STANDARDIZED SCORES AND THE NORMAL DISTRIBUTION

4

Chapter

You will need to use the following from previous chapters:

Symbols
Σ: Summation sign
μ: Population mean
σ: Population standard deviation
$σ^2$: Population variance

Concepts
Percentile ranks
Mathematical distributions
Properties of the mean and standard deviation

A

CONCEPTUAL FOUNDATION

A friend meets you on campus and says, "Congratulate me! I just got a 70 on my physics test." At first, it may be hard to generate much enthusiasm about this grade. You ask, "That's out of 100, right?" and your friend proudly says, "Yes." You may recall that a 70 was not a very good grade in high school, even in physics. But if you know how low exam grades often are in college physics, you might be a bit more impressed. The next question you would probably want to ask your friend is, "What was the average for the class?" Let's suppose your friend says 60. If your friend has long been afraid to take this physics class and expected to do poorly, you should offer congratulations. Scoring 10 points above the mean isn't bad.

On the other hand, if your friend expected to do well in physics and is doing a bit of bragging, you would need more information to know if your friend has something to brag about. Was 70 the highest grade in the class? If not, you need to locate your friend more precisely within the class distribution to know just how impressed you should be. Of course, it is not important in this case to be precise about your level of enthusiasm, but if you were the teacher trying to decide whether your friend should get a B+ or an A−, more precision would be helpful.

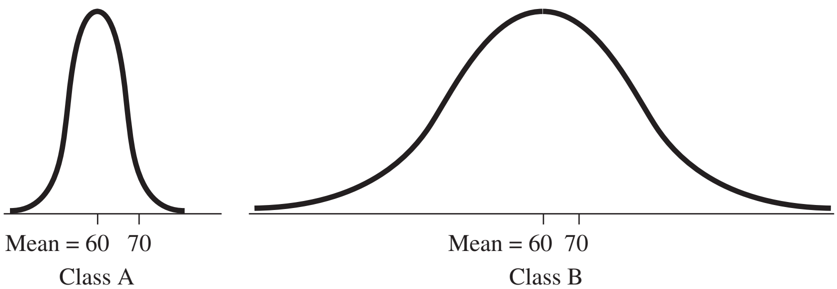
z Scores

To see just how different a score of 70 can be in two different classes, even if both of the classes have a mean of 60, take a look at the two class distributions in Figure 4.1. (To simplify the comparison, I am assuming that the classes are large enough to produce smooth distributions.) As you can see, a score of 70 in class A is excellent, being near the top of the class, whereas the same score is not so impressive in class B, being near the middle of the distribution. The difference between the two class distributions is visually obvious—class B is much more spread out than class A. Having read the previous chapter, you should have an idea of how to quantify this difference in variability. The most useful way to quantify the variability is to calculate the standard deviation (σ). The way the distributions are drawn in Figure 4.1, σ would be about 5 points for class A and about 20 for class B.

An added bonus from calculating σ is that, in conjunction with the mean (μ), σ provides us with an easy and precise way of locating scores in a

Figure 4.1

Distributions of Scores on a Physics Test



distribution. In both classes a score of 70 is 10 points above the mean, but in class A those 10 points represent two standard deviations, whereas 10 points in class B is only half of a standard deviation. Telling someone how many standard deviations your score is above or below the mean is more informative than telling your actual (raw) score. This is the concept behind the *z score*. In any distribution for which μ and σ can be found, any raw score can be expressed as a z score by using Formula 4.1:

$$z = \frac{X - \mu}{\sigma} \qquad \textbf{Formula 4.1}$$

Let us apply this formula to the score of 70 in class A:

$$z = \frac{70 - 60}{5} = \frac{10}{5} = +2$$

and in class B:

$$z = \frac{70 - 60}{20} = \frac{10}{20} = +.5$$

In a compact way, the z scores tell us that your friend's exam score is more impressive if your friend is in class A ($z = +2$) rather than class B ($z = +.5$). Note that the plus sign in these z scores is very important because it tells you that the scores are above rather than below the mean. If your friend had scored a 45 in class B, her z score would have been:

$$z = \frac{45 - 60}{20} = \frac{-15}{20} = -.75$$

The minus sign in this z score informs us that in this case your friend was three quarters of a standard deviation *below* the mean. The *sign* of the z score tells you whether the raw score is above or below the mean; the *magnitude* of the z score tells you the raw score's distance from the mean in terms of standard deviations.

z scores are called standardized scores because they are not associated with any particular unit of measurement. The numerator of the z score is associated with some unit of measurement (e.g., the difference of someone's height from the mean height could be in inches), and the denominator is associated with the same unit of measurement (e.g., the standard deviation for height might be 3 inches), but when you divide the two, the result is dimensionless. A major advantage of standardized scores is that they provide a neutral way to compare raw scores from different distributions. To continue the previous example, suppose that your friend scores a 70 on the physics exam in class B and a 60 on a math exam in a class where μ = 50 and σ = 10. In which class was your friend further above the mean? We have already found that your friend's *z* score for the physics exam in class B was +.5. The *z* score for the math exam would be:

$$z = \frac{60 - 50}{10} = \frac{10}{10} = +1$$

Thus, your friend's *z* score on the math exam is higher than her *z* score on the physics exam, so she seems to be performing better (in terms of class standing on the last exam) in math than in physics.

Finding a Raw Score From a *z* Score

As you will see in the next section, sometimes you want to find the raw score that corresponds to a particular *z* score. As long as you know μ and σ for the distribution, this is easy. You can use Formula 4.1 by filling in the given *z* score and solving for the value of *X*. For example, if you are dealing with class A (as shown in Figure 4.1) and you want to know the raw score for which the *z* score would be −3, you can use Formula 4.1 as follows: $-3 = (X - 60)/5$, so $-15 = X - 60$, so $X = -15 + 60 = 45$. To make the calculation of such problems easier, Formula 4.1 can be rearranged in a new form that I will designate Formula 4.2:

$$X = z\sigma + \mu \qquad \textbf{Formula 4.2}$$

Now if you want to know, for instance, the raw score of someone in class A who obtained a *z* score of −2, you can use Formula 4.2, as follows:

$$X = z\sigma + \mu = -2(5) + 60 = -10 + 60 = 50$$

Note that you must be careful to retain the minus sign on a negative *z* score when working with a formula, or you will come up with the wrong raw score. (In the previous example, $z = +2$ would correspond to a raw score of 70, as compared to a raw score of 50 for $z = -2$.) Some people find negative *z* scores a bit confusing, probably because most measurements in real life (e.g., height, IQ) cannot be negative. It may also be hard to remember that a *z* score of zero is not bad; it is just average (i.e., if $z = 0$, the raw score = μ). Formula 4.2 will come in handy for some of the procedures outlined in Section B. The structure of this formula also bears a strong resemblance to the formula for a confidence interval, for reasons that will be made clear when confidence intervals are defined in Chapter 6.

Sets of *z* Scores

It is interesting to see what happens when you take a group of raw scores (e.g., exam scores for a class) and convert all of them to *z* scores. To keep matters simple, we will work with a set of only four raw scores: 30, 40, 60, and 70. First, we need to find the mean and standard deviation for these numbers. The mean equals (30 + 40 + 60 + 70)/4 = 200/4 = 50. The standard deviation can be found by Formula 3.13B, after first calculating ΣX^2: $30^2 + 40^2 + 60^2 + 70^2 = 900 + 1600 + 3600 + 4900 = 11{,}000$. The standard deviation is found as follows:

$$\sigma = \sqrt{\frac{\sum X^2}{N} - \mu^2} = \sqrt{\frac{11{,}000}{4} - 50^2} = \sqrt{2750 - 2500} = \sqrt{250} = 15.81$$

Each raw score can now be transformed into a z score using Formula 4.1:

$$z = \frac{30 - 50}{15.81} = \frac{-20}{15.81} = -1.265$$
$$z = \frac{40 - 50}{15.81} = \frac{-10}{15.81} = -.6325$$
$$z = \frac{60 - 50}{15.81} = \frac{+10}{15.81} = +.6325$$
$$z = \frac{70 - 50}{15.81} = \frac{+20}{15.81} = +1.265$$

By looking at these four z scores, it is easy to see that they add up to zero, which tells us that the mean of the z scores will also be zero. This is not a coincidence. The mean for any complete set of z scores will be zero. This follows from Property 1 of the mean, as discussed in the previous chapter: If you subtract a constant from every score, the mean is decreased by the same constant. To form z scores, you subtract a constant (namely, μ) from all the scores before dividing. Therefore, this constant must also be subtracted from the mean. But because the constant being subtracted *is* the mean, the new mean is μ (the old mean) minus μ (the constant), or zero.

It is not obvious what the standard deviation of the four z scores will be, but it will be instructive to find out. We will use Formula 3.13B again, substituting z for X:

$$\sigma_z = \sqrt{\frac{\sum z^2}{N} - \mu_z^2}$$

The term that is subtracted is the mean of the z scores squared. But as you have just seen, the mean of the z scores is always zero, so this term drops out. Therefore, $\sigma_z = \sqrt{\Sigma z^2/N}$. The term $\Sigma\, z^2$ equals $(-1.265)^2 + (-.6325)^2 + (.6325)^2 + (1.265)^2 = 1.60 + .40 + .40 + 1.60 = 4.0$. Therefore, σ equals $\sqrt{\Sigma z^2/N} = \sqrt{(4/4)} = \sqrt{1} = 1$. As you have probably guessed, this is also no coincidence. The standard deviation for a complete set of z scores will always be 1. This follows from two of the properties of the standard deviation described in the last chapter. Property 1 implies that subtracting the mean (or any constant) from all the raw scores will not change the standard deviation. Then, according to Property 2, dividing all the scores by a constant will result in the standard deviation being divided by the same constant. The constant used for division when creating z scores *is* the standard deviation, so the new standard deviation is σ (the old standard deviation) divided by σ (the constant divisor), which always equals 1.

Properties of *z* Scores

I have just derived two important properties that apply to any set of z scores: (1) the mean will be zero, and (2) the standard deviation will be 1. Now we must consider an important limitation of z scores. I mentioned that z scores can be useful in comparing scores from two different distributions (in the example discussed earlier in the chapter, your friend performed relatively better in math than in physics). However, the comparison is reasonable only if the two distributions are similar in shape. Consider the distributions for classes D and E, shown in Figure 4.2. In the negatively skewed distribution of class D, a z score of +2 would put you very near the top of the distribution. In class E, however, the positive skewing implies that although there may

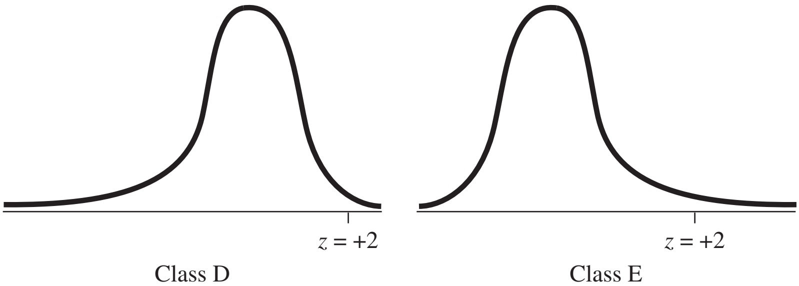


Figure 4.2

Comparing z Scores in Differently Skewed Distributions

not be a large percentage of scores above $z = +2$, there are some scores that are much higher.

Another property of z scores is relevant to the previous discussion. Converting a set of raw scores into z scores will not change the shape of the original distribution. For instance, if all the scores in class E were transformed into z scores, the distribution of z scores would have a mean of zero, a standard deviation of 1, and exactly the same positive skew as the original distribution. We can illustrate this property with a simple example, again involving only four scores: 3, 4, 5, 100. The mean of these scores is 28, and the standard deviation is 41.58 (you should calculate these yourself for practice). Therefore, the corresponding z scores (using Formula 4.1) are −.6, −.58, −.55, and +1.73. First, note the resemblance between the distribution of these four z scores and the distribution of the four raw scores: In both cases there are three numbers close together with a fourth number much higher. You can also see that the z scores add up to zero, which implies that their mean is also zero. Finally, you can calculate $\sigma = \sqrt{\sum z^2/N}$ to see that $\sigma = 1$.

