03.01.2019

SETS

A set is a collection of well-defined objects enclosed in curly brackets.
Usually we denote sets by capital letters A, B, C, ...
Objects of the set are known as elements and we denote them as a, b, c, x, y, z, ...

Example: $A = \{1, 2, 3\}$

If an element a is in the set A then we denote it as $a \in A$

Example: $1 \in A$, $2 \in A$, $3 \in A$, $4 \notin A$

Empty set

A set that does not have any element is known as empty set and it is denoted by ϕ

Representation of a set

A set can be represented in two forms - Roster form and set-builder ~~Roster form~~ form.

In Roster form each element of the set is listed whereas in set-builder form we describe elements by their common properties.

Roaster form

$$A = \{1, 2, 3, 4, 5\}$$

set-builder form

$$A = \{x : x \in \mathbb{N}, x \leq 5\}$$

$A = \{x : x^2 = 4\}$ $\qquad$ $A = \{2, -2\}$

Subset

Let A be any set which is non-empty. ~~is said to be~~ A set B is said to be a subset of A if each element of B is also an element of A.

It is denoted as $B \subset A$

Example: $A = \{1, 2, 3, 4, 5\}$

$B = \{1, 2\}$

$B \subset A$

That means if $B \subset A$

it implies if $x \in B$ then $x \in A$

Equality of two sets

Two sets A and B are said to be equal if $A \subset B$ and $B \subset A$ and we write $A = B$

Note:

Empty set ϕ is a subset of all sets and $A \subset A$

Power set

Let X be a non empty set $X \neq \phi$ then power set of X is collection of all subsets of X and it is denoted by $P(X)$.

That means $P(X) = \{A : A \subset X\}$

Example $X = \{1, 2\}$

$P(X) = \{\phi, X, \{1\}, \{2\}\}$

$X = \{a, b, c\}$

$P(X) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$

Number of elements in a set A is denoted by $|A|$ or $\#A$ and it is called cardinality of set A.

Remark:

If a set X has n elements then cardinality or number of elements in power set, $|P(X)| = 2^n$

operations on sets

1. union

Let A and B are two non-empty sets

then union of A and B is denoted by $A \cup B$ and it is defined as

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Example: $A = \{1, 2\}$ $B = \{1, 2, 3\}$ $A \cup B = \{1, 2, 3\}$

$A = \{1, 2, 4\}$ $B = \{1, 2, 3\}$ $A \cup B = \{1, 2, 3, 4\}$

2. Intersection

Let A and B are two non-empty sets.

The intersection of A and B is denoted by $A \cap B$ and it is defined as

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Example: $A = \{1, 2, 4\}$

$B = \{1, 2, 3\}$

$A \cap B = \{1, 2\}$

3. Difference of sets

Let A and B are two non-empty sets.

Then A difference B is denoted by $A - B$ and it is defined as

$$A - B = \{x : x \in A \text{ and } x \notin B\}$$

Example: $A = \{1, 2, 4\}$

$B = \{1, 2, 3\}$

$A - B = \{4\}$

$B - A = \{3\}$

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universal set

In particular context we have to deal with subsets of basic set which is relevant to that context. This basic set is called universal set and is usually denoted by U.

Example: while studying number systems we can consider set of natural numbers ~~as~~

as universal set and set of prime numbers, even numbers, odd numbers can be taken as subsets.

complement of a set

let U be universal set and let $A \subset U$

then complement of A is denoted by A' or A^c and it is defined as $A^c = \{x \in U : x \notin A\}$

symmetric Difference

Let A and B are two non-empty sets

symmetric Difference of A and B is defined as

$A \oplus B = (A - B) \cup (B - A)$

Show that $A - B = A \cap B^c$

Proof :

