MA1200 CALCULUS AND BASIC LINEAR ALGEBRA I

LECTURE: CG1

Chapter 6
Limits, Continuity and Differentiability

Limit of a function at a point

The limit of a function $f(x)$ at a point $x = a$ is the value that $f(x)$ is approaching as x gets closer and closer to a.

We use the notation

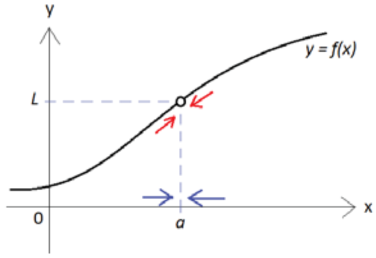
$$\boxed{\lim_{x \to a} f(x) = L}$$

to denote that "the value of $f(x)$ gets arbitrarily close to L as x approaches a". Here, x approaches a from both the left and the right of a.

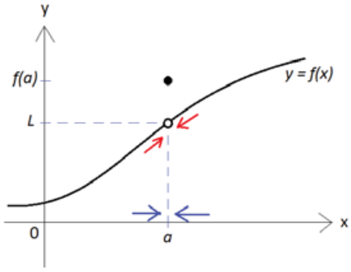
A more formal definition of $\lim_{x \to a} f(x) = L$ is that the difference between $f(x)$ and L can be made arbitrarily small when x is <u>sufficiently close to</u> but <u>different</u> from a.

Remarks:

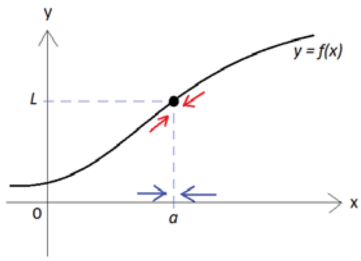
(i) $\lim_{x \to a} f(x)$ may exist even if f is not defined at $x = a$.



(ii) $\lim_{x \to a} f(x)$ may exist even if $f(a) \neq \lim_{x \to a} f(x)$.



(iii) If $f(a) = \lim_{x \to a} f(x)$, then $f(x)$ is said to be **continuous** at $x = a$ (i.e. there is no break at $x = a$.)



(iv) If $\lim_{x \to a} f(x) = \infty$ (or $-\infty$), we say that the limit $\lim_{x \to a} f(x)$ **does not exist (DNE)**.

E.g. $\lim_{x \to 0} \frac{1}{x^2} = \infty$, so the limit does not exist.

Example of Remark (i):

$\boxed{\lim_{x\to a} f(x) \text{ may exist even if } f \text{ is not defined at } x = a.}$

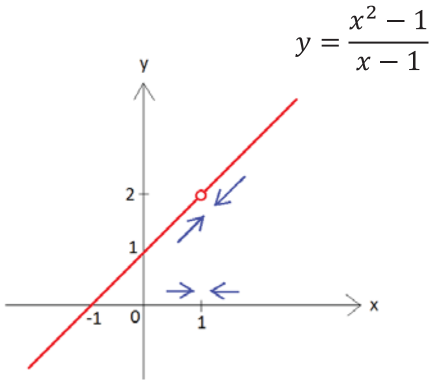
Consider the limit $\lim_{x\to 1}\frac{x^2-1}{x-1}$.

The function $\frac{x^2-1}{x-1}$ is not defined at $x = 1$. To evaluate the above limit, we consider the values of x approaching to 1, but not at $x = 1$, i.e. we assume $x \neq 1$.

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}\frac{(x-1)(x+1)}{x-1} \underbrace{=}_{\because x\neq 1} \lim_{x\to 1}(x+1) = 1+1 = 2$$

$\left(\frac{0}{0}\text{ form}\right)$

We see that the value of $\frac{x^2-1}{x-1}$ approaches 2 as x approaches 1.



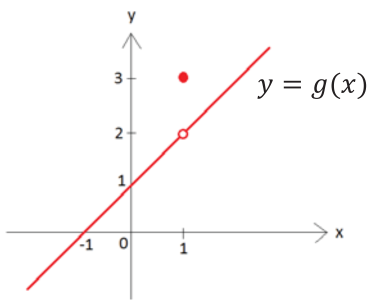
Example of Remark (ii):

$\boxed{\lim_{x\to a} f(x) \text{ may exist even if } f(a) \neq \lim_{x\to a} f(x)}$.

