Heat Advection and Diffusion

1 Heat Diffusion Through Idealized Rod

The diffusion of heat through a perfectly insulated, heat-conducting rod is described by linear PDE

$$\frac{\partial \mathbf{T}}{\partial t}(x,t) = \frac{\partial^2 \mathbf{T}}{\partial x^2}(x,t), \quad t \geq 0, \quad x \in [0,1], \tag{1}$$

with Neumann boundary conditions

$$\frac{\partial \mathbf{T}}{\partial x}(0,t) = 0 \text{ and } \frac{\partial \mathbf{T}}{\partial x}(1,t) = \mathbf{u}(t),$$

and measured output

$$\mathbf{y}(t) = \mathbf{T}(0,t)$$

where $\mathbf{T}(\mathbf{x},t)$ is the temperature at distance x from origin and time t, $\mathbf{u}$ is the input function (supplied heat) and measured output $\mathbf{y}(t) = \mathbf{T}(0,t)$.

1.1 Transfer Function Derivation

To compute the transfer function, we apply Laplace transforms on (1) and we get

$$s\hat{\mathbf{T}}(x,s) = \frac{\partial^2 \hat{\mathbf{T}}}{\partial x^2}(x,s) \tag{2}$$

Applying the Laplace transforms on the boundary conditions and the measured output give

$$\frac{\partial \hat{\mathbf{T}}}{\partial x}(0,s) = 0, \quad \frac{\partial \hat{\mathbf{T}}}{\partial x}(1,s) = \hat{\mathbf{u}}(s), \text{ and } \hat{\mathbf{y}}(t) = \hat{\mathbf{T}}(0,s).$$

The general solution for PDE (4) is

$$\hat{\mathbf{T}}(x,s) = Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x}$$

Using the boundary conditions

$$\frac{\partial \hat{\mathbf{T}}}{\partial x}(0,s) = 0 \Longrightarrow A\sqrt{s} - B\sqrt{s} = 0 \Longrightarrow A = B.$$

$$\frac{\partial \hat{\mathbf{T}}}{\partial x}(1,s) = \hat{\mathbf{u}}(s) \Longrightarrow A\sqrt{s}(e^{\sqrt{s}} - e^{-\sqrt{s}}) = \hat{\mathbf{u}}(s) \Longrightarrow A = \frac{\hat{\mathbf{u}}(s)}{\sqrt{s}(e^{\sqrt{s}} - e^{-\sqrt{s}})}.$$

Therefore, the solution is

$$\hat{\mathbf{T}}(x,s) = \frac{\hat{\mathbf{u}}(s)}{\sqrt{s}(e^{\sqrt{s}} - e^{-\sqrt{s}})}(e^{\sqrt{s}x} + e^{-\sqrt{s}x})$$

$$= \frac{\hat{\mathbf{u}}(s)\cosh(\sqrt{s}x)}{\sqrt{s}\sinh(\sqrt{s})}.$$

Finally,

$$\hat{\mathbf{y}}(s) = \hat{\mathbf{T}}(0,s) = \frac{\hat{\mathbf{u}}(s)}{\sqrt{s}\sinh(\sqrt{s})},$$

and the transfer function is

$$\mathbf{Z}(s) = \frac{\hat{\mathbf{y}}(s)}{\hat{\mathbf{u}}(s)} = \frac{1}{\sqrt{s}\sinh(\sqrt{s})}.$$

1.2 Poles of Transfer Function

The poles of the transfer function $\mathbf{Z}(s)$ are the roots of the denominator polynomial $\sqrt{s}\sinh(\sqrt{s})$, i.e. $s = 0$ or

$$\begin{aligned}\sinh(\sqrt{s}) &= 0\\ \left(\frac{e^{\sqrt{s}} - e^{-\sqrt{s}}}{2}\right) &= 0\\ e^{\sqrt{s}} &= e^{-\sqrt{s}}\end{aligned}$$

Taking natural logarithm on both sides gives

$$\sqrt{s} = -\sqrt{s} + 2k\pi i \quad \text{for } k \in \mathbb{Z}$$

It's easy to see that the poles are

$$s = -k^2\pi^2 \quad \text{for } k \in \mathbb{Z}.$$

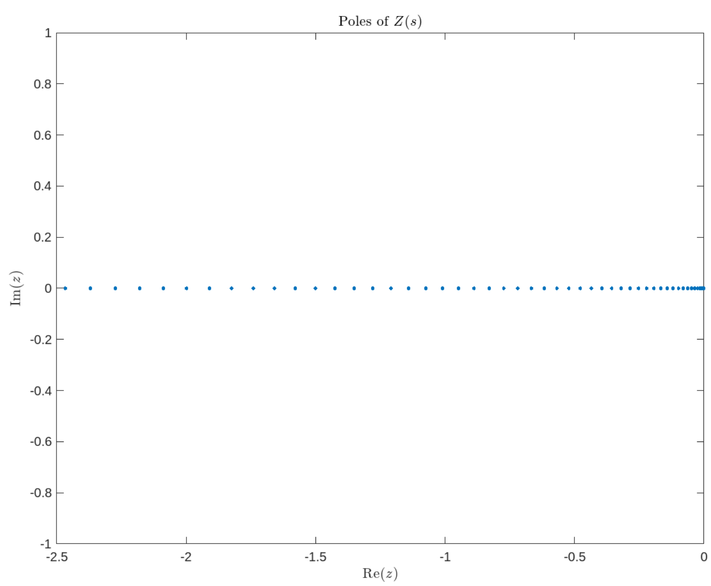


Figure 1: Poles of heat diffusion system (as derived in Sec. 1.2).

