Unit 8

Linearity (with solution)

Remark: Questions 1 and 2 belong to Unit 7.

Question 1: Vector Space

Consider the set of all binary n-vectors, $\{0, 1\}^n$

- Addition of two vectors is defined by

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n),$$

  where the addition of two bits is defined by modulo-2 addition (i.e., logical XOR).

- Scalar multiplication is defined by

$$c(x_1, \dots, x_n) = (cx_1, \dots, cx_n), \quad \text{for } c \in \{0, 1\},$$

  where multiplication of two bits is defined by usual multiplication (i.e., $0 \cdot 0 = 0 \cdot 1 = 0$ and $1 \cdot 1 = 1$).

Is it a vector space?

Q.1 (solution)

- Commutative and associative conditions are satisfied because of the property of XOR.
- Zero condition is satisfied since $x + \mathbf{0} = x$.
- Inverse condition is satisfied since $x + x = \mathbf{0}$.
- Associative and Unitarity conditions for scale multiplication are satisfied due to the property of usual multiplication.
- It is straightforward to check that the two distributive conditions are also satisfied.

Question 2: Subspace

The set of all real polynomials (with usual addition and scalar multiplication) is a vector space.

- The degree of a polynomial is the highest of the degrees of its monomials (individual terms) with ***non-zero*** coefficients.

Is each of the following sets its subspaces? Why?

a) The set of all real polynomials with degree less than n;

b) The set of all real polynomials with degree equal to n.

Q.2 (solution)

a) The set consists of all real polynomials in the form of $p = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$.
(Note: some coefficients a_i may be zero)

- Closed under addition:

  $p + q = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_{n-1} + b_{n-1})x^{n-1}$
  is still in the set.

- Closed under scalar multiplication:

  $cp = (ca_0 + ca_1x + \cdots + ca_{n-1}x^{n-1})$ is still in the set.

- Therefore, it is a subspace.

Q.2 (solution)

b) The set consists of all real polynomials in the form of $p = a_0 + a_1 x + \cdots + a_n x^n, a_n \neq 0$.

- Not closed under addition if $a_n = -b_n$:

  $p + q = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$

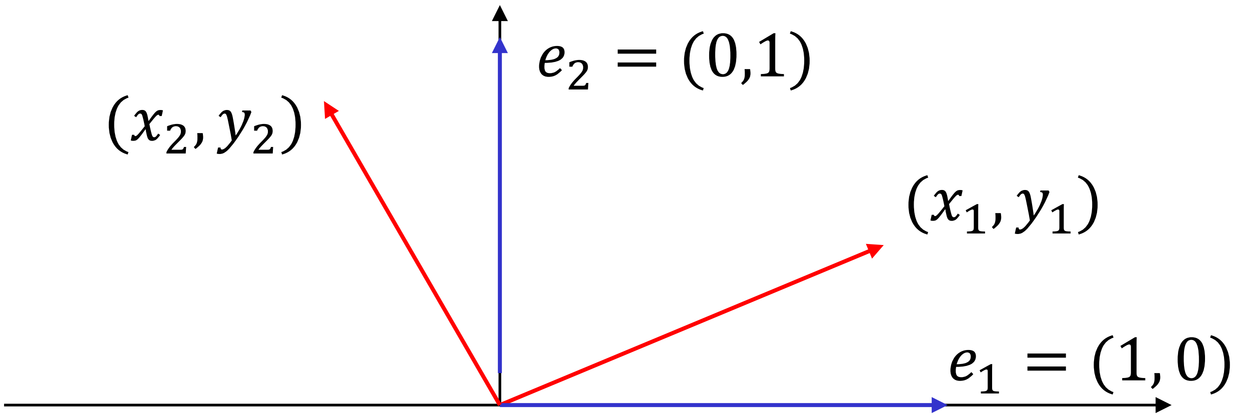
  $= (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_{n-1} - b_{n-1})x^{n-1}$,
  whose degree is at most $n - 1$.

