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Term Structure Movements and Pricing Interest Rate Contingent Claims

THOMAS S. Y. HO and SANG-BIN LEE*

ABSTRACT

This paper derives an arbitrage-free interest rate movements model (AR model). This model takes the complete term structure as given and derives the subsequent stochastic movement of the term structure such that the movement is arbitrage free. We then show that the AR model can be used to price interest rate contingent claims relative to the observed complete term structure of interest rates. This paper also studies the behavior and the economics of the model. Our approach can be used to price a broad range of interest rate contingent claims, including bond options and callable bonds.

INTEREST RATE OPTIONS, CALLABLE bonds, and floating rate notes are a few examples of interest rate contingent claims. They are characterized by their finite lives and their price behavior, which crucially depends on the term structure and its stochastic movements. In recent years, with the increase in interest rate volatility and the prevalent use of the contingent claims, the pricing of these securities has become a primary concern in financial research. The purpose of this paper is to present a general methodology to price a broad class of interest rate contingent claims.

The crux of the problem in pricing interest rate contingent claims is to model the term structure movements and to relate the movements to the assets' prices. Much academic literature has been devoted to this problem. One earlier attempt is that of Pye [15]. He assumed that the interest rates move according to a (Markov) transition probabilities matrix, and he then used the expectation hypothesis to price the expected cash flow of the asset—in his case, a callable bond (Pye [16]). Recently, investigators have focused more on developing equilibrium models.

Cox, Ingersoll, and Ross (CIR) [7] assumed that the short rate follows a mean-reverting process. By further assuming that all interest rate contingent claims are priced contingent on only the short rate, using a continuous arbitrage argument they derived an equilibrium pricing model. Brennan and Schwartz (BS) [2] extended the CIR model to incorporate both short and long rates and studied the pricing of a broad range of contingent claims (BS [1, 3]). In these

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approaches, both the term structure and the contingent claims are derived in an equilibrium context.¹

This paper proposes an alternative approach to these pricing models. We take the term structure as given and derive the feasible subsequent term structure movements. These movements must satisfy certain constraints to ensure that they are consistent with an equilibrium framework. Specifically, the movements cannot permit arbitrage profit opportunities. We shall call these interest rate movements *arbitrage-free rate movements* (AR). When the AR movements are determined, the interest rate contingent claims are then priced by the arbitrage methodology used in CIR and BS. Therefore, our model is a relative pricing model in the sense that we price our contingent claims relative to the observed term structure; we do not endogenize the term structure as the CIR and BS models do.

The main advantage of our approach is that it enables us to utilize the full information of the term structure to price contingent claims. To the extent that the shape of the yield curve should affect the contingent-claim value, our approach can be more applicable. Further, when our model is used to price a straight bond (when viewed as a contingent claim to the term structure movements), the theoretical price is assured to be that determined by the observed term structure. In this way, when we analyze a bond with various (interest rate) provisions, for example, the model can properly isolate the provision value from the value of the underlying bond. Indeed, when the provision has negligible value, the model's bond price is guaranteed to be that given by the term structure. Our methodology also provides the important linkage between contingent-claims pricing and the pricing of straight bonds. As a result, it integrates the relative pricing theory of contingent claims with the established literature on estimating the term structure of interest rates.²

In this paper, we analyze the AR model and show how it is related to other interest rate stochastic processes proposed in previous research. Finally, we present the methodology (using the AR model) to price interest rate contingent claims. Since our methodology is general and has direct applications to many financial problems, this paper should contribute significantly to the fixed-income research.

The paper is organized as follows. Section I presents the basic assumptions of the model and describes the binomial lattice of a term structure movement. Such a movement is shown to determine the stochastic movement of a bond price. Section II derives the AR model. The economics of the term structure movement is analyzed. Section III studies the behavior of the AR model. Section IV presents the methodology of using the AR model to price interest rate contingent claims. Finally, Section V contains the conclusions.

¹There are many subsequent papers following this approach to pricing various interest rate contingent claims. For example, Courtadon [5] priced the interest rate options, and Ramaswamy and Sundaresan [17] analyzed the floating rate notes.

²Many papers have dealt with the problem of estimating the term structure (or pricing default-free straight bonds). Examples are McCulloch [13], Carleton and Cooper [4], Litzenberger and Rolfo [12], and Shea [21].

I. Interest Rate Movements

This section sets up the analytical framework of a term structure movement. We first list the basic assumptions of the model. We then relate the term structure movements to the bond price stochastic process.

A. The Basic Assumptions

- (A1) The market is frictionless. There are no taxes and no transaction costs, and all securities are perfectly divisible.
- (A2) The market clears at discrete points in time, which are separated in regular intervals. For simplicity, we use each period as a unit of time. We define a discount bond of maturity T to be a bond that pays \$1 at the end of the T th period, with no other payments to its holder.
- (A3) The bond market is complete. There exists a discount bond for each maturity n ($n = 0, 1, 2, \dots$).
- (A4) At each time n , there are a finite number of states of the world. For state i , we denote the equilibrium price of the discount bond of maturity T by $P_i^{(n)}(T)$. Note that $P_i^{(n)}(\cdot)$ is a function that relates the price of a discount bond to its maturity. This function is called the *discount function*. Within the context of the model, the discount function completely describes the term structure of interest rates of the i th state at time n .

The discount function $P_i^{(n)}(\cdot)$ must satisfy several conditions. It must be positive since the function represents assets' values. Also, we require that

$$P_i^{(n)}(0) = 1 \quad \text{for all } i, n \quad (1)$$

and

$$\lim_{T \rightarrow \infty} P_i^{(n)}(T) = 0 \quad \text{for all } i, n. \quad (2)$$

Equation (1) says that a discount bond maturing instantaneously must be worth \$1. Equation (2) says that a discount bond with maturity in the distant future must have a negligible value. Assumptions (A1) through (A4) are the standard perfect capital market assumptions in a discrete state-time framework.

