Heat Equation: Reduced Order Modeling

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ELEC 519 Final Project
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Outline

Diffusion

Convection

Heat diffusion

- Consider a perfectly insulated heat-conducting rod

- Diffusion of heat described by linear PDE

$$\frac{\partial \mathbf{T}}{\partial t}(x,t) = \frac{\partial^2 \mathbf{T}}{\partial x^2}(x,t), \quad t \geq 0, \quad x \in [0,1], \tag{1}$$

with Neumann boundary conditions

$$\frac{\partial \mathbf{T}}{\partial x}(0,t) = 0 \text{ and } \frac{\partial \mathbf{T}}{\partial x}(1,t) = u(t),$$

and measured output

$$\mathbf{y}(t) = \mathbf{T}(0,t)$$

where $\mathbf{T}(x,t)$ temp. at distance x from origin and time t.

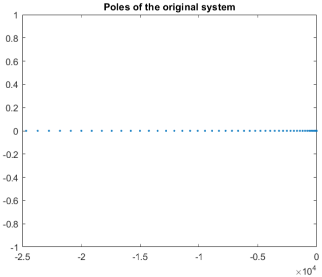
Transfer Function

- To compute transfer function, we first apply Laplace transforms on (1), its boundary conditions, and output
- Solve for $\hat{\mathbf{u}}(s)$ and $\hat{\mathbf{y}}(s)$
- Resulting transfer function

$$\mathbf{Z}(s) = \frac{\hat{\mathbf{y}}(s)}{\hat{\mathbf{u}}(s)} = \frac{1}{\sqrt{s}\sinh(\sqrt{s})}.$$

- Poles are

$$s = -k^2\pi^2 \quad \text{for } k \in \mathbb{Z}.$$



Impulse Response and Approximations

- We can show that impulse response of system is

$$\mathbf{z}(t) = 1 + 2\mathbf{h}(t) \text{ where } \mathbf{h}(t) = \sum_{k=1}^{\infty} (-1)^k e^{-\pi^2 k^2 t}, \quad t \geq 0.$$

- We now investigate 3 different approximations of $\mathbf{z}$
  - Modal approximation

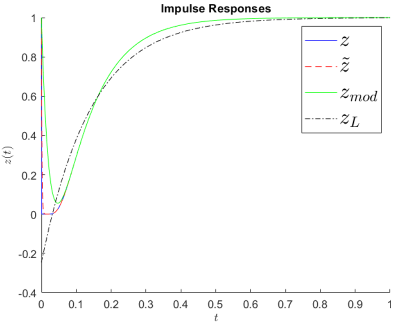
$$\mathbf{z}_{mod}(t) = 1 + 2(-e^{-\pi^2 t} + e^{-4\pi^2 t})$$

  - High-order finite dim. approximation

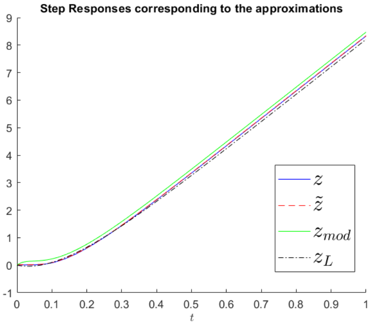
$$\tilde{\mathbf{z}}(t) = 1 + 2\sum_{k=1}^{8} (-1)^k e^{-\pi^2 k^2 t}$$

