SE (Computer Engg) Sem IV (RC 2019-20).
Examination July/August 2021.
Discrete Mathematics [CE 410]
Suggested Solution key.

Part A

1 (a) (i) Reflexive: $\forall a \in A$, $a-a=0=0\cdot 3$, multiple of 3. — (1)

Symmetric: $\forall a, b \in A$, $aRb \Rightarrow a-b$ is multiple of 3 — (1)
i.e. $a-b=3\cdot z$ $z \in \mathbb{Z}$
$\Rightarrow (b-a) = 3(-z) = 3z'$, $z' = -z$. $\therefore bRa$.

Transitive: Let aRb & bRc then $a-b=3z_1$; $b-c=3z_2$ where $z_1, z_2 \in \mathbb{Z}$. $\therefore a-c = 3(z_1+z_2) = 3z$, multiple of 3
$\Rightarrow aRc$. — (1)

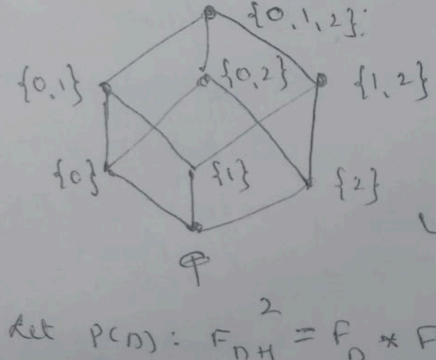
Thus R is an equivalence relation on A.

(ii) The Equivalence classes are as follows:

$[1] = \{x \in A \mid x-1=3z\} = \{x \in A \mid x = 3z+1\} = \{1, 7\} = [7]$
$[2] = \{x \in A \mid x = 3z+2\} = \{2, 5\} = [5]$
$[3] = \{x \in A \mid x = 3z+3\} = \{3\}$
$[4] = \{x \in A \mid x = 3z+4\} = \{4\}$
$[6] = \{x \in A \mid x = 3z+6\} = \{6\}$ (4)

$\therefore$ Partition of A induced by R is $A = [1] \cup [2] \cup [3] \cup [4] \cup [6]$. (1)

b) $P(R) = \{\phi, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$



Minimal Element: ϕ — (1)
Maximal Element: $\{0,1,2\}$ — (2)

— (4)

c) Let $P(n)$: $F_{n+1}^2 = F_n * F_{n+2} - (-1)^n$. — (1)

Basis step: For $n=0$, $LHS = F_1^2 = 1$; $RHS = F_0F_2 - (-1)^0 = 2-1 = 1$
$\Rightarrow LHS = RHS$
$\therefore P(0)$ is true.

— (1)

Induction step: If P(k) is true for some $k \geq 0$,

then

$$F_{k+1} * F_{k+3} - (-1)^{k+1} = F_{k+1} * (F_{k+1} + F_{k+2}) - (-1)^{k+1}$$

$$= F_{k+1}^2 + F_{k+1} * F_{k+2} - (-1)^{k+1}$$

$$= \underline{F_k * F_{k+2} - (-1)^k} + F_{k+1} * F_{k+2} - (-1)^{k+1} \quad \text{using } P(k)$$

$$= (F_k + F_{k+1}) * F_{k+2} - (-1)^k \cdot 0$$

$$= F_{k+2} * F_{k+2}$$

$$= F_{k+2}^2 \quad \text{which is } P(k+1) \qquad \text{—— (4)}$$

$\therefore$ By M.I, P(n) is true $\forall n \geq 0$.

a)

p	0	0	0	0	1	1	1	1
q	0	0	1	1	0	0	1	1
r	0	1	0	1	0	1	0	1
$p \wedge (\sim q \vee r)$	0	0	0	0	1	1	0	1
$p \vee (q \wedge \sim r)$	0	0	0	0	1	1	0	1

—— (5)

