MA1200 CALCULUS AND BASIC LINEAR ALGEBRA I

LECTURE: CG1

Chapter 6
Limits, Continuity and Differentiability

Limit of a function at a point

The limit of a function $f(x)$ at a point $x = a$ is the value that $f(x)$ is approaching as x gets closer and closer to a.

$x \to a$ "x tends to a"
"x approaches a"

We use the notation

$$\boxed{\lim_{x \to a} f(x) = L}$$

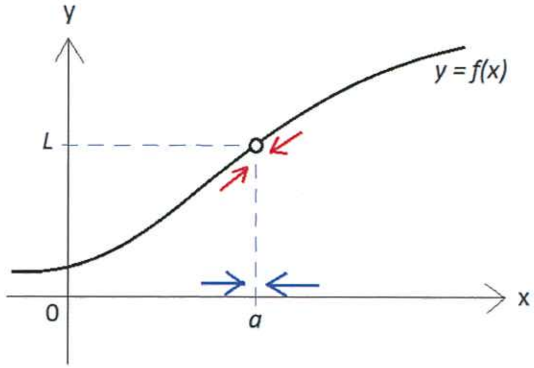
to denote that "the value of $f(x)$ gets arbitrarily close to L as x approaches a". Here, x approaches a from both the left and the right of a.

but $x \neq a$

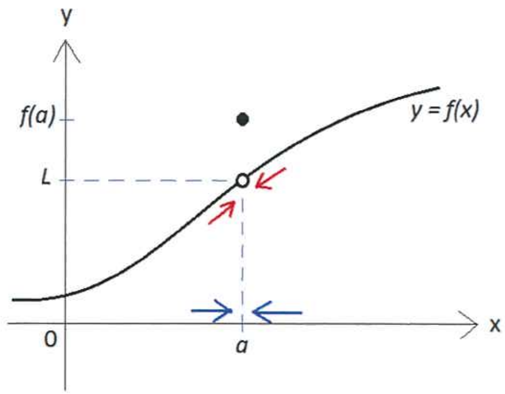
A more formal definition of $\lim_{x \to a} f(x) = L$ is that the difference between $f(x)$ and L can be made arbitrarily small when x is sufficiently close to but different from a.

Remarks:

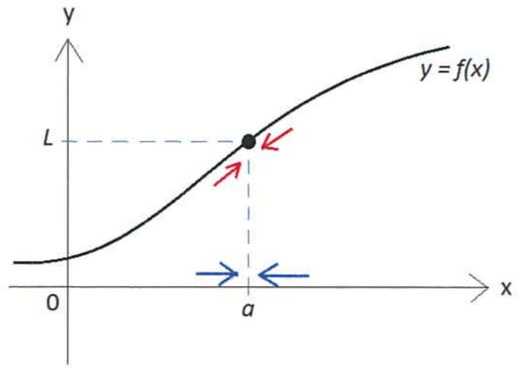
(i) $\lim_{x\to a} f(x)$ may exist even if f is not defined at $x = a$. $x \notin \mathrm{Dom}(f)$



(ii) $\lim_{x\to a} f(x)$ may exist even if $f(a) \neq \lim_{x\to a} f(x)$.



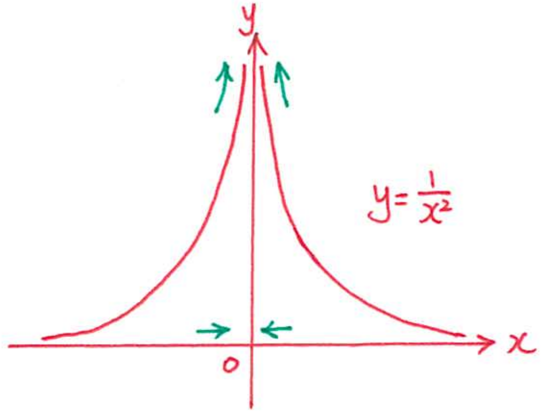
(iii) If $f(a) = \lim_{x \to a} f(x)$, then $f(x)$ is said to be **continuous** at $x = a$ (i.e. there is no break at $x = a$.)



not real number

(iv) If $\lim_{x \to a} f(x) = \infty$ (or $-\infty$), we say that the limit $\lim_{x \to a} f(x)$ **does not exist (DNE)**.

E.g. $\lim_{x \to 0} \frac{1}{x^2} = \infty$, so the limit does not exist.



