Text Classification

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Machine Learning Review 2

CS 287

Quiz: Naive Bayes

Given bag-of-word features

$$\mathcal{F} = \{\texttt{The, movie, was, terrible, rocked, A}\}$$

and two training data points:

Class 1: `The movie was terrible`

Class 2: `The movie rocked`

What is the conditional distribution $P(Y|X)$ of the new example "`terrible movie`" with $\alpha = 0$?

What about "`A terrible movie`" with $\alpha = 1$?

Answer: Naive Bayes (1)

`terrible movie`

Prior is simple: $p(\mathbf{y}) = \begin{bmatrix} 1/2 & 1/2 \end{bmatrix}$

Construct count matrix:

$$\mathbf{F} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

With $\alpha = \epsilon$,

$$p(\mathbf{x}|\mathbf{y}) = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 1/3 & 0 \end{bmatrix}$$

$$p(\mathbf{y}|\mathbf{x}) \propto \begin{bmatrix} 1/2 \times 1/2 \times 1/2 & 1/2 \times 0 \times 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Answer: Naive Bayes (2)

`A terrible movie`

With $\alpha = 1$,

$$\bar{\mathbf{F}} = \begin{bmatrix} 2 & 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 & 2 & 1 \end{bmatrix}$$

$$p(\mathbf{x}|\mathbf{y}) = \begin{bmatrix} 1/5 & 1/5 & 1/5 & 1/5 & 1/10 & 1/10 \\ 2/9 & 2/9 & 1/9 & 1/9 & 2/9 & 1/9 \end{bmatrix}$$

$$\begin{aligned} p(\mathbf{y}|\mathbf{x}) &\propto \begin{bmatrix} 1/2 \times 1/10 \times 1/5 \times 1/5 & 1/2 \times 1/9 \times 1/9 \times 2/9 \end{bmatrix} \\ &\approx \begin{bmatrix} 0.593 & 0.407 \end{bmatrix} \end{aligned}$$

Contents

Review: Multiclass Sentiment

- ⋆⋆⋆⋆

  I visited The Abbey on several occasions on a visit to Cambridge and found it to be a solid, reliable and friendly place for a meal.

- ⋆⋆

  However, the food leaves something to be desired. A very obvious menu and average execution

- ⋆⋆⋆⋆⋆

  Fun, friendly neighborhood bar. Good drinks, good food, not too pricey. Great atmosphere!

Review: Sparse Bag-of-Words Features

Representation is counts of input words,

- $\mathcal{F}$; the vocabulary of the language.
- $\mathbf{x} = \sum_i \delta(f_i)$

Example: Movie review input,

`A sentimental mess`

$$\mathbf{x} = \delta(\texttt{word:A}) + \delta(\texttt{word:sentimental}) + \delta(\texttt{word:mess})$$

$$\mathbf{x}^\top = \begin{bmatrix} 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} = \begin{array}{c l} \left[\begin{matrix} 1 \\ \vdots \\ 1 \\ 1 \end{matrix}\right] & \begin{matrix} \texttt{word:A} \\ \vdots \\ \texttt{word:mess} \\ \texttt{word:sentimental} \end{matrix} \end{array}$$

Review: Output Class Notation

- $\mathcal{C} = \{1, \ldots, d_{\text{out}}\}$; possible output classes
- $c \in \mathcal{C}$; always one true output class
- $\mathbf{y} = \delta(c) \in \mathbb{R}^{1 \times d_{\text{in}}}$; true one-hot output representation

Review: Multiclass Classification

Examples: Yelp stars, etc.

- $d_{\text{out}} = 5$; for examples
- In our notation, one star, two star...

$$\begin{aligned} \star\ c = 1 \quad \mathbf{y} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ vs.} \\ \star\star\ c = 2 \quad \mathbf{y} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix} \ldots \end{aligned}$$

Examples: Word Prediction (Unit 3)

- $d_{\text{out}} > 100,000$;
- In our notation, $\mathcal{C}$ is vocabulary and each c is a word.

$$\begin{aligned} \textit{the}\ c = 1 \quad \mathbf{y} &= \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \text{ vs.} \\ \textit{dog}\ c = 2 \quad \mathbf{y} &= \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \end{bmatrix} \ldots \end{aligned}$$

Review Linear Models for Classification

Linear model,

$$\hat{\mathbf{y}} = f(\mathbf{x}\mathbf{W} + \mathbf{b})$$

- $\mathbf{W} \in \mathbb{R}^{d_{\text{in}} \times d_{\text{out}}}, \mathbf{b} \in \mathbb{R}^{1 \times d_{\text{out}}}$; model parameters
- $f : \mathbb{R}^{d_{\text{out}}} \mapsto \mathbb{R}^{d_{\text{out}}}$; activation function
- Sometimes $\mathbf{z} = \mathbf{x}\mathbf{W} + \mathbf{b}$ informally "score" vector.
- Note $\mathbf{z}$ and $\hat{\mathbf{y}}$ are not one-hot.

