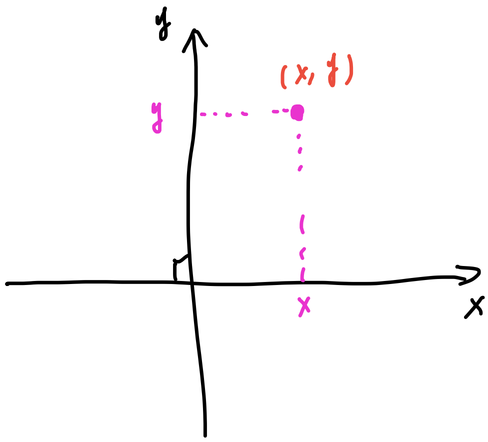
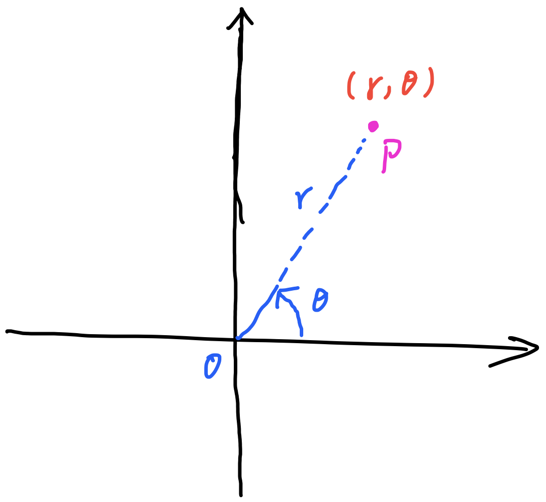
Cartesian coordinates V.S. Polar coordinates in 2D-plane





P (r, θ)

distance from O to P

directed angle from x-axis to OP

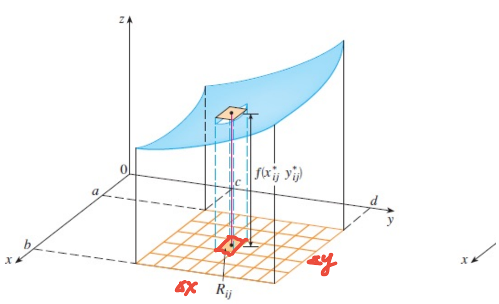
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

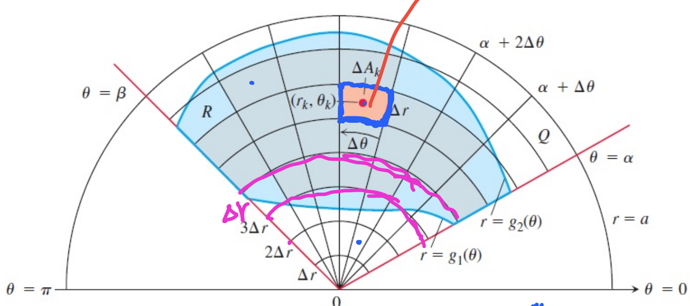
substitution.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta . \end{cases}$$

$$J(r,\theta) = \frac{\partial(x, y)}{\partial(r,\theta)} = r$$

$$\Delta A_k = r\,\Delta r\,\Delta\theta$$





$$\iint_{R^*} f(r,\theta)\, dR^* = \lim_{n\to\infty}\lim_{m\to\infty} \sum_{j=1}^{n}\sum_{i=1}^{m} f(r_i^*, \theta_j^*)\, r_i\, \Delta r_i\, \Delta\theta_j$$

Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$ $\Longleftrightarrow$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

