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Introduction to the Background Field Method

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Paper structure

(the structure I will follow in this presentation)

1. Introduction

2. The BF Formalism
   A. Review of Functional Methods
   B. The BFM
   C. Gauge Theories and the BF gauge

3. BF calculations
   A. Feynman Rules
   B. Renormalization
   C. Calculation of the YM β-function

4. Conclusions

INTRODUCTION TO THE BACKGROUND FIELD METHOD*

BY L. F. ABBOTT**

CERN, Geneva

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The background field approach to calculations in gauge field theories is presented. Conventional functional techniques are reviewed and the background field method is introduced. Feynman rules and renormalization are discussed and, as an example, the Yang--Mills β function is computed.

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Introduction

- BFM is a technique for quantizing gauge field theories without losing explicit gauge invariance… what does this mean?

$\mathcal{L}$ $\Rightarrow$ Feynman Rules

$\Rightarrow$ Compute Physical Quantities

Quantities with no direct physical interpretation will not be gauge invariant

(E.X: Green's Functions)

Which are gauge invariant and independent of the particular gauge chosen

- The BF approach arranges things so that gauge invariance in $\mathcal{L}$ is still present once gauge fixing and ghost terms have been added.

BFM $\rightarrow$ $\mathcal{L}$ $\Rightarrow$ Green's function

$\Rightarrow$ Divergent counterterms

Gauge Invariant!

- The approach of the BFM is do a “field-shifting”: the background field is added to the quantum field in the action (S=QF+BF). After this, the method allows to fix a gauge and evaluate the quantum corrections without breaking the background gauge symmetry.

- The other main feature of the method is that the choice of gauge, for which the effective action will be a gauge invariant of the BF, relates the charge and BF renormalizations.

$$Z_g = Z_A^{-1/2} (1)$$

- Abbott discusses in this first paper the BFM and how to apply it to a pure YM computation. At the end, he gives an explicit example describing the calculation of the YM β-function in the BFM using DReg.

Review of Functional Methods

- As the BFM relies on functional methods in field theory, Abbott does a reviewing of the standard functional techniques.

Green's functions

QFT ➡ *S-matrix*

- Green's functions are determined by taking functional derivatives with respect to the source function J of the generating functional.

$$Z[J]=\int \delta Q \exp i\left[S[Q]+\overbrace{J\cdot Q}^{\int d^4x\,(JQ)}\right] \quad (2)$$

- The Green's functions are define by:

$$\int \delta Q (Q \ldots Q) e^{iS[Q]} = \langle 0 | Q \ldots Q | 0 \rangle = \left(\frac{1}{i} \frac{\delta}{\delta J}\right)^n Z[J] \Big|_{J=0} \quad (3)$$

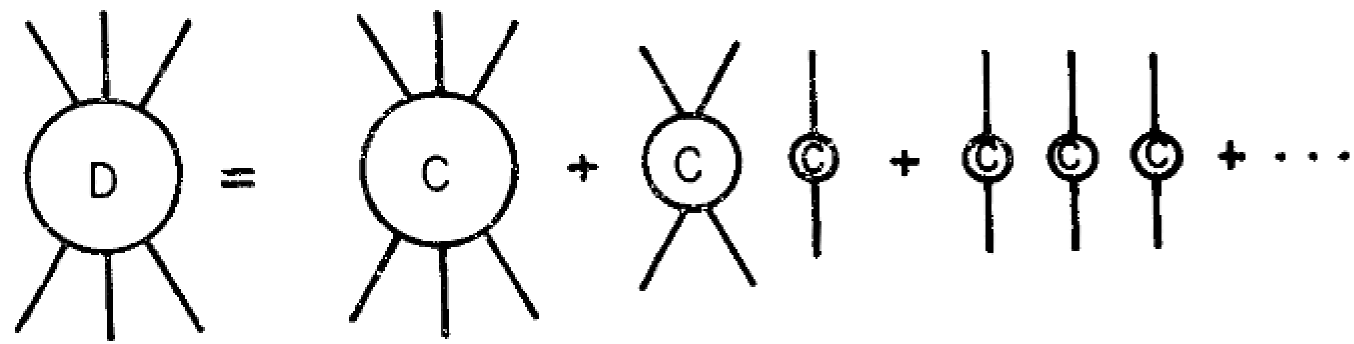


Fig. 1. The relation between connected and disconnected graphs. D = disconnected, C = connected

