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Forecasting Financial Volatilities with Extreme Values: The Conditional Autoregressive Range (CARR) Model

We propose a dynamic model for the high/low range of asset prices within fixed time intervals: the Conditional Autoregressive Range Model (henceforth CARR). The evolution of the conditional range is specified in a fashion similar to the conditional variance models as in GARCH and is very similar to the Autoregressive Conditional Duration (ACD) model of Engle and Russell (1998). Extreme value theories imply that the range is an efficient estimator of the local volatility, e.g., Parkinson (1980). Hence, CARR can be viewed as a model of volatility. Out-of-sample volatility forecasts using the S&P500 index data show that the CARR model does provide sharper volatility estimates compared with a standard GARCH model.

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MODELING THE VOLATILITIES of speculative asset prices has been a central theme in the recent literature of financial economics and econometrics. As a measure of risk, volatility modeling is important to researchers who are

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trying to understand the nature of the dynamics of volatilities. It is also of fundamental importance to policy makers and regulators as it is closely related to the functioning and the stability of financial markets, which has direct links to the functioning and fluctuations of the real economy.

Since the mid-1970s, there has been a remarkable rapid surge and expansion in the market of derivative assets, further reinforcing the concentration of attention on this subject. Hedging funds play important roles in the portfolios of banks: whether commercial or investment, pension funds, insurance companies are very essential in some securities houses. Central bankers now pay close attention to the development of the derivatives market as activities on the off-balance-sheet have increased in their regulated banks and because catastrophic losses have begun to occur at a non-trivial frequency; for example, the episodes of the Barings Bank collapse, the Orange County investment scandal, and the bankruptcy of the Long-term Capital Corporation. Whether such a trend is reversible[1] is debatable, but it is clear that this trend will continue at least in the near future, say 5 to 10 years. Another thing very clear is the fact that what is at stake is increasing dramatically.[2]

A milestone in the theory of derivative assets is the stochastic volatility model of Hull and White (1987). This model formally extends the Black and Scholes (1973) option valuation model to incorporate the time-varying volatilities. Models of stochastic volatilities have surged in finance journals and have been seriously adopted by investment banks, e.g., see Lewis (2000).

It has been known for a long time in statistics that range is a viable measure of the variability of random variables, among other alternatives. Applications and discussions of this measure are common to statisticians of engineering dealing with quality control. Using the application of range in finance is also not a new concept as Mandelbrot (1971) and others employ it to test the existence of long-term dependence in asset prices.[3] The noticeable application of range in the context of financial volatility and in particular to the estimation of volatilities started from the early 1980s. By employing the extreme value theory and some well-known properties of range, Parkinson (1980) forcefully argues and demonstrates the superiority of using range as a volatility estimator as compared with standard methods. Beckers (1983), among others, further extends the range estimator to incorporate information about the opening and closing prices and the treatment of a time-varying drift, as well as other considerations. It is a puzzle, however, that despite the elegant theory and the support of simulation results, the range estimator has performed poorly in empirical studies. See Rogers (1998) for an attempt of resolution and a typical disappointing

1. One of the consequences of the Asian financial crisis in the late 1990s is the reconsideration of central bankers on the pros/cons of the derivative markets. Malaysia and Taiwan are two cases where the regulators have made some drastic policy moves halting the trading of some derivative securities related to foreign exchanges.

2. The above three examples of catastrophic risk, related to derivatives trading, are related to (or has caused) the solvency of a reputable bank, a county government, and unknown number of commercial/investment banks.

3. See Lo (1991) for an extension of the test statistic and a more recent re-investigation of the issue.

conclusion about this puzzle and Cox and Rubinstein (1985) for some conjectures of explanations. Other references include Garman and Klass (1980), Wiggins (1991), Rogers and Satchell (1991), Kunitomo (1992), and more recently Yang and Zhang (2000).

In the last two decades, one of the most phenomenal developments of the literature on empirical finance is the ARCH/GARCH family of models; see Engle (1982), Bollerslev (1986), and Nelson (1991). For a critical review with a thorough survey of the ARCH literature, see Bollerslev, Chou, and Kroner (1992). See also Bollerslev, Engle, and Nelson (1994) for a deeper theoretical treatment. Engle (1995), Rossi (1996), and Jarrow (1999) also provide more references of ARCH models and the linkage to asset pricing models with stochastic volatilities.[4] A competitive volatility model to ARCH is the Stochastic Volatility (henceforth SV) model of Taylor (1986) and Heston (1993). See also Tsay (2001) for discussions of the two branches of the literature. For insightful implementations of GARCH diffusion models to derivative pricing, see Duan (1995, 1997), Ritchken and Trevor (1999), Heston and Nandi (2000), and the recent book by Lewis (2000).

The strength of the ARCH model lies in its flexible adaptation of the dynamics of volatilities and its ease of estimation when compared to the SV models. It is quite interesting that very few have attempted to combine this dynamic modeling strategy with the sharp insight of Parkinson (1980) which states that range is an effective estimator of volatility.[5] Andersen and Bollerslev (1998) report the favorable explanatory power of range in the discussion of the "realized volatilities." Gallant, Hsu, and Tauchen (1999) and Alizadeh, Brandt, and Diebold (2001) incorporate the range into the equilibrium asset price models. Their approaches follow the SV framework. Hence, there is an obvious literature gap between a dynamic model and range is waiting to be filled by our paper. In concurrent work, Brandt and Jones (2002) compare a range-based EGARCH model with the return-based volatility model. They find much better predicting power of the range-based volatility model over the return-based model for out-of-sample forecasts. Their study emphasizes on the model of the log range rather than the level of range using an approximating result from Alizadeh, Brandt, and Diebold (2001) that the log range is approximately normal. It will be useful for future studies to compare the forecast ability between the level vis-a-vis log-range models.

We conjecture that the fundamental reason for the poor empirical performance of range is its failure to capture the dynamic evolution of volatilities. We propose a range-based volatility model: the Conditional Autoregressive Range model (henceforth CARR). By properly modeling the dynamics, range retains its superiority

4. Of the three books of collections of articles, Engle provides reports on the milestones in the ARCH literature; Rossi concentrates on Daniel Nelson's contribution; and Jarrow has the broadest scope in treating ARCH on a relatively equal-footing with the SV approach under a general title of volatility modeling.

5. A noticeable exception is Lin and Rozeff (1994). They introduce the range into the variance equation of a GARCH model and find a significant coefficient for the range; furthermore, the ARCH term becomes insignificant.

in forecasting volatility. We discuss its relationship with an important class of the GARCH family, the standard deviation GARCH. As an empirical illustration, we estimate the CARR model and compare the out-of-sample forecasts of CARR and GARCH using four different measures of volatility as benchmarks for forecasting evaluations.

