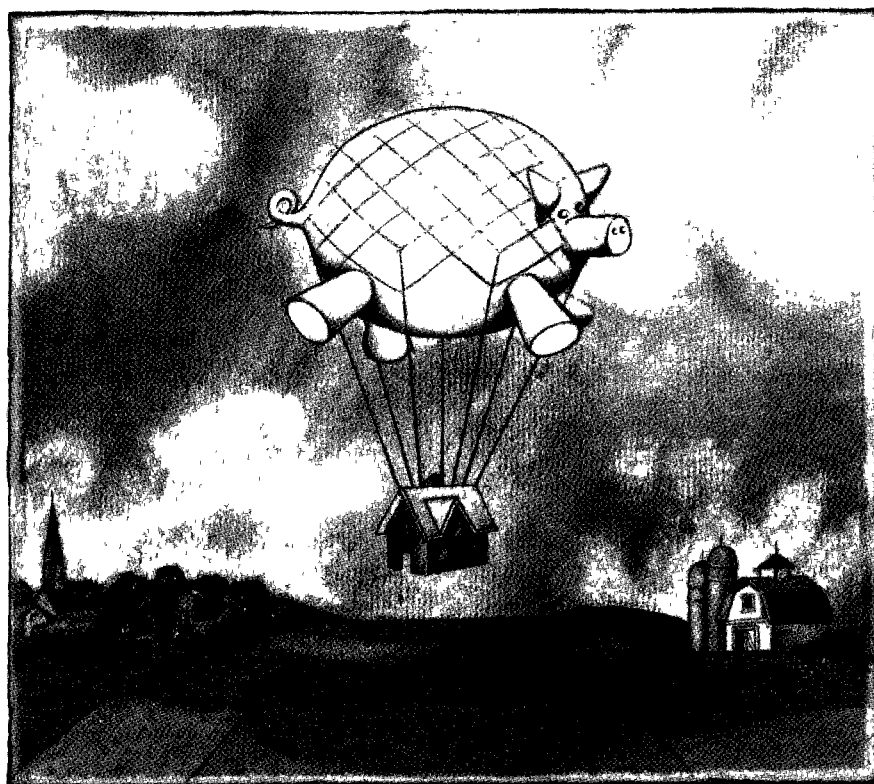


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Commodities in Asset Allocation: A Real-Asset Alternative to Real Estate?

Ernest M. Ankrum and Chris R. Hensel

How to Get Rich Quick Using GAAP

Robert Ferguson and Neal B. Hitzig

A Model for Valuing Bonds and Embedded Options

Andrew J. Kalotay, George O. Williams and Frank J. Fabozzi

Term Structure Forecasts Economic Growth

Campbell R. Harvey

MAY/JUNE 1993

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A Model for Valuing Bonds and Embedded Options

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On their article: "The evolution of bond valuation has progressed from discounting all cash flows at a single rate to discounting each cash flow at its own zero-coupon (spot) rate or, equivalently, by a series of one-period forward rates. When a bond has an embedded option such as a call or put provision, however, consideration must be given to the volatility of forward rates. We present a bond valuation model that allows for the discounting of each cash flow at appropriate volatility-dependent, one-period forward rates. This latest technique allows for valuation of options in a natural, consistent manner."

A Model for Valuing Bonds and Embedded Options

**Andrew J. Kalotay,
George O. Williams and
Frank J. Fabozzi**

One can value a bond by discounting each of its cash flows at its own zero-coupon ("spot") rate. This procedure is equivalent to discounting the cash flows at a series of one-period forward rates. When a bond has one or more embedded options, however, its cash flow is uncertain. If a callable bond is called by the issuer, for example, its cash flow will be truncated.

To value such a bond, one must consider the volatility of interest rates, as their volatility will affect the possibility of the call option being exercised. One can do so by constructing a binomial interest rate tree that models the random evolution of future interest rates. The volatility-dependent one-period forward rates produced by this tree can be used to discount the cash flows of any bond in order to arrive at bond value.

Given the values of bonds with and without an embedded option, one can obtain the value of the embedded option itself. The procedure can be used to value multiple or interrelated embedded options, as

well as stand-alone risk-control instruments such as swaps, swaptions, caps and floors.

In the good old days, bond valuation was relatively simple. Not only did interest rates exhibit little day-to-day volatility, but in the long run they inevitably drifted up, rather than down. Thus the ubiquitous call option on long-term corporate bonds hardly ever required the attention of the financial manager. Those days are gone. Today, investors face volatile interest rates, a historically steep yield curve, and complex bond structures with one or more **embedded options**. The framework used to value bonds in a relatively stable interest rate environment is inappropriate for valuing bonds today. This article sets forth a general model that can be used to value any bond in any interest rate environment.

A Brief History of Bond Valuation

The value of any bond is the present value of its expected cash flows. This sounds simple. Determine the cash flows, and then discount those cash flows at an appropriate rate. In practice, it's not so simple, for two reasons. First, holding aside the possibility of default, it is not easy to determine the cash flows for bonds with embedded options. Because the exercise of options embedded in a bond depends on the future course of interest rates, the cash flow is *a priori* uncertain. The issuer of a callable bond can alter the cash flows to the investor by calling the bond, while the investor in a puttable bond can alter the cash flows by putting the

bond. The future course of interest rates determines when and if the party granted the option is likely to alter the cash flows.

A second complication is determining the rate at which to discount the expected cash flows. The usual starting point is the yield available on Treasury securities. Appropriate spreads must be added to those Treasury yields to reflect additional risks to which the investor is exposed. Determining the appropriate spread is not simple, and is beyond the scope of this article.

The *ad hoc* process for valuing an **option-free bond** (i.e., a bond with no options) once was to discount *all* cash flows at a rate equal to the yield offered on a new, full-coupon bond of the same maturity. Suppose, for example, that one needs to value a 10-year, option-free bond. If the yield to maturity of an **on-the-run** 10-year bond of comparable credit quality is 8%, then the value of the bond under consideration can be taken to be the present value of its cash flows, all discounted at 8%.

According to this approach, the rate used to discount the cash flows of a 10-year, current-coupon bond would be the same rate as that used to discount the cash flow of a 10-year, zero-coupon bond. Conversely, discounting the cash flows of bonds with different maturities would require different discount rates. This approach makes little sense because it does not consider the cash flow characteristics of the bonds. Consider, for example, a portfolio of bonds of similar quality but different maturities. Imagine two equal cash flows occurring, say, five years hence, one coming from a 30-year bond and the

other coming from a 10-year bond. Why should these two cash flows have different discount rates, hence different present values?

