MSBD 5004 Due : 30th Sep.

Assignment #1 Solution

Instructor: Jianfeng CAI Name: StudentName ID: 1

Problem 1.

(a) A norm should have:

① $\|x\|_\infty = \max_{1\le i\le n} |x_i| \geq 0$

if $\max_{1\le i\le n} |x_i| = 0$, $\Leftrightarrow$ $x_i = 0$

② $\|\alpha x\|_\infty = \max_{1\le i\le n} |\alpha x_i| = |\alpha| \max_{1\le i\le n} |x_i| = |\alpha|\cdot\|x\|_\infty$

③ $\|x+y\|_\infty = \max_{1\le i\le n} |x_i+y_i| \le \max_{1\le i\le n} (|x_i|+|y_i|)$

$\le \max_{1\le i\le n} |x_i| + \max_{1\le j\le n} |y_j| = \|x\|_\infty + \|y\|_\infty$

(b) proof:

$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \div \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{\frac{1}{p}}$

denote $|x_k| = \max_{1\le i\le n} |x_i|$. $|x_k|^p \le \sum_{i=1}^{n} |x_i|^p \le n\cdot|x_k|^p$

$\lim_{p\to\infty} (|x_k|^p)^{\frac{1}{p}} \le \lim_{p\to\infty} \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \le \lim_{p\to\infty} (n\cdot|x_k|^p)^{\frac{1}{p}}$

$|x_k| \le \|x\|_p \le |x_k|$

$\therefore \|x\|_p = |x_k| = \max_{1\le i\le n} |x_i|$.

(c) $\|x\|_\infty = \max\limits_{1\le i\le n} |x_i|$, $\|x\|_1 = \sum\limits_{i=1}^{n} |x_i|$

proof:

denote $|x_k| = \max\limits_{1\le i\le n} |x_i| = \|x\|_\infty$.

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| = |x_k| + \sum_{\substack{i=1\\ i\ne k}}^{n} |x_i| \ge |x_k| = \|x\|_\infty$$

$$\|x\|_1 = |x_k| + \sum_{\substack{i=1\\ i\ne k}}^{n} |x_i| \le |x_k| + \sum_{\substack{i=1\\ i\ne k}}^{n} |x_k| = n|x_k| = n\|x\|_\infty$$

$\therefore$ $\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$.

Problem 2

(a) proof:

① $$\|A\|_2 = \max_{\substack{\|x\|_2=1\\ x\in\mathbb{R}^n}} \underset{\text{vector norm}}{\|Ax\|_2} = \max_{x_j} \left(\sum_{i=1}^{m}\Big(\sum_{j=1}^{n} a_{ij}x_j\Big)^2 \right)^{\frac{1}{2}} \ge 0$$

m

$m\times n$ n

$$\|A\|_2 = 0 \iff \max_{x_j}\left(\sum_{i=1}^{n}\Big(\sum_{j=1}^{m} a_{ij}x_j\Big)^2\right)^{\frac{1}{2}} = 0 \underset{(\|x\|_2=1)}{\iff} \forall i,j,\ a_{ij}=0 \iff A = \mathbf{0}\ (\text{0-matrix})$$

② $$\|\alpha A\|_2 = \max_{\substack{\|x\|_2=1\\ x\in\mathbb{R}^n}} \|\alpha Ax\|_2 = |\alpha| \max_{\substack{\|x\|_2=1\\ x\in\mathbb{R}^n}} \|Ax\|_2 = |\alpha|\,\|A\|_2$$

③ $$\|A+B\|_2 = \max_{\substack{\|x\|_2=1\\ x\in\mathbb{R}^n}} \|(A+B)x\|_2 = \max_{\substack{\|x\|_2=1\\ x\in\mathbb{R}^n}} \|Ax+Bx\|_2$$

using property of vector p-norm

$$\le \max_{\substack{\|x\|_2=1\\ x\in\mathbb{R}^n}} \left(\|Ax\|_2 + \|Bx\|_2\right) \le \max_{\substack{\|x\|_2=1\\ x\in\mathbb{R}^n}} \|Ax\|_2 + \max_{\substack{\|x\|_2=1\\ x\in\mathbb{R}^n}} \|Bx\|_2 = \|A\|_2 + \|B\|_2$$

According to ①②③, $\|\cdot\|_2$ is a norm on $\mathbb{R}^{m\times n}$

(b) proof:

① $x=0$: $\|Ax\|_2 = 0 \le \|A\|_2 \|x\|_2 = 0$.

