Example 15

Consider $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to\mathbb{R}$, where $f(x)=\sin x$. Then $f(x)$ is one-to-one and its inverse function is $f^{-1}(x)=\sin^{-1}x$. If $y=\sin^{-1}x$, what is $\frac{dy}{dx}$?

Solution

Method 1: Use the inverse function theorem

$y=\sin^{-1}x \quad\Longrightarrow\quad x=\sin y$

Differentiate both sides with respect to y:

$$\frac{dx}{dy}=\cos y$$

By the **inverse function theorem**,

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}},$$

Do not write $\frac{1}{\cos(\sin^{-1}x)}$

which is only defined when $-1<x<1$.

$\cos^2 y+\sin^2 y=1$

$\Rightarrow \cos y=\pm\sqrt{1-\sin^2 y}$

However, $\cos y>0$ when $y\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

$\therefore$ Take positive $\sqrt{\ }$.

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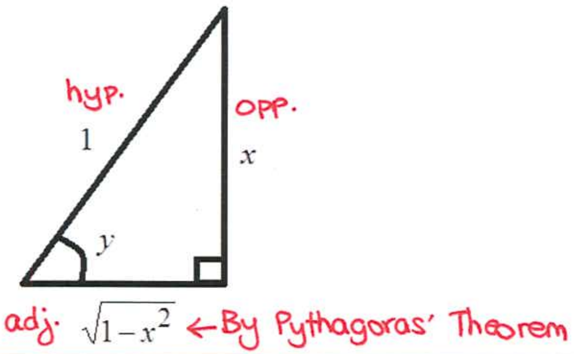
Alternatively, we can deduce the relationship $\cos y = \sqrt{1-x^2}$ by considering the following right-angled triangle:

$$x = \sin y \Rightarrow \sin y = \frac{x}{1} \begin{matrix} \leftarrow \text{opp.} \\ \leftarrow \text{hyp.} \end{matrix}$$

adj.

$$\therefore \cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}.$$

hyp.



Method 2: $\boxed{\textit{Implicit Differentiation}}$

$y = \sin^{-1} x \quad \Longrightarrow \quad x = \sin y$ ← implicit function

Differentiate both sides with respect to x:

$\frac{d}{dx}(\sin y) = \frac{d}{dy}(\sin y) \cdot \frac{dy}{dx}$

$$\underbrace{\frac{dx}{dx}}_{=1} = \cos y \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

Homework: Use the **inverse function theorem** to show that $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$.

(Hint: start with $y = \cos^{-1} x$.)

Example 16

Show that $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$.

Solution

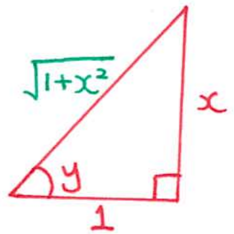
Method 1: Inverse function theorem

Let $y = \tan^{-1} x \quad \Longrightarrow \quad x = \tan y$

Differentiate both sides with respect to $\boldsymbol{y}$:

$$\frac{dx}{dy} = \sec^2 y$$

By the **inverse function theorem**, $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$.



$\tan y = \frac{x}{1}$ ← opp. ← adj.

$\therefore \frac{1}{\sec y} = \cos y$

$= \frac{1}{\sqrt{1+x^2}}$ ← adj. ← hyp.

Method 2: Implicit Differentiation

Let $y = \tan^{-1} x \quad \Longrightarrow \quad x = \tan y$ (which is an **implicit** function)

Differentiate both sides with respect to $\boldsymbol{x}$:

$$1 = \sec^2 y \frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2} .$$

Example 17

Given that $\cosh^2 u - \sinh^2 u = 1$ for all $u \in \mathbb{R}$. Use **implicit differentiation** to show that

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}.$$

Solution

$y = \sinh^{-1} x \Leftrightarrow x = \sinh y$

Diff. both sides w.r.t. x :

$$1 = \cosh y \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$$

using $\cosh y = \sqrt{1+\sinh^2 y}$

since $\cosh y = \frac{1}{2}(e^y + e^{-y}) > 0$ for every $y \in \mathbb{R}$.