Given the drawback of the *ad hoc* approach to bond valuation, greater recognition has been given to the fact that any bond should be thought of as a package of cash flows, with each cash flow viewed as a zero-coupon instrument maturing on the date it will be received. Thus, rather than using a single discount rate, one should use multiple discount rates, discounting each cash flow at its own rate.

One difficulty with implementing this approach is that there may not exist zero-coupon securities from which to derive every discount rate of interest. Even in the absence of zero-coupon securities, however, arbitrage arguments can be used to generate the theoretical zero-coupon rates an issuer would have to pay were it to issue zeros of every maturity. Using these theoretical zero-coupon rates, more popularly referred to as theoretical **spot rates**, the theoretical value of a bond can be determined. When dealer firms began stripping full-coupon Treasury securities in August 1982, the actual prices of Treasury securities began to move toward their theoretical values.¹

Another challenge remains, however—determining the theoretical value of a bond with an embedded option. In the early 1980s, practitioners came to recognize that an option-bearing bond should be viewed as a package of cash flows (i.e., a package of zero-coupon instruments) plus a package of options on those cash flows. For example, a callable bond can be viewed as a package of cash flows plus a package of call options on those cash flows. As such, the position of an investor in a callable bond can be viewed as.

Long a Callable Bond

= Long an Option-Free Bond

+ Short a Call Option on the Bond.

In terms of the value of a callable bond, this means:

Value of Callable Bond

= Value of an Option-Free Bond

– Value of a Call Option

on the Bond.

But this also means that:

Value of an Option-Free Bond

= Value of Callable Bond

+ Value of a Call Option

on the Bond.

An early procedure to determine the fairness of a callable bond's market price was to isolate the implied value of its underlying option-free bond by adding an estimate of the embedded call option's value to the bond's market price.² The former value could be estimated by applying option-pricing theory to interest-rate-sensitive options.³

This insight led to the first generation of valuation models that sought to value a callable bond by estimating the value of the call option. However, estimation of the call option embedded in callable bonds is not that simple. For example, suppose a 20-year bond is not callable for five years, after which time it becomes callable at any time on 30 days' notice. Suppose also that the schedule for the call price begins at 105 and declines to par by the 15th year. This is an extremely complicated call option. It is a deferred American call option for the first five years. Also, its striking, or exercise, price (the call price in the case of a callable bond) is not constant but declines over time. Or consider a corporate sinking-fund bond whose indenture

Glossary

► **Embedded Options:**

Options that are part of the structure of a bond, as opposed to bare options, which trade separately from an underlying security.

► **Forward Rate:**

The interest rate that will prevail for a specified length of time, starting at some future date.

► **On-the-Run Yield Curve:**

The relationship between the yield-to-maturity and maturity for bonds of similar quality trading at par.

► **Option-Free Bond:**

A bond that does not have any embedded options.

► **Spot Rate:**

The interest rate on a zero-coupon instrument.

► **Volatility:**

A measure of interest rate uncertainty that may loosely be regarded as the standard deviation of interest rates over one period, normalized by the level of interest rates.

grants the issuer the right to accelerate the repayment of the principal. This is a partial European call option struck at par, and is extremely difficult to value with an option-pricing model.⁴ When a bond has multiple or interrelated embedded options (e.g., both call and put options) valuation becomes complicated.

The valuation model presented in this article does not rely on an explicit option-pricing model. Rather, it is based on a consistent framework for valuing any bond—option-free or with embedded options. It focuses on discounting each cash flow at an appropriate, **volatility-dependent one-period forward rate**. These rates are derived in a natu-

Table I Yields and Spot Rates

Maturity	Yield to Maturity	Value	Spot Rate
1 Year	3.50%	100	3.500%
2 Years	4.00%	100	4.010%
3 Years	4.50%	100	4.531%

ral, consistent manner. Once the value of a bond with embedded options and the value of its underlying option-free bond are known, the value of the options themselves can be determined.

Spot Rates and Forward Rates

The relationship among the yields to maturity on securities selling at par with the same credit quality but different maturities is called the **on-the-run yield curve**. The yield curve is typically constructed using the maturities and observed yields of Treasury securities. As market participants do not perceive these securities to have any default risk, this government yield curve reflects the effect of maturity alone on yield. However, a theoretical yield curve can be constructed for any issuer.

Spot Rates

The yield curve shows the relationship between the yield on full-coupon securities and maturity. What we are interested in is the relationship between the yield on zero-coupon instruments (i.e., spot rates) and maturity. Pure expectations theory and arbitrage arguments can be used to determine this relationship, called the spot rate curve.⁵

Table I gives, for example, the yield curve (based on current on-the-run yields) for annual-pay, full-coupon par bonds, as well as the spot rate for each year. Now consider an option-free bond with three years remaining to maturity and a coupon rate of 5.25%. The value of this bond is the present value of the cash flows, with each cash flow discounted at the corresponding spot rate. So

the value of our hypothetical bond, assuming a par value of \$100, is:

$$\frac{\$5.25}{(1.03500)} + \frac{\$5.25}{(1.04010)^2} + \frac{\$100 + \$5.25}{(1.04531)^3} = \$102.075$$

This procedure will ascribe a value of 100 to each of the three securities used to define the on-the-run yield curve.

Forward Rates

A forward rate is an interest rate on a security that begins to pay interest at some time in the future. For example, a one-year rate one year forward is an interest rate on a one-year investment beginning one year from today; a one-year rate two years forward is an interest rate on a one-year investment beginning two years from today. Forward rates, like spot rates, can be derived from the yield curve, using arbitrage arguments. Consider once again the hypothetical yield curve used to derive the spot rates. Table II gives the arbitrage-free forward one-year rates.

Note that the forward one-year rate for Year 1 is the rate on a one-year security issued today, the forward one-year rate for Year 2 is the rate on a one-year security issued one year from today, and so forth. In saying that these rates are arbitrage-free, we mean that an investor would be indifferent between, say, a sequence of three one-year investments at these rates and a three-year investment at the on-the-run three-year rate, all else being equal.

