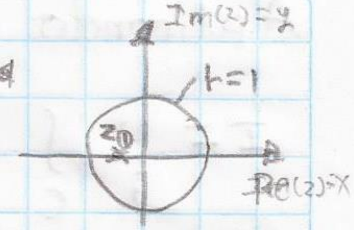
Tarea 1 - Meza Vargas Brandon David - 2CM11

(5) a) $I = \frac{4}{i}\int_C \frac{z}{(z^2+4z+1)^2}\,dz$ si $C: |z| = 1$



sol.

$(z^2 + 4z + 1)$
$\quad a \quad b \quad c$

$z = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(1)}}{2(1)} \Rightarrow z_0 = \frac{-4+\sqrt{12}}{2} = -2+\sqrt{3} \in C$

$z = \frac{-4 \pm \sqrt{12}}{2} \qquad z_0 = \frac{-4-\sqrt{12}}{2} = -2-\sqrt{3} \notin C$

$$I = \frac{4}{i}\int_C \frac{z\,dz}{(z-(-2+\sqrt{3}))^2(z-(-2-\sqrt{3}))^2}$$

Describiendo el integrando como:

$$\frac{z}{(z-(-2+\sqrt{3}))^2(z-(-2-\sqrt{3}))^2} = \frac{\overbrace{\frac{z}{(z-(-2-\sqrt{3}))^2}}^{f(z)}}{(z-(-2+\sqrt{3}))^2}$$

Usando teorema ** $\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$

Con $n=1$; $z_0 = -2+\sqrt{3}$

$$f(z) = \frac{z}{(z+2+\sqrt{3})^2} = \frac{z\,dz}{z^2+7.46z+13.92}$$

Derivamos 1 vez

$$f'(z) = \frac{(z^2+7.46z+13.92)(1) + (2z+7.46)(z)}{(z^2+7.46z+13.92)^2}$$

• Evaluando z_0

$$f'(z_0) = f'(-2+\sqrt{3}) = \frac{((-2+\sqrt{3})^2+7.46(-2+\sqrt{3})+13.92)+(2(-2+\sqrt{3})+7.46)(z_0)}{((-2+\sqrt{3})^2+7.46(-2+\sqrt{3})+13.92)^2}$$

$$= \frac{11.99-1.85}{143.82} = 0.07$$

Aplicando (**)T

$$I = \frac{4}{i}\int_C \frac{z\,dz}{(z^2+4z+1)^2} = \frac{2\pi i}{n!} f'(z_0) = \frac{2\pi i}{1!} f'(-2+\sqrt{3})$$

$$I = \frac{4}{\cancel{i}}(2\pi \cancel{i})(0.07) = \underline{0.64\pi} \text{ //}$$

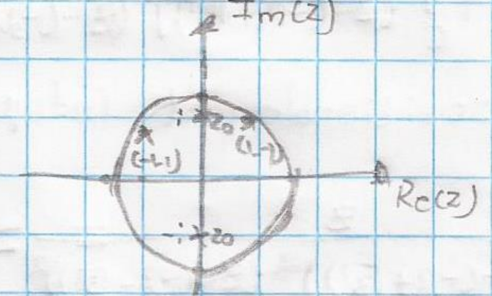
b) $I = \int_C \frac{z\,dz}{(z^2+1)(z^2+2z+2)}$ con $C: |z| = 3/2$

Sol.

