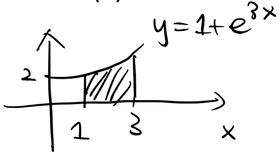
MA1201 Calculus and Basic Linear Algebra II **Problem Set 3** **Application of Integration**

Area of the region

Problem 1

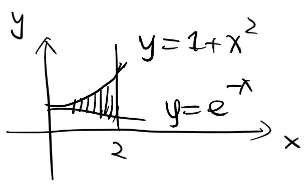
(a) Find the area of the region bounded by the curve $y = 1 + e^{3x}$, x-axis, $x = 1$ and $x = 3$.



$$\text{Area} = \int_1^3 (1+e^{3x})\,dx = x + \frac{1}{3}e^{3x}\Big|_1^3$$

$$= 3 + \frac{1}{3}e^9 - 1 - \frac{1}{3}e^3 = 2 + \frac{1}{3}(e^9 - e^3)$$

(c) Find the area of the region bounded by $y = 1 + x^2$, $y = e^{-x}$, y-axis and $x = 2$.



$$\text{Area} = \int_0^2 (1+x^2 - e^{-x})\,dx = x + \frac{x^3}{3} + e^{-x}\Big|_0^2$$

$$= 2 + \frac{8}{3} + e^{-x} - 0 - 0 - e^{-0} = \frac{4}{3} + e^{-x}.$$

Problem 2

In this problem, we would like to find the area of the region bounded by the curves $y = e^{2x} - 3e^x - 1$ and $y = e^x - 4$ for $-2 \le x \le 2$. In order to find the area, it is important to compare the values between $f_1(x) = e^{2x} - 3e^x - 1$ and $f_2(x) = e^x - 4$ for $-2 \le x \le 2$ so that we can determine the "upper curve" and "lower curve". This problem will show you a general technique (which is taught in the lecture) to achieve this goal.

(a) Find all *critical points* by solving the equation $f_1(x) = f_2(x)$ for $-2 \le x \le 2$. These critical points are the points where the relative magnitude between $f_1(x)$ and $f_2(x)$ changes.

(b) With the critical points obtained in (a), we divide the interval $[-2, 2]$ into several small intervals with the critical points as the "cutoff" points. For each small interval, determine which function ($f_1(x)$ or $f_2(x)$) is larger.
(Hint: You may compare the values by simply substituting some value of x within the small interval.)

(c) Using the information obtained in (b), find the area of the region bounded by the curves $y = e^{2x} - 3e^x - 1$ and $y = e^x - 4$ for $-2 \le x \le 2$.

(a). $f_1(x) = f_2(x)$.

$e^{2x} - 3e^x - 1 = e^x - 4$. $\Rightarrow$ $e^{2x} - 4e^x + 3 = 0$. let $y = e^x$.

$y^2 - 4y - 3 = 0$.

$(y-1)(y-3) = 0$. $\Rightarrow$ $y = 1$ or $y = 3$. $\Rightarrow$ $x = \ln 1 = 0$ or $x = \ln 3$.

x	$-2 \le x \le 0$	$0 < x < \ln 3$	$\ln 3 \le x \le 2$.
$f_1(x)$ vs $f_2(x)$.	$f_1(x) > f_2(x)$.	$f_1(x) < f_2(x)$.	$f_1(x) > f_2(x)$.

$x = -\ln 2$ $\quad f_1(x) = e^{-2\ln 2} - 3e^{-\ln 2} - 1 = -4 + 6 - 1 = 1.$

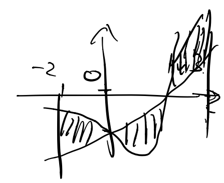
$\quad f_2(x) = e^{-\ln 2} - 4 = -6.$

$x = \ln 2$ $\quad f_1(x) = e^{2\ln 2} - 3e^{\ln 2} - 1 = 4 - 6 - 1 = -3.$

$\quad f_2(x) = e^{\ln 2} - 4 = -2.$

$x = \ln 4$ $\quad f_1(x) = e^{2\ln 4} - 3e^{\ln 4} - 1 = 16 - 12 - 1 = 3.$

$\quad f_2(x) = e^{\ln 4} - 4 = 0.$



conclusion : $f_1(x) > f_2(x)$ for $-2 \le x \le 0$ or $\ln 3 \le x \le 2.$

$f_1(x) < f_2(x)$ for $0 < x < \ln 3.$

(c). Area $= \int_{-2}^{0} (f_1(x) - f_2(x))\,dx + \int_{0}^{\ln 3} (f_2(x) - f_1(x))\,dx + \int_{\ln 3}^{2} (f_1(x) - f_2(x))\,dx$