Spot rates and the arbitrage-free forward rates are related. Specifi-

Table II Forward Rates

Year	One-Year Forward Rate
1	3.500%
2	4.523%
3	5.580%

cally, letting f_t denote the forward one-year rate in year t and z_t the spot t -year rate today, one can show that:

$$\text{Year 1: } z_1 = f_1,$$

$$\text{Year 2: } z_2 = [(1 + f_1)(1 + f_2)]^{1/2} - 1,$$

$$\text{Year 3: } z_3 = [(1 + f_1)(1 + f_2)(1 + f_3)]^{1/3} - 1.$$

Substituting in the forward one-year rates, we can verify this relationship.

$$\text{Year 1: } 3.5\% = 3.5\%,$$

$$\text{Year 2: } [(1.035)(1.04523)]^{1/2} - 1 = 0.0401 = 4.01\%,$$

$$\begin{aligned} \text{Year 3: } & [(1.035)(1.04523) \\ & \cdot (1.05580)]^{1/3} - 1 = 0.04531 \\ & = 4.531\%. \end{aligned}$$

Consequently, discounting cash flows at arbitrage-free forward one-year rates is equivalent to discounting at the current spot rates. This can be verified using the 5.25% coupon bond with three years to maturity. The value of the bond found by discounting at the arbitrage-free forward one-year rates is:

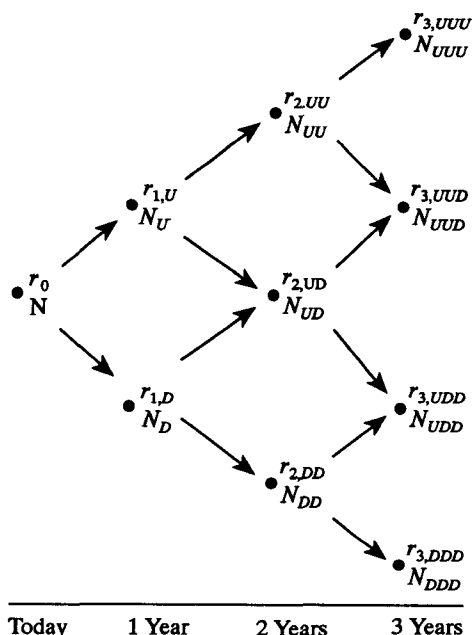
$$\begin{aligned} & \frac{\$5.25}{(1.035)} + \frac{\$5.25}{(1.035)(1.04523)} \\ & + \frac{\$100 + \$5.25}{(1.03500)(1.04523)(1.05580)} \\ & = 102.075. \end{aligned}$$

This is the same value as found earlier.

Interest Rate Volatility

Once we allow for embedded options, consideration must be given to interest rate volatility. This can be done by introducing a *binomial interest rate tree*. This tree is nothing more than a discrete representation of the possi-

Figure A Three-Year Binomial Interest Rate Tree



Given our assumption that the two one-year rates one year forward are equally likely, the relationship between the two is simply:

$$r_{1,U} = r_{1,D}(e^{2\sigma}),$$

where e is the base of the natural logarithm 2.71828. Suppose, for example, that $r_{1,D}$ is 4.074% and σ is 10% per year. Then:

$$r_{1,U} = 4.074\%(e^{2 \cdot 0.10}) = 4.976\%$$

In the second year, the one-year rate has *three* possible values, which we denote as follows:

$r_{2,DD}$ = one-year rate two years forward, assuming rates fall in the first and second years,

$r_{2,UU}$ = one-year rate two years forward, assuming rates rise in the first and second years; and

$r_{2,UD}$ = one-year rate two years forward, assuming rates either rise in the first year and fall in the second year or fall in the first year and rise in the second year.

The relationship between $r_{2,UD}$ and the other two forward rates is as follows.

$$r_{2,UD} = r_{2,DD}(e^{2\sigma})$$

and

$$r_{2,UU} = r_{2,DD}(e^{4\sigma}).$$

For example, if $r_{2,DD}$ is 4.53% and, once again, σ is 10%, then

$$r_{2,UD} = 4.53\%(e^{2 \cdot 0.10}) = 5.532\%$$

and

$$r_{2,UU} = 4.53\%(e^{4 \cdot 0.10}) = 6.757\%.$$

Figure A shows the notation for the binomial interest rate tree over three years. We can simplify

ble evolution over time of the one-period rate based on some assumption about interest rate volatility. The construction of this tree is illustrated below.

Binomial Interest Rate Tree

Figure A shows an example of a binomial interest rate tree. In this tree, the nodes are spaced one year apart in time (left to right). Each node N is labeled with one or more subscripts indicating the path the one-year rate followed in reaching that node. The one-year rate may move to one of two values in the following year: U represents the higher of these two values and D represents the lower. It is important to note that these rates are not necessarily higher or lower than the one-year rate in the preceding year. Thus N_{UU} simply means that the one-year rate followed the upper path in both the first and second years.⁶

Look first at the point denoted N . This is the root of the tree and is nothing more than the current one-year rate, which we denote

by r_0 . What we have assumed in creating this tree is that the one-year rate can take on two possible values one year from today. In what follows, we also assume that these two future rate environments are equally likely. For convenience, we refer to the higher of the two rates in the next period as resulting from a "rise" in rates and the lower as resulting from a "fall" in rates (although we stress that the rate has not necessarily fallen or risen relative to its last value). The tree we construct will be a discrete representation of a lognormal distribution over time of future one-year rates with a certain volatility.

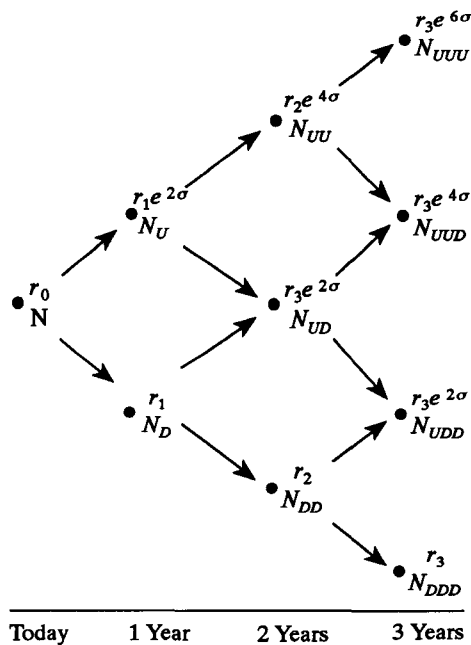
We use the following notation to describe the tree in the first year. Let:

σ = assumed volatility of the forward one-year rate at all times,

$r_{1,D}$ = one-year rate one year forward if rates decline and

$r_{1,U}$ = one-year rate one year forward if rates rise

Figure B Three-Year Binomial Interest Rate Tree with Forward One-Year Rates*



* r_t equals forward one-year rate if rate falls.

the notation by letting r_t denote the one-year rate t years forward if interest rates always decline, since all the other rates t years forward will depend on that rate. Figure B shows the interest rate tree using this simplified notation.

Before we go on to show how to use this binomial interest rate tree to value bonds, let's focus on two issues. First, what does the volatility parameter σ in the expression $e^{2\sigma}$ represent? Second, how do we find the value of the bond at each node?