$z^2+1 = (z+i)(z-i) = 0$

$(z+i) = 0 \Rightarrow z_0 = -i \in C$

$(z-i) = 0 \Rightarrow z_0 = i \in C$



Raices de $(\underset{a}{z^2} + \underset{b}{2z} + \underset{c}{2})$

$z = \frac{-2 \pm \sqrt{4-4(2)}}{2} \Rightarrow z_0 = \frac{-2+i2}{2} = -1+i = (-1,1) \in C$

$z = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm i2}{2} \quad z_0 = \frac{2-i2}{2} = 1-i = (1,-1) \in C$

La integral queda

$$I = \int_C \frac{z\,dz}{(z+i)(z-i)(z+1-i)(z-1+i)}$$

Describiendo el Integrando como

$$\frac{\overbrace{\frac{z}{(z-i)(z+1-i)(z-1+i)}}^{f(z)}}{(z+i)} \text{ - (1)} + \frac{\overbrace{\frac{z}{(z+i)(z+1-i)(z-1+i)}}^{f(z)}}{(z-i)} \text{ - (2)} + \frac{\overbrace{\frac{z}{(z+i)(z-i)(z-1+i)}}^{f(z)}}{(z+1-i)} \text{ - (3)}$$

$$+ \frac{\overbrace{\frac{z}{(z+i)(z-i)(z+1-i)}}^{f(z)}}{(z-1+i)} \text{ - (4)}$$

Aplicando Teorema. $\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$

Para (1);

$$=2\pi i f(-i) = 2\pi i\left(\frac{-i}{(-2i)(-2i+1)(-1)}\right) = 2\pi i\left(\frac{-i}{4+2i}\right)$$

$$= \frac{-2\pi i^2}{2(2+i)} = \frac{\pi}{2+i}$$

Para (2)

$$=2\pi i f(i) = 2\pi i\left(\frac{i}{(2i)(1)(2i-1)}\right) = 2\pi i\left(\frac{i}{-4-2i}\right)$$

$$= \frac{-2\pi}{-2(2+i)} = \frac{\pi}{2+i}$$

Para (3)

$$=2\pi i f(-1+i) = 2\pi i\left(\frac{-1+i}{(-1+2i)(-1)(-2+2i)}\right) = 2\pi i\left(\frac{-1+i}{2+6i}\right)$$

$$= \frac{-2\pi i + 2\pi i^2}{2(1+3i)} = \frac{-2\pi i - 2\pi}{2(1+3i)} = \frac{-\pi(i+1)}{1+3i} \qquad -\pi(i+1)$$

Para (4)

$$=2\pi i f(1-i) = 2\pi i\left(\frac{1-i}{(1)(1-2i)(2-2i)}\right) = 2\pi i\left(\frac{1-i}{-6i-2}\right)$$

$$= -\pi i\left(\frac{1-i}{+3i+1}\right) = \frac{-\pi i + \pi i^2}{+3i+1} = \frac{-\pi(i+1)}{+3i+1} \qquad -2(3i+1)$$

$$\therefore I = \oint_C \frac{z\,dz}{(z^2+1)(z^2+2z+2)} = \frac{\pi}{2+i} + \frac{\pi}{2+i} + \frac{-\pi(i+1)}{1+3i} + \frac{-\pi(i+1)}{+3i+1}$$

$$= \frac{2\pi}{2+i} + \frac{-2\pi i - 2\pi}{3i+1} = \frac{2\pi(i-1)}{(2+i)(3i+1)} = \frac{2\pi(i-1)}{7i-1} \qquad \therefore I = \frac{2\pi(i-1)}{7i-1}$$

(16)

(b) Evalue: $I = \int_0^{2\pi} \frac{d\theta}{\operatorname{sen}\theta - 2\cos\theta + 3}$

sabemos

$$\operatorname{sen}\theta = \frac{z - z^{-1}}{2i}, \quad \cos\theta = \frac{z + z^{-1}}{2}, \quad d\theta = \frac{dz}{iz}$$

$$I = \int_0^{2\pi} \frac{1}{\left(\frac{z - z^{-1}}{2i}\right) - 2\left(\frac{z + z^{-1}}{2}\right) + 3}\left(\frac{dz}{iz}\right) = \int_0^{2\pi} \frac{2\,dz}{2iz\left(\frac{z - z^{-1}}{2i}\right) - 2iz\left(\frac{z + z^{-1}}{2}\right) + 6iz}$$

