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THE VALUATION OF OPTIONS FOR ALTERNATIVE STOCHASTIC PROCESSES*

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This paper examines the structure of option valuation problems and develops a new technique for their solution. It also introduces several jump and diffusion processes which have not been used in previous models. The technique is applied to these processes to find explicit option valuation formulas, and solutions to some previously unsolved problems involving the pricing of securities with payouts and potential bankruptcy.

1. Introduction

One of the central problems of modern finance is that of valuing claims to assets. The major result in this area is the insight of Modigliani and Miller (1958) that, in equilibrium, packages of financial claims which are, in essence, equivalent must command the same price. Modigliani and Miller recognized that in the absence of market imperfections these claims were simply financial tools for offering alternative modes of ownership of the same economic stream of returns. As a consequence, the aggregate value of the claims against the returns of a firm, for example, should be independent of the types of claims issued. Simultaneously with work on the Modigliani-Miller theorems and somewhat independently of it considerable progress was made by Samuelson (1965) and others [see Cootner (1964)] in evaluating stock options, a specialized form of financial claim. This work came to a focus in the major paper by Black and Scholes (1973) in which a complete option pricing model depending only on observable variables was derived. The Black and Scholes option pricing results can in some ways be viewed as an intertemporal analogue of the Modigliani-Miller theory. Although subsequent research has achieved greater

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generality and has been distinctly different in important ways, the underlying theme remains Black and Scholes' observation that in an intertemporal as well as a static setting two things which can be shown to be equivalent must sell for the same price.

It is useful to pursue this a bit further. Modigliani and Miller argue that the financial instruments issued by a firm span the returns stream, i.e., that the total package of claims on a firm, no matter how complex, is equivalent to a simple equity claim on the returns stream. A similar spanning situation is apparent in the Black and Scholes analysis. Black and Scholes assume that the value of the stock follows a particular diffusion process which will be discussed below and, as a consequence, locally in time, a stock and any option written on it will be perfectly correlated and combined with borrowing or lending at the riskless rate a position in one will be a perfect substitute for, or span, a position in the other. In this way the option is, locally, spanned by riskless bonds and the stock, and knowing the value of the stock permits us to value the option (globally by an integration argument). The critical factor in this argument and in any contingent claims valuation model is the precise description of the stochastic process governing the behavior of the basic asset. It is the characteristics of this process that determine the exact nature of the equivalence between packages of claims. The main contribution of this paper is the consideration of some alternative forms of the stochastic process governing stock prices, and the development of an approach to the option valuation problem that connects it directly to the structure of the underlying stochastic process. It will be useful, then, to give a brief and informal discussion of the stochastic processes that have previously been used.

The basic assumption employed by Black and Scholes was that the stock value followed a log-normal diffusion process,

$$\mathrm{d}S/S = \mu \mathrm{d}t + \sigma \mathrm{d}z, \tag{1}$$

where S is the value of the stock, μ is the drift term and z is a Wiener process. Eq. (1) is a short-hand notation for the following stochastic process. Let S_t be the value of the stock at time t. The percentage change in this value in the next instant, from t to $t+\mathrm{d}t$, is

$$\mathrm{d}S/S = (S_{t+\mathrm{d}t} - S_t)/St.$$

By (1) this percentage change is made up of two components, a drift term, $\mu\mathrm{d}t$, which is certain as viewed from time t and a normally distributed stochastic term $\sigma\mathrm{d}z$. The stochastic term is independent of its values in other periods and has mean zero and variance $\sigma^2\mathrm{d}t$. Put simply, eq. (1) says that the percentage change in stock value from t to $t+\mathrm{d}t$ is normally distributed with mean $\mu\mathrm{d}t$ and variance $\sigma^2\mathrm{d}t$. As $\mathrm{d}t$ gets small, then, $S_{t+\mathrm{d}t}$ will not differ much from S_t.

This is the hallmark of a diffusion process; it represents a continuous frictional sort of random walk around a trend term and, in the short run, offers no surprises.

The diffusion processes, though, are only one of two general classes of continuous time stochastic processes. The second type of stochastic process in continuous time is the jump process. A simple jump process can be written in analogy with (1) as

$$\begin{aligned} \mathrm{d}S/S &= \mu \mathrm{d}t + (k-1)\mathrm{d}\pi \\ &= \mu \mathrm{d}t + \left\langle \begin{array}{ll} \xrightarrow{\lambda \mathrm{d}t} & k-1, \\ \xrightarrow[1-\lambda \mathrm{d}t]{} & 0. \end{array} \right. \end{aligned} \tag{2}$$

In eq. (2) π is a continuous time Poisson process, λ is referred to as the intensity of the process and $k-1$ is the jump amplitude. As with (1), eq. (2) is a shorthand notation for the stochastic process that governs the percentage change in the value of the stock on the interval from t to $t+\mathrm{d}t$. Eq. (2) says that this percentage change is composed of a drift term, $\mu \mathrm{d}t$, and a term, $\mathrm{d}\pi$, which with probability $\lambda \mathrm{d}t$ will jump the percentage stock change to $k-1$, possibly random itself, and with probability $1-\lambda \mathrm{d}t$ will do nothing. One possible interpretation is that $\lambda \mathrm{d}t$ represents the instantaneous probability of receiving a packet of information that will cause S to jump.

In contrast to the diffusion process, the jump process (2) follows a deterministic movement upon which are superimposed discrete jumps. Formally, a jump process has sample paths which are discontinuous with probability one, while those of the diffusion process are continuous with probability one. In addition, the jump processes we consider are continuous from the right almost surely, i.e., their discontinuities are simple jumps. Because of the jumps in value the local analysis of Black and Scholes for valuing options does not carry directly over to eq. (2). By assuming that k was fixed, though, Cox and Ross (1975a) showed that a riskless hedge could be formed and used to value options on jump processes.

The intent of this paper is two fold. First, we examine the rationale for assuming that stock value follows (1) or (2) and propose some plausible alternative forms. This will allow us to examine the relationships between the choice of process and the solutions to option problems, while at the same time providing additional models for empirical testing. The second central feature of this paper is the development and application of an intuitive technique, introduced in Cox and Ross (1975a), for finding the solution to option valuation problems. This approach provides new insights into the structure of option valuation problems and its application allows us to solve a previously unresolved problem, the valuation of coupon paying bonds of arbitrary finite maturity, and the complementary problem of valuing an option on a stock with constant dividend payouts. Section 2 introduces the new stochastic processes studied in this paper. Section 3 studies the general option valuation problem and develops the solution technique. Section 4 applies the technique to the processes of section 1, and section 5 briefly summarizes and concludes the paper.

