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symmetry

Article

A Brief Review of Implicit Regularization and Its Connection with the BPHZ Theorem

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Two-loop renormalisation of gauge theories in 4D implicit regularisation and connections to dimensional methods

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Abstract We compute the two-loop β-function of scalar and spinorial quantum electrodynamics as well as pure Yang–Mills and quantum chromodynamics using the background field method in a fully quadridimensional setup using implicit regularization (IREG). Moreover, a thorough comparison with dimensional approaches such as conventional dimensional regularization (CDR) and dimensional reduction (DRED) is presented. Subtleties related to Lorentz algebra contractions/symmetric integrations inside divergent integrals as well as renormalisation schemes are carefully discussed within IREG where the renormalisation constants are fully defined as basic divergent integrals to arbitrary loop order. Moreover, we confirm the hypothesis that momentum routing invariance in the loops of Feynman diagrams implemented via setting well-defined surface terms to zero deliver non-abelian gauge invariant amplitudes within IREG just as it has been proven for abelian theories.

1 Motivations

Unravelling physics beyond the standard model (SM) has entreated theoretical predictions for particle physics precision observables beyond next-to-leading-order (NLO). Such predictions rely on involved Feynman diagram calculations to evaluate scattering amplitudes both in the SM and its extensions. Theoretical models beyond the SM (BSM) can be constructed, for instance, as an extension in the Higgs sector by either changing the number of scalar multiplets or considering the Higgs boson as a composite particle – the so called Composite Higgs Models [1,2]. Supersymmetric and dark matter extensions have also been considered in order to explain SM deviations from experimental results [3] in electroweak precision observables (EWPO) which are known with an accuracy at the per cent level or better [4–6]. On the other hand, precise measurements and calculations of known particles and interactions are just as important to validate, redress, or refute new models. Also, in order to evade from unphysical scale dependence at low order, higher order terms are needed to smooth out such dependence in the resulting, more accurate, predictions. For example, a full N^3LO calculation for QCD corrections to gluon-fusion Higgs boson production was performed in [7] at center-of-mass energy 13 TeV. The considerably low residual theoretical uncertainty ($\approx$ 5–6%) and small sensitivity to scale variation ($\approx$ 2%) superseded earlier results below N^3LO. Because experimental uncertainties are expected to drop below the accuracy of theoretical data, as expected from future experimental measurements at the future circular collider (FCC-e^-e^+) [8], QCD theoretical uncertainties ought to be reduced at many levels so physics BSM can be ultimately ascertained.

Ultraviolet (UV) and infrared (IR) divergences are ubiquitous beyond leading order in S-matrix calculations and must be judiciously removed in order to automated computation codes for the evaluation of Feynman amplitudes. As a by-product of such subtractions, there remain residual dependencies on renormalisation (λ_R) and factorisation (λ_F) scales in the perturbative series that describes a physical observable. The dependence on such scales is expected to diminish after higher terms are taken into account and, at a given order, may in principle be minimised to yield a result the least sensitive to variations in the unphysical parameters [9]. However, the problem of scale setting has been studied extensively and there is no consensus on a procedure valid in general. For a recent account see [10].

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Finally accepted!


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May the four be with you:
Novel IR-subtraction methods to tackle NNLO calculations

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Abstract NNLO

In this report, we present a discussion about different frameworks to perform precise higher-order computations for high-energy physics. These approaches implement novel strategies to deal with infrared and ultraviolet singularities in quantum field theories. A special emphasis is devoted to the local cancellation of these singularities, which can enhance the efficiency of computations and lead to discover novel mathematical properties in quantum field theories.

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arXiv:2012.02567v1

At 2-loop order at similar program can be devised

$$\mathcal{A} = \int_{k_1,k_2} G(p_1,\ldots,p_L,k_1,k_2)H_1(p_1,\ldots,p_L,k_1)H_2(p_1,\ldots,p_L,k_2)$$

3 regimes:

- k1->∞ ; k2->fixed.
- k1->fixed ; k2->∞
- k1->∞ ; k2->∞

$$\frac{1}{(k-p)^2-\mu^2} = \frac{1}{k^2-\mu^2} - \frac{p^2-2p\cdot k}{(k^2-\mu^2)[(k-p)^2-\mu$$

At 2-loop order at similar program can be devised

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$$\mathcal{A}_{k_1\to\infty} = \int_{k_2} \bar{H}_2(p_1,\ldots,p_L,k_2)I_{log}(\lambda^2)$$

$$\mathcal{A}_{k_2\to\infty} = \int_{k_1} \bar{H}_1(p_1,\ldots,p_L,k_1)I_{log}(\lambda^2)$$

$$\mathcal{A}_{k_1\to\infty,k_2\to\infty} = \mathcal{F}(p_1,\ldots,p_L)I_{log}(\lambda^2)$$

- How Implicit Regularization complies with the BPHZ theorem?

Contents

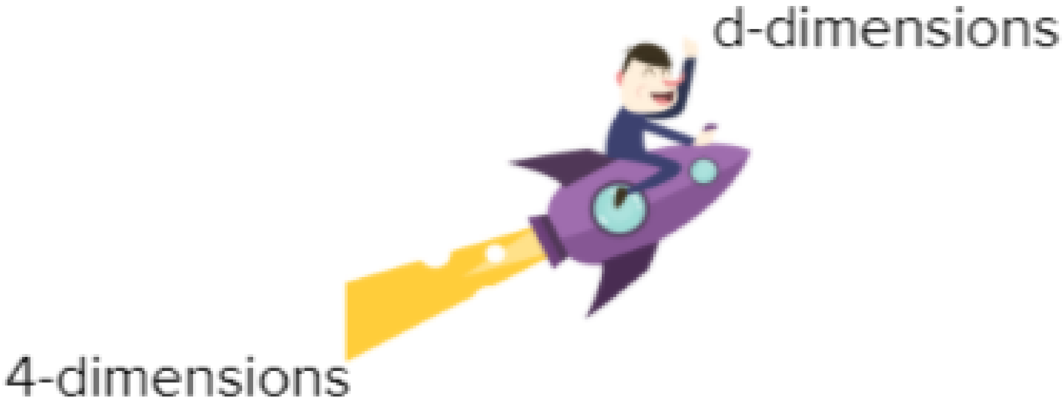
Introduction

- Feynman diagrams calculations -> Divergences at intermediate stages.

$$\boldsymbol{\infty}\ \textbf{in QFT} \left\{ \begin{array}{l} \textbf{UV-div} \\[1em] \text{IR-div} \end{array} \right.$$

- **Naive solution:** regulate the divergences.

