(b,a),(n-1,n-2)})(\psi_{(a,b),(0,1)} - \psi_{(a,b),(1,0)})^{\dagger}}{\lambda_{(a,b)}} \qquad (10)$$

Now we consider the first multiplicand of the numerator of any summand.

$$N_1 = \psi_{(a,b),(n-2,n-1)} - \psi_{(b,a),(n-2,n-1)} - \psi_{(a,b),(n-1,n-2)} + \psi_{(b,a),(n-1,n-2)}$$

The form of eigenfunctions as shown in equation 26 implies that

$$\begin{aligned}
\psi_{(a,b),(n-2,n-1)} &= \psi_{(b,a),(n-1,n-2)} \\
\psi_{(a,b),(n-1,n-2)} &= \psi_{(b,a),(n-2,n-1)}
\end{aligned}$$

So the numerator simplifies to,

$$\begin{aligned}
N_1 &= 2(\psi_{(a,b),(n-2,n-1)} - \psi_{(b,a),(n-2,n-1)}) \qquad (11) \\
&= \frac{4}{n}\{\cos\frac{(n-1.5)a\pi}{n}\cos\frac{(n-.5)b\pi}{n} - \cos\frac{(n-.5)a\pi}{n}\cos\frac{(n-1.5)b\pi}{n}\} \\
&= \frac{2}{n}\{\cos\frac{(n(a+b)-1.5a-.5b)\pi}{n} + \cos\frac{(n(a-b)-1.5a+0.5b)\pi}{n} \\
&\quad - \cos\frac{(n(a+b)-.5a-1.5b)\pi}{n} - \cos\frac{(n(a-b)-.5a+1.5b)\pi}{n}\} \\
&= \frac{2}{n}\{\cos\frac{(n(a+b)-1.5a-.5b)\pi}{n} - \cos\frac{(n(a+b)-.5a-1.5b)\pi}{n} \\
&\quad + \cos\frac{(n(a-b)-1.5a+0.5b)\pi}{n} - \cos\frac{(n(a-b)-.5a+1.5b)\pi}{n}\} \\
&= \frac{4}{n}\{\sin\frac{(n-1)(a+b)\pi}{n}\sin\frac{(a-b)\pi}{2n} + \sin\frac{(n-1)(a-b)\pi}{n}\sin\frac{(a+b)\pi}{2n}\} \\
&= \frac{4}{n}\{\sin((a+b)\pi - \frac{(a+b)\pi}{n})\sin\frac{(a-b)\pi}{2n} + \sin((a-b)\pi - \frac{(a-b)\pi}{n})\sin\frac{(a+b)\pi}{2n}\} \\
&= \frac{4}{n}\{(-1)^{a+b+1}\sin\frac{(a+b)\pi}{n}\sin\frac{(a-b)\pi}{2n} + (-1)^{a-b+1}\sin\frac{(a-b)\pi}{n}\sin\frac{(a+b)\pi}{2n}\} \\
&= \frac{8}{n}\sin\frac{(a-b)\pi}{2n}\sin\frac{(a+b)\pi}{2n}\{(-1)^{a+b+1}\cos\frac{(a+b)\pi}{2n} + (-1)^{a-b+1}\cos\frac{(a-b)\pi}{2n}\}
\end{aligned}$$

However, the parity of $(a+b)$ is always the same as parity of $(a-b)$, which leads us to

$$\begin{aligned}
N_1 &= (-1)^{a+b+1}\frac{8}{n}\sin\frac{(a-b)\pi}{2n}\sin\frac{(a+b)\pi}{2n}\{\cos\frac{(a+b)\pi}{2n} + \cos\frac{(a-b)\pi}{2n}\} \\
&= (-1)^{a+b+1}\frac{16}{n}\sin\frac{(a-b)\pi}{2n}\sin\frac{(a+b)\pi}{2n}\cos\frac{a\pi}{2n}\cos\frac{b\pi}{2n}
\end{aligned}$$

Similarly, the second multiplicand is,

$$
\begin{aligned}
N_2 &= \psi_{(a,b),(0,1)} - \psi_{(a,b),(1,0)} \\
&= \frac{2}{n}\{\cos\frac{a\pi}{2n}\cos\frac{3b\pi}{2n} - \cos\frac{3a\pi}{2n}\cos\frac{b\pi}{2n}\} \\
&= \frac{1}{n}\{\cos\frac{(a+3b)\pi}{2n} + \cos\frac{(a-3b)\pi}{2n} - \cos\frac{(3a+b)\pi}{2n} - \cos\frac{(3a-b)\pi}{2n}\} \\
&= \frac{2}{n}\{\sin\frac{(a+b)\pi}{n}\sin\frac{(a-b)\pi}{2n} + \sin\frac{(a-b)\pi}{n}\sin\frac{(a+b)\pi}{2n}\} \\
&= \frac{8}{n}\sin\frac{(a+b)\pi}{2n}\sin\frac{(a-b)\pi}{2n}\cos\frac{a\pi}{2n}\cos\frac{b\pi}{2n}
\end{aligned}
$$

