MA1200 Calculus and Basic Linear Algebra I
Chapter 5 Exponential and Logarithmic Functions

1 Exponential Functions

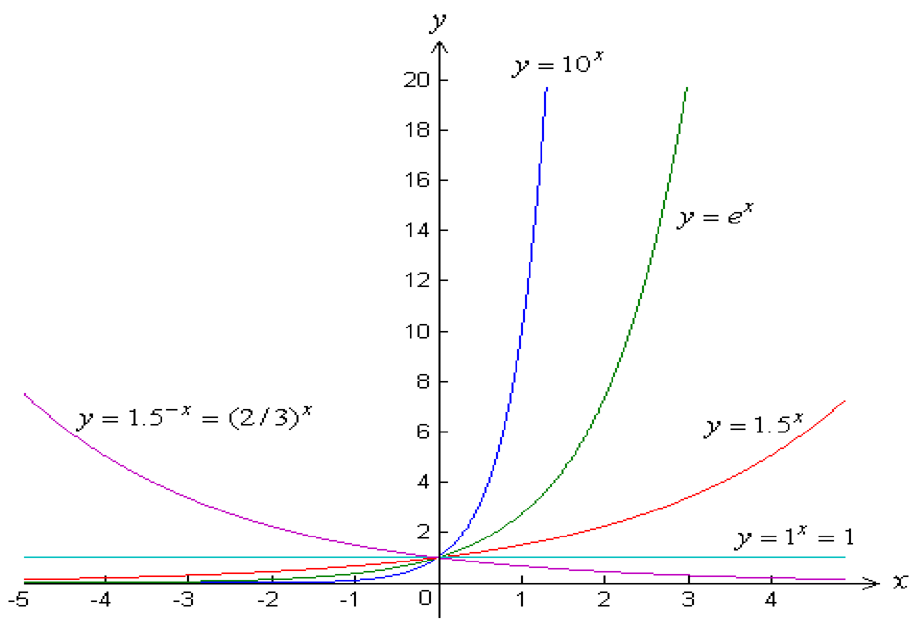
Definition: The **exponential function *f* with base *b*** is defined by

$$f(x) = b^x \qquad \text{or} \qquad y = b^x,$$

where b is a positive constant other than 1 ($b > 0$ and $b \neq 1$) and x is any real number.

e.g. $f(x) = 2^x$, $g(x) = 10^x$, $h(x) = 3^{x+1}$, $k(x) = \left(\frac{1}{2}\right)^x$.

The following shows the graphs of some of the exponential functions.



It should be noted that:

(i) the domain of $f(x) = b^x$ consists of all real numbers, i.e. the domain of $f(x) = b^x$ is **R**. The range of $f(x) = b^x$ consists of all positive real numbers, i.e. the range of $f(x) = b^x$ is $(0, \infty)$.

(ii) for any $b > 0$, the graph $f(x) = b^x$ cuts the *y*-axis at $y = 1$ since $f(0) = b^0 = 1$ for all numbers *b*.

(iii) if $b > 1$, then $f(x) = b^x$ is an increasing function which goes up to the right. The greater the value of *b*, the steeper the increase.

(iv) if $0 < b < 1$, then $f(x) = b^x$ is a decreasing function which goes down to the right. The smaller the value of *b*, the steeper the decrease.

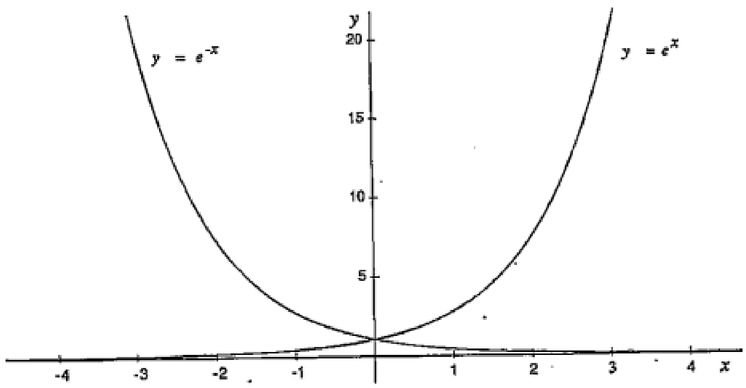
(v) for the exponential function $f(x) = b^x$, each distinct value of output comes from a distinct value of input.