1.3 Partial Fraction Expansion

It can be shown that sinh can be expressed as the infinite product

$$\sinh(s) = s\prod_{n=1}^{\infty}\left(1 + \frac{s^2}{\pi^2 n^2}\right).$$

We can thus express $Z(s)$ as follows

$$\mathbf{Z}(s) = \frac{1}{\sqrt{s}\sinh(\sqrt{s})} = \frac{1}{\sqrt{s}\sqrt{s}\prod_{n=1}^{\infty}\left(1+\frac{s}{\pi^2 n^2}\right)} = \frac{1}{s}\prod_{n=1}^{\infty}\frac{1}{1+\frac{s}{\pi^2 n^2}} = \frac{1}{s}\prod_{n=1}^{\infty}\frac{\pi^2 n^2}{s+\pi^2 n^2}.$$

We want to solve for the partial fraction decomposition of $\mathbf{Z}(s)$,

$$\mathbf{Z}(s) = \sum_{k=0}^{\infty}\frac{\alpha_k}{s+\pi^2 n^2} \quad \text{where } \alpha_k = \mathbf{Z}(s)(s+\pi^2 k^2)|_{s=-\pi^2 k^2}.$$

where the coefficients are computed via the Residue Theorem. For the special case of $k=0$, we get

$$\alpha_0 = \prod_{n=1}^{\infty}\frac{\pi^2 n^2}{0+\pi^2 n^2} = \prod_{n=1}^{\infty} 1 = 1.$$

For $k > 0$, we get

$$\begin{aligned}
\alpha_k &= \frac{1}{-\pi^2 k^2}(\pi^2 k^2)\prod_{n=1}^{\infty}\frac{\pi^2 n^2}{\pi^2 n^2 - \pi^2 k^2} = (-1)\prod_{n=1\neq k}^{\infty}\frac{n^2}{n^2-k^2} = (-1)\left(\prod_{n=1}^{k-1}\frac{n^2}{n^2-k^2}\right)\left(\prod_{n=k+1}^{\infty}\frac{n^2}{n^2-k^2}\right) \\
&= (-1)(-1)^{k-1}\left(\prod_{n=1}^{k-1}\frac{n^2}{k^2-n^2}\right)\left(\prod_{n=k+1}^{\infty}\frac{n^2}{n^2-k^2}\right) = (-1)^k \underbrace{\left(\prod_{n=1}^{k-1}\frac{n^2}{k^2-n^2}\right)}_{A_k}\underbrace{\left(\prod_{n=1}^{\infty}\frac{(n+k)^2}{(n+k)^2-k^2}\right)}_{B_k}.
\end{aligned}$$

We first solve for A_k (assuming $k > 1$):

$$\begin{aligned}
A_k &= \prod_{n=1}^{k-1}\frac{n^2}{k^2-n^2} = \prod_{n=1}^{k-1}\frac{n^2}{(k+n)(k-n)} = \left(\prod_{n=1}^{k-1} n^2\right)\left(\prod_{n=1}^{k-1}\frac{1}{k+n}\right)\left(\prod_{n=1}^{k-1}\frac{1}{k-n}\right) \\
&= (k-1)!^2 \times \frac{k!}{(2k-1)!} \times \frac{1}{(k-1)!} = \frac{k!(k-1)!}{(2k-1)!}.
\end{aligned}$$

To solve for B_k we first consider finite product B_k^m then solve $\lim_{m\to\infty} B_k^m$:

$$\begin{aligned}
B_k^m &= \prod_{n=1}^{m}\frac{(n+k)^2}{(n+k)^2-k^2} = \prod_{n=1}^{m}\frac{(n+k)^2}{n(n+2k)} = \left(\prod_{n=1}^{m}(n+k)\right)^2\left(\prod_{n=1}^{m}\frac{1}{n}\right)\left(\prod_{n=1}^{m}\frac{1}{n+2k}\right) \\
&= \left(\frac{(m+k)!}{k!}\right)^2\left(\frac{1}{m!}\right)\left(\frac{(2k)!}{(m+2k)!}\right) = \left(\frac{(2k)!}{k!k!}\right)\left(\frac{(m+k)!(m+k)!}{m!(m+2k)!}\right).
\end{aligned}$$

Taking the limit we get

$$B_k = \lim_{m\to\infty} B_k^m = \frac{2k!}{k!k!}\lim_{m\to\infty}\frac{(m+k)!}{m!}\frac{(m+k)!}{(m+2k)!} = \frac{2k!}{k!k!}(1) = \frac{2k!}{k!k!}.$$

Finally, for $k > 0$ we get

$$\alpha_k = (-1)^k A_k B_k = (-1)^k\frac{k!(k-1)!}{(2k-1)!}\frac{2k!}{k!k!} = (-1)^k\frac{2k(2k-1)!k!(k-1)!}{k(2k-1)!k!(k-1)!} = 2(-1)^k$$

and, therefore,

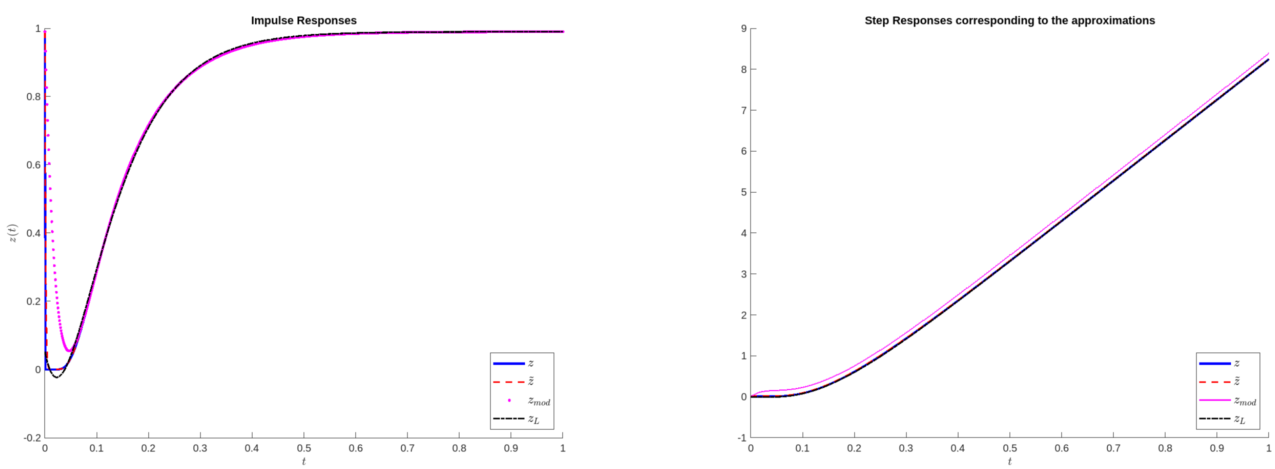
$$\mathbf{Z}(s) = \frac{1}{s} + 2\sum_{k=1}^{\infty}\frac{(-1)^k}{s+k^2\pi^2} \xRightarrow{\mathcal{L}^{-1}\{\cdot\}} \mathbf{z}(t) = 1 + 2\sum_{k=1}^{\infty}(-1)^k e^{-\pi^2 k^2 t}.$$

Letting $\mathbf{h}(t) = \sum_{k=1}^{\infty}(-1)^k e^{-\pi^2 k^2 t}$ we can conclude that $\mathbf{z}(t) = 1 + 2\mathbf{h}(t)$.