$\therefore$ Take positive $\sqrt{\ }$.

Note: $\sinh^{-1} x \neq (\sinh x)^{-1}$.

Example 18

Differentiate each of the following functions with respect to x:

(a) $x^2 \sin^{-1}(x^3+1)$ (b) $\tan^{-1}\left(\frac{1-x}{1+x}\right)$ (c) $\cos^{-1}\left(\frac{1}{1+x^2}\right)$

Solution

product rule

outer: $\sin^{-1}(\cdot)$
inner: x^3+1

(a) $$\frac{d}{dx}[x^2 \sin^{-1}(x^3+1)] = x^2 \cdot \frac{d}{dx}[\sin^{-1}(x^3+1)] + \sin^{-1}(x^3+1)\cdot\frac{d}{dx}(x^2)$$

Not the same as $[\sin(x^3+1)]^{-1}$

$$= x^2 \cdot \frac{1}{\sqrt{1-(x^3+1)^2}} \cdot \frac{d}{dx}(x^3+1) + \sin^{-1}(x^3+1)\cdot 2x$$

$$= x^2 \cdot \frac{1}{\sqrt{1-(x^3+1)^2}} \cdot 3x^2 + 2x\sin^{-1}(x^3+1)$$

$$= \frac{3x^4}{\sqrt{1-(x^3+1)^2}} + 2x\sin^{-1}(x^3+1)$$

(b)
$$\frac{d}{dx}\left[\tan^{-1}\left(\frac{1-x}{1+x}\right)\right] = \frac{1}{1+\left(\frac{1-x}{1+x}\right)^2}\cdot\frac{d}{dx}\left(\frac{1-x}{1+x}\right)$$
← chain rule

outer: $\tan^{-1}(\cdot)$
inner: $\frac{1-x}{1+x}$

$$= \frac{1}{1+\left(\frac{1-x}{1+x}\right)^2}\cdot\frac{(1+x)\frac{d}{dx}(1-x)-(1-x)\frac{d}{dx}(1+x)}{(1+x)^2}$$
← quotient rule

$$= \frac{\cancel{(1+x)^2}}{(1+x)^2+(1-x)^2}\cdot\frac{(1+x)(-1)-(1-x)(1)}{\cancel{(1+x)^2}}$$

$$= \frac{-1-x-1+x}{(1+x)^2+(1-x)^2} = \frac{-2}{(1+\cancel{2x}+x^2)+(1-\cancel{2x}+x^2)} = \frac{-2}{2(1+x^2)} = \frac{-1}{1+x^2}$$

(c)
chain rule
$$\frac{d}{dx}\left[\cos^{-1}\left(\frac{1}{1+x^2}\right)\right] = \frac{-1}{\sqrt{1-\left(\frac{1}{1+x^2}\right)^2}}\cdot\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = \frac{-1}{\sqrt{1-\left(\frac{1}{1+x^2}\right)^2}}\cdot\frac{d}{dx}[(1+x^2)^{-1}]$$

outer: $(\cdot)^{-1}$
inner: $1+x^2$

outer: $\cos^{-1}(\cdot)$
inner: $\frac{1}{1+x^2}$

$$= \frac{-1}{\sqrt{1-\left(\frac{1}{1+x^2}\right)^2}}\cdot(-1)(1+x^2)^{-2}\cdot\frac{d}{dx}(1+x^2)$$

$$= \frac{-1}{\sqrt{1-\left(\frac{1}{1+x^2}\right)^2}}\cdot(-1)(1+x^2)^{-2}\cdot 2x$$

$$= \frac{2x}{(1+x^2)^2\sqrt{1-\left(\frac{1}{1+x^2}\right)^2}} = \frac{2x}{(1+x^2)\sqrt{(1+x^2)^2-1}}$$

Derivatives of Exponential and Logarithmic functions

Recall that exponential and logarithmic functions are the inverse functions of each other.