The paper is organized as follows. We propose and develop the CARR model with some discussions in Section 1. In Section 2, an empirical example is shown using the S&P500 index to estimate the model. In Section 3, we provide out-of-sample forecast comparisons between the CARR and the GARCH model. Section 4 concludes with considerations on future extensions of CARR.

1. MODEL SPECIFICATION, ESTIMATION, AND PROPERTIES

1.1 The Model Specification, Stochastic Volatilities and the Range

Let P_t be the logarithmic price of a speculative asset, possibly driven by a geometric Brownian motion with stochastic volatilities. We focus our analysis in this paper on the range measured at discrete intervals (e.g., daily, weekly) for an asset price with a discrete-path sampled at finer intervals (e.g., every 5 minutes). We define the observed range, as

$$R_t \equiv \mathrm{Max}\{P_\tau\} - \mathrm{Min}\{P_\tau\} , \tag{1}$$

$$\tau = t - 1, t - 1 + \frac{1}{n}, t - 1 + \frac{2}{n}, \ldots, t .$$

The parameter n is the number of intervals used in measuring the price within each range-measured interval, which is normalized to be unity. The bias of range will be a non-increasing function of n. Namely, the finer the sampling interval is of the price path, the more accurate the measured range will be.

Since the price process is in natural logarithm, we can define r_t as the one period ($t - 1$ to t) continuously compounding return,

$$r_t = P_t - P_{t-1} . \tag{2}$$

It is a well-known result in statistics that the range is an estimator of σ_t, the standard deviation of the random variable. From the results of Parkinson (1980) and Lo (1991), the range of any distribution is proportional to its standard deviation.

We hereby posit a dynamic specification, the Conditional Autoregressive Range (or CARR) model for the range:

$$R_t = \lambda_t \varepsilon_t , \tag{3}$$

$$\lambda_t = \omega + \sum_{i=1}^{q} \alpha_i R_{t-i} + \sum_{j=1}^{p} \beta_j \lambda_{t-j} ,$$

$$\varepsilon_t \,\big|\, I_{t-l} \sim f(l, \xi_t) ,$$

where λ_t is the conditional mean of the range based on all information up to time t. The distribution of the disturbance term ε_t, or the normalized range $\varepsilon_t = R_t/\lambda_t$, is assumed to be distributed with a density function $f(.)$ with a unit mean. The coefficients (ω, α_i, β_j) in the conditional mean equation are all positive to ensure positivity of λ_t. The specification of the model has implied some restriction upon the conditional moments of the variable. If the disturbance is i.i.d., then the conditional variance of the range is proportional to the square of its conditional expectation. In fact, this is a property shared by all models with multiplicative errors; see Engle (2002). If it is not i.i.d., then a non-negative distribution with a unit mean and time-varying variance can be specified.

Note that ε_t is positively valued given that both the range R_t and its expected value λ_t are positively valued. A natural choice for the distribution is the exponential as it has non-negative support. Assuming that the distribution follows an exponential distribution with unit mean then the log likelihood function can be written as

$$L(\alpha_i, \beta_j; R_1, R_2, \ldots R_T,) = -\sum_{t=1}^{T}\left[\ln(\lambda_t) + \frac{R_t}{\lambda_t}\right]. \tag{4}$$

Such a model will be called the ECARR model. The second equation in (3) specifies a dynamic structure for λ_t, characterizing the persistence of shocks to the range of speculative prices or what is usually known as volatility clustering as documented by Mandelbrot (1963). The parameters ω, α_i, and β_j characterize the inherent uncertainty in range, the short-term impact effect, and the long-term effect of shocks to the range (or the volatility of return), respectively. The sum of the impact parameters, $\sum_{i=1}^{q}\alpha_i + \sum_{j=1}^{p}\beta_j$, plays a role in determining the persistence of range shocks. See Bollerslev (1986) for a discussion of the parameters in the context of GARCH.

The model is called a Conditional Autoregressive Range model of order (p, q), or CARR(p, q). For the process to be stationary, a condition is that the characteristic roots of the polynomial are outside the unit circle, or $\sum_{i=1}^{q}\alpha_i + \sum_{j=1}^{p}\beta_j < 1$. The unconditional (long-term) mean of range, denoted ω-bar, is calculated as $\omega/(1 - (\sum_{i=1}^{q}\alpha_i + \sum_{j=1}^{p}\beta_j))$.

The equation of the conditional expectation of range can in general be easily extended to incorporate other explanatory variables, namely, $X_{t,l}$, for $l = 1,2 \ldots L$, that are I_{t-1}-adapted.

$$\lambda_t = \omega + \sum_{i=1}^{q}\alpha_i R_{t-i} + \sum_{j=1}^{p}\beta_j \lambda_{t-j} + \sum_{l=1}^{L}\gamma_l X_{t-1,l}. \tag{5}$$

This model is called the CARR model with exogenous variables, or CARRX. It will be called an ECARRX model if the model is estimated with an assumed exponential distribution for the disturbance. Among others, some important exogenous variables are trading volume (see Lamoureux and Lastrapes, 1990, Karpoff, 1987), the lagged returns capturing the leverage effect frequently observed in equity markets, and some seasonal factors that characterize the seasonal pattern within the range interval.

This model is similar to the ACD model of Engle and Russell (1998) for durations between trades, and belongs to the family of Multiplicative Error Model in Engle (2002). Nonetheless, there are essential distinctions between the ACD and the CARR models. First, duration is measured at some random intervals, but the range is measured at fixed intervals; hence, the natures of the variables of interest are different although they share the common property that all observations are positively valued. Secondly, the CARR model is a model for range, but it can also be used as a model for volatility.

1.2 Properties of CARR: Estimation and Relationships with Other Models

The ACD and CARR models have some analogous statistical properties. Furthermore, the CARR model has some unique properties of its own. We illustrate some of the important properties in this subsection. First, a consistent estimation of the parameters can be obtained by the Quasi-Maximum Likelihood Estimation or QMLE method. Engle and Russell (1998) prove that under some regularity conditions, the parameters in the CARR model can be estimated consistently by QMLE in which the density function of the disturbance term ε_t is given by a unit mean exponential density function. See also Engle (2002) for further discussions.

The intuition behind this property relies on the insight that the likelihood function in ACD (and CARR) with an exponential density is identical to the GARCH model with a normal density function, but with some simple adjustments on the specification of the conditional mean. Furthermore, all asymptotic properties of GARCH are applicable to CARR. Given that CARR is a model for the conditional mean, the regularity conditions (e.g., the moment condition) are in fact less stringent than in GARCH. The details of this and some related issues are not dealt with in this paper.

