CITY UNIVERSITY OF HONG KONG

Course code and title : MA1200 Calculus and Basic Linear Algebra I

Session : Semester B, 2017/2018

Time allowed : Three hours

This paper has **SIX** pages (including this cover page).

A brief table of derivatives is attached on pages 5 and 6.

Instructions to candidates:

1. This paper has **EIGHT** questions.
2. Attempt **ALL** questions.
3. Each question carries 13 marks.

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*This is a **closed-book** examination.*

Candidates are allowed to use the following materials/aids:

Non-programmable calculators

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorised materials or aids are found on them.

NOT TO BE TAKEN AWAY

Question 1

Differentiate with respect to x :

(a) $(x^2+3)^5+\dfrac{1}{x+1}$; (2 marks)

(b) $\sqrt{\dfrac{1+\sin x}{1-\sin x}}$; (2 marks)

(c) $\log_e\left(\dfrac{1+x^2}{1-x^2}\right)$; (3 marks)

(d) $\tan^{-1}\left(\dfrac{4\sin x}{5+3\cos x}\right)$; (3 marks)

(e) $\sqrt{\cosh\sqrt{x}}$ for $x \geq 0$. (3 marks)

Question 2

(a) Let $f(x)=x|x|$ for $x \in \mathbf{R}$. Is $f(x)$ differentiable at $x=0$? Give your reason.

(Hint: $|x|=\begin{cases} x \text{ ,if } x \geq 0 \\ -x \text{ ,if } x<0 \end{cases}$.) (6 marks)

(b) Find $\dfrac{dy}{dx}$ when

(i) $y=\dfrac{x(1+x^2)^3e^{-x^2}}{\sqrt{1+x^3}}$, (4 marks)

(ii) $4y^2-3xy+6x+2y-20=0$. (3 marks)

You need not simplify your answers.

Question 3

Evaluate the following limits:

(a) $\lim\limits_{x\to\infty}\dfrac{3x^2-x+5}{4x^3-x^2+7}$; (4 marks)

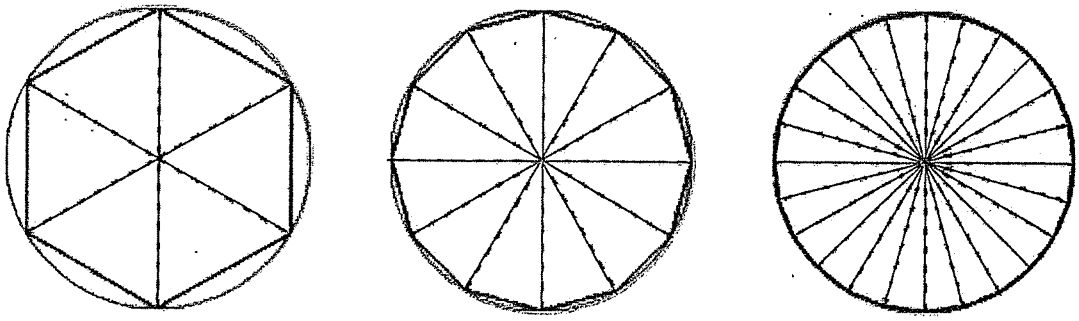
(b) $\lim\limits_{x\to\frac{1}{2}}\dfrac{\cos^2(\pi x)}{2ex-e^{2x}}$; (4 marks)

(c) $\lim\limits_{x\to 0}(x+e^x)^{x^{-1}}$; (5 marks)

Question 4

A former Chinese Mathematician Tsu Chung Chi (429-500 A.D.) proposed a method to compute an approximation to the value of π .

Given a unit circle, he calculated the areas of inscribed regular 6-sided, 12-sided, 24-sided, ... polygons as shown in Figure 1, thus obtaining lower bound for the area of the unit circle ($=\pi$ sq.units).



regular 6-sided polygon regular 12-sided polygon regular 24-sided polygon

Figure 1

Let A_n denotes the area of inscribed regular n-sided polygon as shown in Figure 1.

(a) Find the values of A_6 , A_{12} and A_{24} , correct your answers to 6 decimal places. (10 marks)

(b) Find the value of $\dfrac{64A_{24}-20A_{12}+A_6}{45}$, correct your answer to 5 decimal places. (3 marks)

Question 5

Show that the curve $y=\dfrac{x}{2}+1+\dfrac{1}{2(x+1)}$ has a local minimum at $x=0$.

Sketch the graph of the curve, indicating its particular features such as asymptotes and local extremal points, if any. (13 marks)

Question 6

(a) Express $\dfrac{x^4+x^2+3}{x^4+x}$ in partial fractions.

(Hint: $a^3+b^3=(a+b)(a^2-ab+b^2)$.) (6 marks)

(b) If $y=\log_e x$, find y', y'', y''' and then conjecture the formula for $y^{(m)}$, $m\in\mathbf{N}$.

