Nu verifică ec. => pp. e falsă => ec. nu are răd. rat.
=> $\alpha \notin \mathbb{Q}$

Semінar 2

(4) $\lim_{n\to\infty} \sqrt{n}(\sqrt{n+1}-\sqrt{n}) =$

(1) a) $\sqrt{n} \cdot \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \overset{\lim}{=} \frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\left(\sqrt{1+\underbrace{\frac{1}{n}}_{\to 0}}+1\right)} \overset{\lim}{=} \frac{1}{1+1} = \frac{1}{2}$

b) $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{\cancel{n}\left(1+\frac{\sin n}{n}\right)}{\cancel{n}\left(1+\frac{\cos n}{n}\right)} = 1$

$-1 \le \sin n \le 1 \mid \cdot \frac{1}{n}$

$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$

$\searrow \; \downarrow \; \swarrow$

0

c) $x_n = \frac{(\sqrt{2}+1)^n}{(\sqrt{2})^n+1}$

$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{(\sqrt{2}+1)^n}{(\sqrt{2})^n\left[1+\left(\frac{1}{\sqrt{2}}\right)^2\right]} = \lim_{n\to\infty} \underbrace{\left(1+\frac{1}{\sqrt{2}}\right)^n}_{\to\infty}$

$\cdot \frac{1}{1+\underbrace{\left(\frac{1}{\sqrt{2}}\right)^n}_{\to 0}} = +\infty$

(2) Justificați cu def. val. limitelor:

a) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ a.î.

$\forall n \ge n_0 : \left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon \Rightarrow \left|\frac{1}{\sqrt{n}}\right| < \varepsilon \Rightarrow$

$\Rightarrow \frac{1}{\sqrt{n}} < \varepsilon \Rightarrow \sqrt{n} > \frac{1}{\varepsilon} \Rightarrow n > \frac{1}{\varepsilon^2}$

alegem $n_0 = \left[\frac{1}{\varepsilon^2}\right] + 1 > \frac{1}{\varepsilon^2}$

b) $\lim\limits_{n\to\infty} \frac{n^2}{n+1} = +\infty$

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ a.î. $\forall n \geq n_0 : \frac{n^2}{n+1} > \varepsilon$

$\frac{n^2}{n+1} = \frac{n^2-1+1}{n+1} = \frac{(n-1)\cancel{(n+1)}}{\cancel{n+1}} + \frac{1}{n+1} = n-1+\frac{1}{n+1} >$

$> n-1 \overset{\text{punem}}{>} \varepsilon \Rightarrow n > \varepsilon + 1$

alegem $n_0 = [\varepsilon] + 2$

3. Convergența șirului și calc. limitei (prin metodele: mărginire și monotonie, criteriul cleștelui, subșirul, șir fundamental)

a) $x_n = a^n, a \in \mathbb{R}$

$\lim\limits_{n\to\infty} a^n = 0, \forall a \in (-1, 1)$

$$\lim_{n\to\infty} a^n = \begin{cases} 0, a \in (-1,1) \\ 1, a = 1 \\ +\infty, a > 1 \\ \nexists, a \leq -1 \end{cases} ; a = \frac{1}{b}; b \in (0,1)$$

$a = -2$; $\lim\limits_{n\to\infty} (-2)^n \nexists$

$\lim\limits_{k\to\infty} (-2)^{2k} = +\infty$

$\lim\limits_{k\to\infty} (-2)^{2k+1} = -\infty$ $\neq$

b) $x_n = \frac{2^n}{n!}$

$$\frac{x_{n+1}}{x_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2}{n+1} < 1, \forall n \geq 1$$

$\Rightarrow (x_n)$ descresc.

$x_n > 0, \forall n \in \mathbb{N} \Rightarrow (x_n)$ mărg. inf. $\Bigg| \Rightarrow (x_n)$ convergent

$\Rightarrow \exists L = \lim\limits_{n\to\infty} x_n \in \mathbb{R}$

$x_{n+1} = \frac{2}{n+1} \cdot x_n, \forall n \in \mathbb{N} \Rightarrow \lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} \frac{2}{n+1} \cdot \lim\limits_{n\to\infty} x_n \Rightarrow$

$\Rightarrow L = 0 \cdot \underbrace{L}_{\text{nr. finit}} = 0$

c) $x_n = \sqrt[n]{n}$

$y_n = \sqrt[n]{n} - 1 \geq 0, \forall n \geq 1$

$1 + y_n = \sqrt[n]{n} \Rightarrow (1+y_n)^n = n \Rightarrow$

$\Rightarrow 1 + C_n^1 \cdot y_n + C_n^2 \cdot y_n^2 + \ldots + C_n^n y_n^n = n, \ n \geq 2$

$\Rightarrow \frac{\cancel{n}(n-1)}{2} \cdot y_n^2 < \cancel{n}, \Rightarrow y_n^2 < \frac{2}{n-1} \Rightarrow \underset{0 \leq}{y_n} < \sqrt{\frac{2}{n-1}} \xrightarrow{n\to\infty} 0$