A convenient property for CARR is the ease of estimation. Specifically, the QMLE estimation of the CARR model can be obtained by estimating a GARCH model with a particular specification: specifying a GARCH model for the square root of range without a constant term in the mean equation.[6] This property is related to the above QMLE property by the observation of the equivalence of the exponential distribution's likelihood functions in CARR and ACD and the observation of the normal density in GARCH. It is important to note that the direct application of QMLE will not yield consistent estimates for the covariance matrix of the parameters. The standard errors of the parameters are consistently estimated by the robust method of Bollerslev and Wooldridge (1992).

Notice that although the exponential density specification can yield consistent estimation, it is not efficient. The efficiency result can be attained only if the conditional density is correctly specified. Hence, in our estimation, we also attempt to estimate the model with a more general density function, the Weibull distribution. In this case the log likelihood function can be expressed as

6. See Engle and Russell (1998) for a proof. Hence, any software that is capable of estimating the GARCH model can be used to estimate the CARR model.

$$L(\alpha_i, \beta_j, \theta; R_1, R_2,...R_T,) = \sum_{t=1}^{T} \ln\left(\frac{\theta}{R_t}\right) + \theta \ln\left(\frac{\Gamma(1 + 1/\theta)\, R_t}{\lambda_t}\right) - \left(\frac{\Gamma(1+1/\theta)R_t}{\lambda_t}\right)^{\theta} . \qquad (6)$$

It is important to note that the Weibull distribution includes as a special case the exponential distribution when $\theta = 1$. Otherwise, the transformed error, $(R_t/\lambda_t)^{\theta}$, will have an exponential distribution. This fact can be used in testing the validity of the distribution specifications. A CARR (CARRX) model with the Weibull distribution will be called a WCARR (WCARRX) model.[7]

Another interesting property of the CARR model is its relationship with the GARCH family models. In Taylor (1986) and Davidian and Carroll (1987), a model estimating the standard deviation of stock price is proposed. This model uses the absolute value of the return as an instrument to estimate the volatility of asset prices. We label it the standard deviation GARCH model. It is interesting to notice that the standard deviation GARCH model turns out to be the same as a CARR model if the specification of the mean equation is ignored. This property follows from the observation that, with $n = 1$, the range R_t is identical to the absolute value of the return, r_t, i.e., $R_t = \text{Max}(P_{t-1}, P_t) - \text{Min}(P_{t-1}, P_t) = |P_t - P_{t-1}| = |r_t|$. Hence, the CARR model is directly linked to one of the most useful GARCH models.

There is an issue of fairness concerning the forecast comparison given the consideration of the difference in the information used in the two models. The information set used in CARR includes the GARCH as a subset given that GARCH only uses the closing prices of the interval, say a day, and CARR uses the whole price path in the interval in computing the range variable. Such a comparison is exactly what is made in the static range literature of Parkinson (1980) and others. As a result, a comparison of this should always be interpreted with caution. It would also be interesting to compare the CARR model with a model that utilizes the same information set.

2. AN EMPIRICAL EXAMPLE USING THE S&P500 INDEX

2.1 The Data Set

We collect the daily index data of the Standard and Poors 500 (S&P500) for the sample period from April 26, 1982 to October 17, 2003.[8] For each day, four

7. A feasible alternative estimation method is the GMM using moments in autocorrelations of ranges and their squares. Still another alternative is to take log on both sides of Equation (3), then the current multiplicative specification is transformed into a non-linear regression model with additive innovations. As suggested by Fourgeaud, Gourieroux, and Pradel (1988), a non-linear least squares can be used for estimation. I thank an anonymous referee for raising these points to me.

8. We use a longer sample period starting from the year 1962 in the previous version of the paper. We notice, however, a clear in the structure of the data around April of 1982. We hence decide to start our sample period from the beginning of May 1982. This is to avoid the unnecessary error caused by the changes of the data compilation procedure. Indeed, such a change occurred around the end of April, 1982, according to a telephone conversation with the S&P Inc.

pieces of the price information, open, close, high, low, are reported. The data set is downloaded from the finance subdirectory of the website "Yahoo.com". We estimate the model using both the daily and the weekly frequencies. The estimation results using the daily data are qualitatively the same as the results using weekly data. Some weekday seasonal effects are found for the daily range data. However, to save space without losing much information, we report only the estimation results using the weekly frequency. The daily estimation results are available from the author upon requests.

2.2 *Empirical Estimation of the Weekly Range*

The weekly range is obtained by taking the difference of the highest price (over the five daily highs) and the lowest price (of the five daily lows) that occurred throughout the week. For comparison, we also compute the weekly return series by taking the first difference on the log price series on the closing day of each week. The total number of observations is 1120. Figure 1 presents the plot of the weekly return in the upper panel and the weekly range in the lower panel. Table 1 gives summary statistics of these two series and the series of absolute value of the return. The kurtosis coefficient of the return is 6.38 indicating a strong deviation from the normal distribution. It is interesting to observe the difference in the values of the ACF's and of the Ljung–Box Q statistics for the absolute return and the range series. The Q statistics are 1564.7 for the range and 191.5 for the absolute returns indicating a much stronger degree of persistence in volatility for the range than for the absolute return series. One target in our modeling is to explain away this high degree of persistence by estimating the conditional mean of the range.

The estimation results are reported in Tables 2 and 3 for the ECARR and for the WCARR model, respectively. We estimate variations of the CARR models. Specifically, we estimate four models: CARR(1,1), CARR(2,2), CARRX(1,1)-a, and CARRX(1,1)-b. In the CARRX models, exogenous variables in the conditional range equation include combinations of two variables: a lagged return r_{t-1}, and a lagged absolute return. The lagged return variable is used (in both CARRX specifications) for consideration of the leverage effect of Black (1976) and Nelson (1991). The lagged absolute return is used in the standard deviation GARCH model in the volatility equation. It is included in the CARRX(1,1)-a model to check whether it provides additional information about the volatility in addition to the lagged ranges.

For each of the model estimations, we compute two diagnostic test statistics $Q(12)$ and W2. The W2 statistic is an empirical distribution test by the Cramer–von Mises test. This is based on a comparison of the hypothesized distribution function with the empirical distribution function. For detailed discussion of this testing method and other alternatives, see Stephens (1986). As is stated earlier, the exponential and the Weibull distributions are hypothesized in the ECARR (Table 2) and in the WCARR (Table 3), respectively.

