Physics @ UFABC

The beta-function of non-abelian gauge theories to two-loop order in Implicit Regularization

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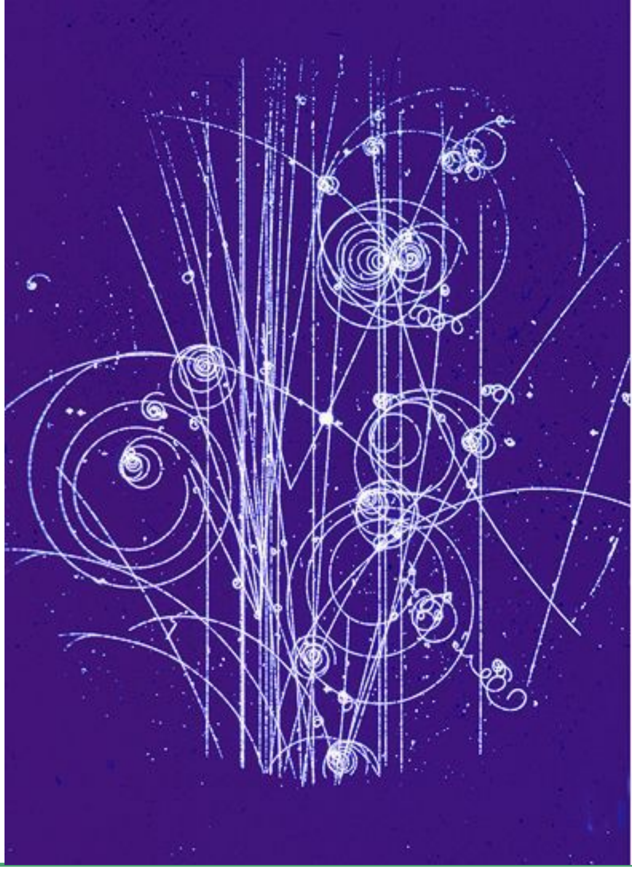
Advisors: Dr. Marcos Sampaio and Dr. Adriano Cherchiglia

September 2020

Outline

1. Motivation.
2. Regularization schemes.
3. Practical approaches to 2-loop calculations.
4. 2-loop applications: results.
5. Conclusions and perspectives.



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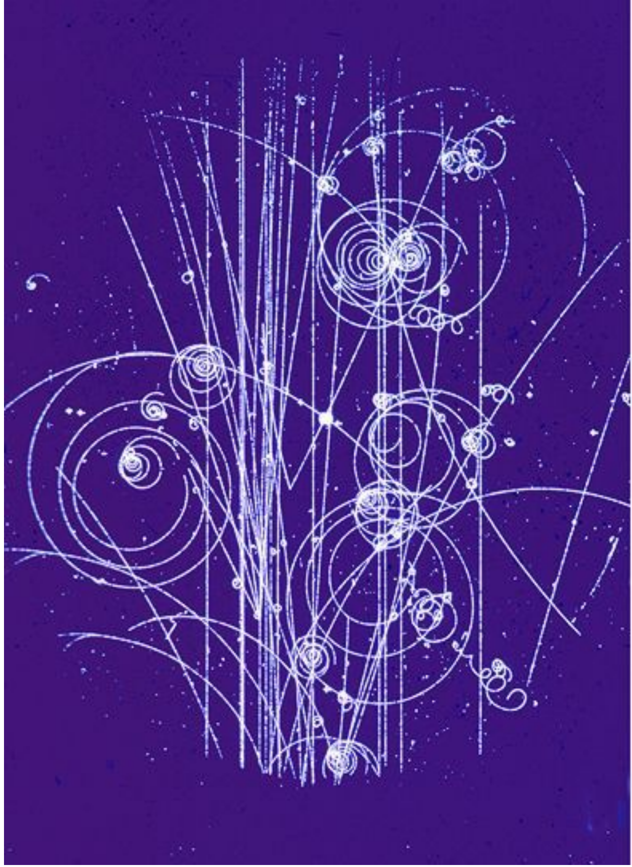
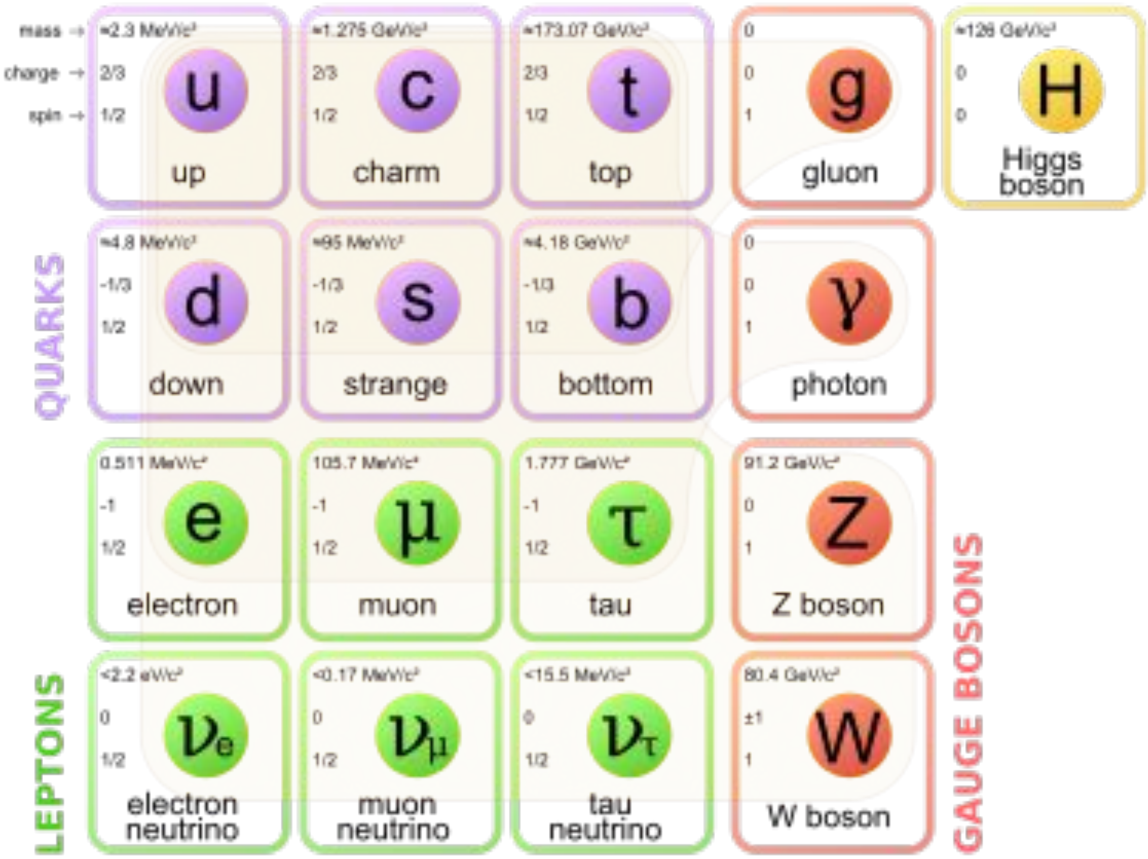


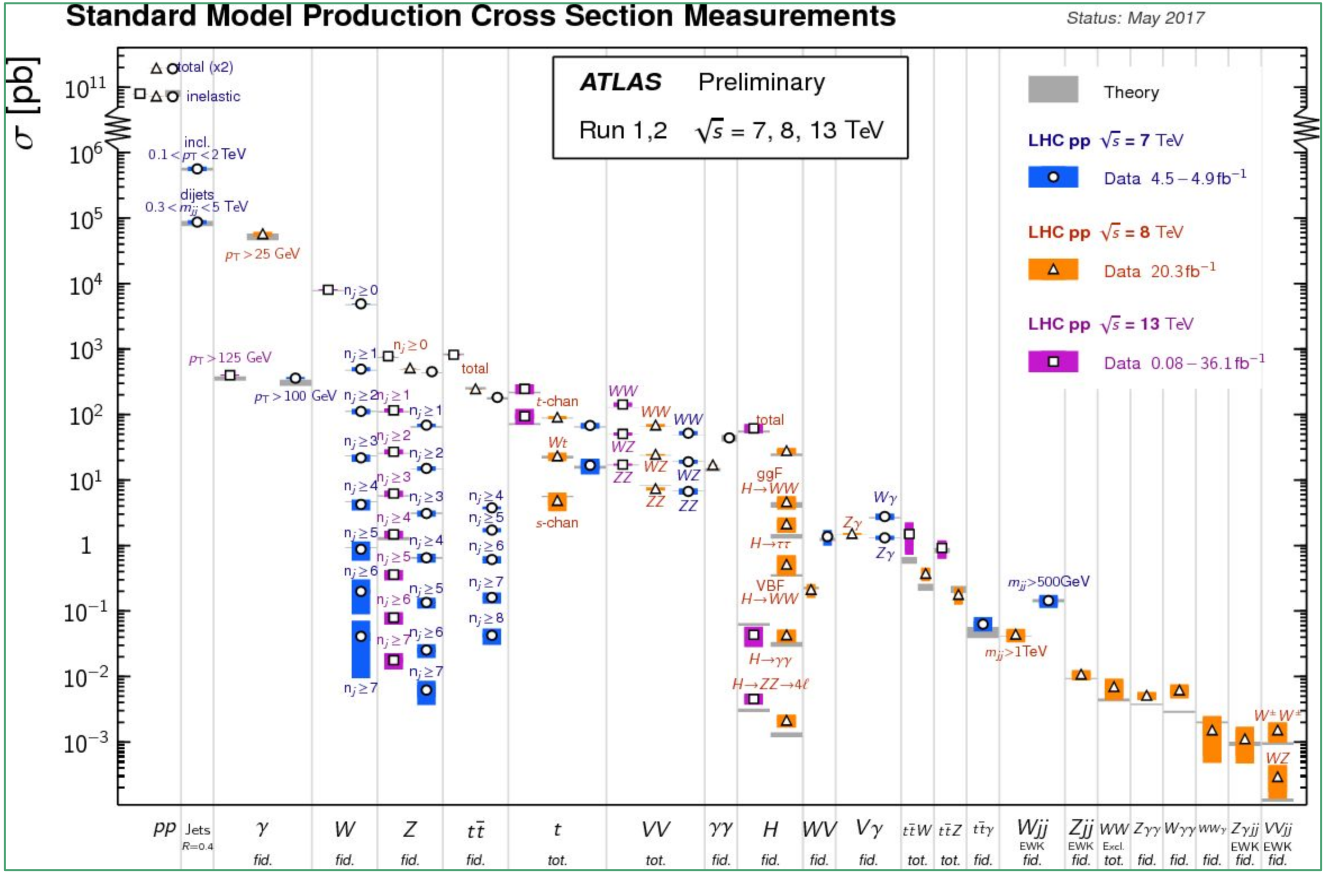
Main references:

- Gnendiger, C. et al. To d, or not to d: recent developments and comparisons of regularization schemes. Eur. Phys. J. C 77, 471
- A. Cherchiglia. et al. Two-loop renormalisation of gauge theories in 4D Implicit Regularisation: transition rules to dimensional methods. [arXiv:2006.10951 [hep-ph]].
- L. Abbott, The Background Field Method Beyond One Loop. Nucl. Phys. B 185 (1981), 189-203 doi:10.1016/0550-3213(81)90371-0

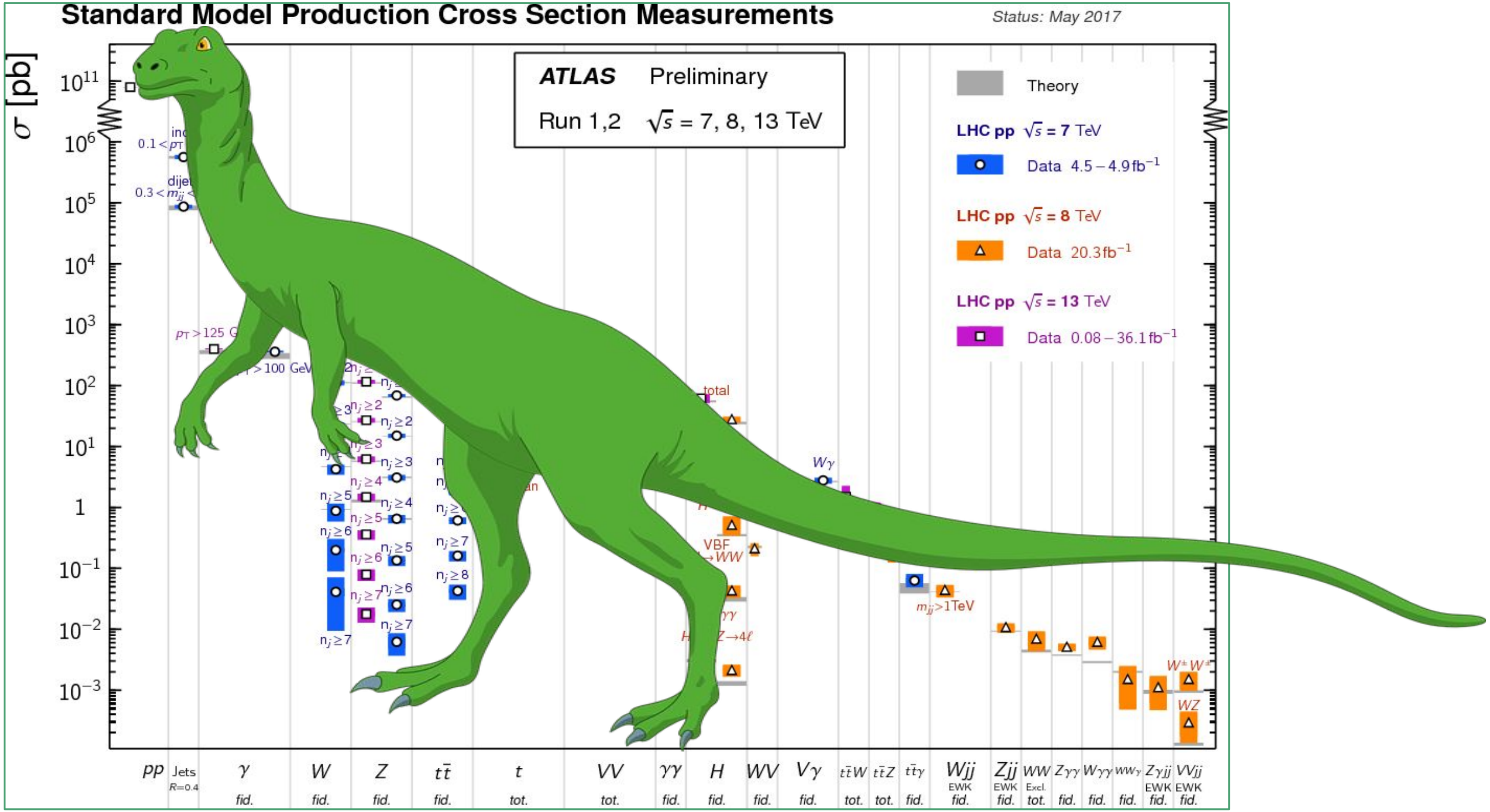
Motivation



Standard Model of particle physics: the assembly of physical laws that governs the behavior of elementary particles.



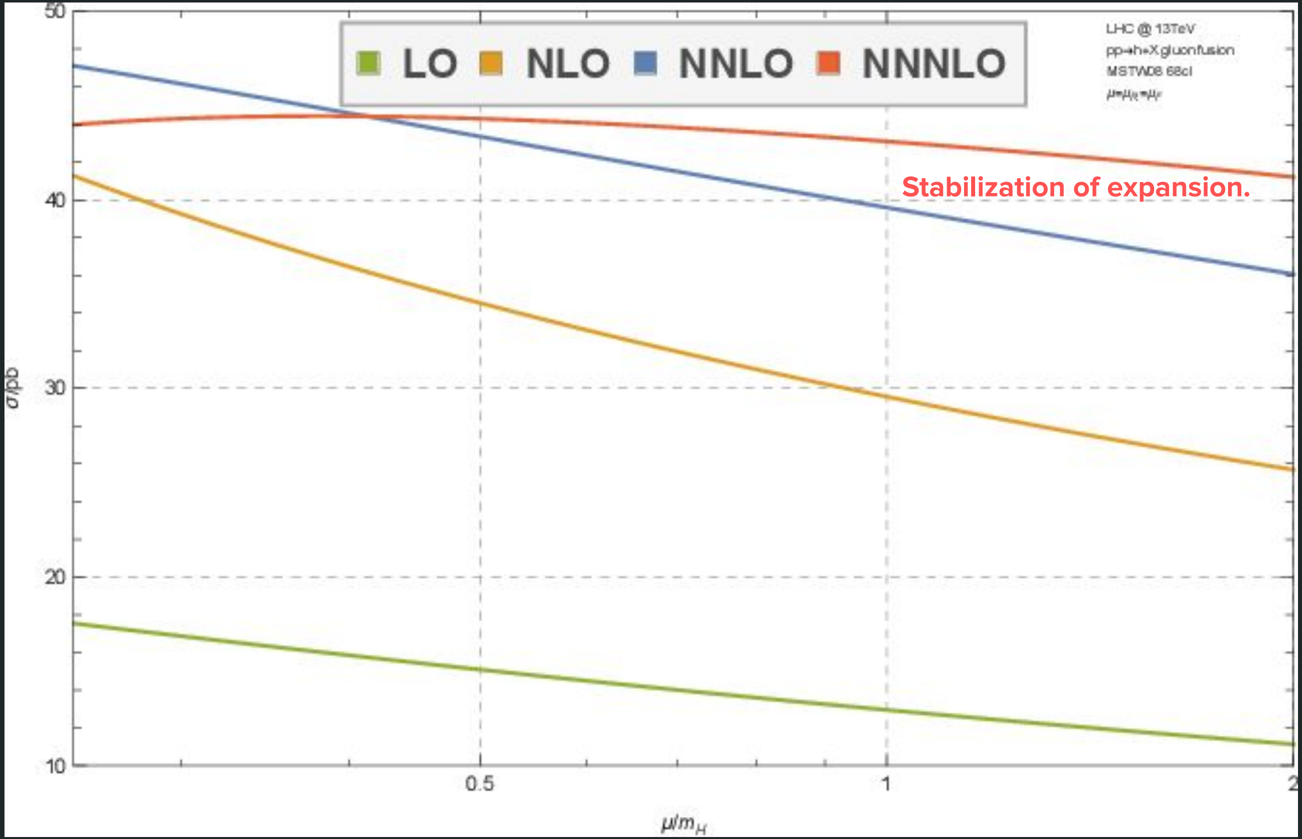
➔ LHC incredibly successful at 7 , 8 & 13 TeV (Runs 1 and 2).



➔ LHC incredibly successful at 7 , 8 & 13 TeV (Runs 1 and II).

Higgs production at N^3LO

(3-loop level)



Anastasiou, Duhr, Dulat, Herzog, Mistlberger (2018) JHEP 1905 (2019) 080

∞ **in QFT**

UV-div → They appear when the momentum of the loop goes to infinity.

IR-div

∞ in QFT

UV-div ⇒ They appear when the momentum of the loop goes to infinity.

