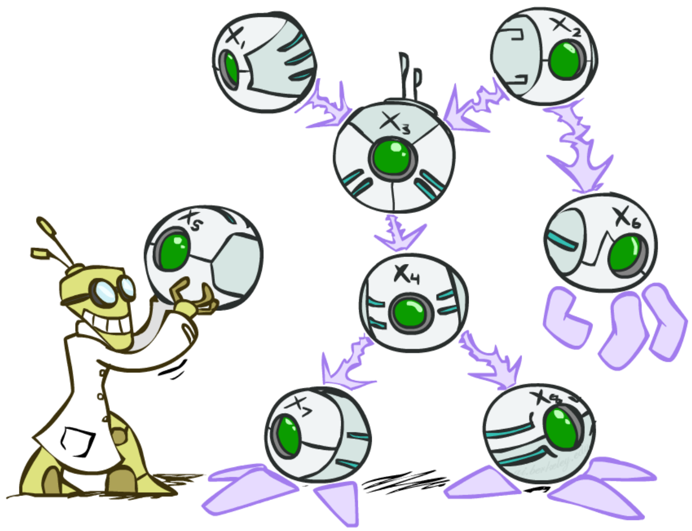
# CS 188: Artificial Intelligence

## Bayes' Nets



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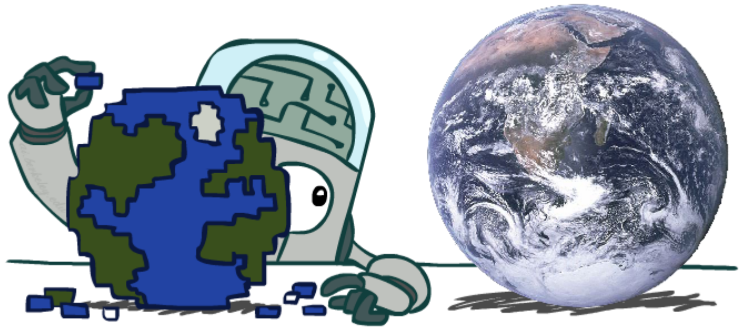
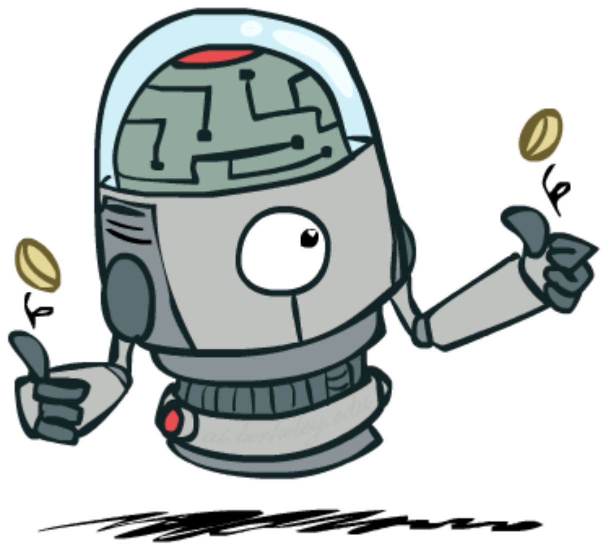
# Probabilistic Models

- Models describe how (a portion of) the world works
- Models are always simplifications
  - May not account for every variable
  - May not account for all interactions between variables
  - "All models are wrong; but some are useful."
    – George E. P. Box



- What do we do with probabilistic models?
  - We (or our agents) need to reason about unknown variables, given evidence
  - Example: explanation (diagnostic reasoning)
  - Example: prediction (causal reasoning)
  - Example: value of information

# Independence



# Independence

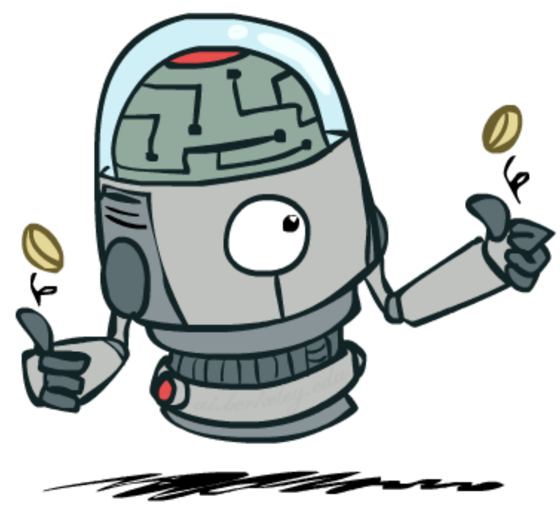
- Two variables are *independent* if:

$$\forall x, y : P(x, y) = P(x)P(y)$$

  - This says that their joint distribution *factors* into a product two simpler distributions
  - Another form:

$$\forall x, y : P(x|y) = P(x)$$

  - We write: $X \perp\!\!\!\perp Y$



- Independence is a simplifying *modeling assumption*
  - *Empirical* joint distributions: at best "close" to independent
  - What could we assume for {Weather, Traffic, Cavity, Toothache}?

