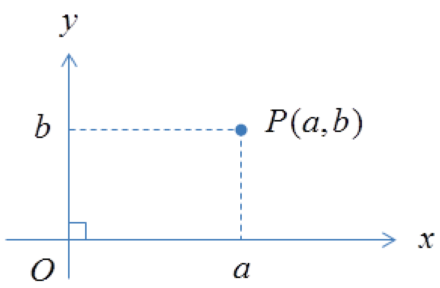
MA1200 CALCULUS AND BASIC LINEAR ALGEBRA

LECTURE: CG1

Chapter 1
Coordinate Geometry and Conic Sections

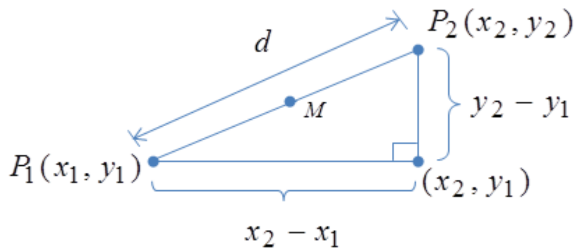
Rectangular / Cartesian coordinate system

In the ***rectangular or Cartesian coordinate system***, a point P in a 2-dimensional plane is represented by an ordered pair (a, b), where a and b are the ***signed distances*** from the point P to a fixed vertical line (the y-axis) and a fixed horizontal line (the x-axis), respectively. The two axes are perpendicular to each other. The point where the two axes cross each other is called the origin, denoted by $O(0,0)$. The values a and b are called the "x-coordinate" and "y-coordinate" of P, respectively.



Two points in a 2-D plane

Given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ written in Cartesian coordinate form. We use a straight line to join the two points and form the line P_1P_2.



- The ***distance*** d between P_1 and P_2 is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (by Pythagoras' Theorem).
- The ***midpoint*** M of the straight line P_1P_2 has Cartesian coordinates $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.
- The ***slope*** m of the straight line P_1P_2 is $m = \frac{y_2-y_1}{x_2-x_1}$ (or equivalently, $m = \frac{y_1-y_2}{x_1-x_2}$).

$m > 0$	$m < 0$	$m = 0$	m is undefined
(Increasing line)	(Decreasing line)	(Horizontal line)	(Vertical line)

Equation of a straight line

Here are four special forms of writing the equation of a straight line:

- ***Point-slope form:*** $\frac{y-y_1}{x-x_1} = m$, where m is the slope of the straight line, and $P(x_1, y_1)$ is a point lying on the straight line.

- ***Two points form:*** $\boxed{\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}}$, where the straight line passes through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

- ***Slope-intercept form:*** $\boxed{y = mx + c}$, where m is the slope of the straight line, and c is the y-intercept of the line.

- ***General form of a straight line:*** $\boxed{Ax + By + C = 0}$, where A, B, C are constants (A and B cannot both equal to zero).

Two special cases of straight lines:

- *Horizontal line*: Its equation is $\boxed{y = k}$, where k is a constant.
  E.g. The equation of the x-axis is $y = 0$.
- *Vertical line*: Its equation is $\boxed{x = k}$, where k is a constant.
  E.g. The equation of the y-axis is $x = 0$.

<u>Two straight lines in a 2-D plane</u>

Given two straight lines L_1 and L_2 with slopes m_1 and m_2, respectively.

We have the following results:

- ***Parallel lines:***

  L_1 is parallel to L_2 (written as $L_1 // L_2$) if and only if $\boxed{m_1 = m_2}$.

- ***Perpendicular lines:***

  L_1 is perpendicular to L_2 (written as $L_1 \perp L_2$) if and only if $\boxed{m_1 m_2 = -1}$ (i.e. $\boxed{m_2 = \frac{-1}{m_1}}$ provided $m_1 \neq 0$).

<u>Example 1</u>

Find the equation of straight line which satisfies each of the following conditions:

(a) joining $P(3,4)$ and $Q(1,-1)$;

(b) perpendicular to the line $L_2: 3x - 2y + 5 = 0$ and cuts the x-axis at $(5,0)$.

<u>*Solution:*</u>

(a) The slope of straight line PQ is $\frac{4-(-1)}{3-1} = \frac{5}{2}$.

$\therefore$ The equation of the required line is given by:

$$\frac{y-4}{x-3} = \frac{5}{2} \Rightarrow 2(y-4) = 5(x-3) \Rightarrow 5x - 2y - 7 = 0$$

<u>Alternatively</u>, we can calculate the equation of the line by using the coordinates of Q instead of P:

$$\frac{y-(-1)}{x-1} = \frac{5}{2} \Rightarrow 2(y+1) = 5(x-1) \Rightarrow 5x - 2y - 7 = 0 \checkmark$$

(b) Let L_1 be the required line. To find the equation of L_1, we should find the slope of L_1 first. Since the required line L_1 is perpendicular to the line L_2, we have

(slope of L_1) × (slope of L_2) $= -1$.

Equation of L_2 is $3x - 2y + 5 = 0 \quad \Rightarrow \quad y = \frac{3}{2}x + \frac{5}{2}$

$\therefore$ Slope of $L_2 = \frac{3}{2}$.

$\therefore$ Slope of $L_1 = \frac{-1}{\text{slope of } L_2} = \frac{-1}{\frac{3}{2}} = -\frac{2}{3}$.

Since L_1 passes through the point $(5, 0)$, the equation of L_1 is given by:

$$\frac{y-0}{x-5} = -\frac{2}{3} \Rightarrow -3y = 2(x-5) \Rightarrow 2x + 3y - 10 = 0.$$

Exercise:

1) Find the coordinates of the foot of the perpendicular from $P(6, 1)$ to the straight line joining $Q(-1, 2)$ and $R(3, -4)$.

2) Find the shortest distance from the origin $O(0, 0)$ to the line joining $A(3, -4)$ to the point of intersection of the line $x - 3y + 6 = 0$ with the y-axis.

Polar coordinate system

The ***polar coordinate system*** is a two-dimensional coordinate system in which each point P in a plane is determined by two quantities:

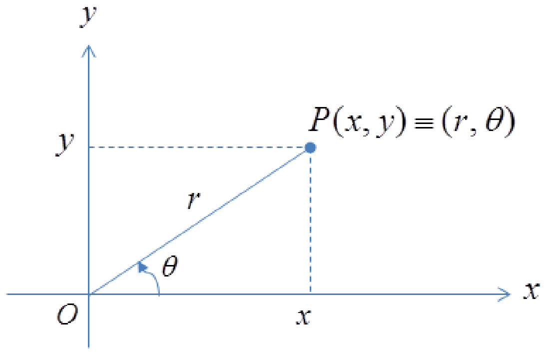
- r = the distance from the pole (i.e. the origin O) to the point P; and
- θ = the angle measured from the polar axis (i.e. the positive x-axis) to the ray OP.

  ($\theta > 0$ for anti-clockwise direction; $\theta < 0$ for clockwise direction.)

We use the notation (r, θ) to describe the location of a point (in 2-D plane) in Polar coordinates.

By using trigonometry and Pythagoras' Theorem, the relations between Polar coordinates (r, θ) and Cartesian coordinates (x, y) are derived as below:

- $x = r\cos\theta$
- $y = r\sin\theta$
- $r^2 = x^2 + y^2$
- $\tan\theta = \frac{y}{x}$



Remark:

Each point in the Cartesian coordinate system is represented by a **unique** pair of x- and y-coordinates, (x, y). However, if we use polar coordinates while we do not set restrictions on the angle θ, there will be more than one pair of Polar coordinates (r, θ) which represent the same point.

For example, the Polar coordinates $(2, 30°)$, $(2, 30° + 360°) = (2, 390°)$, $(2, 30° + 360° \times 2) = (2, 750°)$, $(2, 30° - 360°) = (2, -330°)$, etc., all represent the point with Cartesian coordinates $(\sqrt{3}, 1)$. In other words, the point $P(\sqrt{3}, 1)$ in Cartesian coordinate system can be represented by $(2, 30° + 360°\, n)$ in Polar coordinate system, where n is any integer.

Therefore, we would like to restrict the range of r and θ so that the Polar coordinates (r, θ) of a point would be unique: $\boxed{r \geq 0}$ and $\boxed{-180° < \theta \leq 180°}$.

Conversion from Polar coordinates (r, θ) to Cartesian coordinates (x, y)

We use $\boxed{x = r\cos\theta}$ and $\boxed{y = r\sin\theta}$.

Example 2

Convert each of the following points from polar coordinates to Cartesian coordinates:

$$A(3, 45°), \quad B(7, 120°), \quad C(6, -60°), \quad D(2, 420°).$$

Solution:

For point $A(3, 45°)$, its Cartesian coordinates are given by

$$(3\cos 45°,\ 3\sin 45°) = \left(3 \times \frac{\sqrt{2}}{2},\ 3 \times \frac{\sqrt{2}}{2}\right) = \left(\frac{3\sqrt{2}}{2},\ \frac{3\sqrt{2}}{2}\right).$$

For point $B(7, 120°)$, its Cartesian coordinates are given by

$$(7\cos 120°,\ 7\sin 120°) = \left(7 \times \frac{-1}{2},\ 7 \times \frac{\sqrt{3}}{2}\right) = \left(-\frac{7}{2},\ \frac{7\sqrt{3}}{2}\right).$$

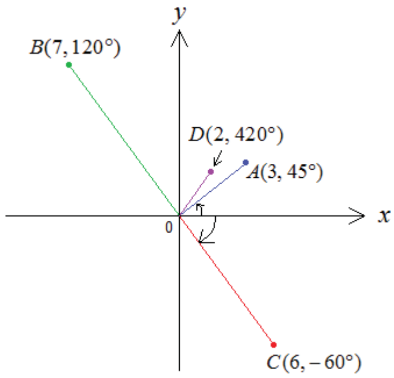
For point $C(6, -60°)$, its Cartesian coordinates are given by

$$(6\cos(-60°),\ 6\sin(-60°)) = \left(6 \times \frac{1}{2},\ 6 \times \frac{-\sqrt{3}}{2}\right) = \left(3,\ -3\sqrt{3}\right).$$

For point $D(2, 420°)$, its Cartesian coordinates are given by

$$(2\cos 420°,\ 2\sin 420°) = \left(2 \times \frac{1}{2},\ 2 \times \frac{\sqrt{3}}{2}\right) = \left(1,\ \sqrt{3}\right).$$

Note: For point D in the previous example, its polar coordinates $(2, 420°)$ are equivalent to $(2, 420° - 360°) = (2, 60°)$.