IR-div

The perturbative key toolkit for precision

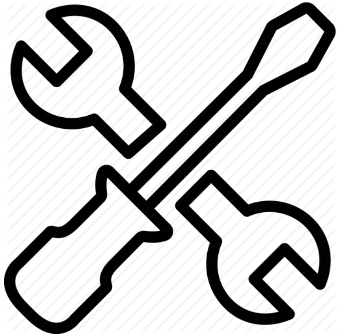
Automation → **LO, NLO, N^{2}LO, N^{3}LO... calculations in SM and BSM**

The perturbative key toolkit for precision

Automation → **LO, NLO, N^{2}LO, N^{3}LO… calculations in SM and BSM**

↓

REGULARIZATION



The perturbative key toolkit for precision

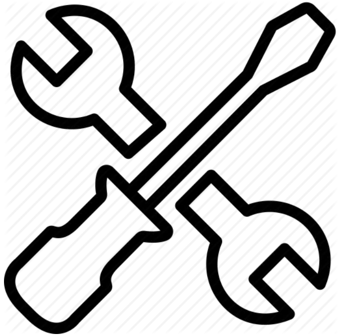
Automation → **LO, NLO, N^2LO, N^3LO... calculations in SM and BSM**

REGULARIZATION

d-dimensions

4-dimensions

DReg: dimensional regularization.
DRED: dimensional reduction.
(traditional regularization schemes)



The perturbative key toolkit for precision

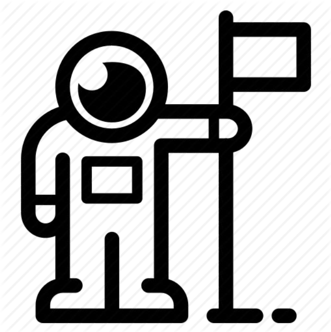
Automation

LO, NLO, N^{2}LO, N^{3}LO... calculations in SM and BSM

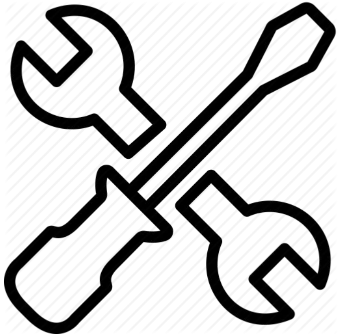
REGULARIZATION

IREG: Implicit Regularization

(Non-dimensional scheme)



4-dimensions



The perturbative key toolkit for precision

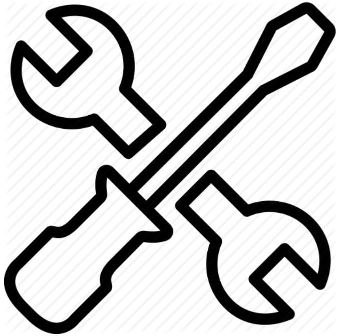
Automation

LO, NLO, N^{2}LO, N^{3}LO... calculations in SM and BSM

REGULARIZATION

IREG: Implicit Regularization

Subtract UV-div



Objectives

Understand how to wisely remove **UV-div** when they arise at 2-loop order for non-abelian gauge theories with IREG.

- N^2LO techniques.
- How the IREG technique works at 2-loop?
- Tools with FeynArts based on IREG.
- Non-abelian theories.
- **β-function** of pure Yang-Mills in a mass-independent subtraction scheme.

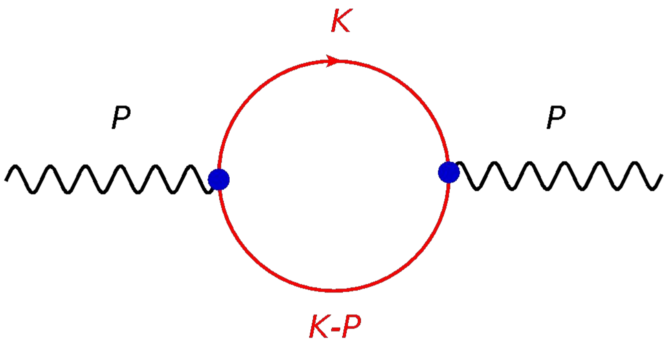
Review of regularization schemes

IREG

- Alternative to traditional dimensional schemes.
- Momentum space.
- **UV-div** as BDI (basic divergent integrals).
- **UV-div** do not depend on physical parameters.
- Parametrization of regularization dependent terms as surface terms.

IREG-**Example**

Vacuum polarization tensor in massless QED at 1-loop in 3+1



$$\Pi_{\tau\nu}(k) = (-)Tr\left[\int_k (-ie\gamma_\nu)\frac{i}{\not{k}}(-ie\gamma_\tau)\frac{i}{\not{k}-\not{p}}\right]$$

IREG-**Example**

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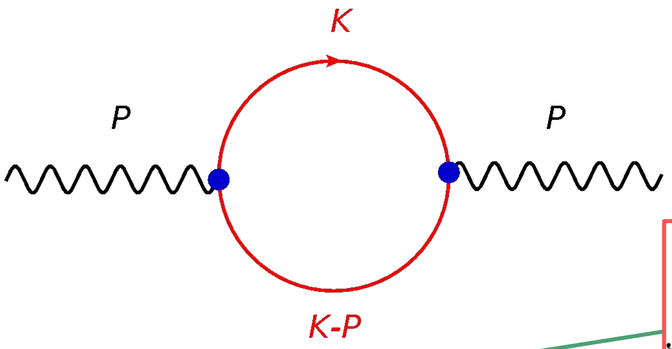
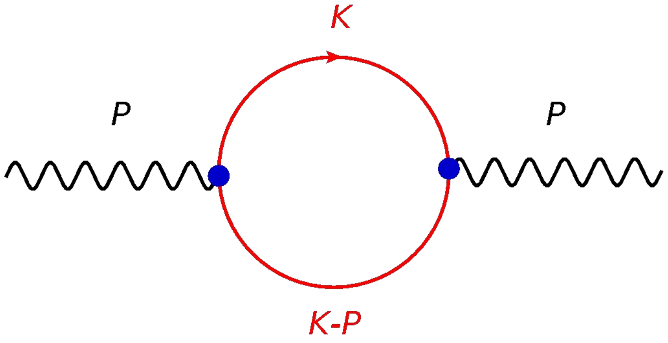


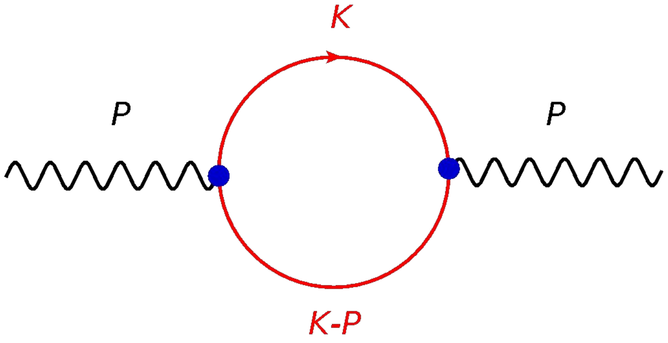
$$\boxed{\int_k = \int_{-\infty}^{+\infty} \frac{d^4k}{(2\pi)^4}}$$

$$\Pi_{\tau\nu}(k) = (-)Tr\left[\boxed{\int_k}(-ie\gamma_\nu)\frac{i}{\not{k}}(-ie\gamma_\tau)\frac{i}{\not{k}-\not{p}}\right]$$



$$D = (k^2)(k-p)^2$$

$$\Pi_{\tau\nu}(k) = -4e^2 \left[2\int_k \frac{k_\tau k_\nu}{D} - p_\tau \int_k \frac{k_\nu}{D} - p_\nu \int_k \frac{k_\tau}{D} - g_{\tau\nu} \int_k \frac{k^2}{D} + g_{\tau\nu} p^\sigma \int_k \frac{k_\sigma}{D} + g_{\tau\nu} \int_k \frac{1}{D} \right]$$



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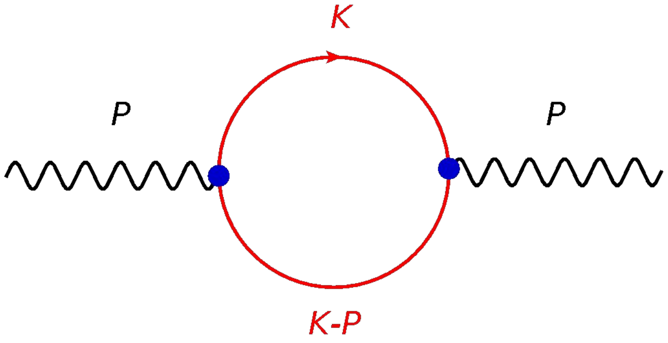
When do we know that a diagram can be regularized?

Superficial degree of divergence Δ in a Feynman Diagram for **UV-div**



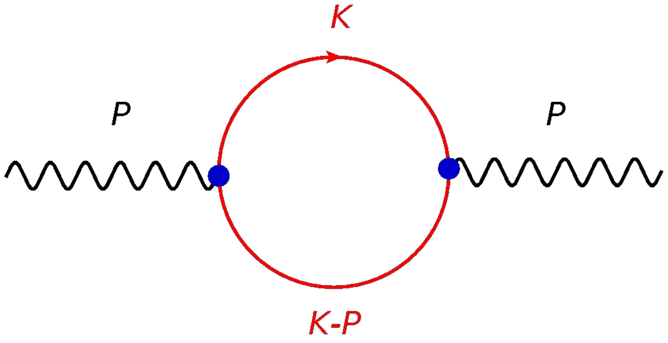
$$\underbrace{D = (k^2)(k-p)^2}_{\text{Power of 4}}$$

$$\Pi_{\tau\nu}(k) = -4e^2 \left[\underbrace{2\int_k \frac{k_\tau k_\nu}{D}}_{\Delta=2} - \underbrace{p_\tau \int_k \frac{k_\nu}{D}}_{\Delta=1} - \underbrace{p_\nu \int_k \frac{k_\tau}{D}}_{\Delta=1} - \underbrace{g_{\tau\nu} \int_k \frac{k^2}{D}}_{\Delta=2} + \underbrace{g_{\tau\nu} p^\sigma \int_k \frac{k_\sigma}{D}}_{\Delta=1} + \underbrace{g_{\tau\nu} \int_k \frac{1}{D}}_{\Delta=0} \right]$$

Power of 4

$$\frac{1}{(2\pi)^2}\int d\Omega \int_0^{+\infty} \frac{k^3\,dk}{k^4} \propto \frac{1}{(2\pi)^2}\int d\Omega \int_0^{+\infty} \frac{dk}{k}$$

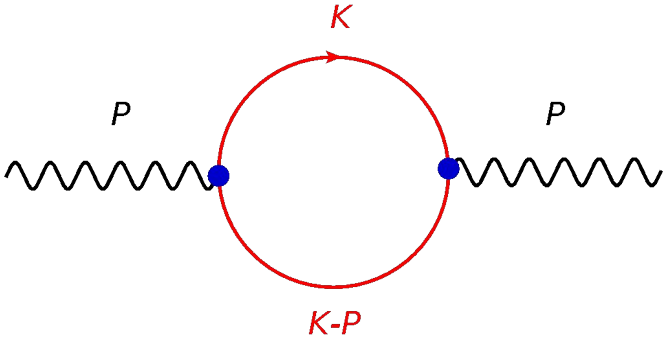
$\Delta =$ **#power in the numerator** **-** **#power in the denominator**



$$D = (k^2)(k-p)^2$$

$$\Pi_{\tau\nu}(k) = -4e^2 \left[2\int_k \frac{k_\tau k_\nu}{D} - p_\tau \int_k \frac{k_\nu}{D} - p_\nu \boxed{\int_k \frac{k_\tau}{D}} - g_{\tau\nu} \int_k \frac{k^2}{D} + g_{\tau\nu} p^\sigma \int_k \frac{k_\sigma}{D} + g_{\tau\nu} \int_k \frac{1}{D} \right]$$

$$I_\tau = \int_k \frac{k_\tau}{D} = \int_k \frac{k_\tau}{(k^2)(k-p)^2}$$



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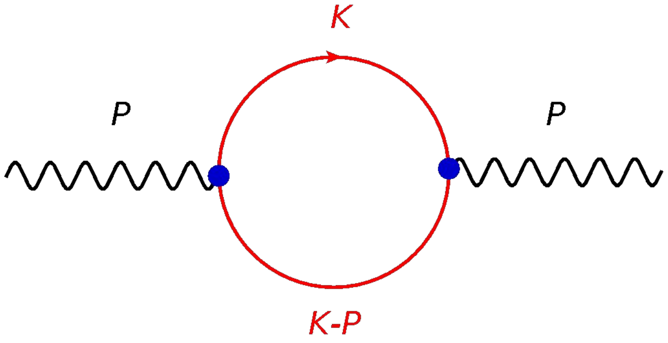
$$I_\tau = \int_k \frac{k_\tau}{D} = \int_k \frac{k_\tau}{(k^2)(k-p)^2}$$

1 $$I_\tau = \boxed{\lim_{\mu^2 \to 0}} \int_k \frac{k_\tau}{(k^2 \boxed{-\mu^2})[(k-p)^2 \boxed{-\mu^2}]}$$



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1 $$I_\tau = \lim_{\mu^2 \to 0} \int_k \frac{k_\tau}{(k^2-\mu^2)[(k-p)^2-\mu^2]}$$

2 $$\frac{1}{(k-p)^2-\mu^2} = \frac{1}{k^2-\mu^2} - \frac{p^2-2p\cdot k}{(k^2-\mu^2)[(k-p)^2-\mu^2]}$$

$$I_\tau = \int_k \frac{k_\tau}{D} = \int_k \frac{k_\tau}{(k^2)(k-p)^2}$$

$$\frac{1}{(k-p)^2 - \mu^2} = \frac{1}{k^2 - \mu^2} - \frac{p^2 - 2p \cdot k}{(k^2 - \mu^2)[(k-p)^2 - \mu^2]}$$

$$I_\tau = \int_k \frac{k_\tau}{D} = \int_k \frac{k_\tau}{(k^2)(k-p)^2}$$

$$\frac{1}{(k-p)^2-\mu^2} = \frac{1}{k^2-\mu^2} - \frac{p^2-2p\cdot k}{(k^2-\mu^2)[(k-p)^2-\mu^2]}$$

$$I_\tau\Big|_{div} = \lim_{\mu^2\to 0}\left[2p^\alpha \int_k \frac{k_\tau k_\alpha}{(k^2-\mu^2)^3} + \int_k \frac{k_\tau}{(k^2-\mu^2)^2}\right]$$

$$I_\tau\Big|_{finite} = -p^2 \lim_{\mu^2\to 0} \int_k \frac{k_\tau}{(k^2-\mu^2)^3} + \lim_{\mu^2\to 0} \int_k k_\tau \frac{p^2-2pk}{(k^2-\mu^2)^3[(k-p)^2-\mu^2]}$$

$$I_\tau = \int_k \frac{k_\tau}{D} = \int_k \frac{k_\tau}{(k^2)(k-p)^2}$$

$$\frac{1}{(k-p)^2-\mu^2} = \frac{1}{k^2-\mu^2} - \frac{p^2-2p\cdot k}{(k^2-\mu^2)[(k-p)^2-\mu^2]}$$

$$I_\tau\Big|_{div} = \lim_{\mu^2\to 0}\left[2p^\alpha \underbrace{\boxed{\int_k \frac{k_\tau k_\alpha}{(k^2-\mu^2)^3}}}_{\text{3 Basic divergent integral (BDI)}} + \underbrace{\int_k \frac{k_\tau}{(k^2-\mu^2)^2}}_{0}\right]$$

BDI

The objective of IREG will be to write the **UV-div** in terms of the Basic Divergent Integrals (BDI).

$$I_{log}(\mu^2) \equiv \int_k \frac{1}{(k^2-\mu^2)^2}$$

$$I_{quad}(\mu^2) \equiv \int_k \frac{1}{(k^2-\mu^2)}$$

$$I_{log}^{\nu_1\nu_2}(\mu^2) \equiv \int_k \frac{k_{\nu_1}k_{\nu_2}}{(k^2-\mu^2)^3}$$

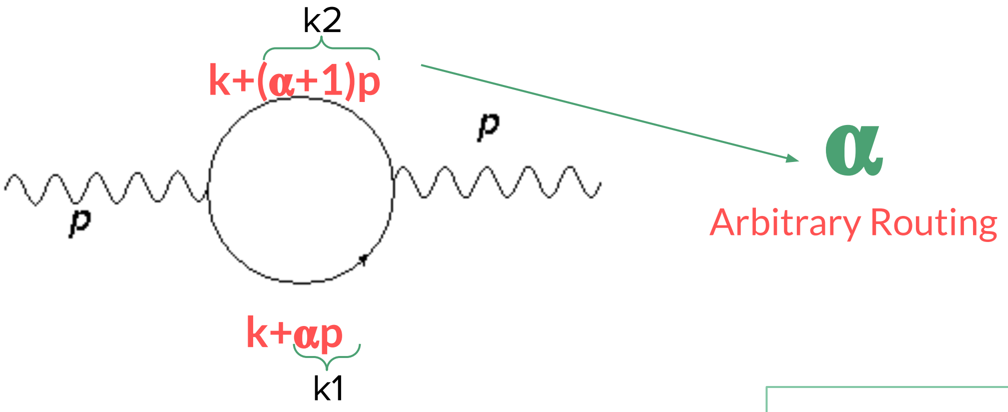
etc...

$$\Upsilon_0^{(1)\mu\nu} = \int_k \frac{\partial}{\partial k_\mu} \frac{k^\nu}{(k^2-\mu^2)^2}$$

$$= 4\left[\frac{g_{\mu\nu}}{4} I_{log}(\mu^2) - I_{log}^{\mu\nu}(\mu^2)\right]$$

Surface terms

Momentum Routing Invariance in a Feynman Diagram



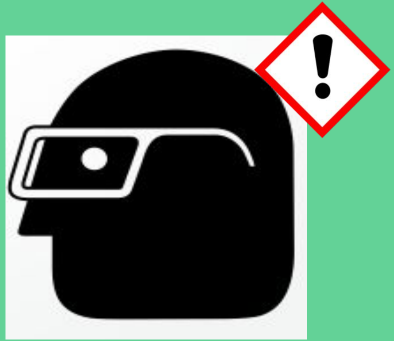
MRI ⟷ **ST=0** ⟷ **Gauge Invariance**

Momentum Routing Invariance in Extended QED: Assuring Gauge Invariance Beyond Tree Level

A. R. Vieira[a,b], A. L. Cherchiglia[c] and Marcos Sampaio[a]

$$I_{log}(\mu^2) \equiv \int_k \frac{1}{(k^2-\mu^2)^2}$$

IR-div



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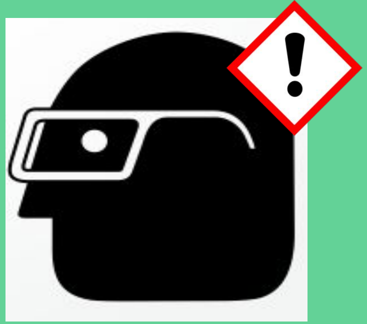
IR-div



$$\boxed{\lim_{\mu^2 \to 0}}$$

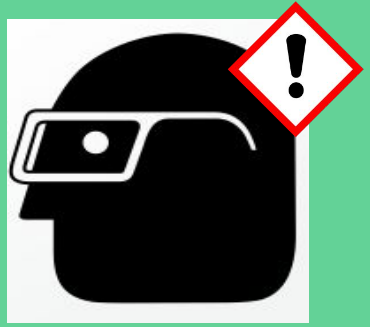
4

Scale relations

$$I_{log}(\mu^2) = I_{log}(\Lambda^2) - \frac{i}{(4\pi)^2} \ln \frac{\Lambda^2}{\mu^2}$$

$$I_{log}(\mu^2) \equiv \int_k \frac{1}{(k^2-\mu^2)^2}$$

IR-div



$$\boxed{\lim_{\mu^2 \to 0}}$$

4

Scale relations

$$I_{log}(\mu^2) = \underbrace{I_{log}(\Lambda^2)} - \overbrace{\frac{i}{(4\pi)^2}}^{b} \ln \frac{\Lambda^2}{\mu^2}$$

It plays the role of the renormalization group scale

- Mass-independent scheme.
- Regularization independent.

