# A Recipie for Induction

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In this document I present some observations and results gained while trying to implement induction in a *minimal* type theory.

## 1 A (minimal) Type Theory

In this section I present the syntax, big-step semantics, and typing rules of a minimal type theory ATT, which I will use as a basis to extend with more axioms.

$$\begin{array}{lcl} s & ::= & \mathcal{U}_n, n \in \mathbb{N} \\ \mathbf{T} & ::= & \lambda v : \mathbf{T}.\mathbf{T} \ \mid \ \Pi_{v:\mathbf{T}}\mathbf{T} \ \mid \ s \ \mid \ \mathbf{T}\mathbf{T} \ \mid \ \mathbf{T} : \mathbf{T} \ \mid \ v \end{array}$$

### 1.1 Typechecking and Evaluation Rules

$$\frac{v : \tau \in \Gamma}{\Gamma \vdash v : \tau} \qquad \frac{i > j}{\Gamma \vdash \mathcal{U}_j : \mathcal{U}_i} \qquad \frac{\Gamma \vdash \tau : \mathcal{U}_i \qquad \Gamma \vdash \tau \rightsquigarrow^{nf} \tau' \qquad \Gamma \vdash e : \tau'}{\Gamma \vdash e : \tau : \tau'}$$

$$\frac{\Gamma \vdash \tau : \mathcal{U}_i \qquad \Gamma \vdash \tau \rightsquigarrow^{nf} \tau' \qquad \Gamma, v : \tau' \vdash e : \upsilon}{\Gamma \vdash \lambda v : \tau.e : \upsilon}$$

$$\frac{\Gamma \vdash \tau : \mathcal{U}_i \qquad \Gamma \vdash \tau \rightsquigarrow^{nf} \tau' \qquad \Gamma, v : \tau' \vdash e : \mathcal{U}_j \qquad k \geq i, j}{\Gamma \vdash \Pi_{v:\tau} e : \mathcal{U}_k}$$

$$\frac{\Gamma \vdash f : \Pi_{v:\tau} e \qquad \Gamma \vdash x : \upsilon \qquad \Gamma \vdash x \rightsquigarrow^{nf} x' \qquad \Gamma \vdash \upsilon \subseteq \tau}{\Gamma \vdash fx : e[v \mapsto x']} \qquad \frac{\Gamma \vdash e : \tau \qquad \Gamma \vdash \tau \subseteq \upsilon}{\Gamma \vdash e : \upsilon} \qquad \frac{i \geq j}{\Gamma \vdash \mathcal{U}_j \subseteq \mathcal{U}_j}$$

$$\frac{\Gamma \vdash \tau_0 \subseteq \tau_1 \qquad \Gamma \vdash \upsilon_0 \subseteq \upsilon_1}{\Gamma \vdash \Pi_{v:\tau_0} \upsilon_0 \subseteq \Pi_{u:\tau_1} \upsilon_1} \qquad \frac{\Gamma \vdash \tau_0 \subseteq \tau_1 \qquad \Gamma \vdash \upsilon_0 \subseteq \upsilon_1}{\Gamma \vdash \lambda v : \tau_0.\upsilon_0 \subseteq \lambda u : \tau_1.\upsilon_1} \qquad \frac{\Gamma \vdash \tau_0 \subseteq \tau_1 \qquad \Gamma \vdash \upsilon_0 \subseteq \upsilon_1}{\Gamma \vdash \tau_0 \upsilon_0 \subseteq \tau_1 \upsilon_1} \qquad \frac{\Gamma \vdash v = u}{\Gamma \vdash v \subseteq u}$$

$$\frac{}{\Gamma \vdash v \rightsquigarrow^{nf} v} \qquad \frac{}{\Gamma \vdash \mathcal{U}_i \rightsquigarrow^{nf} \mathcal{U}_i} \qquad \frac{\Gamma \vdash f \rightsquigarrow^{nf} \lambda v.e \qquad \Gamma \vdash x \rightsquigarrow^{nf} x' \qquad \Gamma \vdash e[v \mapsto x'] \rightsquigarrow^{nf} t}{\Gamma \vdash fx \rightsquigarrow^{nf} t}$$

$$\frac{\Gamma \vdash e \rightsquigarrow^{nf} e'}{\Gamma \vdash e : \tau \rightsquigarrow^{nf} e'} \qquad \frac{\Gamma \vdash e \rightsquigarrow^{nf} e'}{\Gamma \vdash \lambda v : \tau.e \rightsquigarrow^{nf} \lambda v.e'} \qquad \frac{\Gamma \vdash e \rightsquigarrow^{nf} e' \qquad \Gamma \vdash \tau \rightsquigarrow^{nf} \tau'}{\Gamma \vdash \Pi_{v:\tau} e \rightsquigarrow^{nf} \Pi_{v:\tau'} e'}$$

