Question 1

Given that the equation of a conic section is

$$x^2 + 4y^2 - 24y + 20 = 0.$$

(a) Using completing the square, identify the type of the conic section.

(b) Hence, sketch the graph of this conic section, including foci, center and vertices.

Solution:

(a) $x^2 + 4y^2 - 24y + 20 = 0 \Rightarrow x^2 + 4(y^2 - 6y) + 20 = 0$

$$\Rightarrow x^2 + 4[(y-3)^2 - 3^2] + 20 = 0$$

$$\Rightarrow x^2 + 4(y-3)^2 = 16$$

$$\Rightarrow \frac{x^2}{4^2} + \frac{(y-3)^2}{2^2} = 1$$

which is an equation of an ellipse.

(b) $\frac{x^2}{4^2} + \frac{(y-3)^2}{2^2} = 1$ is a "fat" ellipse.

Centre at $(0, 3)$

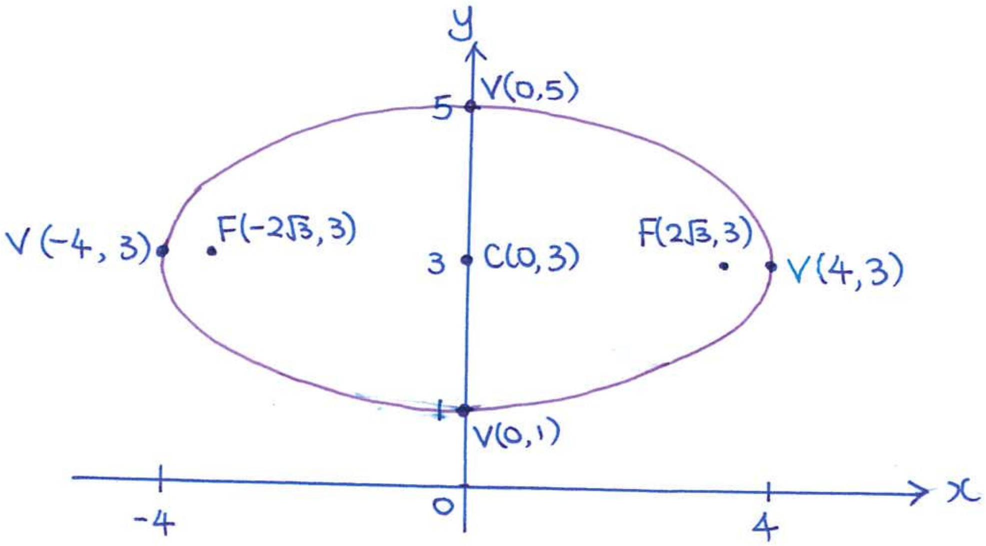
Vertices at $(-4+0, 0+3)$, $(4+0, 0+3)$, $(0+0, -2+3)$, $(0+0, 2+3)$,

i.e. $(-4, 3)$, $(4, 3)$, $(0, 1)$, $(0, 5)$.

$c = \sqrt{4^2 - 2^2} = \sqrt{12} = 2\sqrt{3}$

$\therefore$ Foci at $(-2\sqrt{3}+0, 0+3)$ and $(2\sqrt{3}+0, 0+3)$,

i.e. $(-2\sqrt{3}, 3)$ and $(2\sqrt{3}, 3)$



Question 2

Find the largest possible domains and the ranges of the following functions:

a) $f_1(x) = \ln(1-x^2)$

b) $f_2(x) = \dfrac{1}{10^x - 100}$

Solution:

a) $f_1(x) = \ln(1-x^2)$ is defined when $1-x^2 > 0 \Rightarrow x^2 < 1$

$\Rightarrow -1 < x < 1$

$\therefore \text{Dom}(f_1) = (-1, 1)$

For any $x \in \text{Dom}(f_1)$, $-1 < x < 1 \Rightarrow 0 \leq x^2 < 1$

$\Rightarrow 0 \geq -x^2 > -1$

$\Rightarrow 0 < 1-x^2 \leq 1$

$\Rightarrow -\infty < \ln(1-x^2) \leq \ln 1 = 0$

$\therefore \text{Ran}(f_1) = (-\infty, 0]$

(b) $f_2(x) = \dfrac{1}{10^x - 100}$ is defined when $10^x - 100 \neq 0$.

$$\Rightarrow 10^x \neq 10^2$$

$$\Rightarrow x \neq 2$$

$\therefore \text{Dom}(f_2) = \mathbb{R} \setminus \{2\}$.