Conversion from Cartesian coordinates (x, y) to Polar coordinates (r, θ)

To find r, we use $r = \sqrt{x^2 + y^2}$, since $r \geq 0$.

The relation between θ, x and y is given by $\tan\theta = \frac{y}{x}$. However, special care must be taken when finding θ (where $-180° < \theta \leq 180°$). Please refer to the following example.

Example 3

Convert each of the following points from Cartesian coordinates to polar coordinates:

(a) $A(3,\ \sqrt{3})$, (b) $B(-1,\ \sqrt{3})$, (c) $C(-2,-2)$,

(b) $D(\sqrt{2},-\sqrt{2})$, (e) $E(0,-2)$

Solution: (*Try to locate the points in the xy-plane first. This will help you to find θ.*)

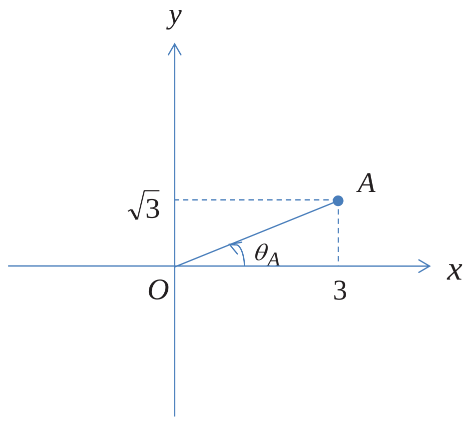
(a) $r_A = \sqrt{3^2+(\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3}$

Notice that $A(3,\ \sqrt{3})$ lies in **Quadrant I**

so that $0° < \theta_A < 90°$.

$\tan\theta_A = \frac{\sqrt{3}}{3} \Rightarrow \theta_A = 30°$

$\therefore$ In Polar coordinates, $A = (2\sqrt{3},\ 30°)$



(b) $r_B = \sqrt{(-1)^2+(\sqrt{3})^2} = \sqrt{4} = 2$

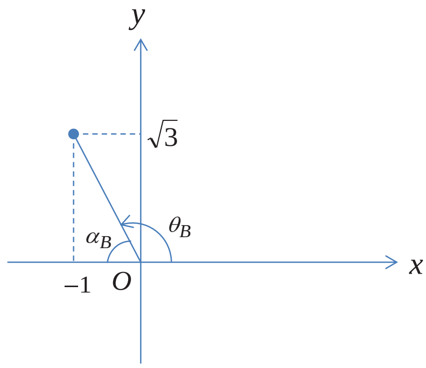
Notice that $B(-1,\ \sqrt{3})$ lies in **Quadrant II**

so that $90° < \theta_B < 180°$.

$\tan\alpha_B = \frac{\sqrt{3}}{1} \Rightarrow \alpha_B = 60°$

$\therefore \theta_B = 180° - 60° = 120°$

$\therefore$ In Polar coordinates, $B = (2,\ 120°)$



(c) $r_C = \sqrt{(-2)^2+(-2)^2} = \sqrt{8} = 2\sqrt{2}$

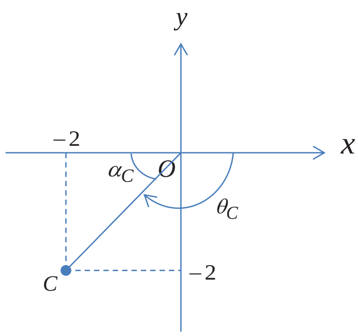
Notice that $C(-2,-2)$ lies in **Quadrant III**

so that $-180° < \theta_C < -90°$.

$\tan\alpha_C = \frac{2}{2} \Rightarrow \alpha_C = 45°$

$\therefore \theta_C = -(180° - 45°) = -135°$

$\therefore$ In Polar coordinates, $C = (2\sqrt{2},\ -135°)$



(d) $r_D = \sqrt{(\sqrt{2})^2 + (-\sqrt{2})^2} = \sqrt{4} = 2$

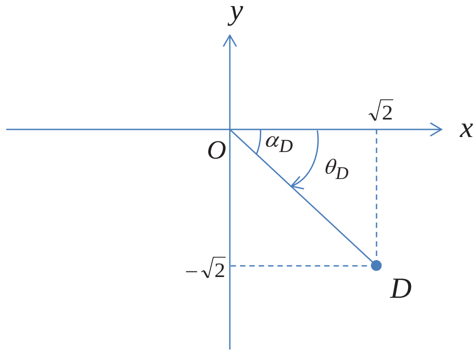
Notice that $D(\sqrt{2}, -\sqrt{2})$ lies in **Quadrant IV**

so that $-90° < \theta_D < 0°$.

$$\tan \alpha_D = \frac{\sqrt{2}}{\sqrt{2}} \Rightarrow \alpha_D = 45°$$

$\therefore \theta_D = -\alpha_D = -45°$

$\therefore$ In Polar coordinates, $D = (2,\ -45°)$

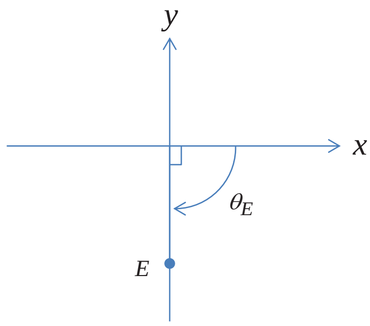


(e) $r_E = \sqrt{(0)^2 + (-2)^2} = \sqrt{4} = 2$

Notice that $E(0, -2)$ lies on the **negative y-axis**

so that $\theta_E = -90°$.

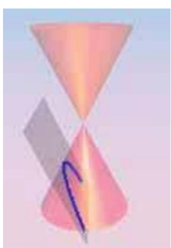
$\therefore$ In Polar coordinates, $E = (2,\ -90°)$



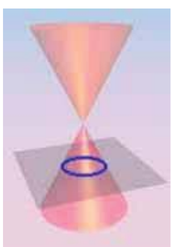
Conic Sections

Conic sections are curves that result from the intersection of a right circular cone with a plane.

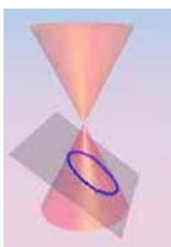
We will study the following four types of conic sections in this chapter.



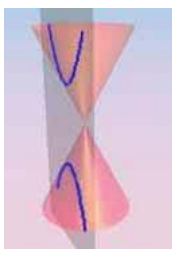
Parabola



Circle



Ellipse



Hyperbola

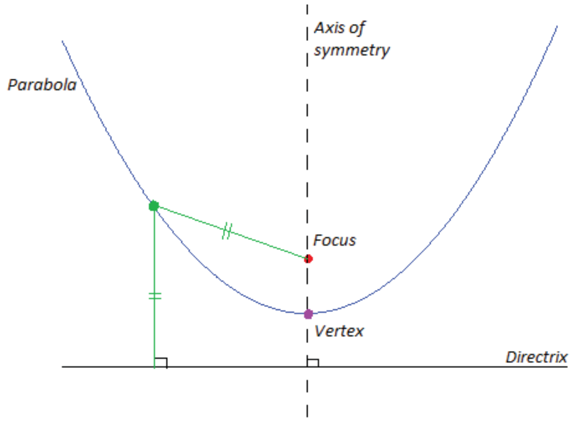
Other cases which are resulted from intersection of a plane with a right-circular cone include:

(i) a single point, (ii) two intersecting lines, (iii) one line, or (iv) no graph at all.

These are called ***degenerate*** cases.

Conic Section Type 1: Parabola

Definition: A ***parabola*** is the set of all points in a plane that are equidistant from a fixed line (the ***directrix***) and a fixed point (the ***focus***).



- The line through the focus perpendicular to the directrix is called the ***axis of symmetry.*** The parabola is symmetric about the axis of symmetry.
- The ***vertex*** of the parabola is the point where the parabola crosses its axis of symmetry. It is on the axis halfway between the focus and the directrix.

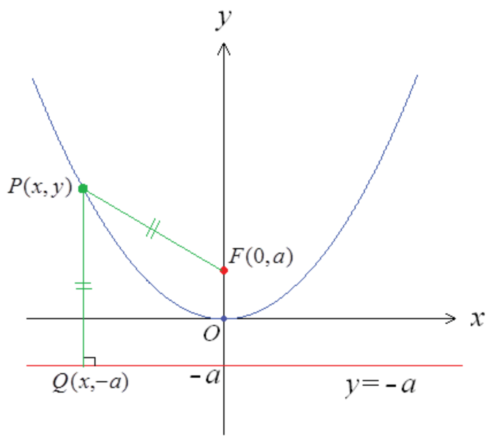
Equation of parabola

Consider the parabola with focus at the point $F(0, a)$ and the directrix $y = -a$, where $a > 0$. The equation of this parabola is given by

$$\boxed{x^2 = 4ay}.$$

Properties of the parabola $x^2 = 4ay$ (for $a > 0$):

- Its axis of symmetry is the y-axis.
- Its vertex is at the origin $O(0,0)$.
- It lies above the x-axis.
- As the value of a increases, the parabola opens wider.