Example of Remark (i):

$$\boxed{\lim_{x \to a} f(x) \text{ may exist even if } f \text{ is not defined at } x = a.}$$

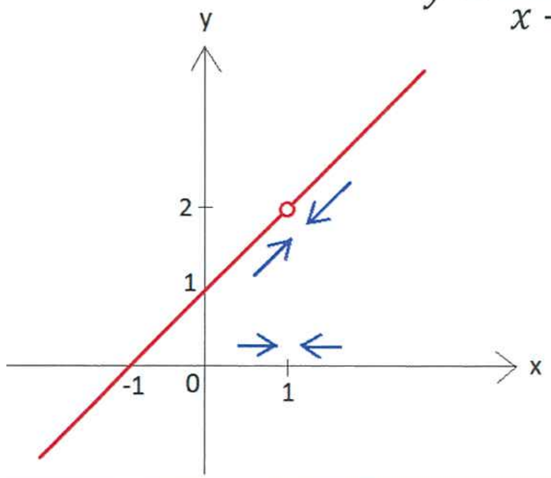
Consider the limit $\lim_{x \to 1} \frac{x^2-1}{x-1}$.

The function $\frac{x^2-1}{x-1}$ is not defined at $x = 1$. To evaluate the above limit, we consider the values of x approaching to 1, but not at $x = 1$, i.e. we assume $x \neq 1$.

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} \underbrace{=}_{\because x \neq 1} \lim_{x \to 1} (x+1) = 1 + 1 = 2$$

$\left(\frac{0}{0} \text{ form}\right)$

$y = \frac{x^2-1}{x-1} = x+1$ for $x \neq 1$



We see that the value of $\frac{x^2-1}{x-1}$ approaches 2 as x approaches 1.

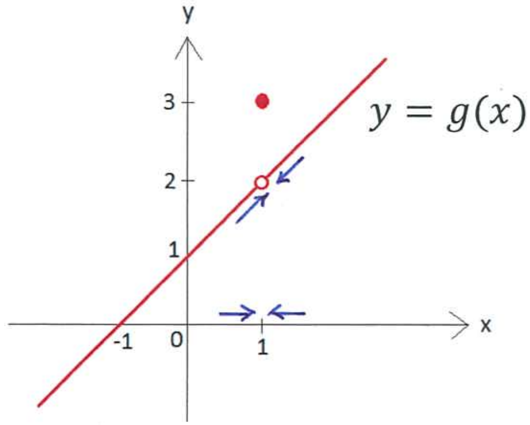
Example of Remark (ii):

$$\boxed{\lim_{x\to a} f(x) \text{ may exist even if } f(a) \neq \lim_{x\to a} f(x)}.$$

Consider the limit of the function $g(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$ at $x = 1$.

$$\lim_{x\to 1} g(x) \underbrace{=}_{\because x\neq 1} \lim_{x\to 1} \frac{x^2-1}{x-1} = \cdots = 2$$

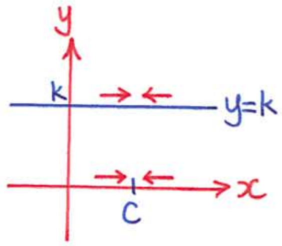
but $g(1) = 3$.

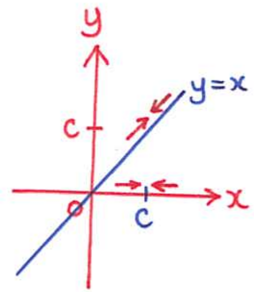


The limit $\lim_{x\to 1} g(x)$ exists but $\lim_{x\to 1} g(x) \neq g(1)$.

Theorems on limits

Let k be a constant, n be a positive integer, and f and g be functions for which $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ exist. Then





(1) $\lim\limits_{x\to c} k = k$, $\lim\limits_{x\to c} x = c$, $\lim\limits_{x\to c} k\,f(x) = k \lim\limits_{x\to c} f(x)$

(2) $\lim\limits_{x\to c}[f(x)+g(x)] = \lim\limits_{x\to c} f(x) + \lim\limits_{x\to c} g(x)$

(3) $\lim\limits_{x\to c}[f(x)-g(x)] = \lim\limits_{x\to c} f(x) - \lim\limits_{x\to c} g(x)$

(4) $\lim\limits_{x\to c}[f(x)\cdot g(x)] = \left(\lim\limits_{x\to c} f(x)\right)\cdot\left(\lim\limits_{x\to c} g(x)\right)$

(5) $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}$, provided $\lim\limits_{x\to c} g(x) \neq 0$

(6) $\lim\limits_{x\to c}[f(x)]^n = \left[\lim\limits_{x\to c} f(x)\right]^n$

(7) $\lim\limits_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to c} f(x)}$, provided $\lim\limits_{x\to c} f(x) \geq 0$ when n is even

Theorem

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ and

$g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$ be two **polynomials**. Then we have the results:

(i) $\lim\limits_{x \to c} f(x) = a_n \cdot c^n + a_{n-1} \cdot c^{n-1} + \cdots + a_1 \cdot c + a_0 = f(c)$.

