Seminarul 2

① Calc. der. de ordinul $n \in \mathbb{N}$ a fct. de mai jos și precizați mult. pe care aceste fct. sunt indefinit derivabile:

a) $f(x) = \sin x$

$$f^{(n)}(x) = \begin{cases} \sin x, & n = 4k \\ \cos x, & n = 4k+1 \\ -\sin x, & n = 4k+2 \\ -\cos x, & n = 4k+3 \end{cases}, k \in \mathbb{N} = \begin{cases} (-1)^k \cdot \sin x, n \\ (-1) \cos x, n = \end{cases}$$

f indef. deriv. pe $\mathbb{R}$

b) $f'(x) = \frac{1}{x+1} = (x+1)^{-1}$

$$f''(x) = \frac{-1}{(x+1)^2} = -(x+1)^{-2}$$

$$f'''(x) = 2(x+1)^{-3}$$

$$f^{(4)}(x) = -2 \cdot 3 (x+1)^{-4}$$

$$f^{(5)}(x) = 2 \cdot 3 \cdot 4 (x+1)^{5}$$

$$f^{(n)}(x) = (-1)^{n+1} (n-1)! \cdot (x+1)^{-n}, \forall n \geq 1$$

f indef. deriv. pe $(-1; +\infty)$

c) Form. lui Leibnitz, $u = u(x)$, $v = v(x)$ $\quad x(x^2-x) e^x$

$$(u \cdot v)^{(n)} = \sum_{k=0}^{n} C_n^k \cdot u^{k} \cdot v^{(n-k)} = \sum_{k=0}^{n} C_n^k (x^2 - x)^{(k)} \cdot$$

$$(e^x)^{(n-k)} = \sum_{k=0}^{n} C_n^0 (x^2 - x) \cdot e^x + C_n^1 (2x-1) e^x +$$

$$+ C_n^2 \cdot 2 \cdot e^x + C_n^3 \cdot 0 = e^x \left[x^2 - x + n(2x-1) + \frac{n(n-1)}{2}\right]$$

$$= e^x \left[x^2 + x(2n-1) + n^2 - 2n\right], \forall n \in \mathbb{N}, \text{ se verifică}$$

$n = 0, n = 1$

f indef. der. pe $\mathbb{R}$

d) $f(x) = \sqrt{1-x}$

$$f'(x) = \frac{-1}{2\sqrt{1-x}} = -\frac{1}{2}\cdot(1-x)^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{2}\cdot\frac{1}{2}\cdot(1-x)^{-\frac{3}{2}}$$

$$f'''(x) = -\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}(1-x)^{-\frac{5}{2}}$$

$$f^{(4)}(x) = -\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}(1-x)^{-\frac{7}{2}}$$

$$f^{(n)}(x) = -\frac{1}{2^n}\cdot(1\cdot3\cdot5\cdot\ldots\cdot(2n-3))\cdot(1-x)^{-\frac{2n-1}{2}} =$$

$$= -\frac{1}{2^n}\cdot(2n-3)!!\,(1-x)^{-\frac{2n-1}{2}}, \forall n \geq 2$$

f indef. deriv. pe $(-\infty, 1)$

Partial

$1-x \geq 0 \Rightarrow x \in (-\infty, 1]$

2) Pt. fct. de la ex. anterior, pt. $x_0 = 0$ și nr. $n \in \mathbb{N}$, det:

a) Pol. lui Taylor de gr. n asociat fct. f în pct x_0

b) Mult. de convergentă a seriei Taylor corespunzătoare

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k; \quad \sum_{k=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n, \; x \in \mathbb{R}$$

1) $f(x) = \sin x$, $f^{(n)}(0) = \begin{cases} 0, & n = 2k \\ (-1)^k, & n = 2k+1 \end{cases}$

$$T_n(x) = f(0) + \frac{f'(0)}{1!}\cdot x + \frac{f''(0)}{2!}\cdot x^2 + \ldots + \frac{f^{(n)}(0)}{n!}\cdot x^n$$

$$= 0 + \frac{x}{1!} + 0 + \frac{x^3}{3!} + 0 + \frac{x^5}{5!} - \ldots + \frac{f^{(n)}(0)}{n!}\cdot x^n$$

$= 2k$, $2k$ d, $k \in$ ... $-2]$ = pt.