Class prediction,

$$\hat{c} = \arg\max_{i \in \mathcal{C}} \hat{y}_i = \arg\max_{i \in \mathcal{C}} (\mathbf{x}\mathbf{W} + \mathbf{b})_i$$

Probabilistic Linear Models

Can estimate a linear model probabilistically,

- Let output be a random variable Y, with sample space $\mathcal{C}$.
- Representation be a random vector X.
- (Simplified frequentist representation)
- Interested in estimating parameters θ,

$$P(Y|X;\theta)$$

Informally we use $p(\mathbf{y} = \delta(c)|\mathbf{x})$ for $P(Y = c|X = \mathbf{x})$.

Contents

Discriminative Model. Conditional Log-Likelihood as Loss

- $(\mathbf{x}_1, \mathbf{y}_1), \dots, (\mathbf{x}_n, \mathbf{y}_n)$; supervised data
- Parameters maximize conditional likelihood of training data.

$$\mathcal{L}(\theta) = -\sum_{i=1}^{n} \log p(\mathbf{y}_i | \mathbf{x}_i; \theta)$$

  For linear models $\theta = (\mathbf{W}, \mathbf{b})$

- (Contrast with generative models like NB, $p(\mathbf{x}_i, \mathbf{y}_i)$)
- Do this by minimizing negative log-likelihood (NLL).

$$\arg\min_{\theta} \mathcal{L}(\theta)$$

The Softmax

Alternative parametrization of probabilistic model.

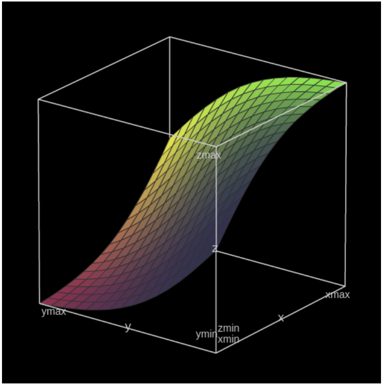
Use a softmax to force a distribution,

$$\text{softmax}(\mathbf{z}) = \frac{\exp(\mathbf{z})}{\sum_c \exp(z_c)}$$

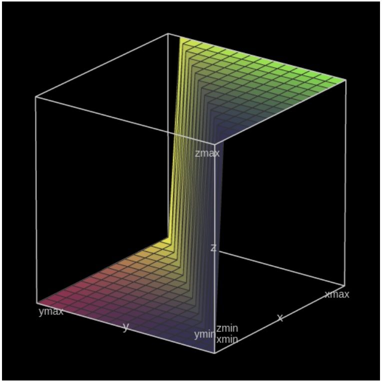
$$\log \text{softmax}(\mathbf{z}) = \mathbf{z} - \log \sum_c \exp(z_c)$$

- Exercise: Confirm always gives a distribution.
- Denominator known as *partition* function (we'll see many times).

Why is it called the softmax?



$$\mathrm{softmax}([x \ \ y]) = \frac{\exp(x)}{\exp(x) + \exp(y)}$$



$$\arg\max([x \ \ y]) = \mathbf{1}(x > y)$$

Multiclass Logistic Regression

- Direct estimation of conditional $p(\mathbf{y} = c|\mathbf{x}; \theta)$

$$\hat{\mathbf{y}} = p(\mathbf{y} = c|\mathbf{x}; \theta) = \mathrm{softmax}(\mathbf{x}\mathbf{W} + \mathbf{b})$$

