

1. Find the eigenvalues and eigenvectors of the following matrices:

$$(a) \begin{pmatrix} 3 & 2 \\ 3 & 4 \end{pmatrix} \quad (b) \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix} \quad (c) \begin{pmatrix} 1 & 0 & \sqrt{2} \\ 0 & 2 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix}.$$

Solution:

(a)

$$\text{Let } A = \begin{pmatrix} 3 & 2 \\ 3 & 4 \end{pmatrix}, \text{ then } A - \lambda I = \begin{pmatrix} 3-\lambda & 2 \\ 3 & 4-\lambda \end{pmatrix} \text{ and}$$

$$\det(A - \lambda I) = (3 - \lambda)(4 - \lambda) - 6 = 12 - 7\lambda + \lambda^2 - 6 = (\lambda - 6)(\lambda - 1).$$

Eigenvalues of A are 1, 6.

$$\text{When } \lambda = 1, \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2 & 2 & 0 \\ 3 & 3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x_1 = -x_2 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \begin{pmatrix} -1 \\ 1 \end{pmatrix}, k \neq 0.$$

$$\text{When } \lambda = 6, \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -3 & 2 & 0 \\ 3 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} -3 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x_1 = \frac{2}{3}x_2 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \begin{pmatrix} \frac{2}{3} \\ 1 \end{pmatrix}, k \neq 0.$$

(b)

$$\text{Let } A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix}, \text{ then } A - \lambda I = \begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 5-\lambda \end{pmatrix} \text{ and}$$

$$\det(A - \lambda I) = (5 - \lambda)[(2 - \lambda)(2 - \lambda) - 1] = (5 - \lambda)[4 - 4\lambda + \lambda^2 - 1] = (5 - \lambda)(\lambda - 3)(\lambda - 1).$$

Eigenvalues of A are 1, 3, 5.

When $\lambda = 1$,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow x_3 = 0, x_1 = -x_2$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, k \neq 0$$

When $\lambda = 3$,

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -1 & 1 & 1 & | & 0 \\ 1 & -1 & 1 & | & 0 \\ 0 & 0 & 2 & | & 0 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 1 & | & 0 \\ 0 & 0 & 2 & | & 0 \\ 0 & 0 & 2 & | & 0 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 1 & | & 0 \\ 0 & 0 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\Rightarrow x_3 = 0, x_1 = x_2 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, k \neq 0$$

When $\lambda = 5$,

$$\begin{pmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & -3 & 1 & | & 0 \\ -3 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 1 & | & 0 \\ 0 & -8 & 4 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\Rightarrow x_2 = \frac{1}{2}x_3, x_1 = 3x_2 - x_3 = \frac{1}{2}x_3 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}, k \neq 0$$

(c)

Let $A = \begin{pmatrix} 1 & 0 & \sqrt{2} \\ 0 & 2 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix}$, then $A - \lambda I = \begin{pmatrix} 1-\lambda & 0 & \sqrt{2} \\ 0 & 2-\lambda & 0 \\ \sqrt{2} & 0 & -\lambda \end{pmatrix}$ and

$$\det(A - \lambda I) = (1-\lambda)(2-\lambda)(-\lambda) - 2(2-\lambda) = (2-\lambda)(-\lambda + \lambda^2 - 2) = (2-\lambda)(\lambda-2)(\lambda+1)$$

Eigenvalues of A are -1, 2.

When $\lambda = -1$,

$$\begin{pmatrix} 2 & 0 & \sqrt{2} \\ 0 & 3 & 0 \\ \sqrt{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2 & 0 & \sqrt{2} & | & 0 \\ 0 & 3 & 0 & | & 0 \\ \sqrt{2} & 0 & 1 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & \sqrt{2} & | & 0 \\ 0 & 3 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\Rightarrow x_2 = 0, x_1 = -\frac{\sqrt{2}}{2}x_3 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ 1 \end{pmatrix}, k \neq 0$$

When $\lambda = 2$,

$$\begin{pmatrix} -1 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -1 & 0 & \sqrt{2} & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \Rightarrow x_1 = \sqrt{2}x_3$$

Let $x_2 = t$, $x_3 = k$, then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \sqrt{2}k \\ t \\ k \end{pmatrix} = \begin{pmatrix} \sqrt{2}k \\ 0 \\ k \end{pmatrix} + \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix} = k \begin{pmatrix} \sqrt{2} \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, where $k^2 + t^2 \neq 0$

□

2. For the matrix in question 1(b), find a matrix P such that $P^{-1}AP = D$, a diagonal matrix with the eigenvalues of A as its elements. Check your solution by evaluating $P^{-1}AP$.

