MA2170 Exercise on Chapter 3 Vectors

1. It is given that $\vec{a}$, $\vec{b}$ are vectors in $\mathbf{R}^3$ with $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$.
Determine whether each of the following is **T**rue of **F**alse.

(a) $\left|\vec{a}+\vec{b}\right| \leq \left|\vec{a}\right| + \left|\vec{b}\right|$ (b) $\left|\vec{a}\right|^3 > 0$

(c) $\text{proj}_{\vec{a}}\vec{b}$ is parallel to $\vec{b}$ (d) $\vec{a} \cdot \vec{b} > 0$

(e) $\left|\vec{a}\times\vec{b}\right| > 0$ (f) $\left(\vec{a}+\vec{b}\right)\cdot\left(\vec{a}+\vec{b}\right) = \vec{a}^2 + 2\vec{a}\vec{b} + \vec{b}^2$

(g) $\left(\vec{a}+\vec{b}\right)\cdot\left(\vec{a}+\vec{b}\right) = \vec{a}\cdot\vec{a} + 2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} = \left|\vec{a}\right|^2 + 2\vec{a}\cdot\vec{b} + \left|\vec{b}\right|^2$

(h) $\left(\vec{a}+\vec{b}\right)\times\left(\vec{a}+\vec{b}\right) = \vec{a}\times\vec{a} + 2\vec{a}\times\vec{b} + \vec{b}\times\vec{b} = 2\vec{a}\times\vec{b}$

(i) The vectors $\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ -1 \\ -\frac{1}{2} \end{pmatrix}$ are linearly dependent.

(j) If $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ is a given non-zero vector and $\vec{x}\cdot\vec{v} = \vec{y}\cdot\vec{v}$, where $\vec{x}$, $\vec{y}$ are vectors in $\mathbf{R}^3$, then $\vec{x} = \vec{y}$.

2. Let $A(2, 3, -4)$, $B(-1, 7, 12)$, $C(0, 2, 6)$ and $D(3, -2, -10)$ are four points in $\mathbf{R}^3$.

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{DC}$. (b) Show that $ABCD$ is a parallelogram.

3. Given the points $P_1(3, 4, -2)$, $P_2(0, 3, -1)$ and $P_3(2, 0, -4)$.

(a) Find $\left|\overrightarrow{P_1P_2}\right|$ and $\left|\overrightarrow{P_1P_3}\right|$. (b) Find the unit vector in the same direction as $\overrightarrow{P_1P_2}$.

(c) Find $\overrightarrow{P_1P_2}\cdot\overrightarrow{P_1P_3}$. (d) Find the angle between $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$.

4. Given $\vec{a} = -2\vec{i} + \vec{j} - 5\vec{k}$. Find the direction cosines of $\vec{a}$.

5. Given the vectors $\vec{a} = \vec{i} + \vec{j} - 2\vec{k}$ and $\vec{b} = 3\vec{j} + 5\vec{k}$. Find

(a) the projection vector of $\vec{a}$ on $\vec{b}$, (b) the projection vector of $\vec{b}$ on $\vec{a}$.

6. Given the vectors $\vec{b} = 2\vec{i} + \vec{j} - 5\vec{k}$ and $\vec{c} = -\vec{i} + 2\vec{j} - \vec{k}$.

(a) Find $\vec{b}\times\vec{c}$.
(b) Find the area of parallelogram with $\vec{b}$ and $\vec{c}$ as the adjacent sides.
(c) Find the area of triangle with $\vec{b}$ and $\vec{c}$ as the adjacent sides.

7. Given the points $A(4, 5, 1)$, $B(4, 2, 1)$ and $C(3, 4, 0)$.

(a) Find the area of parallelogram with AB and AC as the adjacent sides.
(b) Find the area of triangle ABC.

8. Given the vectors $\vec{a} = 3\vec{i} - 2\vec{j} - \vec{k}$, $\vec{b} = -\vec{i} + 2\vec{j} - 5\vec{k}$ and $\vec{c} = -\vec{i} + 2\vec{j} - \vec{k}$.
Find the volume of parallelepiped with $\vec{a}$, $\vec{b}$ and $\vec{c}$ as edges.

9. Find the volume of parallelepiped with AB, AC and AD as edges, where A(2, –1, –4), $B(-1, 0, 4)$, $C(3, -2, 0)$ and $D(2, -1, -2)$.