  - Second order Lyapunov balanced truncation $\mathbf{z}_L = 1 + 2\mathbf{h}_L$

Impulse Response and Step Responses of Approximations

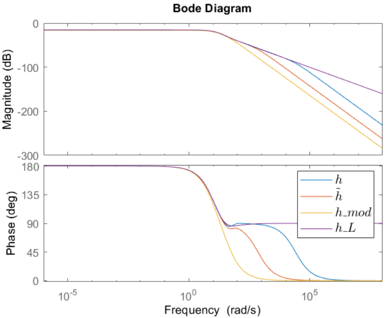


(a) Impulse Responses

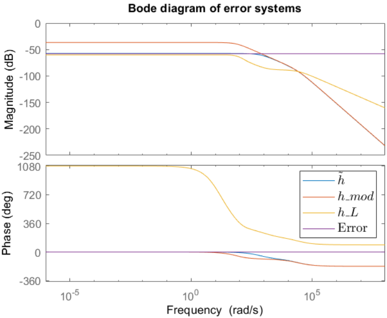


(b) Step Responses

Bode Amplitude Diagrams



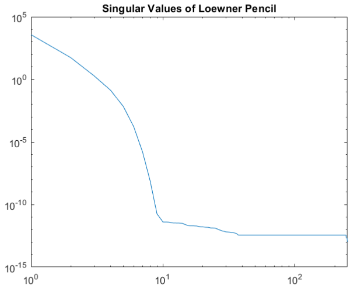
(a) Bode plot of approximants

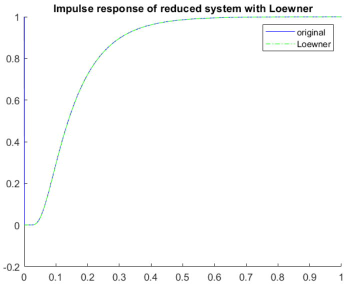


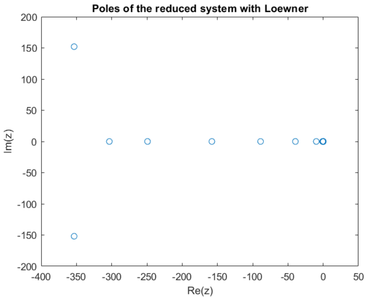
(b) Bode plot of error systems

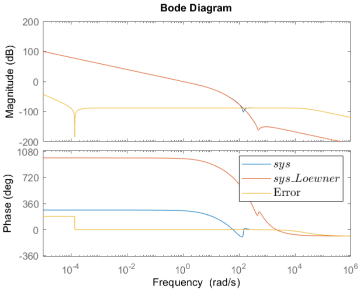
Approximant	$\mathcal{H}_\infty$ norm of error system
$\tilde{\mathbf{h}}$	1.3e-03
$\mathbf{h}_{mod}$	1.4e-02
$\mathbf{h}_L$	1.0e-03 $\leq 1.3e-03$

Loewner Framework (250 interpol. points)

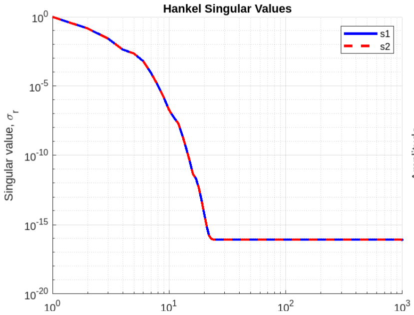


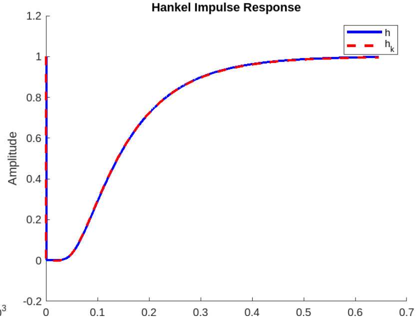


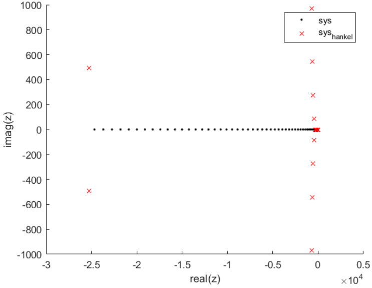


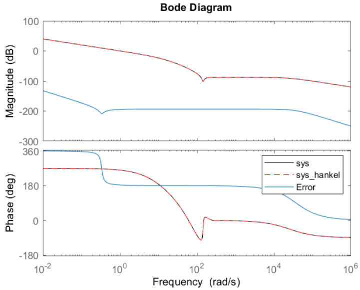


Hankel Framework









Outline

Diffusion

Convection

Heat Diffusion + Convection

- Add a convection coefficient $\eta = 0.01$
- The PDE becomes

$$\frac{\partial \mathbf{T}}{\partial t}(x,t) = \frac{\partial^2 \mathbf{T}}{\partial x^2}(x,t) - 2\eta \frac{\partial \mathbf{T}}{\partial x}(x,t), \quad t \geq 0, \quad x \in [0,1], \quad \eta \geq 0 \tag{2}$$

