MA1200 Calculus and Basic Linear Algebra I

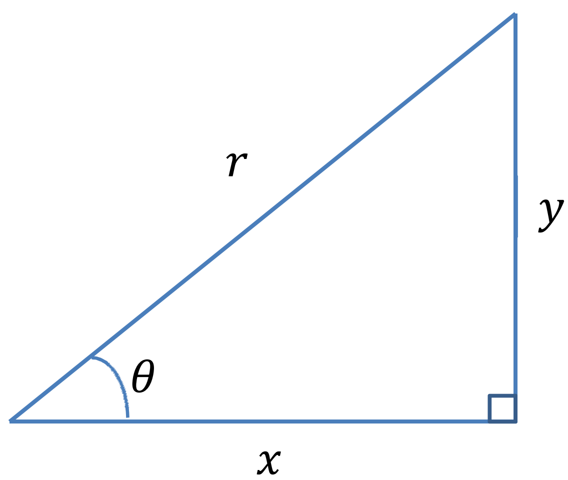
Lecture Note 4

Trigonometric Functions

Trigonometric function and its development

Elementary definition

In elementary trigonometry, the trigonometric functions, denoted by $\cos\theta$, $\sin\theta$ and $\tan\theta$ are defined as the ratios of sides of a right-angled triangle:



$$\cos\theta = \frac{x}{r}$$

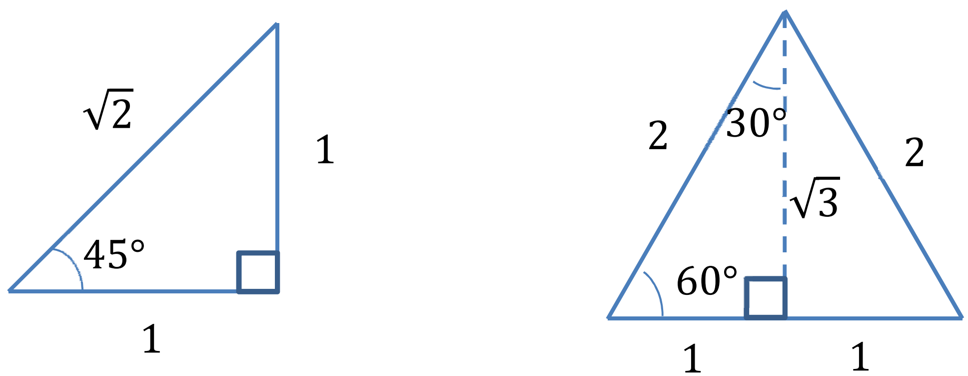
$$\sin\theta = \frac{y}{r}$$

$$\tan\theta = \frac{y}{x}$$

- Here, θ lies between $0°$ and $90°$.

Special Angle Formula

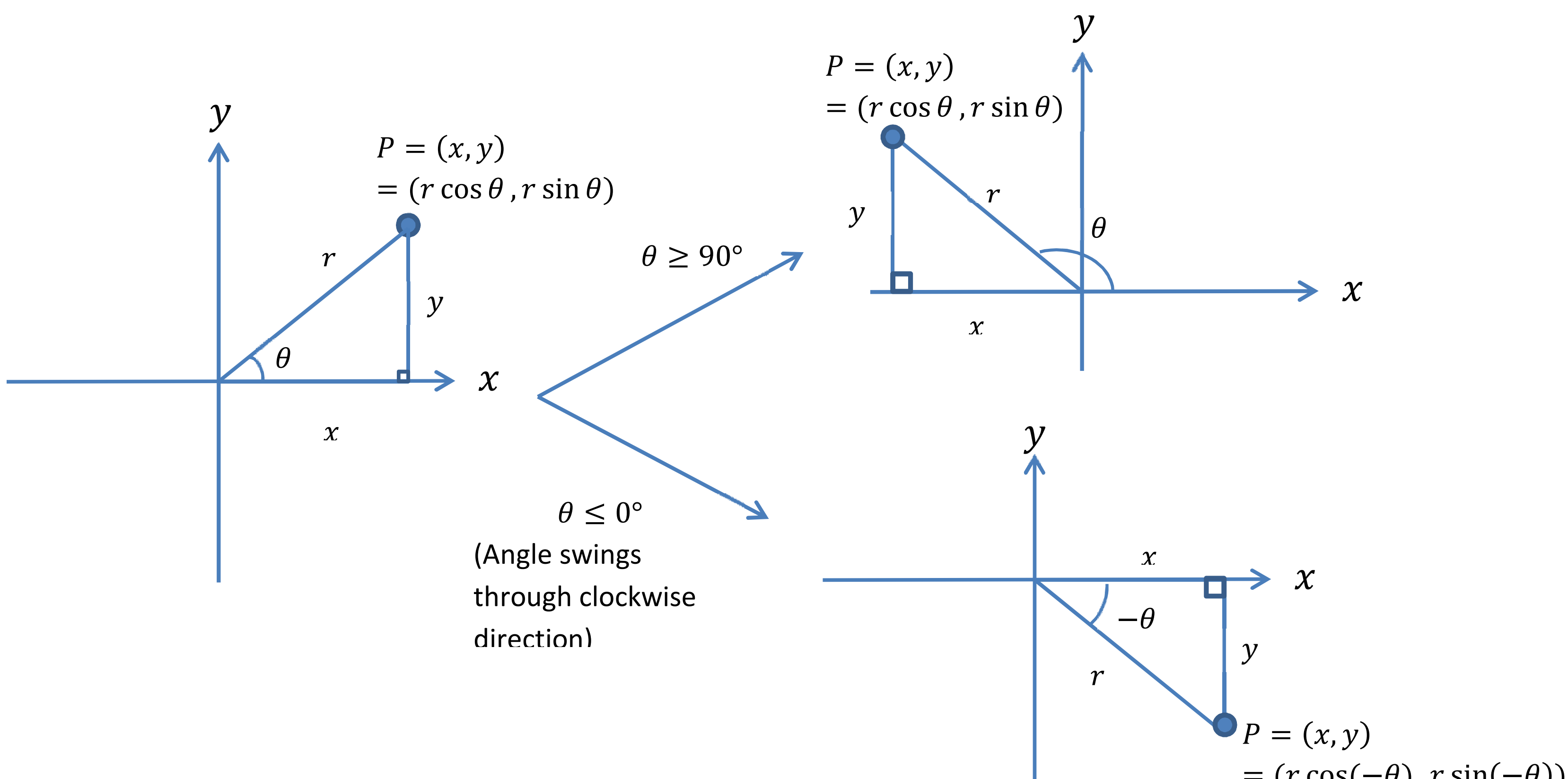
Using the following two triangles, one can obtain the sine, cosine and tangent values of some special angles ($30°$, $45°$ and $60°$).



	$30°$	$45°$	$60°$
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

More Advanced Definition

Later, people modifies the definition of trigonometric functions so that the angle can be taken to be any value (say $\theta \geq 90°$ or $\theta \leq 0°$).