(ii) If $g(c) \neq 0$, then $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}$.

Example 1

(a) $\lim\limits_{x \to 1} (x^2 + x - 6)$ (b) $\lim\limits_{x \to 1} \frac{x^3+1}{x+1}$ (c) $\lim\limits_{x \to -1} \frac{x^3+2}{x+1}$ (d) $\lim\limits_{x \to -1} \frac{x^3+1}{x+1}$

Solution

(a) $\lim\limits_{x \to 1} (x^2 + x - 6) = 1^2 + 1 - 6 = -4$ **(Evaluate this limit by substituting $x = 1$)**

(b) $\lim\limits_{x \to 1} \frac{x^3+1}{x+1} = \frac{1^3+1}{1+1} = \frac{2}{2} = 1$ **(Evaluate this limit by substituting $x = 1$)**

(c) $\lim_{x \to -1} \frac{x^3+2}{x+1}$ $\left(\frac{1}{0}\text{ form}\right)$ **(The function $\frac{x^3+2}{x+1}$ is undefined at $x=-1$ but the numerator is non-zero when $x=-1$)**

$= \frac{(-1)^3+2}{(-1)+1}$ **(Evaluate this limit by substituting $x=-1$)**

$= \frac{1}{0}$ which is undefined. $= \begin{cases} \infty & \text{if } x \text{ approaches } -1 \text{ from the right } (x \to -1^+) \\ -\infty & \text{if } x \text{ approaches } -1 \text{ from the left } (x \to -1^-) \end{cases}$

$\therefore$ The limit $\lim_{x \to -1} \frac{x^3+2}{x+1}$ does not exist.

(d) $\lim_{x \to -1} \frac{x^3+1}{x+1}$ $\left(\frac{0}{0}\text{ form}\right)$ **(The function $\frac{x^3+1}{x+1}$ is undefined at $x=-1$. Both the numerator and denominator are equal to 0 when $x=-1$.)**

$a^3+b^3=(a+b)(a^2-ab+b^2)$

$= \lim_{x \to -1} \frac{\cancel{(x+1)}(x^2-x+1)}{\cancel{x+1}}$ **(Factorize the numerator)**

$\underbrace{=}_{\because x \neq -1} \lim_{x \to -1} (x^2-x+1)$ **(Cancel common factor)**

$= (-1)^2-(-1)+1$ **(Evaluate this limit by substituting $x=-1$)**

$= 3$

Note:

$\frac{0}{1} \longrightarrow$ limit $= 0$

$\frac{1}{0} \longrightarrow$ Not defined $\therefore$ limit does not exist.

$\frac{0}{0} \longrightarrow$ indeterminate form

Examples of $\frac{0}{0}$ form:

Eg. 1:

$$\lim_{x\to 0} \frac{x^2}{x} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} x$$

$$= 0$$

Eg. 2:

$$\lim_{x\to 0} \frac{2x}{x} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} 2$$

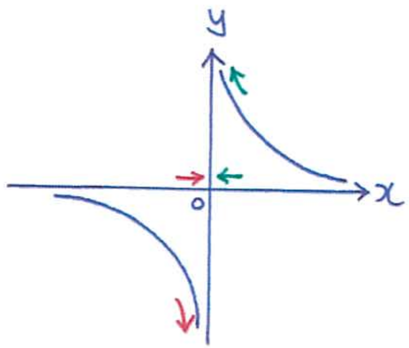
$$= 2$$

Eg. 3:

$$\lim_{x\to 0} \frac{x}{x^2} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{1}{x}$$

$$= \frac{1}{0} \text{ which is undefined.} \quad \left(= \begin{cases} -\infty & \text{if } x \to 0^- \\ \infty & \text{if } x \to 0^+ \end{cases} \right)$$



$$\therefore \lim_{x\to 0} \frac{x}{x^2} \text{ does not exist.}$$

Remark: The $\frac{\mathbf{0}}{\mathbf{0}}$ **form** is known as an **indeterminate form**.

Other indeterminate forms include $\frac{\infty}{\infty}$, $\infty-\infty$, $0\cdot\infty$, 0^0, 1^∞, ∞^0.

