$I \neq 0$ ... $\int_0 (\ldots)' \cos x = -e^{-x}\cos x\Big|_0^\infty -$

$-\int_0^\infty e^{-x}\cdot \sin x\,dx = -e^{-x}\cos x\Big|_0^\infty - \int_0^\infty (-e^{-x})'\sin x\,dx =$

$= -e^{-x}\cos x\Big|_0^\infty - \left[-e^{-x}\sin x\Big|_0^\infty + \int_0^\infty e^{-x}\cdot\cos x\,dx\right] =$

$= \underbrace{\lim_{n\to\infty}(-e^{-x})\cos x\Big|_0^n}_{\lim_{n\to\infty}(-e^{-n}\cos n + 1) = 1} + \underbrace{\lim_{n\to\infty}(-e^{-x})\sin x\Big|_0^n}_{\lim_{n\to\infty}(-e^{-n}\sin n - 0) = 0} - I \Rightarrow 2I = 1$

$\Rightarrow I = \frac{1}{2}$

$\to$ improprie la ambele (calc. primitiva)

f) $\int_0^1 \frac{\ln x}{\sqrt{1-x}}\,dx$

26.11.2021

Seminar 8 (cont.)

• Calc. integrala $\int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx$

$I = \int_{0+0}^{\frac{\pi}{2}} \ln(\sin x)\,dx \underset{x = \frac{\pi}{2} - y}{=} \int_{\frac{\pi}{2}-0}^{0} \ln\left(\sin\left(\frac{\pi}{2} - y\right)\right)(-dy) =$

$dx = -dy$

$x \searrow 0 \Rightarrow y \nearrow \frac{\pi}{2}$

$x = \frac{\pi}{2} \Rightarrow y = 0$

$= \int_0^{\frac{\pi}{2}-0} \ln(\cos y)\,dx$

$2I = I + I = \int_{0+0}^{\frac{\pi}{2}-0} \ln(\sin x)\,dx + \int_{0+0}^{\frac{\pi}{2}-0} \ln(\cos x)\,dx = \int_{0+0}^{\frac{\pi}{2}-0} \ln(\sin x \cdot$

$\cdot \cos x) = \int_{0+0}^{\frac{\pi}{2}-0} \ln\frac{\sin 2x}{2}\,dx = \int_{0+0}^{\frac{\pi}{2}-0} (\ln(\sin 2x) - \ln 2)\,dx =$ ①

$= \underbrace{\int_{0+0}^{\frac{\pi}{2}-0} \ln(\sin 2x)\,dx}_{J} - x\ln 2\Big|_{0+0}^{\frac{\pi}{2}-0} = J - \frac{\pi}{2}\cdot\ln 2 = I - \frac{\pi \ln 2}{2}$ ②

$$y \overset{z=2x}{=} \frac{1}{2}\int_{2\cdot 0+0}^{\pi-0} \ln(\sin z)\,dz = \frac{1}{2}\Big[\underbrace{\int_{0+0}^{\frac{\pi}{2}} \ln(\sin z)\,dz}_{} + \int_{\frac{\pi}{2}}^{\pi-0} \ln(\sin z)\,dz\Big]$$

$z = \pi - u$
$dz = -du$
$z = \frac{\pi}{2},\ u = \frac{\pi}{2}$
$z \nearrow \pi - 0,\ u \searrow 0$

$$= \frac{1}{2}\Big[I + \underbrace{\int_{\frac{\pi}{2}}^{0+0} \ln(\sin u)(-du)}_{I}\Big] = I \quad (1)$$

