2 Linear Time-Invariant Systems

Major References:

- *Chapter 2, Signals and Systems by Alan V. Oppenheim et. al., 2nd edition, Prentice Hall*
- *Chapter 2, Schaum's Outline of Signals and Systems, 2nd Edition, 2010, McGraw-Hill*

2.1 Convolution

2.1.1 Convolution Integral of CT Signal

1. **Definition**
   Convolution Integral of two continuous-time signals $x(t)$ and $y(t)$ is defined by

$$z(t) = x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau. \tag{2.1}$$

   Convolution $x(t) * y(t)$ represents the degree to which $x\&y$ overlap at t as y sweeps across the domain t.

   Step. 1) $y(\tau)$ is time-reversed, then shifted by t; $y(\tau) \to y(-\tau) \to y(t-\tau)$

   Step. 2) $x(\tau)$ and $y(t-\tau)$ are multiplied, then integrated over τ

   Step. 3) Convolution will remain zero as long as $x\&y$ do not overlap

   Step. 4) Sweep $y(t-\tau)$ from $t=-\infty$ to $t=\infty$ to produce the entire output

2. **Properties of the Convolution Integral**
   The convolution integral has the following properties. Refer [Schaum's text, Problem 2.1] for the proof.
   a) Commutative
   $$x(t) * y(t) = y(t) * x(t)$$
   b) Associative
   $$\{x(t) * y_1(t)\} * y_2(t) = x(t) * \{y_1(t) * y_2(t)\}$$
   c) Distributive
   $$x(t) * \{y_1(t) + y_2(t)\} = x(t) * y_1(t) + x(t) * y_2(t)$$

3. **Additional Properties**
   Refer [Schaum's text, Problem 2.2, 2.8] for the proof.
   a) $x(t) * \delta(t) = x(t)$
   b) $x(t) * \delta(t-t_0) = x(t-t_0)$
   c) $x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$
   d) $x(t) * u(t-t_0) = \int_{-\infty}^{t-t_0} x(\tau) d\tau$
   e) If $x(t)$ and $y(t)$ are periodic signals with a common period T, the convolution in (2.1) does not converge. Instead, we define the *periodic convolution* $f(t) = x(t) \circledast y(t)$, where $f(t)$ is periodic with period T.

$$\begin{aligned} f(t) = x(t) \circledast y(t) &= \int_{0}^{T} x(\tau)\, y(t-\tau)\, d\tau \\ &= \int_{a}^{a+T} x(\tau)\, y(t-\tau)\, d\tau \quad \text{for arbitrary } a \end{aligned} \tag{2.2}$$

2.1.2 Convolution Sum of DT Signal

1. **Definition**
   Convolution Sum of two discrete-time sequence $x[n]$ and $y[n]$ is defined by

$$z[n] = x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k] y[n-k] \tag{2.3}$$

Step. 1) $y[k]$ is time-reversed, then shifted by n; $y[k] \to y[-k] \to y[n-k]$
Step. 2) $x[k]$ and $y[n-k]$ are multiplied, then summed over all k
Step. 3) Convolution will remain zero as long as $x \& y$ do not overlap
Step. 4) Sweep $y[n-k]$ from $n=-\infty$ to $n=\infty$ to produce the entire output

2. **Properties of the Convolution Sum**
   The convolution sum has the following properties. Refer [Schaum's text, Problem 2.26] for the proof.
   a) Commutative
   $$x[n] * y[n] = y[n] * x[n]$$
   b) Associative
   $$\{x[n] * y_1[n]\} * y_2[n] = x[n] * \{y_1[n] * y_2[n]\}$$
   c) Distributive
   $$x[n] * \{y_1[n] + y_2[n]\} = x[n] * y_1[n] + x[n] * y_2[n]$$

3. **Additional Properties**
   Refer [Schaum's text, Problem 2.27, 2.31] for the proof.
   a) $x[n] * \delta[n] = x[n]$
   b) $x[n] * \delta[n-n_0] = x[n-n_0]$
   c) $x[n] * u[n] = \sum_{k=-\infty}^{n} x[k]$
   d) $x[n] * u[n-n_0] = \sum_{k=-\infty}^{n-n_0} x[k]$
   e) If $x[n]$ and $y[n]$ are periodic sequence with a common period N, the convolution in (2.3) does not converge. Instead, we define the *periodic convolution* $f[n] = x[n] \circledast y[n]$, where $f[n]$ is periodic with period N.