$(r\cos\theta)^4 + (r\sin\theta)^4 + \cdots = 0$

$\Downarrow$ $\sin^2\theta + \cos^2\theta = 1$

Chapter 3. Multiple Integral

1 Three-Variable Case (Triple Integral):

1.1 Definition

For $w = f(x, y, z)$,

$$\iiint_V f(x,y,z)\,dV = \lim_{n\to\infty}\lim_{m\to\infty}\lim_{s\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{s}\underbrace{f(x_i^*, y_j^*, z_k^*)}_{\text{height}}\underbrace{\Delta x_i \Delta y_j \Delta z_k}_{\text{base}}$$

$$= \text{Volume of 4D solide}$$

1.2 Particular Interpretations:

- (a) If $f(x, y, z) \equiv 1$, then $\iiint_V 1dxdydz =$ **volume** of the region V.
- (b) If the scalar function $\rho(x, y, z)$ gives the density at a point (x, y, z) of the region V, then $\iiint_V \rho(x, y, z)dxdydz =$ **mass** of the region V.
- (c) If the scalar function $\rho(x, y, z)$ gives the charge density at a point (x, y, z) of the region V, then $\iiint_V \rho(x, y, z)dV =$ **total charge** within the region V.
- (d) $\iiint_V f(x,y,z)\,dV =$ Volume of a (4-dimensional) solid below the surface of f and above xyz-space

1.3 Computation of ~~Double~~ Triple Integrals:

Case 1: By iterated integral in some order directly.

$$\begin{cases} dx\,dy\,dz \\ dx\,dz\,dy \\ dy\,dx\,dz \quad ✓ \\ dy\,dz\,dx \\ dz\,dx\,dy \\ dz\,dy\,dx \end{cases}$$

$$\iiint_V f(x,y,z)\,dV = \lim_{n\to\infty}\lim_{m\to\infty}\lim_{s\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{s} f(x_i^*,y_j^*,z_k^*)\,\Delta x_i\,\Delta y_j\,\Delta z_k$$

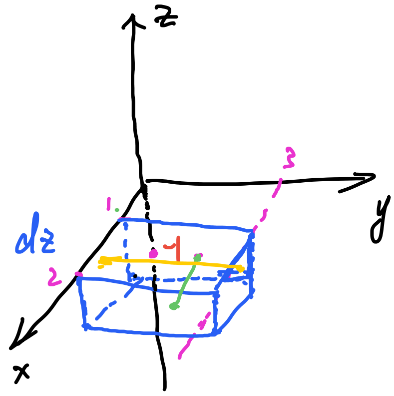
$$= \lim_{s\to\infty}\sum_{k=1}^{s}\left[\lim_{n\to\infty}\sum_{i=1}^{n}\left[\lim_{m\to\infty}\sum_{j=1}^{m} f(x_i^*,y_j^*,z_k^*)\,\Delta y_j\right]\Delta x_i\right]\Delta z_k$$

$$= \lim \sum\left[\lim \sum \int_{y=y_1(x,z)}^{y=y_2(x,z)} f(x_i^*,y,z_k^*)\,dy \quad \Delta x_i\right]\Delta z_k.$$

$$= \lim_{s\to\infty}\sum_{k=1}^{s}\left[\int_{x=x_1(z)}^{x=x_2(z)}\int_{y=y_1(x,z)}^{y=y_2(x,z)} f(x,y,z_k^*)\,dy \quad dx\right]\Delta z_k$$

$$= \int_a^b \int_{x=x_1(z)}^{x=x_2(z)} \int_{y=y_1(x,z)}^{y=y_2(x,z)} f(x,y,z)\,dy\,dx\,dz.$$

Example. The density of a rectangular blocks V bounded by the planes $x = 1, x = 2, y = 0, y = 3, z = -1, z = 0$ is given by the scalar function $\rho(x, y, z) = x(y + 1) - z$. Find the mass of the block.



$$\iiint_V x(y+1)-z \, dV = \int_{-1}^{0}\int_{0}^{3}\int_{1}^{2} [x(y+1)-z] \, dx \, dy \, dz$$

$$= \int_{-1}^{0}\int_{0}^{3} \left[(y+1)\frac{x^2}{2} - zx \right]\Bigg|_{x=1}^{x=2} dy \, dz$$

