

Department of Engineering Science

Part IV Project

Faster Shortest Path Computation for Traffic Assignment

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Abstract

The Traffic Assignment (TA) problem involves the selection the optimal path for every vehicle in a transportation network. One algorithm for solving TA is the Path Equilibration (PE) algorithm. PE requires solving shortest path repeatedly between every origin and destination pair in the network for a large number of iterations, while the edge costs change between iterations. The aim of this project is to find a faster shortest path algorithm for PE. We implement Dijkstra's algorithm and A* search, their bidirectional versions and 8 different versions of priority queue data structures that improve these algorithms' performance. We develop two strategies for using these algorithms in the iterative environment of PE. The first strategy is to avoid next few numbers of iterations when the shortest path of the previous two iterations is the same. The second strategy is to randomly skip the next shortest path calculation, where we hope for the situation where the previous and current iteration is going to be the same. We present experimental results that demonstrate the run time differences between these priority queues, algorithms and strategies and show that A* search algorithm with random skipping strategy has the best performance.

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Chapter 1

Introduction

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1.1 Introduction to traffic modelling

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Nowadays a large portion of people's daily lives involve activities which relate to transportation. For example, most people need to travel between their workplace and residence twice a day, or buy goods from shops that need to be delivered across the city. In the meanwhile, transportation networks expand and improve constantly to cater for people's demand for an efficient transportation network, but the rate of improvement does not confront with the rate of population growth. As a result, the network becomes inefficient, resulting traffic congestion.

Congestion leads to major economical losses due to time delays and increased usage of petrol. Congestion causes air pollution that increases respiratory problems such as asthma, and the exhaust gas exacerbates global warming. Congestion also increases noise pollution and causes frustration, which in turn accelerates traffic accidents. It is important for traffic designers to be able to reduce congestion problems, and eliminate its negative effects. The transportation network can be improved by, for example, introducing road tolls to diverge traffic to less congested roads, or by educating people to use public transport.

Making improvements to the transportation network tends to be very costly, so an better plan is necessary: use the least amount of investment for the greatest change. In order to make better plans for traffic design, different mathematical models have been built to estimate the current and future behaviour of the transportation system. One particular model called the transportation forecasting model is commonly used. The aim of this model is to estimate future traffic usage

when the system is changed. For example, changes to a transport system could include upgrading or adding roads, changing roundabouts to traffic lights, or providing better public transport.

The transportation forecasting model has 4 stages (shown in Figure 1.1): trip generation, trip distribution, mode choice and traffic assignment (Sheffi 1985). In the model, the solution of each stage can be passed to the previous ones to improve transportation forecasting and traffic design. So after numerous iterations of calibration, an accurate model that can estimate future traffic is established. In summary, the model collects traffic demand data and generates origins and destinations for travellers (trip generation). It then calculates the number of trips that are required between each origin and destination (trip distribution), and decides which transportation method should be used for each trip (mode choice). Finally it decides the route (e.g. the shortest path) that each trip needs to travel on (traffic assignment), where traffic flow is estimated for every modelled road in the network. The traffic assignment problem in the last stage of the forecast model is a complicated problem. This is because congestion occurs as traffic flows are assigned onto the network, and it is very difficult to find an equilibrium situation where everybody in the network finds their best route. Generally the equilibrium situation happens when it is assumed that people are selfish, they always try to minimise travel times by travelling along their shortest paths.

not sure

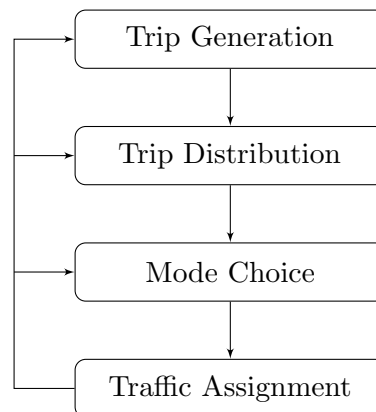


Figure 1.1: Transportation forecasting model

1.2 Purpose of this project

The transportation forecasting model has been implemented in many software packages for traffic design. One key observation from these software is that, the traffic assignment problem takes

hours, or even days to solve. It turns out that, the bottleneck is in the algorithm for solving the shortest path problem. This is because algorithms for solving the traffic assignment problem are usually iterative, where each iteration requires to find millions of shortest paths between every origin and destination in the network. Although each shortest path calculation may take only a fraction of a second, cumulatively the computational time can be very long. Therefore the purpose of this project is to find a faster algorithm for solving the shortest path problem in an iterative environment. As a result, the traffic assignment will be solved faster for larger and more complicated road networks. This will also allow city designers estimate traffic flows further into the future and make better decisions on road network design.

In summary, The algorithms that are going to be experimented are:

- Bellman-Ford's label correcting algorithm,
- Dijkstra's label setting algorithm (using different data structures),
- Bidirectional Dijkstra's algorithm,
- A* search,
- Bidirectional A* search.

We also aim to find and discuss the following techniques that can speed up shortest path calculations in an iterative environment:

- preprocessing algorithms,
- use information from the previous iterations,
- avoid shortest path calculations.

1.3 Structure of the report

Chapter 2 gives a short description of the traffic assignment problem and presents a specific algorithm called the Path Equilibration algorithm that solves it. This algorithm the begin of our experiments for this project. Chapter 3 introduces the shortest path problem and presents some of the well established algorithms that solve it. Strategies for solving the shortest path problem faster in an iterative environment are also described in this chapter. Implementation details are presented in Chapter 4. Chapter 5 shows and discusses results. Finally conclusions are drawn in Chapter 7.

Chapter 2

The Traffic Assignment Problem

In a transportation network, every traveller wishes to travel between different pairs of origins and destinations. As travellers start to travel in the network, congestion happens and travel speeds tend to decrease rapidly due to more and more interactions between the travellers and an increase in traffic volume. This leads to the problem of travellers wishing to find the fastest route to travel on, meanwhile taking account of congestion while every other traveller is trying to do the same. In transportation modelling point of view, we wish to find a traffic flow pattern in the network with a given travel demand between the origin and destination pairs. This is called the Traffic Assignment Problem.

The Traffic Assignment Problem is commonly solved by traffic equilibrium models. The notion of traffic equilibrium was first formalized by Wardrop (1952), where he introduced the postulate of the minimisation of the total travel costs. His first principle states that “the journey times on all routes actually used are equal and less than those which would be experienced by a simple vehicle of any unused route.” The traffic flows that satisfy this principle are referred to as “user optimal” flows, as each traveller chooses the route which is the best for them. On the other hand, we can also find traffic flows that are “system optimal”, which is characterized by Wardrop’s second principle, stating that “the average journey time is minimum”.

In this chapter, the network equilibrium model is first stated. Then a particular solution algorithm for solving such a model is described. Finally the motivation of this project in the context of solving the traffic assignment problem is brief explained.

2.1 The network equilibrium model

In order to formulate a model for solving the traffic assignment problem, two assumptions are made. We assume deterministic traffic flows and fixed demand, and the change of travel time on any link does not depend on the change of traffic flows on the other links. These assumptions may not be realistic but they allow us to formulate the traffic assignment model as a mathematical program.

keep?

Using notations from Florian & Hearn (1995, 2008), we consider a transportation network represented as a graph $G = (N, A)$, where N is a set of nodes and A a set of directed links in the network. The number of vehicles (or flow) on link a is v_a ($a \in A$), and the cost of travelling on a link is given by a link cost function $c_a(v)$ ($a \in A$), where v is the vector of link flows over the entire network.

Let I be the set of origin-destination (O-D) pairs and K_i be the set of cycle-free paths for O-D pair $i \in I$. The origin to destination traffic demands g_i ($i \in I$) are distributed over directed paths $k \in K_i$, and it is assumed $K_i \neq \emptyset$ and $K = \cup_{i \in I} K_i$. Let h_k be the flows on paths k that satisfy the conservation of flow and non-negativity constraints:

$$\sum_{k \in K_i} h_k = g_i \quad \forall i \in I, \quad (2.1)$$

$$h_k \geq 0 \quad \forall k \in K. \quad (2.2)$$

The corresponding link flows v_a are given by:

$$v_a = \sum_{k \in K} \delta_{ak} h_k, \quad \forall a \in A, \quad (2.3)$$

where

$$\delta_{ak} = \begin{cases} 1 & \text{if link } a \text{ is on path } k, \\ 0 & \text{otherwise.} \end{cases} \quad (2.4)$$

With constraints (2.1) - (2.4), the user optimal objective is

$$\min S(v) = \sum_{a \in A} \int_0^{v_a} c_a(x) dx. \quad (2.5)$$

In order to solve for user equilibrium, the (Wardrop 1952) user equilibrium condition is applied:

$$c_k(v^*) - u_i^* \begin{cases} = 0 & \text{if } h_k^* > 0 \\ \geq 0 & \text{if } h_k^* = 0 \end{cases}, \quad \forall k \in K_i, i \in I, \quad (2.6)$$

where

$$c_k(v) = \sum_{a \in A} \delta_{ak} c_a(v), \quad \forall k \in K \quad (2.7)$$

and

$$u_i = \min_{k \in K_i} [c_k(v)], \quad \forall i \in I. \quad (2.8)$$

To elaborate, this condition means that the traffic is in equilibrium when no traveller in the network can find a faster route than the one that is already being travelled on. Furthermore, this condition assumes that the travellers have complete knowledge about the network, they always choose the best route to travel based on the current information about the network. This means the equilibrium is the result of everybody simultaneously attempting to minimize their own travel times.

The *Bureau of Public Roads* (1964) (BPR) link cost function $c_a(v_a)$ is often used to model the travel time on link $a \in A$. The function is given by

$$c_a(v_a) = c_f \left(1 + B \left(\frac{v_a}{C_a} \right)^\alpha \right) \quad (2.9)$$

where B and α are the parameters for the level of congestion that can be experienced, and c_f and C_a are the free-flow travel time and link capacity. It is important to note this cost function only depends on traffic flow on its own link, and it is strictly monotonic, continuous and differentiable. An example of this link cost function is shown in Figure 2.1

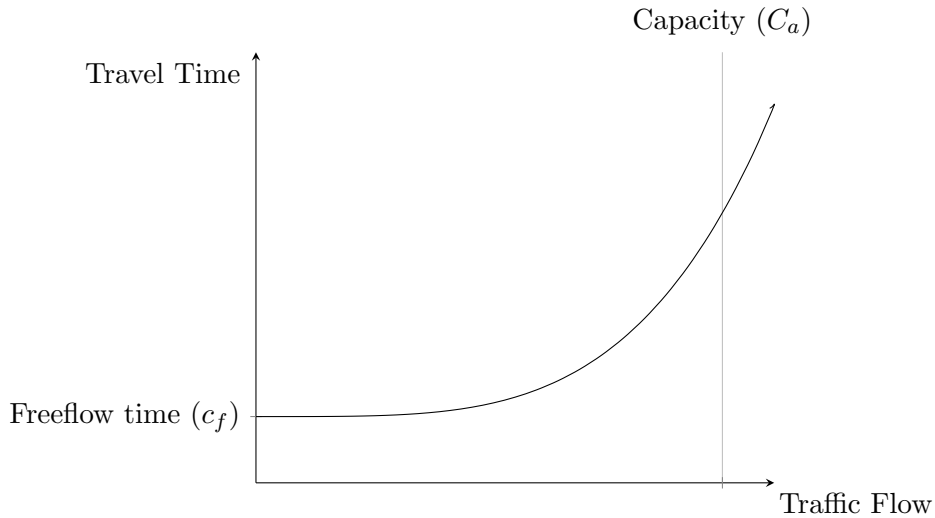


Figure 2.1: Travel time function.

2.2 The path equilibration algorithm

The deterministic symmetric network equilibrium model described in the previous section is equivalent to a convex cost differentiable optimization problem, where a wide range of algorithms

exists for solving such problems. They include, the linear approximation method, the linear approximation with parallel tangents method, the restricted simplicial decomposition method, and the path equilibration algorithm, see Florian & Hearn (1995).

