$n=6p+4 \to x_{6p+4} = \lim_{n\to\infty} \left[\left(1+\frac{2}{6p+5}\right)^{\frac{6p+5}{2}}\cdot ...\right]$ ... $\left(\frac{\sqrt3}{2}\right)$

$\text{LIM}(x_n) = \{e^{\sqrt3}, e^{-\sqrt3}, 1\}$

$\lim_{n\to\infty} \inf x_n = e^{-\sqrt3}$

$\lim_{n\to\infty} \sup x_n = e^{\sqrt3}$

15.10.2021

Seminar 3

ex. sem 2

4) Mult. pct. limită

a) $x_{2k} = (-1)^{2k}\cdot 2k \cdot \underbrace{\sin(k\pi)}_{0} = 0$

$x_{2k+1} = \underbrace{(-1)^{2k+1}}_{-1}\cdot(2k+1)\cdot\underbrace{\sin\left(k\pi+\frac{\pi}{2}\right)}_{(-1)^k} =$

$=(-1)^{k+1}\cdot(2k+1)$

$k=2p \Rightarrow x_{4p+1}=(4p+1)\cdot(-1)^{2p+1}=-(4p+1)\underset{(p\to\infty)}{\longrightarrow}-\infty$

$k=2p+1 \Rightarrow x_{4p+3}=(4p+3)(-1)^{2p+2}=(4p+3)\underset{p\to\infty}{\longrightarrow}\infty$

$\Rightarrow Lim(x_n)=\{-\infty, 0, +\infty\}$ şi $\lim\limits_{n\to\infty} \inf x_n=-\infty$

$\lim\limits_{n\to\infty} \sup x_n=+\infty$

b) analog

Criteriul Stolz - Cesaro: Fie $(a_n)_{n\in\mathbb{N}}$ un şir oarecare de nr. reale şi $(b_n)_{n\in\mathbb{N}}$ un şir strict monoton şi divergent.

Dacă $\exists \lim\limits_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}=l\in\overline{\mathbb{R}} \Rightarrow \exists \lim\limits_{n\to\infty} \frac{a_n}{b_n}=l$

1. Fie $(x_n)_{n\in\mathbb{N}}$ un şir cu termeni strict pozitivi. Dacă $\exists \lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n}=l\in[0,+\infty]$ at. $\lim\limits_{n\to\infty}\sqrt[n]{x_n}=l$.

Reciproca este adevărată? Nu este adevărată! (1)

Alegem $a_n=\ln x_n$, $b_n=n\nearrow+\infty$

$\lim\limits_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}=\lim\limits_{n\to\infty}\frac{\ln x_{n+1}-\ln x}{n+1-n}=$

$\lim\limits_{n\to\infty}\ln\frac{x_{n+1}}{x_n}=\underset{\text{fct. cont.}}{\ln}\left(\lim\limits_{n\to\infty}\frac{x_{n+1}}{x_n}\right)=\ln l$ cu convenţia $\ln 0=-\infty$, $\ln(+\infty)=+\infty$

$\underset{=}{\overset{S-C}{\Rightarrow}} \ln l=\lim\limits_{n\to\infty}\frac{a_n}{b_n}=\lim\limits_{n\to\infty}\frac{\ln x_n}{n}=\lim\limits_{n\to\infty}\frac{1}{n}\cdot\ln x_n=$

$=\lim\limits_{n\to\infty}\ln(x_n)^{\frac{1}{n}}=\ln\left(\underline{\lim\limits_{n\to\infty}\sqrt[n]{x_n}}\right) \Rightarrow \lim\limits_{n\to\infty}\sqrt[n]{x_n}=l$

(1) $\Rightarrow$ alegem $x_n=2^{(-1)^n}$, $\forall n\in\mathbb{N}$

$\lim\limits_{n\to\infty}\sqrt[n]{2^{(-1)^n}}=\lim\limits_{n\to\infty}2^{\frac{(-1)^n}{n}}=2^0=1$

$\lim\limits_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim\limits_{n\to\infty}2^{(-1)^{n+1}-(-1)^n}=\lim\limits_{n\to\infty}2^{(-1)^n}=\lim\limits_{n\to\infty}2^{(-1)^n(-2)}$

