MA1200 Solutions to Practice Exercise on Set and Functions

(1). (a) $A = (-3, 8]$ $B = [-11, -3)$ $D = (5, \infty)$

(b) (i) ϕ (ii) $[-11, 8] \setminus \{-3\}$ (iii) B

(iv) C (v) $(5, 8]$ (vi) $(-3, \infty)$

(c) (i) F (ii) T (iii) T (iv) F (v) T (vi) F

(vii) F (viii) T

(2) (a) Domain $= \mathbb{R}$ Range: $[3, \infty)$

(b) Domain: $\mathbb{R} \setminus \{2\}$ Range $= \mathbb{R} \setminus \{5\}$

(c) Domain: $\mathbb{R}$ Range: $\{-4, 4\}$

(d) Domain: $\mathbb{R}$ Range: $\mathbb{R}$

(e) Domain: $\mathbb{R}$ Range: $[-3, 3]$

(f) Domain: $(-4, 5]$ Range: $[-5, 8)$

(g) Domain: $\{-10, 0, 10\}$ Range: $\{-6, 0, 12\}$

(h) Domain: $\mathbb{R} \setminus \{1\}$ Range: $\{-3\} \cup (+1, \infty)$

(3) (a) $y = x^2 - 9$

Note that it is quadratic fct, $a = 1 > 0$ with vertex $(0, -9)$

Domain : $\mathbb{R}$ Range: $[-9, \infty)$ opens upward

(b). $y = 4 - x$

Note that it is a straight line. (a special poly fct):

Domain $= \mathbb{R}$ Range $= \mathbb{R}$

(c) $y = 5$

Note that it is constant fct. Domain $= \mathbb{R}$ Range $= \{5\}$

(d) $y = x^3$

Note that it is polynomial fct with leading coef $= 1 > 0$

degree $= 3$ (an odd no.) $\therefore$ It goes up on the right and goes down on the left

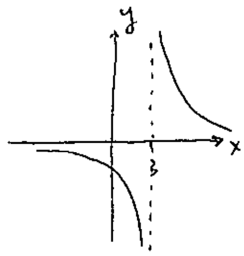
Domain $= \mathbb{R}$ Range: $\mathbb{R}$

(e) $y = x^2 - x - 2 = (x^2 - x + \frac{1}{4}) - \frac{1}{4} - 2 = (x - \frac{1}{2})^2 - \frac{9}{4}$

Note that it is a quadratic fct with $a = 1 > 0$ and vertex $(\frac{1}{2}, -\frac{9}{4})$

Domain $= \mathbb{R}$ Range: [-9/4, infinity) opens upward

(3) (f). $y = \frac{5}{x-3}$



Note that it is rational fct with $x-3=0$ when $x=3$

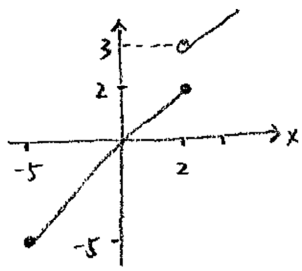
$\frac{5}{x-3} \neq 0$ for all values of x

When x is close to 3, y is close to ∞ (for $x>3$)
and close to $-\infty$ (for $x<3$)

When x is close to ∞ or $-\infty$, y is close to 0.

Domain: $\mathbb{R} \setminus \{3\}$ Range: $\mathbb{R} \setminus \{0\}$

(g). $y = \begin{cases} x & -5 \leq x \leq 2 \\ x+1 & x > 2 \end{cases}$



When $-5 \leq x \leq 2$, $y = x$. it is an identity fct
$\therefore -5 \leq y \leq 2$

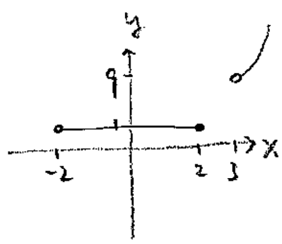
When $x > 2$, $y = x+1$ it is a straight line
$\therefore y > 3$

Domain: $[-5, \infty)$ Range: $[-5, 2] \cup (3, \infty)$

(h) $y = \begin{cases} 1 & -2 \leq x \leq 2 \\ x^2 & x > 3 \end{cases}$

When $-2 \leq x \leq 2$, $y = 1$

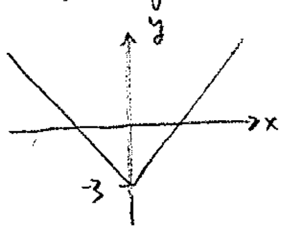
When $x > 3$, $y = x^2$ is quadratic which opens upward
with $y > 9$



