MA1201 Calculus and Basic Linear Algebra II Solution of Problem Set 3

Application of Integration

Problem 1

(a) The required area

$$= \int_1^3 (1+e^{3x})dx = x + \frac{1}{3}e^{3x}|_1^3 = 2 + \frac{1}{3}(e^9 - e^3).$$

(b) The required area

$$= \int_{\frac{1}{2}}^{2} |\ln x| dx = \int_{\frac{1}{2}}^{1} (-\ln x)dx + \int_1^2 \ln x \, dx$$

$$\overset{\substack{u=\ln x \\ dv=dx \\ \Rightarrow v=\int dx = x}}{\cong} -\left(x\ln x\,|_{\frac{1}{2}}^{1} - \int_{\frac{1}{2}}^{1} x\, d(\ln x)\right) + \left(x \ln x\,|_1^2 - \int_1^2 x\, d(\ln x)\right)$$

$$= \frac{1}{2}\ln\left(\frac{1}{2}\right) + \int_{\frac{1}{2}}^{1} 1dx + 2\ln 2 - \int_1^2 1dx = \frac{1}{2}\ln\left(\frac{1}{2}\right) + x|_{\frac{1}{2}}^{1} + 2\ln 2 - x|_1^2$$

$$= \frac{1}{2}\ln\left(\frac{1}{2}\right) + \frac{1}{2} + 2\ln 2 - 1 \overset{\ln\frac{1}{2} = \ln 2^{-1} = -\ln 2}{\cong} \frac{3}{2}\ln 2 - \frac{1}{2}.$$

(c) The required area

$$= \int_0^2 [(1+x^2) - e^{-x}]dx$$

$$= x + \frac{x^3}{3} + e^{-x}|_0^2 = 2 + \frac{8}{3} + e^{-2} - e^{-0}$$

$$= \frac{11}{3} + e^{-2}.$$

(d) The intersection points can be found by solving

$$\begin{cases} y = -x^2 + 2x + 1 \\ x + y = 1 \end{cases} \Rightarrow 1 - x = -x^2 + 2x + 1$$

$$\Rightarrow x^2 - 3x = 0 \Rightarrow x(x-3) = 0$$

$$\Rightarrow x = 0\ (y = 1),\ \ x = 3\ (y = -2).$$

Using the graph, the required area is given by

$$= \int_0^3 [(-x^2 + 2x + 1) - (1 - x)]dx = \int_0^3 (-x^2 + 3x)dx$$

$$= -\frac{x^3}{3} + \frac{3x^2}{2}|_0^3 = \frac{9}{2}.$$

Problem 2

(a) Set $f_1(x) = f_2(x)$, we obtain the following equation

$$e^{2x} - 3e^x - 1 = e^x - 4 \Rightarrow e^{2x} - 4e^x + 3 = 0$$

$$\overset{y=e^x}{\Rightarrow} y^2 - 4y + 3 = 0 \Rightarrow (y-1)(y-3) = 0$$

$$\Rightarrow y = e^x = 1 \quad \text{or} \quad y = e^x = 3$$

$$\Rightarrow x = \ln 1 = 0 \quad \text{or} \quad x = \ln 3.$$

(b) Based on the critical points obtained in (a), we divide the interval $[-2, 2]$ into several parts and compare the values of $f_1(x)$ and $f_2(x)$ in each of the parts:

x	$-2 \le x < 0$	$0 < x < \ln 3$	$\ln 3 < x < 2$
$f_1(x)$ v.s. $f_2(x)$	$f_1(x) > f_2(x)$	$f_1(x) < f_2(x)$	$f_1(x) > f_2(x)$

We conclude from the above analysis that

- $f_1(x) > f_2(x)$ for $-2 < x < 0$ or $\ln 3 < x < 2$
- $f_1(x) < f_2(x)$ for $0 < x < \ln 3$.

(c) Using the result of (b), the required area is found to be

$$= \int_{-2}^{0} (f_1(x) - f_2(x))dx + \int_{0}^{\ln 3} \left(f_2(x) - f_1(x)\right)dx + \int_{\ln 3}^{2} \left(f_1(x) - f_2(x)\right)dx$$
$$= \int_{-2}^{0} (e^{2x} - 4e^x + 3)dx + \int_{0}^{\ln 3} (-e^{2x} + 4e^x - 3)dx + \int_{\ln 3}^{2} (e^{2x} - 4e^x + 3)dx$$
$$= \frac{1}{2}e^{2x} - 4e^x + 3|_{-2}^{0} + \left(-\frac{1}{2}e^{2x} + 4e^x - 3\right)|_0^{\ln 3} + \frac{1}{2}e^{2x} - 4e^x + 3|_{\ln 3}^{2}$$
$$= -\frac{1}{2}e^{-4} + 4e^{-2} + \frac{1}{2}e^4 - 4e^2 - 6\ln 3 + 14.$$

Problem 3

We let $f(x) = (x^2 - x + 1)e^x$ and $g(x) = xe^x$.