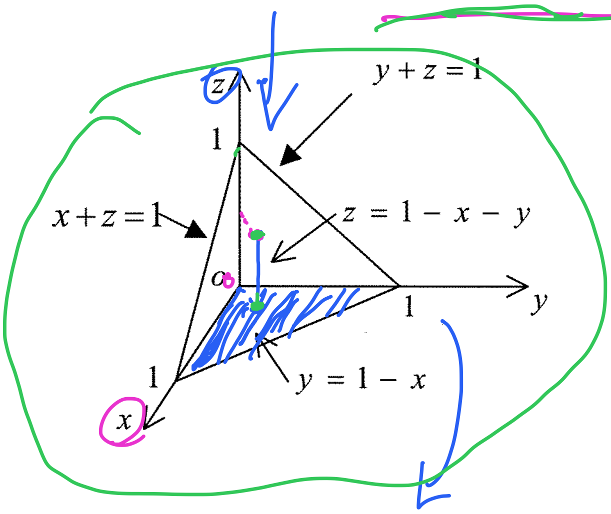
$$= \int_{-1}^{0}\int_{0}^{3} \left[2(y+1) - 2z - \frac{1}{2}(y+1) + z \right] dy \, dz.$$

$$= \cdots$$

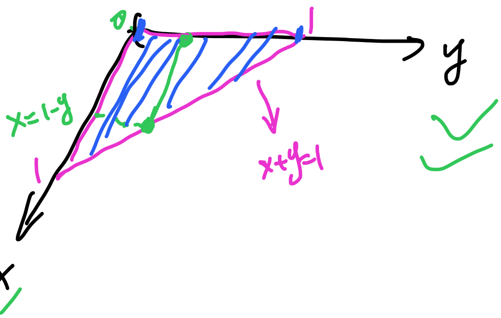
Example. Evaluate $\iiint_V \frac{1}{(x+y+2z+1)^3} dxdydz$ where V is the region enclosed by the planes

