Chapter 4

PRINCIPLE OF MATHEMATICAL INDUCTION

1.1 Introduction

We have studied one method of reasoning, deductive reasoning.

For example, consider the following statements :

(1) $1 + 2 + 3 + ... + 100 = 5050$

(2) $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

(3) Let $n = 100$ in (2). $1 + 2 + 3 + ... + 100 = \frac{(100)(101)}{2} = (50)(101) = 5050$

Here we want to prove that sum of all integers from 1 to 100 is 5050. We have a general result $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$. We take $n = 100$ in it and get the required result. Here, we apply a general principle to deduce a particular result.

Consider (1) If 3 divides product ab, then 3 divides a or 3 divides b. (2) If p is a prime and p divides ab then p divides a or p divides b. (3) Let $p = 3$ in (2) as 3 is a prime. Hence, if 3 divides product ab, then 3 divides a or 3 divides b.

Here also we apply a general principle to deduce a particular result.

(1) Amitabh Bachchan is a good actor.

(2) Actors are awarded national *Padma* honour in their category, if selected.

(3) Amitabh Bachchan was selected and got *Padma* honour.

Here also a similar situation occurs.

But consider the following against this deductive reasoning,

$4 - 1 = 3$ is divisible by 3.

$4^2 - 1 = 15$ is divisible by 3.

$4^3 - 1 = 63$ is divisible by 3.

Here we observe a pattern and we make a conjecture that for every positive integer n, $4^n - 1$ is divisible by 3. So from a particular case, we conjecture a general result. This is not a proof. This inductive assumption has to be proved. All conjectures may not be true. For example, $n^2 - n + 41$ is a prime for $n = 1, 2, 3, ...39$. But for $n = 41$, $41^2 - 41 + 41 = 41^2$ is obviously not a prime. Hence we cannot deduce that $n^2 - n + 41$ is a prime by observing values for $n = 1, 2, 3, ...39$.

So, inductive argument starts from a particular case and by rigorous deduction the conjecture is proved.

The history of this dates back to **Plato**. In 370 B.C. Plato's *parmenides* (Discussions or Dialogues) contained an early example of implicit inductive proof. The early traces of mathematical induction can be found in Euclid's proof that number of primes is infinite. Bhaskara II's *cyclic* method (*Chakravala*) also introduces mathematical induction.

Sorites paradox used the method of descent. He said 10,00,000 grains of sand form a heap. Removing one grain from the heap does not change the situation. So continuing the argument even one grain or no grain also forms a heap !

Around 1000 A.D., **Al-Karaji** introduced mathematical induction for arithmetic sequences in **Al-Fakhri** and proved the binomial theorem and properties of **Pascal's** triangle.

The first explicit formulation of the principle of mathematical induction was given by **Pascal** in *Traité-du-triangle arithmetique* (1665). French mathematician **Fermat** and Swiss mathematician **Jacob Bernoulli** used the principle. The modern rigorous and systematic treatment came only in 19th century with **George Boole, Sanders Peirce, Peano** and **Dedekind.**

1.2 Induction Principle

We start with following principle :

Principle of Induction : If a statement $P(n)$ of natural variable n is true for $n = 1$ and if $P(k)$ is true $\Rightarrow P(k + 1)$ is true, $k \in N$, then $P(n)$ is true, $\forall n \in N$.

Let us be given a statement $P(n)$ involving a natural variable to be true for all natural numbers n. We prove it in two stages :

(1) The basis : We prove it for $n = 1$ (or 0 or the lowest value).

(2) Inductive step : Assuming that the statement holds for some natural number k, prove it for $n = k + 1$.

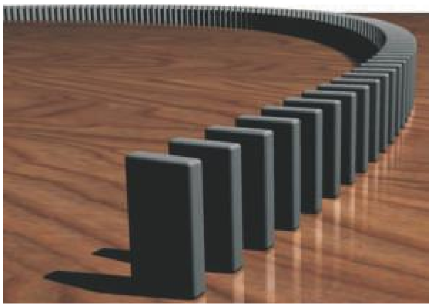
Then $P(n)$ is true for all $n \in N$.

Domino effect : We are presented with a 'long' row of dominos such that,

(1) The first domino will fall.

(2) Whenever a domino falls, its next neighbour will fall.

So it is concluded that all of the dominos will fall.



So the proof is like this. The first statement in an infinite sequence of statements is true and if it is true for some $k \in$ N, it is true for the next value of the variable, then the given sequence of statements is true for all $n \in$ N.

In logical symbols, $(\forall P)\ [P(1) \wedge (\forall k \in N)\ (P(k) \Rightarrow P(k+1))] \Rightarrow (\forall n \in N)[P(n)]$

This can be proved by using **well-ordering principle** which states that every non-empty subset of N has a least element.

Proof : Let S be the set of natural numbers for which P(n) is false. $1 \notin$ S as P(1) is true. If S is non-empty, it has a least element t which is not 1. Let $t = n + 1$. Since t is the least element for which P(t) is false, P(n) is true. Also $P(n) \Rightarrow P(n+1)$. Hence $P(n+1) = P(t)$ is true, a contradiction. Hence S = $\emptyset$.

$\therefore$ P(n) is true, $\forall n \in$ N.

Sometimes paradoxes are created by misuse of the principle.

There is a famous **Polya's** proof that there is no horse of different colour.

Basis : If there is only one horse, there is only one colour and hence P(1) is true.

Induction step : Assume that in any set of n horses, all have the same colour. Consider a set of $n + 1$ horses numbered 1, 2, 3,... $n + 1$. Consider the subsets $\{1, 2, 3,..., n\}$ and $\{2, 3, 4,...n + 1\}$. Each is a set of n horses and therefore they have the same colour and since they are overlapping sets, all $n + 1$ horses have same colour. This argument is true for 1 horse and $n \geq 3$ horses. But for 2 horses the set {1} and {2} are disjoint and the argument falls flat.

1.3 Examples

Now we will apply the principle of mathematical induction to some examples.

Example 1 : Prove $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$, $n \in$ N

Solution : Let P(n) : $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$, $n \in$ N

For $n = 1$, L.H.S. = 1 and R.H.S. = $\frac{1 \times 2}{2} = 1$. Hence, P(1) is true.

