Comparison among different approaches for risk estimation in portfolio theory

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ABSTRACT

This paper proposes new performance measures to be regarded as alternatives for the most popular measures used as criterions for portfolio optimization. In order to prove the forecast ability of new ratios, we analyzed some allocation problems taking into consideration portfolio selection models based on different risk perceptions. In particular, we compare several portfolio selection approaches considering the sample path of the final wealth process for each allocation problem.

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Almost forty years ago, Sharpe [1966] developed a measure, originally termed as reward-to-variability, for evaluating and predicting the performance of mutual fund managers. Subsequently, under the name of Sharpe Ratio, it has become one of the most popular indices widely used in practical applications. Although this ratio is fully compatible with normally distributed returns (or, in general, with elliptical returns), the Sharpe Ratio will lead to incorrect investment decisions when returns present kurtosis and/or skewness (see, among others, Leland [1999], Bernardo and Ledoit [2000]). Specifically, returns on assets exhibit heavy tails and many researchers recognize the limitations of this performance measure (see Ortobelli S., et al [2003]).

In the last fifty years, a variety of different criteria, alternative to the Sharpe Ratio, for optimal portfolio selection have been proposed: MiniMax ratio, Stable Ratio, MAD Ratio, Farinelli-Tibiletti Ratio, Sortino-Satchell Ratio and others (see, among others, Young [1998], Ortobelli S., et al [2003], Farinelli, Tibiletti [2003], Sharpe

[1994], Dowd [2001], Sortino [2000], Pedersen and Satchell [2002], Szegö [2004] Uryasev, [2000]). All of them are theoretically valid and lead to different optimal solutions. Hence, it is not clear how an investor should decide on a criterion.

The purpose of this work is to find such criterion whose application would lead to the correct investment decisions in case of heavy tailed distributed returns. For this reason we propose an ex-post analysis considering historical data over the last four years. In this period the stock market showed that the stock returns are volatile and far from being Gaussian distributed. In addition, we want to suggest further new mean-risk models that show a superior performance with respect to other models in the case of non-Gaussian return distributions. Thus, we introduce new Ratios which take into account a more realistic behavior of financial data.

After the comparative analysis, it became clear which criteria and risk measures are the preferred by the investor. In particular, we show that classical analysis (with the Sharpe Ratio) presents lower performance than other criteria recently proposed and the new proposed ones.

RISK MEASURES AND PERFORMANCE MEASURES

Before Markowitz' analysis, there had been no attempt to quantify risk. Financial risk was considered a correcting factor for expected return. The Markowitz' mean-variance framework was an excellent beginning. Probably, the main innovation introduced by Markowitz was to measure the dependence structure of returns while accounting for their correlation. The dependence structure, represented by the variance covariance matrix, quantifying the impact in the portfolio selection of each asset with respect to the others, marks the importance of the diversification. Markowitz' theory was extended to an equilibrium theory by Sharpe's [1964] Capital Asset Pricing Model (CAPM). The simplicity and the intuitive appeal of the mean-variance analysis and equilibrium results directed the attention toward determining their generality and extendibility. In addition, all the theory found its foundation in arbitrage theory and in stochastic dominance analysis.

Mean-variance theory survived theoretical criticism and empirical rejection. However, in a series of papers Roll ([1977], [1978] and [1979]) was the first to clearly understand the weaknesses of the theory and the empirical deficiencies. The lacks and contrastive results on empirical and theoretical mean variance analysis have represented the main justifications and reasons of the alternative mean-risk models proposed in the last decades. In particular, many researchers have investigated on the "best" risk measure to be applied on portfolio selection (see mean-lower partial moment approach Markowitz [1959], Fishburn [1977], Bawa, [1977], the mean-Gini portfolio analysis Yitzhaki [1982], Shalit, Yitzhaki [1984], Mean-absolute deviation (MAD) approach, Konno and Jamazaki [1991], the mean-stable dispersion analysis Ortobelli et al. [2003], [2004], mean-VaR type models Uryasev [2000], and Giacometti, Ortobelli [2004]). All the above mean-risk approaches admit a Capital Asset Pricing Model and can be easily extended to a multi-parameter framework according to stochastic dominance rules because they are a sub case of the analysis proposed by Ortobelli [2001] and Ortobelli et al. [2003].

In a mean-risk world we can describe the uncertainty of future wealth with two summary statistics: risk and the mean representing reward. Thus, several portfolio selection approaches differ for the risk measure used. In particular, we can distinguish two disjointed categories of risk measures used in portfolio theory: dispersion measures and safety risk measures. These categories differ in their consistency to stochastic dominance orders. In the following, we summarize the most important risk measures proposed and compared in the empirical analysis.

Absolute deviation. The absolute deviation of a random variable is the expected absolute value of the difference between the random variable and its mean. The absolute deviation is a linearizable risk measure consistent with a stochastic dominance order. The primary aim of the mean absolute deviation (MAD) portfolio optimization model, proposed by Konno [1990], Konno, Yamazaki [1991], was to overcome the limits of CAPM model. Thus, they proposed a linear optimization problem based on the absolute deviation that improves and accelerates the computation of optimal portfolios. Consequently, because of its computational simplicity in solving portfolio selection problems, the absolute deviation was intensively used in the last decade.

