$$x_n > \frac{x_n}{1+x_n}, \forall n \in \mathbb{N}$$

Dacă $\sum_{n=1}^{\infty} x_n$ converg $\Rightarrow \sum_{n=1}^{\infty} \frac{x_n}{1+x_n}$ converg (*)

De arătat că dacă $\sum \frac{x_n}{1+x_n}$ conv. $\Rightarrow \sum_{n=1}^{\infty} x_n$ conv?? (temă)

Seminar 5

1. Justificați afirmațiile:

i) $\frac{1}{n+1} < \ln(n+1) - \ln n < \frac{1}{n}, \forall n \in \mathbb{N}^*$

i) Fie $f(x) = \ln x, \forall x > 0$ și aplicăm Th. de medie Lagrange pe $[n, n+1], \forall n \in \mathbb{N}^*$

$\Rightarrow \exists\, c \in (n, n+1)$ a.î. $f'(c) = \frac{f(n+1) - f(n)}{n+1-n} = \frac{\ln(n+1) - \ln n}{1} =$

$= \ln(n+1) - \ln n$

$f'(c) = \frac{1}{c}$, $c \in (n, n+1) \Rightarrow \frac{1}{c} \in \left(\frac{1}{n+1}, \frac{1}{n}\right) \Rightarrow \ln(n+1) - \ln n$

$\Rightarrow \ln(n+1) - \ln n \in \left(\frac{1}{n+1}, \frac{1}{n}\right) \Rightarrow$ i)

ii) $c_{n+1} - c_n = 1 + \frac{1}{2} + \dots + \frac{1}{n+1} - \ln(n+1) - \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n\right) = \frac{1}{n+1} - \ln(n+1) + \ln n < 0$ (cf. i)

$\Rightarrow c_n$ descresc.

$\ln 2 - \ln 1 < 1$

$\ln 3 - \ln 2 < \frac{1}{2}$

$\dots$

$\ln(n+1) - \ln n < \frac{1}{n}$ $(+)$

$\ln(n+1) - \underbrace{\ln 1}_{0} < 1 + \frac{1}{2} + \dots + \frac{1}{n} \Big| - \ln n$

$\ln(n+1) - \ln n < \underbrace{1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n}_{c_n}$, $\forall n \geq 1$

$\Rightarrow 0 <$

$\Rightarrow (c_n)$ mărginit. $\Rightarrow (c_n)$ convergent

Notăm $\lim\limits_{n \to \infty} c_n \overset{not}{=} \gamma$, $\gamma \approx 0{,}57$ const. lui Euler

Ex: Natura s.t.p. $\sum\limits_{n=1}^{\infty} a^{1 + \frac{1}{2} + \dots + \frac{1}{n}}$, $a > 0$

pt. $a = \frac{1}{e}$: $\sum\limits_{n=1}^{\infty} \left(\frac{1}{e}\right)^{1 + \frac{1}{2} + \dots + \frac{1}{n}} \sim \sum\limits_{n=1}^{\infty} \frac{1}{n}$ (au aceeași nat.) $\frac{1}{e^{\ln n}} = \sum\limits_{n=1}^{\infty} \left(\frac{1}{e}\right)^{\ln n}$

C.C. $\lim\limits_{n \to \infty} \frac{\left(\frac{1}{e}\right)^{1 + \frac{1}{2} + \dots + \frac{1}{n}}}{\left(\frac{1}{e}\right)^{\ln n}} = \lim\limits_{n \to \infty} \left(\frac{1}{e}\right)^{c_n} = \left(\frac{1}{e}\right)^{\gamma} \in (0, +\infty)$

$\Rightarrow$ seriile au aceeași natură $\Rightarrow$ serie ~~diverg~~ divergentă

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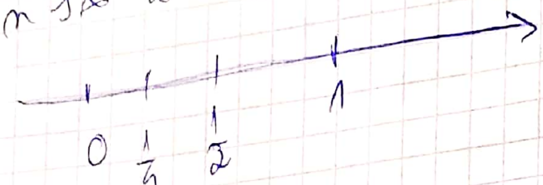
② Determinați mult. pct. de ...

a) $A = \left\{ \frac{1}{2^n} / n \in \mathbb{N} \right\}$

$\lim\limits_{n \to \infty} \frac{1}{2^n} = 0 \Rightarrow 0 \in A'$ $\Rightarrow A' = \{0\}$, $\forall x \in \overline{\mathbb{R}}, \exists (x_n) \subset ...$ cu $\lim\limits_{n\to\infty} x_n = x$



b) $n \in \mathbb{N}$, $x_n = n \to +\infty \Rightarrow +\infty \in A'$ $\Rightarrow A' = \overline{\mathbb{R}}$

