Seminar 12

① Calc. gradientul ∇f, deriv. part. de ord. I și diferențiala def pt.:

a) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^2y^3 + y \sin x - 2z$

$\frac{\partial f}{\partial x} = 2xy^3 + y\cos x$; $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$\frac{\partial f}{\partial y} = 3x^2y^2 + \sin x$

$\frac{\partial f}{\partial z} = \cancel{y} - 2$

$df(x,y,z): \mathbb{R}^3 \to \mathbb{R}$

$df(x,y,z)(u_1,u_2,u_3) = \frac{\partial f}{\partial x}(x,y,z)u_1 + \frac{\partial f}{\partial y}(x,y,z)u_2 + \frac{\partial f}{\partial z}(x,y,z)u_3$

b) $f: (0;+\infty)^2 \to \mathbb{R}$, $f(x,y) = \operatorname{arctg} \frac{x-y}{x+y}$

$\frac{\partial f}{\partial x}(x,y) = \left(\operatorname{arctg}\frac{x-y}{x+y}\right)'_x = \frac{1}{\left(\frac{x-y}{x+y}\right)^2+1} \cdot \left(\frac{x-y}{x+y}\right)'_x =$

$= \frac{\cancel{(x+y)^2}}{(x-y)^2+(x+y)^2} \cdot \frac{(x+y)-(x-y)}{\cancel{(x+y)^2}} = \frac{2y}{2x^2+2y^2} = \frac{y}{x^2+y^2}$

$\frac{\partial f}{\partial y}(x,y) = \left(\operatorname{arctg}\frac{x-y}{x+y}\right)'_y = \frac{1}{\left(\frac{x-y}{x+y}\right)^2+1} \cdot \left(\frac{x-y}{x+y}\right)'_y =$

$= \frac{\cancel{(x+y)^2}}{(x-y)^2+(x+y)^2} \cdot \frac{-1(x+y)-(x-y)}{\cancel{(x+y)^2}} = \frac{-2x}{2x^2+2y^2} = \frac{-x}{x^2+y^2}$

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c) $f:\mathbb{R}^2\to\mathbb{R},\ f(x,y)=x\sqrt{x^2+y^2}$

$$\frac{\partial f}{\partial x}(x,y)=\sqrt{x^2+y^2}+x\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot(x^2+y^2)'_x=\sqrt{x^2+y^2}+$$

$$+\frac{2x^2}{2\sqrt{x^2+y^2}}=\frac{x^2+y^2+x^2}{\sqrt{x^2+y^2}}=\frac{2x^2+y^2}{\sqrt{x^2+y^2}}$$

$\forall (x,y)\in\mathbb{R}^2\setminus\{0_2\}$

$$\frac{\partial f}{\partial y}(x,y)=x\left(\sqrt{x^2+y^2}\right)'_y=x\cdot\frac{1}{2\sqrt{x^2+y^2}}\cdot 2y=\frac{xy}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0}=\lim_{x\to 0}\frac{x\sqrt{x^2}-0}{x}=$$

$$=\lim_{x\to 0}|x|=0$$

$\lim\limits_{x\to 0}\frac{|x|}{x}\nexists$ (poate fi 1 și -1)

$$\frac{\partial f}{\partial y}(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0}=\lim_{y\to 0}\frac{0}{y}=0$$

② Ar. că fct. $f(x,y)=y\cdot\ln(x^2-y^2)$ verifică relația:

$$\frac{1}{x}\cdot\frac{\partial f}{\partial x}(x,y)+\frac{1}{y}\cdot\frac{\partial f}{\partial y}(x,y)=\frac{1}{y^2}f(x,y),\ \forall x>y>0$$

$$\frac{\partial f}{\partial x}(x,y)=\left(y\cdot\ln(x^2-y^2)\right)'_x=0+y\cdot\left(\ln(x^2-y^2)\right)'_x=\frac{2xy}{x^2-y^2}$$

$$\frac{\partial f}{\partial y}(x,y)=\left(y\cdot\ln(x^2-y^2)\right)'_y=\ln(x^2-y^2)+y\cdot\frac{1}{x^2-y^2}(x^2-y^2)'_y=$$

$$=\ln(x^2-y^2)+\frac{-2y^2}{x^2-y^2}$$

$$\Rightarrow\frac{1}{x}\cdot\frac{2xy}{x^2-y^2}+\frac{1}{y}\ln(x^2-y^2)+\frac{1}{y}\cdot\frac{-2y^2}{x^2-y^2}=\frac{1}{y}\ln(x^2-y^2)$$