We first discuss the estimation result in Table 2. It suggests that a simple dynamic structure we consider is satisfactory. In other words, the likelihood functions indicate that $p = 1$ and $q = 1$ for the CARR model is sufficient over the entire data period

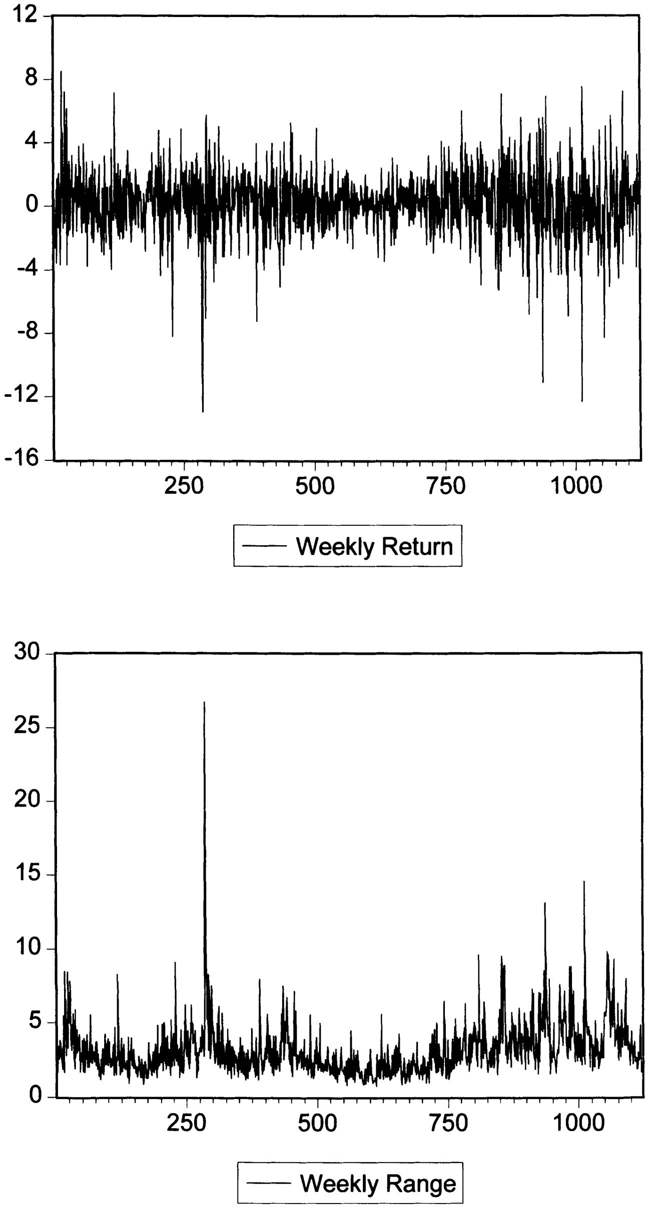

FIG. 1. S&P500 Index Weekly Returns and Ranges, 4/26/1982–10/13/2003

TABLE 1

Summary Statistics for the Returns and Ranges of Weekly S&P500 Index, April 26, 1982 to October 13, 2003

	Return	Absolute return	Range
Mean	0.195	1.675	3.146
Median	0.363	1.331	2.663
Maximum	8.462	13.007	26.698
Minimum	−13.007	0.002	0.707
Standard deviation	2.222	1.472	1.828
Skewness	−0.556	2.309	3.287
Kurtosis	6.382	12.390	30.424
Jarque–Bera	591.3	5109.8	37147.6
Probability	0.000	0.000	0.000
	Auto-Correlation Function (lag)		
ACF (1)	−0.062	0.207	0.530
ACF (2)	0.068	0.101	0.426
ACF (3)	−0.031	0.147	0.386
ACF (4)	−0.037	0.087	0.356
ACF (5)	−0.011	0.064	0.311
ACF (6)	0.082	0.130	0.348
ACF (7)	−0.024	0.142	0.326
ACF (8)	−0.029	0.101	0.285
ACF (9)	−0.012	0.101	0.233
ACF (10)	−0.006	0.105	0.277
ACF (11)	0.059	0.091	0.250
ACF (12)	−0.023	0.083	0.225
$Q(12)$	26.3	191.5	1564.7

spanning more than twenty years.[9] Note also that in the CARR(2,2) estimation results, neither α_2 nor β_2 are significant. This is consistent with the results using the likelihood ratio test. The reduction of the Box–Ljung Q statistics in all four models, when compared to that of the raw range data, is phenomenal. They are reduced from the raw data of 1564.7 (see Table 1) to the levels of 14.6, 14.594, 12.579, and 12.308 for the four models, respectively. They are all insignificant at the 5% level.

The significance level of the leverage effect in the two models with exogenous variables is noteworthy. The absolute values of the t-ratios are 5.11 and 5.62 in model CARRX(1,1)-a and CARRX(1,1)-b, respectively. They correspond to a significance level with p-values less than 0.001%. This contrasts the GARCH literature in which the leverage effect is often reported to be significant at some mediocre level, say 5% but not 1% or 0.001%. Our conjecture about this contrast is the fact that range is observable and the volatility level in GARCH is not observable. Hence, the estimation error in the GARCH model will cause the reduction of the statistical significance of the leverage effect. Another conjecture to be verified is that GARCH uses the variance to measure volatility while CARR uses the range, a proxy for

9. This is consistent with the GARCH literature where a GARCH(1,1) specification is sufficient for a large class of speculative asset returns. See Bollerslev, Chou, and Kroner (1992).

TABLE 2

ESTIMATION OF THE CARR MODEL USING WEEKLY S&P500 INDEX WITH EXPONENTIAL DISTRIBUTION, APRIL 26, 1982 TO OCTOBER 13, 2003

$$R_t = \lambda_t \varepsilon_t$$

$$\lambda_t = \omega + \sum_{i=1}^{q} \alpha_i R_{t-i} + \sum_{j=1}^{p} \beta_j \lambda_{t-j} + \gamma r_{t-1} + \delta |r_{t-1}|$$

$$\varepsilon_t \sim \text{iid} f(.),$$

where R_t is the range and r_t the return. Estimation is carried out using the QMLE method hence it is equivalent to estimating an Exponential CARR(p, q) or an ECARR(p, q) model. LLF is the log likelihood function, $Q(12)$ is the Ljung–Box statistic for auto-correlation test with 12 lags and W2 is the Cramer–von Mises statistic for empirical distribution test. Numbers in parentheses are robust standard errors (p-values) for the model coefficients ($Q(12)$ and W2).