That is,

$$y = a^x \quad \Leftrightarrow \quad x = \log_a y \quad \text{for } x \in \mathbb{R} \text{ and } y \in (0, \infty),$$

where the base $a > 0$ and $a \neq 1$. Now consider the exponential and logarithmic functions with base $a = e$, where $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.71828 \ldots$.

We know that

$$\frac{d}{dx}(e^x) = e^x \quad \text{and} \quad \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Recall the basic properties of natural logarithm:

- ☆ $\ln(ab) = \ln a + \ln b$, where $a, b > 0$.
- ☆ $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$, where $a, b > 0$.
- ☆ $\ln(a^b) = b \ln a$, where $a\ (> 0)$ and b are constants.
- ☆ $\ln\left(e^{f(x)}\right) = f(x) \ln e = f(x)$, since $\ln e = 1$.

Example 19

Find the derivatives of the following functions:

(a) $f(x) = a^x$, where $a > 0$ and $a \neq 1$

(b) $g(x) = \log_a x$, where $a > 0$ and $a \neq 1$

(c) $h(x) = x^a$, where $a \in \mathbb{R}$. (Note that x^a is not an exponential function.)

Solution

(a) $f(x) = a^x = e^{\ln(a^x)} = e^{x \ln a}$ ← change to base e

$$\therefore \; f'(x) = \frac{d}{dx}\left(e^{x \ln a}\right) = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) = e^{x \ln a} \cdot \ln a = a^x(\ln a)$$

(b) $g(x) = \log_a x = \dfrac{\ln x}{\ln a}$ ← change to log. with base e

$$\therefore \; g'(x) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a} \cdot \frac{d}{dx}(\ln x) = \frac{1}{x \ln a}$$

(c) $h(x) = x^a = e^{\ln(x^a)} = e^{a \ln x}$ ← change to base e

$$\therefore \; h'(x) = \frac{d}{dx}\left(e^{a \ln x}\right) = e^{a \ln x} \cdot \frac{d}{dx}(a \ln x) = e^{a \ln x} \cdot \frac{a}{x} = x^a \cdot \frac{a}{x} = ax^{a-1}$$

Example 20

If $f(x)=e^{-\frac{x}{n}}\cos\left(\frac{x}{a}\right)$, find the value of $f(0)+af'(0)$, where $a\neq 0$ and $n\neq 0$ are constants.

Solution

put x=0

$$f(x)=e^{-\frac{x}{n}}\cos\left(\frac{x}{a}\right) \Rightarrow f(0)=e^{0}\cdot\cos 0=1$$

$$f'(x)=e^{-\frac{x}{n}}\cdot\frac{d}{dx}\left[\cos\left(\frac{x}{a}\right)\right]+\cos\left(\frac{x}{a}\right)\cdot\frac{d}{dx}\left(e^{-\frac{x}{n}}\right)$$ ← product rule

$$=e^{-\frac{x}{n}}\cdot\left[-\sin\left(\frac{x}{a}\right)\right]\cdot\frac{1}{a}+\cos\left(\frac{x}{a}\right)\cdot e^{-\frac{x}{n}}\cdot\left(-\frac{1}{n}\right)$$

$$=-\frac{1}{a}e^{-\frac{x}{n}}\sin\left(\frac{x}{a}\right)-\frac{1}{n}e^{-\frac{x}{n}}\cos\left(\frac{x}{a}\right)$$

put x=0

$$\Rightarrow f'(0)=-\frac{1}{a}\underbrace{e^{0}}_{=1}\cdot\underbrace{\sin 0}_{=0}-\frac{1}{n}\underbrace{e^{0}}_{=1}\cdot\underbrace{\cos 0}_{=1}=-\frac{1}{n}$$

$$\therefore\ f(0)+af'(0)=1+a\cdot\left(-\frac{1}{n}\right)=1-\frac{a}{n}$$

Logarithmic Differentiation

This is used to differentiate functions of the form

exponent is a function of x

(i) $\boxed{y = [u(x)]^{v(x)}}$, where $u(x)$ and $\underline{v(x)}$ are both functions of x.

(Here, $u(x)$ could be a non-zero constant or a function of x.)

