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Engineering Science

Faster Shortest Path Computation for Traffic Assignment

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Abstract

Acknowledgement

I acknowledge . . .

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Chapter 1

Introduction

CHECK LATEX LOG!!

probability return infinite

use ‘ we ’ less

write chapter outline brief

1.1 Project Motivation

As the result of ever increasing population, cities worldwide and their road networks are becoming more complicated and hard to navigate, leading to traffic congestions that are more problematic than ever for traffic designers and road users.

show forecast model figure

A traffic model called the transportation forecasting model is built with the aim of reducing congestion and predicting future traffic response when the behaviour of the traffic is changed. This model estimates traffic flows with the following four stages: trip generation, trip distribution, mode choice and traffic assignment. In short, this model generates origins and destinations (or traffic analysis zones) for travellers to travel from and to in different parts of the road network, it then calculates the number of trips that are required for each origin and destination pair and computes the proportion of trips between each pair that use a particular transportation method, in the end it assumes all travellers choose the best trip with the least transportation cost and best transportation method (e.g. shortest path, least travel time or cheapest route) and assigns each traveller to their destination considering traffic congestion.

This traffic assignment (TA) problem in the forecast model is a challenging problem, this is because the problem is solved only when the network reaches user equilibrium, this means no traveller can lower than transportation cost through unilateral action: every traveller will strive to find the shortest path while ignoring all other travellers.

ref user equilibrium - John Glen Wardrop principles of equilibrium

User equilibrium is difficult to find because in traffic assignment, travel times on different roads are modelled as nonlinear functions to capture congestion effects (more traffic flow means slower travel time); as different routes are assigned to the travellers, congestion happen differently for each road in a nonlinear manner, making the result of relocation of travellers hard to calculate.

One method of solving the traffic problem is the Path Equilibration (PE) method (Florian & Hearn 1995). This method initially calculates the shortest paths between each trip origin and destination based on zero-flow travel times, traffic flows are then assigned to these shortest paths and updates the new travel times accordingly. New shortest paths are re-identified and travel flows are re-assigned until user equilibrium is reached.

write about Frank-Wolfe

Both of these methods are iterative methods that require many shortest paths for each trip origin in the network, where the Path Equilibration method requires shortest path calculations between every origin and destination in the road network. It is not difficult to imagine that there would be millions of shortest path calculations if the network has hundreds of origins and destinations and takes some iterations to solve. Each shortest path calculation would also be very hard to solve if the network has a few hundred intersections and a few thousand roads, which is realistic for a real city road network. Sheffi (1985) also states that finding the shortest path is the most computation-intensive component of each iteration compared to other components such as updates and convergence checks that require no more than a few percentages of the total running time. Thus speeding up the shortest path calculation will significantly speed up the traffic assignment algorithms. As a result, traffic forecasting is solved faster for larger and more complicated road networks, predicting longer into the future and allow better designed roads.

1.2 Project Aims

This project aims to embed well known shortest path algorithms that are applicable for traffic assignment methods and find the fastest algorithm. The algorithm that are going to be experimented are:

- Bellman-Ford Label Correcting Algorithm,
- Dijkstra Label Setting Algorithm (using different data structures),
- Bidirectional Dijkstra,
- A* Search,
- Bidirectional A* search.

This project also aims to find and discuss the possibility of preprocessing the network or using data calculated from the previous iteration in traffic assignment methods such that the shortest incomplete

path algorithms have more information to speed up their calculations.

1.3 Report Overview

This report continues in Chapter 2 with the theory behind finding the shortest path under different conditions, and also the description, analysis and pseudocode for each algorithm mentioned in the project aims. Chapter 3 presents the specific implementation details used to give the fastest algorithm possible. Chapter 4 shows and illustrates the results from each algorithm mentioned in the project aims.

incomplete

Chapter 5 discussion chapter 6 conclusion . . .

Chapter 2

Solving the Shortest Path Problem

do I need to show pseudocode for all algos?

Over the years, various algorithms have been developed to address the problem of finding the shortest path in different situations. In this chapter, notations and definitions for the shortest path problem is stated first, the theory for solving the shortest path problem is described next, algorithms that are applicable for road networks are then summarised, including the discussion of their advantages and drawbacks.

Big O analysis for all algorithms

Need to talk about results, what should the reader pay attention to? What should they conclude?

assume a path always exist between an OD pair

2.1 Notations and Definitions

The Shortest Path Problem (SPP) is the problem of finding the shortest path from a given origin to some destination. There are two types of SPP hat are going to be analysed in this chapter: a single-source and a point to point SPP. The Frank-Wolfe algorithm in the TA involves solving the single-source SPP by finding of shortest path going from one origin to every other destinations the network. The Path Equilibration method in the TA Solving the point to point SPP solves from one origin to a specific destination and is used in the Path Equilibration method.

monotonic non-decreasing - correct?