	ECARR(1,1)	ECARR(2,2)	ECARRX(1,1)-a	ECARRX(1,1)-b
LLF	−2204.887	−2204.825	−2199.039	−2199.062
ω	0.139 (0.034)	0.152 (0.097)	0.212 (0.037)	0.207 (0.036)
α_1	0.242 (0.031)	0.262 (0.046)	0.256 (0.033)	0.236 (0.027)
α_2		0.023 (0.183)		
β_1	0.714 (0.034)	0.415 (0.683)	0.697 (0.031)	0.705 (0.031)
β_2		0.252 (0.495)		
γ			−0.096 (0.019)	−0.097 (0.017)
δ			−0.025 (0.038)	
$Q(12)$	14.790 (0.253)	15.081 (0.237)	12.536 (0.404)	12.319 (0.420)
W2	40.355 (0.000)	40.414 (0.000)	41.529 (0.000)	41.479 (0.000)

standard deviation, to measure volatility. In general, the square of a dependent variable often reduces the explanatory power in regression models.[10]

As can be seen from the results of estimation in CARRX(1,1)-a, the lagged absolute return series provides no extra explanatory power over the conditional range in addition to the lagged ranges. This can be viewed as a confirmation of the above discussion that the standard deviation GARCH model is a special case of the CARR model.

The empirical distribution test results indicate clear rejection of the hypothesized exponential distribution. In all four models, the Cramer–von Mises tests are all very large. This indicates that the exponential distribution is not supported by the data. Figure 2 provides a kernel density estimation of the residuals from the CARR(1,1) model. The exponential density function is monotonically declining. The shape of the empirical distribution indicates a clear deviation from the exponential function especially for the small range values, or inliers. By allowing one additional parameter, the Weibull distribution is potentially capable of solving this problem. We now turn to Table 3 for the estimation result using the Weibull specification.

We first notice that the estimates of the parameter of transformation, θ, are in the neighborhood of 2.4–2.5 and are very significantly different from one. Hence, the

10. As is noted by a referee, the negative sign may potentially cause the conditional range to be non-positive. A way to solve this potential problem is to adopt a log-range formulation in the same way as in the EGARCH model. This is done in Brandt and Jones (2002).

TABLE 3

ESTIMATION OF THE CARR MODEL USING WEEKLY S&P500 INDEX WITH WEIBULL DISTRIBUTION APRIL 26, 1982 TO OCTOBER 13, /2003

$$R_t = \lambda_t \varepsilon_t$$

$$\lambda_t = \omega + \sum_{i=1}^{q} \alpha_i R_{t-i} + \sum_{j=1}^{p} \beta_j \lambda_{t-j} + \gamma r_{t-1} + \delta \, |r_{t-1}|$$

$$\varepsilon_t \sim \text{iid} f(.)$$

where R_t is the range and r_t the return. Estimation is carried out using the MLE method assuming a Weibull distribution for the disturbance. LLF is the log likelihood function, $Q(12)$ is the Ljung–Box statistic for auto-correlation test with 12 lags and W2 is the Cramer–von Mises statistic for empirical distribution test. Numbers in parentheses are robust standard errors (p-values) for the model coefficients ($Q(12)$ and W2).

	WCARR(1,1)	WCARR(2,2)	WCARRX(1,1)-a	WCARRX(1,1)-b
LLF	−1810.485	−1810.363	−1781.963	−1782.092
ω	0.180 (0.040)	0.173 (0.146)	0.251 (0.042)	0.256 (0.041)
α_1	0.309 (0.017)	0.314 (0.017)	0.254 (0.031)	0.268 (0.018)
α_2		−0.011 (0.252)		
β_1	0.636 (0.022)	0.615 (0.838)	0.666 (0.026)	0.659 (0.023)
β_2		0.028 (0.546)		
γ			−0.115 (0.010)	−0.115 (0.010)
δ			0.017 (0.027)	
θ	2.403 (0.047)	2.402 (0.048)	2.474 (0.048)	2.473 (0.047)
$Q(12)$	16.889 (0.154)	16.218 (0.181)	14.943 (0.245)	15.196 (0.231)
W2	6.152 (0.000)	6.179 (0.000)	6.208 (0.000)	6.238 (0.000)

data seem to support a Weibull alternative over the null of an exponential distribution. Otherwise, the estimation results are similar to those in Table 2. Specifically, a CARR(1,1) specification is preferred to the alternative of CARR(2,2), the leverage effect is very significant and there is no additional explanatory power provided by

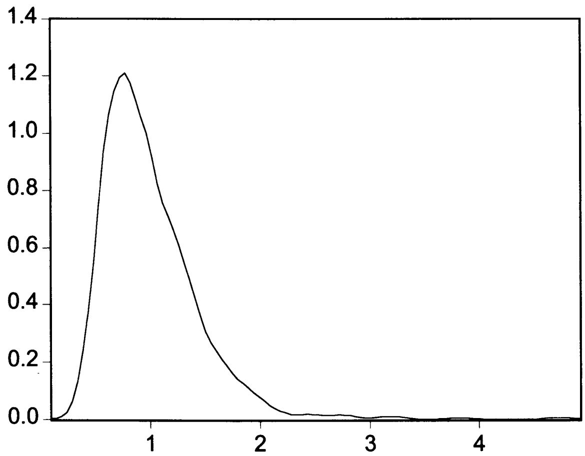

FIG. 2. Residual Density: ECARR(1,1)

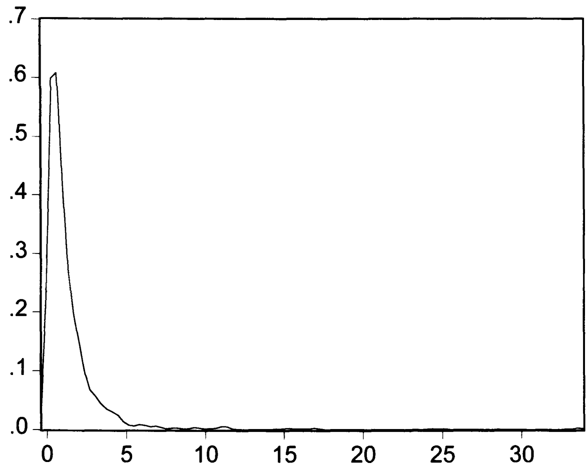

FIG. 3. Transformed Residual Density: WCARR(1,1)

the lagged absolute returns. The Ljung–Box statistics are slightly higher than their counter parts in the exponential function specifications. The Cramer–von Mises statistic, W2 are still significant although they are dramatically reduced in their sizes from the neighborhood of 40–42 to 6.1–6.3. Figure 3 gives the kernel density of the transformed residual $(R_t/\lambda_t)^{\theta}$. It is now much closer to the exponential density function in the sense that the problems of inliers seem to be a lot less serious. There is still however, clear room left for further improvements. This is a potential fruitful topic for future research.

To further check the difference of the two error distribution specifications, we compute the correlation coefficients between the expected ranges for each of the four specifications. These are the correlations of the in-sample forecasts given by the two error specifications. They are 0.994, 0.993, 0.996, and 0.997 for the CARR(1,1), CARR(2,2), CARRX(1,1)-a and CARRX(1,1)-b, respectively. This seems to indicate that the error specification between the two alternatives (exponential or Weibull) does not have much impact on the forecasts of range. This result is consistent with that of Engle and Russell (1998) in the study of dynamic modeling of trading durations.