Seria Taylor: $\sum_{k=0}^{\infty} \frac{f^{(2k+1)}(0)}{(2k+1)!} \cdot x^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot x^{2k+1}$

$\sum a_n \cdot x^n$; $\exists\, r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

Fie $x \in \mathbb{R}^*$ fixat, studiem absolut convergența

$\sum_{k=0}^{\infty} \underbrace{\frac{|x|^{2k+1}}{(2k+1)!}}_{x_k}$ s.t.p., $D = \lim_{k\to\infty} \frac{x_k}{x_{k+1}} = \lim_{k\to\infty} \frac{|x|^{2k+1}}{(2k+1)!} \cdot \frac{(2k+3)!}{|x|^{2k+3}}$

$= \lim_{k\to\infty} \frac{(2k+2)(2k+3)}{|x|^2} = +\infty > 1 \Rightarrow$ seria este conv. $\Rightarrow I = \mathbb{R}$

$\Rightarrow \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot x^{2n+1}, \forall x \in \mathbb{R}$

2) $f(x) = \ln(1+x)$ indef. der. pe $(-1;+\infty)$

$f^{(n)}(0) = (-1)^{n+1}(n-1)!, \forall n \geq 1$

$f(0) = 0$

$T_n(x) = \sum 0 + \sum_{k=1}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=1}^{n} \frac{(-1)^{k+1}(k-1)!}{k!} \cdot x^k =$

$= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \cdot x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + \frac{(-1)^{n+1}}{n} \cdot x^n$

Seria Taylor: $\sum_{n=1}^{\infty} \underbrace{\frac{(-1)^{n+1}}{n}}_{a_n} \cdot x^n$, s.p. (serie de puteri)

raza de conv: $r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\cancel{(-1)^{n+1}} \frac{n+1}{n}\right| = \lim_{n\to\infty} \frac{n+1}{n} =$

$\Rightarrow (-1, 1) \subseteq I \subseteq [-1, 1]$

pt. $x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ seria armonică alternantă conv $\Rightarrow 1 \in I$

pt. $x = -1 \Rightarrow \sum_{n=1}^{\infty} \frac{-1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$ div $\Rightarrow -1 \notin I$

$\Rightarrow I=(-1,1]$, $\ln(x+1)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot x^n, \forall x\in(-1,+1]$

3) Temă: $(x^2-x)\cdot e^x$

(*)

4) $f(x)=\sqrt{1-x}$

$f^{(n)}(0)=-\frac{1}{2^n}\cdot(2n-3)!!, \forall n\geq 2$

$f(0)=1$

$f'(0)=-\frac{1}{2}$

$T_n(x)=1-\frac{1}{2}\cdot x+\sum_{k=2}^{n}\frac{f^{(k)}(0)}{k!}\cdot x^k=1-\frac{x}{2}+\sum_{k=2}^{n}-\frac{(2k-3)!!}{2^k\cdot k!}$

$\cdot x^k$

Seria Taylor: $1-\frac{x}{2}-\underbrace{\sum_{n=2}^{\infty}\frac{(2n-3)!!}{2^n\cdot n!}\cdot x^n}_{\text{mult. de conv.}}$

$a_n=\frac{(2n-3)!!}{2^n\cdot n!}$; $r=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\frac{(2n-3)!!}{2^n\cdot n!}\cdot$

$\cdot\frac{2^{n+1}(n+1)!}{(2n-1)!!}=\lim_{n\to\infty}\frac{2(n+1)}{2n-1}=1\Rightarrow(-1,1)\subseteq I\subseteq[-1,1]$

pt. $x=1\Rightarrow\sum_{n=2}^{\infty}\frac{(2n-3)!!}{2^n\cdot n!}$ s.t.p.; $D=\lim_{n\to\infty}\frac{a_n}{a_{n+1}}$ ⊜ lim