with Neumann boundary conditions

$$\frac{\partial \mathbf{T}}{\partial x}(0,t) = 0 \text{ and } \frac{\partial \mathbf{T}}{\partial x}(1,t) = u(t),$$

and measured output

$$\mathbf{y}(t) = \mathbf{T}(0,t).$$

Discretization

- Partition rod into $N+1$ intervals of length $h = \frac{1}{N+1}$ with variables

$$\mathbf{T}(kh), \quad k = 0, 1, ..., N+1$$

- Discretize using

$$\begin{aligned}\dot{\mathbf{T}}(kh) &= \frac{1}{h^2}\big[\mathbf{T}((k+1)h) - 2\mathbf{T}(kh) + \mathbf{T}((k-1)h)\big] \\ &\quad - \frac{2\eta}{h}\big[\mathbf{T}(kh) - \mathbf{T}((k-1)h)\big], \quad k = 0, 1, ..., N+1\end{aligned}$$

- Include 2 ghost terms $\mathbf{T}(-h)$ and $\mathbf{T}((N+2)h)$ which can be eliminated using the discretized boundary conditions

$$(\mathbf{T}(0) - \mathbf{T}(-h))/h = 0 \quad \& \quad (\mathbf{T}((N+2)h) - \mathbf{T}((N+1)h)/h = \mathbf{u}.$$

- Finally, the measured output is $\mathbf{y} = \mathbf{T}(0)$.

Discretization Continued

- Studying the discretization at $k = 0$, we have

$$
\begin{aligned}
\dot{\mathbf{T}}(0) &= \frac{1}{h^2}\left[\mathbf{T}(h) - 2\mathbf{T}(0) + \mathbf{T}(-h)\right] - \cancelto{0}{\frac{2\eta}{h}\left[\mathbf{T}(0) - \mathbf{T}(-h)\right]} \\
&= \frac{1}{h^2}\left[\mathbf{T}(h) - \mathbf{T}(0)\right]
\end{aligned}
$$

- Similarly with $k = N + 1$.
- Resulting system has the form

$$
\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)
$$

where

$$
\mathbf{A} = \frac{1}{h^2}\begin{bmatrix}
-1 & 1 & 0 & & & & & \\
1+2h\eta & -2(1+h\eta) & 1 & & & & & \\
 & & \ddots & \ddots & \ddots & & & \\
 & & & & & 1+2h\eta & -2(1+h\eta) & 1 \\
 & & & & & 0 & 1+2h\eta & -1-2h\eta
\end{bmatrix},
$$

$$
\mathbf{B} = \frac{1}{h}\begin{bmatrix}0 & 0 & \dots & 0 & 1\end{bmatrix}^T, \mathbf{C} = \begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix}.
$$

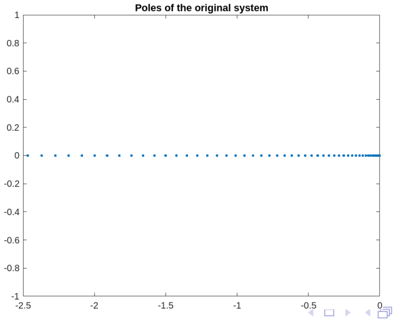
Transfer Function

- Transfer function

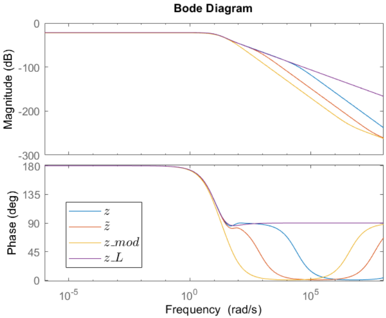
$$\mathbf{Z}(s) = \frac{\hat{\mathbf{y}}(s)}{\hat{\mathbf{u}}(s)} = \frac{1}{se^{\eta}} \cdot \frac{\sqrt{\eta^2 + s}}{\sinh(\sqrt{\eta^2 + s})}.$$