2. Some alternative stochastic processes

In exploring alternative forms it is useful to construct them as jump processes. Aside from the question of whether the 'real world' follows a diffusion or a jump process, even if we use diffusion processes for their analytic conveniences, much of our intuition can be formalized with jump processes. Eq. (2), for example, describes an equity whose value drifts deterministically until a unit of information arrives. Information arrives with probability $\lambda \mathrm{d}t$ and, when it does, the stock value jumps discontinuously by $k-1$ per cent. The diffusion in (1) is the limit of such a process, where information arrives continuously and has only a differential impact.

Eq. (2) is a very special case of the general form for Markov jump processes. If x denotes the current state of the world, then a general jump process is of the form

$$\mathrm{d}S = \mu(x)\,\mathrm{d}t + \begin{cases} \xrightarrow{\lambda(x)\mathrm{d}t} \tilde{k}(x)-1, \\ \xrightarrow[1-\lambda(x)\mathrm{d}t]{} 0, \end{cases} \tag{3}$$

where $\tilde{k}(x)$ has a distribution dependent on the current world state, x. We will assume that $x = S$, i.e., all state information is contained in the current stock value, S. We could, of course, also add a Wiener diffusion term, $\sigma(x)\mathrm{d}z$, to (3) to obtain a more general process, but (3) actually contains the diffusion as a limiting case (see below). The motivation for specializing (3) to (1) or (2), is that they capture two notions. First, they are in relative or percentage terms and there is some intuitive rationale for specifying the stochastic mechanism in percentage terms since this is the form of the returns. Second, by putting the process in percentage terms we can naturally include as the limited liability constraint, $S \geqq 0$. Both (1) and (2) obey this boundary condition. Beyond these not entirely compelling arguments though there does not seem to be terribly much reason for the exclusive use of (1) or (2) and doing so would overlook a number of interesting and equally defensible forms.

Suppose, for example, that in (3) we specialize the intensity, $\lambda(S)$ and the drift, $\mu(S)$, to be proportional to value, λS and μS, and choose the distribution, $k-1$, to be independent of value. Thus,

$$\mathrm{d}S = \mu S\mathrm{d}t + \begin{cases} \xrightarrow{\lambda S\mathrm{d}t} k-1, \\ \xrightarrow[1-\lambda S\mathrm{d}t]{} 0. \end{cases} \tag{4}$$

With the drift term eq. (4) is a generalization of a class of stochastic processes known as birth and death processes. The local mean and variance of (4) are given by

$$E\{\mathrm{d}S\} = [\mu + \lambda E\{k-1\}]S\mathrm{d}t, \tag{5}$$

and

$$V\{\mathrm{d}S\} = \lambda E\{(k-1)^2\}S\mathrm{d}t.$$

To construct a pure birth and death process we ignore the drift in (4) and let k take on two values, $k^+ > 1$ and $k^- < 1$ with respective (conditional) probabilities π^+ and π^-,

$$\mathrm{d}S = \left\langle \begin{matrix} \xrightarrow{\lambda S\mathrm{d}t} \left\langle \begin{matrix} \xrightarrow{\pi^+} k^+ - 1, \\ \xrightarrow[\pi^-]{} k^- - 1 \end{matrix} \right. \\ \xrightarrow[1-\lambda S\mathrm{d}t]{} \end{matrix} \right. = \left\langle \begin{matrix} \xrightarrow{\pi^+ \lambda S\mathrm{d}t} k^+ - 1, \\ \xrightarrow{\pi^- \lambda S\mathrm{d}t} k^- - 1, \\ \xrightarrow[1-\lambda S\mathrm{d}t]{} 0. \end{matrix} \right. \tag{6}$$

Eq. (6) is now an example of a simple birth and death process for a population. Imagine a firm made up of individual units (members of the population) whose sum value (population size) is S. If these units are stochastically independent of each other, we can let $\lambda\mathrm{d}t$ represent the probability of an event occurring for any one unit. An event is, with probability π^+, the 'birth' of $k^+ - 1$ additional units and with probability π^- the 'death' of $1-k^-$ units. For the whole firm (population), then, (6) describes its local movement. If $\mu = 0$ and if $\pi^+ = 1$, then (6) describes a pure birth process and if $\pi^- = 1$, (6) is called a pure death process. Eq. (2) in contrast to (6) describes the stochastic movement of a firm (population) all of whose members are perfectly dependent, that is, when one moves they all move, and the probability of such an event, $\lambda\mathrm{d}t$, is independent of the firm value (population size) although the magnitude is simply proportional.

Another interesting difference between (2) and (6) can be seen by passing to the diffusion limit in (6). The diffusion limit of (2) is the relative process (1) [see Cox and Ross (1975a)]. The limit of (6), though, as $k^+ \to 1$ and $k^- \to 1$ and $\lambda \to \infty$ in the fashion indicated in footnote 1 is a diffusion with instantaneous mean μS and variance $\sigma^2 S$, where μ and σ are given by (5), and μ is not the same as the drift in (4).[1] We could write this in the formalism of the stochastic differential as

$$\mathrm{d}S = \mu S\mathrm{d}t + \sigma\sqrt{S}\,\mathrm{d}z. \tag{7}$$

[1]To derive the diffusion limit (7) it is sufficient to demonstrate that the Kolmogorov backward equation (see discussion below in section 3) for the probability transition function

$$P_{x,y}(t,\tau) \equiv \text{Prob}\,\{S_\tau = y \mid S_t = x\}, \quad \tau > t;$$

Although it is useful to consider this type of diffusion as a limiting case of an economy where firms are compositions of independent units, this interpretation is by no means necessary. Other forms of causation could lead to the same probabilistic description of events. We could in fact consider this diffusion process solely on its own merits as a description of a situation in which changes in state are small and in which the variance of price changes increases with the stock price, but more slowly than (1) so that the variance of the rate of return decreases rather than remaining constant. Considered in this way the process can certainly not be rejected on an a priori basis, and may in many situations be preferable to (1). Unlike (1) it should be noted that the diffusion process represented by (7) does permit $S = 0$, i.e., bankruptcy, to occur with positive probability (even in the absence of stock payouts).

