$= \sum_{k=0}^{n} \left[\frac{1}{(1+a)^k} + \left(1 - \frac{1}{a+1}\right) \right.$

$(1+a)^n = C_n^0 a^n + C_n^1 \cdot a^{n-1} + \ldots + C_n^n$

22.10.2021

Seminar 4

① Studiați natura urm. s.t.p. utilizând criteriile indicate:

1) criteriul comparației:

a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{4n^2-1}}$

Fie seria $\sum_{n=1}^{\infty} \frac{1}{n}$

$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{4n^2-1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{4n^2-1}} = \lim_{n\to\infty} \frac{n}{n\sqrt{4-\frac{1}{n^2}}} = \frac{1}{2} \in (0,+\infty)$

$\Rightarrow$ seriile aceeași natură $\Rightarrow$ seria dată este div

b) $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$

Fie seria $\sum_{n=1}^{\infty} \frac{1}{n^2}$; $\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} n^2 \cdot \ln\left(1+\frac{1}{n^2}\right)$

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$= \lim_{n\to\infty} \ln\left(1+\frac{1}{n^2}\right)^{n^2} = \ln e = 1 \in (0,+\infty) \Rightarrow$ seriile au aceeași natură

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ conv} \Leftrightarrow p>1}$$

$p = 2 > 1 \Rightarrow$ convergentă

ii) consecințe ale crit. lui Kummer

a) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

$x_n = \frac{2^n}{n!}$, $D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{2^n}{n!} \cdot \frac{(n+1)!}{2^{n+1}} =$

$= \lim_{n\to\infty} \frac{n+1}{2} = \infty > 1 \Rightarrow$ seria este convergentă

b) $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{\sqrt{n}}$; $x_n = \left(\frac{1}{2}\right)^{\sqrt{n}}$, $D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} =$

$= \lim_{n\to\infty} \left(\frac{1}{2}\right)^{\sqrt{n}-\sqrt{n+1}} = \lim_{n\to\infty} 2^{\sqrt{n+1}-\sqrt{n}} = \lim_{n\to\infty} 2^{\frac{1}{\sqrt{n+1}+\sqrt{n}}}$

$= 2^0 = 1$ – nu decide

$R = \lim_{n\to\infty} n \cdot \left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n \cdot \left(2^{\frac{1}{\sqrt{n+1}+\sqrt{n}}} - 1\right) =$

$= \lim_{n\to\infty} \frac{2^{\frac{1}{\sqrt{n+1}+\sqrt{n}}} - 1}{\frac{1}{\sqrt{n+1}+\sqrt{n}}} \cdot \frac{n}{\sqrt{n+1}-\sqrt{n}} = \ln 2 \cdot \lim_{n\to\infty} \frac{n}{\sqrt{n+1}+\sqrt{n}} =$

$= \ln 2 \cdot (+\infty) = +\infty > 1 \Rightarrow$ seria este convergentă

c) $\sum_{n=1}^{\infty} \left[\frac{(2n)!!}{(2n+1)!!}\right]^2$; $(2n)!! = 2 \cdot 4 \cdot \ldots \cdot (2n)$

$(2n+1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n+1)$

$x_n = \left[\frac{(2n)!!}{(2n+1)!!}\right]^2$

$D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \left[\frac{(2n)!!}{(2n+1)!!}\right]^2 \cdot \left[\frac{(2n+3)!!}{(2n+2)!!}\right]^2 =$

$= \lim_{n\to\infty} \left(\frac{2n+3}{2n+2}\right)^2 = 1$ – nu decide

$R = \lim_{n\to\infty} n \cdot \left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n \cdot \left[\frac{4n^2+12n+9}{4n^2+8n+4} - 1\right] =$

$$= \lim_{n\to\infty} n \cdot \frac{4n+5}{4n^2+8n+4} = \lim_{n\to\infty} \frac{}{4n^2+8n+4}$$

$$B = \lim_{n\to\infty} (\ln n)\left[n\left(\frac{x_n}{x_{n+1}} - 1\right) - 1\right] = \lim_{n\to\infty} (\ln n)\cdot\left[\frac{4n^2+5n}{4n^2+8n}\right.$$

$$= \lim_{n\to\infty} (\ln n)\cdot \frac{-3n-4}{4n^2+8n+4} = \lim_{n\to\infty} \underbrace{\frac{\ln n}{n}}_{0} \cdot \underline{\frac{-3n^2-4n}{4n^2+8n+4}} ⊜$$