$= \int_{-2}^{0} (e^{2x} - 4e^{x} + 3)\,dx + \int_{0}^{\ln 3} (-e^{2x} + 4e^{x} - 3)\,dx + \int_{\ln 3}^{2} (e^{2x} - 4e^{x} + 3)\,dx.$

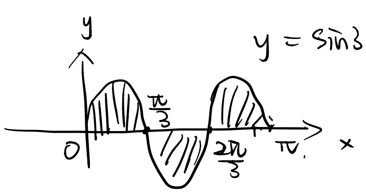
$= \frac{1}{2}e^{2x} - 4e^{x} + 3 \Big|_{-2}^{0} + (-\frac{1}{2}e^{2x} + 4e^{x} - 3)\Big|_{0}^{\ln 3} + \frac{1}{2}e^{2x} - 4e^{x} + 3\Big|_{\ln 3}^{2}.$

$= -\frac{1}{2}e^{-4} + 4e^{-2} + \frac{1}{2}e^{4} - 4e^{2} - 6\ln 3 + 14.$

Volume

Problem 4

(a) Find the volume of the solid generated by rotating the region bounded by $y = \sin 3x$, x-axis for $0 \le x \le \pi$ about the x-axis for one complete revolution.



$y = \sin 3x.$ $\quad V = \pi \int_a^b f^2(x)\,dx.$

$V = \pi \int_0^{\frac{\pi}{3}} \sin^2 3x\,dx + \pi \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin^2 3x\,dx + \pi \int_{\frac{2\pi}{3}}^{\pi} \sin^2 3x\,dx.$

$= \pi \int_0^{\pi} \sin^2 3x\,dx.$

$= \pi \int_0^{\pi} -\frac{1}{2}[\cos(3x+3x) - \cos(3x-3x)]\,dx.$

$= \frac{\pi}{2} \int_0^{\pi} (1 - \cos 6x)\,dx$

$= \frac{\pi}{2}[x - \frac{1}{6}\sin 6x]\Big|_0^{\pi} = \frac{\pi^2}{2} - \frac{1}{6}\sin 6\pi - \frac{1}{6}\sin 0$

$= \frac{\pi^2}{2}$

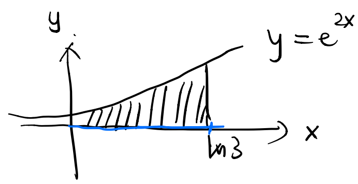
(c) Find the volume of the solid generated by rotating the region bounded by $y = e^{2x}$, x-axis, y-axis and $x = \ln 3$ about

(i) the x-axis for 1 complete revolution.

(ii) the y-axis for 1 complete revolution.

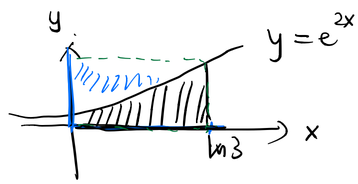
(iii) $y = -1$ for 1 complete revolution.

(iv) $x = -1$ for 1 complete revolution.



(i). $V = \int_0^{\ln 3} \pi (e^{2x})^2 dx = \pi \int_0^{\ln 3} e^{4x} dx$

$= \frac{\pi}{4} e^{4x} \Big|_0^{\ln 3} = 20\pi.$



(ii) $V = \pi \int_c^d f^2(y)\, dy$

$V = \int_0^9 \pi (\ln 3)^2 dy - \int_1^9 \pi \left(\frac{1}{2}\ln y\right)^2 dy$

$= 9\pi(\ln 3)^2 - \frac{\pi}{4}\int_1^9 (\ln y)^2 dy$

$x=0,\ y=1$

$x=\ln 3.\ y=9.$

$y=e^{2x} \Rightarrow x = \frac{1}{2}\ln y.$

let $u=(\ln y)^2$
$dv = dy$
$v=\int dy = y.$

$= 9\pi(\ln 3)^2 - \frac{\pi}{4}\left[y(\ln y)^2\Big|_1^9 - \int_1^9 2 \cdot \ln y \cdot dy\right]$

