Sets and Set Operations

Sets

Generally we are not only interested in asking about the probabilities of **outcomes** but also the probabilities of **events**, which are combinations of outcomes.

Each **event** is a *set*. So we need to know to do operations on sets.

Set Operations

We will use the following basic operations and associated notation:

1. $A \subset B$ (reads "A is a subset of B"):

> **Subset Operator** The **subset operator** $\subset$ is defined for two sets A and B by:
>
> $$A \subset B \text{ if } x \in A \Rightarrow x \in B$$

- Note that $A = B$ is included in $A \subset B$, and
- $A = B$ if and only if $A \subset B$ and $B \subset A$ (this is useful for proofs)

1. $A \cup B$ (reads "A union B" or "A or B"):

> **Union Operator** The **union** of A and B is a set defined by:
>
> $$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

1. $A \cap B$ (reads "A intersect B" or "A and B"):

> **Intersection Operator** The **intersection** of A and B is a set defined by:
>
> $$A \cap B = AB = \{x | x \in A \text{ and } x \in B\}$$

1. A^c or $\overline{A}$ (reads "A complement"):

> **Complement Operator** The **complement** of a set A in a sample space Ω is defined by:
>
> $$A^c = \overline{A} = \{x | x \in \Omega \text{ and } x \notin A\}$$

- One of the fundamental tools for understanding set operations is the **Venn diagram** (Wikipedia entry).
- The following tools are useful to help learn about Venn diagrams and set relations:
  - Khan Academy unit on basic set operations
  - Berkeley Statistics visualization tool for some simple set operations
  - Once you have worked with those for a little bit, you may be ready to test yourself. Start with a test of ability map from mathematical notation to a Venn diagram.
  - Then try to map from normal language on to the Venn diagram

Use Venn diagrams to convince yourself of:

- $(A \cap B) \subset A$ and $(A \cap B) \subset B$
- $\overline{\left(\overline{A}\right)} = A$
- $A \subset B \Rightarrow \overline{B} \subset \overline{A}$
- $A \cup \overline{A} = \Omega$
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$ (DeMorgan's Law 1)
- $\overline{A \cup B} = \overline{A} \cap \overline{B}$ (DeMorgan's Law 2)

1. One of the most important relations for sets is when sets are **mutually exclusive**

Mutually Exclusivity Two sets A and B are said to be **mutually exclusive** or **disjoint** if and only if (iff) $A \cap B = \emptyset$

Note: This is very important! The word "probability" was not used in the mutual exclusive definition: mutually exclusivity is a set relation (regardless of how Wikipedia presents it!).

Classification of Sets by Cardinality

1. $|S|$ (reads "cardinality of the set S"):

Cardinality The **cardinality** of a set S, denoted $|S|$, is the number of elements in that set.

Two sets have the same cardinality if there is a *bijection* (one-to-one mapping) between members of the two sets.

- A bijection is a function that is one-to-one and onto

Using this approach, if there is a bijection from a set S to $\{1, 2, \cdots, N\}$, then $|S| = N$.

For our purposes, we can use a set's cardinality to classify it as either:

- *finite*,

- *countably infinite* (or simply *countable*), or
- *uncountably infinite* (or *uncountable*)

Finite Set A set S is **finite** if $|S| = N < \infty$

Countably Infinite Set A set S is **countably infinite** if $|S| = |\mathbb{Z}|$, i.e., it can be put into one-to-one correspondence with the integers.

- Examples:
  - The Integers, $\mathbb{Z}$
  - The Positive Integers, $\mathbb{Z}^+$
  - The Rationals, $\mathbb{Q}$
    - Watch this video: https://www.youtube.com/watch?v=sLUU6-vokXw

Uncountably Infinite Set A set S is **uncountably infinite** if $|S| > |\mathbb{Z}|$.

- Examples:
  - The Real Line, $\mathbb{R}$
  - The transcendental numbers (i.e., the numbers in $\mathbb{R}$ that are not rationals) or roots of polynomials involving rationals!
  - The Complex Numbers, $\mathbb{C}$
  - **Any interval**, finite or infinite

1. We can formally define an **interval** as follows:

Interval If a and b ($b > a$) are in an **interval** I, then if $a \leq x \leq b$, $x \in I$.

- Intervals can be *open*, *closed*, or *half-open*:
- A closed interval $[a, b]$ contains the endpoints of a and b.
- An open interval (a, b) does not contain the endpoints of a and b.
- An interval can be half-open, such as $(a, b]$, which does not contain a, or $[a, b)$, which does not contain b.
- Intervals can also be either finite, infinite, or partially infinite.
- For our purposes of **assigning probabilities to sets**, we can treat finite and countably infinite sets together.

1. We can classify sets as either *discrete* or *continuous*:

Discrete set A **discrete set** is either finite or countably infinite.

Continuous set A **continuous set** is not countable.