Stable dispersion measure. The stable dispersion measure is the scale parameter of a stable Paretian distribution. Therefore, the mean-stable dispersion approach is a distributional approach based on the assumption that returns are α stable distributed. The adoption of stable modeling in finance is undoubtedly one of the most interesting and promising ideas which has risen in this field. While alternative models based on other non-Gaussian distributions are to be found in the literature, the stable assumption has unique distinctive characteristics that make it an ideal candidate. The Functional

Central Limit Theorem for normalized sums of i.i.d. random variables theoretically justifies the stable Paretian approach. Moreover, the excess kurtosis found in Mandelbrot's [1963] and Fama's [1965] investigations led them to reject the normal assumption and propose the stable Paretian distribution as a statistical model for asset returns. Their early studies on asset returns were carried further by Ziemba [1974], Rachev and Mittnik [2000], among others, and led to a consolidation of the stable Paretian hypothesis. The stable dispersion measure is consistent with a stochastic order and it can be obtained as a function of a centered moment. Thus, in some sense, this risk measure can be seen as a generalization of the absolute deviation.

Gini risk measure. The Gini risk measure, or Gini's mean difference, is a statistic extensively used in measuring income inequality. This index of the random variable variability is based on the expected value of the absolute difference between every pair of realizations. It admits a geometrical interpretation and has been used over the past 80 years by economists to describe economic and social conditions. It was first used in Finance by Samuelson [1969] to prove that it pays to diversify risk. Yitzhaki [1982], Shalit, Yitzhaki [1984], have proposed a mean-Gini portfolio selection theory that is consistent with a stochastic dominance rule and present many analogies with mean-variance analysis. In addition, the portfolio optimization problem with the Gini risk measure is linear. This linearity simplifies the computational complexity of portfolio choice problem and this is the main advantage relative to traditional risk measures. Besides, using the Gini risk measure we can associate with each portfolio a concentration curve that classifies its risk aversion. Therefore, it is possible to identify defensive and aggressive portfolio selection strategies considering the position of their concentration curves.

Lower partial moment. All dispersion measures described above are measures of uncertainty: however uncertainty is not necessarily risk. In contrast to safety risk measures, the dispersion measures consider the positive and the negative deviations from the mean as a potential risk. Thus, in this case, over-performance relative to the mean is penalized just as much as under-performance. In order to overcome this anomaly, Markowitz [1959] proposes the semi variance as risk measure. The natural extension of semi variance is the lower power moment risk measure (see Bawa [1977], Fishburn [1977]) also called *downside risk* or *probability weighted function of deviations below a specified target return*. This risk measure depends on two parameters: i) a power index that is a proxy for the investor's degree of risk aversion; ii) the target rate of return that is the return that must be earned at minimum to accomplish the goal of funding the plan within cost constrain. To the contrary of the lower partial moment, we can consider the *upper partial moment* as a measure of reward used to value differently the excess of return.

VaR. One of the most important tasks of financial institutions is evaluation of exposure to market risks, which arise from variations in prices of equities, commodities, exchange rates, and interest rates. The dependence on market risks can be measured by changes in the portfolio value, or profits and losses. A commonly used methodology for estimation of market risks is the Value at Risk (VaR_ϑ). The main characteristic of VaR_ϑ is that of synthesizing in a single value the possible losses which could occur with a given probability $(1-\vartheta)$ in a given temporal horizon. This feature, together with the (very intuitive) concept of maximum probable loss, allows the non-expert investor to figure out how risky his/her position is and the correcting strategies to adopt. The use of VaR models is rapidly expanding. Financial institutions with significant trading and investment volumes employ the VaR methodology in their risk management operations.

CVaR. In order to select a minimal set of properties that a portfolio statistics has to satisfy to be a coherent risk measure ,Artzner et al. [1999] defined with remarkable financial insight the so called axioms of coherency and found the explicit representation of any coherent risk measure. Even if there is no doubt that provides useful information, is not a coherent risk measure and it cannot offer an exhaustive representation of an investor's preferences. The conditional value at risk (CVaR), would seem a logical escape from this problem. The CVaR, also called expected shortfall or expected tail loss, measures the expected value of portfolio returns given that the Value at Risk has been exceeded. The CVaR is a coherent risk measure and portfolio selection with the expected shortfall can be reduced to a linear optimization problem. Moreover, if we apply the CVaR concept symmetrically to the opposite part of the portfolio returns, we get a measure of portfolio reward.

MiniMax. The MiniMax risk measure was first used in portfolio selection problems by Young [1998]. It represents the maximum loss over all past observations. It is possible to show that the MiniMax risk measure is an extreme sub case of the CVaR. Therefore the MiniMax risk measure satisfies all the properties of the expected tail loss.

Power CVaR. In order to take into account the investor's degree of risk aversion, we have introduced Power CVaR. This new risk measure is the expected value of the lower partial power of the excess return given that some threshold (usually the Value at Risk) has been exceeded. Therefore, this risk measure depends on an additional parameter, *the power index*, which varies in respect to the investor's degree of risk aversion.

For each risk measure, there exists a performance measure to identify superior, ordinary and inferior performance. In particular, the ratio between the "expected excess return" and the relative risk measure is a performance measure $\rho(.)$ that investors wish to maximize. The maximization of the performance measure $\rho(.)$ determines the so called "market portfolio", because it represents the benchmark of the market. Therefore, the market portfolio found for each performance measure $\rho(.)$ is based on a diverse risk perception and sometimes on a different reward perception. Even if most of researchers' attention has been devoted to finding the "right risk measure", our empirical analysis shows that the "right reward" perception is also crucial to determine a desirable risky investment strategy. In order to avoid ambiguities, we don't believe that there exists the "perfect performance measure", because a performance measure depends only on two parameters which summarize the complexity of all admissible choices. However, we believe that some performance measures take into account the common investors' opinions better than others. As a matter of fact, most of the decision makers have some common beliefs. For example, they prefer more than less and are generally risk averse. For this reason most of the risk measures proposed in literature are consistent with at least one stochastic dominance order. Thus, if we minimize a given risk measure (consistent with a stochastic order) for a fixed expected return level, we find a portfolio that is optimal for some investors. This does not mean that this portfolio is the best choice for all the decision makers that want a return with the same mean. Similarly, when we maximize a performance measure, we determine the market portfolio which better represents a predisposition to risk and to invest for a given category of investors. These investors do not necessarily represent all the agents in the market. However, there exists a common interest of all decision makers, that is, to maximize the final wealth. Therefore, in our empirical analysis, we emphasize those performance measures that have maximized the final wealth in the last four years. In particular, we consider the performance ratios whose mathematical description will be given in the appendix.