It is easy to say, but difficult to do.



REGULARIZATION

DReg: dimensional regularization.

DRED: dimensional reduction.

(traditional regularization schemes)

4-dimensions

REGULARIZATION

$$\underbrace{\hspace{6em}}$$

IREG: Implicit Regularization

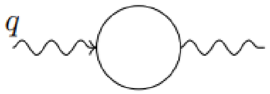
(Non-dimensional scheme)

Renormalization

Renormalization

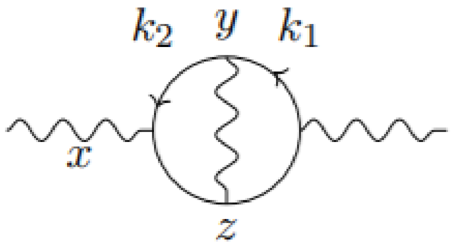
- In renormalizable theories UV-div can be reabsorbed in a finite number of parameters of the theory.

- **But does the renormalization procedure work at all orders?**
  - Mathematically established by the Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ) theorem.



$$\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2), \quad d = 4 - 2\varepsilon.$$

$$\Pi(q^2) \overset{\varepsilon\to 0}{\approx} -\frac{2\alpha}{\pi}\int_0^1 \mathrm{d}x\, x(1-x)\left[\frac{1}{\varepsilon} - \ln\left(m^2 - x(1-x)q^2\right) - \gamma_E + \ln 4\pi\right]$$



$$\sim \alpha(q^2 g^{\mu\nu} - q^\mu q^\nu)(\ln\Lambda^2 + \ln q^2)\ \underbrace{\alpha\ln\Lambda^2}_{\text{vertex corr}}$$

- Show that a regularization scheme called Implicit Regularization (IREG), that works entirely in the physical dimension of the model can be implemented to all orders in perturbation theory (**it complies with the BPHZ theorem).**

$$k^\mu k^\nu \cancel{\to \frac{1}{d}} k^2$$

$$I_{log}^{\nu_1\nu_2}(\mu^2) \equiv \int_k \frac{k_{\nu_1} k_{\nu_2}}{(k^2-\mu^2)^3}$$

$$I_{log}(\mu^2) \equiv \int_k \frac{1}{(k^2-\mu^2)^2}$$

1

Usual Dirac algebra.

2

External momenta.

3

BDI

4

The μ^2 dependence by introducing a scale Λ^2.

$$\frac{1}{(k-p)^2-\mu^2} = \frac{1}{k^2-\mu^2} - \frac{p^2-2p\cdot k}{(k^2-\mu^2)[(k-p)^2-\mu^2]}$$

$$I_{log}(\mu^2) = I_{log}(\Lambda^2) - \frac{i}{(4\pi)^2}$$

IREG rules

BPHZ Algorithm

1-loop: $$\frac{1}{(k-p)^2-\mu^2} = \frac{1}{k^2-\mu^2} - \frac{p^2-2p\cdot k}{(k^2-\mu^2)[(k-p)^2-\mu^2]}$$

l-loop: $$\frac{1}{\underbrace{(k_l-q_i)}_{\{p_1,\dots,p_L,k_{l+1},\dots,k_n\}}{}^2-\mu^2} = \sum_{j=0}^{n_i^{(k_l)}-1} \frac{(-1)^j (q_i^2-2q_i\cdot k_l)^j}{(k_l^2-\mu^2)^{j+1}} + \frac{(-1)^{n_i^{(k_l)}} (q_i^2-2q_i\cdot k_l)^{n_i^{(k_l)}}}{(k_l^2-\mu^2)^{n_i^{(k_l)}} [(k_l-q_i)^2-\mu^2]}$$

How do we identify the order in which integrals must be performed?

We need to adapt previous expression to an arbitrary order when we don't have a natural sequence of how integrals can be done. So, we re-write it to evidence the **UV-div** behaviour of amplitudes when the internal momentum goes to infinity in many ways.

$$\boxed{\frac{1}{(k-p_i)^2-\mu^2} = \sum_{l=0}^{2(n_i^{(k)}-1)} f_l^{(k,p_i)} + \bar{f}^{(k,p_i)}}$$

$$f_l^{(k,p_i)} \equiv \sum_{j=0}^{\lfloor l/2 \rfloor} \Theta(B) \binom{l-j}{j} \frac{(-p_i^2)^j (2p_i\cdot k)^{l-2j}}{(k^2-\mu^2)^{l+1-j}}, \quad \bar{f}^{(k,p_i)} \equiv \frac{(-1)^{n_i^{(k)}} (p_i^2-2p_i\cdot k)^{n_i^{(k)}}}{(k^2-\mu^2)^{n_i^{(k)}} [(k-p_i)^2-\mu^2]}$$

$$\Theta(x) \equiv \begin{cases} 0 & \text{if} \quad x \le 0 \\ 1 & \text{if} \quad x > 0 \end{cases}, \quad B \equiv n_i^{(k)} + j - l, \quad \lfloor x \rfloor \equiv \max\{n \in \mathcal{Z} | n \le x\}.$$

$k^{-(l+2)}$ $n_i^{(k)}$

UV-finiture

1. Identify which propagators depend on the external momenta

2. Obtain the minimum value of all $n_j^{(k_i)}$ necessary to guarantee the finitude of terms that contain $\bar{f}^{\,(k_i,\, p_j)}$ as $k_i \to \infty$ in all possible ways;
3. Isolate the UV-divergent terms, allowing a classification in terms of the different ways that the internal momenta approach infinity to be envisaged;
4. Use the rules of IREG, encoded in steps (a)–(c), in the terms identified in step 3 according to their classification;
5. Set aside the divergent terms that contain $I_{\log}^{(l)}(\lambda^2)$ and apply the procedure again on the ones that do not.