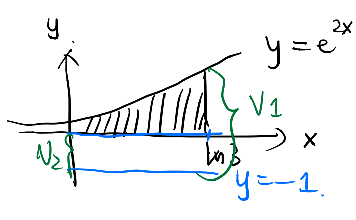
$= 9\pi(\ln 3)^2 - \frac{\pi}{4}\cdot 9(\ln 9)^2 + \frac{\pi}{2}\int_1^9 \ln y\, dy.$

let $u = \ln y$
$dv = dy$
$v = \int dy = y.$

$= 9\pi(\ln 3)^2 - \frac{9}{4}\pi(\ln 9)^2 + \frac{\pi}{2}\left[y\ln y\Big|_1^9 - \int_1^9 1\, dy\right]$

$= 9\pi(\ln 3)^2 - \frac{9}{4}\pi(2\ln 3)^2 + \frac{\pi}{2}\cdot 9 \cdot 2\ln 3 - 8 \cdot \frac{\pi}{2}.$

$= 9\pi\ln 3 - 4\pi.$

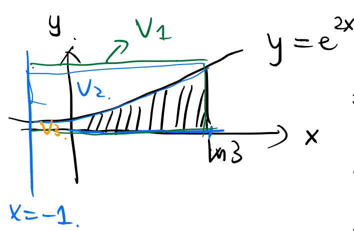


(iii) $V = V_1 - V_2 = \pi\int_0^{\ln 3}(e^{2x}+1)^2 dx - \pi\int_0^{\ln 3} 1^2 dx$

$= \pi\int_0^{\ln 3}(e^{4x} + 2e^{2x} + 1 - 1)\,dx$

$= \pi\left[\frac{1}{4}e^{4x} + e^{2x}\right]\Big|_0^{\ln 3}$

$= 28\pi.$



(iv). $V = V_1 - V_2 - V_3$

$= \int_0^9 \pi(\ln 3+1)^2 dy - \int_1^9 \pi\left(\frac{1}{2}\ln y + 1\right)^2 dy - \int_0^1 \pi(1)^2 dy$

$= \pi(\ln 3+1)^2 9 - \pi - \int_1^9 \pi\left(\frac{1}{4}(\ln y)^2 + \ln y + 1\right)dy$

$= 9\pi(\ln 3+1)^2 - \pi - \frac{\pi}{4}\int_1^9 (\ln y)^2 dy - \pi\int_1^9 \ln y\, dy - \pi\int_1^9 1\, dy$

$= 9\pi\ln 3 + 6\pi.$

<u>Arc Length</u>

Problem 5

(a) Find the arc length of the curve $y = \ln(\sec x)$ for $0 \le x \le \frac{\pi}{4}$.

$$\text{arc length} = \int_0^{\frac{\pi}{4}} \sqrt{1+\left(\frac{d}{dx}\ln(\sec x)\right)^2}\,dx. \qquad S = \int_a^b \sqrt{1+(f'(x))^2}\,dx.$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{1+\left(\frac{1}{\sec x}\cdot\sec x\cdot\tan x\right)^2}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2 x}\,dx = \int_0^{\frac{\pi}{4}} \sec x\,dx = \ln|\sec x+\tan x|\Big|_0^{\frac{\pi}{4}}$$

$$= \ln|1+\sqrt{2}|$$

(c) Find the arc length of the curve $(y-1)^3 = \frac{9}{4}x^2$ for $0 \le x \le \frac{2}{3}(3)^{\frac{3}{2}}$.

$$(y-1)^3 = \frac{9}{4}x^2 \Rightarrow y = 1+\sqrt[3]{\frac{9}{4}x^2} \Rightarrow y = 1+\sqrt[3]{\frac{9}{4}}\cdot x^{\frac{2}{3}}.$$

$$S = \int_0^{\frac{2}{3}(3)^{\frac{3}{2}}} \sqrt{1+\left(\frac{d}{dx}\left(1+\sqrt[3]{\frac{9}{4}}x^{\frac{2}{3}}\right)\right)^2}\,dx.$$

$$= \int_0^{\frac{2}{3}(3)^{\frac{3}{2}}} \sqrt{1+\left(\frac{2}{3}\right)^{\frac{2}{3}}x^{-\frac{2}{3}}}\,dx.$$

$$= \int_0^{\frac{2}{3}(3)^{\frac{3}{2}}}$$