(vi) the graph of $f(x) = b^x$ never touches the *x*-axis.

Question: Plot $f(x) = 2^x$ and $f(x) = \left(\frac{1}{2}\right)^x$ on the same graph. What do you observe?

The Natural Base *e*

Consider $\left(1+\frac{1}{n}\right)^n$ as n gets larger and larger. The value of $\left(1+\frac{1}{n}\right)^n$ will approach to an irrational number approximately equals to 2.718281827. We use the letter e to denote the value of $\left(1+\frac{1}{n}\right)^n$ as n gets larger and larger. This number, e, is called the **natural base**. The function $f(x)=e^x$ is called the **natural exponential function**. The following shows the graph of $y=e^x$ and $y=e^{-x}$.



Example 5.1 Find the domain and range of each of the following functions:

(a) $f(x)=e^x+5$ (b) $g(x)=10^x-3$ (c) $h(x)=2\left(\frac{1}{3}\right)^x$

Solutions

(a) $f(x)=e^x+5$ is well-defined for all real numbers *x*. Therefore, the domain of $f(x)$ is **R**.

Recall that the range of $y=e^x$ consists of all positive real numbers. Therefore, the range of $f(x)=e^x+5$ consists of all real numbers which are greater than 5. i.e. the range of $f(x)$ is $(5, \infty)$.

(b) $g(x)=10^x-3$ is well-defined for all real numbers *x*. Therefore, the domain of $g(x)$ is **R**.

Recall that the range of $y=10^x$ consists of all positive real numbers. Therefore, the range of $g(x)=10^x-3$ consists of all real numbers which are greater than -3. i.e. the range of $g(x)$ is $(-3, \infty)$.

(c) $h(x)=2\left(\frac{1}{3}\right)^x$ is well-defined for all real numbers *x*. Therefore, the domain of $h(x)$ is **R**.

Recall that the range of $y=\left(\frac{1}{3}\right)^x$ consists of all positive real numbers. Therefore, the range of $h(x)=2\left(\frac{1}{3}\right)^x$ consists of all positive real numbers. i.e. the range of $h(x)$ is $(0, \infty)$.

2 Logarithmic Functions

Definition: For $x > 0$ and $b > 0$, $b \neq 1$,

$$y = \log_b x \qquad \text{is equivalent to} \qquad b^y = x.$$

The function $f(x) = \log_b x$ is the **logarithmic function with base *b***. For $y = \log_b x$, y is called the **exponent** and b is called the **base**. $y = \log_b x$ is the **logarithmic form** and $b^y = x$ is the **exponential form**.

Example 5.2 Write each equation in its equivalent exponential form.

(a) $2 = \log_5 x$ (b) $3 = \log_b 64$ (c) $\log_3 7 = y$

Solutions

We use the fact that $y = \log_b x$ means $b^y = x$.

(a) $2 = \log_5 x$ means $5^2 = x$.

(b) $3 = \log_b 64$ means $b^3 = 64$.

(c) $\log_3 7 = y$ means $3^y = 7$.

Example 5.3 Write each equation in its equivalent logarithmic form.