Let P(k) be true i.e. P(n) is true for $n = k$, $k \in$ N.

$$\therefore \quad 1 + 2 + 3 + ... + k = \frac{k(k+1)}{2} \qquad \text{(i)}$$

For $n = k + 1$ we have to prove,

$$1 + 2 + 3 + ... + (k + 1) = \frac{(k+1)(k+2)}{2}$$

Now, $1 + 2 + 3 + ... + (k + 1) = (1 + 2 + 3 + ... + k) + (k + 1)$

$$= \frac{k(k+1)}{2} + (k + 1) \qquad \text{by (i)}$$

$$= (k + 1)\left(\frac{k}{2}+1\right) = \frac{(k+1)(k+2)}{2}$$

Hence, P$(k + 1)$ is true.

$\therefore$ P(1) is true and P(k) is true, $\Rightarrow$ P$(k + 1)$ is true.

$\therefore$ P(n) is true, $\forall n \in$ N by principle of mathematical induction.

Note : This example has historical importance.

Obviously, $1 + 2 + 3 + ... + 100 = 5050$ according to this formula. When this formula was not known, Gauss, at very young age, calculated this by the following method and surprised his teacher Buttner and assistant teacher Bartels.

Let $S = 1 + 2 + 3 + ... + 100$ **(i)**

$\therefore$ $S = 100 + 99 + 98 + ... + 1$ **(ii)**

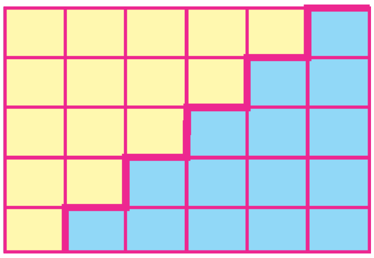
Adding (i) and (ii)

$\therefore$ $2S = (101) + (101) + ...$ 100 times **((i) + (ii))**

$\therefore$ $S = \frac{101 \times 100}{2} = 5050$. This was done in no time !

Let us review a geometric 'proof'.

Consider a rectangle of sides n and $n + 1$ divided into subrectangles of unit sides as shown. The portion under the dark ladder has area $1 + 2 + 3 + ... + n$.



By symmetry the rectangle has area

$$2(1 + 2 + 3 + ... + n) = n(n + 1)$$

$\therefore$ $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

Example 2 : Prove $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$, $n \in$ N

Solution : Let P(n) : $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$, $n \in$ N

Let $n = 1$. L.H.S. $= 1^2 = 1$ and R.H.S. $= \frac{1 \times 2 \times 3}{6} = 1$.

$\therefore$ P(1) is true.

Let P(k) be true, $k \in$ N.

$\therefore$ $1^2 + 2^2 + 3^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$.

Let $n = k + 1$.

$$\begin{aligned}
\therefore \text{ L.H.S.} = 1^2 + 2^2 + 3^2 + ... + k^2 + (k + 1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k + 1)^2 \\
&= (k + 1)\left[\frac{k(2k+1)}{6} + (k+1)\right] \\
&= (k + 1)\left(\frac{2k^2 + k + 6k + 6)}{6}\right) \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)\,(2k+3)}{6} \\
&= \frac{(k+1)(k+1+1)\,(2(k+1)+1)}{6} = \text{R.H.S.}
\end{aligned}$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by principle of mathematical induction.

Example 3 : Prove $1^3 + 2^3 + 3^3 + ... + n^3 = \dfrac{n^2(n+1)^2}{4}$, $n \in$ N

Solution : Let P(n) : $1^3 + 2^3 + 3^3 + ... + n^3 = \dfrac{n^2(n+1)^2}{4}$, $n \in$ N

For $n = 1$, L.H.S. $= 1^3 = 1$ and R.H.S. $= \dfrac{1^2 \times 2^2}{4} = 1$.

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $1^3 + 2^3 + 3^3 + ... + k^3 = \dfrac{k^2(k+1)^2}{4}$

Let $n = k + 1$.

$$\begin{aligned}
\text{L.H.S.} = 1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{(k+1)^2}{4}\,[k^2 + 4(k+1)] \\
&= \frac{(k+1)^2}{4}\,(k^2 + 4k + 4) \\
&= \frac{(k+1)^2\,(k+2)^2}{4} \\
&= \frac{(k+1)^2\,(k+1+1)^2}{4} = \text{R.H.S.}
\end{aligned}$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

(Now onwards we shall abbreviate Principle of Mathematical Induction as P.M.I.)

Example 4 : Prove $1 + 3 + 5 + ... + (2n - 1) = n^2$, $n \in$ N

Solution : Let P(n) : $1 + 3 + 5 + ... + (2n - 1) = n^2$, $n \in$ N

Let $n = 1$. L.H.S. $= 1$ and R.H.S. $= 1^2 = 1$.

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $1 + 3 + 5 + ... + (2k - 1) = k^2$

Let $n = k + 1$.

$$\begin{aligned}
\text{L.H.S.} &= 1 + 3 + 5 + ... + (2k - 1) + (2k + 1) \\
&= k^2 + 2k + 1 \\
&= (k+1)^2 = \text{R.H.S.}
\end{aligned}$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Example 5 : Prove $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1},\ \ n\in \mathrm{N}$

Solution : Let P(n) : $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1},\ \ n\in \mathrm{N}$

Let $n = 1$. L.H.S. $=\frac{1}{1\cdot 2}=\frac{1}{2}$ and R.H.S. $=\frac{1}{2}$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore\ \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$

Let $n = k + 1$.

$$\text{L.H.S.} = \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2} = \text{R.H.S.}$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in \mathrm{N}$ by P.M.I.

Example 6 : Prove $1\cdot 1! + 2\cdot 2! + \ldots + n\cdot n! = (n+1)! - 1,\ \ n \in \mathrm{N}$

Solution : Let P(n) : $1\cdot 1! + 2\cdot 2! + \ldots + n\cdot n! = (n+1)! - 1,\ \ n \in \mathrm{N}$

Let $n = 1$. L.H.S. $= 1\cdot 1! = 1$, R.H.S. $= (1 + 1)! - 1 = 2! - 1 = 1$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore\ 1\cdot 1! + 2\cdot 2! + \ldots + k\cdot k! = (k+1)! - 1$

Let $n = k + 1$.

$$\text{L.H.S.} = 1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \ldots + k\cdot k! + (k+1)(k+1)!$$

$$= (k+1)! - 1 + (k+1)(k+1)!$$

$$= (k+1)!\,[1 + (k+1)] - 1$$

$$= (k+1)!\,(k+2) - 1$$

$$= (k+2)! - 1 = \text{R.H.S.}$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Note : Directly, $n \cdot n! = (n + 1 - 1)\, n! \;= (n + 1)\, n! - n!$

$$= (n + 1)! - n!$$

Let n = 1, 2, 3,... etc. and add

$$1 \cdot 1! + 2 \cdot 2! + 33! + ... + n \cdot n! = (2! - 1!) + (3! - 2!) + (4! - 3!) + ... + ((n + 1)! - n!)$$
$$= (n + 1)! - 1$$

Example 7 : Prove $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right) ... \left(1 + \frac{2n+1}{n^2}\right) = (n + 1)^2, \;\; n \in \text{N}$

Solution : Let P(n) : $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right) ... \left(1 + \frac{2n+1}{n^2}\right) = (n + 1)^2, \;\; n \in \text{N}$

Let n = 1. L.H.S. = $1 + \frac{3}{1} = 4$ and R.H.S. = $(1 + 1)^2 = 2^2 = 4$

$\therefore$ P(1) is true.