Solution:

Let $P = \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$, then $AP = PD$. That is $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, where

$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. Since the eigenvalues of $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix}$ are distinct, columns of $P = \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$ are

linearly independent. It follows that P^{-1} exists and $AP = PD \Rightarrow P^{-1}AP = D$.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 1/2 & 1 & 0 & 0 \\ 1 & 1 & 1/2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1/2 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 1/2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right)$$

We have $P^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. By matrix multiplication, we observe that

$$P^{-1}AP = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & -1 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

□

3. Find a 2×2 matrix A which has eigenvalue 1 with corresponding eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and eigenvalue 3 with corresponding eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Solution:

Let $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. To evaluate P^{-1} , we have $\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array} \right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{array} \right)$. Therefore $P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

$$AP = PD \Rightarrow A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$$

□

4. (a) Using Gaussian elimination, find a matrix X such that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- (b) With the help of the result of (a), find a 4×4 matrix A which has eigenvalues $-1, 0, 0, 1$ with

corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$, which are rows of $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}$.

Solution:

(a)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & | & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 1 & | & 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_4 + r_1} \begin{pmatrix} 1 & 0 & 0 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & | & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{r_3 - r_2} \begin{pmatrix} 1 & 0 & 0 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & | & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & | & 1 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{r_4 - r_3} \begin{pmatrix} 1 & 0 & 0 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & | & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & | & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & | & 1 & 1 & -1 & 1 \end{pmatrix}$$

Therefore,

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix}.$$

(b)

From (a) we have
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix}.$$

Therefore

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}^T \right]^{-1} = \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}^{-1} \right]^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\begin{aligned} A \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \\ A &= \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

□

5. Given that $\xi = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 2 & -1 & 2 \\ 5 & a & 3 \\ -1 & b & -2 \end{pmatrix}$.

- Find a and b .
- Find the eigenvalues and eigenvectors of A .
- Is A diagonalizable? Please give reasons.

Solution:

(a)

$$A\xi = \lambda\xi \Leftrightarrow \begin{pmatrix} 2 & -1 & 2 \\ 5 & a & 3 \\ -1 & b & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \\ -\lambda \end{pmatrix} \Leftrightarrow \begin{pmatrix} -1 \\ 5 + a - 3 \\ -1 + b + 2 \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda \\ -\lambda \end{pmatrix} \Rightarrow \lambda = -1, a = -3, b = 0$$

(b)

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}, \det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & -1 & 2 \\ 5 & -3 - \lambda & 3 \\ -1 & 0 & -2 - \lambda \end{vmatrix} = -(\lambda^3 + 3\lambda^2 + 3\lambda + 1) = -(\lambda + 1)^3$$

For $\lambda = -1$, we have

$$\begin{pmatrix} 3 & -1 & 2 \\ 5 & -2 & 3 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 3 & -1 & 2 & | & 0 \\ 5 & -2 & 3 & | & 0 \\ -1 & 0 & -1 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & | & 0 \\ 3 & -1 & 2 & | & 0 \\ 5 & -2 & 3 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & | & 0 \\ 0 & -1 & -1 & | & 0 \\ 0 & -2 & -2 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Let $x_3 = k$ then $x_2 = -k$, $x_1 = -k$ and $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ is one of the eigenvectors associated with $\lambda = -1$.

(c)

Only one eigenvector is found, A is not diagonalizable. □

6. (a) Find the eigenvalues and corresponding eigenvectors of the symmetric matrix

$$A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \text{ and verify that the eigenvectors are mutually orthogonal.}$$

(b) Let $B = A^5 - 5(A + 3I)^{-1} + 3A^T$. Find the eigenvalues of B .