Another specialization of (3) that is of interest is one where the firm is com-

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for the birth and death process (6) converges to that for the diffusion (7) under an appropriate limiting argument. The backward equation for (6) is

$$-(\partial P_{x,y}/\partial t) = -\lambda x P_{x,y} + \lambda x \pi^+ P_{x+\Delta x,y} + \lambda x \pi^- P_{x-\Delta x,y},$$

where we have set $k^+ - 1 = \Delta x$ and $k^- - 1 = -\Delta x$.

Now, to maintain the instantaneous mean and variance of the diffusion process (7) in the passage to the limit we alter the intensities as $\Delta x \to 0$ in such a fashion as to maintain

$$\lambda x(\Delta x)^2 = \sigma^2 x \quad \text{and} \quad \lambda x(\pi^+ - \pi^-)\Delta x = \mu x; \quad \text{or}$$

$$\lambda = \sigma^2/(\Delta x)^2, \quad \lambda\pi^+ = \tfrac{1}{2}[\sigma^2/(\Delta x)^2 + \mu/\Delta x], \quad \lambda\pi^- = \tfrac{1}{2}[\sigma^2/(\Delta x)^2 - \mu/\Delta x].$$

Passing to the limit as $\Delta x \to 0$ in the backward equation we have the backward equation for the diffusion (7),

$$\begin{aligned} -\partial P_{x,y}/\partial t &= -\lambda x P_{x,y} + \lambda x \pi^+ [P_{x,y} + (\partial P_{x,y}/\partial x)\Delta x + \tfrac{1}{2}(\partial^2 P_{x,y}/\partial x^2(\Delta x)^2)] \\ &\quad + \lambda x \pi^- [P_{x,y} - (\partial P_{x,y}/\partial x)\Delta x + \tfrac{1}{2}(\partial^2 P_{x,y}/\partial x^2)(\Delta x)^2] \\ &= \mu x(\partial P_{x,y}/\partial x) + \tfrac{1}{2}\sigma^2 x(\partial^2 P_{x,y}/\partial x^2). \end{aligned}$$

The derivation of the absolute process (10) from the absolute jump (8) is nearly identical, but in that case we need the drift term. For the absolute process (8) the backward equation is given by

$$-\partial P_{x,y}/\partial t = -\lambda P_{x,y} + \lambda\pi^+ P_{x+\Delta x,y} + \lambda\pi^- P_{x-\Delta x,y} + \mu x(\partial P_{x,y}/\partial x).$$

Using the limiting process $\lambda\pi^+ = \lambda\pi^- = \tfrac{1}{2}(\sigma^2/(\Delta x)^2)$, we can show, as above, that the backward equation converges to the backward equation for the absolute process (10),

$$-\partial P_{x,y}/\partial t = \mu x(\partial P_{x,y}/\partial x) + \tfrac{1}{2}\sigma^2(\partial^2 P_{x,y}/\partial x^2).$$

These derivations are intended to be heuristic and only prove pointwise convergence, but they can be rigorously extended to show uniform convergence. A detailed treatment of a similar argument can be found in Feller (1951a). We should also add that since S is considered to be a value we append to (7) and (10) an absorbing barrier at $S = 0$. This recognizes that both (7) and (10) will drive a positive S to zero with positive probability.

posed of dependent units as in (2), so that intensity, λ, is constant, and where the value increment is also constant. In this case

$$\mathrm{d}S = \mu S\mathrm{d}t + \begin{cases} \xrightarrow{\lambda \mathrm{d}t} \begin{cases} \xrightarrow{\pi^+} k^+ - 1, \\ \xrightarrow{\pi^-} k^- - 1, \end{cases} \\ \xrightarrow{1-\lambda \mathrm{d}t} 0, \end{cases} \tag{8}$$

and we have jettisoned proportionality altogether. This is a case where value grows endogenously at the exponential rate μ, and where jump exogenous increments to value of size, $k-1$, occur with intensity λ. For reference we can call this the absolute process.

The local mean and variance of the absolute process are given by

$$E\{\mathrm{d}S\} = \{\mu S + \lambda[\pi^+(k^+ - 1) + \pi^-(k^- - 1)]\}\,\mathrm{d}t,$$

and

$$V\{\mathrm{d}S\} = \lambda[\pi^+(k^+ - 1)^2 + \pi^-(k^- - 1)^2]\,\mathrm{d}t, \tag{9}$$

in the case where k is constant. If $\pi^- = 0$ the process has limited liability, but if $\pi^- > 0$ there is a positive probability that it will go into default. To preserve limited liability we would, therefore, also have to specify a non-negative lower barrier for S. Taking the diffusion limit of (8) as with (6) (see footnote 1), we obtain

$$\mathrm{d}S = \mu S\mathrm{d}t + \sigma \mathrm{d}z, \tag{10}$$

where μ and σ are given by (9). This process would thus characterize a firm whose increments in value have a constant variance. To impose limited liability, we let the origin be an absorbing barrier, and consider (10) as governing the stock value only as long as this point is not reached. There would again be a positive probability of bankruptcy during any period.

3. Option valuation theory

The structure of the hedging arguments used to obtain valuation formulas for options can be illustrated in a fairly general setting. The first step is to choose a particular stochastic process to govern the price movement of the underlying asset, say a stock with price, S. Let us assume that we can write the random differential movement in S as

$$\mathrm{d}S = \mu_S \mathrm{d}t + \sigma_S \mathrm{d}x_S. \tag{11}$$

As in the examples of section 2, μ_S and σ_S are taken to be functions of the current state of the world, which for simplicity is supposed to be summarized by S and t alone. The (non-anticipating) stochastic term $\mathrm{d}x_S$ is assumed to be either a Wiener diffusion term, $\mathrm{d}z$, or a unit Poisson variable $\mathrm{d}\pi$. If $\mathrm{d}x_S$ is a Poisson term, then we interpret σ_S in (11) to be the random jump amplitude given a jump.

The next step in the argument is to take an instrument whose value is dependent on S, say an option written on the stock, and assume that a sufficiently regular price function exists, $P(S, t)$, which is the option value at time t, given that the stock price at t is equal to S. Postulating such a function permits us (given that μ_S, σ_S and P are sufficiently well behaved mathematically) to derive the differential movement in the option value,

$$\mathrm{d}P = \mu_P \mathrm{d}t + \sigma_P \mathrm{d}x_S. \tag{12}$$

The functions μ_P and σ_P now depend on the unknown function P and the known values of S and t. If $\mathrm{d}x_S$ follows a unit Poisson process σ_P may be a random function whose values depend on the function P and the jump size, σ_S, and it need not be proportional to σ_S.