(ii) $\boxed{y = \dfrac{[u_1(x)]^{a_1} \cdot [u_2(x)]^{a_2} \cdot \dots \cdot [u_n(x)]^{a_n}}{[v_1(x)]^{b_1} \cdot [v_2(x)]^{b_2} \cdot \dots \cdot [v_m(x)]^{b_m}}}$, where $u_1(x), \dots, u_n(x), v_1(x), \dots, v_m(x)$ are functions of x; and $a_1, \dots, a_n, b_1, \dots, b_m$ could be non-zero constants or functions of x.

(product or quotient of many functions of x)

Example 21

Given that $y = (x^2+1)^{\cot x}$. Find $\frac{dy}{dx}$.

Solution

Method 1: Use logarithmic differentiation

$$y = (x^2+1)^{\cot x}$$

Take natural logarithm on both sides:

$\ln(a^b) = b \ln a$

$$\ln y = \ln[(x^2+1)^{\cot x}]$$

$$= (\cot x)\ln(x^2+1) \quad \leftarrow \text{This is an } \textbf{implicit function}.$$

Differentiate both sides w.r.t. x:

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}[(\cot x)\ln(x^2+1)]$$

$\begin{cases} \text{outer: } \ln(\cdot) \\ \text{inner: } x^2+1 \end{cases}$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \underbrace{(\cot x)\cdot\frac{d}{dx}[\ln(x^2+1)] + \ln(x^2+1)\cdot\frac{d}{dx}(\cot x)}_{\text{by product rule}}$$

$\frac{d}{dy}(\ln y)\cdot\frac{dy}{dx}$

$$= (\cot x)\cdot\frac{1}{x^2+1}\cdot\frac{d}{dx}(x^2+1) + \ln(x^2+1)\cdot[-\operatorname{cosec}^2 x]$$

$$= (\cot x)\cdot\frac{1}{x^2+1}\cdot 2x - \ln(x^2+1)\cdot\operatorname{cosec}^2 x$$

Multiply both sides by y and then replace y with $(x^2+1)^{\cot x}$:

$$\begin{aligned}\frac{dy}{dx} &= y\left[(\cot x)\cdot\frac{1}{x^2+1}\cdot 2x - \ln(x^2+1)\cdot\operatorname{cosec}^2 x\right]\\ &= (x^2+1)^{\cot x}\left[(\cot x)\cdot\frac{1}{x^2+1}\cdot 2x - \ln(x^2+1)\cdot\operatorname{cosec}^2 x\right]\end{aligned}$$

Note: For logarithmic differentiation, the right hand side should not contain any y.

Method 2: Use the Table of Derivatives

If $\boxed{y = u^v}$, then $\boxed{\frac{dy}{dx} = v\,u^{v-1}\,\frac{du}{dx} + u^v\,\log_e u\,\frac{dv}{dx}}$, where $\log_e u = \ln u$.

$y = (x^2+1)^{\cot x}$ is of the form $y = u^v$, where $u = x^2+1$ and $v = \cot x$.

Thus, $$\begin{aligned}\frac{dy}{dx} &= (\cot x)(x^2+1)^{\cot x - 1}\,\frac{d}{dx}(x^2+1) + (x^2+1)^{\cot x}\,\log_e(x^2+1)\,\frac{d}{dx}(\cot x)\\ &= (\cot x)(x^2+1)^{\cot x - 1}\cdot(2x) + (x^2+1)^{\cot x}\,\ln(x^2+1)\cdot(-\operatorname{cosec}^2 x)\end{aligned}$$

Example 22

If $y = \left(\frac{a}{x}\right)^{ax}$, find $\frac{dy}{dx}$.

