Af. limitele $\lim$, $\lim$ $f(x_1, x_2)$ ∀ $\lim$ $\lim$

② Det. pct. critice și pct. de extrem local (specificând și tipul acestora) pt. urm. fct.:

a) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = 2x^2 - xy + 2xz - y + y^3 + z^2$

b) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^4 + y^4 - \cancel{2}x^2$

I. pct. critice

$\nabla f(x,y,z) = (4x - y + 2z, -x - 1 + 3y^2, 2x + 2z)$

$$\nabla f(x,y,z) = 0_3 \Rightarrow \begin{cases} 4x - y + 2z = 0 \Rightarrow \cancel{2x + y = 0} \Leftrightarrow y = 2x \\ -x - 1 + 3y^2 = 0 \Rightarrow -x - 1 + 3 \cdot 4x^2 = 0 \\ 2x + 2z = 0 \Rightarrow z = -x \end{cases}$$

$\Rightarrow 4x - y - 2x = 0$

$2x - y = 0 \Rightarrow y = 2x$

$-x - 1 + 12x^2 = 12x^2 - x - 1 = 0$

$\Delta = 1 + 48 = 49$; $\sqrt{\Delta} = 7$

$$\left.\begin{array}{l} x_1 = \frac{1-7}{24} = \frac{-6}{24} = -\frac{1}{4},\ y = -\frac{1}{2},\ z = \frac{1}{4} \\ x_2 = \frac{8}{24} = \frac{1}{3},\ y = \frac{2}{3},\ z = -\frac{1}{3} \end{array}\right\} \Rightarrow$$

$\Rightarrow P_1 = \left(-\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}\right)$ pct. critice

$P_2 = \left(\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right)$

II. Natura pct. critice:

$$H(f)(x,y) = \begin{pmatrix} 4 & 1 & -2 \\ -1 & 6y & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$H(f)(P_2) = \begin{pmatrix} 4 & -1 & +2 \\ -1 & 6\frac{2}{3} & 0 \\ 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -1 & 2 \\ -1 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$\Delta_1 = |4| = 4 > 0$

$$\Delta_2 = \begin{vmatrix} 4 & -1 \\ -1 & 4 \end{vmatrix} = 16 - 1 = 15 > 0$$

$$\Delta_3 = \begin{vmatrix} 4 & -1 & 2 \\ -1 & 4 & 0 \\ 2 & 0 & 2 \end{vmatrix} = 32-16-2=14>0$$

$d^2 f(P_2)$ pozitiv definită $\Rightarrow P_2$ pct. de minim local

$$H(f)(p_1) = \begin{pmatrix} 4 & -1 & 2 \\ -1 & 6\cdot\left(-\frac{1}{2}\right) & 0 \\ 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -1 & 2 \\ -1 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$\Delta_1 = |4| > 0$

$$\Delta_2 = \begin{vmatrix} 4 & -1 \\ -1 & -3 \end{vmatrix} = -12-1 = -13 < 0$$

$\Rightarrow$ criteriul lui Sylvester nu se aplică

$d^2(f)(p_1)(u_1,u_2,u_3) = 4u_1^2 - 3u_2^2 + 2u_3^2 + 2\cdot(-1)\cdot u_1u_2 + 2\cdot 2\cdot u_1u_3 = 4u_1^2 - 3u_2^2 + 2u_3^2 - 2u_1u_2 + 4u_1u_3$

$d^2(f)(p_1)(1,0,0) = 4 > 0$

$d^2(f)(p_1)(0,1,0) = -3 < 0$

$\Rightarrow d^2(f)(p_1)$ indefinită $\Rightarrow$

$\Rightarrow p_1$ pct. şa

b) I. $\nabla f(x,y) = (4x^3-4x, 4y^3)$

$$\nabla f = 0_2 \Rightarrow \begin{cases} 4x^3-4x=0 \Rightarrow x^3-x=0 \Leftrightarrow x(x^2-1)=0 \\ 4y^3=0 \Rightarrow y=0 \end{cases}$$

$x(x+1)(x-1)=0$

$x_1=0,\ x_2=-1,\ x_3=1$

$\Rightarrow (0,0), (-1,0), (1,0)$ pct. critice

II. $$H(f)(x,y) = \begin{pmatrix} 12x^2-4 & 0 \\ 0 & 12y^2 \end{pmatrix}$$

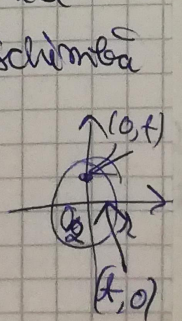
$H(f)(0,0) = \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix}$, $\Delta_2 = 0$ (nu avem crit. lui Sylvester)

$d^2 f(0,0)(u_1,u_2) = -4u_1^2 \le 0$ (Nu este neg. def, nici indefinită – nu-şi schimbă semnul)

$\Rightarrow$ teorema nu se aplică

$f(0,0) = 0$

$f(0,t) = t^4 > 0,\ t \neq 0$; $f(t,0) = t^4 - 2t^2$



[illegible] $\Rightarrow$

$= t^2(t-\sqrt{2})(t+\sqrt{2}) < 0, \forall t \in (0, \sqrt{2})$

$\Rightarrow (0,0)$ pct. șa

$H(f)(\pm 1, 0) = \begin{pmatrix} 8 & 0 \\ 0 & 0 \end{pmatrix}$, $D_2 = 0$ (nu aplic. crit. lui Sylvester)

$d^2 f(\pm 1, 0)(u_1, u_2) = 8u_1^2 \geq 0$, nu este poz. def, nici indef. (nu ia val. neg.)