From this, we have the following definition of trigonometric function:

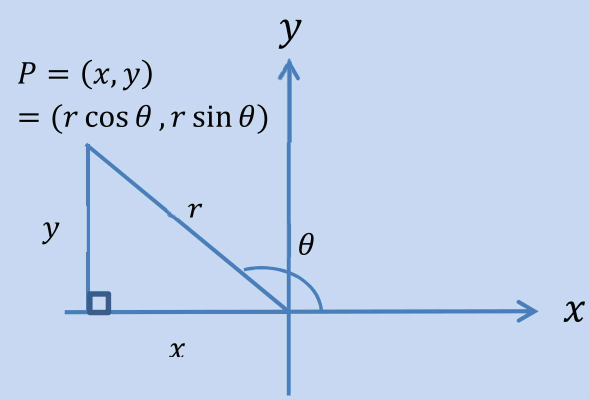
Definition

The *trigonometric functions* $\cos\theta$, $\sin\theta$ and $\tan\theta$ satisfies

$$r\cos\theta = x \Rightarrow \cos\theta = \frac{x}{r},$$

$$r\sin\theta = x \Rightarrow \sin\theta = \frac{y}{r}.$$

$$\tan\theta = \frac{y}{x}$$



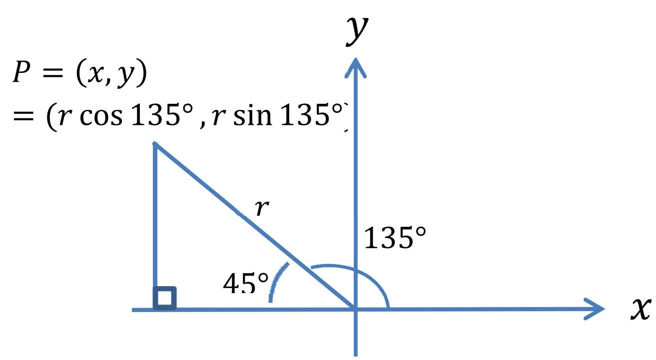
where $r = \sqrt{x^2 + y^2}$. Her the angle θ swings from positive x-axis through anti-clockwise direction (positive θ) or clockwise direction (negative θ).

Remark:

Since x, y can be positive and negative, so $\sin\theta$, $\cos\theta$ and $\tan\theta$ can be either positive or negative.

Example 1

Suppose that we would like to compute $\cos 135°$, $\sin 135°$ and $\tan 135°$, we first consider the following diagram

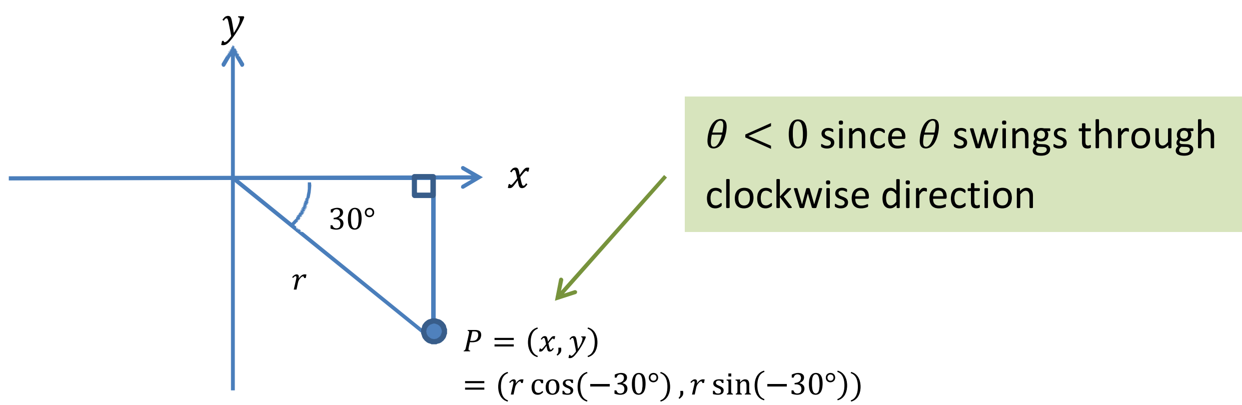


- Given the value r, we express x, y (coordinates of P) in terms of r, one can find that $x = \underbrace{-r\cos 45°}_{negative} = -\frac{r}{\sqrt{2}}$ and $y = r\sin 45° = \frac{r}{\sqrt{2}}$.
- So we have

$$\cos 135° = \frac{x}{r} = -\frac{1}{\sqrt{2}}, \qquad \sin 135° = \frac{y}{r} = \frac{1}{\sqrt{2}}, \qquad \tan 135° = \frac{y}{x} = -1.$$

Example 2

We try to compute $\cos(-30°)$, $\sin(-30°)$ and $\tan(-30°)$. We can consider the following diagram:



- Given the value r, we express x, y (coordinates of P) in terms of r, one can find that $x = \underbrace{r\cos 30°}_{positive} = \frac{\sqrt{3}r}{2}$ and $y = \underbrace{-r\sin 30°}_{negative} = -\frac{r}{2}$.
- So we have

$$\cos(-30°) = \frac{x}{r} = \frac{\sqrt{3}}{2}, \qquad \sin(-30°) = \frac{y}{r} = -\frac{1}{2}, \qquad \tan(-30°) = \frac{y}{x} = -\frac{1}{\sqrt{3}}.$$

Properties of trigonometric functions

Using the definition and the similar method used in Example 1 and 2, one can establish the following properties:

	$90° - \theta$	$90° + \theta$	$180° - \theta$	$180° + \theta$
sin	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$
cos	$\sin\theta$	$-\sin\theta$	$\cos\theta$	$-\cos\theta$
tan	$\frac{1}{\tan\theta}$	$-\frac{1}{\tan\theta}$	$-\tan\theta$	$\tan\theta$

	$270° - \theta$	$270° + \theta$	$360° - \theta$ $(or - \theta)$	$360° + \theta$
sin	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$
cos	$-\sin\theta$	$\sin\theta$	$\cos\theta$	$\cos\theta$
tan	$\frac{1}{\tan\theta}$	$-\frac{1}{\tan\theta}$	$-\tan\theta$	$\tan\theta$

Example 3

Compute $\tan(-135°)$, $\sin(240°)$ and $\cos(510°)$.

☺Solution:

$$\tan(-135°) \overset{\substack{\tan(-\theta) \\ = -\tan\theta}}{\triangleq} -\tan 135° = -\tan(180° - 45°) \overset{\substack{\tan(180°-\theta) \\ = -\tan\theta}}{\triangleq} -(-\tan 45°) = 1.$$

$$\sin 240° = \sin(180° + 60°) \overset{\substack{\sin(180°+\theta) \\ = -\sin\theta}}{\triangleq} -\sin 60° = -\frac{\sqrt{3}}{2}.$$

$$\cos 510° = \cos(360° + 150°) \overset{\substack{\cos(360°+\theta) \\ = \cos\theta}}{\triangleq} \cos(150°)$$

$$\overset{\substack{\cos(180°-\theta) \\ = -\cos\theta}}{\triangleq} \cos(180° - 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}.$$

Example 4

Compute

$$\frac{\cos(360° - A)\sin(90° - A)\tan(A - 180°)\tan(-A)}{\sin(-A)\sin(180° + A)}.$$

☺Solution:

Note that $\tan(A - 180°) = \tan\big(-(180° - A)\big) = -\tan(180° - A) = \tan A$.