<u>Proof of the equation of parabola ($x^2 = 4ay$):</u>

Let $P(x, y)$ be any point on the parabola. According to the definition of parabola, the distance from P to $F(0, a)$ is equal to the distance from P to the nearest point $Q(x, -a)$ on the directrix,

i.e. $$PF = PQ \Rightarrow \sqrt{(x-0)^2 + (y-a)^2} = \sqrt{(x-x)^2 + [y-(-a)]^2}$$
$$\Rightarrow \sqrt{x^2 + (y-a)^2} = \sqrt{(y+a)^2}$$

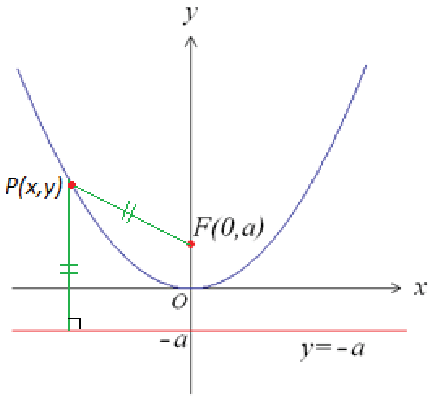
Squaring both sides gives

$$x^2 + (y-a)^2 = (y+a)^2$$
$$\Rightarrow x^2 + y^2 - 2ay + a^2 = y^2 + 2ay + a^2$$
$$\Rightarrow x^2 = 4ay \qquad \square$$

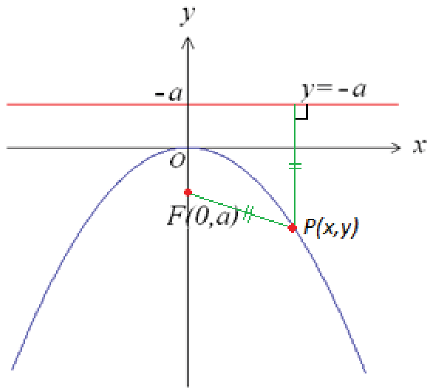
Similarly, we can obtain the standard equations for parabolae with vertices at the origin and foci at (i) $(0, a)$ for $a < 0$; (ii) $(a, 0)$ for $a > 0$; and (iii) $(a, 0)$ for $a < 0$. The results are summarized as follows:

<u>Type I</u>: $\boxed{x^2 = 4ay}$: (∪ or ∩ shape)

<u>Case 1:</u> $a > 0$



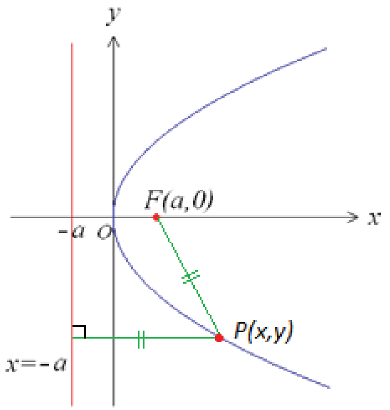
<u>Case 2:</u> $a < 0$



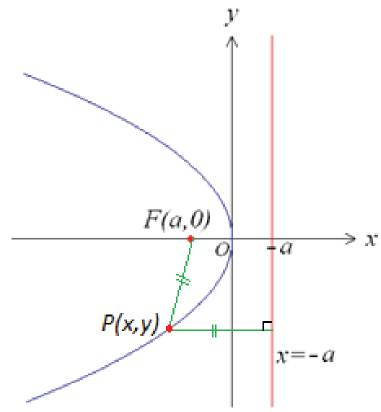
Equation	$x^2 = 4ay$
Vertex	$(0,0)$
Axis of symmetry	y-axis, i.e. $x = 0$
Focus	$F(0, a)$
Directrix	$y = -a$

Type II: $\boxed{y^2 = 4ax}$: ($\subset$ or $\supset$ shape)

Case 1: $a > 0$



Case 2: $a < 0$



Equation	$y^2 = 4ax$
Vertex	$(0,0)$
Axis of symmetry	x-axis, i.e. $y = 0$
Focus	$F(a,0)$
Directrix	$x = -a$

Example 4

For the parabola $x^2 = -8y$, find the coordinates of the focus and the equation of the directrix. Sketch its graph.

Solution:

The equation $x^2 = -8y$ can be written as $x^2 = 4(-2)y$. It is of the form $x^2 = 4ay$, where $a = -2$ (< 0), so the parabola opens downward with vertex at the origin.

The focus of the parabola is at $(0, a)$ i.e. $(0, -2)$.

The directrix is the horizontal line $y = -a$, i.e. $y = 2$.

Sketch:

Example 5

The vertex and the axis of symmetry of a parabola are the origin and the x-axis respectively. If the parabola passes through the point $(6, 3)$, find its equation and sketch its graph.

Solution:

Since the parabola is symmetric about the x-axis and has vertex at the origin, the equation of the parabola is of the form $y^2 = 4ax$.

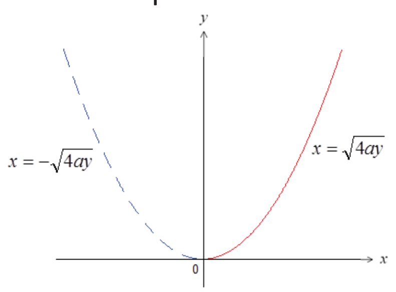
The parabola passes through the point $(6, 3)$, we substitute $x = 6$ and $y = 3$ into the above equation to get $3^2 = 4a(6) \Rightarrow a = \frac{3}{8}$.

Hence the equation of the parabola is $y^2 = 4\left(\frac{3}{8}\right)x$, i.e. $y^2 = \frac{3}{2}x$.

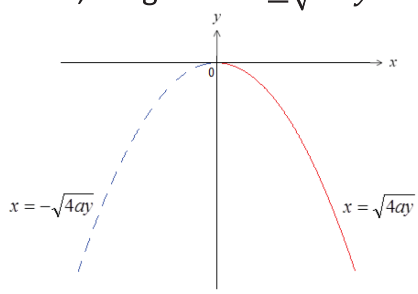
Sketch:

Remarks on the parabola $x^2 = 4ay$ or $y^2 = 4ax$ $(a \neq 0)$:

- Consider the parabola $x^2 = 4ay$. Solving for x, we get $x = \pm\sqrt{4ay}$.

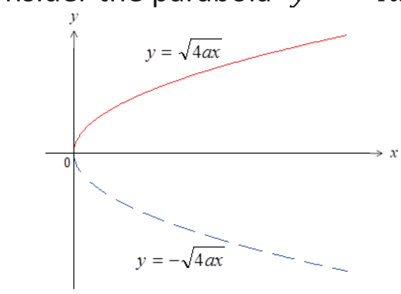


(for $a > 0$)

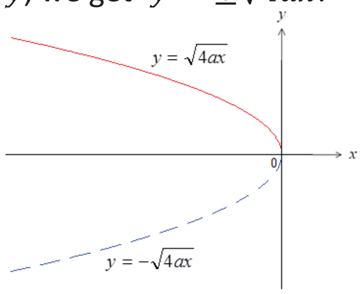


(for $a < 0$)

- Consider the parabola $y^2 = 4ax$. Solving for y, we get $y = \pm\sqrt{4ax}$.



(for $a > 0$)



(for $a < 0$)

General results: *Translation of a point & Translation of the graph of a function*

- When a point $P(x, y)$ is moved (or translated) **h units to the right** and **k units upward**, its new coordinates become $P'(x+h, y+k)$.

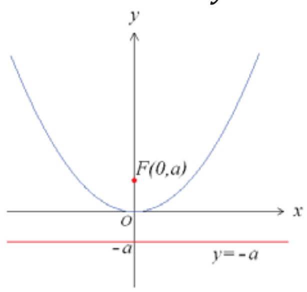
- When the graph of a function $y = f(x)$ is translated **h units to the right** and **k units upward**, the new function becomes $y - k = f(x - h)$. That is, we replace "x" with "$x - h$", and "y" with "$y - k$".

- Similarly, if the graph of a function is translated **p units to the left (i.e. $-p$ units to the right)** and **q units downward (i.e. $-q$ units upward)**, the new function becomes $y - (-q) = f(x - (-p))$, i.e. $y + q = f(x + p)$. That is, we replace "x" with "$x + p$", and "y" with "$y + q$".

Translation of parabolae

Type I parabola: $x^2 = 4ay$

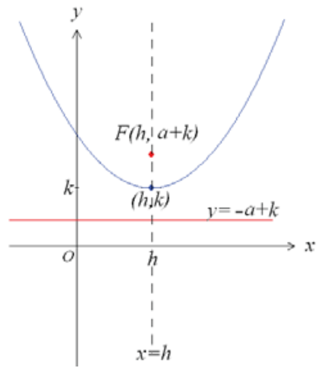
E.g. $(a > 0)$



Translate $x^2 = 4ay$

h units to the right and

k units upward

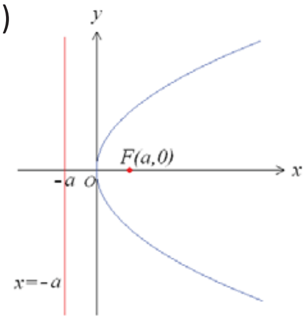


Equation	$x^2 = 4ay$	Translate $x^2 = 4ay$ h units to the right and k units upward	$(x-h)^2 = 4a(y-k)$
Vertex	$(0,0)$		(h,k)
Axis of symmetry	$x = 0$		$x = h$
Focus	$F(0,a)$		$F(h, a+k)$
Directrix	$y = -a$		$y = -a + k$

Remark: The results for the parabola $x^2 = 4ay$ $(a < 0)$ are the same as above except that it is an " ∩ "-shaped curve.