$$f[n] = x[n] \circledast y[n] = \sum_{k=0}^{N-1} x[k] y[n-k] \tag{2.4}$$

[Example 2-1] Evaluate the following convolutions

1. $u(t+a) * u(t+b)$
2. $rect(t/\tau) * rect(t/\tau)$
3. $rect(t/\tau) * u(t)$
4. $x(t) * y(t)$ where $x(t) = \begin{cases} 1 & \text{for } 0 < t < 3 \\ 0 & \text{otherwise} \end{cases}$ and $y(t) = \begin{cases} 1 & \text{for } 0 < t < 2 \\ 0 & \text{otherwise} \end{cases}$
5. $rect(t/\tau) * \delta_T(t)$ where $\tau < T$ and $\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$ is the *unit impulse train*

Refer [Schaum's text, Problem 2.6, 2.7, 2.8]

Solution) Example 2-1. 1) The convolution integral is given by

$$\begin{aligned} u(t+a) * u(t+b) &= \int_{-\infty}^{\infty} u(\tau+a)\, u(-\tau+b+t)\, d\tau = (t+a+b)\, u(t+a+b) \\ &= \begin{cases} \displaystyle\int_{-a}^{b+t} 1 \cdot d\tau = (t+a+b) & \text{if } t+a+b>0 \\ 0 & \text{if } t+a+b<0 \end{cases} \end{aligned} \tag{2.5}$$

where the product of two step functions $u(\tau+a)\, u(-\tau+b+t)$ has a non-zero value at $\tau+a>0$ and $-\tau+b+t>0$. As shown in Fig. 2.1, if $-a<b+t$, then the product of two step functions overlap each other within the interval $-a<\tau<b+t$. If $b+t<-a$, there is no overlap and the integral in (2.5) becomes zero.

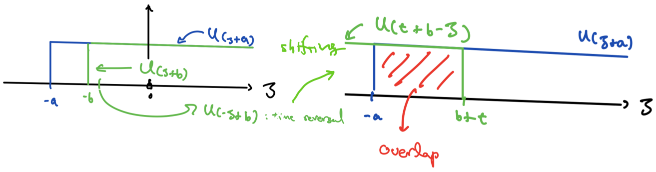


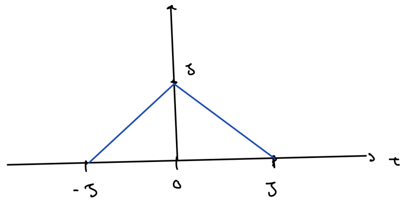
Figure 2.1:

Example 2-1. 2) The convolution can be expanded by expressing the rectangular pulse signal via the step fucntion

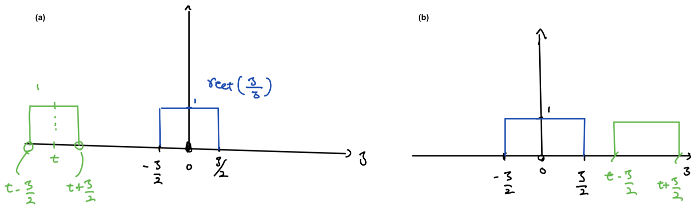
$$\begin{aligned} rect(t/\tau) * rect(t/\tau) &= \left\{u\left(t+\frac{\tau}{2}\right) - u\left(t-\frac{\tau}{2}\right)\right\} * \left\{u\left(t+\frac{\tau}{2}\right) - u\left(t-\frac{\tau}{2}\right)\right\} \\ &= (t+\tau)\, u(t+\tau) - 2tu(t) + (t-\tau)\, u(t-\tau), \end{aligned} \tag{2.6}$$

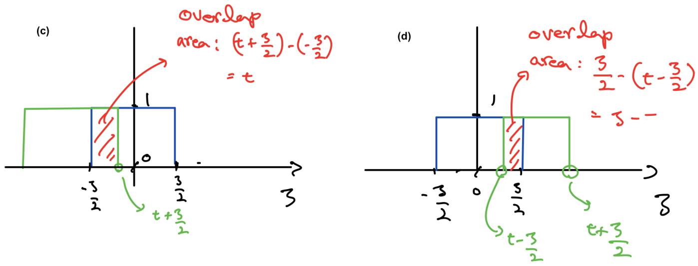
where we used (1.9) in the first equality and the result from [Example 2-1. 1] in the second equality. The last expression has four separate intervals with different values, which is a *triangular pulse signal* with maximum magnitude τ at $t=0$ and width 2τ (two times larger than the rectangular pulse signal $rect(t/\tau)$).

