Bernoulli polynomials in Lean

Ashvni Narayanan

Lean Summer Projects 2021

June 29, 2021

My PhD project

- To formalize Iwasawa Theory Main Conjecture :

$$f_k((1+p)^s - 1) = L_p(\omega^k, s)$$

My PhD project

- To formalize Iwasawa Theory Main Conjecture :

$$f_k((1+p)^s - 1) = L_p(\omega^k, s)$$

- To formalize the p-adic L-functions :

$$L_p(-s,\chi) = \frac{-1}{1-\chi(c) < c >^{s+1}} \int_{(\mathbb{Z}/dp\mathbb{Z})^\times \times (1+p\mathbb{Z}_p)} \chi\omega^{-1}(a) < a >^s \, dE_c$$

My PhD project

- To formalize Iwasawa Theory Main Conjecture :

$$f_k((1+p)^s - 1) = L_p(\omega^k, s)$$

- To formalize the p-adic L-functions :

$$L_p(-s, \chi) = \frac{-1}{1 - \chi(c) < c >^{s+1}} \int_{(\mathbb{Z}/dp\mathbb{Z})^\times \times (1+p\mathbb{Z}_p)} \chi\omega^{-1}(a) < a >^s \, dE_c$$

- To formalize the p-adic integral

My PhD project

- To formalize Iwasawa Theory Main Conjecture :

$$f_k((1+p)^s - 1) = L_p(\omega^k, s)$$

- To formalize the p-adic L-functions :

$$L_p(-s, \chi) = \frac{-1}{1 - \chi(c) < c >^{s+1}} \int_{(\mathbb{Z}/dp\mathbb{Z})^\times \times (1+p\mathbb{Z}_p)} \chi\omega^{-1}(a) < a >^s \, dE_c$$

- To formalize the p-adic integral
- To define the Bernoulli measure

My PhD project

- To formalize Iwasawa Theory Main Conjecture :

$$f_k((1+p)^s - 1) = L_p(\omega^k, s)$$

- To formalize the p-adic L-functions :

$$L_p(-s,\chi) = \frac{-1}{1-\chi(c) < c >^{s+1}} \int_{(\mathbb{Z}/dp\mathbb{Z})^\times \times (1+p\mathbb{Z}_p)} \chi\omega^{-1}(a) < a >^s \, dE_c$$

- To formalize the p-adic integral
- To define the Bernoulli measure
- To define the Bernoulli polynomial

Bernoulli numbers

The Bernoulli numbers are an important number theoretic object. They occur as special values of the Reimann-zeta functions / p-adic L-functions. They are a generalization of Bernoulli numbers.

Bernoulli numbers

The Bernoulli numbers are an important number theoretic object. They occur as special values of the Reimann-zeta functions / p-adic L-functions. They are a generalization of Bernoulli numbers.
The Bernoulli numbers B_n are generating functions given by :

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}$$

Bernoulli numbers

The Bernoulli numbers are an important number theoretic object. They occur as special values of the Reimann-zeta functions / p-adic L-functions. They are a generalization of Bernoulli numbers.
The Bernoulli numbers B_n are generating functions given by :

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}$$

Note that several authors think of Bernoulli numbers B'_n to be defined as :

$$\sum_{n=0}^{\infty} B'_n \frac{t^n}{n!} = \frac{t}{1 - e^{-t}}$$

Bernoulli numbers

The Bernoulli numbers are an important number theoretic object. They occur as special values of the Reimann-zeta functions / p-adic L-functions. They are a generalization of Bernoulli numbers.
The Bernoulli numbers B_n are generating functions given by :

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}$$

Note that several authors think of Bernoulli numbers B'_n to be defined as :

$$\sum_{n=0}^{\infty} B'_n \frac{t^n}{n!} = \frac{t}{1 - e^{-t}}$$

The difference between these two is : $B'_n = (-1)^n B_n$, with $B_1 = \frac{-1}{2}$.

Bernoulli numbers

The Bernoulli numbers are an important number theoretic object. They occur as special values of the Reimann-zeta functions / p-adic L-functions. They are a generalization of Bernoulli numbers.
The Bernoulli numbers B_n are generating functions given by :

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}$$

Note that several authors think of Bernoulli numbers B'_n to be defined as :

$$\sum_{n=0}^{\infty} B'_n \frac{t^n}{n!} = \frac{t}{1 - e^{-t}}$$

The difference between these two is : $B'_n = (-1)^n B_n$, with $B_1 = \frac{-1}{2}$.