When solving SPP for a normal road network, different measurements such as distance and travel exist for the road length. But in traffic assignment, the road length is measured in a monotonic non-decreasing travel time function, which encapsulates information such as traffic flow, road show equation?

capacity and travel speed. This travel time function is always non-negative so taking advantage of this helps the selection of algorithms that uses this property.

Using notations from Cormen et al. (2001) and Klunder & Post (2006) and in the context of transportation networks, we denote $G = (V, E)$ for a weighted, directed graph, where V denotes the set of nodes (origins, destinations, and intersections) and E the set of edges (roads); we say E is a subset of the set $\{(u, v) \,|\, u, v \in V\}$ of all ordered pairs of nodes. We denote the weight function $c : E \to \mathbb{R}$ which assigns a cost (travel time) to any arc $(u, v) \in E$. We write the costs of arc (u, v) as: $c((u, v)) = c_{uv}$.

The path P inside a transportation network has to be a directed simple path, which is a sequence of nodes and arcs $(u_1, (u_1, u_2), u_2, \ldots, (u_{k-1}, u_k), u_k)$ such that $(u_i, u_{i+1}) \in E$ for $i = 1, \ldots, k-1$ and $u_i \neq u_j$ for all $1 \leq i < j \leq k$. Note u_1 is the origin and u_k is the destination of the path P, u_1 and u_k together is called an O-D pair for this path. For simplicity, we denote s to be the source (origin) and t to be the target (destination) for any path P.

In a transportation network, the origins and destinations are often called centroids or zones. They are traffic analysis zones for generating trip demands and supplies and hold information such as household income and employment information, these information helps the understanding of trips that are produced and attracted within the zone. The zones are conceptual nodes in the network and are untravellable, which means a path between two zone nodes must not contain another zone node.

Maybe a picture of the network explain what the zones are.

Through out the report, run-time analysis (big O and other notations) is used to demonstrate the estimation of algorithms running time regarding their input size.

How do I nicely say 'let the reader refer to other resources?' or do I desribe what big O notation is?

2.2 Generic Shortest Path Algorithm (GSP)

A family of algorithms exist for solving SPP with directed non-negative length arcs, in this section we describe the generic case for these algorithms.

This family of algorithms aim at finding a vector $(d_1, d_2, \ldots d_v)$ of distance labels and its corresponding shortest path (Klunder & Post 2006). Each d_v keeps the least distance of any path going from s to v, $d_v = \infty$ if no paths has been found. A shortest path is optimal when it satisfies the following conditions:

$$d_v \leq d_u + c_{uv}, \quad \forall (u, v) \in E, \tag{2.1}$$

$$d_v = d_u + c_{uv}, \quad \forall (u, v) \in P. \tag{2.2}$$

what is P?

The inequalities (2.1) is called Bellman's condition (Bellman 1958). In other words, we wish to find a label vector d which satisfies Bellman's condition for all of the vertices in the graph. To maintain the label vector, the algorithm uses a queue $\mathcal{Q}$ to store the label distances.

In the label vector, a node is said to be unvisited when $d_u = \infty$, scanned when $d_u \neq \infty$ and is still in the queue, and labelled when the node has been retrieved from the queue and its distance label cannot be updated further. If a node is labelled then its distance value is guaranteed to represent the minimal distance from s to t, Bellman's condition must have been satisfied.

In the generic shortest path algorithm, we start by putting the origin node in the queue, and then iteratively find the arc that violates the Bellman's condition (i.e., $d_v > d_u + c_{uv}$), distance labels are set to a value which satisfies condition (2.1) to the corresponding node of that arc. Shortest path going from s to all other nodes in V is found when (2.1) is satisfied for all arcs in E. It may not be obvious but negative costs are permitted in the GSP but not negative cost cycles.

We use p_u to denote the predecessor of node u. The shortest path can be constructed by following the predecessor of the destination node t back to the origin node s. p_s is often set to -1 to indicate it does not have a predecessor.

diagram showing u, v, c_{vw} etc.

Algorithm 1 (Klunder & Post 2006) describes the generic shortest path algorithm mentioned above, with an extra constraint required when solving a TA problem: travelling through zone nodes are not permitted. In essence, this algorithm repeatedly selects node $u \in \mathcal{Q}$ and checks the violation of Bellman's condition for all emanating arcs of node u.