Let $x \in A - B$

This implies $x \in A$ and $x \notin B$

$x \in A$ and $x \in B^c$

$x \in A \cap B^c$

$A - B \subset A \cap B^c$ ——— ①

Let $x \in A \cap B^c$

This implies $x \in A$ and $x \in B^c$

$x \in A$ and $x \notin B$

$x \in A - B$

$A \cap B^c \subset A - B$ ——— ②

Therefore ① and ② imply

$\boxed{A - B = A \cap B^c}$

VENN DIAGRAMS

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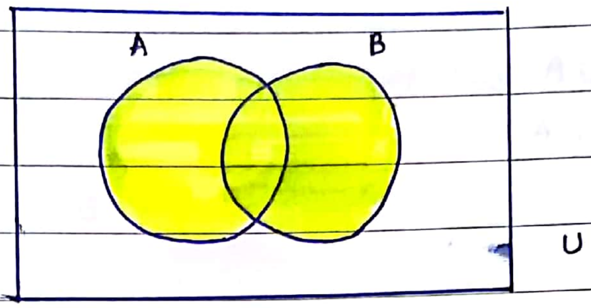
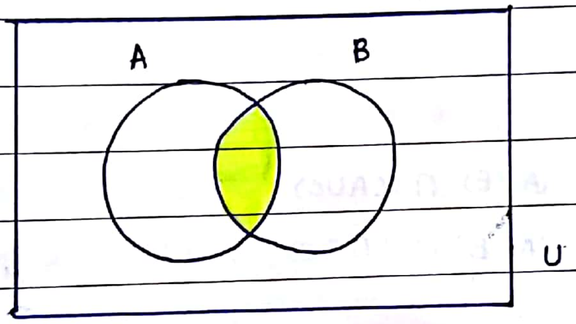
venn diagrams are diagrams used to represent relationships between sets graphically.
universal set is represented by a rectangle and its subsets are represented by circles.

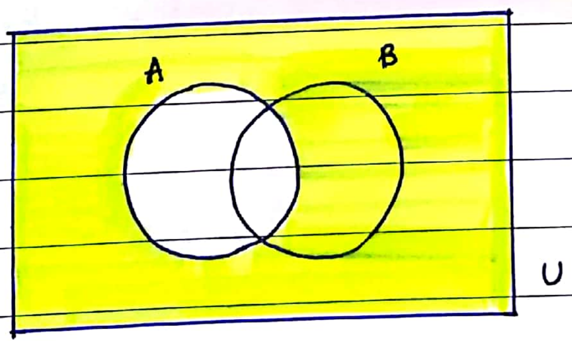
1. $A \cup B$



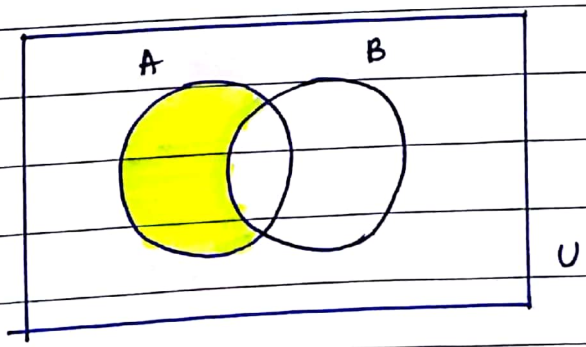
2. $A \cap B$



3. A^c



4. $A - B$



ALGEBRAIC LAWS OF SET OPERATIONS

1. ~~Identity laws~~ Idempotent laws

i) $A \cup A = A$

ii) $A \cap A = A$

2. commutative laws

i) $A \cup B = B \cup A$

ii) $A \cap B = B \cap A$

3. Associative laws

i) $A \cup (B \cup C) = (A \cup B) \cup C$

ii $A \cap (B \cap C) = (A \cap B) \cap C$

4. Distributive laws

i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

5. Involution laws

$(A^c)^c = A$

6. Identity laws

(a) i) $A \cup \phi = A$

ii) $A \cup U = U$

(b) i) $A \cap \phi = \phi$

ii $A \cap U = A$

7. complement laws

i) $A \cup A^c = U$

ii) $A \cap A^c = \phi$

iii) $\phi^c = U$

iv) $U^c = \phi$

9. Absorption law

i) $A \cup (A \cap B) = A$

ii) $A \cap (A \cup B) = A$

8. De-Morgan's laws

i) $(A \cup B)^c = A^c \cap B^c$

ii) $(A \cap B)^c = A^c \cup B^c$

Let A, B and C are three non-empty set then show that

i) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

ii) If $A \cap B = A \cap C$, is $B = C$? Justify.