B. The Binomial Lattice

Now we describe the evolution of the term structure. Initially, we observe the discount function $P(\cdot)$. At the initial time, by convention, we have the 0-state. So we have

$$P(\cdot) = P_0^{(0)}(\cdot). \quad (3)$$

At time 1, the discount function may be specified by two possible functions, $P_1^{(1)}(\cdot)$ and $P_0^{(1)}(\cdot)$. (The superscript denotes the time, and the subscript denotes the state.) Therefore, there are only two states of the world at time 1. When $P_1^{(1)}(\cdot)$ prevails, we say that the upstate is attained; when $P_0^{(1)}(\cdot)$ prevails, the downstate is attained.

Next, consider the second period from time 1 to time 2. We confine each discount function, in either upstate or downstate, to also attain only one of two possible functions. Specifically, conditional on the discount function $P(\cdot)$ attaining $P_1^{(1)}(\cdot)$ at time 1, we require the discount function to be either $P_2^{(2)}(\cdot)$ or $P_1^{(2)}(\cdot)$ at time 2. At this point, we do not specify any particular functional forms for $P_2^{(2)}(\cdot)$ and $P_1^{(2)}(\cdot)$. Similarly, conditional on the discount function $P(\cdot)$ attaining the downstate at time 1, the discount function can then only attain an upstate or a downstate, taking on the functions $P_1^{(2)}(\cdot)$ and $P_0^{(2)}(\cdot)$, respectively. Note that the binomial lattice assumption requires the discount function attained by an upstate followed by a downstate to be equal to the discount function reached by a downstate followed by an upstate.

The stochastic process of the discount function in subsequent periods is described analogously. For the $(n + 1)$ th period, from time n to time $n + 1$, let $P_i^{(n)}(\cdot)$ denote the discount function at time n after i upstate movements and $(n - i)$ downstate movements. We require the discount function to depend only on the number of upstate movements and not on the sequence in which they occur. Thus, the binomial process at time n is specified by

$$\begin{array}{ccc}
 & P_{i+1}^{(n+1)}(\cdot) & \text{upstate} \\
 & \swarrow & \\
 P_i^{(n)}(\cdot) & & \\
 & \searrow & \\
 & P_i^{(n+1)}(\cdot) & \text{downstate.}
 \end{array} \quad (4)$$

A discount function is defined for each time n and state i . This set of discount functions is said to form a binomial lattice. A vertex of this lattice is specified by (n, i) . Notice that, for each time n , there are exactly $(n + 1)$ states ($i = 0, \dots, n$). The term structure can evolve from one vertex to another by different paths, but this will not affect the value of the discount function at the vertex at the end of the path. That is, the discount functions are path independent.

Often, it is more convenient to represent a term structure by the yield curve as opposed to the discount function. Given the discount function $P(T)$, we define the yield curve to be

$$r(T) = \frac{-\ln P(T)}{T}. \quad (5)$$

$r(T)$ is the continuously compounded yield of a discount bond with maturity T .

C. The Binomial Process of Bond Prices

When the term structure evolves in a binomial lattice, the price of each discount bond must follow a binomial process with time-dependent step size. In particular, consider the discount bond with maturity N . Initially, the bond price is, by definition, $P(N)$.

After the first period, the bond shortens its maturity to $(N - 1)$, and therefore, given the discount functions in the upstate and downstate, we can determine the bond prices; they are $P_1^{(1)}(N - 1)$ and $P_0^{(1)}(N - 1)$ in an upstate and downstate, respectively. All subsequent prices are determined analogously. In particular,

after N periods, the bond values are $P_i^{(N)}(0)$ for all states i ($i = 0, \dots, N$). But by equation (1) the bond value is \$1, as one would expect the bond value to be at maturity.

The stochastic price process of a discount bond is depicted in Figure 1. The discount function is depicted in each state and time in a binomial lattice. Figure 1 shows that the discount function always originates from unity. It increases in value in an upstate but drops in value in a downstate. Now consider the three-period bond. Initially, its value is $P(3)$. At time 1, it becomes a two-period bond, and its value can be either $P_1^{(1)}(2)$ or $P_0^{(1)}(2)$. At time 2, this bond becomes a one-period bond, and its value cannot deviate too much from unity in any state of the world and must converge to unity at maturity.

This model of a bond price stochastic process is similar to the binomial process proposed by Cox, Ross, and Rubinstein (CRR) [8] and Rendleman and Bartter [18] for stocks. However, there are two main differences. First, when pricing interest rate contingent claims, in most cases we are concerned with how the prices of discount bonds with different maturities move relative to each other. That is why we focus on the binomial lattice of a term structure rather than a binomial process of a particular bond. Second, in our model, the step size is time dependent to ensure that the bond value converges to unity at maturity.