To compare the values of $f(x)$ and $g(x)$ within $[0, 2]$, we first obtain all critical points by solving

$f(x) = g(x) \Rightarrow (x^2 - x + 1)e^x = xe^x$

$\Rightarrow (x^2 - 2x + 1)e^x = 0 \Rightarrow (x-1)^2 e^x = 0 \Rightarrow x = 1.$

Then we divide the integral into two parts and compare the values of two functions in each part:

x	$0 \le x < 1$	$1 < x < 2$
$f(x)$ v.s. $g(x)$	$f(x) > g(x)$	$f(x) > g(x)$

Therefore, we conclude that $f(x) \ge g(x)$ for ALL x between 0 and 2.

Using the data obtained, the required area is

$$\int_0^2 [f(x) - g(x)]dx = \int_0^2 [(x^2 - x + 1)e^x - xe^x]dx = \int_0^2 (x^2 - 2x + 1)e^x dx$$

$$= \int_0^2 \underbrace{(x-1)^2}_{u}\underbrace{e^x dx}_{dv} \overset{\substack{u=(x-1)^2 \\ dv=e^x dx \\ \Rightarrow v=\int e^x dx = e^x}}{\cong} (x-1)^2 e^x|_0^2 - \int_0^2 e^x d(x-1)^2$$

$$= (e^2 - 1) - 2\int_0^2 \underbrace{(x-1)}_{u}\underbrace{e^x dx}_{dv} \overset{\substack{u=x-1 \\ dv=e^x dx \\ \Rightarrow v=\int e^x dx = e^x}}{\cong} (e^2 - 1) - 2\left[(x-1)e^x|_0^2 - \int_0^2 e^x d(x-1)\right]$$

$$= e^2 - 1 - 2(e^2 + 1) + 2\int_0^2 e^x dx = -e^2 - 3 + 2e^x|_0^2 = e^2 - 5.$$

Problem 4

(a) The required volume

$$= \pi\int_0^{\frac{\pi}{3}} \sin^2 3x\, dx + \pi\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin^2 3x\, dx + \pi\int_{\frac{2\pi}{3}}^{\pi} \sin^2 3x\, dx = \pi\int_0^{\pi} \sin^2 3x\, dx$$
$$= \pi\int_0^{\pi} -\frac{1}{2}[\cos(3x + 3x) - \cos(3x - 3x)]dx$$
$$= \frac{\pi}{2}\int_0^{\pi} (1 - \cos 6x)dx = \frac{\pi}{2}\left[x - \frac{\sin 6x}{6}\right]_0^{\pi} = \frac{\pi^2}{2}.$$

(b) Using the graph, the required volume

$$= \pi \int_0^{\frac{\pi}{2}} (1+\cos x)^2 dx - \pi \int_0^{\frac{\pi}{2}} (1+\cos 3x)^2 dx$$
$$= \pi \int_0^{\frac{\pi}{2}} (2\cos x + \cos^2 x - 2\cos 3x - \cos^2 3x) dx$$
$$= \pi \int_0^{\frac{\pi}{2}} \left(2\cos x + \frac{\cos 2x + 1}{2} - 2\cos 3x - \frac{\cos 6x + 1}{2}\right) dx$$
$$= \pi \int_0^{\frac{\pi}{2}} \left(2\cos x + \frac{\cos 2x}{2} - 2\cos 3x - \frac{\cos 6x}{2}\right) dx$$
$$= \pi \left(2\sin x + \frac{\sin 2x}{4} - \frac{2}{3}\sin 3x - \frac{\sin 6x}{12}\right) \Big|_0^{\frac{\pi}{2}}$$
$$= 2\pi + \frac{2\pi}{3} = \frac{8\pi}{3}.$$

(c) The required volume

(i) $$\int_0^{\ln 3} \pi (e^{2x})^2 dx = \pi \int_0^{\ln 3} e^{4x} dx = \frac{\pi}{4} e^{4x} \Big|_0^{\ln 3} = \frac{80\pi}{4} = 20\pi.$$

