45. $(x-1)^2+(y-2)^2+(z-3)^2=14$

46. $x^2+(y+1)^2+(z-5)^2=4$

47. $(x+2)^2+y^2+z^2=3$

48. $x^2+(y+7)^2+z^2=49$

49. $x^2+y^2+z^2+4x-4z=0 \Rightarrow (x^2+4x+4)+y^2+(z^2-4z+4)=4+4$
$\Rightarrow (x+2)^2+(y-0)^2+(z-2)^2=\left(\sqrt{8}\right)^2 \Rightarrow$ the center is at $(-2,0,2)$ and the radius is $\sqrt{8}$

50. $x^2+y^2+z^2-6y+8z=0 \Rightarrow x^2+(y^2-6y+9)+(z^2+8z+16)=9+16 \Rightarrow (x-0)^2+(y-3)^2+(z+4)^2=5^2$
$\Rightarrow$ the center is at $(0,3,-4)$ and the radius is 5

51. $2x^2+2y^2+2z^2+x+y+z=9 \Rightarrow x^2+\frac{1}{2}x+y^2+\frac{1}{2}y+z^2+\frac{1}{2}z=\frac{9}{2}$
$\Rightarrow \left(x^2+\frac{1}{2}x+\frac{1}{16}\right)+\left(y^2+\frac{1}{2}y+\frac{1}{16}\right)+\left(z^2+\frac{1}{2}z+\frac{1}{16}\right)=\frac{9}{2}+\frac{3}{16} \Rightarrow \left(x+\frac{1}{4}\right)^2+\left(y+\frac{1}{4}\right)^2+\left(z+\frac{1}{4}\right)^2=\left(\frac{5\sqrt{3}}{4}\right)^2$
$\Rightarrow$ the center is at $\left(-\frac{1}{4},-\frac{1}{4},-\frac{1}{4}\right)$ and the radius is $\frac{5\sqrt{3}}{4}$

52. $3x^2+3y^2+3z^2+2y-2z=9 \Rightarrow x^2+y^2+\frac{2}{3}y+z^2-\frac{2}{3}z=3 \Rightarrow x^2+\left(y^2+\frac{2}{3}y+\frac{1}{9}\right)+\left(z^2-\frac{2}{3}z+\frac{1}{9}\right)=3+\frac{2}{9}$
$\Rightarrow (x-0)^2+\left(y+\frac{1}{3}\right)^2+\left(z-\frac{1}{3}\right)^2=\left(\frac{\sqrt{29}}{3}\right)^2 \Rightarrow$ the center is at $\left(0,-\frac{1}{3},\frac{1}{3}\right)$ and the radius is $\frac{\sqrt{29}}{3}$

53. (a) the distance between (x,y,z) and $(x,0,0)$ is $\sqrt{y^2+z^2}$
(b) the distance between (x,y,z) and $(0,y,0)$ is $\sqrt{x^2+z^2}$
(c) the distance between (x,y,z) and $(0,0,z)$ is $\sqrt{x^2+y^2}$

54. (a) the distance between (x,y,z) and $(x,y,0)$ is z
(b) the distance between (x,y,z) and $(0,y,z)$ is x
(c) the distance between (x,y,z) and $(x,0,z)$ is y

55. $|AB|=\sqrt{(1-(-1))^2+(-1-2)^2+(3-1)^2}=\sqrt{4+9+4}=\sqrt{17}$
$|BC|=\sqrt{(3-1)^2+(4-(-1))^2+(5-3)^2}=\sqrt{4+25+4}=\sqrt{33}$
$|CA|=\sqrt{(-1-3)^2+(2-4)^2+(1-5)^2}=\sqrt{16+4+16}=\sqrt{36}=6$
Thus the perimeter of triangle ABC is $\sqrt{17}+\sqrt{33}+6$.

56. $|PA|=\sqrt{(2-3)^2+(-1-1)^2+(3-2)^2}=\sqrt{1+4+1}=\sqrt{6}$
$|PB|=\sqrt{(4-3)^2+(3-1)^2+(1-2)^2}=\sqrt{1+4+1}=\sqrt{6}$
Thus P is equidistant from A and B.

12.2 VECTORS

1. (a) $\langle 3(3), 3(-2)\rangle=\langle 9,-6\rangle$
(b) $\sqrt{9^2+(-6)^2}=\sqrt{117}=3\sqrt{13}$

2. (a) $\langle -2(-2), -2(5)\rangle=\langle 4,-10\rangle$
(b) $\sqrt{4^2+(-10)^2}=\sqrt{116}=2\sqrt{29}$

3. (a) $\langle 3+(-2), -2+5\rangle=\langle 1,3\rangle$
(b) $\sqrt{1^2+3^2}=\sqrt{10}$

4. (a) $\langle 3-(-2), -2-5\rangle=\langle 5,-7\rangle$
(b) $\sqrt{5^2+(-7)^2}=\sqrt{74}$

5. (a) $2\mathbf{u} = \langle 2(3), 2(-2)\rangle = \langle 6, -4\rangle$
$3\mathbf{v} = \langle 3(-2), 3(5)\rangle = \langle -6, 15\rangle$
$2\mathbf{u} - 3\mathbf{v} = \langle 6 - (-4), -4 - 15\rangle = \langle 12, -19\rangle$
(b) $\sqrt{12^2 + (-19)^2} = \sqrt{505}$

6. (a) $-2\mathbf{u} = \langle -2(3), -2(-2)\rangle = \langle -6, 4\rangle$
$5\mathbf{v} = \langle 5(-2), 5(5)\rangle = \langle -10, 25\rangle$
$-2\mathbf{u} + 5\mathbf{v} = \langle -6 + (-10), 4 + 25\rangle = \langle -16, 29\rangle$
(b) $\sqrt{(-16)^2 + 29^2} = \sqrt{1097}$

7. (a) $\frac{3}{5}\mathbf{u} = \left\langle \frac{3}{5}(3), \frac{3}{5}(-2)\right\rangle = \left\langle \frac{9}{5}, -\frac{6}{5}\right\rangle$
$\frac{4}{5}\mathbf{v} = \left\langle \frac{4}{5}(-2), \frac{4}{5}(5)\right\rangle = \left\langle -\frac{8}{5}, 4\right\rangle$
$\frac{3}{5}\mathbf{u} + \frac{4}{5}\mathbf{v} = \left\langle \frac{9}{5} + \left(-\frac{8}{5}\right), -\frac{6}{5} + 4\right\rangle = \left\langle \frac{1}{5}, \frac{14}{5}\right\rangle$
(b) $\sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{14}{5}\right)^2} = \frac{\sqrt{197}}{5}$

8. (a) $-\frac{5}{13}\mathbf{u} = \left\langle -\frac{5}{13}(3), -\frac{5}{13}(-2)\right\rangle = \left\langle -\frac{15}{13}, \frac{10}{13}\right\rangle$
$\frac{12}{13}\mathbf{v} = \left\langle \frac{12}{13}(-2), \frac{12}{13}(5)\right\rangle = \left\langle -\frac{24}{13}, \frac{60}{13}\right\rangle$
$-\frac{5}{13}\mathbf{u} + \frac{12}{13}\mathbf{v} = \left\langle -\frac{15}{13} + \left(-\frac{24}{13}\right), \frac{10}{13} + \frac{60}{13}\right\rangle = \left\langle -3, \frac{70}{13}\right\rangle$
(b) $\sqrt{(-3)^2 + \left(\frac{70}{13}\right)^2} = \frac{\sqrt{6421}}{13}$

9. $\langle 2 - 1, -1 - 3\rangle = \langle 1, -4\rangle$

10. $\left\langle \frac{2+(-4)}{2} - 0, \frac{-1+3}{2} - 0\right\rangle = \langle -1, 1\rangle$

11. $\langle 0 - 2, 0 - 3\rangle = \langle -2, -3\rangle$

12. $\overrightarrow{AB} = \langle 2 - 1, 0 - (-1)\rangle = \langle 1, 1\rangle$, $\overrightarrow{CD} = \langle -2 - (-1), 2 - 3\rangle = \langle -1, -1\rangle$, $\overrightarrow{AB} + \overrightarrow{CD} = \langle 0, 0\rangle$

13. $\left\langle \cos\frac{2\pi}{3}, \sin\frac{2\pi}{3}\right\rangle = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle$

14. $\left\langle \cos\left(-\frac{3\pi}{4}\right), \sin\left(-\frac{3\pi}{4}\right)\right\rangle = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\rangle$

15. This is the unit vector which makes an angle of $120° + 90° = 210°$ with the positive x-axis;
$\langle \cos 210°, \sin 210°\rangle = \left\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2}\right\rangle$

16. $\langle \cos 135°, \sin 135°\rangle = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$

17. $\overrightarrow{P_1P_2} = (2 - 5)\mathbf{i} + (9 - 7)\mathbf{j} + (-2 - (-1))\mathbf{k} = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

18. $\overrightarrow{P_1P_2} = (-3 - 1)\mathbf{i} + (0 - 2)\mathbf{j} + (5 - 0)\mathbf{k} = -4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$

19. $\overrightarrow{AB} = (-10 - (-7))\mathbf{i} + (8 - (-8))\mathbf{j} + (1 - 1)\mathbf{k} = -3\mathbf{i} + 16\mathbf{j}$

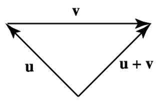
20. $\overrightarrow{AB} = (-1 - 1)\mathbf{i} + (4 - 0)\mathbf{j} + (5 - 3)\mathbf{k} = -2\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

21. $5\mathbf{u} - \mathbf{v} = 5\langle 1, 1, -1\rangle - \langle 2, 0, 3\rangle = \langle 5, 5, -5\rangle - \langle 2, 0, 3\rangle = \langle 5 - 2, 5 - 0, -5 - 3\rangle = \langle 3, 5, -8\rangle = 3\mathbf{i} + 5\mathbf{j} - 8\mathbf{k}$

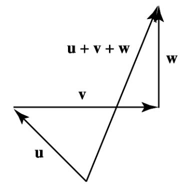
22. $-2\mathbf{u} + 3\mathbf{v} = -2\langle -1, 0, 2\rangle + 3\langle 1, 1, 1\rangle = \langle 2, 0, -4\rangle + \langle 3, 3, 3\rangle = \langle 5, 3, -1\rangle = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

23. The vector $\mathbf{v}$ is horizontal and 1 in. long. The vectors $\mathbf{u}$ and $\mathbf{w}$ are $\frac{11}{16}$ in. long. $\mathbf{w}$ is vertical and $\mathbf{u}$ makes a 45° angle with the horizontal. All vectors must be drawn to scale.

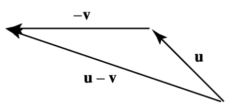
(a)



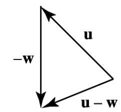
(b)



(c)

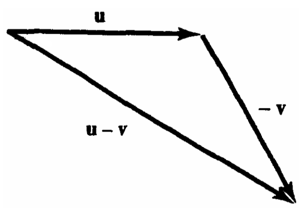


(d)

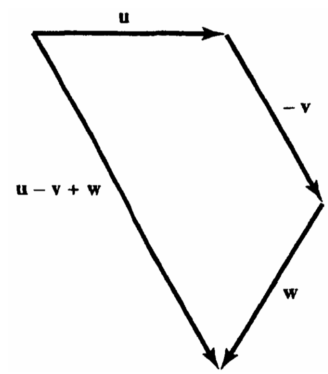


24. The angle between the vectors is 120° and vector $\mathbf{u}$ is horizontal. They are all 1 in. long. Draw to scale.

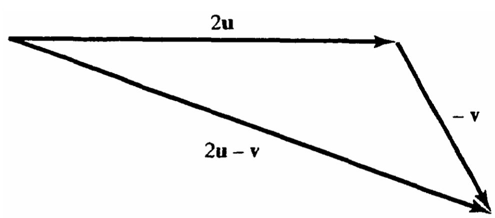
(a)



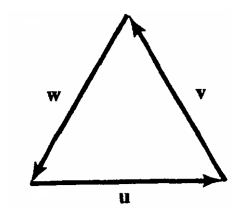
(b)



(c)



(d)



$\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$

25. length $= |2\mathbf{i} + \mathbf{j} - 2\mathbf{k}| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$, the direction is $\frac{2}{3}\,\mathbf{i} + \frac{1}{3}\,\mathbf{j} - \frac{2}{3}\,\mathbf{k} \;\Rightarrow\; 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} = 3\left(\frac{2}{3}\,\mathbf{i} + \frac{1}{3}\,\mathbf{j} - \frac{2}{3}\,\mathbf{k}\right)$

26. length $= |9\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}| = \sqrt{81 + 4 + 36} = 11$, the direction is $\frac{9}{11}\,\mathbf{i} - \frac{2}{11}\,\mathbf{j} + \frac{6}{11}\,\mathbf{k} \;\Rightarrow\; 9\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$
$= 11\left(\frac{9}{11}\,\mathbf{i} - \frac{2}{11}\,\mathbf{j} + \frac{6}{11}\,\mathbf{k}\right)$

27. length $= |5\mathbf{k}| = \sqrt{25} = 5$, the direction is $\mathbf{k} \;\Rightarrow\; 5\mathbf{k} = 5(\mathbf{k})$

28. length $= \left|\frac{3}{5}\,\mathbf{i} + \frac{4}{5}\,\mathbf{k}\right| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$, the direction is $\frac{3}{5}\,\mathbf{i} + \frac{4}{5}\,\mathbf{k} \;\Rightarrow\; \frac{3}{5}\,\mathbf{i} + \frac{4}{5}\,\mathbf{k} = 1\left(\frac{3}{5}\,\mathbf{i} + \frac{4}{5}\,\mathbf{k}\right)$

29. length $= \left|\frac{1}{\sqrt{6}}\,\mathbf{i} - \frac{1}{\sqrt{6}}\,\mathbf{j} - \frac{1}{\sqrt{6}}\,\mathbf{k}\right| = \sqrt{3\left(\frac{1}{\sqrt{6}}\right)^2} = \sqrt{\frac{1}{2}}$, the direction is $\frac{1}{\sqrt{3}}\,\mathbf{i} - \frac{1}{\sqrt{3}}\,\mathbf{j} - \frac{1}{\sqrt{3}}\,\mathbf{k}$
$\Rightarrow\; \frac{1}{\sqrt{6}}\,\mathbf{i} - \frac{1}{\sqrt{6}}\,\mathbf{j} - \frac{1}{\sqrt{6}}\,\mathbf{k} = \sqrt{\frac{1}{2}}\left(\frac{1}{\sqrt{3}}\,\mathbf{i} - \frac{1}{\sqrt{3}}\,\mathbf{j} - \frac{1}{\sqrt{3}}\,\mathbf{k}\right)$