Bernoulli numbers in Lean

Using recursion, B_n is defined in mathlib as $B_n := \sum_{k=0}^{n-1} \binom{n}{k} \frac{B_k}{n-k+1}$:

```
bernoulli' n = 1 - ∑ k : fin n, n.choose k / (n - k + 1)
  * bernoulli' k
```

Bernoulli numbers in Lean

Using recursion, B_n is defined in mathlib as $B_n := \sum_{k=0}^{n-1} \binom{n}{k} \frac{B_k}{n-k+1}$:

```
bernoulli' n = 1 - ∑ k : fin n, n.choose k / (n - k + 1)
  * bernoulli' k
```

and B'_n as $B'_n := (-1)^n B_n$:

```
def bernoulli (n : ℕ) : ℚ := (-1)^n * bernoulli' n
```

Bernoulli polynomials

The Bernoulli polynomials denoted $B_n(X)$, a generalization of the Bernoulli numbers, are generating functions :

$$\sum_{n=0}^{\infty} B_n(X)\frac{t^n}{n!} = \frac{te^{tX}}{e^t - 1}$$

Bernoulli polynomials

The Bernoulli polynomials denoted $B_n(X)$, a generalization of the Bernoulli numbers, are generating functions :

$$\sum_{n=0}^{\infty} B_n(X)\frac{t^n}{n!} = \frac{te^{tX}}{e^t - 1}$$

We now define the Bernoulli polynomials as $B_n(X) := \sum_{i=0}^{n} \binom{n}{i} B_i X^{n-i}$:

```
def bernoulli_poly (n : ℕ) : polynomial ℚ :=
Σ i in range (n + 1), polynomial.monomial (n - i) ((
 bernoulli i) * (choose n i))
```

Bernoulli polynomials in Lean

The following properties of Bernoulli polynomials were proved :

1. $B_0(X) = X$:

```
lemma bernoulli_poly_zero : bernoulli_poly 0 = 1
```

Bernoulli polynomials in Lean

The following properties of Bernoulli polynomials were proved :

1. $B_0(X) = X$:

```
lemma bernoulli_poly_zero : bernoulli_poly 0 = 1
```

2. $B_n(0) = B_n$:

```
lemma bernoulli_poly_eval_zero (n : ℕ) : (
bernoulli_poly n).eval 0 = bernoulli n
```

Bernoulli polynomials in Lean

The following properties of Bernoulli polynomials were proved :

1. $B_0(X) = X$:

```
lemma bernoulli_poly_zero : bernoulli_poly 0 = 1
```

2. $B_n(0) = B_n$:

```
lemma bernoulli_poly_eval_zero (n : ℕ) : (
bernoulli_poly n).eval 0 = bernoulli n
```

3. $B_n(1) = B'_n$:

```
lemma bernoulli_poly_eval_one (n : ℕ) : (
bernoulli_poly n).eval 1 = bernoulli’ n
```

Bernoulli polynomials in Lean

1. $\sum_{k=0}^{n} \binom{n+1}{k} B_k(X) = (n+1)X^n$

```
theorem sum_bernoulli_poly (n : ℕ) :
  ∑ k in range (n + 1), ((n + 1).choose k : ℚ) •
  bernoulli_poly k =
  polynomial.monomial n (n + 1 : ℚ)
```

Bernoulli polynomials in Lean

1. $\sum_{k=0}^{n} \binom{n+1}{k} B_k(X) = (n+1)X^n$

```
  theorem sum_bernoulli_poly (n : ℕ) :
Σ k in range (n + 1), ((n + 1).choose k : ℚ) ·
 bernoulli_poly k =
  polynomial.monomial n (n + 1 : ℚ)
```

2. $\left(\sum_{n=0}^{\infty} B_n(t)\frac{X^n}{n!}\right) * (e^X - 1) = Xe^{tX}$:

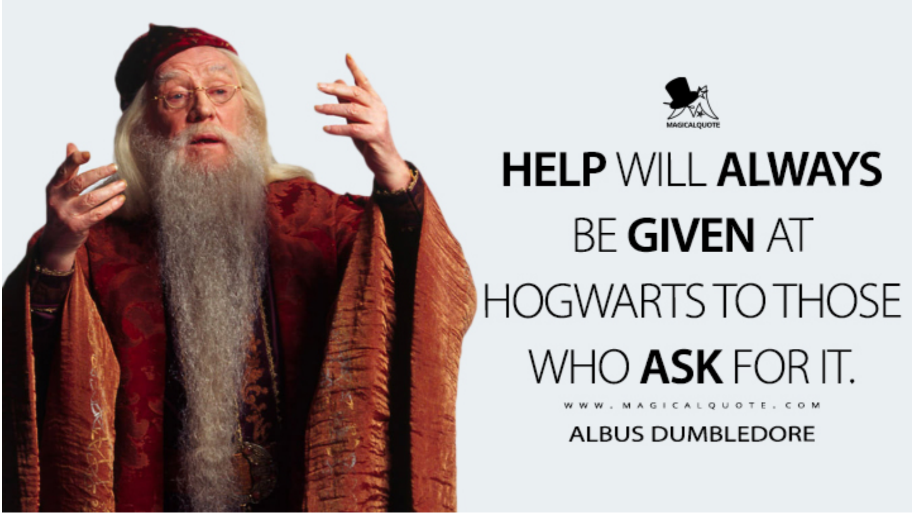
```
  theorem exp_bernoulli_poly' (t : A) :
mk (λ n, aeval t ((1 / n! : ℚ) · bernoulli_poly n)) *
   (exp A - 1) =
X * rescale t (exp A)
```

So..

Come join us on Discord/Zulip!

So..

Come join us on Discord/Zulip!



Thank you!