(see Ch.8 L'Hôpital's rule)

Example 2

Evaluate each of the following limits:

(a) $\displaystyle\lim_{x\to-2}\frac{x^2+x-2}{x^2+5x+6}$ (b) $\displaystyle\lim_{x\to4}\frac{\sqrt{x}-2}{x^2-16}$ (c) $\displaystyle\lim_{x\to2}\frac{4-x^2}{3-\sqrt{x^2+5}}$

Solution

(a) $\displaystyle\lim_{x\to-2}\frac{x^2+x-2}{x^2+5x+6}$ $\left(\frac{\mathbf{0}}{\mathbf{0}}\ \textbf{form}\right)$

$$= \lim_{x\to-2}\frac{\cancel{(x+2)}(x-1)}{\cancel{(x+2)}(x+3)}$$

$$\underbrace{=}_{\because x\neq-2} \lim_{x\to-2}\frac{x-1}{x+3}$$

$$= \frac{(-2)-1}{(-2)+3}$$

$$= -3$$

(b) $\lim_{x\to4}\frac{\sqrt{x}-2}{x^2-16}$ $\left(\frac{\mathbf{0}}{\mathbf{0}}\ \textbf{form}\right)$

$a^2-b^2=(a-b)(a+b)$

$$=\lim_{x\to4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x^2-16)(\sqrt{x}+2)}=\lim_{x\to4}\frac{\cancel{x-4}}{(\cancel{x-4})(x+4)(\sqrt{x}+2)}=\lim_{x\to4}\frac{1}{(x+4)(\sqrt{x}+2)}$$

$$=\frac{1}{(4+4)(\sqrt{4}+2)}=\frac{1}{32}$$

(c) $\lim_{x\to2}\frac{4-x^2}{3-\sqrt{x^2+5}}$ $\left(\frac{\mathbf{0}}{\mathbf{0}}\ \textbf{form}\right)$

$$=\lim_{x\to2}\left(\frac{4-x^2}{3-\sqrt{x^2+5}}\cdot\frac{3+\sqrt{x^2+5}}{3+\sqrt{x^2+5}}\right)$$

$$=\lim_{x\to2}\frac{(4-x^2)(3+\sqrt{x^2+5})}{9-(x^2+5)}$$

$$=\lim_{x\to2}\frac{(\cancel{4-x^2})(3+\sqrt{x^2+5})}{\cancel{4-x^2}}=\lim_{x\to2}\left(3+\sqrt{x^2+5}\right)=3+\sqrt{2^2+5}=6$$

Example 3

Evaluate the limit

$$\lim_{x \to a} \frac{x^n - a^n}{x - a}$$

where n is a positive integer.

Solution

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} \quad \left(\frac{0}{0} \textbf{ form}\right)$$

$$= \lim_{x \to a} \frac{(x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-1})}{x - a}$$

a^0 a^1

$$= \lim_{x \to a} (x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-1})$$

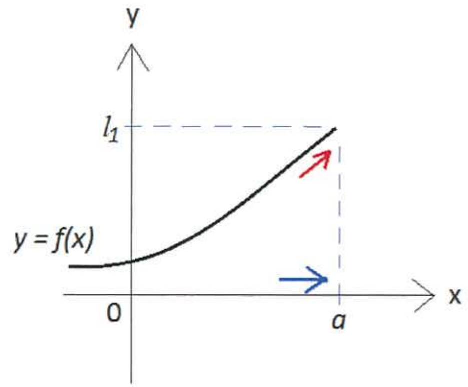
$$= \underbrace{a^{n-1} + a \cdot a^{n-2} + a^2 \cdot a^{n-3} + \cdots + a^{n-1}}_{n\ terms}$$

$$= na^{n-1}$$

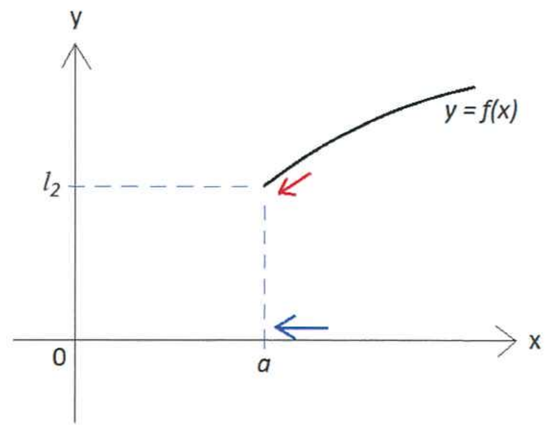
$$\begin{array}{r|l} & x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-1} \\ \hline x-a & x^n \qquad\qquad\qquad\qquad - a^n \\ & x^n - ax^{n-1} \\ \hline & ax^{n-1} \\ & ax^{n-1} - a^2x^{n-2} \\ \hline & a^2x^{n-2} \\ & a^2x^{n-2} - a^3x^{n-3} \\ \hline & a^3x^{n-3} \\ & \vdots \\ \hline & a^{n-1}x - a^n \\ & a^{n-1}x - a^n \\ \hline \end{array}$$