The rules of IREG

1. Symmetric integration must be avoided (source of error).

$$k^\mu k^\nu \xcancel{\rightarrow} \frac{1}{d} k^2$$

2. Fictitious mass μ^2 to avoid spurious IR-div.

3. External momenta. $$\frac{1}{(k-p)^2-\mu^2} = \frac{1}{k^2-\mu^2} - \frac{p^2 - 2p \cdot k}{(k^2-\mu^2)[(k-p)^2-\mu^2]}$$

4. Divergent part in terms of BDI's which may be written as linear combinations of scalar BDIs plus ST.

5. The μ^2 dependence by introducing a scale Λ^2.

Traditional dimensional schemes (DS): DReg and DRED

- DS are based on analytical continuations of the space: 4 ↣ d dimensions.
- In DS **UV-div** manifest as poles: $\frac{1}{\epsilon^n}$
- Dimensional regularization (DReg) analytically continues the integral into d=4-2ε.

$$\int_{-\infty}^{+\infty} \frac{d^4 k_{[4]}}{(2\pi)^4} \to \underbrace{\mu_{\mathrm{DR}}^{4-d}}_{\text{scale}} \int_{-\infty}^{+\infty} \frac{d^d k_{[d]}}{(2\pi)^d}$$

Traditional dimensional schemes (DS): DReg and DRED

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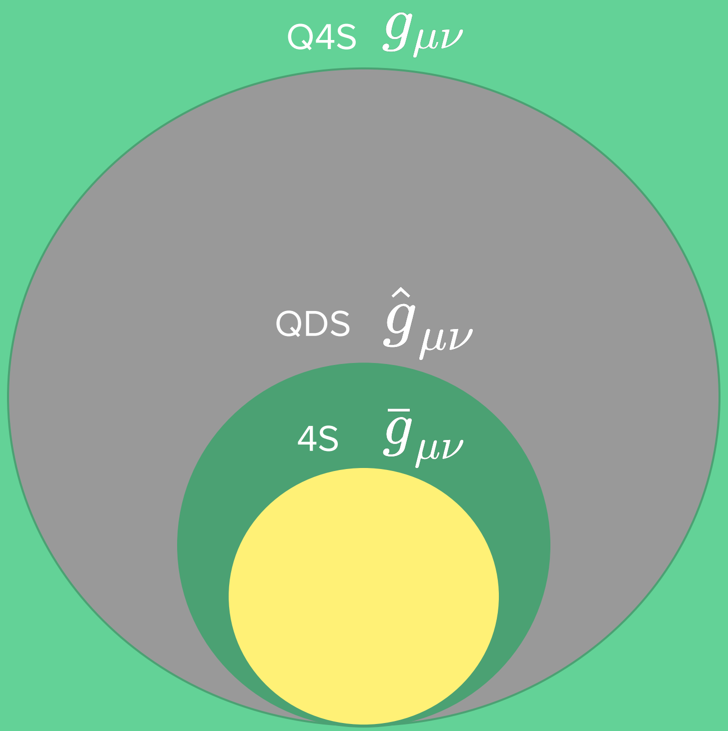
- In DS **UV-div** manifest as poles: $\frac{1}{\epsilon^n}$
- Dimensional regularization (DReg) analytically continues the integral into d=4-2ε.
- Alternative schemes to DReg have been developed, such as dimensional reduction (DRED).
- In DRED we split the boson fields "4" into a boson space "d" plus a boson space N=2ε.

$$\mathscr{A}_4 = \underbrace{\mathscr{A}_d}_{4-2\varepsilon} + \underbrace{\mathscr{A}_N}_{2\varepsilon} \Longrightarrow \mathcal{L}^{(4)} = \mathcal{L}^{(d)} + \underbrace{\mathcal{L}^{(2\epsilon)}}_{\text{New Feynman Rules}}$$

Only gluons that appear inside a divergent loop or phase space integral need to be regularized.



$$Q4S \supset QDS \supset 4S$$

	DReg	DRED
Internal gluon	$\hat{g}_{\mu\nu}$	$g_{\mu\nu}$
External gluon	$\hat{g}_{\mu\nu}$	$g_{\mu\nu}$

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + \underbrace{\tilde{g}_{\mu\nu}}_{2\varepsilon}$$

UV-div

DS

$$\frac{1}{\epsilon^n}$$

Poles when ε→0

UV-div

IREG

$$I_{log}(\mu^2) \equiv \int_k \frac{1}{(k^2-\mu^2)^2}$$

$$I_{quad}(\mu^2) \equiv \int_k \frac{1}{(k^2-\mu^2)}$$

etc...

BDI

UV-div

DS

$$\frac{1}{\epsilon^n}$$

Poles when ε→0

d=4-2ε

?

UV-div

IREG

$$I_{log}(\mu^2) \equiv \int_k \frac{1}{(k^2-\mu^2)^2}$$

$$I_{quad}(\mu^2) \equiv \int_k \frac{1}{(k^2-\mu^2)}$$

etc...

BDI

1-loop
DS

Same!

1-loop
IREG

$$I = \int_k \frac{1}{k^2(k-p)^2} \overset{\text{IREG}}{=} I_{log}(\Lambda^2) - b\ln\left[-\frac{p^2}{\Lambda^2}\right] + 2b.$$

d=4-2ε

$$b\left[\frac{1}{\epsilon} - \gamma_E + \ln(4\pi) + \ln\left(\frac{\mu_{DR}^2}{\Lambda^2}\right)\right]$$

$$I^{IREG}\Big|_d = b\left[\frac{1}{\epsilon} - \gamma_E + \ln(4\pi) - \ln\left[-\frac{p^2}{\boxed{\mu_{DR}^2}}\right] + 2\right]$$

n-loop DS

$$\mathcal{J}_d = b^n \left[\frac{1}{(4\pi)^{-\epsilon}} \left(-\frac{\mu_{DR}^2}{p^2} \right)^{\epsilon} \frac{\Gamma(\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} \right]^n \neq$$

n-loop IREG

$$J = \int_{k_1} \frac{1}{k_1^2(k_1-p)^2} \cdots \int_{k_n} \frac{1}{k_n^2(k_n-p)^2}.$$

$$J^{\text{IREG}} = \left[I_{log}(\Lambda^2) - b \ln\left[-\frac{p^2}{\Lambda^2} \right] + 2b \right]^n$$

d=4-2ε

$$\frac{b}{(4\pi)^{-\epsilon}} \left(\frac{\mu_{DR}^2}{\Lambda^2} \right)^{\epsilon} \Gamma(\epsilon)$$

$$J^{\text{IREG}}\Big|_d = b^n \left[\frac{1}{(4\pi)^{-\epsilon}} \left(-\frac{\mu_{DR}^2}{p^2} \right)^{\epsilon} \Gamma(\epsilon) + 2 \right]^n$$

Practical approaches to 2-loop calculations

Minimal set of rules

- ST's=0.
- Avoid symmetric integration.
- Feynman rules to the **whole** multi-loop diagram.
- Perform contractions and simplify numerator against denominator if possible.

After these, the rules sketched for IREG can be applied.

Algorithm

STAGE 1: FeynArts&FormCalc

Contractions

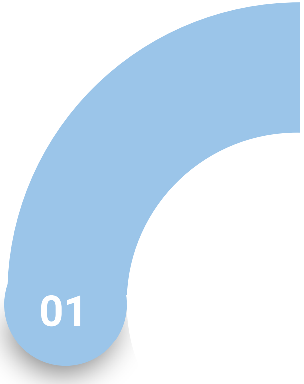
STAGE 2: Regularization

Integral reduction

STAGE 3: Integral Evaluation

FeynArts&FormCalc

We create the topologies and the amplitudes. We do the contractions.



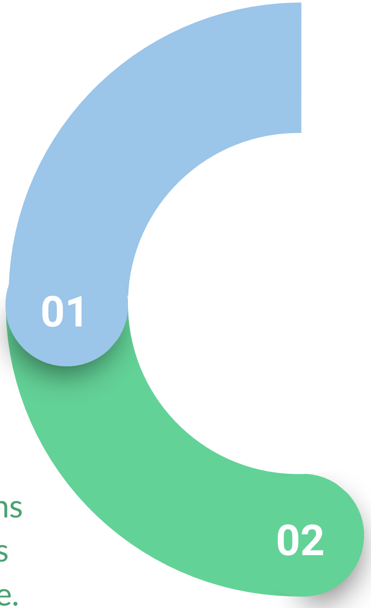
$$\int_{k,q} \frac{k^2}{k^2 q^2 (k-q)^2}\bigg|_{\mathrm{IREG}} = \int_{k,q} \frac{1}{q^2 (k-q)^2}\bigg|_{\mathrm{IREG}}$$

FeynArts&FormCalc

We create the topologies and the amplitudes. We do the contractions.

Regularization

We make the necessary simplifications and reductions for the integrals. This depends of the regularization scheme.



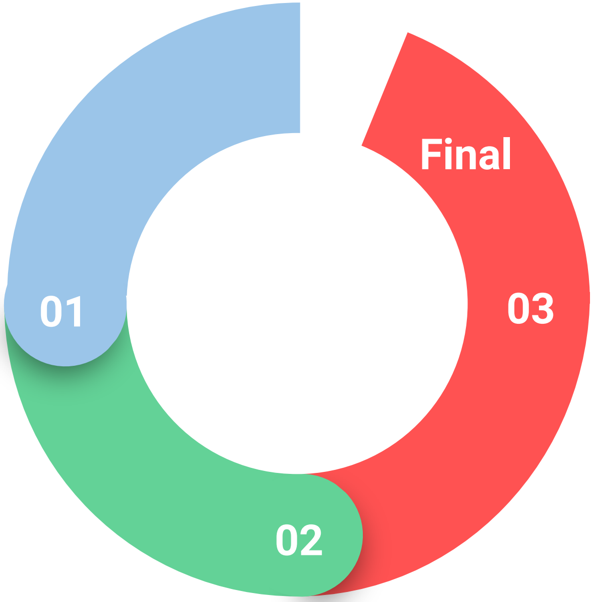
Algorithm

FeynArts&FormCalc

We create the topologies and the amplitudes. We do the contractions.

Regularization

We make the necessary simplifications and reductions for the integrals. This depends on the regularization scheme.

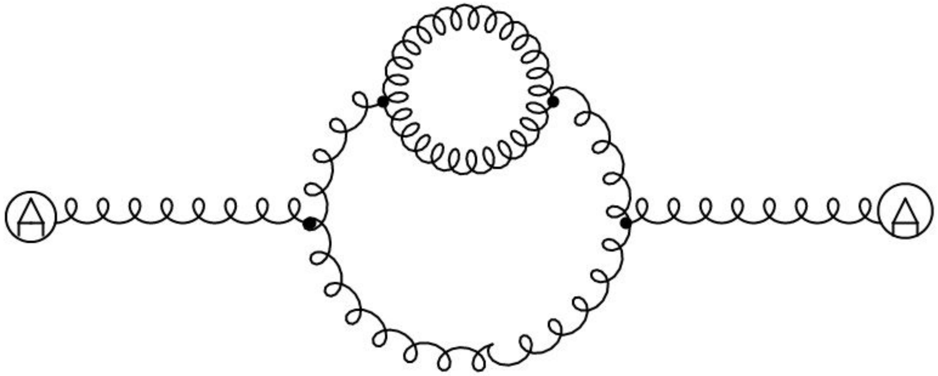


β-function

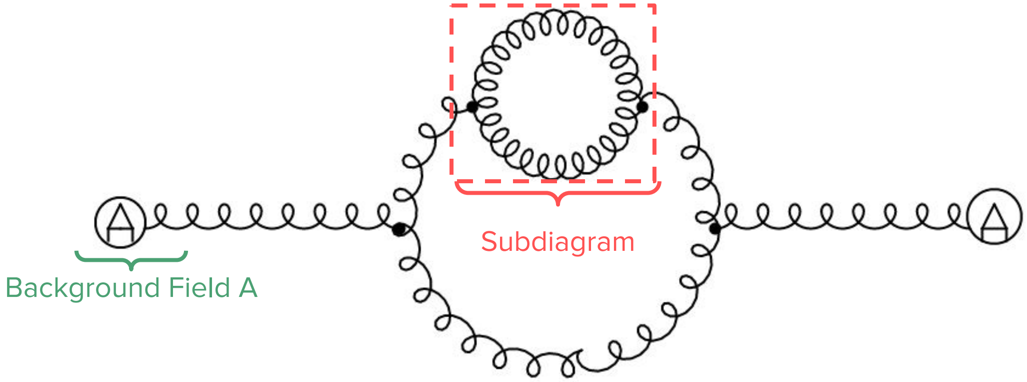
Integral evaluation

According to the regularization method used.

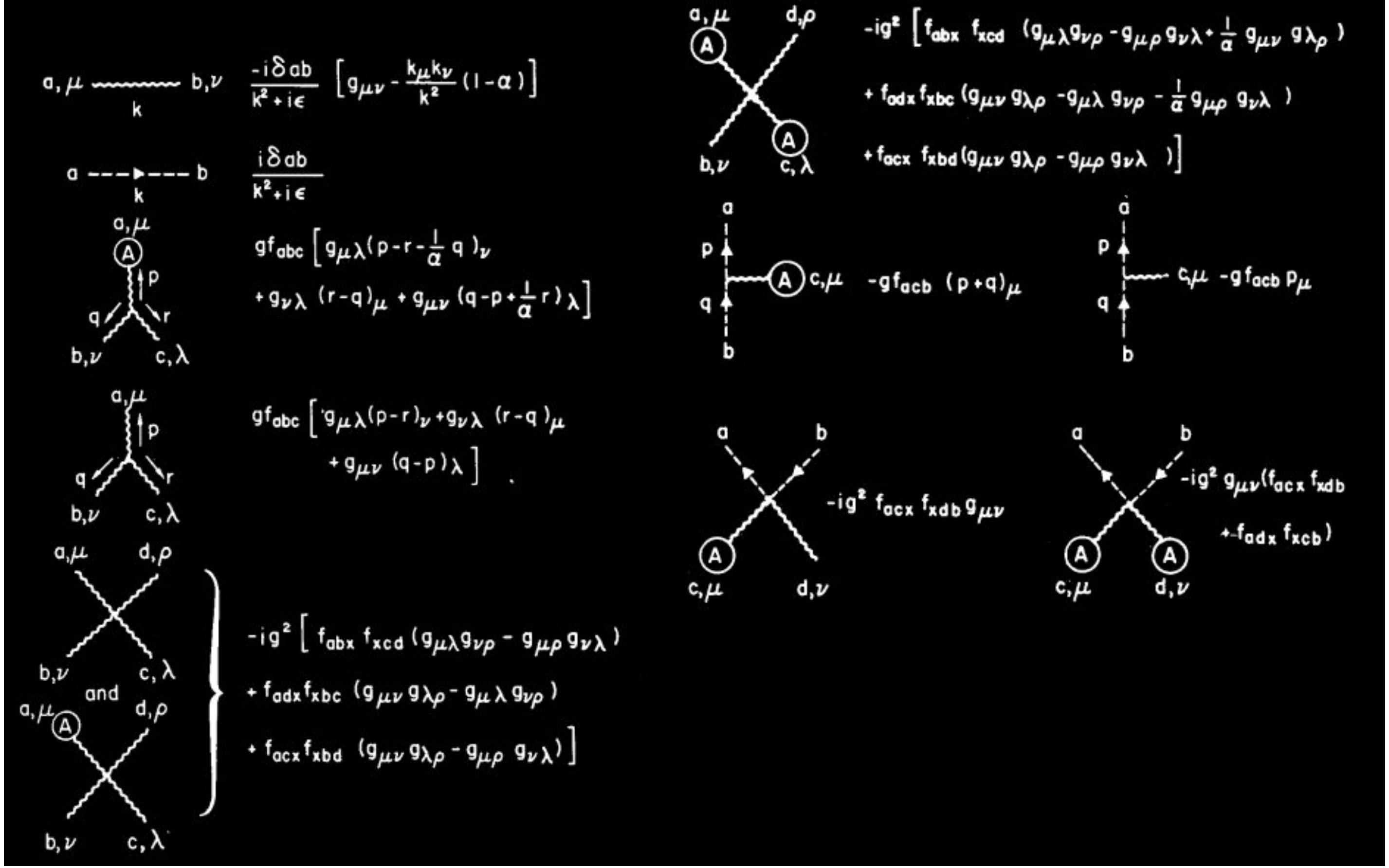
2-loop approach example



2-loop approach example



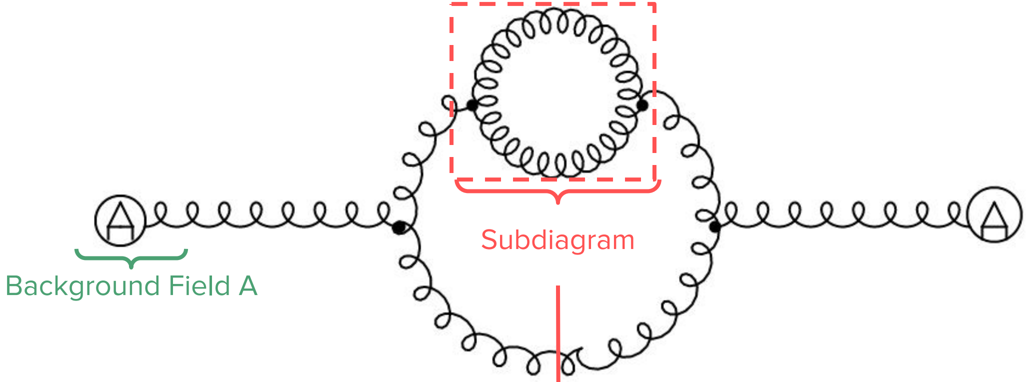
➔ **Stage 1:** Feynman Rules.