30. length $= \left|\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right| = \sqrt{3\left(\frac{1}{\sqrt{3}}\right)^2} = 1$, the direction is $\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$
$\Rightarrow \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k} = 1\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right)$

31. (a) $2\mathbf{i}$ (b) $-\sqrt{3}\mathbf{k}$ (c) $\frac{3}{10}\mathbf{j} + \frac{2}{5}\mathbf{k}$ (d) $6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

32. (a) $-7\mathbf{j}$ (b) $-\frac{3\sqrt{2}}{5}\mathbf{i} - \frac{4\sqrt{2}}{5}\mathbf{k}$ (c) $\frac{1}{4}\mathbf{i} - \frac{1}{3}\mathbf{j} - \mathbf{k}$ (d) $\frac{a}{\sqrt{2}}\mathbf{i} + \frac{a}{\sqrt{3}}\mathbf{j} - \frac{a}{\sqrt{6}}\mathbf{k}$

33. $|\mathbf{v}| = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$; $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{13}\mathbf{v} = \frac{1}{13}(12\mathbf{i} - 5\mathbf{k}) \Rightarrow$ the desired vector is $\frac{7}{13}(12\mathbf{i} - 5\mathbf{k})$

34. $|\mathbf{v}| = \sqrt{\frac{1}{4} + \frac{1}{4} + \frac{1}{4}} = \frac{\sqrt{3}}{2}$; $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k} \Rightarrow$ the desired vector is $-3\left(\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right)$
$= -\sqrt{3}\mathbf{i} + \sqrt{3}\mathbf{j} + \sqrt{3}\mathbf{k}$

35. (a) $3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} = 5\sqrt{2}\left(\frac{3}{5\sqrt{2}}\mathbf{i} + \frac{4}{5\sqrt{2}}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}\right) \Rightarrow$ the direction is $\frac{3}{5\sqrt{2}}\mathbf{i} + \frac{4}{5\sqrt{2}}\mathbf{j} - \frac{1}{\sqrt{2}}\mathbf{k}$
(b) the midpoint is $\left(\frac{1}{2}, 3, \frac{5}{2}\right)$

36. (a) $3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} = 7\left(\frac{3}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}\right) \Rightarrow$ the direction is $\frac{3}{7}\mathbf{i} - \frac{6}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}$
(b) the midpoint is $\left(\frac{5}{2}, 1, 6\right)$

37. (a) $-\mathbf{i} - \mathbf{j} - \mathbf{k} = \sqrt{3}\left(-\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right) \Rightarrow$ the direction is $-\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$
(b) the midpoint is $\left(\frac{5}{2}, \frac{7}{2}, \frac{9}{2}\right)$

38. (a) $2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k} = 2\sqrt{3}\left(\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}\right) \Rightarrow$ the direction is $\frac{1}{\sqrt{3}}\mathbf{i} - \frac{1}{\sqrt{3}}\mathbf{j} - \frac{1}{\sqrt{3}}\mathbf{k}$
(b) the midpoint is $(1, -1, -1)$

39. $\overrightarrow{AB} = (5 - a)\mathbf{i} + (1 - b)\mathbf{j} + (3 - c)\mathbf{k} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \Rightarrow 5 - a = 1, 1 - b = 4$, and $3 - c = -2 \Rightarrow a = 4, b = -3$, and $c = 5 \Rightarrow$ A is the point $(4, -3, 5)$

40. $\overrightarrow{AB} = (a + 2)\mathbf{i} + (b + 3)\mathbf{j} + (c - 6)\mathbf{k} = -7\mathbf{i} + 3\mathbf{j} + 8\mathbf{k} \Rightarrow a + 2 = -7, b + 3 = 3$, and $c - 6 = 8 \Rightarrow a = -9, b = 0$, and $c = 14 \Rightarrow$ B is the point $(-9, 0, 14)$

41. $2\mathbf{i} + \mathbf{j} = a(\mathbf{i} + \mathbf{j}) + b(\mathbf{i} - \mathbf{j}) = (a + b)\mathbf{i} + (a - b)\mathbf{j} \Rightarrow a + b = 2$ and $a - b = 1 \Rightarrow 2a = 3 \Rightarrow a = \frac{3}{2}$ and $b = a - 1 = \frac{1}{2}$

42. $\mathbf{i} - 2\mathbf{j} = a(2\mathbf{i} + 3\mathbf{j}) + b(\mathbf{i} + \mathbf{j}) = (2a + b)\mathbf{i} + (3a + b)\mathbf{j} \Rightarrow 2a + b = 1$ and $3a + b = -2 \Rightarrow a = -3$ and $b = 1 - 2a = 7 \Rightarrow \mathbf{u}_1 = a(2\mathbf{i} + 3\mathbf{j}) = -6\mathbf{i} - 9\mathbf{j}$ and $\mathbf{u}_2 = b(\mathbf{i} + \mathbf{j}) = 7\mathbf{i} + 7\mathbf{j}$

43. If $|x|$ is the magnitude of the x-component, then $\cos 30° = \frac{|x|}{|F|} \Rightarrow |x| = |F| \cos 30° = (10)\left(\frac{\sqrt{3}}{2}\right) = 5\sqrt{3}$ lb
$\Rightarrow \mathbf{F}_x = 5\sqrt{3}\,\mathbf{i}$;
if $|y|$ is the magnitude of the y-component, then $\sin 30° = \frac{|y|}{|F|} \Rightarrow |y| = |F| \sin 30° = (10)\left(\frac{1}{2}\right) = 5$ lb $\Rightarrow \mathbf{F}_y = 5\,\mathbf{j}$.

44. If $|x|$ is the magnitude of the x-component, then $\cos 45° = \frac{|x|}{|F|} \Rightarrow |x| = |F| \cos 45° = (12)\left(\frac{\sqrt{2}}{2}\right) = 6\sqrt{2}$ lb
$\Rightarrow \mathbf{F}_x = -6\sqrt{2}\,\mathbf{i}$ (the negative sign is indicated by the diagram)
if $|y|$ is the magnitude of the y-component, then $\sin 45° = \frac{|y|}{|F|} \Rightarrow |y| = |F| \sin 45° = (12)\left(\frac{\sqrt{2}}{2}\right) = 6\sqrt{2}$ lb
$\Rightarrow \mathbf{F}_y = -6\sqrt{2}\,\mathbf{j}$ (the negative sign is indicated by the diagram)

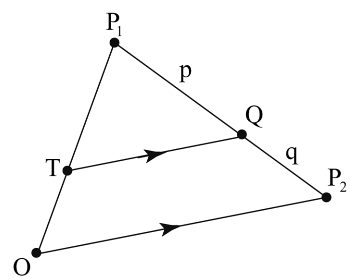
45. 25° west of north is $90° + 25° = 115°$ north of east. $800\langle \cos 155°, \sin 115° \rangle \approx \langle -338.095, 725.046 \rangle$

46. 10° east of south is $270° + 10° = 280°$ "north" of east. $600\langle \cos 280°, \sin 280° \rangle \approx \langle 104.189, -590.885 \rangle$

47. (a) The tree is located at the tip of the vector $\overrightarrow{OP} = (5 \cos 60°)\mathbf{i} + (5 \sin 60°)\mathbf{j} = \frac{5}{2}\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j} \Rightarrow P = \left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$

(b) The telephone pole is located at the point Q, which is the tip of the vector $\overrightarrow{OP} + \overrightarrow{PQ}$
$= \left(\frac{5}{2}\mathbf{i} + \frac{5\sqrt{3}}{2}\mathbf{j}\right) + (10 \cos 315°)\mathbf{i} + (10 \sin 315°)\mathbf{j} = \left(\frac{5}{2} + \frac{10\sqrt{2}}{2}\right)\mathbf{i} + \left(\frac{5\sqrt{3}}{2} - \frac{10\sqrt{2}}{2}\right)\mathbf{j}$
$\Rightarrow Q = \left(\frac{5+10\sqrt{2}}{2}, \frac{5\sqrt{3}-10\sqrt{2}}{2}\right)$

48. Let $t = \frac{q}{p+q}$ and $s = \frac{p}{p+q}$. Choose T on $\overline{OP_1}$ so that $\overline{TQ}$ is parallel to $\overline{OP_2}$, so that $\triangle TP_1Q$ is similar to $\triangle OP_1P_2$. Then $\frac{|OT|}{|OP_1|} = t \Rightarrow \overrightarrow{OT} = t\,\overrightarrow{OP_1}$ so that $T = (t\,x_1, t\,y_1, t\,z_1)$.
Also, $\frac{|TQ|}{|OP_2|} = s \Rightarrow \overrightarrow{TQ} = s\,\overrightarrow{OP_2} = s\langle x_2, y_2, z_2 \rangle$.
Letting $Q = (x, y, z)$, we have that
$\overrightarrow{TQ} = \langle x - t\,x_1, y - t\,y_1, z - t\,z_1 \rangle = s\langle x_2, y_2, z_2 \rangle$
Thus $x = t\,x_1 + s\,x_2$, $y = t\,y_1 + s\,y_2$, $z = t\,z_1 + s\,z_2$.
(Note that if Q is the midpoint, then $\frac{p}{q} = 1$ and $t = s = \frac{1}{2}$
so that $x = \frac{1}{2}x_1 + \frac{1}{2}x_2 = \frac{x_1+x_2}{2}$, $y = \frac{y_1+y_2}{2}$, $z = \frac{z_1+z_2}{2}$ so that this result agress with the midpoint formula.)



49. (a) the midpoint of AB is $M\left(\frac{5}{2}, \frac{5}{2}, 0\right)$ and $\overrightarrow{CM} = \left(\frac{5}{2} - 1\right)\mathbf{i} + \left(\frac{5}{2} - 1\right)\mathbf{j} + (0 - 3)\mathbf{k} = \frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\mathbf{k}$

(b) the desired vector is $\left(\frac{2}{3}\right)\overrightarrow{CM} = \frac{2}{3}\left(\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} - 3\mathbf{k}\right) = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$

(c) the vector whose sum is the vector from the origin to C and the result of part (b) will terminate at the center of mass $\Rightarrow$ the terminal point of $(\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ is the point $(2, 2, 1)$, which is the location of the center of mass

50. The midpoint of AB is $M\left(\frac{3}{2}, 0, \frac{5}{2}\right)$ and $\left(\frac{2}{3}\right)\overrightarrow{CM} = \frac{2}{3}\left[\left(\frac{3}{2} + 1\right)\mathbf{i} + (0 - 2)\mathbf{j} + \left(\frac{5}{2} + 1\right)\mathbf{k}\right] = \frac{2}{3}\left(\frac{5}{2}\mathbf{i} - 2\mathbf{j} + \frac{7}{2}\mathbf{k}\right)$
$= \frac{5}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} + \frac{7}{3}\mathbf{k}$. The terminal point of $\left(\frac{5}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} + \frac{7}{3}\mathbf{k}\right) + \overrightarrow{OC} = \left(\frac{5}{3}\mathbf{i} - \frac{4}{3}\mathbf{j} + \frac{7}{3}\mathbf{k}\right) + (-\mathbf{i} + 2\mathbf{j} - \mathbf{k})$
$= \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{4}{3}\mathbf{k}$ is the point $\left(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}\right)$ which is the location of the intersection of the medians.

51. Without loss of generality we identify the vertices of the quadrilateral such that $A(0, 0, 0)$, $B(x_b, 0, 0)$, $C(x_c, y_c, 0)$ and $D(x_d, y_d, z_d) \Rightarrow$ the midpoint of AB is $M_{AB}\left(\frac{x_b}{2}, 0, 0\right)$, the midpoint of BC is $M_{BC}\left(\frac{x_b + x_c}{2}, \frac{y_c}{2}, 0\right)$, the midpoint of CD is $M_{CD}\left(\frac{x_c + x_d}{2}, \frac{y_c + y_d}{2}, \frac{z_d}{2}\right)$ and the midpoint of AD is $M_{AD}\left(\frac{x_d}{2}, \frac{y_d}{2}, \frac{z_d}{2}\right) \Rightarrow$ the midpoint of $M_{AB}M_{CD}$ is $\left(\frac{\frac{x_b}{2} + \frac{x_c + x_d}{2}}{2}, \frac{y_c + y_d}{4}, \frac{z_d}{4}\right)$ which is the same as the midpoint of $M_{AD}M_{BC} = \left(\frac{\frac{x_b + x_c}{2} + \frac{x_d}{2}}{2}, \frac{y_c + y_d}{4}, \frac{z_d}{4}\right)$.

52. Let $V_1, V_2, V_3, \ldots, V_n$ be the vertices of a regular n-sided polygon and $\mathbf{v}_i$ denote the vector from the center to V_i for $i = 1, 2, 3, \ldots, n$. If $\mathbf{S} = \sum_{i=1}^{n} \mathbf{v}_i$ and the polygon is rotated through an angle of $\frac{i(2\pi)}{n}$ where $i = 1, 2, 3, \ldots, n$, then $\mathbf{S}$ would remain the same. Since the vector $\mathbf{S}$ does not change with these rotations we conclude that $\mathbf{S} = \mathbf{0}$.

53. Without loss of generality we can coordinatize the vertices of the triangle such that $A(0,0)$, $B(b,0)$ and $C(x_c, y_c) \Rightarrow$ a is located at $\left(\frac{b+x_c}{2}, \frac{y_c}{2}\right)$, b is at $\left(\frac{x_c}{2}, \frac{y_c}{2}\right)$ and c is at $\left(\frac{b}{2}, 0\right)$. Therefore, $\overrightarrow{Aa} = \left(\frac{b}{2} + \frac{x_c}{2}\right)\mathbf{i} + \left(\frac{y_c}{2}\right)\mathbf{j}$, $\overrightarrow{Bb} = \left(\frac{x_c}{2} - b\right)\mathbf{i} + \left(\frac{y_c}{2}\right)\mathbf{j}$, and $\overrightarrow{Cc} = \left(\frac{b}{2} - x_c\right)\mathbf{i} + (-y_c)\mathbf{j} \Rightarrow \overrightarrow{Aa} + \overrightarrow{Bb} + \overrightarrow{Cc} = \mathbf{0}$.

54. Let $\mathbf{u}$ be any unit vector in the plane. If $\mathbf{u}$ is positioned so that its initial point is at the origin and terminal point is at (x, y), then $\mathbf{u}$ makes an angle θ with $\mathbf{i}$, measured in the counter-clockwise direction. Since $|\mathbf{u}| = 1$, we have that $x = \cos\theta$ and $y = \sin\theta$. Thus $\mathbf{u} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$. Since $\mathbf{u}$ was assumed to be any unit vector in the plane, this holds for <u>every</u> unit vector in the plane.