Left hand limit / Right hand limit

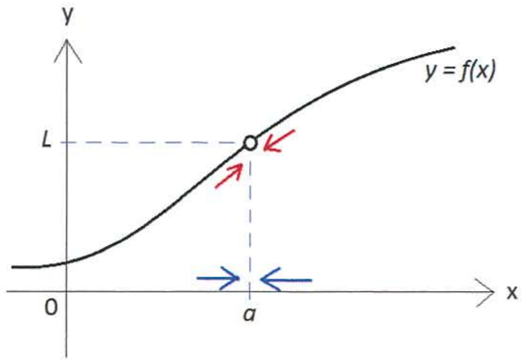
- The **left hand limit** of $f(x)$ at $x = a$ is $\lim_{x \to a^-} f(x) = l_1$ if the value of $f(x)$ approaches l_1 as x approaches a from the left. $x < a$



- The **right hand limit** of $f(x)$ at $x = a$ is $\lim_{x \to a^+} f(x) = l_2$ if the value of $f(x)$ approaches l_2 as x approaches a from the right. $x > a$



- If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$ (where L is a real number), we say that the limit $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x) = L$.



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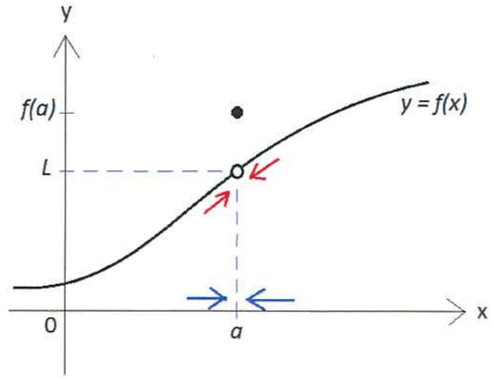

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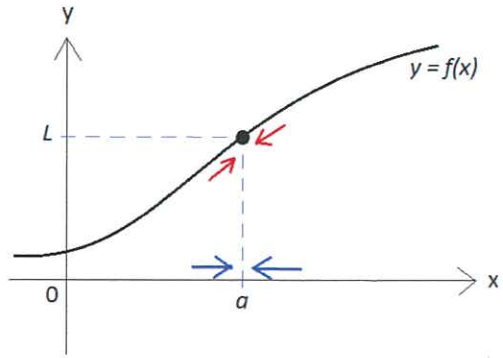
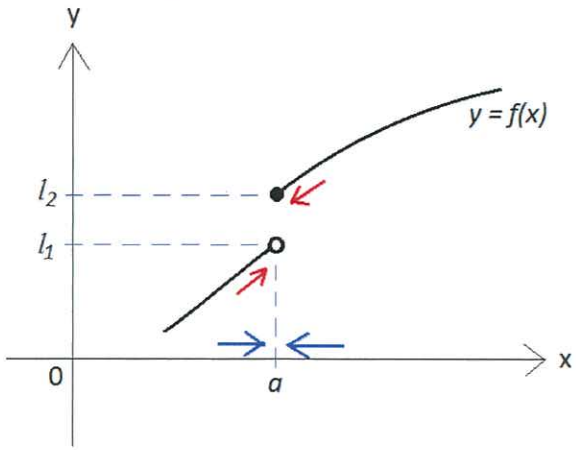


- If $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, the limit $\lim_{x\to a} f(x)$ does not exist.



Example 4

Does the limit of $f(x) = \dfrac{|x|}{x}$ exist at $x = 0$?

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

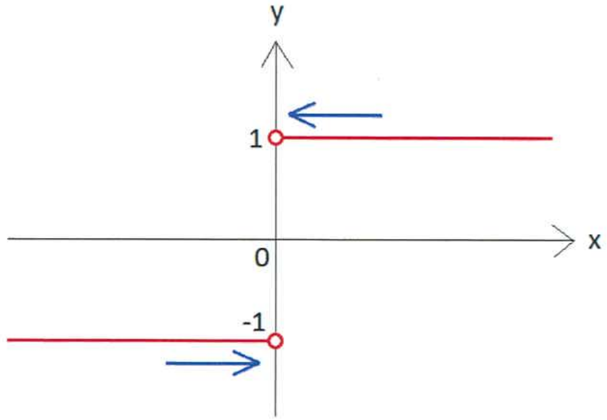
Solution

First note that the function $f(x) = \dfrac{|x|}{x}$ is not defined at $x = 0$.