Algorithm 1 The Generic Shortest Path Algorithm

```
procedure GenericShortestPath($s$)
    $\mathcal{Q} \leftarrow \mathcal{Q} \cup \{s\}$                ▷ initialise queue with source node
    $p_s \leftarrow -1$                                      ▷ origin has no predecessor
    $d_s \leftarrow 0$
    for all $u \in V : u \neq s$ do                 ▷ all nodes are unvisited except the source
        $d_u \leftarrow \infty$
    while $\mathcal{Q} \neq \emptyset$ do
        $u \leftarrow \text{next}(\mathcal{Q})$                             ▷ select next node
        $\mathcal{Q} \leftarrow \mathcal{Q} \setminus \{u\}$
        if $u \neq$ zone then
            for all $v : (u, v) \in E$ do    ▷ check Bellman's condition for all successors of $u$
                if $d_u + c_{vw} < d_v$ then
                    $d_v \leftarrow d_u + c_{vw}$
                    $p_v \leftarrow u$
                    if $v \notin \mathcal{Q}$ then
                        $\mathcal{Q} \leftarrow \mathcal{Q} \cup \{v\}$           ▷ add node $v$ to queue if unvisited
```

Algorithm 1 is generic because of two reasons: the rule for selecting the next node u (the next function in line 8) and the implementation for the queue $\mathcal{Q}$ is unspecified. Different algorithms use different rules and implementations to give either the one-source or the point-to-point shortest path algorithm (Pallottino & Scutellà 1997). The next two sections describes these rules and implementations.

2.3 Label Correcting Algorithm

pseudo code

The GSP is addressed as a label correcting algorithm when the queue is a first in first out (FIFO) queue. Given the arc costs can be negative in the GSP, and in order to satisfy the Bellman's conditions for all arcs, the algorithm has to scan all arcs $|V| - 1$ times, giving a run time of $O(|V||E|)$.

Check if its FIFO or double ended queue

In this algorithm, the distance labels do not get permanently labelled when the next node in the queue is retrieved, another node may 'correct' this node's distance label again, thus the name label correcting algorithm. This algorithm is also called the BellmanFordMoore algorithm credited to Bellman (1958), Ford (1956) and Moore (1959).

2.4 Label Setting Algorithm

The classical algorithm for solving the single-source shortest path problem is the label setting algorithm published by Dijkstra (1959). The algorithm is addressed as label setting because when the next node u is retrieved from the queue, it gets permanently labelled; the shortest path going to this node is solved and the distance label represents the shortest length. In order to achieve label setting, the queue $\mathcal{Q}$ is modified to always have the minimum distance label in front of the queue, hence the algorithm iterates through every node in the graph exactly once, labelling the next node u in the order of non-decreasing distance labels.

The advantage of this algorithm over the label correcting algorithm is that all nodes in the graph are only visited once; the shortest path tree grows radially outward from the source node. It is clear that when the next node in the queue is the destination node, the algorithm can be stopped for the point to point SPP case, which is desirable for the Path Equilibration method.

2.4.1 Priority Queue

The run time performance of the Dijkstra's algorithm depends heavily on the implementation of the queue for storing the scanned nodes, Cormen et al. (2001) suggest the use of a min-priority queue, which is a collection of data structures that always serve elements with higher priorities, in our case they are the nodes with shorter distance labels.

Algorithm 2 shows the use of the min-priority queue in Dijkstra's algorithm. The min-priority queue has 3 main operations: Insert, Extract-Min and Decrease-Key. The Insert operation (line 2 and 17 in Algorithm 2) is used for adding new nodes to the queue, the Extract-Min operation (line 8) is used for getting the element with the minimum distance label and the Decrease-Key is used for updating the distance if the node is already in the queue.

Algorithm 2 Point to Point Dijkstra's Algorithm

```
 1: procedure DIJKSTRA($s, t$)
 2:     Insert($\mathcal{Q}$ , u)                          ▷ initialise priority queue with source node
 3:     $p_s \leftarrow -1$                                ▷ origin has no predecessor
 4:     $d_s \leftarrow 0$
 5:     for all $u \in V : u \neq s$ do                   ▷ all nodes are unvisited except the source
 6:         $d_u \leftarrow \infty$
 7:     while $\mathcal{Q} \neq \emptyset$ do
 8:         $u \leftarrow$ Extract-Min($\mathcal{Q}$)          ▷ select next node with minimum value
 9:         if u = t then
10:             Terminate Procedure                     ▷ finish if next node is the destination
11:         if $u \neq$ zone then
12:             for all $v : (u, v) \in E$ do             ▷ check Bellman's condition for all successors of $u$
13:                 if $d_u + c_{vw} < d_v$ then
14:                     $d_v \leftarrow d_u + c_{vw}$
15:                     $p_v \leftarrow u$
16:                     if $v \notin \mathcal{Q}$ then
17:                         Insert($\mathcal{Q}, v$)          ▷ add node $v$ to queue if unvisited
18:                     else
19:                         Decrease-Key($\mathcal{Q}, v$)    ▷ else update value of $v$ in queue
```

According to Cormen et al. (2001), min-priority queue can implemented via an array or different kinds of min-heap data structure.