SAT, *T*, and IQ Scores

For descriptive purposes, standardized scores that have a mean of zero and a standard deviation of 1.0 may not be optimal. For one thing, about half the z scores will be negative (even more than half if the distribution has a positive skew), and minus signs can be cumbersome to deal with; leaving off a minus sign by accident can lead to a gross error. For another thing, most of the scores will be between 0 and 2, requiring two places to the right of the decimal point to have a reasonable amount of accuracy. Like minus signs, decimals can be cumbersome to deal with. For these reasons, it can be more desirable to standardize scores so that the mean is 500 and the standard deviation is 100. Because this scale is used by the Educational Testing Service (Princeton, New Jersey) to report the results of the Scholastic Assessment Test, standardized scores with $\mu = 500$ and $\sigma = 100$ are often called *SAT scores*. (Recently, ETS changed the scale it uses to report the results of the Graduate Record Examination from the same one as the SAT to one that has a mean of 150 and an *SD* of 10.)

Probably the easiest way to convert a set of raw scores into SAT scores is to first find the z scores with Formula 4.1 and then use Formula 4.3 to transform each z score into an SAT score:

$$\text{SAT} = 100z + 500 \qquad \textbf{Formula 4.3}$$

Thus a z score of −3 will correspond to an SAT score of $100(-3) + 500 = -300 + 500 = 200$. If $z = +3$, the SAT $= 100(+3) + 500 = 300 + 500 = 800$.

(Notice how important it is to keep track of the sign of the z score.) For any distribution of raw scores that is not extremely skewed, nearly all of the z scores will fall between −3 and +3; this means (as shown previously) that nearly all the SAT scores will be between 200 and 800. There are so few scores that would lead to an SAT score below 200 or above 800 that generally these are the most extreme scores given; thus, we don't have to deal with any negative SAT scores. (Moreover, from a psychological point of view, it must feel better to score 500 or 400 on the SAT than to be presented with a zero or negative z score.) Because z scores are rarely expressed to more than two places beyond the decimal point, multiplying by 100 also ensures that the SAT scores will not require decimal points at all. Less familiar to students, but commonly employed for reporting the results of psychological tests, is the *T score*. The T score is very similar to the SAT score, as you can see from Formula 4.4:

$$T = 10z + 50$$ **Formula 4.4**

A full set of T scores will have a mean of 50 and a standard deviation of 10. If z scores are expressed to only one place past the decimal point, the corresponding T scores will not require decimal points.

The choice of which standardized score to use is usually a matter of convenience and tradition. The current convention regarding intelligence quotient (IQ) scores is to use a formula that creates a mean of 100. The Stanford-Binet test uses the formula $16z + 100$, resulting in a standard deviation of 16, whereas the Wechsler test uses $15z + 100$, resulting in a standard deviation of 15.

The Normal Distribution

It would be nice if all variables measured by psychologists had identically shaped distributions because then the z scores would always fall in the same relative locations, regardless of the variable under study. Although this is unfortunately not the case, it is useful that the distributions for many variables somewhat resemble one or another of the well-known mathematical distributions. Perhaps the best understood distribution with the most convenient mathematical properties is the normal distribution (mentioned in Chapter 2). Actually, you can think of the normal distribution as a family of distributions. There are two ways that members of this family can differ. Two normal distributions can differ either by having different means (e.g., heights of men and heights of women) and/or by having different standard deviations (e.g., heights of adults and IQs of adults). What all normal distributions have in common is the same shape—and not just any bell-like shape, but rather a very precise shape that follows an exact mathematical equation (see Advanced Material at the end of Section B).

Because all normal distributions have the same shape, a particular z score will fall in the same relative location on any normal distribution. Probably the most useful way to define relative location is to state what proportion of the distribution is above (i.e., to the right of) the z score and what proportion is below (to the left of) the z score. For instance, if $z = 0$, .5 of the distribution (i.e., 50%) will be above that z score and .5 will be below it. (Because of the symmetry of the normal distribution, the mean and the median fall at the same location, which is also the mode.) A statistician can find the proportions above and below any z score. In fact, these proportions have been found for all z scores expressed to two decimal places (e.g., 0.63, 2.17, etc.) up to some limit, beyond which the proportion on one side is too small to deal with easily. These proportions have been put into tables of the standard normal distribution, such as Table A.1 in Appendix A of this text.

The Standard Normal Distribution

Tables that give the proportion of the normal distribution below and/or above different z scores are called tables of the *standard normal distribution*; the standard normal distribution is just a normal distribution for which $\mu = 0$ and $\sigma = 1$. It is the distribution you get when you transform all of the scores from any normal distribution into z scores. Of course, you could work out a table for any particular normal distribution. For example, a table for a normal distribution with $\mu = 60$ and $\sigma = 20$ would show that the proportion of scores above 60 is .5, and there would be entries for 61, 62, and so forth. However, it should be obvious that it would be impractical to have a table for every possible normal distribution. Fortunately, it is easy enough to convert scores to z scores (or vice versa) when necessary and use a table of the standard normal distribution (see Table A.1 in the Appendix). It is also unnecessary to include negative z scores in the table. Because of the symmetry of the normal distribution, the proportion of scores above a particular positive z score is the same as the proportion below the corresponding negative z score (e.g., the proportion above $z = +1.5$ equals the proportion below $z = -1.5$; see Figure 4.3).

Suppose you want to know the proportion of the normal distribution that falls between the mean and one standard deviation above the mean (i.e., between $z = 0$ and $z = +1$). This portion of the distribution corresponds to the shaded area in Figure 4.4. Assume that all of the scores in the distribution fall between $z = -3$ and $z = +3$. (The fraction of scores not included in this region of the normal distribution is so tiny that you can ignore it without fear of making a noticeable error.) Thus for the moment, assume that the shaded area plus the cross-hatched areas of Figure 4.4 represent 100% of the distribution, or 1.0 in terms of proportions. The question about proportions can now be translated into areas of the normal distribution. If you knew what proportion of the area of Figure 4.4 is shaded, you would know what proportion of the scores in the entire distribution were between $z = 0$ and $z = +1$. (Recall from Chapter 2 that the size of an "area under the curve" represents a proportion of the scores.) The shaded area looks like it is about one third of the entire distribution, so you can guess that in any normal distribution about one third of the scores will fall between the mean and $z = +1$.

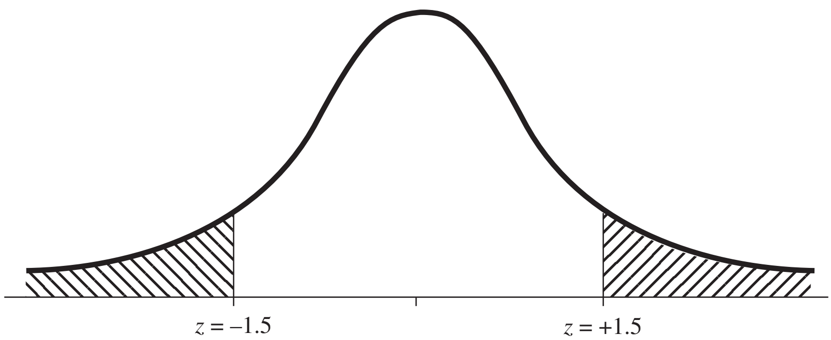


Figure 4.3

Areas Beyond $z = \pm 1.5$

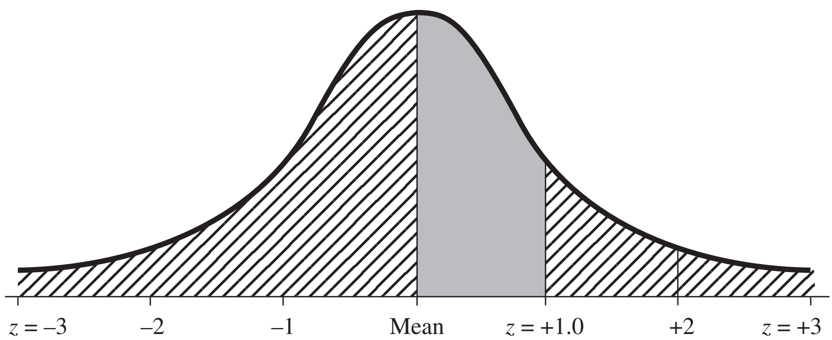


Figure 4.4

Proportion of the Normal Distribution Between the Mean and $z = +1.0$

Table 4.1

z	Mean to z	Beyond z
.98	.3365	.1635
.99	.3389	.1611
1.00	.3413	.1587
1.01	.3438	.1562
1.02	.3461	.1539

Table of the Standard Normal Distribution

Fortunately, you do not have to guess about the relative size of the shaded area in Figure 4.4; Table A.1 can tell you the exact proportion. A small section of that table has been reproduced in Table 4.1. The column labeled "Mean to z" tells you what you need to know. First, go down the column labeled z until you get to 1.00. The column next to it contains the entry .3413, which is the proportion of the normal distribution enclosed between the mean and z when $z = 1.00$. This tells you that the shaded area in Figure 4.4 contains a bit more than one third of the scores in the distribution—it contains .3413. The proportion between the mean and $z = -1.00$ is the same: .3413. Thus about 68% (a little over two thirds) of any normal distribution is within one standard deviation on either side of the mean. Section B will show you how to use all three columns of Table A.1 to solve various practical problems.

Introducing Probability: Smooth Distributions Versus Discrete Events

The main reason that finding areas for different portions of the normal distribution is so important to the psychological researcher is that these areas can be translated into statements about probability. Researchers are often interested in knowing the probability that a totally ineffective treatment can accidentally produce results as promising as the results they have just obtained in their own experiment. The next two chapters will show how the normal distribution can be used to answer such abstract questions about probability rather easily.

Before we can get to that point, it is important to lay out some of the basic rules of probability. These rules can be applied either to discrete events or to a smooth, mathematical distribution. An example of a discrete event is picking one card from a deck of 52 playing cards. Predicting which five cards might be in an ordinary poker hand is a more complex event, but it is composed of simple, discrete events (i.e., the probability of each card). Applying the rules of probability to discrete events is useful in figuring out the likely outcomes of games of chance (e.g., playing cards, rolling dice, etc.) and in dealing with certain types of nonparametric statistics. I will postpone any discussion of discrete probability until Part VII, in which nonparametric statistics are introduced. Until Part VII, I will be dealing with parametric statistics, which are based on measurements that lead to smooth distributions. Therefore, at this point, I will describe the rules of probability only as they apply to smooth, continuous distributions, such as the normal curve.

A good example of a smooth distribution that resembles the normal curve is the distribution of height for adult females in a large population. Finding probabilities involving a continuous variable like height is very different from dealing with discrete events like selecting playing cards. With a deck of cards there are only 52 distinct possibilities. On the other hand, how many different measurements for height are there? It depends on how precisely height is measured. With enough precision, everyone in the population can be determined to have a slightly different height from everyone else. With an infinite population (which is assumed when dealing with the true normal distribution), there are infinitely many different height measurements. Therefore, instead of trying to determine the probability of any particular height being selected from the population, it is only feasible to consider the probability associated with a range of heights (e.g., 60 to 68 inches or 61.5 to 62.5 inches).

Probability as Area Under the Curve

In the context of a continuous variable measured on an infinite population, an "event" can be defined as selecting a value within a particular range of a distribution (e.g., picking an adult female whose height is between 60 and 68 inches). Having defined an event in this way, we can next define the probability that a particular event will occur if we select one adult female at random. The probability of some event can be defined as the proportion of times this event occurs out of an infinite number of random selections from the distribution. This proportion is equal to the area of the distribution under the curve that is enclosed by the range in which the event occurs. This brings us back to finding areas of the distribution. If you want to know the probability that the height of the next adult female selected at random will be between one standard deviation below the mean and one standard deviation above the mean, you must find the proportion of the normal distribution enclosed by $z = -1$ and $z = +1$. I have already pointed out that this proportion is equal to about .68, so the probability is .68 (roughly two chances out of three) that the height of the next woman selected at random will fall in this range. If you wanted to know whether the next randomly selected adult female would be between 60 and 68 inches tall, you would need to convert both of these heights to z scores so that you could use Table A.1 to find the enclosed area according to procedures described in Section B. Probability rules for dealing with combinations of two or more events (e.g., selections from a normal distribution) will be described at the end of Section B.