Consider the limit of the function $g(x) = \begin{cases}\frac{x^2-1}{x-1} & \text{if } x \neq 1\\ 3 & \text{if } x = 1\end{cases}$ at $x = 1$.

$$\lim_{x\to 1} g(x) \underbrace{=}_{\because x\neq 1} \lim_{x\to 1}\frac{x^2-1}{x-1} = \cdots = 2$$

but $g(1) = 3$.



The limit $\lim_{x\to 1} g(x)$ exists but $\lim_{x\to 1} g(x) \neq g(1)$.

Theorems on limits

Let k be a constant, n be a positive integer, and f and g be functions for which $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then

(1) $\lim_{x\to c} k = k$, $\lim_{x\to c} x = c$, $\lim_{x\to c} k\,f(x) = k\lim_{x\to c} f(x)$

(2) $\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

(3) $\lim_{x\to c}[f(x)-g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$

(4) $\lim_{x\to c}[f(x)\cdot g(x)] = \left(\lim_{x\to c} f(x)\right)\cdot\left(\lim_{x\to c} g(x)\right)$

(5) $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided $\lim_{x\to c} g(x) \neq 0$

(6) $\lim_{x\to c}[f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n$

(7) $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}$, provided $\lim_{x\to c} f(x) \geq 0$ when n is even

Theorem

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and

$g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$ be two **polynomials**. Then we have the results:

(i) $\lim_{x\to c} f(x) = a_n\cdot c^n + a_{n-1}\cdot c^{n-1} + \cdots + a_1\cdot c + a_0 = f(c)$.

(ii) If $g(c)\neq 0$, then $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}$.

Example 1

(a) $\lim_{x\to 1}(x^2+x-6)$ (b) $\lim_{x\to 1}\frac{x^3+1}{x+1}$ (c) $\lim_{x\to -1}\frac{x^3+2}{x+1}$ (d) $\lim_{x\to -1}\frac{x^3+1}{x+1}$

Solution

(a) $\lim_{x\to 1}(x^2+x-6) = 1^2+1-6 = -4$ **(Evaluate this limit by substituting $x=1$)**

(b) $\lim_{x\to 1}\frac{x^3+1}{x+1} = \frac{1^3+1}{1+1} = \frac{2}{2} = 1$ **(Evaluate this limit by substituting $x=1$)**

(c) $\lim_{x \to -1} \frac{x^3+2}{x+1}$ $\left(\frac{1}{0} \text{ form}\right)$ **(The function $\frac{x^3+2}{x+1}$ is undefined at $x = -1$ but the numerator is non-zero when $x = -1$)**

$= \frac{(-1)^3+2}{(-1)+1}$ **(Evaluate this limit by substituting $x = -1$)**

$= \frac{1}{0}$ which is undefined.

$\therefore$ The limit $\lim_{x \to -1} \frac{x^3+2}{x+1}$ <u>does not exist</u>.

(d) $\lim_{x \to -1} \frac{x^3+1}{x+1}$ $\left(\frac{0}{0} \text{ form}\right)$ **(The function $\frac{x^3+1}{x+1}$ is undefined at $x = -1$. Both the numerator and denominator are equal to 0 when $x = -1$.)**

$= \lim_{x \to -1} \frac{(x+1)(x^2-x+1)}{x+1}$ **(Factorize the numerator)**

$\underbrace{=}_{\because x \neq -1} \lim_{x \to -1} (x^2 - x + 1)$ **(Cancel common factor)**

$= (-1)^2 - (-1) + 1$ **(Evaluate this limit by substituting $x = -1$)**

$= 3$

<u>**Remark:**</u> The $\frac{0}{0}$ **form** is known as an **indeterminate form**.

