Frequency dependent circuits

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21/03/22

Introduction

Context

Circuits, and electricity more generally underpins the operation of our modern existence: from computation and automation to permitting construction and engineering on scales previously unimaginable. Modern electronic systems are complex, mainly relying on digital systems, that is two state electronic systems which are either off or on, these systems are still fundamentally voltages and currents: a voltage of 0 V is recorded as "off" and a voltage of 5 V is recorded as "on". As there is an ever-constant driver for devices do things faster and more efficiently, one might wonder what limitations exist, notably, is there a physical limit on how quickly one can do something in an electrical system, and what is the origin of this limit.

Background

When looking at circuits, the most important quantities are those of voltage (V) and current (I) which describe the difference in electrical potential between two points and the movement of charge respectively. Circuits confine the broader problem of free charges and applied potentials to set conduction pathways, which have circuit elements. For example, a battery or power supply can provide a potential difference, and by connecting the terminals of different potentials with a wire, the flow of charge is confined to the wire. A fundamental relation in the proportionality between voltage and current, which makes intuitive sense as if the difference in energy between two points increases, more charge will seek to find the low-energy state. The constant of proportionality between the voltage and current is the resistance (R) which describes how difficult it is for charge to travel through the system, with conductors having a low resistance (roughly $1\ \Omega$), whereas insulators have a very high resistance ($10^6\ \Omega$).

The system above has one critical assumption: the applied potential remains constant. If the voltage changes with time, which includes a system with a fixed voltage that is turned on or off using a switch, the system becomes dynamic. In the case of a constant voltage that is switched, the system is *transient* until a steady state is reached, and a time-varying voltage will produce a dynamic system. From Maxwell's equations, we know a time varying electric potential - and thus field - will produce a magnetic field and vice versa. Depending on the geometry of circuit components, they may provide pathways for the storage of magnetic or electric energy, for example, in coils of wire (a solenoid) or conductors separated by insulation (a capacitor) respectively. These components are termed *reactive*, as how they respond to a given potential depends on the frequency of the potential, and it is this phenomenon that we are going to investigate in this experiment.

Plan

We want to investigate the behaviour of circuits when driven at different frequencies. This means that we will have to make a circuit where the frequency of the input potential can be varied, and then have some way of measuring how the circuit reacts. With this system, we can then collect data at a range of frequencies and compare this to the behaviour we would expect

22/03/22
Partner: Revolio Clockberg

Experiment

We are going to need to build circuits which have some non-trivial frequency response, but first, we need to figure out how to measure the response! What does that even mean?

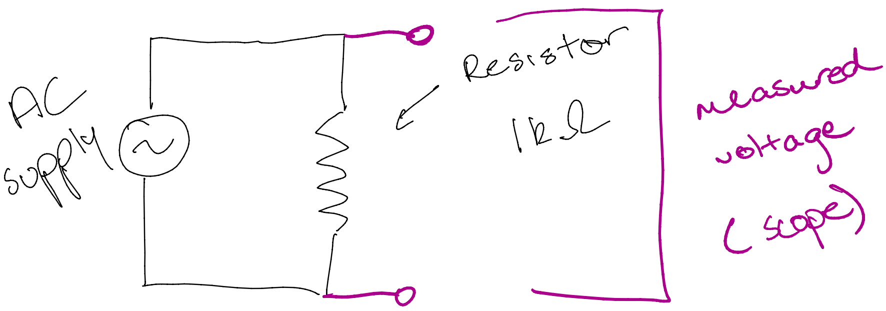
After having a chat with Revolio and Jessica (our demonstrator) we concluded that the best way to proceed was to build a circuit and measure the voltage drop across a given component as a function of frequency, as this will provide a way to directly measure the "activity" of the circuit. Another idea was to measure the current in the circuit, but we decided that measurement of the voltage is:

- More configurable: as voltage measurements are taken in parallel rather than in series, it means that we can test individual elements or collections of elements
- More convenient: we have access to a digital oscilloscope, which explicitly measure voltages, and our current measurement device would be an ammeter designed for many amps, which in this case is much less precise, but also the oscilloscope provides a way to accurately record time dependent data.

Note that in the above statement when I say that the ammeter is less precise, this is not a problem with the instrument, just we will be looking at small (mA) currents and the ammeters are designed for many amp. Oscilloscopes can "zoom", so this is a better way of doing it.

