Chapter 4 Solutions

4.1-1. (a)

$$x(t) = u(t) - u(t-1)$$

$$\begin{aligned} X(s) = \int_0^1 e^{-st}\,dt &= -\frac{e^{-st}}{s}\Bigg|_0^1 \\ &= -\frac{1}{s}[e^{-s} - 1] \\ &= \frac{1}{s}[1 - e^{-s}] \end{aligned}$$

Note that the result is valid for all values of s; hence the region of convergence is the entire s-plane. The abscissa of convergence is $\sigma_0 = -\infty$.

(b)

$$x(t) = te^{-t}u(t)$$

$$\begin{aligned} X(s) = \int_0^\infty te^{-t}e^{-st}\,dt &= \int_0^\infty te^{-(s+1)}\,dt \\ &= -\frac{e^{-(s+1)t}}{(s+1)^2}\left[-(s+1)t - 1\right]_0^\infty \\ &= \frac{1}{(s+1)^2} \end{aligned}$$

provided that $e^{-(s+1)\infty} = 0$ or Re $(s+1) > 0$. Hence the abscissa of convergence is Re $(s) > -1$ or $\sigma_0 > -1$.

(c)

$$x(t) = t\cos\omega_0 t\, u(t)$$

$$\begin{aligned} X(s) &= \int_0^\infty t\cos\omega_0 t e^{-st}\,dt \\ &= \frac{1}{2}\left\{\int_0^\infty [te^{(j\omega_0 - s)t} + te^{-(j\omega_0+s)t}]\,dt\right\} \\ &= \frac{1}{2}\left[\frac{1}{(s-j\omega_0)^2} + \frac{1}{(s+j\omega_0)^2}\right] \qquad \text{Re } (s) > 0 \end{aligned}$$

$$= \frac{s^2 - \omega_0^2}{(s^2 + \omega_0^2)^2}$$

(d)

$$x(t) = (e^{2t} - 2e^{-t})u(t)$$

$$\begin{aligned}
X(s) &= \int_0^\infty (e^{2t} - 2e^{-t})e^{-st}\,dt \\
&= \int_0^\infty e^{2t}e^{-st}\,dt - 2\int_0^\infty e^{-t}e^{-st}\,dt \\
&= \int_0^\infty e^{-(s-2)t}\,dt - 2\int_0^\infty e^{-(s+1)t}\,dt \\
&= \frac{1}{s-2} - \frac{2}{s+1}
\end{aligned}$$

We get the first term only if Re $s > 2$, and we get the second term only if Re $(s) > -1$. Both conditions will be satisfied if Re $(s) > 2$ or $\sigma_0 > 2$. Hence:

$$X(s) = \frac{1}{s-2} - \frac{2}{s+1} \qquad \text{for} \quad \sigma_0 > 2$$

(e)

$$x(t) = \cos\omega_1 t \cos\omega_2 t\, u(t) = \left[\frac{1}{2}\cos(\omega_1 + \omega_2)t + \frac{1}{2}\cos(\omega_1 - \omega_2)t\right] u(t)$$

$$\begin{aligned}
X(s) &= \frac{1}{2}\int_0^\infty \cos(\omega_1 + \omega_2)te^{-st}\,dt + \frac{1}{2}\int_0^\infty \cos(\omega_1 - \omega_2)te^{-st}\,dt \\
&= \frac{1}{2}\left[\frac{s}{s^2 + (\omega_1 + \omega_2)^2} + \frac{s}{s^2 + (\omega_1 - \omega_2)^2}\right]
\end{aligned}$$

provided that Re $(s) > 0$.

(f)

$$x(t) = \cosh(at)u(t)$$

$$\begin{aligned}
X(s) &= \frac{1}{2}\left[\int_0^\infty e^{at}e^{-st}\,dt + \int_0^\infty e^{-at}e^{-st}\,dt\right] \\
&= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\,dt + \int_0^\infty e^{-(s+a)t}\,dt\right] \\
&= \frac{s}{s^2 - a^2} \qquad\qquad \text{Re } s > |a|
\end{aligned}$$

(g)

$$x(t) = \sinh(at)u(t)$$

$$X(s) = \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\,dt - \int_0^\infty e^{-(s+a)t}\,dt\right]$$

$$= \frac{a}{s^2 - a^2} \qquad \text{Re } s > |a|$$

(h)

$$\begin{aligned}
x(t) &= e^{-2t}\cos(5t+\theta)u(t) \\
&= \frac{1}{2}\left[e^{-2t+j(5t+\theta)} + e^{-2t-j(5t+\theta)}\right] \\
&= \frac{1}{2}e^{j\theta}e^{-(2-j5)t} + \frac{1}{2}e^{-j\theta}e^{-(2+j5)t}
\end{aligned}$$

Hence $\quad X(s) = \frac{1}{2}e^{j\theta}\left(\frac{1}{s+2-j5}\right) + \frac{1}{2}e^{-j\theta}\left(\frac{1}{s+2+j5}\right)$

This is valid if Re $(s) > -2$ for both terms; hence

$$X(s) = \frac{(s+2)\cos\theta - 5\sin\theta}{s^2+4s+29}$$

4.1-2. (a)

$$X(s) = \int_0^1 te^{-st}\,dt = \frac{e^{-st}}{s}(-st-1)\bigg|_0^1 = \frac{1}{s^2}(1 - e^{-s} - se^{-s})$$

(b)

$$X(s) = \int_0^\pi \sin t\, e^{-st}\,dt = \frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\bigg|_0^\pi = \frac{1+e^{-\pi s}}{s^2+1}$$

(c)

$$\begin{aligned}
X(s) &= \int_0^1 \frac{t}{e}e^{-st}\,dt + \int_1^\infty e^{-t}e^{-st}\,dt = \frac{1}{e}\int_0^1 te^{-st}\,dt + \int_1^\infty e^{-(s+1)t}\,dt \\
&= \frac{e^{-st}}{es}(-st-1)\bigg|_0^1 - \frac{1}{s+1}e^{-(s+1)}\bigg|_1^\infty \\
&= \frac{1}{es^2}(1 - e^{-s} - se^{-s}) + \frac{1}{s+1}e^{-(s+1)}
\end{aligned}$$

4.1-3. (a)

$$\begin{aligned}
X(s) &= \frac{2s+5}{s^2+5s+6} = \frac{2s+5}{(s+2)(s+3)} = \frac{1}{s+2} + \frac{1}{s+3} \\
x(t) &= (e^{-2t} + e^{-3t})u(t)
\end{aligned}$$

(b)

$$X(s) = \frac{3s+5}{s^2+4s+13}$$

Here $A = 3$, $B = 5$, $a = 2$, $c = 13$, $b = \sqrt{13-4} = 3$.

$$r = \sqrt{\frac{117+25-60}{13-4}} = 3.018 \qquad \theta = \tan^{-1}(\frac{1}{9}) = 6.34°$$

$$x(t) = 3.018e^{-2t}\cos(3t + 6.34°)u(t)$$

(c)

$$X(s) = \frac{(s+1)^2}{s^2 - s - 6} = \frac{(s+1)^2}{(s+2)(s-3)}$$

This is an improper fraction with $b_n = b_2 = 1$. Therefore

$$\begin{aligned} X(s) &= 1 + \frac{a}{s+2} + \frac{b}{s-3} = 1 - \frac{0.2}{s+2} + \frac{3.2}{s-3} \\ x(t) &= \delta(t) + (3.2e^{3t} - 0.2e^{-2t})u(t) \end{aligned}$$

(d)

$$X(s) = \frac{5}{s^2(s+2)} = \frac{k}{s} + \frac{2.5}{s^2} + \frac{1.25}{s+2}$$

To find k set $s = 1$ on both sides to obtain

$$\frac{5}{3} = k + 2.5 + \frac{5}{12} \quad \Longrightarrow \quad k = -1.25$$

and

$$\begin{aligned} X(s) &= -\frac{1.25}{s} + \frac{2.5}{s^2} + \frac{1.25}{s+2} \\ x(t) &= 1.25(-1 + 2t + e^{-2t})u(t) \end{aligned}$$

(e)

$$X(s) = \frac{2s+1}{(s+1)(s^2+2s+2)} = \frac{-1}{s+1} + \frac{As+B}{s^2+2s+2}$$

Multiply both sides by s and let $s \to \infty$. This yields

$$0 = -1 + A \quad \Longrightarrow \quad A = 1$$

Setting $s = 0$ on both sides yields

$$\frac{1}{2} = -1 + \frac{B}{2} \quad \Longrightarrow \quad B = 3$$

$$X(s) = -\frac{1}{s+1} + \frac{s+3}{s^2+2s+2}$$

In the second fraction, $A = 1$, $B = 3$, $a = 1$, $c = 2$, $b = \sqrt{2-1} = 1$.

$$r = \sqrt{\frac{2+9-6}{2-1}} = \sqrt{5} \qquad \theta = \tan^{-1}\left(\frac{-2}{1}\right) = -63.4^\circ$$

$$x(t) = [-e^{-t} + \sqrt{5}e^{-t}\cos(t - 63.4^\circ)]u(t)$$

(f)

$$X(s) = \frac{s+2}{s(s+1)^2} = \frac{2}{s} + \frac{k}{s+1} - \frac{1}{(s+1)^2}$$

To compute k, multiply both sides by s and let $s \to \infty$. This yields

$$0 = 2 + k + 0 \quad \Longrightarrow \quad k = -2$$

and

$$\begin{aligned} X(s) &= \frac{2}{s} - \frac{2}{s+1} - \frac{1}{(s+1)^2} \\ x(t) &= [2 - (2+t)e^{-t}]u(t) \end{aligned}$$

(g)

$$X(s) = \frac{1}{(s+1)(s+2)^4} = \frac{1}{s+1} + \frac{k_1}{s+2} + \frac{k_2}{(s+2)^2} + \frac{k_3}{(s+2)^3} - \frac{1}{(s+2)^4}$$

Multiplying both sides by s and let $s \to \infty$. This yields

$$0 = 1 + k_1 \quad \Longrightarrow \quad k_1 = -1$$

$$\frac{1}{(s+1)(s+2)^4} = \frac{1}{s+1} - \frac{1}{s+2} + \frac{k_2}{(s+2)^2} + \frac{k_3}{(s+2)^3} - \frac{1}{(s+2)^4}$$

Setting $s = 0$ and -3 on both sides yields

$$\begin{aligned} \frac{1}{16} = 1 - \frac{1}{2} + \frac{k_2}{4} + \frac{k_3}{8} - \frac{1}{16} \quad &\Longrightarrow \quad 4k_2 + 2k_3 = -6 \\ -\frac{1}{2} = -\frac{1}{2} + 1 + k_2 - k_3 - 1 \quad &\Longrightarrow \quad k_2 - k_3 = 0 \end{aligned}$$

Solving these two equations simultaneously yields $k_2 = k_3 = -1$. Therefore

$$\begin{aligned} X(s) &= \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2} - \frac{1}{(s+2)^3} - \frac{1}{(s+2)^4} \\ x(t) &= [e^{-t} - (1 + t + \frac{t^2}{2} + \frac{t^3}{6})e^{-2t}]u(t) \end{aligned}$$

Comment: This problem could be tackled in many ways. We could have used Eq. (B.47b), or after determining first two coefficients by Heaviside method, we could have cleared fractions. Also instead of letting $s = 0$ and -3, we could have selected any other set of values. However, in this case these values appear most suitable for numerical work.

(h)

$$X(s) = \frac{s+1}{s(s+2)^2(s^2+4s+5)} = \frac{(1/20)}{s} + \frac{k}{s+2} + \frac{(1/2)}{(s+2)^2} + \frac{As+B}{s^2+4s+5}$$

Multiplying both sides by s and let $s \to \infty$ yields

$$0 = \frac{1}{20} + k + A \quad \Longrightarrow \quad k + A = -\frac{1}{20}$$

Setting $s = 1$ and -1 yields

$$\begin{aligned} \tfrac{2}{90} = \tfrac{1}{20} + \tfrac{k}{3} + \tfrac{1}{18} + \tfrac{A+B}{10} \quad &\Longrightarrow \quad 20k + 6A + 6B = -5 \\ 0 = -\tfrac{1}{20} + k + \tfrac{1}{2} + \tfrac{-A+B}{2} \quad &\Longrightarrow \quad 20k - 10A + 10B = -9 \end{aligned}$$

Solving these three equations in k, A and B yields $k = -\frac{1}{4}$, $A = \frac{1}{5}$ and $B = -\frac{1}{5}$. Therefore

$$X(s) = \frac{1/20}{s} - \frac{1/4}{s+2} + \frac{(1/2)}{(s+2)^2} + \frac{1}{5}(\frac{s-1}{s^2+4s+5})$$

For the last fraction in parenthesis on the right-hand side $A = 1$, $B = -1$, $a = 2$,

$c = 5$, $b = \sqrt{5-4} = 1$.

$$r = \sqrt{\frac{5+1+4}{5-4}} = \sqrt{10} \qquad \theta = \tan^{-1}(\frac{3}{1}) = 71.56°$$

$$x(t) = [\frac{1}{20} - \frac{1}{4}(1-2t)e^{-2t} + \frac{\sqrt{10}}{5}e^{-2t}\cos(t + 71.56°)]u(t)$$

(i)

$$X(s) = \frac{s^3}{(s+1)^2(s^2+2s+5)} = \frac{k}{s+1} - \frac{1/4}{(s+1)^2} + \frac{As+B}{s^2+2s+5}$$

Multiply both sides by s and let $s \to \infty$ to obtain

$$1 = k + A$$

Setting $s = 0$ and 1 yields

$$0 = k - \tfrac{1}{4} + \tfrac{B}{5} \qquad \Longrightarrow \qquad 20k + 4B = 5$$

$$\tfrac{1}{32} = \tfrac{k}{2} - \tfrac{1}{16} + \tfrac{A+B}{8} \qquad \Longrightarrow \qquad 16k + 4A + 4B = 3$$

Solving these three equations in k, A and B yields $k = \frac{3}{4}$, $A = \frac{1}{4}$ and $B = -\frac{5}{2}$.

$$X(s) = \frac{3/4}{s+1} - \frac{1/4}{(s+1)^2} + \frac{1}{4}(\frac{s-10}{s^2+2s+5})$$

For the last fraction in parenthesis, $A = 1$, $B = -10$, $a = 1$, $c = 5$, $b = \sqrt{5-1} = 2$.

$$r = \sqrt{\frac{5+100+20}{5-1}} = 5.59 \qquad \theta = \tan^{-1}(\frac{11}{4}) = 70°$$

Therefore

$$\begin{aligned} x(t) &= [(\frac{3}{4} - \frac{1}{4}t)e^{-t} + \frac{5.59}{4}e^{-t}\cos(2t + 70°)]u(t) \\ &= [\frac{1}{4}(3-t) + 1.3975\cos(2t + 70°)]e^{-t}u(t) \end{aligned}$$

4.2-1. (a)

$$x(t) = u(t) - u(t-1)$$

and

$$\begin{aligned} X(s) &= \mathcal{L}[u(t)] - \mathcal{L}[u(t-1)] \\ &= \frac{1}{s} - e^{-s}\frac{1}{s} \\ &= \frac{1}{s}(1 - e^{-s}) \end{aligned}$$

(b)

$$x(t) = e^{-(t-\tau)}u(t-\tau)$$

$$X(s) = \frac{1}{s+1}e^{-s\tau}$$

(c)

$$x(t) = e^{-(t-\tau)}u(t) = e^{\tau}e^{-t}u(t)$$

$$\text{Therefore} \qquad X(s) = e^{\tau}\frac{1}{s+1}$$

(d)

$$x(t) = e^{-t}u(t-\tau) = e^{-\tau}e^{-(t-\tau)}u(t-\tau)$$

Observe that $e^{-(t-\tau)}u(t-\tau)$ is $e^{-t}u(t)$ delayed by τ. Therefore

$$X(s) = e^{-\tau}\left(\frac{1}{s+1}\right)e^{-s\tau} = \left(\frac{1}{s+1}\right)e^{-(s+1)\tau}$$

(e)

$$\begin{aligned} x(t) &= te^{-t}u(t-\tau) = (t-\tau+\tau)e^{-(t-\tau+\tau)}u(t-\tau) \\ &= e^{-\tau}\left[(t-\tau)e^{-(t-\tau)}u(t-\tau) + \tau e^{-(t-\tau)}u(t-\tau)\right] \end{aligned}$$

Therefore

$$\begin{aligned} X(s) &= e^{-\tau}\left[\frac{1}{(s+1)^2}e^{-s\tau} + \frac{\tau}{(s+1)}e^{-s\tau}\right] \\ &= \frac{e^{-(s+1)\tau}\left[1+\tau(s+1)\right]}{(s+1)^2} \end{aligned}$$

(f)

$$x(t) = \sin\omega_0(t-\tau)u(t-\tau)$$

Note that this is $\sin\omega_0 t$ shifted by τ; hence

$$X(s) = \left(\frac{\omega_0}{s^2+\omega_0^2}\right)e^{-s\tau}$$

(g)

$$x(t) = \sin\omega_0(t-\tau)u(t) = [\sin\omega_0 t\cos\omega_0\tau - \cos\omega_0 t\sin\omega_0\tau]u(t)$$

$$X(s) = \frac{\omega_0\cos\omega_0\tau - s\sin\omega_0\tau}{s^2+\omega_0^2}$$

(h)

$$\begin{aligned} x(t) &= \sin\omega_0 t\, u(t-\tau) = \sin[\omega_0(t-\tau+\tau)]u(t-\tau) \\ &= \cos\omega_0\tau\sin[\omega_0(t-\tau)]u(t-\tau) + \sin\omega_0\tau\cos[\omega_0(t-\tau)]u(t-\tau) \end{aligned}$$

Therefore

$$X(s) = \left[\cos\omega_0\tau\left(\frac{\omega_0}{s^2+\omega_0^2}\right) + \sin\omega_0\tau\left(\frac{s}{s^2+\omega_0^2}\right)\right]e^{-s\tau}$$

4.2-2. (a)

$$x(t) = t[u(t) - u(t-1)] = tu(t) - (t-1)u(t-1) - u(t-1)$$

$$X(s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}$$

(b)

$$x(t) = \sin t\, u(t) + \sin(t-\pi)\, u(t-\pi)$$

$$X(s) = \frac{1}{s^2+1}(1 + e^{-\pi s})$$

(c)

$$\begin{aligned} x(t) &= t[u(t) - u(t-1)] + e^{-t}u(t-1) \\ &= tu(t) - (t-1)u(t-1) - u(t-1) + e^{-1}e^{-(t-1)}u(t-1) \end{aligned}$$

Therefore

$$X(s) = \frac{1}{s^2}(1 - e^{-s} - se^{-s}) + \frac{e^{-s}}{e(s+1)}$$

4.2-3. (a)

$$X(s) = \frac{(2s+5)e^{-2s}}{s^2+5s+6} = \hat{X}(s)e^{-2s}$$

It is clear that $x(t) = \hat{x}(t-2)$.

$$\begin{aligned} \hat{X}(s) &= \frac{2s+5}{s^2+5s+6} = \frac{2s+5}{(s+2)(s+3)} = \frac{1}{s+2} + \frac{1}{s+3} \\ \hat{x}(t) &= (e^{-2t} + e^{-3t})u(t) \\ x(t) &= \hat{x}(t-2) = [e^{-2(t-2)} + e^{-3(t-2)}]u(t-2) \end{aligned}$$

(b)

$$X(s) = \frac{s}{s^2+2s+2}e^{-3s} + \frac{2}{s^2+2s+2} = X_1(s)e^{-3s} + X_2(s)$$

where

$$X_1(s) = \frac{s}{s^2+2s+2} \quad \left\{ \begin{array}{l} A = 1,\, B = 0,\, a = 1,\, c = 2,\, b = 1 \\ r = \sqrt{2},\, \theta = \tan^{-1}(1) = \pi/4 \end{array} \right.$$

$$\begin{aligned} x_1(t) &= \sqrt{2}e^{-t}\cos(t + \frac{\pi}{4}) \\ X_2(s) &= \frac{2}{s^2+2s+2} \quad \text{and} \quad x_2(t) = 2e^{-t}\sin t \end{aligned}$$

Also

$$\begin{aligned} x(t) &= x_1(t-3) + x_2(t) \\ &= \sqrt{2}e^{-(t-3)}\cos(t - 3 + \frac{\pi}{4})u(t-3) + 2e^{-t}\sin t\, u(t) \end{aligned}$$

(c)

$$\begin{aligned} X(s) &= \frac{(e)e^{-s}}{s^2-2s+5} + \frac{3}{s^2-2s+5} \\ &= e\frac{1}{s^2-2s+5}e^{-s} + \frac{3}{s^2-2s+5} \\ &= eX_1(s)e^{-s} + X_2(s) \end{aligned}$$

where

$$\begin{aligned} X_1(s) &= \frac{1}{s^2-2s+2} \quad \text{and} \quad x_1(t) = \frac{1}{2}e^t \sin 2t\, u(t) \\ X_2(s) &= \frac{3}{s^2-2s+2} \quad \text{and} \quad x_2(t) = \frac{3}{2}e^t \sin 2t\, u(t) \end{aligned}$$

Therefore

$$\begin{aligned} x(t) &= e x_1(t-1) + x_2(t) \\ &= \frac{e}{2}e^{(t-1)} \sin 2(t-1) u(t-1) + \frac{3}{2}e^t \sin 2t\, u(t) \end{aligned}$$

(d)

$$\begin{aligned} X(s) = \frac{e^{-s}+e^{-2s}+1}{s^2+3s+2} &= (e^{-s}+e^{-2s}+1)\left[\frac{1}{s^2+3s+2}\right] \\ &= (e^{-s}+e^{-2s}+1)\left[\frac{1}{s+1} - \frac{1}{s+2}\right] \end{aligned}$$

$$X(s) = (e^{-s}+e^{-2s}+1)\hat{X}(s)$$

where $\hat{X}(s) = \frac{1}{s+1} - \frac{1}{s+2}$ and $\hat{x}(t) = (e^{-t} - e^{-2t})u(t)$

Moreover

$$\begin{aligned} x(t) &= \hat{x}(t-1) + \hat{x}(t-2) + \hat{x}(t) \\ &= [e^{-(t-1)} - e^{-2(t-1)}]u(t-1) + [e^{-(t-2)} - e^{-2(t-2)}]u(t-2) + (e^{-t} - e^{-2t})u(t) \end{aligned}$$

4.2-4. (a)

$$g(t) = x(t) + x(t-T_0) + x(t-2T_0) + \cdots$$

and

$$\begin{aligned} G(s) &= X(s) + X(s)e^{-sT_0} + X(s)e^{-2sT_0} + \cdots \\ &= X(s)[1 + e^{-sT_0} + e^{-2sT_0} + e^{-3sT_0} + \cdots \\ &= \frac{X(s)}{1-e^{-sT_0}} \qquad |e^{-sT_0}| < 1 \text{ or } \operatorname{Re} s > 0 \end{aligned}$$

(b)

$$\begin{aligned} x(t) &= u(t) - u(t-2) \quad \text{and} \quad X(s) = \frac{1}{s}(1-e^{-2s}) \\ G(s) &= \frac{X(s)}{1-e^{-8s}} = \frac{1}{s}\left(\frac{1-e^{-2s}}{1-e^{-8s}}\right) \end{aligned}$$

4.2-5. Pair 2

$$u(t) = \int_{0^-}^{t} \delta(\tau)\, d\tau \Longleftrightarrow \frac{1}{s}(1) = \frac{1}{s}$$

Pair 3

$$tu(t) = \int_{0^-}^{t} u(\tau)\, d\tau \Longleftrightarrow \frac{1}{s}(\frac{1}{s}) = \frac{1}{s^2}$$

Pair 4: Use successive integration of $tu(t)$

Pair 5: From frequency-shifting (4.23), we have

$$u(t) \Longleftrightarrow \frac{1}{s} \quad \text{and} \quad e^{\lambda t}u(t) \Longleftrightarrow \frac{1}{s-\lambda}$$

Pair 6: Because

$$tu(t) \Longleftrightarrow \frac{1}{s^2} \quad \text{and} \quad te^{\lambda t}u(t) \Longleftrightarrow \frac{1}{(s-\lambda)^2}$$

Pair 7: Apply the same argument to $t^2u(t)$, $t^3u(t)$, ..., and so on.
Pair 8a:

$$\cos bt\, u(t) = \frac{1}{2}(e^{jbt} + e^{-jbt})u(t) \Longleftrightarrow \frac{1}{2}\left(\frac{1}{s-jb} + \frac{1}{s+jb}\right) = \frac{s}{s^2+b^2}$$

Pair 8b: Same way as the pair 8a.
Pair 9a: Application of the frequency-shift property (4.23) to pair 8a $\cos bt\, u(t) \Longleftrightarrow \frac{s}{s^2+b^2}$ yields

$$e^{-at}\cos bt\, u(t) \Longleftrightarrow \frac{s+a}{(s+a)^2+b^2}$$

Pair 9b: Similar to the pair 9a.
Pairs 10a and 10b: Recognize that

$$re^{-at}\cos(bt+\theta) = re^{-at}[\cos\theta\cos bt - \sin\theta\sin bt]$$

Now use results in pairs 9a and 9b to obtain pair 10a. Pair 10b is equivalent to pair 10a.

4.2-6. (a) (i)

$$\begin{aligned} \frac{dx}{dt} &= \delta(t) - \delta(t-2) \\ sX(s) &= 1 - e^{-2s} \\ X(s) &= \frac{1}{s}(1 - e^{-2s}) \end{aligned}$$

(ii)

$$\begin{aligned} \frac{dx}{dt} &= \delta(t-2) - \delta(t-4) \\ sX(s) &= e^{-2s} - e^{-4s} \\ X(s) &= \frac{1}{s}(e^{-2s} - e^{-4s}) \end{aligned}$$

(b)

$$\begin{aligned} \frac{dx}{dt} &= u(t) - 3u(t-2) + 2u(t-3) \\ sX(s) &= \frac{1}{s} - \frac{3}{s}e^{-2s} + \frac{2}{s}e^{-3s} \quad [x(0^-) = 0] \\ X(s) &= \frac{1}{s^2}(1 - 3e^{-2s} + 2e^{-3s}) \end{aligned}$$

4.2-7. $X(s) = e^{-3}e^{-s}\left[\frac{s^2}{(s+1)(s+2)}\right] = e^{-3}e^{-s}\left[1 + \frac{-3s-2}{(s+1)(s+2)}\right] = e^{-3}e^{-s}\left[1 + \frac{1}{(s+1)} + \frac{-4}{(s+2)}\right]$.
Thus,

$$x(t) = e^{-3}\left[\delta(t-1) + e^{-(t-1)}u(t-1) - 4e^{-2(t-1)}u(t-1)\right].$$

4.2-8. First, note that the n^{th} derivative of $\frac{1}{s+a}$ is $\frac{(-1)^n n!}{(s+1)^{n+1}}$. Thus, rewrite the transform as $X(s) = \frac{1}{(s+1)^{13}} = \frac{1}{12!}\frac{12!}{(s+1)^{13}} = \frac{1}{12!}\frac{d^{12}}{ds^{12}}\left(\frac{1}{s+1}\right)$. Since $\sigma > -1$, the time-domain signal $x(t)$ must be right sided. Repeated use of the differentiation in s property provides the resulting inverse transform.

$$x(t) = \frac{1}{12!}(-t)^{12}e^{-t}u(t) = \frac{t^{12}}{12!}e^{-t}u(t).$$

4.2-9. (a) Using the differentiation in s property,

$$\mathcal{L}\left[tx(t)\right] = -\frac{d}{ds}X(s).$$

(b) $y(t) = tx(t) = t\frac{1}{t}u(t) = u(t)$. Thus, $Y(s) = \int_{-\infty}^{\infty} u(t)e^{-st}dt = \int_0^{\infty} e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_{t=0}^{\infty}$. For $\sigma > 0$, this simplifies to $Y(s) = \frac{1}{s}$.