$$= \int_0^{2\pi} \frac{2dz}{z^2 - 1 - iz^2 - i + 6iz} = \int_0^{2\pi} \frac{2dz}{z^2\underbrace{(1-2i)}_{a} + \underbrace{6iz}_{b} \underbrace{-1-2i}_{c}}$$

obteniendo los polos

$$z = \frac{-6i \pm \sqrt{(6i)^2 - 4(1-2i)(-1-2i)}}{2(1-2i)} = \frac{-6i \pm \sqrt{-16}}{2(1-2i)} = \frac{-6i \pm 4i}{2(1-2i)}$$

$$z_0 = \frac{-6i + 4i}{2(1-2i)} = \frac{2-i}{5}, \quad z_1 = \frac{-6i - 4i}{2(1-2i)} = 2 - i$$

tenemos que $z_0 \in C$ y $z_1 \notin C$

calculamos el residuo de z_0

$$R_{z_0} = \lim_{z \to z_0}\left[\left(z - \left(\frac{2-i}{5}\right)\right) \frac{2}{\left(z - \frac{2-i}{5}\right)\left(z - (2-i)\right)}\right]$$

$$= \lim_{z \to z_0}\left[\frac{2}{z - 2 + i}\right] = \frac{2}{\frac{2-i}{5} - 2 + i} = \frac{2}{\odot}$$

Aplicando L' Hopital

$$R_{z_0} = \lim_{z \to z_0} \left[\frac{\frac{d}{dz}\left(2z - (4-2i)/5\right)}{\frac{d}{dz}\left((1-2i)z^2 + 6iz - 1 - 2i\right)} \right] = \lim_{z \to z_0} \left[\frac{2}{2((1-2i)z + 3i)} \right]$$

$$= \lim_{z \to z_0} \left[\frac{1}{(1-2i)z + 3i} \right] = \frac{1}{(1-2i)\left(\frac{2-i}{5}\right) + 3i} \left(\frac{5}{5}\right)$$

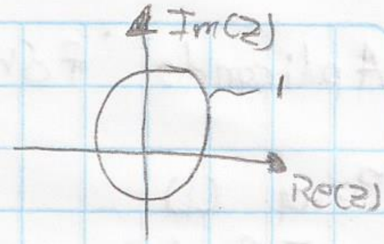
$$= \frac{5}{2 - i - 4i - 2 + 15i} = \frac{5}{10i} = \frac{1}{2i}$$

Así tenemos

$$I = \int_0^{2\pi} \frac{d\theta}{\operatorname{sen}\theta - 2\cos\theta + 3} = 2\pi i \left(\frac{1}{2i}\right) = \pi$$

$$\therefore \underline{I = \pi}$$

$$\text{(e)}\quad I = \int_0^{2\pi} \frac{\cos 3\theta}{10-8\cos\theta}\, d\theta$$



$$\cos 3\theta = \frac{z^3+z^{-3}}{2} \ ; \ d\theta = \frac{dz}{iz} \ ; \cos\theta = \frac{z+z^{-1}}{2}$$

$$I = \int_0^{2\pi} \frac{\left(\frac{z^3+z^{-3}}{2}\right)}{10-8\left(\frac{z+z^{-1}}{2}\right)} \left(\frac{dz}{iz}\right) = \int_0^{2\pi} \frac{2\left(\frac{z^3+z^{-3}}{2}\right)}{2\left(10-8\left(\frac{z+z^{-1}}{2}\right)\right)} \frac{dz}{iz}$$

$$= \int_0^{2\pi} \frac{z^3+z^{-3}}{20-8z-8z^{-1}} \frac{dz}{iz} = \int_0^{2\pi} \frac{z^3\left(z^3+z^{-3}\right)}{z^3\left(20-8z-8z^{-1}\right)} \frac{dz}{iz}$$

$$= \frac{1}{i}\int_0^{2\pi} \frac{z^6+1 \ dz}{z^3\left(20z-8z^2-8\right)} = -i \int_0^{2\pi} \frac{z^6+1 \ dz}{\frac{z^3}{-8}\left(z^2-\frac{5}{2}z+1\right)} = \frac{i}{8}\int_0^{2\pi} \frac{z^6+1 \ dz}{z^3\underbrace{\left(z^2-\frac{5}{2}z+1\right)}_{*}}$$