Using the properties, we get

$$\frac{\cos(360° - A)\sin(90° - A)\tan(A - 180°)\tan(-A)}{\sin(-A)\sin(180° + A)}$$

$$= \frac{\cos A\,(\cos A)(\tan A)(-\tan A)}{(-\sin A)(-\sin A)} = \frac{-\cos^2 A\,\frac{\sin^2 A}{\cos^2 A}}{\sin^2 A} = -1.$$

Three new trigonometric functions

The secant, cosecant and cotangent functions, denoted by $\sec\theta$, $\csc\theta$ and $\cot\theta$ respectively, are defined as the *reciprocal* of cosine function ($\cos\theta$), sine function ($\sin\theta$) and tangent function ($\tan\theta$) respectively, namely:

$$\sec\theta = \frac{1}{\cos\theta}, \qquad \csc\theta = \frac{1}{\sin\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

- The domains of these functions are all θ such that the denominator is non-zero.
- We compute the values of these functions by using the above definition and the properties of $\sin\theta$, $\cos\theta$, $\tan\theta$.
  For example:

$$\sec 60° = \frac{1}{\cos 60°} = \frac{1}{1/2} = 2.$$

$$\cot(315°) = \frac{1}{\tan 315°} = \frac{1}{\tan(360° - 45°)} = \frac{1}{-\tan 45°} = -1.$$

Radian Measure – Another way to measure angles

Different from other element functions such as polynomial, exponential function and logarithm function. The trigonometric functions require the "special number" $\theta°$ to generate the values of $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$ and $\cot\theta$.

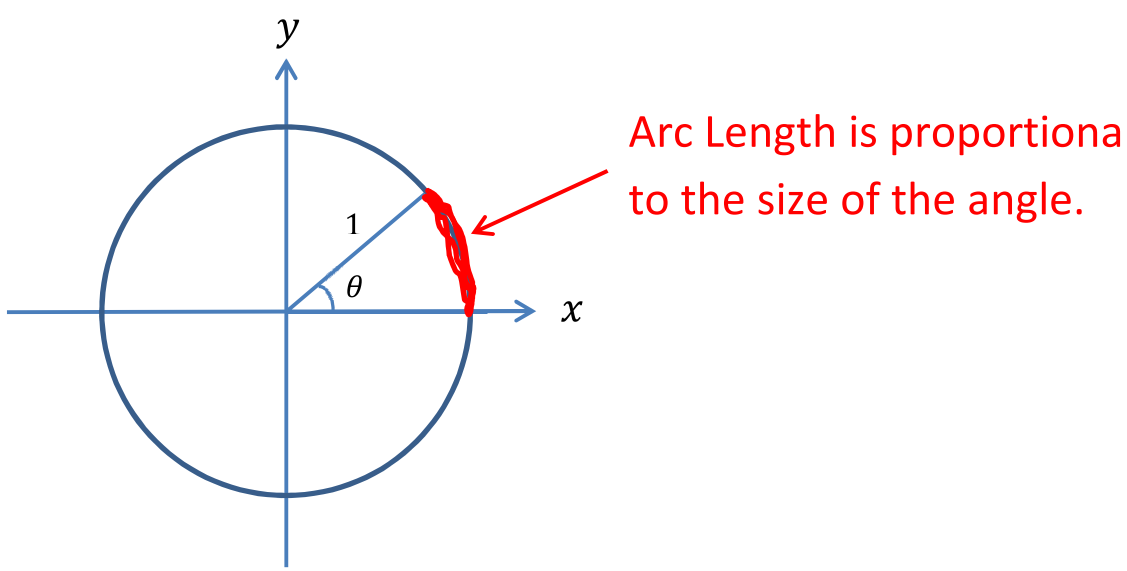
However, the number $\theta°$ is not compatible with other functions: One cannot compute the values of $5^{45°}$ or $(30°)^2$ etc.. In order word, one cannot combine the trigonometric functions and other functions in this manner.

One may ask whether it is possible to input a real number into trigonometric functions. In other word, we ask

Can we measure an angle using a real number??

Such measure is called *radian measure*.

We consider the following graph:



- In the figure, we observe that one can measure the angle using the arc length (which is the real number).

- A radian measure, denoted by rad., is the measure of the angle subtended at the centre of a unit circle (radius $= 1$) by an arc length.

Relationship between degree measure and radial measure

Note that

$$360° \text{ (degree)} = 1 \text{ complete circle} = 2\pi \text{ (radian)}.$$

$$\Rightarrow 1°(degree) = \frac{\pi}{180}(radian).$$

Example 5

$$\cos\frac{\pi}{4} \overset{1°=\frac{\pi}{180}rad}{\cong} \cos\left(\frac{\frac{\pi}{4}}{\frac{\pi}{180}}\right)° = \cos 45° = \frac{1}{\sqrt{2}}.$$

$$\sin\left(-\frac{5\pi}{3}\right) = \sin\left(\frac{-\frac{5\pi}{3}}{\frac{\pi}{180}}\right)° = \sin(-300°) = -\sin 300° = -\sin(360° - 60°)$$

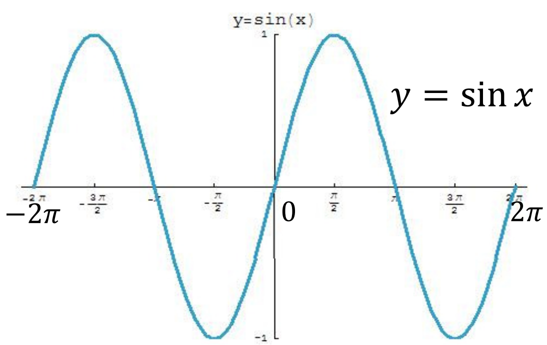
$$= -(-\sin 60°) = \sin 60° = \frac{\sqrt{3}}{2}.$$

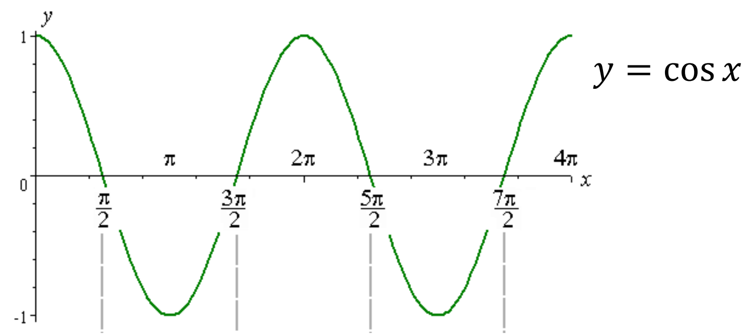
Graph of trigonometric functions

In this section, we investigate the graph of various trigonometric functions.