### 1.2 The Type $\mathcal{W}$ of Well-Founded Trees

This is the $\mathcal{W}$ axiom for well-founded trees as proposed by Martin-Löf[Mar79]:

$$\frac{\Gamma \vdash S : \mathcal{U} \qquad \Gamma \vdash P : S \to \mathcal{U}}{\Gamma \vdash \mathcal{W}(S,P) : \mathcal{U}} \qquad \frac{\Gamma \vdash s : S \qquad \Gamma \vdash p : Ps \to \mathcal{W}(S,P)}{\Gamma \vdash \sup(s,p) : \mathcal{W}(S,P)}$$

$$\frac{\begin{array}{c}\Gamma \vdash w : \mathcal{W}(S,P) \qquad \Gamma \vdash S : \mathcal{U} \qquad \Gamma \vdash P : S \to \mathcal{U} \\ \Gamma \vdash C : \mathcal{W}(S,P) \to \mathcal{U}x \qquad \Gamma \vdash \Upsilon : \Pi_{s:S}[\Pi_{f:Ps \to \mathcal{W}(S,P)}\Pi_{p:Ps}C(fp) \to C\sup(s,f)] \to \Pi_{x:\mathcal{W}(S,P)}Cx\end{array}}{\Gamma \vdash \mathrm{rec}(w,\Upsilon) : Cw}$$

$$\frac{\Gamma \vdash \Upsilon xu(\lambda y.\mathrm{rec}(uy,c)) \rightsquigarrow^{nf} e}{\Gamma \vdash \mathrm{rec}(\sup(x,u),\Upsilon) \rightsquigarrow^{nf} e}$$

Unfortunately, the $\mathcal{W}$-type fails to be useful in practice without a degree of extensionality[AM06], which may be undesirable. To this end, I formuate an alternative representation of $\mathcal{W}$-types appropriate for intensional type theories, such as ATT.

## 1.3 The Type $\mathcal{V}$ of Well-Formed Trees

### 1.3.1 Naturals, Bounded Naturals, and Vectors

These axioms just contain the usual definitions of $\mathbb{N}$ and $n$-bounded naturals $\mathbb{F}_n$ with recursive eliminators. I also add $n$-ary tuple types, $T^n$ - this can be implemented using $\Sigma$-types and $\mathrm{rec}_{\mathbb{N}}$, but I include it axiomatically for simplicity.

$$\frac{}{\Gamma \vdash \mathbb{N} : \mathcal{U}_n} \qquad \frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash Sn : \mathbb{N}} \qquad \frac{}{\Gamma \vdash Z : \mathbb{N}} \qquad \frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash \mathbb{F}_n : \mathcal{U}} \qquad \frac{\Gamma \vdash n : \mathbb{N} \qquad \Gamma \vdash m : \mathbb{F}_n}{\Gamma \vdash S_n m : \mathbb{F}_{Sn}} \qquad \frac{\Gamma \vdash n : \mathbb{N}}{\Gamma \vdash Z_n : \mathbb{F}_{Sn}}$$

$$\frac{\Gamma \vdash n : \mathbb{N} \qquad \Gamma \vdash T : \mathcal{U}}{\Gamma \vdash T^n : \mathcal{U}} \qquad \frac{\Gamma \vdash n : \mathbb{N} \qquad \Gamma \vdash T : \mathcal{U} \qquad \Gamma \vdash t : T \qquad \Gamma \vdash v : T^n}{\Gamma \vdash \mathrm{cons}(T,n,t,v) : T^{Sn}} \qquad \frac{\Gamma \vdash T : \mathcal{U}}{\Gamma \vdash \mathrm{unit}(T) : T^Z}$$

$$\frac{\Gamma \vdash P : \mathbb{N} \to \mathcal{U}_n \qquad \Gamma \vdash z : PZ \qquad \Gamma \vdash s : \Pi_{n:\mathbb{N}}Pn \to P(Sn) \qquad \Gamma \vdash n : \mathbb{N}}{\Gamma \vdash \mathrm{rec}_{\mathbb{N}}(P,z,s,n) : Pn}$$