Previous JC (08/03/21) :

$$\mathcal{A} = \int_{k_1,k_2} G(p_1,\ldots,p_L,k_1,k_2)H_1(p_1,\ldots,p_L,k_1)H_2(p_1,\ldots,p_L,k_2)$$

$$\Downarrow$$

$$\mathcal{A}_{k_1\to\infty} = \int_{k_2} \bar{H}_2(p_1,\ldots,p_L,k_2)I_{log}(\lambda^2)$$

$$\mathcal{A}_{k_2\to\infty} = \int_{k_1} \bar{H}_1(p_1,\ldots,p_L,k_1)I_{log}(\lambda^2)$$

$$\mathcal{A}_{k_1\to\infty,k_2\to\infty} = \mathcal{F}(p_1,\ldots,p_L)I_{log}(\lambda^2)$$

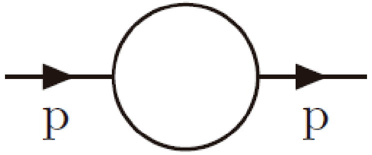
May the four be with you:
Novel IR-subtraction methods to tackle NNLO calculations

arXiv:2012.02567v1

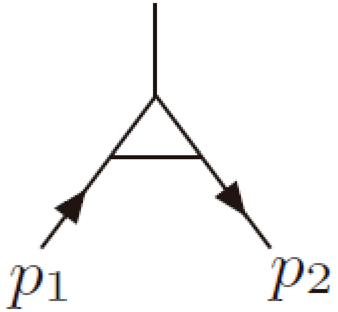
Selected Example: phi3 in d=6

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - \frac{m^2}{2}\phi^2}_{\mathcal{L}_0} + \underbrace{\frac{g}{3!}\phi^3}_{\mathcal{L}_I}$$

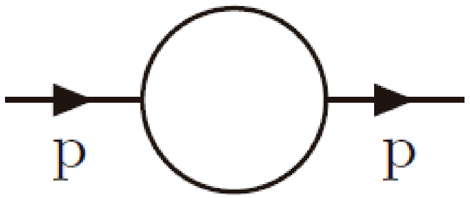
$$\boxed{\Delta(p^2) = \frac{1}{m^2 - p^2 - \Pi(p^2)}}$$



$$\Gamma(p,q) = g(1 + \Lambda(p,q))$$



Example:



Notation

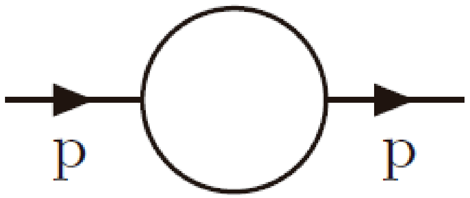
$$\int\limits_k \equiv \int \frac{d^6 k}{(2\pi)^6}$$

$$\Xi^{(1)} \equiv \frac{g^2}{2} \int\limits_k \frac{1}{k^2} \frac{1}{(k-p)^2} = \lim_{\mu^2 \to 0} \frac{g^2}{2} \int\limits_k \frac{1}{(k^2-\mu^2)} \frac{1}{[(k-p)^2-\mu^2]}$$

$$\frac{\Xi^{(1)}}{g^2} = \frac{1}{2} \int\limits_k \frac{1}{(k^2-\mu^2)} \left[\sum_{l=0}^{2(n^{(k)}-1)} f_l^{(k,\,p)} + \bar{f}^{(k,\,p)} \right]$$

$k^{-(l+2)}$ $n_i^{(k)}$

Example:



Notation

$$\int_k \equiv \int \frac{d^6k}{(2\pi)^6}$$

$$b_{2n} \equiv \frac{i}{(4\pi)^n}\frac{(-1)^n}{\Gamma(n)}$$

$$\frac{\Xi^{(1)}}{g^2} = \frac{1}{2}\int_k \frac{1}{(k^2-\mu^2)}\left[\sum_{l=0}^{2(n^{(k)}-1)} f_l^{(k,\,p)} + \bar{f}^{\,(k,\,p)}\right]$$

$k^{-(l+2)}$ $\qquad$ $n^{(k)} = 3$

1. Quadratic divergence

$$\int_k \frac{f_0^{(k,\,p)}}{(k^2-\mu^2)} = \int_k \frac{1}{(k^2-\mu^2)^2},$$

2. Linear divergence

$$\int_k \frac{f_1^{(k,\,p)}}{(k^2-\mu^2)} = \int_k \frac{2p\cdot k}{(k^2-\mu^2)^3},$$

3. Logarithmic divergence

$$\int_k \frac{f_2^{(k,\,p)}}{(k^2-\mu^2)} = \int_k \frac{1}{(k^2-\mu^2)^3}\left[\frac{(2p\cdot k)^2}{(k^2-\mu^2)} - p^2\right] = \boxed{-\frac{p^2}{3} I_{\log}(\mu^2)}$$

$$\boxed{\begin{aligned}\frac{1}{2}\int_k \frac{f_3^{(k,\,p)} + f_4^{(k,\,p)} + \bar{f}^{\,(k,\,p)}}{(k^2-\mu^2)} &= \frac{1}{2}\int_k \frac{1}{(k^2-\mu^2)^4}\left[-4p^2(p\cdot k) + p^4 - \frac{(p^2-2p\cdot k)^3}{(k-p)^2-\mu^2}\right]\\ &= \frac{p^2 b_6}{6}\ln\left(-\frac{p^2}{\mu^2}\right) - \frac{4p^2 b_6}{9} + O(\mu^2).\end{aligned}}$$

Example:

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - \frac{m^2}{2}\phi^2}_{\mathcal{L}_0} + \underbrace{\frac{g}{3!}\phi^3}_{\mathcal{L}_I}$$

Notation

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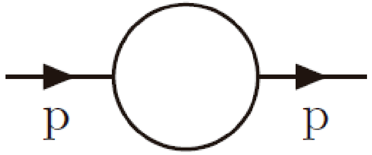
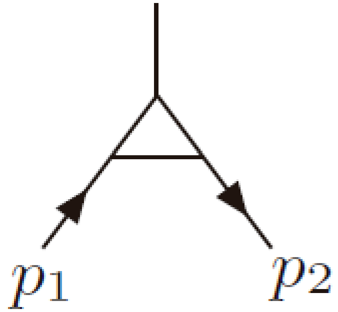
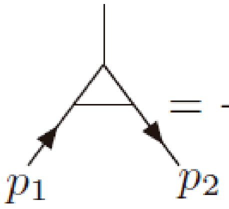