Real Distributions Versus the Normal Distribution

It is important to remember that this text is dealing with methods of *applied* statistics. We are taking theorems and laws worked out by mathematicians for ideal cases and applying them to situations involving humans or animals or even abstract entities (e.g., hospitals, cities, etc.). To a mathematical statistician, a population has nothing to do with people; it is simply an infinite set of numbers that follow some distribution. Usually some numbers are more popular in the set than others, so the curve of the distribution is higher over those numbers than others (although you can have a uniform distribution, in which all of the numbers are equally popular). These distributions are determined by mathematical equations. On the other hand, the distribution that a psychologist is dealing with (or speculating about) is a set of numbers that is not infinite. The numbers would come from measuring each individual in some very large, but finite, population (e.g., adults in the United States) on some variable of interest (e.g., need for achievement, ability to recall words from a list). Thus, we can be sure that such a population of numbers will not follow some simple mathematical distribution exactly. This leads to some warnings about using Table A.1, which is based on a perfect, theoretical distribution: the normal distribution.

First, we can be sure that even if the human population were infinite, none of the variables studied by psychologists would produce a perfect normal distribution. I can state this with confidence because the normal distribution never ends. No matter what the mean and standard deviation, the normal distribution extends infinitely in both directions. On the other hand, the measurements psychologists deal with have limits. For instance, if a psychophysiologist is studying the resting heart rates of humans, the distribution will have a lowest and a highest value and therefore will differ from a true normal distribution. This means that the proportions found in Table A.1 will not apply exactly to the variables and populations in the problems of Section B. However, for many real-life distributions the deviations from Table A.1 tend to be small, and the approximation involved can be a very useful tool. More importantly, when you group many scores together before finding the distribution, the distribution tends to look like the normal distribution, even if the distribution of individual scores does not. Because experiments are usually done with groups rather than individuals, the normal distribution plays a pervasive role in evaluating the results of experiments. This is the topic I will turn to next. But first, one more warning.

Because z scores are usually used only when a normal distribution can be assumed, some students get the false impression that converting to z scores somehow makes any distribution more like the normal distribution. In fact, as I pointed out earlier, converting to z scores does not change the shape of a distribution at all. Certain transformations *will* change the shape of a distribution (as described in Section C of this chapter), and in some cases will normalize the distribution, but converting to z scores is not one of them. (The z score is a linear transformation, and linear transformations don't change the shape of the distribution. These kinds of transformations will be discussed further in Chapter 9.)

z Scores as a Research Tool

You can use z scores to locate an individual within a normal distribution and to see how likely it is to encounter scores randomly in a particular range. However, of interest to psychological research is the fact that determining how unusual a score is can have more general implications. Suppose you know that heart rate at rest is approximately normally distributed, with a mean of 72 beats per minute (bpm) and a standard deviation of 10 bpm. You also know that a friend of yours, who drinks an unusual amount of coffee every day—five cups—has a resting heart rate of 95 bpm. Naturally, you suspect that the coffee is related to the high heart rate, but then you realize that some people in the ordinary population must have resting heart rates just as high as your friend's. Coffee isn't necessary as an explanation of the high heart rate because there is plenty of variability within the population based on genetic and other factors. Still, it may seem like quite a coincidence that your friend drinks so much coffee *and* has such a high heart rate. How much of a coincidence this really is depends in part on just how unusual your friend's heart rate is. If a fairly large proportion of the population has heart rates as high as your friend's, it would be reasonable to suppose that your friend was just one of the many with high heart rates that have nothing to do with coffee consumption. On the other hand, if a very small segment of the population has heart rates as high as your friend's, you must believe either that your friend happens to be one of those rare individuals who naturally have a high heart rate or that the coffee is elevating his heart rate. The more unusual your friend's heart rate, the harder it is to believe that the coffee is not to blame.

You can use your knowledge of the normal distribution to determine just how unusual your friend's heart rate is. Calculating your friend's z score (Formula 4.1), we find:

$$z = \frac{X - \mu}{\sigma} = \frac{95 - 72}{10} = \frac{23}{10} = 2.3$$

From Table A.1, the area beyond a z score of 2.3 is only about .011, so this is quite an unusual heart rate; only a little more than 1% of the population has heart rates that are as high or higher. The fact that your friend drinks a lot of coffee could be just a coincidence, but it also suggests that there may be a connection between drinking coffee and having a high heart rate (such a finding may not seem terribly shocking or interesting, but what if you found an unusual association between coffee drinking and some serious medical condition?).

The above example suggests an application for z scores in psychological research. However, a researcher would not be interested in finding out whether coffee has raised the heart rate of one particular individual. The more important question is whether coffee raises the heart rates of humans in general. One way to answer this question is to look at a random series of individuals who are heavy coffee drinkers and, in each case, find out how unusually high the heart rate is. Somehow all of these individual probabilities would have to be combined to decide whether these heart rates are just too unusual to believe that the coffee is uninvolved. There is a simpler way to attack the problem. Instead of focusing on one individual at a time, psychological researchers usually look at a group of subjects as a whole. This is certainly not the only way to conduct research, but because of its simplicity and widespread use, the group approach is the basis of statistics in introductory texts, including this one.

Sampling Distribution of the Mean

It is at this point in the text that I will begin to shift the focus from individuals to groups of individuals. Instead of the heart rate of an individual, we can talk about the heart rate of a group. To do so we have to find a single heart rate to characterize an entire group. Chapter 3 showed that the mean, median, and mode are all possible ways to describe the central tendency of a group, but the mean has the most convenient mathematical properties and leads to the simplest statistical procedures. Therefore, for most of this text, the mean will be used to characterize a group; that is, when I want to refer to a group by a single number, I will use the mean.

A researcher who wanted to explore the effects of coffee on resting heart rate might begin by assembling a group of heavy coffee drinkers and find the mean of their heart rates. Then the researcher could see if the group mean was unusual or not. However, to evaluate how unusual a group mean is, you cannot compare the group mean to a distribution of individuals. You need, instead, a distribution of groups (all the same size). This is a more abstract concept than a population distribution that consists of individuals, but it is a critical concept for understanding the statistical procedures in the remainder of this text.

If we know that heart rate has a nearly normal distribution with a mean of 72 and a standard deviation of 10, what can we expect for the mean heart rate of a small group? There is a very concrete way to approach this question. First, you have to decide on the size of the groups you want to deal with—this makes quite a difference, as you will soon see. For our first example, let us say that we are interested in studying groups that have 25 participants each. So we take 25 people at random from the general population and find the mean heart rate for that group. Then we do this again and again, each time recording the mean heart rate. If we do this many times, the mean heart rates will start to pile up into a distribution. As we approach an infinite number of group means, the distribution becomes smooth and continuous. One convenient property of this distribution of means is that it will be a normal distribution, provided that the variable has a normal distribution for the individuals in the population.

Because the groups that we have been hypothetically gathering are supposed to be random samples of the population, the group means are called *sample means* and are symbolized by $\overline{X}$. The distribution of sample means is called a *sampling distribution*. More specifically, it is called the *sampling distribution of the mean*. (Had we been taking the median of each group of 25 and piling up these medians into a distribution, it would be called the sampling distribution of the *median*.) Just as the population distribution gives us a picture of how the individuals are spread out on a particular variable, the sampling distribution shows us how the sample means (or medians or whatever is being used to summarize each sample) would be spread out if we grouped the population into very many samples. To make things simple, I will assume for the moment that we are always dealing with variables that have a normal distribution in the population. Therefore, the sampling distribution of the mean will always be a normal distribution, which implies that we need only know its mean and standard deviation to know everything about it.

First, consider the mean of the sampling distribution of the mean. This term may sound confusing, but it really is very simple. The mean of all the group means will always be the same as the mean of the individuals (i.e., the population mean, μ). It should make sense that if you have very many random samples from a population, there is no reason for the sample means to be more often above or below the population mean. For instance, if you are looking at the average heights for groups of men, why should the average heights of the groups be any different from the average height of individual men? However, finding the standard deviation of the sampling distribution is a more complicated matter. Whereas the standard deviation of the individuals within each sample should be roughly the same as the standard deviation of the individuals within the population as a whole, the standard deviation of the sample means is a very different kind of thing.

Standard Error of the Mean

The means of samples do not vary as much as the individuals in the population. To make this concrete, consider again a very familiar variable: the height of adult men. It is obvious that if you were to pick a man off the street at random, it is somewhat unlikely that the man would be over 6 feet tall, but not very unlikely (in some countries, the chance would be better than .2). On the other hand, imagine selecting a group of 25 men *at random* and finding their average height. The probability that the 25 men would average over 6 feet in height is extremely small. Remember that the group was selected at random. It is not difficult to find 25 men whose average height is over 6 feet tall (you might start at the nearest basketball court), but if the selection is truly random, men below 5 feet 6 inches will be just as likely to be picked as men over 6 feet tall. The larger the group, the smaller the chance that the group mean will be far from the population mean (in this case, about 5 feet 9 inches). Imagine finding the average height of men in each of the 50 states of the United States. Could the average height of men in Wisconsin be much different from the average height of men in Pennsylvania or Alabama? Such extremely large groups will not vary much from each other or from the population mean. That sample means vary less from each other than do individuals is a critical concept for understanding the statistical procedures in most of this book. The concept is critical because we will be judging whether groups are unusual, and the fact that groups vary less than individuals do implies that it takes a smaller deviation for a group to be unusual than for an individual to be unusual. Fortunately, there is a simple formula that can be used to find out just how much groups tend to vary.

Because sample means do not vary as much as individuals, the standard deviation for the sampling distribution will be less than the standard deviation for a population. As the samples get larger, the sample means are clustered more closely, and the standard deviation of the sample means therefore gets smaller. This characteristic can be expressed by a simple formula, but first I will introduce a new term. The standard deviation of the sampling distribution of the mean is called the *standard error of the mean* and is symbolized as $\sigma_{\overline{X}}$. For any particular sampling distribution, all of the samples must be the same size, symbolized by n (for the number of observations in each sample). How many different random samples do you have to select to make a sampling distribution? The question is irrelevant because nobody really creates a sampling distribution this way. The kinds of sampling distributions that I will be discussing are mathematical ideals based on drawing an infinite number of samples all of the same size. (This approach creates a sampling distribution that is analogous to the population distribution, which is based on an infinite number of individuals.)

Now I can show how the standard error of the mean decreases as the size of the samples increases. This relationship is expressed as Formula 4.5:

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} \qquad \textbf{Formula 4.5}$$

To find the standard error, you start with the standard deviation of the population and then divide by the square root of the sample size. This means that for any given sample size, the more the individuals vary, the more the groups will vary (i.e., if σ gets larger, $\sigma_{\overline{X}}$ gets larger). On the other hand, the larger the sample size, the *less* the sample means will vary (i.e., as n increases, $\sigma_{\overline{X}}$ decreases). For example, if you make the sample size 4 times larger, the standard error is cut in half (e.g., σ is divided by 5 for a sample size of 25, but it is divided by 10 if the sample size is increased to 100).

Sampling Distribution Versus Population Distribution

In Figure 4.5, you can see how the sampling distribution of the mean compares with the population distribution for a specific case. We begin with the population distribution for the heights of adult men. It is a nearly normal distribution with $\mu = 69$ inches and $\sigma = 3$ inches. For $n = 9$, the sampling distribution of the mean is also approximately normal. We know this because there is a statistical law that states that if the population

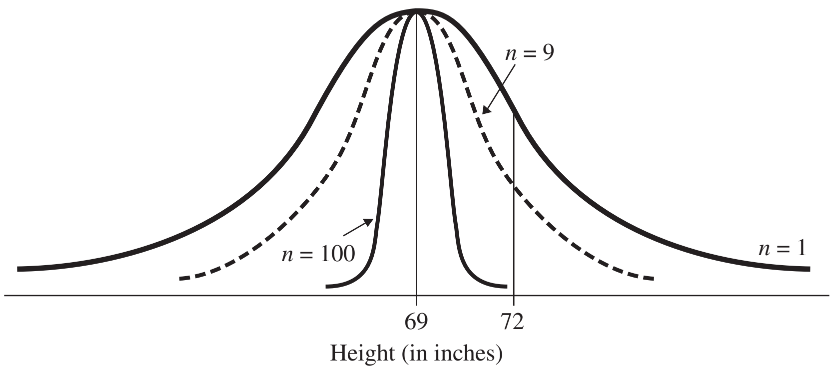


Figure 4.5

Sampling Distributions of the Mean for Different Sample Sizes

distribution is normal, the sampling distribution of the mean will also be normal. Moreover, there is a theorem that states that when the population distribution is not normal, the sampling distribution of the mean will be closer to the normal distribution than the population distribution (I'm referring to the Central Limit Theorem, which will be discussed further in the next chapter). So, if the population distribution is close to normal to begin with (as in the case of height for adults of the same gender), we can be sure that the sampling distribution of the mean for this variable will be very similar to the normal distribution.