The economic argument that leads to a formula for pricing the option is based on the presence of a third asset that earns a riskless instantaneous interest rate, r, which we will take to be a constant rate at which individuals can borrow and lend freely. We also will assume that the stock, S, can be sold short with the seller receiving the proceeds, and that there are no transactions costs or taxes. Most importantly, we make the competitive assumption that agents act as though they cannot influence r or any price. Under these assumptions it is easy to show that all riskless assets must earn the riskless rate, r, to prevent arbitrage.

While it is possible to solve option problems for random jumps to more than one value as in the birth and death process of (6) when $\pi^+, \pi^- \neq 0$, to do so requires the introduction of additional stocks either to support the hedging argument as in Cox and Ross (1975b), or the use of Ross's (1973) arbitrage argument to obtain an approximate formula as in Merton (1975). To avoid either of these possibilities we will further assume that if $\mathrm{d}x_S$ is a Poisson process, the jump amplitude σ_S (and σ_P) is a non-random function at a jump. It follows that there is a hedge portfolio of the stock, S, and its option, P, such that

$$\alpha_S \sigma_S (\mathrm{d}x_S/S) + \alpha_P \sigma_P (\mathrm{d}x_S/P) = 0,$$

or

$$\alpha_S(\sigma_S/S) + \alpha_P(\sigma_P/P) = 0, \tag{13}$$

where α_S and α_P are the portfolio weights in the stock and the option respectively. Such a hedge portfolio is riskless and must have a rate of return

$$\alpha_S(\mu_S/S) + \alpha_P(\mu_P/P) = (\alpha_S + \alpha_P)r, \tag{14}$$

the return at the riskless rate. From (13) and (14) we obtain the fundamental option valuation equation

$$(\mu_P - rP)/\sigma_P = (\mu_S - rS)/\sigma_S. \tag{15}$$

The valuation equation, thus, reduces to the familiar statement that the risk premium divided by the scale of risk has to be the same for the stock and its option. As a mathematical matter, eq. (15) is usually a differential-difference equation and together with the terms of the option we can hopefully apply some available mathematical techniques to solve it.

For example, with the Black and Scholes log-normal diffusion (1), the valuation eq. (15) takes the form

$$\tfrac{1}{2}\sigma^2 S^2 P_{SS} + rSP_S - rP = -P_t. \tag{16}$$

Using the boundary condition for a European call option

$$P(S, T) = \max\{S-E, 0\},$$

where E is the exercise price, Black and Scholes were able to transform (16) to the heat equation of physics and solve it in closed form.

In Cox and Ross (1975a), however, a systematic technique for solving the valuation equation was introduced that exploits the economic structure of the problem and provides further insight into the structure of option valuation problems in general. The fact that we could use a hedging argument to derive (15) and the argument that $P(S, t)$ exists uniquely means that given S and t the value of the option, P, does not depend directly on the structure of investors' preferences. Investors' preferences and demand conditions in general enter the valuation problem only in so far as they determine the equilibrium parameter values. No matter what preferences are, as long as they determine the same relevant parameter values, they will also value the option identically. In the Black and Scholes case, for example, (16) does not depend on μ and the only relevant parameters for the pricing problem are r and σ. To solve (15), then, we need only find the equilibrium solution for P in some world where preferences are given and consistent with the specified parameter values; the solution obtained will then be preference free.

A convenient choice of preferences for many problems (although one can envision problems where another preference structure might be more suitable) is risk neutrality. In such a world equilibrium requires that the expected returns on both the stock and the option must equal the riskless rate. For the stock, then

$$E\left\{\frac{S_T}{S_t}\,\middle|\, S_t\right\} = e^{r(T-t)}. \tag{17}$$

Similarly, if we are considering a general European option with boundary value,

$$P(S, T) = h(S), \tag{18}$$

then, at time t,

$$E\left\{\frac{P(S_T, T)}{P}\,\middle|\, S_t\right\} = \frac{1}{P}E\{h(S_T) \mid S_t\} = e^{r(T-t)},$$

or

$$\begin{aligned} P(S, t) &= e^{-r(T-t)}E\{h(S_T) \mid S_t\} \\ &= e^{-r(T-t)} \int h(S_T)\mathrm{d}F(S_T, T \mid S_t, t), \end{aligned} \tag{19}$$

where $F(S_T, T \mid S_t, t)$ is the probability distribution of the stock price at time T, S_T, given the stock price at time $t < T$, S_t. Eq. (19), with (17), provides the solution to the option valuation problem. Eq. (17) is used to satisfy any special features of the parameter set that are implied by the hedging equation.[2]

From (19) it is apparent that if we know the cumulative probability distribution of the stock process we can value the option. The converse is generally true as well. In the case of European calls, for example, the general option pricing formula (19) for arbitrary exercise prices, E, involves knowing all of the right semi-moments of the terminal stock distribution, given that (17) is satisfied. This is, however, equivalent to knowing the distribution itself.[3] In other words, the option valuation problem is really equivalent to the problem of determining the distribution of the stock variable, S, whose movement is governed by the postulated process (11). This establishes an important link between the option valuation problem and the fundamentals of stochastic processes.

It is well known that the probability transition functions, $F(S_T, T \mid S_t, t)$, satisfy two central equations, the forward (or Fokker–Planck) equation and the Kolomogorov backward equations [see Feller (1966)]. The backward equations

[2]Some awareness of this technique appears in previous work. Black and Scholes, in an earlier version of their seminal paper, first found the solution to (16) by setting $\mu = r$ in Sprenkle's (1961) formula for the option value. Merton, in Samuelson (1973), also noted that setting $\alpha = \beta = r$ in Samuelson's $\alpha - \beta$ model gave the Black and Scholes solution.