1. $y = \sin x$ and $y = \cos x$

- The functions are both periodic with 2π.
- The ranges of two functions are both $[-1,1]$. (i.e. $-1 \leq \sin x \leq 1$ and $-1 \leq \cos x \leq 1$)

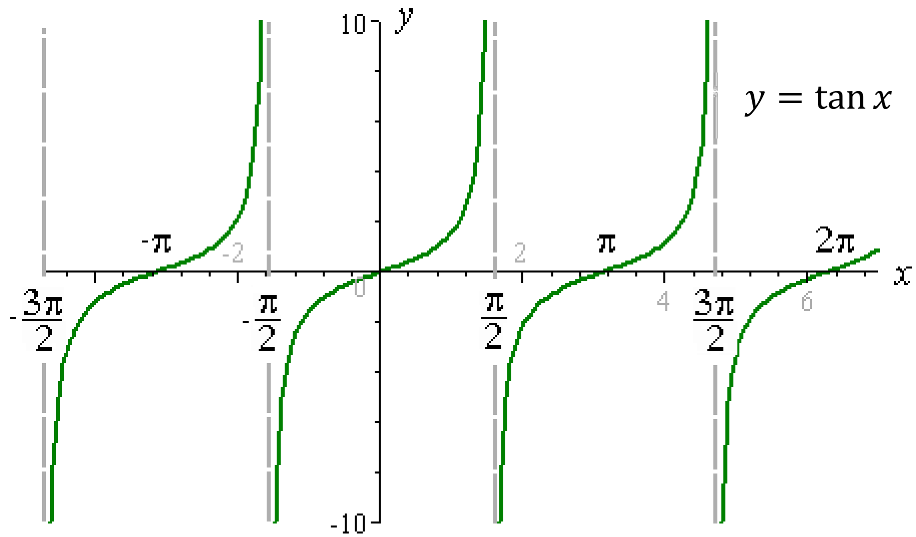




2. $y = \tan x \;\; (= \frac{\sin x}{\cos x})$

- The function is undefined at $x = \frac{\pi}{2}$ (or $90°$), $\frac{3\pi}{2}$ (or $270°$), $\frac{5\pi}{2}$ (or $450°$),… and $x = -\frac{\pi}{2}$ (or $-90°$), $x = -\frac{3\pi}{2}$ (or $270°$), $x - \frac{5\pi}{2}$ (or $-450°$).

  (i.e. when the denominator $\cos x = 0$)
- The function is periodic with period π (NOT 2π!!!)
- The range of $y = \tan x$ is $\mathbb{R}$ (or $(-\infty, \infty)$).



Operations of trigonometric functions

It is important to know some identities of trigonometric functions so that we can do some algebra manipulations on them. There are three types of such identities:

- Relationship between $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$ and $\cot\theta$.
- Compound Angle Formulae (i.e. $\cos(A+B) = ??$)
- Sum to product formulae (i.e. $\sin A - \sin B$) and product to sum formulae (i.e. $\sin A \sin B$).

Theorem 1 (Basic Identities)

1. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ *and* $\sin^2\theta + \cos^2\theta = 1$
2. $1 + \tan^2\theta = \sec^2\theta$
3. $1 + \cot^2\theta = \csc^2\theta$

Theorem 2 (Compound Angle Theorem)

Addition

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Subtraction

$$\cos(A-B) = \cos A\, cosB + \sin A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

where A and B can be measured in either degree or radian.

With compound angle formula, one can compute the cosine/ sine value

Example 6

Compute $\sin 15°$ and $\tan 75°$

☺Solution:

$$\sin 15° = \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{(\sqrt{3}-1)\sqrt{2}}{(2\sqrt{2})\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

$$\tan 75° = \tan(45° + 30°) = \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1\left(\frac{1}{\sqrt{3}}\right)} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1}\left(\frac{\sqrt{3}+1}{\sqrt{3}+1}\right) = \frac{4+2\sqrt{3}}{2} = 2 + \sqrt{3}.$$

Example 7

(a) Show that for any θ, we have $\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$. (It is called half-angle formula).

(b) Using (a) and the fact that $\cos 15° = \frac{\sqrt{6}+\sqrt{2}}{4}$, find the value of $\cos 7.5°$.

☺Solution:

(a) Using compound angle formula with $A = B = \frac{\theta}{2}$, we have

$$\cos\theta = \cos\left(\frac{\theta}{2}+\frac{\theta}{2}\right) = \cos\frac{\theta}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\sin\frac{\theta}{2}$$

$$= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \overset{\sin^2 A+\cos^2 A=1}{\cong} \cos^2\frac{\theta}{2} - \left(1-\cos^2\frac{\theta}{2}\right) = 2\cos^2\frac{\theta}{2} - 1.$$

Rearranging the terms, we finally have

$$\Rightarrow \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}.$$

(b) Using (a) with $\theta = 15°$, we have

$$\cos^2 7.5° = \frac{1+\cos 15°}{2} = \frac{1+\frac{\sqrt{6}+\sqrt{2}}{4}}{2} = \frac{4+\sqrt{6}+\sqrt{2}}{8}.$$

Since $0° < 7.5° < 90°$, so $\cos 7.5° > 0$. Therefore,

$$\Rightarrow \cos 7.5° = \pm\sqrt{\frac{4+\sqrt{6}+\sqrt{2}}{8}} = \sqrt{\frac{4+\sqrt{6}+\sqrt{2}}{8}}.$$

☺Important Note:

In the last step, we still keep " $\pm$ " sign when taking square root since $\cos\theta$ can be positive or negative. The final sign depends on the magnitude of the θ. In the last example, we take " $+$ " sign since $7.5°$ is in "first quadrant". Consider another example, say $\cos 105°$, then

$$\cos^2 105° = \frac{1+\cos 210°}{2} \Rightarrow \cos 105° = -\sqrt{\frac{1+\cos 210°}{2}} = -\sqrt{\frac{2-\sqrt{3}}{4}}.$$

Example 8 (Finding $\cos 36°$)

(a) Using compound angle formula, show that

$$\cos 2\theta = 2\cos^2\theta - 1 \quad \text{and} \quad \cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

(b) For $\theta = 36°$, we have

$$\cos 3\theta = \cos 108° = \cos(180° - 2\theta).$$

Using this fact and the result of (a), find the value of $\cos 36°$.