$x = 0, y = 0, z = 0, x + y + z = 1.$ $\Rightarrow z = 1 - x - y$

$\Rightarrow x = 1 - y - z$

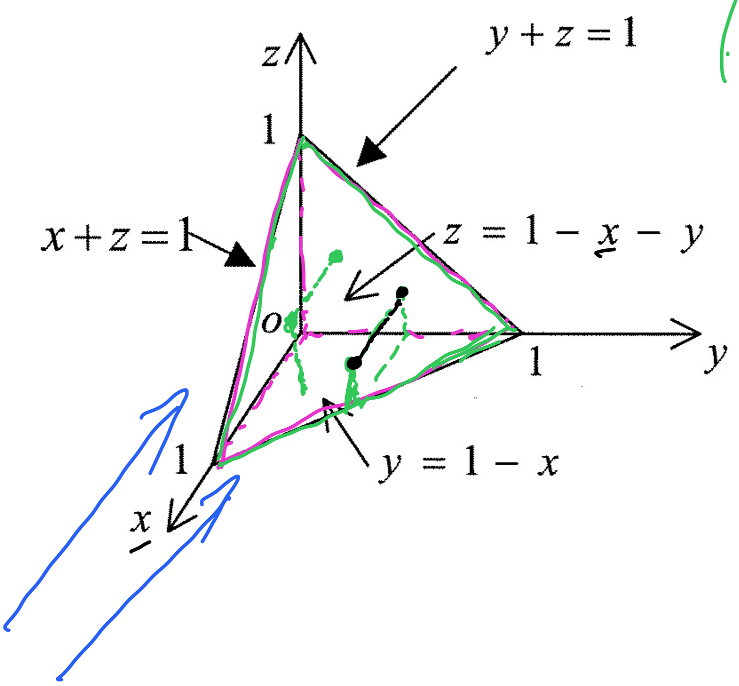


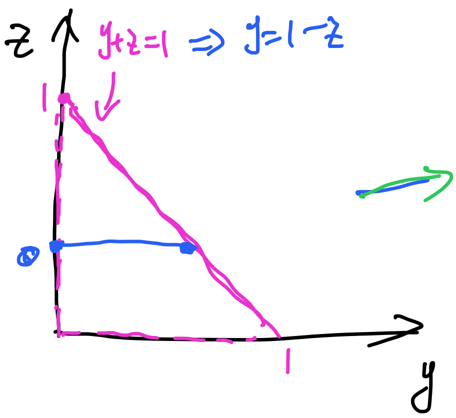
$$V = \left\{ \begin{array}{l} 0 \le z \le 1 - x - y \\ 0 \le x \le 1 - y \\ 0 \le y \le 1 \end{array} \right\}$$



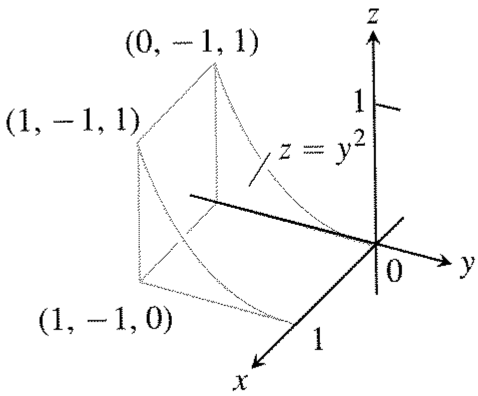
$$\iiint_V \frac{1}{(x+y+2z+1)^3} dV = \int_0^1 \int_0^{1-y} \int_0^{1-x-y} \frac{1}{\cdots} dz\, dx\, dy$$

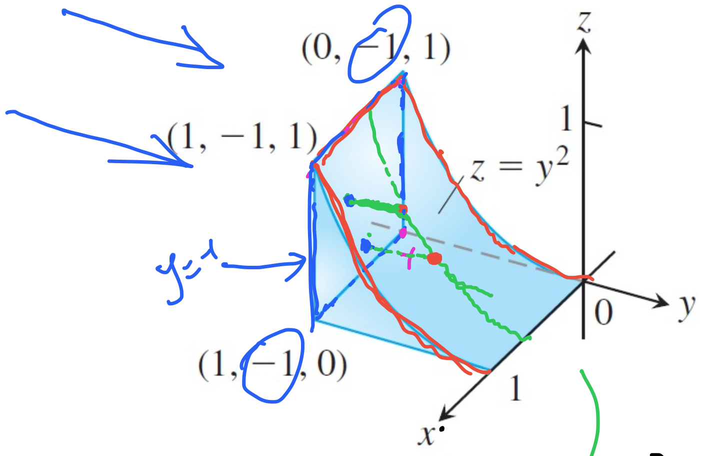
$$= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} \frac{1}{(x+y+2z+1)^3} dx\, dy\, dz.$$





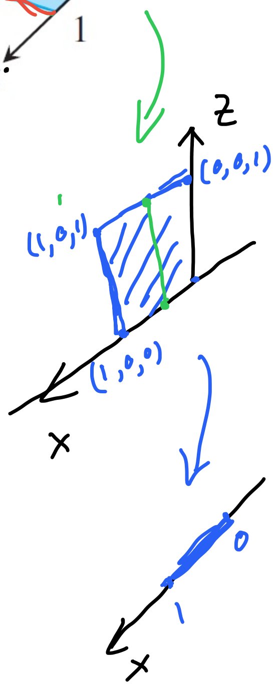
Example. With the aid of the following figure, change the order of the iterated integral $\int_0^1 \left[\int_{-1}^0 \left(\int_0^{y^2 = z} f(x,y,z)\,dz\right) dy\right] dx$ to an equivalent iterated integral with order $dydzdx$.