$$\frac{\begin{array}{c}\Gamma \vdash P : \Pi_{n:\mathbb{N}}\mathbb{F}_n \to \mathcal{U} \\ \Gamma \vdash z : \Pi_{n:\mathbb{N}}P(Sn)Z_n \qquad \Gamma \vdash s : \Pi_{n:\mathbb{N}}\Pi_{f:\mathbb{F}_n}Pnf \to P(Sn)(S_n f) \qquad \Gamma \vdash n : \mathbb{N} \qquad \Gamma \vdash f : \mathbb{F}_n\end{array}}{\Gamma \vdash \mathrm{rec}_{\mathbb{F}}(n,P,z,s,f) : Pnf}$$

$$\frac{\begin{array}{c}\Gamma \vdash T : \mathcal{U} \qquad \Gamma \vdash P : \Pi_{n:\mathbb{N}}T^n \to \mathcal{U} \\ \Gamma \vdash z : PZ(\mathrm{unit}(T)) \qquad \Gamma \vdash s : \Pi_{n:\mathbb{N}}\Pi_{v:T^n}\Pi_{t:T}Pnv \to P(Sn)(\mathrm{cons}(T,Sn,t,v) \qquad \Gamma \vdash n : \mathbb{N} \qquad \Gamma \vdash v : T^n\end{array}}{\Gamma \vdash \mathrm{rec}_{T^n}(P,z,s,v) : Pnv}$$

$$\frac{\Gamma \vdash z \rightsquigarrow^{nf} z'}{\Gamma \vdash \mathrm{rec}_{\mathbb{N}}(P,z,s,Z) \rightsquigarrow^{nf} z'} \qquad \frac{\Gamma \vdash s(\mathrm{rec}_{\mathbb{N}}(P,z,s,n)) \rightsquigarrow^{nf} p}{\Gamma \vdash \mathrm{rec}_{\mathbb{N}}(P,z,s,Sn) \rightsquigarrow^{nf} p} \qquad \frac{\Gamma \vdash zn \rightsquigarrow^{nf} z'}{\Gamma \vdash \mathrm{rec}_{\mathbb{F}}(Sn,P,z,s,Z_n) \rightsquigarrow^{nf} z'}$$

$$\frac{\Gamma \vdash sn(\mathrm{rec}_{\mathbb{F}}(n,P,z,s,f)) \rightsquigarrow^{nf} p}{\Gamma \vdash \mathrm{rec}_{\mathbb{F}}(Sn,P,z,s,S_n f) \rightsquigarrow^{nf} p} \qquad \frac{\Gamma \vdash z \rightsquigarrow^{nf} z'}{\Gamma \vdash \mathrm{rec}_{T^Z}(P,z,s,\mathrm{unit}(T)) \rightsquigarrow^{nf} z'}$$

$$\frac{\Gamma \vdash snvt(\mathrm{rec}_{T^n}(P,z,s,v)) \rightsquigarrow^{nf} z'}{\Gamma \vdash \mathrm{rec}_{T^{Sn}}(P,z,s,\mathrm{cons}(T,n,t,v)) \rightsquigarrow^{nf} z'}$$

I now define the usual $\Sigma$-types (existential quantifiers), which are needed for the definition of $\mathcal{V}$:

$$\frac{\Gamma \vdash A : \mathcal{U} \qquad \Gamma \vdash B : A \to \mathcal{U}}{\Gamma \vdash \Sigma(A,B) : \mathcal{U}} \qquad \frac{\Gamma \vdash A : \mathcal{U} \qquad \Gamma \vdash B : A \to \mathcal{U} \qquad \Gamma \vdash a : A \qquad \Gamma \vdash p : Ba}{\Gamma \vdash \exists(A,B,a,p) : \Sigma(A,B)}$$

$$\frac{\begin{array}{c}\Gamma \vdash A : \mathcal{U} \\ \Gamma \vdash B : A \to \mathcal{U} \qquad \Gamma \vdash P : \Sigma(A,B) \to \mathcal{U} \qquad \Gamma \vdash f : \Pi_{x:A}\Pi_{p:Ba} P\exists(A,B,x,p) \qquad \Gamma \vdash p : \Sigma(A,B)\end{array}}{\Gamma \vdash \mathrm{elim}_{\Sigma}(A,B,f,p) : Pp}$$

$$\frac{\Gamma \vdash fab \rightsquigarrow^{nf} e}{\Gamma \vdash \mathrm{elim}_{\Sigma}(A,B,f,\exists(a,b)) \rightsquigarrow^{nf} e}$$