In the array implementation, the distance labels are stored in an array where the n^{th} position gives the distance value for node n. Each Insert and Decrease-Key operation in this implementation takes $O(1)$ time, and each Extract-Min takes $O(V)$ time (searching through the entire array), giving a overall time of $O(V^2 + E)$.

A min-heap is a tree-based data structure which satisfies the min-heap property: the value of each node is smaller or equal to the value of its child nodes. Cormen et al. (2001) shows that if the graph is sufficiently sparse (in particular $E = o(V^2/\log(V))$, the Dijkstra's algorithm can be improved with a binary min-heap. In this implementation, the binary tree takes $O(V)$ time, Extract-Min takes $O(\log(V))$ time for $|V|$ operations and Decrease-Key takes $O(\log(V))$ time for each $|E|$. The total running time is therefore $O((V + E)\log(V))$, which improves the array implementation.

The running time can be improved further using a Fibonacci heap developed by Fredman & Tarjan

(1987). Where historically, the development of Fibonacci heaps was motivated by the observation that Dijkstra's algorithm typically makes many more Decrease-Key calls than Extract-Min calls. In Fibonacci heap, each of the $|V|$ Extract-Min operations take $O(\log(V))$ amortized time, and each of the $|E|$ Decrease-Key operations take only $O(1)$ amortized time, which gives a total running time of $O(V \log(V) + E)$.

2.5 Bidirectional Label Setting Algorithm

The Dijkstra's algorithm can be imagined to be searching radially outward like a circle with the origin in the centre and destination on the boundary. Likewise, Dijkstra's algorithm can be used on the reverse graph (all arcs reversed in the graph) from the destination node. Thus Dijkstra's algorithm can be run on the origin and destination simultaneously at the same time. The motivation for doing this is because the number of scanned nodes can be reduced when searching bidirectionally: two smaller circles growing outward radially in stead of a larger one. It is common to conclude that the shortest path is found when the two searches meet somewhere in the centre, but this is not actually the case as there may exist another arc connecting the two frontiers of the searches that has a shorter path. The correct termination criteria was first designed and implementation by Pohl (1971) based on researches presented by Dantzig (1963), Nicholson (1966) and Dreyfus (1969). Klunder & Post (2006) summarises the procedure and algorithm (Algorithm 3) for the termination criteria presented by Pohl (1971).

illustrate this

In Algorithm 3, two independent Dijkstra's algorithms are alternatively run on the forward and reverse graph (forward and backward algorithm), the algorithms terminate when a node is permanently labelled in both directions. Once the algorithms have terminated, the correct shortest path is found by looking for a arc connecting the frontiers of the two searches that may yield a shorter path. This extra condition increases the run time significantly because we have to search for all arcs that connect all labelled nodes in the forward search to all labelled nodes in the backward search.

Show theorem and proof?

Note in Algorithm 3, $\mathcal{R}^s$ is the subset of nodes that are permanently labelled from s with labels d_v^s in the forward search, and $\mathcal{R}^t$ is the subset of nodes that are permanently labelled from s with labels d_v^t in the backward search.

Algorithm 3 Bidirectional Label Setting Algorithm

```
1: procedure BIDIRECTIONAL($s, t$)
2:     Execute one iteration of the forward algorithm. If the next node $u$ is labelled permanently
       by the backward algorithm ($u \in \mathcal{R}^t$), go to step 3. Else, go to step 2.
3:     Execute one iteration of the backward algorithm. If the next node $u$ is labelled permanently
       by the forward algorithm ($u \in \mathcal{R}^s$), go to step 3. Else, goto step 1.
4:     Find $\min\{\min\{d_v^s + c_{vw} + d_w^t | v \in \mathcal{R}^s, w \in \mathcal{R}^t, (v,w) \in E\}, d_u^s + d_u^t\}$, which gives the
       correct shortest path between $s$ and $t$.
```

In recent years, Goldberg & Werneck (2005) improved the bidirectional search using a better termination condition. The termination condition and its method is described as follows. During the forward and backward search, we maintain the length of the shortest path seen so far, μ, and its corresponding path. Initially, $\mu = \infty$. When an arc (v, w) is scanned by the forward search and w has already been scanned in the reverse search (or vice versa), we know the shortest $s - v$ and $w - t$ path have lengths d^s_v and d^t_w respectively. If $\mu > d^s_v + c_{vw} + d^t_w$ then this path is shorter than the one detected before, so we update u and its path accordingly. The algorithm terminates when the search in one direction selects a node already scanned in the other direction.