By taking the binomial lattice approach, we ensure that a bond price stochastic

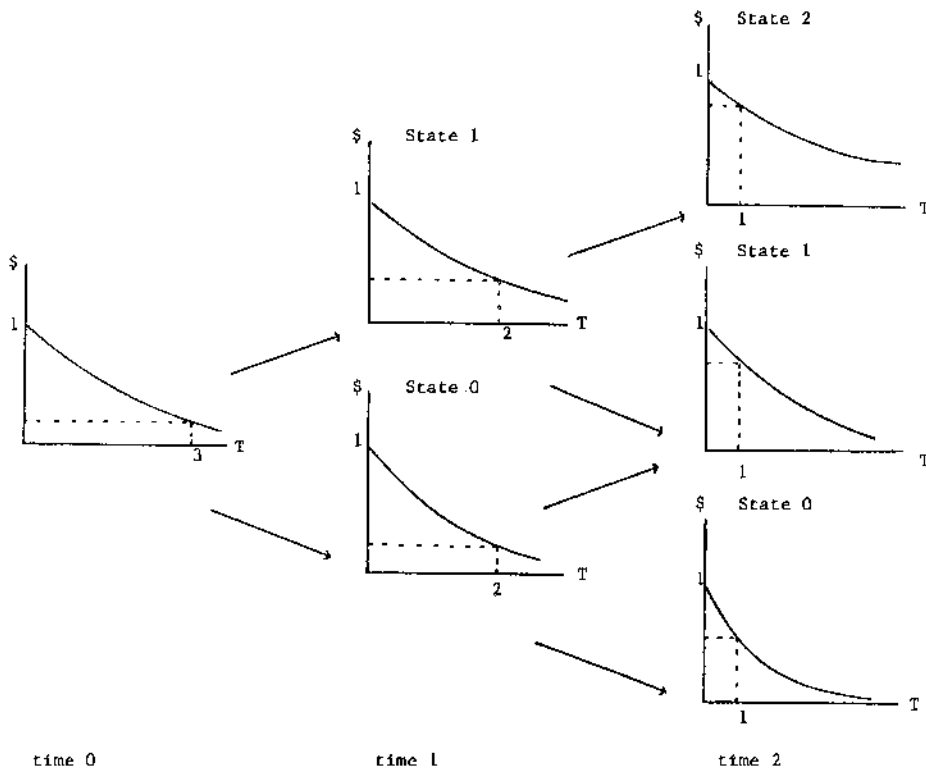


Figure 1. The Binomial Lattice of the Term Structure Movement

process has the following characteristics. The bond price uncertainty is small at the two extreme time points: for the time horizon in the immediate future and near bond maturity. The price uncertainty is large for time horizons away from these two endpoints. We achieve this by isolating the two effects. The first effect is the resolution of uncertainty of the term structure. As the time horizon lengthens, we are more uncertain about the term structure configuration. (There are more variations of the term structure when n is large.) The second effect is that the bond price uncertainty must decrease when the time horizon approaches maturity since the bond price cannot significantly deviate from unity when the maturity is short. Now consider a particular bond. As the time horizon increases, uncertainty of the term structure configuration increases, leading to larger bond price variance. However, at the same time, the bond has shorter maturity in the future, and the latter effect (the maturity effect) prevails. When the time horizon is sufficiently distant in the future, the latter effect may dominate the former, leading to a decrease in bond price uncertainty. We can now compare our model with that of Schaefer and Schwartz [20]. In their approach, they seek to model the stochastic process of a bond price. In their case, they must specify a process such that its variance is time dependent.³ Here, we separate the two effects and model them separately.

II. The Arbitrage-Free Rate Movements—the AR Model

In Section I, we introduced the binomial lattice of a term structure. This section introduces the necessary constraints on the term structure movement such that the movement is consistent with an arbitrage-free environment. We also introduce some simplifying restrictions so that we can develop a procedure to construct a “desirable” term structure movement.⁴

A. Perturbation Functions $h(T)$ and $h^*(T)$

At any n th period and i th state, we have a discount function $P_i^{(n)}(T)$. If everyone perceives no interest rate risk over the next period, then the term structure in the upstate must equal the downstate at time $n + 1$. Furthermore, the discount function must be the implied forward discount function $F_i^{(n)}(T)$ to avoid any arbitrage opportunities. That is,

$$F_i^{(n)}(T) = P_i^{(n+1)}(T) = P_{i+1}^{(n+1)}(T) = \frac{P_i^{(n)}(T+1)}{P_i^{(n)}(1)} \quad T = 0, 1, \dots \quad (6)$$

³ For an Itô process to specify a bond price stochastic movement, the instantaneous variance must necessarily be time dependent. This observation was made in Merton [14]. Schaefer and Schwartz [20] analyze a particular process where the instantaneous variance is proportional to the underlying bond duration.

⁴ In order to exposit the basic idea of an arbitrage-free rate movement clearly here, we confine ourselves to a special case. We only consider the case when the perturbation functions (in Section IIA) $h(T)$ and $h^*(T)$ and the implied binomial probability (in Section IIB) π are independent of state i and time n . In general, h^* , h , and π can be dependent on n and i , and a more general class of AR models can be determined. (See Ho and Lee [11].)

In a certainty world, if the next period discount function differs from $F_i^{(n)}(T)$, then investors can realize arbitrage profits. Therefore, in modeling term structure uncertainty, we are concerned with how the discount function is perturbed from the implied forward function in the following period. For this reason, we define two functions, called *perturbation functions*, $h(T)$ and $h^*(T)$ such that, in the upstate,

$$P_{i+1}^{(n+1)}(T) = \frac{P_i^{(n)}(T+1)}{P_i^{(n)}(1)} h(T) \quad (7)$$

and, in the downstate,

$$P_i^{(n+1)}(T) = \frac{P_i^{(n)}(T+1)}{P_i^{(n)}(1)} h^*(T). \quad (8)$$

The perturbation functions specify the deviations of the discount functions from the implied forward function. Thus, roughly, they specify the difference between the upstate and downstate prices over the next period. When $h(T)$ is significantly greater than unity for all values of T , then all the bond prices will rise substantially in the upstate. Analogously, when $h^*(T)$ is less than unity for all values of T , all the bond prices will fall in the downstate. Equations (1) and (2) impose the following conditions on the functions h and h^* . They must both be positive and, also,

$$h(0) = h^*(0) = 1. \quad (9)$$

Equation (9) follows directly from equations (1), (7), and (8).