The numerator of the term coresponding to (a,b) entry is then,

$$
\begin{aligned}
N_{(a,b)} &= N_1.N_2 \\
&= (-1)^{a+b+1}\frac{128}{n^2}\sin^2\frac{(a-b)\pi}{2n}\sin^2\frac{(a+b)\pi}{2n}\cos^2\frac{a\pi}{2n}\cos^2\frac{b\pi}{2n} \qquad (12)
\end{aligned}
$$

So, the value of V is,

$$
\begin{aligned}
V &= \sum_{\substack{0\le a<b\le n-1\\ a\neq b}} \frac{(-1)^{a+b+1}\frac{128}{n^2}\sin^2\frac{(a-b)\pi}{2n}\sin^2\frac{(a+b)\pi}{2n}\cos^2\frac{a\pi}{2n}\cos^2\frac{b\pi}{2n}}{4-2\cos(\frac{a\pi}{n})-2\cos(\frac{b\pi}{n})} \\
&= \sum_{0\le a<b\le n-1}(-1)^{a+b+1}\frac{32}{n^2}\sin^2\frac{(a-b)\pi}{2n}\sin^2\frac{(a+b)\pi}{2n}\cos^2\frac{a\pi}{2n}\cos^2\frac{b\pi}{2n}(1-\frac{\cos(\frac{a\pi}{n})+\cos(\frac{b\pi}{n})}{2})^{-1} \\
&= \sum_{0\le a<b\le n-1}(-1)^{a+b+1}\frac{32}{n^2}\sin^2\frac{(a-b)\pi}{2n}\sin^2\frac{(a+b)\pi}{2n}\cos^2\frac{a\pi}{2n}\cos^2\frac{b\pi}{2n}(1-\cos\frac{(a+b)\pi}{2n}\cos\frac{(a-b)\pi}{2n})^{-1} \\
&= \sum_{0\le a<b\le n-1}(-1)^{a+b+1}\frac{32}{n^2}\sin^2\frac{(a-b)\pi}{2n}\sin^2\frac{(a+b)\pi}{2n}\cos^2\frac{a\pi}{2n}\cos^2\frac{b\pi}{2n}\sum_{0\le i}\left(\cos\frac{(a+b)\pi}{2n}\cos\frac{(a-b)\pi}{2n}\right)^i \\
&= \sum_{0\le a<b\le n-1}\frac{32}{n^2}\sin{(a+b-.5)\pi}\sin^2\frac{(a-b)\pi}{2n}\sin^2\frac{(a+b)\pi}{2n}\cos^2\frac{a\pi}{2n}\cos^2\frac{b\pi}{2n}\sum_{0\le i}\left(\cos\frac{(a+b)\pi}{2n}\cos\frac{(a-b)\pi}{2n}\right)^i \qquad (13)
\end{aligned}
$$

An alternate expression can be obtained, where the pairings of terms corresponding to (a,b) and (b,a) is not done. We have,

$$
\begin{aligned}
\psi_{(a,b),(n-2,n-1)} &= \frac{2}{n}\cos\frac{(n-1.5)a\pi}{n}\cos\frac{(n-0.5)b\pi}{n} \qquad (14) \\
&= \frac{1}{n}\{\cos\frac{((a+b)n-(1.5a+0.5b))\pi}{n} + \cos\frac{((a-b)n-(1.5a-0.5b))\pi}{n}\} \\
&= \frac{1}{n}\{\cos\left((a+b)\pi-\frac{(3a+b)\pi}{2n}\right) + \cos\left((a-b)\pi-\frac{(3a-b)\pi}{2n}\right)\} \\
&= \frac{1}{n}\{(-1)^{a+b}\cos\frac{(3a+b)\pi}{2n} + (-1)^{a-b}\cos\frac{(3a-b)\pi}{2n}\} \qquad (15)
\end{aligned}
$$

again, using the fact that $a+b$ and $a-b$ have the same parity,

$$
\begin{aligned}
\psi_{(a,b),(n-2,n-1)} &= \frac{1}{n}\{(-1)^{a+b}\cos\frac{(3a+b)\pi}{2n} + (-1)^{a-b}\cos\frac{(3a-b)\pi}{2n}\} \\
&= \frac{(-1)^{a+b}}{n}\{\cos\frac{(3a+b)\pi}{2n} + \cos\frac{(3a-b)\pi}{2n}\} \\
&= \frac{2.(-1)^{a+b}}{n}\{\cos\frac{3a\pi}{2n}\cos\frac{b\pi}{2n}\}
\end{aligned} \tag{16}
$$