Circuit design and construction

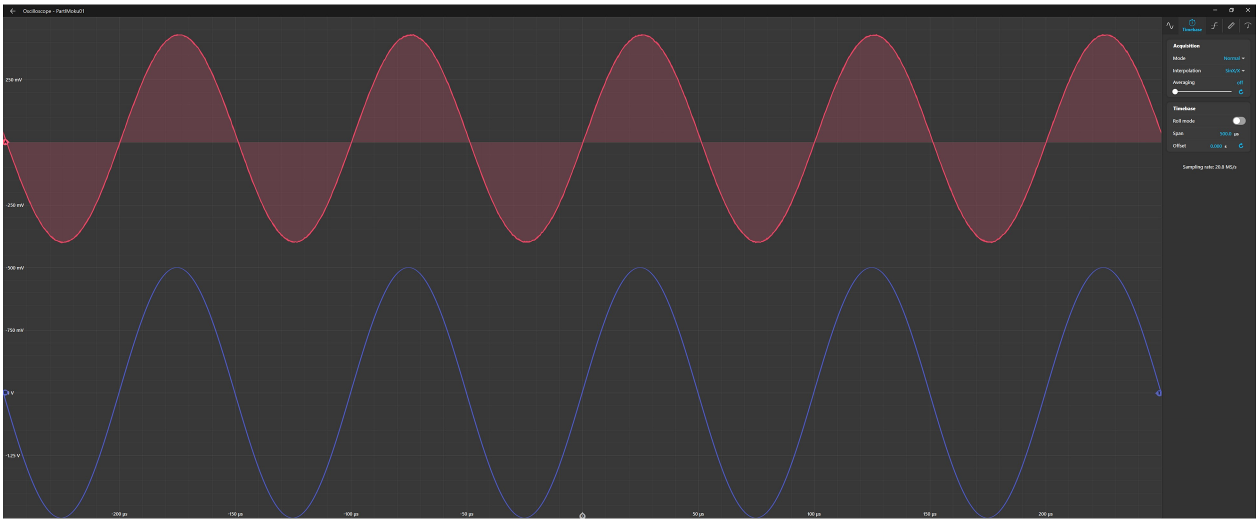
I have not really had any experience building circuits, so I want to ensure I don't do anything silly. In the first instance, we are going to build the simplest circuit possible:



This is an AC supply connected directly into a resistor, and we are measuring the voltage across the resistor. As a resistor has no frequency dependence, we would expect that Ohm's law, $V = IR$ should hold.

The explicit setup was using a Moku:Go, which I had never seen before: it is some kind of multi-device, but we were able to use the documentation on the lab website to get it running: we were using the device in oscilloscope mode, which has a signal generator built in, so we could both create and measure the waveform, which is pretty neat!

As a test, we set a 10 kHz sinusoid as the input waveform and the observed the output:



Oscilloscope trace of an AC signal as measured across a resistor

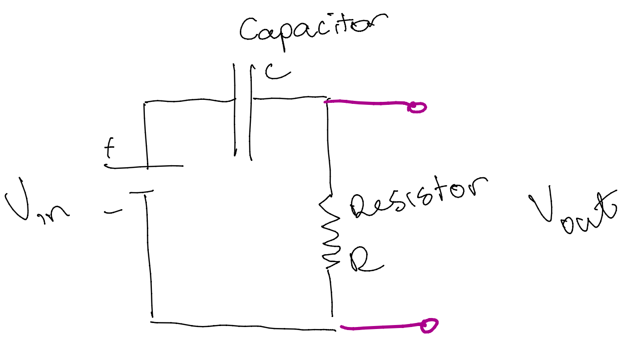
The above is a direct screenshot from the Moku:Go software, and the top (red) signal is the measured waveform, with the bottom (blue) waveform being the output waveform. It is clear that the waveforms are mostly identical, although there is a slight voltage drop, which must be due to the resistance of the wires and any connections made in the circuit: the resistor is the only element and all voltage must dropped across this, and since this isn't the case, there must be other resistive elements in the circuit.

We repeated the above measurements for a range of frequencies between 0 and 1 MHz, and at all points the input and output signals remained identical but for a slightly lower amplitude on the measured signal. Once again, as the resistance is independent of frequency, this is to be expected.

We now moved to make the simplest circuit which has a reactive component, the RC circuit, which is comprised of a resistor and a capacitor. What do we expect will happen when we do this? Well, by definition, in a DC circuit the voltage drop across a capacitor (V_C) is given by

$$V_C = \frac{q}{C}$$

where q is the charge and C is the capacitence. So given the circuit below:



it must be true that

$V_{in} = V_C + V_R$

$$= \frac{q}{C} + IR$$

$$\Rightarrow I = \frac{1}{R}\left(V_{in} - \frac{q}{C}\right) = \frac{dQ}{dt}$$

$$\therefore \int_{q_0}^{q} \frac{dq'}{q' - V_{in}C} = -\frac{1}{RC}\int_0^t dt$$

$$\Rightarrow \log\left(\frac{q - V_{in}C}{q_0 - V_{in}C}\right) = -\frac{t}{RC}$$

This is what a capacitor Does!