$$\begin{cases} 0 & \text{if } t<-\tau \\ t+\tau & \text{if } -\tau<t<0 \\ t+\tau-2t=\tau-t & \text{if } 0<t<\tau \\ \tau-t+t-\tau=0 & \text{if } t>\tau \end{cases}$$



(b) can also be solved using the direct definition of the convolution where we multiply one original signal to another signal that is time reversed, then time-shifted by t, which is plotted below.





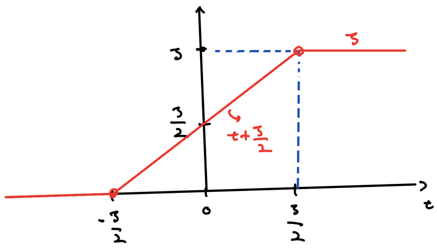
- For sub-figure (a) and (b), there is no overlap between two signals, hence the convolution is zero. These cases correspond to the condition $t+\frac{\tau}{2} < -\frac{\tau}{2}$ and $t-\frac{\tau}{2} > \frac{\tau}{2}$, *i.e.*, $t < -\tau$ for sub-figure (a) and $t > \tau$ for (b).
- For sub-figure (c), if $t+\frac{\tau}{2} > -\frac{\tau}{2}$ and $t-\frac{\tau}{2} < -\frac{\tau}{2}$, the overlap area is $t+\frac{\tau}{2} - \left(-\frac{\tau}{2}\right) = t$, which is the convolution in the interval $-\tau < t < 0$.
- For sub-figure (d), if $t-\frac{\tau}{2} < \frac{\tau}{2}$ and $t+\frac{\tau}{2} > \frac{\tau}{2}$, the overlap area is $\frac{\tau}{2} - \left(t-\frac{\tau}{2}\right) = \tau - t$, which is the convolution in the interval $0 < t < \tau$.

Example 2-1. 3) The convolution can be expanded in terms of the step fucntion as follows

$$rect\,(t/\tau) * u(t) = \left\{u\left(t+\frac{\tau}{2}\right) - u\left(t-\frac{\tau}{2}\right)\right\} * u(t) = \left(t+\frac{\tau}{2}\right)u\left(t+\frac{\tau}{2}\right) - \left(t-\frac{\tau}{2}\right)u\left(t-\frac{\tau}{2}\right), \tag{2.7}$$

where we applied [Example 2-1. 1] in the second equality. The last expression has three separate intervals with different values as follows

$$\begin{cases} 0 & \text{if } t < -\dfrac{\tau}{2} \\ t+\dfrac{\tau}{2} & \text{if } -\dfrac{\tau}{2} < t < \dfrac{\tau}{2} \\ t+\dfrac{\tau}{2} - \left(t-\dfrac{\tau}{2}\right) = \tau & \text{if } t > \dfrac{\tau}{2} \end{cases}$$

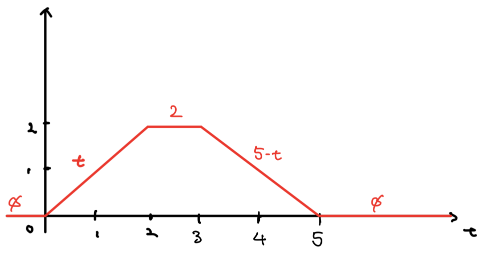


Example 2-1. 4) The convolution can be expanded in terms of the step fucntion as follows

$$\begin{aligned} x(t) * y(t) &= \{u\,(t) - u\,(t-3)\} * \{u\,(t) - u\,(t-2)\} \\ &= tu(t) - (t-3)\,u\,(t-3) - (t-2)\,u\,(t-2) + (t-5)\,u\,(t-5)\,, \end{aligned} \tag{2.8}$$

where we applied [Example 2-1. 1] in the second equality. The last expression has five separate intervals with different values as follows

$$\begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 < t < 2 \\ t-(t-2) = 2 & \text{if } 2 < t < 3 \\ 2-(t-3) = 5-t & \text{if } 3 < t < 5 \\ 5-t-(t-5) = 0 & \text{if } t > 5 \end{cases}$$