For this project, we focus on the path equilibration algorithm.

why? reformat to a new section for solving TAP

The general approach of the algorithm is equivalent to Gauss-Seidel decomposition (an iterative method for solving a linear system of equations). In a step of the algorithm, path flows $h_k \forall k \in K_i$ between a single O-D pair i are considered by keeping the flows of all other O-D pairs fixed. The algorithm iteratively solves each O-D pair sub-problem until any of them cannot be improved.

The sub-problem for solving each O-D pair i is another fixed-demand network equilibrium problem.

$$\min \sum_{a \in A} \int_0^{v_a^i + \bar{v}_a} c_a(x) dx \quad (2.10)$$

$$\text{s.t.} \quad \sum_{k \in K_i} h_k = \bar{g}_i, \quad i \in I, \quad (2.11)$$

$$h_k \geq 0, \quad k \in K_i, \quad (2.12)$$

where flows \bar{v}_a are fixed:

$$\bar{v}_a = \sum_{i' \neq i} \sum_{k \in K_{i'}} \delta_{ak} h_k, \quad (2.13)$$

and flows v_a^i are varied:

$$v_a^i = \sum_{k \in K_i} \delta_{ak} h_k. \quad (2.14)$$

The Gauss-Seidel decomposition (or ‘cyclic decomposition’ by O-D pair) is stated as follows.

‘cyclic decomposition’ by O-D pair (Florian & Hearn 1995)

Step 0. Given initial solution, set $l = 0$, $l' = 0$.

Step 1. If $l' = |I|$, stop; otherwise set $l = l \bmod |I| + 1$ and continue.

Step 2. If the current solution is optimal for the i th sub-problem (2.10)-(2.14), set $l' = l' + 1$ and return to step 1; otherwise solve the l th sub-problem, update the path flows, set $l' = 0$ and return to step 1.

The path equilibration algorithm for solving (2.10)-(2.14) finds the solution of the traffic assignment problem by balancing path flows between each O-D pair. One such algorithm, proposed

by Dafermos (1971), finds the shortest and longest path and equalizes the travel times between them. Let $K_i^+ = \{k \in K_i | h_k > 0\}$ be the set of paths with positive flows.

The algorithm for solving each O-D pair i is stated as follows.

Path Equilibration Algorithm (Florian & Hearn 1995)

Step 0. All-or-nothing assignment: find an initial solution v_a^i ; $c_a = c_a(v_a^i + \bar{v}_a)$ and the initial K_i^+ .

Step 1. Compute the costs of the currently used paths c_k , $k \in K_i^+$. Find k_1 such that $c_{k_1} = \min_{k \in K_i^+} [c_k]$ and k_2 such that $c_{k_2} = \max_{k \in K_i^+} [c_k]$.

If $(c_{k_2} - c_{k_1}) \leq \epsilon$, go to step 4; otherwise define the direction $d_{k_1} = (c_{k_2} - c_{k_1})$ for path flow k_1 and $d_{k_2} = (c_{k_1} - c_{k_2})$ for path flow k_2

Step 2. Find the step size λ which redistributes the flow $h_{k_1} + h_{k_2}$ between the paths k_1 and k_2 in such a way that their costs become equal, that is, solve

$$\min_{\lambda} \sum_{a \in A} \int_0^{v_a^i + \lambda y_a + \bar{v}_a} c_a(x) dx \quad (2.15)$$

$$\text{s.t. } 0 \leq \lambda \leq \left(\frac{-h_{k_2}}{d_{k_2}} \right), \quad (2.16)$$

$$\text{where } y_a = \delta_{ak_1} d_{k_1} - \delta_{ak_2} d_{k_2}. \quad (2.17)$$

Step 3. Using the λ obtained, update $h_k = h_k + \lambda d_k$, $k = \{k_1, k_2\}$; $v_a^i = v_a^i + \lambda y_a$; $c_a = c_a(v_a^i + \bar{v}_a)$.

Step 4. Compute the shortest path \tilde{k} with cost $\tilde{s}_k = \min_{k \in K_i^+} [c_k]$; if $\tilde{s}_k < \min_{k \in K_i^+} [c_k]$, then the path \tilde{k} is added to the set of kept paths, $K_i^+ = K_i^+ \cup \tilde{k}$ and return to step 1; otherwise stop.

fix this, see page 8 Andrea fix

In step 0, a so called all-or-nothing assignment is performed for each of the O-D pairs, where it finds the shortest path and assigns all traffic flows along that path. In step 1 and 2, the algorithm finds the two paths that have the minimum and maximum cost, and balances the flow between them to equalize their costs. These two steps are equivalent to solving the Wardrop equilibrium shown in Equations (2.6)-(2.8). In step 4, the shortest path between the O-D pair is computed and added to the set of used paths for the all-or-nothing assignment and Wardrop equilibrium.

rewrite

Now if we are given a large network and assume it requires many iterations to find the optimal solution, it can be shown that a large number of shortest path calculations are needed. For example, given the algorithm takes 20 iterations to solve a small network with 100,000 O-D pairs, it can take more than 6 minutes to solve if each shortest path calculation consumes 0.01 second. In reality, networks may contain millions of O-D pairs. Thus for this project we wish to investigate and find a faster shortest path algorithm, and also investigate other ways of improving it for the traffic assignment problem.

2.3 Convergence and stopping criterion

For completeness, the convergence criterion for the traffic assignment is discussed in this section. It is known that the objective function of the problem (Equation 2.5) is convex (any local minimum is also the global minimum), so the convergence of the Gauss-Seidel strategy is ensured (Florian & Hearn 2008). The convergence criterion of traffic assignment algorithms is normally measured by so called relative gap (Rose et al. 1988). Rose et al. (1988) states that “the relative gap is expressed by the difference between the current value of the objective function and the lower bound as a percentage of the current objective function.” Here the objective function is the user equilibrium (UE) solution of the traffic assignment problem. And the lower bound refers to the all-or-nothing (AON) assignment in step 0 of the path equilibrium algorithm.

The relative gap (RGAP) is computed as

$$\text{RGAP} = \frac{\text{UE} - \text{AON}}{\text{UE}} \quad (2.18)$$

$$= \frac{\sum_{a \in A} v_a c_a - \sum_{i \in I} g_i c_{\tilde{k}_i}}{\sum_{a \in A} v_a c_a}. \quad (2.19)$$

The AON solution is the sum of travel times $c_{\tilde{k}_i}$ of each traffic demand g_i travelling on their shortest path \tilde{k}_i . The UE solution is the sum of travel times over the entire network. These two values are improved from iteration to iteration during the path equilibration algorithm until they become identical, i.e. RGAP is converging towards 0. The traffic assignment problem is solved when RGAP is 0, but normally we stop the algorithm at some tolerance such as 10^{-6} .

it doesn't decrease monotonically though...

It is worth to notice that the speed of convergence is highly dependent on how small we set the relative gap, as well as the size and complexity of the network and number of supply and demand nodes. A smaller relative gap will result in more iterations for the traffic assignment algorithms.

Chapter 3

Solving the Shortest Path Problem

Over the years, various algorithms have been developed to address the problem of finding the shortest path. This chapter states notation and definitions for the shortest path problem and discusses the theory of solving it. Algorithms that are applicable for the traffic assignment problem are summarised, including the discussion of their advantages and drawbacks.

see comment p.23 Andrea fix for what to add here

3.1 Notations and definitions

The Shortest Path Problem (SPP) is the problem of finding the shortest path from a given origin to some destination. There are two types of SPP that are going to be analysed in this chapter: a single-source and a point-to-point SPP. More emphasis is going to be put on the point-to-point SPP required by the path equilibration algorithm described in the previous chapter.

Now we present the notation mainly borrowed from Cormen et al. (2001) and Klunder & Post (2006). We denote $G = (N, A)$ a weighted, directed graph, where N is the set of nodes (origins, destinations, and intersections) and A the set of arcs (road). We say A is a subset of the set $\{(u, v) \mid u, v \in N\}$ of all ordered pairs of nodes. We denote the link (arc) cost function $c : A \rightarrow \mathbb{R}$ which assigns a cost (travel time) to any arc $(u, v) \in A$. We write the costs of link (u, v) as: $c((u, v)) = c_{uv}$.

The path P inside a transportation network has to be a directed simple path, which is a sequence of nodes and edges $(u_1, (u_1, u_2), u_2, \dots, (u_{k-1}, u_k), u_k)$ such that $(u_i, u_{i+1}) \in A$ for $i = 1, \dots, k-1$ and $u_i \neq u_j$ for all $1 \leq i < j \leq k$. Note u_1 is the origin and u_k is the destination of the path P , u_1 and u_k together is called an O-D pair for this path. For simplicity, we denote s to be the source (origin) and t to be the target (destination) for any path P .

In traffic assignment, the origin and destination nodes used for traffic supply and demand are referred as zone nodes. The zones are conceptual nodes that are untravellable in the network,

which means a path between two zone nodes must not pass through another zone node. Figure 3.1 demonstrates how a zone node behaves under different conditions. If the zone is an origin node, then only out-going arcs are allowed. If it is a destination node, then only in-coming arcs are allowed. And if it is neither, then no arcs can pass through it, otherwise an incorrect path is going to be created.

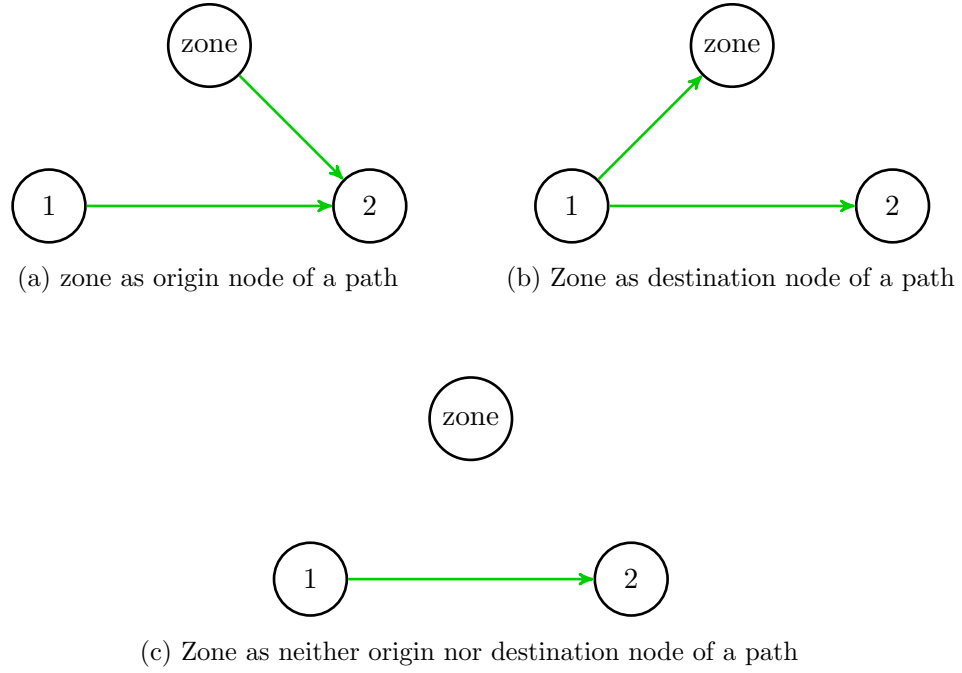


Figure 3.1: Behaviour of arcs on zone nodes acting as origin, destination or neither.

The following sections will use big O, $f \in O(g) : f \leq g$, $f = O(g)$: f does not grow asymptotically faster than g .

3.2 Generic shortest path algorithm

A family of algorithms exists for solving the shortest path problem. In this section the generic case for these algorithms is described.

This family of algorithms aims to find a distance label vector (d_1, d_2, \dots, d_v) , and the corresponding shortest path between each origin and destination (Klunder & Post 2006). Each d_v tracks the least distance of the path going from the origin s to a node v . We denote $d_v = \infty$ if no path P

from s to v has been found. A shortest path is optimal when it satisfies the following conditions:

$$d_v \leq d_u + c_{uv}, \quad \forall (u, v) \in A, \quad (3.1)$$

$$d_v = d_u + c_{uv}, \quad \forall (u, v) \in P. \quad (3.2)$$

The inequalities (3.1) are called Bellman's condition (Bellman 1958). To solve the shortest path problem, we wish to find a label vector d which satisfies Bellman's condition for all of the nodes in the graph. In the generic shortest path algorithm, it continuously finds a node that violates Bellman's condition (3.1) and updates its distance label with the path that has a shorter distance that connects to it. All shortest paths connecting the origin s to all other nodes in N are guaranteed to be found when both Equation (3.1) and (3.2) hold. It is important to note that arcs with negative costs are permitted, but the graph must not contain negative cycles.