1.4 Approximant Comparisons

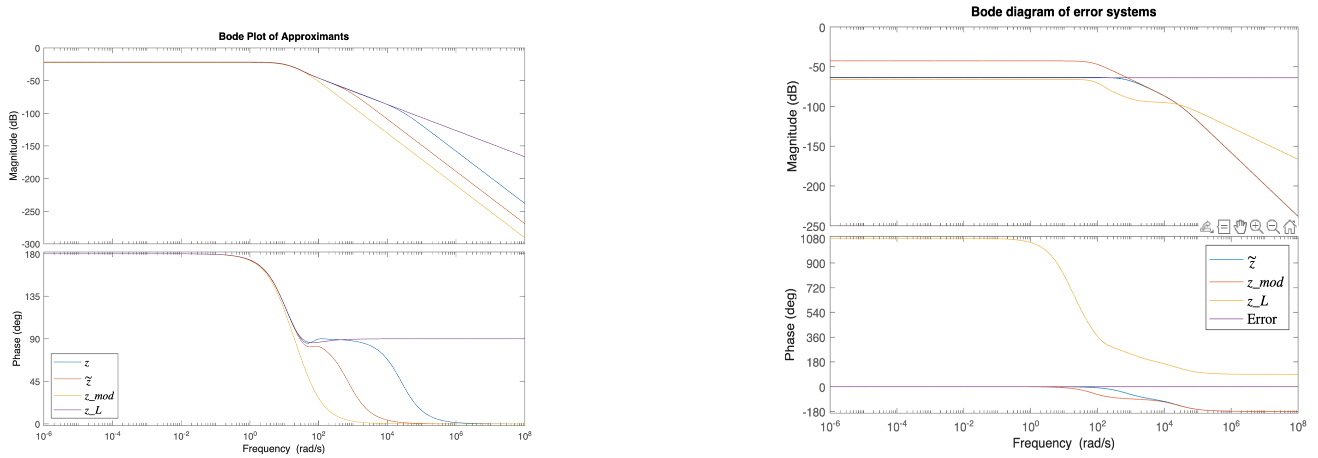
We now investigate 3 different approximations of $\mathbf{z}$

- Modal approximation
$$\mathbf{z}_{mod}(t) = 1 + 2(-e^{-\pi^2 t} + e^{-4\pi^2 t})$$
- High-order finite dimensional approximation
$$\tilde{\mathbf{z}}(t) = 1 + 2\sum_{k=1}^{8}(-1)^k e^{-\pi^2 k^2 t}$$
- Second order Lyapunov balanced truncation $\mathbf{z}_L = 1 + 2\mathbf{h}_L$: Recall that the impulse response can be expressed in terms of state space equation matrices $A, B,$ and C as $h(t) = Ce^{At}B$. By inspection, these matrices are
$$A = \mathrm{diag}(-k^2\pi^2)_{k=1}^{50}, \quad C = \begin{bmatrix} 2 & 2 & 2 & 2 & ... \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 & -1 & 1 & ... \end{bmatrix}^\top.$$
This will be the system (sys$(A, B, C, 0)$ in MATLAB)that we apply second order balanced truncation on. More details are in the code.



(a) (b)

Figure 2: Impulse (a) and step (b) responses of the heat diffusion system along with high-order (order 8), modal (order 2), and balanced truncation (order 2) approximants.



(a) (b)

Figure 3: Bode plots of (a) the original systems and its approximants and (b) the error systems.

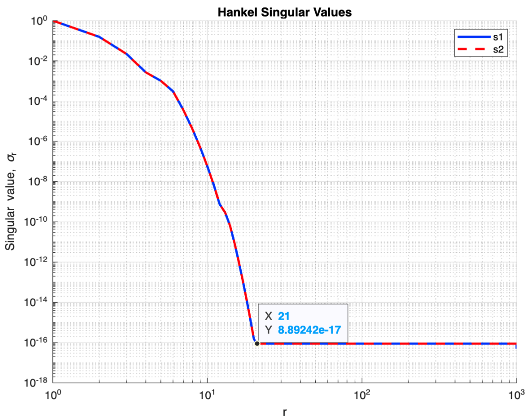
We can note the the impulse and step responses for the approximants overlap except at the beginning where there is a jump from 1 to 0 in the impulse response.

Approximant	$\mathcal{H}_\infty$ norm of error system
$\tilde{\mathbf{h}}$	1.3e−03
$\mathbf{h}_{mod}$	1.4e−02
$\mathbf{h}_L$	1.0e−03 $\leq$ 1.3e−03

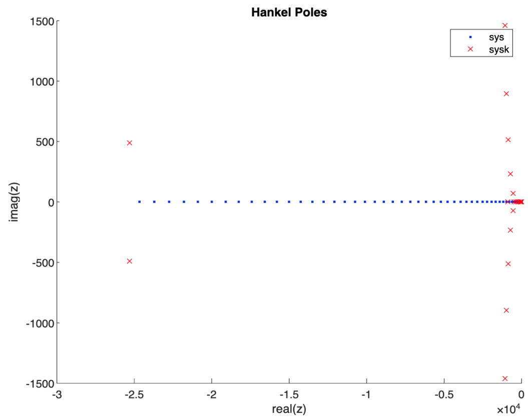
The approximant using the Lyapunov balanced truncation method performs the best and also satisfies the apriori error bound which is twice the sum of the discarded singular values

$$2 \sum_{i=(k+1)}^{n} \sigma_i$$

1.5 Reduced Order Modeling using Hankel and Loewner

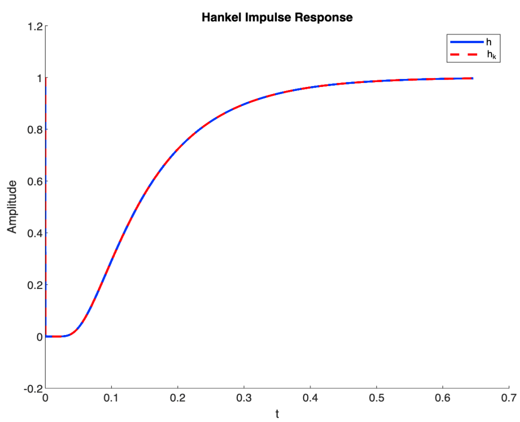


(a)