☺Solution

(a) Using the compound angle formula with $A = B = \theta$, we have

$$\cos 2\theta = \cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$$
$$= \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1.$$

Using the compound angle formula with $A = 2\theta$ and $B = \theta$, we have

$$\cos 3\theta = \cos(2\theta + \theta) = \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$$
$$= (2\cos^2\theta - 1)\cos\theta - (\sin\theta\cos\theta + \cos\theta\sin\theta)\sin\theta$$
$$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$$
$$= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta = 4\cos^3\theta - 3\cos\theta.$$

(b) For $\theta = 36°$, we have $\cos 3\theta = \cos 108° = \cos(180° - 2\theta)$. Expanding the terms using the result of (a), we have

$$\overbrace{4\cos^3\theta - 3\cos\theta}^{\cos 3\theta} = \overbrace{-2\cos^2\theta + 1}^{\cos(180°-2\theta)=-\cos 2\theta}$$

$$\Rightarrow 4\cos^3\theta + 2\cos^2\theta - 3\cos\theta - 1 = 0 \ldots\ldots (*)$$

Let $y = \cos\theta = \cos 36°$, then the equation becomes

$$4y^3 + 2y^2 - 3y - 1 = 0 \Rightarrow (y+1)(4y^2 - 2y - 1) = 0.$$

$$\Rightarrow y + 1 = 0 \ \ or \ \ 4y^2 - 2y - 1 = 0.$$

$$\Rightarrow y = -1 \quad or \ \ y = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(4)(-1)}}{2(4)} = \frac{1 \pm \sqrt{5}}{4}.$$

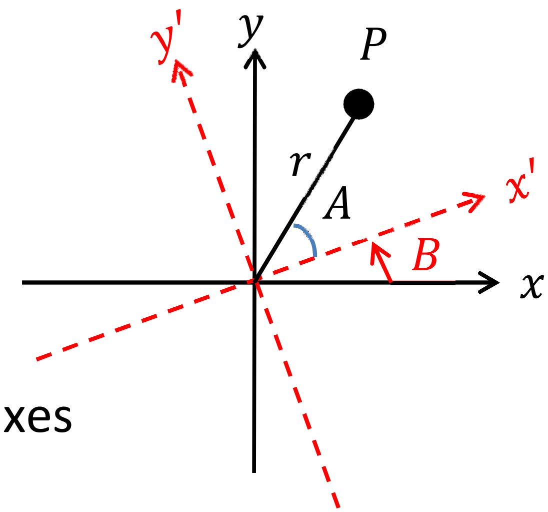
$$\Rightarrow \cos 36° = -1 \quad or \ \ \cos 36° = \frac{1+\sqrt{5}}{4} \quad or \ \ \cos 36° = \frac{1-\sqrt{5}}{4}.$$

Since $0° < 36° < 90°$, then $\cos 36° > 0$. So we conclude that

$$\cos 36° = \frac{1+\sqrt{5}}{4} \approx 0.809.$$

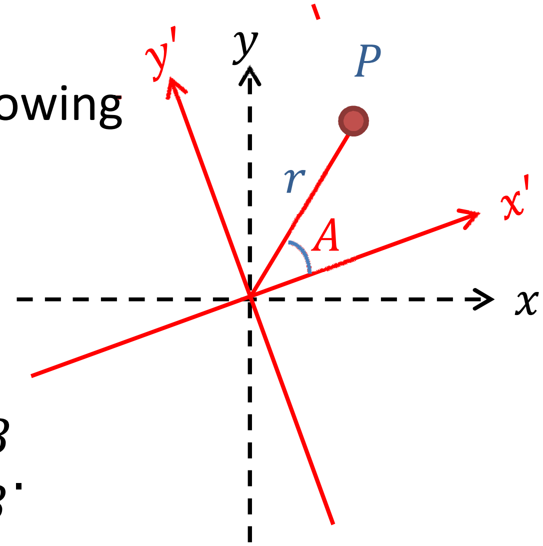
Proof of Theorem 2 (Compound Angle Formula)

We consider the figures on R.H.S.. Note that



- The (x, y)-coordinate of P is
  $(x, y) = (r\cos(A+B)\,, r\sin(A+B))\,.$
- The (x', y')-coordinate of P is
  $(x', y') = (r\cos A\,, r\sin A).$

Since the $x'y'$-axes is obtained by rotating xy-axes by an angle B through anti-clockwise direction.



Thus the two coordinates are related by the following transformation formula:

$$\begin{cases} x = x'\cos B - y'\sin B \\ y = y'\cos B + x'\sin B \end{cases}$$

$$\Rightarrow \begin{cases} r\cos(A+B) = r\cos A\cos B - r\sin A\sin B \\ r\sin(A+B) = r\sin A\cos B + r\cos A\sin B \end{cases}.$$

$$\Rightarrow \begin{cases} \cos(A+B) = \cos A \cos B - \sin A \sin B \\ \sin(A+B) = \sin A \cos B + \cos A \sin B \end{cases}.$$

To compute $\tan(A+B)$, note that

$$\begin{aligned} \tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \\ &= \frac{\dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B - \sin A \sin B}{\cos A \cos B}} = \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}} \\ &= \frac{\tan A + \tan B}{1 - \tan A \tan B}. \end{aligned}$$

To compute $\cos(A-B)$, $\sin(A-B)$ and $\tan(A-B)$, one can see that

$$\begin{aligned} \cos(A-B) &= \cos\big(A+(-B)\big) = \cos A \cos(-B) - \sin A \sin(-B) \\ &= \cos A \cos B - \sin A\,(-\sin B) = \cos A \cos B + \sin A \sin B. \end{aligned}$$

The formula of $\sin(A-B)$ and $\tan(A-B)$ can be derived in a similar fashion.

Theorem 3 (Sum-to-product Formula)

1. $\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

2. $\sin A - \sin B = 2\sin\dfrac{A-B}{2}\cos\dfrac{A+B}{2}$

3. $\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$

4. $\cos A - \cos B = -2\sin\dfrac{A+B}{2}\sin\dfrac{A-B}{2}$

Example 9

$$\sin 75° - \sin 15° = 2\cos\frac{75° + 15°}{2}\sin\frac{75° - 15°}{2} = 2\cos 45°\sin 30° = \frac{1}{\sqrt{2}}.$$

$$\frac{\cos 50° + \cos 10°}{\cos 20°} = -\frac{2\left(\cos\frac{50° + 10°}{2}\cos\frac{50° - 10°}{2}\right)}{\cos 20°} = \frac{2\cos 30°\cos 20°}{\cos 20°} = \sqrt{3}$$

Example 10

Simplify

$$\frac{\sin 5\theta + \sin(180° - 3\theta)}{\cos 5\theta + \cos(180° - 3\theta)}, \qquad \frac{\sin(x+y) + \sin(x-y)}{\sin(x+y) - \sin(x-y)}$$

☺Solution:

$$\frac{\sin 5\theta + \sin(180° - 3\theta)}{\cos 5\theta + \cos(180° - 3\theta)} = \frac{\sin 5\theta + \sin 3\theta}{\cos 5\theta - \cos 3\theta} = \frac{2\sin\frac{5\theta + 3\theta}{2}\cos\frac{5\theta - 3\theta}{2}}{-2\sin\frac{5\theta + 3\theta}{2}\sin\frac{5\theta - 3\theta}{2}}$$

$$= -\frac{\sin 4\theta \cos\theta}{\sin 4\theta \sin\theta} = -\frac{1}{\tan\theta} = -\cot\theta.$$

$$\frac{\sin(x+y) + \sin(x-y)}{\sin(x+y) - \sin(x-y)} = \frac{2\sin\left[\frac{(x+y)+(x-y)}{2}\right]\cos\left[\frac{(x+y)-(x-y)}{2}\right]}{2\cos\left[\frac{(x+y)+(x-y)}{2}\right]\sin\left[\frac{(x+y)-(x-y)}{2}\right]}$$

$$= \frac{2\sin x\cos y}{2\cos x\sin y} = \frac{\tan x}{\tan y}.$$

Proof of Theorem 3 (Sum-to-product formula and Product-to-sum formula)