③ Studiați existența der. parțiale în origine și a der. după direcție în origine pt.

$$f:\mathbb{R}^2\to\mathbb{R},\ f(x,y)=\begin{cases}\frac{x^2y}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \ldots$

analog $\frac{\partial f}{\partial y}(0,0) = 0$

$\forall v = (v_1, v_2) \in \mathbb{R}^2$, $f_v'(0,0) = \lim_{t\to 0} \frac{f((0,0)+t\cdot(v_1,v_2)) - f}{t} =$

$= \lim_{t\to 0} \frac{f(tv_1, tv_2)}{t} = \lim_{t\to 0} \frac{1}{t} \cdot \frac{(tv_1)^2 \cdot tv_2}{(tv_1)^4 + (tv_2)^2} =$

$= \lim_{t\to 0} \frac{\cancel{t^3} \cdot v_1^2 \cdot v_2}{\cancel{t^3}(t^2 \cdot v_1^4 + v_2^2)} = \begin{cases} \frac{v_1^2}{v_2}, & v_2 \neq 0 \\ 0, & v_2 = 0 \end{cases}$

Cazuri particulare:

$v = e^1 = (1,0) \Rightarrow f'_{e^1}(0,0) = 0$

$v = e^2 = (0,1) \Rightarrow f'_{e^2}(0,0) = 0$

(4) Calc. d.p. ~~ale~~ ale fct. compuse $g\circ f$, unde $f: \mathbb{R}^2 \to \mathbb{R}^2$,

$f(x,y) = (\underbrace{x e^y + x e^{-y}}_{f^1}, \underbrace{x e^y - x e^{-y}}_{f^2})$ și $g = g(u,v): \mathbb{R}^2 \to \mathbb{R}$

este o fct. oarecare de clasă C^1 pe $\mathbb{R}^2$.

der. parț. sunt fct. cont.

$\nabla(g\circ f)(x,y) = \nabla g(f(x,y)) \odot J(f)(x,y)$ (înm. matr.)

$\left(\frac{\partial(g\circ f)}{\partial x}(x,y), \frac{\partial(g\circ f)}{\partial y}(x,y)\right) = \left(\frac{\partial g}{\partial u}(f(x,y)), \frac{\partial g}{\partial v}(f(x,y))\right)$

$J(f)(x,y) = \begin{pmatrix} \frac{\partial f^1}{\partial x}(x,y) & \frac{\partial f^1}{\partial y}(x,y) \\ \frac{\partial f^2}{\partial x}(x,y) & \frac{\partial f^2}{\partial y}(x,y) \end{pmatrix} =$

$= \begin{pmatrix} e^y + e^{-y} & x e^y - x e^{-y} \\ e^y - e^{-y} & x\cdot e^y + x\cdot e^{-y} \end{pmatrix}$

④ $\begin{pmatrix} e^y - e^{-y} & xe^y + xe^{-y} \end{pmatrix}$ $(a,a) - (1,-1) \begin{pmatrix} 2 & 0 \\ -1 & \alpha \end{pmatrix}$

$a = 2+1 = 3$

$\Rightarrow \frac{\partial(g \circ f)}{\partial x}(x,y) = \frac{\partial g}{\partial u} f(x,y) \cdot (e^y + e^{-y}) + \frac{\partial g}{\partial v}(f(x,y)) \cdot$

$\cdot (e^y - e^{-y})$

$\frac{\partial(g \circ f)}{\partial y}(x,y) = \frac{\partial g}{\partial v}(x e^y - x e^{-y})$

⑤ Expl. ec: $u \cdot \frac{\partial g}{\partial u}(u,v) + v \cdot \frac{\partial g}{\partial v}(u,v) = \sqrt{u^2 + v^2}$,

$\forall (u,v) \in (0,+\infty)^2$ în rel. $(x,y) \in (0;+\infty) \times (0; \frac{\pi}{2})$, f.

transformarea $u = x \cdot \cos y$, $v = x \sin y$.