It can be easily seen that truth value of both the given compound statements are same. Therefore, they are logically equivalent —— (1)

b) $7 \equiv 7 \pmod{12}$; $75 \equiv 3 \pmod{12}$; $29 \equiv 5 \pmod{12}$;
$37 \equiv 1 \pmod{12}$; $37^2 \equiv 1 \pmod{12}$; $53 \equiv 5 \pmod{12}$;
$\therefore 53^3 \equiv 5^3 \pmod{12} \equiv 5 \pmod{12}$; $539 \equiv 11 \pmod{12}$;
$1269 \equiv 9 \pmod{12}$ $\therefore 1269^3 \equiv 9^3 \pmod{12} \equiv 9 \pmod{12}$ —— (4)

$\therefore 7 \times 75 \times 29 \equiv 105 \pmod{12} \equiv 9 \pmod{12}$

$37^2 \times 53^3 \times 539 \equiv 7 \pmod{12}$;

$\therefore$ Given integer $\equiv 3 \pmod{12}$ —— (2)

c). Let $A = \{1, 2, 3, 4\}$ $B = \{a, b, c, d, e\}$. Define a f: $f: A \to B$ by $f(1) = a$, $f(2) = b$, $f(3) = c$, $f(4) = d$ Then f is injective but not surjective. $\Big\}$ — 2

——— (2) reasons

d) Let $A = \{1, 2, 3\}$, $B = \{2\}$, $C = \{2, 3\}$ $\Big\}$ (4)

Then $A \cap B = \{2\}$ & $A \cap C = \{2, 3\} \Rightarrow A \cap B \subseteq A \cap C$ but $B \neq C$

3 a) $55 = 3(17) + 4$; $17 = 4(4) + 1$; $4 = 4(1) + 0$ $\Big\}$ — (3)

$\therefore gcd(55, 17) = 1$.

$\therefore 1 = 13(17) - 4(55)$.

$\therefore$ Manish must fill his 17 ounce container 13 times & empty the content (for the first 12 times) in to the larger container. Before he fills 17 ounce container for the 13th time, Manish has $12(17) - 3(55) = 39$ ounce in the 55 ounce container. He empties the larger container whenever it is full. After he fills the smaller container for the 13th time he will empty $55 - 39 = 16$ ounces from this container filling the larger container. Thus exactly one ounce will be left in the smaller container. (4)

b) $[(p \vee q) \wedge \neg\{\neg p \wedge (\neg q \vee \neg r)\}] \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$

$\equiv [(p \vee q) \wedge \{p \vee (q \wedge r)\}] \vee [\neg p \wedge (\neg q \vee \neg r)]$.

$\equiv [(p \vee q) \wedge (p \vee q) \wedge (p \vee r)] \vee [\neg(p \vee q) \vee \neg(p \vee r)]$ (6)

$\equiv [(p \vee q) \wedge (p \vee r)] \vee \neg[(p \vee q) \wedge (p \vee r)]$

$\equiv [p \vee (q \wedge r)] \vee [\neg(p \vee (q \wedge r))]$

$= T_0$

c $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

(i) Reflexive: $\forall a \in D_{36}$, $a | a$ ——— (1)

(ii) Antisymmetric: If aRb & bRa, $a, b \in D_{36}$ ——— (2)

$b = \cancel{a\ldots}\, az'$; $a = bz'$ $\quad z, z' \in \mathbb{Z}$.

$= bzz' \Rightarrow zz' = 1 \Rightarrow z = z' = 1$

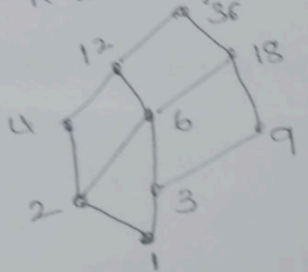
$\therefore b = a$

~~Transitive~~ : Transitive: If aRb & bRc, $a, b, c \in D_{36}$,

$b = az_1$; $c = bz_2 = az_1z_2$, $z_1 z_2 \in \mathbb{Z}$. —— (2)

$\Rightarrow a|c \Rightarrow aRc$

Hence R is a partial order & hence (D_{36}, R) is a poset



—— (2)

Part B.

4 a). Let U: Set of all positive integers from 1 to 2000.