$\Rightarrow \lim\limits_{n\to\infty} y_n = 0 \Rightarrow \lim\limits_{n\to\infty} x_n = 1$

d) $x_n = \left(1 + \frac{1}{n}\right)^n$

$x_n = 1 + C_n^1 \cdot \frac{1}{n} + C_n^2 \cdot \frac{1}{n^2} + \ldots + C_n^n \cdot \frac{1}{n^n} = 1 +$

$+ n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \ldots +$

$+ \frac{n(n-1)\ldots(n-n+1)}{n!} \cdot \frac{1}{n^n} = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}$

$\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{n-1}{n}\right)$ (*)

$$x_{n+1} = 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \dots +$$
$$+ \underbrace{\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)}_{>0} \quad (\Rightarrow)$$
$$\frac{1}{n!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)$$

$(\Rightarrow)$ $x_n < x_{n+1}, \forall n \in \mathbb{N}^* \Rightarrow (x_n)$ cresc.

$$x_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} \quad (1)$$

$$\frac{1}{n!} = \frac{1}{1\cdot 2\cdot 3\cdot \ldots \cdot n} < \frac{1}{\underbrace{1\cdot 2\cdot 2\cdot \ldots \cdot 2}_{n-1}} = \frac{1}{2^{n-1}}, \forall n \geq 3$$

$$(1) \; 1 + \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 1 + 2\cdot\underbrace{\left(1-\frac{1}{2^n}\right)}_{<1} < 3, \forall n \geq 1 \Rightarrow$$

$\Rightarrow (x_n)$ mărg. sup. $\Rightarrow (x_n)$ convergent

$\rightarrow$ Notăm $\lim\limits_{n\to\infty} x_n = e \simeq 2,71$ (nr. Euler)

e) $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}$

$\left.\begin{array}{l} x_n < 3, \forall n \in \mathbb{N} \\ (x_n) \text{ cresc.} \end{array}\right\} \Rightarrow (x_n)$ convergent

Notăm $L = \lim\limits_{n\to\infty} x_n$

Fie $y_n = \left(1+\frac{1}{n}\right)^n \to e$

$\Rightarrow y_n \leq x_n, \forall n \geq 1 \xRightarrow{n\to\infty} \boxed{e \leq L}$

Fie $p \in \mathbb{N}^*$ fixat și $n \in \mathbb{N}$, $n > p$

$\xRightarrow{(\alpha)} y_n > 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \dots + \frac{1}{p!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{p-1}{n}\right)$

$\xRightarrow{n\to\infty} e \geq 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{p!} = x_p, \forall p \in \mathbb{N}^*$

$\xRightarrow{p\to\infty} \boxed{e \geq L} \Rightarrow \underline{L = e}$

f) $x_n = \frac{\sin(1!)}{1\cdot 2} + \frac{\sin(2!)}{2\cdot 3} + \dots + \frac{\sin(n!)}{n(n+1)}$ → Temă: exemplu numeric

(x_n) fundamental $\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ a.î. $\forall n \geq n_0$, $\forall p \in \mathbb{N}: |x_{n+p} - x_n| < \varepsilon$

$$|x_{n+p} - x_n| = \left| \frac{\sin(1!)}{1\cdot 2} + \frac{\sin(2!)}{2\cdot 5} + \dots + \frac{\sin(n+p)!}{(n+p)(n+p+1)} - \left(\frac{\sin 1!}{1\cdot 2} + \frac{\sin 2!}{2\cdot 3} + \dots + \frac{\sin n!}{n(n+1)}\right) \right| =$$

$$= \left| \frac{\sin(n+1)!}{(n+1)(n+2)} + \dots + \frac{\sin(n+p)!}{(n+p)(n+p+1)} \right| < \frac{|\sin(n+1)!|}{(n+1)(n+2)} + \dots + \frac{|\sin(n+p)!|}{(n+p)(n+p+1)} \leq \frac{1}{(n+1)(n+2)} + \dots + \frac{1}{(n+p)(n+p+1)} =$$

$$= \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\frac{1}{n+2} - \frac{1}{n+3}\right) + \dots + \left(\frac{1}{n+p} - \frac{1}{n+p+1}\right) =$$

$$= \frac{1}{n+1} - \frac{1}{n+p+1} < \frac{1}{n+1} \underset{\text{punem}}{<} \varepsilon \Rightarrow n+1 > \frac{1}{\varepsilon} \Rightarrow n > \frac{1}{\varepsilon}$$

→ alegem ~~ε~~ $n_0 = \left[\frac{1}{\varepsilon}\right] \Rightarrow (x_n)$ fundamental $\Rightarrow (x_n)$ convergent $\Rightarrow$ ~~(x_n)~~ $\exists L = \lim_{x\to\infty} x_n \in \mathbb{R}$