$$\Rightarrow q(t) = V_{in}C + (q_0 - V_{in}C)e^{-t/RC}$$

$$V_{in}C = q_{max} \leftarrow$$ maximum charge that circuit can hold

$$\therefore q(t) = q_0 e^{-t/RC} + q_{max}\left(1 - e^{-t/RC}\right)$$

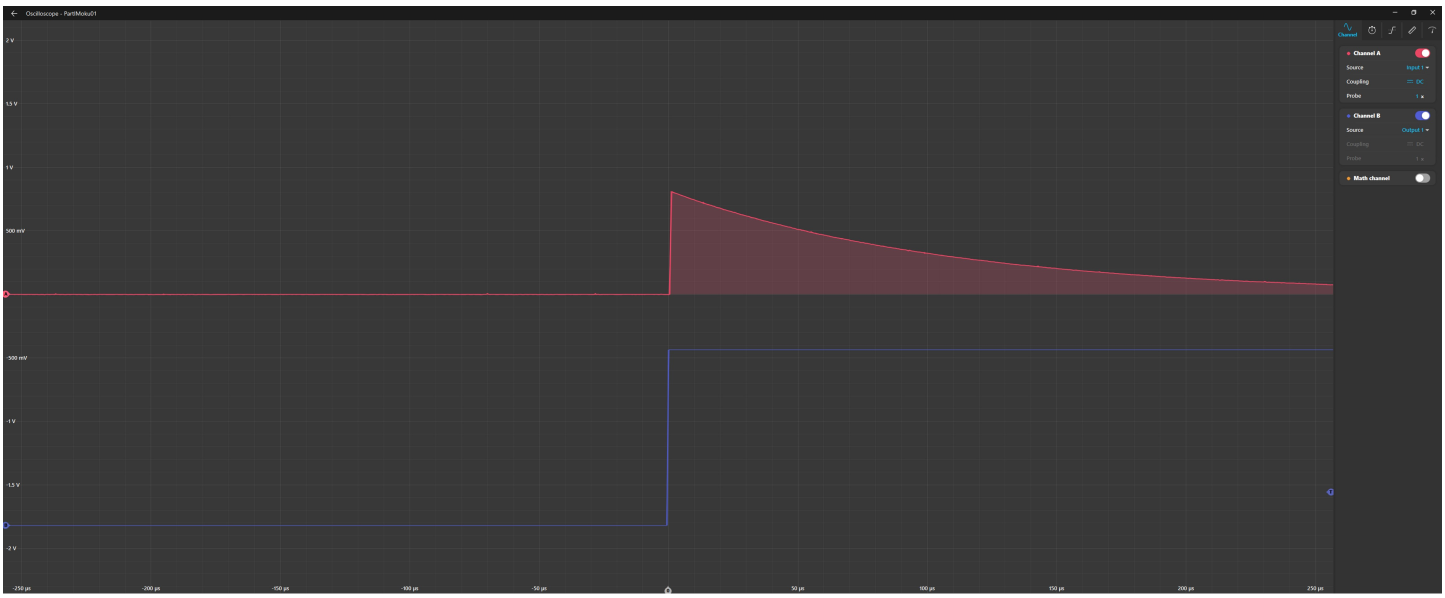
and from this we have

$$I(t) = (I_{max} - I_0)e^{-t/RC} \qquad \left(I = \frac{dq}{dt}\right)$$

this is the general case, so we just want to check a specific case and fix the initial conditions. Let's try charging, namely, turning the circuit on, such that $q_0 = 0$. Then we would expect that

$$I(t) = I_{max}e^{-\frac{t}{RC}}$$
$$V(t) = V_{in}\left(1 - e^{-\frac{t}{RC}}\right)$$

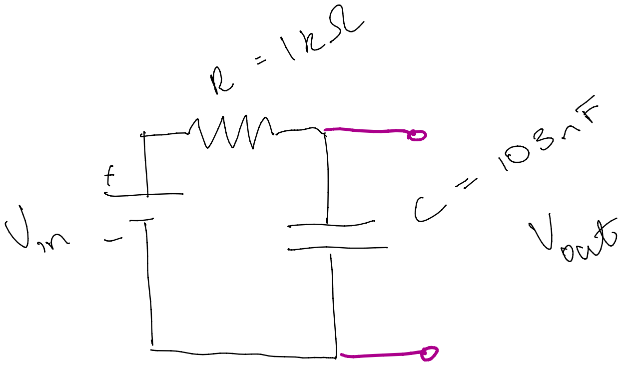
So let's give it a whirl. The circuit as set shown above was set up, and shown below (red) is the measured voltage.