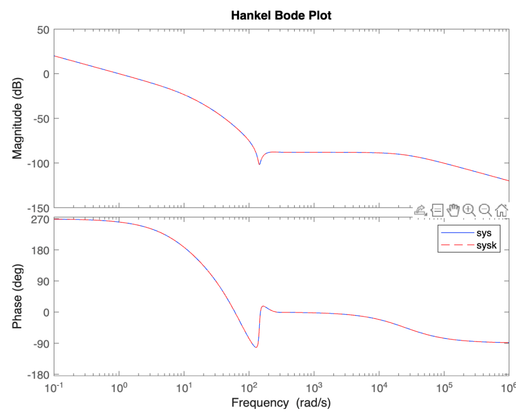


(b)

Figure 4: (a) Hankel singular values of the system and (b) the poles of the reduced-order Hankel system (order 21).



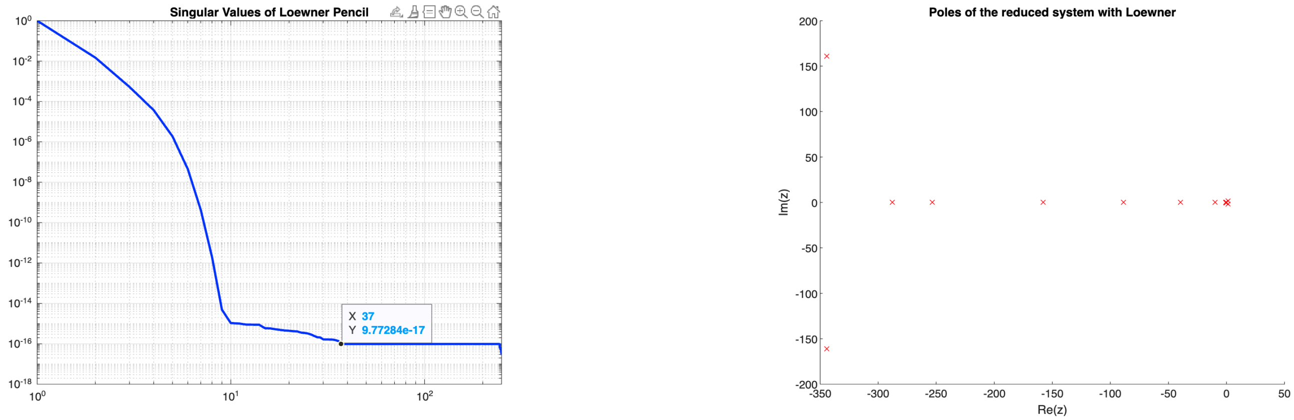
(a)



(b)

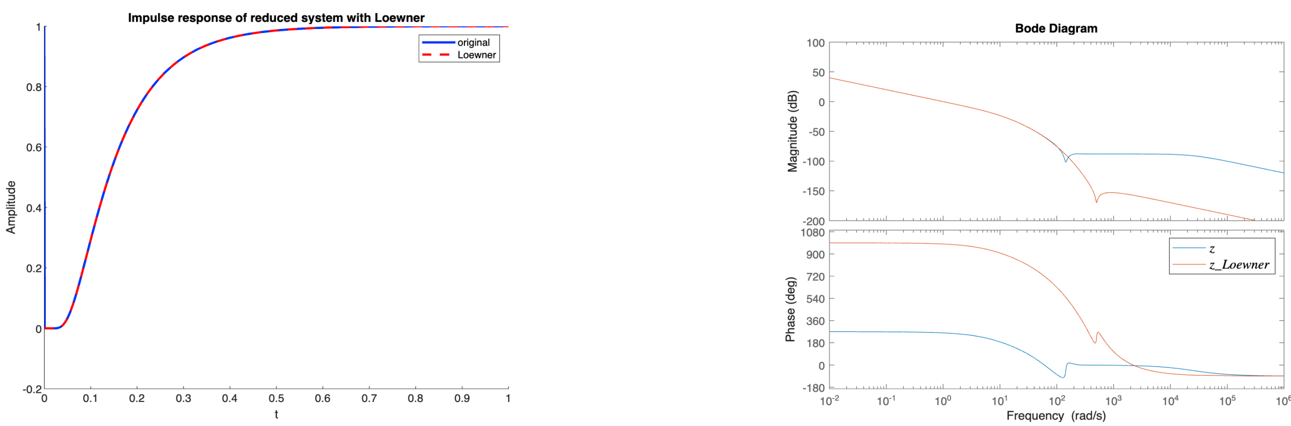
Figure 5: Impulse (a) and frequency (b) response of system and its Hankel approximation (order 21).

The Hankel method produces an essentially perfect reconstruction of the original system. This is due in large part to the high sampling rate (5 kHz) used.



(a) (b)

Figure 6: (a) Loewner singular values of the system and (b) the poles of the reduced-order Loewner system (order 37).



(a) (b)

Figure 7: Impulse (a) and frequency (b) response of system and its Loewner approximation (order 37).

The Loewner method produces a near-perfect reconstruction of the system, as can be seen in Fig. 6(b), deviating from the original system only at $t = 0$, where the system is 1 and the Loewner approximation is 0. This imperfection is reflected in the frequency response in which the original and Loewner systems deviate at large frequencies.