Raíces de $*$

$$z = \frac{\frac{5}{2} \pm \sqrt{\left(-\frac{5}{2}\right)^2 - 4(1)(1)}}{2(1)}$$

$$z = \frac{\frac{5}{2} \pm \sqrt{\frac{25}{4}-4}}{2}$$

$$z = \frac{\frac{5}{2} \pm \sqrt{\frac{9}{4}}}{2}$$

$$\Rightarrow \quad z_0 = \frac{\frac{5}{2}+\frac{3}{2}}{2} = 2 \notin C$$

$$z_0 = \frac{\frac{5}{2}-3/2}{2} = \frac{1}{2} \in C$$

Así tenemos:

$$I = \frac{i}{8}\int_0^{2\pi} \frac{z^6+1 \ dz}{z^3(z-2)(z-1/2)} = \frac{i}{8}\int_0^{2\pi} \frac{\overset{①}{\frac{z^6+1 \ dz}{(z-2)(z-1/2)}}}{z^3} + \frac{i}{8}\int_0^{2\pi} \frac{\overset{②}{\frac{z^6+1 \ dz}{z^3(z-2)}}}{z-1/2}$$

Aplicando fórmulas de la integral de cauchy

Para (1)

$n = 2 \; ; z_0 = 0$

$$f(z) = \frac{z^6+1}{z^2 - \frac{5}{2}z + 1} \quad \text{Derivamos 2 veces}$$

$$f'(z) = \frac{(z^2 - 5/2\, z + 1)(6z^5) - (2z - 5/2)(z^6+1)}{(z^2 - 5/2\, z + 1)^2}$$

$$= \frac{4z^7 - \frac{25}{2}z^6 + 6z^5 - 2z + 5/2}{(z^2 - 5/2\, z + 1)^2 \to u}$$

$$f''(z) = \frac{(z^2 - 5/2\, z + 1)^2 (28z^6 - 75z^5 + 30z^4 - 2) - (4z^7 - \frac{25}{2}z^6 + 6z^5 - 2z + 5/2)\frac{du}{dx}}{((z^2 - 5/2\, z + 1)^2)^2}$$

$$f''(z) = \frac{(z^2 - 5/2\, z + 1)^2 (28z^6 - 75z^5 + 30z^4 - 2) - (4z^7 - \frac{25}{2}z^6 + 6z^5 - 2z + 5/2)(2(z^2 - 5/2\, z + 1)(2z - 5/2))}{((z^2 - 5/2\, z + 1)^2)^2}$$

$$f''(z_0) = f''(0) = \frac{(-2) - (5/2)(2)(-5/2)}{1} = -2 + 25/2 = \frac{21}{2}$$

Para (2)

$$F(z) = \frac{z^6 + 1\, dz}{z^3(z-2)} \quad z_0 = 1/2$$

$$F(1/2) = \frac{(1/2)^6 + 1}{(1/2)^3(1/2 - 2)} = \frac{\frac{65}{64}}{-3/16} = \frac{-65}{12}$$

$$\therefore I = \frac{i}{8}(2\pi i)\left(\frac{21}{2}\right) + \frac{i}{8}(2\pi i)\left(-\frac{65}{12}\right) = \frac{i}{8}\left(2\pi i\left(\frac{21}{2} - \frac{65}{12}\right)\right)$$

$$= \frac{i}{8}\left(2\pi i\left(\frac{61}{12}\right)\right) = \frac{2\pi i^2\left(\frac{61}{12}\right)}{8} = \frac{-\pi\left(\frac{61}{6}\right)}{8} = \boxed{\frac{-\pi 61}{48}}$$