(c) Combining the previous two parts yields $-\frac{d}{ds}X(s) = \frac{1}{s}$. Thus,

$$X(s) = -\int \frac{1}{s}ds = -\ln(s).$$

4.3-1. (a)

$$\begin{aligned}
(s^2+3s+2)Y(s) &= s(\frac{1}{s}) \\
Y(s) &= \frac{1}{s^2+3s+2} = \frac{1}{s+1} - \frac{1}{s+2} \\
y(t) &= (e^{-t} - e^{-2t})u(t)
\end{aligned}$$

(b)

$$(s^2Y(s) - 2s - 1) + 4(sY(s) - 2) + 4Y(s) = (s+1)\frac{1}{s+1}$$

or

$$(s^2+4s+4)Y(s) = 2s+10$$

and

$$\begin{aligned}
Y(s) &= \frac{2s+10}{s^2+4s+4} = \frac{2s+10}{(s+2)^2} = \frac{2}{s+2} + \frac{6}{(s+2)^2} \\
y(t) &= (2+6t)e^{-2t}u(t)
\end{aligned}$$

(c)

$$(s^2Y(s) - s - 1) + 6(sY(s) - 1) + 25Y(s) = (s+2)\frac{25}{s} = 25 + \frac{50}{s}$$

or

$$(s^2+6s+25)Y(s) = s + 32 + \frac{50}{s} = \frac{s^2+32s+50}{s}$$

and

$$\begin{aligned}
Y(s) &= \frac{s^2+32s+50}{s(s^2+6s+25)} = \frac{2}{s} + \frac{-s+20}{s^2+6s+25} \\
y(t) &= [2 + 5.836e^{-3t}\cos(4t - 99.86^\circ)]u(t)
\end{aligned}$$

4.3-2. (a) All initial conditions are zero. The zero-input response is zero. The entire response found in Prob4.3-2a is zero-state response, that is

$$y_{zs}(t) = (e^{-t} - e^{-2t})u(t)$$

$$y_{zi}(t) = 0$$

(b) The Laplace transform of the differential equation is

$$(s^2Y(s) - 2s - 1) + 4(sY(s) - 2) + 4Y(s) = (s+1)\frac{1}{s+1}$$

or
$$(s^2 + 4s + 4)Y(s) - (2s + 9) = 1$$

or
$$(s^2 + 4s + 4)Y(s) = \underbrace{2s+9}_{\text{i.c. terms}} + \underbrace{1}_{\text{input}}$$

$$\begin{aligned}
Y(s) &= \underbrace{\frac{2s+9}{s^2+4s+4}}_{\text{zero-input}} + \underbrace{\frac{1}{s^2+4s+4}}_{\text{zero-state}} \\
&= \underbrace{\frac{2}{s+2} + \frac{5}{(s+2)^2}}_{\text{zero-input}} + \underbrace{\frac{1}{(s+2)^2}}_{\text{zero-state}} \\
y(t) &= \underbrace{(2+5t)e^{-2t}}_{\text{zero-input}} + \underbrace{te^{-2t}}_{\text{zero-state}}
\end{aligned}$$

(c) The Laplace transform of the equation is

$$(s^2Y(s) - s - 1) + 6(sY(s) - 1) + 25Y(s) = 25 + \frac{50}{s}$$

or

$$(s^2 + 6s + 25)Y(s) = \underbrace{s+7}_{\text{i.c. terms}} + \underbrace{25 + \frac{50}{s}}_{\text{input}}$$

$$\begin{aligned}
Y(s) &= \underbrace{\frac{s+7}{s^2+6s+25}}_{\text{zero-input}} + \underbrace{\frac{25s+50}{s(s^2+6s+25)}}_{\text{zero-state}} \\
&= \left(\frac{s+7}{s^2+6s+25}\right) + \left(\frac{2}{s} + \frac{-2s+13}{s^2+6s+25}\right) \\
y(t) &= \underbrace{[\sqrt{2}e^{-3t}\cos(4t - \frac{\pi}{4})]}_{\text{zero-input}} + \underbrace{[2 + 5.154e^{-3t}\cos(4t - 112.83^\circ)]}_{\text{zero-state}}
\end{aligned}$$

4.3-3. (a) Laplace transform of the two equations yields

$$\begin{aligned}
(s+3)Y_1(s) - 2Y_2(s) &= \frac{1}{s} \\
-2Y_1(s) + (2s+4)Y_2(s) &= 0
\end{aligned}$$

Using Cramer's rule, we obtain

$$Y_1(s) = \frac{s+2}{s(s^2+5s+4)} = \frac{s+2}{s(s+1)(s+4)} = \frac{1/2}{s} - \frac{1/3}{s+1} - \frac{1/6}{s+4}$$

$$Y_2(s) = \frac{1}{s(s^2+5s+4)} = \frac{1}{s(s+1)(s+4)} = \frac{1/4}{s} - \frac{1/3}{s+1} + \frac{1/12}{s+4}$$

and

$$\begin{aligned} y_1(t) &= (\frac{1}{2} - \frac{1}{3}e^{-t} - \frac{1}{6}e^{-4t})u(t) \\ y_2(t) &= (\frac{1}{4} - \frac{1}{3}e^{-t} + \frac{1}{12}e^{-4t})u(t) \end{aligned}$$

If $H_1(s)$ and $H_2(s)$ are the transfer functions relating $y_1(t)$ and $y_2(t)$, respectively to the input $x(t)$, thus

$$H_1(s) = \frac{s+2}{s^2+5s+4} \quad \text{and} \quad H_2(s) = \frac{1}{s^2+5s+4}$$

(b) The Laplace transform of the equations are

$$\begin{aligned} (s+2)Y_1(s) - (s+1)Y_2(s) &= 0 \\ -(s+1)Y_1(s) + (2s+1)Y_2(s) &= 0 \end{aligned}$$

Application of Cramer's rule yields

$$\begin{aligned} Y_1(s) &= \frac{s+1}{s(s^2+3s+1)} = \frac{s+1}{s(s+0.382)(s+2.618)} = \frac{1}{s} - \frac{0.724}{s+0.382} - \frac{0.276}{s+2.618} \\ Y_2(s) &= \frac{s+2}{s(s^2+3s+1)} = \frac{s+2}{s(s+0.382)(s+2.618)} = \frac{2}{s} - \frac{1.894}{s+0.382} - \frac{0.1056}{s+2.618} \end{aligned}$$

$$H_1(s) = \frac{s+1}{s^2+3s+1} \quad \text{and} \quad H_2(s) = \frac{s+2}{s^2+3s+1}$$

$$\begin{aligned} y_1(t) &= (1 - 0.724e^{-0.382t} - 0.276e^{-2.618t})u(t) \\ y_2(t) &= (2 - 1.894e^{-0.382t} - 0.1056e^{-2.618t})u(t) \end{aligned}$$

4.3-4. At $t = 0$, the inductor current $y_1(0) = 4$ and the capacitor voltage is 16 volts. After $t = 0$, the loop equations are

$$\begin{aligned} 2\frac{dy_1}{dt} - 2\frac{dy_2}{dt} + 5y_1(t) - 4y_2(t) &= 40 \\ -2\frac{dy_1}{dt} - 4y_1(t) + 2\frac{dy_2}{dt} + 4y_2(t) + \int_{-\infty}^{t} y_2(\tau)\,d\tau &= 0 \end{aligned}$$

If

$$\begin{aligned} y_1(t) &\Longleftrightarrow Y_1(s), \quad \tfrac{dy_1}{dt} = sY_1(s) - 4 \\ y_2(t) &\Longleftrightarrow Y_2(s), \quad \tfrac{dy_2}{dt} = sY_2(s) \end{aligned}$$

$$\int_{-\infty}^{t} y_2(\tau)\,d\tau \Longleftrightarrow \frac{1}{s}Y_2(s) + \frac{16}{s}$$

Laplace transform of the loop equations are

$$\begin{aligned} 2(sY_1(s) - 4) - 2sY_2(s) + 5Y_1(s) - 4Y_2(s) &= \frac{40}{s} \\ -2(sY_1(s) - 4) - 4Y_1(s) + 2sY_2(s) + 4Y_2(s) + \frac{1}{s}Y_2(s) + \frac{16}{s} &= 0 \end{aligned}$$

Or

$$\begin{aligned}(2s+5)Y_1(s)-(2s+4)Y_2(s) &= 8+\frac{40}{s}\\ -(2s+4)Y_1(s)+(2s+4+\frac{1}{s})Y_2(s) &= -8-\frac{16}{s}\end{aligned}$$

Cramer's rule yields

$$\begin{aligned}Y_1(s) &= \frac{4(6s^2+13s+5)}{s(s^2+3s+2.5)} = \frac{8}{s}+\frac{16s+28}{s^2+3s+2.5}\\ y_1(t) &= [8+17.89e^{-1.5t}\cos(\frac{t}{2}-26.56°)]u(t)\\ Y_2(s) &= \frac{20(s+2)}{(s^2+3s+2.5)}\\ y_2(t) &= 20\sqrt{2}e^{-1.5t}\cos(\frac{t}{2}-\frac{\pi}{4})u(t)\end{aligned}$$

4.3-5. (a) $\frac{5s+3}{s^2+11s+24}$

(b) $\frac{3s^2+7s+5}{s^3+6s^2-11s+6}$

(c) $\frac{3s+2}{s(s^3+4)}$

(d) $\frac{1}{s+1}$

4.3-6. (a) $\frac{d^2y}{dt^2}+3\frac{dy}{dt}+8y(t)=\frac{dx}{dt}+5x(t)$

(b) $\frac{d^3y}{dt^3}+8\frac{d^2y}{dt^2}+5\frac{dy}{dt}+7y(t)=\frac{d^2x}{dt^2}+3\frac{dx}{dt}+5x(t)$

(c) $\frac{d^2y}{dt^2}-2\frac{dy}{dt}+5y(t)=5\frac{d^2x}{dt^2}+7\frac{dx}{dt}+2x(t)$

4.3-7. (a) (i) $X(s)=\frac{10}{s}$

$$\begin{aligned}Y(s) &= \frac{10(2s+3)}{s(s^2+2s+5)} = \frac{6}{s}+\frac{-6s+8}{s^2+2s+5}\\ y(t) &= [6+9.22e^{-t}\cos(2t-130.6°)]u(t)\end{aligned}$$

(ii) $x(t)=u(t-5)$ and $X(s)=\frac{1}{s}e^{-5s}$

$$\begin{aligned}Y(s) &= \frac{2s+3}{s(s^2+2s+5)}e^{-5s} = \left[\frac{0.6}{s}+\frac{1}{10}\left(\frac{-6s+8}{s^2+2s+5}\right)\right]e^{-5s}\\ y(t) &= \frac{1}{10}\{6+9.22e^{-(t-5)}\cos[2(t-5)-130.6°]\}u(t-5)\end{aligned}$$

(b) $\ddot{y}(t)+2\dot{y}(t)+5y(t)=2\dot{x}(t)+3x(t)$

4.3-8. (a) $X(s)=\frac{1}{s(s+1)}$

$$Y(s)=\frac{1}{(s+1)(s^2+9)}=\frac{0.1}{s+1}+\frac{re^{j\theta}}{s+j3}+\frac{re^{-j\theta}}{s-j3} \qquad r=\frac{1}{3\sqrt{10}},\ \theta=-161.56°$$

$$y(t)=0.1e^{-t}+\frac{1}{3\sqrt{10}}\cos(3t-161.56°)$$

(b)

$$\ddot{y}(t) + 9y(t) = \dot{x}(t)$$

4.3-9. (a) (i) $X(s) = \frac{1}{s+3}$ and

$$\begin{aligned} Y(s) &= \frac{s+5}{(s+3)(s^2+5s+6)} = \frac{s+5}{(s+2)(s+3)^2} = \frac{3}{s+2} - \frac{3}{s+3} - \frac{2}{(s-3)^2} \\ y(t) &= (3e^{-2t} - 3e^{-3t} - 2te^{-3t})u(t) \end{aligned}$$

(ii) $X(s) = \frac{1}{s+4}$

$$\begin{aligned} Y(s) &= \frac{s+5}{(s+2)(s+3)(s+4)} = \frac{3/2}{s+2} - \frac{2}{s+3} + \frac{1/2}{(s+4)} \\ y(t) &= \frac{3}{2}e^{-2t} - 2e^{-3t} + \frac{1}{2}e^{-4t}u(t) \end{aligned}$$

(iii) The input here is the input in (ii) delayed by 5 secs. Therefore $X(s) = \frac{1}{s+4}e^{-5s}$

$$\begin{aligned} Y(s) &= \frac{s+5}{(s+2)(s+3)(s+4)}e^{-5s} = [\frac{3/2}{s+2} - \frac{2}{s+3} + \frac{1/2}{(s+4)}]e^{-5s} \\ y(t) &= \frac{3}{2}e^{-2(t-5)} - 2e^{-3(t-5)} + \frac{1}{2}e^{-4(t-5)}]u(t-5) \end{aligned}$$

(iv) The input here is equal to the input in (ii) multiplied by e^{20} because $e^{-4(t-5)} = e^{20}e^{-4t}$. Therefore the output is equal to the output in (ii) multiplied by e^{20}.

$$y(t) = e^{20}[\frac{3}{2}e^{-2t} - 2e^{-3t} + \frac{1}{2}e^{-4t}]u(t)$$

(v) The input here is equal to the input in (iii) multiplied by e^{-20} because $e^{-4t}u(t-5) = e^{-20}e^{-4(t-5)}u(t-5)$. Therefore

$$y(t) = e^{-20}[\frac{3}{2}e^{-2(t-5)} - 2e^{-3(t-5)} + \frac{1}{2}e^{-4(t-5)}]u(t-5)$$

(b) $(D^2 + 2D + 5)y(t) = (2D + 3)x(t)$

4.3-10. Although this problem can be solved with Laplace transforms, it is easier to solve in the time domain. Since the system step response is $s(t) = e^{-t}u(t) - e^{-2t}u(t)$, the system impulse response is $h(t) = \frac{d}{dt}s(t) = -e^{-t}u(t) + \delta(t) + 2e^{-2t}u(t) - \delta(t) = (2e^{-2t} - e^{-t})u(t)$. The input $x(t) = \delta(t-\pi) - \cos(\sqrt{3})u(t)$ is just a sum of a shifted delta function and a scaled step function. Since the system is LTI, the output is quickly computed using just $h(t)$ and $s(t)$. That is,

$$y(t) = h(t-\pi) - \cos(\sqrt{3})s(t) = (2e^{-2(t-\pi)} - e^{-(t-\pi)})u(t-\pi) - \cos(\sqrt{3})(e^{-t} - e^{-2t})u(t).$$

4.3-11. (a) Let $H(s)$ be the system transfer function.

$$Y(s) = X(s)H(s)$$

Consider an input $x_1(t) = \dot{x}(t)$. Then $X_1(s) = sX(s)$. If the output is $y_1(t)$ and its transform is $Y_1(s)$, then

$$Y_1(s) = X_1(s)H(s) = sX(s)H(s) = sY(s)$$

This shows that $y_1(t) = dy/dt$.

(b) Using similar argument we show that for the input $\int_0^t x(\tau)\,d\tau$, the output is $\int_0^t y(\tau)\,d\tau$. Because $u(t)$ is an integral of $\delta(t)$, the unit step response is the integral of the unit impulse response $h(t)$.

4.3-12. (a) (i)

$$H(s) = \frac{s+5}{s^2+3s+2} = \frac{s+5}{(s+1)(s+2)}$$

Both characteristic roots, -1 and -2 are in LHP. Hence the system is BIBO (and asymptotically) stable.

(ii)

$$H(s) = \frac{s+5}{s^2(s+2)}$$

The characteristic roots are 0, 0, -2. There are repeated roots on imaginary axis. Hence the system is BIBO (and asymptotically) unstable.

(iii)

$$H(s) = \frac{s(s+2)}{s+5}$$

Although the characteristic root -5 is in LHP, because $M > N$, the system is BIBO unstable.

(iv)

$$H(s) = \frac{s+5}{s(s+2)}$$

The roots are 0, and -2. One of the roots is on the imaginary axis which makes the system BIBO unstable (but marginally stable).

(v)

$$H(s) = \frac{s+5}{s^2-2s-3} = \frac{s+5}{(s-3)(s+1)}$$

The roots are 3 and -1. One root in RHP makes system BIBO unstable (and also asymptotically unstable).

(b) (i)

$$(D^2+3D+2)y(t) = (D+3)x(t)$$

or

$$(D+1)(D+2)y(t) = (D+3)x(t)$$

The system transfer function is

$$H(s) = \frac{s+3}{(s+1)(s+2)}$$

The characteristic roots are -1 and -2 (both in LHP). Hence the system is asymptotically and BIBO stable.

(ii)

$$(D^2 + 3D + 2)y(t) = (D + 1)x(t)$$

or

$$(D + 1)(D + 2)y(t) = (D + 1)x(t)$$

The system transfer function is

$$H(s) = \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2}$$

The characteristic roots are -1 and -2 (both in LHP). The only pole of H(s) is at -2. Hence the system is asymptotically and BIBO stable.

(iii)

$$(D^2 + D - 2)y(t) = (D - 1)x(t)$$

or

$$(D - 1)(D + 2)y(t) = (D - 1)x(t)$$

The system transfer function is

$$H(s) = \frac{s-1}{(s-1)(s+2)} = \frac{1}{s+2}$$

The system's characteristic roots at 1 and -2 makes system asymptotically unstable. But the only pole of H(s) is at -2, which makes system BIBO stable.

(iv)

$$(D^2 - 3D + 2)y(t) = (D - 1)x(t)$$

or

$$(D - 1)(D - 2)y(t) = (D - 1)x(t)$$

The system transfer function is

$$H(s) = \frac{s-1}{(s-1)(s-2)} = \frac{1}{s-2}$$

The characteristic roots are 1 and 2. The only pole of H(s) is at 2. Hence the system is asymptotically and BIBO unstable.

4.4-1. Figure S4.4-1 shows the transformed network. The loop equations are

$$\begin{aligned}
(1 + \frac{1}{s})Y_1(s) - \frac{1}{s}Y_2(s) &= \frac{1}{(s+1)^2} \\
-\frac{1}{s}Y_1(s) + (s + 1 + \frac{1}{s})Y_2(s) &= 0
\end{aligned}$$

$$\begin{bmatrix} \frac{s+1}{s} & -\frac{1}{s} \\ -\frac{1}{s} & \frac{s^2+s+1}{s} \end{bmatrix} \begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{(s+1)^2} \\ 0 \end{bmatrix}$$

Cramer's rule yields

$$Y_2(s) = \frac{1}{(s+1)^2(s^2+2s+2)} = \frac{1}{(s+1)^2} - \frac{1}{s^2+2s+2}$$

$$v_0(t) = y_2(t) = (te^{-t} - \frac{1}{2}e^{-t}\sin t)u(t)$$

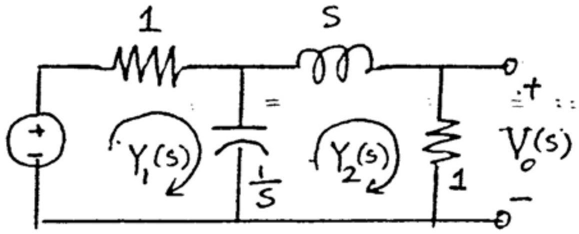


Figure S4.4-1

4.4-2. Before the switch is opened, the inductor current is 5A, that is $y(0) = 5$. Figure S4.4-2b shows the transformed circuit for $t \geq 0$ with initial condition generator. The current $Y(s)$ is given by

$$\begin{aligned} Y(s) &= \frac{(10/s)+5}{3s+2} = \frac{5s+10}{s(3s+2)} = \frac{5}{3}[\frac{3}{s} - \frac{2}{s+(2/3)}] \\ y(t) &= (5 - \frac{10}{3}e^{-2t/3})u(t) \end{aligned}$$

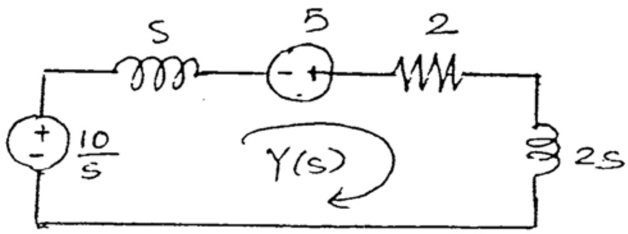


Figure S4.4-2

4.4-3. The impedance seen by the source $x(t)$ is

$$Z(s) = \frac{Ls(1/Cs)}{Ls+(1/Cs)} = \frac{Ls}{LCs^2+1} = \frac{Ls{\omega_0}^2}{s^2+{\omega_0}^2}$$

The current $Y(s)$ is given by

$$Y(s) = \frac{X(s)}{Z(s)} = \frac{s^2+{\omega_0}^2}{Ls{\omega_0}^2}X(s)$$

(a)

$$X(s) = \frac{As}{s^2+{\omega_0}^2}, \qquad Y(s) = \frac{A}{L{\omega_0}^2} \quad \text{and} \quad y(t) = \frac{A}{L{\omega_0}^2}\delta(t)$$

(b)

$$X(s) = \frac{A\omega_0}{s^2+{\omega_0}^2}, \qquad Y(s) = \frac{A}{L\omega_0 s} \quad \text{and} \quad y(t) = \frac{A}{L\omega_0}u(t)$$

4.4-4. At $t = 0$, the steady-state values of currents y_1 and y_2 is $y_1(0) = 2$, $y_2(0) = 1$. Figure S4.4-4 shows the transformed circuit for $t \geq 0$ with initial condition generators. The loop equations are

$$\begin{aligned}(s+2)Y_1(s) - Y_2(s) &= 2 + \frac{6}{s}\\ -Y_1(s) + (s+2)Y_2(s) &= 1\end{aligned}$$

Cramer's rule yields

$$\begin{aligned}Y_1(s) &= \frac{2s^2 + 11s + 12}{s(s+1)(s+3)} = \frac{4}{s} - \frac{3/2}{s+1} - \frac{1/2}{s+3}\\ Y_2(s) &= \frac{s^2 + 4s + 6}{s(s+1)(s+3)} = \frac{2}{s} - \frac{3/2}{s+1} + \frac{1/2}{s+3}\end{aligned}$$

$$\begin{aligned}y_1(t) &= (4 - \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t})u(t)\\ y_2(t) &= (2 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t})u(t)\end{aligned}$$

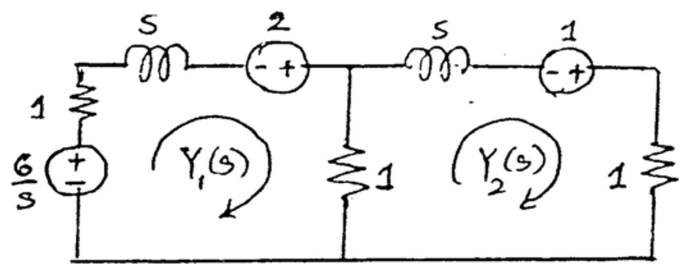


Figure S4.4-4

4.4-5. The current in the 2H inductor at $t = 0$ is 10A. The transformed circuit with initial condition generators is shown in Figure S4.4-5 for $t \geq 0$.

$$Y_1(s) = \frac{\frac{10}{s} + 20}{3s + \frac{1}{s} + 1} = \frac{20s + 10}{3s^2 + s + 1} = \frac{20}{s}[\frac{s + 0.5}{s^2 + \frac{1}{3s} + \frac{1}{3}}]$$

Here $A = 1$, $B = 0.5$, $a = \frac{1}{6}$, $c = \frac{1}{3}$, $b = \frac{\sqrt{11}}{6}$

$$r = \sqrt{\frac{15}{11}} = 1.168 \qquad \theta = \tan^{-1}(\frac{-2}{\sqrt{11}}) = -31.1^\circ$$

$$\begin{aligned}y_1(t) &= \frac{20}{3}(1.168)e^{-t/6}\cos(\frac{\sqrt{11}}{6}t - 31.1^\circ)u(t)\\ &= 7.787e^{-t/6}\cos(\frac{\sqrt{11}}{6}t - 31.1^\circ)u(t)\end{aligned}$$

The voltage $v_s(t)$ across the switch is

$$
\begin{aligned}
V_s(s) &= (s+\frac{1}{s})Y(s) = (\frac{s^2+1}{s})(\frac{20s+10}{3s^2+s+1}) = \frac{20}{s}\frac{(s^2+1)(s+0.5)}{s(s^2+\frac{1}{3s}+\frac{1}{3})} \\
&= \frac{20}{3}\left[1+\frac{3/2}{s}+\frac{1}{6}\frac{-8s+1}{s^2+1/3s+1/3}\right] \\
v_s(t) &= \frac{20}{3}\delta(t)+[10+9.045e^{-t/6}\cos(\frac{\sqrt{11}}{6}t-152.2°)]u(t)
\end{aligned}
$$

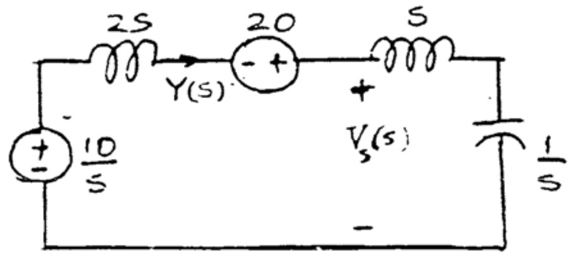


Figure S4.4-5

4.4-6. Figure S4.4-6 shows the transformed circuit with mutually coupled inductor replaced by their equivalents (see Figure 4.14b). The loop equations are

$$
\begin{aligned}
(s+1)Y_1(s) - 2sY_2(s) &= \frac{100}{s} \\
-2sY_1(s) + (4s+1)Y_2(s) &= 0
\end{aligned}
$$

Cramer's rule yields

$$Y_2(s) = \frac{40}{(s+0.2)}$$

and

$$v_0(t) = y_2(t) = 40e^{-t/5}u(t)$$

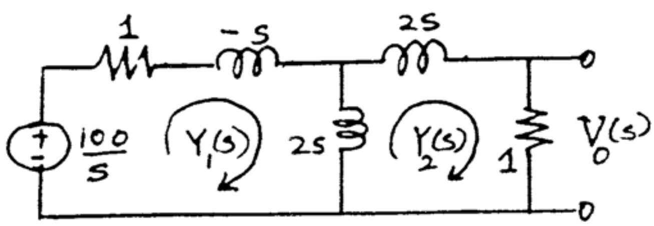


Figure S4.4-6

4.4-7. Figure S4.4-7 shows the transformed circuit with parallel form of initial condition generators. The admittance $W(s)$ seen by the source is

$$W(s) = \frac{13}{s} + s + 4 = \frac{s^2+4s+13}{s}$$

The voltage across terminals a b is

$$V_{ab}(s) = \frac{I(s)}{W(s)} = \frac{\frac{1}{s}+3}{\frac{s^2+4s+13}{s}} = \frac{3s+1}{s^2+4s+13}$$

Also

$$V_0(s) = \frac{1}{2}V_{ab}(s) = \frac{3s+1}{2(s^2+4s+13)}$$

and $$v_0(t) = 1.716e^{-2t}\cos(3t+29°)u(t)$$

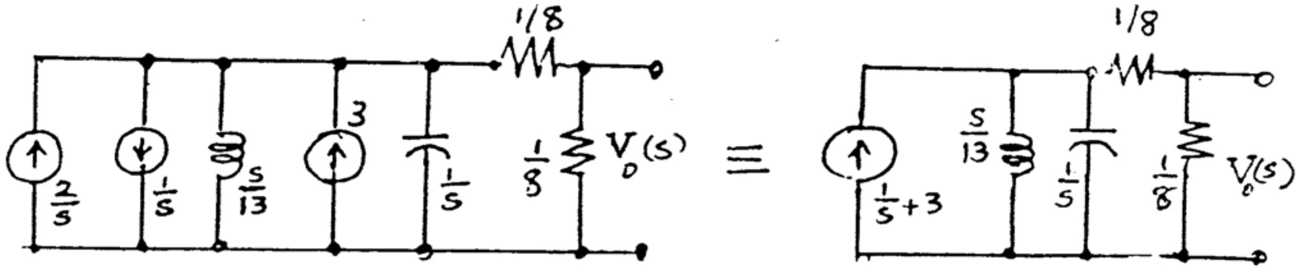


Figure S4.4-7

4.4-8. The capacitor voltage at $t = 0$ is 10 volts. The inductor current is zero. The transformed circuit with initial condition generators is shown for $t > 0$ in Figure S4.4-8. To determine the current $Y(s)$, we determine $Z_{ab}(s)$, the impedance seen across terminals ab:

$$Z_{ab}(s) = \frac{1}{1+\left(\frac{1}{2+\frac{s+2}{s+3}}\right)} = \frac{3s+8}{4s+11}$$

Also
$$\begin{aligned} Y(s) &= \frac{\frac{90}{s}}{\frac{5}{s}+\left(\frac{3s+8}{4s+11}\right)} \\ &= \frac{90(4s+11)}{3s^2+28s+55} \\ &= \frac{30(4s+11)}{s^2+\frac{28}{3}s+\frac{55}{3}} \\ &= \frac{30(4s+11)}{(s+2.8)(s+6.53)} \\ &= -\frac{1.61}{s+2.8}+\frac{121.61}{s+6.53} \end{aligned}$$

and $$y(t) = [121.61e^{-6.53t} - 1.61e^{-2.8t}]u(t)$$

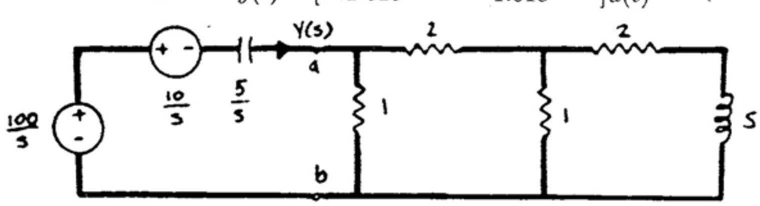


Figure S4.4-8

4.4-9. Figure S4.4-9 shows the transformed circuit (with noninverting op amp replaced by its equivalent as shown in Figure 4.16) from Figure S4.4-9a

$$V_0(s) = KV_1(s) = K\frac{1}{Cs}R + \frac{1}{Cs}X(s) = \frac{Ka}{s+a} \qquad a = \frac{1}{RC}$$

Therefore $\qquad H(s) = \dfrac{Ka}{s+a} \qquad a = \dfrac{1}{RC}, \quad K = 1 + \dfrac{R_b}{R_a}$