Background Field-Method Feynman's Rules

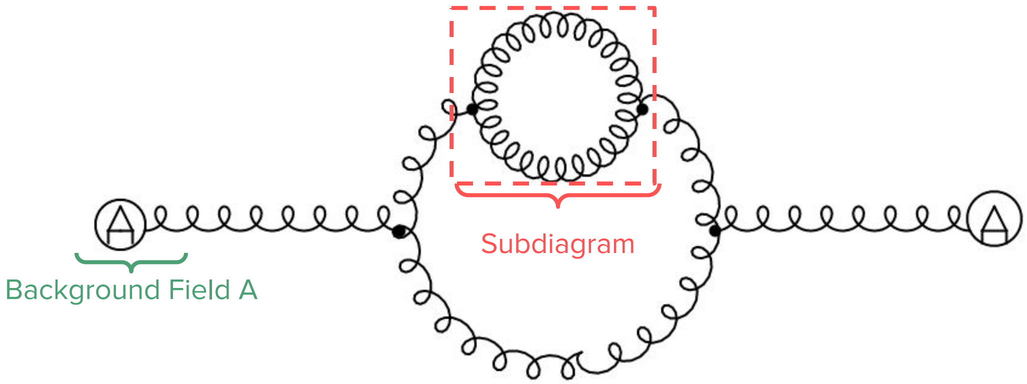
Ref: L. Abbott, The Background Field Method Beyond One Loop.

2-loop approach example



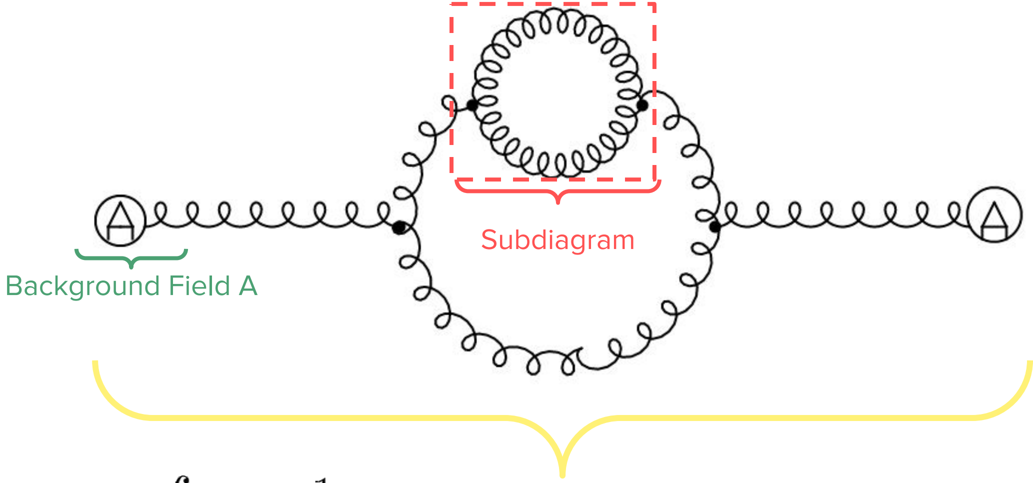
$$\pi^{mn}_{\mu\nu}(l) = -g^2 C_A \delta^{mn} \frac{1}{2} \int_l \frac{1}{l^2(l-k)^2} [2k_\nu k_\mu + 5k_\mu l_\nu + k_\nu l_\mu - 10 l_\nu l_\mu - g^{\nu\mu}(5k^2 + 2l^2 - 2k \cdot l)].$$

2-loop approach example



➔ **Stage 1:** Feynman Rules. **Don't** evaluate the integral of the subdiagram. Apply Feynman's rules back to the main diagram and add subdiagram result.

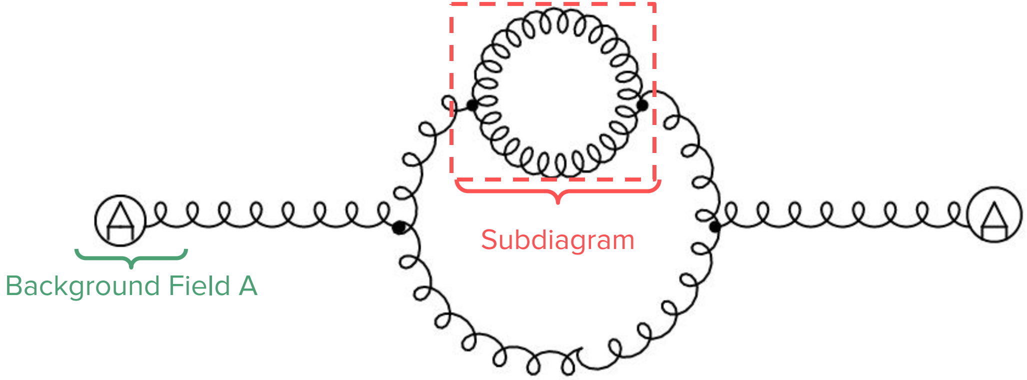
2-loop approach example



$$\Pi^{\lambda\sigma}_{ls} = +ig^4 C_A^2 \delta^{sl} \int_k \frac{1}{k^4(k-p)^2} \times$$

$$\xi_{\mu\nu\lambda\sigma} \times \left[\frac{1}{2} \int_l \frac{1}{l^2(l-k)^2} (2k_\nu k_\mu + 5k_\mu l_\nu + k_\nu l_\mu - 10 l_\nu l_\mu - g^{\nu\mu}(5k^2 + 2l^2 - 2k\cdot l)) \right]$$

2-loop approach example



- **Stage 1:** Feynman Rules. **Don't** evaluate the integral of the subdiagram. Perform Contractions and simplifications.

$$\int_k \frac{k_\lambda k_\sigma}{k^4(k-p)^2} \int_l \frac{l^2}{l^2(l-k)^2} \propto \int_k \frac{k_\lambda k_\sigma}{k^4(k-p)^2} \int_l \frac{1}{(l-k)^2}$$

- **Stage 2:** Regularization!

The Background Field Method (BFM)

- BFM is a technique for quantizing gauge field theories without losing explicit gauge invariance.

$\mathcal{L}$ $\Rightarrow$ Feynman Rules

$\Rightarrow$ Compute physical quantities

The Background Field Method (BFM)

- BFM is a technique for quantizing gauge field theories without losing explicit gauge invariance.

Quantities with no direct physical interpretation will not be gauge invariant

$\mathcal{L}$ $\Rightarrow$ Feynman Rules

$\Rightarrow$ Compute physical quantities

Which are gauge invariant and independent of the particular gauge chosen

- The BFM approach arranges things so that gauge invariance in $\mathcal{L}$ is still present once gauge fixing and ghost terms have been added.

BFM $\rightarrow$ $\mathcal{L}$ $\Rightarrow$ Green's Functions

$\Rightarrow$ Divergent counterterms

Gauge Invariant

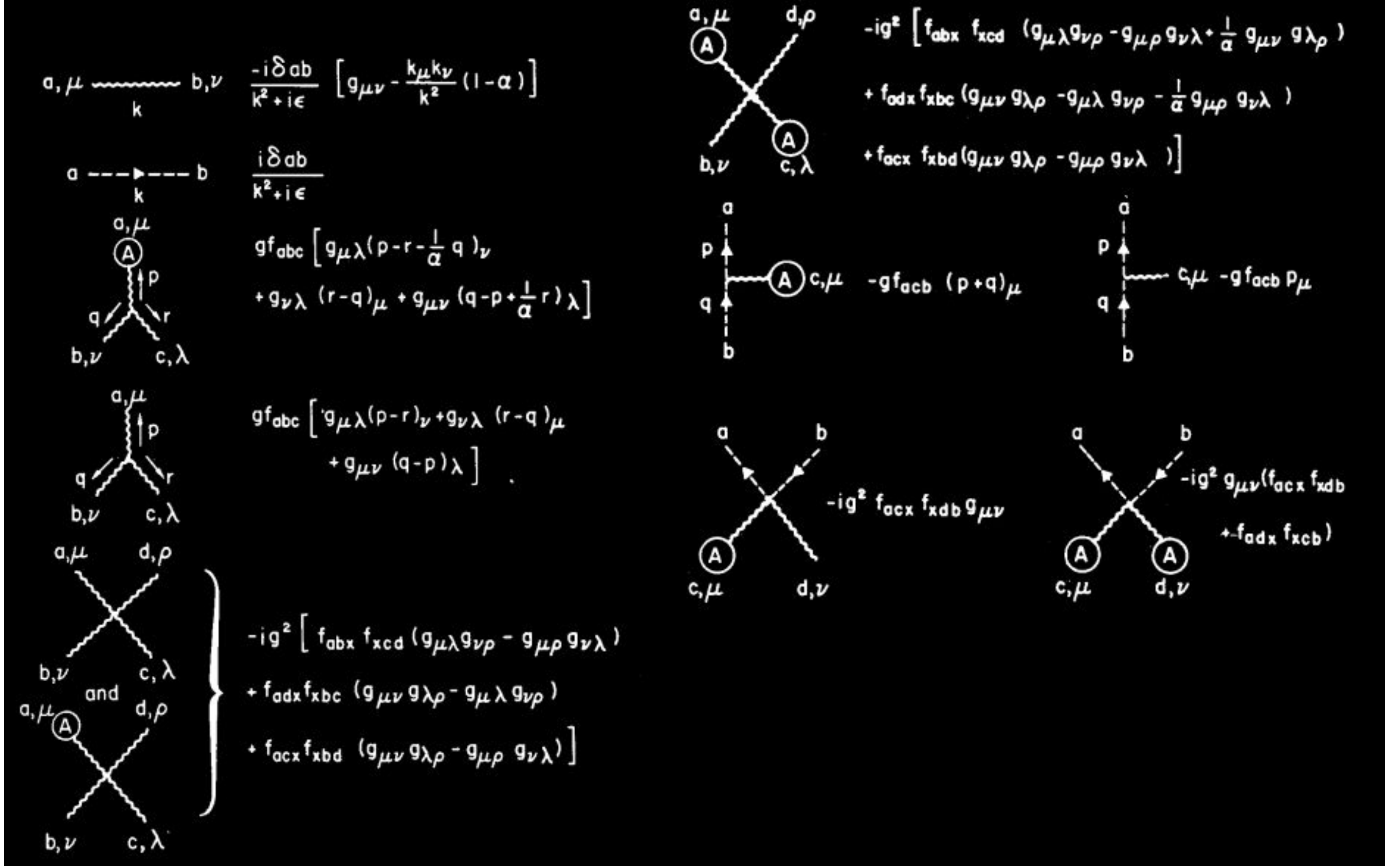
The Background Field Method (BFM)

- The approach of the BFM is to do a “field-shifting”: the background field is added to the quantum field in the action (S=QF+BF). After this, the method allows to fix a gauge and evaluate the quantum corrections without breaking the background gauge symmetry.

$$Z(J) = \int \mathcal{D}_Q \det\left(\frac{\delta G^a}{\delta w^b}\right) \exp^{i(S^{YM}(Q) - \frac{1}{2\alpha} G\cdot G + J\cdot Q}$$

$$\Big\downarrow \text{BFM}$$

$$\tilde{Z}(J, A) = \int \mathcal{D}_Q \det\left(\frac{\delta \tilde{G}^a}{\delta w^b}\right) \exp^{i(S^{YM}(Q+A) - \frac{1}{2\alpha} \tilde{G}\cdot \tilde{G} + J\cdot Q}$$



Background Field-Method Feynman's Rules

Ref: L. Abbott, The Background Field Method Beyond One Loop.

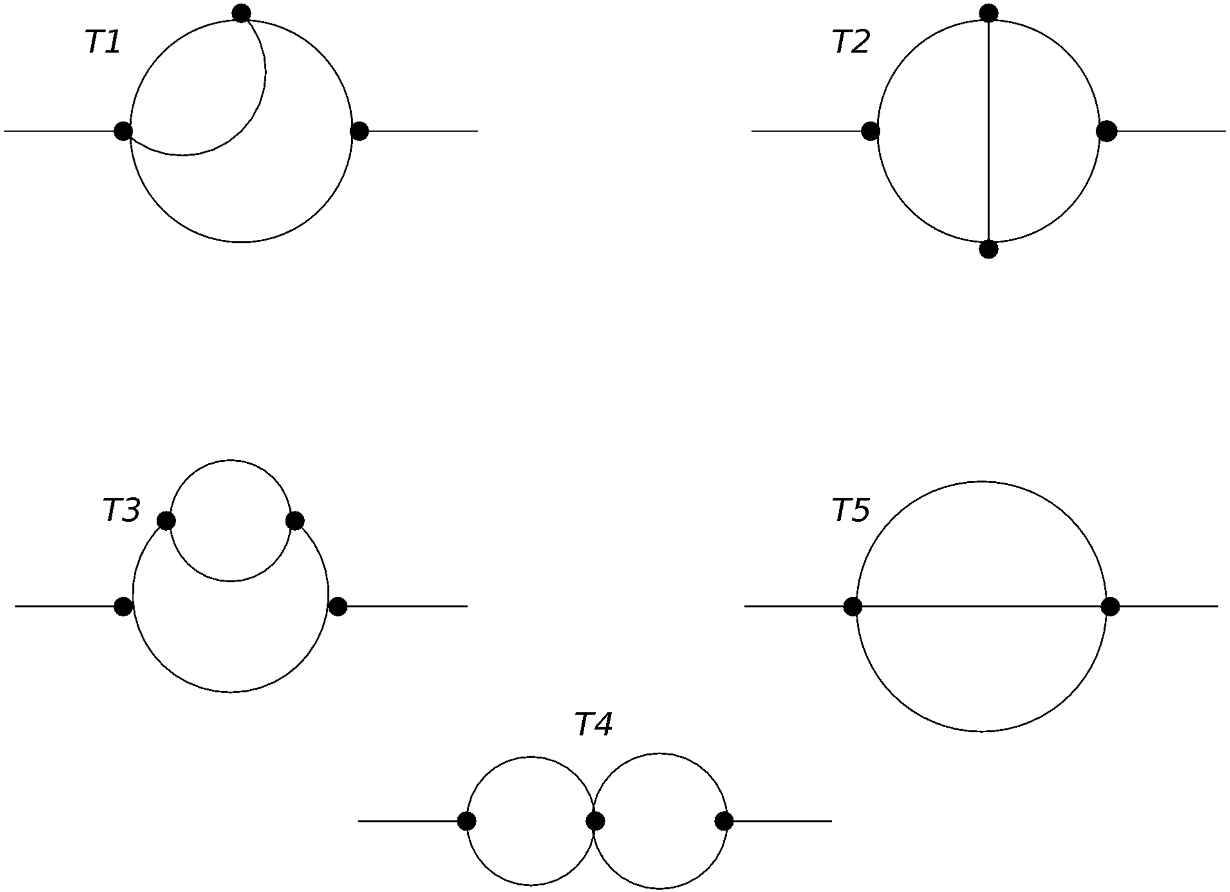
The renormalization program in the BFM

- We redefine a new set of variables: $\hat{A}_o = Z_{\hat{A}}\hat{A}_r; \quad g_o = Z_g g_r; \quad \alpha_o = Z_\alpha \alpha_r$
- Q field don't need to be renormalized (only appears in loops).
- Infinities in the action must take the gauge invariant form of a divergent constant times the gauge field strength tensor.

$$F^a_{\mu\nu}F^{a\mu\nu} = Z_A^{\frac{1}{2}}[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \underbrace{\boxed{Z_g Z_A^{\frac{1}{2}}}}_{Z_g = Z_A^{-\frac{1}{2}}} g f^{abc} A^b_\mu A^c_\nu]$$

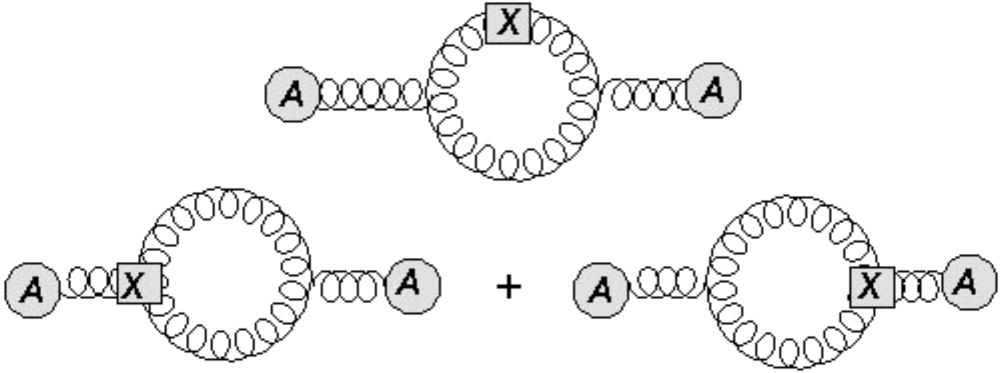
We only need the
two-point functions as
the abelian theories!

$$Z_g = Z_A^{-\frac{1}{2}}$$



The renormalization program in the BFM

- We redefine a new set of variables: $\hat{A}_o = Z_{\hat{A}} \hat{A}_r; \quad g_o = Z_g g_r; \quad \boxed{\alpha_o = Z_\alpha \alpha_r}$
- We adopted in our calculations the Feynman gauge α = 1, therefore it is necessary to include counterterms related to gauge-fixing renormalisation.