(a) $12^2 = r$ (b) $b^3 = 8$ (c) $e^a = 9$

Solutions

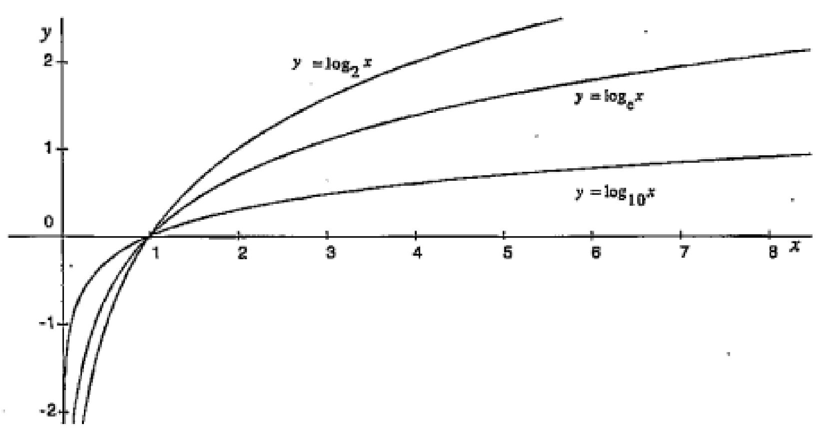
We use the fact that $b^y = x$ means $y = \log_b x$.

(a) $12^2 = r$ means $2 = \log_{12} r$.

(b) $b^3 = 8$ means $3 = \log_b 8$.

(c) $e^a = 9$ means $a = \log_e 9$.

We may think of taking logarithm as the reverse operation of exponentiation. The following shows the graph of $y = \log_b x$ for various values of *b*.



It can be observed that

(i) $y = \log_b x$ cuts the *x*-axis at $x = 1$ for each case.

(ii) *x* can never be negative. The domain of $y = \log_b x$ is $(0, \infty)$.

(iii) The range of $y = \log_b x$ covers all the real numbers. i.e. the range of $y = \log_b x$ is **R**.

When operating with logarithms, there are several useful rules:

(i) For any real number *n*, $\log_b b^n = n$.

(ii) For any real number $N > 0$, $b^{\log_b N} = N$.

(iii) For any real number $M > 0$ and *n*, $\log_b M^n = n\log_b M$.

(iv) For any real number $M > 0$ and $N > 0$, $\log_b(MN) = \log_b M + \log_b N$.

(v) For any real number $M > 0$ and $N > 0$, $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$.

(vi) For any real number $M > 0$, $b > 1$ and $N > 1$, $\log_N M = \dfrac{\log_b M}{\log_b N}$.

Further, if the *base* is equal to 10, then it is called the **common logarithm**. e.g. $\log_{10} 3$, $\log_{10} 1000$ are common logarithm. For simplicity, if the base is 10 (i.e. the common logarithm), the value of the base 10 is always omitted. e.g. $\log 3$, $\log 1000$.

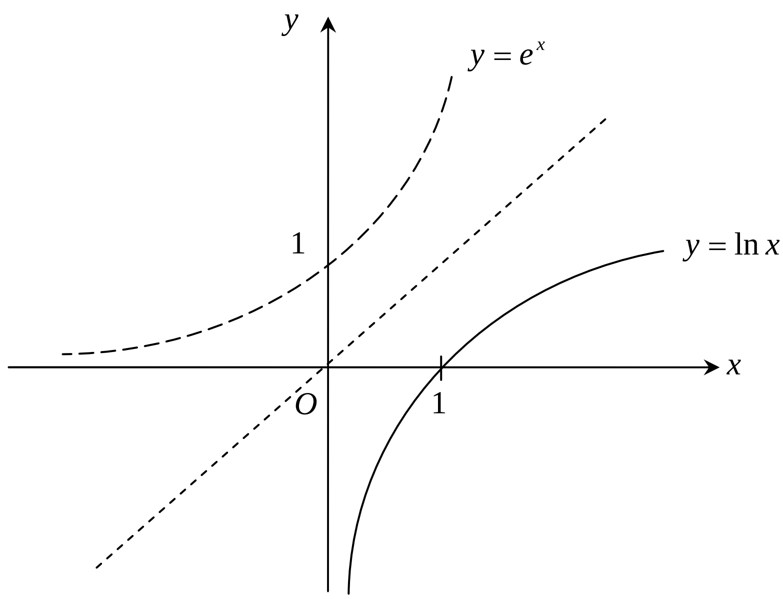
If the base is chosen to be the natural number *e* ($\approx 2.71828182\ldots$), then it is called the **natural logarithm**. We define the symbol $\ln$ to be the natural logarithm, e.g. $\ln 4 = \log_e 4$.