Volatility and Standard Deviation

The standard deviation of the one-year rate one year forward is equal to $r_0\sigma$.⁷ The standard deviation is a statistical measure of volatility. This means that volatility is measured relative to the current level of rates. For example, if σ is 10% and the one-year rate (r_0) is 4%, then the standard deviation of the one-year rate one

year forward is 4% times 10%, which equals 0.4%, or 40 basis points. However, if the current one-year rate is 12%, the standard deviation of the one-year rate one year forward would be 12% times 10%, or 120 basis points

Determining the Value at a Node

To find the value of the bond at a node, first calculate the bond's value at the two nodes to the right of the node of interest. For example, in Figure B, to determine the bond's value at node N_U , values at nodes N_{UU} and N_{UD} must be determined. (Hold aside for now how we get these two values, as we will see, the process involves starting from the last year in the tree and working backward to get the final solution, so these two values will be known.)

What we are saying, in effect, is that the value of a bond at a given node will depend on *future* cash flows. We can separate the contribution from future cash flows

into (1) the coupon payment one year from now and (2) the bond's value one year from now. The former is known. The latter depends on whether the one-year rate rises or falls in the coming year. The two nodes to the right of any given node show the bond's value when rates rise or fall. Suppose we intend to sell the bond at the next node date. The cash flow we will receive on that future date will be either (1) the bond's value if rates rise plus the coupon payment or (2) the bond's value if rates fall plus the coupon payment. The bond's value at N_U , then, will be either its value at N_{UU} plus the coupon payment or its value at N_{UD} plus the coupon payment, discounted.

In calculating the present value of expected future cash flows, the appropriate discount rate is the one-year rate at the node where we seek the value. Now, there are two future values to discount—the value if the one-year rate rises over the coming year and the value if it falls. We assume both outcomes are equally likely, so we can simply average the two future values. Assuming that the one-year rate is r_* at the node being valued, and letting

V = the bond's value at the node in question,

V_U = the bond's value in one year if the one-year rate rises,

V_D = the bond's value in one year if the one-year rate falls and

C = the coupon payment,

the cash flow at a node is either.

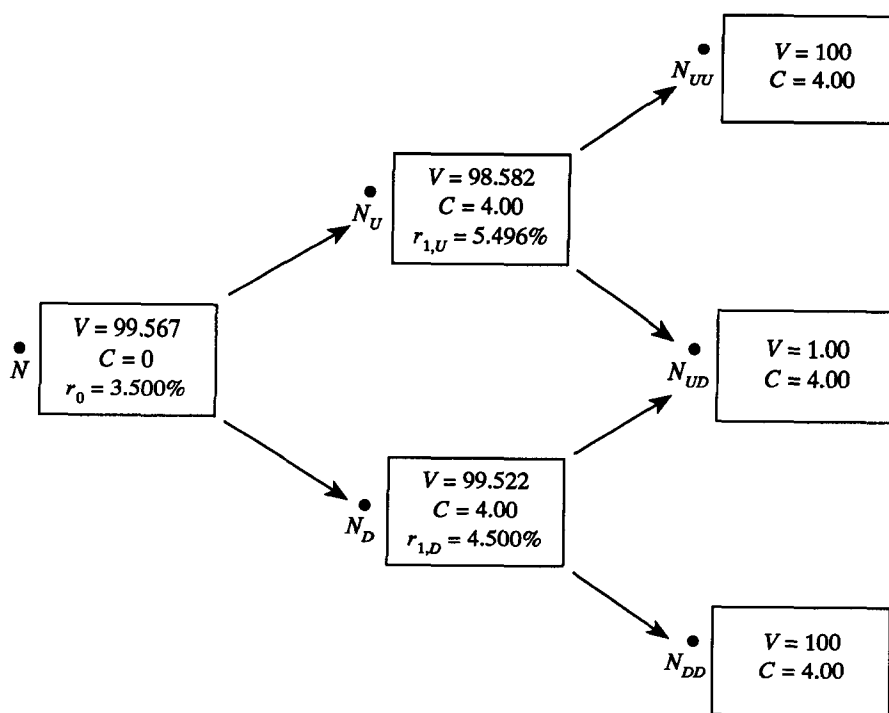
$$V_U + C$$

if the one-year rate rises or

$$V_D + C$$

if the one-year rate falls. The present values of these two cash flows, using the one-year rate r_* at the node, are

Figure C Finding the Forward One-Year Rates for Year One
Using Two-Year, 4%, On-the-Run Issue: First Trial



$$\frac{V_U + C}{1 + r_*}$$

if the one-year rate rises and

$$\frac{V_D + C}{1 + r_*}$$

if the one-year rate falls. The value of the bond at the node is then found as follows:

$$V = \frac{1}{2} \left[\frac{V_U + C}{1 + r_*} + \frac{V_D + C}{1 + r_*} \right]$$

Constructing a Binomial Interest Rate Tree

To see how to construct a binomial interest rate tree, assume on-the-run yields are as given in Table I and that volatility, σ , is 10%. We construct a two-year model that correctly values a two-year bond with a 4% coupon at 100.

Figure C shows a binomial interest rate tree that gives the cash flow at each node. The root rate

for the tree, r_0 , is simply the current one-year rate, 3.5%. This rate was chosen because it will value a one-year bond with a 3.5% coupon at 100.

There are two possible one-year rates one year forward—one if rates fall and one if rates rise. What we want to find are the two forward rates consistent with the volatility assumption that result in a value of 100 for a 4% full-coupon, two-year bond. While one can construct a simple algebraic expression for these two rates, the formulation becomes increasingly complex beyond the first year. Furthermore, because one may want to implement this procedure on a computer, the natural approach is to find the rates by an iterative process (i.e., trial-and-error). The steps are described below.

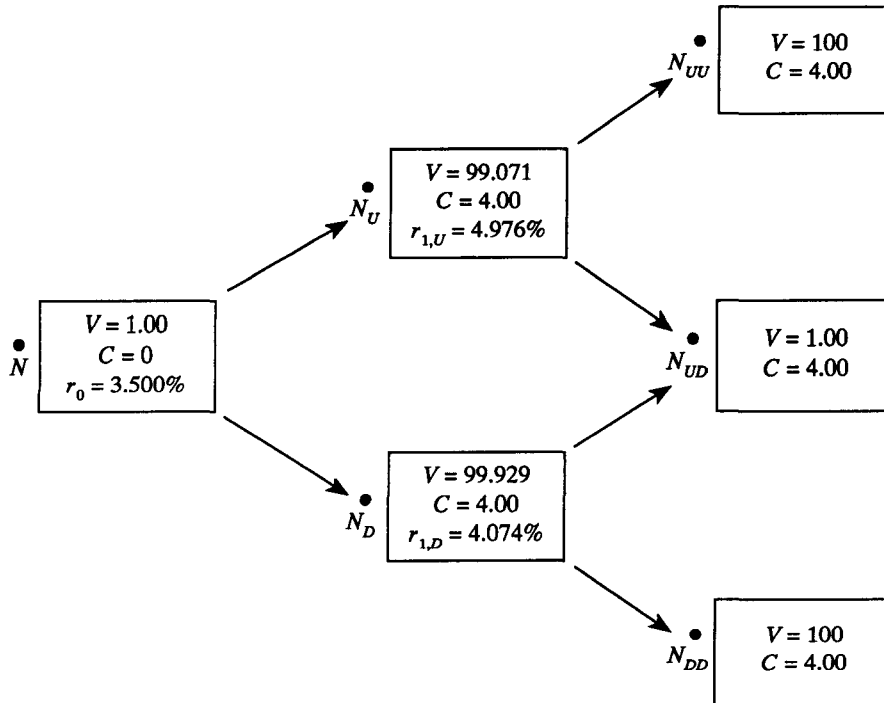
Step 1: Select a value for r_1 . Recall that r_1 is the one-year rate one year forward if rates fall. In this first trial, we arbitrarily selected a value of 4.5%.