Example 2-1. 5)

$$rect\,(t/\tau) * \left[\sum_{n=-\infty}^{\infty} \delta\,(t-nT)\right] = \sum_{n=-\infty}^{\infty} rect\,(t/\tau) * \delta\,(t-nT) = \sum_{n=-\infty}^{\infty} rect\left(\frac{t-nT}{\tau}\right) \tag{2.9}$$

where we used distributive property in the second equality and $x(t) * \delta\,(t-t_0) = x\,(t-t_0)$ in the last equality.

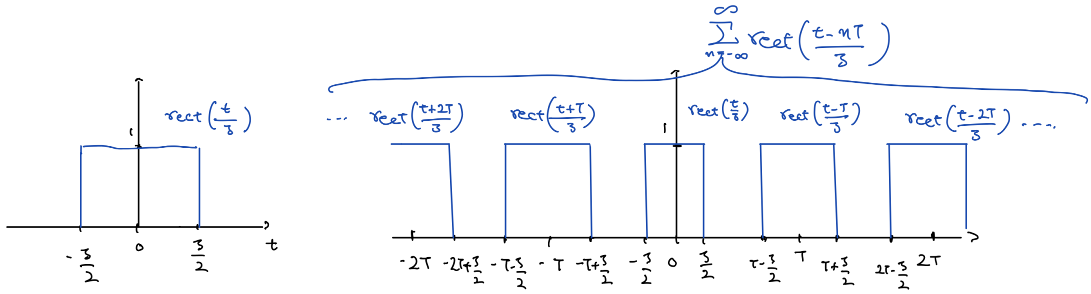


Figure 2.2:

As shown in Fig. 2.2, the time-shifted rectangular pulse signal do not overlap each other as far as $\tau < T$ is satisfied. However, if $\tau > T$, then the time-shifted rectangular pulse signals will overlap each other.

Important Convolution Pairs

1. $u(t+a) * u(t+b) = (t+a+b)\,u(t+a+b)$
2. $rect(t/\tau) * rect(t/\tau) = \begin{cases} 0 & \text{if } t < -\tau \\ \tau + t & \text{if } -\tau < t < 0 \\ \tau - t & \text{if } 0 < t < \tau \\ 0 & \text{if } t > \tau \end{cases}$

[Example 2-2] Evaluate the following convolutions

1. $x(t) * y(t)$ where $x(t) = u(t)$ and $y(t) = e^{-\alpha t}u(t)$, $\alpha > 0$
2. $x(t) * y(t)$ where $x(t) = e^{-\alpha t}u(t)$ and $y(t) = e^{\alpha t}u(-t)$, $\alpha > 0$

Refer [Schaum's text, Problem 2.4, 2.5]

Solution) Example 2-2. 1)

$$x(t) * y(t) = \int_{-\infty}^{\infty} e^{-\alpha\tau} u(\tau)\, u(t-\tau)\, d\tau = \begin{cases} \int_0^t e^{-\alpha\tau} d\tau = \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right), & \text{if } t > 0 \\ 0, & \text{if } t < 0 \end{cases} \tag{2.10}$$

$$= \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right) u(t)$$

Example 2-2. 2)

$$x(t) * y(t) = \int_{-\infty}^{\infty} e^{\alpha\tau} u(-\tau)\, e^{-\alpha(t-\tau)} u(t-\tau)\, d\tau \tag{2.11}$$

where the product of two step functions $u(-\tau)\,u(t-\tau)$ is determined by the magnitude of t as shown in Fig. 2.3. Then (2.11) can be derived as follows

$$\begin{cases} e^{-\alpha t} \int_{-\infty}^{0} e^{2\alpha\tau} d\tau = \frac{1}{2\alpha} e^{-\alpha t}, & \text{if } t > 0 \\ e^{-\alpha t} \int_{-\infty}^{t} e^{2\alpha\tau} d\tau = \frac{1}{2\alpha} e^{\alpha t}, & \text{if } t < 0 \end{cases} \Rightarrow \frac{1}{2\alpha} e^{-\alpha|t|} \tag{2.12}$$