Solution

$$y = \left(\frac{a}{x}\right)^{ax}$$

$\ln(a^b) = b \ln a$

$\ln(\frac{a}{b}) = \ln a - \ln b$

Take natural logarithm on both sides:

$$\ln y = \ln\left[\left(\frac{a}{x}\right)^{ax}\right] = ax \ln\left(\frac{a}{x}\right) = ax\,(\ln a - \ln x)$$

Differentiate both sides w.r.t. x:

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}[ax(\ln a - \ln x)]$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \underbrace{ax \cdot \frac{d}{dx}(\overset{\text{Constant}}{\boxed{\ln a}} - \ln x) + (\ln a - \ln x)\cdot\frac{d}{dx}(ax)}_{\text{by product rule}}$$

$$= ax \cdot \left(0 - \frac{1}{x}\right) + (\ln a - \ln x)\cdot a$$

$$\Rightarrow \quad \frac{dy}{dx} = \left(\frac{a}{x}\right)^{ax}\left[-a + a\ln\left(\frac{a}{x}\right)\right]$$

Example 23

Given that $y = \frac{x}{(x-1)(x-2)(x-3)}$. Find $\frac{dy}{dx}$.

Solution

Method 1: Use Product rule and Quotient rule ← long calculation, tedious!

Not recommended.

Method 2: Use logarithmic differentiation ← more convenient!

$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

$\ln(ab) = \ln a + \ln b$

Take natural logarithm on both sides:

$$\ln y = \ln\left[\frac{x}{(x-1)(x-2)(x-3)}\right] = \ln x - \ln(x-1) - \ln(x-2) - \ln(x-3)$$

Differentiate both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x-1} - \frac{1}{x-2} - \frac{1}{x-3}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{x} - \frac{1}{x-1} - \frac{1}{x-2} - \frac{1}{x-3}\right)$$

$$= \frac{x}{(x-1)(x-2)(x-3)}\left(\frac{1}{x} - \frac{1}{x-1} - \frac{1}{x-2} - \frac{1}{x-3}\right)$$

Example

Differentiate $2^{\sqrt{x}}$ with respect to x.

Solution

Common mistake:

$$\frac{d}{dx}\left(2^{\sqrt{x}}\right) \neq \sqrt{x}\, 2^{\sqrt{x}-1}$$

Cannot use the result:

$$\frac{d}{dx}(x^p) = p x^{p-1}$$

Let $y = 2^{\sqrt{x}}$. Take natural logarithm on both sides:

$$\begin{aligned} \ln y &= \ln\left(2^{\sqrt{x}}\right) \\ &= \sqrt{x}\, \underbrace{\ln 2}_{\text{constant}} \end{aligned}$$

Differentiate both sides w.r.t. x:

$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{dx} &= \ln 2 \cdot \frac{d}{dx}\left(x^{\frac{1}{2}}\right) \\ &= (\ln 2) \cdot \frac{1}{2} x^{-\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{dy}{dx} &= y\left[(\ln 2) \cdot \frac{1}{2} x^{-\frac{1}{2}}\right] \\ &= 2^{\sqrt{x}} \cdot \frac{\ln 2}{2\sqrt{x}} \end{aligned}$$

Example 24

Given that $y=\frac{(x+2)^3\,(x^2+4)^{\frac{1}{2}}\,e^{\sin 2x}}{(3x+5)^2\,(4x^3+1)}$. Find $\frac{dy}{dx}$.

Solution

We use **logarithmic differentiation**.

$$y=\frac{(x+2)^3\,(x^2+4)^{\frac{1}{2}}\,e^{\sin 2x}}{(3x+5)^2\,(4x^3+1)}$$

Take natural logarithm on both sides:

$$\ln y=\ln\left[\frac{(x+2)^3\,(x^2+4)^{\frac{1}{2}}\,e^{\sin 2x}}{(3x+5)^2\,(4x^3+1)}\right]$$

