A Recipie for Induction

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In this document I present some observations and results gained while trying to implement induction in a *minimal* type theory.

1 A (minimal) Type Theory

In this section I present the syntax, big-step semantics, and typing rules of a minimal type theory ATT.

$$
\begin{array}{rcl}
s & ::= & \mathcal{U}_n, n \in \mathbb{N} \\
\mathbf{T} & ::= & \lambda v : \mathbf{T}.\mathbf{T} \;\mid\; \Pi_{v:\mathbf{T}}\mathbf{T} \;\mid\; s \;\mid\; \mathbf{T}\mathbf{T} \;\mid\; \mathbf{T} : \mathbf{T} \;\mid\; v
\end{array}
$$

1.1 Typechecking and Evaluation Rules

$$\frac{v : \tau \in \Gamma}{\Gamma \vdash v : \tau} \qquad \frac{i > j}{\Gamma \vdash \mathcal{U}_j : \mathcal{U}_i} \qquad \frac{\Gamma \vdash \tau : \mathcal{U}_i \quad \Gamma \vdash \tau \leadsto^{nf} \tau' \quad \Gamma \vdash e : \tau'}{\Gamma \vdash e : \tau : \tau'}$$

$$\frac{\Gamma \vdash \tau : \mathcal{U}_i \quad \Gamma \vdash \tau \leadsto^{nf} \tau' \quad \Gamma, v : \tau' \vdash e : \upsilon}{\Gamma \vdash \lambda v : \tau.e : \upsilon}$$

$$\frac{\Gamma \vdash \tau : \mathcal{U}_i \quad \Gamma \vdash \tau \leadsto^{nf} \tau' \quad \Gamma, v : \tau' \vdash e : \mathcal{U}_j \quad k \geq i, j}{\Gamma \vdash \Pi_{v:\tau} e : \mathcal{U}_k}$$

$$\frac{\Gamma \vdash f : \Pi_{v:\tau} e \quad \Gamma \vdash x : \upsilon \quad \Gamma \vdash x \leadsto^{nf} x' \quad \Gamma \vdash \upsilon \subseteq \tau}{\Gamma \vdash fx : e[v \mapsto x']} \qquad \frac{\Gamma \vdash e : \tau \quad \Gamma \vdash \tau \subseteq \upsilon}{\Gamma \vdash e : \upsilon} \qquad \frac{i \geq j}{\Gamma \vdash \mathcal{U}_j \subseteq \mathcal{U}_j}$$

$$\frac{\Gamma \vdash \tau_0 \subseteq \tau_1 \quad \Gamma \vdash \upsilon_0 \subseteq \upsilon_1}{\Gamma \vdash \Pi_{v:\tau_0}\upsilon_0 \subseteq \Pi_{u:\tau_1}\upsilon_1} \qquad \frac{\Gamma \vdash \tau_0 \subseteq \tau_1 \quad \Gamma \vdash \upsilon_0 \subseteq \upsilon_1}{\Gamma \vdash \lambda v : \tau_0.\upsilon_0 \subseteq \lambda u : \tau_1.\upsilon_1} \qquad \frac{\Gamma \vdash \tau_0 \subseteq \tau_1 \quad \Gamma \vdash \upsilon_0 \subseteq \upsilon_1}{\Gamma \vdash \tau_0\upsilon_0 \subseteq \tau_1\upsilon_1} \qquad \frac{\Gamma \vdash v = u}{\Gamma \vdash v \subseteq u}$$

$$\frac{}{\Gamma \vdash v \leadsto^{nf} v} \qquad \frac{}{\Gamma \vdash \mathcal{U}_i \leadsto^{nf} \mathcal{U}_i} \qquad \frac{\Gamma \vdash f \leadsto^{nf} \lambda v.e \quad \Gamma \vdash x \leadsto^{nf} x' \quad \Gamma \vdash e[v \mapsto x'] \leadsto^{nf} t}{\Gamma \vdash fx \leadsto^{nf} t}$$

$$\frac{\Gamma \vdash e \leadsto^{nf} e'}{\Gamma \vdash e : \tau \leadsto^{nf} e'} \qquad \frac{\Gamma \vdash e \leadsto^{nf} e'}{\Gamma \vdash \lambda v : \tau.e \leadsto^{nf} \lambda v.e'} \qquad \frac{\Gamma \vdash e \leadsto^{nf} e' \quad \Gamma \vdash \tau \leadsto^{nf} \tau'}{\Gamma \vdash \Pi_{v:\tau} e \leadsto^{nf} \Pi_{v:\tau'} e'}$$