Domain: $[-2, 2] \cup (3, \infty)$

Range: $\{1\} \cup (9, \infty)$

(i). $y = |x| - 3 = \begin{cases} -x-3 & x < 0 \\ x-3 & x \geq 0 \end{cases}$



When $x < 0$, $y = -x-3$ is a straight line with $y > -3$

When $x \geq 0$, $y = x - 3$ is a straight line with $y \geq -3$

Domain: $\mathbb{R}$ Range: $[-3, \infty)$

(j). $f(x) = \sqrt{25 - x^2}$

Note that $25 - x^2$ is quadratic expression with a maximum value 25 when $x = 0$. At $x=0$, $y = \sqrt{25 - 0^2} = 5$. (max)

$f(x)$ is defined when $25 - x^2 \geq 0$

$$x^2 \leq 25$$
$$-5 \leq x \leq 5$$

$\therefore$ Domain: $[-5, 5]$ Range: $[0, 5]$

(3)(k) $f(x)=\sqrt{x^2-4x+8}$

$x^2-4x+8 = x^2-4x+4+4 = (x-2)^2+4$

it is quadratic expression with minimum value 4 when $x=2$.

At $x=2$, $f(2)=\sqrt{4}=2$

$f(x)$ is defined when $(x-2)^2+4 \geq 0$ i.e. x can be any real numbers.

$\therefore$ Domain: $\mathbb{R}$ Range: $[2,\infty)$

(l) $f(x)=\sqrt{x^2-4x-21}$

$x^2-4x-21 = x^2-4x+4-25 = (x-2)^2-25$

it is quadratic expression with minimum value when $x=2$

$f(x)$ is defined when $(x-2)^2-25 \geq 0$

$(x-2)^2 \geq 25$

x-2>=5 or x-2<=-5

x>=7 or x<=-3

$\therefore$ Domain: R\(-3,7) Range: $[0,\infty)$

(4). $f(x)=\frac{3}{x-2}$ and $g(x)=\frac{6}{x-5}$

(a) $f(x)$ is defined if $x-2\neq 0$ i.e. $x\neq 2$

$\therefore$ Domain: $\mathbb{R}\setminus\{2\}$

(b) $g(x)$ is defined if $x-5\neq 0$ i.e. $x\neq 5$

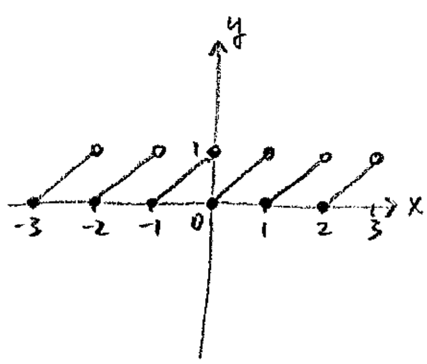
$\therefore$ Domain: $\mathbb{R}\setminus\{5\}$

(c) $f(x)+g(x) = \frac{3}{x-2}+\frac{6}{x-5} = \frac{9x-27}{(x-2)(x-5)}$

it is defined if $(x-2)(x-5)\neq 0$ i.e. $x\neq 2$ and $x\neq 5$

$\therefore$ Domain: $\mathbb{R}\setminus\{2,5\}$

(5). (a)



(b) Domain: $\mathbb{R}$

Range: $[0,1)$

(6) Given $f(x)=\frac{5}{x-3}$ and $g(x)=x^2-6x+18$

(a) Note that $f(x)$ is defined iff $x-3\neq 0$, i.e. $x\neq 3$

$\therefore$ Domain of $f(x)$ is $\mathbb{R}\setminus\{3\}$

(b). Note that $g(x)$ is quadratic fct

$\therefore$ Domain of $g(x)$ is $\mathbb{R}$

(c) $f(g(x)) = \frac{5}{x^2-6x+15}$

Note that $f(g(x))$ is defined for all real numbers of x.