We prove the case for $\sin A + \sin B$ and $\sin A - \sin B$. Recall the compound angle formula:

$$\sin(X+Y) = \sin X \cos Y + \cos X \sin Y \ \ldots\ldots (1)$$

$$\sin(X-Y) = \sin X \cos Y - \cos X \sin Y \ \ldots\ldots (2)$$

By $(1)+(2)$, we have

$$\sin(X+Y) + \sin(X-Y) = 2\sin X \cos Y \ \ldots\ldots (3)$$

By $(1)-(2)$, we have

$$\sin(X+Y) - \sin(X-Y) = 2\cos X \sin Y \ \ldots\ldots (4)$$

Let $A = X+Y$ and $B = X-Y$, then $X = \frac{A+B}{2}$ and $Y = \frac{A-B}{2}$. From (3), (4),

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}, \qquad \sin A - \sin B = 2\sin\frac{A-B}{2}\cos\frac{A+B}{2}.$$

The derivation of $\cos A + \cos B$ and $\cos A - \cos B$ is similar and left as exercise.

Inverse trigonometric functions

- Given the value of θ, one can obtain the values of $y = \sin\theta\,, y = \cos\theta$ and $y = \tan\theta$. Conversely, given the value of y, one would like to find the corresponding value of θ such that
$$y = \sin\theta\,, y = \cos\theta\,, y = \tan\theta.$$

- Suppose the solution exists, the θ obtained is called *the inverse* of the trigonometric functions and is denoted by
$$\theta = \sin^{-1} y\,, \qquad \theta = \cos^{-1} y\,, \qquad \theta = \tan^{-1} y.$$

☺Note:

One has to be careful that $\sin^{-1} y \neq \frac{1}{\sin y}$. In mathematics, $\sin^{-1} y$ represents the inverse of the function. One can use $(\sin y)^{-1}$ to represent $\frac{1}{\sin y}$.

Important features of inverse trigonometric functions

- **The domain of $\sin^{-1} y$ and $\cos^{-1} y$ can only be $[-1, 1]$ (not $\mathbb{R}$) . The domain of $\tan^{-1} y$ is $\mathbb{R}$.**
  Reason: The range of $\sin\theta$ and $\cos\theta$ are both $[-1,1]$. In particular, one cannot find any solution for $\sin\theta = 2$. On the other hand, the range of $y = \tan\theta$ is $\mathbb{R}$.

- **The *principle* ranges of $\sin^{-1} y$, $\cos^{-1} y$ and $\tan^{-1} y$ are chosen to be $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (for $\sin^{-1} y$), $[0, \pi]$ (for $\cos^{-1} y$) and $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (for $\tan^{-1} y$) respectively.**
  Reason: In general, the equation $y = \sin\theta$, $y = \cos\theta$ and $y = \tan\theta$ have more than 1 (or infinitely many) solutions. For example, when we compute $\sin^{-1}\frac{1}{2}$, one can see that the equation $\sin\theta = \frac{1}{2}$ has infinitely many solutions (say $\theta = \cdots - 330°, -210°, 30°, 150°, 390°,\ldots$).

In order to settle this problem, one has to restrict the range of θ so that $\sin^{-1}\frac{1}{2}$. In this example, one can restrict θ to be within $[-90°, 90°]$ or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so that $\sin^{-1}\frac{1}{2} = 30°$.

If one restrict θ to be within $[90°, 270°]$, then $\sin^{-1}\frac{1}{2} = 150°$. Hence the final value of $\sin^{-1}\frac{1}{2}$ depends on the range chosen. In most of the applications, one chooses $[-90°, 90°]$ as the range of $\sin^{-1} y$. So $[-90°, 90°]$ is called principal range.

- Since $\sin^{-1} y$, $\cos^{-1} y$ and $\tan^{-1} y$ are inverse of $\sin\theta$, $\cos\theta$ and $\tan\theta$. So we expect that

$$\begin{aligned} \sin(\sin^{-1} y) &= y, & \sin^{-1}(\sin\theta) &= \theta \\ \cos(\cos^{-1} y) &= y, & \cos^{-1}(\cos\theta) &= \theta \\ \tan(\tan^{-1} y) &= y, & \tan^{-1}(\tan\theta) &= \theta. \end{aligned}$$

Example 11

Compute $\sin^{-1}(\cos 390°)$ and $\tan^{-1}(\tan 225°)$. Express your answer in principal range.

☺Solution:

$$\sin^{-1}(\cos 390°) = \sin^{-1}(\cos 30°) = \sin^{-1}\frac{\sqrt{3}}{2} = 60°.$$

$$\begin{aligned}\tan^{-1}(\tan 225°) = \tan^{-1}(\tan(180° + 45°)) &= \tan^{-1}(\tan 45°) = \tan^{-1} 1 \\ &= 45°\end{aligned}$$

☺Note:

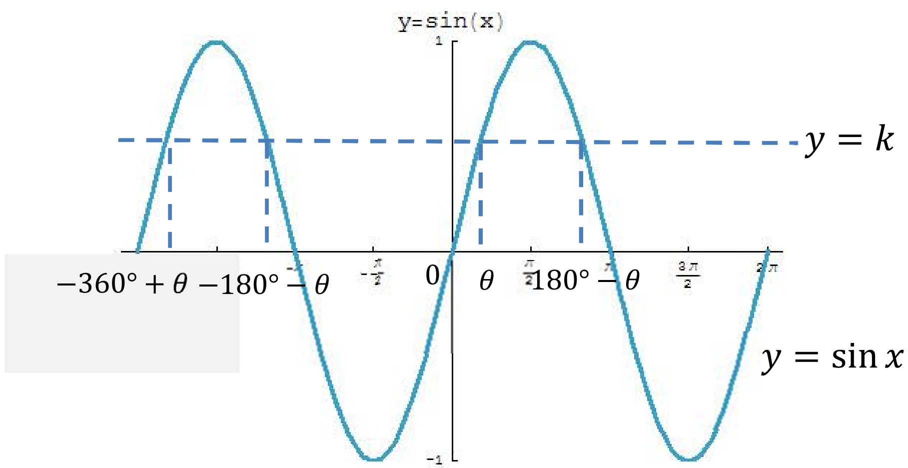
In the last part, we see that $\tan^{-1}(\tan 225°) = 45° \neq 225°$. It is simply because the selection of range of $\tan^{-1} y$. One can have $\tan^{-1}(\tan 225°) = 225°$ if the range is chosen to be $[90°, 270°]$.

General Solution of Trigonometric Equations

In our previous discussion about the inverse of trigonometric function, we have pointed out that the equation $\sin x = \frac{1}{2}$ has infinitely many solutions. In this section, we shall discuss the *general solution* of such equation.