Compared to the population distribution, the sampling distribution of the mean will have the same mean but a smaller standard deviation (i.e., standard error). The standard error for height when n is 9 is 1 inch ($\sigma_{\overline{X}} = \sigma/\sqrt{n} = 3/\sqrt{9} = 3/3 = 1$). For $n = 100$, the sampling distribution becomes even narrower; the standard error equals 0.3 inch. Notice that the means of groups tend to vary less from the population mean than do the individuals and that large groups vary less than small groups.

Referring to Figure 4.5, you can see that it is not very unusual to pick a man at random who is about 72 inches, or 6 feet, tall. This is just one standard deviation above the mean, so nearly one in six men is 6 feet tall or taller. On the other hand, to find a group of nine randomly selected men whose average height is over 6 feet is quite unusual; such a group would be three standard errors above the mean. This corresponds to a z score of 3, and the area beyond this z score is only about .0013. And to find a group of 100 randomly selected men who averaged 6 feet or more in height would be extremely rare indeed; the area beyond $z = 10$ is too small to appear in standard tables of the normal distribution. Section B will illustrate the various uses of z scores when dealing with both individuals and groups.

A SUMMARY

1. To localize a score within a distribution or compare scores from different distributions, *standardized scores* can be used. The most common standardized score is the z *score*. The z score expresses a raw score in terms of the mean and standard deviation of the distribution of raw scores. The magnitude of the z score tells you how many standard deviations away from the mean the raw score is, and the sign of the z score tells you whether the raw score is above (+) or below (−) the mean.
2. If you take a set of raw scores and convert each one to a z score, the mean of the z scores will be zero and the standard deviation will be 1. The shape of the distribution of z scores, however, will be exactly the same as the shape of the distribution of raw scores.
3. z scores can be converted to *SAT scores* by multiplying by 100 and then adding 500. SAT scores have the advantages of not requiring minus

signs or decimals to be sufficiently accurate. *T scores* are similar but involve multiplication by 10 and the addition of 50.

4. The *normal distribution* is a symmetrical, bell-shaped mathematical distribution whose shape is precisely determined by an equation (see Advanced Material at the end of Section B). The normal distribution is actually a family of distributions, the members of which differ according to their means and/or standard deviations.
5. If all the scores in a normal distribution are converted to z scores, the resulting distribution of z scores is called the *standard normal distribution*, which is a normal distribution that has a mean of zero and a standard deviation of 1.
6. The proportion of the scores in a normal distribution that falls between a particular z score and the mean is equal to the amount of area under the curve between the mean and z, divided by the total area of the distribution (defined as 1.0). This proportion is the probability that one random selection from the normal distribution will have a value between the mean and that z score. The areas between the mean and z and the areas beyond z (into the tail of the distribution) are given in Table A.1.
7. Distributions based on real variables measured in populations of real subjects (whether people or not) can be similar to, but not exactly the same as, the normal distribution. This is because the true normal distribution extends infinitely in both the negative and positive directions.
8. Just as it is sometimes useful to determine if an individual is unusual with respect to a population, it can also be useful to determine how unusual a group is compared to other groups that could be randomly selected. The group mean (more often called the *sample mean*) is usually used to summarize the group with a single number. To find out how unusual a sample is, the sample mean ($\overline{X}$) must be compared to a distribution of sample means, called, appropriately, the *sampling distribution of the mean*.
9. This sampling distribution could be found by taking very many samples from a population and gathering the sample means into a distribution, but there are statistical laws that tell you just what the sampling distribution of the mean will look like if certain conditions are met. If the population distribution is normal, and the samples are *independent random samples*, all of the same size, the sampling distribution of the mean will be a normal distribution with a mean of μ (the same mean as the population) and a standard deviation called the *standard error of the mean*.
10. The larger the sample size, n, the smaller the standard error of the mean, which is equal to the population standard deviation divided by the square root of n.

EXERCISES

*1. If you convert each score in a set of scores to a z score, which of the following will be true about the resulting set of z scores?
 a. The mean will equal 1.
 b. The variance will equal 1.
 c. The distribution will be normal in shape.
 d. All of the above.
 e. None of the above.

2. The distribution of body weights for adults is somewhat positively skewed—there is much more room for people to be above average than below. If you take the mean

weights for random groups of 10 adults each and form a new distribution, how will this new distribution compare to the distribution of individuals?

a. The new distribution will be more symmetrical than the distribution of individuals.
b. The new distribution will more closely resemble the normal distribution.
c. The new distribution will be narrower (i.e., have a smaller standard deviation) than the distribution of individuals.
d. All of the above.
e. None of the above.

*3. Assume that the mean height for adult women (μ) is 65 inches, and that the standard deviation (σ) is 3 inches.
a. What is the z score for a woman who is exactly 5 feet tall? Who is 5 feet 5 inches tall?
b. What is the z score for a woman who is 70 inches tall? Who is 75 inches tall? Who is 64 inches tall?
c. How tall is a woman whose z score for height is −3? −1.33? −0.3? −2.1?
d. How tall is a woman whose z score for height is +3? +2.33? +1.7? +.9?

4. a. Calculate μ and σ for the following set of scores and then convert each score to a z score: 64, 45, 58, 51, 53, 60, 52, 49.
b. Calculate the mean and standard deviation of these z scores. Did you obtain the values you expected? Explain.

*5. What is the SAT score corresponding to
a. $z = -0.2$?
b. $z = +1.3$?
c. $z = -3.1$?
d. $z = +1.9$?

6. What is the z score that corresponds to an SAT score of
a. 520?
b. 680?
c. 250?
d. 410?

*7. Suppose that the verbal part of the SAT contains 30 questions and that μ = 18 correct responses, with σ = 3. What SAT score corresponds to
a. 15 correct?
b. 10 correct?
c. 20 correct?
d. 27 correct?

8. Suppose the mean for a psychological test is 24 with σ = 6. What is the T score that corresponds to a raw score of
a. 0?
b. 14?
c. 24?
d. 35?

*9. Use Table A.1 to find the area of the normal distribution between the mean and z, when z equals
a. .18
b. .50
c. .88
d. 1.25
e. 2.11

10. Use Table A.1 to find the area of the normal distribution beyond z, when z equals
a. .09
b. .75
c. 1.05
d. 1.96
e. 2.57

11. Assuming that IQ is normally distributed with a mean of 100 and a standard deviation of 15, describe completely the sampling distribution of the mean for a sample size (n) equal to 20.

*12. If the population standard deviation (σ) for some variable equals 17.5, what is the value of the standard error of the mean when
a. $n = 5$?
b. $n = 25$?
c. $n = 125$?
d. $n = 625$?

If the sample size is cut in half, what happens to the standard error of the mean for a particular variable?

13. a. In one college, freshman English classes always contain exactly 20 students. An English teacher wonders how much these classes are likely to vary in terms of their verbal scores on the SAT. What would you expect for the standard deviation (i.e., standard error) of class means on the verbal SAT?
b. Suppose that a crew for the space shuttle consists of seven people, and we are interested in the average weights of all possible shuttle crews. If the standard deviation for weight is 30 pounds, what is the standard deviation for the mean weights of shuttle crews (i.e., the standard error of the mean)?

*14. If for a particular sampling distribution of the mean we know that the standard error is 4.6, and we also know that σ = 32.2, what is the sample size (n)?

B

BASIC STATISTICAL PROCEDURES

As you have seen, z scores can be used for descriptive purposes to locate a score in a distribution. Later in this section, I will show that z scores can also be used to describe groups, although when we are dealing with groups, we usually have some purpose in mind beyond pure description. For now, I want to expand on the descriptive power of z scores when dealing with a population of individuals and some variable that follows the normal distribution in that population. As mentioned in Chapter 2, one of the most informative ways of locating a score in a distribution is by finding the percentile rank (PR) of that score (i.e., the percentage of the distribution that is below that score). To find the PR of a score within a small set of scores, the techniques described in Chapter 2 are appropriate. However, if you want to find the PR of a score with respect to a very large group of scores whose distribution resembles the normal distribution (and you know both the mean and standard deviation of this reference group), you can use the following procedure.

Finding Percentile Ranks

I'll begin with the procedure for finding the PR of a score that is above the mean of a normal distribution. The variable we will use for the examples in this section is the IQ of adults, which has a fairly normal distribution and is usually expressed as a standardized score with $\mu = 100$ and (for the Stanford-Binet test) $\sigma = 16$. To use Table A.1, however, IQ scores will have to be converted back to z scores. I will illustrate this procedure by finding the PR for an IQ score of 116. First find z using Formula 4.1:

$$z = \frac{116 - 100}{16} = \frac{16}{16} = +1.0$$

Next, draw a picture of the normal distribution, always placing a vertical line at the mean ($z = 0$), and at the z score in question ($z = +1$, for this example). The area of interest, as shown by the crosshatching in Figure 4.6, is the portion of the normal distribution to the left of $z = +1.0$. The entire crosshatched area does not appear as an entry in Table A.1 (although some standard normal tables include a column that would correspond to the shaded area). Notice that the crosshatched area is divided in two portions by the mean of the distribution. The area to the left of the mean is always half of the normal distribution and therefore corresponds to a proportion of .5. The area between the mean and $z = +1.0$ can be found in Table A.1 (under "Mean to z"), as demonstrated in Section A. This proportion is .3413. Adding .5 to .3413, we get .8413, which is the proportion represented by the

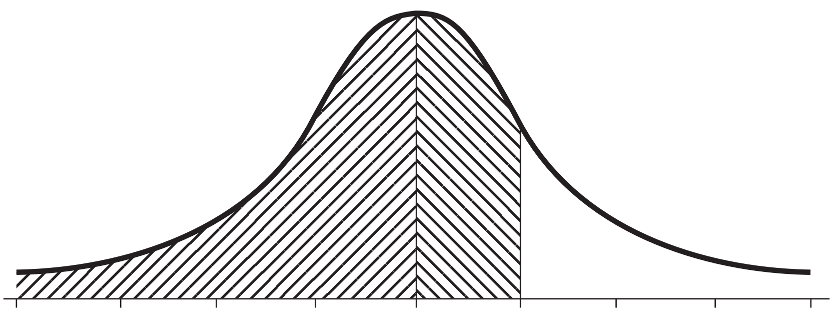
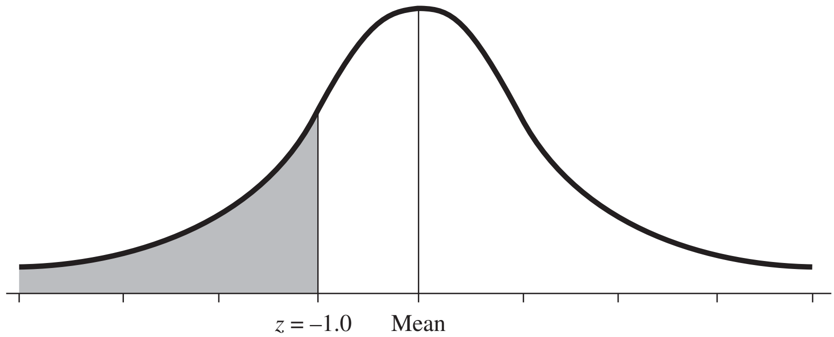


Figure 4.6

Percentile Rank: Area Below $z = +1.0$

Figure 4.7

Area Beyond $z = -1.0$



crosshatched area in Figure 4.6. To convert a proportion to a percentage, we need only multiply by 100. Thus the proportion .8413 corresponds to a PR of 84.13. Now we know that 84.13% of the population have IQ scores lower than 116. I emphasize the importance of drawing a picture of the normal distribution to solve these problems. In the problem above, it would have been easy to forget the .5 area to the left of the mean without a picture to refer to.