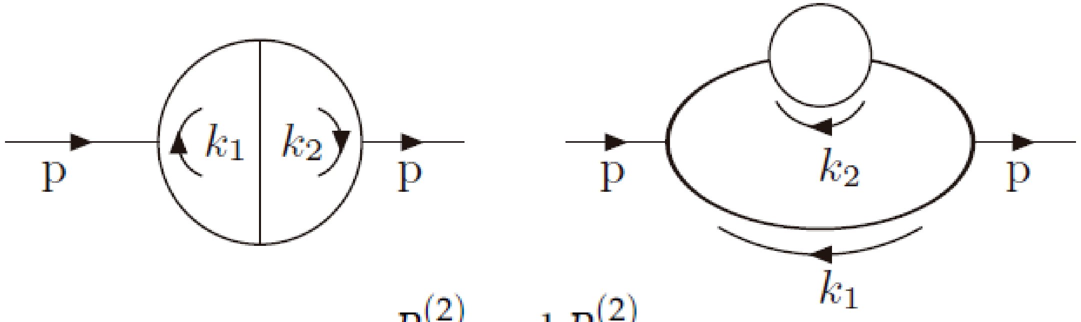
$$\boxed{\Gamma(p,q) = g(1 + \Lambda(p,q))}$$



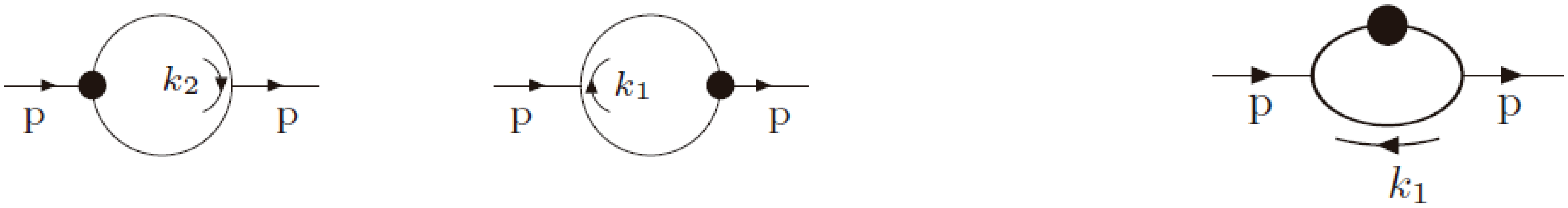


$$= -g^3\left[I_{log}(\lambda^2) - b_6 \ln(-\frac{p_1^2}{\lambda^2}) + 2b_6 - h(p_1, p_2)\right]$$

2-loop example



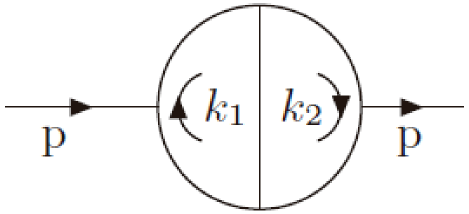
$P_A^{(2)}$ and $P_B^{(2)}$



1. Identify which propagators depend on the external momenta

2-loop example

$P_A^{(2)}$



$$\frac{\Xi_A^{(2)}}{ig^4} = \frac{1}{2}\int_{k_1 k_2} \Delta(k_1)\boxed{\Delta(k_1 - p)}\Delta(k_1 - k_2)\Delta(k_2)\boxed{\Delta(k_2 - p)}, \quad \text{where} \quad \Delta(k_i) \equiv \frac{1}{k_i^2 - \mu^2}.$$

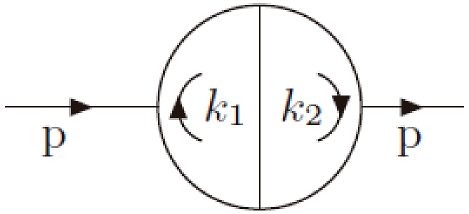
$$\frac{1}{(k - p_i)^2 - \mu^2} = \sum_{l=0}^{2(n_i^{(k)}-1)} f_l^{(k,\, p_i)} + \bar{f}^{(k,\, p_i)},$$

$$\int_{k_1 k_2} \Delta(k_1)\Delta(k_1 - k_2)\Delta(k_2)\left[\sum_{l=0}^{2(n^{(k_1)}-1)} f_l^{(k_1,\, p)} + \bar{f}^{(k_1,\, p)}\right]\left[\sum_{m=0}^{2(n^{(k_2)}-1)} f_m^{(k_2,\, p)} + \bar{f}^{(k_2,\, p)}\right].$$

$n_i^{(k)}$?

2-loop example

$P_A^{(2)}$



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$$\int_{k_1k_2} \Delta(k_1)\Delta(k_1-k_2)\Delta(k_2)\bar{f}^{(k_1,\,p)}\left[\sum_{m=0}^{2(n^{(k_2)}-1)} f_m^{(k_2,\,p)} + \bar{f}^{(k_2,\,p)}\right]$$
$$= \int_{k_1k_2}\Delta(k_1)\Delta(k_1-k_2)\Delta(k_2)\bar{f}^{(k_1,\,p)}\Delta(k_2-p).$$

$$\int_{k_1 k_2} \Delta(k_1)\Delta(k_1 - k_2)\Delta(k_2)\bar{f}^{(k_1,\,p)}\Delta(k_2 - p). \tag{30}$$

We want to assure the finitude of the above integral as $k_1 \to \infty$. Two cases must be considered:

1. Finitude as $k_1 \to \infty$ and k_2 fixed: $n^{(k_1)} > 0$,
2. Finitude as $k_1 \to \infty$ and $k_2 \to \infty$: $n^{(k_1)} > 2$,

which allows us to conclude that $n^{(k_1)}$ should be at least 3. Similarly, we obtain $n^{(k_2)} = 3$.