1.2 The Type $\mathcal{W}$ of Well-Formed Trees

$$\frac{\Gamma \vdash S : \mathcal{U}_u \qquad \Gamma \vdash P : S \to \mathcal{U}_v}{\Gamma \vdash \mathcal{W}(S,P) : \mathcal{U}_w} \qquad\qquad \frac{\Gamma \vdash s : S \qquad \Gamma \vdash p : Ps \to \mathcal{W}(S,P)}{\Gamma \vdash \sup(s,p) : \mathcal{W}(S,P)}$$

$$\frac{\begin{array}{c}\Gamma \vdash w : \mathcal{W}(S,P) \qquad \Gamma \vdash S : \mathcal{U}_u \qquad \Gamma \vdash P : S \to \mathcal{U}_v \\ \Gamma \vdash C : \mathcal{W}(S,P) \to \mathcal{U}_x \qquad \Gamma \vdash \Upsilon : \Pi_{s:S}[\Pi_{f:Ps \to \mathcal{W}(S,P)} \Pi_{p:Ps} C(fp) \to C \sup(s,f)] \to \Pi_{x:\mathcal{W}(S,P)} Cx\end{array}}{\Gamma \vdash \mathrm{rec}(w, \Upsilon) : Cw}$$

$$\frac{\Gamma \vdash \Upsilon x u(\lambda y.\mathrm{rec}(uy, c)) \rightsquigarrow^{nf} e}{\Gamma \vdash \mathrm{rec}(\sup(x,u), \Upsilon) \rightsquigarrow^{nf} e}$$

1.3 The Type $\mathcal{V}$ of Well-Formed Trees

1.3.1 Naturals, Bounded Naturals, and Vectors

These axioms just contain the usual definitions of $\mathbb{N}$ and n-bounded naturals $\mathbb{N}_n$ with recursive eliminators. I also add n-ary tuple types, T^n - this can be implemented using Σ-types and $\mathrm{rec}_{\mathbb{N}}$, but I include it axiomatically for simplicity.

$$\frac{}{\Gamma \vdash \mathbb{N} : \mathcal{U}_n} \qquad \frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash Sn : \mathbb{N}} \qquad \frac{}{\Gamma \vdash Z : \mathbb{N}} \qquad \frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash \mathbb{F}_n : \mathcal{U}_b} \qquad \frac{\Gamma \vdash n : \mathbb{N} \qquad \Gamma \vdash m : \mathbb{F}_n}{\Gamma \vdash S_n m : \mathbb{F}_{Sn}} \qquad \frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash Z_n : \mathbb{F}_{Sn}}$$

$$\frac{\Gamma \vdash n : \mathbb{N} \qquad \Gamma \vdash T : \mathcal{U}_v}{\Gamma \vdash T^n : \mathcal{U}_v} \qquad \frac{\Gamma \vdash n : \mathbb{N} \qquad \Gamma \vdash T : \mathcal{U}_v \qquad \Gamma \vdash t : T \qquad \Gamma \vdash v : T^n}{\Gamma \vdash \mathrm{cons}(T, n, t, v) : T^{Sn}} \qquad \frac{\Gamma \vdash T : \mathcal{U}_v}{\Gamma \vdash \mathrm{unit}(T) : T^Z}$$

$$\frac{\Gamma \vdash P : \mathbb{N} \to \mathcal{U}_n \qquad \Gamma \vdash z : PZ \qquad \Gamma \vdash s : \Pi_{n:\mathbb{N}} Pn \to P(Sn) \qquad \Gamma \vdash n : \mathbb{N}}{\Gamma \vdash \mathrm{rec}_{\mathbb{N}}(P, z, s, n) : Pn}$$