Algorithms for solving the SPP generally use some kind of queue \mathcal{Q} to store the label distances d . In the label vector, a node is said to be

- *unvisited* when $d_u = \infty$,
- *scanned* when $d_u \neq \infty$ and it is still in the queue,
- *labelled* when the node has been retrieved from the queue and its distance label cannot be improved further.

If a node is labelled then its distance value is guaranteed to represent the minimal distance from s to t .

To keep track of the shortest path found so far for node u , we denote p_u the predecessor of node u . The shortest path can be constructed by following the predecessor of the destination node t back to the origin node s . We set $p_s = -1$ to indicate that node s does not have a predecessor.

Algorithm 1 (Klunder & Post 2006) describes the generic shortest path algorithm mentioned above, with an extra constraint required when solving a TA problem: travelling through zone nodes is not permitted. In essence, this algorithm repeatedly selects node $u \in \mathcal{Q}$ and updates its distance label if Bellman's condition is violated for all its emanating edges.

Algorithm 1 is generic because the rule for selecting the next node u (the 'next' function in line 8) and the implementation of the queue \mathcal{Q} are unspecified. Different algorithms use different rules and implementations to give either the one-source or the point-to-point shortest path algorithm (Pallottino & Scutellà 1997). The rules and implementations are described in the following sections to give concrete algorithms for solving the SPP.

Algorithm 1 The Generic Shortest Path Algorithm

```

1: procedure GENERICSHORTESTPATH( $s$ )
2:    $\mathcal{Q} \leftarrow \mathcal{Q} \cup \{s\}$  ▷ initialise queue with source node
3:    $p_s \leftarrow -1$  ▷ origin has no predecessor
4:    $d_s \leftarrow 0$ 
5:   for all  $u \in N : u \neq s$  do ▷ all nodes are unvisited except the source
6:      $d_u \leftarrow \infty$ 
7:   while  $\mathcal{Q} \neq \emptyset$  do
8:      $u \leftarrow \text{next}(\mathcal{Q})$  ▷ select next node
9:      $\mathcal{Q} \leftarrow \mathcal{Q} \setminus \{u\}$ 
10:    if  $u$  is not zone or  $u = s$  then
11:      for all  $v : (u, v) \in A$  do ▷ check Bellman's condition for all successors of  $u$ 
12:        if  $d_u + c_{uv} < d_v$  then
13:           $d_v \leftarrow d_u + c_{uv}$ 
14:           $p_v \leftarrow u$ 
15:          if  $v \notin \mathcal{Q}$  then
16:             $\mathcal{Q} \leftarrow \mathcal{Q} \cup \{v\}$  ▷ add node  $v$  to queue if unvisited

```

3.3 Bellman-Ford-Moore algorithm

When some specific strategy is applied to maintain the queue \mathcal{Q} and arc costs are allowed to have negative values, the generic shortest path algorithm is addressed as the label correcting algorithm, or Bellman-Ford-Moore algorithm (credited to Bellman (1958), Ford (1956) and Moore (1959)).

The algorithm is addressed as the label correcting algorithm because the distance labels do not get permanently labelled when a node u is retrieved from the queue, the distance label of node u may be ‘corrected’ by some other node v when v is retrieved from the queue and re-updates u .

not sure

One specific strategy for maintaining the queue is described in Sheffi (1985).

ref THE D'ESOPPO-PAPE ALGORITHM, implementation and efficiency of moore-algorithms for the shortest route problem

This strategy is shown to be very effective for computing shortest path on transportation networks. It avoids duplicating computation by not physically moving (or reordering) nodes in the queue,

as well as not adding nodes to the queue if they are already in it. The nodes in the queue are simply processed from front to end. Scanned nodes are firstly added to the end of the queue, and if the scanned node is already in the queue, then the node is put in front of the queue so they can be processed first. This strategy transforms the queue into a *double-ended queue* (or *deque*).

In order to satisfy the Bellman's condition for all edges, the algorithm has to scan all edges $|N| - 1$ times, resulting in a run time of $O(|N||A|)$.

3.4 Dijkstra's algorithm

The classic algorithm for solving the single-source shortest path problem is the label setting algorithm published by Dijkstra (1959). The algorithm is addressed as label setting because when the next node u is retrieved from the queue, it gets permanently labelled, The shortest path going to this node is solved and the distance label represents the length of its shortest path. In order to achieve label setting, it is assumed that all arc costs are non-negative, and queue is modified to always have the minimum distance label in the front. This modification allows the algorithm to visit every node in the graph exactly once, where the next node is labelled in the order of non-decreasing distance labels.

The advantage of this algorithm is that, when the next labelled node is the destination node, the algorithm can be stopped for the point-to-point shortest path problem. This reduces the total run time as the algorithm does not have to scan the entire graph, which is desirable for the Path Equilibration algorithm described in the previous chapter.

3.4.1 Priority queues

draw a min-heap tree?

The run time performance of Dijkstra's algorithm depends heavily on the implementation of the queue for storing the scanned nodes. Cormen et al. (2001) suggests the use of priority queues. Priority queues are a collection of data structures that they keep track of elements using an extra priority attribute. This allows an element with a higher priority to be processed before an element with a lower priority when iterating through the queue. In the shortest path problem, the priority is measured by the distance labels, where smaller distance labels have higher priority. Priority queues that have smaller values come first are addressed as min-priority queues, whereas larger values come first are addressed as max-priority queues.

Algorithm 2 shows the use of the min-priority queue in Dijkstra's algorithm. The min-priority queue has three main operations: Insert, Extract-Min and Decrease-Key. The Insert operation (line 2 and 17) is used for adding new nodes to the queue. The Extract-Min operation (line 8) is used for getting the element with the minimum distance label. The Decrease-Key (line 19) is used for updating the distance label if the node is already in the queue.

Algorithm 2 Point to Point Dijkstra's Algorithm

```

1: procedure DIJKSTRA( $s, t$ )
2:   Insert( $\mathcal{Q}, u$ )                                 $\triangleright$  initialise priority queue with source node
3:    $p_s \leftarrow -1$                                  $\triangleright$  origin has no predecessor
4:    $d_s \leftarrow 0$ 
5:   for all  $u \in N : u \neq s$  do                     $\triangleright$  all nodes are unvisited except the source
6:      $d_u \leftarrow \infty$ 
7:   while  $\mathcal{Q} \neq \emptyset$  do
8:      $u \leftarrow \text{Extract-Min}(\mathcal{Q})$                  $\triangleright$  select next node with minimum value
9:     if  $u = t$  then
10:      Terminate Procedure                             $\triangleright$  finish if next node is the destination
11:     if  $u$  is not zone or  $u = s$  then
12:       for all  $v : (u, v) \in A$  do                 $\triangleright$  check Bellman's condition for all successors of  $u$ 
13:         if  $d_u + c_{uv} < d_v$  then
14:            $d_v \leftarrow d_u + c_{uv}$ 
15:            $p_v \leftarrow u$ 
16:           if  $v \notin \mathcal{Q}$  then
17:             Insert( $\mathcal{Q}, v$ )                         $\triangleright$  add node  $v$  to queue if unvisited
18:           else
19:             Decrease-Key( $\mathcal{Q}, v$ )                     $\triangleright$  else update value of  $v$  in queue

```

According to Cormen et al. (2001), a min-priority queue can be implemented via an array, a binary min-heap or a binary search tree, where each implementation gives different run time performances.

In the array implementation, the distance labels are stored in an array where the n^{th} position gives the distance value for node n . Each Insert and Decrease-Key operation in this implementation takes $O(1)$ time, and each Extract-Min takes $O(|N|)$ time (searching through the entire array), giving a overall time of $O(|N|^2 + |A|)$.

A binary min-heap is a binary tree that satisfies the min-heap property: the value of each node is

smaller or equal to the value of its child nodes. Cormen et al. (2001) shows that the performance of the Dijkstra's algorithm can be improved with a binary min-heap if the graph is sufficiently sparse (in particular $A = o(|N|^2 / \log(|N|))$ ¹). In this implementation, each Insert and Extract-Min operation takes $O(\log(|N|))$ time in total of $|N|$ times, and the Decrease-Key operation takes $O(\log(|N|))$ time in total of $|A|$ times. The total running time of Dijkstra's algorithm using min-priority is therefore $O((|N| + |A|) \log(|N|))$, which is an improvement compared to the array implementation.

The running time can be further improved using a Fibonacci heap developed by Fredman & Tarjan (1987). Historically, the development of the Fibonacci heap was motivated by the observation that Dijkstra's algorithm typically does more Decrease-Key operations compared to Extract-Min operations. In Fibonacci heap, each of the $|N|$ Extract-Min operation takes $O(\log(|N|))$ amortized time and each of the $|A|$ Decrease-Key operation takes only $O(1)$ amortized time². The total running time is therefore $O(|N| \log(|N|) + |A|)$, which is a further improvement.

Min-priority queue can also be implemented as a binary search tree. In a binary search tree, the worst case for insertion, deletion and search for an element all have $O(\log(|N|))$ time. Dijkstra's algorithm can easily be modified to accommodate a binary search tree: when label distance of a node needs to be updated, we remove that node from the tree and insert a new one with the updated value (this is analogous to the Decrease-Key operation). Dijkstra's algorithm using a binary search runs $O((|N| + |A|) \log(|N|))$ in the worst case, which is the same compared to the min-binary heap implementation. The advantage of using a binary search tree is that we do not have to keep track of information about whether a node is in the queue, this is because when performing the Decrease-Key operation, we simply delete the node and add a new node with the updated value.

Say I will be comparing all these

3.5 Bidirectional Dijkstra's algorithm

Dijkstra's algorithm can be imagined to be searching radially outward in a circle with the origin in the centre and destination on the boundary. Likewise, Dijkstra's algorithm can be used on the reverse graph (all edges reversed in the graph) from the destination node. Thus Dijkstra's

¹The little-o notation gives the strict upper bound, i.e. the size of A grows asymptotically slower than $|N|^2 / \log(|N|)$. $f \in o(g) : f < g$

²Amortized time gives how much time is taken in total when an operation is repeated a million times. For example with different inputs, run time is averaged between the worst-case and the best-case.

algorithm can be started from the origin and destination at the same time. The motivation for doing this is because the number of scanned nodes can be reduced when searching bidirectionally: two smaller circles growing outward radially instead of a larger one. Figure 3.2 demonstrates the difference in search area between Dijkstra's algorithm and its bidirectional version on a planar graph using Euclidean distances. It is easy to see that the total search area of the bidirectional version is a lot smaller.

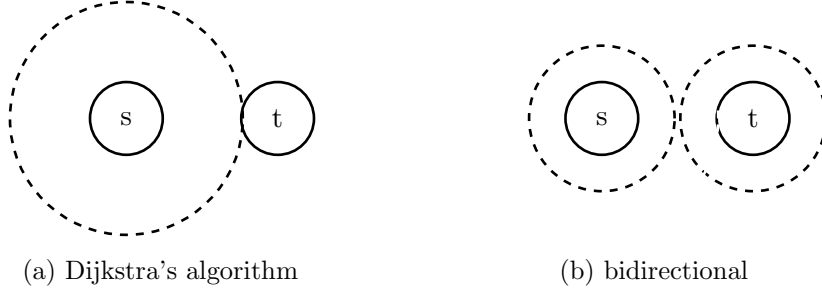


Figure 3.2: Difference between the scanned area of label setting and its bidirectional version

It is common to conclude that the shortest path is found when the two searches meet somewhere in the middle, but this is not actually the case. There may exist another arc connecting the two frontiers of the searches that has a shorter path (see Klunder & Post (2006) for the proof). The correct termination criteria was first designed and implementation by Pohl (1971) based on research presented by Dantzig (1963), Nicholson (1966) and Dreyfus (1969). The procedure and algorithm are summarised in Klunder & Post (2006), and the termination criteria is presented by Pohl (1971).