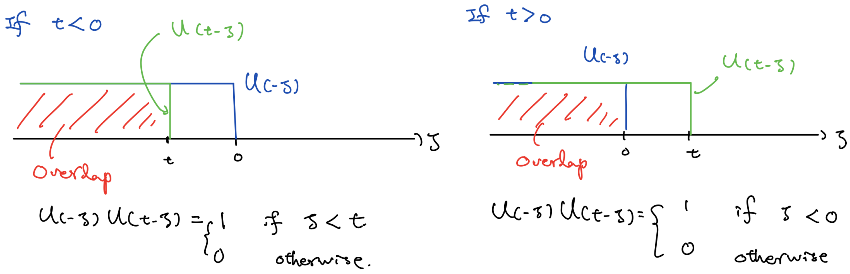


Figure 2.3:

2.2 LTI System Response

Linear Time-Invariant (LTI) System, represented by $\mathbf{T}\{\cdot\}$, satisfy the following two attributes.

- Linearity: $\mathbf{T}\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} = \alpha_1 \mathbf{T}\{x_1(t)\} + \alpha_2 \mathbf{T}\{x_2(t)\}$
- Time-Invariance: $\mathbf{T}\{x(t-t_0)\} = y(t-t_0)$

2.2.1 Response of a CT LTI System

1. **Impulse Response**
   - *Impulse Response* is defined as the output of a system when the input is a impulse signal $\delta(t)$.

$$h(t) = \mathbf{T}\{\delta(t)\} \tag{2.13}$$

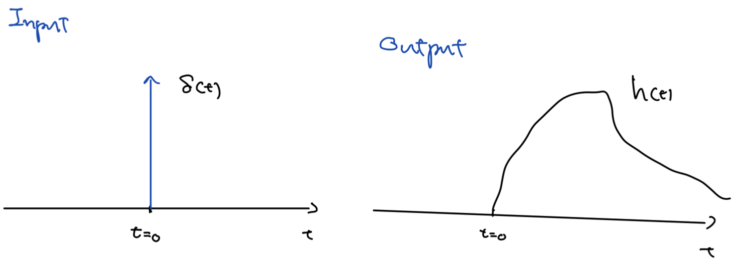


Figure 2.4:

- *The output of any CT-LTI system is the convolution of the input $x(t)$ with the impulse response $h(t)$*

$$y(t) = x(t) * h(t) \tag{2.14}$$

Proof) Arbitrary input $x(t)$ can be expressed in terms of the imulse signal $\delta(t)$ using convolution

$$x(t) = x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau \tag{2.15}$$

USing the properties of an LTI system, the output to an arbitrary input $x(t)$ can be expressed as

$$\begin{aligned} y(t) = \mathbf{T}\{x(t)\} &= \mathbf{T}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau\right\} \\ &= \int_{-\infty}^{\infty} x(\tau)\mathbf{T}\{\delta(t-\tau)\}\,d\tau = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = x(t) * h(t), \end{aligned} \tag{2.16}$$

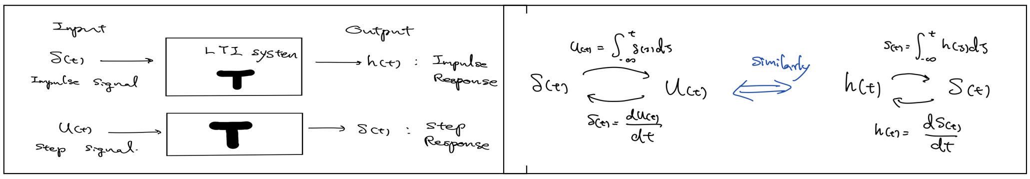
by using (2.15) in the second equality, linearity in the third, and time-invariance in the fourth equality.

2. **Step Response**
   - *Step Response* is defined as the output of a system when the input is a step signal $u(t)$.

$$s(t) = \mathbf{T}\{u(t)\} \tag{2.17}$$

   - *Step response* can be obtained by integrating the impulse response $h(t)$. Similarly, the impulse response $h(t)$ can be determined by differentiating the step response $h(t)$.

$$s(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\,d\tau = \int_{-\infty}^{t} h(\tau)d\tau \quad \Leftrightarrow \quad h(t) = \frac{ds(t)}{dt} \tag{2.18}$$