3. OUT-OF-SAMPLE VOLATILITY FORECAST COMPARISON

To gauge the differences in the forecasting power between CARR and GARCH, we perform out-of-sample forecasts and make comparisons using different methods. Our benchmark GARCH model is a symmetric model with conditional normal distribution. Hence, on equal ground, we use an ECARR model instead of a WCARR or an ECARRX model in the forecast comparison. We choose h, the forecast horizons to be from 1 week to 50 weeks. Rolling sample estimations are made to estimate the parameters for an ECARR(1,1) model and for a standard GARCH(1,1) model. For the ECARR(1,1) model, the weekly range series is used for estimation to make forecasts for the ranges. For the GARCH(1,1) model the weekly return series is used and the forecasts for conditional variances are made. In each of the sample estimations, 972 weeks of data prior to the forecast interval are used. The first end date is December 4, 2000 and the last end date is October 28, 2002, the first end date +99 weeks. For each forecast horizon, 100 out-of-sample forecasts are made and forecasts for all horizons are made for the same 100 end dates. For the one-step forecasts, the first forecast is on December 11, 2000 and the last is on November 4, 2002. The first 50-step forecast is made on November 19, 2001 and the last made on October 13, 2003.

We use four measures of the ex post volatility: the sum of squared daily returns (SSDR), weekly return squared (WRSQ), weekly range (WRNG), and absolute weekly return (AWRET).[11] The measure SSDR is obtained by aggregating the squared daily returns within each week; see Poterba and Summers (1986) for one of the first serious attempts in computing monthly volatilities using this procedure. This method is adopted by French, Schwert, and Stambaugh (1987) and recently by Andersen et al. (2000). In the latter work, it is named the "realized volatility."

Out of the four "measured volatilities" (denote MV_t), the first two measure the variance while the last two measure the standard deviation with and without a scale adjustment. It is clear that a GARCH model should be good in forecasting the variable WRSQ, because it is precisely the variable used in the variance equation of the GARCH model. Similarly, ECARR should have advantages in forecasting WRNG for exactly the same reason. Given the difference in the target forecasts in the four measures of volatilities, we conduct transformations on the estimated volatility from the two models for FVs, the forecasted volatilities. In other words, the GARCH volatility forecasts, FV(GARCH), are the conditional variances of the return series in forecasting SSDR and WRSQ, but they are the conditional standard deviations (by taking the square root of the conditional variances) in forecasting WRNG and AWRET. Similarly, for the ECARR model, the expected (or the conditional mean of) range is used in forecasting WRNG and AWRET, while a "squared" expected range is used in forecasting SSDR and WRSQ.

We compute the root-mean-squared-errors (RMSE) and the mean-absolute-errors (MAE) i.e.,

$$\mathrm{RMSE}(m, h) = \left[T^{-1} \sum_{t=1}^{T} \left(\mathrm{MV}_{t+h} - \mathrm{FV}_{t+h}(m)\right)^2\right]^{0.5}, \tag{7}$$

$$\mathrm{MAE}(m, h) = T^{-1} \sum_{t=1}^{T} \left(|\mathrm{MV}_{t+h} - \mathrm{FV}_{t+h}(m)|\right), \tag{8}$$

11. In this paper, we do not compare the evaluation of the forecasts of the daily volatility, because a serious comparison of such would require the use of intra-daily data and it is beyond the scope of this paper.

TABLE 4

Out-of-sample Forecast Comparison for ECARR and GARCH

This table computes the root-mean-squared-errors (RMSE) and the mean-absolute-errors (MAE) using the following equations:

$$\mathrm{RMSE}(m,h)=\left[T^{-1}\sum_{t=1}^{T}\left(\mathrm{MV}_{t+h}-\hat{\mathrm{M}}\mathrm{V}_{t+h}(m)\right)^{2}\right]^{0.5}$$

$$\mathrm{MAE}(m,h)=T^{-1}\sum_{t=1}^{T}\left(|\mathrm{MV}_{t+h}-\hat{\mathrm{M}}\mathrm{V}_{t+h}(m)|\right),$$

where T = 100, m = ECARR, GARCH. The four measured volatilities (MV_t): SSDR, WRSQ, WRNG, AWRET are the sum of squared daily returns over the week, the weekly return squared, the weekly range, and the absolute weekly return, respectively. An ECARR(1,1) model is fitted for the weekly range series and a GARCH(1,1) is fitted for the weekly return series. The data used are S&P500 stock index from April 26, 1982 to October 13, 2003. Rolling samples of 972 observations are used in fitting the two models and 100 observations are made for the out-of-sample forecasts.

	SSDR		WRSQ		WRNG		AWRET	
Horizon	ECARR	GARCH	ECARR	GARCH	ECARR	GARCH	ECARR	GARCH
				RMSE				
1	9.262	11.329	18.999	19.310	1.956	2.263	2.019	2.058
2	9.955	11.820	19.242	19.653	2.055	2.358	2.044	2.091
4	11.225	12.597	19.565	19.792	2.238	2.452	2.074	2.106
8	11.231	12.397	19.524	19.800	2.394	2.561	2.085	2.125
13	11.593	12.675	19.598	19.760	2.480	2.595	2.103	2.131
				MAE				
1	6.759	8.015	9.619	9.878	1.374	1.640	1.492	1.485
2	7.328	8.440	9.708	9.995	1.442	1.687	1.499	1.493
4	7.835	8.653	9.474	10.072	1.605	1.724	1.473	1.483
8	7.745	8.715	9.027	9.984	1.690	1.864	1.433	1.495
13	7.431	8.853	8.922	10.061	1.752	1.919	1.420	1.496

where $T = 100$, $\mathrm{MV}_t = \mathrm{SSDR}_t$, WRSQ_t, WRNG_t, or AWRET_t. $\mathrm{FV}_t(m)$ are forecasted volatilities using model m, and m stands for model ECARR or GARCH.

To save space we only report cases with h = 1, 2, 4, 8, and 13 weeks. Results for longer horizons (h = 26 and 50) are available in the working paper. Table 4 gives the result of these two forecast evaluation criteria. Both criteria give almost unanimous support for the ECARR model over GARCH. For RMSE, the values are smaller for the ECARR model for 20 out of the 20 (four measures and five different horizons) cases. For MAE, again, for all cases, the ECARR model has smaller values than the GARCH model. A closer examination of the evaluation reveals that the differences in the performance of the two models are more obvious when SSDR and WRNG are used for the measured volatility and for shorter horizons. Given the fact that SSDR and WRNG use more information (daily) than WRSQ and AWRET (weekly information), it is not surprising that they contain less noise and will yield more precise pictures in forecast comparisons.