[3]We can sketch a formal proof of this proposition. We only need to show that the semi-moments determine the distribution. Suppose that two distributions, F and G, have the same semimoments, or equivalently, the same option values for all exercise prices, E. The family of functions, $f_E(S) = \max\{S - E, 0\}$, generates a lattice, K (closed under addition and multiplication by constants), on compact sets on the line which contains the constant functions and separates points. The lattice structure is immediate and for $E' > E$, $f_E(S) - f_{E'}(S) = E' - E$, $S \geqq E'$, i.e., a constant. By the Stone–Weierstrass theorem, then, on compact sets, the lattice, K, is dense in the continuous functions and since F and G agree on K, it follows from the Helly–Bray lemma that they agree on all continuous functions.

describe the way in which $F(S_T, T \mid S_t, t)$ is altered as the initial time, t, is changed. For example, the backward equation for the diffusion process (1) is given by

$$\tfrac{1}{2}\sigma^2 S^2 F_{SS} + \mu S F_S + F_t = 0, \tag{20}$$

where $S_t = S$, and $F(S_T, T \mid S, t)$ must satisfy (20) for all values of (S_T, T). In a risk-neutral world, from (17), the drift on the stock $\mu = r$. Suppose, then, we consider the backward equation (20) with $\mu = r$. Transforming this equation by substituting (19) we obtain (16), the Black and Scholes option valuation equation. In general, if (17) can be satisfied, the option valuation equation (15), is the transform (19) of the Kolmogorov backward equation for the transition probability function, F. The operational significance of these observations is simply that we can solve the option valuation problem only for those cases where we know the probability distribution of the terminal stock value.

The next sections illustrates these techniques by applying them to the option valuation problems for the stochastic processes introduced above. In the final section we are able to obtain an important new result, the valuation of options on stocks paying dividends, by the application of these techniques to the square root process (7).

4. Option valuation problems

4.1. Alternative jump processes

In this section we will explore the option valuation problem for some of the jump processes considered in section 2. As in section 3 we restrict the general form (3) to the case of a single post jump value, $k(S, t)$.

Our problem is to value a call option on S with an expiration date T at which time the holder receives max $\{S_T - E, 0\}$. We will initially assume that the stock pays no dividends so that it would never be optimal to exercise an American call before the expiration date T and it will, therefore, be valued as a European call [see Merton (1973)]. To solve this problem we specialize the hedging argument to the jump case. The local return on the stock is given by (3) and the option follows a perfectly dependent process

$$\mathrm{d}P = \begin{cases} \xrightarrow{\lambda S \mathrm{d}t} P(S+k-1, t) - P(S, t), \\ \xrightarrow[1-\lambda S \mathrm{d}t]{} P_t \mathrm{d}t + \mu P_S \mathrm{d}t, \end{cases} \tag{21}$$

where λ is an arbitrary function.

By forming a hedge portfolio of the stock and the option with weights α_S and α_P respectively chosen so that

$$\alpha_S\left[\frac{k-1}{S}\right]+\alpha_P\left[\frac{P(S+k-1,t)-P(S,t)}{P(S,t)}\right] = 0, \tag{22}$$

the hedge position will be riskless. It follows that if r is the (instantaneous) riskless rate of interest, then

$$\alpha_S\left(\frac{\mu}{S}\right)+\alpha_P\left(\frac{P_t+\mu P_S}{P}\right) = (\alpha_S+\alpha_P)\, r, \tag{23}$$

i.e., the hedge must be equivalent to a riskless short bond to prevent arbitrage possibilities. Combining (22) and (23) we have that $P(S,t)$ must satisfy the difference-differential equation version of (15),

$$\mu P_S+\left[\frac{\mu-rS}{1-k}\right]P(S+k-1,t)+\left[\frac{r[k-1+S]-\mu}{1-k}\right]P = -P_t, \tag{24}$$

where μ and k are functions of S and t.

An important feature of (24) and, consequently, the resulting option formulas is that they are independent of the choice of λ, the process intensity. This characteristic feature of option valuation formulas for jump processes was first shown for process (2) [see Cox and Ross (1975a)], and it is easy to see by the hedging argument that the intensity, quite generally, plays no role in the valuation since the hedge position depends only on the jump size. In fact, by setting $\mu(S,t) = \mu S$ and $k(S,t) = S(k-1)+1$, (24) becomes the option pricing relation [equation (10) in Cox and Ross (1975a)] for process (2). We can, now, use (24) to study a variety of alternative jump processes.

Example 1

Consider, first, a pure birth process without drift,

$$\mathrm{d}S = \begin{cases} \xrightarrow{\lambda S \mathrm{d}t} k-1, \\ \xrightarrow[1-\lambda S\mathrm{d}t]{} 0. \end{cases} \tag{25}$$

In this case eq. (24) specializes to

$$r(k-1)^{-1}S[P(S+(k-1),t)-P(S,t)]-rP(S,t) = -P_t(S,t), \tag{26}$$

with

$$P(S,T) = \max\{S-E,0\}.$$

To solve (26) we use the technique described in section 3. [As a check, it is not difficult to verify that (26) is the transformed backward equation for the process (25).] In a risk-neutral world the expected returns on both the stock and the option must equal the riskless rate, and (17) becomes

$$E\left[\frac{S_T}{S_0}\right] = e^{\lambda(k-1)(T-t)} = e^{r(T-t)}, \tag{27}$$

or

$$\lambda(k-1) = r,$$

where we have used a familiar result from the theory of birth processes. To obtain the expected return on the option we have to use the distribution function for S_T, which is simply the distribution for a scaled pure birth process [see, e.g., Feller (1966)]. It follows that,

$$\begin{aligned} E\left[\frac{P_T}{P}\right] &= \frac{1}{P} E[\max\{S_T - E, 0\}] \\ &= \frac{1}{P} \sum_{S_T \geq E} (S_T - E) \left[\begin{matrix} \dfrac{S_t}{k-1} - 1 \\ \\ \dfrac{S_t}{k-1} - 1 \end{matrix}\right] (e^{-r(T-t)})^{S_t/(k-1)} \\ &\quad \times (1 - e^{-r(T-t)})^{(S_T - S_t)/(k-1)}, \end{aligned} \tag{28}$$

and using (27) and the required equality with the riskless return, we obtain

$$\begin{aligned} P(S,t) = S &\sum_{j \geq [E/(k-1)+2]} B\left(j; \frac{S}{k-1} + 1, e^{-r(T-t)}\right) \\ &- Ee^{-r(T-t)} \sum_{j \geq [E/(k-1)+1]} B\left(j; \frac{S}{k-1}, e^{-r(T-t)}\right), \end{aligned} \tag{29}$$

where

$$B(j; x, q) = \binom{j-1}{x-1} q^x (1-q)^{j-x},$$

the negative binomial density, $\binom{j-1}{x-1}$ denotes $\Gamma(j)/\Gamma(x)\Gamma(j-x+1)$, and $[y]$ is the largest integer not exceeding y. This example illustrates the way in which (17) is used in the solution technique. From the hedging argument the intensity λ does not affect the option valuation. In a risk-neutral market $\lambda = r/(k-1)$, which allows us to eliminate λ from the option valuation (29). It is important to realize that this does not imply that we are only solving the valuation problem when

(27) is satisfied. On the contrary, for a given r and k the solution is independent of λ and for any λ the solution will be identical to the solution when (27) holds.