$x_n = \frac{1}{2^n} \to 0 \Rightarrow 0 \in A'$

③ Verificați dacă fct. urm. își ating val. extreme și det. aceste val.:

$f: A \to \mathbb{R}$, $f(A)$ – img.
$\inf f(A)$, $\sup f(A)$ } val. extreme
se ating dacă $\exists x_1, x_2 \in A$
$f(x_1) = \inf f(A) = \min f(A)$
$f(x_2) = \sup f(A) = \max f(A)$

a) $f: (-1,1) \to \mathbb{R}$, $f(x) = \ln \frac{1-x}{1+x}$

$A = (-1, 1)$ – $\boxed{\text{Th. lui Weierstrass nu se aplică}}$

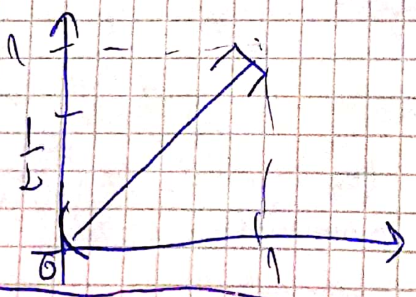
$\lim\limits_{\substack{x \nearrow 1 \\ (st)}} f(x) = \lim\limits_{\substack{x \nearrow 1 \\ (drept)}} \ln \frac{1-x}{1+x} =$

$= \ln \lim\limits_{x \nearrow 1} \frac{1-x}{1+x} = \ln 0_+ = -\infty$

$\lim\limits_{x \searrow -1} f(x) = \lim\limits_{x \searrow -1} \ln \frac{1-x}{1+x} = \ln \frac{2}{0_+} = \ln \infty = +\infty$

$\Rightarrow f(A) = \mathbb{R}$, $\inf f(A) = -\infty$ – nu se ating
$\sup f(A) = +\infty$

b) $f: [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} \frac{1}{2}, & x = 0 \\ x, & x \in (0,1] \end{cases}$



$A = [0, 1]$

$f(A) = (0, 1]$

~~$\inf f(A) =$~~

$\inf f(A) = 0$ – nu se atinge

$\sup f(A) = 1 = f(1)$

$\boxed{\text{Nu se poate apl. W. pt. că nu e cont. în 0 (la dreapta lat.)}}$

c) $f:[-1,1]\to\mathbb{R}$, $f(x)=x\sqrt{1-x^2}$

f este cont. pe $[-1,1] \xRightarrow{T.W.}$ f își atinge val. extreme pe $[-1,1]$

$x\in[-1,1] \Rightarrow x=\sin t,\ t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$f(x)=f(\sin t)=\sin t\cdot\sqrt{1-\sin^2 t}=\sin t\cdot|\underbrace{\cos t}_{\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)}|=$

$=\sin t\cos t=\frac{1}{2}\sin(2t)\in\left[-\frac{1}{2},\frac{1}{2}\right]$

$A=[-1,1]$

$\Rightarrow f(A)=\left[-\frac{1}{2},\frac{1}{2}\right]\Rightarrow \inf f(A)=-\frac{1}{2}=f\left(-\frac{\sqrt{2}}{2}\right),\ \sup f(A)=\frac{1}{2}=f\left(\frac{\sqrt{2}}{2}\right)$

$\sin(2t)=1\Rightarrow 2t=\frac{\pi}{2}\Rightarrow t=\frac{\pi}{4},\ x=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$

4) $f:(a,b)\to\mathbb{R}$ fct. derivabilă pe (a,b). Au loc. af.

a) f cresc. pe (a,b) $\Leftrightarrow f'(x)\geq 0,\ \forall x\in(a,b)$

b) f descresc. pe (a,b) $\Leftrightarrow f'(x)\leq 0,\ \forall x\in(a,b)$

c) ~~st. descresc~~ Dacă $f'(x)>0,\ \forall x\in(a,b)\Rightarrow f$ st. cresc.

d) Dacă $f'(x)<0,\ \forall x\in(a,b)\Rightarrow f$ st. descresc. pe (a,b)

a) f cresc pe $(a,b)\Leftrightarrow\forall x,y\in(a,b)$ cu $x<y$: $f(x)\leq f(y)$

"$\Rightarrow$" $f'(x)=\lim_{y\to x}\underbrace{\frac{f(y)-f(x)}{y-x}}_{\geq 0}\geq 0,\ \forall x\in(a,b)$

~~"$\Leftarrow$" avem că f cresc. pe (a,b): $x<y$, $x,y\in(a,b)$~~

$\boxed{\text{"}\Leftarrow\text{"}}$ avem că $f'(x)\geq 0,\ \forall x\in(a,b)$, fie $y>x$

Aplicăm th. de medie a lui Lagr. pe $[x,y]$: $\exists c\in(x,y)$:

$f'(c)=\frac{f(y)-f(x)}{y-x}\geq 0\Rightarrow f(y)-f(x)\geq 0\Rightarrow f$ cresc.