$$-1 \le y \le \sqrt{z}$$

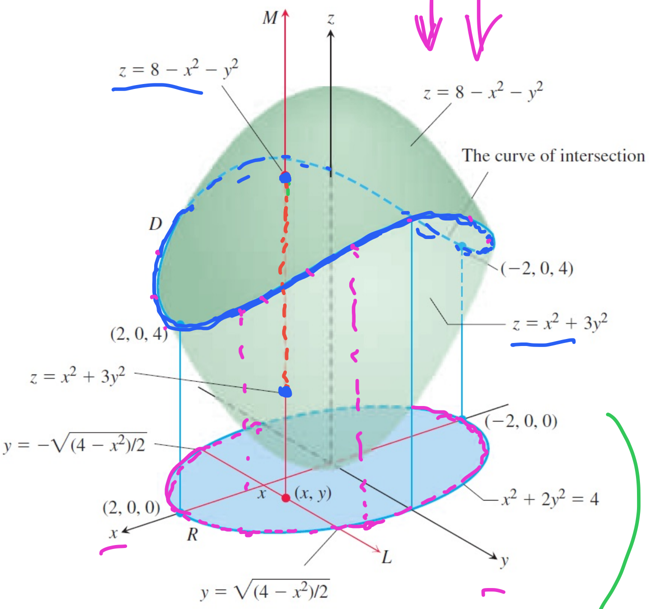
$$\begin{cases} 0 \le z \le 1 \\ 0 \le x \le 1 \end{cases}$$



$$\iiint_V f(x,y,z)\,dV = \int_0^1 \int_0^1 \int_{-1}^{\sqrt{z}} f(x,y,z)\,dy\,dz\,dx$$

Example. Find the volume of the solid D enclosed by $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

$dz\,dy\,dx$



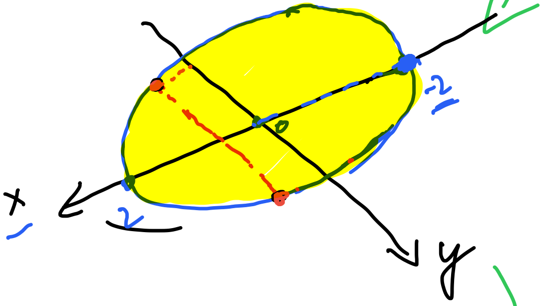
$$x^2+3y^2 \le z \le 8-x^2-y^2$$

$$x^2+3y^2 = 8-x^2-y^2$$

$$\Rightarrow 2x^2+4y^2=8$$

$$\Rightarrow y^2 = \frac{8-2x^2}{4}$$

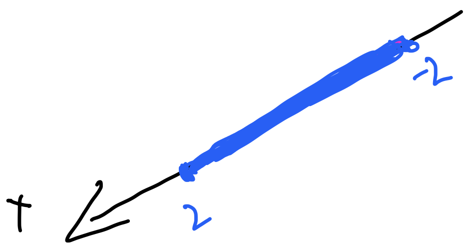
$$\Rightarrow y = \pm\sqrt{\frac{4-x^2}{2}}$$



$$-\sqrt{\frac{4-x^2}{2}} \le y \le \sqrt{\frac{4-x^2}{2}}$$

$$2x^2 \le 8-\cancel{0}^2$$

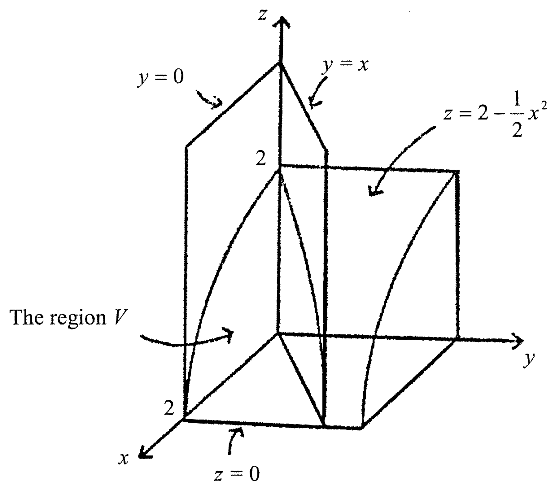
$$-2 \le x \le 2$$



$$\iiint_D 1\,dV = \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{x^2+3y^2}^{8-x^2-y^2} 1\,dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8-x^2-y^2-x^2-3y^2)\,dy\,dx = 6\pi$$