A: " " " " " " divisible by 2

B: " " " " " " divisible by 3.

C: " " " " " " divisible by 5.

D: " " " " " divisible by 7.

$|U| = 2000$, $|A| = \lfloor \frac{2000}{2} \rfloor = 1000$, $|B| = 666$, $|C| = 400$,
$|D| = 285$, $|AB| = 333$, $|AC| = 200$, $|AD| = 142$, $|BC| = 133$
$|BD| = 95$, $|CD| = 57$, $|ABC| = 66$, $|ABD| = 47$, $|BCD| = 19$ (2)
$|ABCD| = 9$.

(i) $|\bar{A}\bar{B}\bar{C}| = |U| - |A \cup B \cup C| = 534$ (2)

(ii) $|D - A - B - C| = |D| - |DA| - |DB| - |DC| + |DAB| + |DAC| + |DBC| - |DABC|$

$= 76.$ —— (2)

b) Let k be one of 6 people.

Let X : Set of those which are friends of k

" " " " enemies of k

By Pigeonhole Principle, either x or y has at least $\lfloor \frac{5-1}{2} \rfloor + 1 = 3$ people. Suppose x has 3 people. If two of them are friends then the two with k are 3 mutual friends.
If not, then x has 3 mutual enemies. Alternately suppose y has 3 people. If two of them are enemies, then the two with k are 3 mutual enemies. If not, then y has 3 mutual friends. (4)

b) Let a_n: no of bit strings of length n have no two consecutive 1's. —— (2)

$a_1 = 2, \quad a_2 = 3.$

Let $n \geq 3$.

Case(i): Suppose bit string is ending with 1. $\boxed{|0|1}$
Then we should have 0 in the $(n-1)^{th}$ place. —— (2)
∴ required strings are of length n-2 satisfying given conditions & there are a_{n-2} such bit strings

Case(ii): Suppose bit string is ending with 0 $\boxed{|0}$
Then we have strings of length n-1 satisfying given conditions & there are a_{n-1} such bit strings —— (2)

$\therefore a_n = a_{n-1} + a_{n-2}, \quad n \geq 3, \quad a_1 = 2. \; a_2 = 3.$

d) (i) no of internal vertices $= \frac{817-1}{5-1} = 204$ —— (2)

(ii) " " leaves $= \frac{(6-1)733+1}{6} = 611$ —— (2)

5 a). The incidence matrices of G_1 and G_2 are

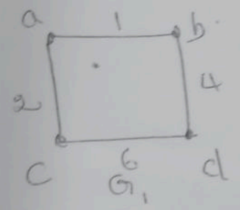
$$A(G_1) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \; \& \; A(G_2) = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$ —(2+2)

(cont)

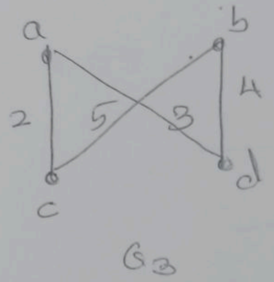
First 3 columns of the above matrices are same. After interchanging u_5 & u_6 rows also we get diffrent last column of both matrices. ——— (2)

Since $A(G_2) \neq A(G_1)$, $G_1 \not\cong G_2$

(b).

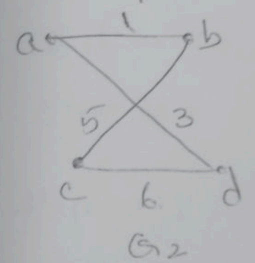


wt = 13 units



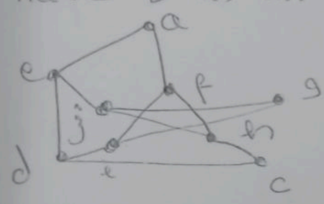
wt = 14 units

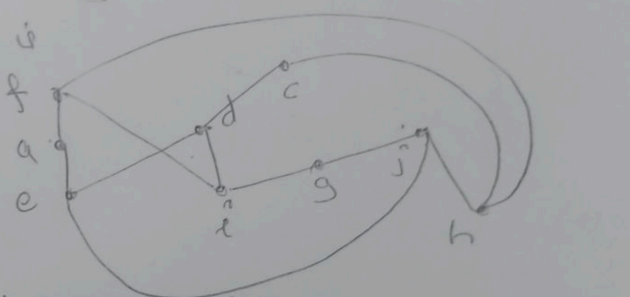
(3+1).