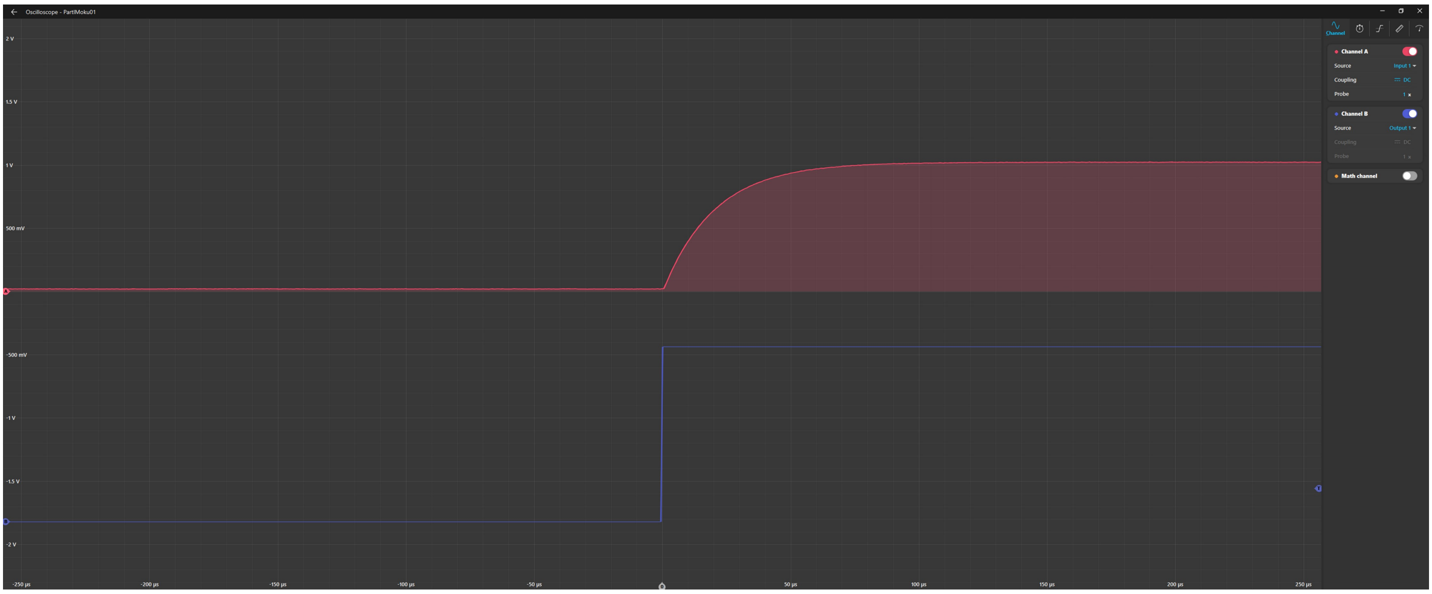


Signal as measured across a resistor in series with a capacitor, with the input voltage switching on

This is not at all what I would expect! The voltage instantaneously jumps up, and then decays, which is more or less the opposite than that which we predicted above. Why would a current start flowing and then stop? I need to think about this.

So I realise the mistake: the circuit as drawn above has a measurement of the voltage drop across the resistor, but I calculated the expected voltage drop over the capacitor. Altering the circuit to the following:

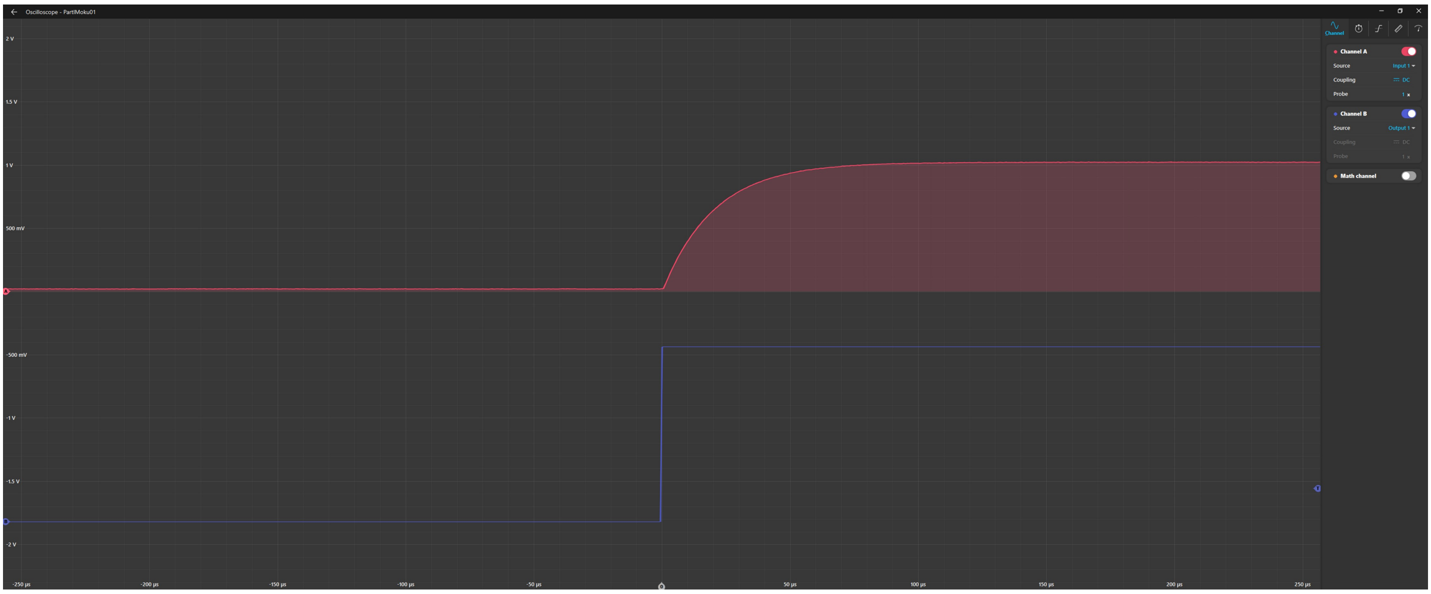




Signal as measured across a capacitor in series with a resistor, with the input voltage switching on

The Moku:Go was able to measure a rise time of roughly 44 μs, and the rise time is related to the RC constant, which is this case is $10^3 \times 103 \times 10^{-9} \approx 100\ \mu$s. According to the [Moku:Go oscilloscope manual](), the rise time measures the time taken to rise from 10% to 90% of the maximum value, which when computed means that the rise time $\approx 2.2 \times RC$, so the measured RC value is roughly 20 μs. Whilst this is not meant to be a rigorous measurement - rather just a sanity check that we understand what is happening - we have an issue. Why are they different?

Thinking about things a little more, it is clear that there is a connection between the Moku:Go input and output. I think this might be an issue: if the resistance of the output channel is lower than the resistance of the resistor, then the circuit would be made with the external capacitor and then internally in the Moku:Go. A simple was to test this would be to disconnect the resistor, and see if there is a signal. I am going to try that now.



Signal as measured across a capacitor without a resistor, with the input voltage switching on

Very problematically, the signal is identical with or without the resistor! What is going on?!

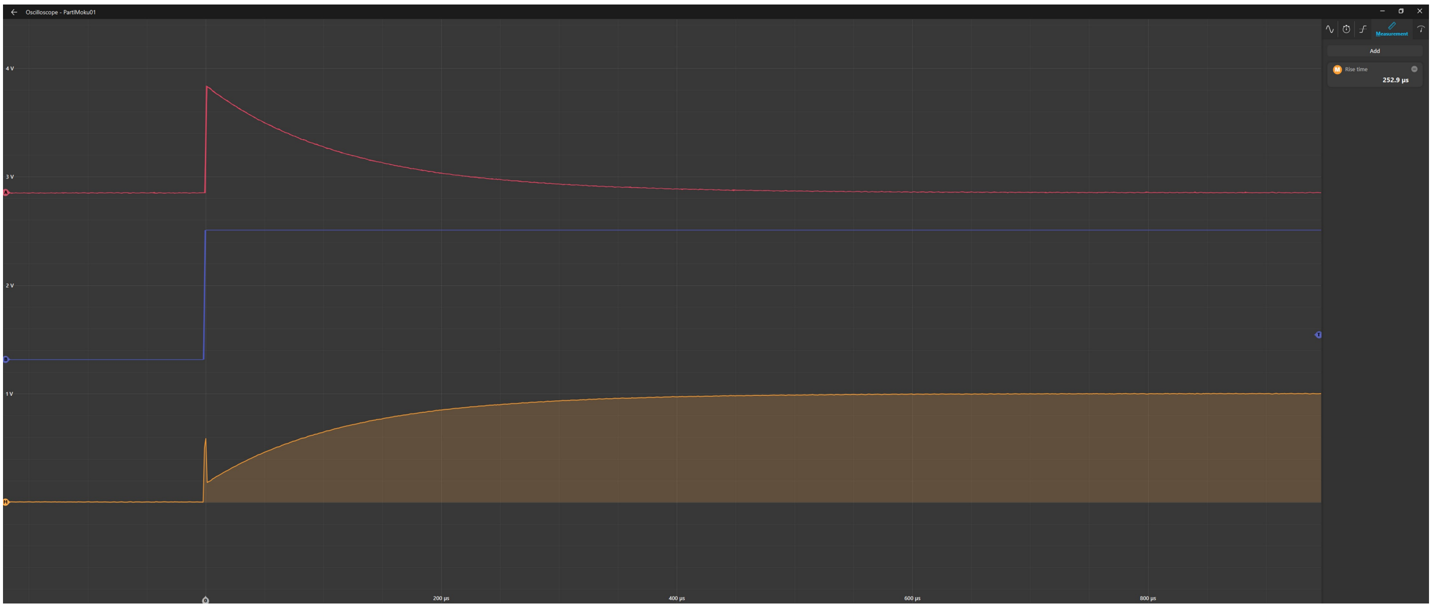
After some discussion, it would seem as though my earlier thoughts were probably on the right track: the connection between the input and output of the Moku:Go is definitely causing issues. It is a good time to discuss something lurking in the background of this prac: impedance. Impedance is in some sense a generalised impedance (maybe?) My understanding is that whilst resistance occurs because of material properties - how difficult it is for electrons to move through/on a material - whereas impedance occurs due to the geometry of the conductor, and thus energy storage in electric and magnetic fields. Mathematically, impedance is given by