<u>**Example 2**</u>

Evaluate each of the following limits:

(a) $\lim_{x \to -2} \frac{x^2 + x - 2}{x^2 + 5x + 6}$ (b) $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x^2 - 16}$ (c) $\lim_{x \to 2} \frac{4 - x^2}{3 - \sqrt{x^2 + 5}}$

<u>*Solution*</u>

(a) $\lim_{x \to -2} \frac{x^2 + x - 2}{x^2 + 5x + 6}$ $\left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x \to -2} \frac{(x+2)(x-1)}{(x+2)(x+3)}$

$\underbrace{=}_{\because x \neq -2} \lim_{x \to -2} \frac{x-1}{x+3}$

$= \frac{(-2) - 1}{(-2) + 3}$

$= -3$

(b) $\lim_{x\to 4}\frac{\sqrt{x}-2}{x^2-16}$ $\left(\frac{0}{0}\text{ form}\right)$

$$=\lim_{x\to 4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x^2-16)(\sqrt{x}+2)}=\lim_{x\to 4}\frac{x-4}{(x-4)(x+4)(\sqrt{x}+2)}=\lim_{x\to 4}\frac{1}{(x+4)(\sqrt{x}+2)}$$

$$=\frac{1}{(4+4)(\sqrt{4}+2)}=\frac{1}{32}$$

(c) $\lim_{x\to 2}\frac{4-x^2}{3-\sqrt{x^2+5}}$ $\left(\frac{0}{0}\text{ form}\right)$

$$=\lim_{x\to 2}\left(\frac{4-x^2}{3-\sqrt{x^2+5}}\cdot\frac{3+\sqrt{x^2+5}}{3+\sqrt{x^2+5}}\right)$$

$$=\lim_{x\to 2}\frac{(4-x^2)(3+\sqrt{x^2+5})}{9-(x^2+5)}$$

$$=\lim_{x\to 2}\frac{(4-x^2)(3+\sqrt{x^2+5})}{4-x^2}=\lim_{x\to 2}\left(3+\sqrt{x^2+5}\right)=3+\sqrt{2^2+5}=6$$

Example 3

Evaluate the limit

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}$$

where n is a positive integer.

Solution

$\lim_{x\to a}\frac{x^n-a^n}{x-a}$ $\left(\frac{0}{0}\text{ form}\right)$

$$=\lim_{x\to a}\frac{(x-a)(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\cdots+a^{n-1})}{x-a}$$

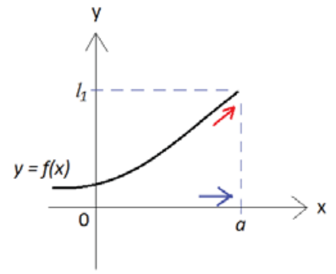
$$=\lim_{x\to a}(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\cdots+a^{n-1})$$

$$=\underbrace{a^{n-1}+a\cdot a^{n-2}+a^2\cdot a^{n-3}+\cdots+a^{n-1}}_{n\ terms}$$

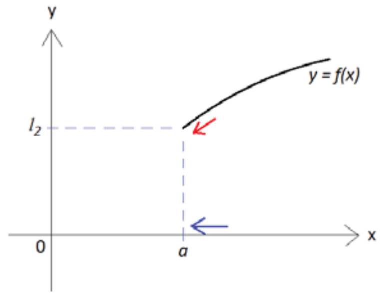
$$=na^{n-1}$$

Left hand limit / Right hand limit

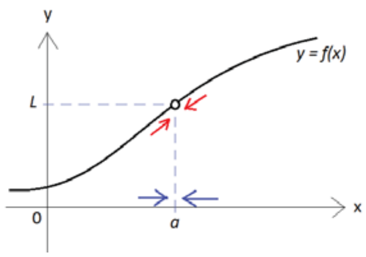
- The **left hand limit** of $f(x)$ at $x=a$ is $\lim\limits_{x\to a^-} f(x)=l_1$ if the value of $f(x)$ approaches l_1 as x approaches a from the left.



- The **right hand limit** of $f(x)$ at $x=a$ is $\lim\limits_{x\to a^+} f(x)=l_2$ if the value of $f(x)$ approaches l_2 as x approaches a from the right.



- If $\lim\limits_{x\to a^-} f(x)=\lim\limits_{x\to a^+} f(x)=L$ (where L is a real number), we say that the limit $\lim\limits_{x\to a} f(x)$ exists and $\lim\limits_{x\to a} f(x)=L$.