Step 2: Determine the corresponding value for the one-year rate one year forward if rates rise. As explained earlier, this rate equals $r_1 e^{2\sigma}$. Because r_1 is 4.5%, the forward one-year rate if rates rise is 5.496% ($= 4.5\% e^{2 \cdot 0.10}$). This value is reported in Figure C at node N_U .

Step 3: Compute the bond's value in each of the two interest rate states one year from now, using the following steps.

- Determine the bond's value two years from now. Since we are using a two-year bond, the bond's value is its maturity value of \$100 plus its final coupon payment of \$4, or \$104.
- Calculate the bond's present value at node N_U . The appropriate discount rate is the forward one-year rate assuming rates rise, or 5.496% in our example. The present value is \$98.582 ($= \$104 / 1.05496$). This is the value of V_U referred to earlier.

Figure D Forward One-Year Rates for Year One
Using Two-Year, 4%, On-the-Run Issue



c. Calculate the bond's present value at node N_D . The appropriate discount rate is the forward one-year rate assuming rates fall, or 4.5%. The present value is \$99 522 (= \$104/1.045) and is the value of V_D .

Step 4: Add the coupon to the cash flows V_U and V_D computed in Step 3 to get the total cash flow at N_U and N_D , respectively. Then calculate the average of these two cash flows, using the assumed value of r_0 , 3.5%, for r_* .

$$\begin{aligned}
 V &= \frac{1}{2} \left[\frac{V_U + C}{1 + r_0} + \frac{V_D + C}{1 + r_0} \right] \\
 &= \frac{1}{2} \left[\frac{\$98.562 + \$4.000}{1.035} + \frac{\$99.522 + \$4.000}{1.035} \right] \\
 &= (\$99.113 + \$100.021)/2 \\
 &= \$99.567
 \end{aligned}$$

Step 5: Compare the value obtained in Step 4 with the target value of \$100. If the two values are the same, then the r_1 used in this trial is the one we seek. It is the forward one-year rate to be used in the binomial interest rate tree for a decline in rates, as well as the rate to determine the rate to be used if rates rise. If, however, the value found in Step 4 is not equal to the target value of \$100, our assumed value is not consistent with the volatility assumption and the yield curve. In this case, repeat the five steps with a different value for r_1 .

If we use 4.5% for r_1 , a value of \$99.567 results in Step 4. This is smaller than the target value of \$100. Therefore, 4.5% is too large. The five steps must be repeated, with a smaller value for r_1 . It turns out that the correct value for r_1 in this example is 4.074%. The corresponding binomial interest rate tree is shown in Figure D. Steps 1 through 5, using the correct rate, are as follows

Step 1: Select a value of 4.074% for r_1 .

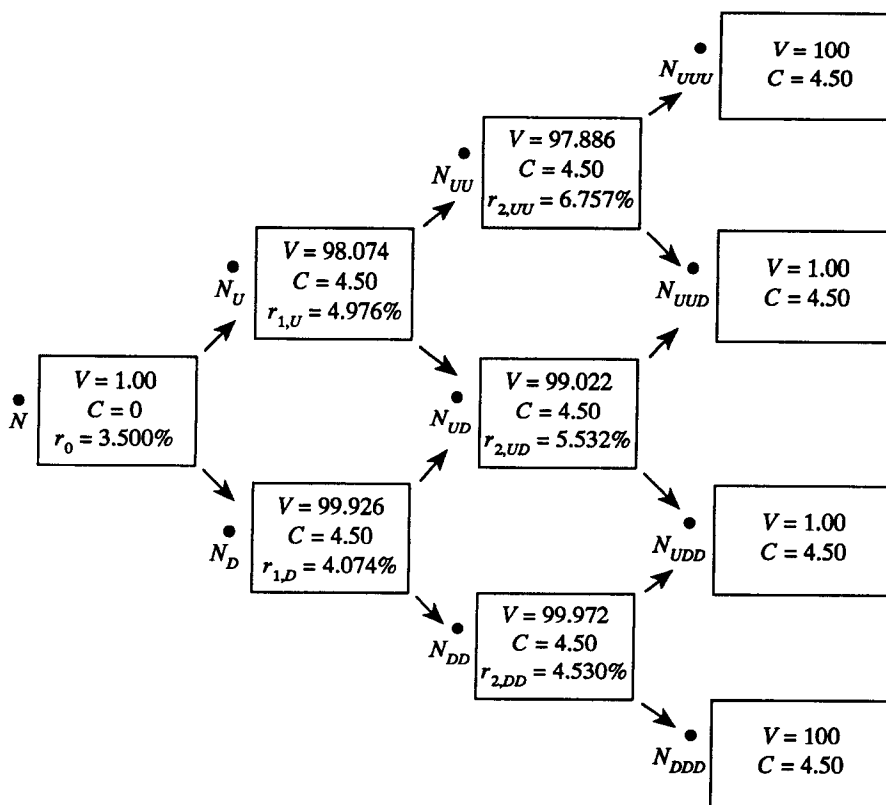
Step 2: The corresponding value for the forward one-year rate if rates rise is 4.976% (= 4.074% $e^{2 \cdot 0.10}$).

Step 3: The bond's value one year from now is determined from the bond's value two years from now—\$104, just as in the first trial, the bond's present value at node N_U — V_U , or \$99.071 (= \$104/1.04976); and the bond's present value at node N_D — V_D or \$99.929 (= \$104/1.04074).