Similarly for the circuit in Figure P4.4-9b, we can show (see Figure S4.4-9)

$$H(s) = \frac{Ks}{s+a}$$

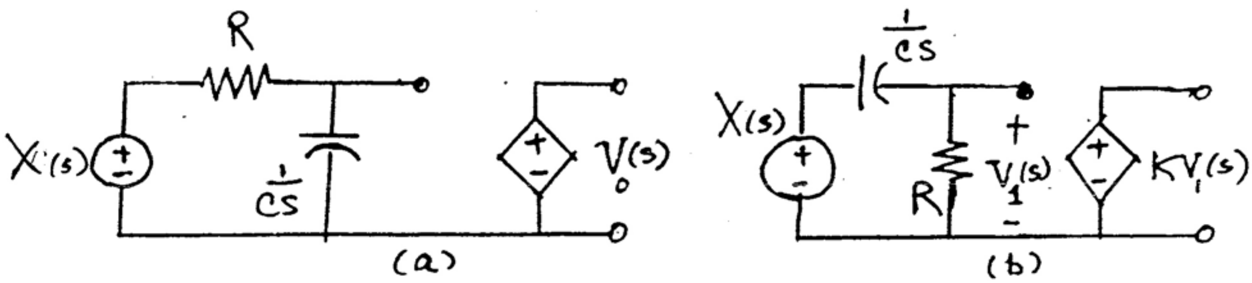


Figure S4.4-9

4.4-10. Figure S4.4-10 shows the transformed circuit. The op amp input voltage is $V_x(s) \simeq 0$. The loop equations are

$$\begin{aligned} I_1(s) + (\frac{6}{s} + 1)[I_1(s) - I_2(s)] &= X(s) \\ -\frac{3}{s}I_2(s) + (\frac{6}{s} + \frac{3}{2})[I_1(s) - I_2(s)] &= 0 \end{aligned}$$

Cramer's rule yields

$$I_1(s) = \frac{s(s+6)}{s^2+8s+12}X(s), \qquad I_2(s) = \frac{s(s+4)}{s^2+8s+12}$$

$$Y(s) = -\frac{1}{2}[I_1(s) - I_2(s)] = \frac{-s}{s^2+8s+12}X(s)$$

The transfer function $\qquad H(s) = \dfrac{-s}{s^2+8s+12}$

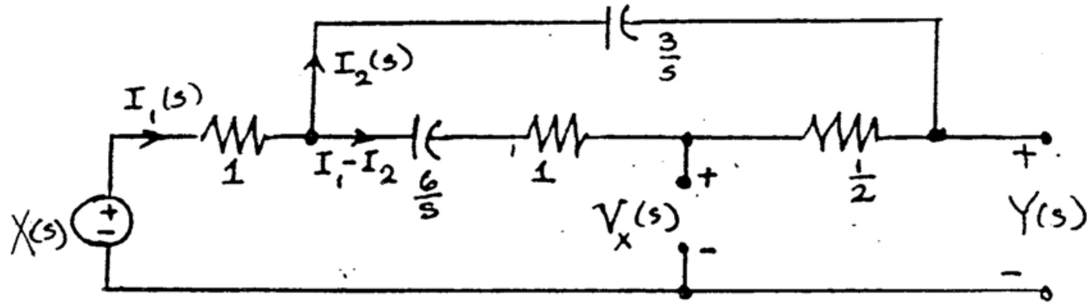


Figure S4.4-10

4.4-11. (a) (i)

$$\begin{aligned} Y(s) &= \frac{6s^2+3s+10}{s(2s^2+6s+5)} \\ y(0^+) &= \lim_{s\to\infty} sY(s) = 3 \end{aligned}$$

$$y(\infty) = \lim_{s\to 0} sY(s) = 2$$

(ii)

$$\begin{aligned}
Y(s) &= \frac{6s^2+3s+10}{(s+1)(2s^2+6s+5)} \\
y(0^+) &= \lim_{s\to\infty} sY(s) = 3 \\
y(\infty) &= \lim_{s\to 0} sY(s) = 0
\end{aligned}$$

(b) (i)

$$Y(s) = \frac{s^2+5s+6}{s^2+3s+2}$$

This Y(s) is not strictly proper. We can express it as

$$Y(s) = 1 + \frac{2s+4}{s^2+3s+2}$$

Hence
$$y(0^+) = \lim_{s\to\infty} \frac{s(2s+4)}{s^2+3s+2} = 2$$

and
$$y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{s^3+5s^2+6s}{s^2+3s+2} = 0$$

(ii)

$$Y(s) = \frac{s^3+4s^2+10s+7}{s^2+2s+3}$$

Because Y(s) is improper, we shall find its strictly proper component.

$$Y(s) = (s+2) + \frac{s+1}{s^2+2s+3}$$

Hence
$$y(0^+) = \lim_{s\to\infty} s\left(\frac{s+1}{s^2+2s+3}\right) = 1$$

$$y(\infty) = \lim_{s\to 0} s\left(\frac{s^3+4s^2+10s+7}{s^2+2s+3}\right) = 0$$

4.5-1. (a) At first glance, we are tempted to answer the question in affirmative. Let us verify the reality.

(b) The loop equations are

$$4I_1 - 2I_2 = X(s)$$

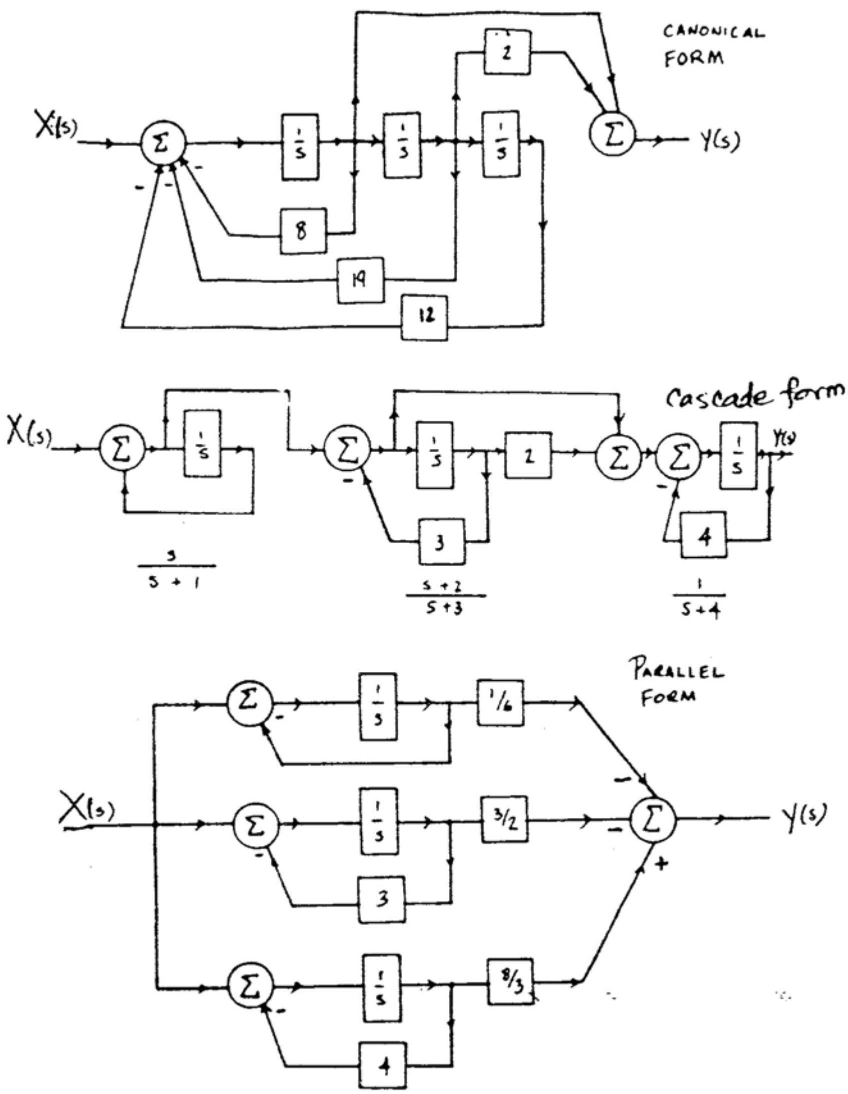

CANONICAL FORM
X(s)
Y(s)
2
8
19
12
Cascade form
X(s)
Y(s)
2
3
4
s / s + 1
s + 2 / s + 3
1 / s + 4
PARALLEL FORM
X(s)
Y(s)
1/6
3/2
8/3
3
4

FIGURE S4.5-1

$$-2I_1 + 4I_2 = 0$$

Cramer's rule yields
$$I_2(s) = \frac{1}{6}X(s)$$

and
$$Y(s) = I_2(s) = \frac{1}{6}X(s)$$

Therefore $H(s) = \frac{1}{6}$ not $\frac{1}{4}$.

(c) In this case ($R_3 = R_4 = 20000$)

$$\begin{aligned} 4I_1 - 2I_2 &= X(s) \\ -2I_1 + 40002I_2 &= 0 \end{aligned}$$

Cramer's rule yields

$$\begin{aligned} I_2(s) &= \frac{1}{80002}X(s) \\ Y(s) &= 20000I_2(s) = \frac{20000}{80002}X(s) = 0.249994X(s) \end{aligned}$$

In this case $H(s)$ is very close to 1/4. This is because the second ladder section causes a negligible load on the first. Let $R_3 = R_4 = R$. In this case, as $R \to \infty$, we observe that $H(s) \to 1/4$. The second ladder causes no loading in this case. The Cascade rule applies only when the successive subsystems do not load the preceding subsystems.

4.5-2. The transfer function of the two paths are e^{-st} and $ae^{-s(T+\tau)}$. The two paths are in parallel. Hence the transfer function of this communication channel is

$$\begin{aligned} H(s) &= e^{-sT} + ae^{-s(T+\tau)} \\ &= e^{-sT}(1 + ae^{-s\tau}) \end{aligned}$$

For distortionless transmission, it is adequate to undo only the term $(1 + ae^{-s\tau})$ in H(s) because e^{-sT} represents pure delay. Clearly, we need an equalizer with transfer function

$$H_{eq}(s) = \frac{1}{1 + ae^{-sT}}$$

Comparing this form with the transfer function of the feedback system in Eq. (4.59) or Figure 4.18d, it is immediately obvious that such an equalizer can be realized by the following system

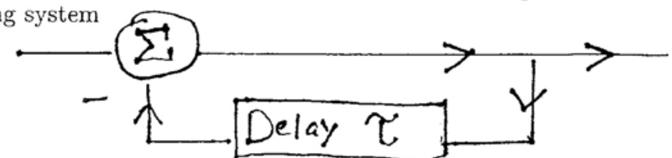


Figure S4.5-2

When this equalizer is placed in cascade with the communication channel, the effective transfer function is given by

$$H_c(s) = \frac{e^{-sT}(1 + ae^{-s\tau})}{1 + ae^{-s\tau}} = e^{-sT}$$

The effective system represents a pure delay of T seconds, which makes it distortionless. Moreover, the equalizer is realizable.

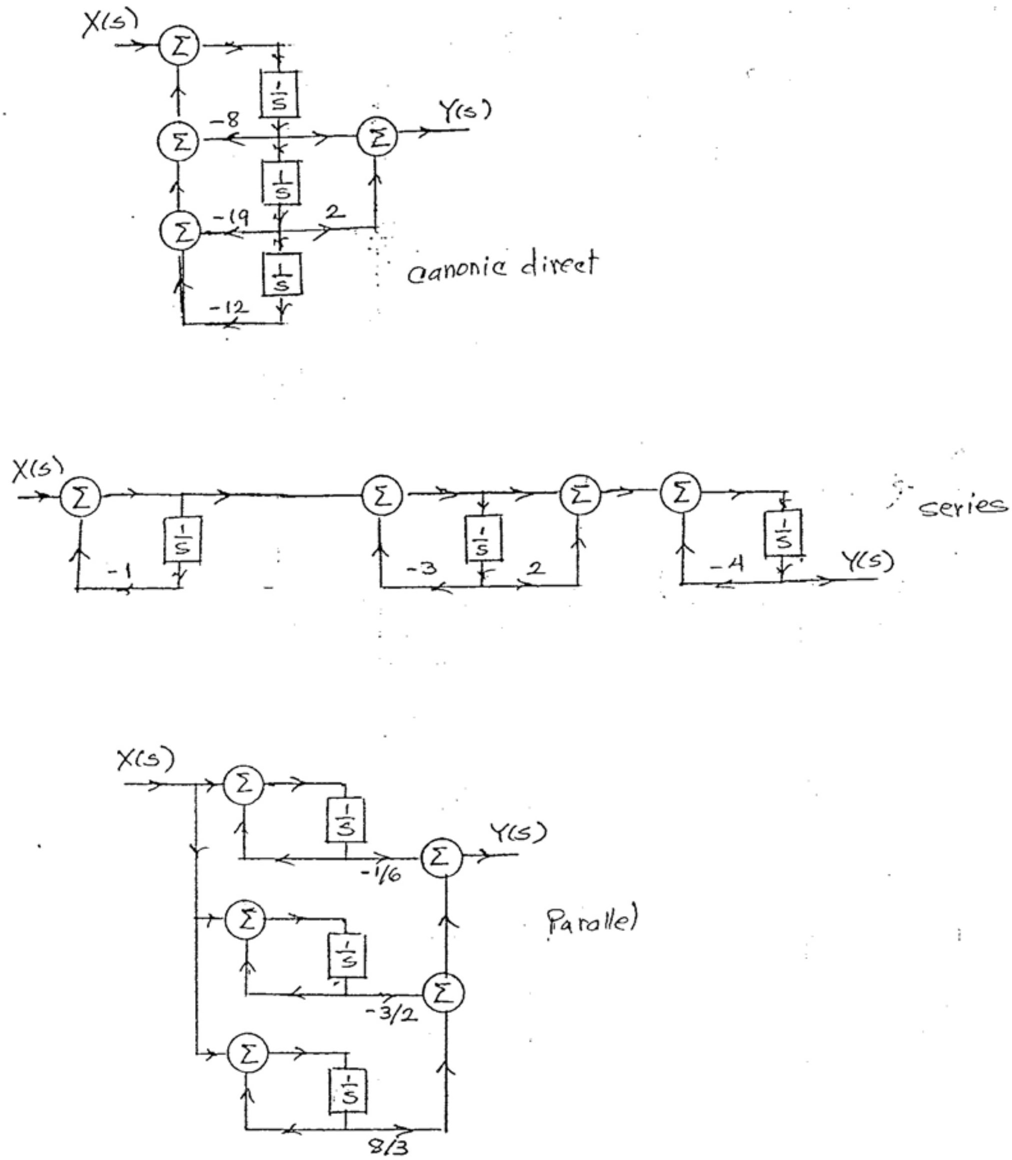


Figure S4.6-1

4.5-3. (a) The system transfer function is

$$H(s) = \frac{\frac{1}{s-1}}{1+\frac{2}{s-1}} = \frac{1}{s+1}$$

The system is BIBO stable.

(b) The system transfer function is

$$H(s) = \frac{\frac{K}{s(s+2)(s+4)}}{1+\frac{K}{s(s+2)(s+4)}} = \frac{1}{s^3+6s^2+8s+K}$$

(i) We can verify that for $K = 10$, all the roots are in LHP and hence the system is BIBO stable.

(ii) For $K = 50$, we can verify that two roots are in RHP and on LHP. Hence the system is BIBO stable.

(iii) For $K = 48$, we verify that two roots are on imaginary axis at $\pm j\sqrt{8}$ and one is in the LHP. Hence the system is BIBO unstable (but marginally stable).

4.6-1.

$$H(s) = \frac{s^2+2s}{s^3+8s^2+19s+12} = \left(\frac{s}{s+1}\right)\left(\frac{s+2}{s+3}\right)\left(\frac{1}{s+4}\right) = \frac{-1/6}{s+1} - \frac{3/2}{s+3} + \frac{8/3}{s+4}$$

$$\text{Also} \qquad H(s) = \frac{b_0s^3+b_1s^2+b_2s+b_3}{s^3+a_1s^2+a_2s+a_3}$$

with $a_3 = 12$, $a_2 = 19$, $a_1 = 8$, and $b_3 = 0$, $b_2 = 2$, $b_1 = 1$, $b_0 = 0$. Figure S4.6-1 shows the canonical, series and parallel realizations.

4.6-2. The transposed version of the realizations for the transfer function in Prob. 4.6-1 are shown in Figure S4.6-2.

4.6-3. (a)

$$\begin{aligned} H(s) &= \frac{3s(s+2)}{(s+1)(s^2+2s+2)} = \frac{3s^2+6s}{s^3+3s^2+4s+2} \\ &= \left(\frac{3s}{s+1}\right)\left(\frac{s+2}{s^2+2s+2}\right) = -\frac{3}{s+1} + \frac{6s+6}{s^2+2s+2} \end{aligned}$$

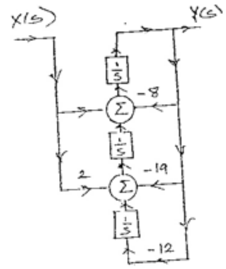
For the canonical form, we have $a_3 = 2$, $a_2 = 4$, $a_1 = 3$, and $b_3 = 0$, $b_2 = 6$, $b_1 = 3$, $b_0 = 0$. Figure S4.6-3a shows a canonical, cascade and parallel realizations.

(b)

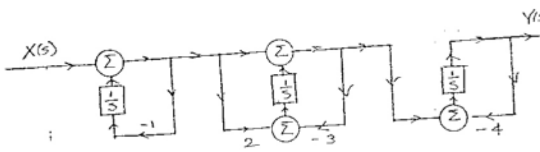
$$\begin{aligned} H(s) &= \frac{2s-4}{(s+2)(s^2+4)} = \frac{2s-4}{s^3+2s^2+4s+8} \\ &= \frac{2(s-2)}{s^3+2s^2+4s+8} = \left(\frac{s-2}{s+2}\right)\left(\frac{2}{s^2+4}\right) = -\frac{1}{s+2} + \frac{s}{s^2+4} \end{aligned}$$

For a canonical forms, we have $a_3 = 8$, $a_2 = 4$, $a_1 = 2$, and $b_3 = -4$, $b_2 = 2$, $b_1 = 0$, $b_0 = 0$. Figure S4.6-3b shows a canonical, cascade and parallel realizations.

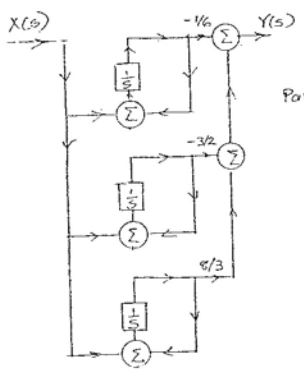
FIG. S4.6-2



Canonic direct

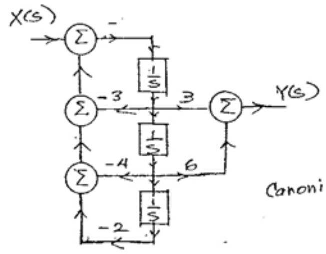


series

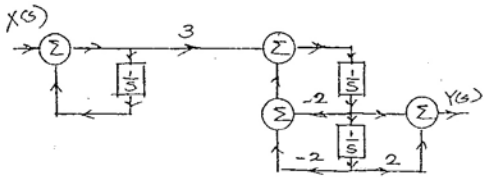


Parallel

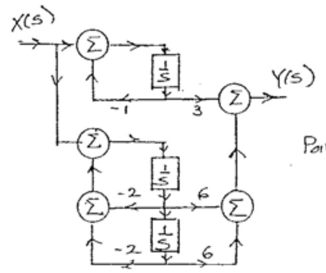
Fig S4.6-3a



Canonic direct

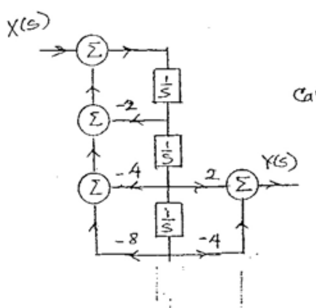


series

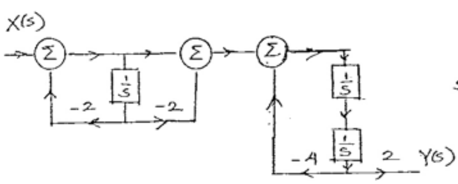


Parallel

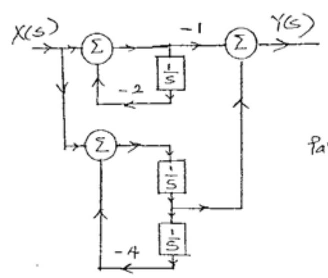
Fig. S4.6-3b



Canonic direct

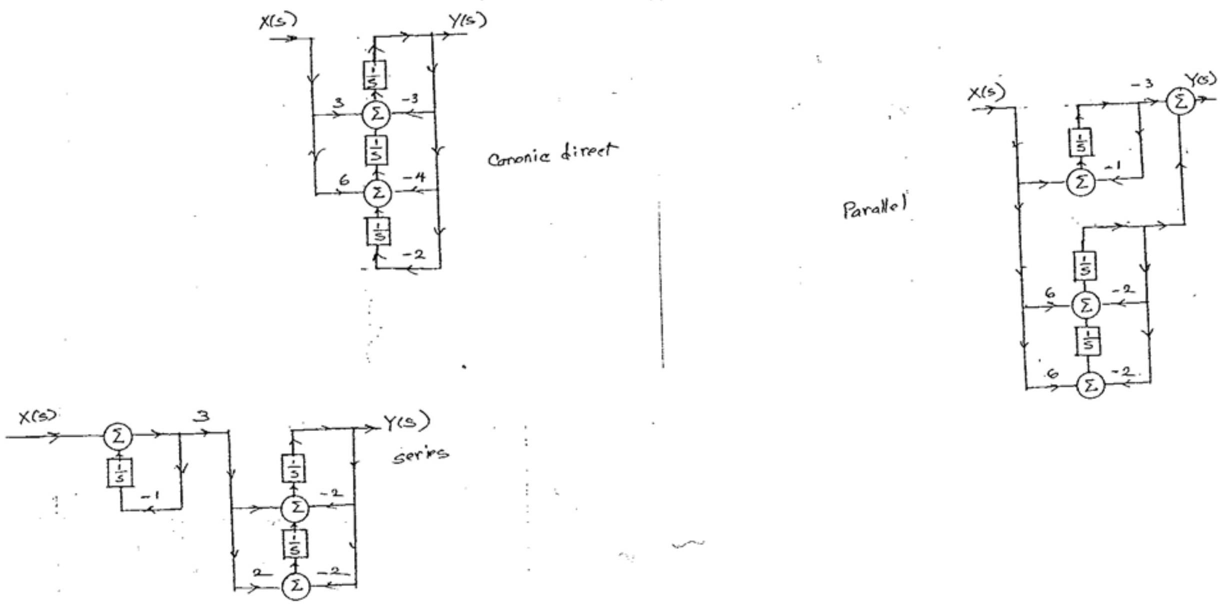


series



Parallel

Fig. S 4-6-4a



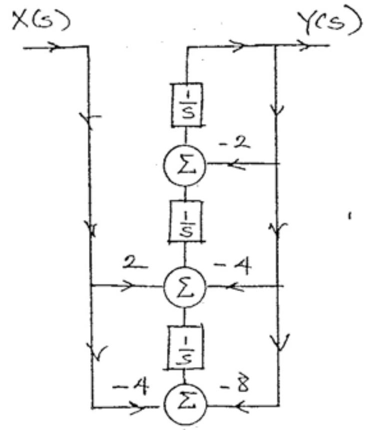
4.6-4. The transposed version of the realizations for the transfer function in Prob. 4.6-3 are shown in Figure S4.6-4.

4.6-5.

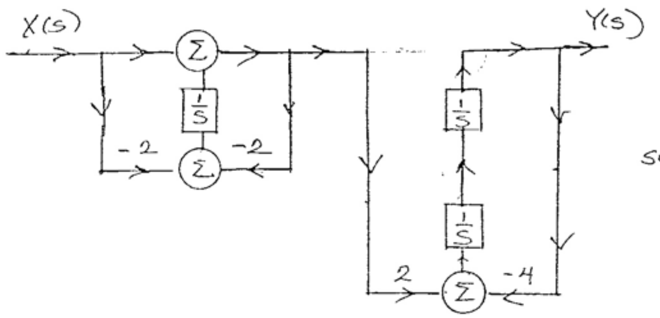
$$
\begin{aligned}
H(s) &= 7\frac{2s+3}{5(s^4+7s^3+16s^2+12s)} = \frac{0.4s+0.6}{s^4+7s^3+16s^2+12s} \\
&= \left(\frac{1}{s}\right)\left(\frac{1}{s+2}\right)\left(\frac{1}{s+2}\right)\left(\frac{0.4s+0.6}{s+3}\right) = \frac{\frac{1}{20}}{s} - \frac{\frac{1}{4}}{s+2} + \frac{\frac{1}{10}}{(s+2)^2} + \frac{\frac{1}{5}}{s+3}
\end{aligned}
$$

Figure S4.6-5 shows a canonical, cascade and parallel realizations.

4.6-6. The transposed version of the realizations for the transfer function in Prob. 4.6-5 are shown in Figure S4.6-6.



canonic direct



series

Parallel

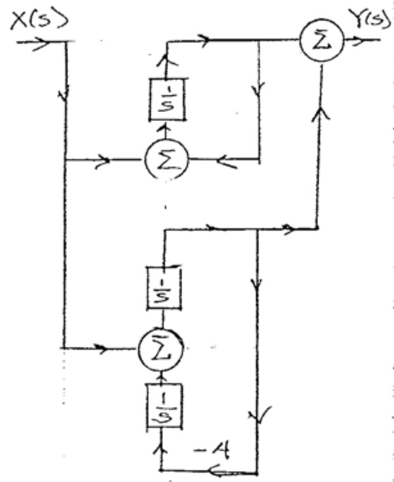
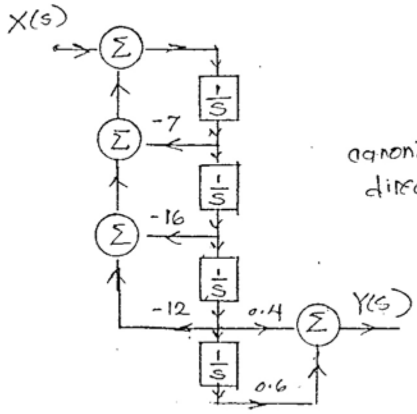
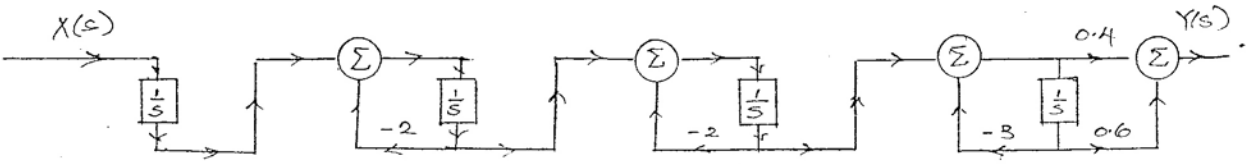


FIG. S 4.6-4b

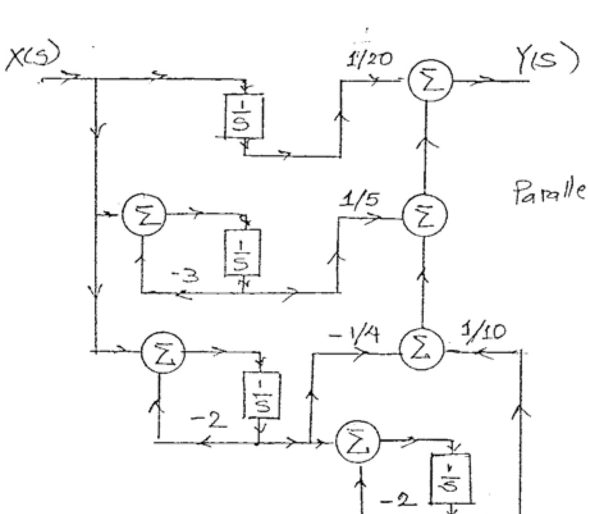
Fig S4.6-5



canonic direct

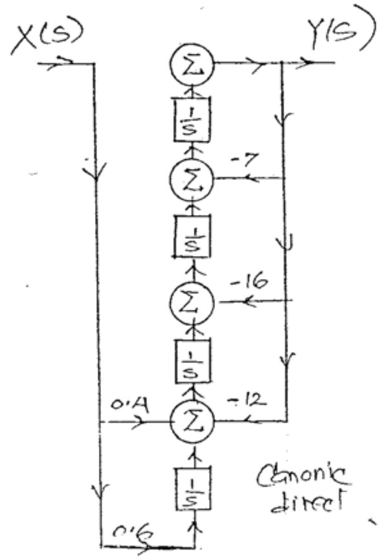


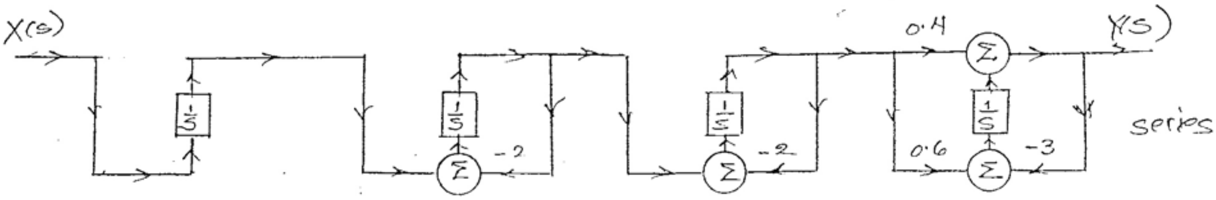
series

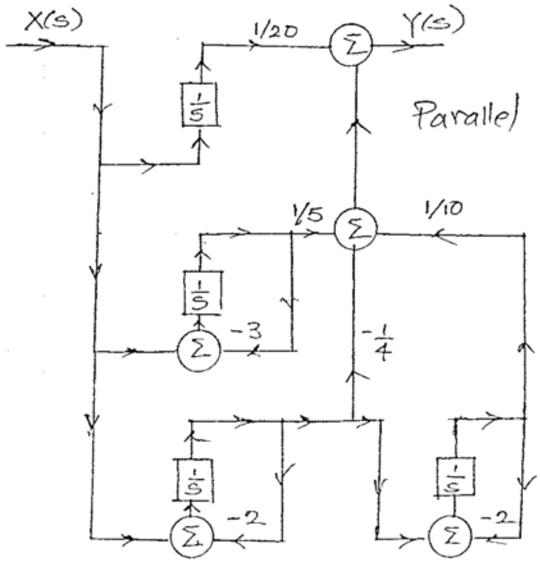


Parallel

Fig. S 4.6-6


X(s)
Y(s)
-7
-16
0.4
-12
0.6
Canonic direct


X(s)
-2
-2
0.4
Y(s)
0.6
-3
series


X(s)
1/20
Y(s)
Parallel
1/5
1/10
-3
$-\frac{1}{4}$
-2
-2

4.6-7.

$$H(s) = \frac{s(s+1)(s+2)}{(s+5)(s+6)(s+8)} = \frac{s^3+3s^2+2s}{s^3+19s^2+118s+240} = 1 - \frac{20}{s+5} + \frac{60}{s+6} - \frac{56}{s+8}$$

For a canonical form $a_3 = 24$, $a_2 = 118$, $a_1 = 19$, and $b_3 = 0$, $b_2 = 2$, $b_1 = 3$, $b_0 = 1$. Figure S4.6-7 shows a canonical, cascade and parallel realizations.