AN EMPIRICAL COMPARISON

In order to propose an ex-post comparison among several performance ratios, we suppose to invest every day all wealth in the portfolio that maximizes a given performance measure $\rho(.)$. This market portfolio is the solution of the optimization problem (1) found in the appendix. Then we compare the sample paths of the final wealth obtained with the different approaches.

We consider daily data in the period from 01.27.1999 to 06.30.2003. In particular, we examine optimal allocation among the riskless return LIBOR and 9 returns on the German market (Adidas_salomon AG, Basf AG, Bayerishe Motoren Werke AG, Continental AG, Bayer AG, Hoechst AG, Fresenius Medical Care AG, MAN AG, Henkel KGAA). We distinguish two cases:

a) borrowing or lending is not allowed ($r_f = 0$)

b) the riskless r_f is represented by daily observations of LIBOR.

Generally in literature, different definitions of return $r_{i,t}$ (return of *i*-th asset at time *t*) are used. Observe that the results in the above optimization problems could change assuming returns $r_{i,t} = \frac{S_{i,t} - S_{i,t-1}}{S_{i,t-1}}$ (Case A) (where $S_{i,t}$ is the price of *i*-th asset at time *t*) or continuously compounded returns $r_{i,t} = \log\left(\frac{S_{i,t}}{S_{i,t-1}}\right)$ (Case B). Thus, we analyze the two cases separately.

Case A

Let us suppose that decision makers invest their wealth purchasing the market portfolio. Considering that each investor has a diverse risk perception and a different excess return perception, the market portfolio is determined by maximizing one of the performance measures in the appendix. In order to compare the efficiency of alternative performance measures, we consider the sample path of the final wealth obtained from the several approaches.

We assume that the investor has an initial wealth of $W_0 = 1$ on 01.27.2000. Thus, every day and for each performance measure $\rho(.)$, we have to solve the optimization problem (1) using the observations from the last year (the last 250 observations). Once we determine the market portfolio $x'_{M(n)}$, the final wealth after *n* days is given by

$$W_n = W_{n-1}(1 + x'_{M(n)}\, r_{(n)}) \qquad (2)$$

where $r_{(n)}$ is the vector of returns after *n* days from 01.27.2000. We consider the cases of borrowing or lending are not allowed, i.e. $r_f = 0$, and the case r_f =LIBOR.

Exhibit 1 presents graphs of the predicted final wealth of the market portfolio for the best ratio and for the Sharpe Ratio as well as values of the final wealth of the portfolio W_{870} for some ratios (after 870 capitalizations) in the case $r_f = 0$. Exhibit 2 presents graphs of the predicted final wealth of the market portfolio for the best ratio and for the Sharpe Ratio and values of the final wealth of the portfolio W_{870} for some Ratios (after 870 capitalizations) in the case where we assume r_f =LIBOR.

These *Exhibits* show that the R1-Ratio and the RG-Ratio sample paths dominate the Sharpe Ratio sample path. This means that investing the wealth in the market portfolio every day, we obtain a greater wealth assuming either the R1-Ratio approach or the RG-Ratio approach. It follows that the R1-Ratio and RG-Ratio models show a superior performance with respect to the classic Sharpe Ratio. In contrast, the other approaches do not seem to diverge significantly from the mean-variance of one, even though we observe that the optimal portfolio weights are significantly different. This result implicitly supports the hypothesis that the new Ratios above mentioned capture the distributional behavior of the data (typically the component of risk due to heavy tails) better than the classic mean-variance model.

Case B

As for case A, we examine optimal allocation among the riskless return r_f and 9 returns on the German market during the period between 01.27.1999 and 06.30.2003 (1120 observations). We consider the cases where borrowing or lending are not allowed, (i.e. $r_f = 0$) and the case r_f =LIBOR.

We assume that the investor has an initial wealth equal to 1 and initial cumulative return $CR_0 = 0$ in data from 01.27.2000. Thus, every day and for each performance measure $\rho(.)$, we have to solve the optimization problem (1) in the appendix using the observations of the last year (the last 250 observations). Once we determine the market portfolio $x'_{M(n)}$, the cumulative return CR_n after *n* days is given by

$$CR_n = CR_{n-1} + x'_{M(n)} r_{(n)} \tag{3}$$

where $r_{(n)}$ is the vector of continuously compounded returns after *n* days from 01.27.2000.

Exhibit 3 presents sample paths of cumulative returns for the best ratio and for the Sharpe Ratio and values of cumulative returns for some ratios (after 870 capitalizations) in the case where $r_f = 0$. Exhibit 3 shows that R1-Ratio, always presents better performance than the Sharpe Ratio and yields the maximum total realized return (received using R1-Ratio) at the end of the examined period is equal to 9.583%.

In order to understand the relevance of this result, we compare the cumulative returns of R1-Ratio with one of the most representative German indexes, namely DAX 30. Then we consider the sample path of cumulative return for DAX 30 during the examined period. That is $CR_n(DAX\,30) = \log\left(\frac{S^{dax}(n)}{S^{dax}(0)}\right)$ where $S^{dax}(0)$ is the price of DAX 30 in data from 01.27.2000 and $S^{dax}(n)$ is the price of the German index after *n* days from this data. Exhibit 4 shows that the forecast, based on the R1-Ratio, gives the best results during the entire period, realizing higher returns than DAX 30. Thus, if an agent has invested all his/her wealth in DAX 30 on 01.27.2000, at the end of the period on 06.30.2003, he/she has lost 16.606% of his/her initial wealth. Instead, if he/she calibrates the portfolio using the R1-Ratio every day, he/she will realize a return of 9.583%. Clearly, this analysis does not consider the transaction costs and the taxes. However, we could observe that adding proportional transaction costs, we still obtain a positive return using a strategy based on the R1-Ratio.