$\Rightarrow$ teorema nu se aplică

$f(\pm 1, 0) = -1$

$f(x,y) = x^4 + y^4 - 2x^2 + 1 - 1 = (x^2-1)^2 + y^4 - 1 \geq -1, \forall (x,y) \in \mathbb{R}^2$

$\Rightarrow (1,0), (-1,0)$ pct. de minim (global)

(35) Det. pct. de extrem condiționat (specificând tipul acestora) și val. extreme ale fct. f relativ la mulț. S indicată (știind că aceasta este compactă)

$f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = (1-x)(1-y)$, $S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

f continuă, S compactă $\xRightarrow{T.W.}$ f își atinge val. extreme $\Longrightarrow$ f are pct. de minim și maxim cond. rel. la S

$F(x,y) = x^2 + y^2 - 1 \Rightarrow S = \{(x,y) \in \mathbb{R} \mid F(x,y) = 0\}$

$L(x,y,\mu) = f(x,y) + \mu F(x,y) = (1-x)(1-y) + \mu(x^2+y^2-1) =$

$= 1 - y - x + xy + \mu x^2 + \mu y^2 - \mu$

$L(x,y,\mu) = (-1 + y + 2\mu x, -1 + x + 2\mu y, x^2 + y^2 - 1) = 0$,

$$\begin{cases} -1 + y + 2\mu x = 0 \\ -1 + x + 2\mu y = 0 \\ x^2 + y^2 - 1 = 0 \end{cases} \overset{\ominus}{\Rightarrow} y - x + 2\mu(x-y) = 0$$

$(x-y)(-1+2\mu) = 0$

I. $\underbrace{x-y}_{=0} \Rightarrow x = y$

$x^2 + x^2 - 1 = 0 \Rightarrow 2x^2 = 1$; $x = \pm\frac{\sqrt{2}}{2} = y$

$\mu_1 = \frac{1+\sqrt{2}}{2}$, $\mu_2 = \frac{1-\sqrt{2}}{2}$

II. $2\mu - 1 = 0 \Rightarrow \mu = \frac{1}{2}$

$-1 + y + x = 0$

$\Rightarrow (1-y)^2 + y^2 - 1 = 0$

$\Rightarrow (1-y)^2 + y^2 - 1 = 0$

$1 - 2y + y^2 + y^2 - 1 = 0$

$2y^2 - 2y = 0$

$2y(y-1) = 0 \Rightarrow y_1 = 0 \Rightarrow x_1 = 1$

$y_2 = 1 \Rightarrow x_2 = 0$

$\Rightarrow \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1+\sqrt{2}}{2}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1-\sqrt{2}}{2}\right), \left(1, 0, \frac{1}{2}\right), \left(0, 1, \frac{1}{2}\right)$

pct. critice ale lui L

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \left(1 - \frac{1}{\sqrt{2}}\right)\left(1 - \frac{1}{\sqrt{2}}\right) = \left(1 - \frac{1}{\sqrt{2}}\right)^2$

$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \left(1 + \frac{1}{\sqrt{2}}\right)^2 = \max \left(1 + \frac{1}{\sqrt{2}}\right)^2$

$f(1,0) = 0$ = min lui ~~f pe S~~ $f(S)$

$f(0,1) = 0$

$\Rightarrow (1,0), (0,1)$ pct. de min. condiționat

$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ pct. de max. condiționat

4) Det. val. extreme ale urm. fct. relativ la mulț. S indicată:

a) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x + 2y + 3z$; $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$

S compactă, f cont. $\overset{T.W.}{\Rightarrow}$ f are pct. de min și max cond. relative la S

~~S compactă~~

S închisă $\Rightarrow S = (\text{int } S) \cup (\text{fr } S)$ (interiorul lui S reunit cu frontiera)

I. $(x,y) \in \text{int } S$

$\nabla f(x,y,z) = (1,2,3) \neq 0_3 \Rightarrow$ f nu are pct. critice $\Rightarrow$

$\Rightarrow$ f nu are pct. de extrem în int S

II. $(x,y) \in \text{fr } S$; $\text{fr } S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$

$F(x,y,z) = x^2 + y^2 + z^2 - 1$

$L(x,y,z,\lambda) = f(x,y,z) + \lambda F(x,y,z) = x + 2y + 3z +$
$+ \lambda(x^2 + y^2 + z^2 - 1)$

$\nabla L = 0_4 \Rightarrow \begin{cases} 1 + 2\lambda x = 0 & \Rightarrow x = -\frac{1}{2\lambda} \\ 2 + 2\lambda y = 0 & \Rightarrow y = -\frac{2}{2\lambda} \\ 3 + 2\lambda z = 0 & \Rightarrow z = -\frac{3}{2\lambda} \\ x^2 + y^2 + z^2 = 1 \end{cases}$

$\Rightarrow \left(-\frac{1}{2\lambda}\right)^2 + \left(-\frac{2}{2\lambda}\right)^2 + \left(-\frac{3}{2\lambda}\right)^2 = 1$

$\Rightarrow 2\lambda = \pm\sqrt{14}$

$\Rightarrow \left(\frac{-1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \sqrt{14}\right), \left(\frac{+1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, -\sqrt{14}\right)$ pct. critice ale lui L

$f\left(\frac{-1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right) = -\sqrt{14} = \min f(\mathcal{J} \cap S) = \min f(S)$

$\sqrt{14} = \max f(\mathcal{J} \cap S) = \max f(S)$