4.6-8. The transposed version of the realizations for the transfer function in Prob. 4.6-7 are shown in Figure S4.6-8.

4.6-9.

$$\begin{aligned} H(s) &= \frac{s^3}{(s+1)^2(s+2)(s+3)} = \frac{s^3}{s^4+7s^3+17s^2+17s+6} \\ &= \left(\frac{s}{s+1}\right)\left(\frac{s}{s+1}\right)\left(\frac{s}{s+2}\right)\left(\frac{1}{s+3}\right) = -\frac{8}{s+2} + \frac{\frac{27}{4}}{s+3} + \frac{\frac{9}{4}}{s+1} - \frac{\frac{1}{2}}{(s+1)^2} \end{aligned}$$

Figure S4.6-9 shows a canonical, cascade and parallel realizations.

4.6-10. The transposed version of the realizations for the transfer function in Prob. 4.6-9 are shown in Figure S4.6-10.

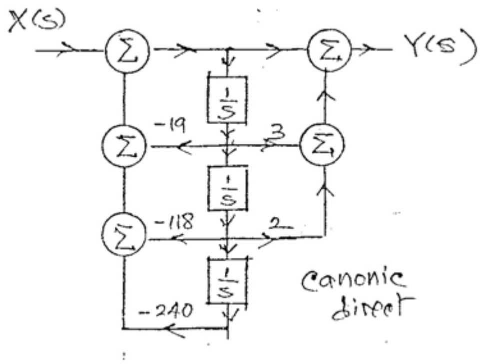
4.6-11.

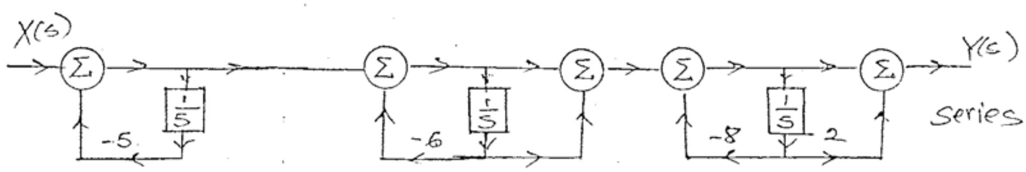
$$\begin{aligned} H(s) &= \frac{s^3}{(s+1)(s^2+4s+13)} = \frac{s^3}{s^3+5s^2+17s+13} \\ &= \left(\frac{s}{s+1}\right)\left(\frac{s^2}{s^2+4s+13}\right) = -\frac{0.1}{s+1} + \frac{s^2-0.9s+1.3}{s^2+4s+13} = 1 - \frac{0.1}{s+1} - \frac{4.9s+11.7}{s^2+4s+13} \end{aligned}$$

Figure S4.6-11 shows a canonical, cascade and parallel realizations.

4.6-12. The transposed version of the realizations for the transfer function in Prob. 4.6-11 are shown in Figure S4.6-12.

Fig S 4·6-7


X(s)
Y(s)
-19
3
-118
2
-240
canonic
direct


X(s)
-5
-6
-8
2
Y(s)
series

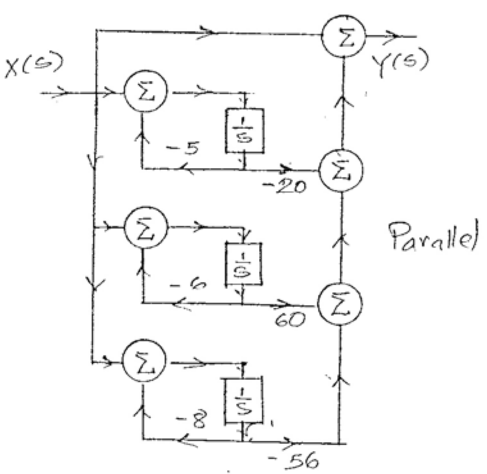
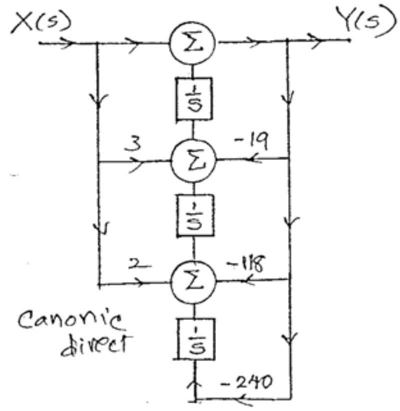
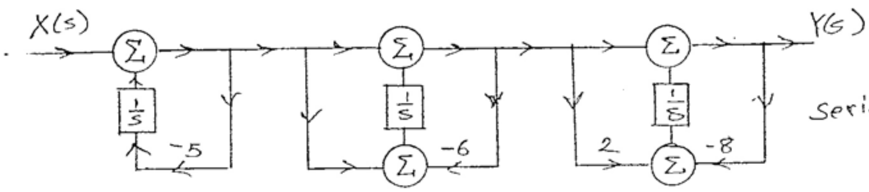

X(s)
Y(s)
-5
-20
-6
60
-8
-56
Parallel

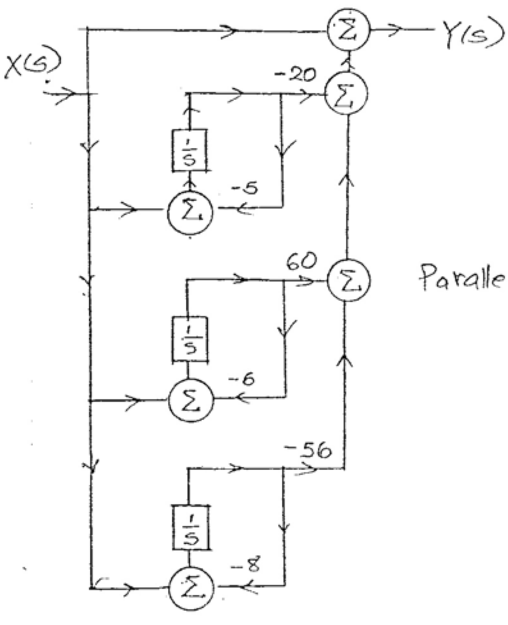
Fig. S4.6-8.



Canonic direct

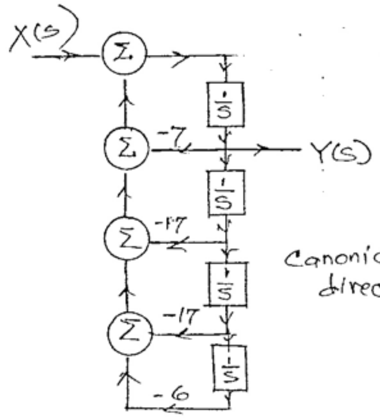


Series

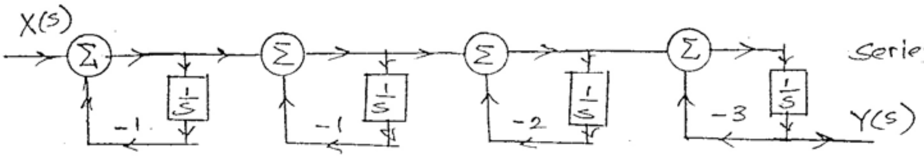


Parallel

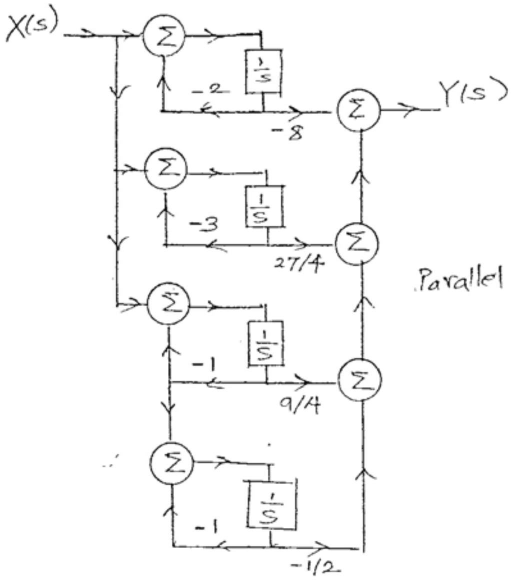
Fig. S 4.6-9



Canonic direct

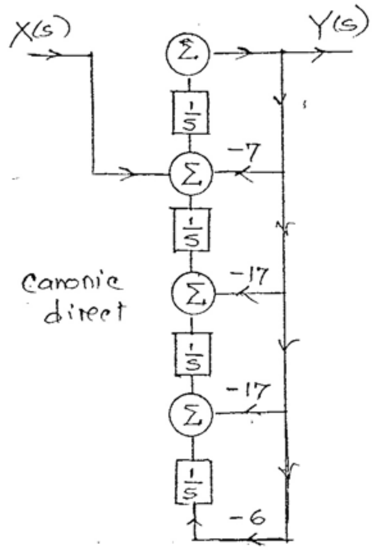


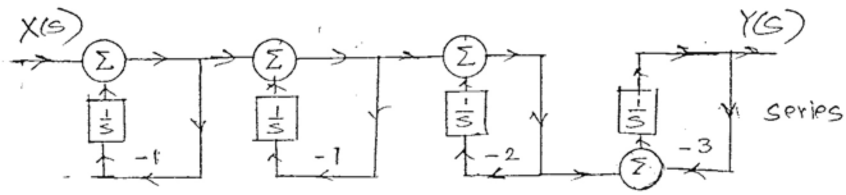
Series



Parallel

Fig. S 4.6-10


X(s)
Y(s)
-7
-17
-17
-6
canonic direct


X(s)
Y(s)
-1
-1
-2
-3
series

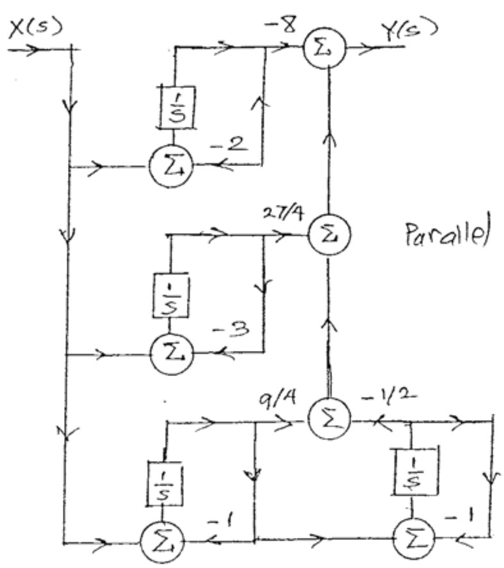
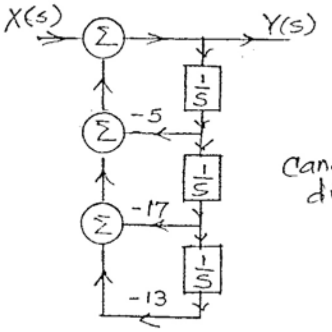
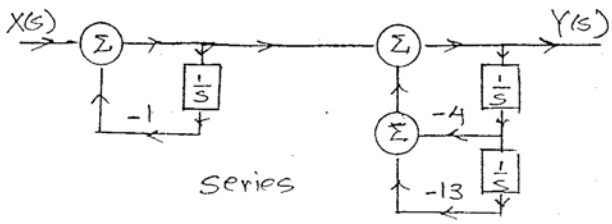

X(s)
Y(s)
-8
-2
27/4
Parallel
-3
9/4
-1/2
-1
-1

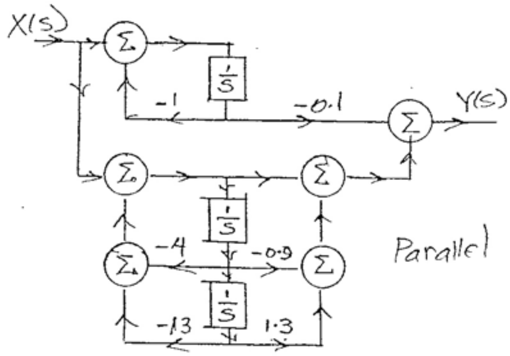
Fig. S 4.6-11



canonic direct

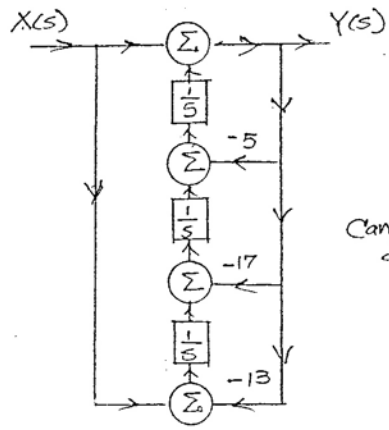


series

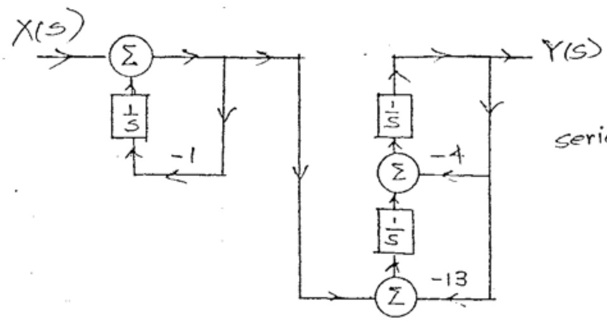


Parallel

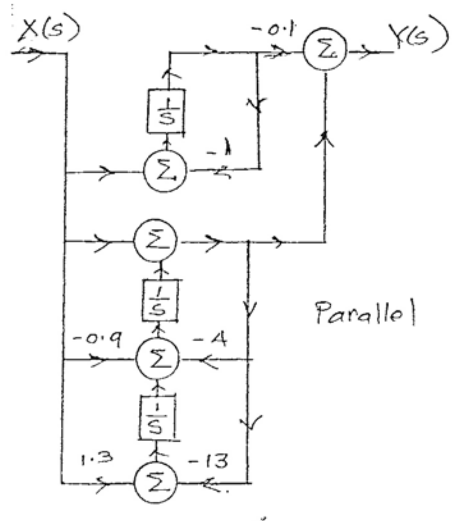
Fig. S 4.6-12



Canonic direct



series



Parallel

4.6-13. Application of Eq. (4.59) to Figure P4.6-13a yields

$$H_1(s) = \frac{\frac{1}{(s+a)^2}}{1 + \frac{b^2}{(s+a)^2}} = \frac{1}{(s+a)^2 + b^2}$$

Figure P4.6-13b is also a feedback loop with forward gain $G(s) = \frac{1}{s+a}$ and the loop gain $\frac{b^2}{(s+a)^2}$. Therefore

$$H_2(s) = \frac{\frac{1}{s+a}}{1 + \frac{b^2}{(s+a)^2}} = \frac{s+a}{(s+a)^2 + b^2}$$

The output in Figure P4.6-13c is the same of $B - aA$ times the output of Figure P4.6-13a and A times the output of Figure P4.6-13b. Therefore its transfer function is

$$\begin{aligned} H(s) &= (B - aA)H_1(s) + AH_2(s) \\ &= \frac{B - aA}{(s+a)^2 + b^2} + \frac{A(s+a)}{(s+a)^2 + b^2} \\ &= \frac{As + B}{(s+a)^2 + b^2} \end{aligned}$$

4.6-14. These transfer functions are readily realized by using the arrangement in Figure 4.28 by a proper choice of $Z_f(s)$ and $Z(s)$.

(i) In Figure S4.6-14a

$$\begin{aligned} Z_f(s) &= \frac{\frac{R_f}{C_f s}}{R_f + \frac{1}{C_f s}} = \frac{1}{C_f(s+a)} \qquad a = \frac{1}{R_f C_f} \\ Z(s) &= R \end{aligned}$$

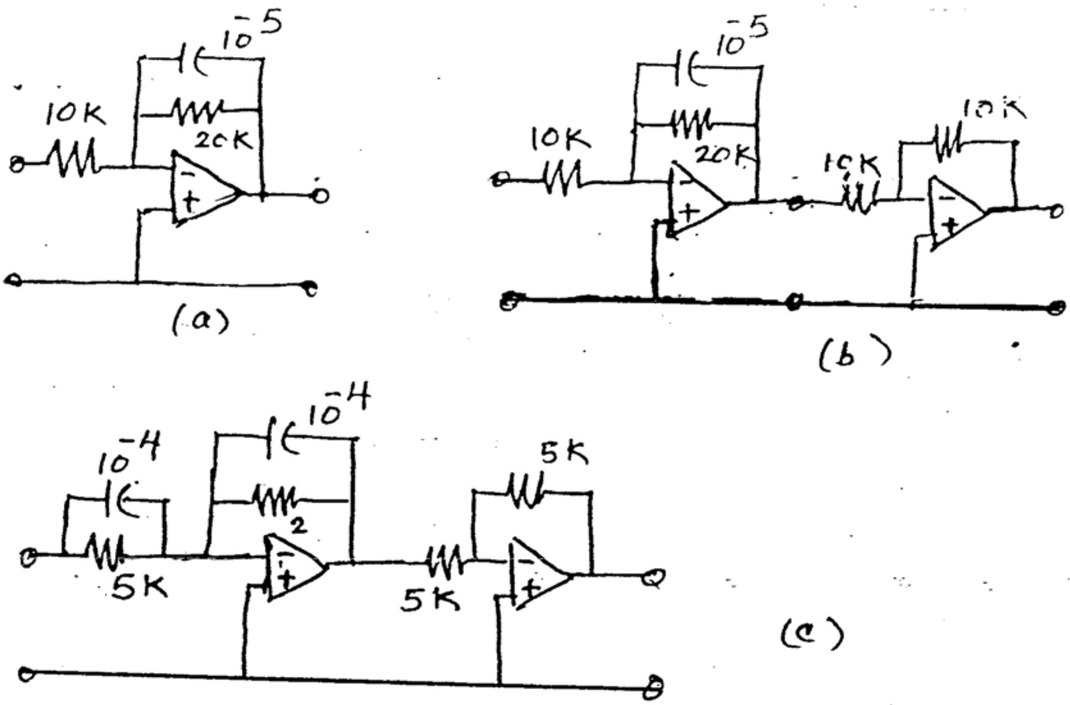


Figure S4.6-14

and
$$H(s) = -\frac{Z_f(s)}{Z(s)} = -\frac{k}{s+a} \qquad k = \frac{1}{RC_f}, \quad a = \frac{1}{R_fC_f}$$

Choose $R = 10,000$, $R_f = 20,000$ and $C_f = 10^{-5}$. This yields $k = 10$ and $a = 5$. Therefore
$$H(s) = \frac{-10}{s+5}$$

(ii) This is same as (i) followed by an amplifier of gain -1 as shown in Figure S4.6-14b.

(iii) For the first stage in Figure S4.6-14c (see Exercise E4.13, Figure 4.32b),
$$Z_f(s) = \frac{1}{C_f(s+a)} \qquad a = \frac{1}{R_fC_f}$$
$$Z(s) = \frac{1}{C(s+b)} \qquad b = \frac{1}{RC}$$

and
$$H(s) = -\frac{Z_f(s)}{Z(s)} = -\frac{C}{C_f}(\frac{s+b}{s+a})$$

Choose $C = C_f = 10^{-4}$, $R = 5000$, $R_f = 2000$. This yields
$$H(s) = -(\frac{s+2}{s+5})$$

This is followed by an op amp of gain -1 as shown in Figure S4.6-14c. This yields
$$H(s) = \frac{s+2}{s+5}$$

4.6-15. One realization is given in Figure S4.6-14c. For the other realization, we express $H(s)$ as
$$H(s) = \frac{s+2}{s+5} = 1 - \frac{3}{s+5}$$

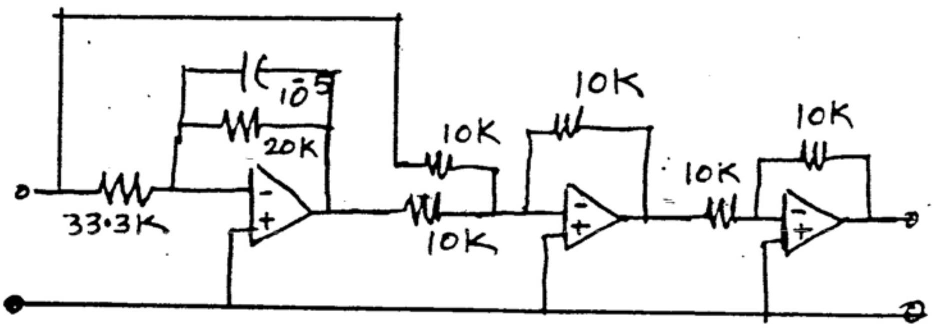


Figure S4.6-15

We realize $H(s)$ as a parallel combination of $H_1(s) = 1$ and $H_2(s) = -3/(s+5)$ as shown in Figure S4.6-15. The second stage serves as a summer for which the inputs are the input and output of the first stage. Because the summer has a gain -1, we need a third stage of gain -1 to obtain the desired transfer functions.

4.6-16. Canonical realization of $H(s)$ is shown in Figure S4.6-16. Observe that this is identical to $H(s)$ in Example 4.22 with a minor difference. Hence the op amp circuit in Figure 4.31c can be used for our purpose with appropriate changes in the element values. The last summer input resistors now are $\frac{100}{3}\,k\Omega$ and $\frac{100}{7}\,k\Omega$ instead of $50\,k\Omega$ and $20\,k\Omega$.

4.6-17. We follow the procedure in Example 4.22 with appropriate modifications. In this case $a_2 = 13$, $a_1 = 4$, and $b_2 = 2$, $b_1 = 5$, and $b_0 = 1$ (in Example 4.22, we have $a_2 = 10$, $a_1 = 4$, and $b_2 = 5$, $b_1 = 2$, and $b_0 = 0$). Because b_0 is nonzero here, we have one more feedforward connection. Figure S4.6-17 shows the development of the suitable realization.

4.7-1. (a) In this case,

$$H(j\omega) = \frac{\omega_c}{j\omega + \omega_c} \quad \Longrightarrow |H(j\omega)| = \frac{\omega_c}{\sqrt{\omega^2 + \omega_c^2}}$$

The dc gain is $H(0) = 1$ and the gain at $\omega = \omega_c$ is $1/\sqrt{2}$, which is -3 dB below the dc gain. Hence, the 3-dB bandwidth is ω_c. Also the dc gain is unity. Hence, the gain-bandwidth product is ω_c.
We could derive this result indirectly as follows. The system is a lowpass filter with a single pole at $\omega = \omega_c$. The dc gain is $H(0) = 1$ (0 dB). Because, there is a single pole at ω_c (and no zeros), there is only one asymptote starting at $\omega = \omega_c$ (at a rate -20 dB/dec.). The break point is ω_c, where there is a correction of -3 dB. Hence, the amplitude response at ω_c is 3 dB below 0 dB (the dc gain). Thus, the 3-dB bandwidth of this filter is ω_c.

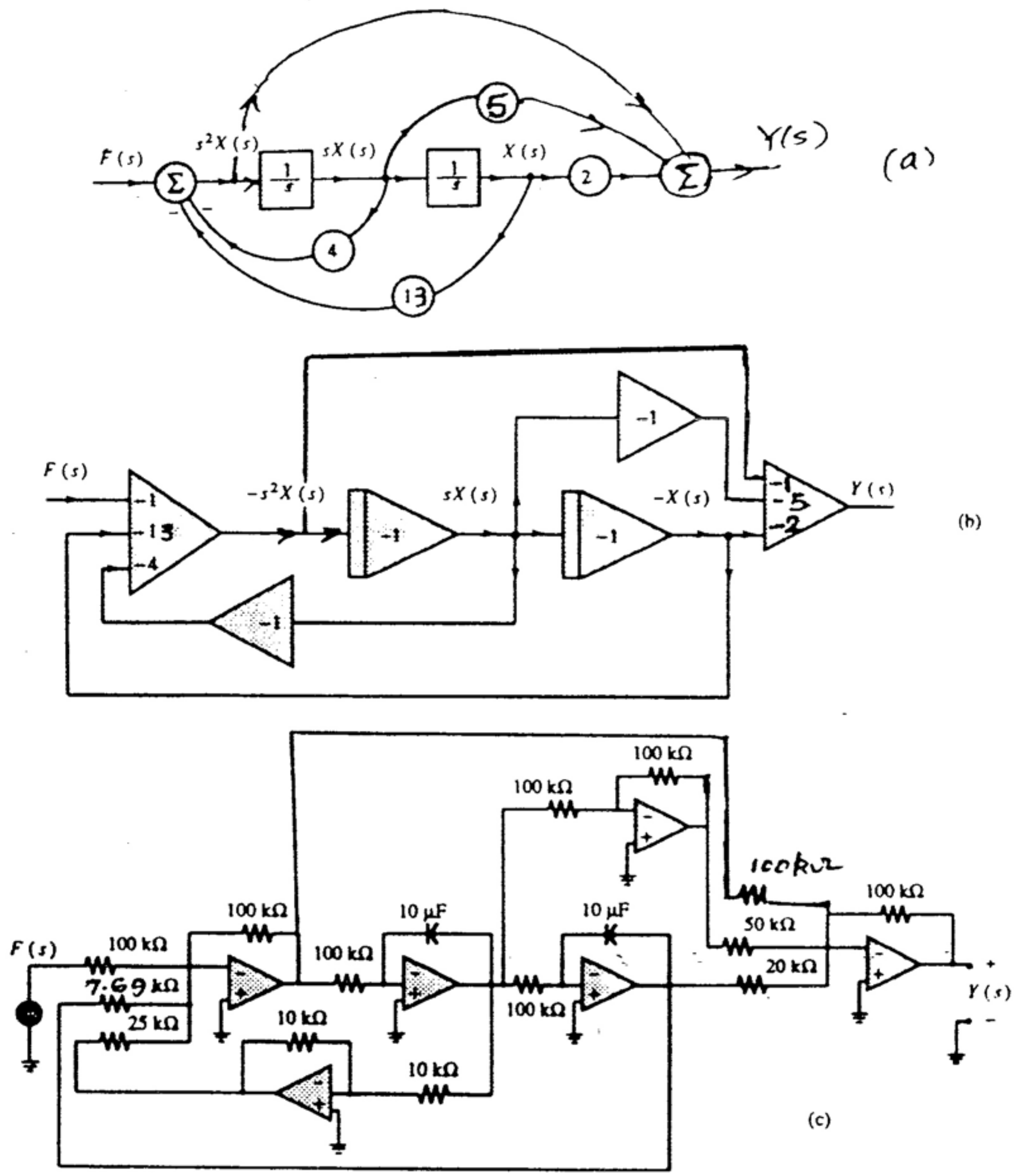


FIGURE S4.6-16

(b) The transfer function of this system is

$$H(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{\frac{\omega_c}{s+\omega_c}}{1 + \frac{9\omega_c}{s+\omega_c}} = \frac{\omega_c}{s + 10\omega_c}$$

We use the same argument as in part (a) to deduce that the dc gain is 0.1 and the 3-dB bandwidth is $10\omega_c$. Hence, the gain-bandwidth product is ω_c.