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \ln\sqrt[n]{n} = \ln 1 = 0$$

$$⊜\ 0\cdot\left(-\frac{3}{4}\right) = 0 < 1 \Rightarrow$$ seria este divergentă

iii) criteriul radicalului

$$\sum_{n=1}^{\infty} \frac{n^2}{\left(2+\frac{1}{n}\right)^n};\quad C = \lim_{n\to\infty}\sqrt[n]{x_n} = \lim_{n\to\infty}\sqrt[n]{\frac{n^2}{\left(2+\frac{1}{n}\right)^n}} = \lim_{n\to\infty}\frac{n^{\frac{2}{n}}}{2+\frac{1}{n}}$$

$$x_n = \frac{n^2}{\left(2+\frac{1}{n}\right)^n}\ \Bigg|\ = \lim_{n\to\infty}\frac{\left(\sqrt[n]{n}\right)^2}{2+\frac{1}{n}} = \frac{1}{2} < 1 \Rightarrow$$ seria este conv

iv) criteriul condensării; (x_n) să descrească, $x_n \geq 0$

$$\sum_{n=2}^{\infty}\frac{1}{n\cdot(\ln n)^p},\ p>0 \qquad \Rightarrow \sum_{n=2}^{\infty} x_n \sim \sum_{n=1}^{\infty} 2^n\cdot x_{2^n}$$

$$x_n = \frac{1}{n\cdot(\ln n)^p}$$

$$\frac{x_{n+1}}{x_n} = \frac{n(\ln n)^p}{(n+1)[\ln(n+1)]^p} = \frac{n}{n+1}\cdot\left[\frac{\ln n}{\ln(n+1)}\right]^p < 1 \Rightarrow (x_n)$$ de

$$\sum_{n=1}^{\infty} 2^n\cdot x_{2^n} = \sum_{n=1}^{\infty}\cancel{2^n}\cdot\frac{1}{\cancel{2^n}\cdot(\ln 2^n)^p} = \sum_{n=1}^{\infty}\frac{1}{(n\cdot\ln 2)^p} = \frac{1}{(\ln 2)^p}\sum_{k=}^{\infty}$$

Caz part. $p = 1 \Rightarrow \sum_{n=2}^{\infty}\frac{1}{n\cdot\ln n}$ divergentă

$$\underset{\substack{\searrow\\ C}}{\frac{1}{n^2}} < \underset{\substack{\downarrow\\ D}}{\frac{1}{n\cdot\ln n}} < \underset{\substack{\swarrow\\ D}}{\frac{1}{n}}$$

② Studiați convergența și absolut convergența urm. serii cu termeni oarecare.

a) $\sum_{n=0} (-1) \cdot \frac{\quad}{3^n}$

$a_n = \frac{2n+1}{3^n} > 0$, $\frac{a_{n+1}}{a_n} = \frac{2n+3}{3^{n+1}} \cdot \frac{3^n}{2n+1} = \frac{2n+3}{6n+3} < 1$

$\leq 1] =$

$\Rightarrow (a_n)$ descresc.

$\lim_{n\to\infty} a_n = L$, căci (a_n) convergent

$a_{n+1} = \frac{2n+3}{6n+3} \cdot a_n \Big|^{n\to\infty} \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{2n+3}{6n+3} \cdot \lim_{n\to\infty} a_n$

$\Rightarrow L = \frac{1}{3} \cdot L \Rightarrow L = 0 \xrightarrow[\text{Leibniz}]{\text{crit.}} \sum_{n=0}^{\infty} (-1)^n \cdot a_n$ convergentă

$=$

$\sum_{n=0}^{\infty} \left| (-1)^n \cdot \frac{2n+1}{3^n} \right| = \sum_{n=0}^{\infty} \frac{2n+1}{3^n}$ s.t.p. (serie cu term. poz.)

ntă

$D = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{6n+3}{2n+3} = 3 > 1 \Rightarrow$ seria convergentă $\Rightarrow$

$\Rightarrow$ seria dată este abs. convergentă

b) $\sum_{n=1} \frac{\sin n}{2^n}$

Studiem absolut convergența

$\sum_{n=1}^{\infty} \left| \frac{\sin n}{2^n} \right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{2^n}$ s.t.p.