The perturbation on the bond price depends on the maturity, and we therefore let h and h^* be functions of T . To construct a binomial lattice of the term structure movement, we need only to specify the set of perturbation functions $h(T)$, $h^*(T)$ and the initial discount function $P(T)$.

B. The Implied Binomial Probability π

Given a binomial lattice of a term structure movement, we also need to ensure that there is no arbitrage profit to be made in forming arbitrary portfolios of the discount bonds. Specifically, if we take any two discount bonds with different maturities and construct a portfolio of these two bonds such that the portfolio realizes a risk-free return over the next period, then the risk-free rate must be the return of a one-period discount bond. This arbitrage-free condition imposes a restriction on the perturbation functions at each vertex (n, i) . The method to calculate the risk-free hedge is similar to that of CRR, and the details of the arguments are given in Appendix A. The result shows that, when the bond price upward movement is significant, the bond price downward movement must also be sizable such that the weighted average of the movements is the same across all bonds. Specifically, we have

$$\pi h(T) + (1 - \pi)h^*(T) = 1 \quad \text{for } n, i > 0 \quad (10)$$

for some constant π independent of time T and the initial discount function

$P(T)$, but possibly dependent on state and time. π will be called the *implied binomial probability*.

The implied binomial probability can be understood in the Cox, Ross, and Rubinstein context of binomial option pricing. Indeed, equation (10) can be rewritten as

$$P_i^{(n)}(T) = [\pi P_{i+1}^{(n+1)}(T-1) + (1-\pi)P_i^{(n+1)}(T-1)]P_i^{(n)}(1). \quad (11)$$

Equation (11) says that the bond price equals the "expected" bond value at the end of the period discounted by the prevailing one-period bond rate if we interpret π to be a binomial probability. For this reason, the implied binomial probability π is the "risk-neutral" probability of CRR within their model's context. To interpret π , it is more useful to rewrite equation (11) as

$$\pi = (r-d)/(u-d),$$

where r is the one-period bond return and u and d are the bond returns for upstate and downstate, respectively.⁵ Therefore, π measures the extent of the downstate return as a percentage of the total spread between upstate and downstate returns. For large π , the model says that the price change for the next period is mainly a price decrease. Similarly, when π is small (approximately zero), the price change is dominated by a price rise. Equation (10) shows that, if one cannot arbitrage using the discount bond, then this ratio must be the same for all bonds.

C. The Path-Independent Condition

In constructing the binomial lattice, we assume a discount function evolving from one state to another depending only on the number of the upward movements and not on the sequence in which they occur. This restriction is equivalent to imposing a constraint on the perturbation functions (h and h^*) and the implied binomial probability (π) such that, at any time n and state i , an upward move followed by a downward move of a bond price equals a downward move followed by an upward move of its price.

To investigate the implication of this constraint, consider the discount function $P_i^{(n)}(T)$ at time n and state i . Using equations (7) and (8), a direct calculation shows that, from an upward move followed by a downward move, we get

$$P_{i+1}^{(n+2)}(T) = \frac{P_i^{(n)}(T+2)}{P_i^{(n)}(2)} \frac{h(T+1)h^*(T)}{h(1)}. \quad (12)$$

By a similar calculation, we can analyze the case of a downward movement followed by an upward movement and show that the resulting discount function is

$$P_{i-1}^{(n+2)}(T) = \frac{P_i^{(n)}(T+2)}{P_i^{(n)}(2)} \frac{h^*(T+1)h(T)}{h^*(1)}. \quad (13)$$

⁵ We use the notations (r , u , d) here in order to be consistent with those in Cox, Ross, and Rubinstein (CRR). In our case, $r = 1/P_i^{(n)}(1)$, $u = P_{i+1}^{(n+1)}(T-1)/P_i^{(n)}(T)$, and $d = P_{i-1}^{(n+1)}(T-1)/P_i^{(n)}(T)$. Then, the above expression, derived also in CRR, follows in a straightforward manner from equation (11).

The path-independent condition implies that

$$h(T+1)h^*(T)h^*(1) = h^*(T+1)h(T)h(1). \quad (14)$$

But, by equation (10), we can eliminate h^* , and we obtain

$$h(T+1)[1 - \pi h(T)][1 - \pi h(1)] = (1 - \pi)h(1)h(T)[1 - \pi h(T+1)]. \quad (15)$$

We simplify equation (15) and obtain, for $T \geq 1$,

$$\frac{1}{h(T+1)} = \frac{\delta}{h(T)} + \gamma, \quad (16)$$

where δ is some constant such that

$$h(1) = 1/(\pi + (1 - \pi)\delta) \quad (17)$$

and

$$\gamma = \frac{\pi(h(1) - 1)}{(1 - \pi)h(1)}. \quad (18)$$

Equation (16) is a first-order linear difference equation, and its general solution is given by

$$h(T) = \frac{1}{\pi + c\delta^T} \text{ for some constant } c.$$

But, by equation (9), we require that $h(0) = 1$, and, therefore, the initial condition determines the unique solution

$$h(T) = \frac{1}{\pi + (1 - \pi)\delta^T} \text{ for } T \geq 0. \quad (19)$$

Note that equation (17) becomes a special case of equation (19).

From equation (10), we get

$$h^*(T) = \frac{\delta^T}{\pi + (1 - \pi)\delta^T}. \quad (20)$$

For given constants π and δ , the AR model is well defined by equations (7), (8), (19), and (20).