- As the disconnected pieces of Green's functions don't contribute to the S-matrix, it's better to work only with connected Green's functions.

$$W[J] = -i \ln(Z[J]) \quad (4)$$

- Abbott shows graphically how disconnected pieces can be removed just by taking ln(Z[J]). For that, he constructed the first few Green functions from previous Eq.(4)

$$W[J] = -i \ln(Z[J])$$

$$\frac{\delta W[J]}{\delta J} = \frac{\delta}{\delta J}(-i \ln(Z[J])) = -i \frac{1}{Z[J]} \frac{\delta}{\delta J} Z[J] = \frac{1}{\langle 0|0\rangle} \langle 0|Q|0\rangle \quad (5)$$

For the 2nd derivative:

$$\frac{1}{i} \frac{\delta^2 W[J]}{\delta J^2} = \left[\frac{\langle 0|T\{QQ\}|0\rangle}{\langle 0|0\rangle} - \left(\frac{\langle 0|Q|0\rangle}{\langle 0|0\rangle} \right)^2 \right] \quad (6)$$

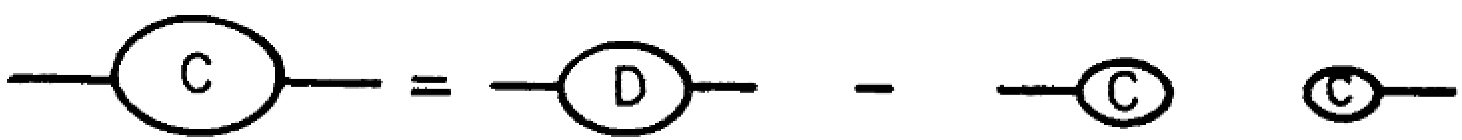


Fig. 2. The relation between the connected (C) and disconnected (D) two-point functions

- We can further simplify these expressions by write the connected Green's functions in terms of 1PI.

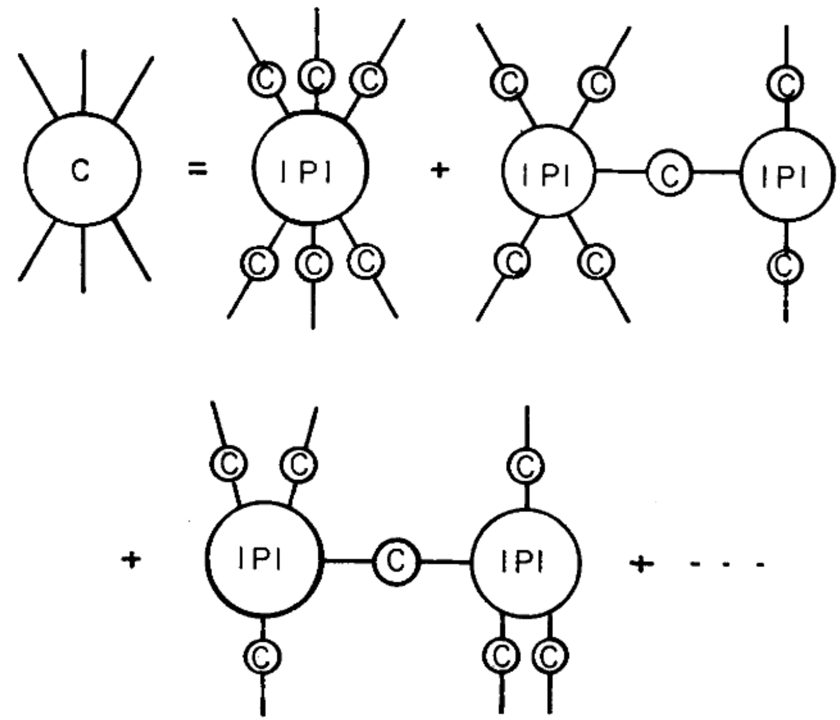


Fig. 4. The relation between connected (C) and 1PI graphs

- The 1PI Green's function are generated by the effective action.