As we could expect, the final wealth increases when we add the riskless asset LIBOR. Exhibit 5 presents sample paths of cumulative returns for the best ratio and for the Sharpe Ratio and values of predicted total realized returns for some ratios (after 870 capitalizations) in the case we assume r_f =LIBOR. As you can see the ratio which presents the best performance is the RG-Ratio. However, Exhibit 6 shows that the R1-Ratio sometimes yields a better performance than the RG-Ratio. In any case, both ratios (R1-Ratio, RG-Ratio) realize a cumulative return greater than that of the German index. As a matter of fact, at the end of the examined period, the cumulative return obtained using R1-Ratio is 12.933%, while the return received using RG-Ratio is 17.375%.

CONCLUDING REMARKS

This work proposes and compares alternative portfolio selection models, based on different risk measures. In order to suggest the best criterion for portfolio optimization, we have conducted a comparative analysis on several recently proposed performance measures including the most popular one, the Sharpe Ratio.

Our empirical analysis of these ratios confirms that the Sharpe Ratio forecasts are not as accurate as performance measures proposed in literature such as the MiniMax

Ratio and the Farinelli-Tibiletti Ratio. Among the alternative models proposed, we suggest the use two new performance measures (R1-Ratio and RG-Ratio) as a criterion for portfolio optimization, as they yield the best performance for investors in the examined period. However, further empirical analyses with a more general theoretical discussion will be the subject of future research.

APPENDIX: PERFORMANCE RATIOS

This appendix describes different performance ratios used in portfolio selection problems. Particularly, we analyze the problem of optimal allocation among *n*+1 assets: *n* of those assets are risky with returns (continuously compounded) $r = [r_1, r_2, ..., r_n]'$, and the (*n*+1)-th asset is the risk-free asset (or a market index) with return r_f. No short selling is allowed, i.e., the portfolio risky weights x_i for every $i = 1,...,n$ and the riskless weight $\lambda = 1 - \sum_{i=1}^{n} x_i$ are greater than or equal to zero. We assume that all portfolios are uniquely determined by the mean and a risk measure consistent with some stochastic dominance order. Under these assumptions and further regularity hypotheses, all risk measures are equivalent. In addition, the investors will choose an optimal portfolio which is the linear combination of the riskless asset and an optimal risky portfolio. The optimal risky portfolio is given by the portfolio that maximizes the performance measure $\rho(.)$. Generally, this measure is a ratio between the “expected excess return” and a risk measure of portfolio return. Thus, for any performance measure $\rho(.)$, we compute the "market portfolio" $x_M{}'r$ that is the solution of the following optimization problem:

$$\max_x \rho\left(x'r\right) \quad s.t. \quad \sum_{i=1}^{n} x_i = 1;\ x_i \geq 0;\ \ i = 1,\ldots,n \tag{1}$$

However, for different performance measures we generally obtain different optimal risky portfolios because all the admissible choices are not uniquely identified by only two parameters. Therefore, the market portfolio composition $x'_M = [x_{1,M},\ldots,x_{n,M}]$ found for each performance measure $\rho(.)$ is based on a diverse risk perception and sometimes on a different reward perception. Alternatively, we could propose a portfolio selection theory depending on more than two parameters (see Ortobelli [2001]), but the complexity of the choice problem improves. In particular, we take into account the following performance ratios.

1) Sharpe Ratio (see Sharpe [1966], [1994]). The Sharpe ratio is the ratio between the expected excess return and its standard deviation. That is,

$$\rho(x'r) = \frac{E(x'r - r_f)}{STD_{x'r-r_f}} \tag{2}$$

where the risk measure $STD_{x'r-r_f}$ is the standard deviation of portfolio $x'r - r_f$.

2) Stable Ratio (see Ortobelli, et al. [2003]). The stable ratio is the ratio between the expected excess return and its stable dispersion measure. That is,

$$\rho(x'r) = \frac{E(x'r - r_f)}{\sigma_{x'r-r_f}} \tag{4}$$

where the stable risk measure is $\sigma_{x'r-r_f} = \sqrt{x'Qx}$ and the dispersion matrix $Q = [q_{ij}]$ can be estimated with the following formulas:

$$\overline{q}_{ij} = \left(\overline{q}_{jj}\right)^{\frac{1}{2}} A(1) \frac{1}{T} \sum_{k=1}^{T} \tilde{r}_{i,k} \operatorname{sgn}(\tilde{r}_{j,k}) \tag{5}$$

$$\overline{q}_{jj} = \left(A(p) \frac{1}{T} \sum_{k=1}^{T} \left| \tilde{r}_{j,k} \right|^{p} \right)^{2/p} \tag{6}$$

In these equations $\tilde{r}_{j,k} = \left(r_j - r_f\right)_k - E(r_j - r_f)$ is the *k-th* centered observation of *j-th* excess return, $A(p) = \dfrac{\Gamma(1-p/2)\sqrt{\pi}}{2^p\, \Gamma\left(\dfrac{1+p}{2}\right)\Gamma(1-p/\alpha)}$, $p \in [0,\alpha)$ is computed in order to minimize the rate of convergence of asset return series, $\alpha = \dfrac{1}{n}\sum_{i=1}^{n} \alpha_i$ is the index of stability of return vector and α_i is the index of stability of the *i-th* asset estimated using a maximum likelihood estimator.