- Poles are

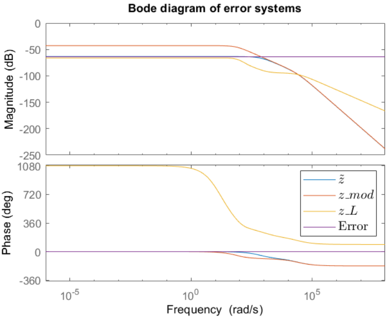
$$s = 0 \text{ and } s = -\eta^2 - k^2\pi^2 \quad \text{for } k \in \mathbb{Z}$$



Bode Amplitude Diagrams



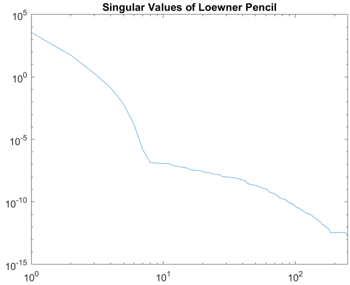
(a) Bode plot of approximants

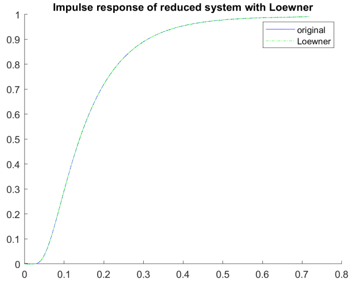


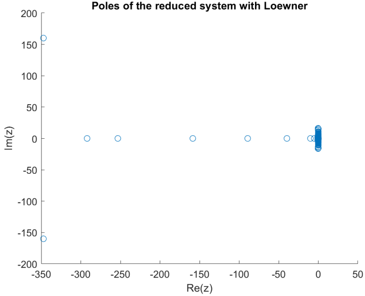
(b) Bode plot of error systems

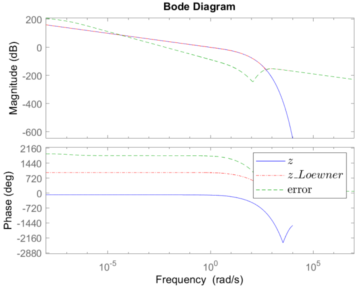
Approximant	$\mathcal{H}_\infty$ norm of error system
$\tilde{\mathbf{h}}$	7e-04
$\mathbf{h}_{mod}$	7.3e-03
$\mathbf{h}_L$	5.0e-04 $\leq 6.3e-04$

Loewner Framework

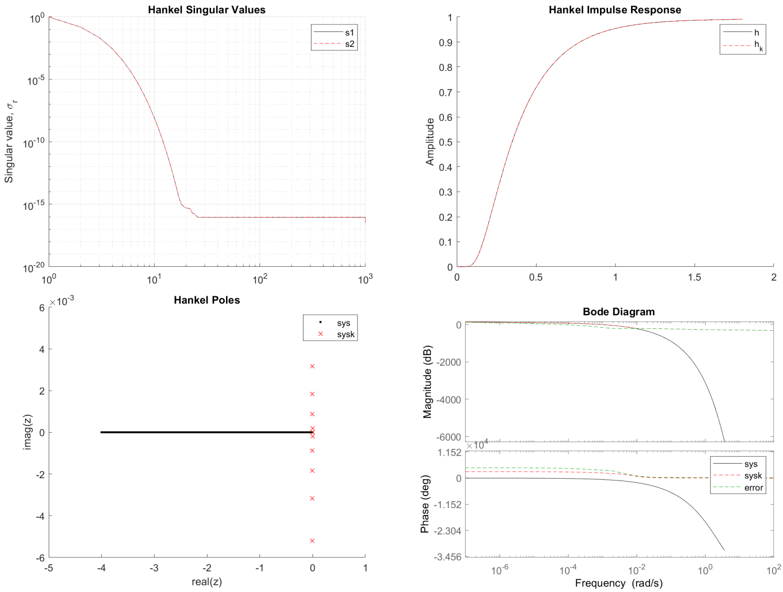








Hankel Framework



Thank you!