② $x \ne 0$: Let $x_0 = \frac{x}{\|x\|_2}$, which means each row of $x_0 \in (0,1)$

thus $\|Ax_0\|_2 \le \|A\|_2$, because Ax_0 is part of A.

$$\|Ax\|_2 = \| A \cdot x_0 \cdot \|x\|_2 \|_2 = \|x\|_2 \|Ax_0\|_2 \le \|x\|_2 \|A\|_2$$

(c) $\|\overset{\substack{m\times p}}{\overset{m\times n\ n\times p}{AB}}\|_2 = \max\limits_{\substack{\|x\|_2=1 \\ x\in\mathbb{R}^p}} \|A\underline{Bx}\|_2 \overset{\text{using (b)}}{\le} \max\limits_{\substack{\|x\|_2=1 \\ x\in\mathbb{R}^p}} \|A\|_2 \|Bx\|_2 = \|A\|_2 \max\limits_{\substack{\|x\|_2=1 \\ x\in\mathbb{R}^p}} \|Bx\|_2$

$= \|A\|_2 \|B\|_2$

$\boxed{\text{Problem 3}}$ $\sum_{i=1}^{m} |a_i - b|$

proof: denote $c_1, c_2, \cdots, c_m$ as a sorted list of $\{a_i\}_{i=1}^m$,

where $c_1 \le c_2 \le \cdots \le c_m$, then:

$\sum_{i=1}^m |a_i - b| = \sum_{i=1}^m |c_i - b|$

① if $b > c_m$ or $b < c_1$,

$\sum_{i=1}^m |c_i - b| = m\cdot b - \sum_{i=1}^m c_i$ $\quad\quad$ $\sum_{i=1}^m |c_i - b| = \sum_{i=1}^m c_i - m\cdot b$

it's obvious that $\sum_{i=1}^m |a_i - b|$ can't be smallest.

② if $b \in [c_1, c_m]$:

$$\sum_{i=1}^m |c_i - b| = |b-c_1| + |b-c_2| + \cdots + |c_{m-1} - b| + |c_m - b|$$

$$= \underbrace{(|b-c_1| + |c_m - b|)}_{\ge |c_m - c_1|} + \underbrace{(|b-c_2| + |c_{m-1} - b|)}_{\ge |c_{m-1} - c_2|} + \cdots$$

$\Rightarrow c_1 \le b \le c_m$ $\quad\quad$ $\Rightarrow c_2 \le b \le c_{m-1}$ $\quad\quad$ $\cdots$

$\begin{cases} \text{if } m \% 2 = 0 : c_{\frac{m}{2}} \le b \le c_{\frac{m}{2}+1} \\ \text{if } m \% 2 = 1 : b = c_{\frac{m+1}{2}} \end{cases}$ $\Rightarrow$ b is the median of $\{a_i\}_{i=1}^m$.

Problem 4

(a) K-means: $\min \sum_{i \in G_j} \| x_i - z_j \|_2^2$

$$\Rightarrow z_j = \frac{1}{|G_j|} \sum_{i \in G_j} x_i$$

if $\forall i$, $x_i \geq 0$, then $z_j \geq 0$

(b) if $\forall i$, $1^T x_i = 1$,

then $1^T z_j = \frac{1}{|G_j|} \sum_{i \in G_j} 1^T x_i = \frac{1}{|G_j|} \sum_{i \in G_j} 1 = \frac{|G_j|}{|G_j|} = 1.$

(c) $(z_j)_i = \frac{1}{|G_j|} \sum_{k \in G_j} x_{ki}$

① $(z_j)_i = 1$, which represents all $x_{ki} = 1$: all i-th entries of x_k equal to 1.

② $(z_j)_i = 0$, which represents all $x_{ki} = 0$: all i-th entries of x_k equal to 0.

③ $(z_j)_i \in (0,1)$, which represents some $x_{ki} = 0$ and others $= 1$: some of the i-th entries of x_k equal to 0, others equal to 1.

The closer $(z_j)_i$ is to 1, the more $x_{ki} = 1$.