$\ln(ab) = \ln a + \ln b$

$\ln(\frac{a}{b}) = \ln a - \ln b$

$$=\ln[(x+2)^3]+\ln\left[(x^2+4)^{\frac{1}{2}}\right]+\ln\left(e^{\sin 2x}\right)-\ln[(3x+5)^2]-\ln(4x^3+1)$$

$$=3\ln(x+2)+\frac{1}{2}\ln(x^2+4)+\sin 2x-2\ln(3x+5)-\ln(4x^3+1)$$

Differentiate both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = 3\cdot\frac{1}{x+2}\cdot\frac{d(x+2)}{dx} + \frac{1}{2}\cdot\frac{1}{x^2+4}\cdot\frac{d(x^2+4)}{dx} + \cos 2x\cdot\frac{d(2x)}{dx}$$

$$-2\cdot\frac{1}{3x+5}\cdot\frac{d(3x+5)}{dx} - \frac{1}{4x^3+1}\cdot\frac{d(4x^3+1)}{dx}$$

$$= \frac{3}{x+2}\cdot(1) + \frac{1}{2}\cdot\frac{1}{x^2+4}\cdot(2x) + \cos 2x\cdot(2) - \frac{2}{3x+5}\cdot(3) - \frac{1}{4x^3+1}\cdot(4\cdot 3x^2)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{3}{x+2} + \frac{x}{x^2+4} + 2\cos 2x - \frac{6}{3x+5} - \frac{12x^2}{4x^3+1}\right]$$

$$= \left[\frac{(x+2)^3\,(x^2+4)^{\frac{1}{2}}\,e^{\sin 2x}}{(3x+5)^2\,(4x^3+1)}\right]\left[\frac{3}{x+2} + \frac{x}{x^2+4} + 2\cos 2x - \frac{6}{3x+5} - \frac{12x^2}{4x^3+1}\right]$$

Homework: If $y = \left(x+\frac{1}{x}\right)^{x^2}$, find $\frac{dy}{dx}$ by using logarithmic differentiation. Check your answer by using the table of derivatives. [*Ans.*: $\frac{dy}{dx} = \left(x+\frac{1}{x}\right)^{x^2}\left[2x\ln\left(x+\frac{1}{x}\right) + \frac{x(x^2-1)}{x^2+1}\right]$]

Example 25

cannot be simplified

$\ln(a+b) \neq \ln a + \ln b$

Given that $y = (\cos x)^x + 3^x$. Find $\frac{dy}{dx}$.

Solution

$\boxed{\text{Note that } \ln y = \ln[(\cos x)^x + 3^x] \neq \ln[(\cos x)^x] + \ln(3^x).}$

Let $y_1 = (\cos x)^x$ and $y_2 = 3^x$.

Then $\ln y_1 = \ln[(\cos x)^x] = x\ln(\cos x) \cdots\cdots (1)$

and $\ln y_2 = \ln(3^x) = x\ln 3 \cdots\cdots (2)$.

Differentiate both sides of (1) w.r.t. x:

$$\frac{1}{y_1}\cdot\frac{dy_1}{dx} = x\cdot\frac{d}{dx}[\ln(\cos x)] + \ln(\cos x)\cdot\frac{d(x)}{dx}$$

$$= x\cdot\frac{1}{\cos x}\cdot\underbrace{\frac{d}{dx}(\cos x)}_{=-\sin x} + \ln(\cos x)\cdot 1$$

$$= -x\tan x + \ln(\cos x)$$

$$\Rightarrow \frac{dy_1}{dx} = (\cos x)^x\,[-x\tan x + \ln(\cos x)]$$

Differentiate both sides of (2) w.r.t. x:

$$\frac{1}{y_2} \cdot \frac{dy_2}{dx} = \ln 3$$

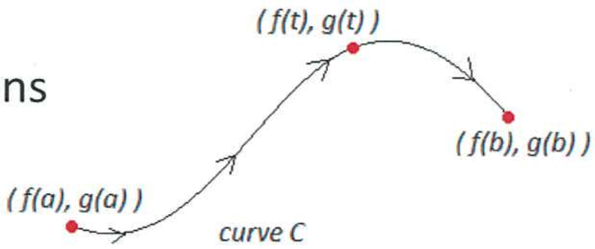
$$\Rightarrow \frac{dy_2}{dx} = 3^x \ln 3$$

$\because \; y = y_1 + y_2$

$$\therefore \; \frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx} = (\cos x)^x \left[-x \tan x + \ln(\cos x)\right] + 3^x \ln 3$$

Differentiation of Parametric Equations

Suppose that a curve C is described by the parametric equations



$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad t \in [a, b]$$

where $f(t)$ and $g(t)$ are differentiable functions of t, and t is a parameter.