3. Isolate the UV-divergent terms, allowing a classification in terms of the different ways that the internal momenta approach infinity to be envisaged;

$$\int_{k_1 k_2} \Delta(k_1)\Delta(k_1-k_2)\Delta(k_2)\left[\sum_{l=0}^{2(n^{(k_1)}-1)} f_l^{(k_1,\,p)} + \bar{f}^{(k_1,\,p)}\right]\left[\sum_{m=0}^{2(n^{(k_2)}-1)} f_m^{(k_2,\,p)} + \bar{f}^{(k_2,\,p)}\right].$$

case $k_1 \to \infty$ and k_2 fixed

case $k_2 \to \infty$ and k_1 fixed

$k_1 \to \infty$ and $k_2 \to \infty$ simultaneously

Isolate the UV-divergent terms,

case $k_1 \to \infty$ and k_2 fixed

$$\int_{k_1 k_2} \Delta(k_1)\Delta(k_1-k_2)\Delta(k_2)\left[\sum_{l=0}^{2(n^{(k_1)}-1)} f_l^{(k_1,p)} + \bar{f}^{(k_1,p)}\right]\left[\underbrace{\sum_{m=0}^{2(n^{(k_2)}-1)}}_{\boxed{n^{(k_2)}=3}} f_m^{(k_2,p)} + \bar{f}^{(k_2,p)}\right].$$

$$\int_{k_1 k_2} \Delta(k_1)\Delta(k_2)\Delta(k_1-k_2)\underbrace{f_l^{(k_1,p)}}_{\boxed{k^{-(l+2)}}}\left[\sum_{m=0}^{4} f_m^{(k_2,p)} + \bar{f}^{(k_2,p)}\right]$$

$$A_1^{\Xi} \equiv \int_{k_1 k_2} \Delta(k_1)\Delta(k_2)\Delta(k_1-k_2) f_0^{(k_1,p)}\left[\sum_{m=0}^{4} f_m^{(k_2,p)} + \bar{f}^{(k_2,p)}\right]$$

$$= \int_{k_1 k_2} \Delta^2(k_1)\Delta(k_1-k_2)\Delta(k_2)\Delta(k_2-p).$$

Isolate the UV-divergent terms,

case $k_2 \to \infty$ and k_1 fixed

$$
\begin{aligned}
A_2^{\Xi} &\equiv \int_{k_1 k_2} \Delta(k_1)\Delta(k_2)\Delta(k_1 - k_2) f_0^{(k_2,\, p)} \left[\sum_{l=0}^{4} f_l^{(k_1,\, p)} + \bar{f}^{(k_1,\, p)} \right] \\
&= \int_{k_1 k_2} \Delta^2(k_2)\Delta(k_1 - k_2)\Delta(k_1)\Delta(k_1 - p).
\end{aligned}
$$

A_1^{Ξ} and A_2^{Ξ}

have the same structure.

Isolate the UV-divergent terms,

$k_1 \to \infty$ and $k_2 \to \infty$ simultaneously

$$\int_{k_1 k_2} \Delta(k_1)\Delta(k_1-k_2)\Delta(k_2)\left[\sum_{l=0}^{2(n^{(k_1)}-1)} f_l^{(k_1,\,p)} + \bar{f}^{(k_1,\,p)}\right]\left[\sum_{m=0}^{2(n^{(k_2)}-1)} f_m^{(k_2,\,p)} + \bar{f}^{(k_2,\,p)}\right].$$

$$\underbrace{\int_{k_1 k_2} \Delta(k_1)\Delta(k_2)\Delta(k_1-k_2) f_l^{(k_1,\,p)} f_m^{(k_2,\,p)}}$$

Cases $l=0$ and $m=0,1,2$ are already contained in A_1^{Ξ}

Cases $m=0$ and $l=0,1,2$ are part of A_2^{Ξ}

$$\boxed{l=m=1}\quad \boxed{\begin{aligned} A_3^{\Xi} &\equiv \int_{k_1 k_2} \Delta(k_2)\Delta(k_1-k_2)\Delta(k_1) f_1^{(k_1,\,p)} f_1^{(k_2,\,p)} \\ &= \int_{k_1 k_2} \Delta^3(k_1)\Delta(k_1-k_2)\Delta^3(k_2)(2p\cdot k_1)(2p\cdot k_2). \end{aligned}}$$

$$A_4^{\Xi} \equiv \int\limits_{k_1 k_2} \Delta(k_2)\Delta(k_1 - k_2)\Delta(k_1) f_0^{(k_1,\, p)} f_0^{(k_2,\, p)} = \int\limits_{k_1 k_2} \Delta^2(k_2)\Delta(k_1 - k_2)\Delta^2(k_1)$$

which must be subtracted since it was counted twice.

In summary, the divergent terms are the following:

1. Case $k_1 \to \infty$ and k_2 is fixed

$$A_1^{\Xi} = \int\limits_{k_1 k_2} \Delta^2(k_1)\Delta(k_1 - k_2)\Delta(k_2)\Delta(k_2 - p),$$

2. Case $k_2 \to \infty$ and k_1 is fixed

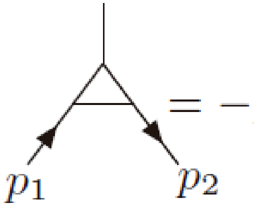
$$A_2^{\Xi} = \int\limits_{k_1 k_2} \Delta^2(k_2)\Delta(k_1 - k_2)\Delta(k_1)\Delta(k_1 - p),$$

3. Case $k_1 \to \infty$ and $k_2 \to \infty$ simultaneously

$$A_3^{\Xi} = \int\limits_{k_1 k_2} \Delta^3(k_1)\Delta(k_1 - k_2)\Delta^3(k_2)(2p \cdot k_1)(2p \cdot k_2).$$

the divergent content of $\Xi_A^{(2)}$ amounts to $A_1^{\Xi} + A_2^{\Xi} + A_3^{\Xi} - A_4^{\Xi}$

$$\int_{k_i} \Delta^2(k_i)\Delta(k_i - k_j), \quad i,j = 1,2 \ \text{and} \ i \neq j \quad =$$



$$= -g^3\left[I_{log}(\lambda^2) - b_6 \ln(-\tfrac{p_1^2}{\lambda^2}) + 2b_6 - h(p_1, p_2)\right]$$

$$p_1 \to k_j \qquad\qquad p_2 = 0$$

$$A_i^{\Xi} = \bar{A}_i^{\Xi} + \alpha_i^{\Xi}, \qquad i,j = 1,2 \quad \text{and} \quad i \neq j$$

$$\bar{A}_i^{\Xi} \equiv \int_{k_j} \Delta(k_j)\Delta(k_j - p)\left[I_{\log}(\lambda^2)\right], \longrightarrow \bar{A}_i^{\Xi} \ (i = 1,2)$$

$$\alpha_i^{\Xi} \equiv b_6 \int_{k_j} \Delta(k_j)\Delta(k_j - p)\left[2 - \ln\left(-\frac{k_j^2 - \mu^2}{\lambda^2}\right)\right].$$

$$b_{2n} \equiv \frac{i}{(4\pi)^n}\frac{(-1)^n}{\Gamma(n)}$$

$$A_3^{\Xi} = \int_{k_1 k_2} \Delta^3(k_1)\Delta(k_1 - k_2)\Delta^3(k_2)(2p \cdot k_1)(2p \cdot k_2).$$

$$\bar{\alpha}_3^{\Xi} \equiv A_3^{\Xi} = b_6 p^2 \left[\frac{I_{\log}(\lambda^2)}{3} \right]$$