III. The AR Model

This section analyzes the AR model. We compare it with other models of interest rate movements. Equations (19) and (20) show that the AR model is uniquely determined by two constants (π and δ). Section IIB has given the intuitive explanation of π . δ determines the spread between the two perturbation functions h and h^* . The larger the spread, the greater the interest rate variability. $h(\cdot)$ is a concave function that increases monotonically, asymptotic to $1/\pi$. $h^*(\cdot)$ is a convex function that decreases monotonically to zero. For this model, at any state-time, consider a bond with maturity T . The bond's upstate price relative to the implied forward price is $h(T)$; therefore, the longer the maturity, the larger the price change. For short-term bonds, the price change is negatively related to δ . For long-term bonds, the price change is quite constant across different

maturities and is $(1/\pi)$. δ is a parameter that affects the bond volatility, and it is inversely related to the term structure uncertainty. According to equation (17), we see that $0 \leq \delta \leq 1$. When $\delta = 1$, we reduce to the certainty case.

Vasicek [22], Dothan [9], Richard [19], and Cox, Ingersoll, and Ross [7] have proposed equilibrium models of the term structure. In these models, all the discount bonds are priced relative to the stochastic short rate in such a way that there are no arbitrage opportunities in trading the discount bonds. Since the AR model also requires all bonds to be priced relative to a bond, and hence to a specific interest rate, it would provide valuable insight to show how bonds are priced relative to the short rate and how the model may be identified with these single-state-variable models.

The short rate, within the context of our model, is the rate of a one-period discount bond. Now we wish to determine the stochastic process of the short rate. First, we need to express the discount function at any time n and state i in terms of the initial discount function. This can be achieved by applying equations (7) and (8) recursively backwards. This procedure gives the final expression

$$P_i^{(n)}(T) = \frac{P(T+n)}{P(n)} \times \frac{h^*(T+n-1)h^*(T+n-2) \dots h^*(T+i)h(T+i-1) \dots h(T)}{h^*(n-1)h^*(n-2) \dots h^*(i)h(i-1) \dots h(1)}. \quad (21)$$

Using equation (20), equation (21) can be simplified to

$$P_i^{(n)}(T) = \frac{P(T+n)h(T+n-1)h(T+n-2) \dots h(T)\delta^{T(n-1)}}{P(n)h(n-1)h(n-2) \dots h(1)}. \quad (22)$$

Equation (22) gives the explicit expression of the discount function in each state-time. In the special case of a one-period bond ($T=1$), the bond price is

$$P_i^{(n)}(1) = \frac{P(n+1)\delta^{n-i}}{P(n)(\pi + (1-\pi)\delta^n)}. \quad (23)$$

Also, the one-period rate $r_i^{(n)}(1)$, according to equation (5), is

$$\begin{aligned} r_i^{(n)}(1) &= -\ln P_i^{(n)}(1) \\ &= \ln \left[\frac{P(n)}{P(n+1)} \right] + \ln(\pi \delta^{-n} + (1-\pi)) + i \ln \delta. \end{aligned} \quad (24)$$

Thus far, we have not assigned any binomial probability to the binomial lattice. If we further assume that the probability is q , it follows directly that, for each n , $r_i^{(n)}(1)$ is a binomial distribution in i , with mean μ given by

$$\mu = \ln \left[\frac{P(n)}{P(n+1)} \right] + \ln(\pi \delta^{-n} + (1-\pi)) + nq \ln \delta. \quad (25)$$

By simplifying, we get

$$\mu = \ln \left[\frac{P(n)}{P(n+1)} \right] + \ln(\pi \delta^{-(1-q)n} + (1-\pi)\delta^{qn}). \quad (26)$$

Also, the variance is given by

$$\text{var} = nq(1 - q)(\ln \delta)^2. \quad (27)$$

Note that the first term of equation (26) is the implied forward one-period bond rate. Equation (26) says that the expected rate is the implied forward rate plus a bias. The bias would naturally disappear when there is no uncertainty ($\delta = 1$). The variance of the one-period rate depends only on δ . As expected, the variance is negatively related to δ .

This reformulation enables us to compare our model with the single-factor term structure models. Here, our short-rate stochastic process depends on the information of the initial term structure. In this sense, our term structure movements incorporate the full information of the initial term structure. By way of comparison, single-factor models specify the short-rate movements exogenously without using the full information of the initial term structure.

The two approaches differ because they differ in their purposes. The one-factor models seek to endogenize the equilibrium term structure. To do so, they try to determine the short-rate process that can generate a meaningful equilibrium term structure. We take the initial term structure as given and determine the short-rate movement to price the contingent claims. When the discount bonds, viewed as contingent claims, are priced by this term structure movement, they fit the initial discount function. (This assertion will be proved in the following section.) Therefore, our term structure movement is not used to determine the equilibrium term structure. Rather, it is assured to be consistent with the term structure and is used to price other contingent claims.

A. Properties of the AR Model

To gain insight into the AR model, it is useful to view discount function movements in terms of shifts in the yield curve. To this end, we analyze equation (5). At the initial time,

$$r(T) = \frac{-\ln P(T)}{T}.$$

In the next period, in the upstate, using equation (7) we have

$$r_1^{(1)}(T) = -\frac{1}{T} \ln \frac{P(T+1)}{P(1)} - \frac{1}{T} \ln(h(T)). \quad (28)$$

Also, in the downstate, using equation (8) we have

$$r_0^{(1)}(T) = -\frac{1}{T} \ln \frac{P(T+1)}{P(1)} - \frac{1}{T} \ln[h^*(T)]. \quad (29)$$

Using the behavior of the functions $h(T)$ and $h^*(T)$, we see that the AR model imposes certain restrictions on the yield curve movements. When short rates attain a higher (lower) level, the long rates also attain a higher (lower) level. The movements are made relative to the implied forward yield curve. Since the implied forward yield curve can take on any shape (we do not impose any condition on the initial term structure), the yield curve in subsequent periods, in principle, can take on any shape.