⊜ $\lim_{n\to\infty}\frac{2n+2}{2n-1}=1$ - nu decide

$R=\lim_{n\to\infty}n\left(\frac{a_n}{a_{n+1}}-1\right)=\lim_{n\to\infty}n\cdot\left(\frac{2n+2}{2n-1}-1\right)=\lim_{n\to\infty}\frac{3n}{2n-1}=$

$=\frac{3}{2}>1\Rightarrow$ convergentă $\Rightarrow 1\in I$

pt. $x=-1 \Rightarrow \sum_{n=2}^{\infty} \frac{(2n-3)!!}{2^n \cdot n!} (-1)^n$ este absolut conv.

$\Rightarrow -1 \in I \Rightarrow I = [-1, 1]$

$$\Rightarrow \boxed{\sqrt{1-x} = 1 - \frac{x}{2} - \sum_{n=2}^{\infty} \frac{(2n-3)!!}{2^n \cdot n!} \cdot x^n, \forall x \in [-1, 1]}$$

③ Utilizând operațiile cu serii de puteri, justificați egalitățile

a) $\sum_{n=0}^{\infty} (-1)^n (n+1) \cdot x^n = \frac{1}{(1+x)^2}, \forall x \in (-1, 1)$

$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n \cdot x^n, \forall x \in (-1,1) \Big|'$ | derivăm

$$\Rightarrow -\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} [(-1)^n \cdot x^n]' = \sum_{n=1}^{\infty} n(-1)^n \cdot x^{n-1} \overset{n-1=m}{=}$$

$= \sum_{m=0}^{\infty} (m+1)(-1)^{m+1} \cdot x^m \,|\, (-1)$

$\Rightarrow \frac{1}{1+x^2} = \sum_{m=0}^{\infty} (m+1)(-1)^m \cdot x^m$, raza de conv este $r = 1$

$\frac{(m+1)(-1)^m}{\not\to 0}$ (nu tinde la 0)

$\Rightarrow (-1,1) \subseteq I \subseteq [-1,1]$, pt. $x=1$ și $x=-1$ sunt divergente

$\Rightarrow I = (-1, 1)$

b) $1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^n = \frac{1}{\sqrt{1-x}}, \forall x \in [-1, 1)$

$-\frac{1}{2} \cdot \frac{1}{\sqrt{1-x}} = 0 - \frac{1}{2} - \sum_{n=2}^{\infty} \frac{(2n-3)!!}{2^n \cdot n!} \cdot n \cdot x^{n-1} = -\frac{1}{2} - \sum_{n=2} \frac{(2n-3)}{2^n (n-1)}$

$\cdot x^{n-1} \overset{n-1=m}{=} -\frac{1}{2} - \sum_{m=1}^{\infty} \frac{(2m-1)!!}{2^{m+1} \cdot m!} \cdot x^m \,\Big|\, \cdot (-2)$

(7) Det. mulț. de convergență a seriei de puteri $\sum_{n=1}^{\infty} \frac{1}{n^2}(x-1)^2$

$\sum_{n=0}^{\infty} a_n (x-x_0)^2$; $a_n = \frac{1}{n^2}$, $x_0 = 1$

$$r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 1$$

$(0,2) \subseteq I \subseteq [0,2]$

$x=0$; $\sum_{n=1}^{\infty} \frac{(-1)^2}{n^2}$, $a_n = \frac{1}{n^2} \searrow 0 \overset{\text{C.L.}}{\Longrightarrow}$ serie conv. $\Rightarrow 0 \in I$

$x=2$; $\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv , $p=2>1$ $\Rightarrow 2 \in I$

$\Rightarrow I = [0,2]$

Seminarul 7

(1) Determinați mulț. de convergență a seriei de