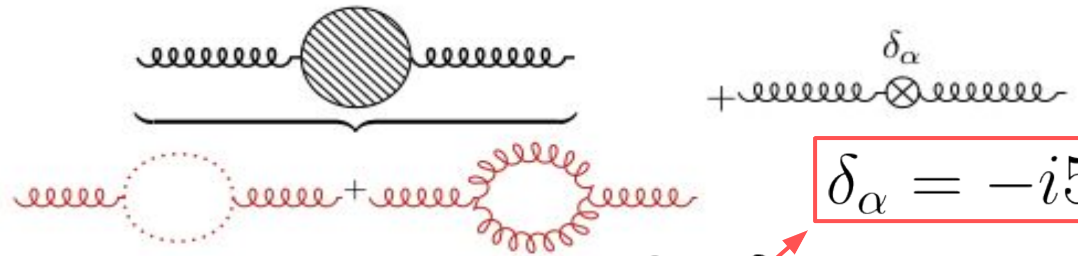


Gauge-fixing renormalization diagrams

$$\underbrace{Z_\alpha = (1 + \overbrace{\delta_\alpha}^{\text{Counterterm}})}_{\text{Redefinition}}$$

The renormalization program in the BFM

- We redefine a new set of variables: $\hat{A}_o = Z_{\hat{A}} \hat{A}_r; \quad g_o = Z_g g_r; \quad \boxed{\alpha_o = Z_\alpha \alpha_r}$
- We adopted in our calculations the Feynman gauge α = 1, therefore it is necessary to include counterterms related to gauge-fixing renormalisation.
- For the determination of the renormalization factor:



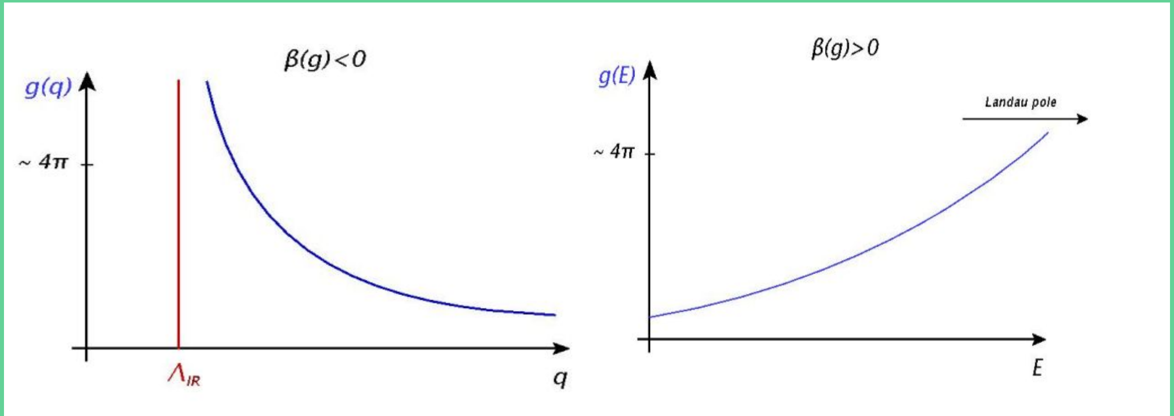
$$\boxed{\delta_\alpha = -i5g^2 C_A I_{log}(\Lambda^2)}$$

$$\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2\alpha_r}\left[\partial_\mu Q^\mu_a + g_r f^{abc}(\hat{A}_r)^b_\mu Q^c_\mu\right]^2 + \frac{\delta_\alpha}{2\alpha_r}\left[\partial_\mu Q^\mu_a + g_r f^{abc}(\hat{A}_r)^b_\mu Q^c_\mu\right]^2$$

The β-function at 2-loop

- Physical predictions in our theories cannot depend on: $\boxed{\mu_{DR}}$ $\boxed{\Lambda^2}$
- Physical parameters need to be independent. We do that using the renormalization group.
- The β-function is define as: $\beta = \lambda \frac{\partial}{\partial \lambda} g_R$

QCD
β-function



QED
β-function

The β-function at 2-loop

- The β-function is define as:

$$\beta = \lambda \frac{\partial}{\partial \lambda} g_R$$

- The renormalization constant of the BF field goes at 2-loops as:

$$Z_A = 1 + \underbrace{A_1}_{\text{1-loop}} \tilde{g_R}^2 + \underbrace{A_2}_{\text{2-loop}} \tilde{g_R}^4$$

$$\boxed{\beta = \frac{g_R}{2} \lambda \frac{\partial}{\partial \lambda} \left[A_1 \tilde{g_R}^2 + \left(A_2 - \frac{A_1^2}{2} \tilde{g_R}^4 \right) \right]}$$

DS β-function

$$\beta = \frac{d-4}{2} g_R \left[A_1 \tilde{g}_R^2 + 2 A_2 \tilde{g}_R^4 \right]$$

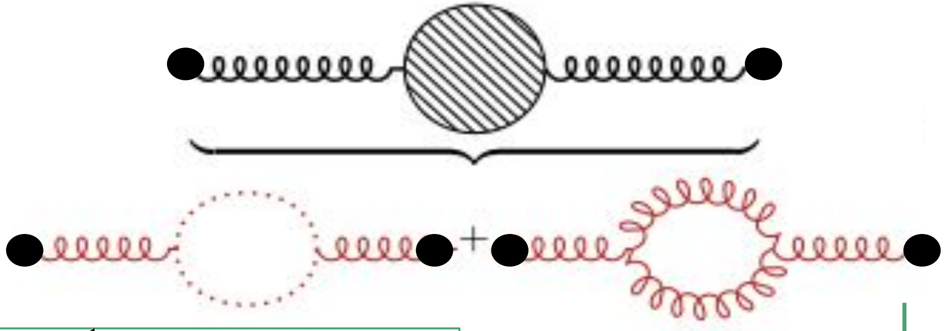
IREG β-function

$$\beta = -g_R \left[-\frac{g_R^2}{2} \Lambda \frac{\partial}{\partial \Lambda} A_1 - \frac{g_R^4}{2} \Lambda \frac{\partial}{\partial \Lambda} A_2 \right]$$

2-loop applications: results

Pure Yang-Mills at 1-loop with IREG

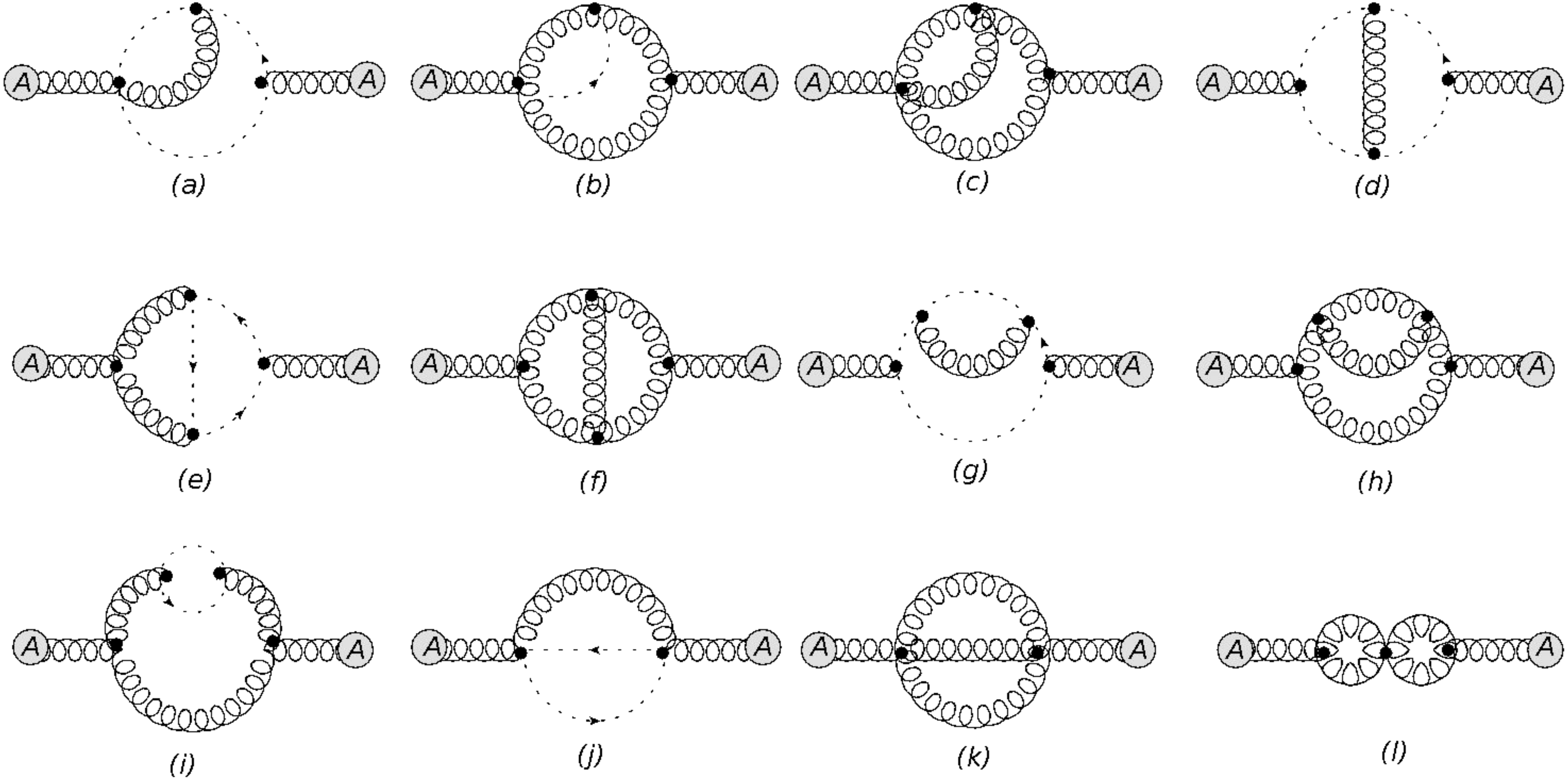
- We only need the gluon self-energy diagrams.



$$\boxed{+g^2 C_A \delta_{mn} \times [(+\frac{1}{3}k^2 g_{\mu\nu} I_{log}(\Lambda^2)) - (\frac{1}{3}k_\mu k_\nu I_{log}(\Lambda^2))]}$$

$$\boxed{+g^2 C_A \delta_{mn} \times [(\frac{10}{3}k^2 g_{\mu\nu} I_{log}(\Lambda^2)) - (\frac{10}{3}k_\mu k_\nu I_{log}(\Lambda^2))]}$$

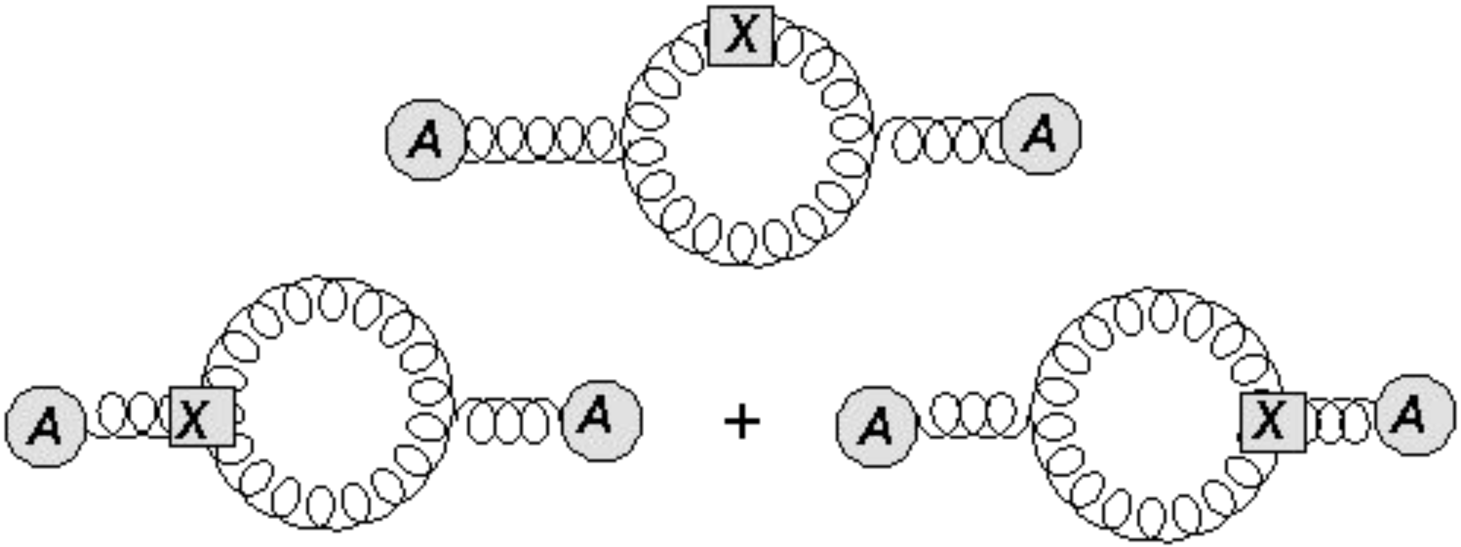
$$Z_{\hat{A}} = 1 + g^2 C_A \frac{11}{3} I_{log}(\Lambda^2)\Lambda^2 \Rightarrow \beta = -\frac{11 C_A}{3}\frac{g^3}{(4\pi)^2}$$



Pure Yang-Mills at 2-loop

Results on table 4.1 for IREG and 4.3 for DS

$$\boxed{\frac{i g_s^4 C_A^2 \delta^{ab}}{(4\pi)^4}\left[A g_{\mu\nu} p^2 - B p_\mu p_\nu\right]}$$



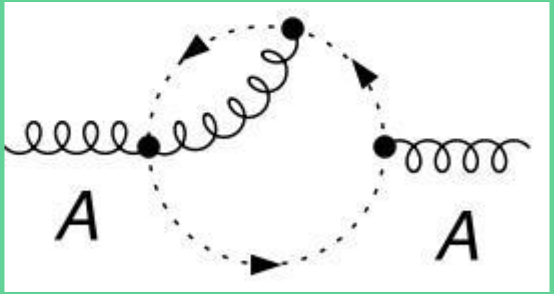
Gauge-fixing renormalization diagrams

(Remember, we chose Feynman gauge α = 1)

Results on table 4.2 for IREG and 4.4 for DS

IREG

(a)

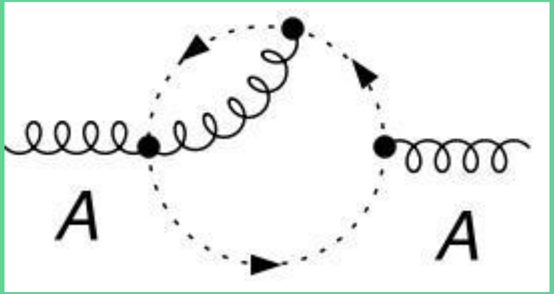


$$\frac{ig_s^4 C_A^2 \delta^{ab}}{(4\pi)^4}\left(\frac{1}{3b} I_{log}^{(2)}(\Lambda^2) - \frac{1}{3b^2} I_{log}^2(\Lambda^2) + \frac{1}{3b}\rho_{IREG}\right)\left[g_{\mu\nu}p^2 - p_\mu p_\nu\right] +$$

$$\left[\frac{-29}{18b} I_{log}(\Lambda^2) g_{\mu\nu} p^2 - \frac{-17}{18b} I_{log}(\Lambda^2) p_\mu p_\nu\right]$$

IREG

(a)



Apart from minus a sign and b, they have the same coefficient

Non-local term

$$\rho_{IREG} = I_{log}(\Lambda^2)\ln\left[-\frac{p^2}{\Lambda^2}\right]$$

$$\frac{ig_s^4 C_A^2 \delta^{ab}}{(4\pi)^4}\left(\boxed{\frac{1}{3b}} I_{log}^{(2)}(\Lambda^2) \boxed{-\frac{1}{3b^2}} I_{log}^2(\Lambda^2) + \boxed{\frac{1}{3b}}\rho_{IREG}\right)\left[g_{\mu\nu}p^2 - p_\mu p_\nu\right] +$$

$$\underbrace{\boxed{\left[\frac{-29}{18b} I_{log}(\Lambda^2) g_{\mu\nu}p^2 - \frac{-17}{18b} I_{log}(\Lambda^2) p_\mu p_\nu\right]}}$$

Not transverse individually, just in the sum of all diagrams

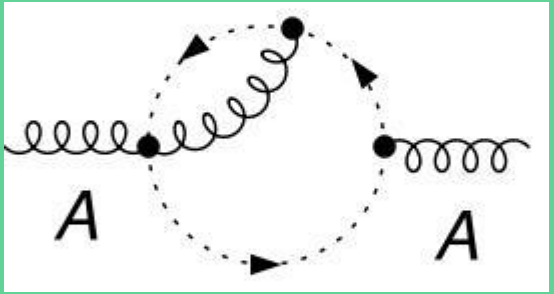
Was this correlation expected? yes

$$\frac{ig_s^4 C_A^2 \delta^{ab}}{(4\pi)^4}\left(-\frac{12}{b}I_{log}^{(2)}(\Lambda^2)+\frac{6}{b^2}I_{log}^2(\Lambda^2)+\frac{70}{3b}I_{log}(\Lambda^2)\right)\left[g_{\mu\nu}p^2-p_\mu p_\nu\right]$$

Total of the divergent contributions at 2-loop with IREG

DS

(a)



$$\frac{ig_s^4 C_A^2 \delta^{ab}}{(4\pi)^4}\left[\underbrace{\left(\boxed{-\frac{1}{6\epsilon^2}}+\frac{-13+\boxed{4\rho}}{12\epsilon}\right)}_{\mathbf{A}} g_{\mu\nu}p^2 - \overbrace{\left(\boxed{-\frac{1}{6\epsilon^2}}+\frac{-9+\boxed{4\rho}}{12\epsilon}\right)}^{\mathbf{B}} p_\mu p_\nu\right]$$

$$\rho = \gamma_E - \ln 4\pi + \ln(p^2/\mu_{DR}^2)$$

Non-local term

$$\frac{i g_s^4 C_A^2 \delta^{ab}}{(4\pi)^4} \left(\frac{17}{3\epsilon} \right) \left[g_{\mu\nu} p^2 - p_\mu p_\nu \right]$$

Total of the divergent contributions at 2-loop with DS

Summary of results

$$Z_A = 1 + \frac{g^2}{(4\pi)^2} Z_A^{(1)} + \frac{g^4}{(4\pi)^4} Z_A^{(2)}$$

Renormalisation function of the external gauge boson

Summary of results

$$\beta = -g_R \left[\beta_0 \left(\frac{\tilde{g}_R}{4\pi} \right)^2 + \beta_1 \left(\frac{\tilde{g}_R}{4\pi} \right)^4 \right]$$

Expansion of the β-function in the adimensional coupling constant

Summary of results

$$Z_A = 1 + \frac{g^2}{(4\pi)^2} Z_A^{(1)} + \frac{g^4}{(4\pi)^4} Z_A^{(2)}$$

Renormalisation function of the external gauge boson

Summary of results

$$\beta = -g_R \left[\beta_0 \left(\frac{\tilde{g}_R}{4\pi} \right)^2 + \beta_1 \left(\frac{\tilde{g}_R}{4\pi} \right)^4 \right]$$

Expansion of the β-function in the adimensional coupling constant

DS β-function

$$\beta = \frac{d-4}{2} g_R \left[A_1 \tilde{g}_R^2 + 2 A_2 \tilde{g}_R^4 \right]$$

IREG β-function

$$\beta = -g_R \left[-\frac{g_R^2}{2} \Lambda \frac{\partial}{\partial \Lambda} A_1 - \frac{g_R^4}{2} \Lambda \frac{\partial}{\partial \Lambda} A_2 \right]$$

(just to remember!)

^

DS

$$Z_A^{(1)}|_{\text{DS}} = \frac{11C_A}{3\epsilon}, \quad Z_A^{(2)}|_{\text{DS}} = \frac{17C_A^2}{3\epsilon}$$

$$\beta_0 = \epsilon Z_A^{(1)}, \quad \beta_1 = 2\epsilon Z_A^{(2)}$$

IREG

$$Z_A^{(1)}|_{\text{IREG}} = \frac{11}{3b} C_A I_{log}(\Lambda^2)$$

$$Z_A^{(2)}|_{\text{IREG}} = \left(-\frac{12}{b} I_{log}^{(2)}(\Lambda^2) + \frac{6}{b^2} I_{log}^2(\Lambda^2) + \frac{70}{3b} I_{log}(\Lambda^2)\right) C_A^2$$

$$\beta_0 = -\frac{1}{2}\Lambda \frac{\partial}{\partial \Lambda} Z_A^{(1)}$$

$$\beta_1 = -\frac{1}{2}\Lambda \frac{\partial}{\partial \Lambda} Z_A^{(2)}$$

$$\beta|_{\mathrm{DS}} = -g_R \left[\frac{11}{3} C_A \left(\frac{\tilde{g}_R}{4\pi} \right)^2 + \frac{34}{3} C_A^2 \left(\frac{\tilde{g}_R}{4\pi} \right)^4 \right]$$

$$\beta|_{\mathrm{IREG}} = -g_R \left[\frac{11}{3} C_A \left(\frac{\tilde{g}_R}{4\pi} \right)^2 + \frac{34}{3} C_A^2 \left(\frac{\tilde{g}_R}{4\pi} \right)^4 \right]$$

Conclusions and Perspectives

Conclusions

- The approach we used to handle the UV-div is through the technique of IREG.
- We calculated the β-function to 2-loop order for a non-abelian theory in a quadri-dimensional framework.
- We have obtained the renormalization constants by conducting the subtraction of subdivergences within IREG and compared with DReg and DRED.
- Evaluating the BDI's of IREG in 4-2ε dimensions at the end of the calculation does not yield the same residues for the DS poles of arbitrary orders.
- We have shown that a systematic summation among different contributions from Feynman graphs and counterterms renders an identical result for DReg and DRED.