Solution:

(a)

$$A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad A - \lambda I = \begin{pmatrix} 4 - \lambda & -2 & 0 \\ -2 & 1 - \lambda & 0 \\ 0 & 0 & 3 - \lambda \end{pmatrix}.$$

$$\det(A - \lambda I) = (3 - \lambda)[(4 - \lambda)(1 - \lambda) - 4] = (3 - \lambda)[4 - 5\lambda + \lambda^2 - 4] = (3 - \lambda)(\lambda - 5)\lambda$$

\therefore eigenvalues of $A = 3, 5, 0$

When $\lambda = 3$

$$\begin{pmatrix} 1 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \begin{cases} x_1 = 2x_2 \\ 2x_1 = -2x_2 \end{cases} \Rightarrow x_1 = x_2 = 0, \quad x_3 = k.$$

$$\text{Eigenvectors: } k \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \vec{x_1}, \quad k \neq 0. \text{ We choose the eigenvector } \vec{x_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

When $\lambda = 0$

$$\begin{pmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x_3 = 0, \quad -2x_1 + x_2 = 0, \quad x_1 = \frac{x_2}{2} = k.$$

$$\text{Eigenvectors: } k \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \vec{x_2}, \quad k \neq 0. \text{ We choose the eigenvector } \vec{x_2} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

When $\lambda = 5$

$$\begin{pmatrix} -1 & -2 & 0 \\ -2 & -4 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x_3 = 0, \frac{-x_1}{2} = x_2 = k.$$

Eigenvectors: $k \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \vec{x}_3, k \neq 0$. We choose the eigenvector $\vec{x}_3 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$.

Observe that $\vec{x}_1 \cdot \vec{x}_2 = 0, \vec{x}_2 \cdot \vec{x}_3 = 0, \vec{x}_1 \cdot \vec{x}_3 = 0$.

(b)

Consider $B = A^5 - 5(A + 3I)^{-1} + 3A^T$

Let $M = [\vec{x}_1, \vec{x}_2, \vec{x}_3] = \begin{pmatrix} 0 & 1 & -2 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Then $AM = MD$, where $D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

Therefore $A = MDM^{-1}$. In addition, we have

$$A^5 = MD^5M^{-1} = M \begin{pmatrix} 3^5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5^5 \end{pmatrix} M^{-1} \Leftrightarrow A^5M = M \begin{pmatrix} 3^5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5^5 \end{pmatrix}. \text{ So } A^5 \text{ has eigenvalues } 3^5, 0, 5^5 \text{ and their}$$

corresponding eigenvectors are the corresponding columns of M . Similarly,

$$A + 3I = MDM^{-1} + M(3I)M^{-1} = M(D + 3I)M^{-1} = M \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{pmatrix} M^{-1},$$

$$(A + 3I)^{-1} = \left[M \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{pmatrix} M^{-1} \right]^{-1} = (M^{-1})^{-1} \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{pmatrix}^{-1} M^{-1} = M \begin{pmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{8} \end{pmatrix} M^{-1}$$

$$5(A + 3I)^{-1} = M \begin{pmatrix} \frac{5}{6} & 0 & 0 \\ 0 & \frac{5}{3} & 0 \\ 0 & 0 & \frac{5}{8} \end{pmatrix} M^{-1}, \quad 3A^T = 3A = M(3D)M^{-1} = M \begin{pmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 15 \end{pmatrix} M^{-1}$$

It follows that

$$B = A^5 - 5(A + 3I)^{-1} + 3A^T$$

$$= M \left[\begin{pmatrix} 3^5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5^5 \end{pmatrix} - \begin{pmatrix} \frac{5}{6} & 0 & 0 \\ 0 & \frac{5}{3} & 0 \\ 0 & 0 & \frac{5}{8} \end{pmatrix} + \begin{pmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 15 \end{pmatrix} \right] M^{-1} = M \begin{pmatrix} 3^5 - \frac{5}{6} + 9 & 0 & 0 \\ 0 & -\frac{5}{3} & 0 \\ 0 & 0 & 5^5 - \frac{5}{8} + 15 \end{pmatrix} M^{-1}$$

So

$$BM = M \begin{pmatrix} 3^5 - \frac{5}{6} + 9 & 0 & 0 \\ 0 & -\frac{5}{3} & 0 \\ 0 & 0 & 5^5 - \frac{5}{8} + 15 \end{pmatrix}$$