3) MAD Ratio (see Konno, Yamazaki [1991]). The MAD ratio is the ratio between the expected excess return and its absolute deviation. That is,

$$\rho(x'r) = \frac{E(x'r - r_f)}{\sigma_{x'r - r_f}} \tag{7}$$

where the risk measure MAD is defined $\sigma_{x'r-r_f} = \dfrac{1}{T}\sum_{k=1}^{T} \left| \left(x'r - r_f\right)_k - E\left(x'r - r_f\right) \right|$ and $\left(x'r - r_f\right)_k$ points out the *k*-th observation of excess return $\left(x'r - r_f\right)$.

4) Gini Ratio (see Yitzhaki [1982] Shalit, Yitzhaki [1984]). The Gini ratio is the ratio between the expected excess return and its Gini risk measure. That is

$$\rho(x'r) = \frac{E(x'r - r_f)}{\sigma_{x'r - r_f}} \tag{8}$$

where the Gini risk measure is defined as $\sigma_{x'r-r_f} = \frac{1}{T(T-1)} \sum_{k=1}^{T} \sum_{t>k}^{T} \left| \sum_{i=1}^{n} x_i \left((r_i - r_f)_t - (r_i - r_f)_k \right) \right|$

and $(r_i - r_f)_k$ points out the k-th observation of the i-th excess return $(r_i - r_f)$.

5) Sortino-Satchell Ratio (see Sortino [2000], Sortino, Satchell, [2001] Pedersen, Satchell [2002]) The Sortino-Satchell ratio is the ratio between the expected excess return and its lower partial moment at a given minimum acceptable return (MAR). That is

$$\rho(x'r) = \frac{E(x'r - r_f)}{\sigma_{x'r}(t)} \tag{10}$$

where $\sigma_{x'r}(t) = \sqrt[q]{E\left((t - x'r)_+^q\right)} = \left(\frac{1}{T} \sum_{k=1}^{T} \left(t - x'r_{(k)}\right)_+^q \right)^{1/q}$. In the following analysis we suppose $t = r_f / 2$, q=1 and $\sigma_{x'r}(t) = \left(\frac{1}{T} \sum_{k=1}^{T} \left(t - x'r_{(k)}\right)_+ \right)$.

6) $VaR_{99\%}$ **Ratio (see Favre and Galeano [2002], Rachev S., Martin D., Siboulet F. [2003])** The $VaR_{99\%}$ Ratio is the ratio between the expected excess return and its Value at risk. That is,

$$\rho(x'r) = \frac{E(x'r - r_f)}{VaR_{99\%}\left(x'r - r_f\right)} \tag{11}$$

where $VaR_{99\%}(x'r - r_f)$ is the opposite of 1% quantile of excess return distribution, that is $P(x'r - r_f \leq -VaR_{99\%}(x'r - r_f)) = 0.01$.

7) $CVaR_{99\%}$ **Ratio (STARR Ratio) (see Rachev S., Martin D., Siboulet F. [2003])** The $CVaR_{99\%}$ Ratio is the ratio between the expected excess return and its conditional value at risk. That is,

$$\rho(x'r) = \frac{E(x'r - r_f)}{CVaR_{99\%}(x'r - r_f)} \tag{12}$$

where $CVaR_{99\%}(x'r - r_f) = -\frac{1}{[0.01 * n]} \sum_{(x'r - r_f)_k \leq -VaR_{99\%}(x'r - r_f)} (x'r - r_f)_k$ and

$VaR_{99\%}(x'r - r_f)$ is computed as above.

8) MiniMax Ratio (see Young [1998]). The MiniMax Ratio is the ratio between the expected excess return and its MiniMax risk measure. This ratio can be seen as a sub case of CVaR ratio. That is,

$$\rho(x'r) = \frac{E(x'r - r_f)}{MM_{x'r - r_f}} \tag{3}$$

where the MiniMax risk measure is given by $MM_{x'r - r_f} = \max_{1 \leq t \leq T} (r_f - x'r)_t$ and $(r_f - x'r)_t$ is the t-th observation of $(r_f - x'r)$.

Among the performance measures that are also characterized for their different excess return perception we recall the Farinelli-Tibiletti ratio, and some new risk performance measures.

9) Farinelli-Tibiletti Ratio (see Farinelli, Tibiletti [2003a-b]) The Farinelli-Tibiletti Ratio is a generalization of Omega index (see Sortino, Satchell, [2001] and the references therein). It is the ratio between an upper partial moment and a lower partial moment. That is,

$$\rho(x'r) = \frac{\sqrt[p]{E\left((x'r - t_1)_+^p\right)}}{\sqrt[q]{E\left((x'r - t_2)_-^q\right)}} \tag{9}$$

where $\sqrt[p]{E\left((x'r-t)_+^p\right)} = \left(\frac{1}{T}\sum_{k=1}^{T}\left(x'r_{(k)}-t\right)_+^p\right)^{1/p}$, $\left(x'r_{(k)}-t\right)_+^p = \left(\max(x'r_{(k)}-t,0)\right)^p$ and

$\sqrt[q]{E\left((x'r-t)_-^q\right)} = \left(\frac{1}{T}\sum_{k=1}^{T}\left(x'r_{(k)}-t\right)_-^q\right)^{1/q}$, $\left(x'r_{(k)}-t\right)_-^q = \left(\max(t-x'r_{(k)},0)\right)^q$. We use

$t_1 = r_f = t_2$, $p = \alpha/2$, $q = \alpha/2$ and α is the index of stability of return vector computed as for the stable ratio.