2 Heat Advection and Diffusion Through Idealized Rod

Assume now that convection is also taking place.

$$\frac{\partial \mathbf{T}}{\partial t}(x,t) = \frac{\partial^2 \mathbf{T}}{\partial x^2}(x,t) - 2\eta \frac{\partial \mathbf{T}}{\partial x}(x,t), \quad t \geq 0, \quad x \in [0,1], \quad \eta \geq 0 \tag{3}$$

with Neumann boundary conditions

$$\frac{\partial \mathbf{T}}{\partial x}(0,t) = 0 \text{ and } \frac{\partial \mathbf{T}}{\partial x}(1,t) = u(t),$$

and measured output

$$\mathbf{y}(t) = \mathbf{T}(0,t).$$

2.1 Transfer Function Derivation

To compute the transfer function, we apply Laplace transforms on (3) and we get

$$s\hat{\mathbf{T}}(x,s) = \frac{\partial^2 \hat{\mathbf{T}}}{\partial x^2}(x,s) - 2\eta \frac{\partial \hat{\mathbf{T}}}{\partial x}(x,s) \tag{4}$$

Applying the Laplace transforms on the boundary conditions and the measured output give

$$\frac{\partial \hat{\mathbf{T}}}{\partial x}(0,s) = 0, \quad \frac{\partial \hat{\mathbf{T}}}{\partial x}(1,s) = \hat{\mathbf{u}}(s), \text{ and } \hat{\mathbf{y}}(t) = \hat{\mathbf{T}}(0,s).$$

The general solution for PDE (4) is

$$\hat{\mathbf{T}}(x,s) = Ae^{(\eta - \sqrt{\eta^2+s})x} + Be^{(\eta + \sqrt{\eta^2+s})x}$$

Using the boundary conditions with

$$\frac{\partial \hat{\mathbf{T}}}{\partial x}(x,s) = A(\eta - \sqrt{\eta^2+s})e^{(\eta - \sqrt{\eta^2+s})x} + B(\eta + \sqrt{\eta^2+s})e^{(\eta + \sqrt{\eta^2+s})x},$$

we have

$$\begin{aligned}
\frac{\partial \hat{\mathbf{T}}}{\partial x}(0,s) = 0 &\Longrightarrow A(\eta - \sqrt{\eta^2+s}) + B(\eta + \sqrt{\eta^2+s}) = 0 \Longrightarrow A = \frac{B(\eta + \sqrt{\eta^2+s})^2}{s}. \\
\frac{\partial \hat{\mathbf{T}}}{\partial x}(1,s) = \hat{\mathbf{u}}(s) &\Longrightarrow A(\eta - \sqrt{\eta^2+s})e^{(\eta - \sqrt{\eta^2+s})} + B(\eta + \sqrt{\eta^2+s})e^{(\eta + \sqrt{\eta^2+s})} = \hat{\mathbf{u}}(s) \\
&\Longrightarrow -B(\eta + \sqrt{\eta^2+s})e^{(\eta - \sqrt{\eta^2+s})} + B(\eta + \sqrt{\eta^2+s})e^{(\eta + \sqrt{\eta^2+s})} = \hat{\mathbf{u}}(s) \\
&\Longrightarrow B(\eta + \sqrt{\eta^2+s})\left[e^{(\eta - \sqrt{\eta^2+s})} - e^{(\eta + \sqrt{\eta^2+s})}\right] = \hat{\mathbf{u}}(s) \\
&\Longrightarrow B(\eta + \sqrt{\eta^2+s})e^{\eta}(2\sinh\sqrt{\eta^2+s}) = \hat{\mathbf{u}}(s) \\
&\Longrightarrow B = \frac{\hat{\mathbf{u}}(s)}{(\eta + \sqrt{\eta^2+s})e^{\eta}(2\sinh(\sqrt{\eta^2+s}))}
\end{aligned}$$

Finally, we note that

$$\begin{aligned}
\hat{\mathbf{y}}(t) = \hat{\mathbf{T}}(0,s) &= A + B \\
&= \frac{B(\eta + \sqrt{\eta^2+s})^2}{s} + B \\
&= B\Big[\frac{(\eta + \sqrt{\eta^2+s})^2 + s}{s}\Big] \\
&= \frac{\hat{\mathbf{u}}(s)}{(\eta + \sqrt{\eta^2+s})e^{\eta}(2\sinh(\sqrt{\eta^2+s}))}\Big[\frac{(\eta + \sqrt{\eta^2+s})^2 + s}{s}\Big] \\
&= \frac{\hat{\mathbf{u}}(s)[\eta^2 + \eta\sqrt{\eta^2+s} + s]}{(\eta + \sqrt{\eta^2+s})se^{\eta}\sinh(\sqrt{\eta^2+s})} \\
&= \frac{\hat{\mathbf{u}}(s)(\sqrt{\eta^2+s})}{se^{\eta}\sinh(\sqrt{\eta^2+s})}
\end{aligned}$$

and the transfer function is

$$\mathbf{Z}(s) = \frac{\hat{\mathbf{y}}(s)}{\hat{\mathbf{u}}(s)} = \frac{1}{se^{\eta}} \cdot \frac{\sqrt{\eta^2+s}}{\sinh(\sqrt{\eta^2+s})}.$$

2.2 Poles of Transfer Function

The poles of the transfer function $\mathbf{Z}(s)$ are the roots of the denominator polynomial $se^{\eta} \cdot \sinh(\sqrt{\eta^2+s})$, i.e. $s = 0$ or

$$\begin{aligned}
\sinh(\sqrt{\eta^2+s}) &= 0 \\
\left(\frac{e^{\sqrt{\eta^2+s}} - e^{-\sqrt{\eta^2+s}}}{2}\right) &= 0 \\
e^{\sqrt{\eta^2+s}} = (e^{-\sqrt{\eta^2+s}})(e^{2k\pi i}) & \quad \text{for } k \in \mathbb{Z}
\end{aligned}$$

Taking the natural logarithm on both sides gives

$$\begin{aligned}
\sqrt{\eta^2+s} = -\sqrt{\eta^2+s} + 2k\pi i & \quad \text{for } k \in \mathbb{Z} \\
\sqrt{\eta^2+s} = k\pi i & \quad \text{for } k \in \mathbb{Z}
\end{aligned}$$

It's easy to see that the poles are

$$s = -\eta^2 - k^2\pi^2 \quad \text{for } k \in \mathbb{Z}$$

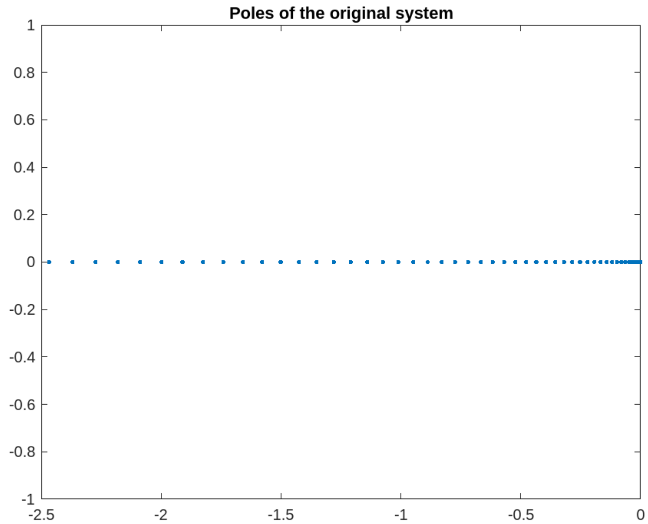


Figure 8: Poles of heat convection system (as derived in Sec. 2.2).