Step 4: Add the coupon to both V_U and V_D and average the present values of the resulting cash flows:

$$\begin{aligned}
 V &= \frac{1}{2} \left[\frac{V_U + C}{1 + r_0} + \frac{V_D + C}{1 + r_0} \right] \\
 &= \frac{1}{2} \left[\frac{\$99.071 + \$4.000}{1.035} + \frac{\$99.929 + \$4.000}{1.035} \right]
 \end{aligned}$$

Figure E Forward One-Year Rates for Year Two
Using Three-Year, 4.5%, On-the-Run Issue



$$\begin{aligned}
 & + \frac{\$99.929 + \$4.000}{1.035} \Big] \\
 & = (\$99,586 + \$100,414)/2 \\
 & = \$100.000
 \end{aligned}$$

Step 5: Since the average present value is equal to the target value of \$100, r_1 is 0.04074 or 4.074%.

We're not done. Suppose we want to "grow" the tree for one more year—that is, we need to determine r_2 . We will use a three-year, on-the-run 4.5% coupon bond to get r_2 . The same five steps are used in an iterative process to find the one-year rates two years forward. Our objective now is to find the value of r_2 that will produce a value of \$100 for the 4.5% on-the-run bond and that will be consistent with (1) a volatility assumption of 10%, (2) a current forward one-year rate of 3.5% and (3) the two forward rates one

year from now of 4.074% and 4.976%. The desired value of r_2 is 4.53%. Figure E shows the completed binomial interest rate tree. Using this binomial interest rate tree, we can value a one-year, two-year or three-year bond, whether option-free or not.

Using the Binomial Tree

To illustrate how to use the binomial interest rate tree, consider an option-free 5.25% bond with two years remaining to maturity. Assume that the issuer's on-the-run yield curve is the one given in Table I so that the appropriate binomial interest rate tree is the one in Figure E. Figure F shows the various values obtained in the discounting process. It produces a bond value of \$102.075.

It is important to note that this value is identical to the bond value found earlier by discounting at either the zero-coupon rate

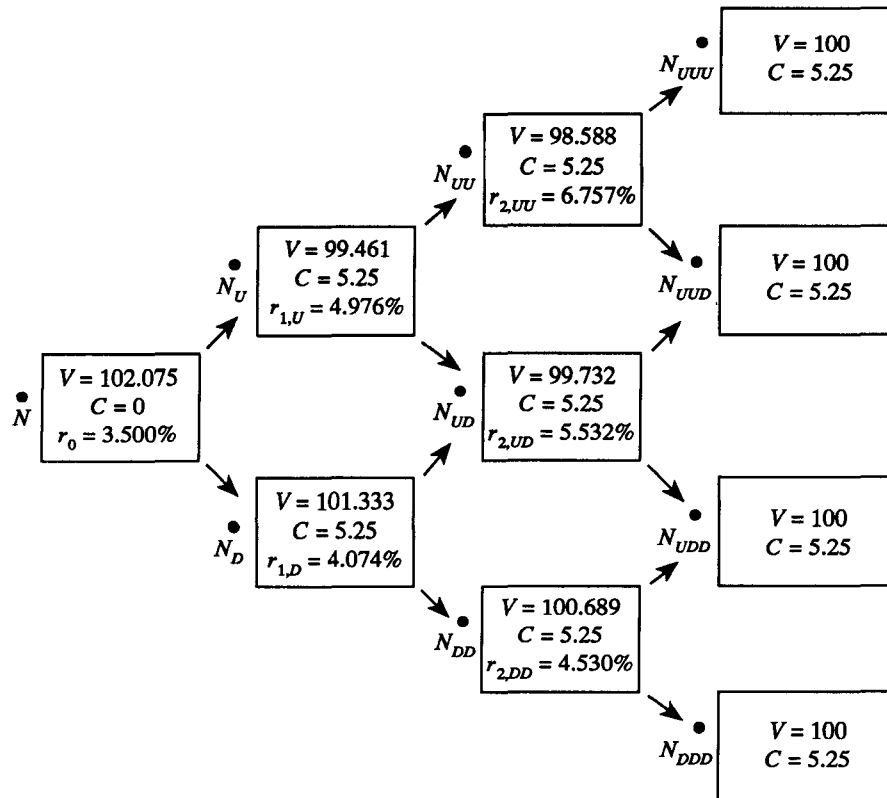
or the forward one-year rate. We should expect to find this result, as our bond is option-free. This clearly demonstrates that, for an option-free bond, the tree-based valuation model is consistent with the standard valuation model.

Valuing a Callable Bond

The binomial interest rate tree can also be applied to callable bonds. The valuation process proceeds in the same fashion as in the case of an option-free bond, with one exception: When the call option can be exercised by the issuer, the bond's value at a node must be changed to reflect the lesser of its value if it is not called (i.e., the value obtained by applying the recursive valuation formula described above) or the call price

Consider a 5.25% bond with three years remaining to maturity that has an embedded call option

Figure F Valuing an Option-Free Bond with Three Years to Maturity and a Coupon of 5.25%



exercisable in years one and two at \$100. Figure G shows the value at each node of the binomial interest rate tree. The discounting process is identical to that shown in Figure F, except that at two nodes, N_D and N_{DD} , the values from the recursive valuation formula (\$101.002 at N_D and \$100.689 at N_{DD}) exceed the call price (\$100); they have therefore been struck out and replaced with \$100. The value for this callable bond is \$101.432

Note that the technique we have just illustrated allows the direct valuation of a bond with an embedded option. While others have implemented interest rate processes and valuation models on discrete tree structures, they have generally valued options directly, not as an integral part of the underlying security.⁸ The distinction becomes even more important when confronted with a pack-

age of interrelated embedded options, such as those found in sinking fund bonds.⁹

Determining the Call Option Value

From our earlier discussion of the relationships between the value of a callable bond, the value of a noncallable bond and the value of the call option, we know that:

Value of Call Option

= Value of Option-Free Bond

– Value of Callable Bond

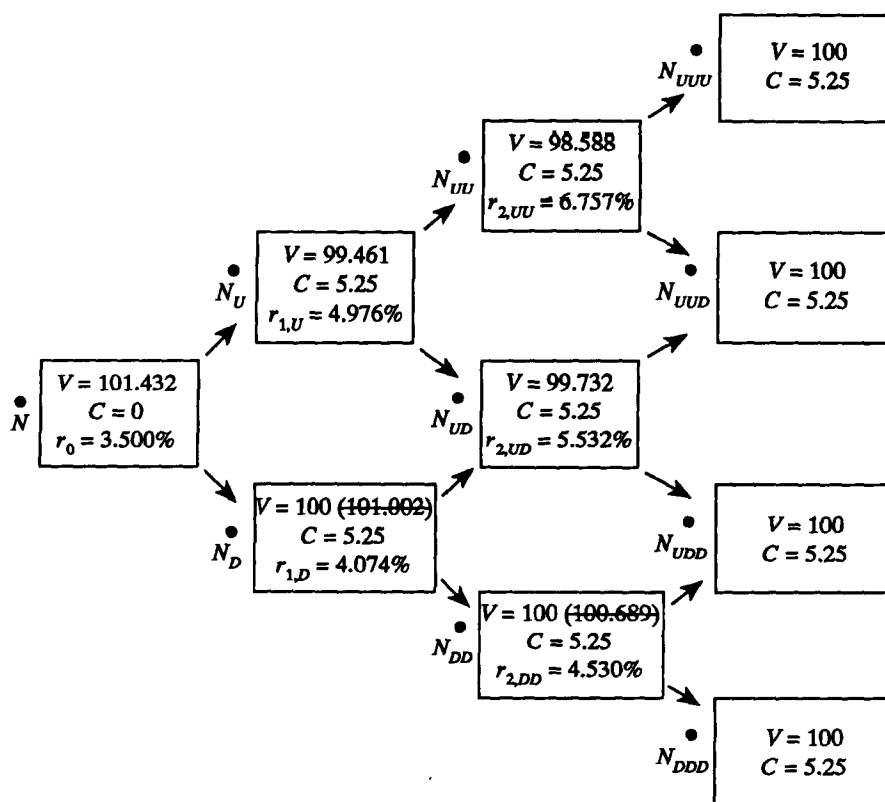
But we have just seen how to determine the values of both a noncallable bond and a callable bond. The difference between the two values is the value of the call option. In our example, the value of the noncallable bond is

\$102.075 and the value of the callable bond is \$101.432, so the value of the call option is \$0.643.

Transaction Costs

When an issuer calls a bond, the debt is generally refinanced. The issuer will, in general, incur transaction costs in the refinancing. These costs may be injected into the valuation model by scaling up the call price appropriately. It would be imprudent for an issuer to exercise a call option unless the net-present-value savings realized by calling the bond represented a substantial fraction of the forfeited option value (i.e., unless the intrinsic value of the call option dominated its incremental time value). Indeed, the option-exercise rule embodied in the valuation model presented here assumes that an option will be exercised *only* if its incremental time value is zero.

Figure G Valuing a Callable Bond with Three Years to Maturity and a Coupon of 5.25%, Callable in Years One and Two at 100



By including transaction costs in every option-exercise decision, one obtains an adjusted option value that does not penalize refunding. One must consider, however, what will happen at the final maturity. Debt is rarely extinguished at maturity; it is generally refinanced, again with transaction expenses. Failure to include these costs at maturity will penalize the decision to exercise the call option on a bond close to maturity

Other Embedded Options

The bond valuation framework presented here can be used to analyze securities such as puttable bonds, options on interest rate swaps, caps and floors on floating-rate notes, and the optional accelerated redemption granted to an issuer in fulfilling sinking fund requirements. Let's consider a puttable bond. Suppose that a 5.25% bond with three years remaining to maturity has a

put option exercisable annually in years one and two at \$100. Assume that the appropriate binomial interest rate tree for this issuer is the one in Figure F. Figure H shows the binomial interest rate tree with the bond values altered at two nodes (N_{UU} and N_{UD}) because the bond values at these two nodes are less than \$100, the value at which the bond can be put. The value of this puttable bond is \$102.523.

As the value of a nonputtable bond can be expressed as the value of a puttable bond minus the value of a put option on that bond:

Value of Put Option

$$= \text{Value of Option-Free Bond} \\ - \text{Value of Puttable Bond.}$$

In our example, the value of the puttable bond is \$102.523 and the value of the corresponding non-

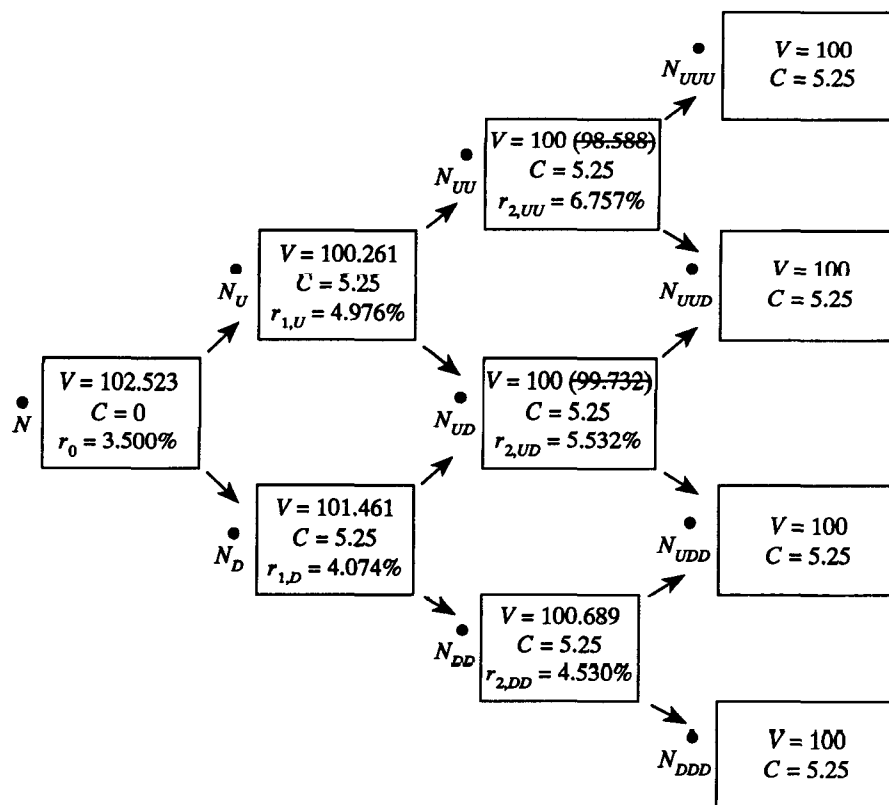
puttable bond is \$102.075, so the value of the put option is $-\$0.448$. The negative sign indicates the issuer has sold the option or, equivalently, the investor has purchased the option.

The same general framework can be used to value a bond with multiple or interrelated embedded options. The bond values at each node are simply altered to reflect the exercise of any of the options.

OAS, Effective Duration and Effective Convexity

Our model determines the theoretical value of a bond. For example, if the observed market price of the three-year, 5.25% callable bond is \$101 and the theoretical value is \$101.432, the bond is underpriced by \$0.432. Bond market participants, however, prefer to think not in dollar terms, but in terms of a yield spread: A "cheap" bond trades at

Figure H Valuing a Puttable Bond with Three Years to Maturity and a Coupon of 5.25%, Puttable in Years One and Two at 100



a higher spread and a “rich” bond at a lower spread relative to some basis.