12.3 THE DOT PRODUCT

<u>NOTE</u>: In Exercises 1-8 below we calculate $\text{proj}_\mathbf{v}\,\mathbf{u}$ as the vector $\left(\frac{|\mathbf{u}|\cos\theta}{|\mathbf{v}|}\right)\mathbf{v}$, so the scalar multiplier of $\mathbf{v}$ is the number in column 5 divided by the number in column 2.

	$\mathbf{v}\cdot\mathbf{u}$	$\lvert\mathbf{v}\rvert$	$\lvert\mathbf{u}\rvert$	$\cos\theta$	$\lvert\mathbf{u}\rvert\cos\theta$	$\text{proj}_\mathbf{v}\,\mathbf{u}$
1.	-25	5	5	-1	-5	$-2\mathbf{i}+4\mathbf{j}-\sqrt{5}\mathbf{k}$
2.	3	1	13	$\frac{3}{13}$	3	$3\left(\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{k}\right)$
3.	25	15	5	$\frac{1}{3}$	$\frac{5}{3}$	$\frac{1}{9}(10\mathbf{i}+11\mathbf{j}-2\mathbf{k})$
4.	13	15	3	$\frac{13}{45}$	$\frac{13}{15}$	$\frac{13}{225}(2\mathbf{i}+10\mathbf{j}-11\mathbf{k})$
5.	2	$\sqrt{34}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}\sqrt{34}}$	$\frac{2}{\sqrt{34}}$	$\frac{1}{17}(5\mathbf{j}-3\mathbf{k})$
6.	$\sqrt{3}-\sqrt{2}$	$\sqrt{2}$	3	$\frac{\sqrt{3}-\sqrt{2}}{3\sqrt{2}}$	$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}$	$\frac{\sqrt{3}-\sqrt{2}}{2}(-\mathbf{i}+\mathbf{j})$
7.	$10+\sqrt{17}$	$\sqrt{26}$	$\sqrt{21}$	$\frac{10+\sqrt{17}}{\sqrt{546}}$	$\frac{10+\sqrt{17}}{\sqrt{26}}$	$\frac{10+\sqrt{17}}{\sqrt{26}}(-5\mathbf{i}+\mathbf{j})$
8.	$\frac{1}{6}$	$\frac{\sqrt{30}}{6}$	$\frac{\sqrt{30}}{6}$	$\frac{1}{5}$	$\frac{1}{\sqrt{30}}$	$\frac{1}{5}\left\langle\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\right\rangle$

9. $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{(2)(1)+(1)(2)+(0)(-1)}{\sqrt{2^2+1^2+0^2}\sqrt{1^2+2^2+(-1)^2}}\right) = \cos^{-1}\left(\frac{4}{\sqrt{5}\sqrt{6}}\right) = \cos^{-1}\left(\frac{4}{\sqrt{30}}\right) \approx 0.75$ rad

10. $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{(2)(3)+(-2)(0)+(1)(4)}{\sqrt{2^2+(-2)^2+1^2}\sqrt{3^2+0^2+4^2}}\right) = \cos^{-1}\left(\frac{10}{\sqrt{9}\sqrt{25}}\right) = \cos^{-1}\left(\frac{2}{3}\right) \approx 0.84$ rad

11. $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{\left(\sqrt{3}\right)\left(\sqrt{3}\right)+(-7)(1)+(0)(-2)}{\sqrt{\left(\sqrt{3}\right)^2+(-7)^2+0^2}\sqrt{\left(\sqrt{3}\right)^2+(1)^2+(-2)^2}}\right) = \cos^{-1}\left(\frac{3-7}{\sqrt{52}\sqrt{8}}\right)$

$= \cos^{-1}\left(\frac{-1}{\sqrt{26}}\right) \approx 1.77$ rad

12. $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{(1)(-1)+\left(\sqrt{2}\right)(1)+\left(-\sqrt{2}\right)(1)}{\sqrt{(1)^2+\left(\sqrt{2}\right)^2+\left(-\sqrt{2}\right)^2}\,\sqrt{(-1)^2+(1)^2+(1)^2}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{5}\,\sqrt{3}}\right)$
$= \cos^{-1}\left(\frac{-1}{\sqrt{15}}\right) \approx 1.83$ rad

13. $\overrightarrow{AB} = \langle 3, 1\rangle$, $\overrightarrow{BC} = \langle -1, -3\rangle$, and $\overrightarrow{AC} = \langle 2, -2\rangle$. $\overrightarrow{BA} = \langle -3, -1\rangle$, $\overrightarrow{CB} = \langle 1, 3\rangle$, $\overrightarrow{CA} = \langle -2, 2\rangle$.
$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{BA}\right| = \sqrt{10}$, $\left|\overrightarrow{BC}\right| = \left|\overrightarrow{CB}\right| = \sqrt{10}$, $\left|\overrightarrow{AC}\right| = \left|\overrightarrow{CA}\right| = 2\sqrt{2}$,
Angle at A $= \cos^{-1}\left(\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\,\left|\overrightarrow{AC}\right|}\right) = \cos^{-1}\left(\frac{3(2)+1(-2)}{\left(\sqrt{10}\right)\left(2\sqrt{2}\right)}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.435°$
Angle at B $= \cos^{-1}\left(\frac{\overrightarrow{BC}\cdot\overrightarrow{BA}}{\left|\overrightarrow{BC}\right|\,\left|\overrightarrow{BA}\right|}\right) = \cos^{-1}\left(\frac{(-1)(-3)+(-3)(-1)}{\left(\sqrt{10}\right)\left(\sqrt{10}\right)}\right) = \cos^{-1}\left(\frac{3}{5}\right) \approx 53.130°$, and
Angle at C $= \cos^{-1}\left(\frac{\overrightarrow{CB}\cdot\overrightarrow{CA}}{\left|\overrightarrow{CB}\right|\,\left|\overrightarrow{CA}\right|}\right) = \cos^{-1}\left(\frac{1(-2)+3(2)}{\left(\sqrt{10}\right)\left(2\sqrt{2}\right)}\right) = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.435°$

14. $\overrightarrow{AC} = \langle 2, 4\rangle$ and $\overrightarrow{BD} = \langle 4, -2\rangle$. $\overrightarrow{AC}\cdot\overrightarrow{BD} = 2(4)+4(-2) = 0$, so the angle measures are all 90°.

15. (a) $\cos\alpha = \frac{\mathbf{i}\cdot\mathbf{v}}{|\mathbf{i}|\,|\mathbf{v}|} = \frac{a}{|\mathbf{v}|}$, $\cos\beta = \frac{\mathbf{j}\cdot\mathbf{v}}{|\mathbf{j}|\,|\mathbf{v}|} = \frac{b}{|\mathbf{v}|}$, $\cos\gamma = \frac{\mathbf{k}\cdot\mathbf{v}}{|\mathbf{k}|\,|\mathbf{v}|} = \frac{c}{|\mathbf{v}|}$ and
$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{a}{|\mathbf{v}|}\right)^2 + \left(\frac{b}{|\mathbf{v}|}\right)^2 + \left(\frac{c}{|\mathbf{v}|}\right)^2 = \frac{a^2+b^2+c^2}{|\mathbf{v}|\,|\mathbf{v}|} = \frac{|\mathbf{v}|\,|\mathbf{v}|}{|\mathbf{v}|\,|\mathbf{v}|} = 1$
(b) $|\mathbf{v}| = 1 \Rightarrow \cos\alpha = \frac{a}{|\mathbf{v}|} = a$, $\cos\beta = \frac{b}{|\mathbf{v}|} = b$ and $\cos\gamma = \frac{c}{|\mathbf{v}|} = c$ are the direction cosines of $\mathbf{v}$

16. $\mathbf{u} = 10\mathbf{i} + 2\mathbf{k}$ is parallel to the pipe in the north direction and $\mathbf{v} = 10\mathbf{j} + \mathbf{k}$ is parallel to the pipe in the east direction. The angle between the two pipes is $\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) = \cos^{-1}\left(\frac{2}{\sqrt{104}\,\sqrt{101}}\right) \approx 1.55 \text{ rad} \approx 88.88°$.

17. $\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right) + \left(\mathbf{u} - \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right) = \frac{3}{2}(\mathbf{i}+\mathbf{j}) + \left[(3\mathbf{j}+4\mathbf{k}) - \frac{3}{2}(\mathbf{i}+\mathbf{j})\right] = \left(\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}\right) + \left(-\frac{3}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} + 4\mathbf{k}\right)$, where $\mathbf{v}\cdot\mathbf{u} = 3$ and $\mathbf{v}\cdot\mathbf{v} = 2$

18. $\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right) + \left(\mathbf{u} - \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right) = \frac{1}{2}\mathbf{v} + \left(\mathbf{u} - \frac{1}{2}\mathbf{v}\right) = \frac{1}{2}(\mathbf{i}+\mathbf{j}) + \left[(\mathbf{j}+\mathbf{k}) - \frac{1}{2}(\mathbf{i}+\mathbf{j})\right] = \left(\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right) + \left(-\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \mathbf{k}\right)$, where $\mathbf{v}\cdot\mathbf{u} = 1$ and $\mathbf{v}\cdot\mathbf{v} = 2$

19. $\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right) + \left(\mathbf{u} - \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right) = \frac{14}{3}(\mathbf{i}+2\mathbf{j}-\mathbf{k}) + \left[(8\mathbf{i}+4\mathbf{j}-12\mathbf{k}) - \left(\frac{14}{3}\mathbf{i} + \frac{28}{3}\mathbf{j} - \frac{14}{3}\mathbf{k}\right)\right]$
$= \left(\frac{14}{3}\mathbf{i} + \frac{28}{3}\mathbf{j} - \frac{14}{3}\mathbf{k}\right) + \left(\frac{10}{3}\mathbf{i} - \frac{16}{3}\mathbf{j} - \frac{22}{3}\mathbf{k}\right)$, where $\mathbf{v}\cdot\mathbf{u} = 28$ and $\mathbf{v}\cdot\mathbf{v} = 6$

20. $\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right) + \left(\mathbf{u} - \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v}\right) = \frac{1}{1}(\mathbf{A}) + \left[(\mathbf{i}+\mathbf{j}+\mathbf{k}) - \left(\frac{1}{1}\right)\mathbf{A}\right] = (\mathbf{i}) + (\mathbf{j}+\mathbf{k})$, where $\mathbf{v}\cdot\mathbf{u} = 1$ and $\mathbf{v}\cdot\mathbf{v} = 1$; yes

21. The sum of two vectors of equal length is *always* orthogonal to their difference, as we can see from the equation $(\mathbf{v}_1+\mathbf{v}_2)\cdot(\mathbf{v}_1-\mathbf{v}_2) = \mathbf{v}_1\cdot\mathbf{v}_1 + \mathbf{v}_2\cdot\mathbf{v}_1 - \mathbf{v}_1\cdot\mathbf{v}_2 - \mathbf{v}_2\cdot\mathbf{v}_2 = |\mathbf{v}_1|^2 - |\mathbf{v}_2|^2 = 0$

22. $\overrightarrow{CA}\cdot\overrightarrow{CB} = (-\mathbf{v}+(-\mathbf{u}))\cdot(-\mathbf{v}+\mathbf{u}) = \mathbf{v}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{u} = |\mathbf{v}|^2 - |\mathbf{u}|^2 = 0$ because $|\mathbf{u}| = |\mathbf{v}|$ since both equal the radius of the circle. Therefore, $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are orthogonal.

23. Let $\mathbf{u}$ and $\mathbf{v}$ be the sides of a rhombus $\Rightarrow$ the diagonals are $\mathbf{d}_1 = \mathbf{u} + \mathbf{v}$ and $\mathbf{d}_2 = -\mathbf{u} + \mathbf{v}$ $\Rightarrow$ $\mathbf{d}_1 \cdot \mathbf{d}_2 = (\mathbf{u} + \mathbf{v}) \cdot (-\mathbf{u} + \mathbf{v}) = -\mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 - |\mathbf{u}|^2 = 0$ because $|\mathbf{u}| = |\mathbf{v}|$, since a rhombus has equal sides.

24. Suppose the diagonals of a rectangle are perpendicular, and let $\mathbf{u}$ and $\mathbf{v}$ be the sides of a rectangle $\Rightarrow$ the diagonals are $\mathbf{d}_1 = \mathbf{u} + \mathbf{v}$ and $\mathbf{d}_2 = -\mathbf{u} + \mathbf{v}$. Since the diagonals are perpendicular we have $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$ $\Leftrightarrow$ $(\mathbf{u} + \mathbf{v}) \cdot (-\mathbf{u} + \mathbf{v}) = -\mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = 0$ $\Leftrightarrow$ $|\mathbf{v}|^2 - |\mathbf{u}|^2 = 0 \Leftrightarrow (|\mathbf{v}| + |\mathbf{u}|)(|\mathbf{v}| - |\mathbf{u}|) = 0$ $\Leftrightarrow$ $(|\mathbf{v}| + |\mathbf{u}|) = 0$ which is not possible, or $(|\mathbf{v}| - |\mathbf{u}|) = 0$ which is equivalent to $|\mathbf{v}| = |\mathbf{u}|$ $\Rightarrow$ the rectangle is a square.

25. Clearly the diagonals of a rectangle are equal in length. What is not as obvious is the statement that equal diagonals happen only in a rectangle. We show this is true by letting the adjacent sides of a parallelogram be the vectors $(v_1\mathbf{i} + v_2\mathbf{j})$ and $(u_1\mathbf{i} + u_2\mathbf{j})$. The equal diagonals of the parallelogram are $\mathbf{d}_1 = (v_1\mathbf{i} + v_2\mathbf{j}) + (u_1\mathbf{i} + u_2\mathbf{j})$ and $\mathbf{d}_2 = (v_1\mathbf{i} + v_2\mathbf{j}) - (u_1\mathbf{i} + u_2\mathbf{j})$. Hence $|\mathbf{d}_1| = |\mathbf{d}_2| = |(v_1\mathbf{i} + v_2\mathbf{j}) + (u_1\mathbf{i} + u_2\mathbf{j})|$ $= |(v_1\mathbf{i} + v_2\mathbf{j}) - (u_1\mathbf{i} + u_2\mathbf{j})|$ $\Rightarrow$ $|(v_1 + u_1)\mathbf{i} + (v_2 + u_2)\mathbf{j}| = |(v_1 - u_1)\mathbf{i} + (v_2 - u_2)\mathbf{j}|$ $\Rightarrow$ $\sqrt{(v_1 + u_1)^2 + (v_2 + u_2)^2} = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2}$ $\Rightarrow$ $v_1^2 + 2v_1u_1 + u_1^2 + v_2^2 + 2v_2u_2 + u_2^2$ $= v_1^2 - 2v_1u_1 + u_1^2 + v_2^2 - 2v_2u_2 + u_2^2$ $\Rightarrow$ $2(v_1u_1 + v_2u_2) = -2(v_1u_1 + v_2u_2)$ $\Rightarrow$ $v_1u_1 + v_2u_2 = 0$ $\Rightarrow$ $(v_1\mathbf{i} + v_2\mathbf{j}) \cdot (u_1\mathbf{i} + u_2\mathbf{j}) = 0$ $\Rightarrow$ the vectors $(v_1\mathbf{i} + v_2\mathbf{j})$ and $(u_1\mathbf{i} + u_2\mathbf{j})$ are perpendicular and the parallelogram must be a rectangle.