Perspectives

- To pursuit the IREG program to apply it to precision calculations, it is important to evaluate the quark mass anomalous dimension.
- The anomalous dimensions describe the deviation of the scaling laws of various operators considered as a function of the running scales.
- To complete our 2-loop project, we will perform the calculation of the 2-loop quark mass anomalous dimension (γ_m-function) in QCD within a mass-independent regularization scheme in the BFM.
- This quantity depends on the renormalization scheme use.

Perspectives

- This result will be derived by a direct calculation of the relevant Feynman diagrams in IREG.

$$\gamma_m^{\bar{MS}}(\mu) = \frac{\mu}{m^{\bar{MS}}}\frac{\partial m^{\bar{MS}}}{\partial \mu} = -\beta^{\bar{MS}}(\mu)\frac{\partial \ln Z_m^{\bar{MS}}}{\partial g^{\bar{MS}}}$$

$$\gamma_m^{\bar{MS}}(\Lambda^2) = 2\frac{\Lambda^2}{m^{\bar{MS}}}\frac{\partial m^{\bar{MS}}}{\partial \Lambda^2} = -\beta^{\bar{MS}}(\Lambda^2)\frac{\partial \ln Z_m^{\bar{MS}}}{\partial g^{\bar{MS}}}$$

Thank you!

Questions

Backup slides

Basic information

To whom it may concern:

- **Undergraduate Program:** Licenciada en Física-Universidad Central de Venezuela.
- **Master Program:** M2 Noyaux, particules, astroparticules et cosmologie-Université Paris Diderot.
- **Graduate Program:** Doutorado em Física-Universidade do ABC.
- **Starting date of the PhD-Expected completion date:** February 19 (2018) to May (2022).
- **Credits:** 114 credits completed out of 124 credits corresponding to mandatory subjects. Relevant courses included: Quantum Field Theory I and II; Quantum Mechanics I,II and III, and Electrodynamics.
- **Estágios De Docência:** Estágio Docente Supervisionado I-Done.
  Estágio Docente Supervisionado II-Not yet enrolled.
- **Proficiency in English Language:** Approved
- **Articles in progress (pre-print):** Two-loop renormalisation of gauge theories in 4D Implicit Regularisation: transition rules to dimensional methods. A. Cherchiglia, D. C. Arias-Perdomo, A. R. Vieira, M. Sampaio and B. Hiller.
- **Works in progress:** Calculation of the massive QCD γ_m-function in the Background Field Method (BFM) in IReg and off-shell renormalization of QCD at 2-loops.

LO

Born level cross-section

NLO

1-loop level cross-section

NNLO

2-loop level cross-section

$$\hat{\sigma} \propto |A|^2$$

$$A = \alpha_s^n \left(A^{\text{tree}} + \frac{\alpha_s}{2\pi} A^{1-\text{loop}} + \left(\frac{\alpha_s}{2\pi}\right)^2 A^{2-\text{loop}} + \ldots \right)$$

Understanding the difficulties

01

Ultraviolet (**UV-div**) and **infrared** (**IR-div**) divergences are all-over beyond leading order in S-matrix calculations and must be wisely removed in order to automated computation codes for the evaluation of Feynman amplitudes.

02

A general cross section in QCD usually includes short and long-distance behaviour and thus it is not computable directly in perturbation theory.

$$\mathcal{M}_n\left(\frac{\mu}{\lambda_{IR}}, \frac{p_i}{\lambda_{UV}}, \alpha_s(\lambda_{UV})\right) = Z\left(\frac{\mu}{\lambda_{IR}}, \frac{p_i}{\lambda_{IR}}, \alpha_s(\lambda_{IR})\right) H_n\left(\frac{p_i}{\lambda_{UV}}, \frac{\lambda_{IR}}{\lambda_{UV}}, \alpha_s(\lambda_{UV})\right)$$

Superficial degree of UV-div divergence

$$\Delta = dl - 2p$$

$$l = \underbrace{p}_{p = \frac{nv - N}{2}} - v + 1$$

$$\Delta = d + \left[n\left(\frac{d-2}{2}\right) - d \right] v - \left(\frac{d-2}{2}\right) N$$

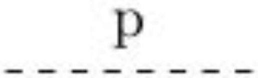
A simpler example: the renormalization of ϕ^4

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4$$

$$\phi_0 = Z^{\frac{1}{2}}\phi_R, \quad m_0^2 = Z_m m_R^2, \quad g_0 = Z_g g_R$$

$$Z_\phi = 1 + A, \quad Z_\phi Z_m = 1 + B, \quad Z_\phi^2 Z_g = 1 + C$$

$$\mathcal{L}_{\phi_R^4} = \underbrace{\frac{1}{2}(\partial_\mu \phi_R)^2 - \frac{1}{2}m_R^2\phi_R^2 - \frac{g_R}{4!}\phi_R^4}_{\text{Free lagrangian and interaction term}} + \underbrace{\frac{1}{2}A(\partial_\mu \phi_R)^2 - \frac{1}{2}Bm_R^2\phi_R^2 - \frac{g_R}{4!}C\phi_R^4}_{\text{Counterterms lagrangian}}$$



$$= \frac{i}{p^2 - m^2}.$$

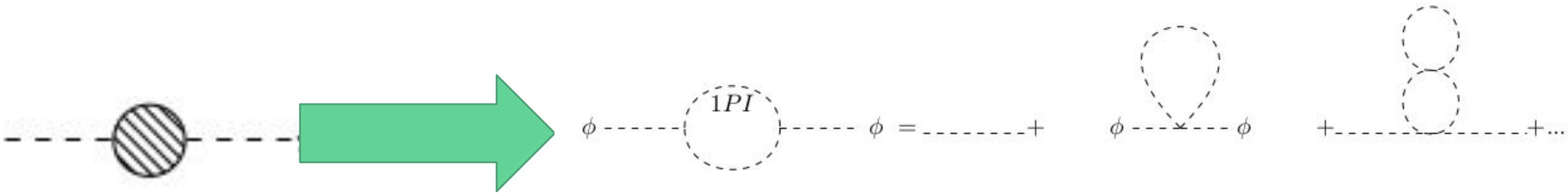
$$= i(Ap^2 - Bm_R^2),$$

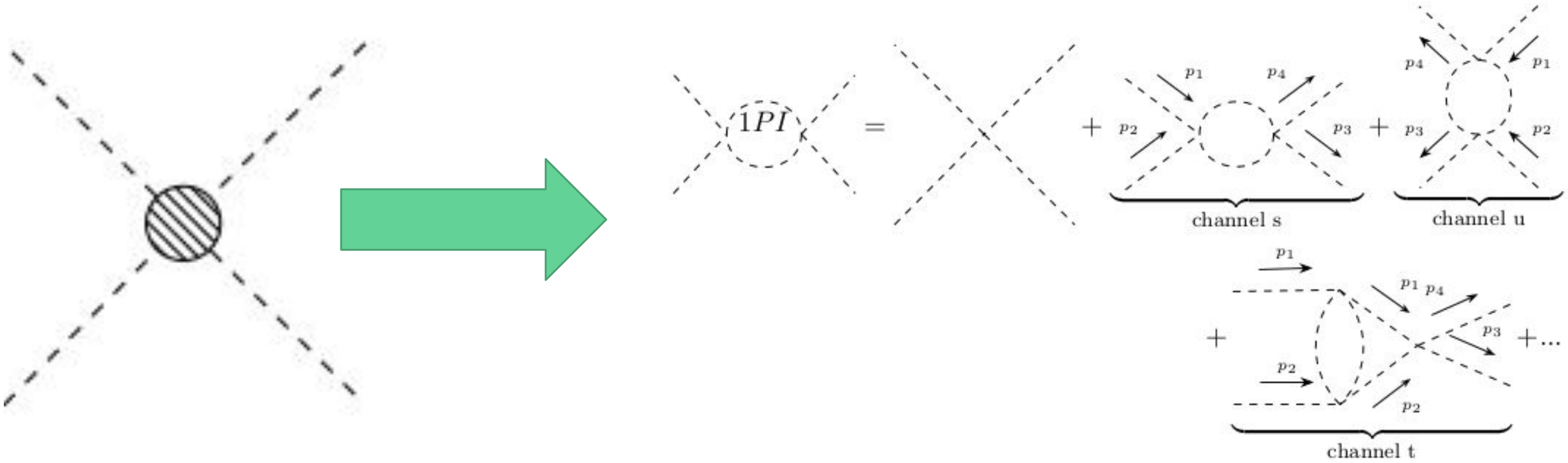
$$= -ig.$$

$$= -ig_R C$$

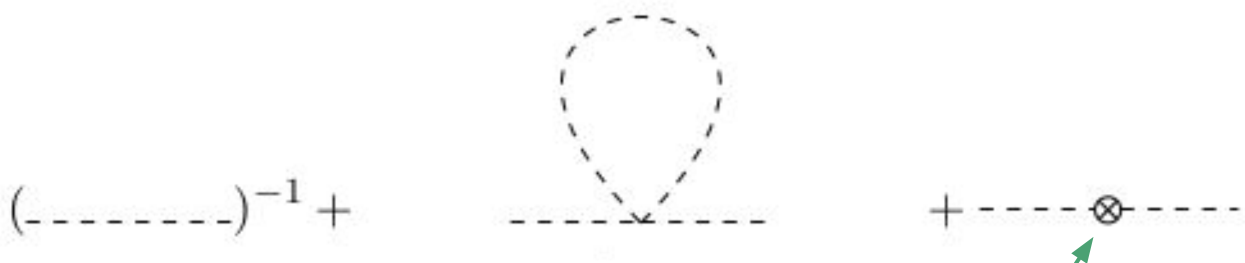
Rules from the Free lagrangian and interaction term

Rules from the counterterms lagrangian



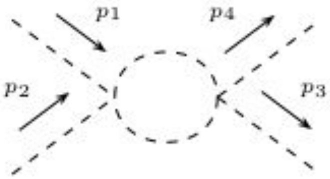


The only divergent amplitudes we can have for ϕ^4



After, we chose the renormalization scheme

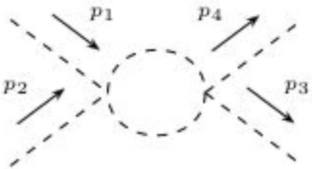
Dimensional Regularization application example



$$= \frac{1}{2} g^2 \mu^{4-d} \int \frac{d^dk}{(2\pi)^d} \frac{1}{\underbrace{(k^2 - m^2)}_{\mathrm{B}} \underbrace{(k-p)^2 - m^2]}_{\mathrm{A}}}$$

Using the Feynman parameters:

$$\frac{1}{A_1 A_2 \ldots A_n} = \int_{-\infty}^{+\infty} dx_1 \ldots dx_n \delta\left(\sum x_i - 1\right) \frac{(n-1)!}{[x_1 A_1 + x_2 A_2 + \ldots + x_n A_n]^n}$$



$$= \frac{g\mu^{4-d}}{2} \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 - \Delta)^2}.$$

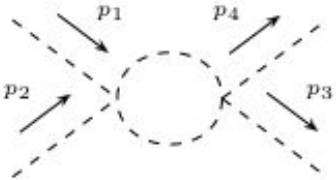
Here we did a shift
into a new
momentum
k -> k+px

$$\Delta^2 = m^2 - p^2 x(1-x)$$

Using:

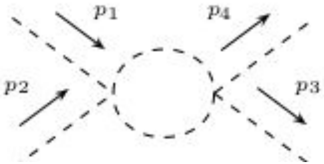
$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2 - \Delta]^n} = \frac{(-1)^n}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n - d/2}$$

setting $d = 4 - 2\epsilon$



$$= \frac{g\mu^{2\epsilon}}{2} \int_0^1 dx \frac{i\Gamma(\epsilon)}{(4\pi)^2} (4\pi)^\epsilon \frac{1}{\Delta^\epsilon}.$$

$$a^\epsilon = \exp^{\epsilon \ln a} + \mathcal{O}(\epsilon) \qquad\qquad \Gamma(\epsilon) \equiv \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)$$



$$= \frac{ig^2}{32\pi^2\epsilon} - \frac{ig^2}{32\pi^2} \int_0^1 dx \left[\ln\left(\frac{m^2 - x(1-x)s}{4\pi\mu^2} \right) + \gamma_E \right] + \mathcal{O}(\epsilon)$$

Dimensional Reduction (DRED)

- Modifications on DReg can lead to additional terms at the Lagrangian level, such as the **ε-scalar** particles.

$$\mathcal{A}_4 = \underbrace{\mathcal{A}_d}_{4\text{-}2\epsilon} + \underbrace{\mathcal{A}_N}_{2\epsilon} \qquad \mathcal{L}^{(4)} = \mathcal{L}^{(d)} + \mathcal{L}^{(2\epsilon)}$$

$$\begin{aligned}\mathcal{L}^{(\varepsilon)} = & -\tfrac{1}{2}(\partial^i \mathcal{G}^\rho_a)^2 + g f_{abc}(\partial^i \mathcal{G}^\rho_a) G_{bi} \mathcal{G}_{c\rho} - g (T_a)_{kl} \mathcal{G}^\rho_a \bar{\psi}_k \gamma_\rho \psi_l \\ & -\tfrac{1}{2} g^2 f_{abc} f_{ade} G^i_b \mathcal{G}^\rho_c G_{di} \mathcal{G}_{e\rho} \\ & -\tfrac{1}{4} g^2 f_{abc} f_{ade} \mathcal{G}^\rho_b \mathcal{G}^\sigma_c \mathcal{G}_{d\rho} \mathcal{G}_{e\sigma}. \qquad (2.8)\end{aligned}$$

Körner, J.G., Tung, M.M. Dimensional reduction methods in QCD. *Z. Phys. C - Particles and Fields* 64, 255–265 (1994). https://doi.org/10.1007/BF01557396

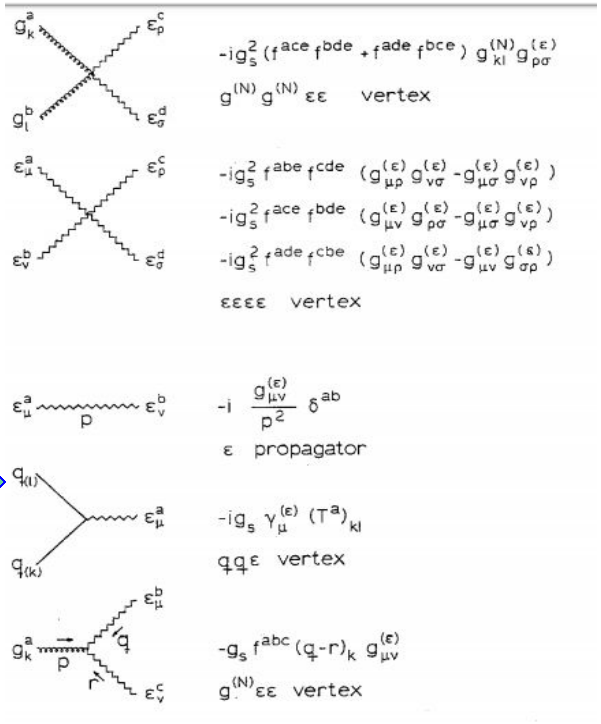
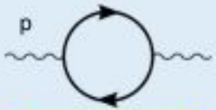


Fig. 1. Feynman rules for ε-scalars (all momenta flow into the vertices)

Momentum Routing Invariance and Gauge Invariance



$\boxed{k_1 - k_2 = p}$

Final result:

$$\Pi_{\mu\nu} = \tilde{\Pi}_{\mu\nu} + 4\Big(\boxed{\Upsilon^2_{\mu\nu}} - \frac{1}{2}(k_1^2 + k_2^2)\boxed{\Upsilon^0_{\mu\nu}} + \frac{1}{3}(k_1^\alpha k_1^\beta + k_2^\alpha k_2^\beta + k_1^\alpha k_2^\beta)\boxed{\Upsilon^0_{\mu\nu\alpha\beta}}$$
$$- (k_1 + k_2)^\alpha (k_1 + k_2)_\mu \boxed{\Upsilon^0_{\nu\alpha}} - \frac{1}{2}(k_1^\alpha k_1^\beta + k_2^\alpha k_2^\beta) g_{\mu\nu}\boxed{\Upsilon^0_{\alpha\beta}}\Big) \quad \text{where}$$

$\boxed{\Upsilon = 0 \quad \mathrm{MRI}}$

It depends explicitly of k1-k2 -> routing invariance

$$\tilde{\Pi}_{\mu\nu} = \frac{4}{3}\Big((k_1 - k_2)^2 g_{\mu\nu} - (k_1 - k_2)_\mu (k_1 - k_2)_\nu\Big)\left(I_{log}(\mu^2) - \underbrace{\frac{i}{(4\pi)^2}\left(\frac{5}{3} + \ln\frac{-(k_1-k_2)^2}{\mu^2}\right)}_{\text{IR diverg. eliminated}}\right)$$

Scale relation

Ward Identity: $(k_1 - k_2)_\mu \Pi_{\mu\nu} = 0$ **(massless photon)**

Background Field Method (BFM)-details

$$S^{YM}(Q) = -{}^1\!/_4 \int d^4x (F^a_{\mu\nu})^2$$

$$Z(J) = \int \mathcal{D}_Q \det\left(\frac{\delta G^a}{\delta w^b}\right) \exp^{i(S^{YM}(Q) - {}^1\!/_{2\alpha} G\cdot G + J\cdot Q)},$$

$$G \cdot G \equiv \int d^4x G^a G^a.$$

$$J \cdot Q \equiv \int d^4x J^a_\mu Q^a_\mu$$

Review of Functional Methods

- The BFM relies on functional methods in field theory. Therefore it is necessary to do a reviewing of the standard functional techniques.