5' - 3 - 37

① Evalúe: $I = \int_{-\infty}^{\infty} \frac{dx}{x^2+a^2}$ si $a>0$

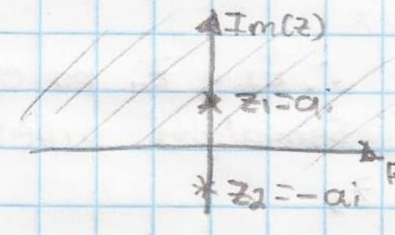
Sol.

$f(x) = \frac{1}{x^2+a^2} \Rightarrow f(z) = \frac{1}{z^2+a^2}$

singularidades

$z^2+a^2 = 0 \Rightarrow z^2 = -a^2 \therefore z_1 = ai$

$z_2 = -ai$



$\therefore z_1$ está en el semiplano superior

$\therefore z_2$ No está en el semiplano superior

Cálculo de los residuos para z_1 con $n=1$ $\quad Res(f(z), z_0) = \lim_{z\to z_0} (z-z_0) f(z)$

$R_{z_1} = R_{ai} = \lim_{z\to ai} \left[(z-ai) \left(\frac{1}{z^2+a^2} \right) \right]$

$= (ai-ai) \left(\frac{1}{(ai)^2+a^2} \right) = \frac{0}{(a^2i^2)+a^2} = \frac{0}{-a^2+a^2} = \frac{0}{0}$ Indeterminación

Por L'Hopital

$R_{z_1} = \lim_{z\to ai} \left\{ (1-0) \frac{1}{2z} \right\} = \lim_{z\to ai} \left[\frac{1}{2z} \right] = \frac{1}{2(ai)} = \frac{1}{2ai} \therefore R_{z_1} = \frac{1}{2ai}$

Sabemos que

$R_{z_2} = 0$ No esta en el semiplano superior

por el T. del Residuo:

$I = 2\pi i [R_{z_1} + R_{z_2}] = 2\pi i \left[\frac{1}{2ai} + 0 \right] = \frac{\pi}{a} \therefore I = \frac{\pi}{a}$

5'-3-39 ③ Evalúe: $I = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx$

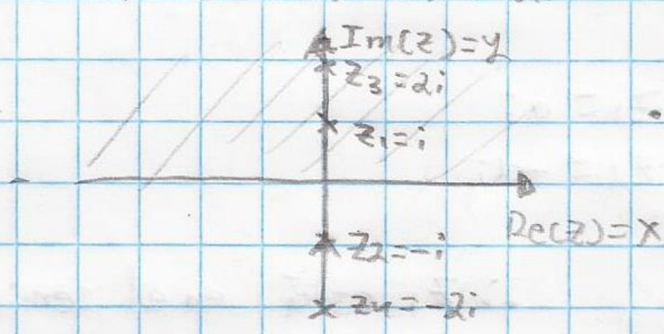
Sol.

$$F(x) = \frac{x^2}{(x^2+1)(x^2+4)} \Rightarrow f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$$

singularidades

$(z^2+1) = 0 \Rightarrow z_1 = i \,;\, z_2 = -i$

$(z^2+4) = 0 \Rightarrow z_3 = 2i \,;\, z_4 = -2i$



$\therefore z_1$ y z_3 estan en el semiplano superior

$\therefore z_2$ y z_4 no estan en el semiplano superior

calculo de residuos para z_1 y z_3

$$R_{z_1} = R_i = \lim_{z\to i}\left[(z-i)\left(\frac{z^2}{(z^2+1)(z^2+4)}\right)\right]$$