It is even easier to find the PR of a score below the mean if you use the correct column of Table A.1. Suppose you want to find the PR for an IQ of 84. Begin by finding z:

$$z = \frac{84 - 100}{16} = \frac{-16}{16} = -1.0$$

Next, draw a picture and shade the area to the left of $z = -1.0$ (see Figure 4.7). Unlike the previous problem, the shaded area this time consists of only one section, which *does* correspond to an entry in Table A.1. First, you must temporarily ignore the minus sign of the z score and find 1.00 in the first column of Table A.1. Then look at the corresponding entry in the column labeled “Beyond z,” which is .1587. This is the proportion represented by the shaded area in Figure 4.7 (i.e., the area to the left of $z = -1.0$). The PR of 84 = .1587 × 100 = 15.87; only about 16% of the population have IQ scores less than 84.

The area referred to as Beyond z (in the third column of Table A.1) is the area that begins at z and extends *away* from the mean in the direction of the closest tail. In Figure 4.7, the area between the mean and $z = -1.0$ is .3413 (the same as between the mean and $z = +1.0$), and the area beyond $z = -1$ is .1587. Notice that these two areas add up to .5000. In fact, for any particular z score, the entries for Mean to z and Beyond z will add up to .5000. You can see why by looking at Figure 4.7. The z score divides one half of the distribution into two sections; together those two sections add up to half the distribution, which equals .5.

Finding the Area Between Two z Scores

Now we are ready to tackle more complex problems involving two different z scores. I’ll start with two z scores on opposite sides of the mean (i.e., one z is positive and the other is negative). Suppose you have devised a teaching technique that is not accessible to someone with an IQ below 76 and would be too boring for someone with an IQ over 132. To find the proportion of the population for whom your technique would be appropriate, you must first find the two z scores and locate them in a drawing.

$$z = \frac{76 - 100}{16} = \frac{-24}{16} = -1.5$$
$$z = \frac{132 - 100}{16} = \frac{32}{16} = +2.0$$

From Figure 4.8 you can see that you must find two areas of the normal distribution, both of which can be found under the column "Mean to z." For $z = -1.5$ you ignore the minus sign and find that the area from the mean to z is .4332. The corresponding area for $z = +2.0$ is .4772. Adding these two areas together gives a total proportion of .9104. Your teaching technique would be appropriate for 91.04% of the population.

Finding the area enclosed between two z scores becomes a bit trickier when both of the z scores are on the same side of the mean (i.e., both are positive or both are negative). Suppose that you have designed a remedial teaching program that is only appropriate for those whose IQs are below 80 but would be useless for someone with an IQ below 68. As in the problem above, you can find the proportion of people for whom your remedial program is appropriate by first finding the two z scores and locating them in your drawing.

$$z = \frac{80 - 100}{16} = \frac{-20}{16} = -1.25$$
$$z = \frac{68 - 100}{16} = \frac{-32}{16} = -2.0$$

The shaded area in Figure 4.9 is the proportion you are looking for, but it does not correspond to any entry in Table A.1. The trick is to notice that if you take the area from $z = -2$ to the mean and remove the section from $z = -1.25$ to the mean, you are left with the shaded area. (You could also find the area beyond $z = -1.25$ and then remove the area beyond $z = -2.0$.) The area between $z = 2$ and the mean was found in the previous problem to be .4772. From this we subtract the area between $z = 1.25$ and the mean, which is .3944. The proportion we want is .4772 − .3944 = .0828. Thus the remedial teaching program is suitable for use with 8.28% of the population. Note that you cannot subtract the two z scores and then find an area corresponding to the difference of the two z scores; z scores just don't work that way (e.g., the area between z scores of 1 and 2 is much larger than the area between z scores of 2 and 3).

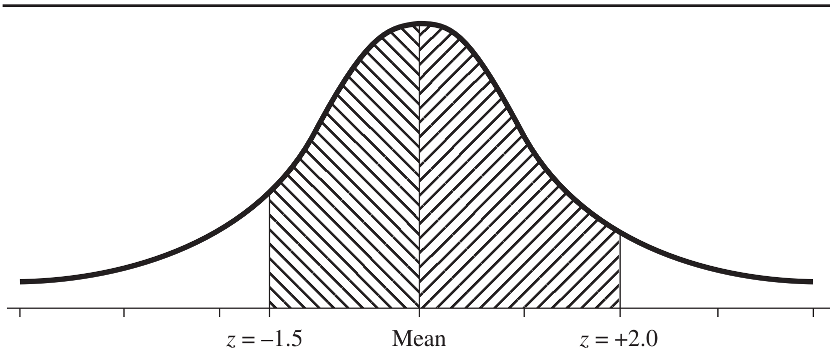
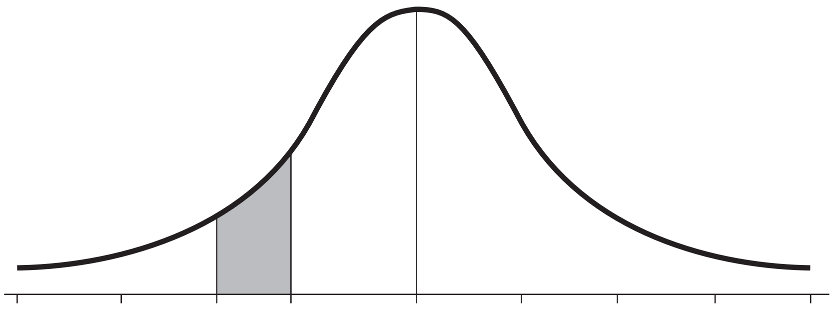


Figure 4.8

The Area Between Two z Scores on Opposite Sides of the Mean

Figure 4.9

The Area Between Two z Scores on the Same Side of the Mean



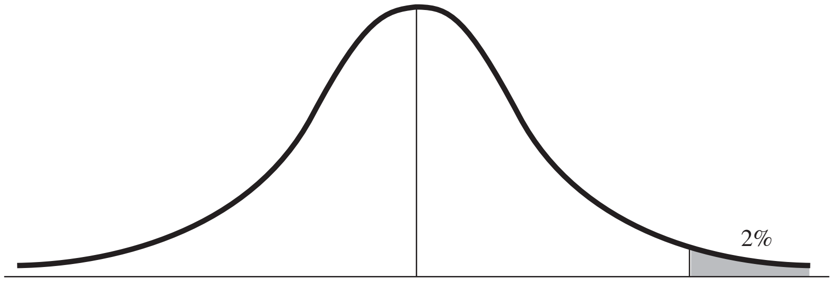
Finding the Raw Scores Corresponding to a Given Area

Often a problem involving the normal distribution will be presented in terms of a given proportion, and it is necessary to find the range of raw scores that represents that proportion. For instance, a national organization called MENSA is a club for people with high IQs. Only people in the top 2% of the IQ distribution are allowed to join. If you were interested in joining and you knew your own IQ, you would want to know the minimum IQ score required for membership. Using the IQ distribution from the problems above, this is an easy question to answer (even if you are not qualified for MENSA). However, because you are starting with an area and trying to find a raw score, the procedure is reversed. You begin by drawing a picture of the distribution and shading in the area of interest, as in Figure 4.10. (Notice that the score that cuts off the upper 2% is also the score that lands at the 98th percentile; that is, you are looking for the score whose PR is 98.) Given a particular area (2% corresponds to a proportion of .0200), you cannot find the corresponding IQ score directly, but you can find the z score using Table A.1. Instead of looking down the z column, you look for the area of interest (in this case, .0200) first and then see which z score corresponds to it. From Figure 4.10, it should be clear that the shaded area is the *area beyond* some as yet unknown z score, so you look in the "Beyond z" column for .0200. You will not be able to find this exact entry, as is often the case, so look at the closest entry, which is .0202. The z score corresponding to this entry is 2.05, so $z = +2.05$ is the z score that cuts off (about) the top 2% of the distribution. To find the raw score that corresponds to $z = +2.05$, you can use Formula 4.2:

$$X = z\sigma + \mu = +2.05(16) + 100 = 32.8 + 100 = 132.8$$

Figure 4.10

Score Cutting Off the Top 2% of the Normal Distribution



Rounding off, you get an IQ of 133—so if your IQ is 133 or above, you are eligible to join MENSA.

Areas in the Middle of a Distribution

One of the most important types of problems involving normal distributions is to locate a given proportion in the middle of a distribution. Imagine an organization called MEZZA, which is designed for people in the middle range of intelligence. In particular, this organization will only accept those in the middle 80% of the distribution—those in the upper or lower 10% are not eligible. What is the range of IQ scores within which your IQ must fall if you are to be eligible to join MEZZA? The appropriate drawing is shown in Figure 4.11. From the drawing you can see that you must look for .1000 in the column labeled "Beyond z." The closest entry is .1003, which corresponds to $z = 1.28$. Therefore, $z = +1.28$ cuts off (about) the upper 10%, and $z = -1.28$ the lower 10% of the distribution. Finally, both of these z scores must be transformed into raw scores, using Formula 4.2:

$$X = -1.28(16) + 100 = -20.48 + 100 = 79.52$$
$$X = +1.28(16) + 100 = +20.48 + 100 = 120.48$$

Thus (rounding off) the range of IQ scores that contain the middle 80% of the distribution extends from 80 to 120.

From Score to Proportion and Proportion to Score

The above procedures relate raw scores to areas under the curve, and vice versa, by using z scores as the intermediate step, as follows:

Raw score $\leftrightarrow$ (Formula) $\leftrightarrow$ z score $\leftrightarrow$ (Table A.1) $\leftrightarrow$ Area

When you are given a raw score to start with, and you are looking for a proportion or percentage, you move from left to right in the preceding diagram. A raw score can be converted to a z score using Formula 4.1. Then an area (or proportion) can be associated with that z score by looking down the appropriate column of Table A.1. Drawing a picture will make it clear which column is needed. When given a proportion or percentage to start with, you move from right to left. First, use Table A.1 backwards (look up the area in the appropriate column to find the z score), and then use Formula 4.2 to transform the z score into a corresponding raw score.

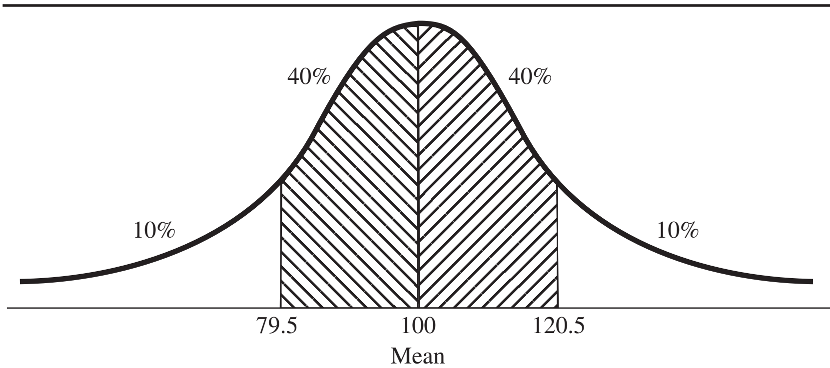


Figure 4.11

Scores Enclosing the Middle 80% of the Normal Distribution

Describing Groups

You can use z scores to find the location of one group with respect to all other groups of the same size, but to do that you must work with the sampling distribution of the mean. The z score has to be modified only slightly for this purpose, as you will see. As an example of a situation in which you may want to compare groups, imagine the following. Suppose there is a university that encourages women's basketball by assigning all of the female students to one basketball team or another at random. Assume that each team has exactly the required five players. Imagine that a particular woman wants to know the probability that the team she is assigned to will have an average height of 67 inches or more. I will show how the sampling distribution of the mean can be used to answer that question.

We begin by assuming that the heights of women at this large university form a normal distribution with a mean of 65 inches and a standard deviation of 3 inches. Next, we need to know what the distribution would look like if it were composed of the means from an infinite number of basketball teams, each with five players. In other words, we need to find the sampling distribution of the mean for $n = 5$. First, we can say that the sampling distribution will be a normal one because we are assuming that the population distribution is (nearly) normal. Given that the sampling distribution is normal, we need only specify its mean and standard deviation.