The market convention has been to think of a yield spread as the difference between the yield to maturity on a particular bond and the yield on a Treasury bond or other benchmark security of comparable maturity. This is inappropriate, however, because yield to maturity is just a conversion of a dollar price into a yield, given the contractual cash flows, and may have little to do with the underlying yield curve that determines price. A step in the right direction is to determine a discounting spread over the issuer’s spot or forward rate curve. In terms of our binomial interest rate tree, one seeks the constant spread that, when added to all the forward rates on the tree, makes the theoretical value equal to the market price. This quantity is called the *option-adjusted spread* (OAS). It is “option-adjusted” in

that a bond fairly priced relative to an issuer’s yield curve will have an OAS of zero whether the bond is option-free or has embedded options.

Investors also want to know the sensitivity of a bond’s price to changes in interest rates. Unwarranted emphasis has been placed on modified duration, which is a measure of the sensitivity of a bond’s price, not to changes in interest rates, but to changes in the bond’s yield. It has the further shortcoming of ignoring any dependence of cash flows on interest rate levels, hence is unsuited for bonds with embedded options. The correct duration measure—called *effective duration*—quantifies price sensitivity to small changes in interest rates while simultaneously allowing for changing cash flows. In terms of our binomial interest rate tree, price response to changing interest rates is found by shifting the tree up and down by a few basis

points. Any measure of convexity should also be derived from changes in the yield curve, rather than the yield to maturity. *Effective convexity* recognizes the dependence of cash flows on interest rates.

If $V(0)$ is the value of a bond, including accrued interest, $V(\Delta)$ is its value when the tree is shifted up by Δ basis points and $V(-\Delta)$ is its value when the tree is shifted down by a similar amount. Then:

Effective Duration

$$= \frac{V(-\Delta) - V(\Delta)}{2V(0)\Delta}$$

and

Effective Convexity

$$= \frac{V(-\Delta) - 2V(0) + V(\Delta)}{V(0)\Delta^2}$$

After-Tax Valuation

Certain market participants, particularly corporations, must value bonds on an after-tax basis. The standard approach is to discount after-tax cash flows at after-tax rates. One begins by multiplying the entire on-the-run yield curve by $(1 - \text{Tax Rate})$. Using the resulting after-tax yield curve, one can determine either a set of zero-coupon rates or a set of forward one-period rates, which can then be used to discount after-tax cash flows. As in the case of pretax valuation, however, this discounting methodology does not take the volatility of interest rates into account, hence is unable to capture the value of any embedded options.

Using a binomial tree for after-tax valuation would seem to be simple. There is a significant catch, however. If an after-tax binomial interest rate tree is constructed by simply multiplying each one-period rate on a pretax tree by $(1 - \text{Tax Rate})$, the after-tax expected cost of an option-free bond is dependent on volatility. If one constructs an interest rate tree based on an after-tax yield curve—so that the after-tax discounted present value of the after-tax cash flows of a bond is independent of volatility—the implied underlying pretax interest rate process is different from that obtained directly from the pretax yield curve.

There is another interesting implication of the after-tax valuation of option-bearing bonds. Consider for a moment the pretax valuation of a callable bond. The valuation process described here is essentially an optimization procedure designed to produce the lowest possible value for an investor or, equivalently, the lowest possible expected cost of servicing the outstanding debt for the issuer. It follows, then, that any other set of option-exercise decisions can only increase the value of the bond to the investor (who, we assume, is not a taxpayer). But suppose that the valuation pro-

cess is carried out so as to minimize the issuer's after-tax cost of servicing the debt. In this case, what's good for the goose is also good for the gander. The reason is that this is a three-player game, with the third player being the taxman.

The Challenge of Implementation

To transform the basic interest rate tree into a practical tool requires several refinements. For one thing, the spacing of the node lines in the tree must be much finer, particularly if American options are to be valued. However, the fine spacing required to value short-dated securities becomes computationally inefficient if one seeks to value, say, 30-year bonds. While one can introduce time-dependent node spacing, caution is required; it is easy to distort the term structure of volatility. Other practical difficulties include the management of cash flows that fall between two node lines.

Footnotes

- 1 For an illustration of coupon stripping and why the theoretical price of a Treasury bond will be equal to the present value of the cash flows discounted at the theoretical spot rates, see Chapter 10 in F J Fabozzi, *Bond Markets, Analysis and Strategies* (Englewood Cliffs, NJ: Prentice Hall, 1993).
- 2 See, for example, M. L. Dunetz and J. Mahoney, "Using Duration and Convexity in the Analysis of Callable Bonds," *Financial Analysts Journal*, May/June 1988, and F J Fabozzi, *Fixed Income Mathematics* (Chicago: Probus Publishing, 1988).
- 3 See F Black, E Derman and W Toy, "A One-Factor Model of Interest Rates and its Application to Treasury Bond Options," *Financial Analysts Journal*, January/February 1990, J Hull and A White, "Valuing Derivative Securities Using the Explicit Finite Difference Method," *Journal of Financial and Quantitative Analysis* 25 (1990), pp 87-100, J Hull and A White, "Pricing Interest-rate Derivative Securities," *Review of Financial Studies* 3 (1990), pp 573-92, T S Y Ho and S B Lee, "Term Structure Movements and Pricing Interest Rate

Contingent Claims," *Journal of Finance* 41 (1986), pp 1011-29, and R Dattatreya and F J Fabozzi, "A Simplified Model for Valuing Debt Options," *Journal of Portfolio Management* 15 (1989), pp 64-73.

- 4 For an explanation of these options and their valuation using the model presented in this article, see A J Kalotay and G O Williams, "The Valuation and Management of Bonds with Sinking Fund Provisions," *Financial Analysts Journal*, March/April 1992.
- 5 The method used is called "bootstrapping." For an illustration of how this is done, see Chapter 10 in Fabozzi, *Bond Markets, Analysis and Strategies*, op cit.
- 6 Note that this tree is said to be recombining, so that N_{UD} is equivalent to N_{DU} in the second year, and in the third year N_{UUD} is equivalent to both N_{UDU} and N_{DUU} . We have simply selected one label for a node rather than cluttering up the exhibit with unnecessary information.
- 7 This can be seen by noting that $e^{2\sigma} \approx 1 + 2\sigma$. Then the standard deviation of forward one-period rates is

$$\frac{re^{2\sigma} - r}{2} \approx \frac{r + 2\sigma r - r}{2} = \sigma r$$

- 8 See, for example, Black, Derman and Toy, "A One-Factor Model of Interest Rates and its Application to Treasury Bond Options," op cit. Like ours, their model is a one-factor (the short-term interest rate) process represented on a binomial tree. However, theirs takes as inputs a zero-coupon yield curve and its volatility term structure. The stochastic differential equations underlying the two interest rate processes are also different.
- 9 See Kalotay and Williams, "The Valuation and Management of Bonds with Sinking Fund Provisions," op cit.