$= \lim_{n\to\infty} 2^{2(-1)^{n+1}} \nexists$

$\lim_{n\to\infty} (-1)^{n+1} \nexists$

(2) Calculați limita șirurilor:

a) $y_n = \dfrac{1+\frac{1}{2}+\dots+\frac{1}{n}}{\ln n}$

$a_n = 1+\frac{1}{2}+\dots+\frac{1}{n}$; $b_n = \ln n \nearrow +\infty$ (cresc.)(lim)

$\lim_{n\to\infty} \dfrac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \dfrac{\left(1+\frac{1}{2}+\dots+\frac{1}{n+1}\right)-\left(1+\frac{1}{2}+\dots+\frac{1}{n}\right)}{\ln(n+1)-\ln n} =$

$= \lim_{n\to\infty} \dfrac{\frac{1}{n+1}}{\cancel{n+1}\ln\frac{n+1}{n}} = \lim_{n\to\infty} \dfrac{1}{(n+1)\ln\frac{n+1}{n}} = \lim_{n\to\infty} \dfrac{1}{\ln\left(1+\frac{1}{n}\right)^{n+1}} =$

$= \lim_{n\to\infty} \dfrac{1}{\ln\left(1+\frac{1}{n}\right)^{n}\cdot\left(1+\frac{1}{n}\right)} = \dfrac{1}{\ln e} = 1 \overset{S.C.}{\Longrightarrow} \lim_{n\to\infty} y_n =$

$= \lim_{n\to\infty} \dfrac{a_n}{b_n} = 1$

b) $y_n = \sqrt[n]{n!}$

$x_n = n!$

$\lim_{n\to\infty} \dfrac{x_{n+1}}{x_n} = \lim_{n\to\infty} \dfrac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = +\infty \Rightarrow \lim_{n\to\infty} y_n =$

$= \lim_{n\to\infty} \sqrt[n]{x_n} = +\infty$

c) $y_n = \dfrac{\sqrt[n]{n!}}{n}$; $y_n = \sqrt[n]{\dfrac{n!}{n^n}}$; fie $x_n = \dfrac{n!}{n^n}$

$\lim_{n\to\infty} \dfrac{x_{n+1}}{x_n} = \lim_{n\to\infty} \dfrac{(n+1)!}{(n+1)^{n+1}} \cdot \dfrac{n^n}{\cancel{n!}} = \lim_{n\to\infty} \dfrac{n+1\cdot n^n}{(n+1)^{n+1}} =$

$= \lim_{n\to\infty} \dfrac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(\dfrac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(1-\dfrac{1}{n+1}\right)^n = \lim_{n\to\infty} \left[\left(1-\dfrac{1}{n+1}\right)^{-(n+1)}\right]^{\frac{-n}{n+1}}$

$= \cancel{e^{-n}}\ e^{\lim_{n\to\infty} \frac{-n}{n+1}} = e^{-1} \xrightarrow{\text{prop}(1)} \lim_{n\to\infty} y_n = \lim_{n\to\infty} \sqrt[n]{x_n} = e^{-1}$

3) Pt. șirurile de mai jos:

$a_n = \sum_{k=1}^{n} \frac{1+(-1)^k}{2}$, $b_n = n$, $\forall n \in \mathbb{N}^*$

calculați val. limitelor $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$, și $\lim_{n\to\infty} \frac{a_n}{b_n}$.

$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n+1} \frac{1+(-1)^k}{2} - \sum_{k=1}^{n} \frac{1+(-1)^k}{2}}{n+1-n} =$

$= \lim_{n\to\infty} \frac{1+(-1)^{n+1}}{2} \nexists$

$a_n = 0+1+0+1+\dots+\frac{1+(-1)^n}{2} = \begin{cases} \frac{n}{2}, & n \text{ par} \\ \frac{n-1}{2}, & n \text{ impar} \end{cases}$

$\left.\begin{array}{l} \lim_{k\to\infty} \frac{a_{2k}}{b_{2k}} = \lim_{k\to\infty} \frac{k}{2k} = \frac{1}{2} \\ \lim_{k\to\infty} \frac{a_{2k+1}}{b_{2k+1}} = \lim_{k\to\infty} \frac{k}{2k+1} = \frac{1}{2} \end{array}\right\} \Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{2}$

$\lim \left(\frac{a_n}{b_n}\right) = \frac{1}{2}$

$\boxed{-8}$ Contrazice acest lucru criteriul Stolz - Cesaro? Nu

Reciproca nu este adevărată!