$\therefore$ Domain of $f(g(x))$ is $\mathbb{R}$

(d). $g(f(x)) = \left(\frac{5}{x-3}\right)^2-6\left(\frac{5}{x-3}\right)+18$

Note that $g(f(x))$ is defined iff $x-3\neq 0$ i.e. $x\neq 3$

$\therefore$ Domain of $g(f(x))$ is $\mathbb{R}\setminus\{3\}$

(7). (a). Given $f(x)=\sin(2x)+5x^3$

$f(-x)=\sin[-2x]+5(-x)^3 = -\sin(2x)-5x^3$

$= -[\sin(2x)+5x^3] = -f(x)$

$\therefore$ $f(x)$ is odd.

(b). $f(x) = \tan x-3$

$f(-x)= \tan(-x)-3 = -\tan x-3$

which is neither $f(x)$ ~~or~~ nor $-f(x)$.

$\therefore$ $f(x)$ is neither odd nor even

(c) $f(x)=\frac{\cos x}{x^2}$

$f(-x)=\frac{\cos(-x)}{(-x)^2} = \frac{\cos x}{x^2} = f(x)$

$\therefore$ $f(x)$ is an even fct.

(d) $f(x)=|-3x|+5 = 3|-x|+5 = 3|x|+5$

$f(-x)=|3x|+5 = 3|x|+5 = f(x)$

$\therefore$ $f(x)$ is even.

8.
(a)
$x^2-4x-5=(x-5)(x+1)=0 \Rightarrow x=-1$ or $x=5$
The largest domain of $f(x)=R\setminus\{-1,5\}$
(b)
$x-1=0 \Rightarrow x=1$
The largest domain of $\phi(x)=R\setminus\{1\}$

9.
(a)
$$(f+g)(x)=f(x)+g(x)=x^3+2+\frac{2}{x-1}$$
The largest domain of $(f+g)(x)$ is equal to the intersection of the largest domain of $f(x)$ and the largest domain of $g(x)$.
Since the largest domain of $f(x)=R$ and the largest domain of $g(x)=R\setminus\{1\}$, the largest domain of $(f+g)(x)=R\cap R\setminus\{1\}=R\setminus\{1\}$.
(b)
$$\left(\frac{g}{f}\right)(x)=\frac{g(x)}{f(x)}=\frac{\frac{2}{x-1}}{x^3+2}=\frac{2}{(x-1)(x^3+2)}$$
The largest domain of $\left(\frac{g}{f}\right)(x)$ is equal to the intersection of the largest domain of $g(x)$ and the largest domain of $f(x)$ minus $\{x: f(x)=0\}$.
Since the largest domain of $f(x)=R$ and $\{x: f(x)=0\}=\{x: x^3+2=0\}=\{(-2)^{1/3}\}$ and also the largest domain of $g(x)=R\setminus\{1\}$, the largest domain of $\left(\frac{g}{f}\right)(x)$ is $\left(R\setminus\{(-2)^{1/3}\}\right)\cap\left(R\setminus\{1\}\right)=R\setminus\{1,(-2)^{1/3}\}$.
(c)
$$(g\circ f)(x)=g(f(x))=g(x^3+2)=\frac{2}{x^3+2-1}=\frac{2}{x^3+1}$$
Solve the equation $x^3+2=1 \Leftrightarrow x^3+1=0 \Leftrightarrow x=-1$.
The largest domain of $(g\circ f)(x)=g(f(x))=\frac{2}{x^3+1}$ is :
$R\setminus\{-1\}$

(d)
$$(f\circ g)(x)=f(g(x))=f\left(\frac{2}{x-1}\right)=\left(\frac{2}{x-1}\right)^3+2$$
The largest domain of $(f\circ g)(x)=\left(\frac{2}{x-1}\right)^3+2$ is:
$R\setminus\{1\}$

10.
(a)
Let both f and g be even, i.e. $f(-x) = f(x), g(-x) = g(x)$.

Then $(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$

So we have the sum of two even functions is an even function.
(b)
Let both f and g be odd, i.e. $f(-x) = -f(x), g(-x) = -g(x)$.

Then $(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -[f(x) + g(x)] = -(f+g)(x)$

So we have the sum of two odd functions is an odd function.
(c)
Let both f and g be even, i.e. $f(-x) = f(x), g(-x) = g(x)$.

Then $(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$

So we have the product of two even functions is an even function.
(d)
Let both f and g be odd, i.e. $f(-x) = -f(x), g(-x) = -g(x)$.