$\Rightarrow (1) \ I = -\frac{\pi}{2}\ln 2$

(2) St. convergența integralelor improprii

a) $\displaystyle\int_0^{3-0} \frac{x^3+1}{\sqrt{9-x^2}}\,dx$

$f : [0,3) \to [0,+\infty),\ f(x) = \dfrac{x^3+1}{\sqrt{9-x^2}}$

P1:

$$\lambda = \lim_{x \nearrow 3} (3-x)^p \cdot f(x) = \lim_{x\nearrow 3} (3-x)^p \cdot \frac{x^3+1}{\sqrt{(3-x)(3+x)}} = \lim_{x\nearrow 3} (3-x)^{p-\frac{1}{2}} \cdot \frac{x^3+1}{\sqrt{3+x}}$$

- $p < 1,\ \lambda < +\infty \Rightarrow$ int. conv.
- $p \geq 1$ și $\lambda > 0 \Rightarrow$ int. div.

alegem $p = \frac{1}{2} < 1 \Rightarrow \lambda = \lim_{x\nearrow 3} \dfrac{x^3+1}{\sqrt{3+x}} = \dfrac{28}{\sqrt{6}} < +\infty$

$\Rightarrow \displaystyle\int_0^3 \frac{x^3+1}{\sqrt{9-x^2}}\,dx$

(alegere proastă $p = 1$, deoarece $\lambda = 0 \to$ nu putem det. conv.)

b) $\displaystyle\int_{0+0}^{+\infty-0} \frac{\operatorname{arctg} x}{x}\,dx =$ (improprie la ambele capete)

$$= \underbrace{\int_{0+0}^{1} \frac{\operatorname{arctg} x}{x}\,dx}_{I_1} + \underbrace{\int_1^{\infty} \frac{\operatorname{arctg} x}{x}\,dx}_{I_2} =$$

$f^{*}:[0,1]\to[0,+\infty),\ f^{*}(x)=\begin{cases}\frac{\operatorname{arctg} x}{x}, & x\in(0,1]\\ 1, & x=0\end{cases} \Rightarrow i_1=\int_0^1 f^{*}(x)\,dx \in \mathbb{R}$ ⇔

⇒ f^{*} este prel. prin cont. a lui f la $[0,1]$

⇔ i_1 conv. (int. Riemann)

pt i_2: $f:[1,+\infty)\to[0,+\infty),\ f(x)=\frac{\operatorname{arctg} x}{x}$

$\lambda=\lim_{x\to\infty} x^p\cdot f(x)=\lim_{x\to\infty} x^{p-1}\cdot\operatorname{arctg} x \underset{p}{=} \lim_{x\to\infty}\operatorname{arctg} x=\frac{\pi}{2}>0$

alegem $p=1$

- $p>1,\ \lambda<+\infty \Rightarrow$ int. conv.
- $p<1,\ \lambda>0 \Rightarrow$ int. div.

⇒ i_2 divergentă (adică are val. $+\infty$)

⇒ i_1+i_2 divergentă

c) $\int_{0+0}^{\pi-0} x\ln(\sin x)\,dx = \underbrace{\int_{0+0}^{\frac{\pi}{2}} -x\cdot\ln(\sin x)\,dx}_{i_1} + (-1)\underbrace{\int_{\frac{\pi}{2}}^{\pi-0} -x\ln(\sin x)\,dx}_{i_2} =$

p3

$=-i_1-i_2$

pt i_1: $f:\left(0,\frac{\pi}{2}\right]\to[0,+\infty),\ f(x)=-x\cdot\ln(\sin x)$

$\lambda=\lim_{x\searrow 0}(+x)^p\cdot f(x)=\lim_{x\searrow 0} -x^{p+1}\cdot\ln(\sin x)$

- alegem $p=0 \Rightarrow \lambda=\lim_{x\searrow 0} -x\cdot\ln(\sin x)=\lim_{x\searrow 0}\frac{-\ln(\sin x)}{\frac{1}{x}} \overset{\frac{\infty}{\infty}\ l'H}{=}$

$=\lim_{x\searrow 0}\frac{-\frac{\cos x}{\sin x}}{-\frac{1}{x^2}}=\lim_{x\searrow 0}\frac{x^2\cos x}{\sin x}=\lim_{x\searrow 0}\underbrace{\frac{x}{\sin x}}_{1}(x\cdot\cos x)=1\cdot 0=0$