For any $x \in (-\infty, 2)$, $x < 2 \Rightarrow 0 < 10^x < 100 \Rightarrow -100 < 10^x - 100 < 0 \Rightarrow \dfrac{1}{10^x - 100} < -\dfrac{1}{100}$

For any $x \in (2, \infty)$, $x > 2 \Rightarrow 10^x > 100 \Rightarrow 10^x - 100 > 0 \Rightarrow \dfrac{1}{10^x - 100} > 0$

$\therefore \text{Ran}(f_2) = (-\infty, -0.01) \cup (0, \infty)$

Question 3

Resolve into partial fractions $\dfrac{x^3-2x^2-3x-5}{x^3-1}$.

Solution:

By long division,

improper rational function

$$\frac{x^3-2x^2-3x-5}{x^3-1} = \underbrace{1}_{\text{polynomial}} + \underbrace{\frac{-2x^2-3x-4}{x^3-1}}_{\text{proper rational function}}$$

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad\quad\quad 1 \\ x^3-1 & \overline{)\, x^3-2x^2-3x-5} \\ & \;\; x^3 \quad\quad\quad\quad\quad -1 \\ & \overline{\quad\quad -2x^2-3x-4} \end{array}$$

Consider $\dfrac{-2x^2-3x-4}{x^3-1} = \dfrac{-2x^2-3x-4}{(x-1)(x^2+x+1)} \equiv \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1}$

$\therefore \; -2x^2-3x-4 = A(x^2+x+1) + (Bx+C)(x-1)$

Put $x=1$: $-9 = 3A \Rightarrow A=-3$

Compare coeff. of x^2 : $-2 = A+B \Rightarrow B=1$

Compare constant terms : $-4 = A-C \Rightarrow C=1$

$\therefore \; \dfrac{x^3-2x^2-3x-5}{x^3-1} = 1 - \dfrac{3}{x-1} + \dfrac{x+1}{x^2+x+1}$

Question 4

(a) It is given that $\sin A = \frac{4}{5}$ and $\cos B = \frac{5}{13}$. Without using a calculator, show that one possible value of $\sin(A+B)$ is $\frac{56}{65}$, and find all the other possible values.

(Hint: $\sin(A+B) = \sin A \cos B + \cos A \sin B$)

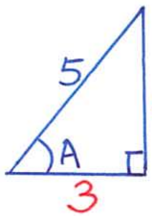
(b) (i) Prove that $\tan x + \cot x = 2\operatorname{cosec}(2x)$.

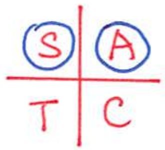
(ii) Find the general solution, in radians, of the equation

$$1 + 2\operatorname{cosec}(2x) = \cot x.$$

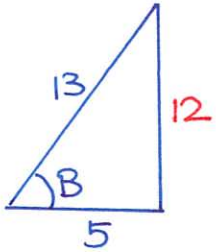
Solution

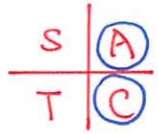
(a) $\sin A = \frac{4}{5}\ (>0)$ $\therefore$ A is in Quad. I or II.





$\cos B = \frac{5}{13}\ (>0)$ $\therefore$ B is in Quad. I or IV.