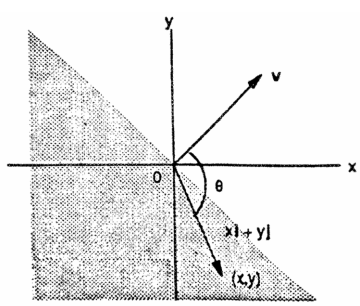
26. If $|\mathbf{u}| = |\mathbf{v}|$ and $\mathbf{u} + \mathbf{v}$ is the indicated diagonal, then $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u} = |\mathbf{u}|^2 + \mathbf{v} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{v} + |\mathbf{v}|^2$ $= \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = (\mathbf{u} + \mathbf{v}) \cdot \mathbf{v}$ $\Rightarrow$ the angle $\cos^{-1}\left(\frac{(\mathbf{u}+\mathbf{v})\cdot\mathbf{u}}{|\mathbf{u}+\mathbf{v}|\,|\mathbf{u}|}\right)$ between the diagonal and $\mathbf{u}$ and the angle $\cos^{-1}\left(\frac{(\mathbf{u}+\mathbf{v})\cdot\mathbf{v}}{|\mathbf{u}+\mathbf{v}|\,|\mathbf{v}|}\right)$ between the diagonal and $\mathbf{v}$ are equal because the inverse cosine function is one-to-one. Therefore, the diagonal bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

27. horizontal component: $1200\cos(8^\circ) \approx 1188$ ft/s; vertical component: $1200\sin(8^\circ) \approx 167$ ft/s

28. $|\mathbf{w}|\cos(33^\circ - 15^\circ) = 2.5$ lb, so $|\mathbf{w}| = \frac{2.5\text{ lb}}{\cos 18^\circ}$. Then $\mathbf{w} = \frac{2.5\text{ lb}}{\cos 18^\circ}\langle \cos 33^\circ, \sin 33^\circ\rangle \approx \langle 2.205, 1.432\rangle$

29. (a) Since $|\cos\theta| \le 1$, we have $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\,|\cos\theta| \le |\mathbf{u}|\,|\mathbf{v}|(1) = |\mathbf{u}|\,|\mathbf{v}|$.
    (b) We have equality precisely when $|\cos\theta| = 1$ or when one or both of $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{0}$. In the case of nonzero vectors, we have equality when $\theta = 0$ or π, i.e., when the vectors are parallel.

30. $(x\mathbf{i} + y\mathbf{j}) \cdot \mathbf{v} = |x\mathbf{i} + y\mathbf{j}|\,|\mathbf{v}|\cos\theta \le 0$ when $\frac{\pi}{2} \le \theta \le \pi$. This means (x, y) has to be a point whose position vector makes an angle with $\mathbf{v}$ that is a right angle or bigger.



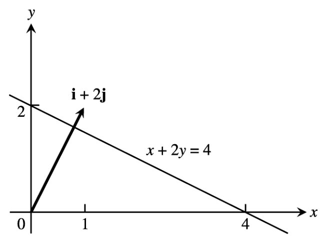
31. $\mathbf{v} \cdot \mathbf{u}_1 = (a\mathbf{u}_1 + b\mathbf{u}_2) \cdot \mathbf{u}_1 = a\mathbf{u}_1 \cdot \mathbf{u}_1 + b\mathbf{u}_2 \cdot \mathbf{u}_1 = a|\mathbf{u}_1|^2 + b(\mathbf{u}_2 \cdot \mathbf{u}_1) = a(1)^2 + b(0) = a$

32. No, $\mathbf{v}_1$need not equal $\mathbf{v}_2$. For example, $\mathbf{i} + \mathbf{j} \ne \mathbf{i} + 2\mathbf{j}$ but $\mathbf{i} \cdot (\mathbf{i} + \mathbf{j}) = \mathbf{i} \cdot \mathbf{i} + \mathbf{i} \cdot \mathbf{j} = 1 + 0 = 1$ and $\mathbf{i} \cdot (\mathbf{i} + 2\mathbf{j}) = \mathbf{i} \cdot \mathbf{i} + 2\mathbf{i} \cdot \mathbf{j} = 1 + 2 \cdot 0 = 1$.

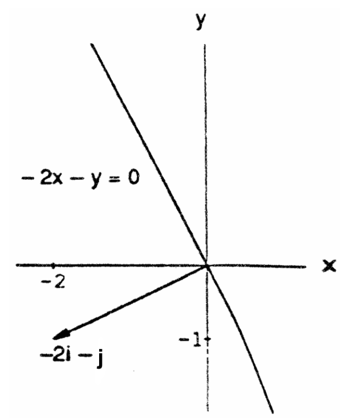
33. $P(x_1, y_1) = P\left(x_1, \frac{c}{b} - \frac{a}{b}x_1\right)$ and $Q(x_2, y_2) = Q\left(x_2, \frac{c}{b} - \frac{a}{b}x_2\right)$ are any two points P and Q on the line with $b \neq 0$
$\Rightarrow\ \overrightarrow{PQ} = (x_2 - x_1)\mathbf{i} + \frac{a}{b}(x_1 - x_2)\mathbf{j}\ \Rightarrow\ \overrightarrow{PQ} \cdot \mathbf{v} = \left[(x_2 - x_1)\mathbf{i} + \frac{a}{b}(x_1 - x_2)\mathbf{j}\right] \cdot (a\mathbf{i} + b\mathbf{j}) = a(x_2 - x_1) + b\left(\frac{a}{b}\right)(x_1 - x_2)$
$= 0\ \Rightarrow$ $\mathbf{v}$ is perpendicular to $\overrightarrow{PQ}$ for $b \neq 0$. If $b = 0$, then $\mathbf{v} = a\mathbf{i}$ is perpendicular to the vertical line $ax = c$.
Alternatively, the slope of $\mathbf{v}$ is $\frac{b}{a}$ and the slope of the line $ax + by = c$ is $-\frac{a}{b}$, so the slopes are negative reciprocals $\Rightarrow$ the vector $\mathbf{v}$ and the line are perpendicular.

34. The slope of $\mathbf{v}$ is $\frac{b}{a}$ and the slope of $bx - ay = c$ is $\frac{b}{a}$, provided that $a \neq 0$. If $a = 0$, then $\mathbf{v} = b\mathbf{j}$ is parallel to the vertical line $bx = c$. In either case, the vector $\mathbf{v}$ is parallel to the line $ax - by = c$.

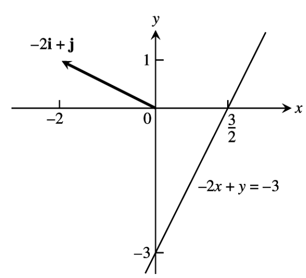
35. $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$ is perpendicular to the line $x + 2y = c$;
$P(2, 1)$ on the line $\Rightarrow\ 2 + 2 = c\ \Rightarrow\ x + 2y = 4$



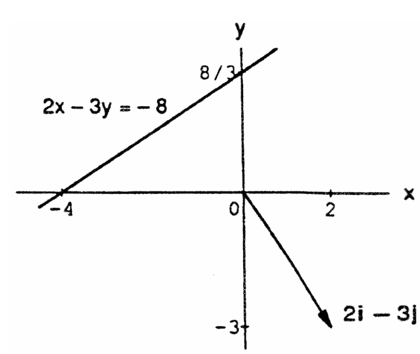
36. $\mathbf{v} = -2\mathbf{i} - \mathbf{j}$ is perpendicular to the line $-2x - y = c$;
$P(-1, 2)$ on the line $\Rightarrow\ (-2)(-1) - 2 = c$
$\Rightarrow\ -2x - y = 0$



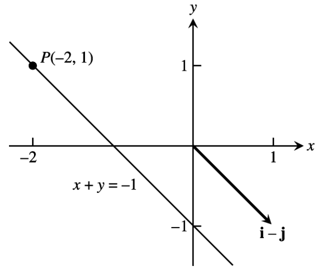
37. $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$ is perpendicular to the line $-2x + y = c$;
$P(-2, -7)$ on the line $\Rightarrow\ (-2)(-2) - 7 = c$
$\Rightarrow\ -2x + y = -3$



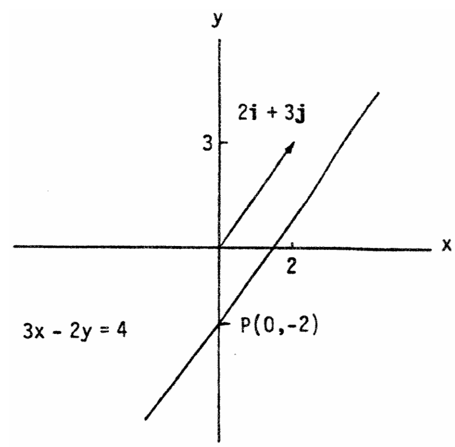
38. $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$ is perpendicular to the line $2x - 3y = c$;
$P(11, 10)$ on the line $\Rightarrow\ (2)(11) - (3)(10) = c$
$\Rightarrow\ 2x - 3y = -8$



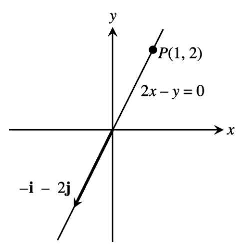
39. $\mathbf{v} = \mathbf{i} - \mathbf{j}$ is parallel to the line $-x - y = c$; $P(-2, 1)$ on the line $\Rightarrow -(-2) - 1 = c \Rightarrow -x - y = 1$ or $x + y = -1$.



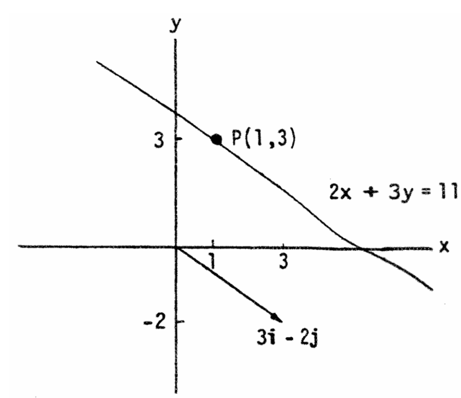
40. $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$ is parallel to the line $3x - 2y = c$; $P(0, -2)$ on the line $\Rightarrow 0 - 2(-2) = c \Rightarrow 3x - 2y = 4$



41. $\mathbf{v} = -\mathbf{i} - 2\mathbf{j}$ is parallel to the line $-2x + y = c$; $P(1, 2)$ on the line $\Rightarrow -2(1) + 2 = c \Rightarrow -2x - y = 0$ or $2x - y = 0$.



42. $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$ is parallel to the line $-2x - 3y = c$; $P(1, 3)$ on the line $\Rightarrow (-2)(1) - (3)(3) = c$ $\Rightarrow -2x - 3y = -11$ or $2x + 3y = 11$



43. $P(0, 0)$, $Q(1, 1)$ and $\mathbf{F} = 5\mathbf{j} \Rightarrow \overrightarrow{PQ} = \mathbf{i} + \mathbf{j}$ and $\mathbf{W} = \mathbf{F} \cdot \overrightarrow{PQ} = (5\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j}) = 5 \text{ N} \cdot \text{m} = 5 \text{ J}$

44. $\mathbf{W} = |\mathbf{F}|$ (distance) $\cos\theta = (602{,}148 \text{ N})(605 \text{ km})(\cos 0) = 364{,}299{,}540 \text{ N} \cdot \text{km} = (364{,}299{,}540)(1000) \text{ N} \cdot \text{m}$ $= 3.6429954 \times 10^{11} \text{ J}$

45. $\mathbf{W} = |\mathbf{F}| \left|\overrightarrow{PQ}\right| \cos\theta = (200)(20)(\cos 30^\circ) = 2000\sqrt{3} = 3464.10 \text{ N} \cdot \text{m} = 3464.10 \text{ J}$

46. $\mathbf{W} = |\mathbf{F}| \left|\overrightarrow{PQ}\right| \cos\theta = (1000)(5280)(\cos 60^\circ) = 2{,}640{,}000 \text{ ft} \cdot \text{lb}$

In Exercises 47-52 we use the fact that $\mathbf{n} = a\mathbf{i} + b\mathbf{j}$ is normal to the line $ax + by = c$.