2.

$\frac{|\sin n|}{2^n} \leq \frac{1}{2^n}$, $\forall n \in \mathbb{N}$

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ convergentă (seria geometrică) $\Bigg| \xrightarrow{\text{C.C.}} \sum_{n=1}^{\infty} \frac{|\sin n|}{2^n}$ conv.

$\frac{1}{n^p}$ conv. $\Leftrightarrow p > 1$

$\Rightarrow$ seria dată este absolut conv.

③ (CRITERIUL RAP. PT. ȘIRURI)

Fie $(x_n)_{n\in\mathbb{N}}$ un șir cu termeni strict pozitivi pt. care

$\exists \lim_{n\to\infty} \boxed{\frac{x_{n+1}}{x_n}} \overset{\frac{x_n}{x_{n+1}}}{=} l$. Au loc afirmațiile:

i) Dacă $l > 1 \Rightarrow \lim_{n\to\infty} x_n = 0$

Fie s.t.p. $\sum_{n=0}^{\infty} x_n$, $D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = l > 1 \Rightarrow \exists x_n$ conv. $\Rightarrow$

$\Rightarrow \lim\limits_{n\to\infty} x_n = 0$ ⊛

ii) Dacă $l \leq 1 \Rightarrow \lim\limits_{n\to\infty} x_n = +\infty$

Fie s.t.p. $\sum\limits_{n=0}^{\infty} \frac{1}{x_n}$, $l = \lim\limits_{n\to\infty} \frac{\frac{1}{x_n}}{\frac{1}{x_{n+1}}} = \lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1}{l} > 1$

$\Rightarrow \sum \frac{1}{x_n}$ conv. $\Rightarrow \lim\limits_{n\to\infty} \frac{1}{x_n} = 0 \Rightarrow \lim\limits_{n\to\infty} x_n = +\infty$

Ex: $\lim\limits_{n\to\infty} \frac{3^n \cdot n!}{n^n}$?

Fie $x_n = \frac{3^n \cdot n!}{n^n}$, $l = \lim\limits_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim\limits_{n\to\infty} \frac{3^n \cdot n!}{n^n} \cdot \frac{(n+1)^{n+1}}{3^{n+1} \cdot (n+1)!} =$

$= \lim\limits_{n\to\infty} \frac{(n+1)^n}{n^n \cdot 3} = \frac{1}{3} \lim\limits_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \frac{e}{3} < 1$

$\Rightarrow \lim\limits_{n\to\infty} x_n = +\infty$

④ Fie $\sum\limits_{n=1}^{\infty} x_n$ s.t.p. Arătați că:

$\sum\limits_{n=1}^{\infty} x_n \sim \sum\limits_{n=1}^{\infty} \frac{x_n}{1+x_n}$

$x_n > \frac{x_n}{1+x_n}$, $\forall n \in \mathbb{N}$, $x_n \geq 0$

Dacă $\sum\limits_{n=1}^{\infty} x_n$ converg $\Rightarrow \sum\limits_{n=1}^{\infty} \frac{x_n}{1+x_n}$ converg ⊛

De arătat că dacă $\sum \frac{x_n}{1+x_n}$ conv. $\Rightarrow \sum\limits_{n=1}^{\infty} x_n$ conv ?? (temă)

Seminar 5

① Justificați afirmațiile:

i) $\frac{1}{n+1} < \ln(n+1) - \ln n < \frac{1}{n}$, $\forall n \in \mathbb{N}^*$

i) Fie $f(x) = \ln x$, $\forall x > 0$ și aplicăm Th. de medie Lagrange pe $[n, n+1]$ $\forall n \in \mathbb{N}^*$