The z Score for Groups

The mean of the sampling distribution is the same as the mean of the population, that is, μ. For this example, $\mu = 65$ inches. The standard deviation of the sampling distribution of the mean, called the standard error of the mean, is given by Formula 4.5. For this example the standard error, $\sigma_{\overline{X}}$, equals:

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{3}{\sqrt{5}} = \frac{3}{2.24} = 1.34$$

So, the sampling distribution of the mean in this case is an approximately normal distribution with a mean of 65 and a standard deviation (i.e., standard error) of 1.34. Now that we know the parameters of the distribution of the means of groups of five (e.g., the basketball teams), we are prepared to answer questions about any particular group, such as the team that includes the inquisitive woman in our example. Because the sampling distribution is normal, we can use the standard normal table to determine, for example, the probability of a particular team having an average height greater than 67 inches. However, we first have to convert the particular group mean of interest to a z score—in particular, a z score with respect to the sampling distribution of the mean, or more informally, a z score for groups. The z score for groups closely resembles the z score for individuals and is given by Formula 4.6:

$$z = \frac{\overline{X} - \mu}{\sigma_{\overline{X}}}$$ **Formula 4.6**

in which $\sigma_{\overline{X}}$ is a value found using Formula 4.5.

To show the parallel structures of the z score for individuals and the z score for groups, Formula 4.1 for the individual z score follows:

$$z = \frac{X - \mu}{\sigma}$$

Comparing Formula 4.1 with Formula 4.6, you can see that in both cases we start with the score of a particular individual (or mean of a particular sample), subtract the mean of those scores (or sample means), and then divide by the standard deviation of those scores (or sample means). Note that we could put a subscript X on the σ in Formula 4.1 to make it clear that that formula is dealing with individual scores, but unless we are dealing with more than one variable at a time (as in Chapter 9), it is common to leave off the subscript for the sake of simplicity.

In the present example, if we want to find the probability that a randomly selected basketball team will have an average height over 67 inches, it is necessary to convert 67 inches to a z score for groups, as follows:

$$z = \frac{67 - 65}{1.34} = 1.49$$

The final step is to find the area beyond $z = 1.49$ in Table A.1; this area is approximately .068. As Figure 4.12 shows, most of the basketball teams have mean heights that are less than 67 inches; an area of .068 corresponds to fewer than 7 chances out of 100 (or about 1 out of 15) that the woman in our example will be on a team whose average height is at least 67 inches.

Using the z score for groups, you can answer a variety of questions about how common or unusual a particular group is. For the present example, because the standard error is 1.34 and the mean is 65, we know immediately that a little more than two thirds of the basketball teams will average between 63.66 inches (i.e., 65 − 1.34) and 66.34 inches (i.e., 65 + 1.34) in height. Teams with average heights in this range would be considered fairly common, whereas teams averaging more than 67 inches or less than 63 inches in height would be relatively uncommon.

The most common application for determining the probability of selecting a random sample whose mean is unusually small or large is to test a research hypothesis, as you will see in the next chapter. For instance, you could gather a group of heavy coffee drinkers, find the average heart rate

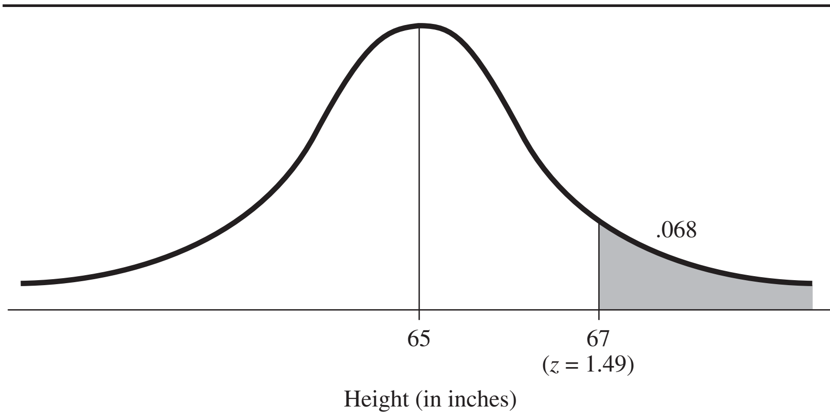


Figure 4.12

Area of the Sampling Distribution Above a z Score for Groups for that group, and use the preceding procedures to determine how unusual it would be to find a random group (the same size) with an average heart rate just as high. The more unusual the heart rate of the coffee-drinking group turns out to be, the more inclined you would be to suspect a link between coffee consumption and heart rate. (Remember, however, that the observation that heavy coffee drinkers do indeed have higher heart rates does not imply that drinking coffee *causes* an increase in heart rate; there are alternative explanations that can only be ruled out by a *true* experiment, in which the experimenter decides at random who will be drinking coffee and who will not.) However, the area that we look up beyond a particular z score does not translate very well to a statement about probability, unless we can make a few important assumptions about how the sample was selected, and about the population it was selected from. I will discuss these assumptions in the next chapter, in the context of drawing inferential conclusions from z scores. In the meantime, I will conclude this section by explaining some important rules of probability.

Probability Rules

As I pointed out in Section A, statements about areas under the normal curve can be translated directly to statements about probability. For instance, if you select one person at random, the probability that that person will have an IQ between 76 and 132 is about .91, because that is the amount of area enclosed between those two IQ scores, as we found earlier (see Figure 4.8). To give you a more complete understanding of probability and its relation to problems involving distributions, I will lay out some specific rules. To represent the probability of an event symbolically, I will write $p(\text{A})$, where A stands for some event. For example, $p(\text{IQ} > 110)$ stands for the probability of selecting someone with an IQ greater than 110.

Rule 1

Probabilities range from 0 (the event is certain *not* to occur) to 1 (the event is *certain* to occur) or from 0 to 100 if probability is expressed as a percentage instead of a proportion. As an example of $p = 0$, consider the case of adult height. The distribution ends somewhere around $z = -15$ on the low end and $z = +15$ on the high end. So for height, the probability of selecting someone for whom z is greater than +20 (or less than −20) is truly zero. An example of $p = 1$ is the probability that a person's height will be between $z = -20$ and $z = +20$ [i.e, $p(-20 < z < +20) = 1$].

Rule 2: The Addition Rule

If two events are *mutually exclusive*, the probability that either one event *or* the other will occur is equal to the sum of the two individual probabilities. Stated as Formula 4.7, the addition rule for mutually exclusive events is:

$$p(\text{A or B}) = p(\text{A}) + p(\text{B})$$ **Formula 4.7**

Two events are mutually exclusive if the occurrence of one rules out the occurrence of the other. For instance, if we select one individual from the IQ distribution, this person cannot have an IQ that is both above 120.5 and also below 79.5—these are mutually exclusive events. As I demonstrated in the discussion of the hypothetical MEZZA organization, the probability of each of these events is .10. We can now ask: What is the probability that a randomly selected individual will have an IQ above 120.5 *or* below 79.5? Using Formula 4.7, we simply add the two individual probabilities: .1 + .1 = .2. In terms of a single distribution, two mutually exclusive events are represented by two areas under the curve that do *not* overlap. (In contrast, the area from $z = -1$ to $z = +1$ and the area above $z = 0$ are *not* mutually exclusive because they *do* overlap.) If the areas do not overlap, we can simply add the two areas to find the probability that an event will be in one area *or* the other. The addition rule can be extended easily to any number of events, if all of the events are mutually exclusive (i.e., no event overlaps with any other). For a set of mutually exclusive events the probability that one of them will occur, p(A or B or C, etc.), is the sum of the probabilities for each event, that is, $p(\text{A}) + p(\text{B}) + p(\text{C})$, and so on.

The Addition Rule for Overlapping Events

The addition rule must be modified if events are not mutually exclusive. If there is some overlap between two events, the overlap must be subtracted after the two probabilities have been added. Stated as Formula 4.8, the addition rule for two events that are *not* mutually exclusive is:

$$p(\text{A or B}) = p(\text{A}) + p(\text{B}) - p(\text{A and B}) \qquad \textbf{Formula 4.8}$$

where p(A and B) represents the overlap (the region where A and B are both true simultaneously). For example, what is the probability that a single selection from the normal distribution will be either within one standard deviation of the mean *or* above the mean? The probability of the first event is the area between $z = -1$ and $z = +1$, which is about .68. The probability of the second event is the area above $z = 0$, which is .5. Adding these we get .68 + .5 = 1.18, which is more than 1.0 and therefore impossible. However, as you can see from Figure 4.13, these events are not mutually exclusive; the area of overlap corresponds to the interval from $z = 0$ to $z = +1$. The area of overlap, that is, p(A and B), equals about .34, and because it is actually being added in twice (once for each event), it must be subtracted once from the total. Using Formula 4.8, we find that p(A or B) = .68 + .5 − .34 = 1.18 − .34 = .84 (rounded off).

A Note About Exhaustive Events

Besides being mutually exclusive, two events can also be *exhaustive* if one or the other *must* occur (together they exhaust all the possible events). For

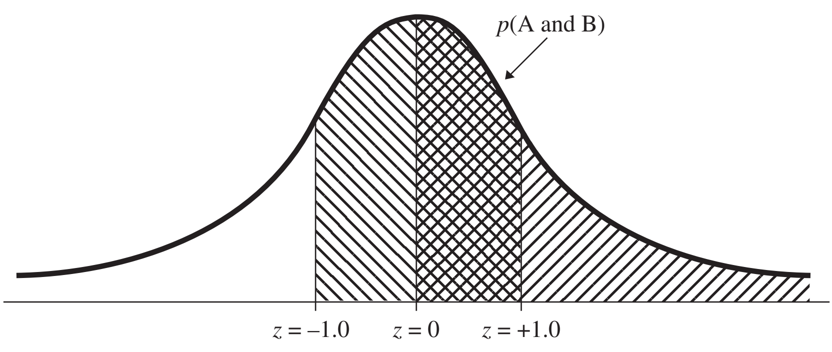


Figure 4.13

The Area Corresponding to Two Overlapping Events instance, consider the event of being above the mean and the event of being below the mean; these two events are not only mutually exclusive, they are exhaustive as well. The same is true of being within one standard deviation from the mean and being at least one standard deviation away from the mean. When two events are both mutually exclusive and exhaustive, one event is considered the *complement* of the other, and the probabilities of the two events must add up to 1.0. If the events A and B are mutually exclusive and exhaustive, we can state that $p(\text{B}) = 1.0 - p(\text{A})$.

Just as two events can be mutually exclusive but not exhaustive (e.g., $z > +1.0$ and $z < -1.0$), two events can be exhaustive without being mutually exclusive. For example, the two events $z > -1.0$ and $z < +1.0$ are exhaustive (there is no location in the normal distribution that is not covered by one event or the other), but they are not mutually exclusive; the area of overlap is shown in Figure 4.14. Therefore, the two areas represented by these events will not add up to 1.0, but rather somewhat more than 1.0.

Rule 3: The Multiplication Rule

If two events are *independent*, the probability that both will occur (i.e., A *and* B) is equal to the two individual probabilities multiplied together. Stated as Formula 4.9, the multiplication rule for independent events is:

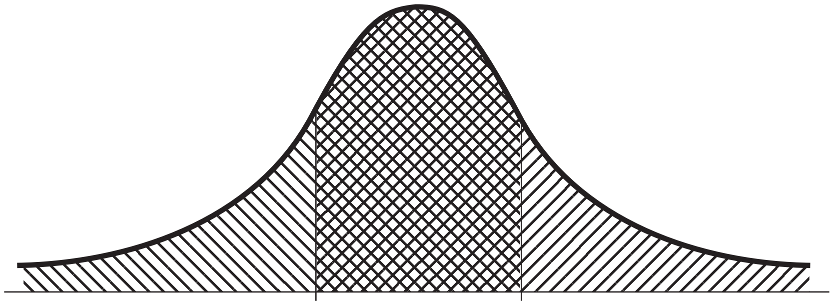
$$p(\text{A and B}) = p(\text{A})p(\text{B}) \qquad \textbf{Formula 4.9}$$

Two events are said to be independent if the occurrence of one in no way affects the probability of the other. The most common example of independent events is two flips of a coin. As long as the first flip does not damage or change the coin in some way—and it's hard to imagine how flipping a coin could change it—the second flip will have the same probability of coming up heads as the first flip. (If the coin is unbiased, $p(\text{H}) = p(\text{T}) = .5$.) Even if you have flipped a fair coin and have gotten 10 heads in a row, the coin will not be altered by the flipping; the chance of getting a head on the eleventh flip is still .5. It may seem that after 10 heads, a tail would become more likely than usual, so as to even out the total number of heads and tails. This belief is a version of the *gambler's fallacy*; in reality, the coin has no memory—it doesn't keep track of the previous 10 heads in a row. The multiplication rule can be extended easily to any number of events that are all independent of each other ($p(\text{A and B and C, etc.}) = p(\text{A})p(\text{B})p(\text{C})$, etc.).