<u>General Solution of $\sin\theta = k$ where $k \in [-1,1]$</u>

One can visualize the solution by considering the following diagram:



- One can observe that if θ is one of the solution, then $180° - \theta$ is also the solution since $\sin(180° - \theta) = \sin\theta = k$.

- Since $y = \sin\theta$ is periodic with period $360°$ (or 2π), this implies that

$$k = \sin(180° - \theta) = \sin(180° - \theta - 360°) = \sin(-180° - \theta)$$

$$k = \sin\theta = \sin(\theta - 360°) = \sin(-360° + \theta)$$

  This implies that $-180° - \theta$ and $-360° + \theta$ are also the solution of $\sin x = k$.

- Using this argument repeatedly, we find that the solution of $\sin x = k$ are

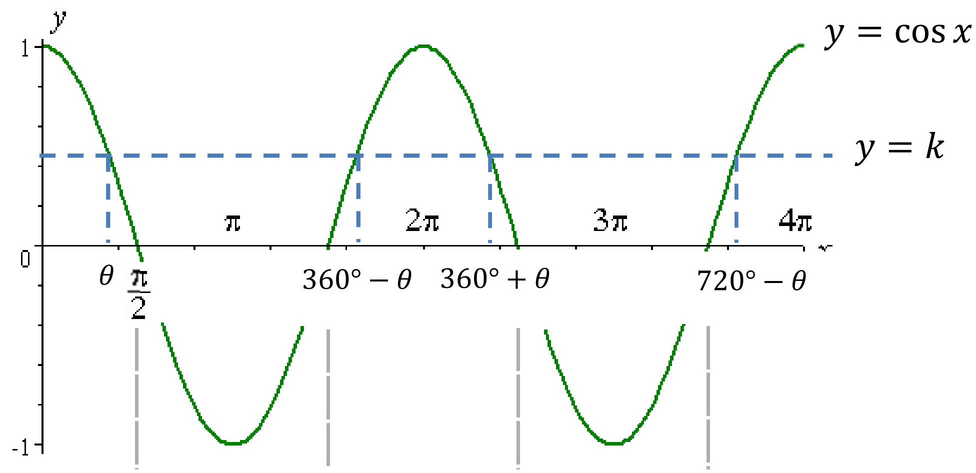
$$\ldots - 360° + \theta, -180° + \theta, \theta, 180° - \theta, 360° + \theta, 540° - \theta$$

  By observing the pattern of the solution, we conjecture the general solution of $\sin x = k$ is given by

$$x = 180°n + (-1)^n\theta, \quad (or \;\; x = n\pi + (-1)^n\theta)$$

  where $\theta = \sin^{-1} k$ (can be obtained from calculator) and n is integer.

General Solution of $\cos x = k$



- One can observe that if θ is one of the solution, then $360° - \theta$ is also the solution since $\cos(360° - \theta) = \cos\theta = k$.

- Since $y = \cos\theta$ is periodic with period $360°$ (or 2π), this implies that

$$k = \cos\theta = \cos(360° + \theta)$$

$$k = \cos(360° - \theta) = \cos(360° - \theta + 360°) = \cos(720° - \theta)$$

This implies that $360° + \theta$ and $720° - \theta$ are also the solution of $\cos x = k$.

- Using this argument repeatedly, we find that the solution of $\sin x = k$ are

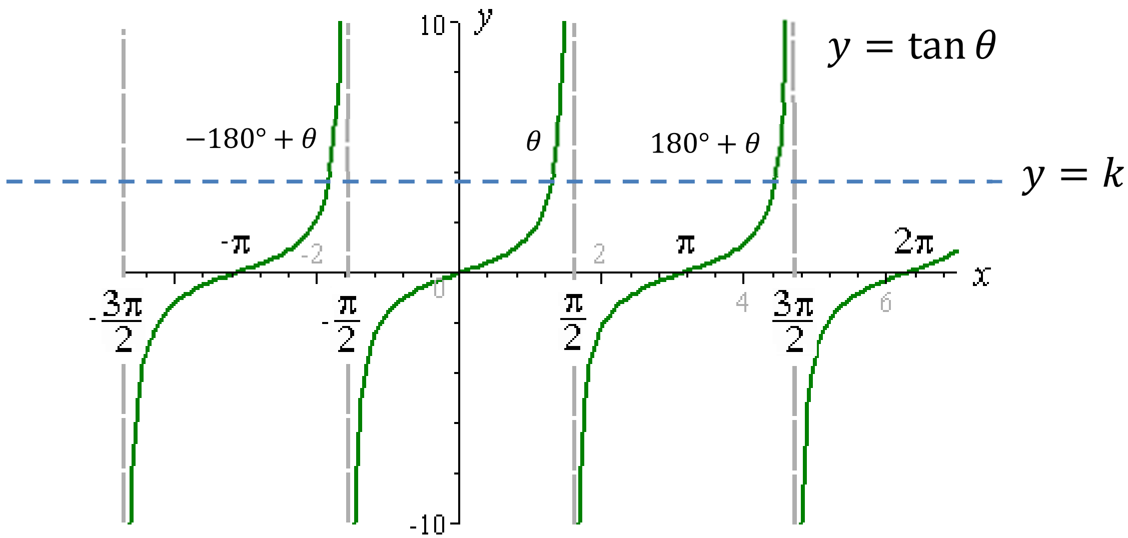
$$\ldots - 360° - \theta, -360° + \theta, -\theta, \theta, 360° - \theta, 360° + \theta, 720° - \theta$$

By observing the pattern of the solution, we conjecture the general solution of $\cos x = k$ is given by

$$x = 360° n \pm \theta,$$

where $\theta = \cos^{-1} k$ (can be obtained from calculator) and n is integer.

General Solution of $\tan x = k$



- If θ is the solution of $\tan x = k$, using the periodicity of $y = \tan x$, one can see that $\ldots, -360° + \theta, -180° + \theta, 180° + \theta, 360 + \theta, \ldots$ are all solution of $\tan x = k$.

  By observing the pattern of the solution, we conjecture that the general solution of $\tan x = k$ is given by

$$x = 180°n + \theta,$$

  where $\theta = \tan^{-1} k$ (can be obtained from calculator) and n is integer.

Summary (General Solution of Trigonometric Equation)

(i) The general solution of $\sin x = k$ (where $-1 \leq k \leq 1$) is given by

$$x = 180°n + (-1)^n \sin^{-1} k, \qquad n \; is \; integer.$$

(ii) The general solution of $\cos x = k$ (where $-1 \leq k \leq 1$) is given by

$$x = 360°n \pm \cos^{-1} k, \qquad n \; is \; integer.$$

(iii) The general solution of $\tan x = k$ (where $k \in \mathbb{R}$) is given by

$$x = 180°n + \tan^{-1} k, \quad n \; is \; integer$$

Example 12

Solve the equation

$$\sin\frac{x}{3}=\frac{\sqrt{3}}{2}, \qquad \tan 2x=-\sqrt{3}.$$

Express your answer in *radian* measure.