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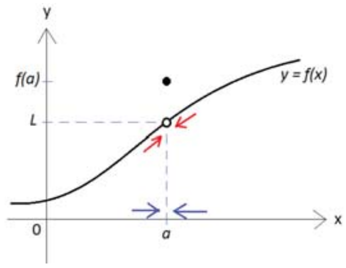

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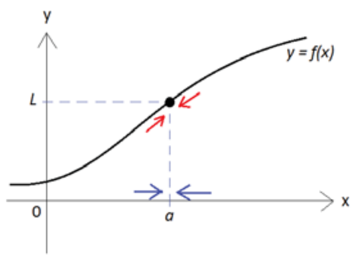
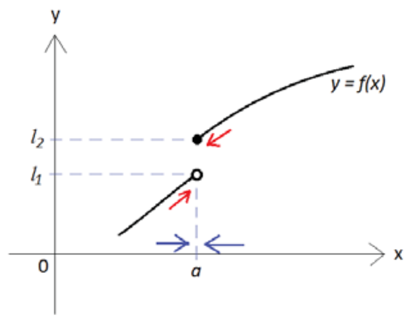


- If $\lim\limits_{x\to a^-} f(x)\neq\lim\limits_{x\to a^+} f(x)$, the limit $\lim\limits_{x\to a} f(x)$ does not exist.



Example 4

Does the limit of $f(x) = \frac{|x|}{x}$ exist at $x = 0$?

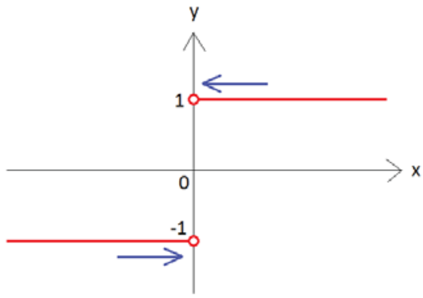
Solution

First note that the function $f(x) = \frac{|x|}{x}$ is not defined at $x = 0$.

Rewrite the function as

$$f(x) = \frac{|x|}{x} = \begin{cases} \frac{x}{x} & \text{if } x > 0 \\ \frac{-x}{x} & \text{if } x < 0 \end{cases}$$

$$= \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$



Left hand limit: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} (-1) = -1$

Right hand limit: $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} (1) = 1$

Since $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$, the limit $\lim_{x \to 0} f(x)$ does not exist.

Example 5

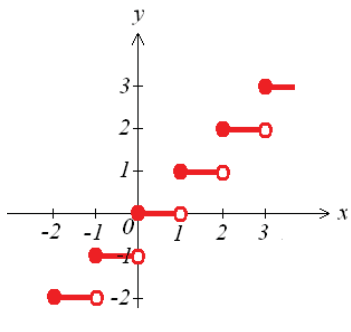
Consider the function $f(x) = [x] =$ "greatest integer $\leq x$" (the greatest integer function), find the limits (a) $\lim_{x \to 0.5} f(x)$ and (b) $\lim_{x \to 1} f(x)$ if they exist.

Solution:

$f(x) = [x] =$ "greatest integer $\leq x$".

E.g. $f(2.8) = [2.8] = 2$, $f(1) = [1] = 1$.

Its graph is shown on the right.



(a) $\because \lim_{x \to 0.5^-} f(x) = 0 = \lim_{x \to 0.5^+} f(x)$,

$\therefore \lim_{x \to 0.5} f(x)$ exists and $\lim_{x \to 0.5} f(x) = 0$.

(b) Consider the function in the neighborhood of $x = 1$.

$$[x] = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 \leq x < 2 \end{cases}.$$

Since $\lim_{x \to 1^-} f(x) = 0 \neq 1 = \lim_{x \to 1^+} f(x)$, the limit $\lim_{x \to 1} f(x)$ does not exist.

Example 6

Consider the function $f(x) = \begin{cases} 2x\sin\left(\frac{x}{2}\right) & \text{if } x \le \pi \\ \frac{x^2-\pi^2}{x-\pi} & \text{if } x > \pi. \end{cases}$

Does the limit $\lim\limits_{x\to\pi} f(x)$ exist? Find the value of the limit if it exists.