Hence, or otherwise, find the formula for $(x^2\log_e x)^{(n)}$, $n=3,4,5,\ldots$

(Hint: Leibnitz' rule: For any functions u and v whose derivatives up to the nth order exist, $(uv)^{(n)}={}_nC_0u^{(n)}v^{(0)}+{}_nC_1u^{(n-1)}v^{(1)}+{}_nC_2u^{(n-2)}v^{(2)}+\ldots+{}_nC_ru^{(n-r)}v^{(r)}+\ldots+{}_nC_nu^{(0)}v^{(n)}$, where ${}_nC_r=\dfrac{n!}{(n-r)!r!}$, $u^{(0)}=u$, $v^{(0)}=v$ and $u^{(r)}$, $v^{(r)}$ are the rth derivatives of u and v, respectively, for $r=1,2,3,\ldots,n$.) (7 marks)

Question 7

(a) Show that the cubic equation
$5x^3 + 50x^2 - 232x + 131 = 0$ ------ (*)
has a real root in $(0,1)$. (5 marks)

(b) Use any method or combination of methods to compute the roots of equation (*). Correct your answers to 6 decimal places.
(Hint: Newton iterative scheme for the solution of $f(x) = 0$ is $x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}$, for $k = 0, 1, 2, \ldots$)
(8 marks)

Question 8

(a) Show that if $y = \left(\dfrac{2}{1+e^x}\right)^{\frac{1}{2}}$ then $2(1+e^{-x})\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = 0$.

By repeated differentiation of this result, find the values of $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$, $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$ at $x = 0$ and hence obtain the Maclaurin series of $\left(\dfrac{2}{1+e^x}\right)^{\frac{1}{2}}$ in ascending powers of x as far as the term in x^3. (7 marks)

(b) Define $T_n(x) = \cos(n \cos^{-1} x)$, $x \in [-1, 1]$, for $n = 0, 1, 2, \ldots$.

Show that

(i) $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ for $n \geq 1$, (3 marks)
(Hint: $\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$, put $\theta = \cos^{-1} x$, then $\cos\theta = x$.)

(ii) $(1-x^2)\dfrac{\mathrm{d}^2T_n(x)}{\mathrm{d}x^2} - x\dfrac{\mathrm{d}T_n(x)}{\mathrm{d}x} + n^2T_n(x) = 0$. (3 marks)

Short Table of Derivatives of $y = f(u)$ with respect to x, where u is a function of x

Functions, $y = f(u)$	**Derivative of y with respect to x**
$y = c$, where c is a constant.	$\frac{dy}{dx} = 0$
$y = cu$, where c is a constant.	$\frac{dy}{dx} = c\frac{du}{dx}$
$y = u^p$, where p is a constant.	$\frac{dy}{dx} = pu^{p-1}\frac{du}{dx}$
$y = u + v$	$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = f(u)$, where u is a function of x.	$\frac{dy}{dx} = \frac{d f(u)}{du} \cdot \frac{du}{dx}$, the chain rule
$y = \log_a u$, $a > 0$.	$\frac{dy}{dx} = \frac{1}{u}\log_a e\frac{du}{dx}$
$y = a^u$, $a > 0$.	$\frac{dy}{dx} = a^u \log_e a\frac{du}{dx}$
$y = e^u$	$\frac{dy}{dx} = e^u\frac{du}{dx}$
$y = u^v$	$\frac{dy}{dx} = vu^{v-1}\frac{du}{dx} + u^v \log_e u\frac{dv}{dx}$
$y = \sin u$	$\frac{dy}{dx} = \cos u\frac{du}{dx}$
$y = \cos u$	$\frac{dy}{dx} = -\sin u\frac{du}{dx}$
$y = \tan u$	$\frac{dy}{dx} = \sec^2 u\frac{du}{dx}$
$y = \cot u$	$\frac{dy}{dx} = -\mathrm{cosec}^2 u\frac{du}{dx}$
$y = \sec u$	$\frac{dy}{dx} = \sec u \tan u\frac{du}{dx}$
$y = \mathrm{cosec}\, u$	$\frac{dy}{dx} = -\mathrm{cosec}\, u \cot u\frac{du}{dx}$
$y = \sin^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y = \cos^{-1} u$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y = \tan^{-1} u$	$\frac{dy}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$

Functions, $y = f(u)$	**Derivative of** y **with respect to** x
$y = \cot^{-1} u$	$\frac{dy}{dx} = -\frac{1}{1+u^2}\frac{du}{dx}$
$y = \sec^{-1} u$	$\frac{dy}{dx} = \frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$
$y = \operatorname{cosec}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$
$y = \sinh u$	$\frac{dy}{dx} = \cosh u\frac{du}{dx}$
$y = \cosh u$	$\frac{dy}{dx} = \sinh u\frac{du}{dx}$
$y = \tanh u$	$\frac{dy}{dx} = \operatorname{sech}^2 u\frac{du}{dx}$
$y = \coth u$	$\frac{dy}{dx} = -\operatorname{cosech}^2 u\frac{du}{dx}$
$y = \operatorname{sech} u$	$\frac{dy}{dx} = -\operatorname{sech} u \tanh u\frac{du}{dx}$
$y = \operatorname{cosech} u$	$\frac{dy}{dx} = -\operatorname{cosech} u \coth u\frac{du}{dx}$
$y = \sinh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$
$y = \cosh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$
$y = \tanh^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$
$y = \coth^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$
$y = \operatorname{sech}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$
$y = \operatorname{cosech}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{\lvert u\rvert\sqrt{u^2+1}}\frac{du}{dx}$

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