Let P(k) be true.

$$\therefore \quad \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right) ... \left(1 + \frac{2k+1}{k^2}\right) = (k + 1)^2$$

Let $n = k + 1$.

$$\text{L.H.S.} = \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right) ... \left(1 + \frac{2k+1}{k^2}\right)\left(1+\frac{2k+3}{(k+1)^2}\right)$$

$$= (k + 1)^2 \times \left(\frac{k^2 + 2k + 1 + 2k + 3}{(k+1)^2}\right)$$

$$= k^2 + 4k + 4$$

$$= (k + 2)^2 = \text{R.H.S.}$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Note : Directly, $\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right) ... \left(1+\frac{2n+1}{n^2}\right)$

$$= \frac{4}{1} \cdot \frac{9}{4} \cdot \frac{16}{9} ... \frac{(n+1)^2}{n^2} = (n + 1)^2$$

Example 8 : Prove $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + ... + n \cdot 2^n = (n - 1)2^{n+1} + 2, \; n \in \text{N}$

(This type of series is called **arithmetico geometric** series.)

Solution : Let P(n) : $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + ... + n \cdot 2^n = (n - 1)2^{n+1} + 2, \; n \in \text{N}$

Let n = 1. L.H.S. = 2 and R.H.S. = 0 + 2 = 2

$\therefore$ P(1) is true.

Let P(k) be true.

Hence, $1 \cdot 2 + 2 \cdot 2^2 + ... + k \cdot 2^k = (k-1)2^{k+1} + 2$

Let $n = k + 1$.

L.H.S. $= 1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + ... + k \cdot 2^k + (k+1)2^{k+1}$

$= (k-1)2^{k+1} + 2 + (k+1)2^{k+1}$

$= (k - 1 + k + 1)2^{k+1} + 2$

$= 2k \cdot 2^{k+1} + 2$

$= k \cdot 2^{k+2} + 2 =$ R.H.S.

$\therefore$ P$(k + 1)$ is true.

$\therefore$ P(k) is true $\Rightarrow$ P$(k+1)$ is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Example 9 : Prove $a + ar + ar^2 + ... + ar^{n-1} = \dfrac{a(r^n - 1)}{r-1}$ **$(r \neq 1), n \in$ N**

Solution : Let P(n) : $a + ar + ar^2 + ... + ar^{n-1} = \dfrac{a(r^n - 1)}{r-1}$ **$(r \neq 1), n \in$ N**

Let $n = 1$. L.H.S. $= a$ and R.H.S. $= \dfrac{a(r-1)}{r-1} = a$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $a + ar + ar^2 + ... + ar^{k-1} = \dfrac{a(r^k - 1)}{r-1}$

Let $n = k + 1$.

L.H.S. $= a + ar + ar^2 + ... + ar^{k-1} + ar^k$

$$= \frac{a(r^k - 1)}{r-1} + ar^k$$

$$= a\left(\frac{r^k - 1}{r-1} + r^k\right)$$

$$= a\,\frac{r^k - 1 + r^k(r-1)}{r-1}$$

$$= a\,\frac{(r^k - 1 + r^{k+1} - r^k)}{r-1}$$

$$= \frac{a(r^{k+1} - 1)}{r-1} = \text{R.H.S.}$$

$\therefore$ P$(k + 1)$ is true.

$\therefore$ P(k) is true $\Rightarrow$ P$(k+1)$ is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Example 10 : Prove $3^{2n+2} - 8n - 9$ is divisible by 8, $n \in$ N

Solution : Let P(n) : $3^{2n+2} - 8n - 9$ is divisible by 8, $n \in$ N

Let $n = 1$. $3^4 - 8 - 9 = 81 - 8 - 9 = 64$ is divisible by 8.

Let P(k) be true. Hence $3^{2k+2} - 8k - 9$ is divisible by 8.

Let $n = k + 1$.

Now, $3^{2k+4} - 8(k+1) - 9$ **$(2(k+1) + 2 = 2k + 4)$**

$= 3^{2k+2} \cdot 3^2 - 8k - 8 - 9$

$= 3^{2k+2}(8+1) - 8k - 8 - 9$ **$(3^2 = 9 = 8 + 1)$**

$= 8 \cdot 3^{2k+2} + 3^{2k+2} - 8k - 8 - 9$

$= 3^{2k+2} - 8k - 9 + 8(3^{2k+2} - 1)$

Now, 8 divides $3^{2k+2} - 8k - 9$ **by $p(k)$**

Also, 8 divides $8(3^{2k+2} - 1)$

$\therefore$ 8 divides $3^{2k+2} - 8k - 9 + 8(3^{2k+2} - 1)$

$\therefore$ $3^{2(k+1)+2} - 8(k+1) - 9$ is divisible by 8.

$\therefore$ P($k + 1$) is true.

$\therefore$ P(k) is true $\Rightarrow$ P($k + 1$) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Note : Obviously,

$3^{2n+2} - 8n - 9 = (3^2)^{n+1} - 1 - 8n - 8$

$= (3^2 - 1)((3^2)^n + (3^2)^{n-1} + \ldots + 1) - 8n - 8$ **(Example 9)**

$= 8(3^{2n} + 3^{2n-2} + \ldots + 1) - 8n - 8$ is divisible by 8.

Another Method :

P(n) : $3^{2n+2} - 8n - 9$ is divisible by 8, $n \in$ N

For $n = 1$, $3^{2+2} - 8(1) - 9 = 64$ is divisible by 8.

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $3^{2k+2} - 8k - 9$ is divisible by 8.