☺Solution:

$$\sin\frac{x}{3}=\frac{\sqrt{3}}{2}\Rightarrow\frac{x}{3}=n\pi+(-1)^n\sin^{-1}\frac{\sqrt{3}}{2}\Rightarrow\frac{x}{3}=n\pi+(-1)^n\frac{\pi}{3}$$

$$\Rightarrow x=3n\pi+(-1)^n\pi$$

$$\tan 2x=-\sqrt{3}\Rightarrow 2x=n\pi+\tan^{-1}\left(-\sqrt{3}\right)\Rightarrow 2x=n\pi+\left(-\frac{\pi}{3}\right)$$

$$\Rightarrow x=\frac{n\pi}{2}-\frac{\pi}{6}.$$

Example 13

Solve

$$2\cos^2 x + 5\cos x - 3 = 0.$$

☺Solution:

Let $y = \cos x$, then the equation becomes

$$2y^2 + 5y - 3 = 0 \Rightarrow (2y-1)(y+3) = 0 \Rightarrow y = \frac{1}{2} \;\; or \;\; y = -3$$

$$\Rightarrow \cos x = \frac{1}{2} \quad or \quad \cos x = -3$$

(*The solution $\cos x = -3$ is rejected since $-1 \le x \le 1$)

$$\Rightarrow x = 2n\pi \pm \cos^{-1}\frac{1}{2} = 2n\pi \pm \frac{\pi}{3} \;\; (or\ x = 360°n \pm 60°).$$

where n is integer.

Extra Example 1 (Some useful formula)

(a) Using compound angle formulae, show that

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}, \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta.$$

(b) Using the result of (a), show that

$$\sin^2 x\cos^2 x = \frac{1}{8} - \frac{1}{8}\cos 4x.$$

☺Solution

(a) Using compound angle formula with $A = B = \theta$, we have

$$\cos 2\theta = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$$

$$= \begin{cases} \cos^2\theta - (1-\cos^2\theta) \\ (1-\sin^2\theta) - \sin^2\theta \end{cases} = \begin{cases} 2\cos^2\theta - 1 \\ 1 - 2\sin^2\theta \end{cases}.$$

Rearranging the term for each case, we get

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}, \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}.$$

On the other hand, we have

$$\sin 2\theta = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\cos\theta\sin\theta$$

$$\Rightarrow \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta.$$

(b) Note that

$$\sin^2 x\cos^2 x = (\sin x\cos x)^2 = \left(\frac{1}{2}\sin 2x\right)^2$$

$$= \frac{1}{4}\sin^2 2x = \frac{1}{4}\left(\frac{1-\cos 2(2x)}{2}\right)$$

$$= \frac{1}{8}(1-\cos 4x).$$

Extra Example 2

Find the general solution of the equation

$$\cos 3\theta = \cos(60° - 2\theta),$$

$$\sin 5\theta = \cos(80° + 3\theta).$$

☺Solution:

Note that

$\cos 3\theta = \cos(60° - 2\theta)$

$\Rightarrow 3\theta = 360°n \pm \cos^{-1}[\cos(60° - 2\theta)] = 360°n \pm (60° - 2\theta)$

$\Rightarrow 3\theta = 360°n + (60° - 2\theta)$ or $3\theta = 360°n - 60° + 2\theta$

$\Rightarrow 5\theta = 360°n + 60°$ or $\theta = 360°n - 60°$

$\Rightarrow \theta = 72°n + 12°$ or $\theta = 360°n - 60°$

where n is integer.

On the other hand, using the identity $\cos\theta = \sin(90° - \theta)$, we get

$$\sin 5\theta = \cos(80° + 3\theta)$$

$$\Rightarrow \sin 5\theta = \sin(90° - (80° + 3\theta)) = \sin(10° - 3\theta)$$

$$\Rightarrow 5\theta = 180°n + (-1)^n \sin^{-1}\sin(10° - 3\theta)$$

$$\Rightarrow 5\theta = 180°n + (-1)^n(10° - 3\theta)$$

$$\Rightarrow 5\theta + (-1)^n 3\theta = 180°n + (-1)^n 10°$$

$$\Rightarrow \begin{cases} 5\theta - 3\theta = 180°n - 10° & if \;\; n\; is\; odd \\ 5\theta + 3\theta = 180°n + 10° & if\; n\; is\; even \end{cases}.$$

$$\Rightarrow \begin{cases} \theta = 90°n - 5° & if \;\; n\; is\; odd \\ \theta = 22.5°n + 1.25° & if\; n\; is\; even \end{cases}.$$

Theorem 4 (Product to Sum Formula)

1. $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

2. $\sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$

3. $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

Example

$$\cos \overbrace{3\theta}^{B} \sin \overbrace{5\theta}^{A} = \frac{1}{2}[\sin(5\theta + 3\theta) + \sin(5\theta - 3\theta)] = \frac{1}{2}(\sin 8\theta + \sin 2\theta).$$

$$\begin{aligned}
&\sin(x-y)\sin(x+y) \\
&\qquad = \frac{1}{2}\left[\cos\big((x-y)+(x+y)\big) - \cos\big((x-y)-(x+y)\big)\right] \\
&\qquad = \frac{1}{2}(\cos 2x - \cos(-2y)) = \frac{1}{2}(\cos 2x - \cos 2y).
\end{aligned}$$

Proof of Theorem 4 (Product-to-sum Formula)

To establish the 1st and 2nd identities, we consider the compound angle formulae for cosine function:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \text{ , …… (1)}$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B \text{ …… (2)}$$

By adding (1) to (2), we have

$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$\Rightarrow \cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)].$$

By subtracting (2) from (1), we have

$$-2\sin A \sin B = \cos(A+B) - \cos(A-B),$$

$$\Rightarrow \sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)].$$

To establish the last identity, we consider the compound angle formula for sine function:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \;\ldots\ldots (3)$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B \;\ldots\ldots (4)$$

By adding (3) to (4), we have

$$2\sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$\Rightarrow \sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)].$$

About the use of sum-to-product and product-to-sum formulae

When we handle the algebraic computation involving trigonometric functions, it is often easier if one can express the expression as the product/quotient of trigonometric functions using sum-to-product formula:

$$\underbrace{\frac{\sin 5\theta + \sin 3\theta}{\cos 5\theta - \cos 3\theta}}_{\substack{difficult\ to \\ calculate}} = \frac{2\sin\frac{5\theta+3\theta}{2}\cos\frac{5\theta-3\theta}{2}}{-2\sin\frac{5\theta+3\theta}{2}\sin\frac{5\theta-3\theta}{2}} = \underbrace{-\frac{\sin 4\theta\cos\theta}{\sin 4\theta\sin\theta}}_{\substack{easy\ to \\ calculate}}$$

On the other hand, when we try to do some calculus (differentiation or integration) to the functions which are combination of trigonometric functions, it would be easier to rewrite the expression into the sum of trigonometric functions:

$$\sin^2\theta\cos^3\theta = \underbrace{\frac{1}{8}\cos\theta - \frac{1}{16}\cos 5\theta - \frac{1}{16}\cos 3\theta}_{easy\ for\ calculus}.$$