(ii) The required volume
$$= \int_0^9 \pi (\ln 3)^2 dy - \int_1^9 \pi \left(\frac{1}{2}\ln y\right)^2 dy$$
$$= \pi(\ln 3)^2 \int_0^9 1 dy - \frac{\pi}{4}\int_1^9 \underbrace{(\ln y)^2}_{u} \underbrace{dy}_{dv}$$
$$\overset{\substack{u=(\ln y)^2 \\ dv=dy \\ \Rightarrow v=\int dy = y}}{\triangleq} \pi(\ln 3)^2 y|_0^9 - \frac{\pi}{4}\left[y(\ln y)^2|_1^9 - \int_1^9 y d(\ln y)^2\right]$$
$$= 9(\ln 3)^2\pi - \frac{9\pi(\ln 9)^2}{4} + \frac{\pi}{2}\int_1^9 \underbrace{\ln y}_{u} \underbrace{dy}_{dv}$$
$$\overset{\substack{u=\ln y \\ dv=dy \\ \Rightarrow v=\int dy = y}}{\triangleq} 9(\ln 3)^2\pi - \frac{9\pi(2\ln 3)^2}{4} + \frac{\pi}{2}\left(y\ln y\,|_1^9 - \int_1^9 y d(\ln y)\right)$$
$$= \frac{\pi}{2}\left(9\ln 9 - \int_1^9 1 dy\right) = \frac{9\pi}{2}\ln 9 - \frac{\pi}{2} y|_1^9 = \frac{9\pi}{2}\ln 9 - 4\pi = 9\pi\ln 3 - 4\pi.$$

(iii) The required volume
$$= \pi \int_0^{\ln 3} (e^{2x}+1)^2 dx - \pi \int_0^{\ln 3} (1)^2 dx$$
$$= \pi \int_0^{\ln 3} (e^{4x} + 2e^{2x}) dx = \pi \left[\frac{e^{4x}}{4} + e^{2x}\right]_0^{\ln 3} = \frac{81\pi}{4} - \frac{\pi}{4} + 9\pi - \pi = 28\pi.$$

(iv) Using similar method as in (iii), the required volume is given by
$$V = \int_0^9 \pi(\ln 3 + 1)^2 dy - \int_0^1 \pi (1)^2 dy - \int_1^9 \pi\left(\frac{1}{2}\ln y + 1\right)^2 dy$$
$$= \pi(\ln 3+1)^2 y|_0^9 - \pi\, y|_0^1 - \frac{\pi}{4}\int_1^9 \underbrace{(\ln y)^2}_{u} \underbrace{dy}_{dv} - \pi \int_1^9 \underbrace{\ln y}_{u} \underbrace{dy}_{dv} - \pi \int_1^9 1 dy$$

$$= 9\pi(\ln 3+1)^2 - \pi - \frac{\pi}{4}\left[y(\ln y)^2|_1^9 - \int_1^9 yd(\ln y)^2\right] - \pi\left(y\ln y\,|_1^9 - \int_1^9 yd(\ln y)\right) - \pi y|_1^9$$
$$= 9\pi(\ln 3+1)^2 - 9\pi - \frac{9\pi(\ln 9)^2}{4} + \frac{\pi}{2}\int_1^9 \underbrace{\ln y}_{u}\,\underbrace{dy}_{dv} - 9\pi\ln 9 + \pi\int_1^9 1dy$$
$$= 9\pi(\ln 3+1)^2 - 9\pi - \frac{9\pi(2\ln 3)^2}{4} + \frac{\pi}{2}\left(y\ln y\,|_1^9 - \int_1^9 yd(\ln y)\right) - 9\pi\ln 9 + \pi y|_1^9$$
$$= 9\pi(\ln 3)^2 + 18\pi\ln 3 + 9\pi - 9\pi - 9\pi(\ln 3)^2 + \frac{\pi}{2}\left(9\ln 9 - \int_1^9 1dy\right) - 9\pi\ln 9 + 8\pi$$
$$= 18\pi\ln 3 + \frac{\pi}{2}(9\ln 9 - y|_1^9) - 9\pi\ln 9 + 8\pi$$
$$= 18\pi\ln 3 + \frac{9\pi}{2}(2\ln 3) - 4\pi - 9\pi(2\ln 3) + 8\pi = 9\pi\ln 3 + 4\pi.$$

(d) (i) Note that the intersection point can be found by solving

$$\begin{cases} y = \sin x \\ y = \dfrac{1}{2} \end{cases} \Rightarrow \sin x = \frac{1}{2}$$
$$\Rightarrow x = \frac{\pi}{6}\ \left(y = \frac{1}{2}\right)\ \text{ or }\ x = \frac{5\pi}{6}\ \left(y = \frac{1}{2}\right).$$