$$= \lim_{z\to i}\left[(\cancel{z-i})\left(\frac{z^2}{(\cancel{z-i})(z+i)(z-2i)(z+2i)}\right)\right]$$

$$= \lim_{z\to i}\left[\frac{z^2}{(z+i)(z-2i)(z+2i)}\right] = \frac{i^2}{(i+i)(i-2i)(i+2i)} = \frac{1}{(2i)(-i)(3i)}$$

$$= \frac{i^2}{6i^2(-i)} = -\frac{1}{6i}$$

Así mismo para R_{z_3}

$$R_{z3} = R_{2i} = \lim_{z \to 2i} \left[(z-2i) \left(\frac{z^2}{(z^2+1)(z^2+4)} \right) \right] =$$

$$= \lim_{z \to 2i} \left[\cancel{(z-2i)} \left(\frac{z^2}{(z-i)(z+i)\cancel{(z-2i)}(z+2i)} \right) \right] = \lim_{z \to 2i} \left[\frac{z^2}{(z-i)(z+i)(z+2i)} \right]$$

$$= \frac{(2i)^2}{(2i-i)(2i+i)(2i+2i)} = \frac{-4}{(i)(3i)(4i)} = \frac{-4}{-12i} = \frac{1}{3i}$$

Sabemos que $R_{z2} = R_{z4} = 0$ No estan en el semiplano Superior

Por el T. del Residuo

$$I = 2\pi i \left[R_{z1} + R_{z3} \right] = 2\pi i \left[-\frac{1}{6i} + \frac{1}{3i} \right] = 2\pi i \left[\frac{1}{6i} \right] = \frac{\pi}{3}$$

$$\therefore \; I = \frac{\pi}{3}$$

5.1-3-42 ①

ⓐ $I = \int_{-\infty}^{\infty} \frac{x \operatorname{sen} mx}{x^2+a^2}\, dx$ con $a \neq 0$

Sabiendo $\int_{-\infty}^{\infty} f(x) \operatorname{sen} mx\, dx = \operatorname{Im}\left(2\pi i \sum \operatorname{Res}[f(z)e^{imz}, z_k]\right)$ — ⊛

$f(x) = \frac{x}{x^2+a^2} \Rightarrow f(z) = \frac{z}{z^2+a^2}$

Singularidades

$z^2+a^2 = 0 \Rightarrow z_1 = ai$ esta en el semiplano superior

$z_2 = -ai$ no esta en el semiplano superior

De acuerdo a ⊛

$\operatorname{Res}[f(z)e^{imz}, z_0] = \lim_{z\to ai}\left[(z-ai)\left(\frac{e^{imz} z}{z^2+a^2}\right)\right]$

$= \lim_{z\to ai}\left[(z-ai)\left(\frac{e^{imz}\, z}{(z-ai)(z+ai)}\right)\right] = \lim_{z\to ai}\left(\frac{e^{imz}\, z}{z+ai}\right)$

$= \frac{e^{i^2 ma}\, ai}{2ai} = \frac{e^{-ma}}{2}$ $\therefore \operatorname{Res}[f(z)e^{imz}, z_0] = \frac{e^{-ma}}{2}$

$I = \operatorname{Im}\left(2\pi i\left(\frac{e^{-ma}}{2}\right)\right) = \frac{\pi}{e^{ma}}$

~~5~~ (b) $I = \int_{-\infty}^{\infty} \frac{x \cos mx}{x^2+a^2}\, dx$ con $a>0$

Sabiendo

$$\int_{-\infty}^{\infty} f(x) \cos mx \, dx = Re\left(2\pi i \sum Res(f(z)\, e^{imz}, z_k)\right)$$

y también sabiendo el residuo del ploblema 5'-3-42 (a)

$$Res[f(z)\, e^{imz}, z_0] = \frac{e^{-ma}}{2}$$

tenemos

$$I = Re\left(2\pi i \left(\frac{e^{-ma}}{2}\right)\right) = 0$$