Then $(fg)(-x) = f(-x)g(-x) = [-f(x)][-g(x)] = f(x)g(x) = (fg)(x)$

So we have the product of two odd functions is an even function.
(e)
Let f be even and g be odd, i.e. $f(-x) = f(x), g(-x) = -g(x)$.

Then $(fg)(-x) = f(-x)g(-x) = [f(x)][-g(x)] = -f(x)g(x) = -(fg)(x)$

So we have the product of an even function and an odd function is an odd function.

11.
<u>Proof</u>:
(a)
Let $G(x) \equiv F(x) - F(-x)$. Then $G(-x) = F(-x) - F(-(-x)) = F(-x) - F(x) = -(F(x) - F(-x)) = -G(x)$

It follows that $G(x) \equiv F(x) - F(-x)$ is odd.
(b)
Let $G(x) \equiv F(x) + F(-x)$. Then $G(-x) = F(-x) + F(-(-x)) = F(-x) + F(x) = F(x) + F(-x) = G(x)$

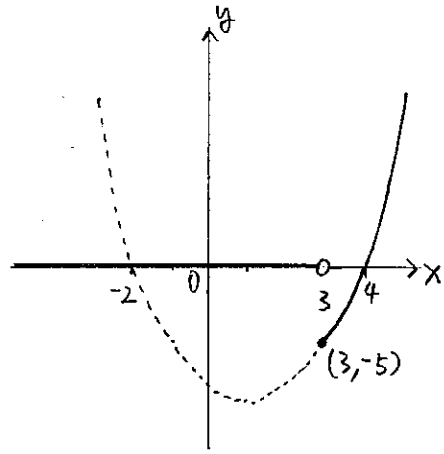
It follows that $G(x) \equiv F(x) + F(-x)$ is even.
(c)

$$F(x) = \frac{1}{2}F(x) + \frac{1}{2}F(-x) + \frac{1}{2}F(x) - \frac{1}{2}F(-x) = \frac{1}{2}(F(x) + F(-x)) + \frac{1}{2}(F(x) - F(-x))$$

Let $G(x) \equiv \frac{1}{2}(F(x) + F(-x)), H(x) \equiv \frac{1}{2}(F(x) - F(-x))$.

Then, $F(x) = G(x) + H(x)$, where $G(x)$ is an even function and $H(x)$ is an odd function.

(12)



(a) $f(x) = (x^2-2x-8)\,u_3(x)$

$$= \begin{cases} 0\cdot(x^2-2x-8), & x<3 \\ 1\cdot(x^2-2x-8), & x\geq 3 \end{cases}$$

$$= \begin{cases} 0, & x<3 \\ x^2-2x-8, & x\geq 3 \end{cases}$$

$\therefore$ The sketch of graph is shown on left.

(b). Since $f(x)$ is defined for all x,
$\therefore$ Largest possible domain of $f(x)$ is $\mathbb{R}$.

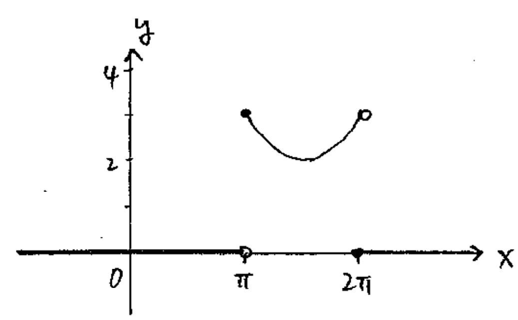
(c) From the graph,
Largest possible range of $f(x)$ is $[3, \infty)$

(13). (a). $f(x) = \left(u_\pi(x) - u_{2\pi}(x)\right)(3+\sin x)$

$$= \begin{cases} (0-0)(3+\sin x), & x<\pi \\ (1-0)(3+\sin x), & \pi\leq x<2\pi \\ (1-1)(3+\sin x), & x\geq 2\pi \end{cases}$$

$$= \begin{cases} 0, & x<\pi \\ 3+\sin x, & \pi\leq x<2\pi \\ 0, & x\geq 2\pi \end{cases}$$

The sketch of graph is shown on right.



(b). Since $f(x)$ is defined for all x,
$\therefore$ Largest possible domain of $f(x)$ is $\mathbb{R}$.

(c). From the graph,
Largest possible range of $f(x)$ is $\{0\}\cup[2,3]$