Eigenvalues of B are :

$$3^5 - \frac{5}{6} + 9 = 251.1666, -\frac{5}{3} = -1.6666, 5^5 - \frac{5}{8} + 15 = 3139.375$$

Remark: Let $D = P^{-1}AP$, where P is non-singular. We observe that:

$$\begin{aligned} \det(D - \lambda I) &= \det(P^{-1}AP - \lambda I) = \det(P^{-1}AP - \lambda P^{-1}IP) \\ &= \det[P^{-1}(A - \lambda I)P] = \det(P^{-1}) \det(A - \lambda I) \det(P) = \det(A - \lambda I) \end{aligned}$$

Thus, A has the same eigenvalues as D . We also notice that the eigenvalues of D are just the main diagonal entries.

□

□

7. Consider $A = \begin{pmatrix} 8 & 2 & 4 & 12 & 1 \\ 4 & 1 & 2 & 6 & 0.5 \\ 12 & 3 & 6 & 18 & 1.5 \\ 6 & 1.5 & 3 & 9 & 0.75 \\ 18 & 4.5 & 9 & 27 & 2.25 \end{pmatrix}$

- Find rank A .
- Find a column vector \vec{x} and a row vector \vec{y}^T , where $\vec{x}, \vec{y} \in R^5$ such that $A = \vec{x}\vec{y}^T$.
- Show that 0 is an eigenvalue of A and find its corresponding independent eigenvectors.
- Does A have eigenvalues other than 0? If yes, find those eigenvalues and the corresponding independent eigenvectors.
- Find an invertible matrix P and a diagonal matrix D such that $AP = PD$.
- Find all eigenvalues of $A^2 + 3I_5$, where I_5 is the 5×5 unit matrix.

Solution:

(a)

Column 2 of A is a nonzero column and all other columns of A are scalar multiple of it.

$$\text{So rank } A = \text{rank} \begin{pmatrix} 8 & 2 & 4 & 12 & 1 \\ 4 & 1 & 2 & 6 & 0.5 \\ 12 & 3 & 6 & 18 & 1.5 \\ 6 & 1.5 & 3 & 9 & 0.75 \\ 18 & 4.5 & 9 & 27 & 2.25 \end{pmatrix} = 1$$

(b)

$$\begin{pmatrix} 8 & 2 & 4 & 12 & 1 \\ 4 & 1 & 2 & 6 & 0.5 \\ 12 & 3 & 6 & 18 & 1.5 \\ 6 & 1.5 & 3 & 9 & 0.75 \\ 18 & 4.5 & 9 & 27 & 2.25 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1.5 \\ 4.5 \end{pmatrix} (4 \ 1 \ 2 \ 6 \ 0.5)$$

(c)

Since rank $A = 1 < 5$, the equation $\begin{pmatrix} 8 & 2 & 4 & 12 & 1 \\ 4 & 1 & 2 & 6 & 0.5 \\ 12 & 3 & 6 & 18 & 1.5 \\ 6 & 1.5 & 3 & 9 & 0.75 \\ 18 & 4.5 & 9 & 27 & 2.25 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ has nontrivial solutions

and any nontrivial solutions of the homogeneous equations are eigenvectors of A corresponding to the eigenvalue 0.

$$\left(\begin{array}{ccccc|c} 8 & 2 & 4 & 12 & 1 & 0 \\ 4 & 1 & 2 & 6 & 0.5 & 0 \\ 12 & 3 & 6 & 18 & 1.5 & 0 \\ 6 & 1.5 & 3 & 9 & 0.75 & 0 \\ 18 & 4.5 & 9 & 27 & 2.25 & 0 \end{array} \right) \sim \left(\begin{array}{ccccc|c} 8 & 2 & 4 & 12 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

Consider $8x_1 + 2x_2 + 4x_3 + 12x_4 + x_5 = 0$.