This example also reveals an important feature of solutions to valuation problems in general and jump problems in particular. At points where (29) and the solutions below are not differentiable they cannot, of course, satisfy differential equations of the form of (26). The paradox is resolved by modifying (26) appropriately. At points of non-differentiability, P_t, for example, will not in general capture the true time component of the change, or gradient, in option value. The hedging argument, on the other hand, will use this time gradient and the result is a slight generalization of the differential equations. Our solutions are everywhere correct for these generalized equations. This point is discussed in greater detail in Cox and Ross (1975a and b).

Unfortunately, though, there is no general solution for (24) available, and our technique of evaluating the solution for a risk-neutral world cannot avoid this difficulty. As a consequence, even seemingly straightforward generalizations of the results we do have can become formidable.

Example 2

Suppose we try to extend our option valuation results to the case of the pure jump process, (25), augmented by a proportional drift term, μS, as in (4). This is an important extension because by the techniques employed in footnote 1, (4), like a birth and death process, can be made to converge to the square root diffusion, (7).

To apply our technique to obtain the solution for the differential valuation equation, (24), requires us to know the distribution of S_T/S_t so as to be able to calculate the semi-moment $E\{\max\{S_T-E,\ 0\}\}$, which gives the option value. Unfortunately, though, the addition of the deterministic drift term greatly complicates this problem. The reason is that the process is now non-homogeneous in time in the sense that the probability of a jump in the next instant depends not only on the number of past jumps, but also on when they occurred, i.e., their timing. Without going into the messy details it can be shown, though, that, given (4), the density function of S_T is

$$\operatorname{Prob}\{S_T \varepsilon (x, x+\mathrm{d}x)\} = \sum_{n=0}^{\infty} \left(\frac{\lambda(k-1)}{\mu}\right)^n \mathrm{e}^{(\lambda/\mu)[S_t+n(k-1)-x]} \times \int_{A_n} \prod_{i=1}^{n} \left(S_t - \sum_{j=1}^{i=1} x_j\right) x_i^{-2} \mathrm{d}x_{ij},$$

where

$$A_n \equiv \{(x_1, \ldots, x_n) \mid \sum_i x_i = S_t - x\mathrm{e}^{-\mu(T-t)},\ -(k-1) \leqq x_1 \leqq \ldots \leqq x_n \leqq -(k-1)\,\mathrm{e}^{-\mu(T-t)}\}. \tag{30}$$

We can now use (19) and (30) to evaluate the option value, but we will no longer have a closed form solution. [The integrals in (30) do not appear to be readily available in closed form, but (30) can be approximated for computation purposes.]

Example 3

As a final jump example that illustrates one of the hazards of applying the solution technique consider applying it to the option valuation problem for the absolute process (8) without drift. To avoid limited liability problems and to permit one stock hedging suppose that $\Pi^+ = 1$, so that this is a pure growth jump. Simply solving (17), and using the fact that the number of jumps in $[t, T]$ is Poisson distributed, would give

$$E\left\{\frac{S_T}{S_t}\right\} = \frac{\lambda(k-1)}{S_t}(T-t) = e^{r(T-t)},$$

which solves for the omitted parameter, λ, as a function of S_t, the current stock value. This, however, violates the originally postulated absolute process with λ independent of current stock value. In other words, the assumed process is inconsistent in a risk neutral market. Nevertheless, we can still value an option on such a process by noting that the hedging differential equation is given by (26) just as for the pure birth process. This must be the case, since the intensity parameter plays no role in the valuation, other than through (27). It follows, then, that the solution to this problem will be the same as that for the birth process, (29) without drift, and can be found from (30) with a drift term. Even though the absolute process is inconsistent in a risk-neutral market, the differential valuation equation is the same as that for the birth process which is consistent with risk neutrality. This permits us to value the absolute process given whatever structure of market preferences and other assets which will support it in equilibrium. This inconsistency with risk neutrality does not apply to an absolute process with a symmetric two point jump and proportional drift, but consideration of this two jump case cannot be done in the context of single stock hedging.

We now turn to the option problems for the diffusion limits introduced in section 3.

4.2. *Alternative diffusion processes*

The first step is to derive the differential equation which the option value must follow for all diffusion processes and then specialize for our two cases.[4] In this section we will explicitly consider a stock or firm which makes payouts.[5]

[4]This development follows along the lines given by Black and Scholes and Merton (1974) for the diffusion process (1).

[5]Payouts can also be readily introduced into the jump processes, but this may in some cases greatly complicate the solution.

Suppose that the stock price is governed by

$$\mathrm{d}S = \mu(S, t)\,\mathrm{d}t + \sigma(S, t)\,\mathrm{d}z, \tag{31}$$

where $\mu(S, t)$ and $\sigma^2(S, t)$ are, respectively, the instantaneous mean and variance of the diffusion process. Applying Ito's lemma [see McKean (1969)], the option price will follow,

$$\mathrm{d}P = [P_t + \mu(S, t)P_S + \tfrac{1}{2}\sigma^2(S, t)P_{SS}]\,\mathrm{d}t + P_S\sigma(S, t)\,\mathrm{d}z. \tag{32}$$

Suppose also that each unit of the stock pays out in dividends the continuous stream $b(S, t)$. Consider a portfolio in which we hold a unit of the option, some fraction α_S of a unit of stock, and some amount of borrowing or lending such that the aggregate investment is zero. If we choose α_S to be $-P_S$ then the portfolio will have no stochastic component and to prevent arbitrage its local mean must be zero.

This means that in each instant the three sources of change in the portfolio, the deterministic part of the price changes in the stock and option, the risk-free return on the lending (or borrowing), and payouts received (or made in restitution in the case of short sales) must exactly offset each other. From the above we have that the net deterministic price change component is $\frac{1}{2}\sigma^2(S, t)P_{SS} + P_t$, the return on the bond position in $rSP_S - rP$, and the restitution required for dividend payments made to the stock while held short is $-b(S, t)P_S$. Collecting these terms yields the differential equation form of (15),

$$\tfrac{1}{2}\sigma^2(S, t)P_{SS} + [rS - b(S, t)]P_S - rP = -P_t. \tag{33}$$

With the diffusion processes, then, the stochastic assumptions enter the valuation equation only in the determination of the coefficient of the second derivative term, as would be expected from the earlier discussion about the relation between the valuation equation and the Kolmogorov backward equation for the process in question. Also we can note the convenience of the choice of risk-neutral preferences is not affected by payouts, since the risk neutrality would simply require that the instantaneous mean total return on the stock be rS, so that the required mean price change would be $\mu(S, t) = rS - b(S, t)$.

