Stochastic Frontier Analysis

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Part 1

Aigner, D., C. A. Lovell, P. Schmidt. 1977. Formulation and Estimation of Stochastic Frontier Production Models. *Journal of Econometrics*(6) 21-37.

Introduction

- Prior to ALS (Aigner, Lovell, and Schmidt 1977) and MvdB (Meeusen and Van den Broeck 1977), the estimation of parametric production functions started with the theoretical representation of a production function

$$y_i = f(x_i, \beta)$$

where y_i represents the *maximal* amount of output y_i obtainable from inputs x_i and production technology $f(x_i, \beta)$

- Estimation followed mathematical programming techniques (Aigner and Chu 1968), maximizing

$$\sum_{i=1}^{n}(y_i - f(x_i, \beta))$$

or

$$\sum_{i=1}^{n}(y_i - f(x_i, \beta))^2$$

s.t. $y_i \leq f(x_i, \beta)$.

Raises two questions:

1. How does one explain differences in y_i for identical x_i?
2. What accounts for a firm producing below (or above) the $f(x_i, \beta)$ frontier?

- Answer: *Measurement error*
- But this fails to address the stochastic nature of production, long realized by economists and highlighted by the pioneering theoretical work of Farrell (1957).
- ALS and MvdB sought to operationalize the theoretical framework of Farrell (1957), allowing for the estimation of a stochastic production frontier, where firms could operate below the frontier for two reasons:
  1. Technical inefficiency
  2. Statistical noise (measurement error)
- How is technical inefficiency defined?

Technical Inefficiency
Intuitively, technical inefficiency is the amount by which all inputs can be proportionally reduced without a reduction in output

[Graph]

Stochastic Frontier

With the idea of technical inefficiency in mind, consider the following parametric equation:

$$y_i = f(x_i, \beta) + \varepsilon_i$$

where $\varepsilon_i = v_i - u_i$ for $i = 1, \ldots n$ (firms) and

- v_i is a symmetric error term accounting for statistical noise
- u_i is a non-negative term accounting for technical inefficiency

Each firm's output must lie on or below its frontier, $y_i \leq f(x_i, \beta) + v_i$, which can vary randomly across firms or over time.

Checking for the initial presence of TE

- Observe that if $u_i = 0$, then $\varepsilon_i = v_i$, implying that the error term is symmetric, which does not support the presence of technical inefficiency
- However, if $u_i > 0$, then ε_i should be negatively skewed
- SFA should start with a simple test (Schmidt and Lin 1984) of the presence of TE in the data. Consider the test statistic:

$$(b_1)^{(1/2)} = \frac{m_3}{(m_2)^{(3/2)}}$$

where m_2 and m_3 are the second and third sample moments of the OLS residuals of the previous model. $m_3 < 0$ indicates technical efficiency may be present, and $m_3 > 0$ is a sign that your model may be misspecified.

Checking for the initial presence of TE

As a quick note, the distribution for $(b_1)^{(1/2)}$ is not widely distributed, so it's more common to test the statistic:

$$b^{alt} = \frac{m_3}{(6m_2^3/n)^{1/2}} \sim N(0,1)$$

Estimation

Maximum likelihood is the preferred technique, representing an increase in efficiency over OLS. Of course, that means we require a variety of assumptions about the standard errors:

$$
\begin{aligned}
&E(v_i) = 0 \\
&E(v_i^2) = \sigma_v^2 \\
&E(v_i v_j) = 0 \text{ for all } i \neq j \\
&E(u_i^2) = \text{constant} \\
&E(u_i u_j) = 0 \text{ for all } i \neq j
\end{aligned}
$$

(*"Corrected" OLS (COLS), GMM, and Bayesian methods have been used as well*)

For maximum likelihood, we require parametric assumptions about the two disturbance terms. ALS use a normal distribution for the symmetric disturbance and a half-normal distribution for the technical inefficiency term:

$$v_i \sim^{iid} N(0, \sigma_v^2)$$
$$u_i \sim^{iid} N^+(0, \sigma_u^2)$$

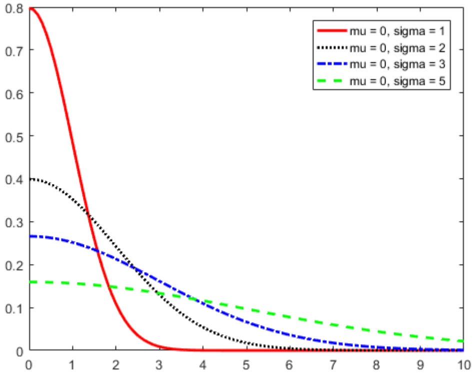
Other popular choices for the inefficiency term are:

1. Truncated normal
2. Exponential
3. Gamma

In practice, half-normal is the default choice, and the remaining distributions are often used as robustness checks

Half-normal density

- Negative values set to zero, postitive values follow the right-half of a normal distribution



Half-normal density

- Note that the parameters μ and σ^2 in the half-normal distribution $N^+(\mu, \sigma^2)$ are *not* the mean and variance!
- The density is given by

$$f(x;\sigma) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

- The mean is

$$E(x) = \frac{\sigma\sqrt{2}}{\sqrt{\pi}}$$

- The variance is

$$V(x) = \sigma^2\left(1 - \frac{2}{\pi}\right)$$

- And the density has support over all $x \in [0, \infty)$

Reparameterization

Reparameterize variance terms by defining $\gamma = \sigma_u^2/\sigma^2$, where $\sigma^2 = \sigma_u^2 + \sigma_v^2$. Benefits:

- Reduces search area of γ, $\{\gamma \in (0,1)\}$
- Easy interpretation: $\gamma \to 1$ implies more of the variation is attributed to inefficiency, and $\gamma \to 0$ implies more of the variation due to statistical noise

Likelihood

With the reparameterization, Battese and Corra (1977) demonstrate that the log-likelihood function can be written:

$$\ln \mathcal{L} = -\frac{n}{2}\ln\left(\frac{\pi}{2}\right) - \frac{n}{2}\ln(\sigma^2) + \sum_{i=1}^{n}\ln(1-\Phi(z_i)) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - x_i\beta)^2$$

where

$$z_i = \frac{y_i - x_i\beta}{\sigma}\sqrt{\frac{\gamma}{1-\gamma}}$$

Recall the rule that the density of a sum of random variables, $f(z)$, where $Z = X + Y$ and $f(x)$ and $g(y)$ are the resp. densities, is given by the convolution

$$(f * g)(z) = \int_{-\infty}^{\infty} f(z-y)g(y)dy$$

Algorithm

Estimation of the stochastic frontier follows a three-step algorithm:

1. Obtain OLS estimates from $y_i = f(x_i, \beta) + v_i$
2. Adjust intercept β_0 and σ^2 for bias, and iterate $\gamma \in (0, 1)$ over the likelihood function to identify a preferred starting value.

$$\hat{\sigma}^2 = \frac{n-k}{n}\left(\frac{\pi}{\pi - 2\gamma}\right)$$

$$\hat{\beta}_0 = \hat{\beta}_0(OLS) + \sqrt{\frac{2\gamma\hat{\sigma}^2}{\pi}}$$

3. Use the values from step 2 as the starting values in a $k+2$ dimensional nonlinear maximization problem.

Firm-Level Technical Efficiency Estimates

Most common output oriented measure of technical efficiency is the ratio of observed output to the corresponding stochastic frontier output (Coelli et al. 2005):

$$TE_i = \frac{q_i}{f(x_i, \beta) + v_i} = \frac{f(x_i, \beta) + v_i - u_i}{f(x_i, \beta) + v_i}$$

When the dependent variable is logged (CD, TL)*, TE_i reduces to the convenient:

$$TE_i = exp(-u_i)$$

*I am unaware of *any* study that does not utilize a logged dependent variable in SFA

Estimator for TE_i

There are several estimators of TE_i based on the previous derivation (c.f. Jondrow et al. 1982). One of the more popular forms was developed by Battese and Coelli (1988), who used the conditional density $p(u_i|q_i)$ to derive

$$\hat{TE}_i = E(\exp(-u_i)|q_i) = \left[\Phi\left(\frac{u_i^*}{\sigma_*} - \sigma_*\right) / \left(\frac{u_i^*}{\sigma_*}\right)\right] \exp\left(\frac{\sigma_*^2}{2} - u_i^*\right)$$

where $u_i^* = -(\ln q_i - x_i\beta)\hat{\sigma}_u^2/\hat{\sigma}^2$, and $\hat{\sigma}_*^2 = \hat{\sigma}_v^2\hat{\sigma}_u^2/\hat{\sigma}^2$. Note that

$$\hat{\sigma}_u^2/\hat{\sigma}^2 = \hat{\gamma}$$
$$\hat{\sigma}_*^2 = \hat{\sigma}^2\hat{\gamma}(1-\hat{\gamma})$$

Part 2

Key, Nigel and Stacy Sneeringer. 2014. Potential Effects of Climate Change on the Productivity of U.S. Dairies. *Journal of Econometrics*(6) 21-37.