Solution

LHL: $\lim\limits_{x\to\pi^-} f(x) = \lim\limits_{x\to\pi^-} 2x\sin\left(\frac{x}{2}\right) = 2\pi \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} = 2\pi$

RHL: $\lim\limits_{x\to\pi^+} f(x) = \lim\limits_{x\to\pi^+} \frac{x^2-\pi^2}{x-\pi}$ $\left(\frac{0}{0}\text{ form}\right)$

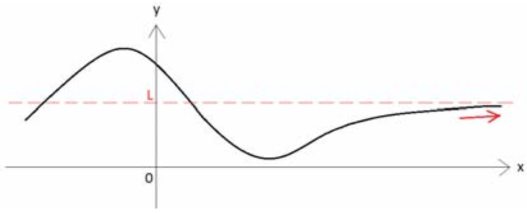
$$= \lim_{x\to\pi^+} \frac{(x-\pi)(x+\pi)}{x-\pi} = \lim_{x\to\pi^+}(x+\pi) = \pi+\pi = 2\pi$$

$\because \lim\limits_{x\to\pi^-} f(x) = \lim\limits_{x\to\pi^+} f(x) = 2\pi$

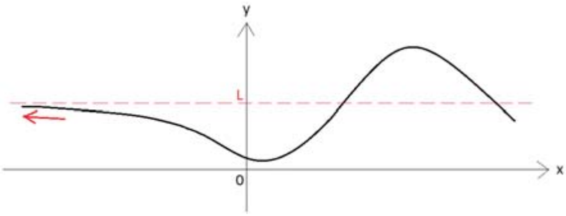
$\therefore \lim\limits_{x\to\pi} f(x)$ exists and $\lim\limits_{x\to\pi} f(x) = 2\pi$

Limit at infinity

- $\boxed{\lim\limits_{x\to\infty} f(x) = L}$ means "as x increases indefinitely, $f(x)$ tends to L".



- $\boxed{\lim\limits_{x\to-\infty} f(x) = L}$ means "as x decreases indefinitely, $f(x)$ tends to L".



Useful results:

(i) $\lim\limits_{x\to\infty} \frac{1}{x} = 0 \quad \Rightarrow \quad \lim\limits_{x\to\infty} \frac{1}{x^n} = 0$ for $n > 0$

(ii) $\lim\limits_{x\to-\infty} \frac{1}{x} = 0 \quad \Rightarrow \quad \lim\limits_{x\to-\infty} \frac{1}{x^n} = 0$ for $n > 0$ and whenever x^n is defined for $x < 0$.

Example 7

Evaluate each of the following limits:

(a) $\lim\limits_{x\to\infty}\dfrac{5x^4-2x^3+4x-1}{2x^4+3x^2+4}$ (b) $\lim\limits_{x\to-\infty}\dfrac{3x^2-5x+4}{4x-3}$ (c) $\lim\limits_{x\to\infty}\dfrac{x^3+7x-2}{5x^4+6x^3-1}$

Solution

(a) $\lim\limits_{x\to\infty}\frac{5x^4-2x^3+4x-1}{2x^4+3x^2+4}$ $\left(\frac{\infty}{\infty}\text{ form, which is an indeterminate form}\right)$

$$=\lim_{x\to\infty}\frac{\frac{5x^4-2x^3+4x-1}{x^4}}{\frac{2x^4+3x^2+4}{x^4}}$$

(Divide both the numerator and denominator by x^p where p = degree of denominator)

$$=\lim_{x\to\infty}\frac{5-\overset{\to 0}{\frac{2}{x}}+\overset{\to 0}{\frac{4}{x^3}}-\overset{\to 0}{\frac{1}{x^4}}}{2+\underset{\to 0}{\frac{3}{x^2}}+\underset{\to 0}{\frac{4}{x^4}}}=\frac{5-0+0-0}{2+0+0}$$