2.3 Partial Fraction Expansion

As seen previously, sinh can be expressed as the infinite product

$$\sinh(s) = s\prod_{n=1}^{\infty}\left(1+\frac{s^2}{\pi^2 n^2}\right)$$

We can thus express $Z(s)$ as follows

$$Z(s) = \frac{1}{se^{\eta}}\cdot\frac{\sqrt{s+\eta^2}}{\sqrt{s+\eta^2}\prod_{n=1}^{\infty}\left(1+\frac{s+\eta^2}{\pi^2 n^2}\right)} = \frac{1}{se^{\eta}}\cdot\prod_{n=1}^{\infty}\left(\frac{1}{1+\frac{s+\eta^2}{\pi^2 n^2}}\right) = \frac{1}{se^{\eta}}\cdot\prod_{n=1}^{\infty}\left(\frac{\pi^2 n^2}{\pi^2 n^2+s+\eta^2}\right)$$

We want to solve for the partial fraction decomposition of $e^{\eta}Z(s)$,

$$e^{\eta}Z(s) = \frac{\alpha_0}{s} + \sum_{k=1}^{\infty}\frac{\alpha_k}{s+\eta^2+k^2\pi^2}$$

Leveraging the Residue Theorem we can compute the coefficients as

$$\alpha_0 = se^{\eta}Z(s)|_{s=0} \quad \text{and} \quad \alpha_k = (s+\eta^2+\pi^2k^2)e^{\eta}Z(s)|_{s=-\eta^2-k^2\pi^2}$$

For the special case of $k=0$, we get

$$\alpha_0 = se^{\eta}Z(s)|_{s=0} = \frac{\sqrt{\eta^2+s}}{\sinh(\sqrt{\eta^2+s})}\Big|_{s=0} = \frac{\eta}{\sinh\eta}$$

For $k > 0$, we get

$$\begin{aligned}
\alpha_k = (s+\eta^2+k^2\pi^2)e^{\eta}Z(s)|_{s=-\eta^2-k^2\pi^2} &= \frac{\pi^2k^2}{s}\cdot\prod_{n=1\neq k}^{\infty}\left(\frac{\pi^2n^2}{\pi^2n^2+s+\eta^2}\right)\Big|_{s=-\eta^2-\pi^2k^2}\\
&= -\frac{\pi^2k^2}{\eta^2+\pi^2k^2}\cdot\prod_{n=1\neq k}^{\infty}\left(\frac{\pi^2n^2}{\pi^2n^2-\pi^2k^2}\right) = -\frac{\pi^2k^2}{\eta^2+\pi^2k^2}\cdot\prod_{n=1\neq k}^{\infty}\left(\frac{n^2}{n^2-k^2}\right)\\
&= -\frac{\pi^2k^2}{\eta^2+\pi^2k^2}\cdot\left(\prod_{n=1}^{k-1}\left(\frac{n^2}{n^2-k^2}\right)\right)\left(\prod_{k+1}^{\infty}\left(\frac{n^2}{n^2-k^2}\right)\right)\\
&= -\frac{\pi^2k^2}{\eta^2+\pi^2k^2}(-1)^{k-1}\left(\prod_{n=1}^{k-1}\left(\frac{n^2}{k^2-n^2}\right)\right)\left(\prod_{k+1}^{\infty}\left(\frac{n^2}{n^2-k^2}\right)\right)\\
&= \frac{\pi^2k^2}{\eta^2+\pi^2k^2}(-1)^{k}\underbrace{\left(\prod_{n=1}^{k-1}\left(\frac{n^2}{k^2-n^2}\right)\right)}_{A_k}\underbrace{\left(\prod_{k+1}^{\infty}\left(\frac{n^2}{n^2-k^2}\right)\right)}_{B_k}
\end{aligned}$$

We have previously shown that

$$A_k = \frac{k!(k-1)!}{(2k-1)!},\quad B_k = \frac{2k!}{k!k!}.$$

Therefore, for $k > 0$ we get

$$a_k = \frac{\pi^2k^2}{\eta^2+\pi^2k^2}(-1)^kA_kB_k = \frac{\pi^2k^2}{\eta^2+\pi^2k^2}(-1)^k\frac{k!(k-1)!2k!}{(2k-1)!k!k!} = \frac{\pi^2k^2}{\eta^2+\pi^2k^2}(-1)^k\frac{2k(2k-1)!k!(k-1)!}{k(2k-1)!k!(k-1)!}$$

$$a_k = 2(-1)^k\frac{\pi^2k^2}{\eta^2+\pi^2k^2}$$

2.4 Deriving State and Output Equations From Discretization

The above problem is to be solved by discretizing (3) in space as follows: Assume the rod is partitioned into $N+1$ intervals of length $h = \frac{1}{N+1}$ with $N+2$ discrete variables

$$\mathbf{T}(kh),\quad k = 0,1,...,N+1$$

Now, the discretized PDE is

$$\dot{\mathbf{T}}(kh) = \frac{1}{h^2}\big[\mathbf{T}((k+1)h) - 2\mathbf{T}(kh) + \mathbf{T}((k-1)h)\big] - \frac{2\eta}{h}\big[\mathbf{T}(kh) - \mathbf{T}((k-1)h)\big],\quad k = 0,1,...,N+1$$

This will include 2 ghost terms $\mathbf{T}(-h)$ and $\mathbf{T}((N+2)h)$ which can be eliminated using the discretized boundary conditions

$$(\mathbf{T}(0) - \mathbf{T}(-h))/h = 0 \quad \& \quad (\mathbf{T}((N+2)h) - \mathbf{T}((N+1)h)/h = \mathbf{u}.$$

Finally, the measured output is $\mathbf{y} = \mathbf{T}(0)$.