10) Rachev Ratio (R-Ratio) The R-Ratio is the ratio between the CVaR of the opposite of excess return at a given confidence level and the CVaR of the of excess return at another confidence level. That is,

$$\rho(x'r) = \frac{CVaR_{(1-\alpha)\%}(r_f - x'r)}{CVaR_{(1-\beta)\%}(x'r - r_f)} \qquad (13)$$

or, alternatively,

$$\rho(x'r) = \frac{ETL_{\alpha\%}(r_f - x'r)}{ETL_{\beta\%}(x'r - r_f)} \qquad (14)$$

where α and β are in [0,1]. Here, if X is a return of a given portfolio, then L=-X represents the relative loss and $ETL_{\alpha\%}(r) = E(L / L > VaR_{\alpha\%})$ is the expected tail loss where $VaR_{\alpha\%}$ is defined by $P(L > VaR_{\alpha\%}) = \alpha$, and α is in (0,1). In particular, we consider this ratio for different parameters α and β:

Rachev Ratio1 (when $\alpha = \beta = 0.01$) (R1-Ratio)

Rachev Ratio 2 (when $\alpha = \beta = 0.05$) (R2-Ratio)

Rachev Ratio 3 (when $\alpha = 0.5, \beta = 0.01$) (R3-Ratio).

11) Rachev Generalized Ratio (RG-Ratio). The RG-Ratio is the ratio between the power CVaR of the opposite of excess return at a given confidence level and the power CVaR of the excess return at another confidence level. That is,

$$\rho(x'r) = \frac{ETL_{(\gamma,\alpha\%)}(r_f - x'r)}{ETL_{(\delta,\beta\%)}(x'r - r_f)} \tag{15}$$

where $ETL_{(\gamma,\alpha\%)}(X) = E((\max(L,0)^{\gamma} / L > VaR_{\alpha\%})$ is the Power CVaR of X and γ is a positive constant. We suggest to analyze Rachev Generalized Ratio with parameters $\gamma = \delta = \frac{\alpha}{2}$, $\alpha = \beta = 0.5$.

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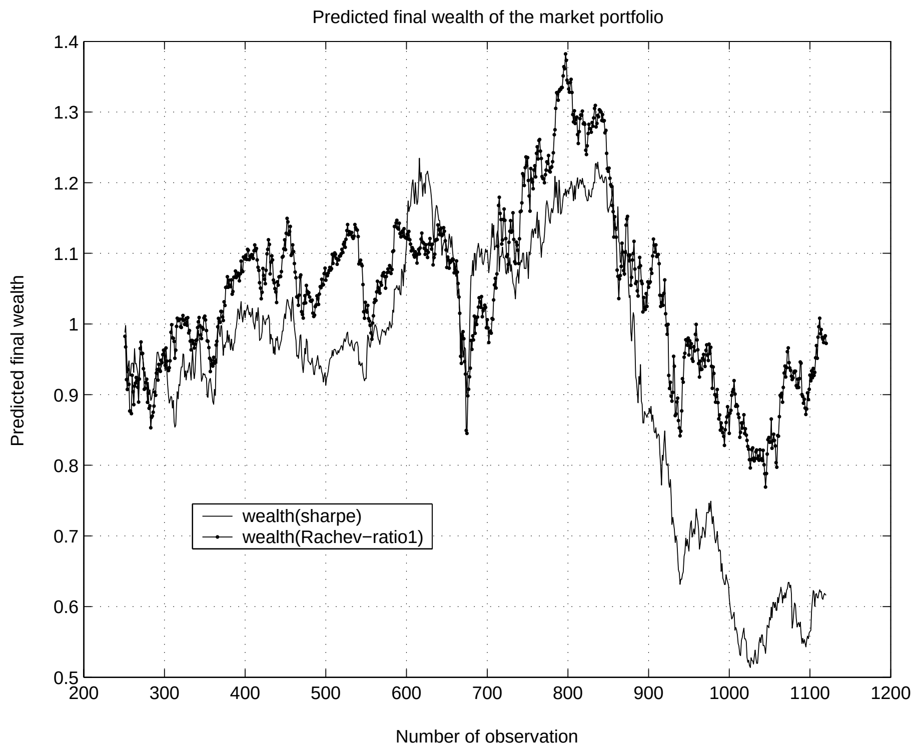
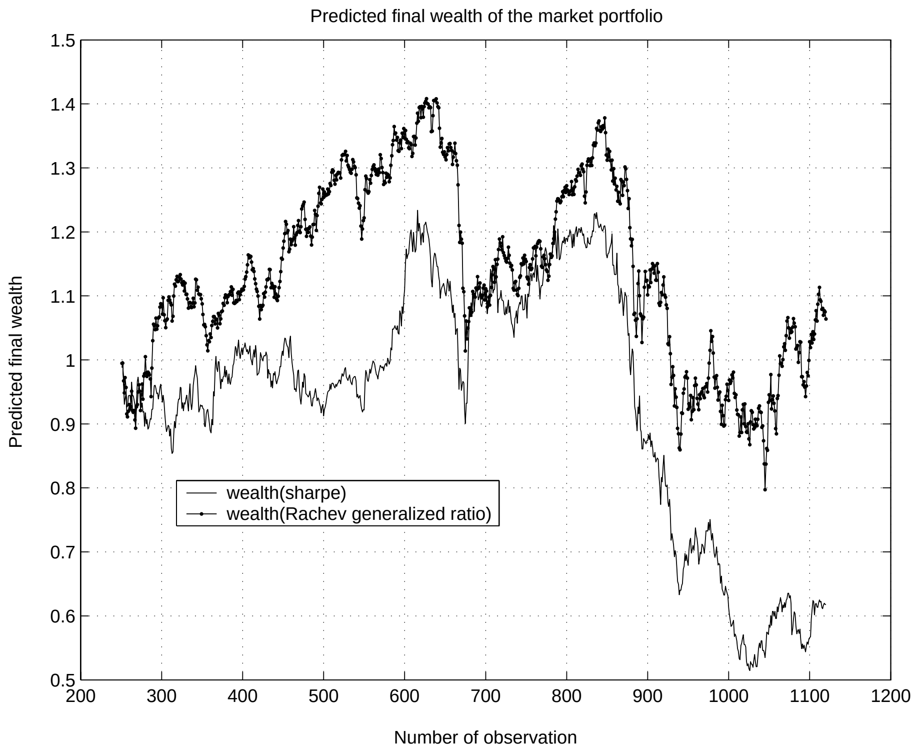
Exhibit 1