3. **Properties of CT LTI System**
   - **Memoryless or Memory System:** The output of a memoryless system depends only on the present input, so that the input-output relationship can be represented in the form $y(t) = Kx(t)$. Since $h(t)$ is the system output for an impulse signal input $\delta(t)$, the impulse response $h(t)$ can be expressed as $h(t) = K\delta(t)$. Hence, the following statement holds

$$\text{If } h(t_0) \neq 0 \text{ for } t_0 \neq 0\text{, then it is a LTI system with memory.} \tag{2.19}$$

   - **Causality:** For a causal system, the output at a present instant do not anticipate input from future instants. In other words, the input that occurs before $t < t_0$ can only determine the output with $t < t_0$.

$$x(t),\ t < t_0 \quad \Leftrightarrow \quad y(t),\ t < t_0 \tag{2.20}$$

   Thus, in a causal system, it is impossible to obtain an output before an input is applied. Since the impulse response is defined as $h(t) = \mathbf{T}\{\delta(t)\}$, the impulse response $h(t)$ is zero for $t < 0$

$$\delta(t) = 0,\ t < 0 \quad \Leftrightarrow \quad h(t) = 0,\ t < 0 \tag{2.21}$$

   Due to (2.21), the output of a causal LTI system can be expressed as follows

$$\text{Causal LTI System, Arbitrary Input:} \quad y(t) = \int_{0}^{\infty} h(\tau)x(t-\tau)\,d\tau = \int_{-\infty}^{t} x(\tau)h(t-\tau)\,d\tau \tag{2.22}$$

   Furthermore, if we define a *causal signal* to achieve the following condition

$$x(t) = 0, \quad t < 0, \tag{2.23}$$

   the output of causal signal input on a causal LTI system can be expressed as follows

$$\text{Causal LTI System, Causal Input:} \quad y(t) = \int_{0}^{t} h(\tau)x(t-\tau)\,d\tau = \int_{0}^{t} x(\tau)h(t-\tau)\,d\tau \tag{2.24}$$

   - **Stability:** An LTI system is stable if its impulse response $h(t)$ is absolutely integrable.

$$\int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty \tag{2.25}$$

Proof) If $|x(t)| \le k_1 < \infty$ for any t, then the output can be bounded as follows

$$\begin{aligned} |y(t)| &= |\int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau| \le \int_{-\infty}^{\infty} |h(\tau)\, x(t-\tau)| d\tau \\ &= \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)| d\tau \le k_1 \int_{-\infty}^{\infty} |h(\tau)| d\tau \end{aligned} \tag{2.26}$$

If $\int_{-\infty}^{\infty} |h(\tau)| d\tau \le k_2 < \infty$, then $|y(t)| \le k_1 \times k_2 < \infty$ and the system is stable.

2.2.2 Response of a DT LTI System

1. **Impulse Response**
   - *Impulse Response* is the output of a system when the input is a impulse signal $\delta[n]$

$$h[n] = \mathbf{T}\{\delta[n]\} \tag{2.27}$$

   - *The output of any DT-LTI system is the convolution of the input $x[n]$ with the impulse response $h[n]$*

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \tag{2.28}$$

2. **Step Response**
   - *Step Response* is the system output for a step signal input $u[n]$, *i.e.*, $s[n] = \mathbf{T}\{u[n]\}$
   - *Step response* can be obtained by calculating the cumulative sum of the impulse response $h[n]$. Similarly, the impulse response $h[n]$ can be determined by differentiating the step response $h[n]$.

$$\begin{aligned} s[n] &= h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k] u[n-k] = \sum_{k=-\infty}^{n} h[k], \\ h[n] &= s[n] - s[n-1] \end{aligned} \tag{2.29}$$

3. **Properties of DT LTI System**
   - **Memory System**

$$\text{If } h[n_0] \ne 0 \text{ for } n_0 \ne 0, \text{ then it is a LTI system with memory.} \tag{2.30}$$

   - **Causality:** The causality condition for a DT-LTI system is given by

$$h[n] = 0, \;\; n < 0 \tag{2.31}$$

   Due to (2.31) and the definition of causal signal, the output of a causal LTI system can be expressed as

$$\begin{aligned} &\text{Causal LTI System, Arbitrary Input:} \quad y[n] = \sum_{k=0}^{\infty} h[k] x[n-k] = \sum_{k=-\infty}^{n} x[k] h[n-k], \\ &\text{Causal LTI System, Causal Input:} \quad y[n] = \sum_{k=0}^{n} h[k] x[n-k] = \sum_{k=0}^{n} x[k] h[n-k] \end{aligned} \tag{2.32}$$

   - **Stability:** An LTI system is stable if its impulse response $h(t)$ is absolutely summable.