$p<1,\ \lambda<+\infty \Rightarrow i_1$ conv.

pt. i_2: $f:\left[\frac{\pi}{2},\pi\right)\to[0,+\infty),\ f(x)=-x\cdot\ln(\sin x)$

$\lambda=\lim_{x\nearrow\pi}(\pi-x)^p\cdot f(x)=\lim_{x\nearrow\pi}(\pi-x)^p(-x)\ln(\sin x)=$

$$\underset{\substack{\pi \\ \pi - x = y}}{=} \lim_{y \searrow 0} y^p \cdot (y-\pi) \cdot \ln(\sin(\pi - y)) = -\pi \lim_{y \searrow 0} y^p \cdot \ln(\sin y)$$

• alegem $p = \frac{1}{2} < 1 \Rightarrow \lambda = 0 \underset{<+\infty}{}$ (calc. la curs) $\Rightarrow I_2$ conv.

$\Rightarrow I_1, I_2$ conv.

Tip EXAMEN ③ Studiați conv. integralei improprii

$$I(\alpha) = \int_0^1 \left(\frac{x}{1-x}\right)^\alpha dx, \ \alpha \in \mathbb{R}$$

Și calc. val lui $I\left(\frac{1}{2}\right)$

Cazul $\alpha > 0 \Rightarrow I(\alpha) = \int_0^{1-0} \left(\frac{x}{1-x}\right)^\alpha dx$

$f: [0,1) \to [0,+\infty), f(x) = \left(\frac{x}{1-x}\right)^\alpha$

$$\lambda = \lim_{x \nearrow 1} (1-x)^p \cdot f(x) = \lim_{x \nearrow 1} (1-x)^p \cdot \frac{x^\alpha}{(1-x)^\alpha} = \lim_{x \nearrow 1} (1-x)^{p-\alpha} \cdot \underbrace{x^\alpha}_{\to 1}$$

$$= \lim_{x \nearrow 1} (1-x)^{p-\alpha}$$

• alegem $p = \alpha \Rightarrow \underline{\lambda = 1}$ ($\lambda < +\infty$, $\lambda > 0$)

• dacă $p < 1$ ($\alpha < 1$) $\Rightarrow$ int. conv.

• dacă $p \geq 1$ ($\alpha \geq 1$) $\Rightarrow$ int. div.

Cazul $\alpha < 0$: $\Rightarrow I(\alpha) = \int_{0+0}^1 \left(\frac{x}{1-x}\right)^\alpha dx = \int_{0+0}^1 \left(\frac{1-x}{x}\right)^{-\alpha} dx$

p^3

$f: (0;1] \to [0;+\infty), f(x) = \left(\frac{x}{1-x}\right)^\alpha$

$$\lambda = \lim_{x \searrow 0} x^p \cdot f(x) = \lim_{x \searrow 0} \frac{x^{p+\alpha}}{\underbrace{(1-x)^\alpha}_{\to 1}} = \lim_{x \searrow 0} x^{p+\alpha}$$

• alegem $p = -\alpha \Rightarrow \lambda = 1$

• dacă $p < 1$ ($\alpha > -1$) $\Rightarrow$ int. conv.