Rewrite the function as

$$f(x) = \frac{|x|}{x} = \begin{cases} \dfrac{x}{x} & \text{if } x > 0 \\ \dfrac{-x}{x} & \text{if } x < 0 \end{cases}$$

$$= \begin{cases} \boxed{1} & \text{if } x > 0 \\ \boxed{-1} & \text{if } x < 0 \end{cases}$$



Since f(x) has different formulas when x<0 and x>0, we consider LHL & RHL separately.

LHL

Left hand limit: $\lim\limits_{x \to 0^-} f(x) \overset{x<0}{=} \lim\limits_{x \to 0^-} \dfrac{|x|}{x} = \lim\limits_{x \to 0^-} (\boxed{-1}) = -1$

RHL

Right hand limit: $\lim\limits_{x \to 0^+} f(x) \overset{x>0}{=} \lim\limits_{x \to 0^+} \dfrac{|x|}{x} = \lim\limits_{x \to 0^+} (\boxed{1}) = 1$

Since $\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$, the limit $\lim\limits_{x \to 0} f(x)$ does not exist.

Example 5

(see Ch.2)

Consider the function $f(x) = [x] =$ "greatest integer $\leq x$" (the greatest integer function), find the limits (a) $\lim_{x\to 0.5} f(x)$ and (b) $\lim_{x\to 1} f(x)$ if they exist.

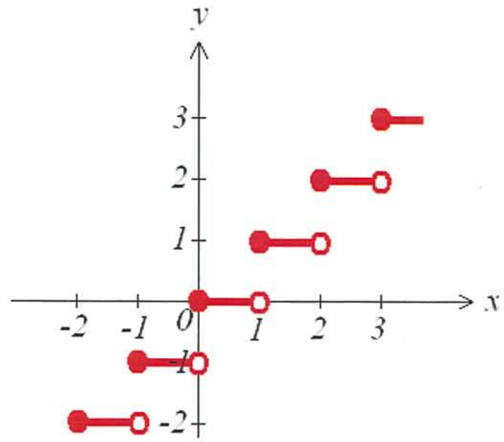
Solution:

take the next smaller integer if x is not an integer

$f(x) = [x] =$ "greatest integer $\leq x$".

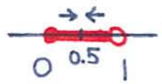
E.g. $f(2.8) = [2.8] = 2$, $f(1) = [1] = 1$.

Its graph is shown on the right.



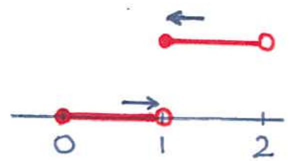
(a) $\because \lim_{x\to 0.5^-} f(x) = 0 = \lim_{x\to 0.5^+} f(x)$,

$\therefore \lim_{x\to 0.5} f(x)$ exists and $\lim_{x\to 0.5} f(x) = 0$.



(b) Consider the function in the neighborhood of $x = 1$.

$$[x] = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 \leq x < 2 \end{cases}.$$



Since $\lim_{x\to 1^-} f(x) = 0 \neq 1 = \lim_{x\to 1^+} f(x)$, the limit $\lim_{x\to 1} f(x)$ does not exist.

$x<1$ $x>1$

Example 6

Consider the function $f(x) = \begin{cases} 2x\sin\left(\frac{x}{2}\right) & \text{if } x \le \pi \\ \frac{x^2-\pi^2}{x-\pi} & \text{if } x > \pi. \end{cases}$

Since the formula of f(x) changes at $x=\pi$, we evaluate LHL & RHL separately.

Does the limit $\lim_{x\to\pi} f(x)$ exist? Find the value of the limit if it exists.

Solution

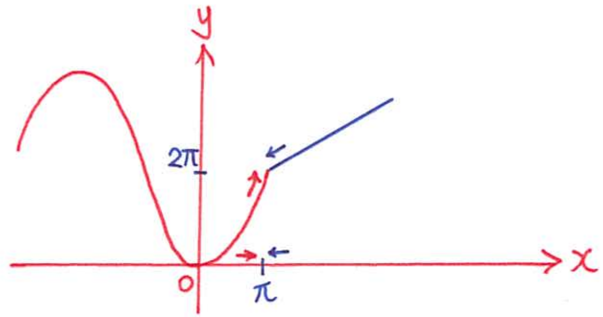
LHL: $\lim_{x\to\pi^-} f(x) \overset{x<\pi}{=} \lim_{x\to\pi^-} 2x\sin\left(\frac{x}{2}\right) = 2\pi \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} = 2\pi$

RHL: $\lim_{x\to\pi^+} f(x) \overset{x>\pi}{=} \lim_{x\to\pi^+} \frac{x^2-\pi^2}{x-\pi}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to\pi^+} \frac{\cancel{(x-\pi)}(x+\pi)}{\cancel{x-\pi}} = \lim_{x\to\pi^+}(x+\pi) = \pi+\pi = 2\pi$$

