CSC 249/449 Machine Vision: Homework 3
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Problem 1
Draw the perpendicular from $P(\mu,\nu)$ to the $l: ax+by+c=0$.
$k_l = -\dfrac{a}{b}$
$\Rightarrow k_{PQ} = \dfrac{b}{a}$
$\Rightarrow PQ: y-\nu = \dfrac{b}{a}(x-\mu)$
$\begin{cases} y-\nu = \dfrac{b}{a}(x-\mu) \\ ax+by+c=0 \end{cases} \quad \Rightarrow Q(x,y) \begin{cases} x = \dfrac{b^2\mu - ab\nu - ac}{a^2+b^2} \\ y = \dfrac{a^2\nu - ab\mu - bc}{a^2+b^2} \end{cases}$
Therefore, $d = \left|PQ\right| = \sqrt{(\dfrac{b^2\mu - ab\nu - ac}{a^2+b^2} - \mu)^2 + (\dfrac{a^2\nu - ab\mu - bc}{a^2+b^2} - \nu)^2} = \dfrac{\left|a\mu + b\nu + c\right|}{\sqrt{a^2+b^2}}$
When $a^2+b^2=1, d = \left|a\mu+b\nu+c\right|$.

Problem 2
I derived Hough transform for circle case.
A circle centered at (a,b) with radius r is represented by

$$(x-a)^2 + (y-b)^2 = r^2$$

The Hough space for this equation is (a,b,r).
It can be represented as

$$a = x - r * \cos\theta$$
$$b = y - r * \sin\theta$$

The accumulator array is $A(r,a,b)$. It will record votes for coordinates and radius when looping through edge points. By setting a threshold and region size, it will only determine the best matching circle within a region. Therefore, there will be only one circle with the same center and radius. It is able to detect concentric circles. However, it will take too much computation by looping through angles and radius. The cv2.HoughCircles function will simplify computation by using gradient information, but it fails to detect concentric circles. It only detected one circle from the image. This is due to the algorithm it used. It draws lines along the magnitude orientation of edge points and determine the center of circles based on the lines' interaction. Circles with the same center (x, y) coordinates are considered identical, and only one of them with the largest radius is selected as the "best" fit. However, making minDist unreasonably small cannot solve the problem because many false circles due to noises and biases will also be detected. The drawback they have in common is when the outline of a circle is blurry, the circle is hard to detect. We can see it from the figures below. The cv2.HoughCircle failed to detect larger circles. My implementation also failed though it can detect concentric circles. Moreover, the input edge image will also have impact on the result of Hough transformation.

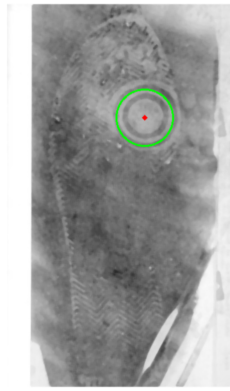

Fig1. Result of cv2.HoughCircle

Fig2. Result of my implementation(blur kernel size 3 & blur kernel size 5)

Problem 4

Use properties of trace and take first derivative w.r.t. W of $E = \left|\left|Y - WX\right|\right|^2 + \lambda\left|\left|W\right|\right|^2$ and set it to 0.
Trace properties I used are:
(1) $Tr(A + B) = TrA + TrB$
(2) $Tr(aA) = aTrA$, a is a constant
(3) $TrA = TrA^T$
(4) $Tr(AB) = Tr(BA)$
(5) $\nabla_A Tr(AB) = B^T$
(6) $\nabla_A Tr(ABA^TC) = CAB + C^TAB^T$
According to Frobenius norm, we get

$$\begin{aligned} E &= Tr((Y - WX)^T(Y - WX)) + \lambda Tr(W^TW) \\ &= Tr((Y^T - X^TW^T)(Y - WX)) + \lambda Tr(W^TW) \\ &= Tr(Y^TY - Y^TXW - X^TW^TY + X^TW^TWX) + \lambda Tr(W^TW) \end{aligned}$$

Use (1), (2), (3) and (4), we get

$$E = Tr(Y^TY - 2Y^TXW + X^TW^TWX + \lambda W^TW)$$

Take first derivative w.r.t. W, we get

$$\begin{aligned} \nabla_W E &= \nabla_W Tr(-2Y^TXW + X^TW^TWX + \lambda W^TW) \\ &= \nabla_W Tr((XX^T + \lambda I)W^TW) - 2\nabla_W Tr(Y^TXW) \end{aligned}$$

Use (5) and (6), for $\nabla_W Tr((XX^T + \lambda I)W^T W)$ part, we consider it as $\nabla_W Tr(WIW^T(XX^T + \lambda I))$, we get

$$\begin{aligned}
\nabla_W E &= (XX^T + \lambda I)WI + (XX^T + \lambda I)^T WI^T - 2\nabla_W Tr(Y^T XW) \\
&= 2(XX^T + \lambda I)W - 2YX^T
\end{aligned}$$

Let $\nabla_W E = 0$, we get

$$W = YX^T(XX^T + \lambda I)^{-1}$$