47. $\mathbf{n}_1 = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = 2\mathbf{i} - \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{6-1}{\sqrt{10}\,\sqrt{5}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

48. $\mathbf{n}_1 = -\sqrt{3}\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = \sqrt{3}\mathbf{i} + \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{-3+1}{\sqrt{4}\,\sqrt{4}}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

49. $\mathbf{n}_1 = \sqrt{3}\mathbf{i} - \mathbf{j}$ and $\mathbf{n}_2 = \mathbf{i} - \sqrt{3}\mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{\sqrt{3}+\sqrt{3}}{\sqrt{4}\,\sqrt{4}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$

50. $\mathbf{n}_1 = \mathbf{i} + \sqrt{3}\mathbf{j}$ and $\mathbf{n}_2 = \left(1-\sqrt{3}\right)\mathbf{i} + \left(1+\sqrt{3}\right)\mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right)$
$= \cos^{-1}\left(\frac{1-\sqrt{3}+\sqrt{3}+3}{\sqrt{1+3}\,\sqrt{1-2\sqrt{3}+3+1+2\sqrt{3}+3}}\right) = \cos^{-1}\left(\frac{4}{2\sqrt{8}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

51. $\mathbf{n}_1 = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{n}_2 = \mathbf{i} - \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{3+4}{\sqrt{25}\,\sqrt{2}}\right) = \cos^{-1}\left(\frac{7}{5\sqrt{2}}\right) \approx 0.14 \text{ rad}$

52. $\mathbf{n}_1 = 12\mathbf{i} + 5\mathbf{j}$ and $\mathbf{n}_2 = 2\mathbf{i} - 2\mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{24-10}{\sqrt{169}\,\sqrt{8}}\right) = \cos^{-1}\left(\frac{14}{26\sqrt{2}}\right) \approx 1.18 \text{ rad}$

53. The angle between the corresponding normals is equal to the angle between the corresponding tangents. The points of intersection are $\left(-\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$. At $\left(-\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ the tangent line for $f(x) = x^2$ is $y - \frac{3}{4} = f'\left(-\frac{\sqrt{3}}{2}\right)\left(x - \left(-\frac{\sqrt{3}}{2}\right)\right) \Rightarrow y = -\sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{3}{4} \Rightarrow y = -\sqrt{3}x - \frac{3}{4}$, and the tangent line for $f(x) = \left(\frac{3}{2}\right) - x^2$ is $y - \frac{3}{4} = f'\left(-\frac{\sqrt{3}}{2}\right)\left(x - \left(-\frac{\sqrt{3}}{2}\right)\right) \Rightarrow y = \sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{3}{4} = \sqrt{3}x + \frac{9}{4}$. The corresponding normals are $\mathbf{n}_1 = \sqrt{3}\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = -\sqrt{3}\mathbf{i} + \mathbf{j}$. The angle at $\left(-\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ is $\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right)$ $= \cos^{-1}\left(\frac{-3+1}{\sqrt{4}\,\sqrt{4}}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$, the angle is $\frac{\pi}{3}$ and $\frac{2\pi}{3}$. At $\left(\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ the tangent line for $f(x) = x^2$ is $y = \sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{3}{4} = \sqrt{3}x + \frac{9}{4}$ and the tangent line for $f(x) = \frac{3}{2} - x^2$ is $y = -\sqrt{3}\left(x + \frac{\sqrt{3}}{2}\right) + \frac{3}{4}$ $= -\sqrt{3}x - \frac{3}{4}$. The corresponding normals are $\mathbf{n}_1 = -\sqrt{3}\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = \sqrt{3}\mathbf{i} + \mathbf{j}$. The angle at $\left(\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ is $\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{-3+1}{\sqrt{4}\,\sqrt{4}}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$, the angle is $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

54. The points of intersection are $\left(0, \frac{\sqrt{3}}{2}\right)$ and $\left(0, -\frac{\sqrt{3}}{2}\right)$. The curve $x = \frac{3}{4} - y^2$ has derivative $\frac{dy}{dx} = -\frac{1}{2y} \Rightarrow$ the tangent line at $\left(0, \frac{\sqrt{3}}{2}\right)$ is $y - \frac{\sqrt{3}}{2} = -\frac{1}{\sqrt{3}}(x-0) \Rightarrow \mathbf{n}_1 = \frac{1}{\sqrt{3}}\mathbf{i} + \mathbf{j}$ is normal to the curve at that point. The curve $x = y^2 - \frac{3}{4}$ has derivative $\frac{dy}{dx} = \frac{1}{2y} \Rightarrow$ the tangent line at $\left(0, \frac{\sqrt{3}}{2}\right)$ is $y - \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{3}}(x-0)$ $\Rightarrow \mathbf{n}_2 = -\frac{1}{\sqrt{3}}\mathbf{i} + \mathbf{j}$ is normal to the curve. The angle between the curves is $\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right)$ $= \cos^{-1}\left(\frac{-\frac{1}{3}+1}{\sqrt{\frac{1}{3}+1}\,\sqrt{\frac{1}{3}+1}}\right) = \cos^{-1}\left(\frac{\left(\frac{2}{3}\right)}{\left(\frac{4}{3}\right)}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$ and $\frac{2\pi}{3}$. Because of symmetry the angles between the curves at the two points of intersection are the same.

55. The curves intersect when $y = x^3 = \left(y^2\right)^3 = y^6 \Rightarrow y = 0$ or $y = 1$. The points of intersection are $(0,0)$ and $(1,1)$. Note that $y \ge 0$ since $y = y^6$. At $(0,0)$ the tangent line for $y = x^3$ is $y = 0$ and the tangent line for

$y = \sqrt{x}$ is $x = 0$. Therefore, the angle of intersection at $(0, 0)$ is $\frac{\pi}{2}$. At $(1, 1)$ the tangent line for $y = x^3$ is $y = 3x - 2$ and the tangent line for $y = \sqrt{x}$ is $y = \frac{1}{2}x + \frac{1}{2}$. The corresponding normal vectors are $\mathbf{n}_1 = -3\mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = -\frac{1}{2}\mathbf{i} + \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$, the angle is $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

56. The points of intersection for the curves $y = -x^2$ and $y = \sqrt[3]{x}$ are $(0, 0)$ and $(-1, -1)$. At $(0, 0)$ the tangent line for $y = -x^2$ is $y = 0$ and the tangent line for $y = \sqrt[3]{x}$ is $x = 0$. Therefore, the angle of intersection at $(0, 0)$ is $\frac{\pi}{2}$. At $(-1, -1)$ the tangent line for $y = -x^2$ is $y = 2x + 1$ and the tangent line for $y = \sqrt[3]{x}$ is $y = \frac{1}{3}x - \frac{2}{3}$. The corresponding normal vectors are $\mathbf{n}_1 = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{n}_2 = \frac{1}{3}\mathbf{i} - \mathbf{j} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right)$

$$= \cos^{-1}\left(\frac{\frac{2}{3}+1}{\sqrt{5}\sqrt{\frac{1}{9}+1}}\right) = \cos^{-1}\left(\frac{\left(\frac{5}{3}\right)}{\frac{\sqrt{5}\sqrt{10}}{3}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}, \text{ the angle is } \frac{\pi}{4} \text{ and } \frac{3\pi}{4}.$$

12.4 THE CROSS PRODUCT

1. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & -1 \\ 1 & 0 & -1 \end{vmatrix} = 3\left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right) \Rightarrow$ length $= 3$ and the direction is $\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$;

   $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -3\left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right) \Rightarrow$ length $= 3$ and the direction is $-\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$

2. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ -1 & 1 & 0 \end{vmatrix} = 5(\mathbf{k}) \Rightarrow$ length $= 5$ and the direction is $\mathbf{k}$

   $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -5(\mathbf{k}) \Rightarrow$ length $= 5$ and the direction is $-\mathbf{k}$

3. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix} = \mathbf{0} \Rightarrow$ length $= 0$ and has no direction

   $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = \mathbf{0} \Rightarrow$ length $= 0$ and has no direction

4. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix} = \mathbf{0} \Rightarrow$ length $= 0$ and has no direction

   $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = \mathbf{0} \Rightarrow$ length $= 0$ and has no direction

5. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 0 \\ 0 & -3 & 0 \end{vmatrix} = -6(\mathbf{k}) \Rightarrow$ length $= 6$ and the direction is $-\mathbf{k}$

   $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 6(\mathbf{k}) \Rightarrow$ length $= 6$ and the direction is $\mathbf{k}$

6. $\mathbf{u} \times \mathbf{v} = (\mathbf{i} \times \mathbf{j}) \times (\mathbf{j} \times \mathbf{k}) = \mathbf{k} \times \mathbf{i} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \mathbf{j} \Rightarrow$ length $= 1$ and the direction is $\mathbf{j}$

   $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -\mathbf{j} \Rightarrow$ length $= 1$ and the direction is $-\mathbf{j}$

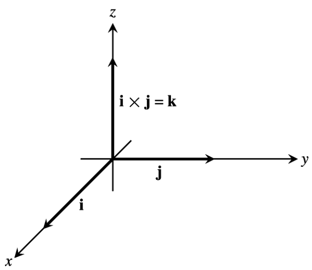
7. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & -2 & -4 \\ 2 & 2 & 1 \end{vmatrix} = 6\mathbf{i} - 12\mathbf{k} \Rightarrow$ length $= 6\sqrt{5}$ and the direction is $\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{k}$

   $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = -(6\mathbf{i} - 12\mathbf{k}) \Rightarrow$ length $= 6\sqrt{5}$ and the direction is $-\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{k}$

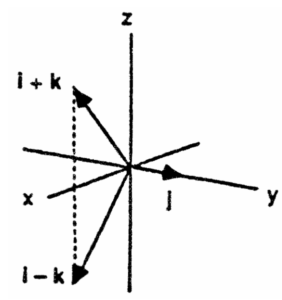
8. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{3}{2} & -\frac{1}{2} & 1\\ 1 & 1 & 2\end{vmatrix}=-2\mathbf{i}-2\mathbf{j}+2\mathbf{k} \Rightarrow$ length $=2\sqrt{3}$ and the direction is $-\frac{1}{\sqrt{3}}\mathbf{i}-\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k}$

$\mathbf{v}\times\mathbf{u}=-(\mathbf{u}\times\mathbf{v})=-(-2\mathbf{i}-2\mathbf{j}+2\mathbf{k}) \Rightarrow$ length $=2\sqrt{3}$ and the direction is $\frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}-\frac{1}{\sqrt{3}}\mathbf{k}$

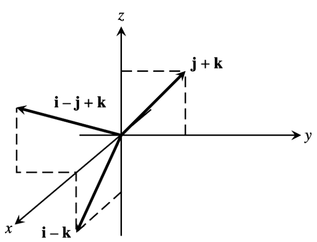
9. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & 0\\ 0 & 1 & 0\end{vmatrix}=\mathbf{k}$



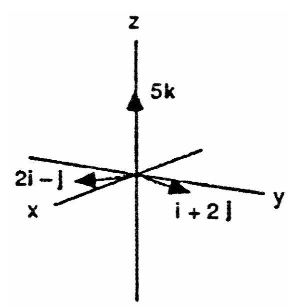
10. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & -1\\ 0 & 1 & 0\end{vmatrix}=\mathbf{i}+\mathbf{k}$



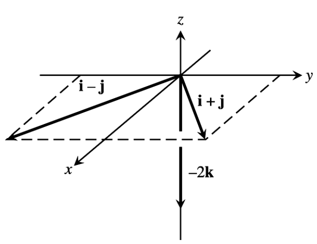
11. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & -1\\ 0 & 1 & 1\end{vmatrix}=\mathbf{i}-\mathbf{j}+\mathbf{k}$



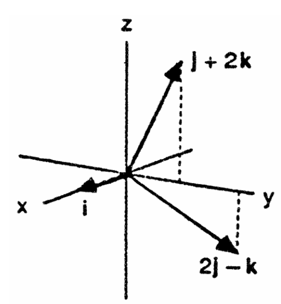
12. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -1 & 0\\ 1 & 2 & 0\end{vmatrix}=5\mathbf{k}$



13. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 0\\ 1 & -1 & 0\end{vmatrix}=-2\mathbf{k}$



14. $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 1 & 2\\ 1 & 0 & 0\end{vmatrix}=2\mathbf{j}-\mathbf{k}$



15. (a) $\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & -3\\ -1 & 3 & -1\end{vmatrix}=8\mathbf{i}+4\mathbf{j}+4\mathbf{k} \Rightarrow \text{Area}=\frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|=\frac{1}{2}\sqrt{64+16+16}=2\sqrt{6}$

(b) $\mathbf{u}=\pm\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|}=\pm\frac{1}{\sqrt{6}}(2\mathbf{i}+\mathbf{j}+\mathbf{k})$

16. (a) $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2 \\ 2 & -2 & 0 \end{vmatrix} = 4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} \Rightarrow \text{Area} = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}\sqrt{16+16+4} = 3$

(b) $\mathbf{u} = \pm \frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|} = \pm \frac{1}{3}(2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

17. (a) $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{j} \Rightarrow \text{Area} = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}\sqrt{1+1} = \frac{\sqrt{2}}{2}$

(b) $\mathbf{u} = \pm \frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|} = \pm \frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) = \pm \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$

18. (a) $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -1 \\ 1 & 0 & -2 \end{vmatrix} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} \Rightarrow \text{Area} = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}\sqrt{4+9+1} = \frac{\sqrt{14}}{2}$

(b) $\mathbf{u} = \pm \frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|} = \pm \frac{1}{\sqrt{14}}(2\mathbf{i} + 3\mathbf{j} + \mathbf{k})$

19. If $\mathbf{u} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{v} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, and $\mathbf{w} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, then $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

$\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$ and $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ which all have the same value, since the interchanging of two pair of rows in a determinant does not change its value $\Rightarrow$ the volume is

$|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs}\begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8$

20. $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs}\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ -1 & 2 & -1 \end{vmatrix} = 4$ (for details about verification, see Exercise 19)

21. $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs}\begin{vmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = |-7| = 7$ (for details about verification, see Exercise 19)

22. $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = \text{abs}\begin{vmatrix} 1 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 4 & -2 \end{vmatrix} = 8$ (for details about verification, see Exercise 19)

23. (a) $\mathbf{u} \cdot \mathbf{v} = -6$, $\mathbf{u} \cdot \mathbf{w} = -81$, $\mathbf{v} \cdot \mathbf{w} = 18 \Rightarrow$ none

(b) $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 1 \\ 0 & 1 & -5 \end{vmatrix} \neq \mathbf{0}$, $\mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 1 \\ -15 & 3 & -3 \end{vmatrix} = \mathbf{0}$, $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -5 \\ -15 & 3 & -3 \end{vmatrix} \neq \mathbf{0}$

$\Rightarrow$ **u** and **w** are parallel

24. (a) $\mathbf{u} \cdot \mathbf{v} = 0$, $\mathbf{u} \times \mathbf{w} = 0$, $\mathbf{u} \cdot \mathbf{r} = -3\pi$, $\mathbf{v} \cdot \mathbf{w} = 0$, $\mathbf{v} \cdot \mathbf{r} = 0$, $\mathbf{w} \cdot \mathbf{r} = 0 \Rightarrow \mathbf{u} \perp \mathbf{v}$, $\mathbf{u} \perp \mathbf{w}$, $\mathbf{v} \perp \mathbf{w}$, $\mathbf{v} \perp \mathbf{r}$ and $\mathbf{w} \perp \mathbf{r}$

(b) $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&-1\\-1&1&1\end{vmatrix}\neq\mathbf{0},\ \mathbf{u}\times\mathbf{w}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&-1\\1&0&1\end{vmatrix}\neq\mathbf{0},\ \mathbf{u}\times\mathbf{r}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&-1\\-\frac{\pi}{2}&-\pi&\frac{\pi}{2}\end{vmatrix}=\mathbf{0}$