The required volume

$$V = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \pi(\sin x)^2 dx - \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \pi\left(\frac{1}{2}\right)^2 dx = \pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin^2 x\,dx - \frac{\pi}{4}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 1dx$$
$$= \pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1-\cos 2x}{2}dx - \frac{\pi}{4}x|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} = \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - \frac{\pi}{4}\left(\frac{2\pi}{3}\right) = \frac{\pi^2}{6} + \frac{\sqrt{3}\pi}{4}.$$

(ii) Using the graph, the required volume by the Disk method,

$$V = \pi\int_{\frac{1}{2}}^{1} (\pi - \sin^{-1} y)^2 dy - \pi\int_{\frac{1}{2}}^{1} (\sin^{-1} y)^2 dy = \pi\int_{\frac{1}{2}}^{1} (\pi^2 - 2\pi\sin^{-1} y)dy$$
$$= \pi^3\int_{\frac{1}{2}}^{1} 1dy - 2\pi^2\int_{\frac{1}{2}}^{1} \underbrace{\sin^{-1} y}_{u}\,\underbrace{dy}_{dv}$$
$$\overset{\substack{u=\sin^{-1}y \\ dv=dy \\ \Rightarrow v=\int dy=y}}{=} \pi^3 y|_{\frac{1}{2}}^{1} - 2\pi^2\left(y\sin^{-1} y\,|_{\frac{1}{2}}^{1} - \int_{\frac{1}{2}}^{1} y\,d(\sin^{-1} y)\right)$$
$$= \frac{\pi^3}{2} - 2\pi^2\left(\frac{\pi}{2} - \frac{\pi}{12} - \int_{\frac{1}{2}}^{1} \frac{y}{\sqrt{1-y^2}}dy\right)$$
$$\overset{z=1-y^2}{=} \frac{\pi^3}{2} - \frac{5\pi^3}{6} - \pi^2\int_{\frac{3}{4}}^{0} \frac{1}{\sqrt{z}}dz = -\frac{2\pi^3}{6} - \pi^2\left[2\sqrt{z}\right]_{\frac{3}{4}}^{0} = -\frac{\pi^3}{3} + \sqrt{3}\pi^2.$$

Alternatively by the Shell method,

$$V = 2\pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} x\left(\sin x - \frac{1}{2}\right)dx = -2\pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} xd\cos x\ - \frac{\pi}{2}[x^2]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$
$$= -2\pi\left([x\cos x]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \cos x\,dx\right) - \frac{\pi}{2}\left(\frac{25\pi^2}{36} - \frac{\pi^2}{36}\right)$$
$$= -2\pi\left(\frac{5\pi}{6}\left(-\frac{\sqrt{3}}{2}\right) - \frac{\pi}{6}\left(\frac{\sqrt{3}}{2}\right) - [\sin x]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\right) - \frac{\pi^3}{3}$$

$$= -2\pi\left(-\frac{\sqrt{3}\pi}{2} - \left[\frac{1}{2} - \frac{1}{2}\right]\right) - \frac{\pi^3}{3} = \sqrt{3}\pi^2 - \frac{\pi^3}{3}.$$

(iii) Using the graph, the required volume is given by

$$V = \pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(\sin x - \frac{1}{2}\right)^2 dx = \pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin^2 x\, dx - \pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin x\, dx + \frac{\pi}{4}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 1 dx$$

$$= \pi\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1-\cos 2x}{2} dx + \pi\cos x\,\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} + \frac{\pi}{4}x\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} = \pi\left(\frac{x}{2} - \frac{\sin 2x}{4}\right)\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - \pi\sqrt{3} + \frac{\pi^2}{6}$$

$$= \frac{\pi^2}{2} - \frac{3\sqrt{3}}{4}\pi.$$

Problem 5

(a) The arc length

$$= \int_0^{\frac{\pi}{4}} \sqrt{1 + \left(\frac{d}{dx}\ln(\sec x)\right)^2}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{1 + \left(\frac{\sec x \tan x}{\sec x}\right)^2}\, dx = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \sec x\, dx = \ln|\sec x + \tan x|\,\Big|_0^{\frac{\pi}{4}} = \ln\left|1 + \sqrt{2}\right|.$$

(b) The arc length

$$= \int_0^5 \sqrt{1 + \left(\frac{d}{dx}\left(\frac{1}{3}x^{\frac{3}{2}}\right)\right)^2}\, dx = \int_0^5 \sqrt{1 + \frac{x}{4}}\, dx = \frac{1}{\frac{1}{4}}\frac{\left(1 + \frac{x}{4}\right)^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^5 = \frac{8}{3}\left(\frac{27}{8} - 1\right) = \frac{19}{3}.$$