Case 1: A and B are in Quad. I (i.e. $0 < A < \frac{\pi}{2}$ and $0 < B < \frac{\pi}{2}$)

$\cos A = \frac{3}{5}\ (>0)$ and $\sin B = \frac{12}{13}\ (>0)$

$\therefore \sin(A+B) = \sin A \cos B + \cos A \sin B = \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) = \frac{56}{65}$

Case 2: A is in Quad. I and B is in Quad. IV (i.e. $0 < A < \frac{\pi}{2}$ and $-\frac{\pi}{2} < B < 0$)

$\cos A = \frac{3}{5}\ (>0)$ and $\sin B = -\frac{12}{13}\ (<0)$

$\therefore \sin(A+B) = \sin A \cos B + \cos A \sin B = \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) = \frac{-16}{65}$

Case 3: A is in Quad. II and B is in Quad. I (i.e. $\frac{\pi}{2} < A < \pi$ and $0 < B < \frac{\pi}{2}$)

$\cos A = -\frac{3}{5}$ (<0) and $\sin B = \frac{12}{13}$ (>0)

$\therefore$ $\sin(A+B) = \sin A \cos B + \cos A \sin B = \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right) = \frac{-16}{65}$

Case 4: A is in Quad. II and B is in Quad. IV. (i.e. $\frac{\pi}{2} < A < \pi$ and $-\frac{\pi}{2} < B < 0$)

$\cos A = -\frac{3}{5}$ (<0) and $\sin B = -\frac{12}{13}$ (<0)

$\therefore$ $\sin(A+B) = \sin A \cos B + \cos A \sin B = \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(-\frac{3}{5}\right)\left(-\frac{12}{13}\right) = \frac{56}{65}$

(b) (i) $$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$

$$= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x}$$

$$= \frac{1}{\frac{1}{2}\sin 2x}$$

$$= 2\operatorname{cosec}(2x)$$

$\therefore \tan x + \cot x = 2\operatorname{cosec}(2x)$

(ii) $1 + 2\operatorname{cosec} 2x = \cot x \Rightarrow 1 + \tan x + \cot x = \cot x$, by (i)

$\Rightarrow \tan x = -1$

$\therefore$ The general solution is

$$x = n\pi + \tan^{-1}(-1)$$

$$= n\pi - \frac{\pi}{4} \quad \text{for } n \in \mathbb{Z}.$$

Question 5

Solve the equation: $\cos(2x) = 5\cos x - 3$.

Solution:

$$\cos(2x) = 5\cos x - 3 \;\Rightarrow\; \cos^2 x - \sin^2 x = 5\cos x - 3$$

$$\Rightarrow\; \cos^2 x - (1 - \cos^2 x) = 5\cos x - 3$$

$$\Rightarrow\; 2\cos^2 x - 5\cos x + 2 = 0$$

$$\Rightarrow\; (2\cos x - 1)(\cos x - 2) = 0$$

$$\Rightarrow\; \cos x = \frac{1}{2} \quad \text{or} \quad \cos x = 2$$

(no solution, since range of $\cos x$ is $[-1, 1]$.)

$\therefore$ The general solution is

$$x = 2n\pi \pm \cos^{-1}\left(\frac{1}{2}\right)$$

$$= 2n\pi \pm \frac{\pi}{3}, \quad n \in \mathbb{Z}$$

Question 6

Solve the equation $2\log_{10} x = 1 + \log_{10}\left(\frac{2(2x+5)}{5}\right)$.

Solution:

$$2\log_{10} x = 1 + \log_{10}\left(\frac{2(2x+5)}{5}\right)$$

$$\Rightarrow \quad 2\log_{10} x - \log_{10}\left(\frac{2(2x+5)}{5}\right) = 1$$

$$\Rightarrow \quad \log_{10}\left(\frac{x^2}{\frac{2(2x+5)}{5}}\right) = 1$$

$$\Rightarrow \quad \frac{5x^2}{2(2x+5)} = 10$$

$$\Rightarrow \quad 5x^2 - 40x - 100 = 0$$

$$\Rightarrow \quad 5(x^2 - 8x - 20) = 0$$

$$\Rightarrow \quad 5(x-10)(x+2) = 0$$

$$\Rightarrow \quad x = 10 \quad \text{or} \quad x = -2 \ (\text{rejected} \because x > 0)$$

$\therefore$ The solution is $x = 10$.

The equation is valid when

$x > 0$ and $\frac{2(2x+5)}{2} > 0$

$\Rightarrow x > 0$ and $x > \frac{-5}{2}$

$\Rightarrow x > 0$