$$\frac{\begin{array}{c}\Gamma \vdash P : \Pi_{n:\mathbb{N}} \mathbb{F}_n \to \mathcal{U}_b \\ \Gamma \vdash z : \Pi_{n:\mathbb{N}} P(Sn) Z_n \qquad \Gamma \vdash s : \Pi_{n:\mathbb{N}} \Pi_{f:\mathbb{F}_n} Pnf \to P(Sn)(S_n f) \qquad \Gamma \vdash n : \mathbb{N} \qquad \Gamma \vdash f : \mathbb{F}_n\end{array}}{\Gamma \vdash \mathrm{rec}_{\mathbb{F}}(n, P, z, s, f) : Pnf}$$

$$\frac{\begin{array}{c}\Gamma \vdash T : \mathcal{U}_v \qquad \Gamma \vdash P : \Pi_{n:\mathbb{N}} T^n \to \mathcal{U}_v \\ \Gamma \vdash z : PZ(\mathrm{unit}(T)) \qquad \Gamma \vdash s : \Pi_{n:\mathbb{N}} \Pi_{v:T^n} \Pi_{t:T} Pnv \to P(Sn)(\mathrm{cons}(T, Sn, t, v) \qquad \Gamma \vdash n : \mathbb{N} \qquad \Gamma \vdash v : T^n\end{array}}{\Gamma \vdash \mathrm{rec}_{T^n}(P, z, s, v) : Pnv}$$

$$\frac{\Gamma \vdash z \rightsquigarrow^{nf} z'}{\Gamma \vdash \mathrm{rec}_{\mathbb{N}}(P, z, s, Z) \rightsquigarrow^{nf} z'} \qquad \frac{\Gamma \vdash s(\mathrm{rec}_{\mathbb{N}}(P, z, s, n)) \rightsquigarrow^{nf} p}{\Gamma \vdash \mathrm{rec}_{\mathbb{N}}(P, z, s, Sn) \rightsquigarrow^{nf} p} \qquad \frac{\Gamma \vdash zn \rightsquigarrow^{nf} z'}{\Gamma \vdash \mathrm{rec}_{\mathbb{F}}(Sn, P, z, s, Z_n) \rightsquigarrow^{nf} z'}$$

$$\frac{\Gamma \vdash sn(\mathrm{rec}_{\mathbb{F}}(n, P, z, s, f)) \rightsquigarrow^{nf} p}{\Gamma \vdash \mathrm{rec}_{\mathbb{F}}(Sn, P, z, s, S_n f) \rightsquigarrow^{nf} p} \qquad \frac{\Gamma \vdash z \rightsquigarrow^{nf} z'}{\Gamma \vdash \mathrm{rec}_{T^Z}(P, z, s, \mathrm{unit}(T)) \rightsquigarrow^{nf} z'}$$

$$\frac{\Gamma \vdash snvt(\mathrm{rec}_{T^n}(P, z, s, v)) \rightsquigarrow^{nf} z'}{\Gamma \vdash \mathrm{rec}_{T^{Sn}}(P, z, s, \mathrm{cons}(T, n, t, v)) \rightsquigarrow^{nf} z'}$$

I now define the usual Σ-types (existential quantifiers), which are needed for the definition of $\mathcal{V}$:

$$\frac{\Gamma \vdash A : \mathcal{U}_s \qquad \Gamma \vdash B : A \to \mathcal{U}_s}{\Gamma \vdash \Sigma(A, B) : \mathcal{U}_s} \qquad \frac{\Gamma \vdash A : \mathcal{U}_s \qquad \Gamma \vdash B : A \to \mathcal{U}_s \qquad \Gamma \vdash a : A \qquad \Gamma \vdash p : Ba}{\Gamma \vdash \exists(A, B, a, p) : \Sigma(A, B)}$$

$$\frac{\begin{array}{c}\Gamma \vdash A : \mathcal{U}_s \\ \Gamma \vdash B : A \to \mathcal{U}_s \qquad \Gamma \vdash P : \Sigma(A, B) \to \mathcal{U}_s \qquad \Gamma \vdash f : \Pi_{x:A}\Pi_{p:Ba} P\exists(A, B, x, p) \qquad \Gamma \vdash p : \Sigma(A, B)\end{array}}{\Gamma \vdash \mathrm{elim}_\Sigma(A, B, f, p) : Pp}$$

$$\frac{\Gamma \vdash fab \rightsquigarrow^{nf} e}{\Gamma \vdash \mathrm{elim}_\Sigma(A, B, f, \exists(a, b)) \rightsquigarrow^{nf} e}$$