10. Find the volume of tetrahedron with $\vec{a} = \vec{i} - 3\vec{j} + \vec{k}$, $\vec{b} = \vec{i} + 3\vec{j} - 2\vec{k}$ and $\vec{c} = -\vec{i} - \vec{k}$ as edges.

11. Determine whether each of the following sets of vectors are coplanar.
(a) $\vec{a} = 3\vec{i} - 2\vec{j} - \vec{k}$, $\vec{b} = 3\vec{j} + 5\vec{k}$ and $\vec{c} = -\vec{i} + 2\vec{j} - \vec{k}$
(b) $\vec{a} = 2\vec{i} - 3\vec{j} + \vec{k}$, $\vec{b} = -\vec{i} + 2\vec{j} - 5\vec{k}$ and $\vec{c} = 3\vec{i} - 4\vec{j} - 3\vec{k}$

12. Determine whether $\vec{a} = \vec{i} + \vec{j} - 2\vec{k}$, $\vec{b} = 3\vec{j} + 5\vec{k}$ and $\vec{c} = -\vec{i} - \vec{k}$ are linearly independent.

13. Given that A(2, 3, –1), B(2, 0, –1) and C(1, 2, –2) are three points on a plane.
(a) Find $\vec{n}$, one unit vector which is perpendicular to the plane.
(b) Using the $\vec{n}$ obtained in (a), find the distance between the point $D(0, 1, 0)$ and the plane.

Answers

1.

(a)	T	(b)	T	(c)	F	(d)	F	(e)	F
(f)	F	(g)	T	(h)	F	(i)	T	(j)	F

2(a)(b) $\overrightarrow{AB} = \begin{bmatrix} -3 \\ 4 \\ 16 \end{bmatrix} = \overrightarrow{DC}$

3.(a) $\left|\overrightarrow{P_1P_2}\right| = \sqrt{11}$, $\left|\overrightarrow{P_1P_3}\right| = \sqrt{21}$ (b) $-\frac{3}{\sqrt{11}}\vec{i} - \frac{1}{\sqrt{11}}\vec{j} + \frac{1}{\sqrt{11}}\vec{k}$ (c) 5 (d) 70.8°

4. $-\frac{2}{\sqrt{30}}$, $\frac{1}{\sqrt{30}}$ and $-\frac{5}{\sqrt{30}}$. 5. (a) $-\frac{21}{34}\vec{j} - \frac{35}{34}\vec{k}$ (b) $-\frac{7}{6}\vec{i} - \frac{7}{6}\vec{j} + \frac{7}{3}\vec{k}$

6. (a) $9\vec{i} + 7\vec{j} + 5\vec{k}$ (b) $\sqrt{155}$ square units (c) $\frac{\sqrt{155}}{2}$ square units

7. (a) $3\sqrt{2}$ square units (b) $\frac{3\sqrt{2}}{2}$ square units

8. 16 cubic units 9. 4 cubic units 10. 1.5 cubic units

11. (a) $\vec{a} \cdot \vec{b} \times \vec{c} = -32 \neq 0$ $\therefore$ $\vec{a}$, $\vec{b}$ and $\vec{c}$ are not coplanar.
(b) $\vec{a} \cdot \vec{b} \times \vec{c} = 0$ $\therefore$ $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

12. $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly independent.

13. (a) $\frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{k}$ or $-\left(\frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{k}\right)$ (b) $\frac{3\sqrt{2}}{2}$ units

First Exercises on Chapter 4 Matrix Algebra

Part A

1. Compute each of the following determinants.

(a) $\begin{vmatrix} 1 & -3 & 4 \\ 3 & 2 & 0 \\ -2 & 0 & -5 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & c & -b \\ -c & 1 & a \\ b & -a & 1 \end{vmatrix}$ (c) $\begin{vmatrix} 0 & 0 & 4 & -2 \\ 0 & 2 & 3 & 7 \\ 3 & -1 & 5 & -2 \\ 0 & 0 & 0 & 5 \end{vmatrix}$

2. It is given that $A = \begin{pmatrix} 2 & 1 & -1 \\ -2 & 3 & 1 \\ 4 & 5 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -1 \\ 0 & 1 & 3 \end{pmatrix}$.

(a) Evaluate $A+B$ and AB.

(b) Show that

(i) $\det(A^T) = \det(A)$;

(ii) $\det(AB) = \det(A)\det(B)$;

(iii) $\det(A+B) \neq \det(A) + \det(B)$.

3. Given $A = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 3 & 4 \\ 1 & 2 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(a) Compute A^2 and A^3.