$$\Gamma[\overline{Q}] = W[J] - \underbrace{J\cdot\overline{Q}}_{Q \neq \overline{Q} \equiv \frac{\delta W}{\delta J}} \quad (7)$$ ¿

- He didn't show that the effective action and the vacuum expectation value of Q are the 1PI Green's functions. But he study the first cases as he did for the connected and disconnected graphs.

$$\frac{\delta\Gamma[\overline{Q}]}{\delta\overline{Q}} = -J \quad (9)$$

Quantum-mechanical field equation for the vacuum expectation value of Q

For the 2nd derivative:

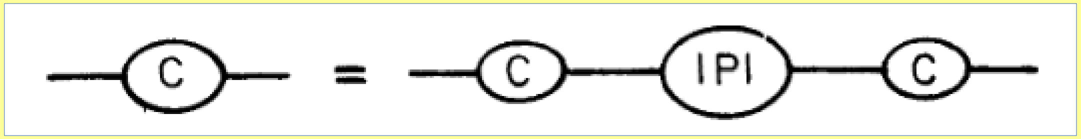
$$\frac{\delta^2\Gamma[\overline{Q}]}{\delta\overline{Q^2}} = \frac{-\delta J}{\delta\overline{Q}} \quad (10)$$

The inverse derivative of the effective action is the propagator:

$$\frac{-\delta J}{\delta\overline{Q}} = -\left[\frac{\delta}{\delta J}\left(\frac{\delta W}{\delta J}\right)\right]^{-1} = \left[\frac{-\delta^2 W}{\delta J^2}\right] = iD^{-1} \quad (11)$$

- The 1PI 2-point function is the inverse of the propagator. The full propagator is obtained from the 1PI 2-point function by dressing the two external legs with propagators.

$$D = D \frac{1}{i} \frac{\delta^2 \Gamma}{\delta \overline{Q}^2} D(12)$$



- This continual amputation of the external propagators is what keeps the diagrams 1PI.

- Abbott continues these examples for the 3-point and 4-point function.

The Background Field Method

$$\Gamma[\overline{Q}] \Rightarrow \mathcal{S}\text{-}matrix$$

- The BFM compute the effective action. Abbott will consider (for this part of the paper) non-gauge theories. For them, the BFM is identical to the conventional "field-shifting" method.

Let us remember: $$Z[J] = \int \delta Q \exp i[S[Q] + J \cdot Q]$$

And define the analogous quantity:

Classical action

$$\widetilde{Z}[J, \Phi] = \int \delta Q \exp i[S[Q+\Phi] + J \cdot Q] \quad (13)$$

Alternative source

Arbitrary BF Φ

By the same analogy: $$W[J] = -i \ln(Z[J]) \rightarrow \widetilde{W}[J, \Phi] = -i \ln(\widetilde{Z}[J, \Phi]) \quad (14)$$

$$\widetilde{Z}[J,\Phi]=\int \delta Q \exp i[S[Q+\Phi]+J\cdot Q]$$

$$\widetilde{W}[J,\Phi]=-i\ln(\widetilde{Z}[J,\Phi])$$

$$\overline{Q}=\frac{\delta W}{\delta J}\rightarrow\widetilde{Q}=\frac{\delta\widetilde{W}}{\delta J}(15)$$

$$\Gamma[\overline{Q}]=W[J]-J\cdot\overline{Q}\rightarrow\widetilde{\Gamma}[\widetilde{Q},\Phi]=\widetilde{W}[J,\Phi]-J\cdot\widetilde{Q}(16)$$

What is the point of all these definitions ?

Compute the effective action!