Type II parabola: $y^2 = 4ax$

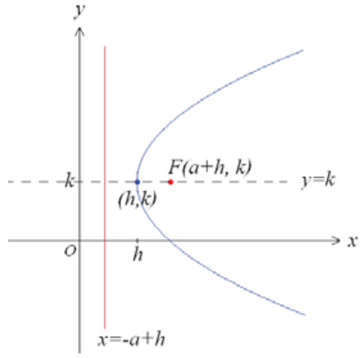
E.g. $(a > 0)$



Translate $y^2 = 4ax$

h units to the right

and k units upward



Equation	$y^2 = 4ax$	Translate	$(y-k)^2 = 4a(x-h)$
Vertex	$(0,0)$	$y^2 = 4ax$	(h,k)
Axis of symmetry	$y = 0$	h units to the right	$y = k$
Focus	$F(a,0)$	and k units upward	$F(a+h,k)$
Directrix	$x = -a$	$\Longrightarrow$	$x = -a+h$

Remark: The results for the parabola $y^2 = 4ax$ $(a < 0)$ are the same as above except that it is a " $\supset$ "-shaped curve.

Additional exercises on parabola:

1. Consider the parabola with equation $(x+2)^2 = -y + 9$.
   (a) Rewrite the equation into the standard form of parabola.
   (b) Find the coordinates of its vertex and focus, and the equation of its directrix.
   (c) Sketch its graph with the vertex, focus and directrix clearly shown.

2. Find the equation of the parabola with focus at the point $(-2, 3)$ and directrix $x = -6$. Express your answer to its standard form. Sketch the graph of the parabola, with the coordinates of its vertex clearly shown.

Quadratic equation $y = ax^2 + bx + c$:

Consider the parabola with equation $(x-h)^2 = 4p(y-k)$, for $p \neq 0$. Clearly, this is a "∪"-shaped or "∩"-shaped curve, depending on the sign of p. If we rearrange this equation, we have

$$(x-h)^2 = 4p(y-k)$$
$$\Longrightarrow x^2 - 2xh + h^2 = 4py - 4pk$$
$$\Longrightarrow 4py = x^2 - 2xh + h^2 + 4pk$$
$$\Longrightarrow y = \underbrace{\frac{1}{4p}}_{=a} x^2 + \underbrace{\left(-\frac{h}{2p}\right)}_{=b} x + \underbrace{\frac{h^2+4pk}{4p}}_{=c}$$

Putting $a = \frac{1}{4p}$, $b = -\frac{h}{2p}$ and $c = \frac{h^2+4pk}{4p}$, we have the **quadratic equation** $y = ax^2 + bx + c$, where $a(\neq 0), b, c$ are constants.

Properties of the graph of $y = ax^2 + bx + c$:

- It is a parabola which **opens upward** when $a > 0$; and **opens downward** when $a < 0$.
- Intersection of parabola with x-axis ($y = 0$):

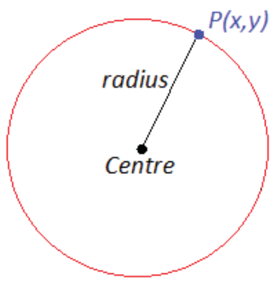
  Solving $ax^2 + bx + c = 0$, we have $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ (called the **quadratic equation formula**), which gives the x-coordinate(s) of the point(s) where the parabola touches or cuts the x-axis. Depending on the sign of $b^2 - 4ac$, there are 3 different cases:

  - It intersects the x-axis at 2 distinct points iff (if and only if) $b^2 - 4ac > 0$;
  - It touches the x-axis at 1 point if and only iff $b^2 - 4ac = 0$;
  - It does not cut the x-axis iff $b^2 - 4ac < 0$.

(Details will be discussed in Chapter 3.)

Conic Section Type 2: Circle

Definition: A ***circle*** is the set of all points in a plane that the distance of the point from a fixed point is a constant. The fixed point is called the ***centre*** and the fixed distance is called the ***radius*** of the circle.



Equation of circle

The equation of a circle with centre at the origin $O(0,0)$ and radius r is given by

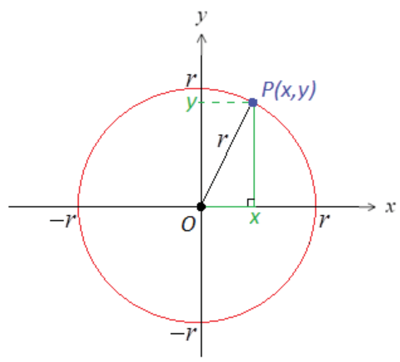
$$\boxed{x^2+y^2=r^2}.$$

Proof :

Let $P(x,y)$ be any point on the circle. According to the definition of circle, the distance from P to *the centre* $O(0,0)$ is equal to a fixed value r. By Pythagoras Theorem,

$$OP=\sqrt{(x-0)^2+(y-0)^2}=r \Rightarrow \sqrt{x^2+y^2}=r$$

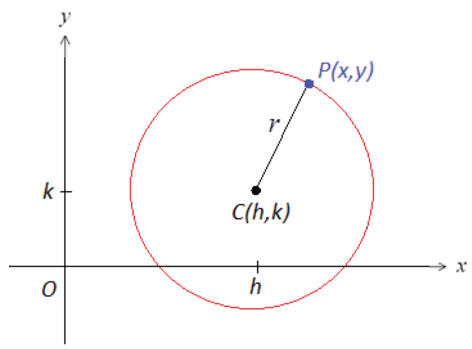
$$\Rightarrow x^2+y^2=r^2$$



Translation of circle

If the circle $x^2+y^2=r^2$ is translated h units to the right and k units upward, then it becomes **a circle with centre at $C(h,k)$ and radius r**. Its equation is given by

$$\boxed{(x-h)^2+(y-k)^2=r^2}.$$



This is the **standard form** of the equation of circle centred at $C(h,k)$ and radius r.

Example 6

Find the equations of the circles with the following centres and radii.

(a) Centre at $(0,2)$ and radius $\sqrt{5}$ units

(b) Centre at $(1,-3)$ and radius 4 units

Solution

(a) The equation of the circle is

$$(x-0)^2+(y-2)^2=(\sqrt{5})^2$$
$$\Rightarrow \quad x^2+(y-2)^2=5$$

(b) The equation of the circle is

$$(x-1)^2+(y-(-3))^2=(4)^2$$
$$\Rightarrow \quad (x-1)^2+(y+3)^2=16$$

Example 7

Find the centre and radius of the circle represented by each of the following equations:

(a) $(x-2)^2+(y+5)^2=10$, (b) $x^2+y^2+8x-10y-8=0$.

Solution

(a) Idea: Rewrite the equation into the standard form $\boxed{(x-h)^2+(y-k)^2=r^2}$ first.

$(x-2)^2+(y+5)^2=10 \quad \Rightarrow \quad (x-2)^2+(y-(-5))^2=(\sqrt{10})^2$

$\therefore$ The centre of the circle is at $(2,-5)$ and the radius is $\sqrt{10}$ units.

(b) Idea: Use the technique called "***completing the square***" to rewrite the equation into the standard form $(x-h)^2+(y-k)^2=r^2$.

*Recall: **Completing the square***

- $a^2+2ab+c=\underbrace{a^2+2ab+b^2}_{=(a+b)^2}-b^2+c=(a+b)^2-b^2+c$
- $a^2-2ab+c=\underbrace{a^2-2ab+b^2}_{=(a-b)^2}-b^2+c=(a-b)^2-b^2+c$

$$x^2 + y^2 + 8x - 10y - 8 = 0$$

$$\Rightarrow \underbrace{x^2 + 8x}_{=x^2+2(4)x} + \underbrace{y^2 - 10y}_{=y^2-2(5)y} - 8 = 0$$

$$\Rightarrow (x+4)^2 - 4^2 + (y-5)^2 - 5^2 - 8 = 0$$

$$\Rightarrow (x+4)^2 + (y-5)^2 = \underbrace{8 + 4^2 + 5^2}_{=49}$$

$$\Rightarrow (x-(-4))^2 + (y-5)^2 = 7^2$$

$\therefore$ The centre of the circle is at $(-4, 5)$ and the radius is 7 units.

Example 8

Find the equation of the circle centred at $(-1, 3)$ and passing through the point $(2, 5)$.

Solution

Idea: The distance between the point $(2, 5)$ on the circle and the centre $(-1, 3)$ is the radius of the circle.

The radius of the circle is $\sqrt{(2-(-1))^2 + (5-3)^2} = \sqrt{13}$.

Thus the equation of the circle is given by

$$(x-(-1))^2 + (y-3)^2 = (\sqrt{13})^2$$

$$\Rightarrow (x+1)^2 + (y-3)^2 = 13.$$

Example 9

Determine the equation of

(a) an upper half circle with centre at the origin and radius 6 units;

(b) a lower half circle with centre at (1, 3) and radius 5 units.

