Take Home Assignment MA2001 #1

For each of the following questions, write down your solution with details of steps.

Marks will not be given if only final answers are provided.

- 1. Use diagonalization to compute A^8 , where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$.
- 2. Let $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$, $\vec{v_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\vec{v_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Suppose you are told that $\vec{v_1}$ and $\vec{v_2}$ are eigenvectors of A. Use this information to diagonalize A.
- 3. Let

$$A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}.$$

It is known $\lambda = 3$ is one eigenvalue of A. Find linearly independent eigenvectors corresponding to λ .

- 4. Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, $\vec{v_1} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\vec{v_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Verify that $\vec{v_1}$ and $\vec{v_2}$ are eigenvectors of A. Then orthogonally diagonalize A.
- 5. A quadratic form Q in the components x_1, \ldots, x_n of a vector $\vec{x} = [x_1, \ldots, x_n]^{\top}$ with symmetric coefficient matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ is defined to be

$$Q(\vec{x}) := \vec{x}^{\top} A \vec{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j.$$

Use **THREE** methods to determine whether the following quadratic forms in three variables is positive or negative definite or semidefinite, or indefinite.

$$Q(\vec{x}) = 2(x_1^2 + x_2^2 + x_3^2 + x_1x_3).$$

- 6. Matrices are $n \times n$ and vectors are in \mathbb{R}^n . Mark each statement True or False. Justify each answer.
 - (a) If A is diagonalizable, then A is invertible.

- (b) If $A\vec{x} = \lambda \vec{x}$ for some vector \vec{x} , then λ is an eigenvalue of A.
- (c) A matrix A is not invertible if and only if 0 is an eigenvalue of A.
- (d) A number c is an eigenvalue of A if and only if the equation $(A cI)\vec{x} = \vec{0}$ has a nontrivial solution.
- (e) A positive definite quadratic form Q satisfies $Q(\vec{x}) > 0$ for all \vec{x} in \mathbb{R}^n .
- (f) If the eigenvalues of a symmetric matrix A are all positive, then the quadratic form $\vec{x}^T A \vec{x}$ is positive definite.
- 7. (a) Construct a matrix A which is diagonalizable, but not invertible.
 - (b) Construct another matrix B which is invertible, but not diagonalizable.
 - (c) Construct two vectors which are linearly independent, but not orthogonal.
- 8. **Discovery Question**. Read "https://en.wikipedia.org/wiki/Gram-Schmidt process" and use the Gram-Schmidt process to find an orthogonal basis spanning the same space of \mathbb{R}^n as the given of vectors:
 - (a) <1,4,0>,<2,-5,0> in \mathbb{R}^3 .
 - (b) <0,2,1,-1>,<0,-1,1,6>,<0,2,2,3> in \mathbb{R}^4 .