Let us see this by shifting the field Q in the first equation:

$$\widetilde{Z}=\int \delta Q \exp i[S[Q-\Phi+\Phi]+J\cdot(Q-\Phi)]$$

$$\ln(\widetilde{Z}[J,\Phi]=Z[J]\exp(-iJ\cdot\Phi))\times -i$$

$$\widetilde{Q}=\overline{Q}-\Phi(18) \longleftarrow \widetilde{W}[J,\Phi]=W[J]-J\cdot\Phi(17)$$

Using all these previous Eq.

$$\left.\begin{aligned}\tilde{\Gamma}[\tilde{Q},\Phi]&=\tilde{W}[J,\Phi]-J\cdot\tilde{Q}\\ \tilde{W}[J,\Phi]&=W[J]-J\cdot\Phi\\ \tilde{Q}&=\overline{Q}-\Phi\end{aligned}\right\}$$

$$\tilde{\Gamma}[\tilde{Q},\Phi]=\Gamma[\overline{Q}]$$

$$\underbrace{\tilde{\Gamma}[\tilde{Q},\Phi]}_{\textit{the BF effective action}}=\Gamma[\tilde{Q}+\Phi] \quad (18)$$

$$\xrightarrow{\tilde{Q}=0}$$

- Therefore, as you can see in Eq.(19) the BFM allows you to compute the effective action by summing only vacuum graphs.

$$\tilde{\Gamma}[0,\Phi]=\Gamma[\Phi] \quad (19)$$

$$\frac{\delta\tilde{\Gamma}[\tilde{Q},\Phi]}{\delta\tilde{Q}}$$ ⇒ 1PI Green's function in the presence of a BF

$$\tilde{\Gamma}[0,\Phi]$$ ⇒ Generates no graphs with external lines

How we compute the effective action? Abbott’s approach is to treat the BF Φ perturbatively:

1. Consider 1PI vacuum graphs of Q with Φ fields appearing as external lines.
2. Then you can use S[Q+Φ] to compute the Feynman Rules.

$\mathcal{S}$ ⇒ Q^2 → Ordinary Q field propagators

$\mathcal{S}$ ⇒ Non-quadratic parts → Interactions

Gauge theories and the BFM

- This section is for gauge theories where you must choose a gauge. In this case, Abbott applies the BF approach for a non-Abelian gauge theory.

$$Z[J]=\int \delta Q \exp i[S[Q]+J\cdot Q]$$

$$Z[J]=\int \delta Q \, det\left[\frac{\delta G^a}{\delta w^b}\right]\exp i[S[Q]-\frac{1}{2\alpha}\overbrace{G\cdot G}^{\int d^4 x G^a G^a}+\overbrace{J\cdot Q}^{\int d^4 x J^a_\mu Q^a_\mu}]\quad(20)$$

Generating functional for a non-Abelian gauge theory

- Now, Q is a gauge field and the action:

$$S=\frac{-1}{4}\int d^4 x (F^a_{\mu\nu})^2 \quad (21)$$

$$F^a_{\mu\nu}=\partial_\mu Q^a_\nu-\partial_\nu Q^a_\mu+g f^{abc} Q^b_\mu Q^c_\nu \quad (22)$$

- The basic idea of the BFM, as we saw before, is to write the gauge field Q as a sum of a BF and a quantum fluctuation. The Z[J] for a non-abelian gauge theory is invariant under the infinitesimal gauge transformation:

$$\delta Q_\mu^a = -f^{abc} w^b Q_\mu^c + \frac{1}{g} \partial_\mu w^a \quad (23)$$

- Now with this information: which is the background field generating functional for a gauge theory? We need to expand the Z[J] expression doing Q=Q+A.

$$\tilde{Z}[J,\Phi] = \int \delta Q \exp i [S[Q+\Phi] + J\cdot Q]$$

$$Z[J,A] = \int \delta Q \, det\left[\frac{\delta \tilde{G}^a}{\delta w^b}\right] \exp i\left[S[Q+A] - \frac{1}{2\alpha}\tilde{G}\cdot\tilde{G} + J\cdot Q\right] \quad (24)$$

Z[J,A] is invariant under: $\delta Q_\mu^a = -f^{abc} w^b (Q_\mu^a + A_\mu^c) + \frac{1}{g}\partial_\mu w^a \quad (25)$

- Shifting the integration variable in the functional integral, gives analogous expressions for the gauge theory as Abbott did for the non-gauge one.