$\therefore$ $3^{2k+2} - 8k - 9 = 8m$ where $m \in$ N **(i)**

Now, Let $n = k + 1$,

$3^{2(k+1)+2} - 8(k+1) - 9 = 3^{2k+2} \times 3^2 - 8k - 8 - 9$

$= (8k + 9 + 8m)9 - 8k - 8 - 9$ **(From (i))**

$= 72k + 81 + 72m - 8k - 8 - 9$

$= 64k + 72m + 64$

$= 8(8k + 9m + 8)$ is divisible by 8.

$\therefore$ P($k + 1$) is true.

$\therefore$ P(k) is true $\Rightarrow$ P($k + 1$) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Example 11 : Prove $2002^{2n+1} + 2003^{2n+1}$ is divisible by 4005, $n \in N$

Solution : Let P(n) : $2002^{2n+1} + 2003^{2n+1}$ is divisible by 4005, $n \in N$

Let $n = 1$.

$$2002^3 + 2003^3 = (2002 + 2003)\,[(2002)^2 - (2002)(2003) + (2003)^2]$$
$$= (4005)\,[(2002)^2 - (2002)(2003) + (2003)^2]$$

$\therefore$ $(2002)^3 + (2003)^3$ is divisible by 4005.

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $2002^{2k+1} + 2003^{2k+1}$ is divisible by 4005.

Let $n = k + 1$.

Now, $2002^{2(k+1)+1} + 2003^{2(k+1)+1}$

$$= 2002^{2k+3} - 2002^{2k+1}(2003)^2 + (2002)^{2k+1} \cdot (2003)^2 + (2003)^{2k+3}$$
$$= (2002)^{2k+1}\,[(2002)^2 - (2003)^2] + (2003)^2\,[(2002)^{2k+1} + (2003)^{2k+1}]$$
$$= -(4005)\,(2002)^{2k+1} + (2003)^2\,[(2002)^{2k+1} + (2003)^{2k+1}]$$

Now, $(2002)^{2k+1}$ $(2003)^{2k+1}$ is divisible by 4005. **(P(k))**

$\therefore$ $(2002)^{2(k+1)+1} + (2003)^{2(k+1)+1}$ is divisible by 4005.

$\therefore$ P($k + 1$) is true.

$\therefore$ P(k) is true $\Rightarrow$ P($k + 1$) is true.

$\therefore$ P(n) is true, $\forall n \in N$ by P.M.I.

Example 12 : Prove $x^{2n} - y^{2n}$ is divisible by $x + y$, $n \in N$

Solution : Let P(n) : $x^{2n} - y^{2n}$ is divisible by $x + y$, $n \in N$

Let $n = 1$.

Then $x^2 - y^2 = (x - y)(x + y)$ and so $x^2 - y^2$ is divisible by $x + y$.

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $x^{2k} - y^{2k}$ is divisible by $x + y$.

Let $n = k + 1$.

$$x^{2(k+1)} - y^{2(k+1)} = x^{2k+2} - x^{2k}y^2 + x^{2k}y^2 - y^{2k+2}$$
$$= x^{2k}(x^2 - y^2) + y^2(x^{2k} - y^{2k})$$
$$= x^{2k}(x - y)(x + y) + y^2(x^{2k} - y^{2k})$$

Now, $x^{2k} - y^{2k}$ is divisible by $(x + y)$. **(P(k))**

$\therefore$ $x^{2(k+1)} - y^{2(k+1)}$ is divisible by $(x + y)$.

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Example 13 : Prove $1^2 + 2^2 + 3^2 + ... + n^2 > \frac{n^3}{3}$, $n \in$ N

Solution : Let P(n) : $1^2 + 2^2 + 3^2 + ... + n^2 > \frac{n^3}{3}$, $n \in$ N

Let $n = 1$. L.H.S. $= 1^2 = 1$, R.H.S. $= \frac{1}{3}$ and $1 > \frac{1}{3}$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $1^2 + 2^2 + 3^2 + ... + k^2 > \frac{k^3}{3}$

Let $n = k + 1$.

Now, $1^2 + 2^2 + 3^2 + ... + k^2 + (k + 1)^2 > \frac{k^3}{3} + (k + 1)^2$ **(i)**

Now, $\frac{k^3}{3} + (k + 1)^2 = \frac{1}{3}(k^3 + 3k^2 + 6k + 3)$

$= \frac{1}{3}(k^3 + 3k^2 + 3k + 1 + 3k + 2)$

$> \frac{1}{3}(k^3 + 3k^2 + 3k + 1)$ as $\frac{1}{3}(3k + 2) \geq \frac{5}{3} > 0$

$\therefore$ $\frac{k^3}{3} + (k + 1)^2 > \frac{1}{3}(k + 1)^3$ **(ii)**

$\therefore$ $1^2 + 2^2 + 3^2 + ... + (k + 1)^2 > \frac{1}{3}(k + 1)^3$ **(by (i) and (ii))**

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Note : $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3 + 3n^2 + n}{6} > \frac{2n^3}{6} = \frac{n^3}{3}$, $n \in$ N

Example 14 : Prove $1 + 2 + 3 + ... + n < \frac{1}{8}(2n + 1)^2$, $n \in$ N

Solution : Let P(n) : $1 + 2 + 3 + ... + n < \frac{1}{8}(2n + 1)^2$, $n \in$ N

Let $n = 1$. L.H.S. $= 1$, R.H.S. $= \frac{1}{8}(3)^2 = \frac{9}{8}$ and $1 < \frac{9}{8}$

$\therefore$ P(1) is true.

Let P(k) be true.

$\therefore$ $1 + 2 + 3 + ... + k < \frac{1}{8}(2k + 1)^2$

Add $k + 1$ on both the sides.

$\therefore$ $1 + 2 + 3 + ... + k + (k + 1) < \frac{1}{8}(2k + 1)^2 + (k + 1)$ **(i)**

Now, $\frac{1}{8}(2k+1)^2 + (k+1) = \frac{1}{8}(4k^2 + 4k + 1 + 8k + 8)$

$= \frac{1}{8}(4k^2 + 12k + 9)$

$\therefore \quad \frac{1}{8}(2k+1)^2 + (k+1) = \frac{1}{8}(2k+3)^2$ **(ii)**

$\therefore \quad 1 + 2 + 3 + ... + (k+1) < \frac{1}{8}(2k+3)^2$ **(by (i) and (ii))**

$\therefore \quad$ P$(k+1)$ is true.

$\therefore \quad$ P(k) is true $\Rightarrow$ P$(k+1)$ is true.

$\therefore \quad$ P(n) is true, $\forall n \in$ N by P.M.I.