• dacă $p \geq 1$ ($\alpha \leq -1$) $\Rightarrow$ int. div.

cazul $\alpha = 0$

$$i(0) = \int_0^1 \left(\frac{x}{1-x}\right)^0 dx = \int_0^1 1\, dx = x\Big|_0^1 = 1$$

$\Rightarrow$ int conv. (nu este nici improprie)

<u>În concluzie</u>:

$i(\alpha)$ conv $\Leftrightarrow \alpha \in (-1,1)$

$$i\left(\frac{1}{2}\right) = \int_0^{1-0} \sqrt{\frac{x}{1-x}}\, dx$$

dacă nu era în $(-1,1)$ integrala era $+\infty$ și nu mai trebuia calculată

Notăm $\sqrt{\frac{x}{1-x}} = t \geq 0 \Rightarrow \frac{x}{1-x} = t^2 \Rightarrow x = t^2 - t^2 \cdot x \Rightarrow x = \frac{t^2}{1+t^2}$

$$dx = \left(\frac{t^2}{1+t^2}\right)' dt = \frac{2t(1+t^2) - t^2 \cdot 2t}{(1+t^2)^2} dt = \frac{2t}{(1+t^2)^2} dt$$

$x = 0 \Rightarrow t = 0$

$x \to 1 \Rightarrow t \to \infty$

$$\Rightarrow i\left(\frac{1}{2}\right) = \int_0^{+\infty} t \frac{2t}{(1+t^2)^2} dt = \int_0^{\infty} t \cdot \left(-\frac{1}{1+t^2}\right)' dt \ominus$$

$$\left(\frac{1}{1+t^2}\right)' = \frac{-2t}{(1+t^2)^2}$$

$$\ominus \left(-\frac{t}{1+t^2}\right)\Big|_0^{\infty} + \underbrace{\int_0^{\infty} \frac{1}{1+t^2} dt}_{\operatorname{arctg} t|_0^{\infty}} = \lim_{u \to \infty} \frac{-t}{1+t^2}\Big|_0^u + \lim_{u \to \infty} \operatorname{arctg}\Big|_0^u =$$

$$= \lim_{u \to \infty} \frac{-u}{1+u^2} + \lim_{u \to \infty} \operatorname{arctg} u = 0 + \frac{\pi}{2} = \frac{\pi}{2}$$

4) (fct Γ gamma) Cons. integr. improprie

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} \cdot e^{-x} dx, \alpha \in \mathbb{R}$$

Dem urm. prop:

a) $\Gamma(\alpha)$ este conv, $\forall \alpha > 0$

$$\Gamma(\alpha) = \underbrace{\int_{0+0}^{1} x^{\alpha-1} \cdot e^{-x} dx}_{I_1} + \underbrace{\int_1^{\infty} x^{\alpha-1} e^{-x} dx}_{I_2}$$

pt I_1: $0<x\le 1 \Rightarrow e^x>1 \Rightarrow \frac{x^{\alpha-1}}{e^x}<x^{\alpha-1}$

$$\int_{0+0}^{1} x^{\alpha-1}\,dx = \frac{x^\alpha}{\alpha}\Big|_{0+0}^{1} = \frac{1}{\alpha}<+\infty \Rightarrow \text{int. convergentă}$$

crit. comp. (sub formă de ineg.) $\Rightarrow I_1$ conv.

Pt I_2: $f:[1,+\infty)\to(0,+\infty)$, $f(x)=x^{\alpha-1}\cdot e^{-x}$

$$\lambda=\lim_{x\to\infty} x^p\cdot x^{\alpha-1}\cdot e^{-x}=\lim_{x\to\infty} x^{p+\alpha-1}\cdot e^{-x}=0$$ (indiferent de exponentul lui x)

alegem $p=2>1$, $\lambda<+\infty \Rightarrow I_2$ conv.

$\Rightarrow \Gamma(\alpha)$ conv.

03.12.2021

Seminar 10 (Sem 8 ex)

cont. ext. ant.

a) $\Gamma(n+1)=n!$, $\forall n\in\mathbb{N}$ tema

) Expr. cu aj. fct. Γ val. urm.:

) $$\int_0^\infty e^{-x^2}dx \underset{\substack{x^2=t,\ x=\sqrt{t} \\ dx=\frac{1}{2\sqrt{t}}dt}}{=} \int_0^\infty e^{-t}\cdot\frac{1}{2\sqrt{t}}\,dt =$$