Seminar 11

ex sem 10

① Studiați existența limitelor de fct.

a) $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{\sqrt{1+xy}-1} \overset{xy=t}{=} \lim\limits_{t\to 0} \frac{t}{\sqrt{1+t}-1} \overset{\text{l'H}}{=} \lim\limits_{t\to 0} \frac{1}{\frac{1}{2\sqrt{1+t}}} =$

$= \lim\limits_{t\to 0} 2\sqrt{1+t} = 2$

b) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

limite iterate: $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} \frac{x^2-y^2}{x^2+y^2}\right) = 1$

$\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0} \frac{x^2-y^2}{x^2+y^2}\right) = -1 \neq 1$

$\Rightarrow \nexists \lim\limits_{(x,y)\to 0_2} \frac{x^2-y^2}{x^2+y^2}$

altfel: $a^n = \left(\frac{1}{n}, \frac{1}{n}\right) \to (0,0)$, $n\to\infty$

$b^n = \left(\frac{1}{n}, 0\right) \to (0,0)$

$\lim\limits_{n\to\infty} a^n = 0$, $\lim\limits_{n\to\infty} b^n = 1$ $\left(\lim\limits_{n\to\infty} \frac{\frac{1}{n}^2-0^2}{\left(\frac{1}{n}\right)^2+0^2}\right)$

$\Rightarrow$ concluzia (nu $\exists$ lim)

c) $\lim\limits_{(x,y)\to(\infty,\infty)} \frac{x^2+y^2}{x^4+y^4} = ?$

$f(x,y) = \frac{x^2+y^2}{x^4+y^4}$

$|f(x,y)-0| = \frac{x^2+y^2}{x^4+y^4} = \frac{x^2}{x^4+y^4} + \frac{y^2}{x^4+y^4} \leq \frac{x^2}{x^4} + \frac{y^2}{y^4} =$

$= \frac{1}{x^2} + \frac{1}{y^2} \to 0$, $(x,y)\to(\infty,\infty)$

$\Rightarrow \lim_{(x,y)\to(\infty,\infty)} f(x,y) = 0$

d) $\lim_{(x,y)\to(0,0)} \dfrac{x \cdot \sin(x^2-y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \dfrac{\sin(x^2-y^2)}{x^2-y^2} \odot$

$\left(\lim_{t\to 0} \dfrac{\sin t}{t} = 1\right)$

$\odot \dfrac{x(x^2-y^2)}{x^2+y^2} = \lim_{\substack{t\to 0 \\ x^2-y^2=t}} \dfrac{\sin t}{t} \cdot \lim_{(x,y)\to(0,0)} \dfrac{x(x^2-y^2)}{x^2+y^2}$

$g(x,y) = \dfrac{x(x^2-y^2)}{x^2+y^2}, \ |g(x,y)-0| = \dfrac{|x(x^2-y^2)|}{x^2+y^2} =$

$= \dfrac{|x^3 - xy^2|}{x^2+y^2} \le \dfrac{|x^3| + |-xy^2|}{x^2+y^2} = \dfrac{x^2 \cdot |x|}{x^2+y^2} + \dfrac{y^2 \cdot |x|}{x^2+y^2} \le$

$\le 2|x| \to 0, \ (x,y)\to(0,0) \Rightarrow \lim_{(x,y)\to(0,0)} g(x,y) = 0$

e) $\lim_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{xy} =$

$A = \{(x,y)\in\mathbb{R}^2 \ / \ x \neq 0 \text{ și } y \neq 0\}$

$a^n = \left(\frac{1}{n}, \frac{1}{n}\right) \to (0,0), \ \lim_{n\to\infty} f(a^n) = \lim_{n\to\infty} \dfrac{\frac{1}{n^3}+\frac{1}{n^3}}{\frac{1}{n^2}} =$

$= \lim_{n\to\infty} \dfrac{2}{n} = 0$

$b^n = \left(\frac{1}{n}, \frac{1}{n}\right) \to (0,0)$

$\lim_{n\to\infty} f(b^n) = \lim_{n\to\infty} \dfrac{\frac{1}{n^6}+\frac{1}{n^3}}{\frac{1}{n^3}} = \lim_{n\to\infty} \dfrac{1+n^3}{n^3} = 1$

sunt diferite $\Rightarrow \nexists \lim_{(x,y)\to 0_2} f(x,y)$

f) $\lim\limits_{(x,y)\to(1,1)} \frac{(x-1)(y-1)}{xy-1}$

$n\in\mathbb{N}$

$x-1=u$

$y-1=v$

$\lim\limits_{(u,v)\to(0,0)} \frac{u\cdot v}{(u+1)(v+1)-1} = \lim\limits_{(u,v)\to(0,0)} \underbrace{\frac{u\cdot v}{uv+u+v}}_{g(u,v)} =$

$a^n = \left(\frac{1}{n}, \frac{1}{n}\right)\to(0,0),\ n\to\infty;\ g(a^n) = \frac{\frac{1}{n^2}}{\frac{1}{n^2}+\frac{2}{n}} = \frac{1}{2n+1}$

$b^n = \left(\frac{1}{n}, -\frac{1}{n}\right)\to(0,0);\ n\to\infty;\ g(b^n) = \frac{-\frac{1}{n^2}}{-\frac{1}{n^2}+0} = 1\to 1$

$\Rightarrow \nexists \lim\limits_{(u,v)\to(0,0)} g(u,v)$

g) $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{(x+y+z)^2}{x^2+y^2+z^2}$ (temă) (limita nu $\exists$)

2) Se dă fct. $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y) = \begin{cases} x\cdot\cos\frac{1}{y^2} + y\cdot\cos\frac{1}{x^2}, & xy\neq 0 \\ 0, & xy=0 \end{cases}$