The bidirectional search algorithm is shown in Algorithm 3. Two independent Dijkstra's algorithms are alternatively run on the forward and reverse graph (forward and backward algorithm). The algorithm terminates when some node is permanently labelled in both directions. Once the algorithm has terminated, the correct shortest path is found by looking for an arc connecting the frontiers of the two searches that may yield a shorter path. This extra requirement increases the run time significantly, where searches are needed for all edges that connect all labelled nodes in the forward search to all labelled nodes in the backward search.

Note in Algorithm 3, \mathcal{R}^s contains the set of nodes v in the forward search that are permanently labelled from the origin node s , \mathcal{R}^s have corresponding label distances d_v^s . Similarly, \mathcal{R}^t contains the set of nodes v in the backward search that are permanently labelled from the destination node t , \mathcal{R}^t have corresponding label distances d_v^t .

Algorithm 3 Bidirectional Dijkstra's Algorithm

- 1: **procedure** BIDIRECTIONAL(s, t)
 - 2: Execute one iteration of the forward algorithm. If the next node u is already labelled permanently by the backward algorithm ($u \in \mathcal{R}^t$), go to step 3. Else, go to step 2.
 - 3: Execute one iteration of the backward algorithm. If the next node u is already labelled permanently by the forward algorithm ($u \in \mathcal{R}^s$), go to step 3. Else, goto step 1.
 - 4: Find $\min\{\min\{d_v^s + c_{vw} + d_w^t \mid v \in \mathcal{R}^s, w \in \mathcal{R}^t, (v, w) \in A\}, d_u^s + d_u^t\}$, which gives the length of the correct shortest path between s and t .
-

In recent years, Goldberg & Werneck (2005) improved the bidirectional algorithm using a better termination condition, where step 3 of Algorithm 3 is embedded during the searches. The termination condition is summarized as follows. During the forward and backward search, we maintain an extra variable, μ , to present the length of shortest path seen so far during the forward and backward search. Initially $\mu = \infty$. When arc (v, w) is visited by the forward search and node w has been scanned in the backward search, or vice versa, we know the shortest path $s - v$ and $w - t$ have lengths d_v^s and d_w^t respectively. During the search, if $\mu > d_v^s + c_{vw} + d_w^t$ then the current connected path $s - v - w - t$ is shorter than the one before, so we update $\mu = d_v^s + c_{vw} + d_w^t$. The algorithm terminates when a node is permanently labelled in both directions, where μ gives the length of the shortest path.

Goldberg, Harrelson, Kaplan & Werneck (2006) showed and proved a stronger termination condition on top of the previous one. The searches can be stopped if the sum of the two top priority queue values is greater than μ ,

$$\text{top}_f + \text{top}_r \geq \mu,$$

where top_f and top_r are the next minimum distance labels that have not been labelled in the forward and backward search.

Overall, the bidirectional version of Dijkstra's algorithm can be faster than the single direction one if it is implemented correctly using the best termination criterion.

also depends on the network?

3.6 A* Search

Dijkstra's algorithm can be imagined as growing the shortest path tree radially out from the origin, the location of the destination does not affect how the shortest path tree is grown. In fact, heuristic estimates can be used to guide the shortest path tree towards the destination, forming an ellipsoid shape. The use of heuristic estimates was first described by Hart et al. (1968), where the algorithm is given the name A* search. A* search is a goal directed search where the direction of search is aimed towards the destination.

Let h_v be the heuristic estimate for the shortest path distance between node v to destination t . We apply Bellman's conditions such that an optimal solution exists, that is

$$h_v \leq h_u + c_{uv} \quad \forall (u, v) \in A, \quad (3.3)$$

$$h(t) = 0. \quad (3.4)$$

Although h_v is a heuristic function, optimal solutions (i.e. shortest path) can still be obtained under the following conditions. Hart et al. (1968) state that the heuristic estimate function h must be admissible and consistent. Admissible means that the heuristic must never over-estimate the distance to the goal. Consistent means that the estimated path length of a node to its destination must not be greater than the estimated length from its predecessors (i.e. Equation (3.3) and (3.4)). These two conditions mean that if the heuristic estimate is smaller than or equal to the actual distance going to that destination, then the optimal shortest path can always be found. In graphs that are measured by Euclidean distances, the heuristic estimate h_u is the straight Euclidean distance between node u and destination t .

To implement A* search, Dijkstra's algorithm is modified. When a node v is about to be added to the priority queue, the heuristic estimate h_v is calculated. Compared to Dijkstra's algorithm, instead of inserting node v with its distance label d_v , we now insert with $d_v + h_v$. By doing so, nodes that are closer to the destination are now labelled first.

3.6.1 A* search in traffic assignment

In our traffic assignment problem, geographical coordinates and Euclidean distances cannot be used. This is because the length of the arcs are determined by the BPR link cost function, where it determines the travel time on the link based on the traffic flow. It is important to note here that in the path equilibration algorithm for solving the traffic assignment problem, the traffic flow changes in each iteration of the algorithm and so does the travel time. When the traffic

assignment is getting solved, there are only two ways to obtain the travel time estimate from any given node to the destination. They are either to use the travel time from the previous iteration, or use the zero-flow travel times.

It is obvious that using travel times from the previous iteration does not work. This is because traffic flows can decrease from one iteration to the next. The decreased traffic flows result in non-admissible travel times, where some of the arc travel times from the previous iteration is longer than the current travel times. Using these longer travel times from the previous iteration will overestimate the travel time for the current iteration, and the resulting shortest path is not guaranteed to be optimal.

Another option is to use zero-flow travel times for the heuristic estimates. The estimates h_u can be obtained from computing the shortest path tree for every node u where the traffic flow of the entire network is set to 0. So when solving for the shortest path of any O-D ($s - t$) pair when there are traffic flows, h_u is equivalent to the zero-flow travel time of the shortest path from node u to destination t . These computed zero-flow travel times are both admissible and consistent. This can be shown by analysing the BPR link cost function shown in Figure 2.1. The function is a monotonic non-decreasing function with the lowest value being the zero-flow travel times. As traffic flows change from iteration to iteration, the travel times can never be smaller than the zero-flow travel times so they will never be an overestimate. This means zero-flow travel times can be used for the heuristic estimates, and the shortest path can be guaranteed to be optimal.

Overall, A* search does not improve the worst case time complexity compared to Dijkstra's algorithm, but it can improve the average case by scanning less nodes in the network. It can be seen that when $h_u = 0, \forall u \in N$, A* search is equivalent to Dijkstra's algorithm. A* search has been tested in various studies. In the case of using road networks, Goldberg & Harrelson (2005) conclude that A* search with Euclidean distance estimates does not improve the run time compared to Dijkstra's algorithm when using Euclidean distance estimates. However, it is still worth a try in the context of solving the traffic assignment problem using zero-flow travel times as the heuristic estimates.

3.7 Bidirectional A* search

Since bidirectional search can be applied to Dijkstra's algorithm, it can also be applied to A* search. The bidirectional A* search can be described by two ellipsoids extending from the origin and destination respectively. This is shown in Figure 3.3, where bidirectional A* search potentially has a smaller search area than the unidirectional version.



Figure 3.3: Difference between the scanned area of A* search and its bidirectional version

One may construct the algorithm with the same termination condition described in the bidirectional Dijkstra's algorithm section (Section 3.5), that is to stop the algorithm when the two frontiers of the searches meet. But the problem with A* search is that, it does not label the nodes permanently in the order of their distance from the origin (Klunder & Post 2006). In other words, the forward and backward heuristic estimates are no longer consistent, it can cause either wrong shortest path to be calculated or having the two search frontiers never meet.

The correct strategy for calculating the heuristic estimates and termination criterion is first published by Pohl (1971). The heuristic calculation was later improved by Ikeda et al. (1994).

Ikeda et al. (1994) demonstrates that two distinct admissible heuristic functions π_f and π_r for the forward and backward search are not consistent for the bidirectional search, but their average is both admissible and consistent. Here $\pi_f(v)$ is the estimate of distance from node v to destination t in the forward search and $\pi_r(v)$ is the estimate of distance from origin s to node v in the backward search. The new heuristic functions are:

$$p_f(v) = \frac{1}{2}(\pi_f(v) - \pi_r(v)) + \frac{\pi_r(t)}{2}, \quad (3.5)$$

$$p_r(v) = \frac{1}{2}(\pi_r(v) - \pi_f(v)) + \frac{\pi_f(s)}{2}. \quad (3.6)$$

The constants $\frac{\pi_r(t)}{2}$ and $\frac{\pi_f(s)}{2}$ are added by Goldberg, Harrelson, Kaplan & Werneck (2006) to provide better estimates. These two modified heuristic estimates are now consistent and the frontiers of the two searches are guaranteed to meet. The drawback of this modification is that they now provide worse bounds compared to the original π values, and the search area may now be larger than the unidirectional A* search.

Goldberg, Harrelson, Kaplan & Werneck (2006) showed and proved a better stopping criterion compared to the one published by Pohl (1971), where they extended the bidirectional Dijkstra's termination criterion, bidirectional A* search has to be stopped when

$$\text{top}_f + \text{top}_r \geq \mu + p_r(t). \quad (3.7)$$

Here μ is the best $s - t$ path seen so far during the search, top_f and top_r are the minimum distance labels in the forward and backward search respectively (they are the nodes at the top (front) of the priority queues).

Bidirectional A* search with Euclidean distance heuristic estimates has been tested by different academics, e.g. Klunder & Post (2006) and Goldberg & Werneck (2005). It is not evident whether bidirectional A* search can guarantee significant improvement, as the result is heavily dependent on the configuration of the road network and the quality of the heuristic estimates.

nether the less, we implement and test this for traffic assignment

3.8 Preprocessing algorithms

In the last two decades, extensive research has been done on the idea of speeding up shortest path calculations using pre-calculated data. They include reach-based pruning (Goldberg, Kaplan & Werneck 2006), A* search with landmarks (Goldberg & Harrelson 2005) and Hierarchical search (Ertl 1998, Pearson & Guesgen 1998), etc. These preprocessing techniques generally need to spend some time to pre-calculate the data required for speeding up the subsequent shortest path queries.

The next section discusses the A* search with landmarks, and its drawback when applied to the traffic assignment problem.

3.8.1 A* search with landmarks

From Section 3.6 follows that different methods can be used to estimate the shortest path distance h_u from node u to a specified destination. The landmarks algorithm developed by Goldberg & Harrelson (2005) is another way to obtain the heuristic estimates. First a small set of landmarks need to be positioned in different locations on the graph, where shortest path trees are calculated for every landmark using Dijkstra's algorithm. Heuristic estimates can now be obtained for every node in the graph using the shortest path trees of the landmarks, Figure 3.4 demonstrates the calculation of the estimates using the two triangle inequalities when a landmark is either placed in front of the scanned node v or behind the destination t . The two triangle inequalities are:

$$\text{dist}(v, t) \geq \text{dist}(L, t) - \text{dist}(L, v) \quad (3.8)$$

$$\text{dist}(v, t) \geq \text{dist}(v, L) - \text{dist}(t, L) \quad (3.9)$$

The heuristic estimate to be used during A* search is the maximum over all landmarks:

$$\text{dist}(v, t) \geq \max\{\text{dist}(L, t) - \text{dist}(L, v), \text{dist}(v, L) - \text{dist}(t, L)\}. \quad (3.10)$$

With these formulations, Goldberg & Harrelson (2005) concluded that it is best to place the landmarks in front of the origins or behind the destinations, i.e. around the outer boundary of the graph.

How about data storage of trees? Check Michelle's report on anything regarding under-estimation of time on trees

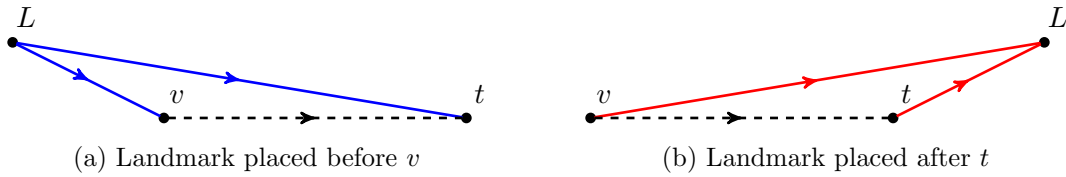


Figure 3.4: Explanatory diagram for triangle inequality

Although the preprocessing stage of the landmarks method can take a long time, it was shown that the shortest path solving times are significantly faster than A* search using Euclidean distance estimates (see results in Goldberg & Harrelson (2005)).