$$\tilde{\Gamma}[\tilde{Q},\Phi]=\Gamma[\tilde{Q}+\Phi] \qquad \tilde{\Gamma}[\tilde{Q},A]=\Gamma[\tilde{Q}+A] \quad (26)$$

$$\tilde{\Gamma}[0,\Phi]=\Gamma[\Phi] \qquad \tilde{\Gamma}[0,A]=\Gamma[A] \quad (27)$$

- The BF effective action with the background gauge field and all the external legs, is a generating functional for 1PI graphs in a particular gauge.
- The BFM retains explicit gauge invariance, there's a choice of the gauge fixing term for which the BF effective action in a non-abelian theory is a gauge invariant functional of A, that gauge choice is:

$$\tilde{G}^a=\partial_\mu Q_\mu^a+gf^{abc}A_\mu^b Q_\mu^c \quad (28)$$

- With this choice Z[J,A] is invariant under:

$$\delta A_\mu^a=-f^{abc}w^b A_\mu^c+\frac{1}{g}\partial_\mu w^a \quad (29)$$

$$\delta J_\mu^a=-f^{abc}w^b J_\mu^c \quad (30)$$

$\tilde{\Gamma}[0,A]=\Gamma[A]$ → It's a gauge invariant function of A. → $\dfrac{\delta\tilde{\Gamma}[0,A]}{\delta A}$ Green's function will obey Ward identities of gauge invariance