Goldberg et al. (2006) showed and proved a stronger termination condition on top of his previous one. The searches can be stopped if the sum of the top priority queue values is greater than μ:

$$\text{top}_f + \text{top}_r \geq \mu$$

Show theorem and proof as well?

where top_f and top_r are the top priority queue values in the forward and reverse search, they the next minimum distance label that have not been labelled.

2.6 A* Search

I have been using the word "heuristic" everywhere, is "potential" function more appropriate or are they interchangeable?

show figure

Up until now, the Dijkstra's algorithm does not take into account the location of the destination, the shortest path tree is grown out radially until the destination is labelled. In a traditional graph where actual distances are used for the distance labels, a heuristic can be used to direct the shortest path tree to grow toward the destination (an ellipsoid in stead of a circle). If the heuristic estimate is the distance from each node to the destination, and the estimate is smaller than or equal to the actual distance going to that destination, then a shortest path can be found. This is called A* search or goad directed search, first described by Hart et al. (1968).

Formally we define the following: Let h_v be a heuristic estimate from node v to t, and apply Bellman's condition such that an optimal solution exist, that is

$$h_v \leq h_u + c_{uv}, \quad \forall (u, v) \in E \tag{2.3}$$

$$h(t) = 0, \tag{2.4}$$

where t is the destination node. This means the heuristic function h must be admissible and consistent (monotonic): it never overestimates the actual path length and the estimated cost of a node reaching its destination node is no greater than the estimated cost of its predecessors. Note a consistent heuristic is also admissible but not the oppossite. Hart et al. (1968) proves if the heuristic function (such as using geographical coordinates and Euclidean distance) is admissible and consistent, then A* is guaranteed to find the correct shortest path with a better time performance by scanning less nodes and arcs.

In the Path Equilibration method, geographical coordinates and Euclidean distances can not be used for the heuristic estimate because a travel time function is used for the length of the arcs. Instead, zero-flow travel time from every node to the destination can be for the heuristic. Zero-flow travel time is admissible and consistent and can be shown by analysing the travel times function (Figure 2.1). The travel times function is a monotonic non-decreasing function with the lowest value being the zero-flow travel times. This means using zero-flow travel times as the heuristic estimate is assured to be admissible as no travel time can be lower than the zero flow travel at any time. The heuristic function is consistent because the travel time from a node to the destination must be no longer than all its predecessors.

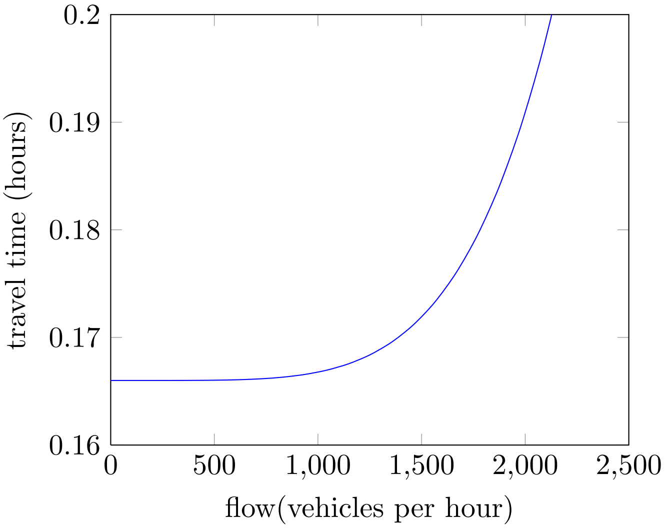


Figure 2.1: Travel time function.

comment and correct graph

mention the LP programming formulation, which may aid the explaintion of Bidirectional A*. See last paragraph in `http://en.wikipedia.org/wiki/Shortest_path_problem`

2.7 Bidirectional A* Search

Bidirectional search can also be applied to A* search, where two ellipsoids are extended from the origin and destination respectively, and construct the shortest path with the same strategy described in section 2.5. But this idea is wrong due to the independence use of the heuristic estimation in the two A* searches, and A* search does not label the nodes permanently in the order of their distance from the origin (Klunder & Post 2006). In short, the heuristic estimations are no longer consistent.