Let $x_2 = a, x_3 = b, x_4 = c, x_5 = d$ then $x_1 = -\frac{1}{4}a - \frac{1}{2}b - \frac{3}{2}c - \frac{1}{8}d$. So

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -0.25a - 0.5b - 1.5c - 0.125d \\ a \\ b \\ c \\ d \end{pmatrix} = a \begin{pmatrix} -0.25 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} -0.5 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1.5 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} -0.125 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \text{ Then}$$

$$\begin{pmatrix} -0.25 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -0.5 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1.5 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -0.125 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ are independent eigenvectors of } A \text{ corresponding to the eigenvalue 0.}$$

(d)

Observe that

$$\begin{pmatrix} 8 & 2 & 4 & 12 & 1 \\ 4 & 1 & 2 & 6 & 0.5 \\ 12 & 3 & 6 & 18 & 1.5 \\ 6 & 1.5 & 3 & 9 & 0.75 \\ 18 & 4.5 & 9 & 27 & 2.25 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1.5 \\ 4.5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1.5 \\ 4.5 \end{pmatrix} (4 \ 1 \ 2 \ 6 \ 0.5) \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1.5 \\ 4.5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1.5 \\ 4.5 \end{pmatrix} (26.25) = 26.25 \begin{pmatrix} 2 \\ 1 \\ 3 \\ 1.5 \\ 4.5 \end{pmatrix}.$$

26.25 is the only nonzero eigenvalue of A and one of its corresponding eigenvector is $\begin{pmatrix} 2 \\ 1 \\ 3 \\ 1.5 \\ 4.5 \end{pmatrix}$.

(e)

Let $P = \begin{pmatrix} -0.25 & -0.5 & -1.5 & -0.125 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1.5 \\ 0 & 0 & 0 & 1 & 4.5 \end{pmatrix}$. Observe that columns of P are independent hence P is

invertible.

See that

$$\begin{pmatrix} 8 & 2 & 4 & 12 & 1 \\ 4 & 1 & 2 & 6 & 0.5 \\ 12 & 3 & 6 & 18 & 1.5 \\ 6 & 1.5 & 3 & 9 & 0.75 \\ 18 & 4.5 & 9 & 27 & 2.25 \end{pmatrix} \begin{pmatrix} -0.25 & -0.5 & -1.5 & -0.125 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1.5 \\ 0 & 0 & 0 & 1 & 4.5 \end{pmatrix} \\ = \begin{pmatrix} -0.25 & -0.5 & -1.5 & -0.125 & 2 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1.5 \\ 0 & 0 & 0 & 1 & 4.5 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 26.25 \end{pmatrix}$$

(f)

Since P is invertible, we have

$$AP = PD \Rightarrow A = PDP^{-1} \Rightarrow A^2 + 3I_5 = PDP^{-1}PDP^{-1} + 3I_5 = PD^2P^{-1} + P3I_5P^{-1} = P(D^2 + 3I_5)P^{-1}$$

$$\Rightarrow (A^2 + 3I_5)P = P(D^2 + 3I_5)$$

, where

$$D^2 + 3I_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 26.25 \end{pmatrix}^2 + \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & (26.25)^2 + 3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 692.0625 \end{pmatrix}$$

Observe that P is invertible, so the eigenvalues of $A^2 + 3I_5$ are 3 and 692.0625.

□

8. Suppose $AP = PD$, where A is 3×3 , $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P = \begin{pmatrix} x & -4/5 & 0 \\ y & 3/5 & 0 \\ z & 0 & 1 \end{pmatrix}$ and P is invertible.

(a) Find the characteristic polynomial, $\det(A - \lambda I)$, of A .

(b) Find all eigenvalues of A .

(c) Suppose the first column of P , $(x \ y \ z)^T$ with $x \geq 0$ is a unit vector and orthogonal to both

$$\begin{pmatrix} -4/5 \\ 3/5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \text{ find } \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

(d) Compute $P^T P$ and then show that $P^T = P^{-1}$.

(e) Find A^n , $n \geq 0$.

Solution:

(a)

$$\begin{aligned} \det(A - \lambda I) &= \det(PDP^{-1} - \lambda I) = \det(PDP^{-1} - P\lambda IP^{-1}) = \det[P(D - \lambda I)P^{-1}] \\ &= \det P \det(D - \lambda I) \det P^{-1} = \det P \det(D - \lambda I) (\det P)^{-1} = \det(D - \lambda I) = \det \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & 1 - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{pmatrix} \\ &= -\lambda(1 - \lambda)^2 = -\lambda(\lambda^2 - 2\lambda + 1) = -\lambda^3 + 2\lambda^2 - \lambda \end{aligned}$$

(b)

$\det(A - \lambda I) = -\lambda(1 - \lambda)^2 \Rightarrow \lambda = 0$ or $\lambda = 1$, so 0 and 1 are all the eigenvalues A has.