4) Scrieți urm. serii infinite cu ajutorul simbolului suma:

a) $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\dots = \sum_{n=1}^{\infty} \frac{1}{2n-1} \sim \sum_{n=1}^{\infty} \frac{1}{n} = +\infty$

b) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots = \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-1/2} = 2$

c) $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2}$

5) Calc. suma urm. serii:

a) $\sum_{n=0}^{\infty} \frac{1}{n!} = 1+\frac{1}{1!}+\dots$

$(n+1)$ $\frac{-n}{n+1}$

$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} = \lim_{n\to\infty} \left(1+\frac{1}{1!}+\frac{1}{2!}+\dots+\frac{1}{n!}\right) = e$

b) $\sum_{n=1}^{\infty} \frac{1}{5^n}$; ~~$S = \lim_{n\to\infty}$~~

$\underbrace{\sum_{n=0}^{\infty} a^n}_{\lim S_n} = \frac{1}{1-a}$, $a \in (-1,1)$

$a = \frac{1}{5} \in (-1,1) \to \sum_{n=0}^{\infty} \frac{1}{5^n} = \frac{1}{1-\frac{1}{5}} = \frac{1}{\frac{4}{5}} = \frac{5}{4}$

$\sum_{n=1}^{\infty} \frac{1}{5^n} = -1 + \sum_{n=0}^{\infty} \frac{1}{5^n} = -1 + \frac{5}{4} = \frac{1}{4}$

c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n-1}}$

$S = \lim_{n\to\infty} \underbrace{S_n} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{k}+\sqrt{k-1}} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sqrt{k}-\sqrt{k-1}}{k-(k-1)} =$

$= \lim_{n\to\infty} \sum_{k=1}^{n} (\sqrt{k}-\sqrt{k-1}) = \lim_{n\to\infty} (\sqrt{1} - 0 + \cancel{\sqrt{2}} - \cancel{\sqrt{1}} + \cancel{\sqrt{3}} - \cancel{\sqrt{2}} + \dots +$

$+ \sqrt{n} - \sqrt{n-1}) = \lim_{n\to\infty} \sqrt{n} = +\infty$

d) $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$ =

$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{4k^2-1} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} =$

$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(2k+1)-(2k-1)}{(2k-1)(2k+1)} \cdot \frac{1}{2} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2} \left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) =$

$= \lim_{n\to\infty} \frac{1}{2} \left(\frac{1}{1} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \dots + \frac{1}{2n-1} - \frac{1}{2n+1}\right) = \lim_{n\to\infty} \frac{1}{2}\left(1 - \underbrace{\frac{1}{2n+1}}_{0}\right) =$

$= \cancel{\lim_{n\to\infty} \frac{1}{2}} (\frac{1}{2}$

e) $\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right) =$

$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=2}^{n} \ln\left(1-\frac{1}{k^2}\right) = \lim_{n\to\infty} \sum_{k=2}^{n} \ln\left(1-\frac{1}{k}\right)\left(1+\frac{1}{k}\right) =$

$= \lim_{n\to\infty} \sum_{k=2}^{n} \left(\ln\left(1-\frac{1}{k}\right) + \ln\left(1+\frac{1}{k}\right)\right) = \lim_{n\to\infty} \sum_{k=2}^{n} \left(\ln \frac{k+1}{k}\right.$

$$\Rightarrow \ln \frac{k+1}{k-1}\Big) = \lim_{n\to\infty}\Big(\ln \cancel{\frac{3}{2}} - \ln \frac{2}{1} + \ln \cancel{\frac{4}{3}} - \ln \cancel{\frac{3}{2}} + \dots + \ln \frac{n+1}{n} -$$

$$- \ln \cancel{\frac{n}{n-1}}\Big) = \lim_{n\to\infty}\Big(-\ln 2 + \ln \frac{n+1}{n}\Big) = -\ln 2$$

f) temă: $\sum_{k=1}^{\infty} \frac{n\cdot 2^{n}}{(n+2)!}$