Solution

(a) The equation of a circle with centre at the origin and radius 6 units is given by

$$x^2 + y^2 = 6^2 \quad \Rightarrow \quad y = \pm\sqrt{36 - x^2}.$$

$\therefore$ The equation of the upper half circle is given by $y = \sqrt{36 - x^2}$.

(b) The equation of a circle with centre at (1, 3) and radius 5 units is

$$\begin{aligned}(x-1)^2 + (y-3)^2 = 5^2 \quad &\Rightarrow \quad (y-3)^2 = 5^2 - (x-1)^2.\\ &\Rightarrow \quad y - 3 = \pm\sqrt{25 - (x-1)^2}\\ &\Rightarrow \quad y = 3 \pm \sqrt{25 - (x-1)^2}.\end{aligned}$$

$\therefore$ The equation of the lower half circle is $y = 3 - \sqrt{25 - (x-1)^2}$.

Example 10

Find the equation of the circle which passes through the points $P(1,2)$, $Q(3,4)$ and $R(7,6)$.

Solution

Method 1:

(Consider the straight lines PQ and PR, and find their perpendicular bisectors.)

The line PQ has slope $\frac{4-2}{3-1} = 1$ and its mid-point is $\left(\frac{1+3}{2}, \frac{2+4}{2}\right) = (2,3)$.

$\therefore$ The perpendicular bisector of PQ is $\frac{y-3}{x-2} = -\frac{1}{1} \Rightarrow y - 3 = -(x-2)$

$$\Rightarrow y = -x + 5 \quad \cdots\cdots (1)$$

The line PR has slope $\frac{6-2}{7-1} = \frac{2}{3}$ and its mid-point is $\left(\frac{1+7}{2}, \frac{2+6}{2}\right) = (4,4)$.

$\therefore$ The perpendicular bisector of PR is $\frac{y-4}{x-4} = -\frac{1}{\frac{2}{3}} \Rightarrow y - 4 = -\frac{3}{2}(x-4)$

$$\Rightarrow y = -\frac{3}{2}x + 10 \quad \cdots\cdots (2)$$

(Next we find the centre of the circle by considering the intersection point of these two

perpendicular bisectors.)

Solving the equations $\begin{cases} y = -x + 5 \ \cdots\cdots (1) \\ y = -\frac{3}{2}x + 10 \ \cdots\cdots (2) \end{cases}$, we have

$$-x + 5 = -\frac{3}{2}x + 10 \quad \Rightarrow \quad \frac{1}{2}x = 5 \quad \Rightarrow \quad x = 10$$

From (1), we get $y = -x + 5 = -10 + 5 = -5$.

$\therefore$ These two perpendicular bisectors meet at $C(10, -5)$, which is the centre of the circle.

(Then we find the radius of the circle and write down the equation of the circle.)

The radius of the circle is given by

$$r = PC = \sqrt{(10-1)^2 + (-5-2)^2} = \sqrt{81 + 49} = \sqrt{130} \text{ units.}$$

$\therefore$ The equation of the required circle is

$$(x - 10)^2 + (y - (-5))^2 = \left(\sqrt{130}\right)^2$$

which reduces to

$$x^2 - 20x + 100 + y^2 + 10y + 25 = 130 \quad \Rightarrow \quad x^2 + y^2 - 20x + 10y - 5 = 0.$$

Method 2:

Let $x^2 + y^2 + Dx + Ey + F = 0$ be the equation of the circle through the points $P(1, 2)$, $Q(3, 4)$ and $R(7, 6)$.

Then we have the three equations:

$$\begin{cases} 1^2 + 2^2 + D + 2E + F = 0 \\ 3^2 + 4^2 + 3D + 4E + F = 0 \\ 7^2 + 6^2 + 7D + 6E + F = 0 \end{cases} \Rightarrow \begin{cases} D + 2E + F = -5 & \dots\dots (1) \\ 3D + 4E + F = -25 & \dots\dots (2) \\ 7D + 6E + F = -85 & \dots\dots (3) \end{cases}$$

$(2) - (1)$: $\quad 2D + 2E = -20 \ \Rightarrow \ D + E = -10 \ \dots\dots (4)$

$(3) - (1)$: $\quad 6D + 4E = -80 \ \Rightarrow \ 3D + 2E = -40 \ \dots\dots (5)$

$3 \times (4) - (5)$: $\quad E = 10$

From (4): $\quad D = -10 - E = -20$

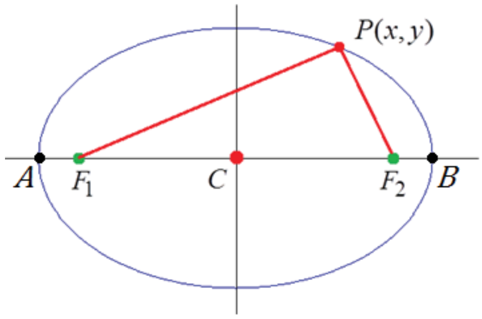
From (1): $\quad F = -5 - D - 2E = -5 - (-20) - 20 = -5$

$\therefore$ The equation of the required circle is

$$x^2 + y^2 - 20x + 10y - 5 = 0.$$

Conic Section Type 3: Ellipse

<u>*Definition*</u>: An ***ellipse*** is the set of all points P in a plane that the <u>*sum*</u> of the distances from P to two fixed points (called the ***foci***) is constant. The midpoint of the segment connecting the foci is the ***centre*** of the ellipse.



- Let F_1 and F_2 be the two foci (the plural of focus).

  Furthermore, let $AC = CB = a$. Then $AB = 2a$.

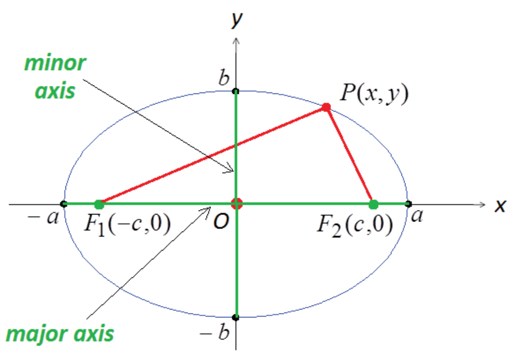
  For any point P on the ellipse, $PF_1 + PF_2$ is a constant, which is equal to $2a$. That is,

$$PF_1 + PF_2 = 2a.$$

Equation of ellipse

The equation of an ellipse with foci at the points $F_1(-c,0)$ and $F_2(c,0)$, where $c > 0$, is given by $\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$.

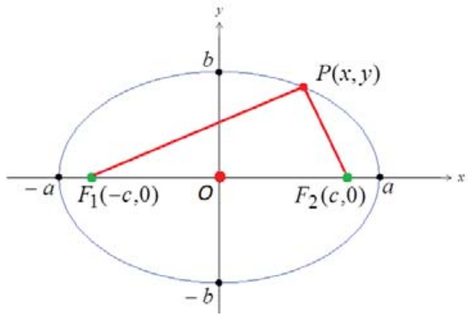
This is the **standard form** of the equation of an ellipse centered at the origin. Note that:



- $a > b > 0$, $\boxed{c^2 = a^2 - b^2}$
- The ***centre*** of this ellipse is at the origin $O(0,0)$.
- This ellipse is symmetrical about the x-axis and y-axis.
- The points $(a,0)$, $(-a,0)$, $(0,b)$ and $(0,-b)$ are called the ***vertices*** of the ellipse.
- The line segment joining the vertices $(a,0)$ and $(-a,0)$ is called the ***major axis***, and the line segment joining the vertices $(0,b)$ and $(0,-b)$ is called the ***minor axis***. The two foci are always on the major axis.

- The sum of the distances from any point on the ellipse to the two foci is $2a$, which is the length of the major axis.
- If $a=b$, the equation of the ellipse becomes $\frac{x^2}{a^2}+\frac{y^2}{a^2}=1 \Rightarrow x^2+y^2=a^2$, which is the equation of a circle.

__Proof of the equation__ $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (where $a>b$):



Let $P(x,y)$ be any point on the ellipse, which has foci at $F_1(-c,0)$ and $F_2(c,0)$, where $c>0$. According to the definition of ellipse, we have

$$PF_1+PF_2=2a$$

$$\Rightarrow \sqrt{(x-(-c))^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2}=2a$$

$$\Rightarrow \sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

$$\Rightarrow \sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$$

Squaring both sides gives

$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

$$\Rightarrow x^2+2cx+c^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2$$

$$\Rightarrow a\sqrt{(x-c)^2+y^2}=a^2-cx$$

Squaring both sides again gives

$$a^2[(x-c)^2+y^2]=a^4-2a^2cx+c^2x^2$$

$$\Rightarrow a^2x^2-2a^2cx+a^2c^2+a^2y^2=a^4-2a^2cx+c^2x^2$$

$$\Rightarrow (a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2)$$

By the triangle inequality,

$$PF_1+PF_2>F_1F_2 \quad \Rightarrow \quad 2a>2c \Rightarrow a>c.$$

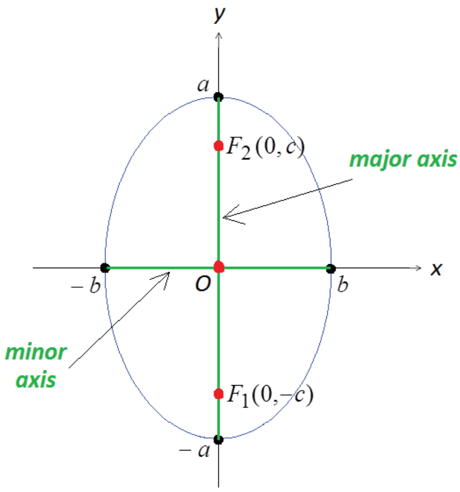
Thus, $a^2>c^2 \Rightarrow a^2-c^2>0$. Let $b^2=a^2-c^2>0$.