The strategy for the correct use of heuristic estimates and termination criterion has first been published by Pohl (1971). The use of heuristic estimates is later improvement by Ikeda et al. (1994) and the termination criterion is improved by Goldberg et al. (2006).

The strategy is as follows. The heuristic estimates need to translated to consistent functions first. We denote $\pi_f(v)$ the estimate on distance from node v to the destination t in the forward search and $\pi_r(v)$ the estimate on distance from origin s to node v in the backward (reverse) search. In general two arbitrary feasible functions π_f and π_r are not consistent, but their average is both feasible and consistent (Ikeda et al. 1994):

illustrate!

$$p_f(v) = \frac{1}{2}(\pi_f(v) - \pi_r(v)) + \frac{\pi_r(t)}{2} \tag{2.5}$$

$$p_r(v) = \frac{1}{2}(\pi_r(v) - \pi_f(v)) + \frac{\pi_f(s)}{2} \tag{2.6}$$

where the two constants $\frac{\pi_r(t)}{2}$ and $\frac{\pi_f(s)}{2}$ are added by Goldberg et al. (2006) to provide better estimates. Note the modified consistent heuristic p provides worse bounds than the original π values.

Finally Goldberg et al. (2006) shows and proves the stopping criterion:

$$\mathrm{top}_f + \mathrm{top}_r \geq \mu + p_r(t), \tag{2.7}$$

where is μ the best $s - t$ path seen fast, top_f the length of the path from s to the top node (minimum distance label) in the forward search priority queue and top_r the length of the path from t to the top node in the backward search priority queue.

the following section is in draft stage

2.8 Preprocessing and More

Preprocessing - trade memory to get faster time. We can either do a fast preprocessing between iterations to make query in each iteration (so combined speed is still faster) or do a long preprocessing at the start and use the computed heuristic values

- A* landmarks and triangle inequality (ALT)
- Reach-based routing
- ALT + Reach
- Geometric Containers
- Arc Flags

If we have more data on the network we can use algorithms that use hierarchies. Consider roads with higher speed first: use a hierarchy of subgraphs.

- Radius search.
- multi-level approach
- highway hierarchies

Above extract from: Speed-Up Techniques for Shortest-Path Computations by Dorothea Wagner, Thomas Willhalm, and Fast Shortest Path Algorithms for Large Road Networks by Faramroze Engineer

We can also do Lifelong Planning A* (LPA*), use heuristic from previous each iteration, but the original paper says only a few percent arc change can boost run time, not idea if it is more than that.

If the edge lengths are whole numbers then we can use multi-level bucket for the priority queue.

Chapter 3

Implementation Details

Be more formal, to colloquial at the moment

The previous chapter have described all the algorithms that are implementation for this report. In this chapter, we seek and research the specific implementation details that make the algorithms run faster.

Note the traffic assignment algorithms have already been implemented by the co-supervisor of this report in a Object Oriented C++ program. The programs includes Frank-Wolfe, Path Equilibration, label correcting algorithm and many more.

3.1 Graph Storage

The graph storage is implemented as a Forward Star data structure. Forward Star stores a network compactly with $O(V + E)$ space. It allows $O(1)$ access for any nodes in the graph and $O(1)$ access for all arcs emanating from a random node, which are the requirements for the generic shortest path algorithms. Information about Forward Star can be found in (Sheffi 1985).

3.2 Priority Queue Implementations

we can also use std::set, which uses binary search tree and provides $O(log(n))$ search and keeps minimum element on top of the tree

Various implementations of the priority queues exist, they include the array based heap implementation (std::priority_queue) from the C++ standard template library (STL) and 6 different variants of heap implementations from the C++ Boost library. Each implementation may have some advantages than the other, for example faster tree balancing, faster Extract-Min or Delete etc.

We first examine the 6 variants of Heap implementations from the C++ Boost Heap Library shown in Table 3.1 (Blechmann 2013). Where N is the number of elements in the Heap tree, and all time complexities are measured in amortized time, i.e. the average run time if the operation is run for a long period of time, average out worse case and best case.

