$$X(s) = \frac{(b_0 s^M + b_1 s^{M-1} + \cdots + b_M) \sim N(s)}{(a_0 s^N + a_1 s^{N-1} + \cdots + a_N s^0) \sim D(s)}$$

$$(M < N) = \frac{N(s)}{\prod_{i=1}^{N} (s - s_{p_i})}$$

Partial Fraction Expansion

- Assuming the poles are simple, the partial fraction expansion of rational $X(s)$ with $M < N$ is given by

$$\boxed{X(s) = \frac{N(s)}{D(s)}} = \frac{b_0 s^M + b_1 s^{M-1} + \cdots + b_M}{(s - s_{p_1})(s - s_{p_2}) \cdots (s - s_{p_N})}$$

$$= \boxed{\frac{A_1}{s - s_{p_1}}} + \boxed{\frac{A_2}{s - s_{p_2}}} + \cdots + \boxed{\frac{A_N}{s - s_{p_N}}} \quad (1)$$

- The coefficients $\{A_l\}$ are computed as

$$A_l = (s - s_{pl}) X(s) \Big|_{s = s_{pl}} \qquad M < N$$

- For $M < N$, $X(s)$ is a proper rational function.

s_{p_i} : roots of $D(s)$, $1 \le i \le N$

Example

- Find the partial-fraction expansion of

$$X(s) = \frac{s+0.5}{(s+1)(s+2)}$$

- Partial-fraction expansion.

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

where

$$A_1 = (s+1)X(s)|_{s=-1} = \cancel{(s+1)} \frac{s+0.5}{\cancel{(s+1)}(s+2)}\Big|_{s=-1} = \frac{s+0.5}{(s+2)}\Big|_{s=-1} = \frac{-0.5}{1} = -0.5$$

$$A_2 = (s+2)X(s)|_{s=-2} = \cancel{(s+2)} \frac{s+0.5}{(s+1)\cancel{(s+2)}}\Big|_{s=-2} = \frac{s+0.5}{(s+1)}\Big|_{s=-2} = \frac{-1.5}{-1} = 1.5$$

We have $\quad X(s) = \dfrac{-0.5}{s+1} + \dfrac{1.5}{s+2}$

$$\underbrace{X(s)}_{\times (s-s_{p_1})} = \underbrace{\frac{N(s)}{\prod_{i=1}^{N}(s-s_{p_i})}}_{(s-s_{p_1})\times} = \left(\frac{A_1}{\underbrace{s-s_{p_1}}} + \frac{A_2}{s-s_{p_2}} + \cdots + \frac{A_N}{s-s_{p_N}}\right)(s-s_{p_1})$$

$$A_1 = \left.(s-s_{p_1})X(s)\right|_{s=s_{p_1}} \quad ①$$

$$\Rightarrow \left.\frac{N(s)}{\prod_{i=2}^{N}(s-s_{p_i})}\right|_{s=s_{p_1}} \quad ②$$

if $s=s_{p_1}$

$$\Rightarrow A_1 + \underbrace{(s-s_{p_1})}_{=0}\left\{\frac{A_2}{s-s_{p_2}} + \cdots + \frac{A_N}{s-s_{p_N}}\right\}$$

$$X(s) = \left(\frac{s+0.5}{\underbrace{(s+1)}(s+\cancel{2})}\right) \times \cancel{(s+2)}$$

$$= \frac{C_1}{\boxed{s+1}} + \frac{C_2}{s+2} = \frac{1.5}{s+2} - \frac{0.5}{s+1}$$

$$C_1 = (\underline{s+1})\,X(s) = \left.\frac{s+0.5}{s+2}\right|_{s=-1}$$

$$= \frac{-0.5}{1} = -0.5$$

$$C_2 = (\underbrace{s+2})\,X(s) = \left.\frac{(s+0.5)}{(s+1)}\right|_{s=-2}$$

$$= \frac{-1.5}{-1} = 1.5.$$

$(2s-5) + \frac{15}{(s+2)} - \frac{2}{(s+1)}$

Partial Fraction Expansion Cont.

- If $M > N$, we need to use long division to express the rational function as the sum of quotient plus a proper rational function.
- Example of long division:

$$X(s) = \frac{2s^3 + s^2 + 2s + 1}{s^2 + 3s + 2} = (2s-5) + \frac{13s+11}{(s+2)(s+1)}$$

Use the long division to convert the rational function to a proper one:

$$\begin{array}{r l}
 & 2s \;-\; 5 \\
s^2 + 3s + 2 & \overline{)\, 2s^3 + s^2 + 2s + 1} \\
 & 2s^3 + 6s^2 + 4s \\
\hline
 & -5s^2 - 2s + 1 \\
 & -5s^2 - 15s - 10 \\
\hline
 & 13s + 11
\end{array}$$

$-26 + 11$

$$\frac{13s+11}{(s+2)(s+1)} = \frac{a_1}{s+2} + \frac{a_2}{s+1}$$

$$a_1 = \frac{13s+11}{s+1}\Big|_{s=-2} = 15$$

$$a_2 = \frac{13s+11}{s+2}\Big|_{s=-1} = \frac{-2}{1} = -2$$

Cont.

After long division, $X(s) = 2s - 5 + \frac{13s+11}{s^2+3s+2} = 2s - 5 + \frac{13s+11}{(s+1)(s+2)}$

Taking partial fraction expansion of the second term,

$$X(s) = 2s - 5 + \frac{A_1}{s+1} + \frac{A_2}{s+2},$$

where

$$A_1 = (s+1)X(s)|_{s=-1} = \frac{13s+11}{s+2}|_{s=-1} = -2,$$

$$A_2 = (s+2)X(s)|_{s=-2} = \frac{13s+11}{s+1}|_{s=-2} = 15.$$

Finally, the partial fraction expansion of $X(s)$ is

$$X(s) = 2s - 5 + \frac{-2}{s+1} + \frac{15}{s+2}$$