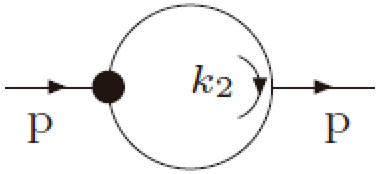
$$\bar{\alpha}_i^{\Xi} \equiv \int_{k_j} \Delta(k_j) f_2^{(k_j,\, p)} \left[-b_6 \ln\left(-\frac{k_j^2 - \mu^2}{\lambda^2} \right) + 2b_6 \right] = b_6 p^2 \left[\frac{I_{\log}^{(2)}(\lambda^2)}{3} - \frac{8}{9} I_{\log}(\lambda^2) \right].$$

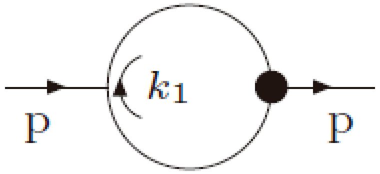
the divergent content of $\Xi_A^{(2)}$ amounts to $A_1^{\Xi} + A_2^{\Xi} + A_3^{\Xi} - A_4^{\Xi}$

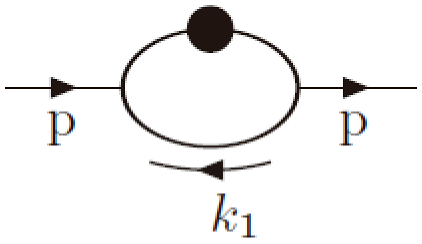
$$\frac{\Xi_A^{(2)\infty}}{ig^4} \equiv \frac{1}{2}\left(\bar{\alpha}_1^{\Xi} + \bar{\alpha}_2^{\Xi} + \bar{\alpha}_3^{\Xi} + \bar{A}_1^{\Xi} + \bar{A}_2^{\Xi}\right)$$

$$\bar{A}_i^{\Xi} \equiv \int_{k_j} \Delta(k_j)\Delta(k_j - p)\left[I_{\log}(\lambda^2)\right]$$

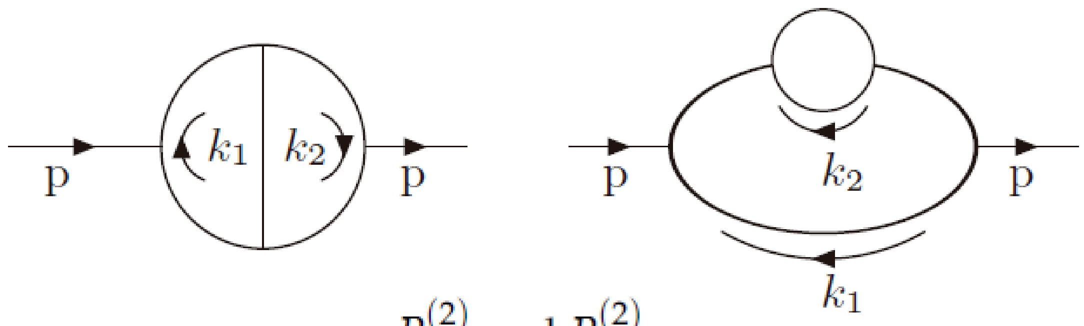
Bogoliubov's recursion formula





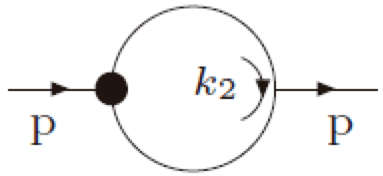


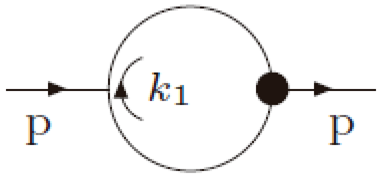
2-loop example

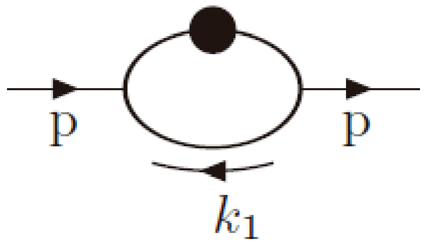


$P_A^{(2)}$ and $P_B^{(2)}$

$$\bar{A}_i^{\Xi} \equiv \int_{k_j} \Delta(k_j)\Delta(k_j - p)\left[I_{\log}(\lambda^2)\right]$$







$$\frac{ig^4}{2}\int_{k_2} \Delta(k_2)\Delta(k_2 - p)\left[-I_{\log}(\lambda^2)\right] = \frac{ig^4}{2}\left(-\bar{A}_1^{\Xi}\right)$$

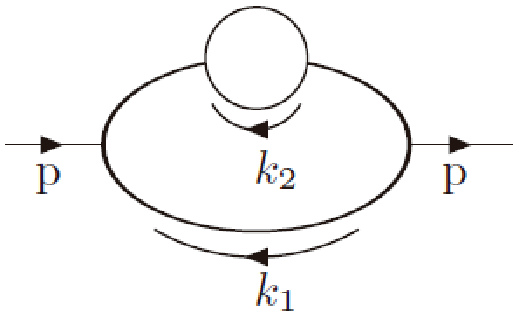
MS in IREG

$$\frac{ig^4}{2}\int_{k_1} \Delta(k_1)\Delta(k_1 - p)\left[-I_{\log}(\lambda^2)\right] = \frac{ig^4}{2}\left(-\bar{A}_2^{\Xi}\right)$$

$$\frac{\Xi_A^{(2)}}{ig^4} \equiv \frac{b_6 p^2}{6}\left[2 I_{\log}^{(2)}(\lambda^2) - \frac{13}{3} I_{\log}(\lambda^2) + \text{finite}\right]$$