To gain further insight into the difference of the two competing volatility models, we follow the approach of Mincer and Zarnowitz (1969) in running the regressions:

$$\mathrm{MV}_{t+h} = a + b\,\mathrm{FV}_{t+h}\,(\mathrm{ECARR}) + u_{t+h}\,, \tag{9}$$

$$\mathrm{MV}_{t+h} = a + c\,\mathrm{FV}_{t+h}\,(\mathrm{GARCH}) + u_{t+h}\,. \tag{10}$$

A test of the unbiasedness of the forecasted volatility FV_t(CARR) (FV_t(GARCH)) can be performed by a joint test of $a = 0$ and $b = 1$ ($c = 1$). Given the consideration of the scale factor in the CARR model as discussed above, FV_t(ECARR) will not have a coefficient of unity even if it is unbiased.[12] We hence focus mainly on the comparison of predictive powers of the two competing models. The heteroskedasticity-autocorrelation-consistent standard errors are computed using the Newey–West (1987) procedure. As for the lag length specification, we follow the suggestion in their work by choosing it to be $(4(T/100)^{2/9})$. Further adjustments are made for parameter estimation error by adopting the correcting procedure suggested by West and McCracken (1998). Specifically, the standard errors are multiplied by a quantity called λ-hat in their work. Given that we adopt rolling-samples method, λ-hat $= 1 - (\pi^2/3)$, where $\pi = 100/972$, is the ratio of number of predictions to the QMLE estimation sample size. We also calculate the R-squared values for each regression to gauge the explanatory power of the regressors. Table 5 gives the results of the Mincer–Zarnowitz regressions using each of the four measured volatilities as the volatility proxy. To save space, we only report cases with 1-week, 2-week, and 8-week ahead forecasts.

The results of the regression-based comparison are very interesting. As is consistent with the results in Table 4, the two noisy measured volatilities, WRSQ and AWRET, are difficult to forecast; the R-squared are all less than 0.03. The other two better proxies, SSDR and WRNG, yield much higher R-squared values up to 0.317 and 0.224, respectively. Further, for all four volatility proxies and for $h = 1$ and $h = 2$, ECARR dominates GARCH in producing higher R-squared values and higher t-ratios with the "right" sign. For the one-step forecast and for the two better volatility proxies, the difference in R-squared is in the range of about six to eight times. For the case $h = 8$, neither model predictions have significant explanatory powers and the dominance of ECARR over GARCH is less obvious. These results are consistent with those of Day and Lewis (1992), Andersen and Bollerslev (1998), and Brandt and Jones (2002).

To determine the relative information content of the two volatility forecasts we also run a forecast encompassing regression:

$$\mathrm{MV}_{t+h} = a + b\,\mathrm{FV}_{t+h}\,(\mathrm{ECARR}) + c\,\mathrm{FV}_{t+h}\,(\mathrm{GARCH}) + u_{t+h}\,. \tag{11}$$

12. By multiplying the constant to the expected range, we can obtain an unbiased estimator for the standard deviation. However, we do not do this in this paper, given that the focus is more on the comparison of the two forecasts simultaneously.

TABLE 5

Out-of-sample Predictive Power for ECARR and GARCH Forecasts

This table performs the Mincer/Zarnowitz regressions. The dependent variable is one of the four measured volatilities (MV): SSDR, WRSQ, WRNG, and AWRET. The independent variable is the out-of-sample forecasts of the volatility using either the ECARR(1,1) model on ranges or the GARCH(1,1) model on returns. Numbers in parentheses are heteroscedasticity-autocorrelation consistent standard errors using the Newey–West procedure and also corrected for parameter estimation error proposed by West and McCracken (1998). The data used are S&P500 stock index from April 26, 1982 to October 13, 2003. Rolling samples of 972 observations are used in fitting the two models and 100 observations are made for the out-of-sample forecasts.

$$MV_{t+h} = a + b\, FV_{t+h}\,(\text{ECARR}) + u_{t+h}$$

$$MV_{t+h} = a + c\, FV_{t+h}\,(\text{GARCH}) + u_{t+h}$$

Horizon	Intercept	FV(ECARR)	FV(GARCH)	*R*-squared
		SSDR		
1	−0.12 (2.07)	0.54 (0.12)		0.317
	6.27 (3.39)		0.54 (0.46)	0.043
2	1.64 (2.10)	0.47 (0.14)		0.214
	8.65 (3.10)		0.27 (0.38)	0.011
8	8.83 (4.44)	0.12 (0.24)		0.007
	13.66 (3.33)		−0.32 (0.31)	0.017
		WRSQ		
1	5.73 (3.19)	0.17 (0.11)		0.011
	9.12 (3.12)		0.01 (0.25)	0.000
2	7.78 (3.50)	0.06 (0.10)		0.001
	12.90 (4.37)		−0.44 (0.27)	0.010
8	12.87 (5.93)	−0.22 (0.25)		0.008
	15.05 (5.19)		−0.67 (0.38)	0.023
		WRNG		
1	0.88 (0.68)	0.86 (0.16)		0.224
	2.72 (1.09)		0.66 (0.37)	0.039
2	1.41 (0.74)	0.74 (0.17)		0.154
	3.84 (1.17)		0.26 (0.37)	0.006
8	4.20 (1.66)	0.09 (0.38)		0.037
	6.42 (1.29)		−0.65 (0.41)	0.090
		AWRET		
1	1.16 (0.55)	0.25 (0.12)		0.023
	1.95 (0.65)		0.11 (0.22)	0.001
2	1.58 (0.57)	0.15 (0.11)		0.008
	2.71 (0.75)		−0.16 (0.21)	0.003
8	2.75 (1.10)	−0.12 (0.25)		0.003
	3.52 (0.90)		−0.43 (0.27)	0.020

The standard errors are computed as in Equations (9) and (10) described above. Under the null of encompassing, the *t*-ratio of the encompassed model can be used for the encompassing test. Under situations when no model is encompassing, West (2001) shows that construction of confidence intervals and test statistics can lead to wildly inaccurate inference. As is defined above, the ratio of number of predictions to the QMLE estimation sample size, $\pi = 100/972$, in our application. He shows that as $\pi \to 0$ and $T \to \infty$, it becomes legitimate to conduct inference using the usual

TABLE 6

ENCOMPASSING TESTS FOR ECARR AND GARCH FORECASTS

This table performs the forecast encompassing regressions. The dependent variable is one of the four measured volatilities (MV): SSDR, WRSQ, WRNG, and AWRET. The two independent variables are the out-of-sample forecasts of the volatility using the ECARR(1,1) model on ranges and the GARCH(1,1) model on returns. Numbers in parentheses are heteroscedasticity-autocorrelation consistent standard errors using the Newey–West procedure and also corrected for parameter estimation error proposed by West and McCracken (1998).