(c) The transfer function of this system is

$$H(s) = \frac{G(s)}{1 - G(s)H(s)} = \frac{\frac{\omega_c}{s+\omega_c}}{1 - \frac{0.9\omega_c}{s+\omega_c}} = \frac{\omega_c}{s + 0.1\omega_c}$$

We use the same argument as in part (a) to deduce that the dc gain is 10 and the 3-dB bandwidth is $0.1\omega_c$. Hence, the gain-bandwidth product is ω_c.

(d) Included in previous parts.

4.8-1.

$$\begin{aligned}
H(j\omega) &= \frac{j\omega + 2}{(j\omega)^2 + 5j\omega + 4} = \frac{j\omega + 2}{(4 - \omega^2) + j5\omega} \\
|H(j\omega)| &= \sqrt{\frac{\omega^2 + 4}{(4 - \omega^2)^2 + (5\omega)^2}} = \sqrt{\frac{\omega^2 + 4}{\omega^4 + 17\omega^2 + 16}} \\
\angle H(j\omega) &= \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{5\omega}{4 - \omega^2}\right)
\end{aligned}$$

(a) $x(t) = 5\cos(2t + 30°)$. Here $\omega = 2$ and

$$\begin{aligned}
|H(j2)| &= \sqrt{\frac{2}{25}} = \frac{\sqrt{2}}{5} \\
\angle H(j\omega) &= \tan^{-1} - \tan^{-1}(\infty) = 45° - 90° = -45°
\end{aligned}$$

$$y(t) = 5\frac{\sqrt{2}}{5}\cos(2t + 30° - 45°) = \sqrt{2}\cos(2t - 15°)$$

(b) $x(t) = 10\sin(2t + 45°)$

$$y(t) = 10(\frac{\sqrt{2}}{5})\sin(2t + 45° - 45°) = 2\sqrt{2}\sin 2t$$

(c) $x(t) = 10\cos(3t + 40°)$. Here $\omega = 3$

$$|H(j\omega)| = \sqrt{\frac{13}{250}} = 0.228 \quad \text{and} \quad \angle H(j3) = 56.31° - 108.43° = -52.12°$$

Therefore

$$y(t) = 10(0.228)\cos(3t + 40° - 52.12°) = 2.28\cos(3t - 12.12°)$$

4.8-2.

$$\begin{aligned} H(j\omega) &= \frac{j\omega + 3}{(j\omega + 2)^2} \\ |H(j\omega)| &= \frac{\sqrt{\omega^2 + 9}}{\omega^2 + 4} \quad \text{and} \quad \angle H(j\omega) = \tan^{-1}(\frac{\omega}{3}) - \tan^{-1}(\frac{\omega}{2}) \end{aligned}$$

(a) $x(t) = 10u(t) = 10e^{j0t}u(t)$. Here $\omega = 0$ and $H(j0) = 1$. Therefore

$$y(t) = 1 \times 10e^{j0t}u(t) = 10u(t)$$

(b) $x(t) = \cos(2t + 60°)\,u(t)$. Here $\omega = 2$

$$|H(j2)| = \frac{\sqrt{13}}{8} \quad \text{and} \quad \angle H(j2) = 33.69° - 90° = -56.31°$$

Therefore

$$y(t) = \frac{\sqrt{13}}{8}\cos(2t + 60^\circ - 56.31^\circ)u(t) = \frac{\sqrt{13}}{8}\cos(2t + 3.69^\circ)u(t)$$

(c) $x(t) = \sin(3t - 45^\circ)u(t)$ Here $\omega = 3$ and

$$|H(j3)| = \frac{\sqrt{18}}{13} \quad \text{and} \quad \angle H(j3) = 45^\circ - 112.62^\circ = -67.62^\circ$$

Therefore

$$y(t) = \frac{\sqrt{18}}{13}\sin(3t - 45^\circ - 67.62^\circ)u(t) = \frac{\sqrt{18}}{13}\sin(3t - 112.62^\circ)u(t)$$

(d) $x(t) = e^{j3t}u(t)$

$$y(t) = H(j3)e^{j3t} = |H(j3)|e^{j[3t + \angle H(j3)]}u(t) = \frac{\sqrt{18}}{13}e^{j[3t - 67.62^\circ]}u(t)$$

4.8-3.

$$\begin{aligned} H(j\omega) &= \frac{-(j\omega - 10)}{j\omega + 10} = \frac{10 - j\omega}{10 + j\omega} \\ |H(j\omega)| &= \sqrt{\frac{\omega^2 + 100}{\omega^2 + 100}} = 1 \\ \angle H(j\omega) &= \tan^{-1}(-\frac{\omega}{10}) - \tan^{-1}(\frac{\omega}{10}) = -2\tan^{-1}(\frac{\omega}{10}) \end{aligned}$$

(a) $x(t) = e^{j\omega t}$

$$y(t) = H(j\omega)e^{j\omega t} = |H(j\omega)|e^{j[\omega t + \angle H(j\omega)]} = e^{j[\omega t - 2\tan^{-1}(\omega/10)]}$$

(b) $x(t) = \cos(\omega t + \theta)$

$$y(t) = \cos[\omega t + \theta - 2\tan^{-1}(\frac{\omega}{10})]$$

(c) $x(t) = \cos t$. Here $\omega = 1$

$$\begin{aligned} |H(j1)| &= 1 \\ \angle H(j\omega) &= -2\tan^{-1}(\frac{1}{10}) = -11.42° \\ y(t) &= \cos(t - 11.42°) \end{aligned}$$

(d) $x(t) = \sin 2t$. Here $\omega = 2$

$$\begin{aligned} |H(j2)| &= 1 \\ \angle H(j2) &= -2\tan^{-1}(\frac{2}{10}) = -22.62° \\ y(t) &= \sin(2t - 22.62°) \end{aligned}$$

(e) $x(t) = \cos 10t$. Here $\omega = 10$

$$\begin{aligned} |H(j10)| &= 1 \\ \angle H(j10) &= -2\tan^{-1}(\frac{10}{10}) = -90° \\ y(t) &= \cos(10t - 90°) = \sin 10t \end{aligned}$$

(f) $x(t) = \cos 100t$. Here $\omega = 100$

$$|H(j100)| = 1$$

$$\begin{aligned}\angle H(j100) &= -2\tan^{-1}(\frac{100}{10}) = -168.58^\circ \\ y(t) &= \cos(100t - 168.58^\circ)\end{aligned}$$

4.8-4. (a) From the graph, the two system zeros are at $s = \pm j1.5$. Thus, $s^2 + b_1 s + b_2 = (s+j1.5)(s-j1.5) = s^2 + 2.25$. The two system poles are at $s = -1 \pm j0.5$. Thus, $s^2 + a_1 s + a_2 = (s+1+j0.5)(s+1-j0.5) = s^2 + 2s + 1.25$. At DC, the system function is $H(j0) = -1 = k\frac{b_2}{a_2} = k\frac{2.25}{1.25} = k\frac{9}{5}$. Therefore,

$$k = -\frac{5}{9}, b_1 = 0, b_2 = \frac{9}{4}, a_1 = 2, \text{ and } a_2 = \frac{5}{4}.$$

(b) The DC gain is given as $S(j0) = -1$. Thus, the input of 4 just becomes -4. To compute the output to $\cos(t/2 + \pi/3)$, $H(j0.5)$ is required. Graphically, $|H(j0.5)| = |k|\frac{(1)(2)}{(1)(\sqrt{2})} = \frac{10}{9\sqrt{2}}$ and $\angle H(j0.5) = \pi - \pi/2 + \pi/2 - (0 + \pi/4) = 3\pi/4$. Thus, the output to $\cos(t/2 + \pi/3$ is just $\frac{10}{9\sqrt{2}}\cos(t/2 + \pi/3 + 3\pi/4)$. Thus, the output to $x(t) = 4 + \cos(t/2 + \pi/3)$ is

$$y(t) = -4 + \frac{10}{9\sqrt{2}} \cos(t/2 + 13\pi/12) \approx -4 + 0.7857 \cos(t/2 + 3.4034).$$

4.9-1. (a) The transfer function can be expressed as

$$H(s) = \frac{100}{2 \times 20} \frac{s(\frac{s}{100} + 1)}{(\frac{s}{2} + 1)(\frac{s}{20} + 1)} = 2.5 \frac{s(\frac{s}{100} + 1)}{(\frac{s}{2} + 1)(\frac{s}{20} + 1)}$$

The amplitude response: The horizontal axis where the asymptotes begin is 2.5, which is 7.96 db. We draw the asymptotes at $\omega = 1$ (20 dB/dec.), 2 (-20 dB/dec.), 20 (-20 dB/dec.), and 100 (20 dB/dec.) as shown in Figure S4.9-1a. The corrections are applied at various points as discussed in Examples 4.25 and 4.26. to obtain the Bode plot for amplitude response. We follow the similar procedure for phase response.

(b) The transfer function can be expressed as

$$H(s) = \frac{10 \times 20}{100} \frac{(\frac{s}{10} + 1)(\frac{s}{20} + 1)}{s^2(\frac{s}{100} + 1)} = 2 \frac{(\frac{s}{10} + 1)(\frac{s}{20} + 1)}{s^2(\frac{s}{100} + 1)}$$

The amplitude response: The horizontal axis where the asymptotes begin is 2, which is 6 db. Asymptotes start at $\omega = 1$ (-40 dB/dec.), 10 (20 dB/dec.), 20 (20 dB/dec.), and 100 (-20 dB/dec.). The corrections are applied at various points as discussed in Examples 4.25 and 4.26. to obtain the Bode plot.

(c) The transfer function can be expressed as

$$H(s) = \frac{10 \times 200}{400 \times 1000} \frac{(\frac{s}{10} + 1)(\frac{s}{200} + 1)}{(\frac{s}{20} + 1)^2(\frac{s}{1000} + 1)} = \frac{1}{200} \frac{(\frac{s}{10} + 1)(\frac{s}{200} + 1)}{(\frac{s}{20} + 1)^2(\frac{s}{1000} + 1)}$$

The amplitude response: The horizontal axis where the asymptotes begin is

FIG. S4.9-1

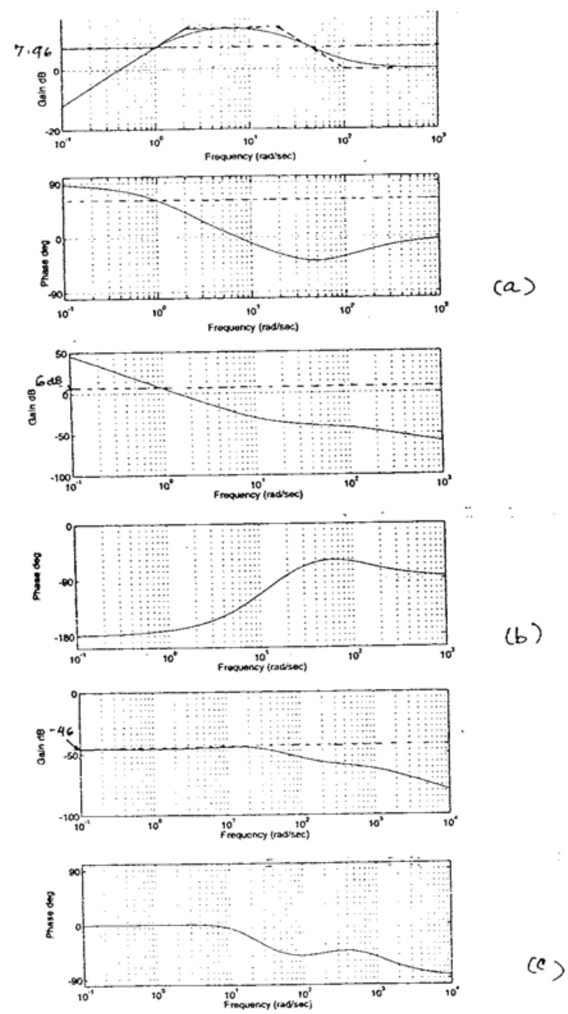


Fig. S4.9-1

1/200, which is -46 db. Asymptotes start at $\omega = 10$ (20 dB/dec.), 20 (-40 dB/dec.), 200 (20 dB/dec.), and 1000 (-20 dB/dec.). The corrections are applied at various points as discussed in Examples 4.25 and 4.26. to obtain the Bode plot.

4.9-2. (a) The transfer function can be expressed as

$$H(s) = \frac{1}{16}\frac{s^2}{(\frac{s}{1}+1)(\frac{s^2}{16}+\frac{s}{4}+1)}$$

The amplitude response: The horizontal axis where the asymptotes begin is 1/16, which is -24 dB. Asymptotes start at $\omega = 1$ (40 dB/dec.), 1 (-20 dB/dec.), 4 (-40 dB/dec.). The corrections are applied at various points as discussed in Examples 4.25 and 4.26. to obtain the Bode plot.

(b) The transfer function can be expressed as

$$H(s) = \frac{1}{100}\frac{s}{(\frac{s}{1}+1)(\frac{s^2}{100}+0.1414s+1)}$$

The amplitude response: The horizontal axis where the asymptotes begin is 1/100, which is -40 dB. Asymptotes start at $\omega = 1$ (20 dB/dec.), 1 (-20 dB/dec.), 10 (-40 dB/dec.). The corrections are applied at various points as discussed in Examples 4.25 and 4.26. to obtain the Bode plot.

(c) The transfer function can be expressed as

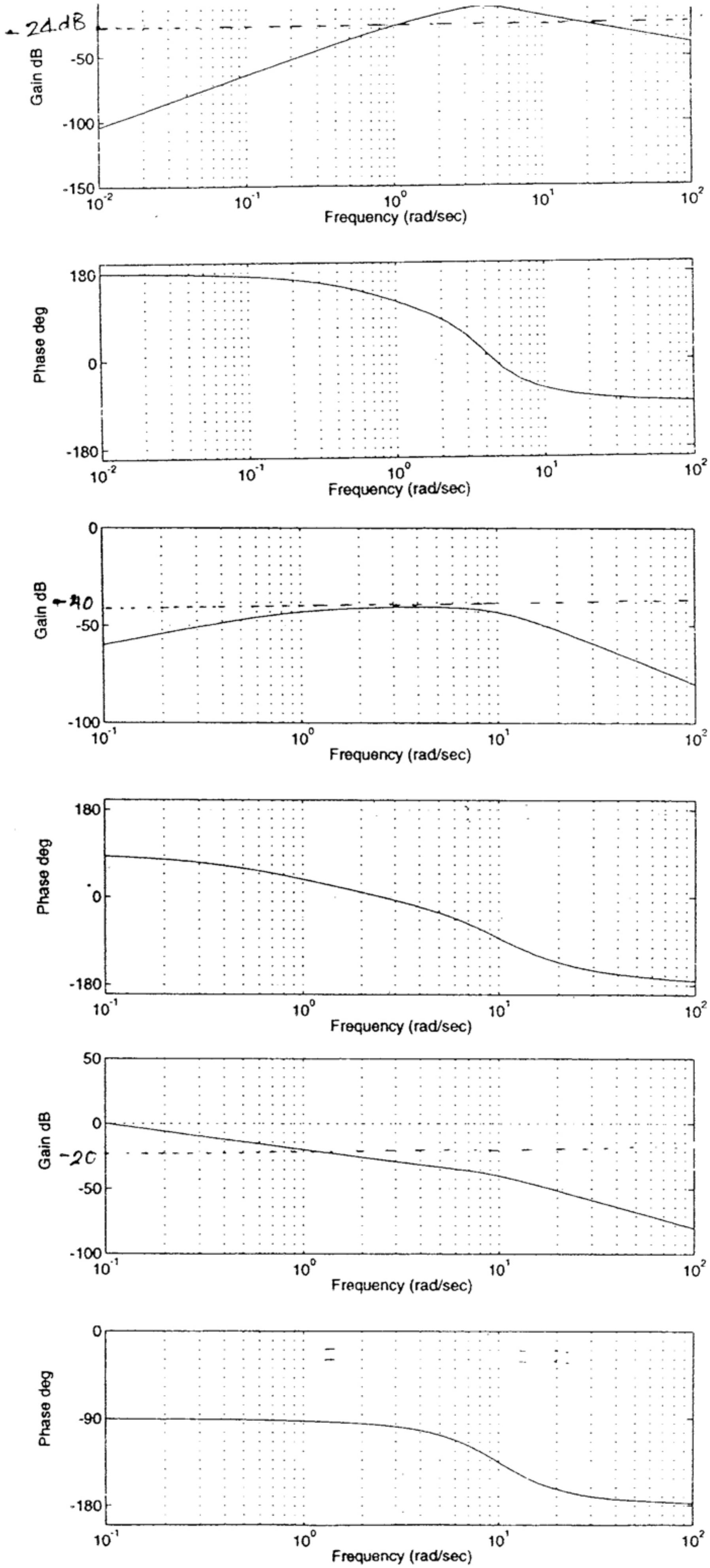
$$H(s) = \frac{10}{100}\frac{\frac{s}{10}+1}{s(\frac{s^2}{100}+0.1414s+1)}$$

The amplitude response: The horizontal axis where the asymptotes begin is 1/10, which is -20 dB. Asymptotes start at $\omega = 1$ (-20 dB/dec.), 10 (20 dB/dec.), 10 (-40 dB/dec.). The corrections are applied at various points as discussed in Examples 4.25 and 4.26. to obtain the Bode plot.

4.9-3. To rise at 20dB per decade, a zero must be present before $\omega = 0.1$. At $\omega = 30$, the magnitude response begins to fall at -20dB per decade. This requires two poles at that frequency: one pole to counteract the previous zero and another pole to cause the -20dB per decade slope. The magnitude response level out at $\omega = 500$, which requires the action of a zero. Thus, a second order system should be sufficient.

$$H(s) = k\frac{s(s+500)}{(s+30)^2}.$$

To determine the constant k, notice that $20\log(|H(j1)|) = 10$ or $|H(j1)| = \sqrt{10} \approx$


-24dB
Gain dB
-50
-100
-150
Frequency (rad/sec)
Phase deg
180
0
-180
Frequency (rad/sec)
0
Gain dB
-40
-50
-100
Frequency (rad/sec)
Phase deg
180
0
-180
Frequency (rad/sec)
50
Gain dB
0
-20
-50
-100
Frequency (rad/sec)
0
Phase deg
-90
-180
Frequency (rad/sec)

(a)

(b)

(c)

FIG. S4.9-2

yields $y(t) = RC_1\dot{v}_{C_1}(t) + v_{C_1}(t)$. In transform domain, this becomes $Y(S) = V_{C_1}(s)\,(1 + RC_1 s)$ or $V_{C_1}(s) = \frac{Y(s)}{1+RC_1 s}$. Combining the KCL and KVL equations yields $\frac{X(s)-C_2 s Y(s)}{C_1 s} = \frac{Y(s)}{1+RC_1 s}$. Simplifying yields $Y(s)\left(RC_1C_2s^2 + C_1 s + C_2 s\right) = X(s)\,(RC_1 s + 1)$. Thus,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{RC_1 s + 1}{RC_1C_2 s^2 + (C_1 + C_2)s}.$$

(b) Notice, $H(s)$ has two poles, one at zero and another at a negative, real number. It also has two zeros, one at infinity and another at a negative, real number. Only plots B and D show evidence of a finite zero as well as a finite pole. Of these, only plot B can have the necessary pole at zero. Thus,

Plot B is the only plot consistent with the system.

(c) At low frequencies, $H(j\omega) \approx \frac{1}{(C_1+C_2)j\omega}$. Thus, R doesn't affect $|H(j\omega)$ at very low frequencies.

(d) At high frequencies, $H(j\omega) \approx \frac{RC_{-}1j\omega}{-RC-1C_2\omega^2} = \frac{-j}{C_2\omega}$. Thus, R doesn't affect $|H(j\omega)$

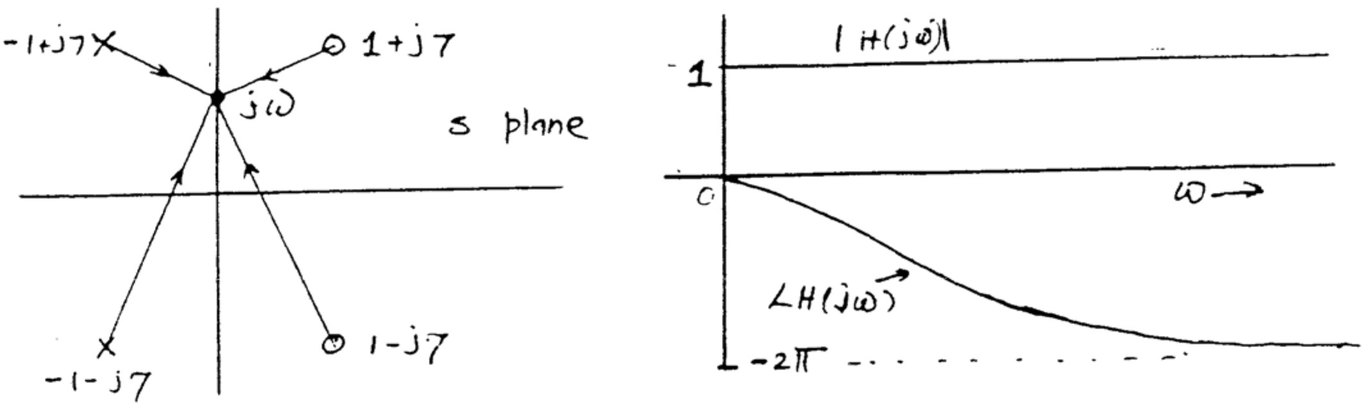


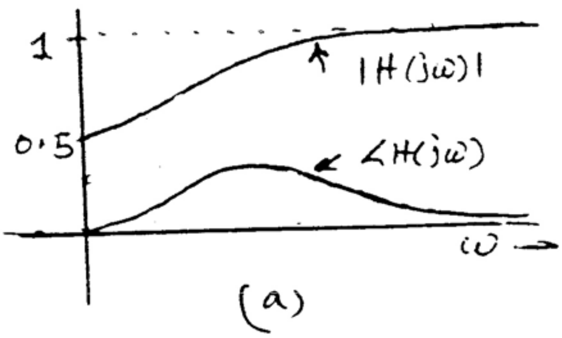
FIG. 4.10-1

at very high frequencies.

4.10-1. We plot the poles $-1 \pm j7$ and $1 \pm j7$ in the s-plane. To find response at some frequency ω, we connect all the poles and zeros to the point $j\omega$ as shown in Figure S4.10-1. Note that the product of distances from the zeros is equal to the product of the distances from the poles for all values of ω. Therefore $|H(j\omega)| = 1$. Graphical argument shows that $\angle H(j\omega)$ (sum of the angles from the zeros$-$ sum of the angles from poles) starts at zero for $\omega = 0$ and then reduces continuously (becomes negative) as ω increases. As $\omega \to \infty$, $\angle H(\omega) \to -2\pi$.

4.10-2. (a) If r and d are the distances of the zero and pole, respectively from $j\omega$, then the amplitude response $|H(j\omega)|$ is the ratio r/d corresponding to $j\omega$. This ratio is 0.5 for $\omega = 0$. Therefore, the dc gain is 0.5. Also the ratio $r/d = 1$ for $\omega = \infty$. Thus, the gain is unity at $\omega = \infty$. Also the angles of the line segments connecting the zero and pole to the point $j\omega$ are both zero for $\omega = 0$, and are both $\pi/2$ for $\omega = \infty$. Therefore $\angle H(j\omega) = 0$ at $\omega = 0$ and $\omega = \infty$. In between the angle is positive as shown in Figure S4.10-2a.

(b) In this case the ratio r/d is 2 for $\omega = 0$. Therefore, the dc gain is 2. Also the ratio $r/d = 1$ for $\omega = \infty$. Thus, the gain is unity at $\omega = \infty$. Also the angles



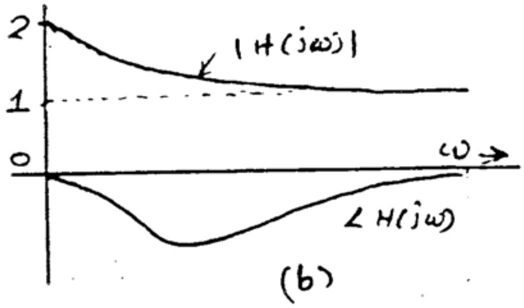


Figure S4.10-2

of the line segments connecting the zero and pole to the point $j\omega$ are both zero for $\omega = 0$, and are both $\pi/2$ for $\omega = \infty$. Therefore $\angle H(j\omega) = 0$ at $\omega = 0$ and $\omega = \infty$. In between the angle is negative as shown in Figure S4.10-2b.

4.10-3. The poles are at $-a \pm j10$. Moreover zero gain at $\omega = 0$ and $\omega = \infty$ requires that there be a single zero at $s = 0$. This clearly causes the gain to be zero at $\omega = 0$. Also because there is one excess pole over zero, the gain for large values of ω is $1/\omega$, which approaches 0 as $\omega \to \infty$. therefore, the suitable transfer function is

$$H(s) = \frac{s}{(s+a+j10)(s+a-j10)} = \frac{s}{s^2+2as+(100+a^2)}$$

The amplitude response is high in the vicinity of $\omega = 10$ provided a is small. Smaller the a, more pronounced the gain in the vicinity of $\omega = 10$. For $a = 0$, the gain at $\omega = 10$ is ∞.

4.10-4. Cynthia is correct. Although the system is all-pass and has $|H(j\omega)| = 1$, the phase response is not zero. Thus, the output generally has different phase than the input. Furthermore, the output can also include transient components that would not be present in the original input.

4.10-5. Both Amy and Jeff are correct. By definition, a zero is any value s that forces $H(s) = 0$

and a pole is any value s that forces $H(s) = \infty$. Thus, the system $H(s) = s = \frac{1}{s^{-1}}$ has both a zero at $s = 0$ and a pole at $s = \infty$. Remember, a rational system function always has the same number of poles and zeros; if $H(s) = s$ has an obvious zero at $s = 0$ there must be a matching pole somewhere, even if it is not finite. By similar argument, the system $H(s) = \frac{1}{s}$ has a pole at $s = 0$ and a zero at $s = \infty$.

4.10-6. At high frequencies, the highest powers of s dominate both the numerator and denominator of $H(s)$. That is, $\lim_{s\to\infty} H(s) = \lim_{s\to\infty} \frac{b_0 s^M}{s^N}$.

Lowpass and bandpass filters both require $\lim_{s\to\infty} H(s) = 0$, which ensures a response of zero at high frequencies. Only $H(s)$ that are strictly proper ($M < N$) yield the required $\lim_{s\to\infty} H(s) = 0$.

Highpass and bandstop filters both require $\lim_{s\to\infty} H(s) = k$, where k is some finite, non-zero constant. Only $H(s)$ that are proper ($M = N$) yield the required $\lim_{s\to\infty} H(s) = b_0 = k$.

The case $M > N$ is not considered, since such systems are not physically practical.

4.10-7. At high frequencies, the highest powers of s dominate both the numerator and denominator of $H(s)$. That is, $\lim_{s\to\infty} H(s) = \lim_{s\to\infty} \frac{b_0 s^M}{s^N}$. Thus, the log magnitude response at high frequencies is given by $\lim_{\omega\to\infty} \log|H(\jmath\omega)| = \log(b_0) + M\log(\omega) - N\log(\omega)$. The fastest attenuation as a function of frequency requires M to be as small as possible. Thus, for a given N, the attenuation rate of an all-pole lowpass filter ($M = 0$) is faster than the attenuation rate of any filter with a finite number of zeros ($M \neq 0$).

4.10-8. No, it is not possible for such a system to function as a lowpass filter. For any choice of ($[k, b_1, b_2, a_1, a_2] \in \mathcal{R}$), the system function $H(s) = k\frac{s^2+b_1 s+b_2}{s^2+a_1 s+a_2}$ is proper. Thus, the system function always has high-frequency gain of k. For $k \neq 0$, the system cannot be lowpass. Furthermore, for $k = 0$ the system becomes a useless "nopass filter" (again, not lowpass).

4.10-9. Nick is more correct than his professor. A cascade of two identical filters, each with system response $H(\jmath\omega)$, gives a total response of $H^2(\jmath\omega)$. Since realizable filters, such as Butterworth filters, are not ideal, the cascade system will tend to have a faster transition band and greater stopband attenuation. In a sense, the resulting fourth-order system really does provide "twice the filtering" of the original second-order system.

Unfortunately, there are also problems with Nick's approach. Simply cascading a designed lowpass filter twice has negative consequences. For example, the cutoff frequency shifts to a lower frequency than desired. As the cascaded RC example in

MATLAB Session 4 suggests, a cascade of low-order filters is inferior to a carefully designed, equivalent-order filter. In general, a fourth-order Butterworth filter performs better than a cascade of two second-order Butterworth filters.

4.10-10. (a) Using tables,

$$H(s) = \frac{1}{s} - \frac{1}{s}e^{-s} = e^{-s/2}\,\frac{e^{s/2} - e^{-s/2}}{s}$$

Substituting $s = j\omega$ yields the frequency response

$$H(j\omega) = e^{-j\omega/2}\,\frac{e^{j\omega/2} - e^{-j\omega/2}}{j\omega} = e^{-j\omega/2}\,\frac{\sin(\omega/2)}{\omega/2} = e^{-j\omega/2}\,\mathrm{sinc}\left(\frac{\omega}{2}\right)$$

The sinc type of frequency response (with a linear phase shift of $-\omega/2$) represents a lowpass system.