i) L.H.S. $= (A - B) \cup (B - A)$

$= (A \cap B^c) \cup (B \cap A^c)$

$= [(A \cap B^c) \cup B] \cap [(A \cap B^c) \cup A^c]$

$= [B \cup (A \cap B^c)] \cap [A^c \cup (A \cap B^c)]$

$= [(B \cup A) \cap (B \cup B^c)] \cap [(A^c \cup A) \cap (A^c \cup B^c)]$

$= [(A \cup B) \cap U] \cap [U \cap (A^c \cup B^c)]$

$= (A \cup B) \cap (A^c \cup B^c)$

$= (A \cup B) \cap \cancel{(A \cup B)^c} (A \cap B)^c$

$= (A \cup B) - (A \cap B)$

$=$ R.H.S.

ii) ~~No A = B~~ $A \cap B = A \cap C$ does not imply $B = C$

$A = \{1, 2\}$

$B = \{\cancel{2}\ 1, 3\}$

$C = \{1\}$

Here, $A \cap B = \{1\} = A \cap C$

But $B \neq C$

iii) If $A \cup B = A \cup C$, is $B = C$? Justify.

$A \cup B = A \cup C$ does not imply $B = C$

$A = \{1, 2\}$

$B = \{3\}$

$C = \{1, 3\}$

Here, $A \cup B = \{1, 2, 3\} = A \cup C$

But $B \neq C$

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Prove that

1. $(A - C) \cap (B - C) = (A \cap B) - C$
2. $A - (B - C) = (A - B) \cup (A \cap C)$
3. $A - (B \cup C) = (A - B) \cap (A - C)$

~~4. Show that B~~

4. Show that $B \subseteq A \Rightarrow B - C \subseteq A - C$

5. Let $P(X)$ be a power set of a set $X \neq \phi$
   i) show that for any two sets A and B
   $P(A \cap B) = P(A) \cap P(B)$
   ii) Is $P(A \cup B) = P(A) \cup P(B)$?
   Justify your answer.

1. L.H.S. $= (A - C) \cap (B - C)$

$= (A \cap C^c) \cap (B \cap C^c)$

$= A \cap C^c \cap B \cap C^c$

$= A \cap B \cap C^c$

$= (A \cap B) \cap C^c$

$= (A \cap B) - C$

$=$ R.H.S.

2. L.H.S. $= A - (B - C)$

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$= A - (B \cap C^c)$

$= A \cap (B \cap C^c)^c$

$= A \cap [B^c \cup (C^c)^c]$ DeMorgan's law

$= A \cap (B^c \cup C)$

$= (A \cap B^c) \cup (A \cap C)$ Distributive property

$= (A - B) \cup (A \cap C) = R.H.S.$

3. L.H.S. $= A - (B \cup C)$

$= A \cap (B \cup C)^c$

$= A \cap (B^c \cap C^c)$

$= \cancel{(A \cap B^c)}\ A \cap B^c \cap C^c$

$= A \cap B^c \cap A \cap C^c$

$= (A - B) \cap (A - C)$

$=$ R.H.S.