Referring back to the original form of equation for $V_{(n-2,n-1)}$,

$$
V_{(n-2,n-1)} = \sum_{0<a,b\leq n-1} \frac{\psi_{(a,b),(n-2,n-1)}(\psi_{(a,b),(0,1)} - \psi_{(a,b),(1,0)})^{\dagger}}{\lambda_{(a,b)}} \tag{17}
$$

which, after putting in values of different expressions, becomes

$$
\begin{aligned}
V_{(n-2,n-1)} &= \sum_{0<a,b\leq n-1} \frac{(-1)^{a+b}\frac{16}{n^2}\sin\frac{(a-b)\pi}{2n}\sin\frac{(a+b)\pi}{2n}\cos\frac{a\pi}{2n}\cos\frac{b\pi}{2n}\cos\frac{3a\pi}{2n}\cos\frac{b\pi}{2n}}{4-2\cos(\frac{a\pi}{n})-2\cos(\frac{b\pi}{n})} \\
&= \frac{16}{n^2}\sum_{0<a,b\leq n-1} \frac{\sin\frac{(a+b+1)\pi}{2}\sin\frac{(a-b)\pi}{2n}\sin\frac{(a+b)\pi}{2n}\cos\frac{a\pi}{2n}\cos\frac{b\pi}{2n}\cos\frac{3a\pi}{2n}\cos\frac{b\pi}{2n}}{4-2\cos(\frac{a\pi}{n})-2\cos(\frac{b\pi}{n})}
\end{aligned} \tag{18}
$$

3

Consider the dual of above stated problem. That is, we have an $n \times n$ grid of one ohm resistances. A one ampere current source-sink pair is connected to the neighbours of a corner vertex and we want estimates on the lower bound on the current flowing through one of the corner vertices adjacent to this one. If the combinatorial laplacian of a grid ghraph is denoted by L, the potentials that appear at each of the nodes because of the current flowing by vector v and let i be the vector containing *net currents* flowing **out** of any node into the network(which means if we are adding a current source of 1 A at some node, then the net current flowing out of that node into the network is 1). The equations of Kirchoff's laws at each node can be succinctly written as,

$$
Lv = i
$$

For now, we will refer to a node in the grid by its cartesian coordinates in its canonical embedding in euclidean plane. In our case, the vector i has all entries 0 except for a 1 at position corresponding to node $(0,0)$ and -1 for node $(0,1)$(i.e. current enters at node $(0,0)$ and leaves from $(0,1)$). We need to estimate the potential difference between nodes $(0,n-2)$ and $(0,n-1)$ given the values of A and i. Denote by $L(\epsilon)$ the matrix $L+\epsilon I$. Note that for small values of ϵ, unlike L, $L(\epsilon)$ is invertible. Call its inverse $G(\epsilon)$. If we denote the row of $G(\epsilon)$ corresponding to node $(0,n-1)$ by $G(\epsilon)_{(0,n-1)}$, then

$$
V_{(0,n-1)} = \lim_{\epsilon\longrightarrow 0}(G(\epsilon)_{(0,n-1)}.i) \tag{19}
$$

$$
V_{(0,n-2)} = \lim_{\epsilon\longrightarrow 0}(G(\epsilon)_{(0,n-2)}.i) \tag{20}
$$

Knowing the value i explicitly, we can write

$$
V_{(i,j)} = \lim_{\epsilon\longrightarrow 0}(G(\epsilon)_{(i,j)}(0,0) - G(\epsilon)_{(i,j)}(0,1))
$$

where $G(\epsilon)_{(i,j)}(x,y)$ is the entry in the row of the node (i,j) and column of node (x,y). The formula for $G(\epsilon)$ is,

$$G(\epsilon) = \Psi\Lambda(\epsilon)^{-1}\Psi^\dagger$$

where Ψ is the unitary matrix containing the eigenvectors of L (and consequently of $L(\epsilon)$ and $G(\epsilon)$) as its columns, $\Psi^\dagger$ is its hermitian and $\Lambda(\epsilon)$ is the diagonal matrix containing the eigenvalues of $L(\epsilon)$.Which means,