(c) Note that

$$(y-1)^3 = \frac{9}{4}x^2 \Rightarrow y = 1 + \sqrt[3]{\frac{9}{4}}x^{\frac{2}{3}}.$$

The arc length

$$= \int_0^{\frac{2}{3}(3)^{\frac{3}{2}}} \sqrt{1 + \left(\frac{d}{dx}\left(1 + \sqrt[3]{\frac{9}{4}}x^{\frac{2}{3}}\right)\right)^2}\, dx = \int_0^{\frac{2}{3}(3)^{\frac{3}{2}}} \sqrt{1 + \left(\frac{2}{3}\right)^{\frac{2}{3}} x^{-\frac{2}{3}} dx}$$

$$= \int_0^{\frac{2}{3}(3)^{\frac{3}{2}}} \sqrt{\frac{x^{\frac{2}{3}} + \left(\frac{2}{3}\right)^{\frac{2}{3}}}{x^{\frac{2}{3}}}}\, dx = \int_0^{\frac{2}{3}(3)^{\frac{3}{2}}} \frac{\sqrt{x^{\frac{2}{3}} + \left(\frac{2}{3}\right)^{\frac{2}{3}}}}{x^{\frac{1}{3}}}\, dx$$

$$\overset{y = x^{\frac{2}{3}} + \left(\frac{2}{3}\right)^{\frac{2}{3}}}{=} \int_{\left(\frac{2}{3}\right)^{\frac{2}{3}}}^{\left(\frac{2}{3}\right)^{\frac{2}{3}}(3)} \frac{3}{2}\sqrt{y}dy = y^{\frac{3}{2}}\Big|_{\left(\frac{2}{3}\right)^{\frac{2}{3}}}^{\left(\frac{2}{3}\right)^{\frac{2}{3}}(3)} = \frac{2}{3}\left(3\sqrt{3} - 1\right).$$

(d) The arc length

$$= \int_0^a \sqrt{1 + \left(\frac{d}{dx}\left(\frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)\right)\right)^2}\, dx = \int_0^a \sqrt{1 + \left(\frac{1}{2}e^{\frac{x}{a}} - \frac{1}{2}e^{-\frac{x}{a}}\right)^2}\, dx$$

$$= \int_0^a \sqrt{1 + \left(\frac{1}{2}e^{\frac{x}{a}} - \frac{1}{2}e^{-\frac{x}{a}}\right)^2}\, dx = \int_0^a \sqrt{\frac{1}{4}e^{\frac{2x}{a}} + \frac{1}{2} + \frac{1}{4}e^{-\frac{2x}{a}}}dx$$

$$= \int_0^a \sqrt{\left(\frac{1}{2}e^{\frac{x}{a}} + \frac{1}{2}e^{-\frac{x}{a}}\right)^2}\, dx = \int_0^a \left(\frac{1}{2}e^{\frac{x}{a}} + \frac{1}{2}e^{-\frac{x}{a}}\right) dx = \frac{a}{2}e^{\frac{x}{a}} - \frac{a}{2}e^{-\frac{x}{a}}|_0^a = \frac{a}{2}\left(e - \frac{1}{e}\right).$$

(e) The arc length

$$= \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{\frac{\pi}{2}} \sqrt{(t\cos t)^2 + (t\sin t)^2}dt = \int_0^{\frac{\pi}{2}} \sqrt{t^2}dt = \int_0^{\frac{\pi}{2}} tdt = \frac{t^2}{2}|_0^{\frac{\pi}{2}} = \frac{\pi^2}{8}.$$

(f) The arc length

$$= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}dt$$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\, dt = \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$$

$$= \sqrt{2}\int_0^{2\pi} \sqrt{1-\cos t}\, dt = \sqrt{2}\int_0^{2\pi} \sqrt{\cos 0 - \cos t}\, dt$$

$$= \sqrt{2}\int_0^{2\pi} \sqrt{-2\sin\left(\frac{0+t}{2}\right)\sin\left(\frac{0-t}{2}\right)}\, dt = \sqrt{2}\int_0^{2\pi} \sqrt{-2\sin\frac{t}{2}\sin\left(-\frac{t}{2}\right)}\, dt$$

$$= 2\int_0^{2\pi} \sqrt{\sin^2\frac{t}{2}}dt = 2\int_0^{2\pi} \sin\frac{t}{2}dt = -4\cos\frac{t}{2}|_0^{2\pi} = 8.$$