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty \tag{2.33}$$

[Example 2-3] Consider a CT-LTI system with step response $s(t) = e^{-t}u(t)$. Determine the output of the system for input signal $x(t) = u(t-1) - u(t-3)$.
Refer [Schaum's text, Problem 2.10]

Solution) Based on the definition of a step response $s(t) = \mathbf{T}\{u(t)\}$, the output of the LTI system is given by

$$y(t) = s(t-1) - s(t-3) = e^{-(t-1)}u(t-1) - e^{-(t-3)}u(t-3), \tag{2.34}$$

where we used linearity and time-invariance of the LTI system in the second equality.

[Example 2-4] Consider a CT-LTI system described by

$$y(t) = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} x(\tau)\, d\tau. \tag{2.35}$$

Find the impulse response $h(t)$ of the system and answer whether this system is causal or not.
Refer [Schaum's text, Problem 2.11]

Solution) Since the impulse response is the output for the impulse signal input, the following equality holds

$$h(t) = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} \delta(\tau)\, d\tau = \begin{cases} \frac{1}{T} & \text{if } t - \frac{T}{2} < 0 \text{ and } t + \frac{T}{2} > 0, \\ 0 & \text{otherwise} \end{cases} \tag{2.36}$$

where we applied the sifting property of $\delta(t)$ in the second equality. The impulse response $h(t)$ is plotted in Fig. 2.5. Since $h(t) \neq 0$ for $-\frac{T}{2} < t < 0$, this system is not causal.

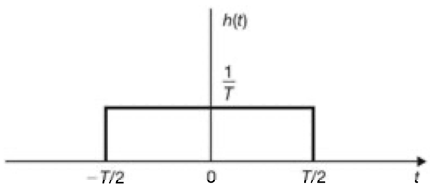


Figure 2.5:

[Example 2-5] Consider CT-LTI systems composed of two component blocks where the impulse response of each block is given by $h_1(t) = e^{-2t}u(t)$ and $h_2(t) = 2e^{-t}u(t)$. For (a) cascade and (b) parallel connection case, find the impulse response $h(t)$ of the overall system and answer whether the overall system is stable or not.
Refer [Schaum's text, Problem 2.14, 2.53]

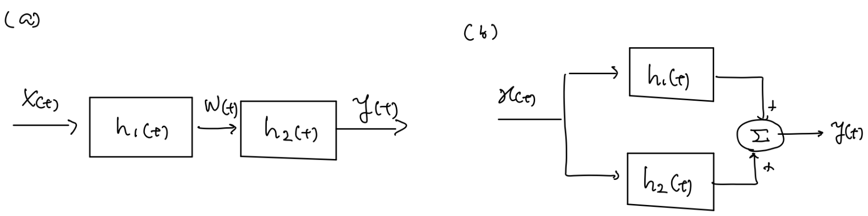


Figure 2.6:

Solution) For (a) cascaded connection case, the impulse response can be derived as follows

$$
\begin{aligned}
h(t) &= h_1(t) * h_2(t) = \int_{-\infty}^{\infty} 2e^{-\tau} u(\tau)\, e^{-2(t-\tau)} u(t-\tau)\, d\tau \\
&= 2e^{-2t} \int_{-\infty}^{\infty} e^{\tau} u(\tau)\, u(t-\tau)\, d\tau = \begin{cases} 2e^{-2t} \displaystyle\int_0^t e^{\tau} d\tau = 2\left(e^{-t} - e^{-2t}\right) & \text{if } t > 0, \\ 0 & \text{otherwise} \end{cases} \\
&= 2\left(e^{-t} - e^{-2t}\right) u(t)
\end{aligned} \tag{2.37}
$$

To check stability, we need to verify whether $h(t)$ is absolutely integrable.

$$
\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \int_0^{\infty} 2\left(e^{-t} - e^{-2t}\right) d\tau = 2\left[\int_0^{\infty} e^{-\tau} d\tau - \int_0^{\infty} e^{-2\tau} d\tau\right] = 1 < \infty \tag{2.38}
$$

For (b) parallel connection case, the overall impulse response is given by

$$
h(t) = h_1(t) + h_2(t) = \left(e^{-2t} + 2e^{-t}\right) u(t), \tag{2.39}
$$

and the stability test can be performed as below

$$
\int_{-\infty}^{\infty} |h(\tau)|\, d\tau = \int_0^{\infty} \left(e^{-2\tau} + 2e^{-\tau}\right) d\tau = \frac{1}{2} + 2 < \infty, \tag{2.40}
$$

which indicates that the parallel connection system in (b) is stable.