Then we have

$$b^2x^2+a^2y^2=a^2b^2 \quad \Rightarrow \quad \frac{b^2x^2}{a^2b^2}+\frac{a^2y^2}{a^2b^2}=\frac{a^2b^2}{a^2b^2} \quad \Rightarrow \boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1} \qquad \square$$

Other type of ellipse (with centre at the origin):

Consider an ellipse with foci at the points $F_1(0,-c)$ and $F_2(0,c)$. Note that both foci lie on the y-axis (instead of the x-axis) and the centre of this ellipse is at the origin. This ellipse has the equation $\boxed{\frac{x^2}{b^2}+\frac{y^2}{a^2}=1}$, where $a>b>0$ and $c^2=a^2-b^2$. The sum of the distances from any point on the ellipse to the two foci is $2a$.



The results of the two types of ellipses are summarized in the following table:

Equation of ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>b>0)$	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ $(a>b>0)$
Centre	$C(0,0)$	$C(0,0)$
Foci	$F_1(-c,0)$ and $F_2(c,0)$, where $c^2=a^2-b^2$	$F_1(0,-c)$ and $F_2(0,c)$, where $c^2=a^2-b^2$
Vertices	$(a,0)$, $(-a,0)$, $(0,b)$ and $(0,-b)$	$(b,0)$, $(-b,0)$, $(0,a)$ and $(0,-a)$
Major axis	Line segment joining $(a,0)$ and $(-a,0)$ on the x-axis	Line segment joining $(0,a)$ and $(0,-a)$ on the y-axis
Minor axis	Line segment joining $(0,b)$ and $(0,-b)$ on the y-axis	Line segment joining $(b,0)$ and $(-b,0)$ on the x-axis
Shape	 "Fat"	 "Thin"

Example 11

For each of the following ellipses, find the coordinates of the vertices and the foci, and sketch its graph: (a) $4x^2 + 9y^2 = 36$ (b) $8x^2 + y^2 = 8$

Solution

(a) Idea: Rearrange the equation into the standard form first. Then decide whether it is a "fat" or "thin" ellipse.

$$4x^2 + 9y^2 = 36 \Rightarrow \frac{4x^2}{36} + \frac{9y^2}{36} = \frac{36}{36} \Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow \frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

(Since 3 > 2, the ellipse is a "fat" one.)

Take $a = 3$ and $b = 2$.

$\therefore$ The vertices of the ellipse are at $(3, 0)$, $(-3, 0)$, $(0, 2)$ and $(0, -2)$.

$c^2 = a^2 - b^2 = 3^2 - 2^2 = 5 \Rightarrow c = \pm\sqrt{5}$. Take $c = \sqrt{5}$ (since $c > 0$).

$\therefore$ The foci of the ellipse are at $F_1(-\sqrt{5}, 0)$ and $F_2(\sqrt{5}, 0)$.

Sketch:

(b) $8x^2 + y^2 = 8 \Rightarrow \frac{8x^2}{8} + \frac{y^2}{8} = \frac{8}{8} \Rightarrow \frac{x^2}{1} + \frac{y^2}{8} = 1 \Rightarrow \frac{x^2}{1^2} + \frac{y^2}{(2\sqrt{2})^2} = 1$

(Since $1 < 2\sqrt{2}$, the ellipse is a "thin" one.)

Take $a = 2\sqrt{2}$ and $b = 1$.

$\therefore$ The vertices of the ellipse are at $(1,0)$, $(-1,0)$, $(0,2\sqrt{2})$ and $(0, -2\sqrt{2})$.

$c^2 = a^2 - b^2 = (2\sqrt{2})^2 - 1^2 = 7 \Rightarrow c = \pm\sqrt{7}$. Take $c = \sqrt{7}$ (since $c > 0$).

$\therefore$ The foci of the ellipse are at $F_1(0, -\sqrt{7})$ and $F_2(0, \sqrt{7})$.

Sketch:

Example 12

Find the equation of the ellipse whose centre is the origin and the ellipse passes through the points $(2\sqrt{2}, 0)$ and $(-2, \sqrt{3})$.

Solution

An ellipse whose centre is at the origin has an equation of the form $\frac{x^2}{p^2}+\frac{y^2}{q^2}=1$, where $p, q > 0$. Since the ellipse passes through the point $(2\sqrt{2}, 0)$, we substitute $x = 2\sqrt{2}$ and $y = 0$ into the above equation and get $\frac{(2\sqrt{2})^2}{p^2}+\frac{0^2}{q^2}=1 \Rightarrow p^2 = (2\sqrt{2})^2$.

Moreover, the ellipse also passes through the point $(-2, \sqrt{3})$, so we substitute $x = -2$ and $y = \sqrt{3}$ into the above equation and get $\frac{(-2)^2}{p^2}+\frac{(\sqrt{3})^2}{q^2}=1$

$$\Rightarrow \frac{(-2)^2}{(2\sqrt{2})^2}+\frac{(\sqrt{3})^2}{q^2}=1 \quad \Rightarrow \quad \frac{4}{8}+\frac{3}{q^2}=1 \quad \Rightarrow \quad \frac{3}{q^2}=\frac{1}{2} \quad \Rightarrow \quad q^2 = 6 = (\sqrt{6})^2.$$

$\therefore$ The equation of the ellipse is $\frac{x^2}{(2\sqrt{2})^2}+\frac{y^2}{(\sqrt{6})^2}=1$.

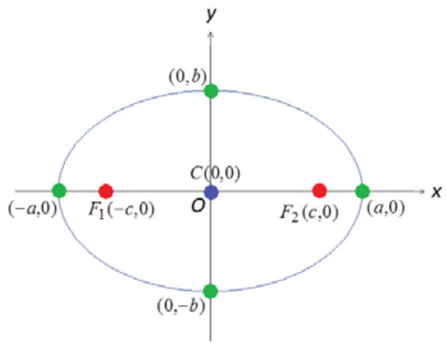
Translation of ellipse

If the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is translated h units to the right and k units upward, then we get **an ellipse with centre at $C(h,k)$**, and the equation of the new ellipse becomes

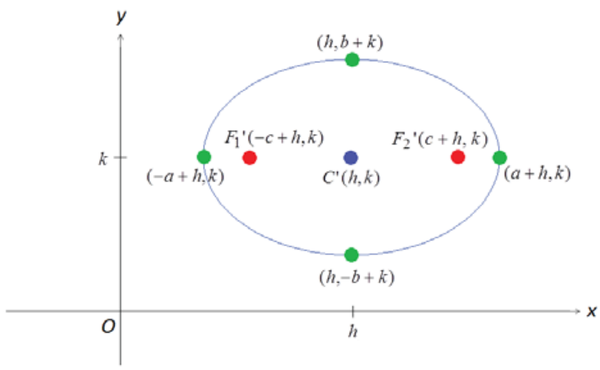
$$\boxed{\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.}$$

That is, we replace "x" with "$x-h$", and "y" with "$y-k$".

Consider the translation of a "fat" ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a > b > 0)$:



Translate $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ h units to the right and k units upward $\Longrightarrow$



	Before translation		After translation
Equation	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	Translate $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ h units to the right and k units upward $\Longrightarrow$	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$
Centre	$C(0,0)$		$C'(h,k)$
Foci	$F_1(-c,0)$ and $F_2(c,0)$, where $c^2=a^2-b^2$		$F_1{}'(-c+h,k)$ and $F_2{}'(c+h,k)$, where $c^2=a^2-b^2$
Vertices	$(a,0)$, $(-a,0)$, $(0,b)$ and $(0,-b)$		$(a+h,k)$, $(-a+h,k)$, $(h,b+k)$ and $(h,-b+k)$

<u>Remark:</u> Similar method could be used to translate "thin" ellipses in horizontal/vertical directions.

<u>Example 13</u>

Consider the equation $4y^2+25x^2-24y+200x+336=0$.

(a) Show that this equation represents an ellipse by rewriting the equation into the standard form of an ellipse.

(b) Find the coordinates of the centre, vertices and foci of this ellipse.

(c) Sketch the graph of this ellipse with the centre, vertices and foci clearly shown.

<u>Solution</u>

(a) Idea: Rewrite the equation into the standard form by "completing the square" first.

$4y^2+25x^2-24y+200x+336=0$

$\Rightarrow \quad 4(y^2-6y)+25(x^2+8x)+336=0$

$\Rightarrow \quad 4[(y-3)^2-3^2]+25[(x+4)^2-4^2]+336=0$

$\Rightarrow \quad 4(y-3)^2-36+25(x+4)^2-400+336=0$

$\Rightarrow \quad 4(y-3)^2+25(x+4)^2=100$

$$\Rightarrow \frac{4(y-3)^2}{100} + \frac{25(x+4)^2}{100} = \frac{100}{100}$$

$$\Rightarrow \frac{(y-3)^2}{25} + \frac{(x+4)^2}{4} = 1$$

$$\Rightarrow \frac{(y-3)^2}{5^2} + \frac{(x-(-4))^2}{2^2} = 1,$$

which represents an ellipse.

(b) Idea: Consider the coordinates of the centre, vertices and foci for the ellipse $\frac{y^2}{5^2} + \frac{x^2}{2^2} = 1$. Then apply appropriate translations to obtain the required ellipse.