Problem 5

proof: denote $z_j^* = \frac{1}{|G_j|} \sum_{i \in G_j} x_i$, $z_1, \cdots, z_k$ is random representative

① : $z_1, \cdots, z_k$ is fixed, find best $G_1, \cdots, G_k$:

let $C^* = \underset{j \in \{1,2,\cdots,k\}}{\arg\min} \|x_i - z_j\|_2^2$, C is any partition,

$$\underline{L(C, z_1, \cdots, z_k)} = \sum_{j=1}^{k} \sum_{i \in C} \|x_i - z_j\|_2^2 \geq \sum_{j=1}^{k} \sum_{i \in C^*} \|x_i - z_j\|_2^2.$$

$$= \underline{L(C^*, z_1, \cdots, z_k)}$$

② $G_1, \cdots, G_j$ is fixed, find best $z_1, \cdots, z_k$:

let $z_k^* = \frac{1}{|G_j|} \sum_{i \in G_j} x_i$, z_k is any representitive,

$$L(C, z_1, \cdots, z_k) = \sum_{i \in C} \|x_i - z_k\|_2^2$$

$$\|x_i - z_k\|_2^2 = \|x_i - z_k^* + z_k^* - z_k\|_2^2 = \|x_i - z_k^*\|^2 + \|z_k^* - z_k\|_2^2 + 2\langle x_i - z_k^*, z_k^* - z_k \rangle$$

$$\geq \|x_i - z_k^*\|_2^2 + 2\langle x_i - z_k^*, z_k^* - z_k \rangle$$

$$\sum_{i \in G_j} \|x_i - z_k\|_2^2 \geq \sum_{i \in G_j} \|x_i - z_k^*\|_2^2 + 2\langle \underbrace{\sum_{i \in G_j} (x_i - z_k^*)}_{=0}, z_k^* - z_k \rangle$$

$$= \sum_{i \in G_j} \|x_i - z_k^*\|_2^2 = L(C, z_1^*, \cdots, z_k^*)$$

$$\underline{L(C, z_1, \cdots, z_k) \geq L(C, z_1^*, \cdots, z_k^*)}.$$

③ $l_n = L(C^{(n+1)}, z_1^{(n)}, \cdots, z_k^{(n)}) \geq L(C^{(n+1)}, z_1^{(n+1)}, \cdots, z_k^{(n+1)})$ using ②

$\geq L(C^{(n+2)}, z_1^{(n+1)}, \cdots, z_k^{(n+1)})$ using ①

$= l_{n+1}$.

$l \geq 0$

Problem 6

proof: $\|a_k - b\|_\infty = \max |a_k - b| = \max(\underbrace{0, \cdots, 0}_{k\text{-times}}, 1, \cdots)$

$\lim_{k\to\infty} \|a_k - b\|_\infty = 1 \neq 0$

Hence $a_k \not\to b$ in l_∞ with $\|\cdot\|_\infty$

Problem 7

(a) $\|c_k \vec{a}_k - c\cdot\vec{a}\| = \|c_k\vec{a}_k - c_k\cdot\vec{a} + c_k\cdot\vec{a} - c\cdot\vec{a}\|$

$= \|c_k(\vec{a}_k - \vec{a}) + (c_k - c)\vec{a}\| \geq 0$

$\lim_{k\to\infty} \|c_k\vec{a}_k - c\cdot\vec{a}\| \leq \|c_k(\vec{a}_k - \vec{a})\| + \|(c_k - c)\vec{a}\|$

$= |c_k| \cdot \|\vec{a}_k - \vec{a}\| + |c_k - c| \cdot \|\vec{a}\|$

$= 0 + 0 = 0$

$\therefore \lim_{k\to\infty} \|c_k\vec{a}_k - c\cdot\vec{a}\| = 0$

$\therefore \{c_k\vec{a}_k\}$ is a convergent sequence in X with limits $c\cdot\vec{a}$

(b)

$0 \leq \|a_k + b_k - (a+b)\| = \|(a_k - a) + (b_k - b)\| \leq \|a_k - a\| + \|b_k - b\|$

$\lim_{k\to\infty} \|a_k + b_k - (a+b)\| \leq \lim_{k\to\infty}(\|a_k - a\| + \|b_k - b\|) = 0$

$\therefore \lim_{k\to\infty} \|a_k + b_k - (a+b)\| = 0$

$\therefore \{a_k + b_k\}$ converges to $a+b$ in X.