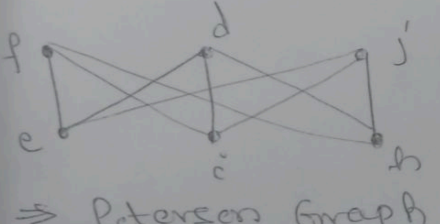
wt = 15 units

(c). Subgraph obtained by deleting b & the 3 edges that have b as an end point is





$d(c) = 2$, $d(a) = 2$. Merging the edges,

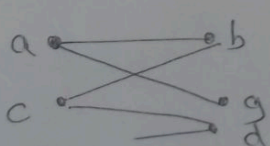


which is $K_{3,3}$ ——— (5)

⇒ Petersen Graph contains $K_{3,3}$ as subgraph. ∴ using Kuratowski's thm, given graph is non planar ——— (1)

(d) (i) example ——— (2)

(ii) C_7 :



but $(f, g) \in E$

∴ C_7 is not bipartite ——— (2).

6(a) For Algorithm steps ——— (4)

shortest paths : $1 \to 2$ with wt 18
: $1 \to 2 \to 3$ " " 27
: $1 \to 4$ " " 15
: $1 \to 4 \to 5$ " " 22
: $1 \to 2 \to 3 \to 6$ " " 55
(3)

(b) Definition ——— (3)

examples ——— (4).

(c). step 0 :

step 1 :

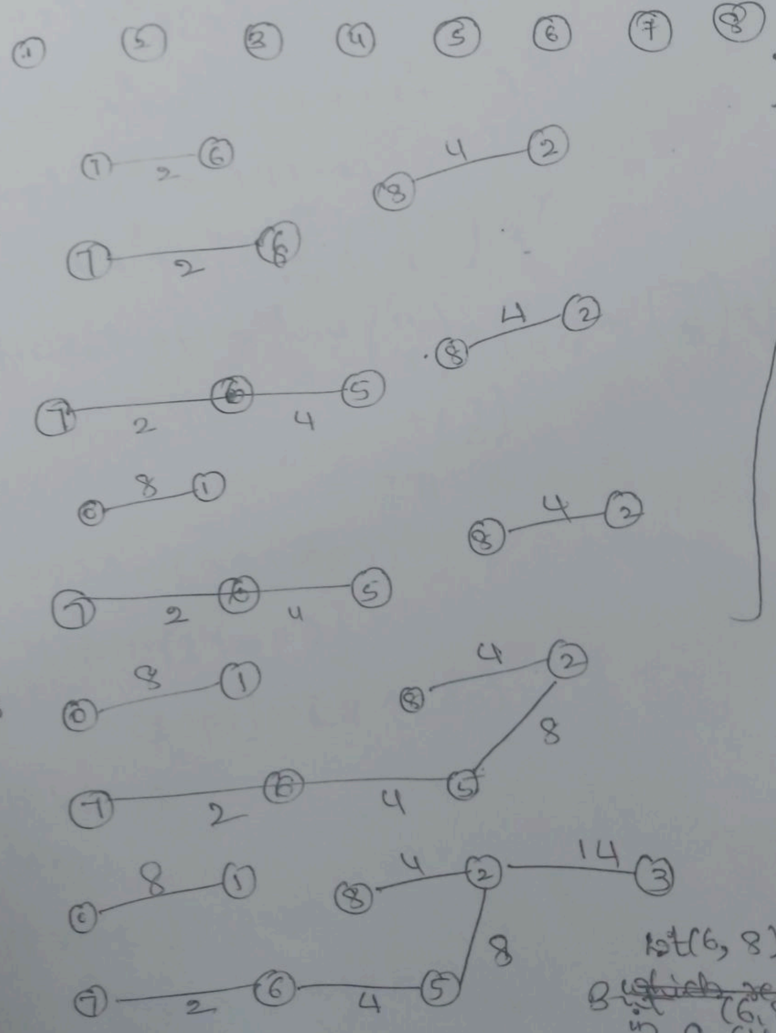
step 2 :

step 3 :

step 4 :

step 5 :

step 6 :