Interlude

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BFM: Feynman Rules

$\widetilde{\Gamma}[0, A]$ Generate 1PI Green's function

Compute using Feynman rules

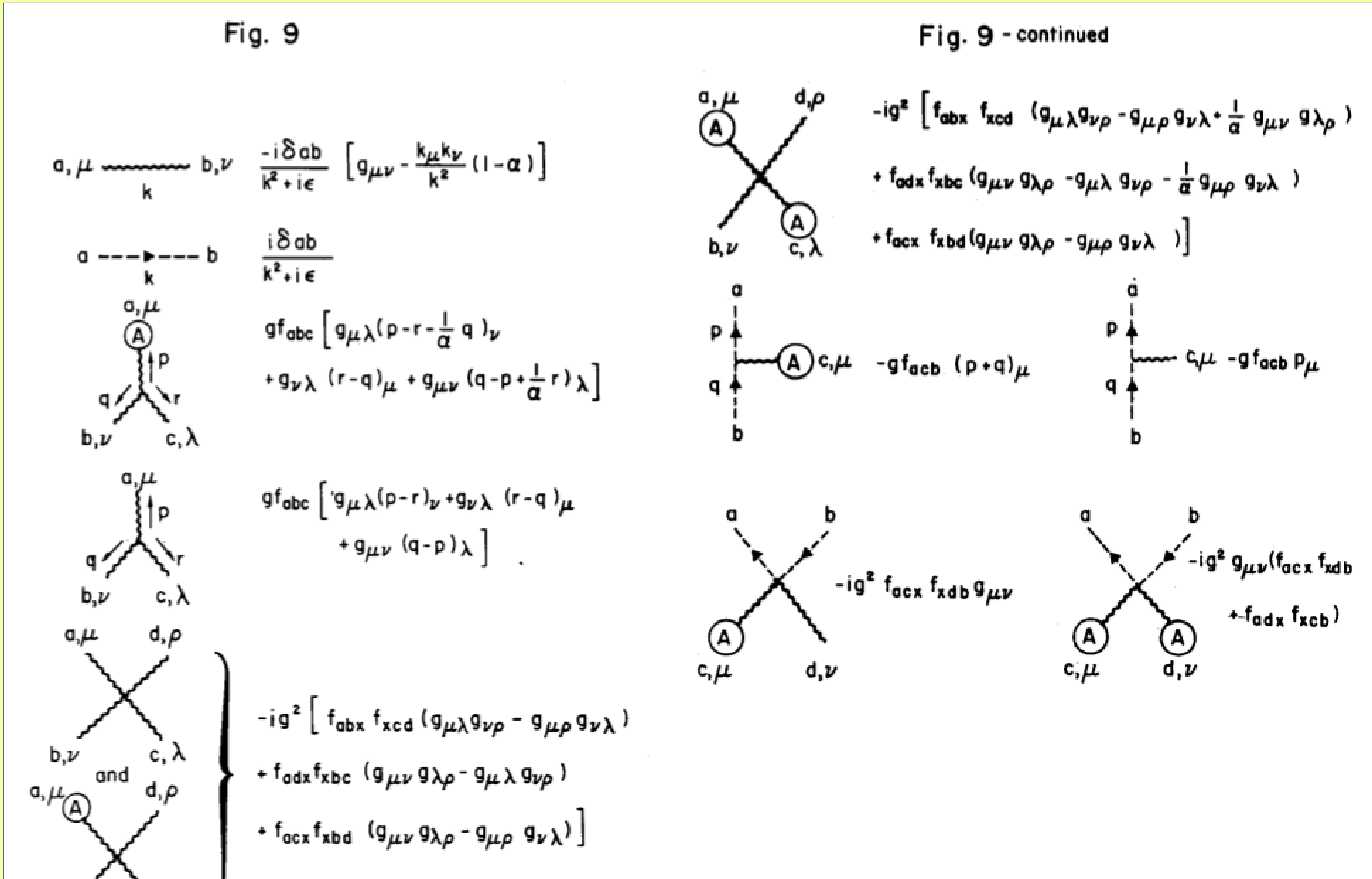


Fig. 9. Feynman rules for background field calculations in Yang-Mills theory. Wavy lines are Q field propagators, wavy lines terminating in an A represent external gauge particles and dashed lines represent ghost propagators

Renormalization

- We are going to find divergences which we must renormalize. Abbott does that with:

c.v

$$(A_\mu)_0 = Z_A^{1/2} A_\mu \quad (40)$$

$$g_0 = Z_g\, g \quad (41)$$

$$\alpha_0 = Z_\alpha\, \alpha \quad (42)$$

Q & Ghost: don't have to be renormalized

(as these fields only appears in loops, when you renormalized them, they cancel with the renormalization factor of the vertices)

$$\tilde{\Gamma}[0, A] \xrightarrow{\tilde{G}^a} \text{invariant}$$

$$Z_g = Z_A^{-1/2}$$

- Infinities in the effective action must take the gauge invariant form of a divergent constant times the gauge field strength tensor.

$$\left(F^a_{\mu\nu}\right)_0 = Z_A^{1/2}\left[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \underbrace{Z_g Z_A^{1/2}}_{Z_g = Z_A^{-1/2}} g f^{abc} A^b_\mu A^c_\nu\right] \quad (43)$$

Abbott says that the easiest way to renormalize a YM theory is to use DReg and MS:

1. All loop momentum integrals $\rightarrow \int d^{4-2\epsilon}x$
2. Write the renormalization constants as sums over poles in ε

$$Z_A = 1 + \sum_{n=1}^{\infty} \frac{Z_A^{(n)}}{\epsilon^n} \quad (44)$$

3. For a dimensionless coupling constant, we must introduce an arbitrary mass parameter → $g_0 = Z_g \mu^\epsilon g$

- Since μ is arbitary, we must require that the bare g be independent of μ which give us the β-function. This function is related to the dependence of the coupling constant renormalization, Zg, on the renormalization mass parameter, μ.

$$\mu\frac{\partial g_0}{\partial\mu}=0=Z_g\mu^{\epsilon}[\epsilon g+g\mu\frac{\partial\ln(Z_g)}{\partial\mu}+\underbrace{\mu\frac{\partial g}{\partial\mu}}_{\beta}](45)$$

$$\beta=-\epsilon g-g\mu\frac{\partial\ln(Z_g)}{\partial\mu}(46)$$

Using the chain result $\mu\frac{\partial}{\partial\mu}=\mu\frac{\partial g}{\partial\mu}\frac{\partial}{\partial g}=\beta\frac{\partial}{\partial g}$ and $Z_g=Z_A^{-1/2}$

$$\beta=-\epsilon g+\frac{1}{2}g\beta\frac{\partial\ln(Z_A)}{\partial g}(47)$$

- There the β-function (in a non-Abelian theory) can be compute **only** with the knowledge of the BF 2-point function. No vertex functions needs to be consider. Only the gauge propagator is needed.