Then the **first derivative** of y w.r.t. x is given by

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}},$$

and the **second derivative** of y w.r.t. x is given by

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[\frac{g'(t)}{f'(t)}\right]}{f'(t)}}.$$

Common mistake:

$$\frac{d^2y}{dx^2} \neq \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$$

Remarks:

rate of change of $\frac{dy}{dx}$ for one unit increase in x

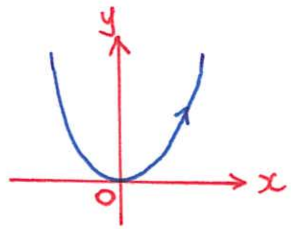
1. The second derivative of y w.r.t. x is the derivative of $\frac{dy}{dx}$ w.r.t. x.

2. For differentiation of parametric equations, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are usually expressed in terms of the parameter t.

Example 26

Given that $\begin{cases} x = 2t \\ y = t^2 \end{cases}$ where $-\infty < t < \infty$, describes a parabola. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Solution



$$x = 2t \quad \Rightarrow \quad \frac{dx}{dt} = 2$$

$$y = t^2 \quad \Rightarrow \quad \frac{dy}{dt} = 2t$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{2} = t \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(t)}{2} = \frac{1}{2}$$

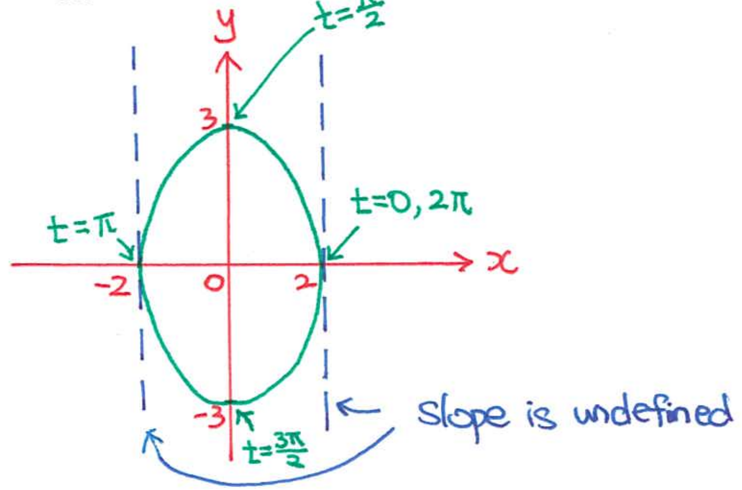
Remark: The parametric equations $\begin{cases} x = 2t \\ y = t^2 \end{cases}$ represent the parabola $y = \dfrac{x^2}{4}$.

Then $\dfrac{dy}{dx} = \dfrac{2x}{4} = \dfrac{x}{2}$ $(= t)$ & $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}\left(\dfrac{x}{2}\right) = \dfrac{1}{2}$

Example 27

$\cos^2 t + \sin^2 t = 1 \Rightarrow \dfrac{x^2}{2^2} + \dfrac{y^2}{3^2} = 1$ ← ellipse

Given that $\begin{cases} x = 2\cos t \\ y = 3\sin t \end{cases}$ where $0 \le t < 2\pi$. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.



Solution

$x = 2\cos t \quad \Rightarrow \quad \dfrac{dx}{dt} = -2\sin t$

$y = 3\sin t \quad \Rightarrow \quad \dfrac{dy}{dt} = 3\cos t$

Then $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{3\cos t}{-2\sin t} = -\dfrac{3}{2}\cot t$ $(t \neq 0, \pi)$

and $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \dfrac{\frac{d}{dt}\left(-\frac{3}{2}\cot t\right)}{-2\sin t} = \dfrac{-\frac{3}{2}(-\operatorname{cosec}^2 t)}{-2\sin t} = -\dfrac{3}{4}\operatorname{cosec}^3 t \quad (t \neq 0, \pi)$