- $\mathbf{W} \in \mathbb{R}^{d_{\mathrm{in}} \times d_{\mathrm{out}}}$, $\mathbf{b} \in \mathbb{R}^{1 \times d_{\mathrm{out}}}$; model parameters
- "Regression" of the distribution.
- Classification still done as,

$$\hat{c} = \arg\max_{c \in \mathcal{C}} (\mathbf{x}\mathbf{W} + \mathbf{b})_c$$

Example: Multiclass Logistic Regression

3 classes, 2 features,

$$\mathbf{W} = \begin{bmatrix} 1 & 2 & -1 \\ -8 & 2 & 3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\mathbf{z} = \mathbf{x}\mathbf{W} + \mathbf{b} = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$$

Log-partition function,

$$\log \sum_c \exp(z_c) = \log(\exp(1) + \exp(2) + \exp(-1)) \approx 2.349$$

$$\log \mathrm{softmax}(\mathbf{z}) \approx \begin{bmatrix} 1 - 2.349 & 2 - 2.349 & -1 - 2.349 \end{bmatrix}$$

$$p(\mathbf{y}|\mathbf{x}) = \begin{bmatrix} 0.259 & 0.705 & 0.035 \end{bmatrix}$$

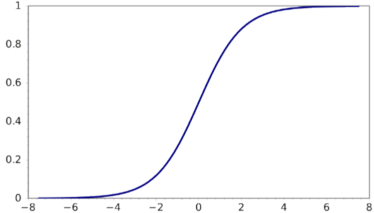
Important Case: Logistic Regression

For binary classification:

$$\begin{aligned}\text{softmax}([z_1 \ \ z_2]) &= \frac{\exp(z_1)}{\exp(z_1)+\exp(z_2)} \\ &= \frac{1}{1+\exp(-(z_1-z_2))} = \sigma(z_1-z_2)\end{aligned}$$

Logistic sigmoid function:

$$\sigma(t) = \frac{1}{1+\exp(-t)}$$



Multiclass Logistic Regression: A Model with Many Names

- Multinomial Logistic Regression
- Log-Linear Model (particularly in NLP)

$$\log \mathrm{softmax}(\mathbf{z}) = \mathbf{z} - \log \sum_{c} \exp(z_c)$$

- Softmax Regression
- Max-Entropy (MaxEnt)

Contents

Cross-Entropy Loss

Define the cross-entropy of two distributions $\mathbf{p}$ and $\hat{\mathbf{p}}$,

$$\text{cross-entropy}(\mathbf{p}, \hat{\mathbf{p}}) = -\sum_{c} p_{c'} \log \hat{p}_{c'}$$

Can use to compare true with prediction,

$$L_{cross-entropy}(\mathbf{y}, \hat{\mathbf{y}}) = -\sum_{c} y_{c'} \log \hat{y}_{c'}$$

Common case, when $y_c = 1$ for class c,

$$L_{cross-entropy}(\mathbf{y}, \hat{\mathbf{y}}) = -\log \hat{y}_c \text{ where } y_c = 1$$

$$\hat{\mathbf{y}} = p(\mathbf{y}_i | \mathbf{x}_i; \theta)$$

Loss Minimization

$$L_{cross-entropy}(\mathbf{y}, \hat{\mathbf{y}}) = -\log \hat{y}_c \text{ where } y_c = 1$$

Equivalent to probabilistic objective:

$$\mathcal{L}(\theta) = -\sum_{i=1}^{n} \log p(\mathbf{y}_i | \mathbf{x}_i; \theta) = \sum_{i=1}^{n} L_{cross-entropy}(\mathbf{y}_i, \hat{\mathbf{y}}_i)$$

And the distribution is parameterized as a softmax,

$$\begin{aligned} L_{cross-entropy}(\mathbf{y}, \hat{\mathbf{y}}) &= -\log \hat{y}_c \\ &= -\log \operatorname{softmax}(\mathbf{xW} + \mathbf{b})_c \\ &= -z_c + \log \sum_{c' \in \mathcal{C}} \exp(z_{c'}) \end{aligned}$$

However, this is much harder to minimize, **no closed form**.