$$Z_R = R$$
$$Z_L = j\omega L$$
$$Z_C = \frac{1}{j\omega C}$$

For a resistor, an inductor, and a capacitor respectively (why on earth do we use j for the complex unit here???). The reason these values are complex, as is always the case, is because phase is important: there will be a phase difference between the voltage and current when such components are present. The reason for this is because the physics of these elements are clearly different. In the case we are currently considering, when a voltage increases, the capacitor will charge, and thus lots of current will flow, but it becomes increasingly difficult to get more charge onto the capacitor electrodes, and thus the current will decrease and eventually cease. Moving from switching circuits from off to on and going to a circuits driven a some frequency, this means that for low frequencies, a capacitor will charge and discharge, but once charged or discharged, no current will flow. A logical extension of this is that if the frequency is high, the capacitor will neither charge or discharge an effectively become invisible. This is pretty neat, one could imagine that this effect would be useful as a frequency filter or something.

Tying this to the problem at hand, from looking in the Moku:Go manual, the output has an impedance of $200\ \Omega$, which is less that the $1\ \mathrm{k}\Omega$ of our resistor, and as they are effective connected in parallel, all current will pass through the output - this is probably not good! But if one calculates what the time constant would be given an impedance of $200\ \Omega$, that would be $.103 \times 200 = 20.6\ \mu$s which leads to a rise time of roughly $45\ \mu$s, which is exactly what we measured... So I guess we figured it out, but that is complicated!

To solve the issue, we are going to go back to monitoring the voltage drop across the resistor - what we had initially - which avoids the oscilloscope and signal generator being connected in parallel, and we just accept that if we want to look at the voltage across the capacitor, we can get this from $V_C = V_{in} - V_R$. With a bit of playing, I realised that we could get the oscilloscope to display exactly this:



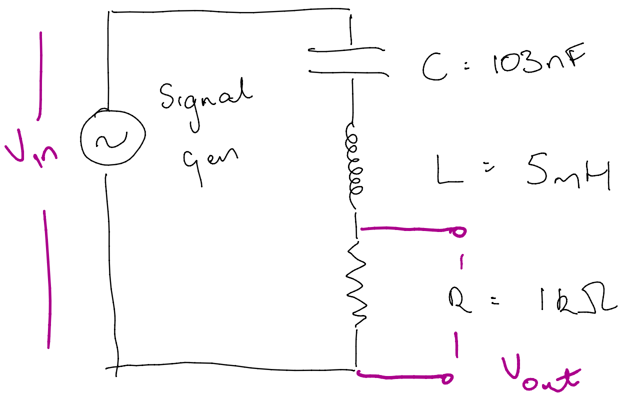
The voltage across a resistor connected in series with a capacitor, with the difference between the input voltage (blue) and the measured voltage (red) shown in yellow.

The bottom curve is exactly what we want, and if we look at the rise time, we see it is $250\ \mu s$ which corresponds to an RC constant of roughly 110 μs , which is more-or-less what we would expect from a circuit with $R = 1\ \mathrm{k}\Omega$ and $C = 103\ \mathrm{nF}$. YAY!