Studying the discretization at $k = 0$, we have

$$\begin{aligned}
\dot{\mathbf{T}}(0) &= \frac{1}{h^2}\big[\mathbf{T}(h) - 2\mathbf{T}(0) + \mathbf{T}(-h)\big] - \cancelto{0}{\frac{2\eta}{h}\big[\mathbf{T}(0) - \mathbf{T}(-h)\big]} \quad \text{(using the first boundary condition)} \\
&= \frac{1}{h^2}\big[\mathbf{T}(h) - \mathbf{T}(0)\big]
\end{aligned}$$

Similarly with $k = N + 1$, we have

$$\begin{aligned}
\dot{\mathbf{T}}(kh) &= \frac{1}{h^2}\big[\mathbf{T}((N+2)h) - 2\mathbf{T}((N+1)h) + \mathbf{T}(Nh)\big] - \frac{2\eta}{h}\big[\mathbf{T}((N+1)h) - \mathbf{T}(Nh)\big] \\
&= \frac{\mathbf{u}}{h} + \frac{1}{h^2}\big[\mathbf{T}(Nh) - \mathbf{T}((N+1)h)\big] - \frac{2\eta}{h}\big[\mathbf{T}((N+1)h) - \mathbf{T}(Nh)\big] \quad \text{(using the second boundary condition)}
\end{aligned}$$

Therefore. we can deduce that the resulting system has the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

where $\mathbf{A}$ is the tridiagonal matrix while $\mathbf{B}, \mathbf{C}$ are multiples of canonical unit vectors:

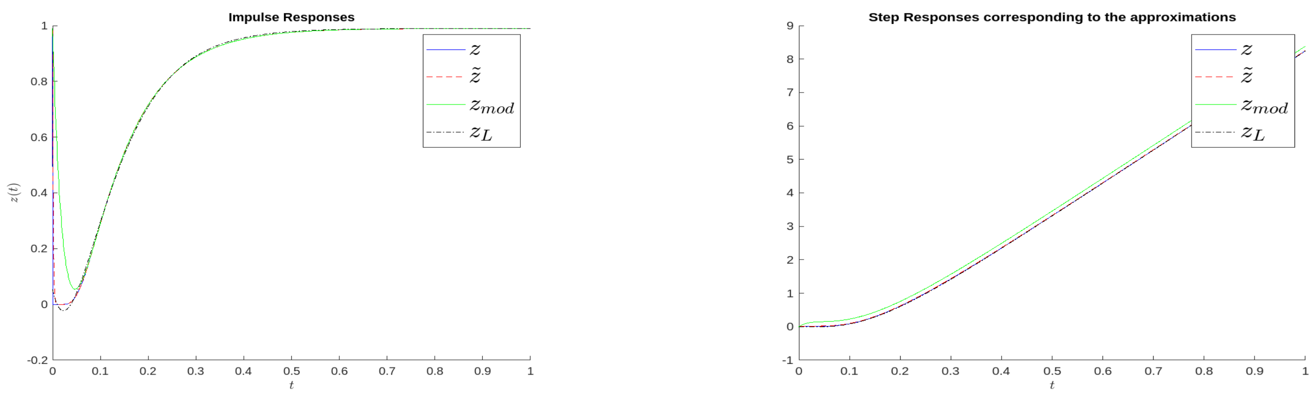
$$\mathbf{A} = \frac{1}{h^2}\begin{bmatrix}
-1 & 1 & 0 & & & & \\
1 + 2h\eta & -2(1+h\eta) & 1 & & & & \\
& & \ddots & \ddots & \ddots & & \\
& & & & 1 + 2h\eta & -2(1+h\eta) & 1 \\
& & & & 0 & 1 + 2h\eta & -1 - 2h\eta
\end{bmatrix}, \quad \mathbf{B} = \frac{1}{h}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}.$$

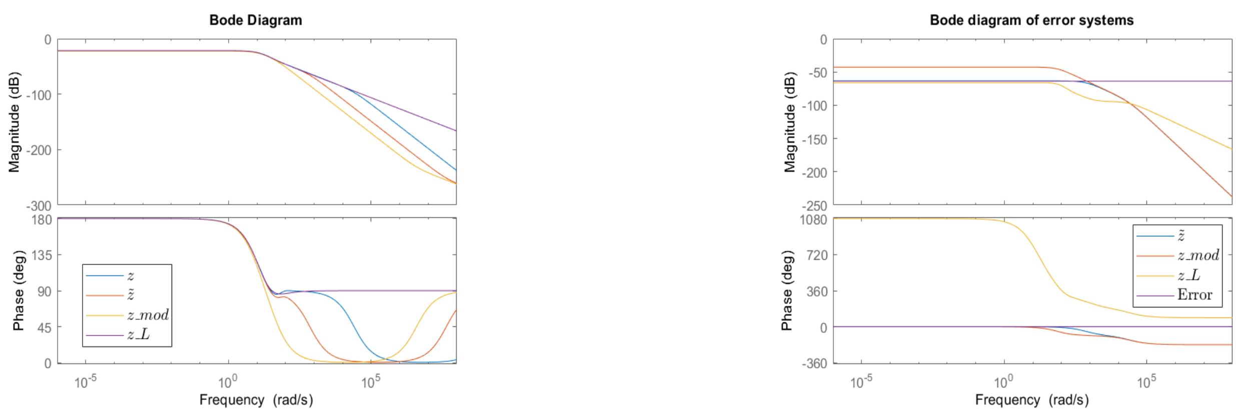
2.5 Approximant Comparisons

Similar to what was done in the previous section, we investigate a high-order finite dimensional approximation and a second order Lyapunov balanced truncation. The results are similar to that in Sec. 2.5, with balanced truncation performing best overall. Frequencies are matched almost exactly, up to about 100 rads/sec. The $\mathcal{H}_\infty$ norm bound is satisfied by the balanced truncation as expected.