(c)

$$\begin{aligned} &\begin{cases} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -4/5 \\ 3/5 \\ 0 \end{pmatrix} = 0 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \\ x^2 + y^2 + z^2 = 1 \\ x \geq 0 \end{cases} \Rightarrow \begin{cases} -\frac{4}{5}x + \frac{3}{5}y = 0 \\ z = 0 \\ x^2 + y^2 + z^2 = 1 \\ x \geq 0 \end{cases} \Rightarrow \begin{cases} y = \frac{4}{3}x \\ z = 0 \\ x^2 + y^2 + z^2 = 1 \\ x \geq 0 \end{cases} \Rightarrow \begin{cases} x^2 + \frac{16}{9}x^2 = 1 \\ x \geq 0 \end{cases} \Rightarrow \begin{cases} x^2 = \frac{9}{25} \\ x \geq 0 \end{cases} \Rightarrow x = \frac{3}{5} \\ \Rightarrow \begin{cases} x = \frac{3}{5} \\ y = \frac{4}{5} \\ z = 0 \end{cases} \end{aligned}$$

(d)

$$\text{Observe that } P^T P = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $P^T = P I = P^T (P P^{-1}) = (P^T P) P^{-1} = I P^{-1} = P^{-1}$.

(e)

$$\begin{aligned}
A &= \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}^T \\
&= \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\Rightarrow A^n &= \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^n \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1^n & 0 \\ 0 & 0 & 1^n \end{pmatrix} \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{4}{5} & 0 \\ 0 & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{16}{25} & -\frac{12}{25} & 0 \\ -\frac{12}{25} & \frac{9}{25} & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

□

9. Let A be a 3×3 matrix with eigenvalues 1, 2, 3.

- (a) (i) Find the characteristic polynomial $|A - \lambda I|$ of A .
- (ii) Determine $|A|$.
- (iii) Is A invertible, why?

If A^{-1} of A exists, the adjoint $\text{adj } A$ of A is defined as the matrix $\text{adj } A \square |A| A^{-1}$.

Suppose $AM = M \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, where M is invertible.

- (b) Find the eigenvalues of $\text{adj } A$.

Let $M = \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $AM = M \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

- (c) (i) Compute MM^T and then find M^{-1} .
- (ii) Find A^{-1} .

(iii) Find $\text{adj } A$.

Solution:

(a)

(i)

$$|A - \lambda I| = -(\lambda - 1)(\lambda - 2)(\lambda - 3).$$

(ii)

$$|A - \lambda I|_{\lambda=0} = |A| = -(-1)(-2)(-3) = 6.$$

(iii)

Since $|A| = 6 \neq 0$, A is invertible.

(b)

$$AM = M \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \Rightarrow A = M \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} M^{-1}$$

$$\Rightarrow A^{-1} = (M^{-1})^{-1} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{-1} M^{-1} = M \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} M^{-1}$$

$$\Rightarrow \text{adj } A = |A| A^{-1} = \underset{|A|=6}{6} M \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} M^{-1} = M \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix} M^{-1}$$

Since M is invertible, the eigenvalues of $\text{adj } A$ are 3, 6, 2.

(c)

(i)

$$MM^T = \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \text{ So } M^{-1} = M^T = \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(ii)

$$\begin{aligned} A^{-1} &= M \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} M^{-1} = \begin{pmatrix} 3/5 & -4/5 & 0 \\ 4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3/10 & -4/5 & 0 \\ 4/10 & 3/5 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 41/50 & -12/50 & 0 \\ -12/50 & 34/50 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} \end{aligned}$$

(iii)

$$\text{adj } A = |A| A^{-1} = 6 \begin{pmatrix} 41/50 & -12/50 & 0 \\ -12/50 & 34/50 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} = \begin{pmatrix} 123/25 & -36/25 & 0 \\ -36/25 & 102/25 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

□

-End-