# Example: Independence?

$P(T)$

| T | P |
|---|---|
| hot | 0.5 |
| cold | 0.5 |

$P_1(T, W)$

| T | W | P |
|---|---|---|
| hot | sun | 0.4 |
| hot | rain | 0.1 |
| cold | sun | 0.2 |
| cold | rain | 0.3 |

$P_2(T, W)$

| T | W | P |
|---|---|---|
| hot | sun | 0.3 |
| hot | rain | 0.2 |
| cold | sun | 0.3 |
| cold | rain | 0.2 |

$P(W)$

| W | P |
|---|---|
| sun | 0.6 |
| rain | 0.4 |

# Example: Independence

- N fair, independent coin flips:

$P(X_1)$

| | |
|---|---|
| H | 0.5 |
| T | 0.5 |

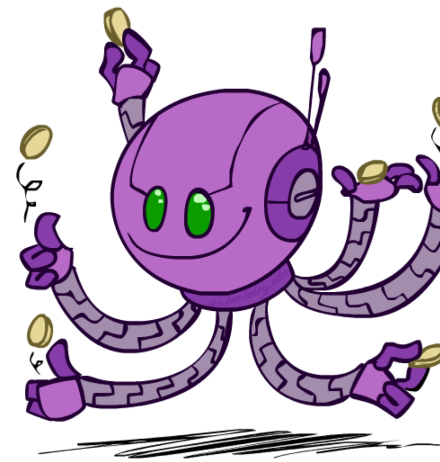
$P(X_2)$

| | |
|---|---|
| H | 0.5 |
| T | 0.5 |

$\cdots$

$P(X_n)$

| | |
|---|---|
| H | 0.5 |
| T | 0.5 |



$P(X_1, X_2, \ldots X_n)$

$2^n$

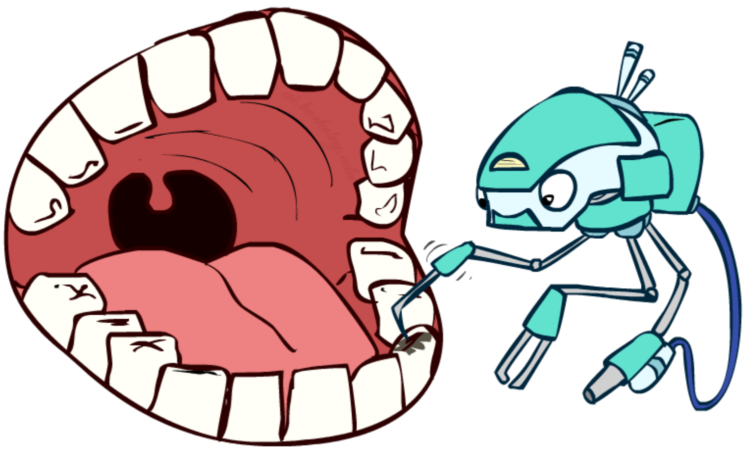
# Conditional Independence

- P(Toothache, Cavity, Catch)

- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:
  - P(+catch | +toothache, +cavity) = P(+catch | +cavity)

- The same independence holds if I don't have a cavity:
  - P(+catch | +toothache, -cavity) = P(+catch| -cavity)

- Catch is *conditionally independent* of Toothache given Cavity:
  - P(Catch | Toothache, Cavity) = P(Catch | Cavity)



- Equivalent statements:
  - P(Toothache | Catch , Cavity) = P(Toothache | Cavity)
  - P(Toothache, Catch | Cavity) = P(Toothache | Cavity) P(Catch | Cavity)
  - One can be derived from the other easily

# Conditional Independence

- Unconditional (absolute) independence very rare (why?)
- *Conditional independence* is our most basic and robust form of knowledge about uncertain environments.
- X is conditionally independent of Y given Z $X \perp\!\!\!\perp Y | Z$

if and only if:

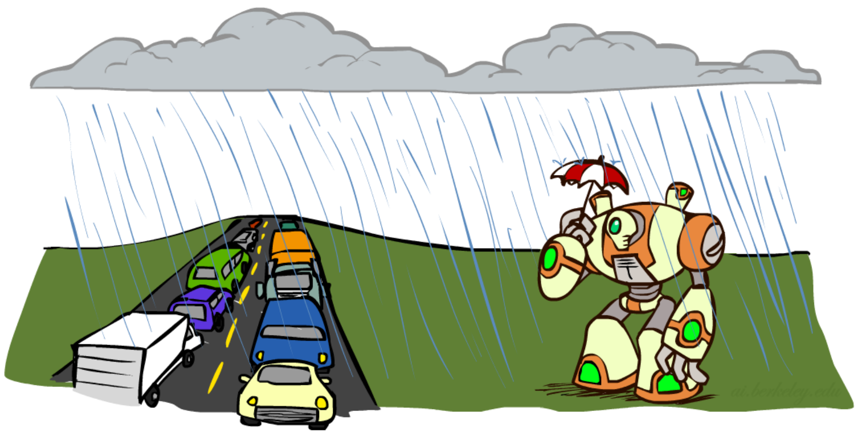
$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$