$\mathbf{v}\times\mathbf{w}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&1&1\\1&0&1\end{vmatrix}\neq\mathbf{0},\ \mathbf{v}\times\mathbf{r}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&1&1\\-\frac{\pi}{2}&-\pi&\frac{\pi}{2}\end{vmatrix}\neq\mathbf{0},\ \mathbf{w}\times\mathbf{r}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&1\\-\frac{\pi}{2}&-\pi&\frac{\pi}{2}\end{vmatrix}\neq\mathbf{0}$

$\Rightarrow$ **u** and **r** are parallel

25. $\left|\overrightarrow{PQ}\times\mathbf{F}\right|=\left|\overrightarrow{PQ}\right|\,|\mathbf{F}|\sin(60^\circ)=\frac{2}{3}\cdot 30\cdot\frac{\sqrt{3}}{2}\text{ ft}\cdot\text{lb}=10\sqrt{3}\text{ ft}\cdot\text{lb}$

26. $\left|\overrightarrow{PQ}\times\mathbf{F}\right|=\left|\overrightarrow{PQ}\right|\,|\mathbf{F}|\sin(135^\circ)=\frac{2}{3}\cdot 30\cdot\frac{\sqrt{2}}{2}\text{ ft}\cdot\text{lb}=10\sqrt{2}\text{ ft}\cdot\text{lb}$

27. (a) true, $|\mathbf{u}|=\sqrt{a_1^2+a_2^2+a_3^2}=\sqrt{\mathbf{u}\cdot\mathbf{u}}$

(b) not always true, $\mathbf{u}\cdot\mathbf{u}=|\mathbf{u}|^2$

(c) true, $\mathbf{u}\times\mathbf{0}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\0&0&0\end{vmatrix}=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}$ and $\mathbf{0}\times\mathbf{u}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&0\\a_1&a_2&a_3\end{vmatrix}=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}$

(d) true, $\mathbf{u}\times(-\mathbf{u})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\-a_1&-a_2&-a_3\end{vmatrix}=(-a_2a_3+a_2a_3)\mathbf{i}-(-a_1a_3+a_1a_3)\mathbf{j}+(-a_1a_2+a_1a_2)\mathbf{k}=\mathbf{0}$

(e) not always true, $\mathbf{i}\times\mathbf{j}=\mathbf{k}\neq-\mathbf{k}=\mathbf{j}\times\mathbf{i}$ for example

(f) true, distributive property of the cross product

(g) true, $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v}=\mathbf{u}\cdot(\mathbf{v}\times\mathbf{v})=\mathbf{u}\cdot\mathbf{0}=0$

(h) true, the volume of a parallelpiped with **u**, **v**, and **w** along the three edges is $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}=(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u}=\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$, since the dot product is commutative.

28. (a) true, $\mathbf{u}\cdot\mathbf{v}=a_1b_1+a_2b_2+a_3b_3=b_1a_1+b_2a_2+b_3a_3=\mathbf{v}\cdot\mathbf{u}$

(b) true, $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}=-\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\b_1&b_2&b_3\\a_1&a_2&a_3\end{vmatrix}=-(\mathbf{v}\times\mathbf{u})$

(c) true, $(-\mathbf{u})\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-a_1&-a_2&-a_3\\b_1&b_2&b_3\end{vmatrix}=-\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}=-(\mathbf{u}\times\mathbf{v})$

(d) true, $(c\mathbf{u})\cdot\mathbf{v}=(ca_1)b_1+(ca_2)b_2+(ca_3)b_3=a_1(cb_1)+a_2(cb_2)+a_3(cb_3)=\mathbf{u}\cdot(c\mathbf{v})=c(a_1b_1+a_2b_2+a_3b_3)$
$=c(\mathbf{u}\cdot\mathbf{v})$

(e) true, $c(\mathbf{u}\times\mathbf{v})=c\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ca_1&ca_2&ca_3\\b_1&b_2&b_3\end{vmatrix}=(c\mathbf{u})\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_1&a_2&a_3\\cb_1&cb_2&cb_3\end{vmatrix}=\mathbf{u}\times(c\mathbf{v})$

(f) true, $\mathbf{u}\cdot\mathbf{u}=a_1^2+a_2^2+a_3^2=\left(\sqrt{a_1^2+a_2^2+a_3^2}\right)^2=|\mathbf{u}|^2$

(g) true, $(\mathbf{u}\times\mathbf{u})\cdot\mathbf{u}=\mathbf{0}\cdot\mathbf{u}=0$

(h) true, $\mathbf{u}\times\mathbf{v}\perp\mathbf{u}$ and $\mathbf{u}\times\mathbf{v}\perp\mathbf{v}\ \Rightarrow\ (\mathbf{u}\times\mathbf{v})\cdot\mathbf{u}=\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v})=0$

29. (a) $\text{proj}_{\mathbf{v}}\,\mathbf{u}=\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}||\mathbf{v}|}\right)\mathbf{v}$ (b) $\pm(\mathbf{u}\times\mathbf{v})$ (c) $\pm((\mathbf{u}\times\mathbf{v})\times\mathbf{w})$ (d) $|(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}|$

30. (a) $(\mathbf{u}\times\mathbf{v})\times(\mathbf{u}\times\mathbf{w})$

(b) $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})=(\mathbf{u}+\mathbf{v})\times\mathbf{u}-(\mathbf{u}+\mathbf{v})\times\mathbf{v}=\mathbf{u}\times\mathbf{u}+\mathbf{v}\times\mathbf{u}-\mathbf{u}\times\mathbf{v}-\mathbf{v}\times\mathbf{v}$
$=\mathbf{0}+\mathbf{v}\times\mathbf{u}-\mathbf{u}\times\mathbf{v}-\mathbf{0}=2(\mathbf{v}\times\mathbf{u})$, or simply $\mathbf{u}\times\mathbf{v}$

(c) $|\mathbf{u}|\,\frac{\mathbf{v}}{|\mathbf{v}|}$ (d) $|\mathbf{u}\times\mathbf{w}|$

31. (a) yes, $\mathbf{u} \times \mathbf{v}$ and $\mathbf{w}$ are both vectors (b) no, $\mathbf{u}$ is a vector but $\mathbf{v} \cdot \mathbf{w}$ is a scalar
(c) yes, $\mathbf{u}$ and $\mathbf{u} \times \mathbf{w}$ are both vectors (d) no, $\mathbf{u}$ is a vector but $\mathbf{v} \cdot \mathbf{w}$ is a scalar

32. $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ is perpendicular to $\mathbf{u} \times \mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ $\Rightarrow$ $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ is parallel to a vector in the plane of $\mathbf{u}$ and $\mathbf{v}$ which means it lies in the plane determined by $\mathbf{u}$ and $\mathbf{v}$. The situation is degenerate if $\mathbf{u}$ and $\mathbf{v}$ are parallel so $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ and the vectors do not determine a plane. Similar reasoning shows that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ lies in the plane of $\mathbf{v}$ and $\mathbf{w}$ provided $\mathbf{v}$ and $\mathbf{w}$ are nonparallel.

33. No, $\mathbf{v}$ need not equal $\mathbf{w}$. For example, $\mathbf{i}+\mathbf{j} \neq -\mathbf{i}+\mathbf{j}$, but $\mathbf{i} \times (\mathbf{i}+\mathbf{j}) = \mathbf{i} \times \mathbf{i} + \mathbf{i} \times \mathbf{j} = \mathbf{0} + \mathbf{k} = \mathbf{k}$ and $\mathbf{i} \times (-\mathbf{i}+\mathbf{j}) = -\mathbf{i} \times \mathbf{i} + \mathbf{i} \times \mathbf{j} = \mathbf{0} + \mathbf{k} = \mathbf{k}$.

34. Yes. If $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, then $\mathbf{u} \times (\mathbf{v} - \mathbf{w}) = \mathbf{0}$ and $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = 0$. Suppose now that $\mathbf{v} \neq \mathbf{w}$. Then $\mathbf{u} \times (\mathbf{v} - \mathbf{w}) = \mathbf{0}$ implies that $\mathbf{v} - \mathbf{w} = k\mathbf{u}$ for some real number $k \neq 0$. This in turn implies that $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot (k\mathbf{u}) = k\,|\mathbf{u}|^2 = 0$, which implies that $\mathbf{u} = \mathbf{0}$. Since $\mathbf{u} \neq \mathbf{0}$, it cannot be that $\mathbf{v} \neq \mathbf{w}$, so $\mathbf{v} = \mathbf{w}$.

35. $\overrightarrow{AB} = -\mathbf{i}+\mathbf{j}$ and $\overrightarrow{AD} = -\mathbf{i}-\mathbf{j}$ $\Rightarrow$ $\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -1 & -1 & 0 \end{vmatrix} = 2\mathbf{k}$ $\Rightarrow$ area $= \left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = 2$

36. $\overrightarrow{AB} = 7\mathbf{i}+3\mathbf{j}$ and $\overrightarrow{AD} = 2\mathbf{i}+5\mathbf{j}$ $\Rightarrow$ $\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & 3 & 0 \\ 2 & 5 & 0 \end{vmatrix} = 29\mathbf{k}$ $\Rightarrow$ area $= \left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = 29$

37. $\overrightarrow{AB} = 3\mathbf{i}-2\mathbf{j}$ and $\overrightarrow{AD} = 5\mathbf{i}+\mathbf{j}$ $\Rightarrow$ $\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 0 \\ 5 & 1 & 0 \end{vmatrix} = 13\mathbf{k}$ $\Rightarrow$ area $= \left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = 13$

38. $\overrightarrow{AB} = 7\mathbf{i}-4\mathbf{j}$ and $\overrightarrow{AD} = 2\mathbf{i}+5\mathbf{j}$ $\Rightarrow$ $\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -4 & 0 \\ 2 & 5 & 0 \end{vmatrix} = 43\mathbf{k}$ $\Rightarrow$ area $= \left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = 43$

39. $\overrightarrow{AB} = -2\mathbf{i}+3\mathbf{j}$ and $\overrightarrow{AC} = 3\mathbf{i}+\mathbf{j}$ $\Rightarrow$ $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 0 \\ 3 & 1 & 0 \end{vmatrix} = -11\mathbf{k}$ $\Rightarrow$ area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{11}{2}$

40. $\overrightarrow{AB} = 4\mathbf{i}+4\mathbf{j}$ and $\overrightarrow{AC} = 3\mathbf{i}+2\mathbf{j}$ $\Rightarrow$ $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 4 & 0 \\ 3 & 2 & 0 \end{vmatrix} = -4\mathbf{k}$ $\Rightarrow$ area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = 2$

41. $\overrightarrow{AB} = 6\mathbf{i}-5\mathbf{j}$ and $\overrightarrow{AC} = 11\mathbf{i}-5\mathbf{j}$ $\Rightarrow$ $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -5 & 0 \\ 11 & -5 & 0 \end{vmatrix} = 25\mathbf{k}$ $\Rightarrow$ area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{25}{2}$

42. $\overrightarrow{AB} = 16\mathbf{i}-5\mathbf{j}$ and $\overrightarrow{AC} = 4\mathbf{i}+4\mathbf{j}$ $\Rightarrow$ $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 16 & -5 & 0 \\ 4 & 4 & 0 \end{vmatrix} = 84\mathbf{k}$ $\Rightarrow$ area $= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = 42$

43. If $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}$, then $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$ and the triangle's area is $\frac{1}{2}|\mathbf{A} \times \mathbf{B}| = \pm\frac{1}{2}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$. The applicable sign is $(+)$ if the acute angle from $\mathbf{A}$ to $\mathbf{B}$ runs counterclockwise in the xy-plane, and $(-)$ if it runs clockwise, because the area must be a nonnegative number.

44. If $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$, $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}$, and $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j}$, then the area of the triangle is $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$. Now,

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 - a_1 & b_2 - a_2 & 0 \\ c_1 - a_1 & c_2 - a_2 & 0 \end{vmatrix} = \begin{vmatrix} b_1 - a_1 & b_2 - a_2 \\ c_1 - a_1 & c_2 - a_2 \end{vmatrix} \mathbf{k} \Rightarrow \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$= \frac{1}{2}|(b_1 - a_1)(c_2 - a_2) - (c_1 - a_1)(b_2 - a_2)| = \frac{1}{2}|a_1(b_2 - c_2) + a_2(c_1 - b_1) + (b_1c_2 - c_1b_2)|$$

$$= \pm\frac{1}{2}\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}.$$ The applicable sign ensures the area formula gives a nonnegative number.

12.5 LINES AND PLANES IN SPACE

1. The direction $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $P(3, -4, -1) \Rightarrow x = 3 + t,\ y = -4 + t,\ z = -1 + t$

2. The direction $\overrightarrow{PQ} = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $P(1, 2, -1) \Rightarrow x = 1 - 2t,\ y = 2 - 2t,\ z = -1 + 2t$

3. The direction $\overrightarrow{PQ} = 5\mathbf{i} + 5\mathbf{j} - 5\mathbf{k}$ and $P(-2, 0, 3) \Rightarrow x = -2 + 5t,\ y = 5t,\ z = 3 - 5t$

4. The direction $\overrightarrow{PQ} = -\mathbf{j} - \mathbf{k}$ and $P(1, 2, 0) \Rightarrow x = 1,\ y = 2 - t,\ z = -t$

5. The direction $2\mathbf{j} + \mathbf{k}$ and $P(0, 0, 0) \Rightarrow x = 0,\ y = 2t,\ z = t$

6. The direction $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $P(3, -2, 1) \Rightarrow x = 3 + 2t,\ y = -2 - t,\ z = 1 + 3t$

7. The direction $\mathbf{k}$ and $P(1, 1, 1) \Rightarrow x = 1,\ y = 1,\ z = 1 + t$

8. The direction $3\mathbf{i} + 7\mathbf{j} - 5\mathbf{k}$ and $P(2, 4, 5) \Rightarrow x = 2 + 3t,\ y = 4 + 7t,\ z = 5 - 5t$

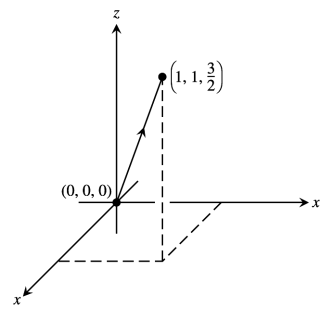
9. The direction $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $P(0, -7, 0) \Rightarrow x = t,\ y = -7 + 2t,\ z = 2t$

10. The direction is $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = -2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $P(2, 3, 0) \Rightarrow x = 2 - 2t,\ y = 3 + 4t,\ z = -2t$

