$\nexists \lim\limits_{n\to\infty} \; 2 \quad \frac{\quad}{\frac{1}{n}}$

$\Rightarrow I = (-1, 1)$

Seminar 8

Sm. 7

1) Evaluați integralele:

a) $\displaystyle\int_0^1 \frac{e^x}{\sqrt{e^{2x}+1}}\,dx$

$\displaystyle\int \frac{u'(x)}{\sqrt{u^2(x)+a^2}} = \ln\left(u(x)+\sqrt{u^2(x)+a^2}\right)+\mathcal{C}$

$u(x) = e^x$, $I = \ln\left(e^x+\sqrt{e^{2x}+1}\right)\Big|_0^1 = \ln\left(e+\sqrt{e^2+1}\right)$

$-\ln(1+\sqrt{2})$

b) $\int_0^2 \max\{x, x^2\}dx \Rightarrow \begin{cases} x, & x\in[0,1] \\ x^2, & x\in[1,2] \end{cases}$

$\Rightarrow \int_0^2 \max\{x,x^2\}dx = \int_0^1 x\,dx + \int_1^2 x^2 dx = \frac{x^2}{2}\Big|_0^1 + \frac{x^3}{3}\Big|_1^2 = \frac{1}{2} -$

$-0 + \frac{8}{3} - \frac{1}{3} = \frac{1}{2} + \frac{7}{3} = \frac{17}{6}$

c) $\int_1^{\sqrt{3}} \frac{\operatorname{arctg} x}{x^2}dx = \int_1^{\sqrt{3}} \left(-\frac{1}{x}\right)' \operatorname{arctg} x\, dx = -\frac{1}{x}\operatorname{arctg} x\Big|_1^{\sqrt{3}} -$

$+\underbrace{\int_1^{\sqrt{3}} \frac{1}{x(x^2+1)}dx}_{J} \Leftrightarrow$

$\Rightarrow 1 \in i \;|\; (-)$

$J = \int_1^{\sqrt{3}}\left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx = \left(\ln x - \frac{1}{2}\ln(x^2+1)\right)\Big|_1^{\sqrt{3}}$

$\Leftrightarrow -\frac{\overset{\frac{\pi}{3}}{\operatorname{arctg}\sqrt{3}}}{\sqrt{3}} + \underset{\frac{\pi}{4}}{\operatorname{arctg} 1} + \ln\sqrt{3} - \frac{1}{2}\ln 4 - (\underset{0}{\ln 1} - \frac{1}{2}\ln 2)$

$= -\frac{\pi}{3\sqrt{3}} + \frac{\pi}{4} + \ln\sqrt{3} - \frac{1}{2}\ln 4 + \frac{1}{2}\ln 2$

d) $\int_{-1}^{1} \sqrt{1-x^2}\,dx$

<u>Subst. trig. pt. int. alg.</u>

i) $\int R(x, \sqrt{a^2-x^2})$ $\quad x = a\sin t$ sau $a\cos t$

ii) $\int R(x, \sqrt{a^2+x^2})$ $\quad x = a\,\operatorname{tg} t$ sau $a\,\operatorname{ctg} t$

iii) $\int R(x^2-a^2)dx$ $\quad x = \frac{a}{\sin t}$ sau $\frac{a}{\cos t}$

$x = \sin t$

$dx = (\sin t)'\,dt = \cos t\,dt$

$x = 1 \rightarrow \sin t = 1 \Rightarrow t = \frac{\pi}{2}$

$x = -1 \Rightarrow \sin t = -1 \Rightarrow t = -\frac{\pi}{2}$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\, \cos t\, dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos t| \cos t\, dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\, dt =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\, dt = \left(\frac{t}{2} + \frac{\sin 2t}{4}\right)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{4} + 0 - \left(-\frac{\pi}{4}\right.$$

$$\left.+0\right) = \frac{\pi}{2}$$

e) $\displaystyle\int_2^4 \frac{\sqrt{x^2-4}}{x}\, dx$

$$x = \frac{2}{\sin t}, \quad dx = \left(\frac{2}{\sin t}\right)' dt = \frac{2' \cdot \sin t - 2 \cdot (\sin t)'}{\sin^2 t}\, dt =$$

$$= \frac{-2\cos t}{\sin^2 t}\, dt$$

$x = 2 \Rightarrow \sin t = 1 \Rightarrow t = \frac{\pi}{2}$

$x = 4 \Rightarrow \sin t = \frac{1}{2} \Rightarrow t = \frac{\pi}{6}$

$$\frac{\sqrt{x^2-4}}{x} = \frac{\sqrt{\frac{4}{\sin^2 t} - 4}}{\frac{2}{\sin t}} = \frac{\sin t}{2} \cdot \sqrt{\frac{4-4\sin^2 t}{\sin^2 t}} = \frac{\sin t}{2} \cdot \sqrt{\frac{4\cos^2 t}{\sin^2 t}} =$$

$$= \frac{\cancel{\sin t}}{\cancel{2}} \cdot \cancel{2} \cdot \frac{\cos t}{\cancel{\sin t}} = \cos t$$

$$\Rightarrow \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \frac{\cos t\, (-2)\cos t}{\sin^2 t}\, dt = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\overbrace{\cos^2 t}^{=1-\sin^2 t}}{\sin^2 t}\, dt = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{1}{\sin^2 t} - 1\right) dt$$

$$= 2\left(\operatorname{ctg} t - t\right)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \ldots$$