or, equivalently, if and only if

$$\forall x, y, z : P(x|z, y) = P(x|z)$$

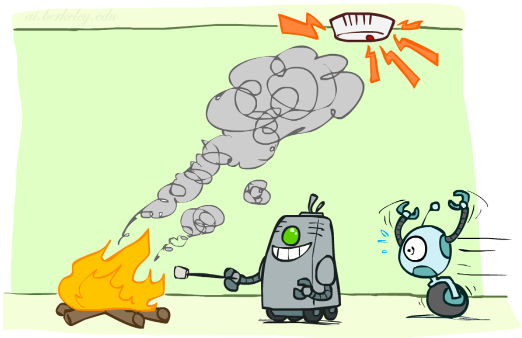
# Conditional Independence

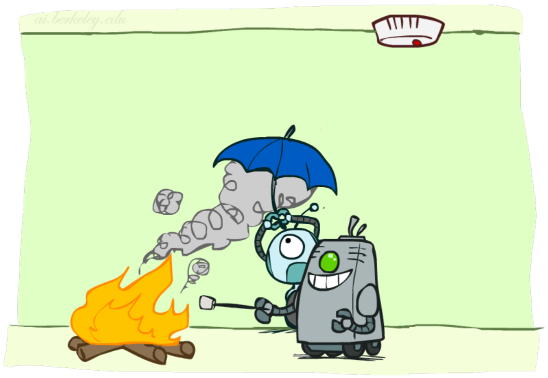
- What about this domain:
  - Traffic
  - Umbrella
  - Raining



# Conditional Independence

- What about this domain:
  - Fire
  - Smoke
  - Alarm

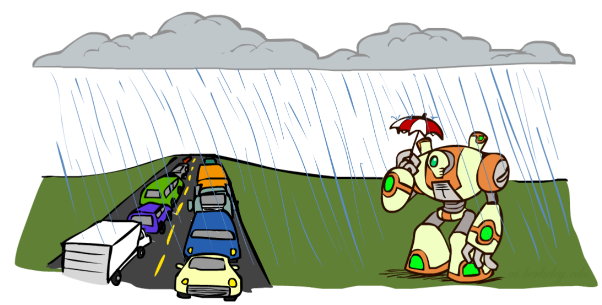




# Conditional Independence and the Chain Rule

- Chain rule: $P(X_1, X_2, \ldots X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)\ldots$

- Trivial decomposition:



$$P(\text{Traffic}, \text{Rain}, \text{Umbrella}) =$$
$$P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain}, \text{Traffic})$$

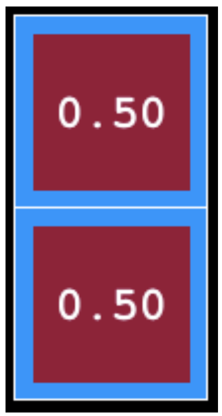
- With assumption of conditional independence:

$$P(\text{Traffic}, \text{Rain}, \text{Umbrella}) =$$
$$P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain})$$

- Bayes' nets / graphical models help us express conditional independence assumptions

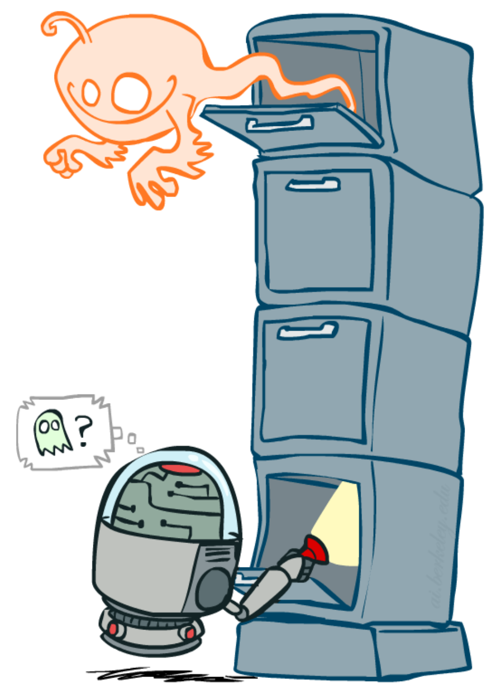
# Ghostbusters Chain Rule

- Each sensor depends only on where the ghost is
- That means, the two sensors are conditionally independent, given the ghost position
- T: Top square is red
  B: Bottom square is red
  G: Ghost is in the top
- Givens:
  $P(+g) = 0.5$
  $P(-g) = 0.5$
  $P(+t \mid +g) = 0.8$
  $P(+t \mid -g) = 0.4$
  $P(+b \mid +g) = 0.4$
  $P(+b \mid -g) = 0.8$