- Not closed under scalar multiplication if $c = 0$:

  $0p = 0$, whose degree is 0.

- Therefore, it is **not** a subspace.

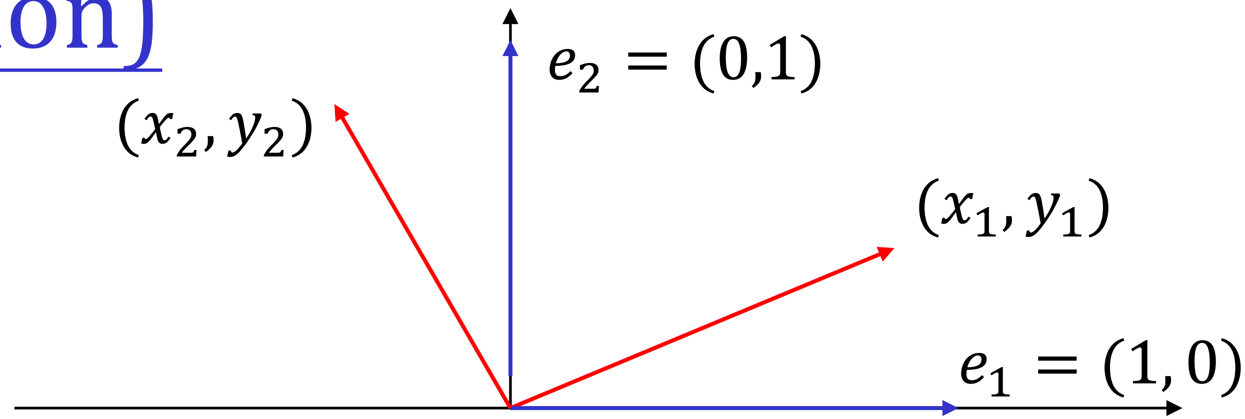
Question 3: Rotation



Consider anti-clockwise rotations of e_1 and e_2 by 30^o.

a) Find (x_1, y_1) and (x_2, y_2).
b) Consider an arbitrary vector $v = (x, y)$. Express v as a linear combination of e_1 and e_2.
c) What is the resultant vector after rotating v by 30^o?
d) What is the corresponding rotation matrix?

Q.3 (solution)



a) $(x_1, y_1) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), (x_2, y_2) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

b) $v = xe_1 + ye_2$

c) $v' = x\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) + y\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ (by linearity)

$= \left(\frac{\sqrt{3}}{2}x - \frac{1}{2}y,\ \frac{1}{2}x + \frac{\sqrt{3}}{2}y\right)$

d) $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

Question 4: Projection

Consider the straight line $y = \frac{x}{2}$ in the 2-dimensional space.

a) Find the matrix that projects any vector to the above line.
b) Hence, find the projection of $(3, 2)$ onto the above line.

Q.4 (solution)

a) Pick a vector of the line: $a = (2,1)$. According to the lecture notes, the projection matrix is

$$P = \frac{aa^T}{a^T a} = \frac{1}{5}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

b) The projection vector is

$$p = Pb = \frac{1}{5}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 16 \\ 8 \end{bmatrix}$$

Question 5: Line Fitting

There are three data points given:

(0, 2), (1, 1), and (3, 2).

a) Find the best line (in the sense of minimum RMS) that fits the three points and passes through the origin.

b) Find the predicted value at $x = 2$.

Q.5 (solution)

- $a^T = [0, 1, 3], b^T = [2, 1, 2]$

a) Let the best line be $\hat{y} = \beta x$, from the lecture notes, we have:

$$\beta = \frac{a^T b}{a^T a} = \frac{0 + 1 + 6}{0 + 1 + 9} = 0.7$$

b) When $x = 2$, the predicted value

$$\hat{y} = 0.7x = 1.4.$$