(b) Compute D^2 and D^3.

(c) What can you observe from the results of (a) and (b)?

4. If the determinant of the matrix $B = \begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 5-\lambda \end{pmatrix}$ is 0, find the values of λ.

5. Without expanding the determinant, find the determinant for the matrix

$$A = \begin{pmatrix} a & b & c \\ 2a & a+2b & a+c \\ -3a & 2a-3b & 2a-c \end{pmatrix}.$$

Part B

1. Determine whether each of the following matrices are in row echelon form

(a) $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 4 & 2 \\ 0 & -1 & 3 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2 & -1 & 0 & 3 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 2 & 9 & 0 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (g) $\begin{bmatrix} 1 & -2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2. For each of the following augmented matrices given, solve the corresponding system of linear equations.

(a) $\left[\begin{array}{ccc|c} 2 & -1 & 0 & 6 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 4 & -8 \end{array}\right]$ (b) $\left[\begin{array}{ccc|c} 4 & 3 & 7 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right]$ (c) $\left[\begin{array}{ccc|c} 2 & -4 & 7 & 6 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & -8 \end{array}\right]$

(d) $\left[\begin{array}{ccc|c} 1 & 0 & -2 & 6 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (e) $\left[\begin{array}{ccc|c} -1 & 2 & 3 & 7 \\ 0 & 0 & 6 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (f) $\left[\begin{array}{ccc|c} 1 & -2 & 5 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(g) $\left[\begin{array}{ccc|c} 1 & -2 & 4 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (h) $\left[\begin{array}{ccc|c} 0 & 2 & -1 & 5 \\ 0 & 0 & 3 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$

3. Determine the solutions of the corresponding system of linear equations and express them in vector form.

(a) $\left[\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 0 & 1 & -1 & 8 \\ 0 & 0 & 2 & -10 \end{array}\right]$ (b) $\left[\begin{array}{ccc|c} 2 & 0 & -1 & 5 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$ c) $\left[\begin{array}{ccc|c} 3 & 2 & -1 & 7 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(d) $\left[\begin{array}{ccc|c} -1 & 2 & -5 & 3 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (e) $\left[\begin{array}{ccc|c} 0 & 2 & -2 & 3 \\ 0 & 0 & 3 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (f) $\left[\begin{array}{ccc|c} 1 & -1 & 3 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(g) $\left[\begin{array}{ccc|c} 0 & 1 & 5 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (h) $\left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ (i) $\left[\begin{array}{cccc|c} 1 & 0 & 2 & -3 & 1 \\ 0 & -1 & 1 & 3 & 8 \\ 0 & 0 & 0 & 2 & -4 \end{array}\right]$

(j) $\left[\begin{array}{cccc|c} 2 & 1 & -1 & -3 & 6 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ (k) $\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 5 \\ 0 & 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ (l) $\left[\begin{array}{cccc|c} 0 & 2 & 1 & -3 & 8 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(m) $\left[\begin{array}{cccc|c} -1 & 3 & -1 & 1 & 0 \\ 0 & 0 & -3 & 5 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ (n) $\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 0 & 4 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

4. Solve each of the following system by Gaussian Elimination and express the solutions in vector form.

(a) $\begin{cases} 5x_1 - x_2 - 3x_3 = 2 \\ -2x_1 - 3x_2 + x_3 = 2 \\ x_1 + x_2 - 2x_3 = -5 \end{cases}$ (b) $\begin{cases} -2x_1 - 10x_2 + 5x_3 = -19 \\ x_1 + 5x_2 - 2x_3 = 8 \\ 3x_1 + 15x_2 - 3x_3 = 15 \end{cases}$ (c) $\begin{cases} 5x_1 + 3x_2 + 2x_3 = 95 \\ 2x_1 + 2x_2 + 4x_3 = 80 \\ 2x_2 + 3x_3 = 40 \end{cases}$

(d) $\begin{cases} x_1 + 5x_2 - 3x_3 = -4 \\ -x_1 - 4x_2 + x_3 = 3 \\ -2x_1 - 7x_2 = 5 \end{cases}$ (e) $\begin{cases} 3x_1 - x_2 + 4x_3 = 9 \\ -x_1 + 2x_2 - 6x_3 = -7 \\ x_1 + 3x_2 + 2x_3 = 0 \end{cases}$ (f) $\begin{cases} -x_1 + 2x_2 + x_3 - 4x_4 = 3 \\ x_1 - 3x_2 - x_3 + 3x_4 = 0 \\ 3x_1 - 2x_2 + 2x_3 + x_4 = 9 \end{cases}$

(g) $\begin{pmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (h) $\begin{pmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

5. For what value(s) of k does the system $\begin{cases} 3x - y - 4z = 11 \\ -x + 2y - z = 13 \\ -2x + kz = -11 \end{cases}$ has unique solution?