(g) The arc length

$$= \int_0^a \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^a \sqrt{(2at)^2 + (2a)^2}dt$$

$$= 2a\int_0^a \sqrt{1+t^2}dt \overset{t=\tan\theta}{\triangleq} 2a\int_0^{\tan^{-1}a} \sqrt{1+(\tan\theta)^2}\sec^2\theta\, d\theta$$

$$= 2a\int_0^{\tan^{-1}a} \sec^3\theta\, d\theta = 2a\left[\frac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|)\right]_0^{\tan^{-1}a}$$

$$= a^2\sqrt{1+a^2} + a\ln\left|a + \sqrt{1+a^2}\right|.$$

Problem 6

(a) The surface area

$$= 2\pi\int_0^2 x^3\sqrt{1 + \left(\frac{d}{dx}x^3\right)^2}\, dx = 2\pi\int_0^2 x^3\sqrt{1+9x^4}dx$$

$$\overset{\substack{y=1+9x^4 \\ \frac{dy}{dx}=36x^3}}{\triangleq} \frac{2\pi}{36}\int_1^{145} \sqrt{y}dy = \frac{\pi}{18}\frac{y^{\frac{3}{2}}}{\frac{3}{2}}|_1^{145} = \frac{\pi}{27}\left(145^{\frac{3}{2}} - 1\right).$$

(b) The surface area

$$= 2\pi\int_0^4 \sqrt{4-x}\sqrt{1 + \left(\frac{d}{dx}\sqrt{4-x}\right)^2}\, dx = 2\pi\int_0^4 \sqrt{4-x}\sqrt{1 + \frac{1}{4(4-x)}}dx$$

$$= \pi \int_0^4 \sqrt{17-4x}dx = \pi\left(-\frac{1}{4}\frac{(17-4x)^{\frac{3}{2}}}{\frac{3}{2}}\right)|_0^4 = \frac{\pi}{6}\left(17^{\frac{3}{2}}-1\right).$$

(c) The surface area

$$I = 2\pi\int_0^1 e^x\sqrt{1+\left(\frac{d}{dx}e^x\right)^2}\,dx = 2\pi\int_0^1 e^x\sqrt{1+e^{2x}}dx$$

Let $e^x = \tan\theta$ (so that $\sqrt{1+e^{2x}} = \sqrt{1+\tan^2\theta} = \sec\theta$),

$$\Rightarrow \frac{d}{d\theta}e^x = \frac{d}{d\theta}\tan\theta \Rightarrow e^x\frac{dx}{d\theta} = \sec^2\theta \Rightarrow dx = \frac{\sec^2\theta}{e^x}d\theta.$$

When $x = 1,\ \theta = \tan^{-1}e$. When $x = 0,\ \theta = \tan^{-1}1 = \frac{\pi}{4}$.

The integral then becomes

$$I = 2\pi\int_{\frac{\pi}{4}}^{\tan^{-1}e} e^x\sqrt{1+e^{2x}}\left(\frac{\sec^2\theta}{e^x}d\theta\right) = 2\pi\int_{\frac{\pi}{4}}^{\tan^{-1}e}\sqrt{1+\tan^2\theta}\sec^2\theta\,d\theta$$

$$= 2\pi\int_{\frac{\pi}{4}}^{\tan^{-1}e}\sec^3\theta\,d\theta = 2\pi\left[\frac{1}{2}(\sec x\tan x + \ln|\sec x+\tan x|)\right]_{\frac{\pi}{4}}^{\tan^{-1}e}$$

$$= \pi\left(e\sqrt{1+e^2} + \ln\left|e+\sqrt{1+e^2}\right|\right) - \pi\left(\sqrt{2}+\ln\left|\sqrt{2}+1\right|\right).$$

(d) The surface area

$$= \left[2\pi\int_1^2 x\sqrt{1+\left(\frac{d}{dx}x\right)^2}\,dx + 2\pi\int_1^2(2-x)\sqrt{1+\left(\frac{d}{dx}(2-x)\right)^2}\,dx\right]$$

$$+\left[2\pi\int_2^3(4-x)\sqrt{1+\left(\frac{d}{dx}(4-x)\right)^2}\,dx + 2\pi\int_2^3(x-2)\sqrt{1+\left(\frac{d}{dx}(x-2)\right)^2}\,dx\right]$$

$$= \left[2\sqrt{2}\pi\int_1^2 x + 2\sqrt{2}\pi\int_1^2(2-x)dx\right] + \left[2\sqrt{2}\pi\int_2^3(4-x)dx + 2\sqrt{2}\pi\int_2^3(x-2)dx\right]$$

$$= 2\sqrt{2}\pi\left(\frac{x^2}{2}\right)|_1^2 + 2\sqrt{2}\pi\left(2x-\frac{x^2}{2}\right)|_1^2 + 2\sqrt{2}\pi\left(4x-\frac{x^2}{2}\right)|_2^3 + 2\sqrt{2}\pi\left(\frac{x^2}{2}-2x\right)|_2^3$$

$$= 2\sqrt{2}\pi\left(\frac{3}{2}+\frac{1}{2}+\frac{3}{2}+\frac{1}{2}\right) = 8\sqrt{2}\pi.$$