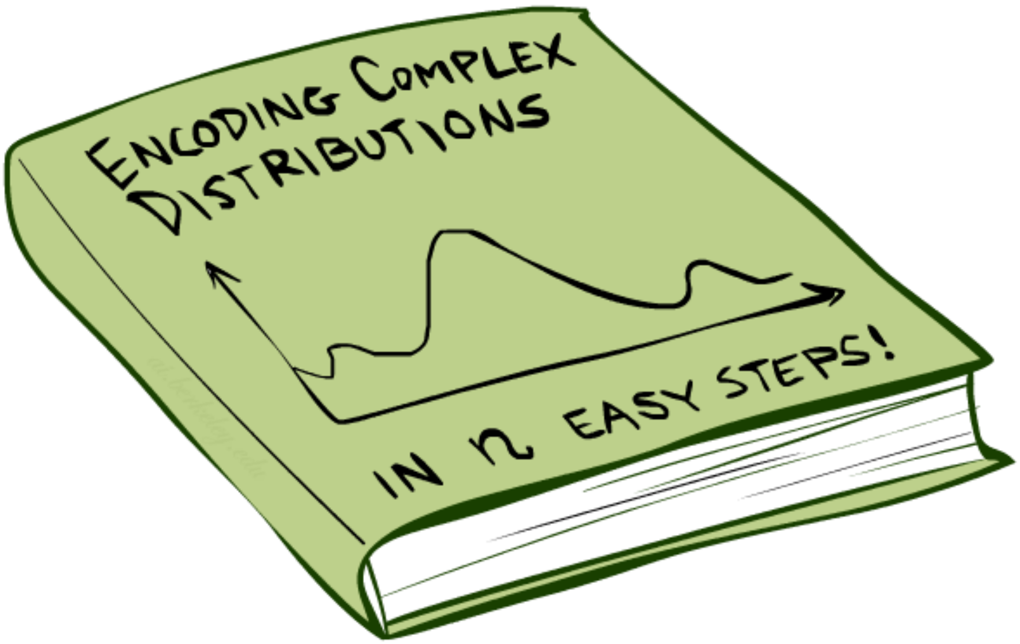



$P(T,B,G) = P(G)\, P(T \mid G)\, P(B \mid G)$

| T | B | G | P(T,B,G) |
|---|---|---|---|
| +t | +b | +g | 0.16 |
| +t | +b | -g | 0.16 |
| +t | -b | +g | 0.24 |
| +t | -b | -g | 0.04 |
| -t | +b | +g | 0.04 |
| -t | +b | -g | 0.24 |
| -t | -b | +g | 0.06 |
| -t | -b | -g | 0.06 |

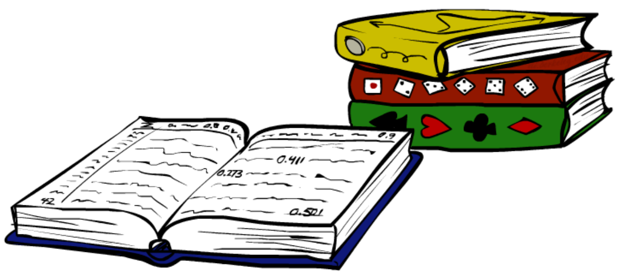


# Bayes' Nets: Big Picture

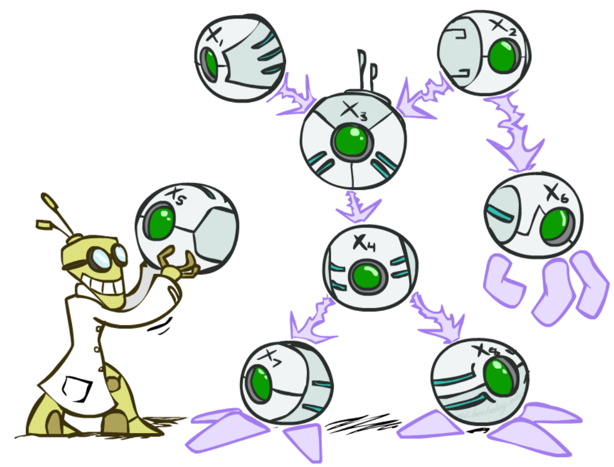


# Bayes' Nets: Big Picture

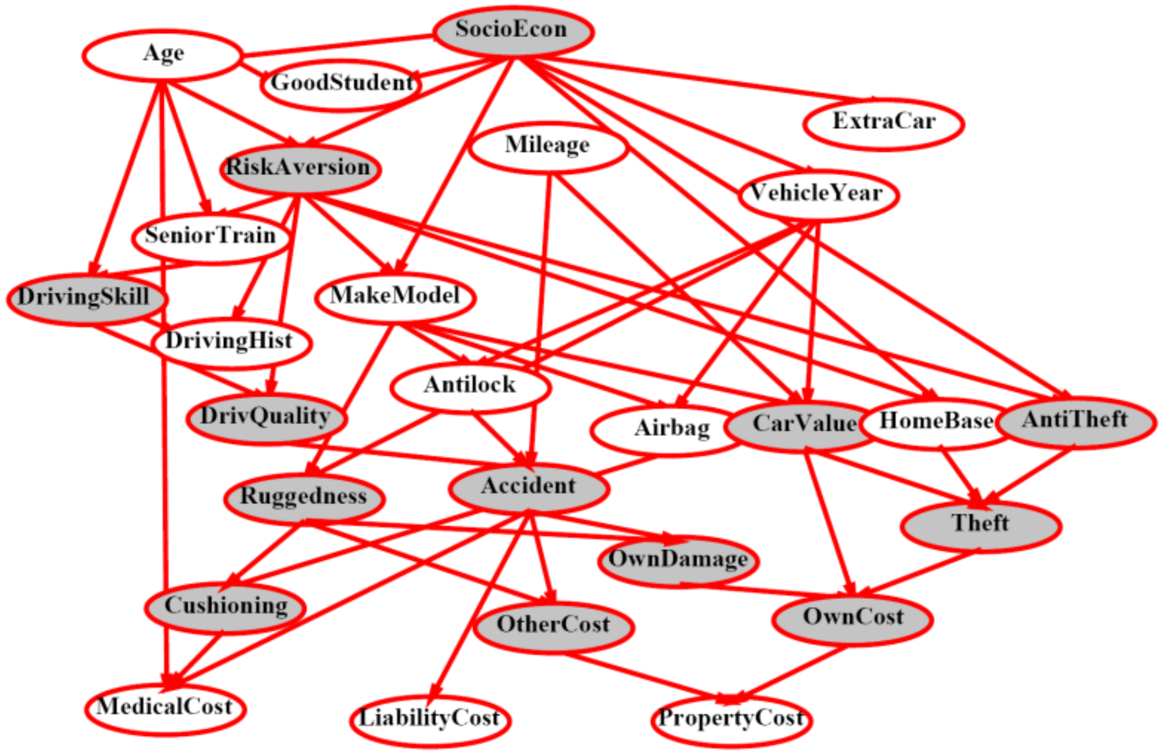
- Two problems with using full joint distribution tables as our probabilistic models:
  - Unless there are only a few variables, the joint is WAY too big to represent explicitly
  - Hard to learn (estimate) anything empirically about more than a few variables at a time