Green's functions

QFT ➡ *S-matrix*

- Green's functions are determined by taking functional derivatives with respect to the source function J of the generating functional.

$$Z[J] = \int \delta Q \exp i\left[S[Q] + \int d^4x J \cdot Q\right]$$

- The Green's functions are defined by:

$$\int \delta(Q\ldots Q)e^{iS[Q]} =< 0|Q\ldots Q|0 >= \left(\frac{1}{i}\frac{\delta}{\delta J}\right)^n Z[J]\Big|_{J=0}$$

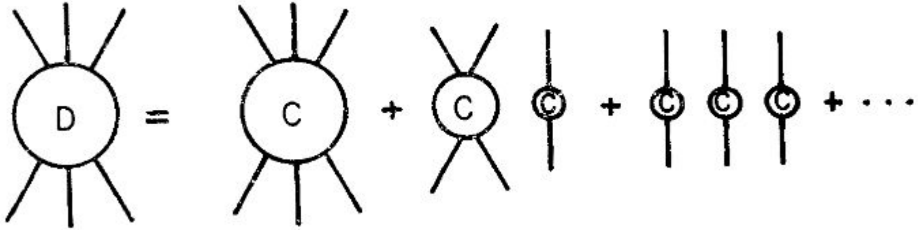


Fig. 1. The relation between connected and disconnected graphs. D = disconnected, C = connected

- As the disconnected pieces of Green's functions don't contribute to the S-matrix, it's better to work only with connected Green's functions.

$$W[J] = -i\ln(Z[J])$$

- We can show graphically how disconnected pieces can be removed just by taking ln(Z[J]). For that, we constructed the first few Green functions from previous Eq.

$$W[J] = -i \ln(Z[J])$$

$$\frac{\delta W[J]}{\delta J} = \frac{\delta}{\delta J}\left(-i \ln(Z[J])\right) = -i \frac{1}{Z[J]} \frac{\delta}{\delta J} Z[J] = \frac{1}{\langle 0|0\rangle} \langle 0|Q|0\rangle$$

For the 2nd derivative:

$$\frac{1}{i} \frac{\delta^2 W[J]}{\delta J^2} = \left[\frac{\langle 0|T\{QQ\}|0\rangle}{\langle 0|0\rangle} - \left| \frac{\langle 0|Q|0\rangle}{\langle 0|0\rangle} \right|^2 \right]$$

Fig. 2. The relation between the connected (C) and disconnected (D) two-point functions

- We can further simplify these expressions by write the connected Green's functions in terms of 1PI.

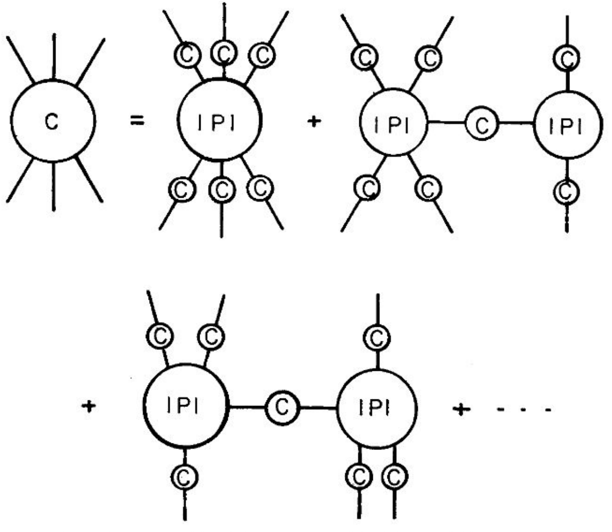


Fig. 4. The relation between connected (C) and 1PI graphs

- The 1PI Green's function are generated by the effective action.

$$\bar{Q} \neq Q = \frac{\delta W}{\delta J}$$

$$\Gamma[\bar{Q}] = W[J] - J \cdot \bar{Q}$$

BFM: non-gauge theory

$$\Gamma[\bar{Q}] \rightarrow \textit{S-matrix}$$

- The BFM compute the effective action. Let us consider a <u>non-gauge</u> theories. For them, the BFM is identical to the conventional "field-shifting" method.

Let us remember: $Z[J] = \int \delta Q \exp i[S[Q] + \int d^4x J \cdot Q]$

And define the analogous quantity:

Classical action

$$\tilde{Z}[J,\Phi] = \int \delta Q \exp i[S[Q+\Phi] + J \cdot Q]$$

Alternative source

Arbitrary BF Φ

By the same analogy: $W[J] = -i \ln(Z[J]) \longrightarrow \tilde{W}[J,\Phi] = -i \ln(Z[\tilde{J},\Phi])$

Using all these definition:

$$\tilde{\Gamma}[\tilde{Q},\Phi]=\tilde{W}[J,\Phi]-J\cdot\tilde{Q}$$
$$\tilde{W}[J,\Phi]=W[J]-J\cdot\Phi$$
$$\tilde{Q}=\bar{Q}-\Phi$$

$$\tilde{\Gamma}[\tilde{Q},\Phi]=\Gamma[\bar{Q}]$$
$$\tilde{\Gamma}[\tilde{Q},\Phi]=\Gamma[\tilde{Q}+\Phi]$$

BF effective action

$$\tilde{Q}=0$$

- As you can see, the BFM allows you to compute the effective action by summing only vacuum graphs.

$$\tilde{\Gamma}[0,\Phi]=\Gamma[\Phi]$$

$$\frac{\delta\tilde{\Gamma}[\tilde{Q},\Phi]}{\delta\tilde{Q}} \longrightarrow$$ 1PI Green's function in the presence of a BF

$$\tilde{\Gamma}[0,\Phi] \longrightarrow$$ Generates no graphs with external lines

How we compute the effective action? Abbott's approach is to treat the BF Φ perturbatively:

1. Consider 1PI vacuum graphs of Q with Φ fields appearing as external lines.
2. Then you can use $S[Q+\Phi]$ to compute the Feynman Rules.

$$S \Rightarrow \begin{cases} Q^2 \longrightarrow \text{Ordinary Q field propagators} \\ \text{Non-quadratic parts} \longrightarrow \text{Interactions} \end{cases}$$

So, what happened with the gauge theories?

- Shifting the integration variable in the functional integral, gives analogous expressions for the gauge theory as we did for the non-gauge one.

$$\tilde{\Gamma}[\tilde{Q},\Phi]=\Gamma[\tilde{Q}+\Phi] \qquad \tilde{\Gamma}[\tilde{Q},A]=\Gamma[\tilde{Q}+A]$$

$$\tilde{\Gamma}[0,\Phi]=\Gamma[\Phi] \qquad \tilde{\Gamma}[0,A]=\Gamma[A]$$

- The BF effective action with the background gauge field and all the external legs, is a generating functional for 1PI graphs in a particular gauge.
- The BFM retains explicit gauge invariance, there's a choice of the gauge fixing term for which the BF effective action in a non-abelian theory is a gauge invariant functional of A, that gauge choice is:

$$\tilde{G}^a=\partial_\mu Q^a_\mu+gf^{abc}A^b_\mu Q^c_\mu$$

- With this choice Z[J,A] is invariant under:

$$\delta A^a_\mu=-f^{abc}w^b A^c_\mu+\frac{1}{g}\partial_\mu w^a$$

$$\delta J^a_\mu=-f^{abc}w^b J^c_\mu$$

So, what happened with the gauge theories?

$$\widetilde{\Gamma}[0,A]=\Gamma[A]$$

$\rightarrow$

It's a gauge
invariant
function of A.

$\rightarrow$

$$\frac{\delta \widetilde{\Gamma}[0,A]}{\delta A}$$

Green's function will obey Ward identities of gauge invariance

BFM: main points

- BFM is useful to compute the effective action of a pure YM theory by expanding the field around a classical BF.
- In the BFM a gauge is choose in order to make Green's functions gauge invariant and respect Ward identities like the ones in QED. These relations makes that the knowledge of 2-point functions is the only one necessary to compute the β-function and simplifies calculations.

The renormalization group equations

Coupling constant

- Green's functions for the bare ϕ^4 lagrangian: $$\Gamma_0^{(n)}(p_1, \ldots, p_n; m_0, g_0)$$

- Green's functions for the ϕ^4 counterterm lagrangian: $$\Gamma^{(n)}(p_1, \ldots, p_n; m_R(\lambda), g_R(\lambda), \lambda)$$

Scaled parameter

Still, they are the same theory.

- Both of them are the same Green's function related by:

$$\underbrace{\Gamma_0^{(n)}(p_1, \ldots, p_n; m_0, g_0)}_{\text{independent of the scaled parameter}} = \underbrace{Z_\Gamma^{-1}}_{\text{Renor. const.}} \underbrace{\Gamma^{(n)}(p_1, \ldots, p_n; m_R(\lambda), g_R(\lambda), \lambda)}_{\text{dependent of the scaled parameter}}$$

$$\underbrace{\Gamma_0^{(n)}(p_1,\ldots,p_n;m_0,g_0)}_{\text{independent of the scaled parameter}} \Longrightarrow \lambda\frac{d\Gamma_0^{(n)}}{d\lambda} = 0$$

$$\lambda\frac{d}{d\lambda}[Z_\Gamma^{-1}\Gamma^{(n)}(p_1,\ldots,p_n;m_R(\lambda),g_R(\lambda),\lambda)] = 0$$

$$Z_\Gamma^{-1} = \left[-\lambda\frac{d\ln Z_\Gamma}{d\lambda} + \lambda\frac{\partial g_i}{\partial\lambda}\frac{\partial}{\partial g_i} + \lambda\frac{\partial m_i}{\partial\lambda}\frac{\partial}{\partial m_i} + \lambda\frac{\partial}{\partial\lambda}\right]\Gamma^{(n)} = 0$$

β-function γm-function

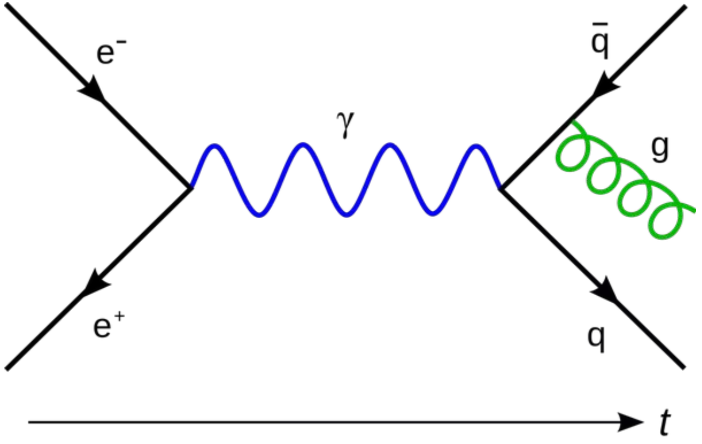
$$\boxed{Z_\Gamma^{-1} = \left[-\lambda \frac{d \ln Z_\Gamma}{d\lambda} + \lambda \frac{\partial g_i}{\partial \lambda} \frac{\partial}{\partial g_i} + \lambda \frac{\partial m_i}{\partial \lambda} \frac{\partial}{\partial m_i} + \lambda \frac{\partial}{\partial \lambda} \right] \Gamma^{(n)} = 0}$$

These partial derivatives equations are the renormalization group equations for renormalized Green functions. The whole group simply tell us about the scale dependence of the parameters in a renormalizable field theory.

Why not this type of contractions?

$$\int_{k,q} \frac{k.p}{k^2(k-p)^2q^2(k-q)^2}\bigg|_{\mathrm{IREG}} \neq \int_{k,q} \frac{k^2+p^2-(k-p)^2}{2k^2(k-p)^2q^2(k-q)^2}\bigg|_{\mathrm{IREG}}$$

IR-div: example



$$\boxed{\int d\Phi_n |\mu|^2}$$

- The gluon emitted in the process has a very low energy, so low that it will not be detected in the detector.
- Even upgrading the detector, it still has an energy limit that it cannot detect.
- At the level of theoretical calculation, this physical limit is not being considered. If so, you would be able to detect any gluon at any energy, no matter how low.
- These divergences are removed in the observables automatically. We don't need to renormalize them.

DS

IREG

$$I = \int_k \frac{1}{k^2(k-p)^2} \overset{\text{IREG}}{=} I_{log}(\Lambda^2) - b\ln\left[-\frac{p^2}{\Lambda^2}\right] + 2b.$$

DS

IREG

$$I = \int_k \frac{1}{k^2(k-p)^2} \overset{\text{IREG}}{=} I_{log}(\Lambda^2) - b\ln\left[-\frac{p^2}{\Lambda^2}\right] + 2b.$$

$$\Big\downarrow{\scriptstyle d=4-2\varepsilon}$$

$$\frac{i(4\pi)^{\epsilon}}{(4\pi)^2}\left(\frac{\mu^{\epsilon}}{\Lambda^{\epsilon}}\right)\Gamma(\epsilon)$$

DS

IREG

$$I = \int_k \frac{1}{k^2(k-p)^2} \overset{\text{IREG}}{=} I_{log}(\Lambda^2) - b\ln\left[-\frac{p^2}{\Lambda^2}\right] + 2b.$$

$$\Big\downarrow \; d=4-2\varepsilon$$

$$\frac{i(4\pi)^\epsilon}{(4\pi)^2}\left(\frac{\mu^\epsilon}{\Lambda^\epsilon}\right)\underbrace{\Gamma(\epsilon)}_{\frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)}$$

DS

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$$I = \int_k \frac{1}{k^2(k-p)^2} \overset{\text{IREG}}{=} I_{log}(\Lambda^2) - b\ln\left[-\frac{p^2}{\Lambda^2}\right] + 2b.$$

d=4-2ε

$$b\left[\frac{1}{\epsilon} - \gamma_E + \ln(4\pi) + \ln\left(\frac{\mu_{DR}^2}{\Lambda^2}\right)\right]$$

DS

IREG

$$I = \int_k \frac{1}{k^2(k-p)^2} \overset{\text{IREG}}{=} I_{log}(\Lambda^2) - b\ln\left[-\frac{p^2}{\Lambda^2}\right] + 2b.$$

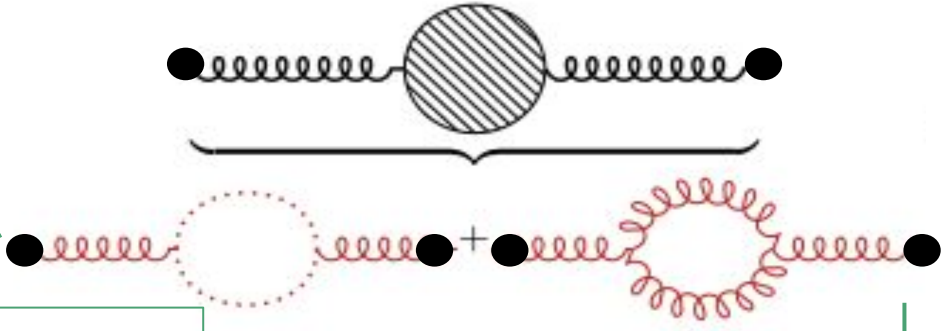
$$\Big\downarrow \; d=4-2\varepsilon$$

$$b\left[\frac{1}{\epsilon} - \gamma_E + \ln(4\pi) + \ln\left(\frac{\mu_{DR}^2}{\Lambda^2}\right)\right]$$

$$I\Big|_d = b\left[\frac{1}{\epsilon} - \gamma_E + \ln(4\pi) - \ln\left[-\frac{p^2}{\boxed{\mu_{DR}^2}}\right] + 2\right].$$

Pure Yang-Mills at 1-loop with DS

- We only need the gluon self-energy diagrams.



$$\boxed{\frac{ig^2 C_A \delta^{ab}}{(4\pi)^2}\left(\frac{1}{3\epsilon}\right)\left[g_{\mu\nu}k^2 - k_\mu k_\nu\right]}$$

$$\boxed{\frac{ig^2 C_A \delta^{ab}}{(4\pi)^2}\left(\frac{10}{3\epsilon}\right)\left[g_{\mu\nu}k^2 - k_\mu k_\nu\right]}$$

$$Z_{\hat{A}} = 1 + \frac{11 C_A}{3\epsilon}\frac{g^2}{(4\pi)^2} \quad\Longrightarrow\quad \beta = -\frac{11 C_A}{3}\frac{g^3}{(4\pi)^2}$$

$$\mathcal{A} = \int_{k,l} \mathcal{F}(l,k,p)$$

$$\mathcal{A} \supset \int_k G(k,p,\mu^2) \int_l F(l,k,p,\mu^2)$$

$$\mathcal{A} = \int_k G(k,p,\mu^2) \left[a_1 I_{log}(\lambda^2) - a_1 b \ln\left[-\frac{k^2}{\lambda^2}\right] + \bar{a}_2 \right]$$

Using: $$\int_k G(k,p,\mu^2) \ln^n \left[-\frac{k^2}{\lambda^2}\right] = A_1 I_{log}^{(n+1)}(\mu^2) + \cdots$$

Thus: $$\mathcal{A} = \boxed{a_1 A_1} \left[I_{log}^2(\lambda^2) - b I_{log}(\lambda^2) \ln\left[-\frac{p^2}{\lambda^2}\right] - b I_{log}^{(2)}(\lambda^2) + \cdots \right]$$

IREG rules are generalisable to 2-loop order

6. At 2-loop order the divergent content can be expressed in terms of BDI.

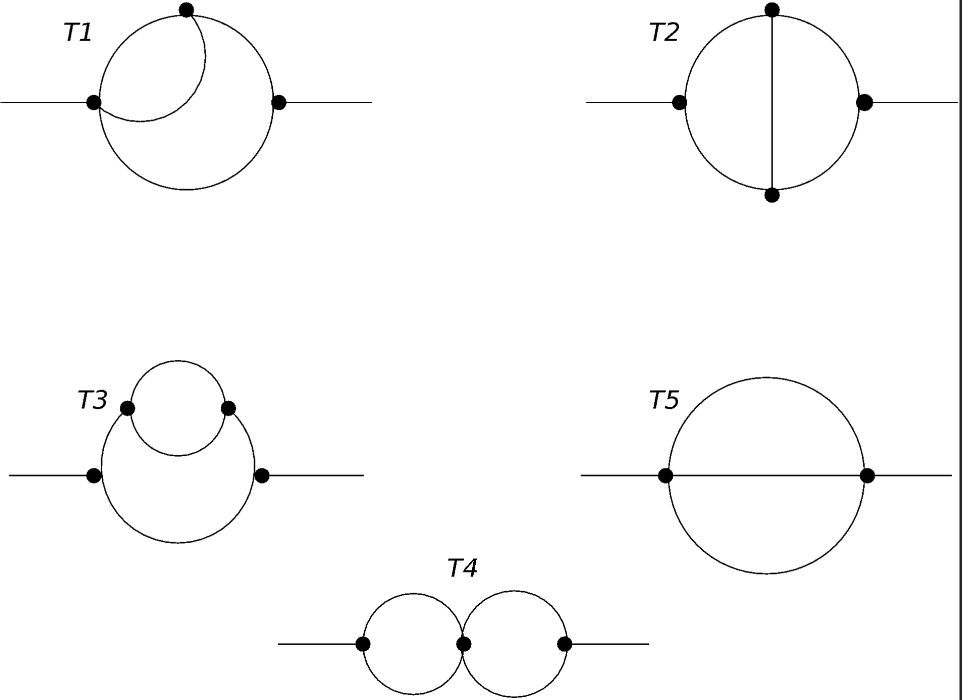
$$I_1^{(2)} = \int_k \frac{1}{k^2(k-p)^2} \ln\left(-\frac{k^2}{\Lambda^2}\right) \qquad I_\mu^{(2)} = \int_k \frac{k_\mu}{k^2(k-p)^2} \ln\left(-\frac{k^2}{\Lambda^2}\right)$$