Problem 7

(a) The required area

$$= \int_0^2(e^x-e^{-x})dx = e^x+e^{-x}|_0^2 = e^2+e^{-2}-2.$$

(b) (i) The required volume

$$= \pi\int_0^2(e^x)^2dx - \pi\int_0^2(e^{-x})^2dx = \pi\int_0^2(e^{2x}-e^{-2x})dx$$

$$= \pi\left(\frac{e^{2x}}{2}+\frac{e^{-2x}}{2}\right)|_0^2 = \pi\left(\frac{e^4}{2}+\frac{e^{-4}}{2}-1\right).$$

(ii) The required volume

$$= \left(\pi\int_{e^{-2}}^1 2^2dy - \pi\int_{e^{-2}}^1(-\ln y)^2dy\right) + \left(\pi\int_1^{e^2}2^2dy - \pi\int_1^{e^2}(\ln y)^2dy\right)$$

$$= \pi \int_{e^{-2}}^{e^2} 2^2 dy - \pi \int_{e^{-2}}^{e^2} \underbrace{(\ln y)^2}_{u} \underbrace{dy}_{dv} = 4\pi y|_{e^{-2}}^{e^2} - \pi \left[y(\ln y)^2|_{e^{-2}}^{e^2} - \int_{e^{-2}}^{e^2} y d(\ln y)^2 \right]$$
$$= 4\pi(e^2 - e^{-2}) - 4\pi(e^2 - e^{-2}) + 2\pi \int_{e^{-2}}^{e^2} y \ln y \, dy = 2\pi \left(y \ln y \,|_{e^{-2}}^{e^2} - \int_{e^{-2}}^{e^2} y d(\ln y) \right)$$
$$= 4\pi(e^2 + e^{-2}) - 2\pi \int_{e^{-2}}^{e^2} 1 dy = 4\pi(e^2 + e^{-2}) - 2\pi y|_{e^{-2}}^{e^2} = 2\pi e^2 + 6\pi e^{-2}.$$

(iii) The required volume
$$= \pi \int_0^2 (e^x + 1)^2 dx - \pi \int_0^2 (e^{-x} + 1)^2 dx = \pi \int_0^2 (e^{2x} + 2e^x - e^{-2x} - 2e^{-x}) dx$$
$$= \pi \left(\frac{e^{2x}}{2} + 2e^x + \frac{e^{-2x}}{2} + 2e^{-x} \right) |_0^2 = \pi \left(\frac{e^4}{2} + 2e^2 + \frac{e^{-4}}{2} + 2e^{-2} - 5 \right).$$

Problem 8

(a) The intersection points can be obtained by solving
$$\begin{cases} y = x^2 \\ x = y^2 \end{cases} \Rightarrow x^2 = \sqrt{x} \Rightarrow \sqrt{x}\left(x^{\frac{3}{2}} - 1\right) = 0$$
$$\Rightarrow x = 0 \ \ (y = 0) \ \text{ or } \ x = 1 \ \ (y = 1).$$
The required area
$$= \int_0^1 (\sqrt{x} - x^2) dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3} |_0^1 = \frac{1}{3}.$$

(b) (i) The required volume
$$= \pi \int_0^1 \left(\sqrt{x}\right)^2 dx - \pi \int_0^1 (x^2)^2 dx = \pi \int_0^1 (x - x^4) dx = \pi \left(\frac{x^2}{2} - \frac{x^5}{5} \right)_0^1 = \frac{3\pi}{10}.$$

(ii) The required volume
$$= \pi \int_0^1 \left(\sqrt{y}\right)^2 dx - \pi \int_0^1 (y^2)^2 dy = \pi \int_0^1 (y - y^4) dy = \pi \left(\frac{y^2}{2} - \frac{y^5}{5} \right)_0^1 = \frac{3\pi}{10}.$$