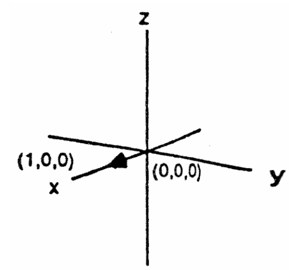
11. The direction $\mathbf{i}$ and $P(0, 0, 0) \Rightarrow x = t,\ y = 0,\ z = 0$

12. The direction $\mathbf{k}$ and $P(0, 0, 0) \Rightarrow x = 0,\ y = 0,\ z = t$

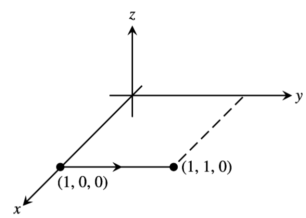
13. The direction $\overrightarrow{PQ} = \mathbf{i} + \mathbf{j} + \frac{3}{2}\mathbf{k}$ and $P(0,0,0) \Rightarrow x = t,$ $y = t, z = \frac{3}{2}t$, where $0 \le t \le 1$



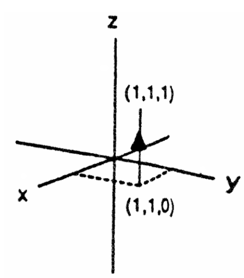
14. The direction $\overrightarrow{PQ} = \mathbf{i}$ and $P(0,0,0) \Rightarrow x = t, y = 0, z = 0,$ where $0 \le t \le 1$



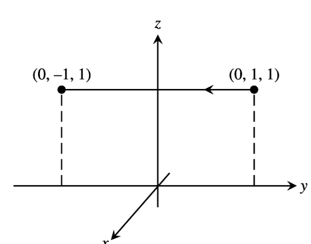
15. The direction $\overrightarrow{PQ} = \mathbf{j}$ and $P(1,1,0) \Rightarrow x = 1, y = 1 + t,$ $z = 0$, where $-1 \le t \le 0$



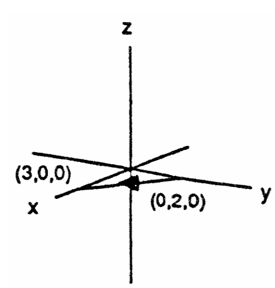
16. The direction $\overrightarrow{PQ} = \mathbf{k}$ and $P(1,1,0) \Rightarrow x = 1, y = 1, z = t,$ where $0 \le t \le 1$



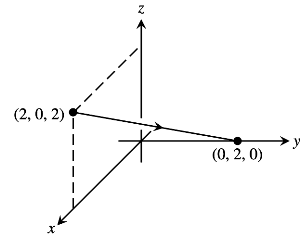
17. The direction $\overrightarrow{PQ} = -2\mathbf{j}$ and $P(0,1,1) \Rightarrow x = 0,$ $y = 1 - 2t, z = 1$, where $0 \le t \le 1$



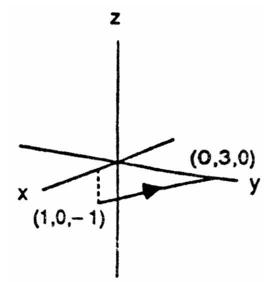
18. The direction $\overrightarrow{PQ} = 3\mathbf{i} - 2\mathbf{j}$ and $P(0,2,0) \Rightarrow x = 3t,$ $y = 2 - 2t, z = 0$, where $0 \le t \le 1$



19. The direction $\overrightarrow{PQ} = -2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $P(2, 0, 2)$
$\Rightarrow\ x = 2 - 2t,\ y = 2t,\ z = 2 - 2t$, where $0 \le t \le 1$



20. The direction $\overrightarrow{PQ} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $P(1, 0, -1)$
$\Rightarrow\ x = 1 - t,\ y = 3t,\ z = -1 + t$, where $0 \le t \le 1$



21. $3(x - 0) + (-2)(y - 2) + (-1)(z + 1) = 0\ \Rightarrow\ 3x - 2y - z = -3$

22. $3(x - 1) + (1)(y + 1) + (1)(z - 3) = 0\ \Rightarrow\ 3x + y + z = 5$

23. $\overrightarrow{PQ} = \mathbf{i} - \mathbf{j} + 3\mathbf{k},\ \overrightarrow{PS} = -\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}\ \Rightarrow\ \overrightarrow{PQ} \times \overrightarrow{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 3 \\ -1 & -3 & 2 \end{vmatrix} = 7\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$ is normal to the plane
$\Rightarrow\ 7(x - 2) + (-5)(y - 0) + (-4)(z - 2) = 0\ \Rightarrow\ 7x - 5y - 4z = 6$

24. $\overrightarrow{PQ} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k},\ \overrightarrow{PS} = -3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}\ \Rightarrow\ \overrightarrow{PQ} \times \overrightarrow{PS} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ -3 & 2 & 3 \end{vmatrix} = -\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is normal to the plane
$\Rightarrow\ (-1)(x - 1) + (-3)(y - 5) + (1)(z - 7) = 0\ \Rightarrow\ x + 3y - z = 9$

25. $\mathbf{n} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k},\ P(2, 4, 5) = (1)(x - 2) + (3)(y - 4) + (4)(z - 5) = 0\ \Rightarrow\ x + 3y + 4z = 34$

26. $\mathbf{n} = \mathbf{i} - 2\mathbf{j} + \mathbf{k},\ P(1, -2, 1) = (1)(x - 1) + (-2)(y + 2) + (1)(z - 1) = 0\ \Rightarrow\ x - 2y + z = 6$

27. $\begin{cases} x = 2t + 1 = s + 2 \\ y = 3t + 2 = 2s + 4 \end{cases}\ \Rightarrow\ \begin{cases} 2t - s = 1 \\ 3t - 2s = 2 \end{cases}\ \Rightarrow\ \begin{cases} 4t - 2s = 2 \\ 3t - 2s = 2 \end{cases}\ \Rightarrow\ t = 0$ and $s = -1$; then $z = 4t + 3 = -4s - 1$
$\Rightarrow\ 4(0) + 3 = (-4)(-1) - 1$ is satisfied $\Rightarrow$ the lines do intersect when $t = 0$ and $s = -1\ \Rightarrow$ the point of intersection is $x = 1,\ y = 2$, and $z = 3$ or $P(1, 2, 3)$. A vector normal to the plane determined by these lines is
$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix} = -20\mathbf{i} + 12\mathbf{j} + \mathbf{k}$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are directions of the lines $\Rightarrow$ the plane containing the lines is represented by $(-20)(x - 1) + (12)(y - 2) + (1)(z - 3) = 0\ \Rightarrow\ -20x + 12y + z = 7$.

28. $\begin{cases} x = t = 2s + 2 \\ y = -t + 2 = s + 3 \end{cases}\ \Rightarrow\ \begin{cases} t - 2s = 2 \\ -t - s = 1 \end{cases}\ \Rightarrow\ s = -1$ and $t = 0$; then $z = t + 1 = 5s + 6\ \Rightarrow\ 0 + 1 = 5(-1) + 6$ is satisfied $\Rightarrow$ the lines do intersect when $s = -1$ and $t = 0\ \Rightarrow$ the point of intersection is $x = 0,\ y = 2$ and $z = 1$ or $P(0, 2, 1)$. A vector normal to the plane determined by these lines is $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & 5 \end{vmatrix}$

$= -6\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are directions of the lines $\Rightarrow$ the plane containing the lines is represented by $(-6)(x-0) + (-3)(y-2) + (3)(z-1) = 0 \Rightarrow 6x + 3y - 3z = 3$.

29. The cross product of $\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $-4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ has the same direction as the normal to the plane $\Rightarrow \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ -4 & 2 & -2 \end{vmatrix} = 6\mathbf{j} + 6\mathbf{k}$. Select a point on either line, such as $P(-1, 2, 1)$. Since the lines are given to intersect, the desired plane is $0(x+1) + 6(y-2) + 6(z-1) = 0 \Rightarrow 6y + 6z = 18 \Rightarrow y + z = 3$.

30. The cross product of $\mathbf{i} - 3\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + \mathbf{j} + \mathbf{k}$ has the same direction as the normal to the plane $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & -1 \\ 1 & 1 & 1 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$. Select a point on either line, such as $P(0, 3, -2)$. Since the lines are given to intersect, the desired plane is $(-2)(x-0) + (-2)(y-3) + (4)(z+2) = 0 \Rightarrow -2x - 2y + 4z = -14 \Rightarrow x + y - 2z = 7$.

31. $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$ is a vector in the direction of the line of intersection of the planes $\Rightarrow 3(x-2) + (-3)(y-1) + 3(z+1) = 0 \Rightarrow 3x - 3y + 3z = 0 \Rightarrow x - y + z = 0$ is the desired plane containing $P_0(2, 1, -1)$

32. A vector normal to the desired plane is $\overrightarrow{P_1P_2} \times \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -2 \\ 4 & -1 & 2 \end{vmatrix} = -2\mathbf{i} - 12\mathbf{j} - 2\mathbf{k}$; choosing $P_1(1, 2, 3)$ as a point on the plane $\Rightarrow (-2)(x-1) + (-12)(y-2) + (-2)(z-3) = 0 \Rightarrow -2x - 12y - 2z = -32 \Rightarrow x + 6y + z = 16$ is the desired plane

33. $S(0, 0, 12)$, $P(0, 0, 0)$ and $\mathbf{v} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} \Rightarrow \overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 12 \\ 4 & -2 & 2 \end{vmatrix} = 24\mathbf{i} + 48\mathbf{j} = 24(\mathbf{i} + 2\mathbf{j})$ $\Rightarrow d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{24\sqrt{1+4}}{\sqrt{16+4+4}} = \frac{24\sqrt{5}}{\sqrt{24}} = \sqrt{5 \cdot 24} = 2\sqrt{30}$ is the distance from S to the line

34. $S(0, 0, 0)$, $P(5, 5, -3)$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} \Rightarrow \overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & -5 & 3 \\ 3 & 4 & -5 \end{vmatrix} = 13\mathbf{i} - 16\mathbf{j} - 5\mathbf{k}$ $\Rightarrow d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{169+256+25}}{\sqrt{9+16+25}} = \frac{\sqrt{450}}{\sqrt{50}} = \sqrt{9} = 3$ is the distance from S to the line

35. $S(2, 1, 3)$, $P(2, 1, 3)$ and $\mathbf{v} = 2\mathbf{i} + 6\mathbf{j} \Rightarrow \overrightarrow{PS} \times \mathbf{v} = \mathbf{0} \Rightarrow d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{0}{\sqrt{40}} = 0$ is the distance from S to the line (i.e., the point S lies on the line)

36. $S(2, 1, -1)$, $P(0, 1, 0)$ and $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \Rightarrow \overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 2 & 2 & 2 \end{vmatrix} = 2\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$ $\Rightarrow d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{4+36+16}}{\sqrt{4+4+4}} = \frac{\sqrt{56}}{\sqrt{12}} = \sqrt{\frac{14}{3}}$ is the distance from S to the line

37. $S(3,-1,4)$, $P(4,3,-5)$ and $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \Rightarrow \overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -4 & 9 \\ -1 & 2 & 3 \end{vmatrix} = -30\mathbf{i} - 6\mathbf{j} - 6\mathbf{k}$

$\Rightarrow d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{900+36+36}}{\sqrt{1+4+9}} = \frac{\sqrt{972}}{\sqrt{14}} = \frac{\sqrt{486}}{\sqrt{7}} = \frac{\sqrt{81\cdot 6}}{\sqrt{7}} = \frac{9\sqrt{42}}{7}$ is the distance from S to the line

38. $S(-1,4,3)$, $P(10,-3,0)$ and $\mathbf{v} = 4\mathbf{i} + 4\mathbf{k} \Rightarrow \overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -11 & 7 & 3 \\ 4 & 0 & 4 \end{vmatrix} = 28\mathbf{i} + 56\mathbf{j} - 28\mathbf{k} = 28(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

$\Rightarrow d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} = \frac{28\sqrt{1+4+1}}{4\sqrt{1+1}} = 7\sqrt{3}$ is the distance from S to the line

39. $S(2,-3,4)$, $x + 2y + 2z = 13$ and $P(13,0,0)$ is on the plane $\Rightarrow \overrightarrow{PS} = -11\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

$\Rightarrow d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{-11-6+8}{\sqrt{1+4+4}}\right| = \left|\frac{-9}{\sqrt{9}}\right| = 3$

40. $S(0,0,0)$, $3x + 2y + 6z = 6$ and $P(2,0,0)$ is on the plane $\Rightarrow \overrightarrow{PS} = -2\mathbf{i}$ and $\mathbf{n} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

$\Rightarrow d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{-6}{\sqrt{9+4+36}}\right| = \frac{6}{\sqrt{49}} = \frac{6}{7}$

41. $S(0,1,1)$, $4y + 3z = -12$ and $P(0,-3,0)$ is on the plane $\Rightarrow \overrightarrow{PS} = 4\mathbf{j} + \mathbf{k}$ and $\mathbf{n} = 4\mathbf{j} + 3\mathbf{k}$

$\Rightarrow d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{16+3}{\sqrt{16+9}}\right| = \frac{19}{5}$

42. $S(2,2,3)$, $2x + y + 2z = 4$ and $P(2,0,0)$ is on the plane $\Rightarrow \overrightarrow{PS} = 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$\Rightarrow d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{2+6}{\sqrt{4+1+4}}\right| = \frac{8}{3}$

43. $S(0,-1,0)$, $2x + y + 2z = 4$ and $P(2,0,0)$ is on the plane $\Rightarrow \overrightarrow{PS} = -2\mathbf{i} - \mathbf{j}$ and $\mathbf{n} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

$\Rightarrow d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{-4-1+0}{\sqrt{4+1+4}}\right| = \frac{5}{3}$

44. $S(1,0,-1)$, $-4x + y + z = 4$ and $P(-1,0,0)$ is on the plane $\Rightarrow \overrightarrow{PS} = 2\mathbf{i} - \mathbf{k}$ and $\mathbf{n} = -4\mathbf{i} + \mathbf{j} + \mathbf{k}$

$\Rightarrow d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{-8-1}{\sqrt{16+1+1}}\right| = \frac{9}{\sqrt{18}} = \frac{3\sqrt{2}}{2}$

45. The point $P(1,0,0)$ is on the first plane and $S(10,0,0)$ is a point on the second plane $\Rightarrow \overrightarrow{PS} = 9\mathbf{i}$, and $\mathbf{n} = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ is normal to the first plane $\Rightarrow$ the distance from S to the first plane is $d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right|$ $= \left|\frac{9}{\sqrt{1+4+36}}\right| = \frac{9}{\sqrt{41}}$, which is also the distance between the planes.