Note : $1 + 2 + 3 + ... + n = \frac{n(n+1)}{2} = \frac{4n^2+4n}{8} < \frac{4n^2+4n+1}{8} = \frac{1}{8}(2n+1)^2$

Example 15 : Prove $(1+x)^n \geq 1 + nx, \quad n \in$ N **$(x > -1)$**

Solution : Let P(n) : $(1+x)^n \geq 1 + nx, \quad n \in$ N

Let $n = 1$. $(1+x)^1 = 1 + x \geq 1 + 1 \cdot x$

$\therefore \quad$ P(1) is true.

Let P(k) be true.

$\therefore \quad (1+x)^k \geq 1 + kx$

Let $n = k + 1$.

Now, $(1+x)^{k+1} = (1+x)^k (1+x)$

$\geq (1+kx)(1+x)$ **(by P(k) and as $x > -1$)**

$\therefore \quad (1+x)^{k+1} \geq 1 + kx + x + kx^2 \geq 1 + kx + x$ as $k \in$ N, $x^2 \geq 0$

$\therefore \quad (1+x)^{k+1} \geq 1 + (k+1)x$

$\therefore \quad$ P$(k+1)$ is true.

$\therefore \quad$ P(k) is true $\Rightarrow$ P$(k+1)$ is true.

$\therefore \quad$ P(n) is true, $\forall n \in$ N by P.M.I.

Example 16 : Prove $1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} \leq 2 - \frac{1}{n}, \quad n \in$ N

Solution : Let P(n) : $1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} \leq 2 - \frac{1}{n}, \quad n \in$ N

Let $n = 1$, L.H.S. $= 1$, R.H.S. $= 2 - 1 = 1$

$\therefore \quad$ P(1) is true.

Let P(k) be true.

$\therefore \quad 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{k^2} \leq 2 - \frac{1}{k}$

Add $\frac{1}{(k+1)^2}$ on both the sides.

Hence, $1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$ **(i)**

$$\text{Now, } 2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \frac{1}{k} + \frac{1}{(k+1)^2} + \frac{1}{k+1} - \frac{1}{k+1}$$

$$= 2 - \frac{1}{k+1} + \frac{1}{(k+1)^2} + \frac{-k-1+k}{k(k+1)}$$

$$= 2 - \frac{1}{k+1} + \frac{1}{(k+1)^2} - \frac{1}{k(k+1)}$$

$$= 2 - \frac{1}{k+1} + \frac{k-k-1}{k(k+1)^2}$$

$$= 2 - \frac{1}{k+1} - \frac{1}{k(k+1)^2}$$

$\therefore \quad 2 - \frac{1}{k} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$ $\qquad$ **$\left(k \in \mathrm{N} \text{ gives } \frac{1}{k(k+1)^2} > 0\right)$ (ii)**

$\therefore \quad 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{(k+1)^2} < 2 - \frac{1}{(k+1)}$ $\qquad$ **(by (i) and (ii))**

$\therefore$ $\quad$ P(k + 1) is true.

$\therefore$ $\quad$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ $\quad$ P(n) is true $\forall n \in \mathrm{N}$ by P.M.I.

Note : Thus however large n, sum $1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}$ is 'bounded' and less than < 2.

Example 17 : Prove $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n, \quad n \in \mathrm{N}$

Solution : Let P(n) : $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n, \quad n \in \mathrm{N}$

Let $n = 1$. L.H.S. $= \binom{1}{0} + \binom{1}{1} = 2$, R.H.S. $= 2^1 = 2$

$\therefore$ $\quad$ P(1) is true.

Let P(k) be true.

$\therefore \quad \binom{k}{0} + \binom{k}{1} + \dots + \binom{k}{k} = 2^k$

Let $n = k + 1$.

$$\text{L.H.S.} = \binom{k+1}{0} + \binom{k+1}{1} + \dots + \binom{k+1}{k} + \binom{k+1}{k+1}$$

$$= \binom{k}{0} + \left(\binom{k}{0} + \binom{k}{1}\right) + \left(\binom{k}{1} + \binom{k}{2}\right) + \dots + \left(\binom{k}{k-1} + \binom{k}{k}\right) + \binom{k}{k}$$

$$\left(\text{as } \binom{k}{0} = \binom{k+1}{0} = 1,\ \binom{k}{k} = \binom{k+1}{k+1} = 1,\ \binom{k}{r} = \binom{k-1}{r-1} + \binom{k-1}{r}\right)$$

$$= 2\left[\binom{k}{0} + \binom{k}{1} + \binom{k}{2} + \dots + \binom{k}{k}\right]$$

$$= 2 \cdot 2^k$$

$$= 2^{k+1} = \text{R.H.S.}$$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

1.4 Some Variants of P.M.I.

Variant 1 : If P(n) is a statement involving natural variable n and if P(k_0) is true for some positive integer k_0 and if the truth of P(k) for some integer $k \geq k_0$ implies the truth of P(k + 1), then P(n) is true $\forall n \in$ N, such that $n \geq k_0$.

Example 18 : Prove $2^n > n^2$; $n \geq 5$, $n \in$ N

Solution : Let P(n) : $2^n > n^2$; $n \geq 5$, $n \in$ N

Let $n = 5$. ($k_0 = 5$), $2^5 = 32$, $5^2 = 25$ and $32 > 25$.

$\therefore$ P(5) is true.

Let P(k) be true for $k \geq 5$. Hence, $2^k > k^2$

Let $n = k + 1$.

Now, $2^{k+1} = 2 \cdot 2^k > 2k^2$ **($2^k > k^2$) (i)**

$$\text{Now, } 2k^2 - (k + 1)^2 = 2k^2 - k^2 - 2k - 1$$
$$= k^2 - 2k + 1 - 2$$
$$= (k + 1)^2 - 2 > 0 \text{ as } k \geq 5$$

$\therefore$ $2k^2 > (k + 1)^2$ **(ii)**

$\therefore$ $2^{k+1} > (k + 1)^2$ **(by (i) and (ii))**

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Variant 2 : Let P(n) be a statement of integer variable n.

If P(1) and P(2) are true and if P(k) and P(k + 1) are true for some positive integer k implies P(k + 2) is also true, then P(n) is true for all $n \in$ N.

Example 19 : Let a_n be a sequence of natural numbers with $a_1 = 5$, $a_2 = 13$ and $a_{n+2} = 5a_{n+1} - 6a_n$ for $n \geq 1$. Prove $a_n = 2^n + 3^n$, $\forall n \in$ N.