2-loop example

$P_B^{(2)}$



$$\frac{\Xi_B^{(2)}}{ig^4} = \frac{1}{2}\int_{k_1k_2} \Delta^2(k_1)\Delta(k_1-p)\Delta(k_2)\Delta(k_1-k_2)$$

$k_2 \to \infty$ and k_1 fixed

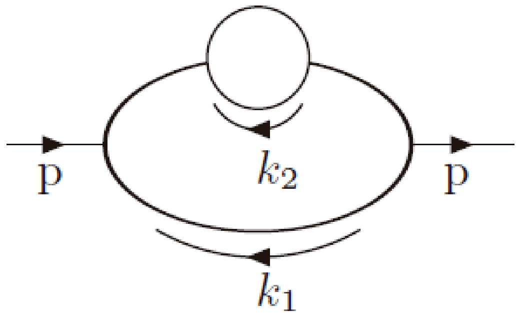
$$\int_{k_1k_2} \Delta^2(k_1)\Delta(k_1-k_2)\Delta(k_2)\left[\sum_{l=0}^{4} f_l^{(k_1,p)} + \bar{f}^{(k_1,p)}\right] = \int_{k_1k_2} \Delta^2(k_1)\Delta(k_1-k_2)\Delta(k_2)\Delta(k_1-p).$$

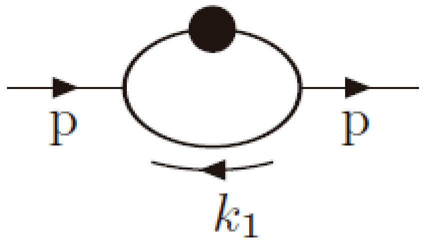
$$\boxed{\begin{aligned} B_1^{\Xi} &= \bar{B}_1^{\Xi} + \beta_1^{\Xi}, \\ \bar{B}_1^{\Xi} &\equiv \int_{k_1} \Delta(k_1)\Delta(k_1-p)\left[-\frac{I_{\log}}{3}(\lambda^2)\right], \\ \beta_1^{\Xi} &\equiv \frac{b_6}{3}\int_{k_1} \Delta(k_1)\Delta(k_1-p)\left[\ln\left(-\frac{k_1^2-\mu^2}{\lambda^2}\right) - \frac{8}{3}\right] \end{aligned}}$$

$$\longrightarrow \quad \beta_1^{\Xi} \equiv \frac{b_6}{3}\int_{k_1} \Delta(k_1)\left[\sum_{l=0}^{2} f_l^{(k_2,p)}\right]\left[\ln\left(-\frac{k_1^2-\mu^2}{\lambda^2}\right)\right] - \frac{8}{3}\Bigg] = -\frac{b_6 p^2}{9}\left[I_{\log}^{(2)}(\lambda^2) - \frac{10}{3}I_{\log}(\lambda^2)\right].$$

2-loop example

$P_B^{(2)}$





$$\frac{\Xi_B^{(2)\infty}}{ig^4} \equiv \frac{1}{2}\left(\bar{\beta}_1^{\Xi} + \bar{B}_1^{\Xi}\right)$$

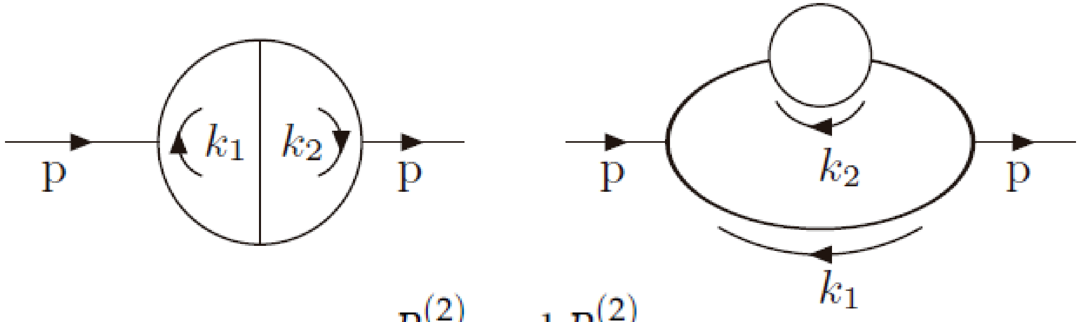
$$\bar{B}_1^{\Xi} \equiv \int_{k_1} \Delta(k_1)\Delta(k_1-p)\left[-\frac{I_{\log}}{3}(\lambda^2)\right],$$

$$\bar{\beta}_1^{\Xi} \equiv \frac{b_6}{3}\int_{k_1}\Delta(k_1)\left[\sum_{l=0}^{2} f_l^{(k_2,\,p)}\right]\Big[\ln\Big(-\frac{k_1^2-\mu^2}{\lambda^2}\Big)\Big] - \frac{8}{3}\Big] = -\frac{b_6 p^2}{9}\left[I_{\log}^{(2)}(\lambda^2) - \frac{10}{3}I_{\log}(\lambda^2)\right].$$

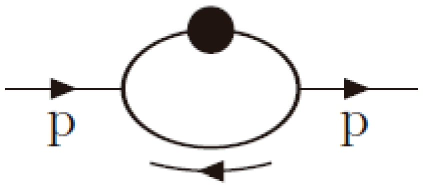
$$\frac{ig^4}{2}\int_{k_1}\Delta(k_1)\Delta(k_1-p)\left[\frac{1}{3}I_{log}(\lambda^2)\right]$$

$$\frac{\Xi_B^{(2)}}{ig^4} \equiv \frac{b_6 p^2}{18}\left[-I_{\log}^{(2)}(\lambda^2) + \frac{10}{3}I_{\log}(\lambda^2) + \text{finite}\right]$$

2-loop example



$P_A^{(2)}$ and $P_B^{(2)}$



$$\frac{\bar{\Xi}_A^{(2)}}{ig^4} \equiv \frac{b_6 p^2}{6}\left[2 I_{\log}^{(2)}(\lambda^2) - \frac{13}{3} I_{\log}(\lambda^2) + \text{finite}\right]$$

$$\frac{\bar{\Xi}_B^{(2)}}{ig^4} \equiv \frac{b_6 p^2}{18}\left[- I_{\log}^{(2)}(\lambda^2) + \frac{10}{3} I_{\log}(\lambda^2) + \text{finite}\right]$$

$$\bar{\Xi}_{\text{div}}^{(2)} \equiv \left(\bar{\Xi}_A^{(2)} + \bar{\Xi}_B^{(2)}\right)_{\text{div}} = ig^4 \frac{p^2}{6}\left[\frac{5b_6}{3} I_{\log}^{(2)}(\lambda^2) - \frac{29 b_6}{9} I_{\log}(\lambda^2)\right]$$