B. Local Expectations Hypothesis and the Term Premium

Following the definition of Cox, Ingersoll, and Ross [6], the T -period term premium is the expected return (holding the bond for one period) of a T -period bond in excess of the one-period bond return. One may expect that the longer-term bonds have higher expected returns, and hence positive term premia. When all bonds have the same expected return (i.e., no term premia exist), we say the *local expectations hypothesis* holds. The following proposition determines the term premium and specifies the necessary conditions for the local expectations hypothesis.

PROPOSITION 1: *The T -period term premium is given by*

$$\tau(T) = \frac{1}{P(1)} \left\{ \left[\frac{(1-q)\delta^T + q}{(1-\pi)\delta^T + \pi} \right] - 1 \right\}, \quad (30)$$

and the local expectations hypothesis holds if and only if $q = \pi$, i.e., if and only if the implied binomial probability (π) is the binomial probability (q).

Proof: Note that the rate of return of the $(T+1)$ -period bond over one period is

$$\left[q \frac{P(T+1)}{P(1)} h(T) + (1-q) \frac{P(T+1)}{P(1)} h^*(T) \right] \div P(T+1).$$

Also, the one-period bond rate of return is $1/P(1)$. By taking the difference between these two rates of return, we get the desired result of equation (30). Now, it follows that the premium is zero if and only if $q = \pi$. Q.E.D.

The T -period term premium is the product of two terms. The first term is the one-period risk-free rate of return. The second term is a function of q , π , and δ . If the binomial probability q is greater than the implied binomial probability π , we have a positive term premium. Long-term bonds would have higher expected returns. This is intuitively clear. π can be viewed as the risk-neutral probability. If the actual probability q for an upstate is greater than π , then the expected return must be higher than the risk-neutral return—hence, a positive term premium. Otherwise, if q is less than π , then the term premium is negative. More importantly, when π is the binomial probability, the local expectations hypothesis must hold.

IV. Pricing of Interest Rate Contingent Claims

This section presents a procedure to price interest rate contingent claims using an AR model. In order to present the pricing procedure clearly, we confine our discussion to pricing only a category of contingent claims. These securities are characterized as follows.

Let C be an interest rate contingent claim. We require that its price $C(n, i)$ be uniquely defined at each vertex (n, i) of the binomial lattice. C has a finite life, and it expires (or matures) at time T , with payoffs given by $\{f(i)\}$, $0 \leq i \leq T$; we call $\{f(i)\}$ the terminal condition and hence

$$C(T, i) = f(i), \quad 0 \leq i \leq T. \quad (31)$$

The contingent claim is also satisfied by its upper bound $U(n, i)$ and lower bound $L(n, i)$ conditions such that

$$L(n, i) \leq C(n, i) \leq U(n, i). \quad (32)$$

Equation (32) contains the boundary conditions. Also, the contingent claim pays $X(n, i)$ to its holder at time n and state i , $1 \leq n < T$.

There are many examples of contingent claims belonging to this category. Interest rate futures (taking marking-to-market into account), both European and American bond options, callable and sinking fund bonds, and interest rate futures options are some of the examples.⁶ These contingent claims differ by the specification of their terminal and boundary conditions and the payoff during their lives.

A. Pricing of the Contingent Claims

The following Lemma is central to our pricing procedure for the above model.

LEMMA (Risk-Neutral Pricing Formula): Consider any interest rate contingent claim $C(n, i)$ that can be bought and sold in a frictionless market environment described by assumptions (A1)–(A5). If no arbitrage profit is to be realized in holding any portfolio of the contingent claim and the discount bonds, the following equation must hold:

$$C(n, i) = [\pi\{C(n+1, i+1) + X(n+1, i+1)\} + (1-\pi)\{C(n+1, i) + X(n+1, i)\}]P_i^{(n)}(1), \quad (33)$$

where $P_i^{(n)}(1)$ is the one-period discount bond price at state-time (n, i) .

The proof of the Lemma is similar to that of Proposition 1, and, for this reason, we give the details in Appendix B.

The Lemma enables us to price the initial contingent claim by the backward substitution procedure—a method often used in recent finance literature. The terminal condition of equation (31) specifies the asset value in all states at time T . Then, equation (33) enables us to determine the arbitrage-free price of the asset at one period before expiration. Let that price be $C^*(T-1, i)$. But the actual market price must satisfy the boundary conditions of equation (32). That is, the market price must be

$$C(T-1, i) = \max[L(T-1, i), \min(C^*(T-1, i), U(T-1, i))]. \quad (34)$$

We now apply this procedure repeatedly, rolling back in time. That is, given the contingent-claim prices at all states at time n , we can always calculate the arbitrage-free contingent-claim prices at time $n-1$ by equation (33). Then, by applying the boundary conditions of equation (32), we can derive the market prices in all states at time $n-1$. After T steps, we reach the asset value at $n=0$, and that is the initial price.

⁶ A floating rate note is an example that does not belong to this category. This is because, at a particular time-state (n, i) , the coupon rate on the note depends on the time path of the term structure movement. Therefore, the floating value is ambiguously defined at each vertex. However, with appropriate adjustments to the pricing procedure, the AR model can still be used to price these instruments.

This recursive procedure has been applied elsewhere—for example, Cox, Ross, and Rubinstein—but there are two interesting features in our case. First, the one-period discount rate, a constant in CRR, is state and time dependent in our model. It is specified by $P_i^{(n)}(1)$ and is endogenized by the AR model. Since we have noted that these one-period rates depend on the initial discount function, we have explicitly shown how the initial term structure affects the contingent-claim pricing.