$$G(\epsilon)_{(p,q)}(i,j) = \frac{1}{n^2\epsilon} + \sum_{\substack{0\le a,b\le n-1\\(a,b)\neq(0,0)}} \frac{\psi_{(a,b),(p,q)}\psi^\dagger_{(a,b),(i,j)}}{\lambda_{(a,b)}+\epsilon}$$

where $\psi_{(a,b),(p,q)}$ is the entry of (p,q) node in the $(a,b)^{th}$ eigenvector. This leads us to,

$$G(\epsilon)_{(p,q)}(i_1,j_1) - G(\epsilon)_{(p,q)}(i_2,j_2) = \sum_{\substack{0\le a,b\le n-1\\(a,b)\neq(0,0)}} \frac{\psi_{(a,b),(p,q)}(\psi_{(a,b),(i_1,j_1)} - \psi_{(a,b),(i_2,j_2)})^\dagger}{\lambda_{(a,b)}+\epsilon}$$

or in particular,

$$V_{(0,n-1)} = \lim_{\epsilon\to 0}\left(\sum_{\substack{0\le a,b\le n-1\\(a,b)\neq(0,0)}} \frac{\psi_{(a,b),(0,n-1)}(\psi_{(a,b),(0,0)} - \psi_{(a,b),(0,1)})^\dagger}{\lambda_{(a,b)}+\epsilon}\right) \tag{21}$$

$$V_{(0,n-1)} = \sum_{\substack{0\le a,b\le n-1\\(a,b)\neq(0,0)}} \frac{\psi_{(a,b),(0,n-1)}(\psi_{(a,b),(0,0)} - \psi_{(a,b),(0,1)})^\dagger}{\lambda_{(a,b)}} \tag{22}$$

The potential difference we want takes the form,

$$V_{(0,n-1)} - V_{(0,n-2)} = \sum_{\substack{0\le a,b\le n-1\\(a,b)\neq(0,0)}} \frac{(\psi_{(a,b),(0,n-1)} - \psi_{(a,b),(0,n-2)})(\psi_{(a,b),(0,0)} - \psi_{(a,b),(0,1)})^\dagger}{\lambda_{(a,b)}} \tag{23}$$

We will now consider the general form of the eigenvalues and eigenvectors of a grid graph. The eigenvalues indexed by ordered pair (i,j) with $0 \le i,j \le n-1$ are,

$$\lambda_{i,j} = 4 - 2\cos(\frac{i\pi}{n}) - 2\cos(\frac{j\pi}{n}) \tag{24}$$

and the corresponding eigenvectors are,

$$\psi_{0,0} = \frac{1}{n}[111\ldots1] \tag{25}$$

$$\psi_{(i,j),(x,y)} = \frac{2}{n}\cos\frac{(x+1/2)i\pi}{n}\cos\frac{(y+1/2)j\pi}{n} \tag{26}$$

Now we consider the first multiplicand of the numerator of any summand.

$$N_1 \quad = \quad \psi_{(a,b),(0,n-1)} - \psi_{(a,b),(0,n-2)}$$

The form of eigenfunctions as shown in equation (26) implies that the numerator simplifies to,

$$
\begin{aligned}
N_1 &= \psi_{(a,b),(0,n-1)} - \psi_{(a,b),(0,n-2)} \qquad (27)\\
&= \frac{2}{n}\{\cos\frac{a\pi}{2n}\cos\frac{(n-.5)b\pi}{n} - \cos\frac{a\pi}{2n}\cos\frac{(n-1.5)b\pi}{n}\}\\
&= \frac{2}{n}\cos\frac{a\pi}{2n}\{\cos\left(b\pi - \frac{.5b\pi}{n}\right) - \cos\left(b\pi - \frac{1.5b\pi}{n}\right)\}\\
&= \frac{(-1)^b 2}{n}\cos\frac{a\pi}{2n}\{\cos\frac{b\pi}{2n} - \cos\frac{3b\pi}{2n}\}\\
&= \frac{(-1)^b 4}{n}\cos\frac{a\pi}{2n}\sin\frac{b\pi}{2n}\sin\frac{b\pi}{n}
\end{aligned}
$$

Similarly, the second multiplicand is,

$$
\begin{aligned}
N_2 &= \psi_{(a,b),(0,0)} - \psi_{(a,b),(0,1)}\\
&= \frac{2}{n}\{\cos\frac{a\pi}{2n}\cos\frac{b\pi}{2n} - \cos\frac{a\pi}{2n}\cos\frac{3b\pi}{2n}\}\\
&= \frac{2}{n}\cos\frac{a\pi}{2n}\{\cos\frac{b\pi}{2n} - \cos\frac{3b\pi}{2n}\}\\
&= \frac{4}{n}\cos\frac{a\pi}{2n}\sin\frac{b\pi}{2n}\sin\frac{b\pi}{n}
\end{aligned}
$$