7. After removing external momenta with the algebraic identity, the divergent integrals are cast as:

$$I_{log}^{(2)}(\mu^2) \equiv \int\limits_k \frac{1}{(k^2-\mu^2)^2} \ln\left(-\frac{k^2-\mu^2}{\Lambda^2}\right) \qquad I_{log}^{(2)\nu_1\cdots\nu_{2r}}(\mu^2) \equiv \int\limits_k \frac{k^{\nu_1}\cdots k^{\nu_{2r}}}{(k^2-\mu^2)^{r+1}} \ln\left(-\frac{k^2-\mu^2}{\Lambda^2}\right)$$

$$I_{quad}^{(2)}(\mu^2) \equiv \int\limits_k \frac{1}{(k^2-\mu^2)} \ln\left(-\frac{k^2-\mu^2}{\lambda^2}\right)$$

etc...



$$\mathcal{A}_{\mathrm{T1}} \propto \int_{k,l} \frac{\mathcal{F}_{T1}^{\alpha\beta}(l,k,p)}{k^2 (k-p)^2 l^2 (l-k)^2}$$

$$\mathcal{A}_{\mathrm{T2}} \propto \int_{k,l} \frac{\mathcal{F}_{T2}^{\alpha\beta}(l,k,p)}{k^2 (k-p)^2 (k-l)^2 l^2 (l-p)^2}$$

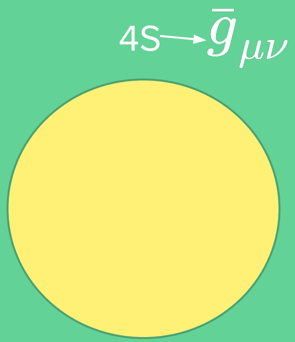
$$\mathcal{A}_{\mathrm{T3}} \propto \int_{k,l} \frac{\mathcal{F}_{T3}^{\alpha\beta}(l,k,p)}{k^4 (k-p)^2 l^2 (l-k)^2}$$

$$\mathcal{A}_{\mathrm{T4}} \propto \int_{k,l} \frac{\mathcal{F}_{T4}^{\alpha\beta}(l,k,p)}{k^2 (k-p)^2 l^2 (l-p)^2}$$

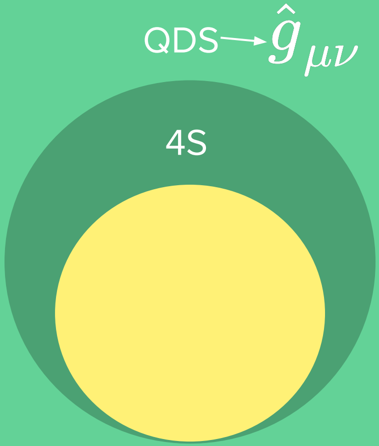
$$\mathcal{A}_{\mathrm{T5}} \propto \int_{k,l} \frac{\mathcal{F}_{T5}^{\alpha\beta}(l,k,p)}{k^2 l^2 (l-k+p)^2}$$

Variants of DS

In order to define new schemes one needs to distinguish three spaces

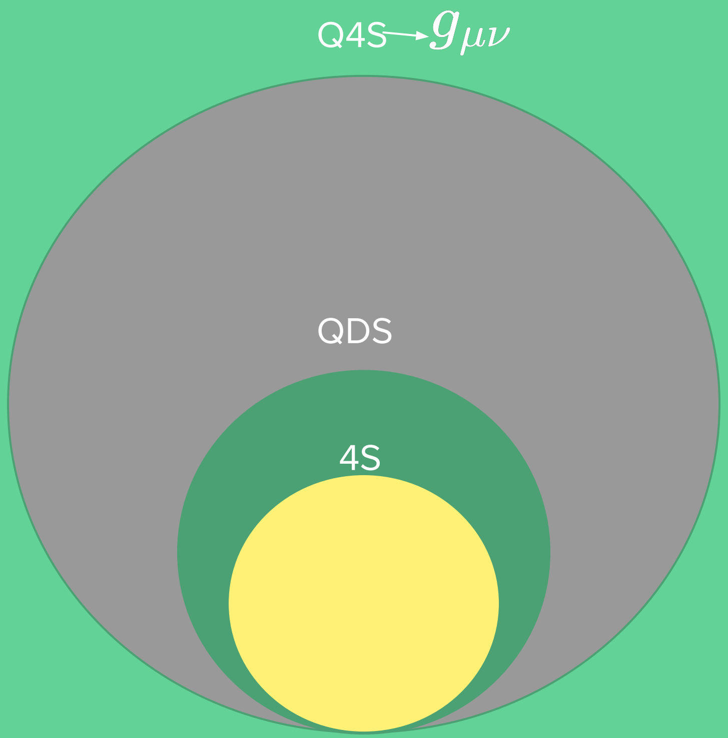


The original 4-dimensional space



The ”quasi-D-dimensional space”

The original 4-dimensional space



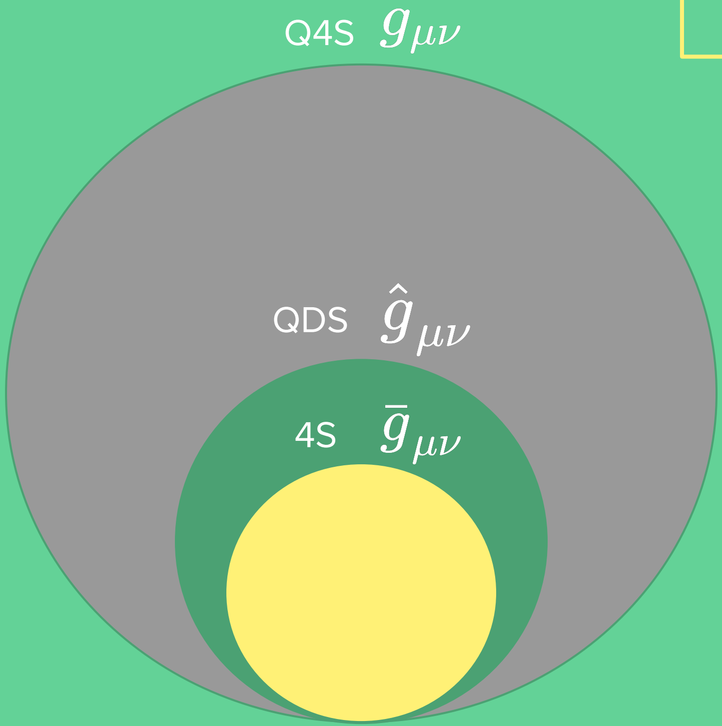
The ”quasi-4-dimensional space”

The ”quasi-D-dimensional space”

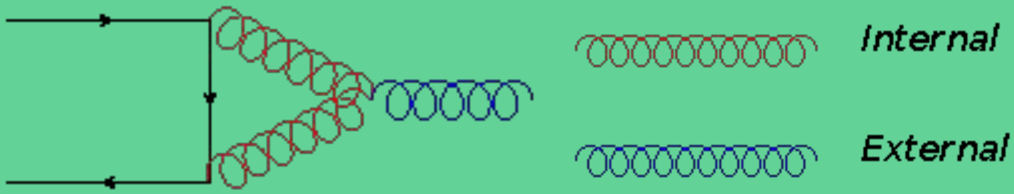
The original 4-dimensional space

Variants of DS

Only gluons that appear inside a divergent loop or phase space integral need to be regularized.



$$Q4S \supset QDS \supset 4S$$



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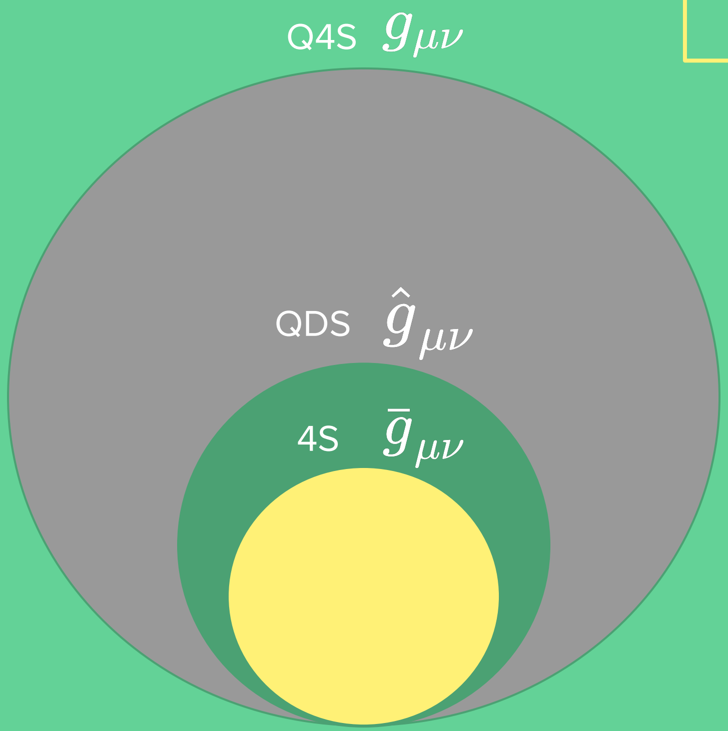

$$Q4S \supset QDS \supset 4S$$

	DReg	DRED
Internal gluon	$\hat{g}_{\mu\nu}$	$g_{\mu\nu}$
External gluon	$\hat{g}_{\mu\nu}$	$g_{\mu\nu}$

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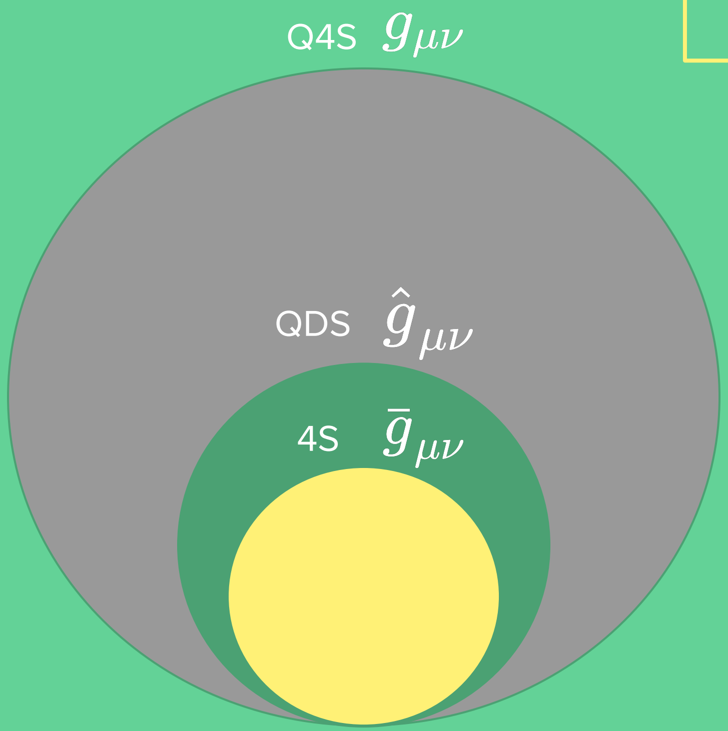


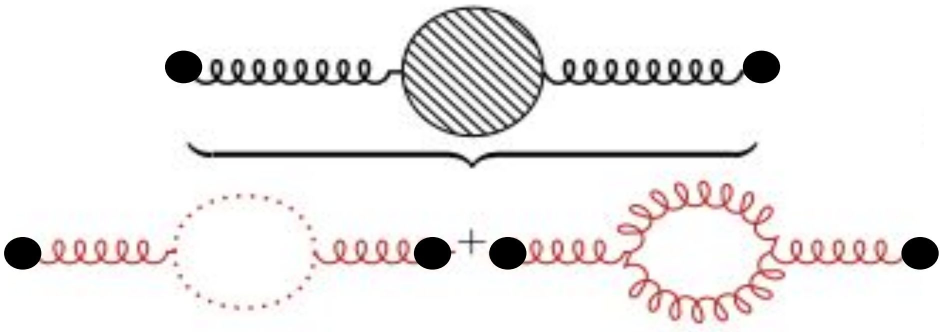
$$Q4S \supset QDS \supset 4S$$

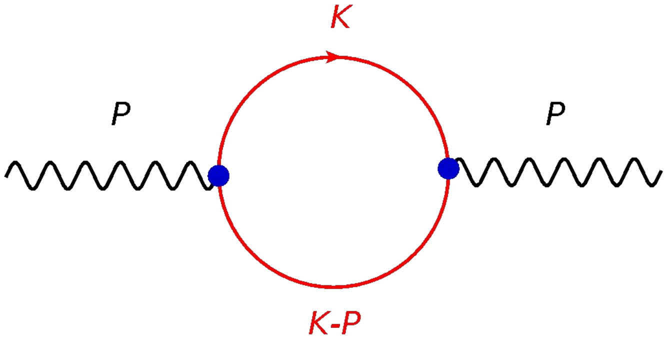
	DReg	DRED
Internal gluon	$\hat{g}_{\mu\nu}$	$g_{\mu\nu}$
External gluon	$\hat{g}_{\mu\nu}$	$g_{\mu\nu}$

$$\boxed{g_{\mu\nu} = \hat{g}_{\mu\nu} + \underbrace{\tilde{g}_{\mu\nu}}_{2\varepsilon}}$$

Pure Yang-Mills at 1-loop

- The β-function can be determined by calculating the renormalization constant of the BF field.
- Therefore, we only need the gluon self-energy diagrams.



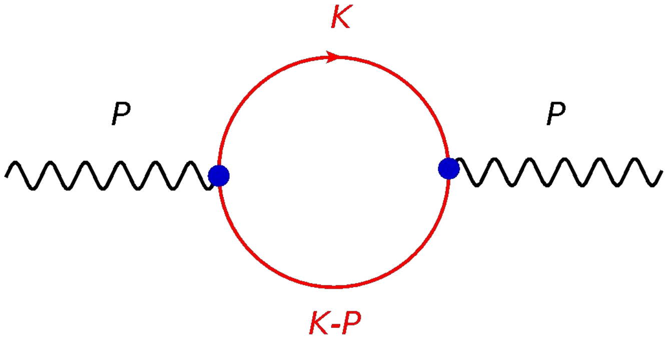


$$\underbrace{D = (k^2)(k-p)^2}_{\text{Power of 4}}$$

Power of 4

$$\Pi_{\tau\nu}(k) = -4e^2 \left[\underbrace{2\boxed{\int_k} \frac{k_\tau k_\nu}{D}}_{\Delta = 2} - p_\tau \underbrace{\int_k \frac{k_\nu}{D}}_{\Delta = 1} - p_\nu \underbrace{\int_k \frac{k_\tau}{D}}_{\Delta = 1} - g_{\tau\nu} \underbrace{\int_k \frac{k^2}{D}}_{\Delta = 2} + g_{\tau\nu} p^\sigma \underbrace{\int_k \frac{k_\sigma}{D}}_{\Delta = 1} + g_{\tau\nu} \underbrace{\int_k \frac{1}{D}}_{\Delta = 0} \right]$$

$\Delta =$ #power in the numerator **-** #power in the denominator



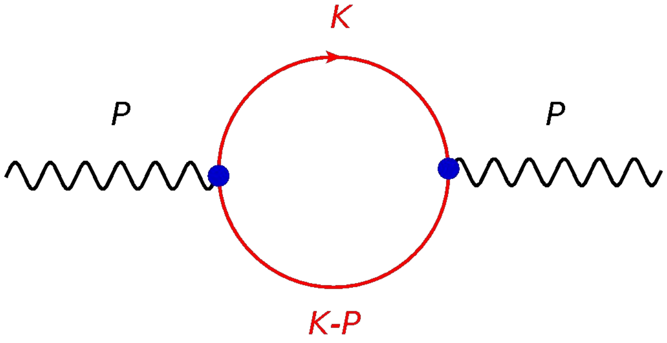
$$D = (k^2)(k-p)^2$$

$$\Pi_{\tau\nu}(k) = -4e^2 \left[2\int_k \frac{k_\tau k_\nu}{D} - p_\tau \int_k \frac{k_\nu}{D} - p_\nu \int_k \frac{k_\tau}{D} - g_{\tau\nu} \int_k \frac{k^2}{D} + g_{\tau\nu} p^\sigma \int_k \frac{k_\sigma}{D} + g_{\tau\nu} \int_k \frac{1}{D} \right]$$

When do we know that a diagram can be regularized?

Traditional dimensional schemes (DS): DReg and DRED

- DS are based on analytical continuations of the space: 4 $\rightarrowtail$ d dimensions.
- In DS **UV-div** manifest as poles: $\frac{1}{\epsilon^n}$
- Dimensional regularization (DReg) analytically continues the integral into d=4-2ε.
- Alternative schemes to DReg have been developed, such as dimensional reduction (DRED).



$$D = (k^2)(k-p)^2$$

$$\Pi_{\tau\nu}(k) = -4e^2 \left[2\underbrace{\int_k \frac{k_\tau k_\nu}{D}}_{\Delta = 2} - p_\tau \underbrace{\int_k \frac{k_\nu}{D}}_{\Delta = 1} - p_\nu \underbrace{\boxed{\int_k \frac{k_\tau}{D}}}_{\Delta = 1} - g_{\tau\nu} \underbrace{\int_k \frac{k^2}{D}}_{\Delta = 2} + g_{\tau\nu} p^\sigma \underbrace{\int_k \frac{k_\sigma}{D}}_{\Delta = 1} + g_{\tau\nu} \underbrace{\int_k \frac{1}{D}}_{\Delta = 0} \right]$$

$$\frac{dI_{log}(m^2)}{dm^2} = -\frac{b_d}{m^2},$$

$$\frac{dI^{\mu\nu}_{log}(m^2)}{dm^2} = -\frac{g^{\mu\nu}}{d}\frac{b_d}{m^2}$$

$$b_d = \frac{i}{(4\pi)^{d/2}}\frac{(-)^{d/2}}{\Gamma(d/2)}$$

$$I_{log}(m^2) = b_d \ln\left(\frac{\Lambda^2}{m^2}\right) + \alpha_1 \qquad I^{\mu\nu}_{log}(m^2) = \frac{g^{\mu\nu}}{d}\left[b_d \ln\left(\frac{\Lambda^2}{m^2}\right) + \alpha_1'\right]$$

$$\Upsilon_0^{\mu\nu} \propto g^{\mu\nu}(\underbrace{\alpha_1 - \alpha_1'}_{\text{arbitrary, reg.dep.}})$$

arbitrary, reg.dep. $\longrightarrow$ Source of symmetry breaking

Symmetric Integration is not valid in general in the integer dimension for *divergent integrals*.

$$\int \mathrm{d}^4 k k^\sigma k_\alpha f(k^2) = \frac{1}{4}\delta^\sigma_\alpha \int \mathrm{d}^4 k k^2 f(k^2).$$

Perez-Victoria (**JHEP 0104 (2001) 032**)