The landmarks method and similar preprocessing techniques have a few drawbacks that need to be considered. The first drawback is that the preprocessing time may be longer than the actual shortest path solving time, so combined running time may take longer.

you'd only do this once for 0-flows times though? And they are only a few SP trees - need to check with Goldberg's results

It also means that the preprocessing stage needs to be re-run whenever there is a physical change in the road network (e.g. a new road is added), which is undesirable for transportation planning because the road network changes constantly to improve congestion. Also both preprocessing and shortest path solving time are heavily dependent on the quantity and placement of the landmarks, which is another optimization problem where different strategies and algorithms need to be studied.

For the equilibration algorithm of the traffic assignment, it is unknown whether landmarks using zero-flow travel times can provide better estimates compared to Euclidean distances.

make this clearer: one could possibly use 0-flows landmarks, but we didn't look into that further

3.9 Techniques for iterative calculations

reformat, see page 23 and 24 in both fixes

before talking about SP at beginning of chapter, should make it VERY clear that in each iteration a static SP is being solved, hence we use standard algos

So far we have only been dealing with solving the shortest path problem on static graphs. In this section we discuss solving the problem in the path equilibration algorithm where the graph dynamically changes its arc costs between iterations.

3.9.1 Avoiding shortest path calculations

All of the shortest algorithms mentioned so far need to fully calculate the shortest path between every O-D pair in each iteration. It turns out that for every O-D pair in the path equilibration algorithm, their shortest path calculation can be avoided to reduce computational time by using solution from the previous iteration in the current iteration. Two situations can occur if we choose to do so. The first situation is when the shortest path between the previous and current iteration are the same, then we have successfully avoided the calculation and reduced the computational time. The second situation is when they are different, then the path equilibration algorithm is not going to converge further for the current iteration, which causes possible increase in total number of iterations and computational time.

rewrite, see page 24 Olga fix

While traffic flows and arc costs change between iterations, if we can prove that shortest path do not change often between iterations, then some strategies can be used to avoid the calculations. The overall computational time is reduced when most shortest path calculations are avoided on the O-D pairs that are not going to change often.

The first strategy is as follows. For each O-D pair, if its shortest path did not change in the last two iterations, then we can delay the next shortest path calculation by a few iterations. This strategy requires prior knowledge of how many iterations there is going to be during a standard run. This is because if we choose to skip more calculations than the total number of iterations, then there will be excessive iterations resulting in wasted time. If we skip too few iterations, then there may not be any impact on the computational time.

The other strategy is to skip shortest path calculations randomly. That is, when it comes to

calculate the shortest path for a given O-D pair, we generate a random number and decide whether to do the calculation based on that number. The advantage of this strategy is that we do not need to know how many iterations the algorithm will take. The disadvantage is that the computational time can vary between different runs, resulting unpredictable run times.

rewrite, see page 24 Olga and Andrea fix

Chapter 4

Implementation Details

The previous chapters have described all the algorithms that are considered for this project. In this chapter, specific implementation details of the algorithms are discussed.

4.1 Traffic assignment implementation

The path equilibration algorithm and other algorithms for solving the traffic assignment problem has already been implemented by Olga Perederieieva, the co-supervisor of this project. The algorithms are implemented in the C++ programming language.

The existing implementation of the path equilibration algorithm uses Bellman-Ford-Moore algorithm for its shortest path calculations as suggested by Sheffi (1985). When solving the traffic assignment problem, the algorithm spends most of its time computing shortest paths. Significant amount of time is also spent on the convergence check step mentioned in Section 2.3, where the algorithm requires to run the Bellman-Ford-Moore algorithm on all of the zones for the all-or-nothing solution.

maybe need profile output support to support this statement?

4.2 Graph storage

To obtain information from the graph when running shortest path algorithms, the storage of the underlying graph has been implemented in such a way that it can provide efficient access to its nodes and arcs. The current implementation uses the Forward Star data structure described in Sheffi (1985). The data structure compactly stores graphs in $O(|N| + |A|)$ spaces, and provides $O(1)$ random access to all of its nodes, it also provides $O(1)$ access to all out-going arcs of a randomly chosen node. Using the Forward Star ensures the run time of accessing the graph can be neglected when analysing the shortest path algorithms.

4.3 Priority queue implementations

As mentioned in the Dijkstra's algorithm section (Section 3.4), the performance of shortest path algorithms is heavily dependent on the implementation of the priority queue data structure. Various priority queue implementations exist, they include:

- `<priority_queue>` from the C++ standard template library,
- `<set>` from the C++ standard template library,
- `<heap>` from the C++ Boost library.

explain notation?

Each priority queue implementation has some advantages and disadvantages. For example some provide faster tree balancing while others provide faster Extract-Min or Delete operation. All of these implementations are going to be studied for this project.

First we examine the six variants of heap implementations from the C++ Boost `<heap>` library shown in Table 4.1. In the table, N is the number of elements in the priority queue and the time complexities are measured in amortized time ¹.

Table 4.1: C++ Boost Heap Implementations with Comparison of Amortized Complexity (Blechnann 2013)

	Insert	Extract-Min	Increase-Key	Decrease-Key
Binary	$O(\log(N))$	$O(\log(N))$	$O(\log(N))$	$O(\log(N))$
Ternary	$O(\log(N))$	$O(\log(N))$	$O(\log(N))$	$O(\log(N))$
Binomial	$O(\log(N))$	$O(\log(N))$	$O(\log(N))$	$O(\log(N))$
Fibonacci	$O(1)$	$O(\log(N))$	$O(1)$	$O(\log(N))$
Pairing	$O(2^{2\log(\log(N))})$	$O(\log(N))$	$O(2^{2\log(\log(N))})$	$O(2^{2\log(\log(N))})$
Skew	$O(\log(N))$	$O(\log(N))$	$O(\log(N))$	$O(\log(N))$

explain briefly what these Heaps are

We are interested in using Boost library Heaps rather than the C++ standard library Heap due to one reason: the Decrease-Key (or Increase-Key) operation. The operation is used to

¹Amortized time gives how much time is taken in total when an operation is repeated a million times. For example with different inputs, run time is averaged between the worst-case and the best-case.

change the distance labels in the priority queue when a node is scanned and updated during the shortest path calculations. The Decrease-Key operation is used for min-heap trees (minimum value on top) and the Increase-key operation is used for max-heap trees (maximum value on top). In Dijkstra's algorithm, nodes are often scanned multiple times in the label updating step, so instead of adding the node again into the Heap tree with a difference value, we can use the Decrease-Key operation. The advantage of using this operation is that we can reduce the size of the Heap tree, which results in a performance improvement as it takes less time to search and insert nodes to smaller heap trees.

In Table 4.1, we observe that the Fibonacci Heap has a very interesting time complexity. It has a constant amortized $O(1)$ time for the Insert and Increase-Key operation. This is very attractive for us, but the problem is that we do not know how much constant time it really uses behind its big O notation. In order to use the Fibonacci $O(1)$ Increase-Key operation, all of the distance labels need to be negated when inserted into the heap because C++ Boost Library heaps are implemented as max-heaps.

Next we examine `<priority_queue>` from the C++ standard library, which provides $O(\log(N))$ Insert and Extract-Min operation. This implementation neither has the Decrease-Key operation nor a way to change node values or delete a node somewhere other than the top of the heap tree. So when solving the shortest path problem, the priority queue is going to contain many nodes several times but with different distance labels. This is not a problem for our shortest path algorithms. When a node is added to the queue more than once with different distance labels, the one with the smaller distance label is always going to be in front of the queue waiting to be labelled first, so when that node is labelled, all the other same nodes with larger distance labels in the queue will not affect find the shortest path, this is because the successors of those nodes have already been assigned with smaller distance labels.

Finally we examine `<set>` from the C++ standard library. A *set* is a data structure used to store unique elements that follow a specific order. In the C++ standard library, it is implemented as a red-black binary search tree. This data structure can be used for our shortest path algorithms because it provides $O(\log(N))$ insert, search and delete operations. For our shortest path algorithms (i.e. Dijkstra's algorithm and A* search), we can modify them to accommodate the unique elements and specific ordering requirement. To meet the unique elements requirement, instead of using the Decrease-key operation whenever a node need to be updated, we simply delete that node and insert the one with the new value. And for the ordering requirement, we can just order the nodes non-decreasingly by their distance labels, so the node with the minimum label always comes first. The advantage of `<set>` compared to `<priority_queue>` is that nodes can be removed from anywhere in the data structure, so `<set>` may be faster than `<priority_queue>`

because the performance of the operations is heavily dependent on the number of nodes in the data structures.

how about storing data? e.g. shortest path trees?

Chapter 5

Experimental Results

In this chapter we present experimental results. All experiments are run under 12.04 Ubuntu Linux on an ASUS K46C laptop, which has four Intel Core i5-3317U CPU and 3.8GiB RAM.

how do I say later I found out I need more RAM so switch to another computer?

Various algorithms (including the path equilibration algorithm) for solving the traffic assignment problem has already been implemented by Olga Perederieieva, the co-supervisor of this project. The algorithms are written in C++, compiled by the g++ compiler using the ‘-O3’ optimization flag.

Bellman was only SP algorithm available, all the results in this chapter are based on newly developed code.

All timed results are measured for a complete run of traffic assignment. All solutions will use 10^{-6} as the relative gap for the stopping criterion.

see top of page 28 Andrea fix

5.1 Road networks

Table 5.1 shows the road networks used for this project, the network details are retrieved from Bar-Gera (2013). The table has an extra column showing the minimum number of iterations the path equilibration algorithm takes to solve the networks. Compared to other networks in the table, Anaheim, Barcelona and Winnipeg are small networks. Chicago Sketch is a medium sized network that is part of the Chicago Regional network. Berlin Center is a medium sized network. Philadelphia and Chicago Regional are two large networks that have over a million number of O-D pairs.

see bottom of page 28 Andrea fix

Table 5.1: Network Problem Data

Network	Category	Nodes	Arcs	Zones	O-D pairs	Iterations
Anaheim	small	378	914	38	1,406	10
Barcelona	small	910	2,522	110	7,922	27
Winnipeg	small	905	2,836	147	4,344	126
Chicago Sketch	medium	546	2,950	387	93,135	25
Berlin Center	medium	12,116	28,376	865	49,688	23
Philadelphia	large	11,864	40,004	1,525	1,149,795	81
Chicago Regional	large	11,192	39,018	1,790	2,296,227	152

5.2 Results on priority queues

In this section we study different priority queues discussed in Section 4.3. The priority queues are used in Dijkstra’s algorithm on the Winnipeg and Chicago Sketch networks. The results are shown in Figure 5.1 and 5.2. The exact numerical results can be found in Appendix A.