Seminar 8 - Integrale improprii

① Evaluați int. improprii:

a) $\int_0^\infty \frac{arctg\, x}{1+x^2}dx = \lim_{v\to\infty}\int_0^v \frac{arctg\, x}{1+x^2}dx =$

$= \lim_{v\to\infty}\int_0^v (arctg\, x)' arctg\, x\, dx = \lim_{v\to\infty} \frac{arctg^2 x}{2}\Big|_0^v =$

$= \lim_{v\to\infty}\frac{arctg^2 v}{2} = \frac{1}{2}\cdot\left(\frac{\pi}{2}\right)^2$

b) $\int_{-1+0}^{1-0}\frac{x+1}{\sqrt{1-x^2}}dx =$ (improprie și în 1 și în −1)

$= \int_{-1+0}^{0}\frac{x+1}{\sqrt{1-x^2}}dx + \int_0^{1-0}\frac{x+1}{\sqrt{1-x^2}}dx = \lim_{u\searrow -1}\int_u^0\frac{x+1}{\sqrt{1-x^2}}dx + \lim_{v\nearrow 1}\int_0^v\frac{x+1}{\sqrt{1-x^2}}dx$ ②

$\int\frac{x+1}{\sqrt{1-x^2}}dx = \int\frac{x}{\sqrt{1-x^2}} + \int\frac{1}{\sqrt{1-x^2}}dx = -\sqrt{1-x^2} + \arcsin x$

$(\sqrt{1-x^2})' = \frac{-x}{1-x^2}$

② $\lim_{u\searrow -1}(-\sqrt{1-x^2}+\arcsin x)\Big|_u^0 + \lim_{v\nearrow 1}(-\sqrt{1-x^2}+\arcsin x)\Big|_0^v =$

$= \lim_{u\searrow -1}(-1+0+\sqrt{1-u^2}-\arcsin u) + \lim_{v\nearrow 1}(-\sqrt{1-v^2}+\arcsin v +$

$+1) = \cancel{-1} + 0 - \left(-\frac{\pi}{2}\right) + 0 + \frac{\pi}{2} + \cancel{1} = \frac{\pi}{2}+\frac{\pi}{2} = \pi$

c) $\int_0^\infty x^n \cdot e^{-x}dx \overset{i_n}{=}, n\in\mathbb{N}$

$i_n = \int_0^\infty x^n\cdot(-e^{-x})'dx = x^n(-e^{-x})\Big|_0^\infty + \int_0^\infty n\cdot x^{n-1}e^{-x}dx =$

$= \lim\limits_{v\to\infty} x(-e^{\ldots})\Big|_0 + n\cdot I_{n-1} = \lim\limits_{v\to\infty}(-v\cdot e^{\ldots} + 0) + n\,I_{n-1}$

$\lim\limits_{v\to\infty} \frac{-v^n}{e^v} \overset{\infty\text{ l'H}}{=} 0$

$= 0 + n\cdot I_{n-1}$

$I_n = n\cdot I_{n-1}, \forall n \geq 1$

$I_0 = \int_0^\infty e^{-x}\,dx = \lim\limits_{v\to\infty}\int_0^v e^{-x}\,dx = \lim\limits_{v\to\infty}(-e^{-x})\Big|_0^v =$

$= \lim\limits_{v\to\infty}(-e^{-v}+1) = 1$

$I_1 = 1\cdot I_0 = 1\cdot 1 = 1$

$I_2 = 2\cdot I_1 = 2\cdot 1 = 2$

$I_3 = 3\cdot I_2 = 3\cdot 2 = 6 = 3!$

$\Rightarrow P(n): I_n = n!, \forall n\in\mathbb{N}$

! Dem. prin ind.

(improprie în 0)

d) $\int_1^{2-0} \frac{1}{\sqrt{x(2-x)}}\,dx \underset{\uparrow}{=} \int_1^{\sqrt{2}-0} \frac{1}{\sqrt{t^2(2-t^2)}}\,2t\,dt =$

$x = t^2, t\geq 0$

$dx = 2t\,dt$

$x = 1 \Rightarrow t = 1$

$x \nearrow 2 \Rightarrow t \nearrow \sqrt{2}$

nu mai este improprie

$= \int_1^{\sqrt{2}-0} \frac{2}{\sqrt{2-t^2}}\,dt = 2\arcsin\frac{t}{\sqrt{2}}\Big|_1^{\sqrt{2}} = 2\underbrace{\arcsin 1}_{\frac{\pi}{2}} - 2\underbrace{\arcsin\ldots}_{\frac{\pi}{4}}$

$= \pi - \frac{\pi}{2} = \frac{\pi}{2}$