Take $a = 5$, $b = 2$. Then $c = \sqrt{a^2 - b^2} = \sqrt{5^2 - 2^2} = \sqrt{21}$.

For the ellipse $\frac{y^2}{5^2} + \frac{x^2}{2^2} = 1$, its centre is at $(0, 0)$;

its vertices are at $(-2, 0)$, $(2, 0)$, $(0, -5)$ and $(0, 5)$;

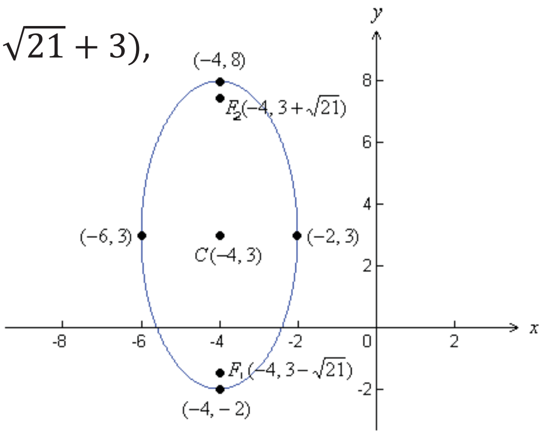
and its foci are at $(0, -\sqrt{21})$ and $(0, \sqrt{21})$.

The graph of the ellipse $\frac{(y-3)^2}{5^2} + \frac{(x-(-4))^2}{2^2} = 1$ is obtained by translating the graph of $\frac{y^2}{5^2} + \frac{x^2}{2^2} = 1$ to the left by 4 units and upward by 3 units.

$\therefore$ For the ellipse $\frac{(y-3)^2}{5^2} + \frac{(x-(-4))^2}{2^2} = 1$:

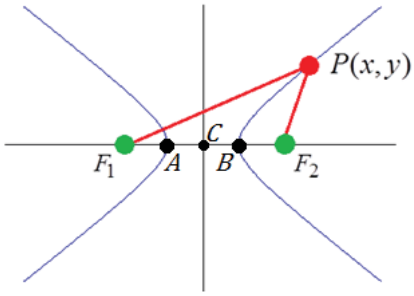
- its **centre** is at $(-4, 3)$.
- its **vertices** are at $(-2-4, 0+3)$, $(2-4, 0+3)$, $(0-4, -5+3)$ and $(0-4, 5+3)$, i.e. $(-6, 3)$, $(-2, 3)$, $(-4, -2)$ and $(-4, 8)$.
- its **foci** are at $(0-4, -\sqrt{21}+3)$ and $(0-4, \sqrt{21}+3)$, i.e. $(-4, 3-\sqrt{21})$ and $(-4, 3+\sqrt{21})$.

(c) The graph of the ellipse $\frac{(y-3)^2}{5^2} + \frac{(x-(-4))^2}{2^2} = 1$ is shown on the right.



Conic Section Type 4: Hyperbola

Definition: A ***hyperbola*** is the set of all points P in a plane that the *difference* of the distances from P to two fixed points (the ***foci***) is a constant.



➢ Let F_1 and F_2 be the two foci.

Furthermore, let $AC = CB = a$. Then $AB = 2a$.

For any point P on the hyperbola, $|PF_1 - PF_2|$ is a constant, which is equal to $2a$.

That is,

$$|PF_1 - PF_2| = 2a.$$

Equation of hyperbola

The equation of a hyperbola with foci at the points $F_1(-c,0)$ and $F_2(c,0)$, where $c > 0$, is

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}.$$

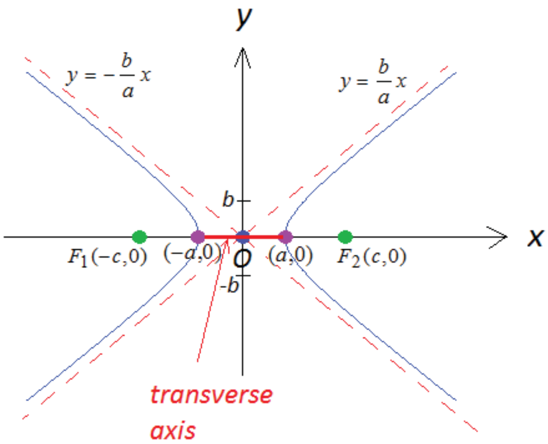
This is the **standard form of the equation of a hyperbola centered at the origin**. Note that:

➢ $a, b > 0$, $\boxed{c^2 = a^2 + b^2}$

➢ This hyperbola is symmetrical about the x-axis and y-axis.

➢ The points $(a,0)$ and $(-a,0)$ are called the ***vertices*** of the hyperbola.

➢ The ***centre*** of this hyperbola (the midpoint of two foci) is the origin $O(0,0)$.



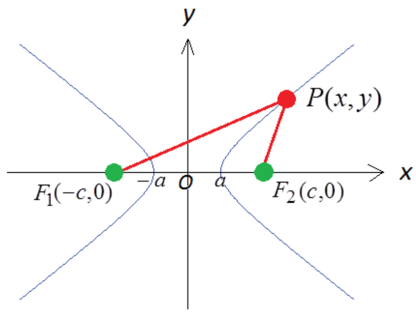
➢ The line segment joining the two vertices is called the ***transverse axis***.

➢ As x gets further away from the origin O, the two branches of the graph approach a pair of intersecting straight lines $y=\frac{b}{a}x$ and $y=-\frac{b}{a}x$, which are called the ***asymptotes*** of the hyperbola.

- The difference of the distances from any point on the hyperbola to the two foci is $2a$, which is the distance between the two vertices.

Proof of the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$:

Let $P(x,y)$ be any point on the hyperbola, which has foci at $F_1(-c,0)$ and $F_2(c,0)$, where $c>0$. According to the definition of hyperbola, we have



$|PF_1-PF_2|=2a$

$\Rightarrow\quad \left|\sqrt{(x-(-c))^2+(y-0)^2}-\sqrt{(x-c)^2+(y-0)^2}\right|=2a$

$\Rightarrow\quad \left|\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}\right|=2a$

Squaring both sides gives

$$(x+c)^2+y^2-2\sqrt{(x+c)^2+y^2}\sqrt{(x-c)^2+y^2}+(x-c)^2\ +y^2=4a^2$$

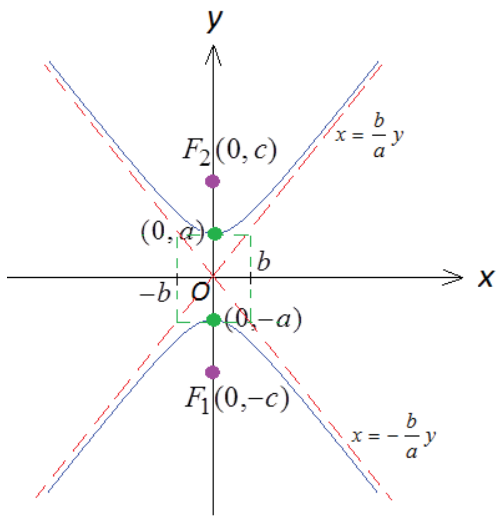
After some calculations, we get $\quad (c^2-a^2)x^2-a^2y^2=a^2(c^2-a^2)$.

Clearly from the graph, $a<c$. Thus, $a^2<c^2 \Rightarrow c^2-a^2>0$.

Let $b^2=c^2-a^2>0$. Then we have

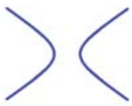
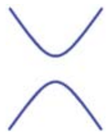
$$b^2x^2-a^2y^2=a^2b^2 \quad\Rightarrow\quad \frac{b^2x^2}{a^2b^2}-\frac{a^2y^2}{a^2b^2}=\frac{a^2b^2}{a^2b^2} \quad\Rightarrow\quad \boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1} \qquad \square$$

<u>Other type of hyperbola (with centre at the origin):</u>



Consider a hyperbola with foci at the points $F_1(0,-c)$ and $F_2(0,c)$. Note that both foci lie on the y-axis (instead of the x-axis) and the centre of this hyperbola is at the origin. This hyperbola has the equation $\boxed{\frac{y^2}{a^2}-\frac{x^2}{b^2}=1}$, where $a,b>0$ and $c^2=a^2+b^2$. The difference of the distances from any point on the hyperbola to the two foci is $2a$.

The two types of hyperbolae are summarized in the following table:

Equation of hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
Centre	$C(0,0)$	$C(0,0)$
Foci	$F_1(-c,0)$ and $F_2(c,0)$, where $c^2 = a^2 + b^2$	$F_1(0,-c)$ and $F_2(0,c)$, where $c^2 = a^2 + b^2$
Vertices	$(a,0)$ and $(-a,0)$	$(0,a)$ and $(0,-a)$
Asymptotes	$y = \pm\frac{b}{a}x$	$y = \pm\frac{a}{b}x$
Shape	 "East-West openings"	 "North-South openings"

Example 14

Arrange the equation $9x^2 - 4y^2 = 144$ into the standard form of hyperbola. Find the coordinates of the centre, vertices and foci of this hyperbola, and sketch its graph.

Solution

$$9x^2 - 4y^2 = 144 \;\Rightarrow\; \frac{9x^2}{144} - \frac{4y^2}{144} = \frac{144}{144} \;\Rightarrow\; \frac{x^2}{16} - \frac{y^2}{36} = 1 \;\Rightarrow\; \frac{x^2}{4^2} - \frac{y^2}{6^2} = 1$$

This is a hyperbola with "East-West openings" and is centred at the origin $(0,0)$.

Its vertices are at $(4,0)$ and $(-4,0)$.

