Competitive Programming Library - Notes

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1 Geometry

1.1 Isomorphism of 2 polygons: Given two polygons with vertices in clockwise/counter-clockwise order, check if they are isomorphic

Solution:

For each polygon, run through their vertices appending, for each three consecutive vertices u,v,w:

- square norm of (u,v) vector
- dot-product of (u,v) and (v,w)
- cross-product of (u,v) and (v,w)
- square norm of (v,w) vector

Now, double the array of one of this polygons and then run some string matching algorithm to find if you have a match with the other one. Complexity: $O(n)$, where n is the number of vertices of the polygons.

1.2 Given two polygons A and B that rotates around points P and Q(respectively) at the same speed and in the same direction, tell if they can collide

Observation: Suppose a reference system where A doesn't rotate. Then, the point Q rotates in the opposite direction around P, while B still rotates around Q. If you draw it, you can see that B doesn't move relative to Q. Thus, we can only rotate B around point P and check the intersection between the circle with center $C = P - Q + B$ and $radius = distance(P, Q)$, for each vertex of B, and all segments of polygon A. If some of them intersect, then there is a collision. Complexity: $O(nm)$, where n and m are the number of vertices of the polygons.

1.3 Area of Planar Polygon in 3D

Let V be the 3D vector that represents the sum of all cross-products between all consecutive points of the polygon. Let N be a unit vector that represents the normal of the polygon (any 2 consecutive points can be used to acquire). The area is given by $A = \frac{1}{2}|N \cdot V|$.

1.4 Centroid of a triangle

Draw a line from each corner that divides the opposite side of the triangle in two equal parts. The intersection of these 3 lines is called the centroid.

1.5 Circumcenter of a triangle

Draw a perpendicular line from each mid-point of the three sides of the triangle. The intersection point is called the circumcenter of the triangle, which is the center of the circumcircle of this triangle.

1.6 Incenter of a triangle

Draw a line from each corner of the triangle, dividing the angle in two equal parts. The intersection point is called the incenter of the triangle, which is the center of the incircle of this triangle.

1.7 Orthocenter of a triangle

Draw a line from each corner of the triangle, making a 90 angle with the opposite side. The intersection point is called the orthocenter of the triangle. It can be outside of the triangle.

1.8 Pick's Theorem: Given a polygon constructed using n vertices with integer coordinates, count the number of integer coordinate points strictly inside this polygon

Solution:

Pick's theorem states that the number of integer coordinates I stricly inside a polygon formed by vertices with integer coordinates is given by $I = (2A - B + 2)/2$, where:

- A = Area of the polygon. We can calculate the area using shoelace's formula.

- B = Number of vertices with integer coordinates on all edges of the polygon. This number B', for each edge, (excluding the endpoints) can be calculated this way:

- $|x' - x''| - 1$, if the edge is parallel to y-axis.
- $|y' - y''| - 1$, if the edge is parralel to x-axis.
- $gcd(|x' - x''|, |y' - y''|) - 1$, otherwise.

2 Math and Number Theory

2.1 Number of ways to make a bracelet with m beads using n colors

Let N be the number of ways to do it. We'll find, using double counting, $X = 6N$. Then, $N = X/6$. We can find X this way:

If we rotate some sequence i times ($i <= m$), then, we get repetitions if the period of the sequence is $gcd(i, m)$. Therefore, $n^{gcd(i,m)}$ sequences are repetitions.

Now, we can do a formula for that:

$X = \sum_{i=1}^{m} n^{gcd(i,m)}$

2.2 Stars and Bars: Given n and k, count the number of ways to divide n stars into k groups (there can exist empty groups)

Solution:

Suppose a string made from a combination of n stars and $k - 1$ bars (we need $k - 1$ divisions to make k groups). Then, we have to choose $k - 1$ positions from $n + k - 1$ to put bars, and the rest will be stars. Then, the number of ways can be expressed by $\binom{n+k-1}{k-1}$.

2.3 Extended Euclidean Algorithm - Solve the equation $ax + by = gcd(a, b)$

Using $g = gcd(a, b)$, assume we found some coefficients (x_1, x_2) for the equation $(b\ mod\ a)x_1 + ay_1 = g$. We can write $(b\ mod\ a) = b - \lfloor \frac{b}{a} \rfloor a$. Substituting this value in the equation gives us $g = bx_1 + a(y_1 - \lfloor \frac{b}{a} \rfloor)$. Then we found that $x = y_1 - \lfloor \frac{b}{a} \rfloor$ and $y = x_1$. Base case is when $a = 0$, where we return $(0, 1)$.

2.4 Chinese Remainder Theorem

Suppose we have a system consisting of two linear congruences:

$$x \equiv a_1 (mod\, n_1)$$
$$x \equiv a_2 (mod\, n_2)$$

We can write them as:

$$x = a_1 + k_1 * n_1$$
$$x = a_2 + k_2 * n_2$$

So, $n_1 * k_1 + n_2(-k_2) = a_2 - a_1$. The system has answer modulo $lcm(n_1, n_2)$ only if $gcd(n_1, n_2)$ divides $a_2 - a_1$. We can merge these two equations to $x \equiv s(mod\, n)$, where $s = a_1 + n_1 * k_1$, k_1 given by the solution of the Diophantine Equation above, and $n = lcm(n_1, n_2)$. We can merge all equations of the system until we remain with only one.

2.5 2-SAT

Suppose we have a logical formula containing clauses with 2 literals each in conjuctive normal form. Given a clause $(x \vee y)$, we can translate it to $(\neg x \to y) \wedge (\neg y \to x)$. Now, just construct a graph with these implications and calculate the SCC's using Tarjan/Kosaraju's Algorithm. After that, you need to check if x and $\not x$ are in the same component, $\forall x$. If it's yes for some literal, then we have a contradiction and the formula can't be satisfied.

2.6 Count Divisors of a Factorial

We need to find the prime divisors of the factorial and their exponents. The prime divisors of $n!$ are all prime numbers smaller or equal than n. Now, to find the exponent of each number we can loop through the powers of the primes dividing n by p^x. The ans is $\prod_p (a_p + 1)$, where a_p is the exponent of p in factorial.