Solution : Let P(n) : If $a_{n+2} = 5a_{n+1} - 6a_n$ for $n \geq 1$, $a_1 = 5$, $a_2 = 13$, then $a_n = 2^n + 3^n$, $\forall n \in$ N.

Let $n = 1$. $a_1 = 5$ and $2^1 + 3^1 = 2 + 3 = 5$. Hence, P(1) is true.

Let $n = 2$. $a_2 = 13$ and $2^2 + 3^2 = 4 + 9 = 13$. Hence, P(2) is true.

Let $a_k = 2^k + 3^k$, $a_{k+1} = 2^{k+1} + 3^{k+1}$ for $k \geq 1$

Now, $a_{k+2} = 5a_{k+1} - 6a_k$

$= 5(2^{k+1} + 3^{k+1}) - 6 \cdot 2^k - 6 \cdot 3^k$

$= 5 \cdot 2^k \cdot 2 + 5 \cdot 3^k \cdot 3 - 6 \cdot 2^k - 6 \cdot 3^k$

$= 2^k (10 - 6) + 3^k (15 - 6)$

$= 2^k \cdot 2^2 + 3^k \cdot 3^2$

$= 2^{k+2} + 3^{k+2}$

$\therefore$ P($k + 2$) is true.

$\therefore$ P(k) is true and P($k + 1$) is true $\Rightarrow$ P($k + 2$) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Note : $a_{n+2} = 5a_{n+1} - 6a_n$ is called a **recurrence relation**. Its solution is $a_n = A\alpha^n + B\beta^n$ where α, β are roots of $x^2 - 5x + 6 = 0$ (5 is co-efficient of a_{n+1}, -6 is co-efficient of a_n)

$\therefore$ $\alpha = 3$, $\beta = 2$

$\therefore$ $a_n = A3^n + B2^n$

$\therefore$ $a_1 = 3A + 2B = 5$; $a_2 = 9A + 4B = 13$. Hence, A = B = 1

$\therefore$ $a_n = 3^n + 2^n$. If $a_{n+2} = l \cdot a_{n+1} - m \cdot a_n$, then α and β are the roots of equation $x^2 - lx + m = 0$.

Miscellaneous Problems :

Example 20 : Prove that any payment of ₹ 4 or more can be made using ₹ 2 and ₹ 5 coins only.

Solution : Let P(n) : Any payment of ₹ 4 or more can be made using ₹ 2 and ₹ 5 coins only. $n \in$ N

For $n = 4$, we require two coins of ₹ 2 to pay ₹ 4. Let the statement be true for $k \geq 4$.

Let $n = k + 1$.

Consider two cases :

(1) If the payment for ₹ k contains a ₹ 5 coin, take it back and give 3, ₹ 2 coins. Hence $k + 6 - 5 = k + 1$ rupees are paid.

(2) If the payment for ₹ k does not contain any ₹ 5 coin, since $k \geq 4$, he must have paid at least two ₹ 2 coins. Take them back and pay one ₹ 5 coin. Hence ₹ $k + 5 - 4 = k + 1$ are paid.

$\therefore$ P($k + 1$) is true.

$\therefore$ P(k) is true $\Rightarrow$ P($k + 1$) is true.

$\therefore$ P(n) be true, for $\forall n \in$ N by P.M.I.

Example 21 : Prove that any integer $n > 23$ can be put in the form $7x + 5y = n$, where $x \in$ N $\cup$ {0}, $y \in$ N $\cup$ {0}.

Solution : Let P(n) : Any integer $n > 23$ can be put in the form $7x + 5y = n$, where $x \in$ N $\cup$ {0}, $y \in$ N $\cup$ {0}.

Let $n = 24$. Then $7 \cdot 2 + 5 \cdot 2 = 24$ is the required form with $x = y = 2$.

Let $7x + 5y = k$ for $k \geq 24$, $x \in N \cup \{0\}$, $y \in N \cup \{0\}$. **(i)**

Now, $5 \cdot 3 - 7 \cdot 2 = 1$ **(ii)**

$\therefore \quad 7(x - 2) + 5(y + 3) = k + 1$ **(Adding (i) and (ii))**

Here $y + 3 \in N \cup \{0\}$ and $x - 2 \in N \cup \{0\}$ if $x \neq 0$ or 1.

Let $x = 0$. Then $5y = k \geq 24$. Thus $y \geq 5$, using (i).

$7 \cdot 3 - 5 \cdot 4 = 1$ and $5y = k$ gives on adding. **(iii)**

$7 \cdot 3 + 5(y - 4) = k + 1$

Here $x = 3 \geq 0$, $y - 4 \geq 0$ **($y \geq 5$)**

$\therefore \quad P(k + 1)$ is true, if $x = 0$

Let $x = 1$. Hence, $7 + 5y = k$, using (i).

Then $5y = k - 7 \geq 17$. Thus $y \geq 4$

$\therefore \quad 7 \cdot 3 - 5 \cdot 4 = 1$ and $7 + 5y = k$ gives on adding. **(iv)**

$7(4) + 5(y - 4) = k + 1$ with $y - 4 \geq 0$ and $x = 4$ **(Adding in (iv))**

$\therefore \quad P(k + 1)$ is true.

$\therefore \quad P(k)$ is true $\Rightarrow P(k + 1)$ is true.

$\therefore \quad P(n)$ is true for $\forall n \in N$ by P.M.I.

Example 22 : (Tower of Hanoi) We have three pegs and a collection of disks of different sizes. Initially they are on top on each other according to their size on the first peg, the largest being on the bottom and the smallest on the top. A move in this game consists of moving disks from one peg to another such that larger disk can never rest on a smaller one. Prove that the number of moves to transfer all disks from first peg to the last peg using the second peg as intermediate is $2^n - 1$, $n \in N$.

Solution : Let $P(n)$: The number of moves to transfer all disks from first peg to the last peg using the second peg as intermediate is $2^n - 1$, $n \in N$.

Let $n = 1$, obviously there is only one move.

$\therefore \quad P(1)$ is true. $2^1 - 1 = 1$. **($p(k)$)**

Suppose there are $2^k - 1$ moves to transfer k disks as required.

First we move top k disks to the second peg using the third peg as the intermediate one. This will take $2^k - 1$ moves. Now move the last disk to the third peg. This is one move. Now move k disks from second peg to the third peg in $2^k - 1$ moves.

$\therefore \quad$ The total number moves is $2^k - 1 + 1 + 2^k - 1 = 2 \cdot 2^k - 1 = 2^{k+1} - 1$

$\therefore \quad P(k + 1)$ is proved.

