Duration & Dimension

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Abstract

In fixed income analysis, duration plays a central role as a proxy for interest rate risk exposure. Although this role relies on the interpretation of duration as (minus) the yield elasticity of the bond price, duration is measured as a bond's present value weighted average time to maturity and expressed in terms of years. Hence duration is regarded as an elasticity with a time dimension. In this paper we resolve this apparent duration paradox and show that duration is a pure number.

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Introduction

The Macaulay duration of a bond is defined as its present value weighted average time to maturity. Macaulay [1938] specified this duration as a descriptive measure for bonds. Since the weighted average time to maturity takes into account the timing of all cash flows and not only of the principal, duration is more meaningful in this respect than a bond's maturity. In interest rate immunization theory, duration characteristics play a central role in relating bond price variability to changes in the yield to maturity (or "the" interest rate). The link between bond price variability and duration was originally discovered by Hicks [1939] and Samuelson [1945], and rediscovered by Fisher [1966] and Fisher & Weil [1971]. Since then, duration has positioned itself firmly in fixed income analysis as a proxy for interest rate risk exposure.

Although Hicks [1939, p.186] and Samuelson [1945, p.19] were interested in duration as a yield elasticity, they equate duration with "average (time) period" in the spirit of Macaulay. Hicks [1939, p.187, fn.I] remarks that "[t]he reader may also find it rather surprising that an elasticity, usually supposed to be a pure number, independent of units, turns out to be equal to a length of time. This is a consequence of compound interest." Although the importance of duration in modern financial theory relies on its interpretation as (minus) an elasticity, duration is still and without exception measured in terms of time periods. For examples we refer to authoritative texts as Fabozzi [2000, Ch.4], Fabozzi [1997, Ch.5], Haugen [2001, pp.395ff] and Sharpe, Alexander & Bailey [1999, pp.424ff], e.g.

Startled by Hick's surprise, the purpose of this paper is to show that duration is indeed independent of time – as we would expect from its definition as an elasticity. The same argument applies to higher order terms in the Taylor series expansion of the bond price, the (hyper-) convexities. The reason for the general confusion on this point is not compound interest but an inaccurate expression of the present value formula. We first review the standard argument to present value calculations and bond duration whereafter we resolve the apparent duration paradox.

The standard argument

Assuming an annual coupon, the value B of an option-free bond on a coupon date (immediately after receiving the coupon) is:

(1) $$B = PV(CF_t) = \sum_{t=1}^{T} \frac{CF_t}{(1+Y)^{t}}$$

where: $PV(\cdot)$ = the present value operator

CF_t = the cash flow to be received at time t

T = the remaining time to maturity in years

t = time index in years

Y = the yield to maturity of the bond per annum.

When the term structure of interest rates is flat for all maturities, Y is "the" interest rate.

The bond's cash flows are:

$$
(2) \qquad \begin{aligned} CF_t &= c \cdot F \qquad \quad for \quad t = 1,\ldots,T-1 \\ CF_T &= (1+c)F \end{aligned}
$$

where: c = the coupon rate per annum

F = the bond's face value.

Duration is defined as minus the (point) elasticity of the bond price with respect to one plus the yield to maturity:

$$
(3) \qquad D = -\frac{dB}{d(1+Y)} \cdot \frac{(1+Y)}{B} = \sum_{t=1}^{T} t \cdot \frac{PV(CF_t)}{B}
$$

Hence, the duration is the present value weighted average time to maturity in years.

When there are n cash flow periods per year, the formulas can easily be adapted by changing the unit of time from one period per year to n periods per year. This can be achieved by applying the following substitutions to the formulas (1), (2) and (3):

$$
(4) \qquad \begin{aligned} &T \to nT \\ &c \to c/n \\ &Y \to Y/n \end{aligned}
$$

As a result, eq.(3) now gives the duration D_n measured in terms of n periods per year. Dividing this duration by n yields the duration in years:

(5) $$D = \frac{D_n}{n}$$

Modified duration is then computed as:

(6) $$D_{\text{mod}} = \frac{D}{(1 + Y / n)}$$

(See for example Fabozzi [2000, Ch.4] or Fabozzi [1997, Ch.5].)