Table 3.1: C++ Boost Heap Implementations with Comparison of Amortized Complexity

	top()	push()	pop()	increase()	decrease()
d-ary (Binary)	O(1)	O(log(N))	O(log(N))	O(log(N))	O(log(N))
d-ary (Ternary)	O(1)	O(log(N))	O(log(N))	O(log(N))	O(log(N))
Binomial	O(1)	O(log(N))	O(log(N))	O(log(N))	O(log(N))
Fibonacci	O(1)	O(1)	O(log(N))	O(1)	O(log(N))
Pairing	O(1)	$O(2^{2*\log(\log(N))})$	O(log(N))	$O(2^{2*\log(\log(N))})$	$O(2^{2*\log(\log(N))})$
Skew	O(1)	O(log(N))	O(log(N))	O(log(N))	O(log(N))

We are interested in using Boost library Heaps rather than the STL library Heap is due to one reason: the decrease (or increase) function. The decrease (or increase) function is referred as the decrease-key (or increase-key) operation mentioned in Section **??**, which updates the value of the key in the Heap tree. Decrease-key is used for a min-heap and increase-key for a max-heap tree. For Dijkstra's algorithm, often nodes are scanned multiple times in the label updating step, instead of adding the node again into the Heap tree, we can use decrease-key on the node, updating its distance label. This means we can reduce the size of the Heap tree and run time by using decrease-key rather than adding the same node with different distance label in the queue again.

In table 3.1, we can observe the Fibonacci Heap has a very interesting time complexity: constant amortized time for the push, pop and increase-key operation time. But the fact is, we do not know how much constant time it really uses behind its big O. It is reported that Fibonacci Heaps only outperforms other Heaps when the graph is very dense, but it is worth to experiment Fibonacci Heap as well as all other Heaps.

this is going to be hard to find a good reference, all reports are from Stackoverflow.com

The STL library Heap is still going to be implemented and tested. The STL library Heap does not support the decrease-key operation but should not be a worry. This is due to the fact that if a node has been added to the Heap more than once, the node with smaller distance label is always going to be removed from the queue and update its successors first, the same node with larger distance label will therefore not update its successors.

C++ Boost Library Heaps are implemented as max-heaps, which means in order to use the Fibonacci O(1) increase-key function, we need to negate the distance labels when we add them into the Heap.

Chapter 4

Results

results interpretation

results speed up in percentages

talk about existing code

4.1 Problem Data and Result Explanation

The problem data for solving the TA problems are retrieved from Transportation Network Test Problems (Bar-Gera 2013).

Through out the report, Table 4.1 is used to show the run time and number of iterations for solving one particular network. In the table, the "OD pairs" column gives the number of pairs of origin and destination in the network. The "zone" column gives the number of traffic zones, in some cases, the nodes in the network also include the traffic zones. The "Run time (seconds)" gives is measured from executing the whole path equilibration algorithm from start to finish. The "Iterations" column gives how many times the whole network gets solved to settle the traffic flows to equilibrium.

need to explain what an iteration where it is introduced

Table 4.1: Network Problem Data

Network	Nodes	Zones	OD pairs	Arcs
SiouxFalls	24	24	528	76
Anaheim	416	38	1406	914
Barcelona	1020	110	7922	2522
Winnipeg	1052	147	4344	2836
ChicagoSketch	933	387	93135	2950

the number of nodes listed in the table includes traffic zones

By examining the network problem data, we can see that the number of OD pairs increase significantly respect to the number of zone nodes, this is important because it indicates how many SPPs need to be solved for each iteration of the PE. We can also roughly tell that these networks are very sparse, as a complete graph (every node is connected to every other node) of 1000 nodes have 499500 arcs ($n(n-1)/2$), and the larger networks in our problem data only have about 0.4% to 0.6% of arcs in a complete graph, this information is useful when we start tuning the algorithms for solving SPP.

mention node degree

Most of the data does not resemble a real world transportation network, for example sometimes all roads have the same speed limit, road type and capacity.

wrong, the smaller data sets have same data but not the large ones

In this report, all problem data are solved on a Intel i5 1.78GHz CPU computer with 4GB RAM, which runs the Ubuntu 12.04 Linux operating system. And the code is compiled with the g++ compiler with the -O3 optimisation flag (i.e. optimise for speed).