In the following we will consider only payout functions of the form $b(S, t) = aS + c$ since this will provide a satisfactory representation for most problems. Also we will consider only European options, although for many constant dividend policies equivalent American options would have the same value, since premature exercising would never be optimal.

Example 4

Let us first examine the case (7) where the variance is proportional to the stock price. From (23), the differential valuation equation becomes

$$\tfrac{1}{2}\sigma^2 S P_{SS} + [(r-a)S - c]P_S - rP = -P_t. \tag{34}$$

We could attack this problem directly by standard analytic methods but it is easier to apply the solution technique used above if the terminal density (in a risk-neutral setting) is already known. Fortunately, this is the case since Feller (1951a, b), in his work on birth and death processes, was led to studying the limiting diffusion case. The density of S_T conditional on S_t is given for $S_T > 0$ by

$$\begin{aligned} f(S_T, T; S_t, t) = {} & \left(\frac{2(r-a)}{\sigma^2(e^{(r-a)(T-t)}-1)}\right)\left(\frac{S_t e^{(r-a)(T-t)}}{S_T}\right)^{\frac{1}{2}(1+2c/\sigma^2)} \\ & \times \exp\left[-\frac{2(r-a)(S_t e^{(r-a)(T-t)}+S_T)}{\sigma^2(e^{(r-a)(T-t)}-1)}\right] \\ & \times I_{1+2c/\sigma^2}\left[\frac{4(r-a)(S_T S_t e^{(r-a)(T-t)})^{\frac{1}{2}}}{\sigma^2(e^{(r-a)(T-t)}-1)}\right], \end{aligned} \tag{35}$$

where $I_q(\cdot)$ is the modified Bessel function of the first kind of order q.[6]

Integrating (35) over the range $S_T > 0$ results in a probability of less than unity. The remaining mass is the probability that $S_t = 0$ for some $t \leqq T$, in which case S was 'absorbed' and remains at zero. Applying our technique we take the expectation of $\max(S_T - E, 0)$ and discount it to time t as in (19) to obtain the valuation formula

$$\begin{aligned} P(S,t) = {} & S e^{-a(T-t)} \sum_{n=0}^{\infty} \frac{(n+1)e^{-y} y^{n+2c/\sigma^2} G(n+2, \theta E)}{\Gamma[n+2+2c/\sigma^2]} \\ & - E e^{-r(T-t)} \sum_{n=0}^{\infty} \frac{e^{-y} y^{n+1+2c/\sigma^2}\, G(n+1, \theta E)}{\Gamma[n+2+2c/\sigma^2]}, \end{aligned} \tag{36}$$

where

$$\theta = \frac{2(r-a)}{\sigma^2[e^{(r-a)(T-t)}-1]},$$

$$y = \theta S e^{(r-a)(T-t)}, \qquad G(m,x) = [\Gamma(m)]^{-1} \int_x^{\infty} e^{-z} z^{m-1} \mathrm{d}z,$$

[6] It is a difficult mathematical question to decide whether a stochastic differential equation can be solved for a non-trivial stochastic process in cases where the coefficients do not have bounded derivatives. In this case, however, (7) was actually derived from the stochastic process given by (35) and no such problems arise. More generally, if we are given the process itself, any increasing transform of it (like P) will itself be a well-defined process with instantaneous mean, $\mu(S,t)P_S + \frac{1}{2}\sigma^2(S,t)P_{SS} + P_t$, and instantaneous variance, $\sigma^2(S,t)P_S^{\,2}$, if it is C^2, and derivable even if it is not [see Feller (1966, p. 326)]. This approach permits us to bypass the Ito processes and any additional regularity conditions they might require.

the complimentary standard gamma distribution function. The value of an option at $S = 0$ is implied by the description of the process and no additional restrictions need to be made. For a process with an absorbing barrier at zero we will obviously have $P(0, t) = 0$.

Example 5

Turning now to (10) where the variance is independent of the price, eq. (33) is specialized to

$$\tfrac{1}{2}\sigma^2 P_{SS} + (r-a)SP_S - rP = -P_t. \tag{37}$$

As with (34) this problem could be handled directly, for example by transformation to the heat equation. It is again easier and more illuminating however to make use of knowledge of the terminal stock distribution. Inspecting the equation we note that it is analogous to the backward equation of the Ornstein–Uhlenbeck process, whose physical origins lay in the study of particles in Brownian motion in the presence of an elastic force. It makes little economic sense for a price with limited liability to reach zero if it can subsequently become positive, so we would wish to use the Ornstein–Uhlenbeck process with an absorbing barrier at zero as with the square root process. In the above we have looked only at the specialization $c = 0$, i.e., only at proportional payouts, since otherwise the corresponding density is not known (and the transformation to the heat equation leads to an as yet unsolved time-dependent boundary problem). For the case considered, the density of S_T conditional on S_t is, for $S_T > 0$,

$$f(S_T, T; S_t, t) = (2\pi Z)^{-\frac{1}{2}}\left[\exp\left(-\frac{[S_T - S_t e^{(r-a)(T-t)}]^2}{2Z}\right)\right.$$
$$\left. -\exp\left(-\frac{-[S_T + S_t e^{(r-a)(T-t)}]^2}{2Z}\right)\right],$$

where

$$Z = \left[\frac{\sigma^2}{2(r-a)}\right][e^{2(r-a)(T-t)} - 1]. \tag{38}$$

Applying (19) yields the valuation formula,

$$\begin{aligned} P(S, t) = {} & (Se^{-a(T-t)} - Ee^{-r(T-t)})N(y_1) \\ & + (Se^{-a(T-t)} + Ee^{-r(T-t)})N(y_2) \\ & + v[n(y_1) - n(y_2)], \end{aligned} \tag{39}$$

where $N(\cdot)$ is the cumulative unit normal distribution function, $n(\cdot)$ is the unit normal density function, and

$$v = \sigma\left(\frac{e^{-2a(T-t)} - e^{-2r(T-t)}}{2(r-a)}\right)^{\frac{1}{2}},$$

$$y_1 = \frac{Se^{-a(T-t)} - Ee^{-r(T-t)}}{v},$$

$$y_2 = \frac{-Se^{-a(T-t)} - Ee^{-r(T-t)}}{v}.$$

The comparative statics associated with parameter changes in (36) and (39) are tedious, but fairly intuitive, and we defer then to subsequent work.