Example. Evaluate $\iiint_V 2xyz\, dV$ where V is the region bounded by the parabolic cylinder $z = 2 - \frac{1}{2}x^2$ and the planes $x = 0$, $y = x$ and $y = 0$, $z = 0$.



Example. Evaluate $\iiint_V xyz\ dV$, where V is the region enclosed by $x^2 + y^2 + z^2 = 1$ and $x \geq 0, z \geq 0, y \geq 0$ and $x = 0, y = 0, z = 0$.

Case 2. Substitution needed first.

For $I = \iiint_V f(x,y,z)dxdydz$, the change of variable $x = x(u,v,w)$, $y = (u,v,w)$, $z = z(u,v,w)$, gives,

$$I = \iiint_{V^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw$$

where $J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}_{3\times 3}$ is the Jacobian of the transformation. V^* is the region in uvw-space corresponding to the region V in xyz-space injectively (one to one) and J must be of one sign in V^*.

Example

$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=\frac{y}{2}+1} \left(\frac{2x-y}{2} + \frac{z}{3}\right)\, dx\, dy\, dz.$$

$$\begin{cases} u = \dfrac{2x-y}{2} \\ v = \dfrac{y}{2} \\ w = \dfrac{z}{3} \end{cases}$$

$$\underbrace{\frac{\partial(u,v,w)}{\partial(x,y,z)}} = \det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ - & - & - \end{pmatrix}$$

$$= \det\begin{pmatrix} 1. & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$= \frac{1}{6}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \left(\frac{1}{6}\right)^{-1} = 6$$

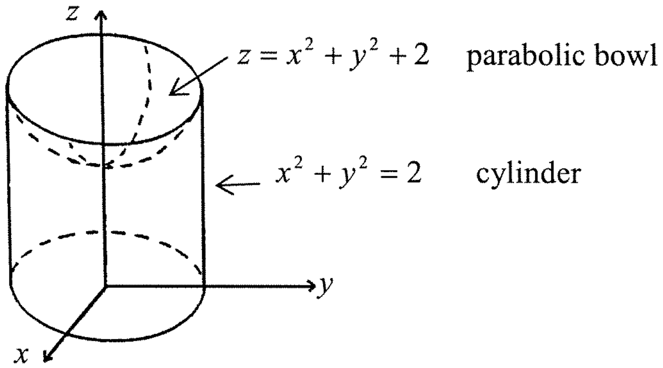
$$\iiint_V f(x,y,z)\, dV = \int\iint f(\quad) \cdot \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw.$$

xyz-boundary	uvw-boundary.	
$x = \frac{y}{2} + 1$	$= v$ ·	

The most popular alternative coordinate systems to rectangular coordinates are **cylindrical polar coordinates** and **spherical polar coordinates**. They induce the popularly used substitutions: cylindrical substitution and spherical substitution.

Cylindrical Polar Coordinates v.s. Rectangular Coordinates:

Example. Find the volume V between the surfaces $x^2 + y^2 = 2, z = x^2 + y^2 + 2$ and the plane $z = 0$.



Example. $\iiint_V z \, dV$ with V enclosed by $x^2 + y^2 = 4, \; z = x^2 + y^2$ and xy-plane.

Spherical Polar Coordinates:

Example.

Find the volume of the "ice cream cone" D cut from the solid sphere $x^2 + y^2 + z^2 = 1$ by the cone $x^2 + y^2 = 3z^2$ for $z \geq 0$.

Example 21

Find an equation for the sphere $x^2 + y^2 + (z - 1)^2 = 1$ under spherical coordinates.