(b) Since $h(t)$ is finite duration, the system has no finite poles. There are, however, an infinite number of finite zeros for $\mathrm{sinc}(\omega/2)$ at $\omega = 2\pi k$ or $s = j2\pi k$, where k is any non-zero integer.

(c) In transform-domain, the inverse system is given by the reciprocal of $H_c(s)$. Thus,

$$H_c^{-1}(s) = \frac{1}{H_c(s)} = e^{s/2}\,\frac{\frac{s}{j2}}{\sin\left(\frac{s}{j2}\right)}$$

The inverse system has no finite zeros and an infinite number of finite poles. Since poles lie on the ω-axis, the inverse system cannot be asymptotically stable. The same approach does not work in the time-domain. That is, $h_c^{-1}(t) \neq \frac{1}{h(t)}$. The impulse response needs to be obtained from an inverse Laplace transform of $H_c^{-1}(s)$. Unfortunately, it is difficult to take the inverse Laplace transform of $H_c^{-1}(s)$; no closed form solution for $h_c^{-1}(t)$ is known to exist.
It is possible to approximate $h_c^{-1}(t)$. Consider the following idea. Replace the denominator $\sin\left(\frac{s}{j2}\right)$ with a truncated Taylor series expansion. The result is a rational approximation to $H_c^{-1}(s)$ that can be inverted using partial fraction expansion techniques. Although not perfect, the result can perform reasonably for many low-frequency inputs.

4.10-11. No, the suggested lowpass to highpass transformation $H_{\mathrm{HP}}(s) = 1 - H_{\mathrm{LP}}(s)$ does not work in general. Although it is possible to relate the ideal magnitude responses according to $|H_{\mathrm{HP}}(j\omega)| = 1 - |H_{\mathrm{LP}}(j\omega)|$, the phase information contained in $H(s)$ generally makes $H_{\mathrm{HP}}(s) \neq 1 - H_{\mathrm{LP}}(s)$.

As an example, consider an ideal lowpass filter described by

$$H_{\mathrm{LP}}(j\omega) = \begin{cases} -1 & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases}.$$

The transformation $1 - H_{\mathrm{LP}}(s)$ is clearly not highpass.

$$1 - H_{\mathrm{LP}}(s) = \begin{cases} 2 & |\omega| \leq \omega_c \\ 1 & |\omega| > \omega_c \end{cases}.$$

4.10-12. (a) Yes, it is possible for the system to output $y(t) = \sin(100\pi t)u(t)$ in response to $x(t) = \cos(100\pi t)u(t)$. Noting $Y(s) = \frac{100\pi}{s^2+(100\pi)^2}$ and $X(s) = \frac{s}{s^2+(100\pi)^2}$, one way to obtain $y(t)$ from $x(t)$ is using the system $H(s) = Y(s)/X(s) = \frac{100\pi}{s^2+(100\pi)^2}\frac{s^2+(100\pi)^2}{s} = \frac{100\pi}{s}$.

(b) Yes, it is possible for the system to output $y(t) = \sin(100\pi t)u(t)$ in response to $x(t) = \sin(50\pi t)u(t)$. Noting $Y(s) = \frac{100\pi}{s^2+(100\pi)^2}$ and $X(s) = \frac{s}{s^2+(50\pi)^2}$, one way to obtain $y(t)$ from $x(t)$ is using the system $H(s) = Y(s)/X(s) = \frac{100\pi}{s^2+(100\pi)^2}\frac{s^2+(50\pi)^2}{s} = \frac{100\pi(s^2+(50\pi)^2)}{s(s^2+(100\pi)^2)}$.

(c) Yes, it is possible for the system to output $y(t) = \sin(100\pi t)$ in response to $x(t) = \cos(100\pi t)$. To do this, the system must have $H(j100\pi) = e^{-j\pi/2}$. That is, the magnitude response at $\omega = 100\pi$ must be unity, and the phase response at $\omega = 100\pi$ must be $-\pi/2$.

(d) No, it is not possible for the system to output $y(t) = \sin(100\pi t)$ in response to $x(t) = \sin(50\pi t)$. In an LTI system, an everlasting sinusoidal input of frequency 50π cannot produce a different frequency output.

4.11-1. (a) Let $x_1(t) = x(t)u(t) = e^t u(t)$ and $x_2(t) = x(t)u(-t) = u(-t)$. Then $X_1(s)$ has a region of convergence $\sigma > 1$. And $X_2(s)$ has a region $\sigma < 0$. Hence there is no common region of convergence for $X(s) = X_1(s) + X_2(s)$.

(b) $x_1(t) = e^{-t}u(t)$, and $X_1(s) = \frac{1}{s+1}$ converges for $\sigma > -1$. Also $x_2(t) = u(-t)$, and $X_2(s) = -\frac{1}{s}$ converges for $\sigma < 0$. Therefore, the strip of convergence is

$$-1 < \sigma < 0$$

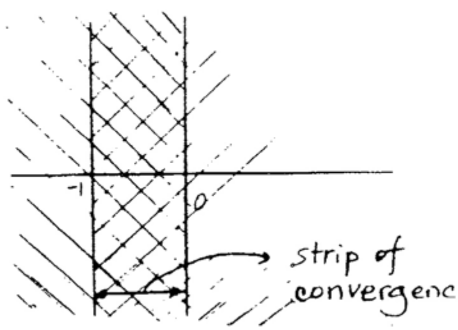


Figure S4.11-1b

(c)

$$\frac{1}{t^2+1}e^{-st} \to 0 \; \Big\} \quad \begin{array}{l} \text{as } t \to \infty \text{ if Re } s \geq 0 \\ \text{as } t \to -\infty \text{ if Re } s \leq 0 \end{array}$$

Hence the convergence occurs at $\sigma = 0$ ($j\omega$-axis)

(d)

$$x(t) = \frac{1}{1+e^t}$$

$$\frac{1}{1+e^t}e^{-st} \to 0 \; \Big\} \quad \begin{array}{l} \text{as } t \to \infty \text{ if Re } s > -1 \\ \text{as } t \to -\infty \text{ if Re } s < 0 \end{array}$$

Hence the region of convergence is $-1 < \sigma < 0$

(e)

$$x(t) = e^{-kt^2}$$

$$e^{-kt^2}e^{-st} \to 0 \left.\begin{matrix} \\ \\ \end{matrix}\right\} \quad \begin{matrix} \text{as } t \to \infty \text{ for any value of } s \\ \text{as } t \to -\infty \text{ for any value of } s \end{matrix}$$

Hence the region of convergence is the entire s-plane.

4.11-2. (a)

$$x(t) = e^{-|t|} = e^{-t}u(t) + e^{t}u(-t) = x_1(t) + x_2(t)$$

$$X_1(s) = \frac{1}{s+1} \qquad \sigma > -1$$

$$x_2(-t) = e^{-t}u(t) \quad \text{and} \quad X_2(-s) = \frac{1}{s+1}$$

$$\text{and} \qquad X_2(s) = \frac{1}{-s+1} \qquad \sigma < 1$$

$$\text{Hence:} \quad X(s) = X_1(s) + X_2(s) = \frac{1}{s+1} + \frac{1}{-s+1} = \frac{-2}{s^2-1} \qquad -1 < \sigma < 1$$

(b)

$$x(t) = e^{-|t|}\cos t = e^{-t}\cos t\, u(t) + e^{t}\cos t\, u(-t) = x_1(t) + x_2(t)$$

$$\text{Hence} \quad X_1(s) = \frac{s+1}{(s+1)^2+1} \quad \text{and} \quad X_2(-s) = \frac{s+1}{(s+1)^2+1} \quad \sigma < 1$$

$$X(s) = X_1(s) + X_2(s) = \frac{s+1}{(s+1)^2+1} - \frac{s-1}{(s-1)^2+1} = \frac{4-2s^2}{s^4-4} \qquad -1 < \sigma < 1$$

(c)

$$x(t) = e^{t}u(t) + e^{2t}u(-t); \quad X_1(s) = \frac{1}{s-1} \quad \sigma > 1 \quad \text{and} \quad X_2(-s) = \frac{1}{s+2}$$

$$X_2(s) = \frac{1}{-s+2} \qquad \sigma < 2.$$

$$\text{Hence} \qquad X(s) = X_1(s) + X_2(s) = \frac{-1}{(s-1)(s-2)} \quad 1 < \sigma < 2$$

(d)

$$x(t) = e^{-tu(t)} = \begin{cases} e^{-t} & \text{for} \quad t > 0 \\ 1 & \text{for} \quad t < 0 \end{cases}$$

$$x_1(t) = e^{-t}u(t), \quad x_2(t) = u(-t). \quad \text{Hence} \quad X_1(s) = \frac{1}{s+1} \quad \sigma > -1$$

$$\text{and} \quad X_2(-s) = \frac{1}{s}, \quad X_2(s) = \frac{-1}{s} \quad \sigma < 0$$

$$\text{and hence:} \qquad X(s) = \frac{1}{s+1} - \frac{1}{s} = \frac{-1}{s(s+1)} \qquad -1 < \sigma < 0$$

(e)

$$x(t) = e^{tu(-t)} = \begin{cases} x_1(t) = 1 & \text{for} \quad t > 0 \\ x_2(t) = e^t & \text{for} \quad t < 0 \end{cases}$$

$$X_1(s) = \frac{1}{s} \qquad \sigma > 0$$

$$X_2(-s) = \frac{1}{s+1} \quad X_2(s) = \frac{1}{-s+1} \quad \sigma < 1$$

and hence: $$X(s) = \frac{1}{s} - \frac{1}{s-1} = \frac{-1}{s(s-1)} \qquad 0 < \sigma < 1$$

(f)

$$x(t) = \cos\omega_0 t\, u(t) + e^t u(-t) = x_1(t) + x_2(t)$$

$$X_1(s) = \frac{s}{s^2 + \omega_0^2} \qquad \sigma > 0$$

and $$X_2(-s) = \frac{1}{s+1}, \qquad X_2(s) = \frac{1}{1-s} \qquad \sigma < 1$$

$$X(s) = X_1(s) + X_2(s) = \frac{-(s+\omega_0^2)}{(s-1)(s^2+\omega_0^2)} \quad 0 < \sigma < 1$$

4.11-3. (a)

$$\begin{aligned} X(s) &= \frac{2s+5}{(s+2)(s+3)} \qquad -3 < \sigma < -2 \\ &= \frac{1}{s+2} + \frac{1}{s+3} \qquad -3 < \sigma < -2 \end{aligned}$$

The pole -2 lies to the right, and the pole -3 lies to the left of the region of convergence; hence the first term represents causal and the second term represents anticausal signal:

$$x(t) = e^{-3t}u(t) - e^{-2t}u(-t)$$

(b)

$$\begin{aligned} X(s) &= \frac{2s-5}{(s-2)(s-3)} \qquad 2 < \sigma < 3 \\ &= \frac{1}{s-2} + \frac{1}{s-3} \qquad 2 < \sigma < 3 \end{aligned}$$

The pole at -2 lies to the left and that at 3 lies to the right of the region of convergence; hence

$$x(t) = e^{2t}u(t) - e^{3t}u(-t)$$

(c)

$$\begin{aligned} X(s) &= \frac{2s+3}{(s+1)(s+2)} \qquad \sigma > -1 \\ &= \frac{1}{s+1} + \frac{1}{s+2} \qquad \sigma > -1 \end{aligned}$$

Both poles lie to the left of the region of convergence, and

$$x(t) = (e^{-t} + e^{-2t})u(t)$$

(d)

$$\begin{aligned} X(s) &= \frac{2s+3}{(s+1)(s+2)} \qquad \sigma < -2 \\ &= \frac{1}{s+1} + \frac{1}{s+2} \qquad \sigma < -2 \end{aligned}$$

Both poles lie to the right of the region of convergence, and hence:

$$x(t) = -(e^{-t} + e^{-2t})u(-t)$$

(e)

$$\begin{aligned} X(s) &= \frac{3s^2 - 2s - 17}{(s+1)(s+3)(s-5)} \qquad -1 < \sigma < 5 \\ &= \frac{1}{s+1} + \frac{1}{s+3} + \frac{1}{s-5} \end{aligned}$$

The poles -1 and -3 lie to the left of the region of convergence, whereas the pole 5 lies to the right:

$$x(t) = (e^{-t} + e^{-3t})u(t) - e^{5t}u(-t)$$

4.11-4.

$$\frac{2s^2 - 2s - 6}{(s+1)(s-1)(s+2)} = \frac{1}{s+1} - \frac{1}{s-1} + \frac{2}{s+2}$$

(a) Re $s > 1$: All poles to the left of the region of convergence. Therefore

$$x(t) = (e^{-t} - e^{t} + 2e^{-2t})u(t)$$

(b) Re $s < -2$: All poles to the right of the region of convergence. Therefore

$$x(t) = (-e^{-t} + e^{t} - 2e^{-2t})u(-t)$$

(c) $-1 <$ Re $s < 1$: Poles -1 and -2 to the left and pole 1 to the right of the region of convergence. Therefore

$$x(t) = (e^{-t} + 2e^{-2t})u(t) + e^{t}u(-t)$$

(d) $-2 <$ Re $s < -1$: Poles -1 and 1 are to the right and pole -2 is to the left of the region of convergence. Therefore

$$x(t) = 2e^{-2t}u(t) + [-e^{-t} + e^{t}]u(-t)$$

4.11-5. (a)

$$x(t) = e^{-\frac{|t|}{2}}, \qquad H(s) = \frac{1}{s+1} \quad \sigma > -1$$

And $$X(s) = \frac{1}{s+0.5} - \frac{1}{s-0.5} \qquad -\frac{1}{2} < \sigma < \frac{1}{2}$$

hence: $$Y(s) = H(s)X(s) = \frac{1}{s+1}\left[\frac{1}{s+0.5} - \frac{1}{s-0.5}\right] \qquad -\frac{1}{2} < \sigma < \frac{1}{2}$$

$$\begin{aligned} Y(s) &= \frac{-2}{s+1} + \frac{2}{s+0.5} + \frac{\frac{2}{3}}{s+1} - \frac{\frac{2}{3}}{s-0.5} \\ &= \frac{-\frac{4}{3}}{s+1} + \frac{2}{s+0.5} - \frac{\frac{2}{3}}{s-0.5} \qquad -\frac{1}{2} < \sigma < \frac{1}{2} \end{aligned}$$

The poles -1 and -0.5, which are to the left of the strip of convergence, yield the causal signal, and the pole 0.5, which is to the right of the strip of convergence, yields the anticausal signal. Hence

$$y(t) = \left(-\frac{4}{3}e^{-t} + 2e^{-t/2}\right)u(t) + \frac{2}{3}e^{t/2}u(-t)$$

(b)

$$x(t) = e^{t}u(t) + e^{2t}u(-t)$$

$$\begin{aligned} X(s) &= \frac{1}{s-1} - \frac{1}{s-2} \qquad 1 < \sigma < 2 \\ &= \frac{-1}{(s-1)(s-2)} \end{aligned}$$

And $$H(s) = \frac{1}{s+1} \qquad \sigma > -1$$

Hence: $$Y(s) = H(s)X(s) = \frac{-1}{(s+1)(s-1)(s-2)} \qquad 1 < \sigma < 2$$

$$Y(s) = \frac{-1/6}{s+1} + \frac{1/2}{s-1} - \frac{1/3}{s-2} \qquad 1 < \sigma < 2$$

Hence $$y(t) = \left(-\frac{1}{6}e^{-t} + \frac{1}{2}e^{t}\right)u(t) + \frac{1}{3}e^{2t}u(-t)$$

(c)

$$x(t) = e^{-t/2}u(t) + e^{-t/4}u(-t)$$

$$X(s) = \frac{1}{s+0.5} - \frac{1}{s+0.25} = \frac{-\frac{1}{4}}{(s+0.5)(s+0.25)} \qquad -\frac{1}{2} < \sigma < \frac{1}{4}$$

Also $$H(s) = \frac{1}{s+1} \qquad \sigma > -1$$

Hence: $$\begin{aligned} Y(s) = H(s)X(s) &= \frac{-\frac{1}{4}}{(s+1)(s+0.5)(s+0.25)} \qquad -\frac{1}{2} < \sigma < \frac{1}{4} \\ &= \frac{-\frac{2}{3}}{s+1} + \frac{2}{s+0.5} - \frac{\frac{4}{3}}{s+0.25} \qquad -\frac{1}{2} < \sigma < \frac{1}{4} \end{aligned}$$

and $$y(t) = \left(-\frac{2}{3}e^{-t} + 2e^{-\frac{t}{2}}\right)u(t) + \frac{4}{3}e^{-\frac{t}{4}}u(-t)$$

(d)

$$x(t) = e^{2t}u(t) + e^{t}u(-t) = x_1(t) + x_2(t)$$

$$\begin{aligned} X_1(s) &= \frac{1}{s-2} && \sigma > 2 \\ X_2(s) &= \frac{-1}{s-1} && \sigma < 1 \end{aligned}$$

$$\text{and} \qquad H(s) = \frac{1}{s+1} \qquad \sigma > -1$$

In this case, there is no region of convergence that is common to $X_1(s)$ and $X_2(s)$. However, each of $X_1(s)$ and $X_2(s)$ have a region of convergence that is common to $H(s)$. Hence the output can be computed by finding the system response to $x_1(t)$ and $x_2(t)$ separately, and then adding these two components. This means we need not worry about the common region of convergence for $X_1(s)$ and $X_2(s)$. Thus:

$$Y(s) = Y_1(s) + Y_2(s) \qquad \text{where}$$

$$\begin{aligned} Y_1(s) = X_1(s)H(s) &= \frac{1}{(s+1)(s-2)} && \sigma > 2 \\ &= \frac{-\frac{1}{3}}{s+1} + \frac{\frac{1}{3}}{s-2} && \sigma > 2 \end{aligned}$$

Observe that both the poles (-1 and 2) are to the left of the region of convergence, hence both terms are causal, and:

$$y_1(t) = \left(-\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t}\right)u(t)$$

$$\begin{aligned} Y_2(s) = X_2(s)H(s) &= \frac{-1}{(s+1)(s-1)} && -1 < \sigma < 1 \\ &= \frac{\frac{1}{2}}{s+1} - \frac{\frac{1}{2}}{s-1} && -1 < \sigma < 1 \end{aligned}$$

The poles -1 and 1 are to the left and the right, respectively, of the strip of convergence. Hence the first term yields causal signal and the second yields anticausal signal. Hence

$$y_2(t) = -\frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{t}u(-t)$$

$$\text{Therefore} \quad y(t) = y_1(t) + y_2(t) = \left(\frac{1}{6}e^{-t} + \frac{1}{3}e^{2t}\right)u(t) + \frac{1}{2}e^{t}u(-t)$$

(e)

$$x(t) = e^{-\frac{t}{4}}u(t) + e^{-\frac{t}{2}}u(-t) = x_1(t) + x_2(t)$$

$$X(s) = X_1(s) + X_2(s)$$

$$\text{where} \quad \begin{aligned} X_1(s) &= \frac{1}{s+0.25} & \sigma > -\frac{1}{4} \\ X_2(s) &= \frac{-1}{s+0.5} & \sigma < -\frac{1}{2} \\ H(s) &= \frac{1}{s+1} & \sigma > -1 \end{aligned}$$

Here also, we have no common region of convergence, for $X_1(s)$ and $X_2(s)$ as in part d. Let $Y(s) = Y_1(s) + Y_2(s)$ where:

$$\begin{aligned} Y_1(s) &= \frac{1}{(s+1)(s+0.25)} & \sigma > -\frac{1}{4} \\ &= \frac{-\frac{4}{3}}{s+1} + \frac{\frac{4}{3}}{s+0.25} & \sigma > -\frac{1}{4} \end{aligned}$$

$$y_1(t) = \left(-\frac{4}{3}e^{-t} + \frac{4}{3}e^{-\frac{t}{4}}\right)u(t)$$

$$\begin{aligned} Y_2(s) &= \frac{-1}{(s+1)(s+0.5)} & -1 < \sigma < -\frac{1}{2} \\ &= \frac{2}{s+1} - \frac{2}{s+0.5} & -1 < \sigma < -\frac{1}{2} \end{aligned}$$

$$\text{and} \quad y_2(t) = 2e^{-t}u(t) + 2e^{-\frac{t}{2}}u(-t)$$

$$\text{Hence} \quad y(t) = y_1(t) + y_2(t) = \left(\frac{2}{3}e^{-t} + \frac{4}{3}e^{-\frac{t}{4}}\right)u(t) + 2e^{-\frac{t}{2}}u(-t)$$

(f)

$$x(t) = e^{-3t}u(t) + e^{-2t}u(-t) = x_1(t) + x_2(t)$$

$$X(s) = X_1(s) + X_2(s)$$

$$\text{where} \quad \begin{aligned} X_1(s) &= \frac{1}{s+3} & \sigma > -3 \\ X_2(s) &= \frac{-1}{s+2} & \sigma < -2 \\ H(s) &= \frac{1}{s+1} & \sigma > -1 \end{aligned}$$

In this case, there is a common region of convergence for $X_1(s)$ and $H(s)$, but there is no region of convergence common to $X_2(s)$ and $H(s)$. Hence the output $y_1(t)$ will be finite but $y_2(t)$ will be ∞.

4.11-6.

$$\begin{aligned} \mathcal{L}\left(r_{xx}(t)\right) &= \int_{-\infty}^{\infty} r_{xx}(t)e^{-st}dt \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(\tau)x(\tau+t)d\tau\right)e^{-st}dt \\ &= \int_{-\infty}^{\infty} x(\tau)\left(\int_{-\infty}^{\infty} x(\tau+t)e^{-st}dt\right)d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)e^{s\tau}X(s)d\tau \\ &= X(s)\int_{-\infty}^{\infty} x(\tau)e^{-\tau(-s)}d\tau \\ R_{xx}(s) &= X(s)X(-s) \end{aligned}$$

4.11-7. For $\sigma < 0$, we know that $\mathcal{L}^{-1}\left[\frac{2}{s}\right] = -2u(-t)$. Additionally, $\mathcal{L}^{-1}[1/2] = \delta/2$. Using properties, $\mathcal{L}^{-1}[s(1/2)] = \frac{d}{dt}(\delta(t)/2)$. Thus,

$$x(t) = -2u(-t) + \frac{d}{dt}(\delta(t)/2).$$

The function $\frac{d}{dt}(\delta(t)/2)$ is called the "unit doublet". Like $\delta(t)$, the unit doublet is not a physically realizable signal. It is a mathematical constructions that is useful, among other things, in finding function deriyatives. Refer to the topic of generalized derivatives.

4.11-8. (a) Yes, $x(t)$ can be left-sided. To be left-sided and absolutely integrable, the signal's region of convergence must: 1) be left-sided, 2) include the ω-axis, and 3) not include any poles. With a pole at $s = \pi$, it is possible to achieve all three necessary conditions. For example, $x(t) = e^{\pi t}u(-t)$ has a pole at $s = \pi$, is absolutely integrable, and is left-sided.

(b) No, $x(t)$ cannot be right-sided. To be right-sided and absolutely integrable, the signal's region of convergence must: 1) be right-sided, 2) include the ω-axis, and 3) not include any poles. The ω-axis cannot be included in a region of convergence that is to the right of the known pole at $s = \pi$.

(c) Yes, $x(t)$ can be two-sided. To be two-sided and absolutely integrable, the signal must: 1) have at least one pole in the right-half plane, 2) have at least one pole in the left-half plane, and 3) have a region of convergence that includes the ω-axis. With a pole at $s = \pi$, these conditions are possible. For example, $x(t) = e^{\pi t}u(-t) + e^{-\pi t}u(t)$ has a pole at $s = \pi$ (and another at $s = -\pi$), is absolutely integrable, and is two-sided.

(d) No, $x(t)$ cannot be finite duration. To be finite duration, the signal's region of convergence must include all finite values of s. However, since a pole is present at $s = \pi$, this point cannot be included in the region of convergence. Thought of another way, a pole at $s = \pi$ implies a signal component of either $e^{\pi t}u(t)$ or $e^{\pi t}u(-t)$, both of which are infinite in duration.

4.11-9. (a) $X_1(s) = \int_{-\infty}^{\infty} x_1(t)e^{-st}dt = \int_{-\infty}^{\infty}(j + e^{jt})u(t)e^{-st}dt = \int_0^{\infty} je^{-st} + e^{t(j-s)}dt = \left(\frac{j}{-s}e^{-st} + \frac{e^{t(j-s)}}{j-s}\right)\Big|_{t=0}^{\infty}$. For $\sigma > 0$, this simplifies to $X_1(s) = \frac{j}{-s}(0 - e^0) + \frac{0-e^0}{j-s}$. Thus,

$$X_1(s) = \frac{j}{s} + \frac{1}{s-j} \text{ for } \sigma > 0.$$

(b) $X_2(s) = \int_{-\infty}^{\infty} x_2(t)e^{-st}dt = \int_{-\infty}^{\infty} j\cosh(t)u(-t)e^{-st}dt = \int_{-\infty}^{0} j\frac{e^t+e^{-t}}{2}e^{-st}dt = \int_{-\infty}^{0} j\frac{e^{t(1-s)}+e^{t(-1-s)}}{2}dt = \left(\frac{je^{t(1-s)}}{2(1-s)} + \frac{je^{t(-1-s)}}{2(-1-s)}\right)\Big|_{t=-\infty}^{0}$. For $\sigma < -1$, this simplifies to $X_2(s) = \frac{j(e^0-0)}{2(1-s)} + \frac{j(e^0-0)}{2(-1-s)} = \frac{-0.5j}{s-1} + \frac{-0.5j}{s+1}$. Thus,

$$X_2(s) = \frac{-js}{s^2-1} \text{ for } \sigma < -1.$$

(c) $X_3(s) = \int_{-\infty}^{\infty} x_3(t)e^{-st}dt = \int_{-\infty}^{\infty}\left(e^{j(\frac{\pi}{4})}u(-t+1) + j\delta(t-5)\right)e^{-st}dt = \int_{-\infty}^{1} e^{j(\frac{\pi}{4})}e^{-st}dt + \int_{-\infty}^{\infty} j\delta(t-5)e^{-st}dt = e^{j(\frac{\pi}{4})}\frac{e^{-st}}{-s}\Big|_{t=-\infty}^{1} + je^{-5s}$. For $\sigma < 0$, this simplifies to $X_3(s) = e^{j(\frac{\pi}{4})}\frac{e^{-s}-0}{-s} + je^{-5s}$. Thus,

$$X_3(s) = -e^{j(\frac{\pi}{4})}\frac{e^{-s}}{s} + je^{-5s} \text{ for } \sigma < 0.$$

(d) $X_4(s) = \int_{-\infty}^{\infty} x_4(t)e^{-st}dt = \int_{-\infty}^{\infty} \left(j^t u(-t) + \delta(t-\pi)\right) e^{-st}dt = \int_{-\infty}^{0} e^{tj\pi/2}e^{-st}dt + \int_{-\infty}^{\infty} \delta(t-\pi)e^{-st}dt = \left.\frac{e^{t(j\pi/2-s)}}{j\pi/2-s}\right|_{t=-\infty}^{0} + je^{-s\pi}$. For $\sigma < 0$, this simplifies to $X_4(s) = \frac{1-0}{j\pi/2-s} + e^{-s\pi}$. Thus,

$$X_4(s) = e^{-s\pi} - \frac{1}{s - j\pi/2} \text{ for } \sigma < 0.$$

4.11-10. (a) To be bounded amplitude, the region of convergence must include the ω-axis. The transfer function has two poles, at $s = \pm 1$, that must be excluded from the region of convergence. Thus, the region of convergence must be

$$-1 < \sigma < 1.$$

(b) Rewrite $H(s)$ as $2^s \frac{s}{(s-1)(s+1)} = e^{\ln(s)}\left(\frac{0.5}{s-1} + \frac{0.5}{s+1}\right)$. Using $-1 < \sigma < 1$, the time-shifting property, and a table of Laplace transform pairs, the inverse transform is found to be

$$h(t) = 0.5e^{-(t+\ln(2))}u(t+\ln(2)) - 0.5e^{t+\ln(2)}u(-(t+\ln(2))).$$

4.M-1. Using program MS4P3:

```
>> MS4P3(20)
ans = 524288          0    -2621440           0     5570560
            0   -6553600           0     4659200           0
     -2050048          0      549120           0      -84480
            0       6600           0        -200           0
            1
```

Thus,

$$\begin{aligned} C_{20}(x) &= 524288x^{20} - 2621440x^{18} + 5570560x^{16} - 6553600x^{14} + 4659200x^{12} + \\ &\quad -2050048x^{10} + 549120x^8 - 84480x^6 + 6600x^4 - 200x^2 + 1 \end{aligned}$$

4.M-2. (a) MATLAB is used for the design. To evaluate filter performance, the magnitude response is plotted over the frequency range ($0 \le f \le 10$kHz).

```
>> N = 12; omega_c = 2*pi*5000;
>> poles = roots([(j*omega_c)^(-2*N),zeros(1,2*N-1),1]);
>> B_poles = poles(find(real(poles)<0));
>> subplot(121), plot(real(B_poles),imag(B_poles),'xk');
>> xlabel('Real'); ylabel('Imag');
>> axis([-4e4 0 -4e4 4e4]); axis equal;
>> A = poly(B_poles); A = A/A(end); B = 1;
>> f = linspace(0,10000,1001);
>> Hmag_B = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>> subplot(122), plot(f,Hmag_B,'k');
>> xlabel('f [Hz]'); ylabel('|H_{LP}(j2\pi f)|');
```

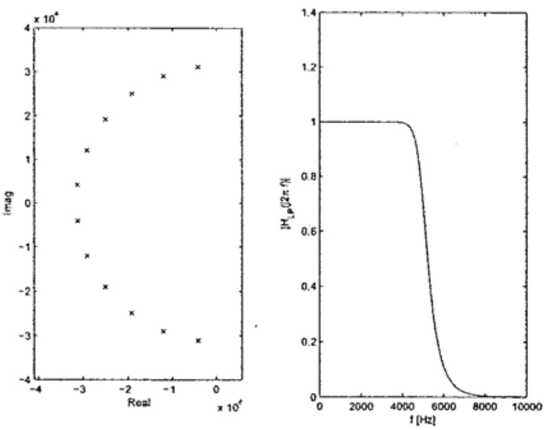


Figure S4.M-2a: Order-12 Butterworth LPF with $\omega_c = 2\pi 5000$.