$\because \lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^+} f(x) = 2\pi$

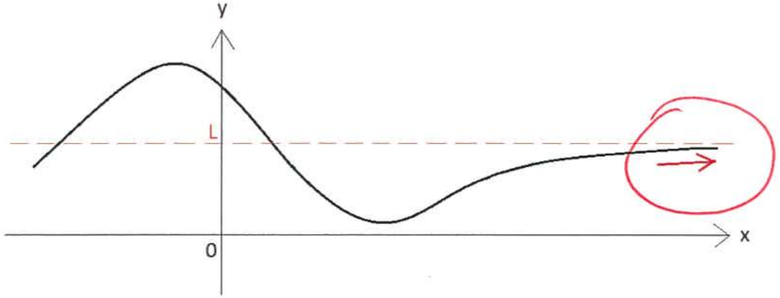
$\therefore \lim_{x\to\pi} f(x)$ exists and $\lim_{x\to\pi} f(x) = 2\pi$



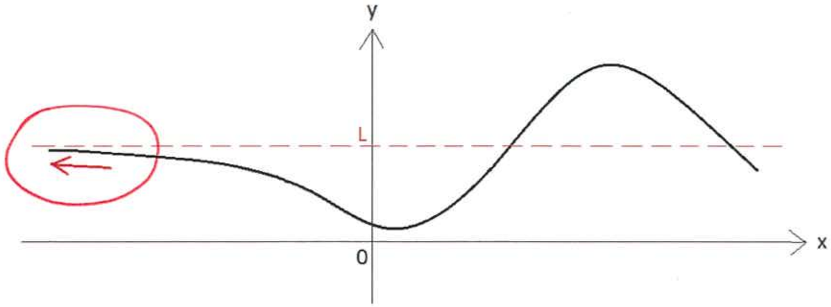
Limit at infinity

➤ $\boxed{\lim_{x \to \infty} f(x) = L}$ means "as x increases indefinitely, $f(x)$ tends to L".

means $+\infty$

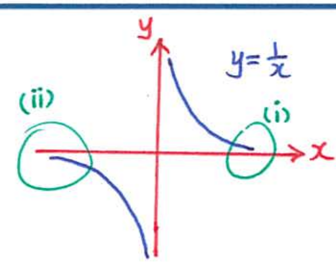


➤ $\boxed{\lim_{x \to -\infty} f(x) = L}$ means "as x decreases indefinitely, $f(x)$ tends to L".



Useful results:

$\because \lim_{x \to \infty} \frac{1}{x^n} = \left(\lim_{x \to \infty} \frac{1}{x}\right)^n = 0^n = 0$



(i) $\lim_{x \to \infty} \frac{1}{x} = 0 \quad \Rightarrow \quad \lim_{x \to \infty} \frac{1}{x^n} = 0$ for $n > 0$

(ii) $\lim_{x \to -\infty} \frac{1}{x} = 0 \quad \Rightarrow \quad \lim_{x \to -\infty} \frac{1}{x^n} = 0$ for $n > 0$ and whenever x^n is defined for $x < 0$.

Example 7

Evaluate each of the following limits:

(a) $\lim\limits_{x\to\infty} \dfrac{5x^4 - 2x^3 + 4x - 1}{2x^4 + 3x^2 + 4}$ (b) $\lim\limits_{x\to-\infty} \dfrac{3x^2 - 5x + 4}{4x - 3}$ (c) $\lim\limits_{x\to\infty} \dfrac{x^3 + 7x - 2}{5x^4 + 6x^3 - 1}$

Solution

(a) $\lim\limits_{x\to\infty} \dfrac{5x^4-2x^3+4x-1}{2x^4+3x^2+4}$ **($\frac{\infty}{\infty}$ form, which is an indeterminate form)**

$$= \lim_{x\to\infty} \frac{\frac{5x^4-2x^3+4x-1}{x^4}}{\frac{2x^4+3x^2+4}{x^4}}$$

(Divide both the numerator and denominator by x^p where p = degree of denominator)

$$= \lim_{x\to\infty} \frac{5 - \overset{0}{\frac{2}{x}} + \overset{0}{\frac{4}{x^3}} - \overset{0}{\frac{1}{x^4}}}{2 + \underset{0}{\frac{3}{x^2}} + \underset{0}{\frac{4}{x^4}}} = \frac{5-0+0-0}{2+0+0}$$