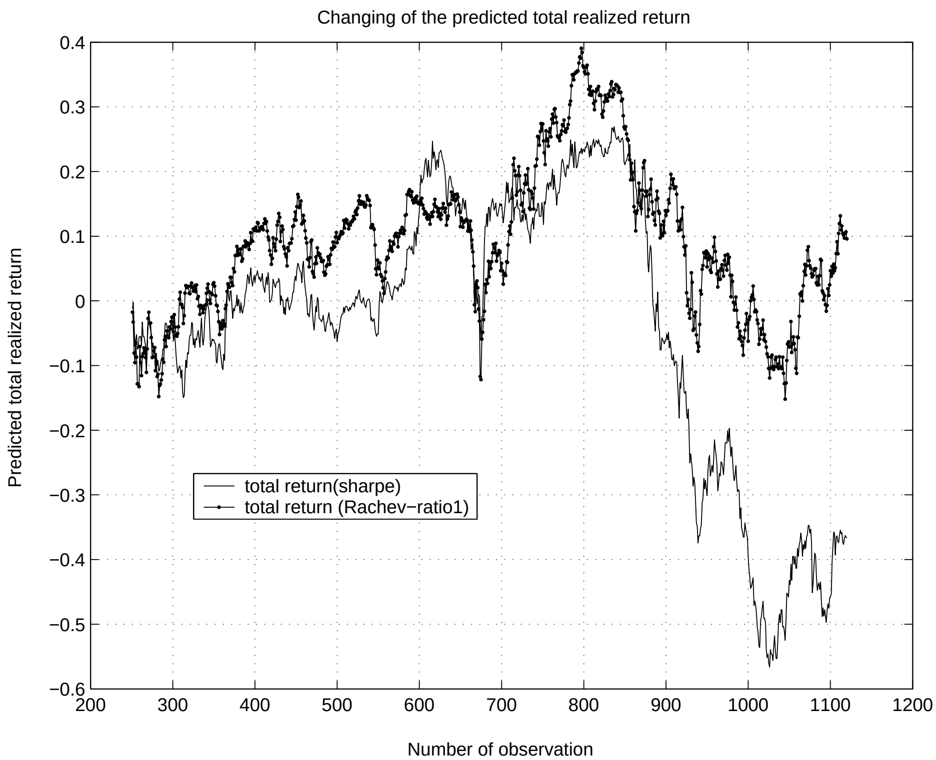
Ratio	*Value of the final wealth of the portfolio at day 1120*
Sharpe Ratio	*0.616240415853423*
Minimax Ratio	*0.804530138619293*
Stable Ratio	*0.578139498466928*
MAD Ratio	*0.628476996045711*
Gini Ratio	*0.625304181881120*
Sortino-Satchell Ratio	*0.617976033633241*
Farinelli-Tibiletti Ratio	*0.839552761793775*
$VAR_{99\%}$ Ratio	*0.605072926895815*
$CVaR_{99\%}$ Ratio (STARR Ratio)	*0.776588630065149*
Rachev Ratio1	*0.972486708253634*
Rachev Generalized Ratio	*0.753680291712404*

Exhibit 2



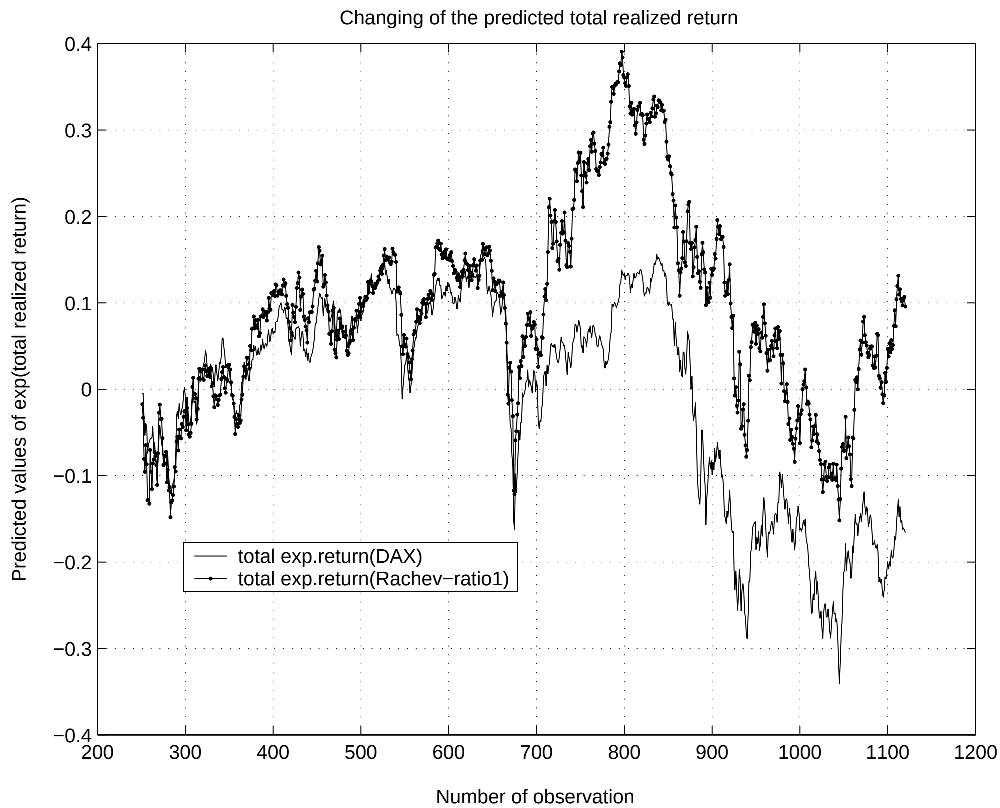
Ratio	*Value of the final wealth of the portfolio at day 1120*
Sharpe Ratio	*0.617451929444473*
Minimax Ratio	*0.775256808505809*
Stable Ratio	*0.536188972808025*
MAD Ratio	*0.638236615004889*
Gini Ratio	*0.634635872560578*
Sortino-Satchell Ratio	*0.617522512797628*
Farinelli-Tibiletti Ratio	*0.989935546412939*
$VAR_{99\%}$ Ratio	*0.630538999681293*
$CVaR_{99\%}$ Ratio (STARR Ratio)	*0.723743228580052*
Rachev Ratio1	*1.006171785779270*
Rachev Generalized Ratio	*1.063973580505089*