6. For what value(s) of k does the system $\begin{cases} x_1 - 3x_2 + x_3 = -4 \\ x_1 - 6x_2 = -9 \\ 2x_1 + 3x_2 + (k+3)x_3 = k+5 \end{cases}$ have

(a) infinitely many solutions?
(b) unique solution?

7. Determine the rank of each of the following matrices.

(a) $\begin{bmatrix} 6 & 30 & -6 \\ -1 & -5 & 2 \\ 2 & 10 & -5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & -2 & -5 \\ 5 & -1 & -3 & 2 \\ -2 & -3 & 1 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -6 & 0 & -9 \\ 2 & 3 & 5 & 7 \\ 1 & -3 & 1 & -4 \end{bmatrix}$

8. Solve each of the following systems by the Cramer's Rule

(a) $\begin{cases} x_1 + 5x_2 + 2x_3 = 39 \\ 2x_1 + 12x_2 + 8x_3 = 112 \\ 5x_1 + 29x_2 + 19x_3 = 269 \end{cases}$ (b) $\begin{cases} 3x - y - 4z = 11 \\ -x + 2y - z = 13 \\ -2x + 3z = -11 \end{cases}$

9. Find the inverse of each of the following matrices (if exists).

(a) $\begin{bmatrix} 2 & -1 & 1 \\ -10 & 4 & -3 \\ 20 & -8 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$ (c) $P = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 3 & 2 \\ 2 & 2 & 4 \\ 0 & 2 & 3 \end{bmatrix}$

10. Let $C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 2 \\ -2 & 3 & -1 \end{pmatrix}$.

(a) Compute $C + 2D$, and show that $-2C^T + D^T = (-2C + D)^T$
(b) Find CD^T and D^TC.

(c) Determine whether each of the matrices CD^T and D^TC in (b) is invertible and find the inverse if it exists.

11. Find the value(s) of k so that the matrix $M = \begin{pmatrix} k+2 & 2k+3 & 0 \\ -4 & k-5 & 2(1-k) \\ 3 & 4 & k-1 \end{pmatrix}$ has no inverse.

12. Let $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$.

(a) Evaluate $A^3 - 5A^2 + 8A - 4I$.

(b) Hence, or otherwise, find A^{-1}.

13. Consider each of the following systems.

(a) $\begin{cases} 2x - y + z = -1 \\ -10x + 4y - 3z = 1 \\ 20x - 8y + 8z = -4 \end{cases}$ (b) $\begin{cases} x - 2z = -1 \\ -3x + y + 4z = -5 \\ 2x - 3y + 4z = 26 \end{cases}$ (c) $\begin{cases} 2x + y = -3 \\ -x - 2y = -9 \end{cases}$ (d) $\begin{cases} 5x + 3y + 2z = 95 \\ 2x + 2y + 4z = 80 \\ 2y + 3z = 40 \end{cases}$

For each of the systems above,

(i) write the system into the form $A\vec{x} = \vec{b}$;

(ii) refer to Question 9, solve the system correspondingly.

14. In each of the following questions, one of the forms (system of linear equations, matrix equation, vector equation, augmented matrix) is presented. Express each of them into the remaining three forms.

e.g. Matrix equation: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 1 \\ -2 & 1 & 2 \\ 3 & 1 & 11 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ Augmented matrix: $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & -2 & 1 & 0 \\ -2 & 1 & 2 & 0 \\ 3 & 1 & 11 & 0 \end{array}\right]$

Vector equation: $x_1\begin{bmatrix} 1 \\ 4 \\ -2 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ -2 \\ 1 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 1 \\ 2 \\ 11 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ System: $\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 4x_1 - 2x_2 + x_3 = 0 \\ -2x_1 + x_2 + 2x_3 = 0 \\ 3x_1 + x_2 + 11x_3 = 0 \end{cases}$