The resulting figures are consistent with a Butterworth design; the poles lie on a semicircle in the left-half s-plane, and the magnitude response exhibits smooth monotonic roll-off.

(b) Modifying program MS4P2, the Sallen-Key component values and magnitude response plots are easily found.

```
>> omega_0 = 5000*2*pi; f = linspace(0,10000,200);
>> psi = [7.5:15:90]*pi/180; Hmag_SK = zeros(6,200);
>> for stage = 1:6,
>>      Q = 1/(2*cos(psi(stage)));
>>      disp(['Stage ',num2str(stage),...
>>              ' (Q = ',num2str(Q),...
>>              '):  R1 = R2 = ',num2str(100000)]);
>>      disp(['        C1 = ',num2str(2*Q/(omega_0*100000)),...
>>              ', C2 = ',num2str(1/(2*Q*omega_0*100000))]);
>>      B = omega_0^2; A = [1 omega_0/Q omega_0^2];
>>      Hmag_SK(stage,:) = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>> end
>> plot(f,Hmag_SK,'k',f,prod(Hmag_SK),'k:')
>> xlabel('f [Hz]'); ylabel('Magnitude Responses')
Stage 1 (Q = 0.50431):  R1 = R2 = 100000
        C1 = 3.2106e-010, C2 = 3.1559e-010
Stage 2 (Q = 0.5412):  R1 = R2 = 100000
        C1 = 3.4454e-010, C2 = 2.9408e-010
Stage 3 (Q = 0.63024):  R1 = R2 = 100000
        C1 = 4.0122e-010, C2 = 2.5253e-010
Stage 4 (Q = 0.82134):  R1 = R2 = 100000
        C1 = 5.2288e-010, C2 = 1.9377e-010
Stage 5 (Q = 1.3066):  R1 = R2 = 100000
        C1 = 8.3178e-010, C2 = 1.2181e-010
Stage 6 (Q = 3.8306):  R1 = R2 = 100000
        C1 = 2.4387e-009, C2 = 4.1548e-011
```

The resulting resistor and capacitor values are realistic. Each Sallen-Key stage implements a complex-conjugate pair of poles. The flattest magnitude response corresponds to the pair of poles that are furthest from the ω-axis, or Stage 1.

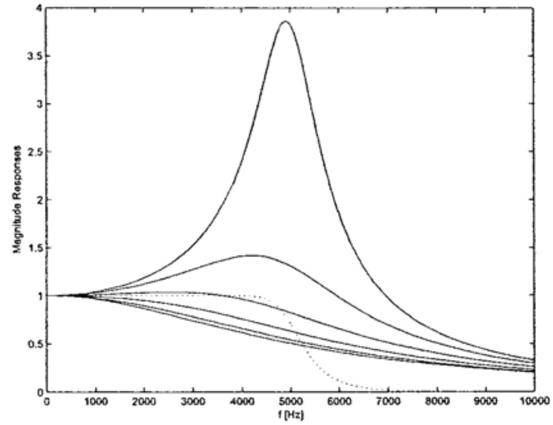


Figure S4.M-2b: Order-12 Butterworth LPF Sallen-Key Stage Responses.

The most peaked magnitude response corresponds to the pair of poles that are closest to the ω-axis, or Stage 6. The remaining stages are ordered in between. The dashed curve is the total magnitude response, and it is exactly the same as the one shown in Figure S4.M-2a.

4.M-3. (a) MATLAB is used for the design. To evaluate filter performance, the magnitude response is plotted over the frequency range ($0 \leq f \leq 10$kHz).

```
>> omega_c = 2*pi*5000; R = 3; N = 12;
>> epsilon = sqrt(10^(R/10)-1);
>> k = [1:N]; xi = 1/N*asinh(1/epsilon); phi = (k*2-1)/(2*N)*pi;
>> C_poles = omega_c*(-sinh(xi)*sin(phi)+j*cosh(xi)*cos(phi));
>> subplot(121), plot(real(C_poles),imag(C_poles),'xk');
>> xlabel('Real'); ylabel('Imag');
>> axis([-4e4 0 -4e4 4e4]); axis equal;
>> A = poly(C_poles);
>> B = A(end)/sqrt(1+epsilon^2);
>> omega = linspace(0,2*pi*10000,2001);
>> Hmag_C = abs(MS4P1(B,A,omega));
>> subplot(122); plot(omega/2/pi,abs(Hmag_C),'k'); grid
>> xlabel('f [Hz]'); ylabel('|H_{LP}(j2\pi f)|');
```

The resulting figures are consistent with a Chebyshev design; the poles lie on an ellipse in the left-half s-plane, passband ripples are equal in height and never exceed $R = 3$dB, there are a total of $N = 12$ maxima and minima in the passband, and the gain rapidly and monotonically decreases after the cutoff frequency of $f_c = 5$kHz.

(b) Modifying program MS4P2, the Sallen-Key component values and magnitude response plots are easily found.

```
>> omega_c = 2*pi*5000; R = 3; N = 12;
>> epsilon = sqrt(10^(R/10)-1);
>> k = [1:N]; xi = 1/N*asinh(1/epsilon); phi = (k*2-1)/(2*N)*pi;
>> C_poles = omega_c*(-sinh(xi)*sin(phi)+j*cosh(xi)*cos(phi));
>> C_poles = C_poles(find(imag(C_poles)>0)); % Quadrant 2 poles
>> f = linspace(0,10000,501); Hmag_SK = zeros(6,501);
```

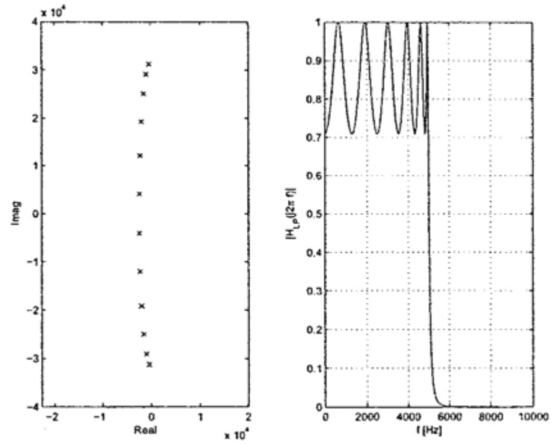


Figure S4.M-3a: Order-12 Chebyshev LPF with $\omega_c = 2\pi 5000$ and $R = 3$dB.

```
>> for stage = 1:6,
>>     omega_0 = abs(C_poles(stage));
>>     psi = pi-angle(C_poles(stage));
>>     Q = 1/(2*cos(psi));
>>     disp(['Stage ',num2str(stage),...
>>             ' (Q = ',num2str(Q),...
>>             '):  R1 = R2 = ',num2str(100000)]);
>>     disp(['          C1 = ',num2str(2*Q/(omega_0*100000)),...
>>             ', C2 = ',num2str(1/(2*Q*omega_0*100000))]);
>>     B = omega_0^2; A = [1 omega_0/Q omega_0^2];
>>     Hmag_SK(stage,:) = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>> end
>> plot(f,20*log10(Hmag_SK),'k',f,20*log10(prod(Hmag_SK)),'k:')
>> xlabel('f [Hz]'); ylabel('Magnitude Responses [dB]')
>> axis([0 10000 -40 40]);
Stage 1 (Q = 51.7057):  R1 = R2 = 100000
        C1 = 3.311e-008, C2 = 3.0961e-012
Stage 2 (Q = 16.4408):  R1 = R2 = 100000
        C1 = 1.1293e-008, C2 = 1.0445e-011
Stage 3 (Q = 8.885):  R1 = R2 = 100000
        C1 = 7.0991e-009, C2 = 2.2482e-011
Stage 4 (Q = 5.247):  R1 = R2 = 100000
        C1 = 5.4474e-009, C2 = 4.9466e-011
Stage 5 (Q = 2.8635):  R1 = R2 = 100000
        C1 = 4.6778e-009, C2 = 1.4262e-010
Stage 6 (Q = 1.0262):  R1 = R2 = 100000
        C1 = 4.359e-009, C2 = 1.0348e-009
```

The resulting resistor and capacitor values are realistic. Each Sallen-Key stage implements a complex-conjugate pair of poles. The most peaked magnitude response corresponds to the pair of poles that are closest to the ω-axis, or Stage 1. The least peaked magnitude response corresponds to the pair of poles that are furthest from the ω-axis, or Stage 6. The remaining stages are ordered in between. The dashed curve is the total magnitude response, and within a gain error of 3dB is exactly the same as the one shown in Figure S4.M-3a. The gain

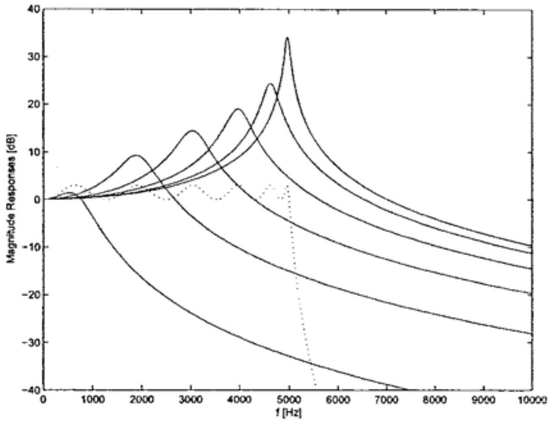


Figure S4.M-3b: Order-12 Chebyshev LPF Sallen-Key Stage Responses.

error occurs since the Salley-Key stages are constrained to unity gain at dc, yet the Chebyshev filter requires gain $\frac{1}{\sqrt{1+\epsilon^2}}$ at dc. This error is easily corrected by adding a gain stage to the circuit.

4.M-4. (a) Using MATLAB, the Sallen-Key component values are easily found.

```
>> omega_0 = 4000*2*pi; NS = 4;
>> psi = [90/(2*NS):90/NS:90]*pi/180;
>> Q = 1./(2*cos(psi));
>> R1 = 1e9/omega_0*ones(1,NS); R2 = R1;
>> C1 = 2*Q./(omega_0*R1); C2 = 1./(2*omega_0*Q.*R2);
>> for stage = 1:NS,
>>     disp(['Stage ',num2str(stage),...
>>            ' (Q = ',num2str(Q(stage)),...
>>            '):  R1 = R2 = ',num2str(R1(stage))]);
>>     disp(['          C1 = ',num2str(C1(stage)),...
>>            ', C2 = ',num2str(C2(stage))]);
>> end
Stage 1 (Q = 0.5098):  R1 = R2 = 39788.7358
        C1 = 1.0196e-009, C2 = 9.8079e-010
Stage 2 (Q = 0.60134):  R1 = R2 = 39788.7358
        C1 = 1.2027e-009, C2 = 8.3147e-010
Stage 3 (Q = 0.89998):  R1 = R2 = 39788.7358
        C1 = 1.8e-009, C2 = 5.5557e-010
Stage 4 (Q = 2.5629):  R1 = R2 = 39788.7358
        C1 = 5.1258e-009, C2 = 1.9509e-010
```

The resulting resistor and capacitor values are realistic.

(b) The transformed Sallen-Key circuit is shown in Figure S4.M-4b. Name the node between capacitors $v(t)$. In transform domain, KCL at the positive terminal of the op-amp yields

$$\frac{Y(s)-V(s)}{\frac{1}{sC_2'}} = -\frac{Y(s)-0}{R_2'}.$$

Solving for $V(s)$ yields

$$V(s) = Y(s)\frac{1+R_2'C_2's}{R_2'C_2's}.$$

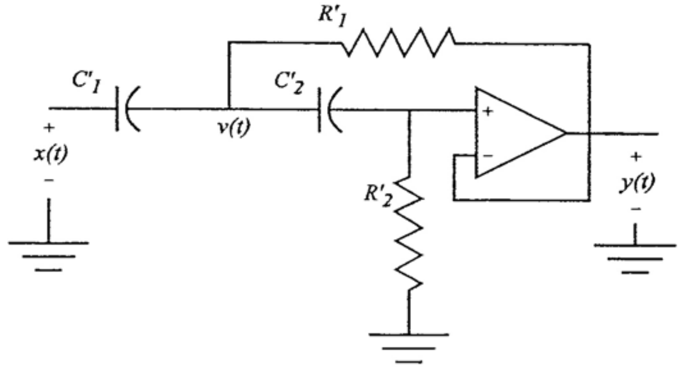


Figure S4.M-4b: $RC - CR$ transformed Sallen-Key circuit.

KCL at node $V(s)$ yields

$$\frac{X(s) - V(s)}{\frac{1}{C_1' s}} + \frac{Y(s) - V(s)}{R_1'} + \frac{Y(s) - V(s)}{\frac{1}{C_2' s}} = 0.$$

Rearranging yields

$$V(s)\left[C_1' s + 1/R_1' + C_2' s\right] = C_1' s X(s) + Y(s)\left[C_2' s + 1/R_1'\right].$$

Substituting the previous expression for $V(s)$ yields

$$Y(s)\frac{1 + R_2' C_2' s}{R_2' C_2' s}\left[C_1' s + 1/R_1' + C_2' s\right] = C_1' s X(s) + Y(s)\left[C_2' s + 1/R_1'\right].$$

Rearranging yields

$$Y(s)\left[\frac{1 + R_2' C_2' s}{R_2' C_2' s}\frac{1 + R_1' C_1' s + R_1' C_2' s}{R_1'} - \frac{1 + R_1' C_2' s}{R_1'}\right] = X(s)\left[C_1' s\right].$$

Following simplification, we get

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2}{s^2 + s\left(\frac{1}{R_2' C_2'} + \frac{1}{R_2' C_1'}\right) + \frac{1}{R_1' R_2' C_1' C_2'}}.$$

(c) MATLAB is used to transform the Butterworth LPF from 4.M-4a.

```
>> R1p = 1./(C1*omega_0); R2p = 1./(C2*omega_0);
>> C1p = 1./(R1*omega_0); C2p = 1./(R2*omega_0);
>> for stage = 1:NS,
>>     disp(['Stage ',num2str(stage),...
>>             ' (Q = ',num2str(Q(stage)),...
>>             '):  C1'' = C2'' = ',num2str(C1p(stage))]);
>>     disp(['          R1'' = ',num2str(R1p(stage)),...
>>             ', R2'' = ',num2str(R2p(stage))]);
>> end
Stage 1 (Q = 0.5098):  C1' = C2' = 1e-009
        R1' = 39024.2064, R2' = 40568.2432
Stage 2 (Q = 0.60134):  C1' = C2' = 1e-009
```

```
        R1' = 33083.1247, R2' = 47853.5056
Stage 3 (Q = 0.89998):  C1' = C2' = 1e-009
        R1' = 22105.4372, R2' = 71617.8323
Stage 4 (Q = 2.5629):  C1' = C2' = 1e-009
        R1' = 7762.3973, R2' = 203950.3311
```

The resulting resistor and capacitor values are realistic.

MATLAB also conveniently computes magnitude responses and pole locations. From $H(s)$, it is clear that all zeros are at zero.

```
>> Hmag_SK = zeros(NS,200); Poles = zeros(NS,2);
>> f = linspace(0,omega_0/pi,200);
>> for stage = 1:NS,
>>     B = [1 0 0];
>>     A = [1,(1/(R2p(stage)*C2p(stage))+1/(R2p(stage)*C1p(stage))),...
>>          1/(R1p(stage)*R2p(stage)*C1p(stage)*C2p(stage))];
>>     Poles(stage,:) = (roots(A)).';
>>     Hmag_SK(stage,:) = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>> end
>> subplot(121), plot(real(Poles(:)),imag(Poles(:)),'kx',0,0,'ko');
>> axis(omega_0*[-1.1, .1 -1.1 1.1]); axis equal;
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> subplot(122), plot(f,Hmag_SK,'k',f,prod(Hmag_SK),'k:');
>> xlabel('f [Hz]'); ylabel('Magnitude Responses');
```

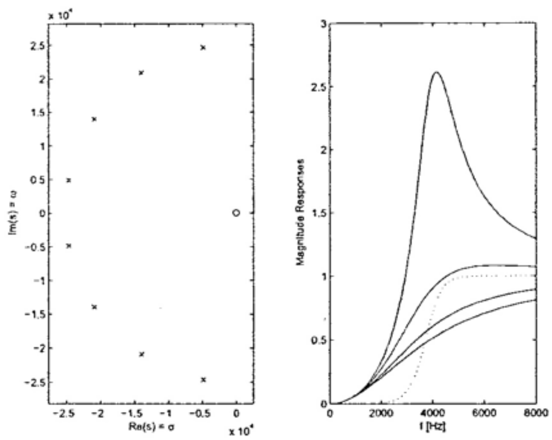


Figure S4.M-4c: Order-8 Butterworth HPF with $\omega_c = 2\pi 4000$.

The overall magnitude response plot looks like a highpass Butterworth filter; the cutoff is correctly located at ω_c and the response is smooth and monotonic. Interestingly, the Butterworth HPF poles look identical to the Butterworth LPF poles. The only difference is seen in the zeros; all the zeros of the LPF are infinite, and all the zeros of the HPF are located at $s = 0$.

4.M-5. (a) Using MATLAB, the Sallen-Key component values are easily found.

```
>> omega_0 = 1500*2*pi; NS = 8;
>> psi = [90/(2*NS):90/NS:90]*pi/180;
>> Q = 1./(2*cos(psi));
>> R1 = 1e9/omega_0*ones(1,NS); R2 = R1;
```

```
>> C1 = 2*Q./(omega_0*R1); C2 = 1./(2*omega_0*Q.*R2);
>> for stage = 1:NS,
>>     disp(['Stage ',num2str(stage),...
>>             ' (Q = ',num2str(Q(stage)),...
>>             '):  R1 = R2 = ',num2str(R1(stage))]);
>>     disp(['         C1 = ',num2str(C1(stage)),...
>>             ', C2 = ',num2str(C2(stage))]);
>> end
Stage 1 (Q = 0.50242):  R1 = R2 = 106103.2954
        C1 = 1.0048e-009, C2 = 9.9518e-010
Stage 2 (Q = 0.5225):  R1 = R2 = 106103.2954
        C1 = 1.045e-009, C2 = 9.5694e-010
Stage 3 (Q = 0.56694):  R1 = R2 = 106103.2954
        C1 = 1.1339e-009, C2 = 8.8192e-010
Stage 4 (Q = 0.64682):  R1 = R2 = 106103.2954
        C1 = 1.2936e-009, C2 = 7.7301e-010
Stage 5 (Q = 0.78815):  R1 = R2 = 106103.2954
        C1 = 1.5763e-009, C2 = 6.3439e-010
Stage 6 (Q = 1.0607):  R1 = R2 = 106103.2954
        C1 = 2.1214e-009, C2 = 4.714e-010
Stage 7 (Q = 1.7224):  R1 = R2 = 106103.2954
        C1 = 3.4449e-009, C2 = 2.9028e-010
Stage 8 (Q = 5.1011):  R1 = R2 = 106103.2954
        C1 = 1.0202e-008, C2 = 9.8017e-011
```

The resulting resistor and capacitor values are realistic.

(b) See the solution to problem 4.M-4b.

(c) MATLAB is used to transform the Butterworth LPF from 4.M-5a.

```
>> R1p = 1./(C1*omega_0); R2p = 1./(C2*omega_0);
>> C1p = 1./(R1*omega_0); C2p = 1./(R2*omega_0);
>> for stage = 1:NS,
>>     disp(['Stage ',num2str(stage),...
>>             ' (Q = ',num2str(Q(stage)),...
>>             '):  C1'' = C2'' = ',num2str(C1p(stage))]);
>>     disp(['         R1'' = ',num2str(R1p(stage)),...
>>             ', R2'' = ',num2str(R2p(stage))]);
>> end
Stage 1 (Q = 0.50242):  C1' = C2' = 1e-009
        R1' = 105592.379, R2' = 106616.6839
Stage 2 (Q = 0.5225):  C1' = C2' = 1e-009
        R1' = 101534.5231, R2' = 110877.6498
Stage 3 (Q = 0.56694):  C1' = C2' = 1e-009
        R1' = 93574.7524, R2' = 120309.2608
Stage 4 (Q = 0.64682):  C1' = C2' = 1e-009
        R1' = 82018.9565, R2' = 137259.8455
Stage 5 (Q = 0.78815):  C1' = C2' = 1e-009
        R1' = 67311.218, R2' = 167251.6057
Stage 6 (Q = 1.0607):  C1' = C2' = 1e-009
        R1' = 50016.7472, R2' = 225082.7957
Stage 7 (Q = 1.7224):  C1' = C2' = 1e-009
        R1' = 30800.1609, R2' = 365514.6265
```

```
Stage 8 (Q = 5.1011):  C1' = C2' = 1e-009
        R1' = 10399.9416, R2' = 1082497.3575
```

The resulting resistor and capacitor values are reasonably realistic.
MATLAB also conveniently computes magnitude responses and pole locations. From $H(s)$, it is clear that all zeros are at zero.

```
>> Hmag_SK = zeros(NS,200); Poles = zeros(NS,2);
>> f = linspace(0,omega_0/pi,200);
>> for stage = 1:NS,
>>     B = [1 0 0];
>>     A = [1,(1/(R2p(stage)*C2p(stage))+1/(R2p(stage)*C1p(stage))),...
>>          1/(R1p(stage)*R2p(stage)*C1p(stage)*C2p(stage))];
>>     Poles(stage,:) = (roots(A)).';
>>     Hmag_SK(stage,:) = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>> end
>> subplot(121), plot(real(Poles(:)),imag(Poles(:)),'kx',0,0,'ko');
>> axis(omega_0*[-1.1, .1 -1.1 1.1]); axis equal;
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> subplot(122), plot(f,Hmag_SK,'k',f,prod(Hmag_SK),'k:');
>> xlabel('f [Hz]'); ylabel('Magnitude Responses');
```

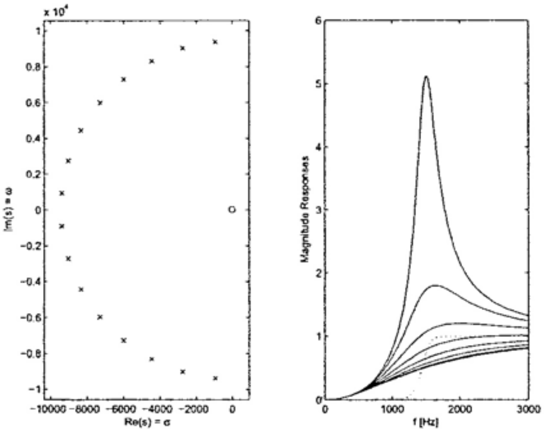


Figure S4.M-5c: Order-16 Butterworth HPF with $\omega_c = 2\pi 1500$.

The overall magnitude response plot looks like a highpass Butterworth filter; the cutoff is correctly located at ω_c and the response is smooth and monotonic. Interestingly, the Butterworth HPF poles look identical to the Butterworth LPF poles. The only difference is seen in the zeros; all the zeros of the LPF are infinite, and all the zeros of the HPF are located at $s = 0$.

4.M-6. (a) Using MATLAB, the Sallen-Key component values are easily found.

```
>> omega_c = 2*pi*4000; R = 3; N = 8;
>> epsilon = sqrt(10^(R/10)-1);
>> k = [1:N]; xi = 1/N*asinh(1/epsilon); phi = (k*2-1)/(2*N)*pi;
>> C_poles = omega_c*(-sinh(xi)*sin(phi)+j*cosh(xi)*cos(phi));
>> C_poles = C_poles(find(imag(C_poles)>0)); % Quadrant 2 poles
>> f = linspace(0,10000,501); Hmag_SK = zeros(6,501);
>> R1 = zeros(N/2,1); R2 = R1; C1 = R1; C2 = R1; Q = R1; omega_0 = R1;
```

```
>> for stage = 1:N/2,
>>     omega_0(stage) = abs(C_poles(stage));
>>     psi = pi-angle(C_poles(stage));
>>     Q(stage) = 1/(2*cos(psi));
>>     R1(stage) = 1e9/omega_c; R2(stage) = R1(stage);
>>     C1(stage) = 2*Q(stage)./(omega_0(stage)*R1(stage));
>>     C2(stage) = 1./(2*omega_0(stage)*Q(stage).*R2(stage));
>>     disp(['Stage ',num2str(stage),...
>>             ' (Q = ',num2str(Q(stage)),...
>>             '):  R1 = R2 = ',num2str(R1(stage))]);
>>     disp(['        C1 = ',num2str(C1(stage)),...
>>             ', C2 = ',num2str(C2(stage))]);
>>     B = omega_0(stage)^2; A = [1 omega_0(stage)/Q omega_0(stage)^2];
>>     Hmag_SK(stage,:) = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>> end
Stage 1 (Q = 22.8704):  R1 = R2 = 39788.7358
        C1 = 4.6343e-008, C2 = 2.215e-011
Stage 2 (Q = 6.8251):  R1 = R2 = 39788.7358
        C1 = 1.6274e-008, C2 = 8.7339e-011
Stage 3 (Q = 3.0798):  R1 = R2 = 39788.7358
        C1 = 1.0874e-008, C2 = 2.8659e-010
Stage 4 (Q = 1.0337):  R1 = R2 = 39788.7358
        C1 = 9.2182e-009, C2 = 2.1569e-009
```

The resulting resistor and capacitor values are realistic.

(b) See the solution to problem 4.M-4b.

(c) MATLAB is used to transform the Chebyshev LPF from 4.M-6a.

```
>> R1p = 1./(C1*omega_c); R2p = 1./(C2*omega_c);
>> C1p = 1./(R1*omega_c); C2p = 1./(R2*omega_c);
>> for stage = 1:N/2,
>>     disp(['Stage ',num2str(stage),...
>>             ' (Q = ',num2str(Q(stage)),...
>>             '):  C1'' = C2'' = ',num2str(C1p(stage))]);
>>     disp(['        R1'' = ',num2str(R1p(stage)),...
>>             ', R2'' = ',num2str(R2p(stage))]);
>> end
Stage 1 (Q = 22.8704):  C1' = C2' = 1e-009
        R1' = 858.5676, R2' = 1796313.2499
Stage 2 (Q = 6.8251):  C1' = C2' = 1e-009
        R1' = 2444.9937, R2' = 455567.9755
Stage 3 (Q = 3.0798):  C1' = C2' = 1e-009
        R1' = 3659.1916, R2' = 138833.4048
Stage 4 (Q = 1.0337):  C1' = C2' = 1e-009
        R1' = 4316.3108, R2' = 18446.8849
```

The resulting resistor and capacitor values possibly realistic; however, there is a fairly large dynamic range between the largest and smallest resistors.

MATLAB also conveniently computes magnitude responses and pole locations. By necessity, the transformation really stretches out the passband; it is therefore important to plot the magnitude response over a broad range of frequencies. To facilitate a reasonable plot, the magnitude response is plotted using both log-magnitude and log-frequency scales. From $H(s)$, it is clear that all zeros are at zero.

```
>> Hmag_SK = zeros(N/2,5001); Poles = zeros(N/2,2);
>> f = logspace(2,5,5001);
>> for stage = 1:N/2,
>>     B = [1 0 0];
>>     A = [1,(1/(R2p(stage)*C2p(stage))+1/(R2p(stage)*C1p(stage))),...
>>          1/(R1p(stage)*R2p(stage)*C1p(stage)*C2p(stage))];
>>     Poles(stage,:) = (roots(A)).';
>>     Hmag_SK(stage,:) = abs(polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f));
>> end
>> subplot(121), plot(real(Poles(:)),imag(Poles(:)),'kx',0,0,'ko');
>> axis equal; ax = axis; axis([1.1*ax]);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> subplot(122),
>> semilogx(f,20*log10(Hmag_SK),'k',f,20*log10(prod(Hmag_SK)),'k:')
>> xlabel('f [Hz]'); ylabel('Magnitude Responses [dB]'); axis tight
>> axis([100 1e5 -40 40]);
```

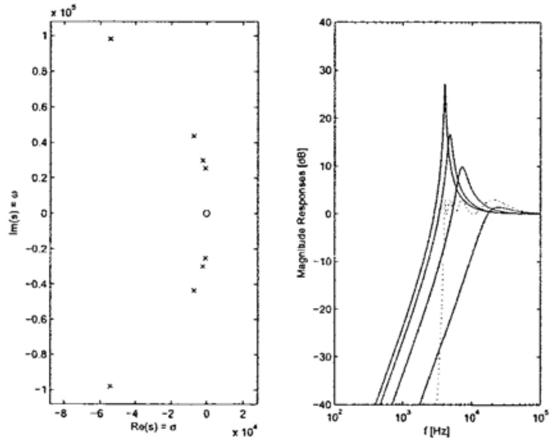


Figure S4.M-6c: Order-8 Chebyshev HPF with $\omega_c = 2\pi4000$ and $R = 3$dB.