The numerator of the term coresponding to (a,b) entry is then,

$$
\begin{aligned}
N_{(a,b)} &= N_1.N_2\\
&= (-1)^b\frac{16}{n^2}\cos^2\frac{a\pi}{2n}\sin^2\frac{b\pi}{2n}\sin^2\frac{b\pi}{n} \qquad (28)
\end{aligned}
$$

So, the value of V is,

$$
\begin{aligned}
V &= \sum_{\substack{0\le a,b\le n-1\\(a,b)\ne(0,0)}} \frac{(-1)^b\frac{16}{n^2}\cos^2\frac{a\pi}{2n}\sin^2\frac{b\pi}{2n}\sin^2\frac{b\pi}{n}}{4-2\cos(\frac{a\pi}{n})-2\cos(\frac{b\pi}{n})}\\
&= \frac{16}{n^2}\sum_{\substack{0\le a,b\le n-1\\(a,b)\ne(0,0)}} \frac{(-1)^b\cos^2\frac{a\pi}{2n}\sin^2\frac{b\pi}{2n}\sin^2\frac{b\pi}{n}}{4-2\cos(\frac{a\pi}{n})-2\cos(\frac{b\pi}{n})}\\
&= \frac{4}{n^2}\sum_{\substack{0\le a,b\le n-1\\(a,b)\ne(0,0)}} \frac{(-1)^b\cos^2\frac{a\pi}{2n}\sin^2\frac{b\pi}{2n}\sin^2\frac{b\pi}{n}}{\sin^2(\frac{a\pi}{2n})+\sin^2(\frac{b\pi}{2n})}
\end{aligned}
$$

Seperating out the summations we get

$$
V = \frac{8}{\pi n}\sum_{0\le b\le n-1} -1^b\sin^2\frac{b\pi}{2n}\sin^2\frac{b\pi}{n}\sum_{0\le a\le n-1}\frac{\cos^2\frac{a\pi}{2n}.(\pi/2n)}{\sin^2(\frac{a\pi}{2n})+\sin^2(\frac{b\pi}{2n})}
$$

consider the following function f,

$$
f = \frac{\cos^2(x)}{1+\sin^2(\frac{b\pi}{2n})-\cos^2(x)}
$$

It is a monotonically decreasing function of $cos(x)$ which in turn is a monotonically decreasing function of x. So f is a decreasing function of x. We will use this property to approximate the following summations value.

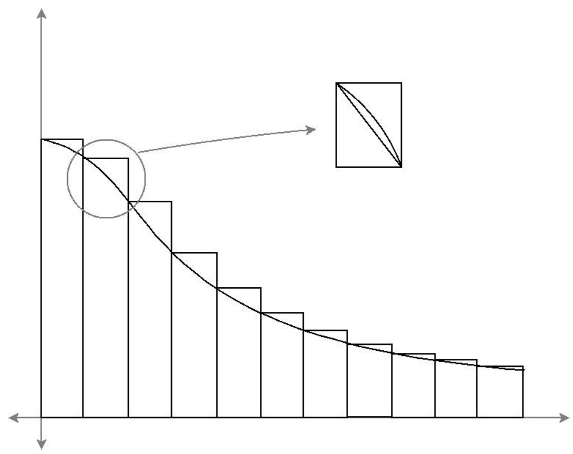

Abbildung 1: Approximating the sum

$$S = \sum_{0 \le a \le n-1} f(a\pi/2n)).(\pi/2n) = \sum_{0 \le a \le n-1} \frac{\cos^2 \frac{a\pi}{2n}.(\pi/2n)}{\sin^2(\frac{a\pi}{2n}) + \sin^2(\frac{b\pi}{2n})}$$