(a) $\left(\begin{array}{ccc|c} 3 & -2 & 0 & 8 \\ 5 & 6 & 1 & 5 \\ -2 & 0 & 3 & 1 \end{array}\right)$ (b) $\begin{bmatrix} 2 & -4 & 0 \\ 3 & 6 & 1 \\ 12 & 5 & 0 \\ 9 & -2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ (c) $x_1\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + x_3\begin{bmatrix} 0 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(d) $\begin{cases} 2x_1 + x_2 - 3x_3 = -1 \\ 9x_1 - 3x_2 + 2x_3 = 0 \\ -7x_1 + x_2 + 3x_3 = 4 \\ 2x_1 + x_2 - 15x_3 = 8 \end{cases}$

15. In each of the following, determine whether the vectors are linearly independent.

(a) $\begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 10 \\ 0 \\ 10 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 9 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -1 \end{pmatrix}$ (c) $\begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 3 \end{pmatrix}$

Answers:

Part A

1. (a) –39 (b) $1+a^2+b^2+c^2$ (c) –120

2. (a) $A+B=\begin{pmatrix} 3 & 4 & 0 \\ 0 & 2 & 0 \\ 4 & 6 & 0 \end{pmatrix}$ $AB=\begin{pmatrix} 4 & 4 & -2 \\ 4 & -8 & -2 \\ 14 & 4 & -10 \end{pmatrix}$

(b)(i) –8 (ii) $\det(AB)=144$, $\det(A)=-8$, $\det(B)=-18$ (iii) $\det(A+B)=0$

3. (a) $A^2=\begin{pmatrix} 7 & 15 & 12 \\ 9 & 20 & 16 \\ 5 & 11 & 9 \end{pmatrix}$ $A^3=\begin{pmatrix} 41 & 90 & 72 \\ 54 & 119 & 96 \\ 30 & 66 & 53 \end{pmatrix}$ (b) $D^2=\begin{pmatrix} 4 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $D^3=\begin{pmatrix} 8 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

(c) D is a diagonal matrix and the results of D^n is to take the nth power of all the entries. However, it is not true for other matrices in general.

4. λ = 1, 3 or 5.

5. $4a^2c$

Part B

1. YES: (b), (d), (e), (f), (g); others are NO

2. (a) $\begin{cases} x_1 = 7 \\ x_2 = 8 \\ x_3 = -2 \end{cases}$ (b) $\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$ (c) No solution (d) $\begin{cases} x_1 = 6-2s \\ x_2 = 5-4s \\ x_3 = s \end{cases}$ (s is any real number)

(e) $\begin{cases} x_1 = 4-2s \\ x_2 = s \\ x_3 = -1 \end{cases}$ (s is any real number) (f) $\begin{cases} x_1 = -3+2s-5t \\ x_2 = s \\ x_3 = t \end{cases}$ (s, t are any real numbers)

(g) No solution (h) $\begin{cases} x_1 = s \\ x_2 = 3 \\ x_3 = -2 \end{cases}$ (s is any real number)

3. In the following, r, s and t are any real numbers.

(a) $\vec{x}=\begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}$ (b) $\vec{x}=\begin{bmatrix} 2.5 \\ 9 \\ 0 \end{bmatrix}+s\begin{bmatrix} 0.5 \\ -4 \\ 1 \end{bmatrix}$ (c) $\vec{x}=\begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}+s\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

(d) $\vec{x}=\begin{bmatrix} 12 \\ 0 \\ -3 \end{bmatrix}+s\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ (e) $\vec{x}=\begin{bmatrix} 0 \\ -1.5 \\ -3 \end{bmatrix}+s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (f) $\vec{x}=\begin{bmatrix} -6 \\ 0 \\ 0 \end{bmatrix}+s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}+t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

(g) $\vec{x}=\begin{bmatrix}0\\-3\\0\end{bmatrix}+s\begin{bmatrix}1\\0\\0\end{bmatrix}+t\begin{bmatrix}0\\-5\\1\end{bmatrix}$ (h) $\vec{x}=\begin{bmatrix}-3\\0\\0\end{bmatrix}+s\begin{bmatrix}2\\1\\0\end{bmatrix}+t\begin{bmatrix}0\\0\\1\end{bmatrix}$ (i) $\vec{x}=\begin{bmatrix}-5\\-14\\0\\-2\end{bmatrix}+s\begin{bmatrix}-2\\1\\1\\0\end{bmatrix}$