5. (i) $P(A \cap B) = P(A) \cap P(B)$

let $x \in P(A \cap B)$

$\Rightarrow \cancel{x \in A \cap B}$

$\Rightarrow x \subset A \cap B$

~~$\Rightarrow x \in A$ and $x \in B$~~

$\Rightarrow x \subset A$ and $x \subset B$

$\Rightarrow x \in P(A)$ and $x \in P(B)$

$\Rightarrow x \in P(A) \cap P(B)$

therefore,

$P(A \cap B) \subset P(A) \cap P(B)$ ——— (i)

let $x \in P(A) \cap P(B)$

$x \in P(A)$ and $x \in P(B)$

$\cancel{x \in}$ $x \subset A$ and $x \subset B$

$x \subset (A \cap B)$

$x \in P(A \cap B)$

therefore ~~P(A∩B)~~ $P(A) \cap P(B) \subset P(A \cap B)$ ——— ②

from ① and ②

$P(A \cap B) = P(A) \cap P(B)$

ii) $P(A) \cup P(B) \neq P(A \cup B)$

Justification

Let $A = \{1\}$, $B = \{2\}$

$P(A) = \{A, \phi\}$, $P(B) = \{B, \phi\}$

$A \cup B = \{1, 2\}$

$P(A \cup B) = \{\phi, A \cup B, \{1\}, \{2\}\} = \{\phi, \{1,2\}, \{1\}, \{2\}\}$

$P(A) \cup P(B) = \{A, B, \phi\} = \{\{1\}, \{2\}, \phi\}$

$P(A) \cup P(B) \subset P(A \cup B)$

But $P(A \cup B) \not\subset P(A) \cup P(B)$

$\{1, 2\} \in P(A \cup B)$ but does not belong to $P(A) \cup P(B)$

4. Show that $B \subseteq A \Rightarrow B - C \subseteq A - C$

Let $x \in (B - C)$

$\Rightarrow x \in B$ and $x \notin C$

Since $B \subseteq A$, $x \in A$

$\Rightarrow x \in A$ and $x \notin C$

$\Rightarrow x \in (A - C)$

$\Rightarrow B - C \subseteq A - C$

cartesian product of sets

Let A and B are two non-empty sets then cartesian product of A and B is denoted by $A \times B$ and is defined as

$$A \times B = \{ (a,b) : a \in A,\ b \in B \}$$

Example : $A = \{1, 2\}$ $B = \{1\}$

$A \times B = \{ (1,1), (2,1) \}$

$A = \{1, 2\}$ $B = \{a, b\}$

$A \times B = \{ (1,a), (1,b), (2,a), (2,b) \}$

Note:

If A set has m elements and set B has n elements then $A \times B$ has $m \times n$ elements.

Let A, B and C are any three non-empty sets then ~~$A \times B \times C$~~

$$A \times B \times C = \{ (a, b, c) : a \in A,\ b \in B,\ c \in C \}$$

In general, $A_1 \times A_2 \times \dots \times A_n = \{ (x_1, x_2, \dots, x_n) : x_i \in A_i \quad i = 1 \text{ to } n \}$

For any set $A \neq 0$

$$A^n = \underbrace{A \times A \times \dots \times A}_{n \text{ times}} = \{ (x_1, x_2, \dots, x_n) : x_i \in A \}$$

1. Let A, B and C are three non-empty sets then prove that

i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

2. Let A, B, P and Q are non-empty sets

then $(A \times B) \cap (P \times Q) = (A \cap P) \times (B \cap Q)$

Proof

b) let $a \in A \times (B \cap C)$

$a = (x, y)$ such that $x \in A$ and $y \in B \cap C$

$x \in A$ and $y \in B$ and $y \in C$

$x \in A$ and $y \in B$ and $x \in A$ and $y \in C$

$(x, y) \in A \times B$ and ~~$x \in$~~ $(x, y) \in A \times C$

$(x, y) \in (A \times B) \cap (A \times C)$

Therefore $A \times (B \cap C) \subset (A \times B) \cap (A \times C)$

Similarly $(A \times B) \cap (A \times C) \subset A \times (B \cap C)$

Therefore $A \times (B \cap C) = (A \times B) \cap (A \times C)$

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Relation

Relation from A to B :

Let A and B are two non-empty sets then a relation from set A to B is any subset of $A \times B$ and it is usually denoted by R.