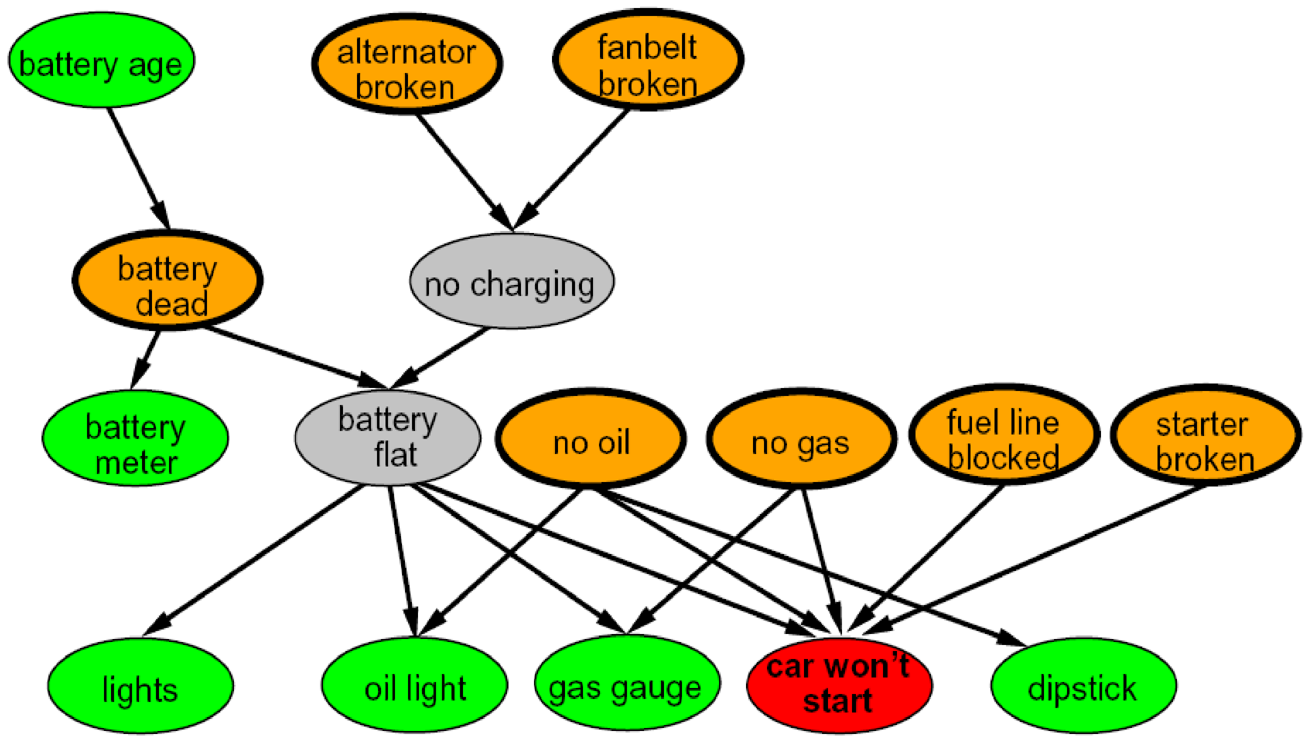
- Bayes' nets: a technique for describing complex joint distributions (models) using simple, local distributions (conditional probabilities)
  - More properly called graphical models
  - We describe how variables locally interact
  - Local interactions chain together to give global, indirect interactions
  - For about 10 min, we'll be vague about how these interactions are specified



# Example Bayes' Net: Insurance

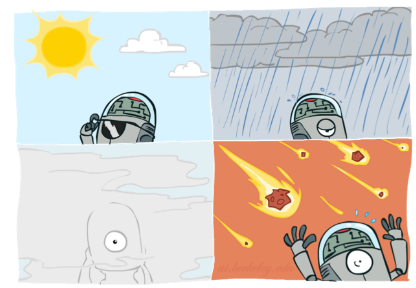


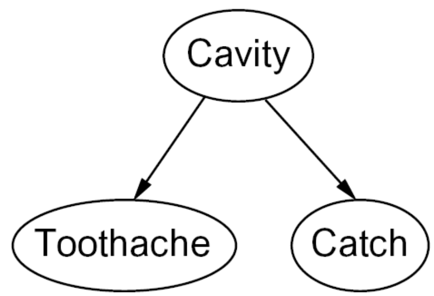
# Example Bayes' Net: Car

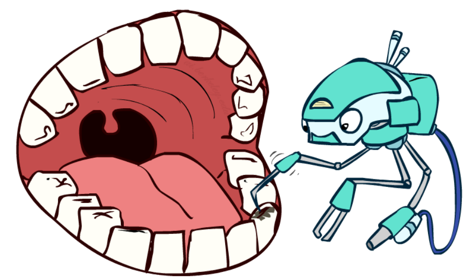


# Graphical Model Notation

- Nodes: variables (with domains)
  - Can be assigned (observed) or unassigned (unobserved)
- Arcs: interactions
  - Similar to CSP constraints
  - Indicate "direct influence" between variables
  - Formally: encode conditional independence (more later)
- For now: imagine that arrows mean direct causation (in general, they don't!)