$$\text{i.e.} \quad e^y = x \Leftrightarrow \quad y = \ln x$$

To plot the graph of $y = \ln x$, we first consider the following procedure:

$$y = \ln x = \log_e x \qquad \therefore \qquad e^y = x \text{ (by definition)}$$

Therefore, the graph of $y = \ln x$ (equivalently $e^y = x$) is the mirror image of the graph $e^x = y$ about the line *y* = *x*.



From the graph, we observe that:

(i) $y = \ln x$ is well-defined for $x > 0$ only. Therefore, the domain of $y = \ln x$ is $(0, \infty)$.

(ii) as *x* increases, $y = \ln x$ increases. Therefore, $y = \ln x$ is an increasing function.

(iii) $\ln 1 = 0$.
(iv) $\ln x < 0$ for $0 < x < 1$.
(v) The range of $y = \ln x$ is **R**.

Remarks:

(i) It is **incorrect** to say that $\frac{\cancel{\log} M}{\cancel{\log} N} = \frac{M}{N}$

(ii) The value of $\log N$ is well-defined only for $N > 0$.

Example 5.4 Find the largest possible domain and largest possible range of each of the following functions.

(a) $f(x) = \log(x+3)$ (b) $g(x) = \ln \frac{1}{x}$

(c) $h(x) = \log \frac{1000}{x-1}$

Solutions

(a) $f(x) = \log(x+3)$ is well-defined for all real numbers *x* that satisfy $x + 3 > 0$, i.e. $x > -3$.
Thus, the largest possible domain of $f(x)$ is $(-3, \infty)$.
The largest possible range of $f(x)$ covers all the real numbers. Thus the range of $f(x)$ is **R**.

(b) $g(x) = \ln \frac{1}{x} = \ln x^{-1} = -\ln x$
$g(x) = -\ln x$ is well-defined for all positive real numbers *x*.
Thus, the largest possible domain of $g(x)$ is $(0, \infty)$.
The largest possible range of $g(x)$ covers all the real numbers. Thus the range of $g(x)$ is **R**.

(c) $h(x) = \log \frac{1000}{x-1} = \log 1000 - \log(x-1) = 3 - \log(x-1)$
$h(x) = 3 - \log(x-1)$ is well-defined for all real numbers *x* that satisfy $x - 1 > 0$, i.e. $x > 1$.
Thus, the largest possible domain of $h(x)$ is $(1, \infty)$.
The largest possible range of $h(x)$ covers all the real numbers. Thus the range of $h(x)$ is **R**.

Example 5.5 Solve each of the following equations:

(a) $2^x = 16$ (b) $3^{x-1} = 81$

(c) $3^x = 17$ (d) $3 \cdot 5^{2x-1} + 2 = 17$

Solutions

(a) $2^x = 16$
$2^x = 2^4$
Taking logarithm on both sides,
$\log 2^x = \log 2^4$
$x \log 2 = 4 \log 2$
$\therefore \quad x = 4$