Este fct. cont în $(0,0)$? Dar în $(1,0)$?

f cont în $(0,0) \overset{def}{\Leftrightarrow} \exists \lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)=0$

$\lim\limits_{(x,y)\to 0_2} \left(x\cos\frac{1}{y^2} + y\cdot\cos\frac{1}{x^2}\right) = 0 \Rightarrow f$ cont. în $(1,0)$

$\left|x\cos\frac{1}{y^2} + y\cos\frac{1}{x^2} - 0\right| \leq |x|\cdot\underbrace{\left|\cos\frac{1}{y^2}\right|}_{\leq 1} + |y|\cdot\underbrace{\left|\cos\frac{1}{x^2}\right|}_{\leq 1} \leq$

$\leq |x|+|y| \to 0 \quad (x,y)\to(0,0)$

$(x,y)\to(1,0)$ ... $y^2 + x \cdots \frac{}{x^2}) = \lim\limits_{(x,y)\to(1,0)} x\cdot\cos\frac{1}{y^2} +$

$+ \lim\limits_{(x,y)\to(1,0)} y\cdot\cos\frac{1}{x^2}$

$= 0$

$a^n = \left(1, \frac{1}{\sqrt{2\pi n}}\right) \to (1,0)$, $n\to\infty$

$g(x,y) = x\cdot\cos\frac{1}{y^2}$, $g(a^n) = \cos(2\pi n) = 1$

$b^n = \left(1, \frac{1}{\sqrt{2\pi n + \frac{\pi}{2}}}\right) \to (1,0)$, $g(b^n) = \cos(2\pi n + \frac{\pi}{2}) = 0$

$\Rightarrow$ limita nu $\nexists$

③ Verificați dacă fct. urm. își ating val. extreme și det. aceste valori

$f: A\to\mathbb{R}$, $A\subseteq\mathbb{R}^n$
$\inf f(A)$, $\sup f(A)$ val. extreme
se ating dacă $\exists$ $x,y\in A$ a.î.
$f(x) = \inf f(A)$
$f(y) = \sup f(A)$

a) $f:(0,+\infty)^2\to\mathbb{R}$, $f(x,y) = \frac{x}{y} + \frac{y}{x}$

$A = (0,+\infty)^2$, $x,y > 0$

$f(x,y) = \underbrace{\frac{x}{y} + \frac{y}{x} \geq 2\sqrt{\frac{x}{y}\cdot\frac{y}{x}}}_{\text{ineg. mediilor}} = 2 = \inf f(A) =$

$= f(1,1)$ – se atinge

(A nu este compactă
A nu este mărginită)

fie $a^n = (2,n)\in A$, $f(2,n) = \frac{2}{n} + \frac{n}{2} \to \infty$, $(n\to\infty) \Rightarrow$

$\Rightarrow \sup f(A) = +\infty$ – nu se atinge

b) $A = B(0_2, 1) = \{(x,y)\in\mathbb{R}^2 \mid \|(x,y)\| < 1\}$

$f: B(0_2,1)\to\mathbb{R}$, $f(x,y) = \frac{1}{x^2+y^2+1}$

$x^2+y^2<1$

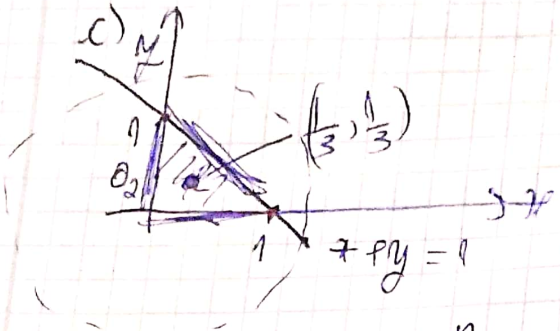
$0 \leq x^2+y^2 < 1 \overset{+1}{\Rightarrow} 1 \leq x^2+y^2+1 < 2 \Rightarrow 1 \geq f(x,y) > \frac{1}{2}$

$\inf f(A) = \frac{1}{2}$, $\sup f(A) = 1 = f(0,0)$

c) $f: A\to\mathbb{R}$, $f(x,y) = xy(1-x-y)$, $A = \{(x,y)\in\mathbb{R}^2 / x \geq 0$,
$y\geq 0$, $x+y\leq 1$

B nu este compactă (A nu este închisă)

c)



A mărg.

A închisă ($\text{fr} A \subseteq A$) $\Rightarrow$ A compactă, f cont. $\Big\}$ $\xRightarrow{\text{T. Weierstrass}}$ f își atinge val. extreme

$f(x,y) = xy(1-x-y)$

$\underbrace{x \geq 0}_{a}, \underbrace{y \geq 0}_{b}, \underbrace{1-x-y \geq 0}_{c}$ $\Big\}$ $\Rightarrow f(x,y) \geq 0 = \inf f(\ldots)$

$\inf f(A) = f(0,0)$ (și încă pct. de pe frontieră) $\leftarrow$ dat. Δ

$\frac{a+b+c}{3} \geq \sqrt[3]{abc} \Rightarrow \frac{1}{3} \geq \sqrt[3]{f(x,y)} \Rightarrow$

$\Rightarrow f(x,y) \leq \frac{1}{27} = \sup f(A) = f\left(\frac{1}{3}, \frac{1}{3}\right)$

Egalitate dacă $a = b = c$

$\Rightarrow x = y = 1-x-y \Rightarrow x = y = \frac{1}{3}$

4) Se dă mult. $A = \{(x,y) \in [-1,1]^2 / x \neq y\}$ și fct.

$f: A \to \mathbb{R}, f(x,y) = \frac{x^2+y^2}{(x-y)^2}$

(MDN