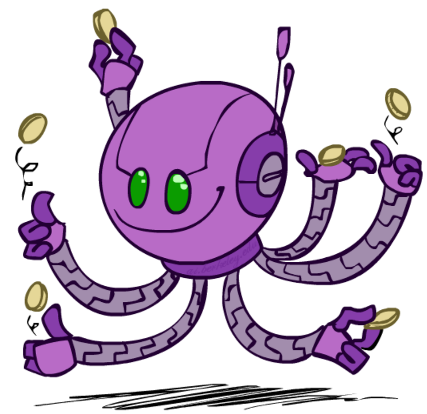





# Example: Coin Flips

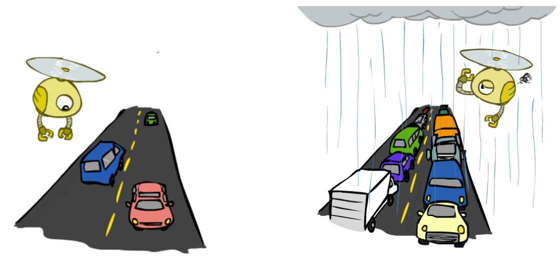
- N independent coin flips

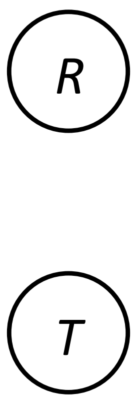
- No interactions between variables: absolute independence

# Example: Traffic

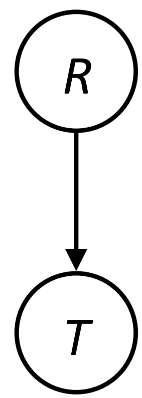
- Variables:
  - R: It rains
  - T: There is traffic



- Model 1: independence



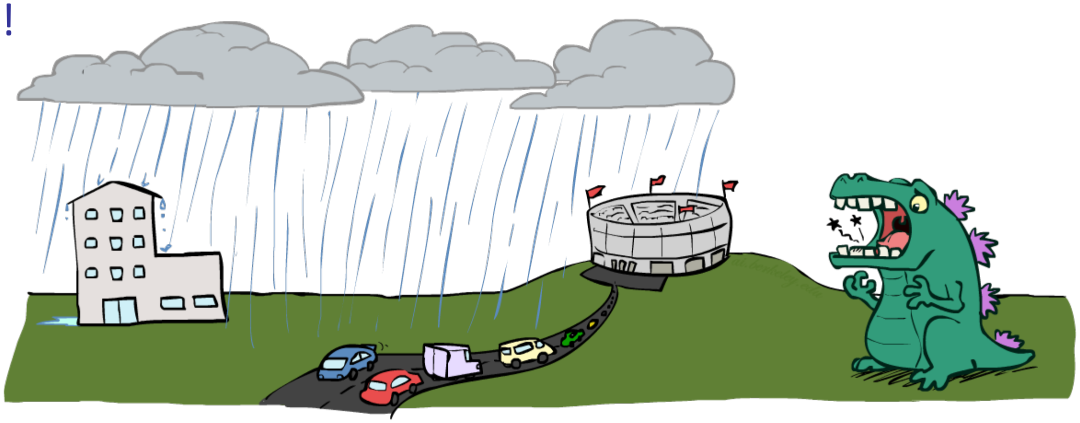

- Model 2: rain causes traffic




- Why is an agent using model 2 better?