The paradox resolved

The valuation formula eq.(1) looks all too familiar and although we all understand the implied valuation recipe, a more scrutinous inspection reveals that it doesn't make any sense at all. The careless definition of the present value in eq.(1) is the source of the interpretation error of duration. In order to show that duration is indeed dimensionless – as we expect from its definition as an elasticity – we must start from the proper general expression for the value of an option-free bond on a coupon date:

(7) $$B = PV(CF_i) = \sum_{i=1}^{N} \frac{CF_i}{(1 + Y/m)^{\,i \cdot \Delta t \cdot m}}$$

where: i = the index number for the cash flows

N = the number of remaining cash flows

m = the compounding frequency within one year

Δt = the time interval (in years) between the cash flow dates.

Hence, the remaining time to maturity in years is $N \cdot \Delta t$, the compounding interval for the yield is m^{-1} (in years) and the number of cash flows per year (the cash flow frequency) is given by $(\Delta t)^{-1}$. In this representation the cash flows are:

$$\begin{aligned} CF_i &= \Delta t \cdot c \cdot F \qquad for \quad i = 1,\ldots,N-1 \\ CF_N &= (1 + \Delta t \cdot c)F \end{aligned} \tag{8}$$

where c is again the coupon rate per annum. For notational simplicity but without loss of generality we have assumed that the cash flow dates are equally spaced; the necessary adjustment for accomodating unevenly spaced cash flows is obvious.

Eq.(7) may look unnecessarily complex, but this expression clearly reveals the dimensions of the relevant parameters. There are two crucial differences with eq.(1). First note that the summation index i now is indeed a pure number and has no longer the dimension time attached. In addition, the compounding frequency m appears explicitly in the expression of the discount factor thus allowing the familiar transformation of discrete compounding to continuous compounding:

(9) $$\lim_{m\to\infty} \left(1+Y/m\right)^{-i\cdot\Delta t\cdot m} = e^{-Y\cdot i\cdot\Delta t}$$

In eq.(1) it is silently and incorrectly assumed that $m=1$. Defining the dimension operator $\dim(\cdot)$, we have:

(10) $$\begin{aligned} \dim(Y) &= year^{-1} \\ \dim(m) &= year^{-1} \\ \dim(\Delta t) &= year \\ \dim(c) &= year^{-1} \end{aligned}$$

Hence, m has the dimension time, so Y/m and $\Delta t\cdot m$ are just pure numbers without any time dimension attached.

Duration is minus the point elasticity of the bond price with respect to one plus the yield to maturity measured over the compounding period m^{-1}. Starting from eq.(7) this yields:

(11) $$D = -\frac{dB}{d\left(1+Y/m\right)}\cdot\frac{\left(1+Y/m\right)}{B} = \sum_{i=1}^{N} i\cdot\Delta t\cdot m\cdot\frac{PV\left(CF_i\right)}{B}$$

and this quantity is indeed dimensionless. The same applies to the Fisher & Weil [1971] duration where the actual spot rates are used as discount rates. Since this duration is measured with respect to the compounding period m^{-1} for the discount rate, we can easily transform it to the duration measured with respect to some compounding period k^{-1} by multiplying it with the dimensionless quantity k/m.

With $m = 1/year$ and $\Delta t = 1\,year$, the valuation formula eq.(7) reduces to eq.(1), provided that it is recognized that t is not a time index but a number index. After all, raising one plus the yield to the power of t years is nonsense. This in turn implies that even the Macaulay-type of duration in eq.(3) when derived from eq.(1) represents a present value weighted average of cash flow numbers and hence is <u>not</u> measured in terms of years. By the same argument it directly follows that convexity is not measured in $(\text{years})^2$ but also represents a pure number.

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