(a) (b)

Figure 9: (a) Impulse responses of original and approximants (b) Step responses of original and approximants



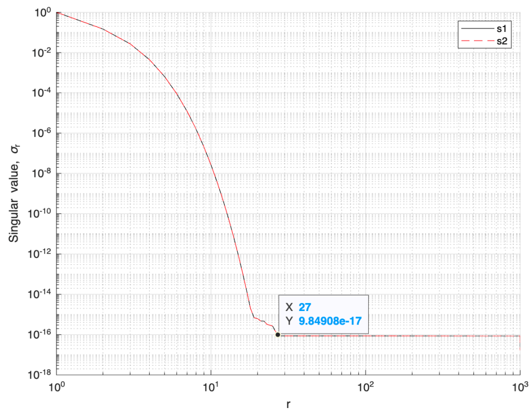
(a) (b)

Figure 10: (a) Bode plot of original and approximants (b) Bode error plot

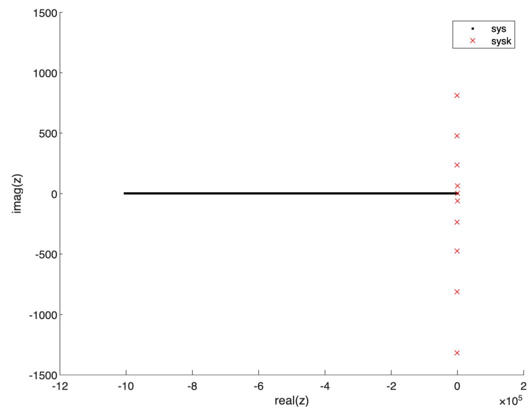
Approximant	$\mathcal{H}_\infty$ norm of error system
$\tilde{\mathbf{h}}$	7e-04
$\mathbf{h}_{mod}$	7.3e-03
$\mathbf{h}_L$	5.0e-04 $\leq 6.3e-04$

2.6 Reduced Order Modeling of Discretized System using Hankel and Loewner

The Hankel method produces a near-perfect reconstruction of the original system up to about 1,000 rads/sec.

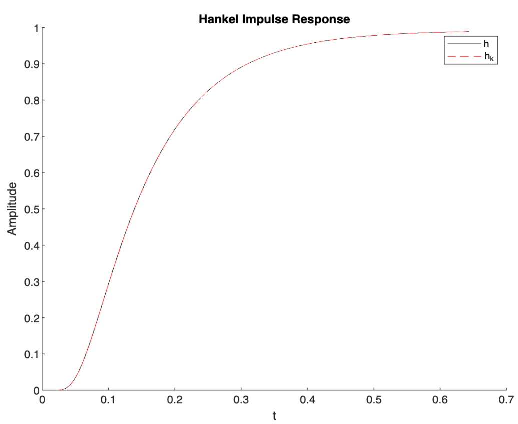


(a)

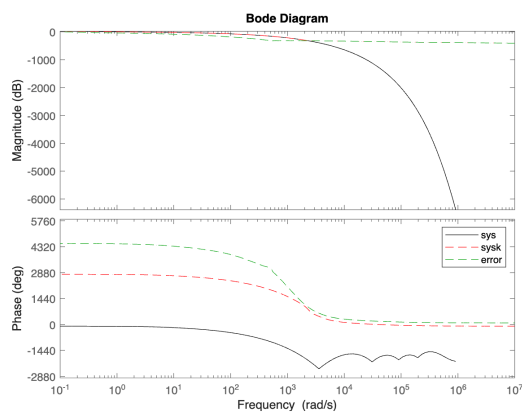


(b)

Figure 11: (a) Hankel singular values of the system and (b) the poles of the reduced-order Hankel system (order 21).



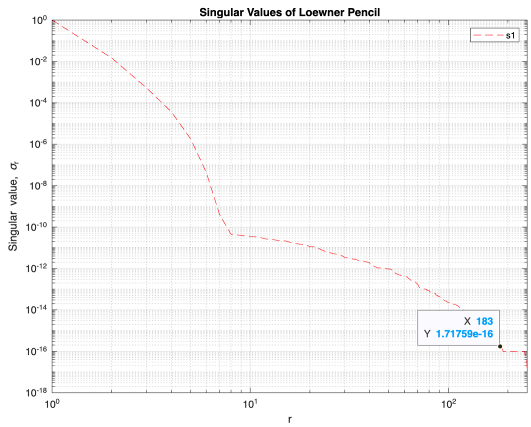
(a)



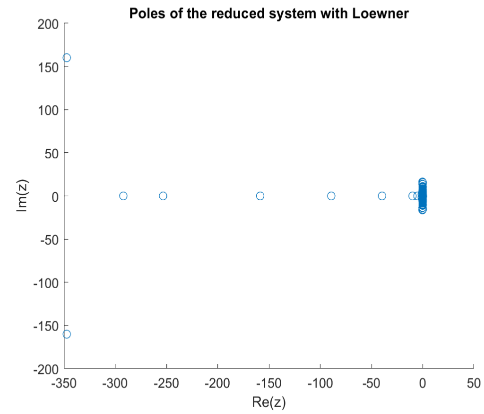
(b)

Figure 12: Impulse (a) and frequency (b) response of system and its Hankel approximation (order 21).

The Loewner method produces a near-perfect reconstruction of the system up to about 1000 rads/sec. The impulse response of the discretization is zero at $t = 0$, unlike the original system impulse response, which the Loewner method is able to replicate.

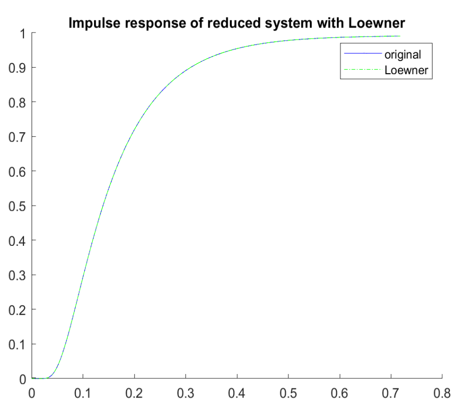


(a)

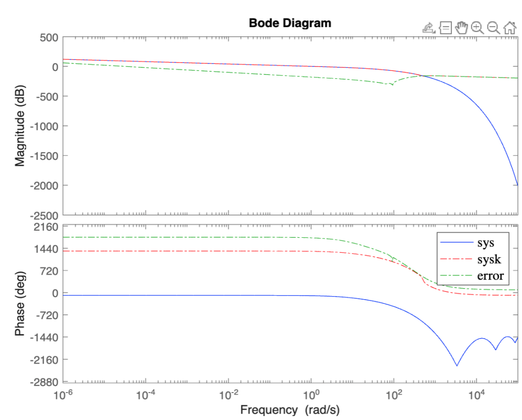


(b)

Figure 13: (a) Loewner singular values of the system and (b) the poles of the reduced-order Loewner system (order 183).



(a)



(b)

Figure 14: Impulse (a) and frequency (b) response of system and its Loewner approximation (order 183).