$\therefore \quad P(k)$ is true $\Rightarrow P(k + 1)$ is true.

$\therefore \quad P(n)$ is true, $\forall n \in N$ by P.M.I.

Example 23 : Prove $\frac{n^{11}}{11} + \frac{n^5}{5} + \frac{n^3}{3} + \frac{62n}{165} \in \mathrm{N}$, $n \in \mathrm{N}$ (to be done after chapter 3)

Solution : Let P(n) : $\frac{n^{11}}{11} + \frac{n^5}{5} + \frac{n^3}{3} + \frac{62n}{165} \in \mathrm{N}$, $n \in \mathrm{N}$

For $n = 1$, $\frac{n^{11}}{11} + \frac{n^5}{5} + \frac{n^3}{3} + \frac{62n}{165} = \frac{15+33+55+62}{165} = \frac{165}{165} = 1$

$\therefore$ P(1) is true.

Let P(k) be true. Hence, $\frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165} \in \mathrm{N}$

Let $n = k + 1$.

Consider $\left(\frac{(k+1)^{11}}{11} + \frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{62(k+1)}{165}\right) - \left(\frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165}\right)$

$$= \frac{1}{11}((k+1)^{11} - k^{11}) + \frac{1}{5}((k+1)^5 - k^5) + \frac{1}{3}((k+1)^3 - k^3) + \frac{62}{165}$$

$$= \frac{1}{11}\left(1 + \binom{11}{1}k + \binom{11}{2}k^2 + \ldots + \binom{11}{10}k^{10}\right) + \frac{1}{5}\left(1 + \binom{5}{1}k + \binom{5}{2}k^2 + \ldots + \binom{5}{4}k^4\right)$$

$$+ \frac{1}{3}\left(1 + \binom{3}{1}k + \binom{3}{2}k^2\right) + \frac{62}{165}$$

$$= \frac{1}{11}\binom{11}{1}k + \frac{1}{11}\binom{11}{2}k^2 + \ldots + \frac{1}{11}\binom{11}{10}k^{10} + \frac{1}{5}\binom{5}{1}k + \frac{1}{5}\binom{5}{2}k^2 + \ldots + \frac{1}{5}\binom{5}{4}k^4$$

$$+ \frac{1}{3}\binom{3}{1}k + \frac{1}{3}\binom{3}{2}k^2 + \frac{1}{11} + \frac{1}{5} + \frac{1}{3} + \frac{62}{165} \qquad \text{(i)}$$

Now, 11, 5, 3 being primes, 11 divides $\binom{11}{r}$ for $r = 1, 2, ..., 10$

5 divides $\binom{5}{r}$ for $r = 1, 2, 3, 4$

3 divides $\binom{3}{r}$ for $r = 1, 2$

and $\frac{1}{11} + \frac{1}{5} + \frac{1}{3} + \frac{62}{165} = 1$

$\therefore$ The R.H.S. in (1) represents a natural number.

Also $\frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165} \in \mathrm{N}$

$\therefore$ $\frac{(k+1)^{11}}{11} + \frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{62(k+1)}{165}$

$= \frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165}$ + a natural number $\in \mathrm{N}$

$\therefore$ P(k + 1) is true.

$\therefore$ P(k) is true $\Rightarrow$ P(k + 1) is true.

$\therefore$ P(n) is true for $\forall n \in \mathrm{N}$ by P.M.I.

Example 24 : There are $2n$ persons in a hall. Some persons handshake with others. There do not exist any three persons who have handshakes with each other. Prove that the number of handshakes is at most n^2.

Solution : Let P(n) : There are $2n$ persons in a hall. Some persons handshake with others. There do not exist any three persons who have handshakes with each other. Then the number of handshakes is at most n^2.

For $n = 1$, there are two persons. Hence there is at most $1 = 1^2$ handshake.

$\therefore$ P(1) is true.

Let P(k) be true.

Let $n = k + 1$.

Now there are $2k + 2$ persons. Choose two persons A and B who have had a handshake.

(If there are no two such persons, number of handshakes is zero which is at most $(k + 1)^2$.

Now the remaining $2k$ persons had at most k^2 handshakes (P(k) is true). A and B have one handshake.

Each of $2k$ persons could shake hands with A or B only as no three persons had handshakes with each other. Hence the number of handshakes is at most

$$k^2 + 1 + 2k = (k + 1)^2$$

$\therefore$ P($k + 1$) is true.

$\therefore$ P(k) is true $\Rightarrow$ P($k + 1$) is true.

$\therefore$ P(n) is true for $\forall n \in$ N by P.M.I.

A paradox :

[**Note :** A paradox is the misinterpretation of a result to arrive at a contradictory result.]

P(n) : A thirsty man can drink n drops of water.

For $n = 1$, obviously a thirsty man would like to drink one drop of water.

If he can drink k drops of water, he can definitely drink $k + 1$ drops of water.

So he can drink any amount water to exhaust all resources of water on the earth !

Exercise 1

Prove the following by the principle of mathematical induction : **(1 to 19) ($n \in$ N)**

1. $1^2 \cdot 2 + 2^2 \cdot 3 + ... + n^2(n + 1) = \frac{n(n+1)(n+2)(3n+1)}{12}$

2. $a + (a + d) + (a + 2d) + ... + (a + (n - 1)d) = \frac{1}{2}n(2a + (n - 1)d)$

3. $\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} + ... + \frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$

4. $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + ... + n(n + 1)(n + 2) = \frac{n(n+1)(n+2)(n+3)}{4}$

5. $1 + 5 + 9 + ... + (4n - 3) = n(2n - 1)$

6. $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^n} = 1 - \left(\frac{1}{2}\right)^n$

7. $\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+n}=\frac{2n}{n+1}$

8. $\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$

9. $1^4+2^4+3^4+\ldots+n^4=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

10. If $a_1=1$, $a_2=1$, $a_n=a_{n-1}+a_{n-2}$, $n \geq 3$, then $a_1+a_2+a_3+\ldots+a_n=a_{n+2}-1$.

11. 41^n-1 is divisible by 40.

12. 4007^n-1 is divisible by 2003.

13. 7^n-6n-1 is divisible by 36.

14. $2\cdot7^n+3\cdot5^n-5$ is a multiple of 24.

15. $11^{n+2}+12^{2n+1}$ is divisible by 133.

16. $n(n+1)(2n+1)$ is divisible by 6.