Note that both a and b cannot be zero simulatneously, so $f(0)$ is bounded by some $O(n^2)$ term. The bars in above figure are the summation and the smooth curve is the integral of f. The box shown above the graph displays the difference between the box and the curve. If the little box corresponds to the bar $a = k$, the height of box is $f(kh) - f((k+1)h)$ and width is $h = \pi/2n$ If we include half of the area of this box alongside the integral, then we make error equivalent to the little sliver between the curve and the diagonal. So we now have three terms, the integral, half the area of each of the boxes and the sum of each of the slivers. So, we get

$$\sum_{0 \le a \le n-1} \frac{\cos^2 \frac{a\pi}{2n}.(\pi/2n)}{\sin^2(\frac{a\pi}{2n}) + \sin^2(\frac{b\pi}{2n})} = \int_0^{\pi/2} \frac{\cos^2(x).dx}{\sin^2(x) + \sin^2(\frac{b\pi}{2n})} + \frac{f(0) - f(\pi/2)}{2}.\frac{\pi}{2n} + E(b)$$

where $E(b)$ is the error of approximating the summation by the integral and first order correction term. The general form of the integral involved is,

$$\int \frac{\cos^2(x).dx}{\sin^2(x) + K} = \sqrt{K^{-1}+1}.\arctan\left(\sqrt{K^{-1}+1}.\tan(x)\right) - x$$

(The formula can be verified by differentiation). Placing the appropriate limits, we get

$$\begin{aligned}
\int_0^{\pi/2} \frac{\cos^2(x).dx}{\sin^2(x) + K} &= \left[\sqrt{K^{-1}+1}.\arctan\left(\sqrt{K^{-1}+1}.\tan(x)\right) - x\right]_0^{\pi/2} \\
&= \left[\sqrt{K^{-1}+1}.\arctan\left(\sqrt{K^{-1}+1}.\tan(x)\right)\right]_{x=\pi/2} - \pi/2 \\
&= \sqrt{K^{-1}+1}.(\pi/2) - \pi/2 \\
&= (\sqrt{K^{-1}+1} - 1)(\pi/2)
\end{aligned}$$

The first order correction term is,

$$\frac{f(0) - f(\pi/2)}{2}.\frac{\pi}{2n} = \frac{K^{-1}\pi}{4n}$$

We have,

$$
\begin{aligned}
V &= \frac{8}{\pi n} \sum_{0 \le b \le n-1} -1^b \sin^2 \frac{b\pi}{2n} \sin^2 \frac{b\pi}{n} \left((\sqrt{K^{-1}+1} - 1)\frac{\pi}{2} + K^{-1}\frac{\pi}{4n} + E(b) \right) \\
&= \frac{8}{\pi n} \sum_{0 \le b \le n-1} -1^b \sin^2 \frac{b\pi}{2n} \sin^2 \frac{b\pi}{n} \left((K^{-1/2}\sqrt{1+K} - 1)\frac{\pi}{2} + K^{-1}\frac{\pi}{4n} + E(b) \right) \\
&= \frac{4}{n} \sum_{0 \le b \le n-1} -1^b \sin^2 \frac{b\pi}{2n} \sin^2 \frac{b\pi}{n} \left(K^{-1/2}(\sqrt{1+K} + K^{-1/2}\frac{1}{2n}) + E(b)\frac{2}{\pi} - 1 \right)
\end{aligned}
$$

Now, ignoring the last terms as insignificant compared to the leading term and putting in the value of $K = \sin(b\pi/2n)$ we get,

$$
V = \frac{4}{n} \sum_{1 \le b \le n-1} -1^b \sin \frac{b\pi}{2n} \sin^2 \frac{b\pi}{n} \left(\sqrt{1 + \sin^2 \frac{b\pi}{2n}} + \frac{1}{2n} \csc \frac{b\pi}{2n} \right)
$$

Note that we have changed the lower limit of the sum from 0 to 1 as the first term is zero anyway. The new limits help in dealing with the terms inside the brackets. much more easily.

For $1 \le b \le n-1$, we have

$$
1 \le \sqrt{1 + \sin^2 \frac{b\pi}{2n}} \le \sqrt{2}
$$

Being bounded between 1 and 1.414, the leading terms will stay almost unaffected by this term. So we simplify further to replace this by some constant c. Also, use the expansion of cosecant function,

$$
\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \ldots
$$

to get,

$$
V = \frac{4}{n} \sum_{1 \le b \le n-1} -1^b \sin \frac{b\pi}{2n} \sin^2 \frac{b\pi}{n} \left(c + \frac{1}{b\pi} + \frac{b\pi}{12n^2} + \ldots \right)
$$

first two terms are going to dominate all the remaining ones, so we neglect them,

$$
V = \frac{4}{n} \sum_{1 \le b \le n-1} -1^b \sin \frac{b\pi}{2n} \sin^2 \frac{b\pi}{n} \left(c + \frac{1}{b\pi} \right)
$$

4 Dealing with alternating sums

Consider a series of the form

$$
S = s_0 - s_1 + s_2 - s_3 + \ldots + s_{2k} - s_{2k-1}
$$

Note that the series we are going to consider always satisfy,

- begins with a positive number
- may or may not end with the a negative entry (here s_{2k+1})
- the terms come from a uniformly continuous function f such that $s_i = f(i)$