(b) $3^{x-1} = 81$

$3^{x-1} = 3^4$

Taking logarithm on both sides,

$\log 3^{x-1} = \log 3^4$

$(x-1)\log 3 = 4\log 3$

$x - 1 = 4$

$\therefore \quad x = 5$

(c) $3^x = 17$

Taking logarithm on both sides,

$\log 3^x = \log 17$

$x\log 3 = \log 17$

$x = \dfrac{\log 17}{\log 3} \approx 2.5789$

(d) $3 \cdot 5^{2x-1} + 2 = 17$

$3 \cdot 5^{2x-1} = 15$

$5^{2x-1} = 5$

Taking logarithm on both sides,

$\log 5^{2x-1} = \log 5$

$(2x-1)\log 5 = \log 5$

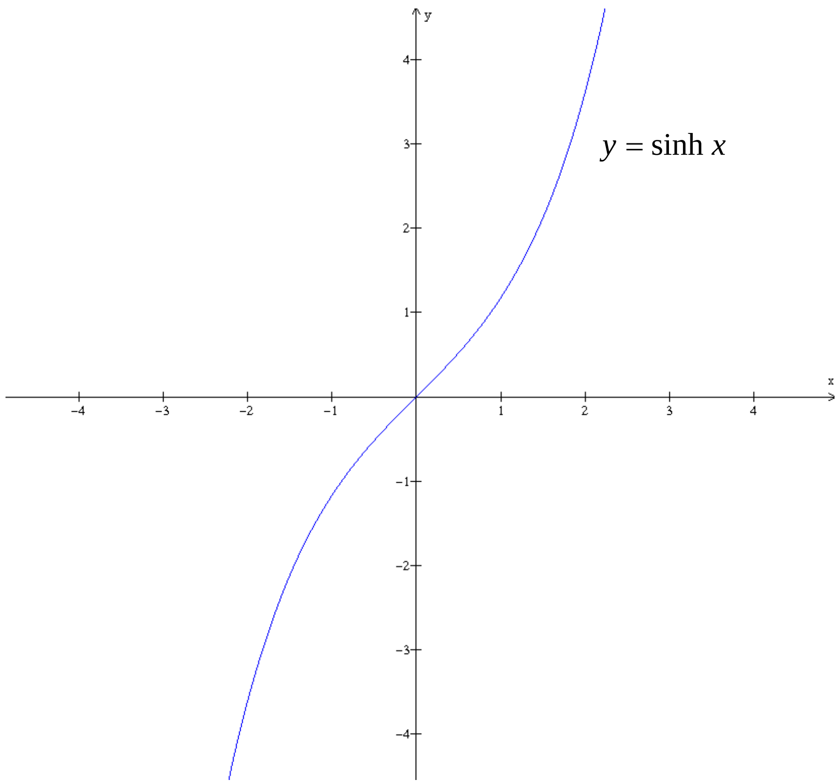
$2x - 1 = 1$

$\therefore \quad x = 1$

3 Hyperbolic Sine and Hyperbolic Cosine Functions

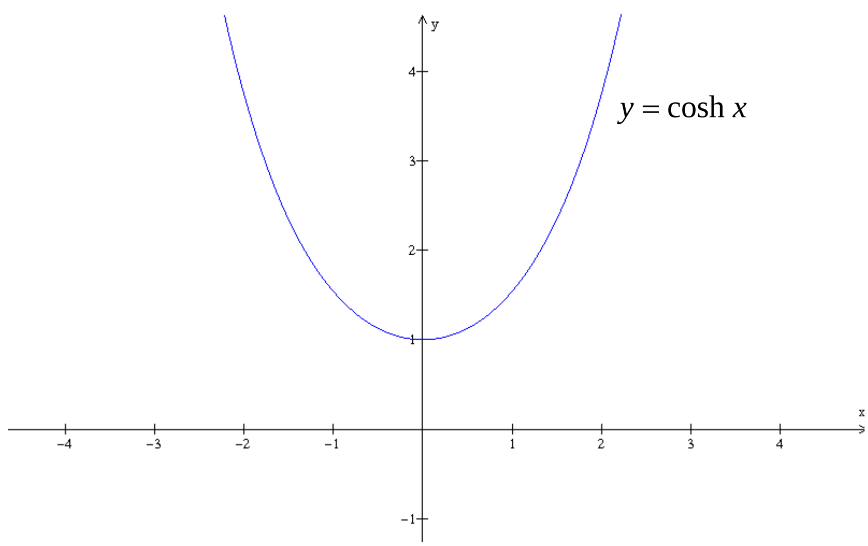
The hyperbolic sine function is defined as:

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$



The hyperbolic cosine function is defined as:

$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$



Question:
What is the largest possible domain and largest possible range of the two functions?
Is/are the functions odd, even or neither of them?