(since $\lim\limits_{x\to\infty} \frac{1}{x^n} = 0$ if n is a positive integer)

$$= \frac{5}{2}$$

(b) $\lim_{x\to-\infty}\frac{3x^2-5x+4}{4x-3}$ $\left(\frac{\infty}{-\infty}\textbf{ form}\right)$

$$=\lim_{x\to-\infty}\frac{\frac{3x^2-5x+4}{x}}{\frac{4x-3}{x}}$$ **(Divide both the numerator and denominator by x)**

$$=\lim_{x\to-\infty}\frac{\overset{-\infty}{3x}-5+\overset{0}{\frac{4}{x}}}{4-\underset{0}{\frac{3}{x}}}=\frac{-\infty}{4}=-\infty$$ not a real number

$\therefore$ The limit $\lim_{x\to-\infty}\frac{3x^2-5x+4}{4x-3}$ does not exist.

(c) $\lim_{x\to\infty}\frac{x^3+7x-2}{5x^4+6x^3-1}$ $\left(\frac{\infty}{\infty}\textbf{ form}\right)$

$$=\lim_{x\to\infty}\frac{\frac{x^3+7x-2}{x^4}}{\frac{5x^4+6x^3-1}{x^4}}$$ **(Divide both the numerator and denominator by x^4)**

$$=\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{7}{x^3}-\frac{2}{x^4}}{5+\frac{6}{x}-\frac{1}{x^4}}=\frac{0+0-0}{5+0-0}=\frac{0}{5}=0$$

Example 8

Evaluate the following limits (a) $\lim_{x\to\infty}\frac{2x-3}{\sqrt{4x^2+7}-5x}$ (b) $\lim_{x\to-\infty}\frac{2x-3}{\sqrt{4x^2+7}-5x}$

Solution

(a) $\lim_{x\to\infty}\frac{2x-3}{\sqrt{4x^2+7}-5x}$ $\left(\frac{\infty}{-\infty}\text{ form}\right)$

$$= \lim_{x\to\infty}\frac{\frac{2x-3}{x}}{\frac{\sqrt{4x^2+7}-5x}{x}} \underbrace{=}_{\because x=\sqrt{x^2}\text{ for } x>0} \lim_{x\to\infty}\frac{2-\frac{3}{x}}{\sqrt{\frac{4x^2+7}{x^2}}-5} = \lim_{x\to\infty}\frac{2-\frac{3}{x}}{\sqrt{4+\frac{7}{x^2}}-5} = \frac{2-0}{\sqrt{4+0}-5} = -\frac{2}{3}$$

$\frac{3}{x}\to 0$, $\frac{7}{x^2}\to 0$

(b) $\lim_{x\to-\infty}\frac{2x-3}{\sqrt{4x^2+7}-5x}$ $\left(\frac{-\infty}{\infty}\text{ form}\right)$

$$= \lim_{x\to-\infty}\frac{\frac{2x-3}{x}}{\frac{\sqrt{4x^2+7}-5x}{x}} \underbrace{=}_{\because x=-\sqrt{x^2}\text{ for } x<0} \lim_{x\to-\infty}\frac{2-\frac{3}{x}}{-\sqrt{\frac{4x^2+7}{x^2}}-5} = \lim_{x\to-\infty}\frac{2-\frac{3}{x}}{-\sqrt{4+\frac{7}{x^2}}-5} = \frac{2-0}{-\sqrt{4+0}-5}$$

$$= -\frac{2}{7}$$

Example 9

Evaluate the limit $\lim_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-x}\right)$.

Solution

$$\lim_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-x}\right)$$ (∞ − ∞ **form, which is an indeterminate form**)

$$=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+3x}-\sqrt{x^2-x}\right)\left(\sqrt{x^2+3x}+\sqrt{x^2-x}\right)}{\sqrt{x^2+3x}+\sqrt{x^2-x}}$$ $\because a^2-b^2=(a-b)(a+b)$

$$=\lim_{x\to\infty}\frac{(x^2+3x)-(x^2-x)}{\sqrt{x^2+3x}+\sqrt{x^2-x}}=\lim_{x\to\infty}\frac{4x}{\sqrt{x^2+3x}+\sqrt{x^2-x}}$$

$$=\lim_{x\to\infty}\frac{\frac{4x}{x}}{\frac{\sqrt{x^2+3x}+\sqrt{x^2-x}}{x}}=\lim_{x\to\infty}\frac{4}{\sqrt{\frac{x^2+3x}{x^2}}+\sqrt{\frac{x^2-x}{x^2}}}=\lim_{x\to\infty}\frac{4}{\sqrt{1+\frac{3}{x}}+\sqrt{1-\frac{1}{x}}}$$

$\because x=\sqrt{x^2}$ for $x>0$

$$=\frac{4}{\sqrt{1+0}+\sqrt{1-0}}=2$$