Consider two independent selections from the IQ distribution. What is the probability that if we choose two people at random, their IQs will both be within one standard deviation of the mean? In this case, the probability

Figure 4.14

The Area Corresponding to Two Exhaustive, but Not Mutually Exclusive, Events



of both individual events is the same, about .68 (assuming we replace the first person in the pool of possible choices before selecting the second; see the next paragraph). Formula 4.9 tells us that the probability of both events occurring jointly, that is, p(A and B), equals (.68)(.68) = .46. When the two events are *not* independent, the multiplication rule must be modified. If the probability of an event changes because of the occurrence of another event, we are dealing with a *conditional probability*.

Conditional Probability

A common example of events that are *not* independent are those that involve successive samplings from a finite population *without replacement*. Let us take the simplest possible case: A bag contains three marbles; two are white and one is black. If you grab marbles from the bag without looking, what is the probability of picking two white marbles in a row? The answer depends on whether you select marbles *with replacement* or *without replacement*. In selecting with replacement, you take out a marble, look at it, and then replace it in the bag before picking a second marble. In this case, the two selections are independent; the probability of picking a white marble is the same for both selections: 2/3. The multiplication rule tells us that when the two events are independent (e.g., sampling *with* replacement), we can multiply their probabilities. Therefore, the probability of picking two white marbles in a row with replacement is (2/3)(2/3) = 4/9, or about .44.

On the other hand, if you are sampling *without* replacement, the two events will *not* be independent because the first selection will alter the probabilities for the second. The probability of selecting a white marble on the first pick is still 2/3, but if the white marble is *not* replaced in the bag, the probability of selecting a white marble on the second pick is only 1/2 (there is one white marble and one black marble left in the bag). Thus the conditional probability of selecting a white marble, *given that* a white marble has already been selected and not replaced, that is, $p(\mathrm{W} \mid \mathrm{W})$, is 1/2 (the vertical bar between the Ws represents the word "given"). To find the probability of selecting two white marbles in a row when not sampling with replacement, we need to use Formula 4.10 (the multiplication rule for dependent events):

$$p(\text{A and B}) = p(\text{A})p(\text{B}|\text{A}) \qquad \textbf{Formula 4.10}$$

In this case, both A and B can be symbolized by W (picking a white marble): $p(\mathrm{W})p(\mathrm{W} \mid \mathrm{W})$ = (2/3) (1/2) = 1/3, or .33 (less than the probability of picking two white marbles when sampling with replacement). The larger the population, the less difference it will make whether you sample with replacement or not. (With an infinite population, the difference is infinitesimally small.) For the remainder of this text I will assume that the population from which a sample is taken is so large that sampling without replacement will not change the probabilities enough to have any practical consequences. Conditional probability will have a large role to play, however, in the logical structure of null hypothesis testing, as described in the next chapter.

B SUMMARY

1. If a variable is normally distributed and you know both the mean and standard deviation of the population, it is easy to find the proportion of the distribution that falls above or below any raw score or between any two raw scores. Conversely, for a given proportion at the top, bottom, or middle of the distribution, you can find the raw score or scores that form the boundary of that proportion.

2. It is easier to work with Table A.1 if you begin by drawing a picture of the normal distribution and then draw a vertical line in the middle and one that corresponds roughly to the z score or area with which you are working.
3. To find the proportion below a given raw score first convert the raw score to a z score with Formula 4.1. If the z score is negative, the area below (i.e., to the left of) the z score is the area in the "Beyond z" column. If the z score is positive, the area below the z score is the area in the "Mean to z" column *plus* .5. Multiply the proportion by 100 to get the percentile rank for that score. To find the proportion above a particular raw score, you can alter the procedure just described appropriately or find the proportion below the score and subtract from 1.0.
4. To find the proportion between two raw scores,
   a. If the corresponding z scores are opposite in sign, you must find two areas—the area between the mean and z for each of the two z scores—and add them.
   b. If both z scores are on the same side of the mean, you can find the area between the mean and z for each and then *subtract* the smaller from the larger area (alternatively, you can find the area beyond z for both and subtract). Reminder: You cannot subtract the two z scores first and then look for the area.
5. To find the raw score corresponding to a given proportion, first draw the picture and shade in the appropriate area. If the area is the top *X%*, convert to a proportion by dividing by 100. Then find that proportion (or the closest one to it) in the "Beyond z" column. The corresponding z score can then be converted to a raw score using Formula 4.2. If the area is the bottom *X%* (i.e., left side of distribution), the procedure is the same, but when you are using Formula 4.2 don't forget that the z score has a negative sign.
6. To find the raw scores enclosing the middle *X%* of the distribution, first subtract *X%* from 100% and then divide by 2 to find the percentage in each tail of the distribution. Convert to a proportion by dividing by 100 and then look for this proportion in the "Beyond z" column of Table A.1 (e.g., if you are trying to locate the middle 90%, you would be looking for a proportion of .05). Transform the corresponding z score to a raw score to find the upper (i.e., right) boundary of the enclosed area. Then, put a minus sign in front of the same z score and transform it to a raw score again to find the lower boundary.
7. If you are working with the mean of a sample instead of an individual score, and you want to know the proportion of random samples of the same size that would have a larger (or smaller) mean, you have to convert your sample mean to a z score for groups. Subtract the population mean from your sample mean, and then divide by the standard error of the mean, which is the population standard deviation divided by the square root of the sample size. The area above or below this z score can be found from Table A.1 just as though the z score were from an individual score in a normal distribution.
8. To use Table A.1 for a z score from a sample mean, you must assume that the sampling distribution of the mean is a normal distribution.
9. The Rules of Probability

   Rule 1: The probability of an event, for example, $p(\text{A})$, is usually expressed as a proportion, in which case $p(\text{A})$ can range from zero (the event A is certain *not* to occur) to 1.0 (the event A *is* certain to occur).

   Rule 2: The addition rule for mutually exclusive events states that if the occurrence of event A precludes the occurrence of event B, the probability of either A *or* B occurring, p(A or B), equals p(A) + p(B). The addition rule must be modified as follows if the events are *not* mutually exclusive: p(A or B) = p(A) + p(B) − p(A and B). Also, if two events are both mutually exclusive and *exhaustive* (one of the two events must occur), p(A) + p(B) = 1.0, and therefore, p(B) = 1.0 − p(A).

Rule 3: The multiplication rule for independent events states that the probability that two independent events will both occur, p(A and B), equals p(A)p(B). Two events are *not* independent if the occurrence of one event changes the probability of the other. The probability of one event, given that another has occurred, is called a *conditional probability*.

10. The probability of two *dependent* events both occurring is given by a modified multiplication rule: The probability of one event is multiplied by the conditional probability of the other event, *given that the first* event has occurred. When you are sampling from a finite population without replacement, successive selections will not be independent. However, if the population is very large, *sampling without replacement* is barely distinguishable from *sampling with replacement* (any individual in the population has an exceedingly tiny probability of being selected twice), so successive selections can be considered independent even without replacement.

Advanced Material: The Mathematics of the Normal Distribution

The true normal curve is determined by a mathematical equation, just as a straight line or a perfect circle is determined by an equation. The equation for the normal curve is a mathematical function into which you can insert any X value (usually represented on the horizontal axis) to find one corresponding Y value (usually plotted along the vertical axis). Because Y, the height of the curve, is a function of X, Y can be symbolized as $f(X)$. The equation for the normal curve can be stated as follows:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

where π is a familiar mathematical constant, and so is e (e = 2.7183, approximately). The symbols μ and σ^2 are called the *parameters* of the normal distribution, and they stand for the ordinary mean and variance. These two parameters determine which normal distribution is being graphed.

The preceding equation is a fairly complex one, but it can be simplified by expressing it in terms of z scores. This gives us the equation for the standard normal distribution and shows the intimate connection between the normal distribution and z scores:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

Because the variance of the standard normal distribution equals 1.0, $2\pi\sigma^2 = 2\pi$, and the power that e is raised to is just $-\frac{1}{2}$ times the z score squared. The fact that z is being squared tells us that the curve is symmetric around zero. For instance, the height for $z = -2$ is the same as the height for $z = +2$ because in both cases, $z^2 = +4$. Because the exponent of e has a minus sign, the function is largest (i.e., the curve is highest) when z is smallest, namely, zero. Thus, the mode of the normal curve (along with the mean and median) occurs in the center, at $z = 0$. Note that the height of the curve is never zero; that is, the curve never touches the X axis. Instead, the curve extends infinitely in both directions, always getting closer to the X axis. In mathematical terms, the X axis is the *asymptote* of the function, and the function touches the asymptote only at infinity.

The height of the curve ($f(X)$, or Y) is called the density of the function, so the preceding equation is often referred to as a *probability density function*. In more concrete terms, the height of the curve can be thought of as representing the relative likelihood of each X value; the higher the curve, the more likely is the X value at that point. However, as I pointed out in Section A, the probability of any *exact* value occurring is infinitely small, so when one talks about the probability of some X value being selected, it is common to talk in terms of a range of X values (i.e., an interval along the horizontal axis). The probability that the next random selection will come from that interval is equal to the proportion of the total distribution that is contained in that interval. This is the area under the curve corresponding to the given interval, and this area can be found mathematically by *integrating* the function over the interval using the calculus. Conveniently, the areas between the mean and various z scores have already been calculated and entered into tables, such as Table A.1. Thanks to modern software, these areas are also easily obtained with great accuracy by statistical calculators on the web, or from statistical packages, like SPSS.

EXERCISES

*1. Suppose that a large Introduction to Psychology class has taken a midterm exam, and the scores are normally distributed (approximately) with $\mu = 75$ and $\sigma = 9$. What is the percentile rank (PR) for a student
   a. Who scores 90?
   b. Who scores 70?
   c. Who scores 60?
   d. Who scores 94?

2. Find the area between
   a. $z = -0.5$ and $z = +1.0$
   b. $z = -1.5$ and $z = +0.75$
   c. $z = +0.75$ and $z = +1.5$
   d. $z = -0.5$ and $z = -1.5$

*3. Assume that the resting heart rate in humans is normally distributed with $\mu = 72$ bpm (i.e., beats per minute) and $\sigma = 8$ bpm.
   a. What proportion of the population has resting heart rates above 82 bpm? Above 70 bpm?
   b. What proportion of the population has resting heart rates below 75 bpm? Below 50 bpm?
   c. What proportion of the population has resting heart rates between 80 and 85 bpm? Between 60 and 70 bpm? Between 55 and 75 bpm?

4. Refer again to the population of heart rates described in Exercise 3:
   a. Above what heart rate do you find the upper 25% of the people? (That is, what heart rate is at the 75th percentile, or third quartile?)
   b. Below what heart rate do you find the lowest 15% of the people? (That is, what heart rate is at the 15th percentile?)
   c. Between which two heart rates do you find the middle 75% of the people?

*5. A new preparation course for the math SAT is open to those who have already taken the test once and scored in the middle 90% of the population. In what range must a test-taker's previous score have fallen for the test-taker to be eligible for the new course?

6. A teacher thinks her class has an unusually high IQ, because her 36 students have an average IQ ($\overline{X}$) of 108. If the population mean is 100 and $\sigma = 15$,
   a. What is the z score for this class?
   b. What percentage of classes ($n = 36$, randomly selected) would be even higher on IQ?

*7. An aerobics instructor thinks that his class has an unusually low resting heart rate.

If $\mu = 72$ bpm and $\sigma = 8$ bpm, and his class of 14 pupils has a mean heart rate ($\overline{X}$) of 66,

a. What is the z score for the aerobics class?
b. What is the probability of randomly selecting a group of 14 people with a mean resting heart rate lower than the mean for the aerobics class?