[Example 2-6] For the following impulse responses, determine whether the given LTI system is causal and stable.

a) $h(t) = e^{-3t} \sin(t)\, u(t)$

b) $h(t) = \delta(t) + e^{-3t} u(t)$

c) $h[n] = \delta[n+1]$

d) $h[n] = \left(-\frac{1}{2}\right)^n u[n-1]$

Solution) For causality, we need to check whether the impulse response has a non-zero value on $t < 0$ (or $n < 0$). Then, it is clear that, except (c), the other systems are all causal system. For stability test, we need to check whether the given impulse response is absolutely integrable (or absolutely summable).

$$
\begin{aligned}
&\text{(a)} \quad \int_0^{\infty} e^{-3t} |\sin(t)|\, dt \le \int_0^{\infty} e^{-3t} dt = \frac{1}{3} < \infty \\
&\text{(b)} \quad \int_{-\infty}^{\infty} |h(\tau)|\, d\tau = 1 + \int_0^{\infty} e^{-3t} dt = \frac{4}{3} < \infty \\
&\text{(c)} \quad \sum_{n=-\infty}^{\infty} |h[n]| = 1 < \infty, \qquad \text{(d)} \quad \sum_{n=1}^{\infty} \left|\left(-\frac{1}{2}\right)^n\right| = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 < \infty
\end{aligned} \tag{2.41}
$$

Hence, the solutions are summarized below.

a) Causal, Stable

b) Causal, Stable

c) Noncausal, Stable

d) Causal, Stable

[Example 2-7] Compute the output $y[n]$ of a DT-LTI system for the given impulse response and the input signals

a) $x[n] = \alpha^n u[n]$ and $h[n] = \beta^n u[n]$

b) $x[n] = u[n]$ and $h[n] = \alpha^n u[n]$, where $0 < \alpha < 1$

Refer [Schaum's text, Problem 2.28, 2.29]

Solution) In (a), the convolution sum between the input signal and the impulse response is given by

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} \alpha^k \beta^{n-k} u[k]u[n-k] \\ &= \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k u[n] = \begin{cases} \beta^n \dfrac{1-(\alpha/\beta)^{n+1}}{1-\alpha/\beta} u[n] = \dfrac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha} u[n] & \text{if } \alpha \neq \beta \\ \beta^n (n+1)\, u[n] & \text{if } \alpha = \beta \end{cases} \end{aligned} \tag{2.42}$$

For (b), we can apply [Example 2-7. a] by substituting $\alpha \leftarrow 1$ and $\beta \leftarrow \alpha$ into (2.42). Then, the output is given by

$$y[n] = \frac{1-\alpha^{n+1}}{1-\alpha} u[n] \tag{2.43}$$

[Example 2-8] Compute the DT convolution of the following sequences

- $x[0] = 0.5, x[1] = 2, x[n] = 0$ otherwise.
- $h[0] = h[1] = h[2] = 1$, $h[n] = 0$ otherwise.

Compute $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$.

Solution) Using the convolution sum $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

$$y[n] = x[0]h[n-0] + x[1]h[n-1] = 0.5h[n] + 2h[n-1] \tag{2.44}$$

Substituting the values of $h[n]$ into $y[n]$, we have

- $y[0] = 0.5h[0] + 2h[0-1] = 0.5 \times 1 = 0.5$
- $y[1] = 0.5h[1] + 2h[0] = 0.5 \times 1 + 2 \times 1 = 2.5$
- $y[2] = 0.5h[2] + 2h[1] = 0.5 \times 1 + 2 \times 1 = 2.5$
- $y[3] = 0.5h[3] + 2h[2] = 0.5 \times 0 + 2 \times 1 = 2$
- $y[n] = 0$ for $n < 0$ and $n \geq 4$.