On both networks, `<priority_queue>` has the best performance and the Binomial heap has the worst performance. Skew heap has the best performance among the six heap implementations from the Boost library. Fibonacci heap has worse performance compared to some of the other implementations despite of its $O(1)$ Insert and Increase-Key operation. `<set>` is significantly slower than `<priority_queue>`.

see page 29 Andrea fix, clarify results discussion

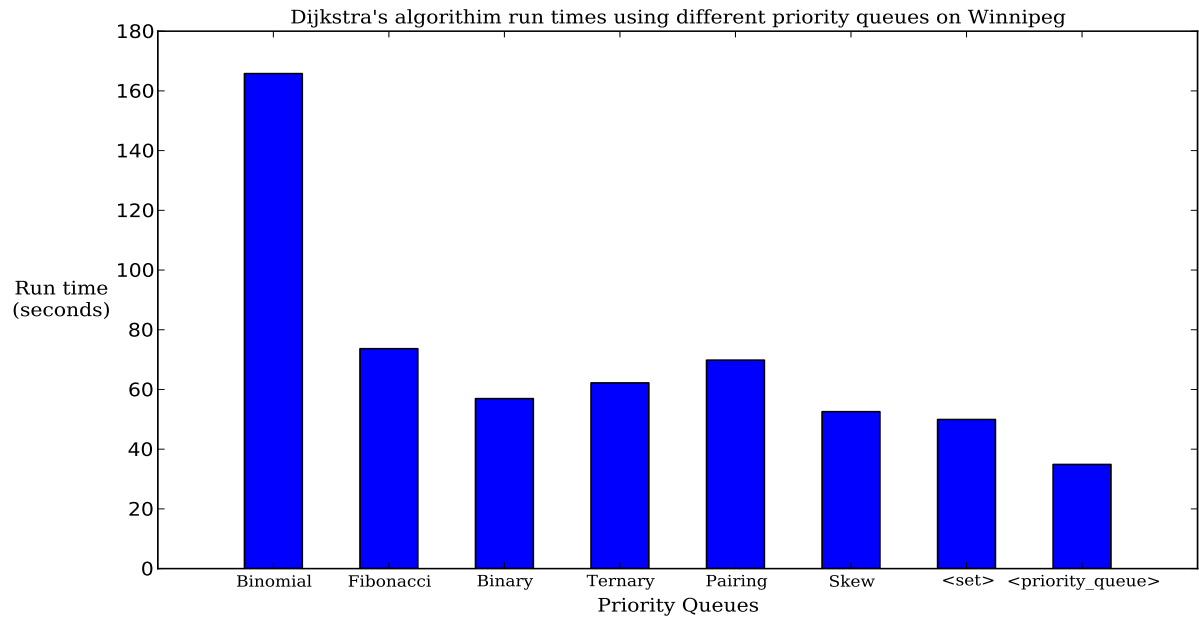


Figure 5.1: Full traffic assignment run times using Dijkstra's algorithm with different priority queues on Winnipeg, with a total number of 547,344 shortest paths solved.

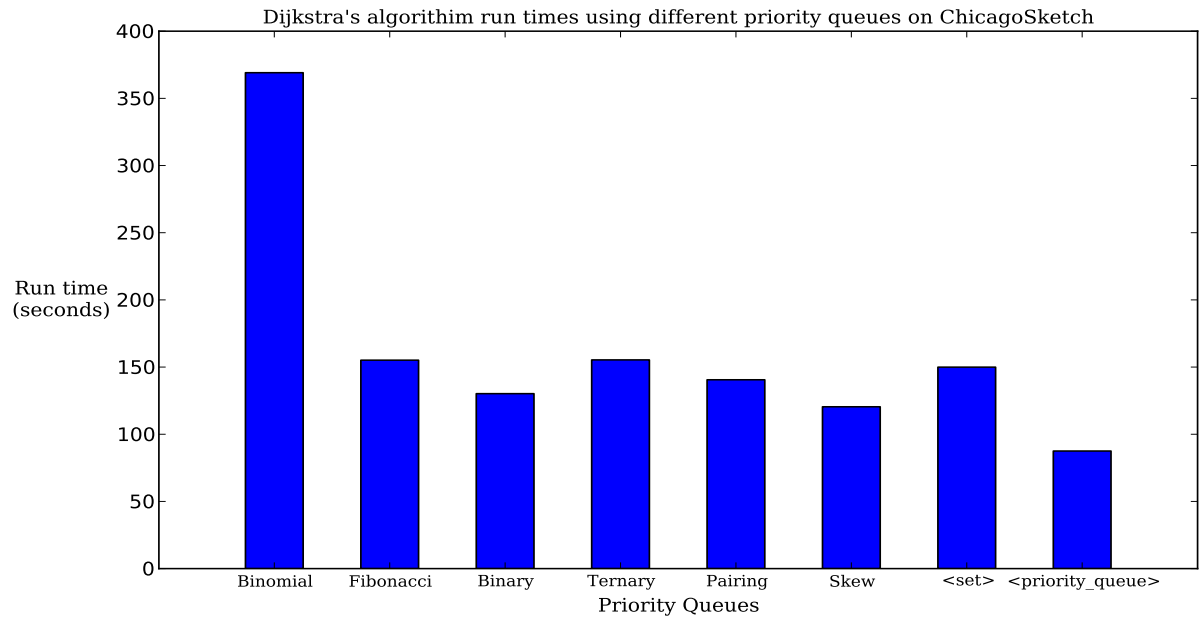


Figure 5.2: Full traffic assignment run times using Dijkstra’s algorithm with different priority queues on Chicago Sketch, with a total number of 2,328,375 shortest paths solved.

5.3 Results on shortest path algorithms

In Section 5.2 we have identified `<priority_queue>` to be the most suitable (efficient) data structure to be used in Dijkstra’s algorithm. Now we use `<priority_queue>`, implement and test the following algorithms:

- Bellman-Ford-Moore algorithm (existing),
- Dijkstra’s algorithm,
- Bidirectional Dijkstra’s algorithm,
- A* search,
- Bidirectional A* search.

Figure 5.3 shows the performance of the mentioned algorithms on the Anaheim, Barcelona, Winnipeg and Chicago Sketch networks (see Appendix B for exact numerical results). The

networks are spaced out on the horizontal axis to show their relative sizes, i.e. the number of O-D pairs.

Bellman-Ford-Moore algorithm has the worst performance while the A* search has the best performance on all networks. The bidirectional versions of Dijkstra's algorithm and A* search are more than twice slower than their unidirectional versions.

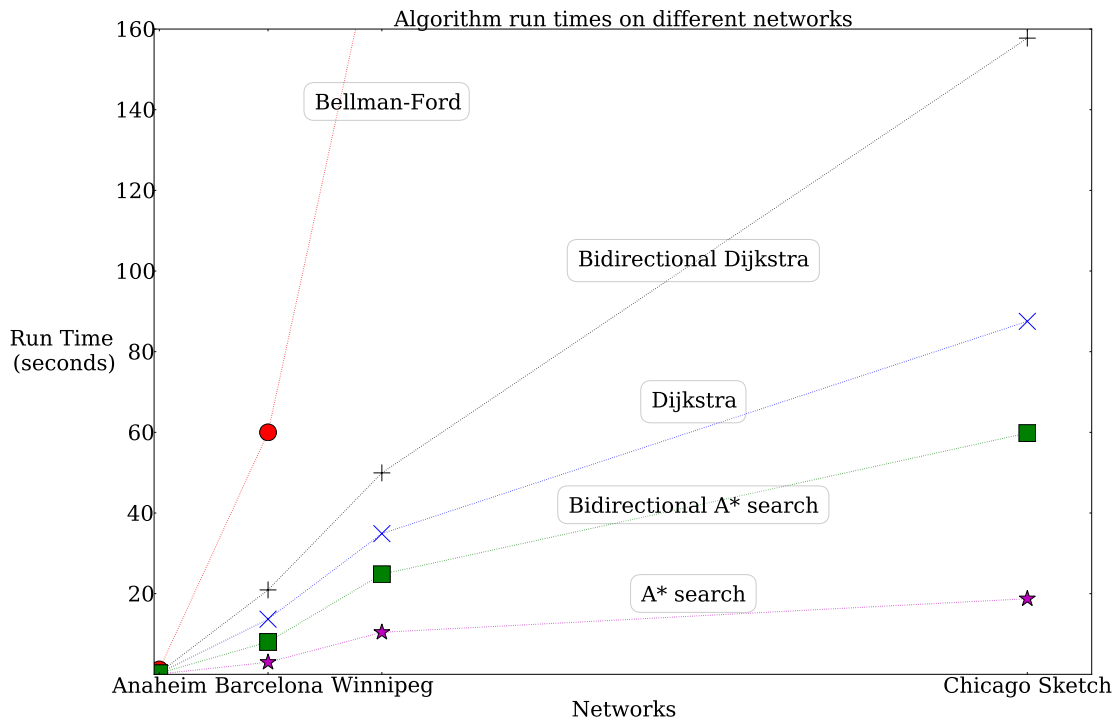


Figure 5.3: Run time performances of different algorithms on different networks

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5.4 Results on avoiding shortest path calculations

In Section 5.3 we have identified A* search to be the most suitable (efficient) algorithm for solving the point-to-point shortest path problem. In this section, we consider avoiding shortest

path calculations in the iterative path equilibration algorithm discussed in Section 3.9.1. A* search is used to generate results for this section.

As mentioned in Section 3.9.1, in order to apply shortest path avoiding strategies, we need to first prove that most of the shortest paths do not change often.

Figure 5.4 shows the histogram for the number of times the shortest path for an O-D pair changed 0,1,2,... times on the Chicago Sketch network. It shows out of 26 iterations, the percentage of O-D pairs that changed their shortest path once, twice, three times etc. The figure shows that 60% of O-D pairs have not changed their shortest path after the initial iteration, and 16% of O-D pairs changed their shortest path only once. This means that after the first iterations, the algorithm spends most of its time changing only a dozen of O-D pairs' shortest path out of 93,135. From these observations, it is assured that run time can be reduced if we avoid shortest path calculations on the paths that do not change between iterations.

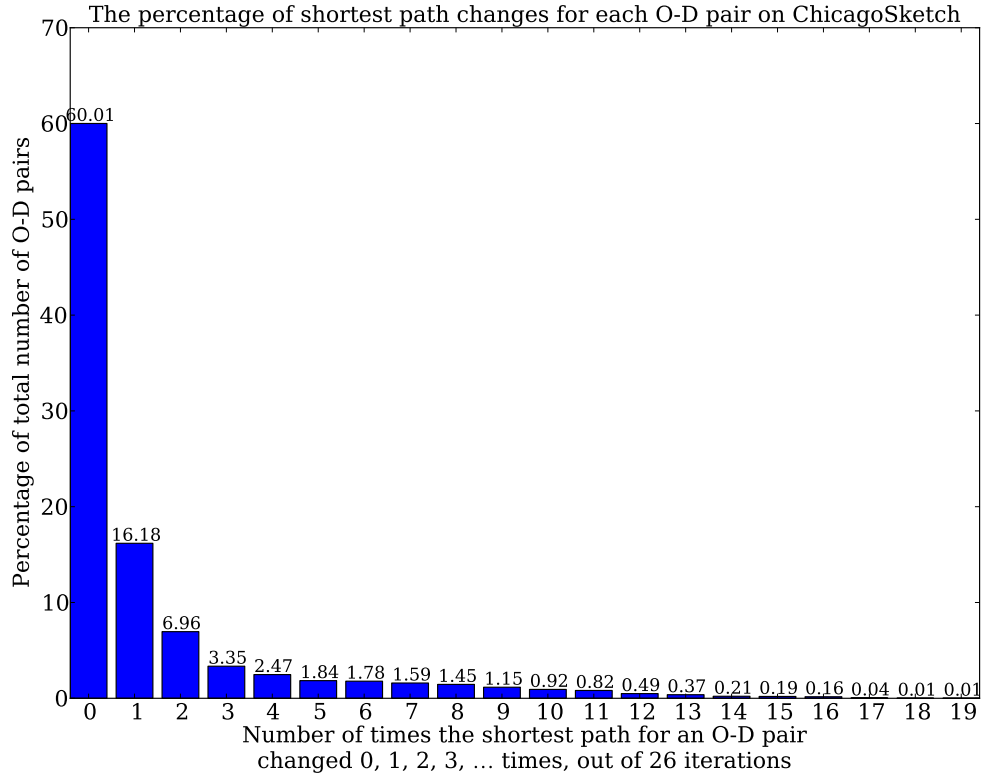


Figure 5.4: The percentage of shortest path change for each O-D pair out of 26 iterations for Chicago Sketch