Exhibit 3



Ratio	*Value of the total realized return of the portfolio at day 1120*
Sharpe Ratio	*-0.366585953821371*
Minimax Ratio	*-0.098776164821371*
Stable Ratio	*-0.437531222524888*
MAD Ratio	*-0.344087473821371*
Gini Ratio	*-0.350627123821371*
Sortino-Satchell Ratio	*-0.359212613821371*
Farinelli-Tibiletti Ratio	*0.010434873821371*
$VAR_{99\%}$ Ratio	*-0.395411563821371*
$CVaR_{99\%}$ Ratio (STARR Ratio)	*-0.142927703821371*
Rachev Ratio1	*0.095833163360289*
Rachev Generalized Ratio	*-0.191643126275180*

Exhibit 4

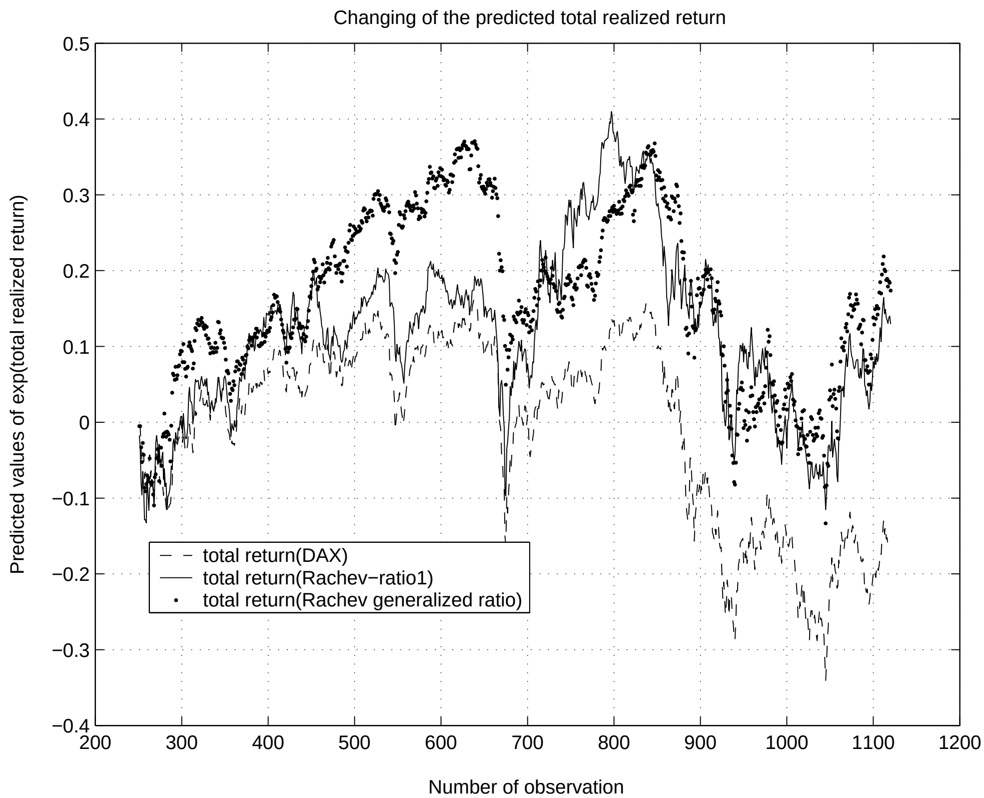


Changing of the predicted total realized return
Predicted values of exp(total realized return)
Number of observation
total exp.return(DAX)
total exp.return(Rachev−ratio1)
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
200
300
400
500
600
700
800
900
1000
1100
1200

Exhibit 5

Ratio	*Value of the total realized return of the portfolio at day 1120*
Sharpe Ratio	*-0.364645423821371*
Minimax Ratio	*-0.135864083821371*
Stable Ratio	*-0.520872263821371*
MAD Ratio	*-0.328669043821371*
Gini Ratio	*-0.335808033821371*
Sortino-Satchell Ratio	*-0.360183193821371*
Farinelli-Tibiletti Ratio	*0.111626853821371*
$VAR_{99\%}$ Ratio	*-0.344402793821371*
$CVaR_{99\%}$ Ratio (STARR Ratio)	*-0.199963063821371*
Rachev Ratio1	*0.129330688520624*
Rachev Generalized Ratio	*0.173757447754619*

Exhibit 6



Changing of the predicted total realized return
Predicted values of exp(total realized return)
Number of observation
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
200
300
400
500
600
700
800
900
1000
1100
1200
total return(DAX)
total return(Rachev−ratio1)
total return(Rachev generalized ratio)