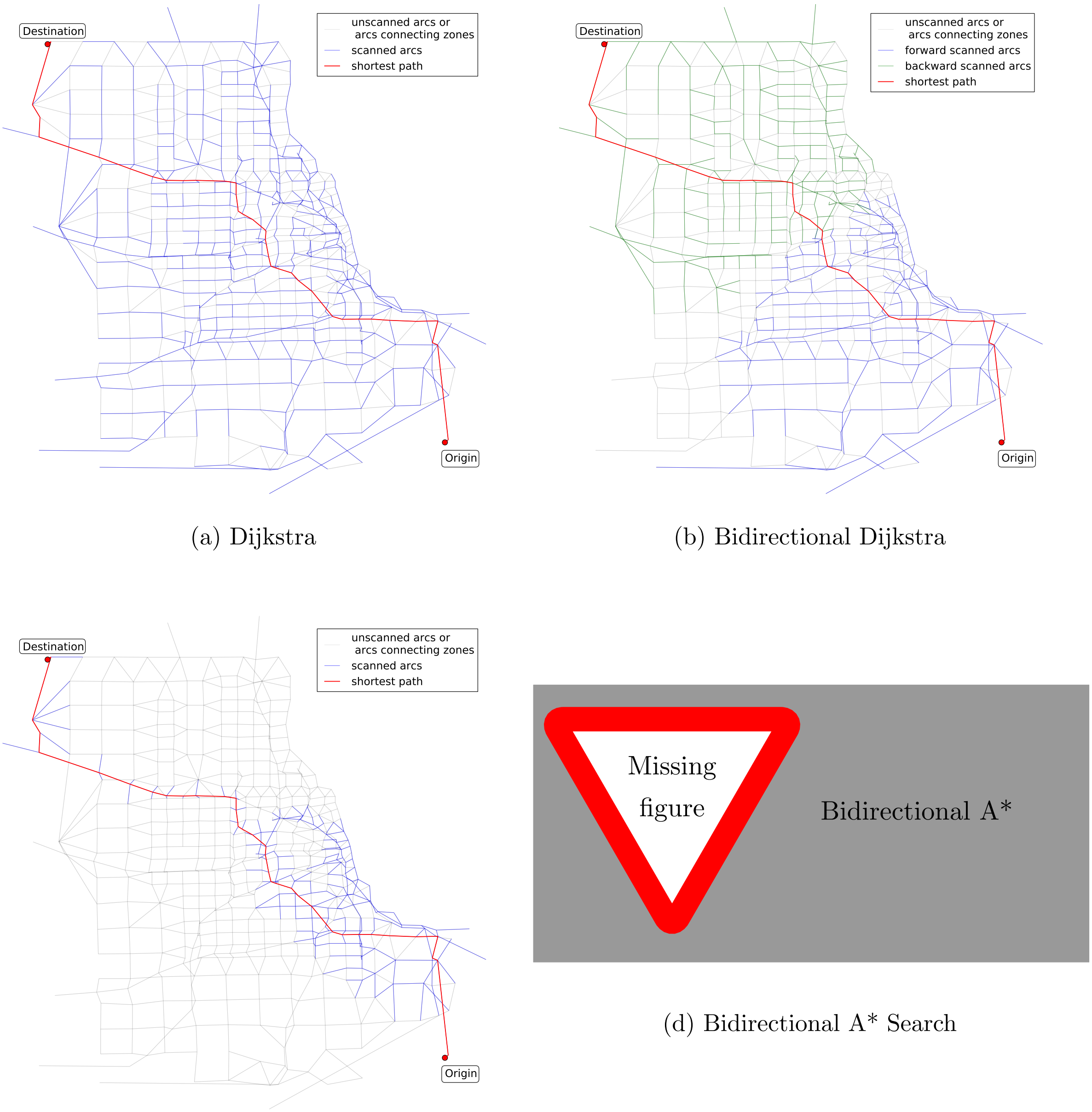
rewrite, this is wrong

The accuracy of all results are checked by comparing the traffic flows from the traffic assignment output, as well as the final shortest path for every OD pairs.

Table 4.2: Results for all test networks. Showing the number of iterations per graph (ITERS), max number of scans (COUNT) and the speed up respect to the label correcting algorithm (SPD).

			Max Scans		Time	
Graph	Algorithm	ITERS	COUNT	SPD	SEC	SPD
SiouxFalls	B	69			0.25	
	AP-D	69			0.24	
	P2P-D	64			0.15	
	Bi-D					
	A*	85			0.16	
	Bi-A*					
Anaheim	B	10			1.20	
	AP-D	10			1.20	
	P2P-D	10			0.67	
	Bi-D					
	A*	10			0.15	
	Bi-A*					
Barcelona	B	28			60.00	
	AP-D	28			43.00	
	P2P-D	27			27.71	
	Bi-D					
	A*	27			6.10	
	Bi-A*					
Winnipeg	B	129			190.00	
	AP-D	129			137.00	
	P2P-D	129			70.00	
	Bi-D					
	A*	128			21.85	
	Bi-A*					
ChicagoSketch	B	25			500.00	
	AP-D	25			541.00	
	P2P-D	25			204.00	
	Bi-D					
	A*	26			42.90	
	Bi-A*					

Result : average number of scans



(a) Dijkstra

(b) Bidirectional Dijkstra

(c) A* Search

(d) Bidirectional A* Search

Figure 4.1: Shortest Path Tree for ChicagoSketch Network with Two Distant OD Pair

draw 2 nodes that are close to each other

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