(iii) The required volume
$$= \pi \int_0^1 \left(\sqrt{x} + 1\right)^2 dx - \pi \int_0^1 (x^2 + 1)^2 dx = \pi \int_0^1 \left(x + 2\sqrt{x} - x^4 - 2x^2\right) dx$$
$$= \pi \left(\frac{x^2}{2} + \frac{4}{3} x^{\frac{3}{2}} - \frac{x^5}{5} - \frac{2}{3} x^3 \right)_0^1 = \frac{29\pi}{30}.$$

(c) The required arc length
$$= \int_0^1 \sqrt{1 + \left(\frac{d}{dx}\sqrt{x}\right)^2} dx + \int_0^1 \sqrt{1 + \left(\frac{d}{dx}x^2\right)^2} dx = \int_0^1 \sqrt{1 + \frac{1}{4x}} dx + \int_0^1 \sqrt{1 + 4x^2} dx$$
$$= \int_0^1 \frac{\sqrt{4x + 1}}{2\sqrt{x}} dx + \int_0^1 \sqrt{1 + 4x^2} dx \overset{y=\sqrt{x}}{\cong} \int_0^1 \sqrt{1 + 4y^2} dy + \int_0^1 \sqrt{1 + 4x^2} dx$$
$$= 2\int_0^1 \sqrt{1 + 4x^2} dx \overset{x=\frac{1}{2}\tan\theta}{\cong} 2\int_0^{\tan^{-1} 2} \sqrt{1 + \tan^2\theta} \left(\frac{1}{2} \sec^2\theta \, d\theta \right)$$

$$=\int_0^{\tan^{-1}2}\sec^3\theta\,d\theta=\frac{1}{2}(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|)_0^{\tan^{-1}2}=\frac{1}{2}\left(2\sqrt{5}+\ln\left|2+\sqrt{5}\right|\right).$$

Problem 9

(a) The required area

$$A=\int_0^1\sqrt{x}dx+\int_1^{\sqrt{2}}\sqrt{2-x^2}dx\overset{\substack{x=\sqrt{2}\sin\theta\\ \frac{dx}{d\theta}=\sqrt{2}\cos\theta}}{\triangleq}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sqrt{2\cos^2\theta}\left(\sqrt{2}\cos\theta\,d\theta\right)$$

$$=\frac{2}{3}+2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^2\theta\,d\theta=\frac{2}{3}+2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\frac{1+\cos 2\theta}{2}\right)d\theta=\frac{2}{3}+\left[\theta+\frac{\sin 2\theta}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}=\frac{\pi}{4}+\frac{1}{6}.$$

(b) (i) The required volume is

$$\pi\int_0^1\left(\sqrt{x}\right)^2dx+\pi\int_1^{\sqrt{2}}\left(\sqrt{2-x^2}\right)^2dx=\pi\int_0^1 xdx+\pi\int_1^{\sqrt{2}}(2-x^2)dx$$

$$=\pi\left[\frac{x^2}{2}\right]_0^1+\pi\left[2x-\frac{x^3}{3}\right]_1^{\sqrt{2}}=\frac{8\sqrt{2}-7}{6}\pi.$$

(ii) The required volume is

$$\pi\int_0^1\left(\sqrt{2-y^2}\right)^2dy-\pi\int_0^1(y^2)^2dy$$

$$=\pi\int_0^1(2-y^2-y^4)dy=\pi\left[2y-\frac{y^3}{3}-\frac{y^5}{5}\right]_0^1=\frac{22\pi}{15}.$$

(c) The surface area is

$$=2\pi\left(\int_0^1\sqrt{x}\sqrt{1+\left(\frac{d}{dx}\sqrt{x}\right)^2}dx+\int_1^{\sqrt{2}}\sqrt{2-x^2}\sqrt{1+\left(\frac{d}{dx}\sqrt{2-x^2}\right)^2}dx\right)$$

$$=2\pi\left(\int_0^1\sqrt{x}\sqrt{1+\frac{1}{4x}}dx+\int_1^{\sqrt{2}}\sqrt{2-x^2}\sqrt{1+\frac{x^2}{2-x^2}}dx\right)$$

$$=2\pi\left(\int_0^1\frac{1}{2}\sqrt{4x+1}dx+\int_1^{\sqrt{2}}\sqrt{2}dx\right)$$

$$=2\pi\left(\left[\frac{1}{12}(4x+1)^{\frac{3}{2}}\right]_0^1+\left[\sqrt{2}x\right]_1^{\sqrt{2}}\right)=\frac{\pi}{6}\left(5^{\frac{3}{2}}-1\right)+2\pi\left(2-\sqrt{2}\right).$$