Derivatives Pricing Course

Lecture 1 – QF main results

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Agenda

- Math needed for risk-neutral pricing.
- Main results of the risk-neutral approach.
- Main results of the PDE approach.
- Greek coefficients. Estimation techniques.

Risk-neutral pricing
First fundamental theorem

- $X(t) = (X_1(t), \dots, X_P(t))^T$ - Price process.
- Strictly positive Ito process used to normalize the asset prices is called a *deflator*.
- Let the deflator be denoted *D(t)* and define the normalized asset process $X^D(t) = \left(\frac{X_1(t)}{D(t)}, \dots, \frac{X_P(t)}{D(t)}\right)^T$
- We say that a measure $\mathbb{Q}$ is an *equivalent martingale measure* induced by *D* if $X^D(t)$ is a $\mathbb{Q}$-martingale.
- Theorem 1.1(Sufficient Condition for No-Arbitrage). If there is a deflator such that the deflated asset price process allows for an equivalent martingale measure, then there is no arbitrage.
- If the deflator is one of the *p* assets, we call it a *numeraire*.

Risk-neutral pricing

Girsanov's theorem

- We need to consider:
  - The conditions on the asset prices that allow for an equivalent martingale measure.
  - The effect on asset dynamics from a change of probability measure.
- $\mathbb{P}$ and $\mathbb{P}(\theta)$ – two measures related by a density:
- $\varsigma^{\theta}(t) = \mathrm{E}_t^{\mathbb{P}}\left(\frac{d\mathbb{P}(\theta)}{d\mathbb{P}}\right)$, where $\frac{d\varsigma^{\theta}(t)}{\varsigma^{\theta}(t)} = -\theta(t)^T dW(t)$
- <u>Theorem 1.2 (Girsanov's Theorem)</u> Suppose that $\varsigma^{\theta}(t)$ is a martingale. Then for all t ϵ [0, T]:

$W^{\theta}(t) = W(t) + \int_0^t \theta(s)ds$ is a Brownian motion under $\mathbb{P}(\theta)$

> *Girsanov's theorem implies that we can shift probability measure to transform an Ito process with a given drift to an Ito process with nearly arbitrary drift*

Risk-neutral pricing

Girsanov's theorem

$$dX(t) = \mu(t)dt + \sigma(t)dW(t)$$

$$\Downarrow$$

$$dX(t) = (\mu(t) - \sigma(t)\theta(t))dt + \sigma(t)dW^{\theta}(t)$$

> *This process could be driftless provided θ satisfies the simple condition*

Risk-neutral pricing

Martingale property

- A T-maturity *derivative security* pays out at time T an $\mathcal{F}_T$–measurable random variable $V(T)$, and makes no payments before T.

- Derivative security is *attainable* if there exists a permissible trading strategy φ such that $V(T) = \varphi(T)^T X(T) = \pi(T)$ a.s.

- If there is no arbitrage: $V(t) = \pi(t)$ for any $t \epsilon [0, T]$.

- Martingale property: $\frac{V(t)}{D(t)} = \mathrm{E}_t{}^{\mathbb{Q}}\left(\frac{V(T)}{D(T)}\right)$

- Theorem 1.3 In the absence of arbitrage, a market is complete if and only if there exists a deflator inducing a unique martingale measure.

PDE approach

BS economy assumptions

- Hedging is continuous
- There are no transaction costs
- Volatility is constant
- There are no arbitrage opportunities
- The underlying is lognormally distributed
- There are no costs associated with borrowing stock for short

PDE approach
BS equation

- Let our option price function $V(t, S(t))$ be enough smooth:

$$dV = \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial V}{\partial S}\sigma S dW$$

- Let's consider the following portfolio Π:
  - Short 1 derivative
  - Long $\frac{\partial V}{\partial S}$ shares
- Portfolio value at time t:

$$\Pi = -V + \frac{\partial V}{\partial S}S$$

PDE approach

BS equation

- Portfolio value increment:

$$d\Pi = -dV + \frac{\partial V}{\partial S} dS$$

- We know how dV and dS look like:

$$d\Pi = \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right) dt$$

- <u>*There is no any stochastic component, no risk:*</u>

$$d\Pi = r\Pi dt$$

PDE approach

BS equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 = rV$$

$\theta = \frac{\partial V}{\partial t}$ - Theta, “time decay” term.