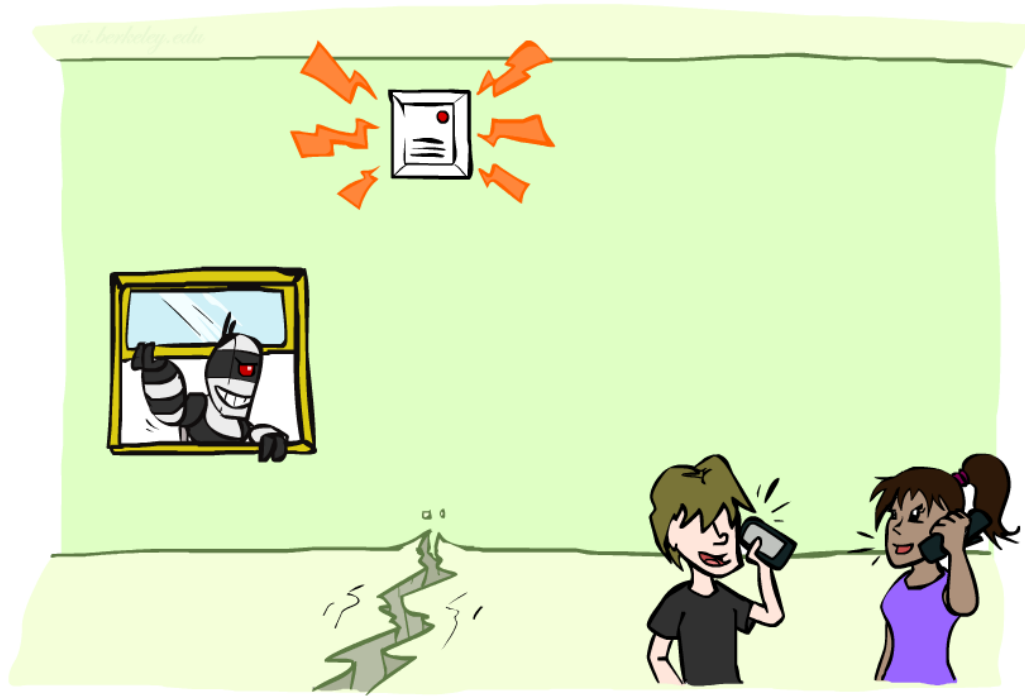
# Example: Traffic II

- Let's build a causal graphical model!
- Variables
  - T: Traffic
  - R: It rains
  - L: Low pressure
  - D: Roof drips
  - B: Ballgame
  - C: Cavity



# Example: Alarm Network

- Variables
  - B: Burglary
  - A: Alarm goes off
  - M: Mary calls
  - J: John calls
  - E: Earthquake!



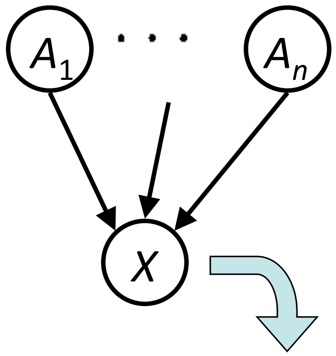
# Bayes' Net Semantics

# Bayes' Net Semantics

- A set of nodes, one per variable X
- A directed, acyclic graph
- A conditional distribution for each node
  - A collection of distributions over X, one for each combination of parents' values

    $$P(X|a_1 \ldots a_n)$$

  - CPT: conditional probability table
  - Description of a noisy "causal" process




$$P(X|A_1 \ldots A_n)$$

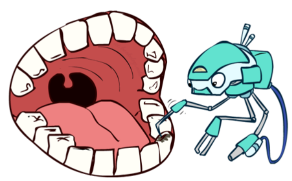
*A Bayes net = Topology (graph) + Local Conditional Probabilities*

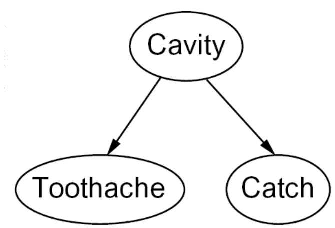
# Probabilities in BNs

- Bayes' nets **implicitly** encode joint distributions
  - As a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | \mathit{parents}(X_i))$$

  - Example:






$P(+\mathit{cavity}, +\mathit{catch}, -\mathit{toothache})$

# Probabilities in BNs

- Why are we guaranteed that setting

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | \textit{parents}(X_i))$$

  results in a proper joint distribution?

- Chain rule (valid for all distributions): $P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | x_1 \ldots x_{i-1})$

- <u>Assume</u> conditional independences: $P(x_i | x_1, \ldots x_{i-1}) = P(x_i | \textit{parents}(X_i))$

  → Consequence: $P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | \textit{parents}(X_i))$

- Not every BN can represent every joint distribution
  - The topology enforces certain conditional independencies

# Example: Coin Flips

$X_1$ $X_2$ $\cdots$ $X_n$

$P(X_1)$

| | |
|---|---|
| h | 0.5 |
| t | 0.5 |

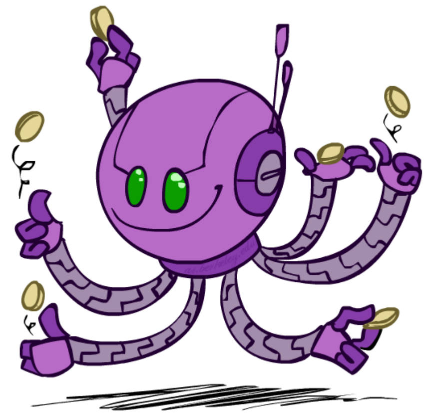
$P(X_2)$

| | |
|---|---|
| h | 0.5 |
| t | 0.5 |

$\cdots$

$P(X_n)$

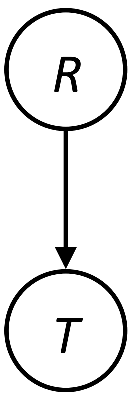
| | |
|---|---|
| h | 0.5 |
| t | 0.5 |



$$P(h, h, t, h) =$$

*Only distributions whose variables are absolutely independent can be represented by a Bayes' net with no arcs.*

# Example: Traffic




$P(R)$

| | |
|---|---|
| +r | 1/4 |
| -r | 3/4 |

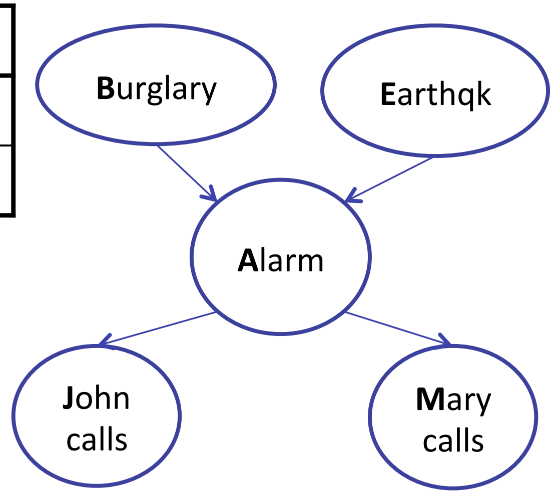
$P(T|R)$

| | | |
|---|---|---|
| +r | +t | 3/4 |
| | -t | 1/4 |
| -r | +t | 1/2 |
| | -t | 1/2 |