$$MV_{t+h} = a + b\,FV_{t+h}\,(\text{ECARR}) + c\,FV_{t+h}\,(\text{GARCH}) + u_{t+h}.$$

Horizon	Intercept	FV(ECARR)	FV(GARCH)	R-squared
		SSDR		
1	2.80 (2.09)	0.73 (0.12)	−0.79 (0.20)	0.368
2	5.07 (2.07)	0.72 (0.17)	−0.96 (0.27)	0.290
4	9.71 (2.66)	0.57 (0.18)	−1.08 (0.40)	0.146
8	11.07 (4.12)	0.42 (0.27)	−0.85 (0.25)	0.071
13	14.03 (4.80)	0.25 (0.31)	−0.78 (0.29)	0.056
		WRSQ		
1	7.71 (3.08)	0.30 (0.18)	−0.53 (0.45)	0.019
2	11.31 (4.13)	0.32 (0.15)	−0.99 (0.45)	0.029
4	14.06 (5.98)	0.10 (0.16)	−0.78 (0.38)	0.022
8	14.78 (6.13)	0.04 (0.24)	−0.73 (0.33)	0.023
13	16.72 (6.46)	−0.25 (0.34)	−0.40 (0.27)	0.024
		WRNG		
1	1.90 (0.65)	1.16 (0.22)	−0.82 (0.32)	0.256
2	2.88 (0.81)	1.21 (0.28)	−1.21 (0.46)	0.224
4	4.17 (1.13)	0.98 (0.31)	−1.29 (0.48)	0.125
8	5.47 (1.54)	0.77 (0.42)	−1.39 (0.33)	0.090
13	7.06 (1.75)	0.21 (0.52)	−1.16 (0.45)	0.090
		AWRET		
1	1.67 (0.68)	0.40 (0.19)	−0.40 (0.37)	0.032
2	2.39 (0.72)	0.41 (0.20)	−0.67 (0.38)	0.033
4	2.85(1.00)	0.25 (0.19)	−0.58 (0.30)	0.019
8	3.30 (1.09)	0.18 (0.27)	−0.61 (0.27)	0.023
13	3.92 (1.13)	−0.11 (0.35)	−0.41 (0.30)	0.026

covariance matrix. The implication is that for sufficiently small π and sufficiently large T, the usual covariance matrix will work fine.[13] See Davidson and MacKinnon (1981) for the inference procedure under the null of encompassing and West (2001) for a way in estimating the variance-covariance matrix under situations when no model is encompassing.

Table 6 gives the results of the encompassing regressions. The dominance of ECARR over the GARCH model is clear. Once the ECARR-predicted-volatility is included, the GARCH-predicted-volatility often becomes insignificant or with wrong signs. It is interesting to observe that the R-squared increases substantially for this

13. Note in particular that Table 1 in West (2001) indicates that for π π = 0.2 the usual standard errors are slightly too small, with nominal 95% confidence intervals have actual coverage of between 85% and 95%. While the example in that Table is much simpler than our application, we hope that the bias is similarly small.

regression comparing with those in Table 5. This indicates that some form of combined forecasts may be able to obtain a higher predictive power, ignoring the fact that negative coefficients are difficult to interpret. For the two better volatility proxies, SSDR and WRNG, the declining pattern of the R-squares over the increase of horizons is obvious. However, even up to 13-week ahead, about 9% of the variations in the volatility can be explained by this regression. It is interesting to note that if the target is the "average" volatility over the next h-horizons, Brandt and Jones (2002) report that the volatility is predictable as far as one year ahead using the range-based volatility model.

As a result of the above forecast evaluations, it is obvious that the ECARR model does provide sharper volatility forecasts than the standard GARCH model. Figure 4 provides a snap shot of the two alternative volatility forecasts together with the measured volatility SSDR. It is interesting to observe that the ECARR model gives a much more adaptive forecast than the GARCH model. This is consistent with the fact that ranges use more information than the close-to-close returns.

4. CONCLUSION

The CARR model provides a simple, yet efficient and natural framework to analyze the volatility dynamics. We have demonstrated empirically that CARR can produce sharper volatility estimates as compared to the commonly adopted model like standard deviation GARCH or GARCH. Further, Monte Carlo analysis is useful to gauge the efficiency gain of CARR over its rival models. Applications of CARR to other frequency of range intervals, say every hour, or every quarter, and other frequencies, will provide further understanding of the performance of the range

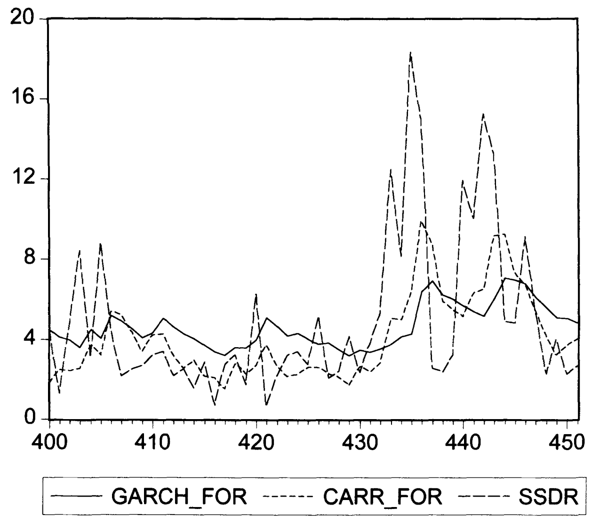

FIG. 4. Volatility Forecasts: CARR vs. GARCH

model. Analyses using other asset prices, e.g., currency, fixed-income securities, and derivative assets, will also be useful. Other generalizations of the CARR model will be worthy subjects of future research, for example, the generalization of the univariate to a multivariate framework, models simultaneously treating the price return and the range data, tests of risk premium hypothesis such as in Chou (1988), long memory CARR models, asymmetric volatility models (see Chou 2004), CARR diffusion models in the spirit of Duan (1995, 1997), and value-at-risk calculations using CARR.

It is suggested in statistics that range is sensitive to outliers. Hence, it is meaningful to consider an extension of CARR by using robust measures of range to replace the standard range. For example, some plausible measures are the next-to-max and the next-to-min, the quantile range and the difference between the average of the top 5% observations and the bottom 5% observations, etc. A closely related analysis is given in Engle and Manganelli (2001).

Following the approaches in the static range literature, the CARR model can also be extended to model a time-varying drift term by incorporating the information in the opening and closing prices. This will lead to models of the volatility process using all four pieces of information: open, close, high, and low. A coherent dynamic model should provide a framework whereby the range gives volatility predictions consistent (identical) with the volatility prediction from the mean return.

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