17. $1\cdot3^1+2\cdot3^2+3\cdot3^3+\ldots+n\cdot3^n=\frac{(2n-1)\,3^{n+1}+3}{4}$

18. $10^n+3\cdot4^{n+2}+5$ is divisible by 9.

19. $\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}\in N$

20. Prove $\frac{(2n)!}{2^{2n}(n!)^2}\leq\frac{1}{\sqrt{3n+1}}$

21. For Lucas' sequence $a_n=a_{n-1}+a_{n-2}$ $(n\geq3)$; $a_1=1$, $a_2=3$, prove $a_n\leq(1.75)^n$.

22. Prove $2^n>n^3$, if $n\geq10$

23. Prove a polygon of n sides has $\frac{n(n-3)}{2}$ diagonals, $n>3$

24. If $a_1=1$, $a_2=1$, $a_n=a_{n-1}+a_{n-2}$, $n\geq3$, then prove that

$$a_n=\frac{1}{\sqrt5}\left(\left(\frac{\sqrt5+1}{2}\right)^n-\left(\frac{1-\sqrt5}{2}\right)^n\right)$$ (This $\{a_n\}$ is called **Fibonacci** sequence.)

25. If $f: N\rightarrow N$, $f(1)=1$, $f(2)=5$, $f(n+1)=f(n)+2f(n-1)$, $n\geq2$

then prove that $f(n)=2^n+(-1)^n$

26. If $f: N\rightarrow N$, $f(1)=1$, $f(n+1)-f(n)=2^n$

then prove that $f(n)=2^n-1$

27. If $a_1=1$, $a_2=1$, $a_n=a_{n-1}+a_{n-2}$; $n\geq3$

then prove that $a_2+a_4+a_6+\ldots+a_{2n}=a_{2n+1}-1$

28. If $a_1=1$, $a_2=11$ and $a_n=2a_{n-1}+3a_{n-2}$; $n\geq3$

then prove that $a_n=2(-1)^n+3^n$ for $n\in N$

29. Prove that every integer $n\geq12$ can be written in the form $7x+3y=n$,

$x\in N\cup\{0\}$, $y\in N\cup\{0\}$

30. Prove that $(3 + \sqrt{5})^n + (3 - \sqrt{5})^n$ is even, $n \in N$.

31. Select a proper option (a), (b), (c) or (d) from given options and write in the box given on the right so that the statement becomes correct :

(1) For P(n) : $2^n < n\,!$, ... is true. $\square$

(a) P(1) (b) P(2) (c) any P(n), $n \in N$ (d) P(4)

(2) For P(n) : $2^n = 0$, ... is true. $\square$

(a) P(1) (b) P(3)

(c) P(10) (d) P(k) $\Rightarrow$ P($k + 1$), $k \in N$

(3) P(n) : $1 + 2 + 3 + ... + (n + 1) = \dfrac{(n+1)(n+2)}{2}$, $n \in N$ $\square$

(a) P(1) requires L.H.S. = 7 = R.H.S.

(b) P(1) requires L.H.S. = 3 = R.H.S.

(c) P(k) $\Rightarrow$ P($k + 1$) is not true for $k \in N$

(d) It is false that P(n) is true, $\forall n \in N$ by P.M.I.

(4) If ...... is true and P(k) is true $\Rightarrow$ P($k + 1$) is true for $k \geq -1$, then P(n) is true for all $n \in N \cup \{0, -1\}$. $\square$

(a) P(-1) (b) P(0) (c) P(1) (d) P(2)

(5) P(n) : Every prime is of the form $2^{2^n} + 1$ is not true, for n = ... $\square$

(a) 1 (b) 2 (c) 0 (d) 5

(6) P(n) : $2^n - 1$ is a prime for n = ... $\square$

(a) 1 (b) 3 (c) 4 (d) 8

(7) P(n) : $n^2 - n + 41$ is a prime, is false for n = ... $\square$

(a) 1 (b) 2 (c) 3 (d) 41

(8) P(n) : $2n + 1$ is a prime, is false for n = ... $\square$

(a) 1 (b) 2 (c) 3 (d) 4

(9) P(n) : $4n + 1$ is a prime, is false for n = ... $\square$

(a) 1 (b) 3 (c) 7 (d) 11

(10) P(n) : $2^n > n^2$ is true for n = ... $\square$

(a) 2 (b) 3 (c) 4 (d) 5

*

Summary

We studied the following points in this chapter :

1. Principle of Induction and Examples

2. Different variants of P.M.I. and applications

Puzzle

There are n people in a room each being put on a hat from amongst at least n white hats and $n - 1$ black hats. They stand in a queue, so that every one can see the colour of the hat of the person standing in front of him. Starting from back we ask the persons in turn, 'Do you know what is the colour of your hat ?' If the first $(n - 1)$ persons say no, the person in the front will say 'Yes the colour of my hat is white.' Prove.

Solution : Let P(n) : If the first $(n - 1)$ persons say no, the person in the front will say yes.

For $n = 1$, there is no black hat $(1 - 1 = 0)$. Hence the first person will say, 'yes, my hat is white.' Suppose the statement is true for $n = k$. Let $n = k + 1$.

See how the man standing in the front would think. Suppose my hat is black. Then excluding me there are k people with at least k white hats and $k - 1$ black hats. By P(k), since the first $(k - 1)$ persons said no, the person behind me must say yes. 'I know the colour of my hat.'

But he said no. So the colour of my hat cannot be black. Hence it is white.

$\therefore$ P($k + 1$) is true.

$\therefore$ P(k) is true $\Rightarrow$ P($k + 1$) is true.

$\therefore$ P(n) is true, $\forall n \in$ N by P.M.I.

Explanation : If $n = 2$, there is one black hat and at least two white hats. If the last person sees a black hat put on by the person in front of him, he would definitely say, 'Yes, colour of my hat is white,' as there is only one black hat. But he is not able to answer. So the first person logically thinks he has put on a white hat and the person behind might have put on a black or a white hat.

B W or B

1 2

Srinivasa Ramanujan (1887-1920) was one of India's greatest mathematical geniuses. He made substantial contributions to the analytical theory of numbers and worked on elliptical functions, continued fractions and infinite series.

In 1990 he began to work on his own on mathematics summing geometric and arithmetic series.

Ramanujan had shown how to solve cubic equations in 1902 and he went to find his own method to solve the quartic.

In 1904 Ramanujan had begun to undertake deep research. He investigated the series $\Sigma\left(\frac{1}{n}\right)$ and calculated Euler's constant to 15 decimal places.

Continuing his mathematical work Ramanujan studied continued fractions and divergent series in 1908.