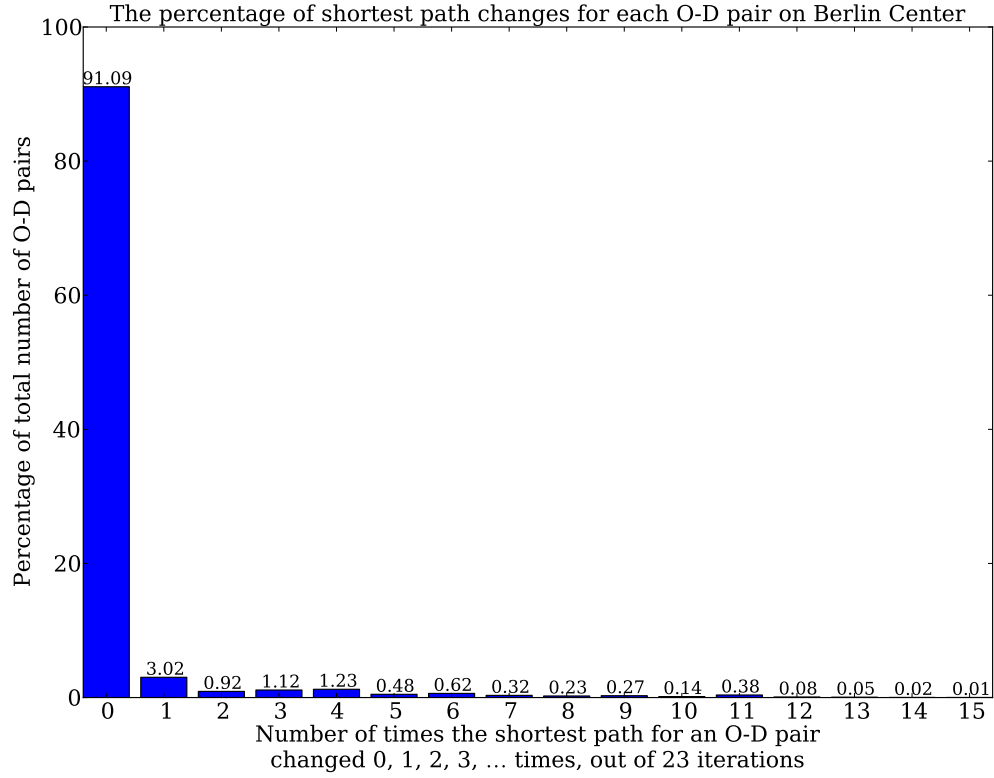


Figure 5.5: The percentage of shortest path change for each O-D pair out of 26 iterations for Chicago Sketch

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We now present the strategy of avoiding a pre-defined number of shortest path calculations for each O-D pair if the previous two iterations result in the identical shortest path trees. The results are shown in Figure 5.6, where the strategy is experimented on the Terrassa and Chicago Sketch network. On the Terrassa network, we choose to avoid the next 15, 25, 50, 100 and 200 iterations of shortest path calculations if the previous two are the same. And on the Chicago Sketch network, we choose to avoid the next 5, 10, 15 and 20 iterations. The strategy worked well on the Terrassa network, where run time is decreased by half for all of the chosen number of avoiding iterations. Due to small size of the network, the run time is not affected even though the total number of iterations increased to 563 when calculations are avoided by 200 iterations. The strategy also worked well on the Chicago Sketch network. Skipping 5 iterations resulted the same 26 iterations and the run time is decreased by 4 seconds. Run times are still reduced in cases where there is an increase in total number of iterations.

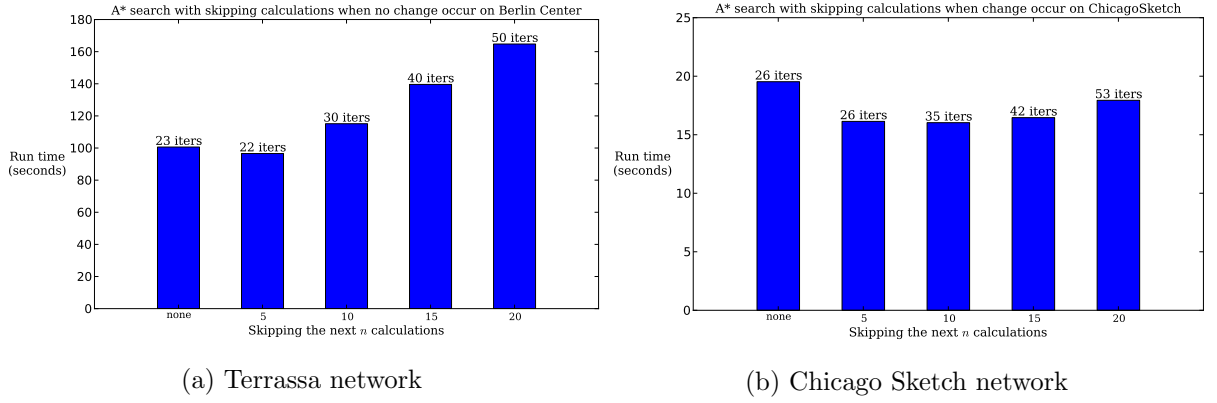


Figure 5.6: Run time for avoiding shortest path calculations if the previous two iteration did not change

The other strategy is to skip the next shortest path calculation randomly. Figure 5.7a shows the results on the Terrassa network, probabilities of 0.3, 0.4, 0.5, 0.6 and 0.7 for skipping the next shortest path calculation. The strategy reduced the run time quite significantly for all probabilities, especially 0.5. Figure 5.7b shows effect of the same strategy on the Chicago Sketch network. This time all run times are reduced slightly, with 0.4 having the most reduction.

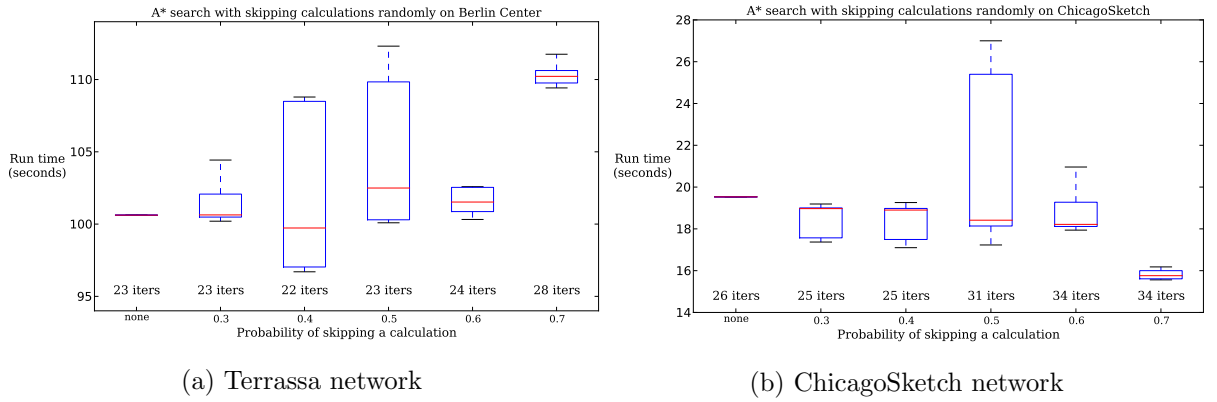


Figure 5.7: Run time for skipping shortest path calculations randomly

The random strategy can be used before knowing the total number of iterations the path equilibration algorithm going to produce, Since only small networks have been tested so far, we now present the random strategy on the Philadelphia and Chicago Regional networks, where they have over a million number of O-D pairs. The run time comparisons are shown in Table 5.2, where the random skipping strategy uses 50% probability to skip a shortest path calculation. The

strategy has a 25% and 27% run time improvement on the Philadelphia and Chicago Regional networks respectively.

Table 5.2: Run time of A* search and the randomly skipping strategy on Philadelphia and Chicago Regional network

	Philadelphia	Chicago Regional
A* search	7.69 hours	33.26 hours
A* search with 50% random skipping	5.75 hours	24.18 hours

see page 34 Andrea fix

Chapter 6

Discussion of Results

The previous chapter presented results that indicated A* search using priority queue from the C++ standard template library has the best performance. The results also show that shortest path calculations can be sped up further in an iterative environment. In this chapter we discuss the details behind the mentioned results, including why some of the algorithms that were expected to perform better failed to do so.

6.1 Priority queues

The priority queue implementation results showed that all of the heap implementations from the Boost library are worse than `<priority_queue>`. The reason behind this can be explained using the Binary heap implementation. It turns out that the implementation of `<priority_queue>` is similar to Binary heap from the Boost library, the only difference is their underlying storage of node information. Nodes are stored using an array in the standard library version, where as the Boost library uses pointers to keep track of the nodes. Due to computer cache coherence, it is known that accessing data from a nearby memory access memory (RAM) locations in a short period of time is faster than accessing from distant memory locations. This is due to cache memory access being much faster than RAM access, and internally a block of memory is pre-fetched into the cache in the hope they will be accessed in a short period of time. In the shortest path algorithms, the Heap tree needs to be searched over and over again in a short period of time when nodes are being scanned and inserted. The standard library version uses an array where data are stored linearly in a nearby location, so it is much faster than the pointer based version where memory is allocated in random locations when nodes are inserted.

6.1.1 Fibonacci heap

Here we discuss the reason behind Fibonacci heap not performing well despite its $O(1)$ amortized time Decrease-Key operation. As described in the two priority queue sections (Section 3.4.1 and

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4.3), the Decrease-Key operation is used to change the distance label of a node when the node is already in the heap. It was discovered that the $O(1)$ time has a very high constant factor, and Fibonacci heap only works well if the underlying graph is large and dense (i.e. every node connects to almost every other node). This discovery comes from the fact that the Decrease-Key operation is only used frequently when the graph is dense, so cumulatively its high constant $O(1)$ time will perform better compared to $O(\log(N))$ time in other heap implementations, where N need to be a large number.

Now we confirm our graph is indeed not dense and the Decrease-Key is not used frequent. We find that all of our graphs are very sparse. The degree of any node of any graph is no more than 5, as it is already really rare to have an intersection with 5 roads connected. The graphs only have about 0.4% to 0.6% of arcs in the corresponding complete graph (every node connects to every other node). We also find that in all of the experimented graphs when using Dijkstra's algorithm, the probability of using Decrease-Key on any node is only around 1 to 5 percent.

6.2 Bidirectional algorithms

In this section we investigate the reason for the worse performances of the bidirectional algorithms compared to their unidirectional versions.

First we examine whether search areas of the algorithms are what is expected. Figure 6.1 shows the shortest path trees of the point-to-point algorithms, where the origin and destination nodes are placed on opposite sides of the Chicago Sketch network. It can be seen that Dijkstra's algorithm scans the entire network. Bidirectional Dijkstra scans almost the entire network with some nodes on the side left out. Bidirectional A* scans a slightly larger region near the origin and destination nodes, and the A* search scans just a few nodes along the shortest path.

remind the reader why that is

The behaviour of the algorithms is shown further in Figure 6.2, where the origin and destination are placed close to each other. It is shown that both Dijkstra's algorithm and its bidirectional version scan almost half of the graph, where bidirectional search scans fewer nodes. A* and its bidirectional search scan a small portion of the graph, and they do not scan the area behind the origin and destination node compared to the Dijkstra's algorithm.

The search areas of the bidirectional Dijkstra's algorithms match what is expected, but not the run times when compared to the unidirectional. The reason for reduction in run time is due to our implementations. In both forward and backward search, the current shortest path μ needs to

be updated every time a node is scanned, and the stopping criterion needs to be checked when a node is labelled. Furthermore, once the algorithm terminates, we need to retrieve the shortest path in both directions by following node predecessors and concatenate them together for the full shortest path. So it is concluded that these additional computations slow down the run times.

For A* search, the bidirectional version always scans more nodes than the unidirectional version. And since the bidirectional version has similar stopping criterion compared to the bidirectional Dijkstra's algorithm, it is easy to understand why bidirectional A* performed worse.