### 1.3.2 $\mathcal{V}$

$\mathcal{V}(n,R)$ is the type of well-founded trees with nodes labelled by $\mathbb{F}_n$ (i.e. it has $n$ constructors), where each node is the supremum of $Rx$ subtrees, where $x : \mathbb{F}_n$. This can be formulated axiomatically as follows:

$$\frac{\Gamma \vdash n : \mathbb{N} \qquad \Gamma \vdash R : \mathbb{F}_n \to \mathbb{N}}{\Gamma \vdash V(n,R) : \mathcal{U}}$$

$\mathrm{sup}(n,R,x,t)$ is the $x$th constructor for $\mathcal{V}(n,R)$, and is the supremum of the subtrees $t$:

$$\frac{\Gamma \vdash n : \mathbb{N} \qquad \Gamma \vdash R : \mathbb{F}_n \to \mathbb{N} \qquad \Gamma \vdash c : \mathbb{F}_n \qquad \Gamma \vdash t : \mathcal{V}(n,R)^{Rc}}{\Gamma \vdash \mathrm{sup}(n,R,c,t) : \mathcal{V}(n,R)}$$

In order to do inductive proofs of some property $P : \mathcal{V}(n,R) \to \mathcal{U}$ on some $\mathrm{sup}(n,R,c,t)$, we need to know that every subtree in $t$ also satisfies $P$, i.e. for every $x$ in $t$ there exists a proof of $Px$; using $\Sigma$ types, this requirement can be encoded as $\Sigma(\mathcal{V}(n,R),P)^{Rc}$.

Given the ability to 'map' over an $n$-ary product, and extract the first element of a $\Sigma$-type,

$$\begin{aligned}
\mathrm{map} :=& \lambda A : \mathcal{U}.\lambda B : \mathcal{U}.\lambda f : A \to B.\lambda n : \mathbb{N}.\lambda v : A^n. \\
& \mathrm{rec}_{A^n}(\lambda m.\lambda x.B^m, \mathrm{unit}(t), \lambda n.\lambda x.\lambda t.\lambda v.cons(B,n,v,(ft)),v) : B^n \\
\mathrm{fst} :=& \lambda A : \mathcal{U}.\lambda B : A \to \mathcal{U}.\lambda p : \Sigma(A,B).\mathrm{elim}_{\Sigma}(A,B,\lambda a.\lambda b.a,p)
\end{aligned}$$

the recursor for $\mathcal{V}(n,R)$ becomes obvious, if verbose:

$$\frac{\begin{array}{c}\Gamma \vdash n : \mathbb{N} \qquad \Gamma \vdash R : \mathbb{F}_n \qquad \Gamma \vdash P : \mathcal{V}(n,R) \to \mathcal{U} \\ \Gamma \vdash f : \Pi_{c:\mathbb{F}_n}\Pi_{s:\Sigma(\mathcal{V}(n,R),P)^{Rc}} P\mathrm{sup}(n,R,c,\mathrm{map}(\Sigma(\mathcal{V}(n,R),P))(\mathcal{V}(n,R))(\mathrm{fst}(\mathcal{V}(n,R))P)(Rc)s) \\ \Gamma \vdash t : \mathcal{V}(n,R)\end{array}}{\Gamma \vdash \mathrm{rec}_{\mathcal{V}}(n,R,P,f,t) : Pt}$$

$$\frac{\Gamma \vdash fc(\mathrm{map}(\mathcal{V}(n,R))(\Sigma(\mathcal{V}(n,R),P))(\lambda t : \mathcal{V}(n,R).\exists(t,\mathrm{rec}_{\mathcal{V}}(n,R,P,f,t)))(Rc)t) \rightsquigarrow^{nf} e}{\Gamma \vdash \mathrm{rec}_{\mathcal{V}}(n,R,P,f,\mathrm{sup}(n,R,c,t)) \rightsquigarrow^{nf} e}$$

## References


[Mar79] Per Martin-Lof. *Constructive Mathematics and Computer Programming*. 1979.

[AM06] Thorsten Altenkirch and Conor McBride. *Towards Observational Type Theory*. 2006.