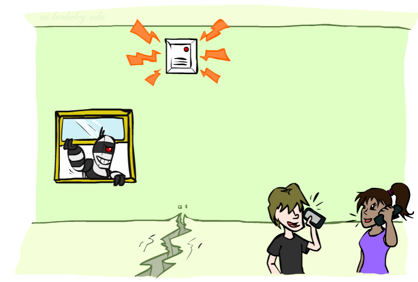
$$P(+r, -t) =$$

# Example: Alarm Network

| B | P(B) |
|---|---|
| +b | 0.001 |
| -b | 0.999 |




| E | P(E) |
|---|---|
| +e | 0.002 |
| -e | 0.998 |



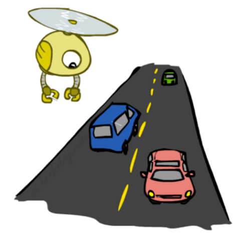
| A | J | P(J\|A) |
|---|---|---|
| +a | +j | 0.9 |
| +a | -j | 0.1 |
| -a | +j | 0.05 |
| -a | -j | 0.95 |

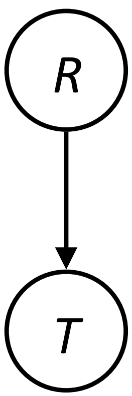
| A | M | P(M\|A) |
|---|---|---|
| +a | +m | 0.7 |
| +a | -m | 0.3 |
| -a | +m | 0.01 |
| -a | -m | 0.99 |

| B | E | A | P(A\|B,E) |
|---|---|---|---|
| +b | +e | +a | 0.95 |
| +b | +e | -a | 0.05 |
| +b | -e | +a | 0.94 |
| +b | -e | -a | 0.06 |
| -b | +e | +a | 0.29 |
| -b | +e | -a | 0.71 |
| -b | -e | +a | 0.001 |
| -b | -e | -a | 0.999 |

# Example: Traffic

- Causal direction






$P(R)$

| | |
|---|---|
| +r | 1/4 |
| -r | 3/4 |

$P(T|R)$

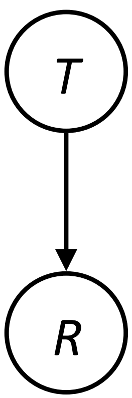
| | | |
|---|---|---|
| +r | +t | 3/4 |
| | -t | 1/4 |
| -r | +t | 1/2 |
| | -t | 1/2 |

$P(T, R)$

| | | |
|---|---|---|
| +r | +t | 3/16 |
| +r | -t | 1/16 |
| -r | +t | 6/16 |
| -r | -t | 6/16 |

# Example: Reverse Traffic

- Reverse causality?




$P(T)$

| | |
|---|---|
| +t | 9/16 |
| -t | 7/16 |

$P(R|T)$

| | | |
|---|---|---|
| +t | +r | 1/3 |
| | -r | 2/3 |
| -t | +r | 1/7 |
| | -r | 6/7 |

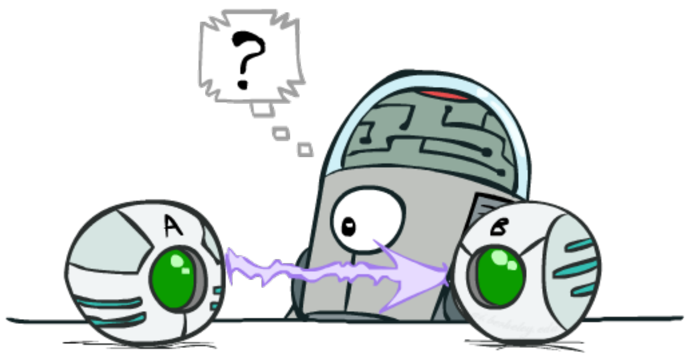
$P(T,R)$

| | | |
|---|---|---|
| +r | +t | 3/16 |
| +r | -t | 1/16 |
| -r | +t | 6/16 |
| -r | -t | 6/16 |

# Causality?

- When Bayes' nets reflect the true causal patterns:
  - Often simpler (nodes have fewer parents)
  - Often easier to think about
  - Often easier to elicit from experts
- BNs need not actually be causal
  - Sometimes no causal net exists over the domain (especially if variables are missing)
  - E.g. consider the variables *Traffic* and *Drips*
  - End up with arrows that reflect correlation, not causation
- What do the arrows really mean?
  - Topology may happen to encode causal structure
  - Topology really encodes conditional independence

$$P(x_i|x_1, \ldots x_{i-1}) = P(x_i|\mathit{parents}(X_i))$$



# Bayes' Nets

- So far: how a Bayes' net encodes a joint distribution
  - First assembled BNs using an intuitive notion of conditional independence as causality
  - Then saw that key property is conditional independence
- Next lecture: how to answer numerical queries (inference)