Det. o fct. g de clasă C^1 ce verifică relația respectivă apoi

$\Rightarrow x \cdot \cos y \cdot \frac{\partial g}{\partial u}(x \cdot \cos y, x \sin y) + x \sin y \frac{\partial g}{\partial v}(x \cos y, x \sin y) \; (*)$ $\cancel{x}$ 1

$\sqrt{u^2 + v^2} = \sqrt{x^2(\cos^2 y + \sin^2 y)} = |x| = x$

Fie $f(x,y) = (x \cdot \cos y, x \sin y)$, $G = g \circ f$

$\frac{\partial G}{\partial x}(x,y) = \frac{\partial g}{\partial u}(f(x,y)) \cdot \frac{\partial f_1}{\partial x}(x,y) + \frac{\partial g}{\partial v}(f(x,y)) \frac{\partial f_2}{\partial x}(x,y)$

$= \frac{\partial g}{\partial u}(x \cos y, x \sin y) \cdot \cos y + \frac{\partial g}{\partial v}(x \cos y, x \sin y) \cdot \sin y$

$= 1 \Rightarrow \frac{\partial G}{\partial x}(x,y) = 1$, $\forall (x,y) \in (0;+\infty) \times (0, \frac{\pi}{2})$

$\Rightarrow G(x,y) = \int 1 \, dx = x + c(y)$

în particular $c(y) = 0 \Rightarrow G(x,y) = x \Rightarrow g(f(x,y)) = x$

$\Rightarrow g(x \cos y, x \sin y) = x \Rightarrow \underline{g(u,v) = \sqrt{u^2 + v^2}}$

⑥ Calc. der. de ord. 2 ale fct:

a) $f:(1;+\infty)\times\mathbb{R}\to\mathbb{R},\ f(x,y)=\ln(x+y^2-1)$

$\frac{\partial f}{\partial x}(x,y)=\frac{1}{x+y^2-1}$; $\frac{\partial f}{\partial y}(x,y)=\frac{1}{x+y^2-1}\cdot 2y$

$\frac{\partial^2 f}{\partial x^2}(x,y)=\left(\frac{1}{x+y^2-1}\right)'_x$

$\frac{\partial^2 f}{\partial y^2}(x,y)=\left(\frac{2y}{x+y^2-1}\right)'_y=\frac{2(x+y^2-1)-4y^2}{(x+y^2-1)^2}=\frac{2x-2y^2}{(x+y^2-1)^2}$

$\frac{\partial^2 f}{\partial x\partial y}(x,y)=\left(\frac{2y}{x+y^2-1}\right)'_x=\frac{-2y}{(x+y^2-1)^2}$

$\frac{\partial^2 f}{\partial y\partial x}(x,y)=\left(\frac{1}{x+y^2-1}\right)'_y=\frac{-2y}{(x+y^2-1)^2}$

b) $f:\mathbb{R}\times(0;+\infty)\to\mathbb{R},\ f(x,y)=x\cdot y\cdot e^{\frac{x}{y}}$

$\frac{\partial f}{\partial x}(x,y)=(x\cdot y\cdot e^{\frac{x}{y}})'_x=y\cdot e^{\frac{x}{y}}+\left(\frac{x}{y}\right)'_x\cdot e^{\frac{x}{y}}\cdot x\cdot y=$

$=y\cdot e^{\frac{x}{y}}+x\cdot y\cdot e^{\frac{x}{y}}\cdot\frac{1}{y}=e^{\frac{x}{y}}(x+y)$

$\frac{\partial f}{\partial y}(x,y)=(x\cdot y\cdot e^{\frac{x}{y}})'_y=x\cdot e^{\frac{x}{y}}+y\cdot x\cdot e^{\frac{x}{y}}\cdot\left(\frac{x}{y}\right)'_y$

$=x\cdot e^{\frac{x}{y}}+x\cdot y\cdot e^{\frac{x}{y}}\cdot\frac{-x}{y^2}=x\cdot e^{\frac{x}{y}}-\frac{x^2}{y}\cdot e^{\frac{x}{y}}=$

$=x\cdot e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)$

$\frac{\partial^2 f}{\partial x^2}(x,y)=\left[e^{\frac{x}{y}}\cdot(x+y)\right]'_x=e^{\frac{x}{y}}\cdot\frac{1}{y}(x+y)+e^{\frac{x}{y}}\cdot 1=$

$=e^{\frac{x}{y}}\cdot\left(\frac{x}{y}+2\right)$

$\frac{\partial^2 f}{\partial y^2}(x,y)=\left[x\,e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)\right]'_y=x\,e^{\frac{x}{y}}\cdot\left(-\frac{x}{y^2}\right)\left(1-\frac{x}{y}\right)+$

$+x\cdot \cancel{e}\, e^{-\frac{x}{y}}\cdot\frac{x}{y^2} = \frac{x}{y^2}\cdot e^{-\frac{x}{y}}\cdot\left[\cancel{-x}+\frac{x^2}{y}\cancel{+x}\right] = \frac{x^3}{y^3}\cdot e^{-\frac{x}{y}}$

Temă: $\dfrac{\partial^2 f}{\partial x\,\partial y}$, $\dfrac{\partial^2 f}{\partial y\,\partial x}$