… and of course: RG

$$\Xi_{ct} = -i\frac{g^2}{6} I_{\log}(\lambda^2) - \frac{g^4}{6}\left[\frac{5b_6}{3} I^{(2)}_{\log}(\lambda^2) - \frac{29b_6}{9} I_{\log}(\lambda^2)\right];$$

$$\Lambda_{ct} = -ig^2 I_{\log}(\lambda^2) - g^4\left[\frac{5b_6}{2} I^{(2)}_{\log}(\lambda^2) - \frac{17b_6}{3} I_{\log}(\lambda^2)\right].$$

- After you have the 1 and 2-loop c.t, you can obtain the renormalization group functions.

$$\phi_o \equiv Z_\phi^{\frac{1}{2}}\phi, \qquad g_o \equiv Z_g g, \quad \Xi_{ct} \equiv Z_\phi - 1, \quad \Lambda_{ct} \equiv Z_g Z_\phi^{\frac{3}{2}} - 1,$$

$$\gamma \equiv \lambda \frac{\partial \ln Z_\phi}{\partial \lambda}, \quad \beta \equiv -g\lambda \frac{\partial \ln Z_g}{\partial \lambda},$$

$$\downarrow$$

$$\boxed{\begin{aligned} \gamma &= \frac{g^2}{6(4\pi)^3} + \frac{13g^4}{216(4\pi)^6} + O(g^6), \\ \beta &= -\frac{3g^3}{4(4\pi)^3} - \frac{125g^5}{144(4\pi)^6} + O(g^6). \end{aligned}}$$

Summary: Gauge theories 1-loop QED

$$i\Pi_{\mu\nu} = -\frac{4e^2}{3}\left[I_{\log}(\lambda^2) - b\ln\left(-\frac{p^2}{\lambda^2}\right) + \frac{5}{3}b\right](g_{\mu\nu}p^2 - p_\mu p_\nu)$$

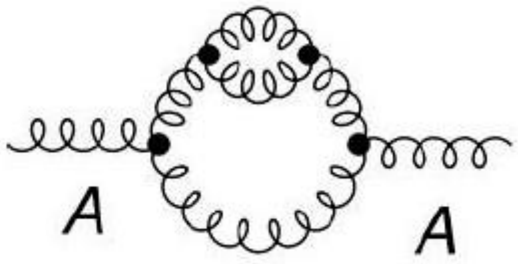
$$i\Sigma(p) = (e)^2 \not{p}\left\{I_{\log}(\lambda^2) - b\ln\left(-\frac{p^2}{\lambda^2}\right) + 2b\right\}$$

$$i\Lambda_\mu = e^3\gamma_\mu I_{\log}(\lambda^2).$$

$$\psi^0 = \sqrt{Z_2}\psi, \quad A_\mu^0 = \sqrt{Z_3}A_\mu, \quad e_0 = Z_e e.$$

$$\Lambda_{ct} = Z_1 - 1, \quad \Sigma_{ct} = Z_2 - 1, \quad \Pi_{ct} = Z_3 - 1, \quad Z_1 = Z_2, \quad Z_3 = 1 + \frac{4}{3}ie^2 I_{\log}(\lambda^2)$$

Two-loop functions



Background Field Method

$$Z_g = Z_A^{-1/2} \longrightarrow Z_A = 1 + \frac{g^2}{(4\pi)^2} Z_A^{(1)} + \frac{g^4}{(4\pi)^4} Z_A^{(2)}$$

for QCD

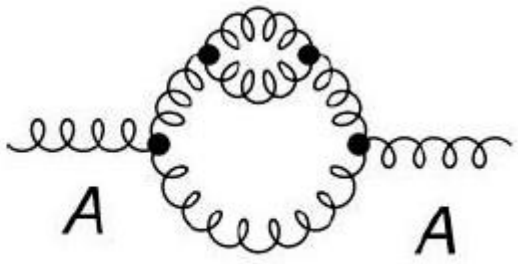
$$Z_A^{(1)} = \left(\frac{11}{b} - \frac{2}{3b} n_f \right) I_{\log}(\lambda^2),$$

$$Z_A^{(2)} = \frac{54}{b^2} \left[I_{\log}^2(\lambda^2) - 2b I_{\log}^{(2)}(\lambda^2) \right] + \left(\frac{210}{b} - \frac{38}{3b} n_f \right) I_{\log}(\lambda^2)$$

for QED

$$Z_A^{(1)} = -\frac{4}{3b} I_{\log}(\lambda^2), \quad Z_A^{(2)} = -\frac{4}{b} I_{\log}(\lambda^2).$$

Two-loop functions



Background Field Method

$$Z_g = Z_A^{-1/2}$$

$$\beta = -g\left[\beta_0\left(\frac{g}{4\pi}\right)^2 + \beta_1\left(\frac{g}{4\pi}\right)^4\right]$$

$$\text{QED}: \qquad \beta_0 = -\frac{4}{3}; \qquad \beta_1 = -4;$$

$$\text{QCD}: \qquad \beta_0 = 11 - \frac{2}{3}n_f; \quad \beta_1 = 102 - \frac{38}{3}n_f.$$

- UV part complies with non-abelian gauge invariance.

4. Concluding Remarks

The purpose of this review was to present in a pedagogical way how the IREG method is implemented to comply with the powerful framework of BPHZ, which is based on the fundamental principles of quantum field theory, unitarity, causality and locality. An algorithm has been shown that delivers the integrals involved in multi-loop amplitudes being decomposed in structures that are identified as the counterterms and divergencies of the order according to the BPHZ scheme. Various examples, ranging from the cubic scalar theory in six space time dimensions, to QED and QCD in the background field method have been worked out, highlighting the procedure. A further benefit of the method is that it automatically delivers all the necessary ingredients to obtain renormalization group functions, of which we have presented the beta functions to two-loop order of the above-mentioned theories, with known universal coefficients.