Review: Calculus!

$$\frac{d(wx)}{dx} = w \qquad \frac{d(u/v)}{dx} = \frac{u'v - uv'}{v^2}$$

$$\frac{d \log u(x)}{dx} = \frac{u'}{u} \qquad \frac{d \exp u(x)}{dx} = u' \exp u$$

- For function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ Jacobian is

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f(\mathbf{x})_1}{\partial x_1} & \dots & \frac{\partial f(\mathbf{x})_1}{\partial x_n} \\ & \ddots & \\ \frac{\partial f(\mathbf{x})_m}{\partial x_1} & \dots & \frac{\partial f(\mathbf{x})_m}{\partial x_n} \end{bmatrix}$$

- In-Class: Compute Jacobian of (1) L, (2) softmax, (3) $\mathbf{z} = \mathbf{x}\mathbf{W} + \mathbf{b}$

Symbolic Gradients

- Partials of $L(\mathbf{y}, \hat{\mathbf{y}})$ for all $j \in \{1, \ldots, d_{\text{out}}\}$ and $y_c = 1$

$$\frac{\partial L(\mathbf{y}, \hat{\mathbf{y}})}{\partial \hat{y}_j} = \begin{cases} -\frac{1}{\hat{y}_j} & j = c \\ 0 & o.w. \end{cases}$$

- Partials of $\hat{\mathbf{y}} = \text{softmax}(\mathbf{z})$

$$\frac{\partial \hat{y}_j}{\partial z_i} = \begin{cases} \hat{y}_i(1 - \hat{y}_i) & i = j \\ -\hat{y}_i \hat{y}_j & i \neq j \end{cases}$$

- Partials of $\mathbf{z} = \mathbf{x}\mathbf{W} + \mathbf{b}$

$$\frac{\partial z_i}{\partial b_{i'}} = \mathbf{1}(i = i') \quad \frac{\partial z_i}{\partial W_{f,i'}} = x_f \mathbf{1}(i = i')$$

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Review: Chain Rule

Assume we have a function and a loss:

$$f : \mathbb{R}^n \to \mathbb{R}^m \quad L : \mathbb{R}^m \to \mathbb{R}$$

Then for $i \in \{1, \ldots, n\}$

$$\frac{\partial L(f(\mathbf{x}))}{\partial x_i} = \sum_{j=1}^{m} \frac{\partial f(\mathbf{x})_j}{\partial x_i} \frac{\partial L(f(\mathbf{x}))}{\partial f(\mathbf{x})_j}$$

Chain Rule

For multiclass logistic regression:

$$\frac{\partial L(\mathbf{y}, \hat{\mathbf{y}})}{\partial z_i} = \sum_j \frac{\partial \hat{y}_j}{\partial z_i} \frac{\mathbf{1}(j = c)}{\hat{y}_j} = \begin{cases} -(1 - \hat{y}_i) & i = c \\ \hat{y}_i & ow. \end{cases}$$

Therefore for parameters,

$$\frac{\partial L}{\partial b_i} = \frac{\partial L}{\partial z_i} \qquad \frac{\partial L}{\partial W_{f,i}} = x_f \frac{\partial L}{\partial z_i}$$

Intuition:

- Nothing happens on correct classification.
- Weight of true features increases based on prob not given.
- Weight of false features decreases based on prob given.

Chain Rule

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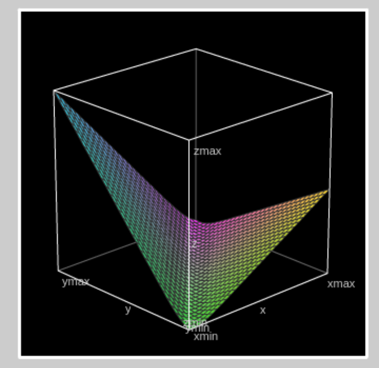
Optimization

Complete objective (without bias):

$$\mathcal{L}(\theta) = -\sum_{i=1}^{n} (\mathbf{x}_i \mathbf{W})_{c_i} + \log \sum_{c'} \exp(\mathbf{x}_i \mathbf{W})_{c'}$$

- First term is linear in **W** (convex)
- Second term is log-sum-exp of a linear functions of **W** (convex).
- Objective is convex, but not strictly convex.
- Hard Exercise: Prove (Boyd and Vandenberghe, 2004 p. 72-74)

Optimization



Loss for 2 classes, only bias, parameters x, y

$$L([x\ \ y]) = -10 \log \text{softmax}([x\ \ y])_1 - 5 \log \text{softmax}([x\ \ y])_2$$

Regularization

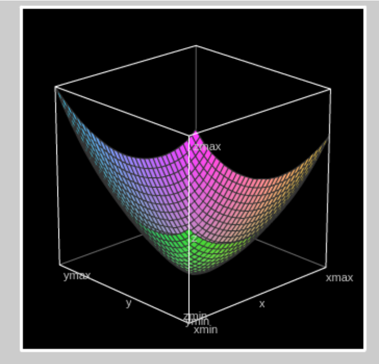
$$\mathcal{L}(\theta) = -\sum_{i=1}^{n} L(\mathbf{y}_i, \hat{\mathbf{y}}_i) + \frac{\lambda}{2}||\theta||_2^2$$

- Objective is strictly convex.