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6.3 A* search with landmarks

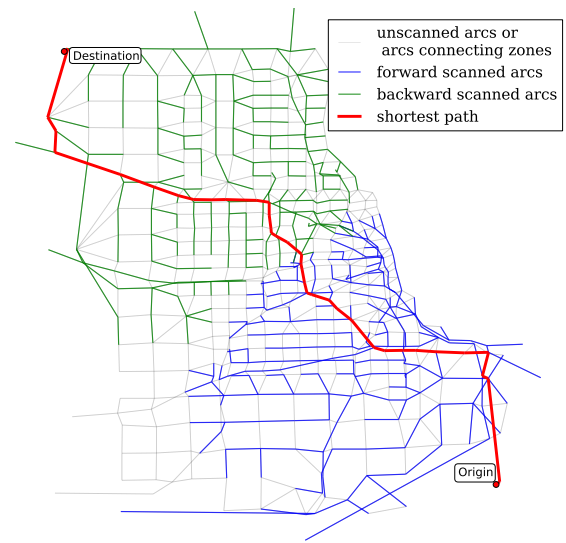
A* search with landmarks algorithm is not implemented for this project because of two reasons. The first reason is due to its sophisticated graph dependent implementation, where we need to either manually or dynamically decide the number of landmarks and their placement locations. The second reason is due to its high chance of not being able to work efficiently, as the algorithm is aimed at geographic node locations and Euclidean distances, not travel times based on traffic flows.

Preprocessing algorithms are not really practical for the traffic assignment problem and the transportation forecasting model. This is because normally these algorithms tend to spend much longer time preprocessing than it takes to just run a standard shortest path algorithm. Since the purpose of the traffic assignment is to modify the network and resolve the problem to see its effect on congestion, it is not ideal to re-run the preprocessing algorithm every time the network is modified.

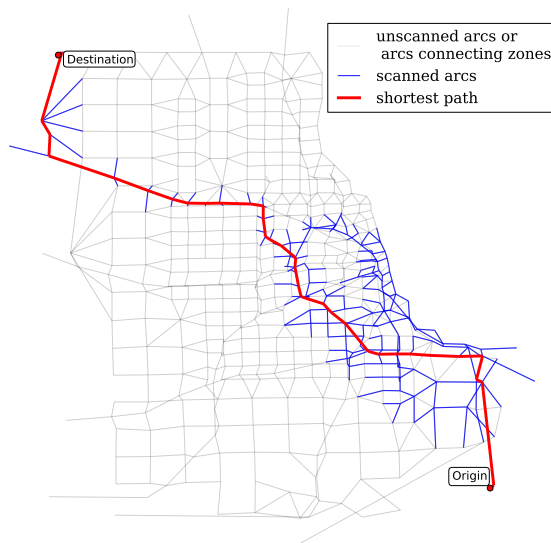
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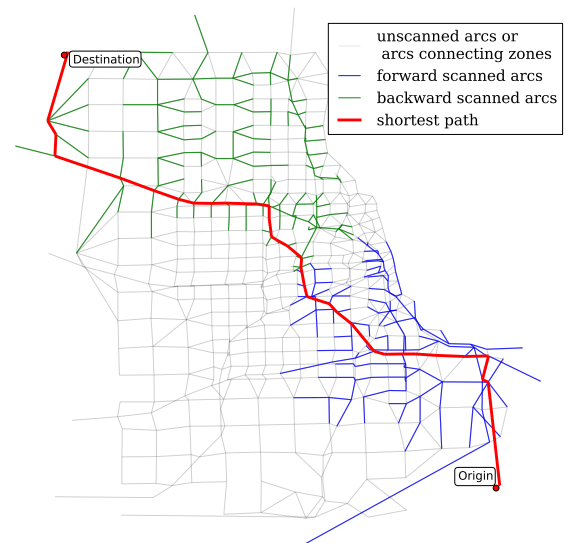
(a) Dijkstra



(b) Bidirectional Dijkstra



(c) A* Search



(d) Bidirectional A* Search

Figure 6.1: Shortest path tree between two distant nodes in the Chicago Sketch network

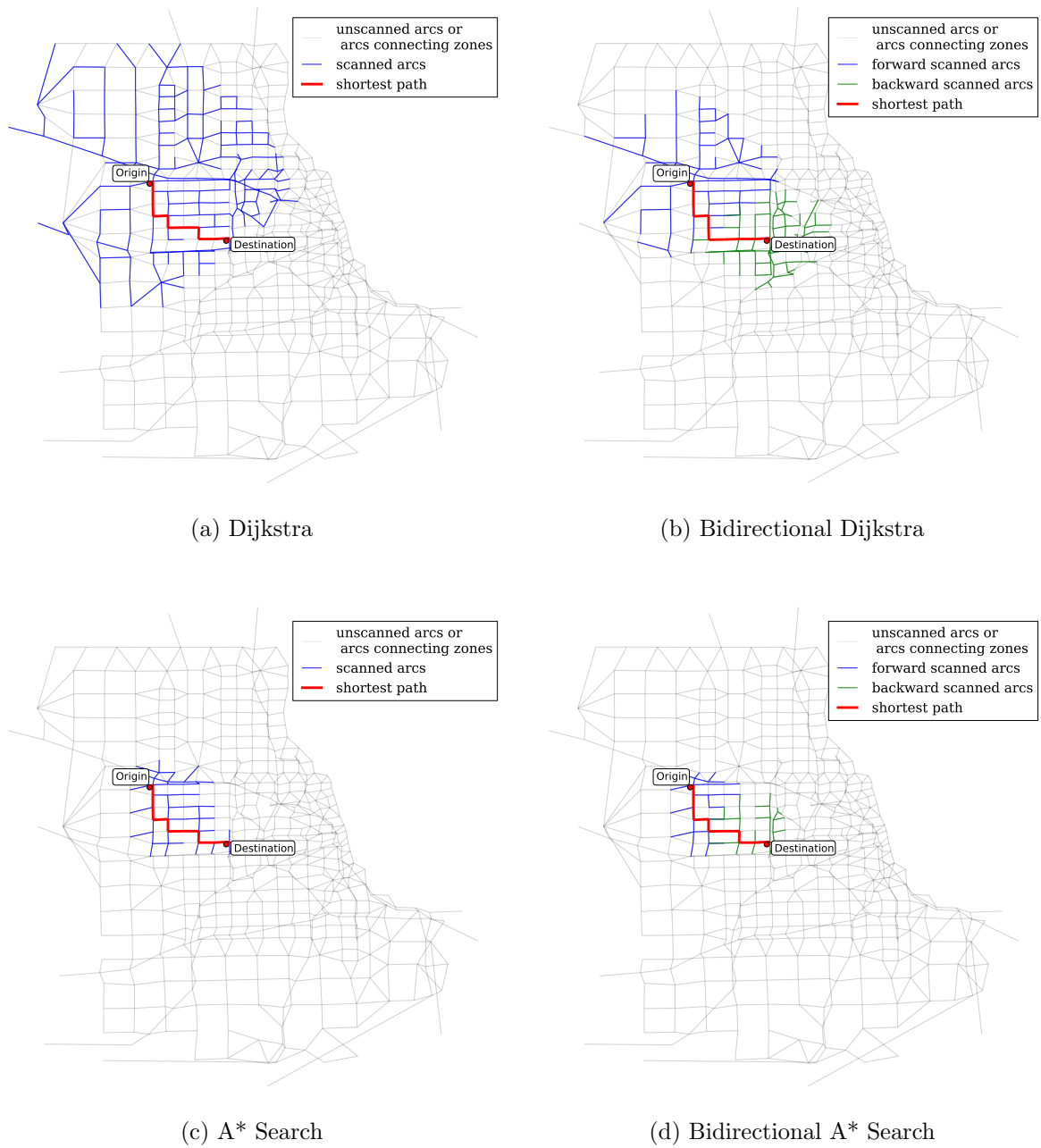


Figure 6.2: Shortest path tree between two close nodes in the Chicago Sketch network

Chapter 7

Summary and Conclusions

To summarise, in this project we have studied the point-to-point shortest path problem embedded in the path equilibration algorithm for solving the traffic assignment problem. We have implemented Dijkstra's algorithm, A* search and their bidirectional versions. We also tested eight different priority queues for the shortest path algorithms. Two strategies were developed to improve the performance of these shortest path algorithms when they are used in the iterative path equilibration algorithm. The first strategy is to avoid the next few number of iterations when the shortest path of the previous two iterations are the same. The second strategy is to randomly skip the next shortest path calculation, with the hope that the shortest path in the current and previous iteration is going to be the same. In addition, we have also investigated the possibility of using preprocessing methods such as A* search with landmarks.

7.1 Conclusions

We now conclude this project with the following results:

- The priority queue implementation from the C++ standard template library has the best performance compared to the six implementations from the C++ Boost library and binary search tree from the C++ standard template library.
- A* search algorithm using zero-flow travel times as heuristic estimates has the best performance.
- Bidirectional versions of Dijkstra's algorithm and A* search have worse performances. Bidirectional Dijkstra is worse because it has to check the stopping criterion at each step. Bidirectional A* is worse because its search area is larger than that of unidirectional A*.
- The strategy of avoiding next few iterations of shortest path calculations is not viable as it is difficult to find the best number of iterations to skip.
- The strategy of avoiding the shortest path calculations randomly is viable. By using A*

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search and 50% random skipping on large networks that require millions of shortest path calculations in each iteration, the run times are further improved by about 25% compared to just using A* search.

- A* search with landmarks is not applicable unless we decide to concentrate on a specific road network, for example the Auckland Regional Transport model.

7.2 Future work

The current A* search algorithm only runs on a single thread. The algorithm can be improved by implementing a multi-threaded version developed by Inam (2009). The algorithm will run extremely fast as it is designed for GPGPU (General Purpose GPU) run on multi processors using many threads concurrently. The main modification of the algorithm is that instead of sequentially updating all out-going arcs from the labelled node, we update them in parallel using multiple threads.

A* search with landmarks algorithm can be investigated if we decide to concentrate on a specific road network. How many landmarks need to be used and where to place them need to be determined experimentally in order to find the best combination of preprocessing run time and actual query time.

SP and random skipping on different TA algorithms

Run PE for a number of iterations, don't add any path until it is equilibrated

one might also consider parallel shortest path runs in TA

Lifelong planning A*:

A family of algorithms exists for dynamically changing graphs, including graphs that have moving nodes or changing arcs. One particular algorithm that tackles the problem of changing arc costs is the Lifelong Planning A*, developed by Koenig et al. (2004). Koenig et al. (2004) showed experimentally that LPA* is more efficient than A* if the change in arc costs is close to the destination. This means Lifelong Planning A* can be used for our problem if we can show that the only changes are close to the destination.

need to be more clearer, see page 24 Olga fix

Appendix A

Priority queue results

Table A.1: Priority queues run time results on Winnipeg and Chicago Sketch network for Figure 5.2

Network	Priority Queue	Iterations	Time (seconds)
Winnipeg	Fibonacci	128	73.66
	Binary	131	56.94
	Ternary	128	62.21
	Skew	127	52.57
	Pairing	131	69.83
	Binomial	127	165.8
	$\langle \text{priority_queue} \rangle$	127	66.89
	$\langle \text{set} \rangle$	127	34.89
ChicagoSketch	Fibonacci	25	155.14
	Binary	25	130.22
	Ternary	25	155.34
	Skew	25	120.45
	Pairing	25	140.55
	Binomial	25	369.10
	$\langle \text{priority_queue} \rangle$	25	145.95
	$\langle \text{set} \rangle$	25	87.52

Appendix B

Shortest path algorithms results

Table B.1: Shortest path algorithms run time results on all test networks for Figure 5.3

Network	Algorithm	Iterations	Time (seconds)
Anaheim	Bellman-Ford	10	1.20
	Dijkstra (priority_queue)	10	0.30
	Dijkstra (set)	10	0.62
	Bidirectional Dijkstra	10	0.43
	A* search	10	0.10
	Bidirectional A* search	10	0.28
Barcelona	Bellman-Ford	28	60.00
	Dijkstra (priority_queue)	27	13.67
	Dijkstra (set)	27	27.36
	Bidirectional Dijkstra	27	20.93
	A* search	28	2.99
	Bidirectional A* search	30	8.02
Winnipeg	Bellman-Ford	129	190.00
	Dijkstra (priority_queue)	127	34.89
	Dijkstra (set)	127	66.89
	Bidirectional Dijkstra	126	49.95
	A* search	126	10.42
	Bidirectional A* search	127	24.81
ChicagoSketch	Bellman-Ford	25	500.00
	Dijkstra (priority_queue)	25	87.52
	Dijkstra (set)	25	149.95
	Bidirectional Dijkstra	25	157.75
	A* search	26	18.75
	Bidirectional A* search	26	59.82

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