Analog b)

pt c) d) $f'(x)>0\Rightarrow f$ st. cresc./descresc.

Exemple de fct. st. cresc., dar derivata nu e st. poz

$f(x)=x^3, \forall x\in\mathbb{R}, f'(0)=0, f'\not>0$

$x<y \Rightarrow x^3<y^3$

5

a) $f:(-1,1)\to\mathbb{R}, f(x)=\ln\frac{1-x}{1+x}=\ln(1-x)-\ln(1+x)$

f derivabilă pe $(-1,1)$, $f'(x)=\frac{-1}{1-x}-\frac{1}{1+x}=\frac{-(1+x)-(1-x)}{1-x^2}=$

$=\frac{-2}{1-x^2}$

$f'(x)\neq 0, \forall x\in(-1,1) \Rightarrow$ ~~[illegible]~~ f nu are pct. de extrem local

b) $f:[0,1]\to\mathbb{R}, f(x)=\begin{cases}\frac{1}{2}, & x=0\\ x, & x\in(0,1]\end{cases}$

$\forall x\in(0,1), f'(x)=1\neq 0 \Rightarrow f$ nu are pct. de extrem în $(0,1)$

$x=0, x=1$

$x=1, 1=f(1)\geq f(x), \forall x\in[0,1] \Rightarrow x=$

$x_0=0$, pct. de maxim local $\Leftrightarrow \exists\,\delta>0$ a.î.

$\forall x\in(-\delta,\delta)\cap[0,1]: \frac{1}{2}=f(0)\geq f(x)$

Alegem $\delta=\frac{1}{2}$

$\forall x\in[0,\frac{1}{2})$; $\frac{1}{2}\geq f(x) \Rightarrow x=0$ pct. de maxim local

c) $f:[-1,1]\to\mathbb{R}, f(x)=x\sqrt{1-x^2}$

f derivabilă pe $(-1,1)$, $f'(x)=\sqrt{1-x^2}+x\cdot\frac{-2x}{2\sqrt{1-x^2}}=$

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$= \sqrt{1-x^2} - \frac{x}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}$

$f'(x)=0 \iff 1-2x^2=0 \iff x^2=\frac{1}{2} \iff x=\pm\sqrt{\frac{1}{2}} \in (-1,1)$

$= \pm\frac{\sqrt{2}}{2}$

x	-1		$-\frac{\sqrt{2}}{2}$		$\frac{\sqrt{2}}{2}$		1
f'		$- \; -$	0	$+ \; +$	0	$- \; - \; - \; -$	
f	0	$\searrow$	$f(-\frac{\sqrt{2}}{2})$	$\nearrow$	$f(\frac{\sqrt{2}}{2})$	$\searrow$	0

$x=-\frac{\sqrt{2}}{2}$, $x=1$ } pct. de minim local

$x=\frac{\sqrt{2}}{2}$, $x=-1$ } pct. de maxim local

⑤ Folosind regula lui L'Hospital, calc. lim.

a) $\lim_{x \to 0} \frac{e-(1+x)^{\frac{1}{x}}}{x}$

$\left[(1+x)^{\frac{1}{x}}\right]' = \left[e^{\ln(1+x)^{\frac{1}{x}}}\right]' = \left[e^{\frac{1}{x}\cdot\ln(1+x)}\right]' = e^{\frac{1}{x}\ln(1+x)} \cdot$

$\cdot \left[\frac{1}{x}\ln(1+x)\right]' = (1+x)^{\frac{1}{x}} \cdot \left[-\frac{1}{x^2}\ln(1+x) + \frac{1}{x(x+1)}\right]$

$\lim_{x \to 0} \frac{e-(1+x)^{\frac{1}{x}}}{x} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \underbrace{-(1+x)^{\frac{1}{x}}}_{} \left[-\frac{1}{x^2}\cdot\ln(x+1) + \frac{1}{x(x+1)}\right] =$

$= -e \cdot \lim_{x\to 0} \frac{-(x+1)\ln(x+1)+x}{x^2+x^3} \overset{\frac{0}{0}}{=} -e \lim_{x\to 0} \frac{-\ln(x+1)-1+1}{2x+3x^2}$

b) $\lim_{x\to\infty} \frac{x^2}{e^x}$, $\alpha \in \mathbb{R}$, parametru

$\lim_{x\to\infty} \frac{x^2}{e^x} \overset{\frac{\infty}{\infty} \, l'H}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{\frac{\infty}{\infty} \, l'H}{=} \lim_{x\to\infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$