RLC circuit

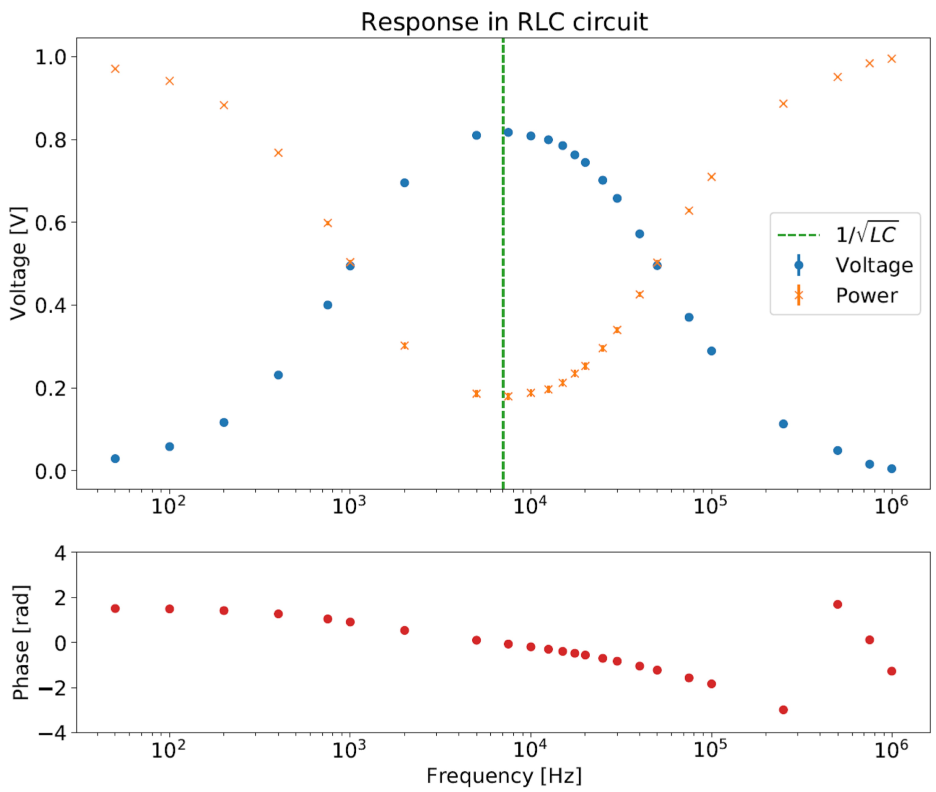
I think we now have a firm handle on charging (and discharging) RC circuits, so we are going to set out to perform our main investigation for this experiment: namely how does a general circuit, with resistance, inductance, and capacitance react with to an input voltage as a function of frequency. In the discussion of impedance above, the effect of a capacitor is significant at lower freqeuncies, and with a similar argument, one can see that the effect of an inductor will be pronounced at high frequencies, so the question will be what does it look like in the region where both a playing a role.

To investigate the RLC circuit, we take the setup the we were using previously, and we can add an inductor in series. In this case, we have an inductor which is marked at $5\ \mathrm{mH}$, and the circuits looks as follows:



Then we can take a series of measurements of the signal as measured across the resistor. The frequency was varied between the limits of where the circuit seemed to respond: at low frequency, there was no voltage - the capacitor would have been held at charge - and the first non-zero amplitude was recorded at a frequency of 50 Hz and the frequency was increased to 1 MHz, but in roughly equally spaced "logarithmic steps", as the resulting plot would need to be plotted on a logarithmic scale to make sense. A peak was observed in the amplitude, so care was taken to trace out the shape of the peak and find the maximum.

The voltage was recorded using the *measure* function of the Moku:Go, which was also used to measure the phase difference between the input and output - and hence the current and voltage. A plot of the measured voltage and phase is shown below:



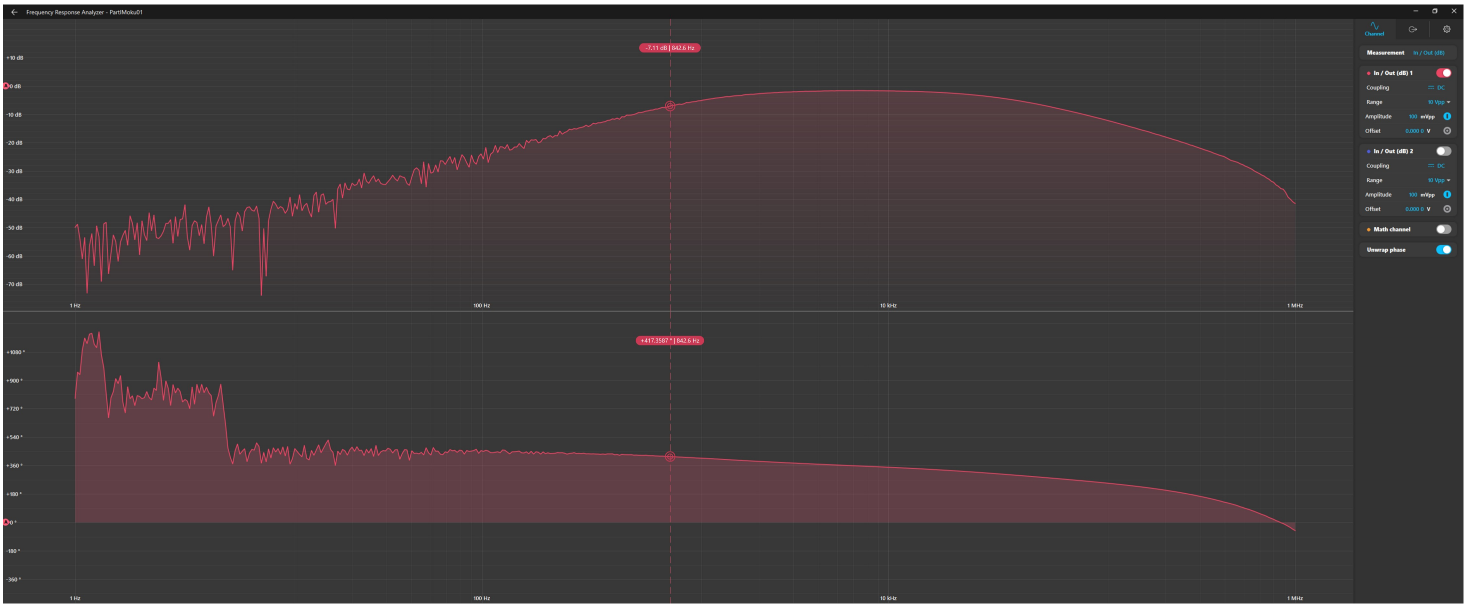
A plot of the response of a series RLC circuit, as measured across the resistor