Without loss of generality, we will consider series with last term positive, the last negative term can then later on be appended to the approximating function of the remaining series. We will use the following notations,

$$\begin{aligned}
\Delta^1(k) &= \Delta(k) = f(k) - f(k-1) \\
\Delta^i(k) &= \Delta^{i-1}(k) - \Delta^{i-1}(k-1) \quad \texttt{for } k \geq i
\end{aligned}$$

Following properties of these Δ functions are going to be useful,

$$\begin{aligned}
\sum_{i=p}^{q} \Delta^k(i) &= \Delta^{k-1}(q) + \Delta^{k-1}(p-1) \\
\lim_{n\to\infty} \frac{\Delta^k(i)}{(\pi/n)^k} &= \frac{\delta^k f}{\delta x^k}
\end{aligned}$$

The second equation is true only in our present context where an increment by value 1 amounts to an increment in the value of x by π/n (or $\pi/2n$, as the case may be).

We start with,

$$\begin{aligned}
S &= f(0) + \sum_{i=1}^{k} \Delta(2i) \\
&= f(0) + \frac{1}{2}\sum_{i=1}^{2k} \Delta(i) + \frac{1}{2}\sum_{i=1}^{k} \Delta^2(2i) \\
&= f(0) + \frac{f(2k) - f(0)}{2} + \frac{1}{2}\Delta^2(2) + \frac{1}{2}\sum_{i=2}^{k} \Delta^2(2i) \\
&= \frac{f(2k) + f(0)}{2} + \frac{\Delta^2(2)}{2} + \frac{1}{4}\sum_{i=3}^{2k} \Delta^2(i) + \frac{1}{4}\sum_{i=2}^{k} \Delta^3(2i) \\
&= \frac{f(2k) + f(0)}{2} + \frac{\Delta^2(2)}{2} + \frac{\Delta^1(2k) - \Delta^1(2)}{4} + \frac{1}{4}\sum_{i=2}^{k} \Delta^3(2i) \\
&= \frac{f(2k) + f(0)}{2} + \frac{\Delta^2(2)}{2} + \frac{\Delta^1(2k) - \Delta^1(2)}{4} + \frac{1}{8}\sum_{i=3}^{2k} \Delta^3(i) + \frac{1}{8}\sum_{i=2}^{k} \Delta^4(2i) \\
&= \frac{f(2k) + f(0)}{2} + \frac{\Delta^2(2)}{2} + \frac{\Delta^1(2k) - \Delta^1(2)}{4} + \frac{\Delta^2(2k) - \Delta^2(2)}{8} + \frac{1}{8}\sum_{i=2}^{k} \Delta^4(2i) \\
&= \frac{f(2k) + f(0)}{2} + \frac{\Delta^2(2)}{2} + \frac{\Delta^1(2k) - \Delta^1(2)}{4} + \frac{\Delta^2(2k) - \Delta^2(2)}{8} + \frac{\Delta^4(4)}{8} + \frac{1}{8}\sum_{i=3}^{k} \Delta^4(2i) \\
&= \frac{f(2k) + f(0)}{2} + \frac{\Delta^2(2)}{2} + \frac{\Delta^1(2k) - \Delta^1(2)}{4} + \frac{\Delta^2(2k) - \Delta^2(2)}{8} + \frac{\Delta^4(4)}{8} + \frac{1}{16}\left(\sum_{i=5}^{2k} \Delta^4(i) + \sum_{i=3}^{k} \Delta^5(2i)\right) \\
&= \frac{f(2k) + f(0)}{2} + \frac{\Delta^2(2)}{2} + \frac{\Delta^1(2k) - \Delta^1(2)}{4} + \frac{\Delta^2(2k) - \Delta^2(2)}{8} + \frac{\Delta^4(4)}{8} + \frac{\Delta^3(2k) - \Delta^3(4)}{16} + \frac{1}{16}\sum_{i=3}^{k} \Delta^5(2i)
\end{aligned}$$