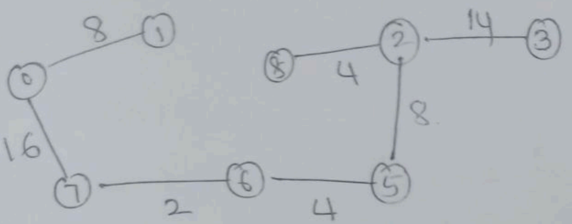


(3)

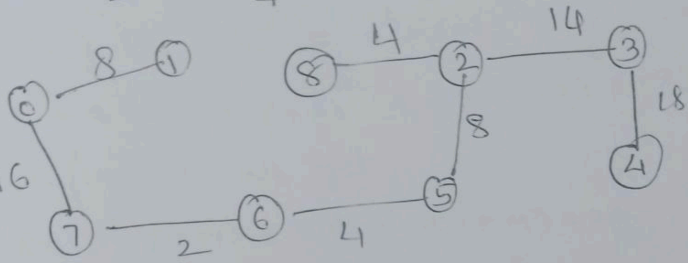
wt(6, 8) = 12.
8 ~~which result~~ ∴ (6, 8) results in a cycle

PTO

step 7 :



step 8 :



(3).

Total wt = 74 units

Part C

7(a) $(x+x^2+x^3+x^4+x^5)(x^2+x^3+x^4+\cdots)^5$

$= x^{11}(1+x+x^2+x^3+x^4)(1+x+x^2+\cdots)^5$

$= x^{11}(1+x+x^2+x^3+x^4)(1-x)^{-5}$

$= x^{11}(1+x+x^2+x^3+x^4)\sum_{r=0}^{\infty}\binom{4+r}{r}x^r$

$\therefore$ The coefficient of x^{18} $= \binom{11}{7}+\binom{10}{6}+\binom{9}{5}+\binom{8}{4}$

$+\binom{7}{3}$

$= 771.$

(b) $p \to ((p\to q)\wedge \sim(\sim q \vee \sim p))$

$\equiv \neg p \vee [(\neg p \vee q)\wedge(q \wedge p)]$ (7)

$\equiv \neg p \vee [\neg p \wedge (q\wedge p)] \vee (q \wedge q \wedge p)$

$\equiv \neg p \vee F \vee (p\wedge q)$

$\equiv \neg p \vee (p \wedge q)$

$\equiv [\neg p \wedge (q \vee \sim q)] \vee (p\wedge q)$

$\equiv (\neg p \wedge q) \vee (\neg p \wedge \sim q) \vee (p \wedge q)$ which is PDNF.

(c). ch eq^n is $x^2+4x+4=0 \Rightarrow x=-2,-2$ (real repeated roots) ——(1)

$\therefore a_n^{(h)} = (A+Bn)(-2)^n$. ——(1)

RHS is of the type αb^n & $b=-2$ is a root of ch eq^n of multiplicity 2, $a_n^{(p)} = n^2A(-2)^n$ ——(1)

$\therefore n^2A(-2)^n + 4A(n-1)^2(-2)^{n-1} + 4A(n-2)^2(-2)^{n-2} = 5(-2)^n$

After simplification, we get

$4An^2 - 8A(n^2-2n+1) + 4A(n^2-4n+4) = 20.$

$\Rightarrow 4A-8A+4A = 0$ (coeff of n^2)

$16A-16A = 0$ (" " n)

$-8A+16A = 20$ (const) ——(2)

$\Rightarrow A = \frac{20}{8} = \frac{5}{2}$ ——(1)

$\therefore a_n^{(p)} = \frac{5}{2}n^2(-2)^n$

Hence G.S is $a_n = (A+Bn)(-2)^n + \frac{5}{2}n^2(-2)^n$ ——(1).