$\Delta = \frac{\partial V}{\partial S}$ - Delta, hedge ratio.

$\Gamma = \frac{\partial^2 V}{\partial S^2}$ - Gamma, “convexity”, rate of change in the delta with respect to changes in the underlying.

- To get the price, one should state boundary conditions, and solve the equation.

PDE approach

BS model advantage

- The equation is easy to solve numerically, by finite-difference methods for example.
- The equation can be generalized to allow for:
  - Dividends
  - Other payoffs
  - Stochastic volatility
  - Jumping stock prices
  - Transaction costs
  - Stochastic interest rates
  - Discontinuous hedging
  - etc.

- *Black-Scholes is remarkably robust model.*

Greek coefficients

Definition

- "Greeks" - a set of risk measures that indicate how exposed an option price is to changes in model parameters/variables.
- Derivative with respect to parameter/variable - $\frac{\partial V}{\partial x}$.

  V – option price

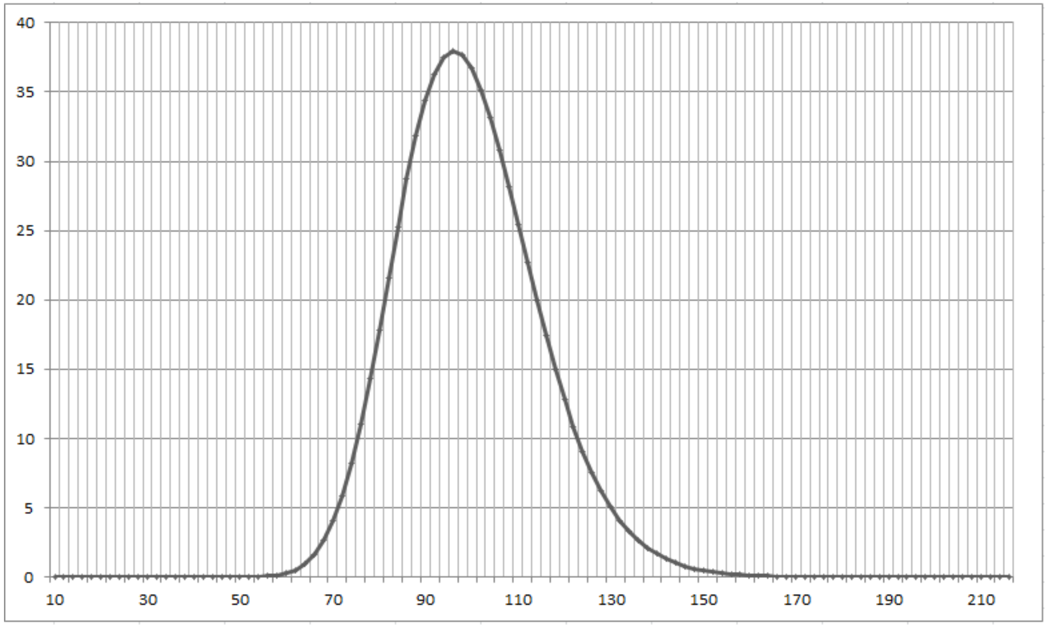
  x – parameter/variable

- Derivatives can be of higher order and also mixed partial.
- Whereas the prices themselves can often be observed in the market, their sensitivities cannot, so accurate calculation of sensitivities is arguably even more important than calculation of prices.

Greek coefficients

Simple example

- Vanilla option Vega - $\frac{\partial V}{\partial \sigma}$, where σ – volatility.



- Explicit formula is available. What can we do if there is no closed-form solution?