Second, in the AR model, the terminal conditions (equation (31)) and boundary conditions (equation (32)) can in turn be specified by interest rate contingent claims. This is because, at each state and time, the prevailing discount function and its subsequent movement are both specified. Therefore, other interest rate contingent claims can in turn be priced to determine the conditions. For example, for interest rate futures options, the futures can first be priced by the AR model, and then the options can be priced by the futures.

When the initial discount function is given by $P(\cdot)$, the price of a discount bond with maturity T is, by definition, $P(T)$. But the discount bond can also be priced by the recursive method described in this section. That is, the discount bond can be viewed as a contingent claim with maturity T , with terminal conditions $f(i) = 1$, and without any lower and upper bounds and interim cashflow.

Hence, the discount bond can be priced by the recursive method described in this section. Appendix C shows that the calculated price is assured to be the observed price $P(T)$. This result is intuitively clear. We use the observed price $P(T)$ to derive an arbitrage-free term structure movement. When we use the AR model to price the discount bond, we should find the price to be $P(T)$; otherwise, we would realize arbitrage opportunities.

This result suggests an important implication. Suppose we know the short-rate (the one-period rate) evolution over time. Specifically, at each time n and state i (at each vertex (n, i) of the binomial lattice), we know the one-period bond price $P_i^{(n)}(1)$ and the implied binomial probability π . In this case, we do not need to know the complete term structure at each vertex, and there is no arbitrage-free condition to be checked. Knowing π and $P_i^{(n)}(1)$ only, according to this section, we can always price interest rate contingent claims. This model of asset pricing depends crucially on the stochastic movement of the short rate, and, for this reason, we call such a model a *one-factor model*.

The one-factor model is analogous to the continuous-time model of CIR, with one subtle difference. The CIR model specifies the actual binomial probability (q) of the short-rate process and the term premium (τ). Here we use the implied binomial probabilities, π , and do not isolate the actual binomial probabilities from the term premium. (Proposition 1 has shown how the implied binomial probability π is related to the actual binomial probability q and the term premium τ in a special case.)

In sum, the AR model is a one-factor model in the sense that all the contingent claims are priced by the short-rate movement. But, in the AR model, the short-rate movement is subject to some constraint. The short rate must evolve in such a way that, when the discount bonds (or portfolio of discount bonds) are priced

as contingent claims (by the backward substitution methodology described in this section), the solution price is guaranteed to be that given by the initial discount function.⁷

What is important about the AR model is its practical implications. Discount bond prices are, in principle, observable, and these prices are fully utilized by the AR model to price the bond contingent claims; in this sense, interest rate contingent claims are priced *relative* to the term structure. In contrast, the one-factor model of, for example, CIR ignores these bond prices in hypothesizing the short-rate stochastic process without ensuring that the model prices of the underlying bonds are those observed in the term structure.

B. Estimating the AR Model

To price any interest rate contingent claim by our approach, we must first specify the parameters of the AR model (π and δ). To estimate π and δ , we take three main steps. First, we need to estimate the discount function at the time of pricing the contingent claims. To this end, we can apply any of the known procedures, including the cubic spline-fitting procedure used by a number of authors—for example, McCulloch [13] or, more recently, Litzenberger and Rolfo [12].

Second, using Section IIIA, we can develop theoretical pricing models of some contingent claims—for example, interest rate options. Since we have already estimated the initial discount function, the only unspecified input to the pricing model are the parameters π and δ . But these two parameters are related to the stochastic movement of the term structure and not to a specific instrument. Therefore, they should be the same for all contingent claims, according to our model.

The third and final step is to use a nonlinear estimation procedure to determine π and δ such that the theoretical prices of a sample of contingent claims can best fit their observed prices. The estimates of π and δ are then used in the AR model to price other contingent claims.⁸

Our approach is similar to that used to estimate the “implied” volatility of a stock option (see, for example, Whaley [23]). For a stock option, the stock return volatility is not directly observable, and, therefore, it can be estimated by the observed option price via a pricing model. This estimated volatility (the implied volatility) is then used to price other options on the same stock. In the case of the AR model, the parameters of the model are not directly observable from the term structure, but their values are reflected in the interest rate contingent claims. For this reason, we can use the observed prices of the contingent claims to estimate the AR parameters via their pricing models derived by the general procedure of Section IVA.

⁷ Ho and Lee [10] have used this procedure to price callable sinking-fund bonds and have shown that the methodology is quite tractable.

⁸ It would be interesting to investigate how the AR model can be extended to a two-factor model where the pricing depends on both the short rate and long rate. But the importance of such extensions will depend partly on the empirical results, and these issues will have to be left for future research.

V. Conclusions

In this paper, we propose an approach to pricing interest rate contingent claims. This approach prices these securities relative to the term structure of interest rates. The crux of the argument hinges on modeling the term structure movement in such a way that the stochastic movement is arbitrage free. In so doing, security prices are derived consistent with a partial equilibrium framework.

We present an arbitrage-free interest rate movement model (AR model). We then show that, if we use the AR model to price a discount bond as a contingent claim, the bond price must equal the price determined by the initial discount function. This consistency enables us to fully utilize the information observed in the term structure to price the contingent claims. Also, in this paper we relate the AR model to other models proposed in previous literature. We show how the initial discount function is used to specify the interest rate stochastic process, something that alternative models do not consider.

This paper is important to financial research because it provides a direct and tractable procedure for pricing a broad range of securities relative to the discount function. These pricing models should have significant implications on fixed-income analysis and management. On the theoretical side, this paper may be viewed as an initial step to acquire insight into pricing the interest rate contingent claims relative to the term structure. More general AR models should be developed to enable us to better understand the "feasible" movements of interest rates. On the empirical side, our paper should enable researchers to test a range of pricing models and the empirical validity of alternative AR models.