4.3. Applications to other securities

While we have focused on options above these same techniques can be applied to a wide range of financial instruments. A convenient approach to valuing corporate securities is to assume that the total value of the firm, V, follows a particular stochastic process and then consider individual securities as functions of the value of the firm and time. The value of the individual securities of any firm whose total value follows a diffusion process, for example, must, then, satisfy an equation of the same form as (33). Unlike options, though, most corporate securities, F, receive payouts, $b'(V, t)$, and eq. (33) must be modified to include this return,

$$\tfrac{1}{2}\sigma^2(V,t)F_{VV} + [rV - b(V,t)]F_V - rF + b'(V,t) = -F_t. \tag{40}$$

The securities of a given firm can be distinguished by their terminal conditions and payouts received. As a concrete example, consider a firm with one stock issue and one bond issue. The bond would have a terminal value of min (B, V), where B is the maturity value of the bond, and would receive a constant payout, say c'. The stock would have a terminal condition of max $(V-B, 0)$ and would receive in dividends $aV+c''$, where $c'+c'' = c$, the total constant portion of payouts.[7]

To value such securities it will be useful to think of the total value of any security as being the value the security would have if it received no payouts, i.e., it only received its terminal return, plus the value of the payouts it will potentially receive. In terms of eq. (40) these two components would correspond,

[7]For the log-normal process (1), Merton (1974) has studied the problem of valuing the pure discount funds of a firm which makes no payouts ($a = c = 0$) and Ingersoll (1975) has discussed a model with proportional payouts ($c = 0$).

respectively, to (i) the solution of (40) without the inhomogeneous term $b'(V, t)$ but with the proper terminal conditions for the security, and (ii) the solution of the full eq. (40) with a zero terminal condition $[F(V, T) = 0]$. If we restrict our attention to payout policies of the form $aV+c$, then we can further break down (ii) into the value of the proportional payout and the value of the constant payout. It is easy to see that the sum of these solutions is the complete value of the security.

Applying our technique, the solution to (i) for any security is simply given by eqs. (19) and (17). Having found the solution to (i) for all securities, by the Modigliani–Miller theorem we can find the value of the total payout stream, $aV+c$, by subtracting the sum of these solutions from V. If the payout received by each security j can be written as a proportion of the total payout $k_j(aV+c)$, $(\sum k_j = 1)$, then it is evident from (40) that the value of (ii) for each security j will be k_j times the value of the total payout to all securities. We can then obtain a complete solution without having to solve (ii) separately. This technique can be used, for example, if $a = 0$, which would be the case of a stock that received constant dividends, or $c = 0$, i.e., the bond was a pure discount bond. In general, though, the securities of the firm receive different proportions of the constant payout, c, and the proportional payout, aV, and it is necessary to have a direct solution to (ii) to value such securities. However, we would only need to value separately the total proportional component and the total constant component, since from (40) the value of the payouts to individual securities can be written as a linear combination of these two terms.

To solve problem (ii) by our techniques note that the value of each point in a payout stream in a risk-neutral world must be its expected value discounled to the present. The total value of the stream can then be obtained in the usuat way by integrating over all points in the stream. Once again, since we have established the hedging eq. (40), this solution will be the correct solution in general, not simply in a risk-neutral world. The expected value of any point in the constant stream, say, at time q, will be c times the probability that the payment will be received. This will be the probability, conditional on the current ($t < q$) value of the firm, that the firm will not be bankrupt at time q. We get this probability by replacing T with q in the terminal density of the value of the firm and then integrating this density with respect to $V_q > 0$. For our process (7), this probability can be obtained from (35) as

$$\sum_{n=0}^{\infty} \frac{e^{-y} y^{n+1+2c/\sigma^2}}{\Gamma[n+2+2c/\sigma^2]} = 1 - G[1+2c/\sigma^2, y], \tag{41}$$

where in y, q replaces T and V replaces S. The value of the entire stream will then be given by integration from t to T, giving

$$\int_t^T c e^{-r(q-t)}[1 - G(1+2c/\sigma^2, \theta V e^{(r-a)(q-t)})] dq. \tag{42}$$

If we knew the solution to (i) for all securities and the solution to (ii) for the constant component, we could value the proportional component by simply subtracting these from V. It is instructive, though, to value the proportional part of the payout stream directly. The expected value of the stream aV_q at each point q can again be obtained from the density and then discounted back to the present. Alternatively, the discounted expected value could be obtained directly in each of our cases from (36) or (39) by replacing S with V and T with q, and setting the exercise price, $E = 0$. We could then find the total value of the proportional component of the stream at time t by integration with respect to q from t to T.

For the case with variance proportional to value, the solution for the proportionate payout is

$$\int_t^T aV\mathrm{e}^{-a(q-t)}\left[\sum_{n=0}^{\infty}\frac{(n+1)\mathrm{e}^{-y}y^{n+2c/\sigma^2}}{\Gamma[n+2+2c/\sigma^2]}\right]\mathrm{d}q, \tag{43}$$

where, again, in y, V replaces S and q replaces T. When $c = 0$ the expression in square brackets equals one, and (43) reduces to simply

$$\int_t^T aV\mathrm{e}^{-a(q-t)}\mathrm{d}q = V[1-\mathrm{e}^{-a(T-t)}]. \tag{44}$$

Applying the analysis to the absolute diffusion (10) where $c = 0$, we find that (44) solves this case as well. In fact, inspection of (40) shows that when only proportional payouts are being made, (44) is the proper valuation for the payout stream for any diffusion process. However, as we have seen, when constant payouts are being made, the valuation of the proportional component will depend on the process being considered.

5. Summary and conclusion

The type of stochastic process determining the movement of the stock is of prime importance in option valuation. At present the workhorse of the option pricing literature has been the log-normal diffusion process. This paper introduced several alternative jump and diffusion processes, and provided solutions for the limiting diffusion cases and for the single-stage forms of the jump processes. The explicit solutions presented have potential empirical applications and a comparative study of them should give additional insight into the structure of security valuation. Aside from the intrinsic value of studying alternative admissible processes, though, a number of important problems involving payouts and bankruptcy which remain intractable for the log-normal are, nevertheless, solvable for some other processes. Throughout, the paper developed and used an economically interpretable technique for solving option problems which has intuitive appeal and should facilitate the solution of other problems in this field.

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