For L2 regularization gradients for multiclass logistic regression:

$$\frac{\partial L}{\partial b_i} = \frac{\partial L}{\partial z_i} + \lambda b_i \qquad \frac{\partial L}{\partial W_{f,i}} = \frac{\partial L}{\partial z_i} + \lambda W_{f,i}$$

Regularized Optimization



Loss for 2 classes, only bias, parameters x, y, $\lambda = 2$

$$L([x\ \ y]) = -10 \log \mathrm{softmax}([x\ \ y])_1 - 5 \log \mathrm{softmax}([x\ \ y])_2 + ||[x\ \ y]||_2^2$$

Gradients-based minimization

Consider one example $(\mathbf{x}, \mathbf{y})$, we compute "forward" and then "backward",

1. Compute scores $\mathbf{z} = \mathbf{x}\mathbf{W} + \mathbf{b}$
2. Compute softmax of scores, $\hat{\mathbf{y}} = \text{softmax}(\mathbf{z})$
3. Compute loss of scores, $L(\mathbf{y}, \hat{\mathbf{y}})$
4. Compute gradient $\frac{\partial L(y,\hat{y})}{\partial \hat{y}_j}$.
5. Compute gradient $\frac{\partial L(y,\hat{y})}{\partial z_i}$.
6. Compute gradient of $\mathbf{b}$ and $\mathbf{W}$,

Gradients-based minimization

Consider one example $(\mathbf{x}, \mathbf{y})$, we compute "forward" and then "backward",

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3. Compute loss of scores, $L(\mathbf{y}, \hat{\mathbf{y}})$
4. Compute gradient $\frac{\partial L(y, \hat{y})}{\partial \hat{y}_j}$.
5. Compute gradient $\frac{\partial L(y, \hat{y})}{\partial z_i}$.
6. Compute gradient of $\mathbf{b}$ and $\mathbf{W}$,

Optimization

In next class we will talk more about gradient-based optimization.

- Here we simply describe an algorithm.

Gradient-Based Optimization: SGD

procedure SGD

while training criterion is not met **do**

Sample a training example $\mathbf{x}_i, \mathbf{y}_i$

Compute the loss $L(\hat{\mathbf{y}}_i, \mathbf{y}_i; \theta)$

Compute gradients $\hat{\mathbf{g}}$ of $L(\hat{\mathbf{y}}_i, \mathbf{y}_i; \theta)$ with respect to θ

$\theta \leftarrow \theta - \eta_k \hat{\mathbf{g}}$

end while

return θ

end procedure

Gradient Descent

while training criterion is not met **do**

  Sample a minibatch of m examples $(\mathbf{x}_1, \mathbf{y}_1), \ldots, (\mathbf{x}_m, \mathbf{y}_m)$

  $\hat{\mathbf{g}} \leftarrow 0$

  for $i = 1 \text{ to } m$ **do**

    Compute the loss $L(\hat{\mathbf{y}}_i, \mathbf{y}_i; \theta)$

    Compute gradients $\mathbf{g}'$ of $L(\hat{\mathbf{y}}_i, \mathbf{y}_i; \theta)$ with respect to θ

    $\hat{\mathbf{g}} \leftarrow \hat{\mathbf{g}} + \frac{1}{m}\mathbf{g}'$

  end for

  $\theta \leftarrow \theta - \eta_k \hat{\mathbf{g}}$

end while

return θ

Pros and Cons of Logistic Regression (Murphy, p 268)

Cons

- Harder to fit versus naive Bayes.
- Must fit all classes together.
- Not a good fit for semi-supervised/missing data cases

Pros

- Better calibrated probability estimates
- Natural handling of feature input
  - Features likely not multinomials
- (For us) extend naturally to neural networks

Softmax Notes: Calculating Log-Sum-Exp

- Calculating $\log \sum_{c' \in \mathcal{C}} \exp(\hat{y}_{c'}$ directly numerical issues.
- Instead $\log \sum_{c' \in \mathcal{C}} \exp(\hat{y}_{c'} - M) + M$ where $M = \max_{c' \in \mathcal{C}} \hat{y}_c'$