$c^2 = a^2 + b^2 = 4^2 + 6^2 = 52 \;\Rightarrow\; c = \sqrt{52} = 2\sqrt{13}$ (Take positive value of c.)

$\therefore$ The foci of this hyperbola are at $(-2\sqrt{13}, 0)$ and $(2\sqrt{13}, 0)$.

Sketch:

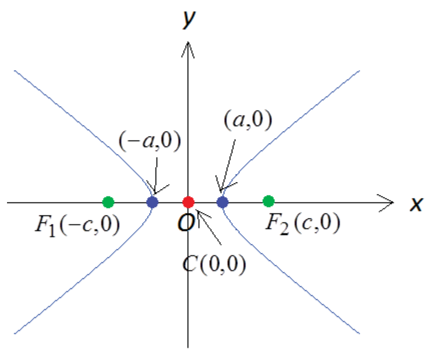
Translation of Hyperbola

Consider the translation of the hyperbola with "East-West openings" $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$:

If the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is translated h units to the right and k units upward, then we get a **hyperbola with centre at $C(h,k)$**, and the equation of the new hyperbola becomes

$$\boxed{\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1}.$$

That is, we replace "x" with "$x-h$", and "y" with "$y-k$".



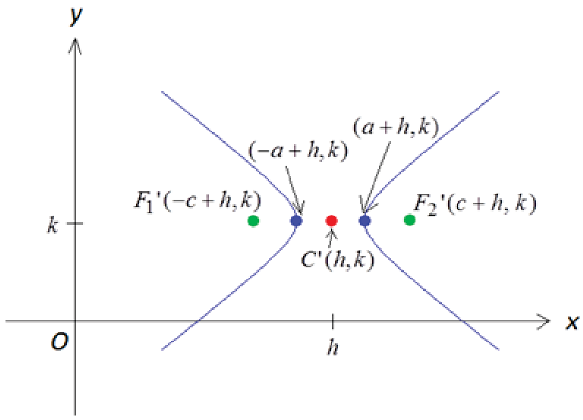
Translate $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

h units to the right

and

k units upward

$\Longrightarrow$



	Before translation		**After translation**
Equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	Translate $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ h units to the right and k units upward $\Longrightarrow$	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1.$
Centre	$C(0,0)$		$C'(h,k)$
Foci	$F_1(-c,0)$ and $F_2(c,0)$, where $c^2=a^2+b^2$		$F_1'(-c+h,k)$ and $F_2'(c+h,k)$, where $c^2=a^2+b^2$
Vertices	$(a,0)$, $(-a,0)$		$(a+h,k)$, $(-a+h,k)$
Asymptotes	$y=\pm\frac{b}{a}x$		$(y-k)=\pm\frac{b}{a}(x-h)$

Remark: Similar method could be done to translate hyperbola with "North-South openings" in horizontal/vertical directions. The equation of the new hyperbola becomes

$$\boxed{\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1}.$$

Example 15

Consider the equation $16x^2 - 9y^2 - 160x - 18y + 247 = 0$.

(a) Show that this equation describes a hyperbola by writing the equation into the standard form of hyperbola.

(b) Find the coordinates of the centre, vertices and foci of this hyperbola.

(c) Sketch the graph of this hyperbola with the centre, vertices and foci clearly shown.

Solution

Classification of conic sections

The four types of conic sections that we have discussed are:

- **Parabola**: $(x-h)^2 = 4a(y-k)$ or $(y-k)^2 = 4a(x-h)$
- **Circle**: $(x-h)^2 + (y-k)^2 = r^2$
- **Ellipse**: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$ $(a > b > 0)$
- **Hyperbola**: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Each of the above equations could be expressed into the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \quad \text{............(*)}$$

where A, C, D, E, F are constants.

(Note that there is no xy term in equation (*).)

If the equation of a conic section is expressed in the form (*), one can identify the type of conic section by first using the technique "***completing the square***" to write the equation into the standard form of the corresponding type [see Example 7(b), 13 and 15].

Example 16

Classify the type of conic section described by each of the following equations by completing the square.

(a) $4x^2 - 16x + 25y^2 - 84 = 0$

(b) $4x^2 + 4y^2 + 8x - 24y + 15 = 0$

Solution

(a) $4x^2 - 16x + 25y^2 - 84 = 0 \quad \Rightarrow \quad 4(x^2 - 4x) + 25y^2 - 84 = 0$

$$\Rightarrow \quad 4[(x-2)^2 - 2^2] + 25y^2 - 84 = 0$$

$$\Rightarrow \quad 4(x-2)^2 + 25y^2 = 100$$

$$\Rightarrow \quad \frac{4(x-2)^2}{100} + \frac{25y^2}{100} = \frac{100}{100}$$

$$\Rightarrow \quad \frac{(x-2)^2}{25} + \frac{y^2}{4} = 1$$

$$\Rightarrow \quad \frac{(x-2)^2}{5^2} + \frac{y^2}{2^2} = 1$$

$\therefore$ The above equation represents an **ellipse**.

Note that this is a "fat" ellipse.

- It is centred at $(2,0)$.
- Its vertices are at $(-5+2,0)$, $(5+2,0)$, $(0+2,-2)$ and $(0+2,2)$,

  i.e. $(-3,0)$, $(7,0)$, $(2,-2)$ and $(2,2)$.

  $c=\sqrt{5^2-2^2}=\sqrt{21}$
- Its foci are at $(-\sqrt{21}+2,0)$ and $(\sqrt{21}+2,0)$.

(b) $4x^2+4y^2+8x-24y+15=0$

$\Rightarrow \quad 4x^2+8x+4y^2-24y+15=0$

$\Rightarrow \quad 4(x^2+2x)+4(y^2-6y)+15=0$

$\Rightarrow \quad 4[(x+1)^2-1^2]+4[(y-3)^2-3^2]+15=0$

$\Rightarrow \quad 4(x+1)^2+4(y-3)^2=25$

$\Rightarrow \quad (x+1)^2+(y-3)^2=\frac{25}{4} \quad \Rightarrow \quad (x-(-1))^2+(y-3)^2=\left(\frac{5}{2}\right)^2$

$\therefore$ The above equation represents a **circle**, centred at $(-1,3)$ and its radius is $\frac{5}{2}$.

Example 17

The following equations represent typical degenerate conic sections. Identify the graph of each equation by completing the square.

(a) $2x^2+y^2-4y+16=0$

(b) $2x^2+4x+y^2-4y+6=0$

Solution

(a) $2x^2+y^2-4y+16=0 \Rightarrow 2x^2+(y-2)^2-2^2+16=0 \Rightarrow 2x^2+(y-2)^2=-12$

Since the LHS $=2x^2+(y-2)^2\ge 0$ for all real values of x and y while the RHS $=-12<0$, the graph contains **no point**.

(b) $2x^2+4x+y^2-4y+6=0 \Rightarrow 2(x^2+2x)+y^2-4y+6=0$

$\Rightarrow 2[(x+1)^2-1^2]+(y-2)^2-2^2+6=0$

$\Rightarrow 2(x+1)^2+(y-2)^2=0$

$\Rightarrow x=-1,\ y=2$ is the only solution.

$\therefore$ The graph contains only **one point** $(-1,2)$.

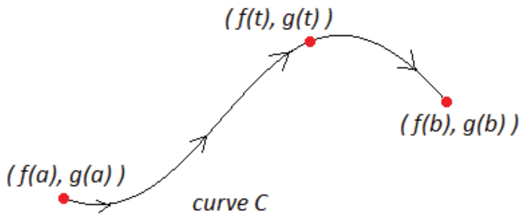
Parametric Equations

Suppose that an object moves around in the xy-plane so that the coordinates of its position at any time t are functions of the variable t:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad a \le t \le b.$$

The point (x, y) moves and traces a curve C as t varies.

These equations are called the ***parametric equations*** for the curve C. The independent variable t is called a **parameter**.



The equations for the four conics (circle, ellipse, parabola, and hyperbola) can be expressed in parametric form.

Type of Conics	Equation in Rectangular Coordinate Form	Equation in Parametric Form
Parabola Type 1	$x^2 = 4ay$	$\begin{cases} x = 2at \\ y = at^2 \end{cases}, \; -\infty < t < \infty$
Parabola Type 2	$y^2 = 4ax$	$\begin{cases} x = at^2 \\ y = 2at \end{cases}, \; -\infty < t < \infty$
Circle	$x^2 + y^2 = r^2$	$\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}, \; 0 \le t \le 2\pi$
Ellipse "Fat"	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \;\; (a > b)$	$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases} \; (a > b),$ $0 \le t \le 2\pi$
Ellipse "Thin"	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \;\; (a > b)$	$\begin{cases} x = b\cos t \\ y = a\sin t \end{cases} \; (a > b),$ $0 \le t \le 2\pi$

Hyperbola (East-West Openings)	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\begin{cases} x = a\sec t \\ y = b\tan t \end{cases}$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and $\frac{\pi}{2} < t < \frac{3\pi}{2}$
Hyperbola (North-South Openings)	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$	$\begin{cases} x = b\tan t \\ y = a\sec t \end{cases}$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and $\frac{\pi}{2} < t < \frac{3\pi}{2}$
Rectangular Hyperbola	$xy = c^2$; c is a constant	$\begin{cases} x = t \\ y = \frac{c^2}{t} \end{cases}$, $-\infty < t < 0$ and $0 < t < \infty$

<u>Remark</u>: In the parametric equations for circles, ellipses and hyperbolae, the parameter t is measured in radians. (π radians = 180°). Topic on radian measure will be discussed in Chapter 4.