(j) $\vec{x}=\begin{bmatrix}5.5\\0\\5\\0\end{bmatrix}+s\begin{bmatrix}0\\0\\-3\\1\end{bmatrix}+t\begin{bmatrix}-0.5\\1\\0\\0\end{bmatrix}$ (k) $\vec{x}=\begin{bmatrix}8\\0\\0\\3\end{bmatrix}+s\begin{bmatrix}-2\\0\\1\\0\end{bmatrix}+t\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$ (l) $\vec{x}=\begin{bmatrix}0\\3\\2\\0\end{bmatrix}+s\begin{bmatrix}0\\1\\1\\1\end{bmatrix}+t\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$

(m) $\vec{x}=\begin{bmatrix}\frac{2}{3}\\0\\-\frac{2}{3}\\0\end{bmatrix}+s\begin{bmatrix}3\\1\\0\\0\end{bmatrix}$ (n) $\vec{x}=r\begin{bmatrix}2\\0\\-2\\0\\1\end{bmatrix}+s\begin{bmatrix}3\\0\\-1\\1\\0\end{bmatrix}+t\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}$

4. In the following, s and t are any real numbers.

(a) $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\-1\\3\end{bmatrix}$ (b) $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\0\\-3\end{bmatrix}+s\begin{bmatrix}-5\\1\\0\end{bmatrix}$ (c) $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}13.5\\0.5\\13\end{bmatrix}$ (d) $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\-1\\0\end{bmatrix}+s\begin{bmatrix}-7\\2\\1\end{bmatrix}$

(e) $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\-1\\0.5\end{bmatrix}$ (f) $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}3\\-3\\6\\0\end{bmatrix}+s\begin{bmatrix}-3\\-1\\3\\1\end{bmatrix}$ (g) $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=t\begin{pmatrix}1\\-1\\1\end{pmatrix}$ (h) $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=s\begin{pmatrix}-1\\1\\0\end{pmatrix}+t\begin{pmatrix}-1\\0\\1\end{pmatrix}$

5. k does not equal to 18/5

6. (a) $k=2$ (b) $k\neq 2$

7. (a) 2 (b) 3 (c) 2

8. (a) $\begin{cases}x_1=2\\x_2=5\\x_3=6\end{cases}$ (b) $\begin{cases}x_1=-2\\x_2=3\\x_3=-5\end{cases}$

9. (a) $\begin{bmatrix}-2&0&\frac{1}{4}\\-5&1&1\\0&1&\frac{1}{2}\end{bmatrix}$ (b) $\begin{bmatrix}8&3&1\\10&4&1\\\frac{7}{2}&\frac{3}{2}&\frac{1}{2}\end{bmatrix}$ (c) $\begin{bmatrix}\frac{2}{3}&\frac{1}{3}\\-\frac{1}{3}&-\frac{2}{3}\end{bmatrix}$ (d) $\begin{bmatrix}0.1&0.25&-0.4\\0.3&-0.75&0.8\\-0.2&0.5&-0.2\end{bmatrix}$

10. (a) $C+2D=\begin{bmatrix}3&0&5\\-4&7&-2\end{bmatrix}$ (b) $CD^T=\begin{bmatrix}3&-3\\0&3\end{bmatrix}$, $D^TC=\begin{bmatrix}1&-2&1\\0&3&0\\2&-1&2\end{bmatrix}$

(c) Inverse of CD^T is: $\begin{bmatrix}\frac{1}{3}&\frac{1}{3}\\0&\frac{1}{3}\end{bmatrix}$ D^TC is not invertable (i.e. inverse of D^TC does not exists)

11. (a) $k = 0$, 1 or -1

12. (a) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $A^{-1} = \begin{bmatrix} \frac{3}{2} & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 \end{bmatrix}$

13. (a)(i) $\begin{bmatrix} 2 & -1 & 1 \\ -10 & 4 & -3 \\ 20 & -8 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -4 \end{bmatrix}$ (ii) $\begin{cases} x = 1 \\ y = 2 \\ z = -1 \end{cases}$ (b)(i) $\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 26 \end{bmatrix}$ (ii) $\begin{cases} x = 3 \\ y = -4 \\ z = 2 \end{cases}$

(c)(i) $\begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ -9 \end{bmatrix}$ (ii) $\begin{cases} x = -5 \\ y = 7 \end{cases}$ (d) (i) $\begin{bmatrix} 5 & 3 & 2 \\ 2 & 2 & 4 \\ 0 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 95 \\ 80 \\ 40 \end{bmatrix}$ (ii) $\begin{cases} x = 13.5 \\ y = 0.5 \\ z = 13 \end{cases}$

15 (a) Yes (b) Yes (c) No (d) Yes