$= (-2)^n\left[A+Bn+\frac{5n^2}{2}\right]$.

8(a) Let p: It is sunny this afternoon

q: It is colder than yesterday

r: we will go swimming

s: we will take a canoe trip.

t: we will be home by sunset. } 1

Given argument is

$\neg p \wedge q$

$r \to p$

$\neg r \to s$

$s \to t$

$\therefore t$ } 1

step 1 : $\neg p \wedge q$ — given premises

step 2 : $\neg p$ — Simplification rule

step 3 : $r \to p$ — given premises

step 4 : $\neg r$ — steps 2, 3, Modus Tollens

step 5 : $\neg r \to s$ — given premises

step 6 : s — step 4, 5, Modus Ponens

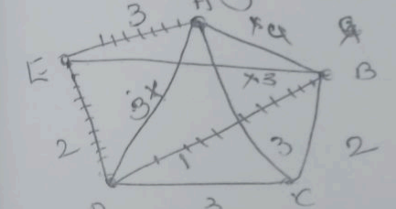
step 7 : $s \to t$ — given premises

step 8 : t — steps 6, 7 & Modus Ponens.

(4)

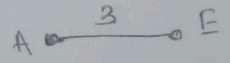
Thus the given argument is valid

(b)

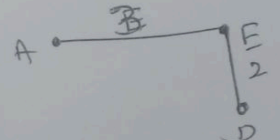


Soln: Let us start with vertex A.

(1)



(2)



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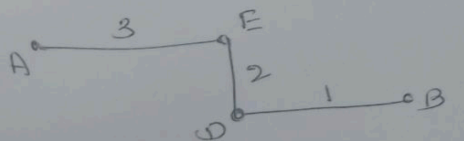

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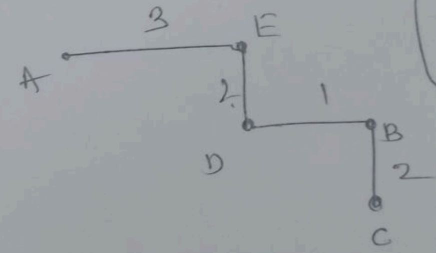

& weight of the minimal spanning tree = 08 units

(6)

(c). (i) $|V| = 2n$, & $|E| = \frac{1}{2} \sum \deg(v) = \frac{1}{2}[4 \times 2 + (2n-4)3]$

$= \frac{1}{2}(6n-4) = \underline{3n-2}$ ——— (2)

Let $P(n) : \lambda(\lambda-1)(\lambda^2-3\lambda+3)^{n-1}$ ——— (1)

(ii) Initialization $n=1$. Then G : K_2 (page 8)

$\therefore P(G,\lambda) = \lambda(\lambda-1)$ $\therefore P(1)$ is true ——— (1)

For $n=2$, $G = C_4$ $\therefore P(G,\lambda) = \lambda(\lambda-1)^3 - \lambda(\lambda-1)(\lambda-2)$.

$\Rightarrow P(2)$ is true

<u>Induction step</u> If $P(k)$ is true for some $k \geq 1$.

then to prove $P(k+1)$ is true. —(1)

Let $G = G_1 \cup G_2$ where G_1 is C_4 & G_2 is the ladder graph with k rungs.

Then $G_1 \cap G_2 = K_2$

$$\therefore P(G,\lambda) = \frac{P(G_1,\lambda)\,P(G_2,\lambda)}{P(K_2,\lambda)} = \frac{[\lambda(\lambda-1)\cdot \lambda^2 - 3\lambda+3]\,[\lambda(\lambda-1)(\lambda^2-3\lambda+3)^{k-1}]}{\lambda(\lambda-1)}$$

$$= \lambda(\lambda-1)(\lambda^2-3\lambda+3)^{k} \quad \text{—(3)}$$

$\Rightarrow P(k+1)$ is true.

Hence by M.I, $P(n)$ is true $\forall n \geq 1$.