1.3.2 $\mathcal{V}$

$\mathcal{V}(n, R)$ is the type of well-founded trees with nodes labelled by $\mathbb{F}n$ (i.e. it has n constructors), where each node is the supremum of Rx subtrees, where $x : finn$. This can be formulated axiomatically as follows:

$$\frac{\Gamma \vdash n : \mathbb{N} \qquad \Gamma \vdash R : \mathbb{F}n \to \mathbb{N}}{\Gamma \vdash V(n, R) : \mathcal{U}_v}$$

$\sup(n, R, x, t)$ is the xth constructor for $\mathcal{V}(n, R)$, and is the supremum of the subtrees t:

$$\frac{\Gamma \vdash n : \mathbb{N} \qquad \Gamma \vdash R : \mathbb{F}n \to \mathbb{N} \qquad \Gamma \vdash c : \mathbb{F}n \qquad \Gamma \vdash t : \mathcal{V}(n, R)^{Rc}}{\Gamma \vdash \sup(n, R, c, t) : \mathcal{V}(n, R)}$$

In order to do inductive proofs of some property $P : \mathcal{V}(n, R) \to \mathcal{U}_v$ on some $\sup(n, R, c, t)$, we need to know that every subtree in t also satisfies P, i.e. for every x in t there exists a proof of Px; using Σ types, this requirement can be encoded as $\Sigma(\mathcal{V}(n, R), P)^{Rc}$.

Given the ability to 'map' over an n-ary product, and extract the first element of a Σ-type,

$$\begin{aligned}\mathrm{map} :=& \lambda A : \mathcal{U}_s.\lambda B : \mathcal{U}_s.\lambda f : A \to B.\lambda n : \mathbb{N}.\lambda v : A^n. \\ &\quad \mathrm{rec}_{A^n}(\lambda m.\lambda x.B^m, \mathrm{unit}(t), \lambda n.\lambda x.\lambda t.\lambda v.cons(B, n, v, (ft)), v) : B^n \\ \mathrm{fst} :=& \lambda A : \mathcal{U}_s.\lambda B : A \to \mathcal{U}_s.\lambda p : \Sigma(A, B).\mathrm{elim}_\Sigma(A, B, \lambda a.\lambda b.a, p)\end{aligned}$$

the recursor for $\mathcal{V}(n, R)$ becomes obvious, if verbose:

$$\frac{\begin{array}{c}\Gamma \vdash n : \mathbb{N} \qquad \Gamma \vdash R : \mathbb{F}n \qquad \Gamma \vdash P : \mathcal{V}(n, R) \to \mathcal{U}_v \\ \Gamma \vdash f : \Pi_{c:\mathbb{F}n}\Pi_{s:\Sigma(\mathcal{V}(n,R),P)^{Rc}} P\sup(n, R, c, \mathrm{map}(\Sigma(\mathcal{V}(n, R), P))(\mathcal{V}(n, R))(\mathrm{fst}(\mathcal{V}(n, R))P)(Rc)s) \\ \Gamma \vdash t : \mathcal{V}(n, R)\end{array}}{\Gamma \vdash \mathrm{rec}_\mathcal{V}(n, R, P, f, t) : Pt}$$

$$\frac{\Gamma \vdash fc(\mathrm{map}(\mathcal{V}(n, R))(\Sigma(\mathcal{V}(n, R), P))(\lambda t : \mathcal{V}(n, R).\exists(t, \mathrm{rec}_\mathcal{V}(n, R, P, f, t)))(Rc)t) \rightsquigarrow^{nf} e}{\Gamma \vdash \mathrm{rec}_\mathcal{V}(n, R, P, f, \sup(n, R, c, t)) \rightsquigarrow^{nf} e}$$