If $(a, b) \in R$, then we write

aRb read as "a relates to b"

Example : Let $A = \{2, 3\}$, $B = \{1, 5\}$

~~to~~ let $R = \{(2,1), (2,5)\} \subset A \times B$ then

$2R1$ and $2R5$

Relation on A

Definition :

Let A be a non-empty set then a relation R on A is a subset of $A \times A$.

Example : $A = \{1, 2, 3\}$

$R = \{(1,1), (1,2)\}$

$R = \{(1,2), (2,1)\}$

Equivalence Relation

1. Reflexive relation :

A relation R on a non-empty set say A is said to be reflexive if ~~aRa for all a ∈ A~~ $a R a \ \forall \ a \in A$

Example : $A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (3,3)\}, (2,1), (1,2)\}$

2. Symmetric relation :

A relation R on a non-empty set say A is said to be symmetric if

$a R b \Rightarrow b R a$

Example : $A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (3,3), (2,1), (1,2)\}$

R is symmetric relation on $A = \{1, 2, 3\}$

3. Transitive relation :

A relation R on a non-empty set say A is said to be transitive if

$a R b$ and $b R c \Rightarrow a R c$

Example : $A = \{1, 2, 3\}$

i) $R = \{(1,1), (2,2), (1,2), (2,1)\}$

ii) $R = \{(1,2), (2,3), (1,3)\}$

* A relation R on a set A is said to be equivalence if it is reflexive, symmetric and transitive.

Let $A = \{1, 2, 3\}$

i) $R = \{(1,1), (2,2), (3,3)\} \cancel{(1,2)}, \cancel{(2,1)}, \cancel{(1,3)}, (1,2), (2,1), (1,3),$

$(3,1), (2,3), \cancel{(3,3)} (3,2)\}$

ii) $R = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$

Equivalence classes

Let R be equivalence relation on a set A and let $a \in A$.
The equivalence classes of A are given as

$[a] = \{ b \in A : aRb \}$

Example: $A = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$

which is equivalence relation

$[1] = \{1, 2\}$

$[3] = \{3\}$

$[2] = \{1, 2\}$

Remark:

Any two equivalence classes are either same or distinct.

The union of all distinct equivalence classes is equal to set A.

Let Z denote set of all integers. Define a relation R on Z as follows

aRb such that $a - b = 3m \quad m \in \mathbb{Z}$

(a) Prove that R is an equivalence relation on Z

(b) Find all distinct equivalence classes.

(a) i) Reflexive

Since for each $a \in \mathbb{Z}$

$a - a = 0 = 0.3 \Rightarrow aRa \ \forall \ a \in \mathbb{Z}$

Therefore R is reflexive.

ii) Symmetric

Let $a, b \in \mathbb{Z}$ such that aRb

$\Rightarrow a - b = 3m, \quad m \in \mathbb{Z}$

$\Rightarrow b - a = -3m$

$\Rightarrow b - a = 3(-m)$

$\Rightarrow$ ~~(b, a)~~ bRa

iii) Transitivity

Let $a, b, c \in \mathbb{Z}$ such that aRb and bRc

$\Rightarrow a - b = 3m$ ~~a~~, $m \in \mathbb{Z}$ ——— ①

~~and~~ $b - c = \cancel{3m}\ 3n$, $n \in \mathbb{Z}$ ——— ②

Adding ① and ②, we get

$a - c = 3(m+n)$

$\Rightarrow aRc$

Hence, the relation R is an equivalence relation on $\mathbb{Z}$

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(b) $[a] = \{ b \in A : aRb \}$