Introduction

- The true nature of production is stochastic, especially in agriculture
- The authors suspect that increased instances of drought, higher average temperatures, and hotter daily maximums may be decreasing technical efficiency in livestock operations, particularly dairies
- The authors specify a model wherein technical efficiency *and* a vector of variables suspected to influence technical efficiency (associated with climate) are estimated simultaneously
- Results indicate that a one unit increase in the annual THI (temperature-humidity index) load is associated with a 3.7 percent reduction in output
- The question for us is: how did they figure this out?

Estimation Strategy

- Objective: Estimate the impact of THI load on technical efficiency
- Starting point: ALS (1977)/MvdB(1977)

$$\ln(q_i) = f(x_i, \beta) + v_i - u_i$$

(where $f(x_i, \beta)$ is parameterized as Translog)

- Recall the deterministic frontier is $f(x_i, \beta)$, the stochastic frontier is $f(x_i, \beta) + v_i$, where v_i is a symmetric random shock, and $u_i \geq 0$ represents inefficiency
- With a logged dependent variable, technical efficiency is represented by

$$TE_i = \frac{q_i}{\exp(f(x_i, \beta) + v_i} = \exp(-u_i)$$

which varies between 0 and 1, where $TE_i = 1$ indicates perfect technical efficiency

Estimation

- Assume default normal/half-normal error specification, define $y_i = \ln(q_i)$ and $f(x_i, \beta) = x_i\beta$, parameterize the log-likelihood function as

$$\ln \mathcal{L}(y_i|\beta, \sigma, \lambda) = \sum_{i=1}^{n} \left(\frac{1}{2} \ln \left(\frac{2}{\pi} \right) - \ln \sigma + \ln \Phi(-w_i) - \frac{\varepsilon_i^2}{2\sigma^2} \right)$$

where

$$\begin{aligned} \sigma^2 &= \sigma_u^2 + \sigma_v^2 \\ \lambda &= \sigma_u/\sigma_v \\ \varepsilon_i &= y_i - x_i\beta \\ w_i &= \varepsilon_i\lambda/\sigma \end{aligned}$$

and $\Phi(\bullet)$ is the standard normal cumulative distribution function

Estimation

Key and Sneeringer employ the Jondrow et al. (1982) version of the expectation of u_i conditional on ε_i:

$$E(u_i|\varepsilon_i) = \frac{\sigma\lambda}{1+\lambda^2}\left(\frac{\phi(w_i)}{1-\Phi(w_i)} - w_i\right)$$

- With this estimate of u_i, how does one calculate the impact of a set of exogenous factors on its determination?
- Two-step estimation? Just estimate the u_i's as normal, and then use it as a dependent variable in a second-stage estimation, regressed on factors thought to have influence
- **No**. Results in biased and inefficient estimates (Wang and Schmidt 2002)

Estimation

A more robust alternative to estimate technical efficiency along with the factors that influence it in a single step. To do this:

- Define the variance of the underlying half-normal distribution of u_i, σ_u^2, as a function of observable factors z_u and a set of parameters δ_u:

$$\sigma_{ui}^2 = \exp(z_{ui}\delta_u)$$

- With this formulation, the factors in z_{ui} directly impact the mean and variance of the inefficiency term u_i, and subsequently, the estimate of technical efficiency (still use Jondrow et al. 1982)

Estimation

Note: This formulation increases the dimensionality of the nonlinear maximization problem by the size of the δ_u vector. The likelihood function is now

$$\ln \mathcal{L}(y_i|\beta, \sigma, \lambda, \delta_u) = \sum_{i=1}^{n} \left(\frac{1}{2} \ln \left(\frac{2}{\pi} \right) - \ln \sigma + \ln \Phi(-w_i) - \frac{\varepsilon_i^2}{2\sigma^2} \right)$$

where

$$\sigma^2 = \exp(z_{ui}\delta_u) + \sigma_v^2$$
$$\lambda = \exp(z_{ui}\delta_u)\sigma_v$$
$$\varepsilon_i = y_i - x_i\beta$$
$$w_i = \varepsilon_i\lambda/\sigma$$

What did Key and Sneeringer find?

- Postulated the impact of THI load, operator education, operator age, operator experience, operation size, and a measure of specialization
- THI = (dry bulb temperature in degrees celsius) + (0.36 x dew point temperature) + 41.2.
- THI load is a measure of the duration and extent above this threshold
- Results: THI load has a large, significant impact on technical efficiency in dairy production. Using 2010 estimates, inefficiency loss from heat stress reduces value by approximately \$1.2 billion/year.
- Climate change simulations: Lost production could get much, much worse depending on the climate simulation model used.