Opening up the deltas gives us,

$$
\begin{aligned}
f &= \frac{f(2k)+f(0)}{2} + \frac{\Delta^2(2)}{2} + \frac{\Delta^1(2k)-\Delta^1(2)}{4} + \frac{\Delta^2(2k)-\Delta^2(2)}{8} + \frac{\Delta^4(4)}{8} + \frac{\Delta^3(2k)-\Delta^3(4)}{16} \\
16f &= 8f(2k)+8f(0)+8\Delta^2(2)+4\Delta^1(2k)-4\Delta^1(2)+2\Delta^2(2k)-2\Delta^2(2)+2\Delta^4(4)+\Delta^3(2k)-\Delta^3(4) \\
16f &= 8f(2k)+4\Delta^1(2k)+2\Delta^2(2k)+\Delta^3(2k)+8f(0)+8\Delta^2(2)-4\Delta^1(2)-2\Delta^2(2)+2\Delta^4(4)-\Delta^3(4) \\
16f &= 8f(2k)+4\Delta^1(2k)+2\Delta^2(2k)+\Delta^2(2k)-\Delta^2(2k-1)+8f(0)+8\Delta^2(2)-4\Delta^1(2)-2\Delta^2(2)+2\Delta^3(4)-2\Delta^3(3)-\Delta^3(4) \\
16f &= 8f(2k)+4\Delta^1(2k)+3\Delta^2(2k)-\Delta^2(2k-1)+8f(0)+8\Delta^2(2)-4\Delta^1(2)-2\Delta^2(2)+\Delta^3(4)-2\Delta^3(3) \\
16f &= 8f(2k)+4\Delta^1(2k)+3\Delta^1(2k)-3\Delta^1(2k-1)-\Delta^1(2k-1)+\Delta^1(2k-2)+8f(0)+8\Delta^2(2)-4\Delta^1(2)-2\Delta^2(2)+\Delta^3(4)- \\
16f &= 8f(2k)+7\Delta^1(2k)-4\Delta^1(2k-1)+\Delta^1(2k-2)+8f(0)+8\Delta^2(2)-4\Delta^1(2)-2\Delta^2(2)+\Delta^3(4)-2\Delta^3(3) \\
16f &= 8f(2k)+7f(2k)-7f(2k-1)-4f(2k-1)+4f(2k-2)+f(2k-2)-f(2k-3)+8f(0)+8\Delta^2(2)-4\Delta^1(2)-2\Delta^2(2)+ \\
16f &= 15f(2k)-11f(2k-1)+5f(2k-2)-f(2k-3)+8f(0)+8\Delta^2(2)-4\Delta^1(2)-2\Delta^2(2)+\Delta^3(4)-2\Delta^3(3) \\
16f &= 15f(2k)-11f(2k-1)+5f(2k-2)-f(2k-3)+\ldots \\
& \quad 8f(0)+8\Delta^2(2)-4\Delta^1(2)-2\Delta^2(2)+\Delta^2(3)-\Delta^2(2)-2\Delta^2(3)+2\Delta^2(2) \\
16f &= 15f(2k)-11f(2k-1)+5f(2k-2)-f(2k-3)+8f(0)-4\Delta^1(2)+7\Delta^2(2)-\Delta^2(3) \\
16f &= 15f(2k)-11f(2k-1)+5f(2k-2)-f(2k-3)+8f(0)-4\Delta^1(2)+7\Delta^1(2)-7\Delta^1(1)-\Delta^1(3)+\Delta^1(2) \\
16f &= 15f(2k)-11f(2k-1)+5f(2k-2)-f(2k-3)+8f(0)+4\Delta^1(2)-7\Delta^1(1)-\Delta^1(3) \\
16f &= 15f(2k)-11f(2k-1)+5f(2k-2)-f(2k-3)+8f(0)+4f(2)-4f(1)-7f(1)+7f(0)-f(3)+f(2) \\
16f &= 15f(2k)-11f(2k-1)+5f(2k-2)-f(2k-3)+15f(0)-8f(1)+5f(2)-f(3)
\end{aligned}
$$

The summation of 5^{th} order differences are bound above,

$$
\begin{aligned}
\frac{1}{16}\sum_{i=3}^{k}\Delta^5(2i) &\leq \frac{k}{16}.\max_k \Delta^5(2i) \\
\frac{1}{16}\sum_{i=3}^{k}\Delta^5(2i) &\leq \frac{1}{16k^4}.\max_k \frac{\Delta^5(2i)}{1/k^5} \\
\frac{1}{16}\sum_{i=3}^{k}\Delta^5(2i) &\leq \frac{1}{16k^4}.\max_x \lim_{k\to\infty} \frac{\Delta^5(2i)}{1/k^5} \\
\frac{1}{16}\sum_{i=3}^{k}\Delta^5(2i) &\leq \frac{1}{16k^4}.\max_x \frac{\delta^5 f}{\delta x^5} \\
\frac{1}{16}\sum_{i=3}^{k}\Delta^5(2i) &\leq \frac{C}{16k^4}
\end{aligned}
$$