46. The line is parallel to the plane since $\mathbf{v} \cdot \mathbf{n} = \left(\mathbf{i} + \mathbf{j} - \frac{1}{2}\mathbf{k}\right) \cdot (\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}) = 1 + 2 - 3 = 0$. Also the point $S(1,0,0)$ when $t = -1$ lies on the line, and the point $P(10,0,0)$ lies on the plane $\Rightarrow \overrightarrow{PS} = -9\mathbf{i}$. The distance from S to the plane is $d = \left|\overrightarrow{PS} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = \left|\frac{-9}{\sqrt{1+4+36}}\right| = \frac{9}{\sqrt{41}}$, which is also the distance from the line to the plane.

47. $\mathbf{n}_1 = \mathbf{i} + \mathbf{j}$ and $\mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{2+1}{\sqrt{2}\,\sqrt{9}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

48. $\mathbf{n}_1 = 5\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{5-2-3}{\sqrt{27}\,\sqrt{14}}\right) = \cos^{-1}(0) = \frac{\pi}{2}$

49. $\mathbf{n}_1 = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{4-4-2}{\sqrt{12}\,\sqrt{9}}\right) = \cos^{-1}\left(\frac{-1}{3\sqrt{3}}\right) \approx 1.76$ rad

50. $\mathbf{n}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{k} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}\,\sqrt{1}}\right) \approx 0.96$ rad

51. $\mathbf{n}_1 = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} + 2\mathbf{j} + \mathbf{k} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{2+4-1}{\sqrt{9}\,\sqrt{6}}\right) = \cos^{-1}\left(\frac{5}{3\sqrt{6}}\right) \approx 0.82$ rad

52. $\mathbf{n}_1 = 4\mathbf{j} + 3\mathbf{k}$ and $\mathbf{n}_2 = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} \Rightarrow \theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{8+18}{\sqrt{25}\,\sqrt{49}}\right) = \cos^{-1}\left(\frac{26}{35}\right) \approx 0.73$ rad

53. $2x - y + 3z = 6 \Rightarrow 2(1-t) - (3t) + 3(1+t) = 6 \Rightarrow -2t + 5 = 6 \Rightarrow t = -\frac{1}{2} \Rightarrow x = \frac{3}{2}, y = -\frac{3}{2}$ and $z = \frac{1}{2}$
$\Rightarrow \left(\frac{3}{2}, -\frac{3}{2}, \frac{1}{2}\right)$ is the point

54. $6x + 3y - 4z = -12 \Rightarrow 6(2) + 3(3+2t) - 4(-2-2t) = -12 \Rightarrow 14t + 29 = -12 \Rightarrow t = -\frac{41}{14} \Rightarrow x = 2, y = 3 - \frac{41}{7}$, and $z = -2 + \frac{41}{7} \Rightarrow \left(2, -\frac{20}{7}, \frac{27}{7}\right)$ is the point

55. $x + y + z = 2 \Rightarrow (1+2t) + (1+5t) + (3t) = 2 \Rightarrow 10t + 2 = 2 \Rightarrow t = 0 \Rightarrow x = 1, y = 1$ and $z = 0$
$\Rightarrow (1, 1, 0)$ is the point

56. $2x - 3z = 7 \Rightarrow 2(-1+3t) - 3(5t) = 7 \Rightarrow -9t - 2 = 7 \Rightarrow t = -1 \Rightarrow x = -1 - 3, y = -2$ and $z = -5$
$\Rightarrow (-4, -2, -5)$ is the point

57. $\mathbf{n}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} + \mathbf{j} \Rightarrow \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\mathbf{i} + \mathbf{j}$, the direction of the desired line; $(1, 1, -1)$ is on both planes $\Rightarrow$ the desired line is $x = 1 - t, y = 1 + t, z = -1$

58. $\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ and $\mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k} \Rightarrow \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$, the direction of the desired line; $(1, 0, 0)$ is on both planes $\Rightarrow$ the desired line is $x = 1 + 14t, y = 2t, z = 15t$

59. $\mathbf{n}_1 = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{n}_2 = \mathbf{i} + \mathbf{j} - 2\mathbf{k} \Rightarrow \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 4 \\ 1 & 1 & -2 \end{vmatrix} = 6\mathbf{j} + 3\mathbf{k}$, the direction of the desired line; $(4, 3, 1)$ is on both planes $\Rightarrow$ the desired line is $x = 4, y = 3 + 6t, z = 1 + 3t$

60. $\mathbf{n}_1 = 5\mathbf{i} - 2\mathbf{j}$ and $\mathbf{n}_2 = 4\mathbf{j} - 5\mathbf{k} \Rightarrow \mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -2 & 0 \\ 0 & 4 & -5 \end{vmatrix} = 10\mathbf{i} + 25\mathbf{j} + 20\mathbf{k}$, the direction of the desired line; $(1, -3, 1)$ is on both planes $\Rightarrow$ the desired line is $x = 1 + 10t, y = -3 + 25t, z = 1 + 20t$

61. <u>L1 & L2</u>: $x = 3 + 2t = 1 + 4s$ and $y = -1 + 4t = 1 + 2s \Rightarrow \begin{cases} 2t - 4s = -2 \\ 4t - 2s = 2 \end{cases} \Rightarrow \begin{cases} 2t - 4s = -2 \\ 2t - s = 1 \end{cases}$
$\Rightarrow -3s = -3 \Rightarrow s = 1$ and $t = 1 \Rightarrow$ on L1, $z = 1$ and on L2, $z = 1 \Rightarrow$ L1 and L2 intersect at $(5, 3, 1)$.

L2 & L3: The direction of L2 is $\frac{1}{6}(4\mathbf{i}+2\mathbf{j}+4\mathbf{k})=\frac{1}{3}(2\mathbf{i}+\mathbf{j}+2\mathbf{k})$ which is the same as the direction $\frac{1}{3}(2\mathbf{i}+\mathbf{j}+2\mathbf{k})$ of L3; hence L2 and L3 are parallel.

L1 & L3: $x=3+2t=3+2r$ and $y=-1+4t=2+r \Rightarrow \begin{cases} 2t-2r=0 \\ 4t-\ \ r=3 \end{cases} \Rightarrow \begin{cases} t-r=0 \\ 4t-\ \ r=3 \end{cases} \Rightarrow 3t=3$ $\Rightarrow t=1$ and $r=1 \Rightarrow$ on L1, $z=2$ while on L3, $z=0 \Rightarrow$ L1 and L2 do not intersect. The direction of L1 is $\frac{1}{\sqrt{21}}(2\mathbf{i}+4\mathbf{j}-\mathbf{k})$ while the direction of L3 is $\frac{1}{3}(2\mathbf{i}+\mathbf{j}+2\mathbf{k})$ and neither is a multiple of the other; hence L1 and L3 are skew.

62. L1 & L2: $x=1+2t=2-s$ and $y=-1-t=3s \Rightarrow \begin{cases} 2t+\ \ s=1 \\ -t-3s=1 \end{cases} \Rightarrow -5s=3 \Rightarrow s=-\frac{3}{5}$ and $t=\frac{4}{5} \Rightarrow$ on L1, $z=\frac{12}{5}$ while on L2, $z=1-\frac{3}{5}=\frac{2}{5} \Rightarrow$ L1 and L2 do not intersect. The direction of L1 is $\frac{1}{\sqrt{14}}(2\mathbf{i}-\mathbf{j}+3\mathbf{k})$ while the direction of L2 is $\frac{1}{\sqrt{11}}(-\mathbf{i}+3\mathbf{j}+\mathbf{k})$ and neither is a multiple of the other; hence, L1 and L2 are skew.

L2 & L3: $x=2-s=5+2r$ and $y=3s=1-r \Rightarrow \begin{cases} -s-2r=3 \\ 3s+\ \ r=1 \end{cases} \Rightarrow 5s=5 \Rightarrow s=1$ and $r=-2 \Rightarrow$ on L2, $z=2$ and on L3, $z=2 \Rightarrow$ L2 and L3 intersect at $(1,3,2)$.

L1 & L3: L1 and L3 have the same direction $\frac{1}{\sqrt{14}}(2\mathbf{i}-\mathbf{j}+3\mathbf{k})$; hence L1 and L3 are parallel.

63. $x=2+2t,\ y=-4-t,\ z=7+3t;\ x=-2-t,\ y=-2+\frac{1}{2}t,\ z=1-\frac{3}{2}t$

64. $1(x-4)-2(y-1)+1(z-5)=0 \Rightarrow x-4-2y+2+z-5=0 \Rightarrow x-2y+z=7;$
$-\sqrt{2}(x-3)+2\sqrt{2}(y+2)-\sqrt{2}(z-0)=0 \Rightarrow -\sqrt{2}x+2\sqrt{2}y-\sqrt{2}z=-7\sqrt{2}$

65. $x=0 \Rightarrow t=-\frac{1}{2},\ y=-\frac{1}{2},\ z=-\frac{3}{2} \Rightarrow \left(0,-\frac{1}{2},-\frac{3}{2}\right);\ y=0 \Rightarrow t=-1,\ x=-1,\ z=-3 \Rightarrow (-1,0,-3);\ z=0$ $\Rightarrow t=0,\ x=1,\ y=-1 \Rightarrow (1,-1,0)$

66. The line contains $(0,0,3)$ and $\left(\sqrt{3},1,3\right)$ because the projection of the line onto the xy-plane contains the origin and intersects the positive x-axis at a $30°$ angle. The direction of the line is $\sqrt{3}\mathbf{i}+\mathbf{j}+0\mathbf{k} \Rightarrow$ the line in question is $x=\sqrt{3}t,\ y=t,\ z=3$.

67. With substitution of the line into the plane we have $2(1-2t)+(2+5t)-(-3t)=8 \Rightarrow 2-4t+2+5t+3t=8$ $\Rightarrow 4t+4=8 \Rightarrow t=1 \Rightarrow$ the point $(-1,7,-3)$ is contained in both the line and plane, so they are not parallel.

68. The planes are parallel when either vector $A_1\mathbf{i}+B_1\mathbf{j}+C_1\mathbf{k}$ or $A_2\mathbf{i}+B_2\mathbf{j}+C_2\mathbf{k}$ is a multiple of the other or when $|(A_1\mathbf{i}+B_1\mathbf{j}+C_1\mathbf{k})\times(A_2\mathbf{i}+B_2\mathbf{j}+C_2\mathbf{k}|=0$. The planes are perpendicular when their normals are perpendicular, or$(A_1\mathbf{i}+B_1\mathbf{j}+C_1\mathbf{k})\cdot(A_2\mathbf{i}+B_2\mathbf{j}+C_2\mathbf{k})=0$.

69. There are many possible answers. One is found as follows: eliminate t to get $t=x-1=2-y=\frac{z-3}{2}$ $\Rightarrow x-1=2-y$ and $2-y=\frac{z-3}{2} \Rightarrow x+y=3$ and $2y+z=7$ are two such planes.

70. Since the plane passes through the origin, its general equation is of the form $Ax+By+Cz=0$. Since it meets the plane M at a right angle, their normal vectors are perpendicular $\Rightarrow 2A+3B+C=0$. One choice satisfying this equation is $A=1,\ B=-1$ and $C=1 \Rightarrow x-y+z=0$. Any plane $Ax+By+Cz=0$ with $2A+3B+C=0$ will pass through the origin and be perpendicular to M.

71. The points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ are the x, y, and z intercepts of the plane. Since a, b, and c are all nonzero, the plane must intersect all three coordinate axes and cannot pass through the origin. Thus, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ describes all planes <u>except</u> those through the origin or parallel to a coordinate axis.

72. Yes. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero vectors parallel to the lines, then $\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$ is perpendicular to the lines.

73. (a) $\overrightarrow{EP} = c\overrightarrow{EP_1} \Rightarrow -x_0\mathbf{i} + y\mathbf{j} + z\mathbf{k} = c\left[(x_1 - x_0)\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}\right] \Rightarrow -x_0 = c(x_1 - x_0)$, $y = cy_1$ and $z = cz_1$, where c is a positive real number

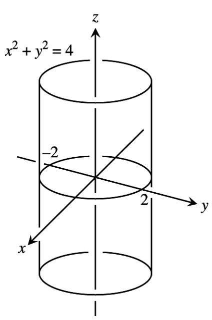
(b) At $x_1 = 0 \Rightarrow c = 1 \Rightarrow y = y_1$ and $z = z_1$; at $x_1 = x_0 \Rightarrow x_0 = 0, y = 0, z = 0$; $\lim\limits_{x_0 \to \infty} c = \lim\limits_{x_0 \to \infty} \frac{-x_0}{x_1 - x_0}$ $= \lim\limits_{x_0 \to \infty} \frac{-1}{-1} = 1 \Rightarrow c \to 1$ so that $y \to y_1$ and $z \to z_1$

74. The plane which contains the triangular plane is $x + y + z = 2$. The line containing the endpoints of the line segment is $x = 1 - t$, $y = 2t$, $z = 2t$. The plane and the line intersect at $\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$. The visible section of the line segment is $\sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = 1$ unit in length. The length of the line segment is $\sqrt{1^2 + 2^2 + 2^2} = 3 \Rightarrow \frac{2}{3}$ of the line segment is hidden from view.

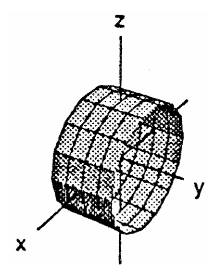
12.6 CYLINDERS AND QUADRIC SURFACES

1. d, ellipsoid
2. i, hyperboloid
3. a, cylinder
4. g, cone
5. l, hyperbolic paraboloid
6. e, paraboloid
7. b, cylinder
8. j, hyperboloid
9. k, hyperbolic paraboloid
10. f, paraboloid
11. h, cone
12. c, ellipsoid

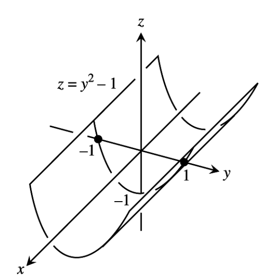
13. $x^2 + y^2 = 4$



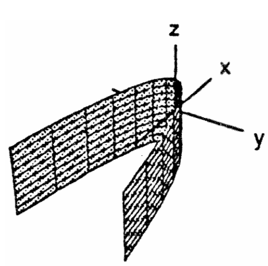
14. $x^2 + z^2 = 4$



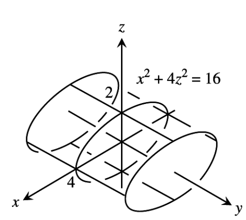
15. $z = y^2 - 1$



16. $x = y^2$



17. $x^2 + 4z^2 = 16$



18. $4x^2 + y^2 = 36$