$$\beta = -\epsilon g + \frac{1}{2} g \beta \frac{\partial \ln(Z_A)}{\partial g}$$

$$\beta\left(1 - \frac{1}{2} g \frac{\partial \ln(Z_A)}{\partial g}\right) = -\epsilon g$$

$$\beta = -\epsilon g \left(1 + \frac{1}{2} g \frac{\partial \ln(Z_A)}{\partial g}\right)$$

To proceed, we need to remember that the BF renormalized constant can be expanded in the dimensionless coupling constant (as a series of poles):

$$Z_A = 1 + \sum_{n=1}^{\infty} \frac{Z_A^{(n)}}{\epsilon^n} \qquad \ln(1+x) \approx x - \frac{x^2}{2} \ldots$$

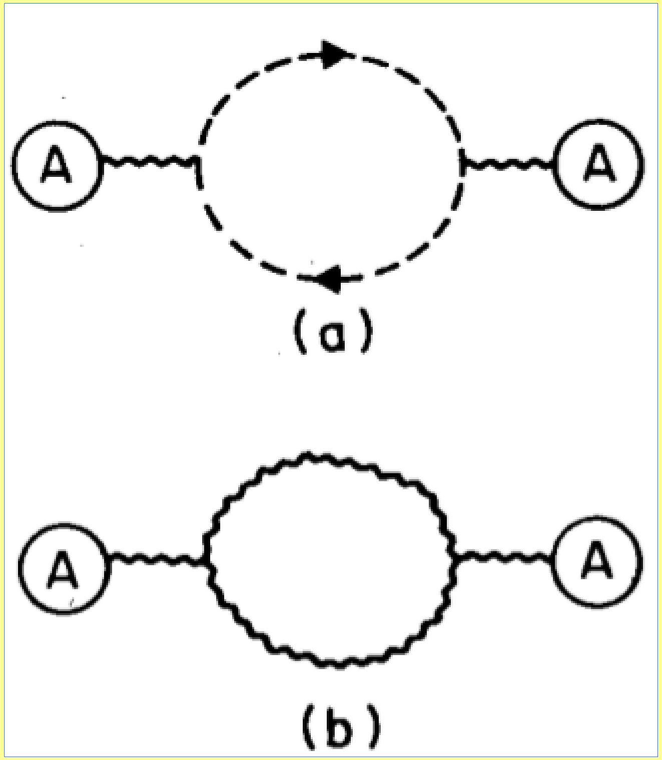
At one-loop level: $\ln(Z_A) = \frac{Z_A^{(1)}}{\epsilon^1}$ Counterterm of the 1st order

Finally: $\beta = -\epsilon g - \left(\frac{1}{2} \frac{g}{\epsilon^1} \frac{\partial}{\partial g} Z_A^{(1)}\right) \epsilon g \xrightarrow{\epsilon \to 0} \beta = \frac{-1}{2} g^2 \frac{\partial Z_A^{(1)}}{\partial g}$ (48)

Calculation of the YM β-function

$$\frac{ig^2 C_A \delta^{ab}}{(4\pi)^2}\left(\frac{1}{3\epsilon}\right)[g_{\mu\nu}k^2 - k_\mu k_\nu] \quad (49)$$

$$\frac{ig^2 C_A \delta^{ab}}{(4\pi)^2}\left(\frac{10}{3\epsilon}\right)[g_{\mu\nu}k^2 - k_\mu k_\nu] \quad (50)$$



$$\beta = \frac{-11 C_A}{3}\frac{g^3}{(4\pi)^2} \quad (51)$$

Conclusions

> The background field formalism now exists for calculations in gauge theories to arbitrary numbers of loops. The method can be used for general discussions about the structure and particularly about the divergences of gauge theories, and for specific calculations. In both of these applications, the retention of explicit gauge invariance made possible by the background field gauge should be extremely useful.
>
> I am very grateful to the organizers of the Cracow School for their hospitality.

- BFM is useful to compute the effective action of a pure YM theory by expanding the field around a classical BF.
- In the BFM a gauge is choose in order to make Green's functions gauge invariant and respect Ward identities like the ones in QED. These relations makes that the knowledge of 2-point functions is the only one necessary to compute the β-function and simplifies calculations.