The pole locations of the transformed Chebyshev filter are dramatically different than the pole locations of the original LPF. The zeros, as expected, are all concentrated at $s = 0$. The overall magnitude response plot looks like a highpass Chebyshev filter; passband ripples are equal in height and never exceed $R = 3$dB, there are a total of $N = 8$ maxima and minima in the passband, and the cutoff is correctly located at $\omega_c = 2\pi4000$.

4.M-7. Factored form is used to plot roots, and standard transfer function form is used to compute magnitude response plots.

(a) Order-6 Butterworth LPF with $\omega_c = 2\pi3500$.

```
>> omega_c = 2*pi*3500;
>> [z,p,k] = butter(6,omega_c,'s');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
```

```
>> axis(omega_c*[-1.1 0.1 -1.1 1.1]); axis equal;
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = butter(6,omega_c,'s');
>> HLP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HLP)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{LP}(j\omega)|');
```

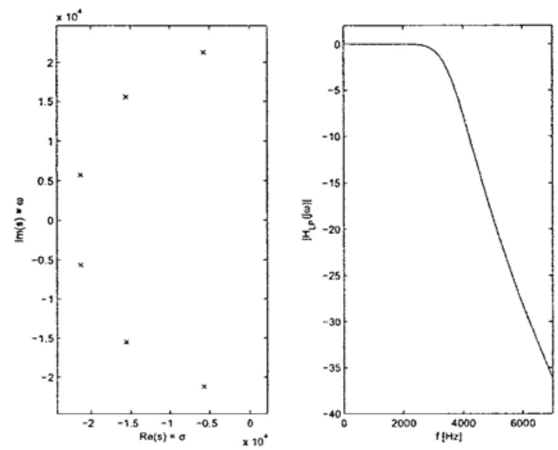


Figure S4.M-7a: Order-6 Butterworth LPF with $\omega_c = 2\pi 3500$.

(b) Order-6 Butterworth HPF with $\omega_c = 2\pi 3500$.

```
>> omega_c = 2*pi*3500;
>> [z,p,k] = butter(6,omega_c,'high','s');
>> subplot(121),plot(real(p),imag(p),'kx',...
     real(z),imag(z),'ko');
>> axis(omega_c*[-1.1 0.1 -1.1 1.1]); axis equal;
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = butter(6,omega_c,'high','s');
>> HHP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HHP)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{HP}(j\omega)|');
```

(c) Order-6 Butterworth BPF with passband between 2kHz and 4kHz. Notice that the command butter requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandpass filter.

```
>> omega_c = [2*pi*2000,2*pi*4000];
>> [z,p,k] = butter(3,omega_c,'s');
>> subplot(121),plot(real(p),imag(p),'kx',...
     real(z),imag(z),'ko');
>> axis(omega_c(2)*[-1.1 0.1 -1.1 1.1]); axis equal;
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = butter(3,omega_c,'s');
```

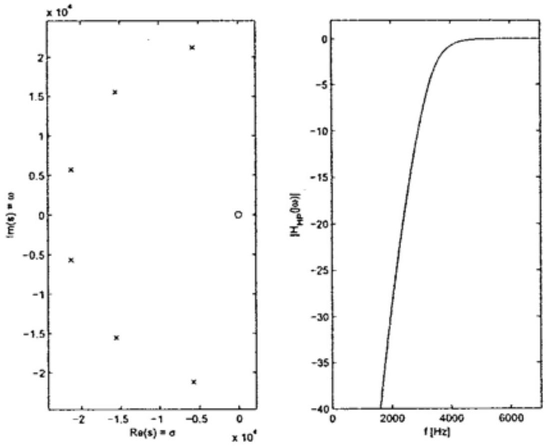


Figure S4.M-7b: Order-6 Butterworth HPF with $\omega_c = 2\pi 3500$.

```
>> HBP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HBP)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{BP}(j\omega)|');
```

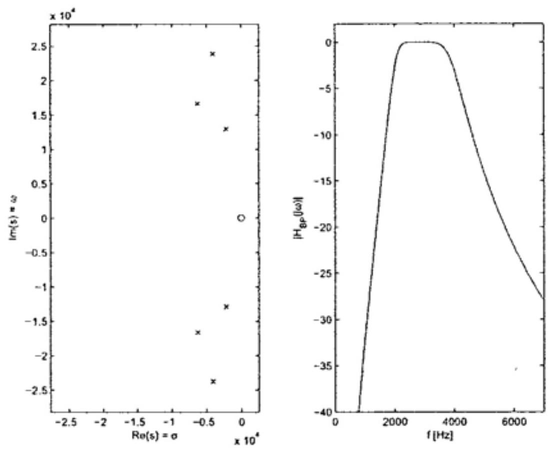


Figure S4.M-7c: Order-6 Butterworth BPF with passband between 2kHz and 4kHz.

(d) Order-6 Butterworth BSF with stopband between 2kHz and 4kHz. Notice that the command butter requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandstop filter.

```
>> omega_c = [2*pi*2000,2*pi*4000];
>> [z,p,k] = butter(3,omega_c,'stop','s');
>> subplot(121),plot(real(p),imag(p),'kx',...
   real(z),imag(z),'ko');
>> axis(omega_c(2)*[-1.1 0.1 -1.1 1.1]); axis equal;
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = butter(3,omega_c,'stop','s');
```

```
>> HBS = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HBS)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{BS}(j\omega)|');
```

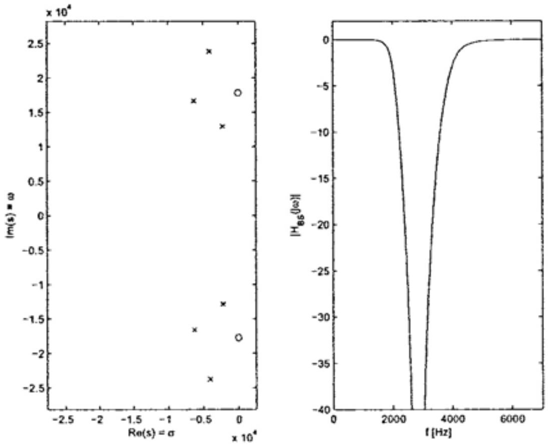


Figure S4.M-7d: Order-6 Butterworth BSF with stopband between 2kHz and 4kHz.

4.M-8. Factored form is used to plot roots, and standard transfer function form is used to compute magnitude response plots.

(a) Order-6 Chebyshev Type I LPF with $\omega_c = 2\pi 3500$.

```
>> omega_c = 2*pi*3500;
>> [z,p,k] = cheby1(6,3,omega_c,'s');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = cheby1(6,3,omega_c,'s');
>> HLP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HLP)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{LP}(j\omega)|');
```

(b) Order-6 Chebyshev Type I HPF with $\omega_c = 2\pi 3500$.

```
>> omega_c = 2*pi*3500;
>> [z,p,k] = cheby1(6,3,omega_c,'high','s');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = cheby1(6,3,omega_c,'high','s');
>> HHP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HHP)),'k');
>> axis([0 7000 -40 2])
```

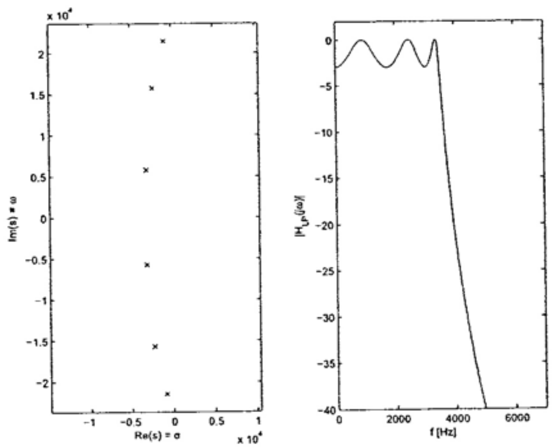


Figure S4.M-8a: Order-6 Chebyshev Type I LPF with $\omega_c = 2\pi 3500$.

```
>> xlabel('f [Hz]'); ylabel('|H_{HP}(j\omega)|');
```

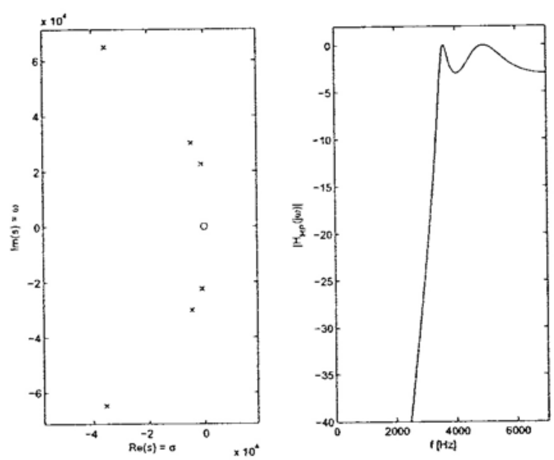


Figure S4.M-8b: Order-6 Chebyshev Type I HPF with $\omega_c = 2\pi 3500$.

(c) Order-6 Chebyshev Type I BPF with passband between 2kHz and 4kHz. Notice that the command `cheby1` requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandpass filter.

```
>> omega_c = [2*pi*2000,2*pi*4000];
>> [z,p,k] = cheby1(3,3,omega_c,'s');
>> subplot(121),plot(real(p),imag(p),'kx',...
     real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = cheby1(3,3,omega_c,'s');
>> HBP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HBP)),'k');
>> axis([0 7000 -40 2])
```

```
>> xlabel('f [Hz]'); ylabel('|H_{BP}(j\omega)|');
```

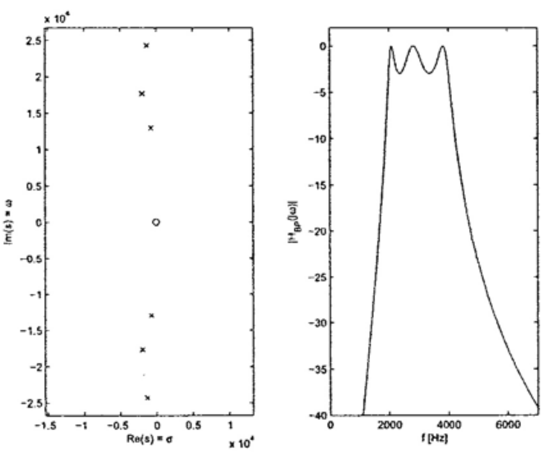


Figure S4.M-8c: Order-6 Chebyshev Type I BPF with passband between 2kHz and 4kHz.

(d) Order-6 Chebyshev Type I BSF with stopband between 2kHz and 4kHz. Notice that the command `cheby1` requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandstop filter.

```
>> omega_c = [2*pi*2000,2*pi*4000];
>> [z,p,k] = cheby1(3,3,omega_c,'stop','s');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = cheby1(3,3,omega_c,'stop','s');
>> HBS = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HBS)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{BS}(j\omega)|');
```

To demonstrate the effect of decreasing the passband ripple, consider magnitude response plots for Chebyshev Type I LPFs with $R_p = \{0.1, 1.0, 3.0\}$.

```
>> omega_c = 2*pi*3500; f = linspace(0,7000,501);
>> [B,A] = cheby1(6,.1,omega_c,'s');
>> HLP1 = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> [B,A] = cheby1(6,1,omega_c,'s');
>> HLP2 = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> [B,A] = cheby1(6,3,omega_c,'s');
>> HLP3 = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> plot(f,20*log10(abs(HLP1)),'k-',...
      f,20*log10(abs(HLP2)),'k--',...
      f,20*log10(abs(HLP3)),'k:');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{LP}(j\omega)|');
>> legend('R_p = 0.1','R_p = 1.0','R_p = 3.0',0);
```

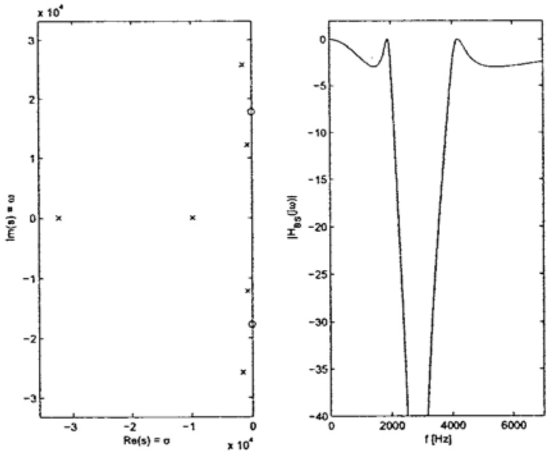


Figure S4.M-8d: Order-6 Chebyshev Type I BSF with stopband between 2kHz and 4kHz.

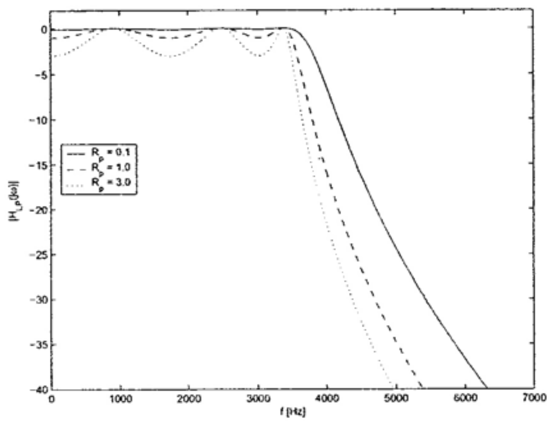


Figure S4.M-8d: Changing R_p for a Chebyshev Type I filter.

Thus, reducing the allowable passband ripple R_p tends to broaden the transition bands of the filter.

4.M-9. Factored form is used to plot roots, and standard transfer function form is used to compute magnitude response plots.

(a) Order-6 Chebyshev Type II LPF with $\omega_c = 2\pi 3500$.

```
>> omega_c = 2*pi*3500;
>> [z,p,k] = cheby2(6,20,omega_c,'s');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = cheby2(6,20,omega_c,'s');
>> HLP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HLP)),'k');
>> axis([0 7000 -40 2])
```

```
>> xlabel('f [Hz]'); ylabel('|H_{LP}(j\omega)|');
```

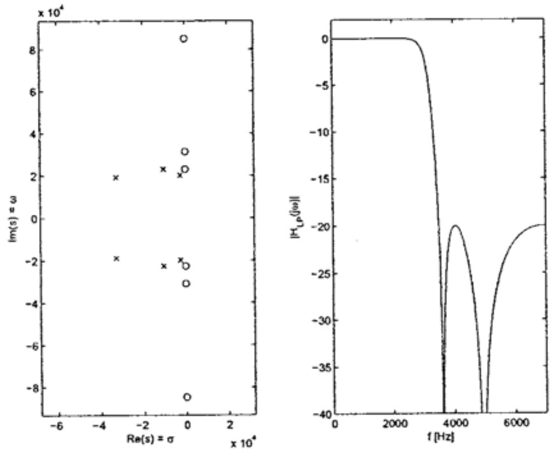


Figure S4.M-9a: Order-6 Chebyshev Type II LPF with $\omega_c = 2\pi 3500$.

(b) Order-6 Chebyshev Type II HPF with $\omega_c = 2\pi 3500$.

```
>> omega_c = 2*pi*3500;
>> [z,p,k] = cheby2(6,20,omega_c,'high','s');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = cheby2(6,20,omega_c,'high','s');
>> HHP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HHP)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{HP}(j\omega)|');
```

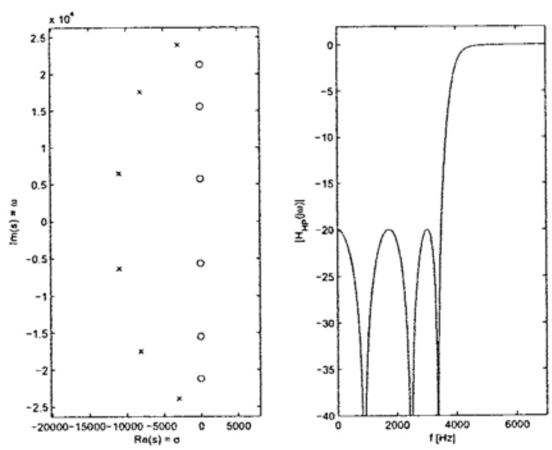


Figure S4.M-9b: Order-6 Chebyshev Type II HPF with $\omega_c = 2\pi 3500$.

(c) Order-6 Chebyshev Type II BPF with passband between 2kHz and 4kHz. Notice that the command cheby2 requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandpass filter.

```
>> omega_c = [2*pi*2000,2*pi*4000];
>> [z,p,k] = cheby2(3,20,omega_c,'s');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = cheby2(3,20,omega_c,'s');
>> HBP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HBP)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{BP}(j\omega)|');
```

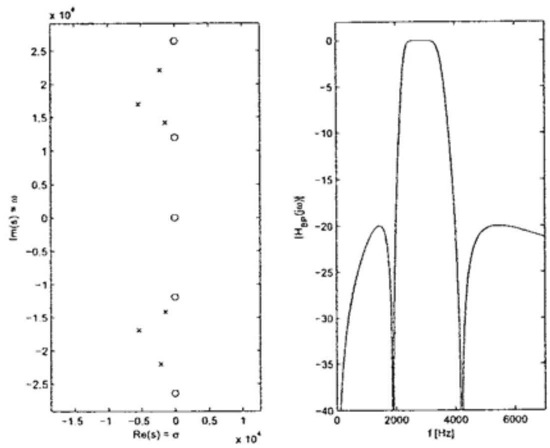


Figure S4.M-9c: Order-6 Chebyshev Type II BPF with passband between 2kHz and 4kHz.

(d) Order-6 Chebyshev Type II BSF with stopband between 2kHz and 4kHz. Notice that the command cheby2 requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandstop filter.

```
>> omega_c = [2*pi*2000,2*pi*4000];
>> [z,p,k] = cheby2(3,20,omega_c,'stop','s');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,501);
>> [B,A] = cheby2(3,20,omega_c,'stop','s');
>> HBS = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HBS)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{BS}(j\omega)|');
```

To demonstrate the effect of increasing R_s, consider magnitude response plots for Chebyshev Type II LPFs with $R_s = \{10, 20, 30\}$.

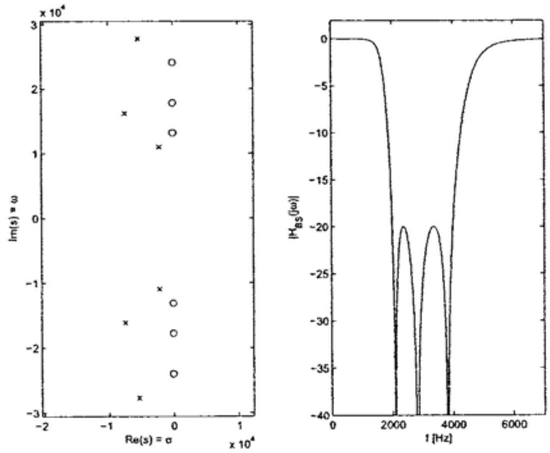


Figure S4.M-9d: Order-6 Chebyshev Type II BSF with stopband between 2kHz and 4kHz.

```
>> omega_c = 2*pi*3500; f = linspace(0,7000,501);
>> [B,A] = cheby2(6,10,omega_c,'s');
>> HLP1 = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> [B,A] = cheby2(6,20,omega_c,'s');
>> HLP2 = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> [B,A] = cheby2(6,30,omega_c,'s');
>> HLP3 = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> plot(f,20*log10(abs(HLP1)),'k-',...
        f,20*log10(abs(HLP2)),'k--',...
        f,20*log10(abs(HLP3)),'k:');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{LP}(j\omega)|');
>> legend('R_s = 10','R_s = 20','R_s = 30',0);
```

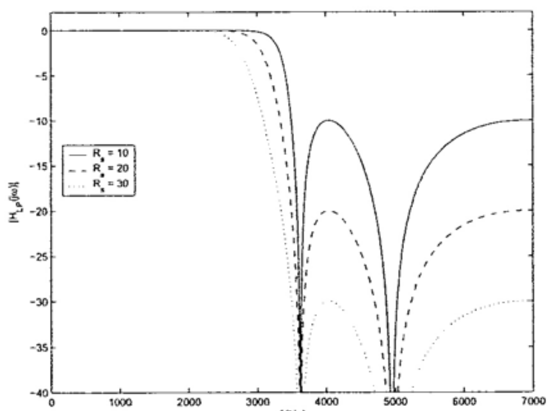


Figure S4.M-9d: Changing R_s for a Chebyshev Type II filter.

Thus, increasing R_s tends to broaden the transition bands of the filter.

4.M-10. Factored form is used to plot roots, and standard transfer function form is used to compute magnitude response plots.

(a) Order-6 Elliptic LPF with $\omega_c = 2\pi 3500$.

```
>> omega_c = 2*pi*3500;
>> [z,p,k] = ellip(6,3,20,omega_c,'s');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,2001);
>> [B,A] = ellip(6,3,20,omega_c,'s');
>> HLP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HLP)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{LP}(j\omega)|');
```

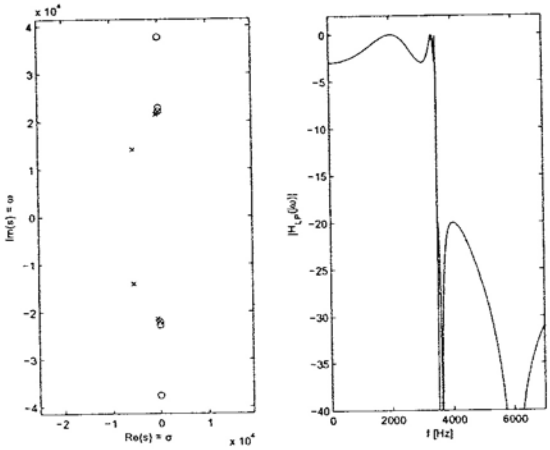


Figure S4.M-10a: Order-6 Elliptic LPF with $\omega_c = 2\pi 3500$.

(b) Order-6 Elliptic HPF with $\omega_c = 2\pi 3500$.

```
>> omega_c = 2*pi*3500;
>> [z,p,k] = ellip(6,3,20,omega_c,'high','s');
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,2001);
>> [B,A] = ellip(6,3,20,omega_c,'high','s');
>> HHP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HHP)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{HP}(j\omega)|');
```

(c) Order-6 Elliptic BPF with passband between 2kHz and 4kHz. Notice that the command `ellip` requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandpass filter.

```
>> omega_c = [2*pi*2000,2*pi*4000];
>> [z,p,k] = ellip(3,3,20,omega_c,'s');
```

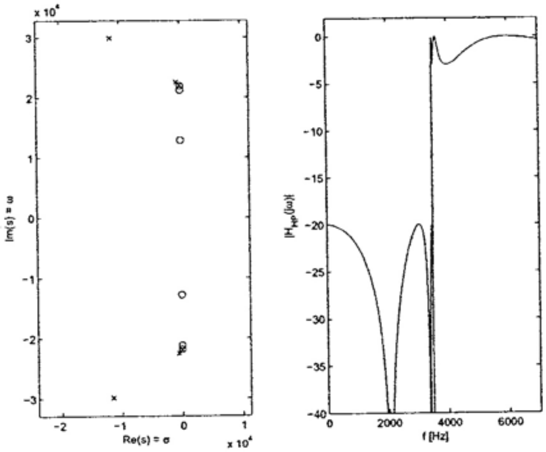


Figure S4.M-10b: Order-6 Elliptic HPF with $\omega_c = 2\pi 3500$.

```
>> subplot(121),plot(real(p),imag(p),'kx',...
   real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,2001);
>> [B,A] = ellip(3,3,20,omega_c,'s');
>> HBP = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HBP)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{BP}(j\omega)|');
```

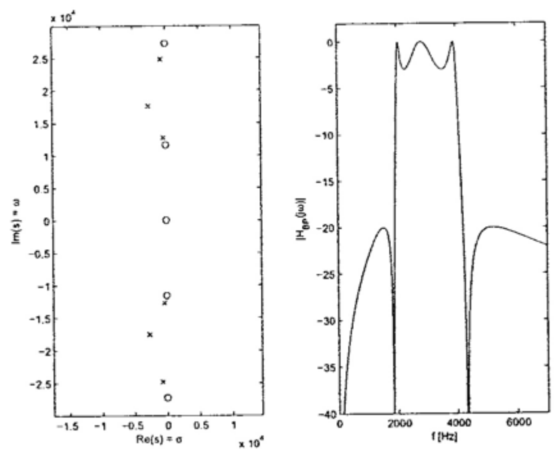


Figure S4.M-10c: Order-6 Elliptic BPF with passband between 2kHz and 4kHz.

(d) Order-6 Elliptic BSF with stopband between 2kHz and 4kHz. Notice that the command `ellip` requires the parameter $N = 3$ to be used to obtain a $(2N = 6)$-order bandstop filter.

```
>> omega_c = [2*pi*2000,2*pi*4000];
>> [z,p,k] = ellip(3,3,20,omega_c,'stop','s');
```

```
>> subplot(121),plot(real(p),imag(p),'kx',...
    real(z),imag(z),'ko');
>> axis equal; axis(1.1*axis);
>> xlabel('Re(s) = \sigma'); ylabel('Im(s) = \omega');
>> f = linspace(0,7000,2001);
>> [B,A] = ellip(3,3,20,omega_c,'stop','s');
>> HBS = polyval(B,j*2*pi*f)./polyval(A,j*2*pi*f);
>> subplot(122),plot(f,20*log10(abs(HBS)),'k');
>> axis([0 7000 -40 2])
>> xlabel('f [Hz]'); ylabel('|H_{BS}(j\omega)|');
```

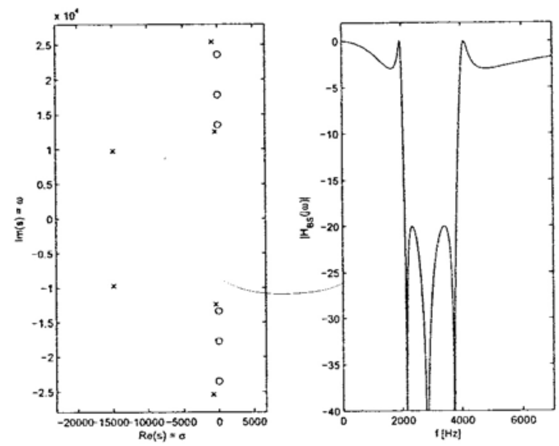


Figure S4.M-10d: Order-6 Elliptic BSF with stopband between 2kHz and 4kHz.

4.M-11. First, the recursion relation $C_N(x) = 2xC_{N-1}(x) - C_{N-2}(x)$ is rewritten as $C_{N+1}(x) = 2xC_N(x) - C_{N-1}(x)$ or $C_{N+1} + C_{N-1} = 2xC_N(x)$.

Letting $\gamma = \cosh^{-1}(x)$ and using Euler's formula, we know $C_N(x) = \cosh\left(N\cosh^{-1}(x)\right) = \cosh\left(N\gamma\right) = \frac{e^{N\gamma}+e^{-N\gamma}}{2}$. Thus, $C_{N+1} + C_{N-1} = \frac{e^{(N+1)\gamma}+e^{-(N+1)\gamma}}{2} + \frac{e^{(N-1)\gamma}+e^{-(N-1)\gamma}}{2} = \frac{e^{N\gamma}(e^{\gamma}+e^{-\gamma})+e^{-N\gamma}(e^{\gamma}+e^{-\gamma})}{2} = 2\frac{(e^{\gamma}+e^{-\gamma})}{2}\frac{(e^{N\gamma}+e^{-N\gamma})}{2} = 2\cosh(\gamma)\cosh(N\gamma)$. Replacing γ yields $C_{N+1} + C_{N-1} = 2\cosh(\cosh^{-1}(x))\cosh(N\cosh^{-1}(x)) = 2xC_N(x)$. Thus,

$$C_{N+1} + C_{N-1} = 2xC_N(x) \text{ or } C_N(x) = 2xC_{N-1}(x) - C_{N-2}(x).$$

4.M-12. Note that $p_k = \sigma_k + j\omega_k = \omega_c\sinh(\xi)\sin(\phi_k) + j\omega_c\cosh(\xi)\cos(\phi_k)$. From the real portion, we know $\sigma_k = \omega_c\sinh(\xi)\sin(\phi_k)$ or $\sin(\phi_k) = \frac{\sigma_k}{\omega_c\sinh(\xi)}$. From the imaginary portion, we know $\omega_k = \omega_c\cosh(\xi)\cos(\phi_k)$ or $\cos(\phi_k) = \frac{\omega_k}{\omega_c\cosh(\xi)}$. From trigonometry, we know $1 = \cos^2(\phi_k) + \sin^2(\phi_k)$. Thus,

$$\left(\frac{\omega_k}{\omega_c\cosh(\xi)}\right)^2 + \left(\frac{\sigma_k}{\omega_c\sinh(\xi)}\right)^2 = 1.$$

This is the equation of an ellipse. Since the Chebyshev poles $p_k = \sigma_k + j\omega_k$ satisfy the equation, they must lie on the ellipse.