Error bars are present on the plot; however, are smaller than the plot points as shown. In this case, the uncertainties come from the observed variation in the measured values, which were observed to be of the order 1% for the voltage and 0.1 degrees for the phase. The notebook used for all calculations can be accessed here.

Also shown on the plot is the proportion of power which is dissipated in the circuit in elements other than the resistor, illustrating that when the voltage drop across the resistor is large, it is necessarily small in the other components. The reason this occurs is exactly as described previously: at low frequencies, the capacitor has a large impedance and hence the bulk of the power is dissipated there, and at high frequencies, the inductor has high impedance and the bulk of the power is dissipated there, and because of the associated phase lead and phase lag between the voltage and current for an inductor and capacitor respectively, there is a magical spot in-between the dominance of the capacitor and inductor where their impedances cancel and all power is dissipated in the resistor. The frequency at which this occurs is the resonant frequency, and from the relations of impedance, we can see that this will occur when $Z_l = Z_C \rightarrow j\omega L = \frac{1}{j\omega C} \rightarrow \omega = \sqrt{\frac{1}{LC}}$, and marked on the plot is the theoretical value at which resonance should occur. There is good agreement between the theory and the observed behaviour, but I note that the peak is very wide, that is the circuit is resonant over a large range of frequencies. The width of this peak is going to be related to the resistance, as the charging/discharging time is related to the product of $R \times C$, so if one was hoping to build a circuit which responded at only a single frequency, this would not be a good design, whereas if you wanted a circuit that responded over a range of frequencies, for example between $1\ \mathrm{kHz}\ - 50\ \mathrm{kHz}$, this would be a good circuit.

I am not entirely clear about the role the phase plays, or what significance it holds, but it is clear that the resonant peak corresponds to zero phase, which makes sense as the impedance of the inductor and capacitor cancel at this point, meaning the impedance is real and thus the phase difference between the current and voltage is zero. Phase values outside this range suggest whether the system is dominated by either capacitance or inductance, but the explicit uses of phase sensitive circuits warrants further investigation.

In finalising our measurements, we discovered that the Moku:Go has a frequency response analyser, so we quickly activated the functionality and the below plot was returned. The plots show almost the exact same behaviour as our manually produced data; however, we note that the y-axis is logarithmic (explaining the disparity in the shape of the curve) and that the data is down to much lower frequency ($1\ \mathrm{Hz}$) and behaviour in this region was not investigated as part of our experiment.



The circuit response as measured with the frequency response function of the Moku:Go

When electrical components are allowed to store energy, that is they have non-trivial geometries such that their capacitance or inductance is non-zero, the response of the circuit is going to be frequency dependent. On the one hand, this is obviously more complicated than the static case, but with complexity comes the opportunity for sophistication. Modelling the charge in these systems allows us to predict the behaviour of a given circuit, and notable phenomena of capacitive charging/discharging on the timescale of RC and resonance occurring at a frequency of $1/\sqrt{LC}$ permit the creation a wonderous devices. For example, one could imagine using a large value of RC to smooth an AC voltage to a DC voltage, or tune an amplifier to amplify only the signals with a given frequency. It is also clear that due to the impendence of a system, if one wants to observe phenomenon on short timescales, ensuring a small impedance of the testing system is critical. This is because evne though the true signal may be "fast", the measuring system will measure a "slow" signal with a timescale dominated by the detection system. The implications are equally true for trying to switch things on really fast, as any capacitance is going to limit how quickly a system can turn on and off.

Reflections

In my last log, it was mentioned that I did not explicitly address the physics of the experiment, explaining what I seeing, rather than commenting on not only what I was seeing, but also why I was seeing it. I have put much effort into focusing on the "why" which hopefully shows through.

It was also noted that I did not provide any explicit analysis or calculations, which I did perform, it was just that everything was performed using python in a jupyter notebook. I have uploaded the notebook to github, which was linked in the text, and can also be accessed here.