Greek coefficients
Estimation techniques

- Finite-difference approximations

- Pathwise Derivative method

- Likelihood Ratio method

Greek coefficients
Finite-difference approximations

- Consider a model that depends on a parameter θ
$$\alpha(\theta) = \mathrm{E}[Y(\theta)]$$

- The derivative estimation problem consists of finding a way to estimate $\alpha'(\theta)$

- Forward-difference estimator:
$$\widehat{\Delta}_F = \frac{Y(\theta + h) - Y(\theta)}{h}$$

Bias $\widehat{\Delta}_F$, using Taylor expansion for $\alpha(\theta + h)$:

$$\mathrm{E}\left[\widehat{\Delta}_F - \alpha'(\theta)\right] = \frac{1}{2}\alpha''(\theta)h + o(h)$$

CMF MSU

Greek coefficients

Finite-difference approximations

- First method is inevitably biased. Try to reduce bias:

- Central-difference estimator:

$$\widehat{\Delta}_C = \frac{Y(\theta + h) - Y(\theta - h)}{2h}$$

$$\text{Bias}\left(\widehat{\Delta}_C\right) = o(h)$$

- The form of the bias would lead us to take smaller values of h to improve accuracy. But the effect of h on bias must be weighed against its effect on variance:

$$\text{Var}\left[\widehat{\Delta}_F\right] = \frac{\text{Var}[Y(\theta + h) - Y(\theta)]}{h^2}$$

Greek coefficients

Pathwise Derivative estimate

- True derivative:

$$Y'(\theta) = \lim_{h \to 0} \frac{Y(\theta + h) - Y(\theta)}{h}$$

- It is an unbiased estimator of $\alpha'(\theta)$ if the interchange of differentiation and expectation is justified.

$$\mathrm{E}\left[\frac{d}{d\theta} Y(\theta)\right] = \frac{d}{d\theta} \mathrm{E}[Y(\theta)]$$

- Method is limited primarily by the requirement of continuity in the discounted payoff as a function of the parameter of differentiation.

- Black-Scholes delta calculation, let:

$$Y = e^{-rT}[S(T) - K]^+$$

$$S(T) = S(0)e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}Z}, Z \sim N(0,1)$$

Greek coefficients

Pathwise Derivative estimate

- Chain rule for differentiation:

$$\frac{dY}{dS(0)} = \frac{dY}{dS(T)}\frac{dS(T)}{dS(0)}$$

- The first derivative:

$$\frac{dY}{dS(T)} = e^{-rT}\mathrm{I}\{S(T) > K\}$$

- The second derivative:

$$\frac{dY}{dS(0)} = e^{-rT}\frac{S(T)}{S(0)}\mathrm{I}\{S(T) > K\}$$

- The expected value of this estimator is indeed BS delta.

Greek coefficients

Likelihood Ratio method

- The *likelihood ratio method* provides an alternative approach to derivative estimation requiring no smoothness at all in the discounted payoff and thus complementing the pathwise method.
- It accomplishes this by differentiating probabilities rather than payoffs.
- Expected discounted payoff:

$$\mathrm{E}_\theta[Y] = \mathrm{E}_\theta[f(X)] = \int f(x) g_\theta(x) dx$$

- Derivative estimator:

$$\frac{d}{d\theta}\mathrm{E}_\theta[Y] = \int f(x) \frac{d}{d\theta} g_\theta(x) dx$$

Greek coefficients

Likelihood Ratio method

- Hence:

$$\frac{d}{d\theta}\mathrm{E}_\theta[Y] = \int f(x)\frac{\dot{g}_\theta(x)}{g_\theta(x)}g_\theta(x)dx = \mathrm{E}_\theta\left[f(X)\frac{\dot{g}_\theta(X)}{g_\theta(X)}\right]$$

- In order to calculate BS delta, one should first take the derivative of lognormal density - $S(T)$ with respect to $S(0)$.

Greek coefficients

Estimation techniques key insights

- Finite-difference approximations
  - Most straightforward.
  - Since results are biased balance between bias and variance are required.
- Pathwise Derivative method
  - Unbiased results.
  - Payoff should be smooth.
- Likelihood Ratio method
  - Unbiased results.
  - Density function should be smooth that is rarely an obstacle.