8. Imagine that a test for spatial ability produces scores that are normally distributed in the population with $\mu = 60$ and $\sigma = 20$.
   a. Between which two scores will you find the middle 80% of the people?
   b. Considering the means of groups, all of which have 25 participants, between what two scores will the middle 80% of these means be?

*9. Suppose that the average person sleeps 8 hours each night and that $\sigma = .7$ hour.
   a. If a group of 50 joggers is found to sleep an average of 7.6 hours per night, what is the z score for this group?
   b. If a group of 200 joggers also has a mean of 7.6, what is the z score for this larger group?
   c. Comparing your answers to parts a and b, can you determine the mathematical relation between sample size and z (when $\overline{X}$ remains constant)?

10. Referring to the information in Exercise 7, if an aerobics class had a mean heart rate ($\overline{X}$) of 62, and this resulted in a group z score of -7.1, how large must the class have been?

*11. Suppose that the mean height for a group of 40 women who had been breastfed for at least the first 6 months of life was 66.8 inches.
   a. If $\mu = 65.5$ inches and $\sigma = 2.6$ inches, what is the z score for this group?
   b. If height had been measured in centimeters, what would the z score be? (*Hint*: Multiply $\overline{X}$, μ, and σ by 2.54 to convert inches to centimeters.)
   c. Comparing your answers to parts a and b, what can you say about the effect on z scores of changing units? Can you explain the significance of this principle?

12. Suppose that the mean weight of adults (μ) is 150 pounds with $\sigma = 30$ pounds. Consider the mean weights of all possible space shuttle crews ($n = 7$). If the space shuttle cannot carry a crew that weighs more than a total of 1190 pounds, what is the probability that a randomly selected crew will be too heavy? (Assume that the sampling distribution of the mean would be approximately normal.)

*13. Consider a normally distributed population of resting heart rates with $\mu = 72$ bpm and $\sigma = 8$ bpm:
   a. What is the probability of randomly selecting someone whose heart rate is either below 58 or above 82 bpm?
   b. What is the probability of randomly selecting someone whose heart rate is either between 67 and 75 bpm, above 80 bpm, or below 60 bpm?
   c. What is the probability of randomly selecting someone whose heart rate is either between 66 and 77 bpm or above 74 bpm?

14. Refer again to the population of heart rates described in the previous exercise:
   a. What is the probability of randomly selecting two people in a row whose resting heart rates are both above 78 bpm?
   b. What is the probability of randomly selecting *three* people in a row whose resting heart rates are all below 68 bpm?
   c. What is the probability of randomly selecting two people, one of whom has a resting heart rate below 70 bpm, while the other has a resting heart rate above 76 bpm?

*15. What is the probability of selecting each of the following at random from the population (assume $\sigma = 16$):
   a. One person whose IQ is either above 110 or below 95?
   b. One person whose IQ is either between 95 and 110 or above 105?
   c. Two people with IQs above 90?
   d. One person with an IQ below 90 and one person with an IQ above 115?

16. An ordinary deck of playing cards consists of 52 different cards, 13 in each of four suits (hearts, diamonds, clubs, and spades).
   a. What is the probability of randomly drawing two hearts in a row if you replace the first card before picking the second?
   b. What is the probability of randomly drawing two hearts in a row if you draw *without* replacement?
   c. What is the probability of randomly drawing one heart and then one spade in two picks *without* replacement?

C ANALYSIS BY SPSS

After SPSS has calculated the mean and standard deviation (*SD*) for a variable in your spreadsheet, you could find the *z* scores for that variable by using **Transform/Compute Variable**. For instance, if you are creating *z* scores for *mathquiz*, you would type the following in the *Numeric Expression* space of the Compute Variable box: **(mathquiz—xx.xx) / yy.yy**, where xx.xx and yy.yy are the values SPSS gave you for the mean and *SD* of *mathquiz*, respectively. It would make sense to name the target variable something like *z_mathquiz*. Note that the standard deviation SPSS gives you is the unbiased one, so your *z* scores will be a bit different from what you would get by using the biased *SD*. Fortunately, this difference is only noticeable when dealing with small samples. A bigger concern is retaining enough digits beyond the decimal point for both the mean and *SD* you use in the Compute box. If your numbers are all less than 1.0, for instance, the two decimals shown in the preceding example would not give you much accuracy. You can avoid this issue by asking SPSS to compute *z* scores for any of your variables automatically using the following four steps.

Creating *z* Scores

1. Select **Descriptive Statistics** from the **Analyze** menu, and then click on **Descriptives** . . .
2. Under the list of variables that appears on the left side of the **Descriptives** dialog box, check the little box that precedes the phrase "Save standardized values as variables."
3. Move over the variables for which you would like to see *z* scores.
4. Click on the **Options** button if you want to see any statistics other than those that are selected by default. When back to the original dialog box, click **OK**.

Two very different things will happen as the result of following the above procedure. First, you will get the usual Descriptives output for your chosen variables (plus any additional statistics you checked in the Options box). Second, for each of those variables, SPSS will have added a new variable at the end of your spreadsheet, containing the *z* scores for that variable. SPSS names the new *z*-score variables by just putting the letter "z" at the beginning of the original name, so, for example, *mathquiz* becomes *zmathquiz*. Note again that these *z* scores will be based on the unbiased standard deviation of your variable. Conveniently, the *un*biased standard deviation of these *z* scores will be 1.0.

Obtaining Standard Errors

If you want to calculate the *standard error of the mean* (SEM) for a particular variable, you just have to divide the standard deviation for that variable by the square root of the size of the sample (*n*). However, if you would like SPSS to do it for you, you can use the preceding list of steps for creating *z* scores with a little modification. Start with step #1, skip step #2, and in step #3 move over the variables for which you would like to see SEMs. At step #4, check the little box for *S.E. mean*. Applied to the variable *mathquiz*, the SPSS results of the steps just described are shown in Table 4.2.

Table 4.2

Descriptive Statistics

	N	Minimum	Maximum	Mean		Std. Deviation
	Statistic	Statistic	Statistic	Statistic	Std. Error	Statistic
mathquiz	85	9	49	29.07	1.028	9.480
Valid N (listwise)	85					

You will find the SEM right next to the Mean, labeled *Std. Error*. From the organization of the output, SPSS is telling you that the *Statistic* for estimating the population mean and its standard error are 29.07 and 1.028, respectively. If you knew the actual population mean, you could subtract it from the mean in the table, and then divide that difference by the Std. Error to obtain the z score for your sample. I will expand on this point in the next chapter. The entry in the lower-left corner of the table, "Valid N (listwise)," tells you how many cases in your dataset have valid (i.e., not missing) values for every variable that you moved over in the Descriptives box.

Obtaining Areas of the Normal Distribution

There are some convenient normal distribution calculators available for free on the web, but in the unlikely case that you are offline and do have access to SPSS, you can use SPSS to obtain areas under the normal distribution with far more accuracy (i.e., more digits past the decimal point) than you could get from a printed table, like Table A.1 in your text. And you can obtain areas beyond z scores that are larger than those included in any printed tables.

1. Start by opening a new (i.e., empty) data sheet, and entering the z score of interest in the first cell. For convenience, you can assign the simple variable name "z" to this first column.
2. Then open the Compute Variable box by selecting Compute (the first choice) from the Transform menu.
3. In the Target Variable space, type an appropriate variable name like "area."
4. In the Numeric Expression space, type "CDFNORM" and then, in parentheses, the name of the first variable—for example, CDFNORM (z). (Note: I am using uppercase letters for emphasis, but SPSS does not distinguish between upper- and lowercase letters in variable or function names).

CDFNORM is a function name that stands for the *C*umulative *D*ensity *F*unction for the *NORM*al distribution; therefore it returns a value equal to all of the area to the left of (i.e., below) the z score you entered. If you multiply this value by 100, you get the percentile rank associated with the z score in question, *if you are dealing with a normal distribution* (this works for both positive and negative z scores, as long as you include the minus sign for any negative z score). Note that the larger the z score you enter in the first cell of your SPSS datasheet, the more "decimals" you will need to display the answer accurately. The fourth column in Variable View lets you set the number of digits that will be displayed to the right of the decimal point—as long as this number is less than the number in the third column (Width) for that variable. For instance, if you are looking for the area associated with a z score between 3 and 4, you will want to set the "decimals" number to at least 6.

Data Transformations

If you create a set of z scores corresponding to one of your variables, the distribution of the z scores will have exactly the same shape as the distribution of the original variable. If you want to change the shape of your distribution, usually to make it resemble the normal distribution, you need to use a transformation that is *not* linear. For example, a transformation that is often used to greatly reduce the positive skew of a distribution is to take the logarithm of each value. This is another task that is best handled by first selecting Compute Variable from the Transform menu. One of the most positively skewed variables in Ihno's data set is *prevmath*, so I'll use that variable for my example. After you have opened the Compute Variable box, type a new variable name, like *log_prevmath* in the Target Variable space, and then type "Lg10 (*prevmath* + 1)" in the Numeric Expression space and click OK. If you look at the distribution of the logs of the *prevmath* scores (and, even better, request a skewness measure), you'll see that it is much less skewed than the distribution for *prevmath*. Note that I had to add 1 to *prevmath*, because there are quite a few scores of zero, and you can't take the log of zero. Note also that "Lg10" yields logs to the base 10, but the natural logs, obtained by typing "Ln" instead of "Lg10," will produce a distribution that has exactly the same shape as do logs to the base 10. Finally, if you wanted to *replace* the original *prevmath* values with their log-transformed values, rather than adding a new variable, you would just type *prevmath* in the Target Variable space. Because this action eliminates the original values of your variable from the spreadsheet, SPSS warns you of this by asking "Change existing variable?" and you can then click OK or Cancel.

EXERCISES

1. Create new variables consisting of the z scores for the anxiety and heart rate measures at baseline in Ihno's data set. Request means and *SD*s of the z-score variables to demonstrate that the means and *SD*s are 0 and 1, respectively, in each case.
2. Create a z-score variable corresponding to the math background quiz score, and then transform the z-score variable to a T score, an SAT score, and an IQ score. Repeat for the stats quiz.
3. Use SPSS to find the following areas under the normal curve (your answer should include six digits past the decimal point):
   a. The area below a z score of +3.1.
   b. The area above a z score of +3.3.
   c. The area below a z score of −3.7.
   d. The area between the mean and a z score of +.542
   e. The area between the mean and a z score of −1.125
4. Use SPSS to find the percentile ranks for the following z scores (your answer should include two digits past the decimal point):
   a. 3.1
   b. 3.3
   c. 3.7
   d. .542
   e. −1.125
5. Find the mean, *SD*, standard error, and skewness for the *phobia* variable. Then, create a new variable that is the square root of the *phobia* variable, and find those statistics again. What happened to the skewness of *phobia* after taking the square root?
6. Find the mean, *SD*, standard error, and skewness for the *statsquiz* variable. Then, create a new variable that is the natural log of the *statsquiz* variable, and find those statistics again. What happened to the skewness of *statsquiz*? Explain the lesson that you learned from this exercise.

KEY FORMULAS

The z score corresponding to a raw score:

$$z = \frac{X - \mu}{\sigma}$$ **Formula 4.1**

The raw score that corresponds to a given z score:

$$X = z\sigma + \mu$$ **Formula 4.2**

The SAT score corresponding to a raw score, if the z score has already been calculated:

$$\text{SAT} = 100z + 500$$ **Formula 4.3**

The T score corresponding to a raw score, if the z score has already been calculated:

$$T = 10z + 50$$ **Formula 4.4**

The standard error of the mean:

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}$$ **Formula 4.5**

The z score for groups (the standard error of the mean must be found first):

$$z = \frac{\overline{X} - \mu}{\sigma_{\overline{X}}}$$ **Formula 4.6**

The addition rule for mutually exclusive events:

$$p(\text{A or B}) = p(\text{A}) + p(\text{B})$$ **Formula 4.7**

The addition rule for events that are *not* mutually exclusive:

$$p(\text{A or B}) = p(\text{A}) + p(\text{B}) - p(\text{A and B})$$ **Formula 4.8**

The multiplication rule for independent events:

$$p(\text{A and B}) = p(\text{A})p(\text{B})$$ **Formula 4.9**

The multiplication rule for events that are *not* independent (based on *conditional probability*):

$$p(\text{A and B}) = p(\text{A})p(\text{B}|\text{A})$$ **Formula 4.10**