$[0] = \{ b \in \mathbb{Z} : 0Rb \}$

$= \{ b \in \mathbb{Z} : 0 - b = 3m, m \in \mathbb{Z} \}$

$= \{ b \in \mathbb{Z} : b = 3m, m \in \mathbb{Z} \}$

$= \{ 3m, m \in \mathbb{Z} \}$

$= \{ \ldots, -6, -3, 0, 3, 6, \ldots \}$

$[1] = \{ b \in \mathbb{Z} ; bR1 \}$

$= \{ b \in \mathbb{Z} : b - 1 = 3m, m \in \mathbb{Z} \}$

$= \{ b \in \mathbb{Z} ; b = 3m+1, m \in \mathbb{Z} \}$

$= \{ 3m+1 : m \in \mathbb{Z} \}$

$= \{ \ldots, -5, -2, 1, 4, 7, \ldots \}$

$[2] = \{ b \in \mathbb{Z}, bR2 \}$

$= \{ b \in \mathbb{Z} : b - 2 = 3m, m \in \mathbb{Z} \}$

$= \{ b \in \mathbb{Z} : b = 3m + 2, m \in \mathbb{Z} \}$

$= \{ 3m+2 : m \in \mathbb{Z} \}$

$= \{ \ldots, -4, -1, 2, 5, \ldots \}$

$[0] \cup [1] \cup [2] = \mathbb{Z}$

Let A be the set of all non-zero integers.

Define a relation on $A \times A$ as

$(a, b) R (c, d)$ iff $ad = bc$

i) show that R is an equivalence relation on A×A.

ii) Find equivalence class of element (1, 2)

(i) (a) Reflexitity

since $ab = ba$

$\Rightarrow (a, b) \, R \, (a, b) \quad \forall \quad (a, b) \in A \times A$

Therefore R is reflexive

(b) Symmetry

Let (a, b) and (c, d) be two elements such that

$(a, b) \, R \, (c, d)$

$\Rightarrow ad = bc$

$\Rightarrow cb = da$

$\Rightarrow (c, d) \, R \, (a, b)$

Therefore R is symmetric

(c) Transitivity

Let (a, b), (c, d), and (e, f) be elements such that

$(a, b) \, R \, (c, d)$ and $(c, d) \, R \, (e, f)$

~~$\Rightarrow ad = bc$ and $cf = de$~~

To show $(a, b) \, R \, (e, f)$

Now $(a, b) \, R \, (c, d) \Rightarrow ad = bc$ —— ①

and $(c, d) \, R \, (e, f) \Rightarrow cf = de$ —— ②

From ① $d = \dfrac{bc}{a}$

Put $d = \dfrac{bc}{a}$ in ②

$cf = \left[\dfrac{bc}{a}\right] e$

$\Rightarrow af = be$

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$\Rightarrow$ $(a, b) R (e, f)$

Therefore, R is transitive.

ii) $[(1,2)] = \{ (a,b) \in A \times A : (1,2) R (a,b) \}$

$= \{ (a,b) : b = 2a \}$

$= \{ (a, 2a) : a \in A \}$

Anti-symmetric Relation

A relation R on a set A is said to be anti-symmetric if

aRb and $bRa \Rightarrow a = b$

Let a and b are such that $a \neq 0$.

We say 'a divides b' and it is denoted as $a \mid b$ and defined as

$\boxed{b = ma}$, $m \in \mathbb{Z}$

$\downarrow$ divides

Partially ordered set or po-set

Let S be a non-empty set.

A relation on S is said to be partial order if it is reflexive, anti-symmetric and transitive and ~~is~~ set together ~~to~~ ~~be~~ with partial order is said to be partially ordered or po-set and usually we denote it as (S, R).

In general, a ~~relation~~ partially ordered set is denoted as ('$\preceq$')

Consider a set of integers $(\mathbb{Z}, \leq)$ is a po-set.

1. Reflexivity

since $a = a \Rightarrow aRa \ \forall a \in \mathbb{Z}$

2. Antisymmetric

Let $aRb \Rightarrow a \leq b$ and $\cancel{b \leq a}$ $bRa \Rightarrow b \leq a \Rightarrow a = b \Rightarrow$ '$\preceq$' is anti-symmetric

3. Transitivity

let a, b and c are such $a \le b$ and $b \le c$

$\Rightarrow a \le c \Rightarrow$ '$\le$' is transitive.

'$\le$' on Z is a partial order $(S, \le)$ is po-set or partially-ordered set.

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