Appendix A

Proof of Equation (10)

Proof: At any time n and state i , we can construct a portfolio of one discount bond with maturity T and ξ discount bonds with maturity t . To simplify our notations, we will drop all the indices (n, i) . Then, the portfolio value is $V = P(T) + \xi P(t)$.

At the end of the period, when an upstate prevails, the portfolio value is, by equation (7),

$$V(\text{upstate}) = [P(T)h(T-1) + \xi P(t)h(t-1)]/P(1). \quad (\text{A1})$$

Similarly, when a downstate prevails, we have

$$V(\text{downstate}) = [P(T)h^*(T-1) + \xi P(t)h^*(t-1)]/P(1). \quad (\text{A2})$$

Choose ξ such that $V(\text{upstate}) = V(\text{downstate})$; then, using equation (A1) and (A2), we can show that

$$\xi = P(T)[h(T-1) - h^*(T-1)]/P(t)[h^*(t-1) - h(t-1)]. \quad (\text{A3})$$

To avoid arbitrage opportunities, this portfolio should yield a one-period discount bond return, i.e., $1/P(1)$. That is,

$$P(T)h^*(T-1) + \xi P(t)h^*(t-1) = P(T) + \xi P(t). \quad (\text{A4})$$

Substituting equation (A3) in equation (A4), we get

$$\begin{aligned} & [1 - h^*(T - 1)]/[h(T - 1) - h^*(T - 1)] \\ &= [1 - h^*(t - 1)]/[h(t - 1) - h^*(t - 1)] \quad \text{for all } T \text{ and } t > 0. \end{aligned} \quad (\text{A5})$$

Equation (A5) can hold for all T and t only if there is a constant π , independent of T and t , such that

$$\frac{1 - h^*(T)}{h(T) - h^*(T)} = \pi. \quad (\text{A6})$$

Equation (A6) can be re-expressed as

$$\pi h(T) + (1 - \pi)h^*(T) = 1 \quad \text{for } T = 0, 1, 2, \dots. \quad (\text{A7})$$

This completes the proof. Note that we require that $h^*(0) = h(0) = 1$, but these conditions are consistent with equation (A7). Q.E.D.

Appendix B

Proof of the Lemma

Proof: For expository reasons, we will ignore the payoffs $X(n, i)$ in this derivation. Incorporating these payoffs is a trivial extension. In state i and time n , consider a discount bond with maturity T (some arbitrary T). We form a risk-free portfolio of this bond and asset C by buying one bond and ξC assets. When an upstate prevails, the portfolio value is

$$V(\text{upstate}) = P_i^{(n)}(T)h(T - 1)/P_i^{(n)} + \xi C(n + 1, i + 1). \quad (\text{B1})$$

Similarly, when the downstate prevails, the portfolio value is

$$V(\text{downstate}) = P_i^{(n)}(T)h^*(T - 1)/P_i^{(n)} + \xi C(n + 1, i). \quad (\text{B2})$$

Since the portfolio is risk free, we require that

$$V(\text{upstate}) = V(\text{downstate}). \quad (\text{B3})$$

In combining equations (B1), (B2), and (B3) and rearranging, we get

$$\begin{aligned} \xi &= P_i^{(n)}(T)[h^*(T - 1) - h(T - 1)] \\ &\quad \times P_i^{(n)}(1)[C(n + 1, i + 1) - C(n + 1, i)]. \end{aligned} \quad (\text{B4})$$

Now the initial portfolio value at time n is

$$V = P_i^{(n)}(T) + \xi C(n, i). \quad (\text{B5})$$

By the arbitrage-free argument, we have

$$V = V(\text{downstate})P_i^{(n)}(1). \quad (\text{B6})$$

Using equations (B4), (B5), and (B6), and also equation (B2), we get the desired results. Q.E.D.

Appendix C

When the term structure movement is generated by the AR model, the recursive procedure in pricing a discount bond gives the price equalling that determined by the initial term structure. Therefore, while the implied binomial probability, π , and the perturbation functions, $h(\cdot)$ and $h^*(\cdot)$, are all used in the recursive procedure, they do not affect the final solution price of a discount bond.

Proof: We prove the theorem by induction on m , the number of periods till maturity of a discount bond. The result clearly holds for $m = 1$. When there is only one period to maturity at any time-state (n, i) , we put $C(n + 1, i + 1) = C(n + 1, i) = 1$. By equation (33), we have

$$C(n, i) = \{\pi + (1 - \pi)\}P_i^{(n)}(1) = P_i^{(n)}(1).$$

Hence, the value of a one-period bond at time (n, i) is $P_i^{(n)}(1)$, as required.

Now suppose the theorem holds for all times to maturity (m) less than some number N . Now consider the case when a discount bond has $N + 1$ periods to maturity at time-state (n, i) . In the next period upstate, the bond would have maturity N . By the induction hypothesis, the recursive procedure would yield the contingent-claim price to be $P_{i+1}^{(n+1)}(N)$. Similarly, for the downstate in the next period, the contingent-claim price must be $P_i^{(n+1)}(N)$.

By equation (33), again, we have

$$C(n, i) = (\pi P_{i+1}^{(n+1)}(N) + (1 - \pi)P_i^{(n+1)}(N))P_i^{(n)}(1). \quad (C1)$$

Using equations (7) and (8) to simplify equation (C1), we can reduce the expression to

$$C(n, i) = P_i^{(n)}(N + 1). \quad (C2)$$

Equation (C2) says that the contingent claim at time-state (n, i) takes the value of the discount bond with maturity $N + 1$. It, thus, completes the induction proof. Note that this result holds even if the AR model is not time and state independent, with h^* , h , and π being functions of i and n . Q.E.D.

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