(since $\lim\limits_{x\to\infty}\frac{1}{x^n}=0$ if n is a positive integer)

$$=\frac{5}{2}$$

(b) $\lim\limits_{x\to-\infty}\frac{3x^2-5x+4}{4x-3}$ $\left(\frac{\infty}{-\infty}\text{ form}\right)$

$$=\lim_{x\to-\infty}\frac{\frac{3x^2-5x+4}{x}}{\frac{4x-3}{x}}$$

(Divide both the numerator and denominator by x)

$$=\lim_{x\to-\infty}\frac{\overset{\to -\infty}{3x}-5+\overset{\to 0}{\frac{4}{x}}}{4-\underset{\to 0}{\frac{3}{x}}}=\frac{-\infty}{4}=-\infty$$

$\therefore$ The limit $\lim\limits_{x\to-\infty}\frac{3x^2-5x+4}{4x-3}$ does not exist.

(c) $\lim\limits_{x\to\infty}\frac{x^3+7x-2}{5x^4+6x^3-1}$ $\left(\frac{\infty}{\infty}\text{ form}\right)$

$$=\lim_{x\to\infty}\frac{\frac{x^3+7x-2}{x^4}}{\frac{5x^4+6x^3-1}{x^4}}$$

(Divide both the numerator and denominator by x^4)

$$=\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{7}{x^3}-\frac{2}{x^4}}{5+\frac{6}{x}-\frac{1}{x^4}}=\frac{0+0-0}{5+0-0}=\frac{0}{5}=0$$

Example 8

Evaluate the following limits (a) $\lim_{x\to\infty}\frac{2x-3}{\sqrt{4x^2+7}-5x}$ (b) $\lim_{x\to-\infty}\frac{2x-3}{\sqrt{4x^2+7}-5x}$

Solution

(a) $\lim_{x\to\infty}\frac{2x-3}{\sqrt{4x^2+7}-5x}$ $\left(\frac{\infty}{-\infty}\text{ form}\right)$

$$=\lim_{x\to\infty}\frac{\frac{2x-3}{x}}{\frac{\sqrt{4x^2+7}-5x}{x}}\underbrace{=}_{\because x=\sqrt{x^2}\text{ for }x>0}\lim_{x\to\infty}\frac{2-\frac{3}{x}}{\sqrt{\frac{4x^2+7}{x^2}}-5}=\lim_{x\to\infty}\frac{2-\frac{3}{x}}{\sqrt{4+\frac{7}{x^2}}-5}=\frac{2-0}{\sqrt{4+0}-5}=-\frac{2}{3}$$

(b) $\lim_{x\to-\infty}\frac{2x-3}{\sqrt{4x^2+7}-5x}$ $\left(\frac{-\infty}{\infty}\text{ form}\right)$

$$=\lim_{x\to-\infty}\frac{\frac{2x-3}{x}}{\frac{\sqrt{4x^2+7}-5x}{x}}\underbrace{=}_{\because x=-\sqrt{x^2}\text{ for }x<0}\lim_{x\to-\infty}\frac{2-\frac{3}{x}}{-\sqrt{\frac{4x^2+7}{x^2}}-5}=\lim_{x\to-\infty}\frac{2-\frac{3}{x}}{-\sqrt{4+\frac{7}{x^2}}-5}=\frac{2-0}{-\sqrt{4+0}-5}$$

$$=-\frac{2}{7}$$

Example 9

Evaluate the limit $\lim_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-x}\right)$.

Solution

$\lim_{x\to\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-x}\right)$ ($\infty-\infty$ **form, which is an indeterminate form**)

$$=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+3x}-\sqrt{x^2-x}\right)\left(\sqrt{x^2+3x}+\sqrt{x^2-x}\right)}{\sqrt{x^2+3x}+\sqrt{x^2-x}}$$

$$=\lim_{x\to\infty}\frac{(x^2+3x)-(x^2-x)}{\sqrt{x^2+3x}+\sqrt{x^2-x}}=\lim_{x\to\infty}\frac{4x}{\sqrt{x^2+3x}+\sqrt{x^2-x}}$$

$$=\lim_{x\to\infty}\frac{\frac{4x}{x}}{\frac{\sqrt{x^2+3x}+\sqrt{x^2-x}}{x}}=\lim_{x\to\infty}\frac{4}{\sqrt{\frac{x^2+3x}{x^2}}+\sqrt{\frac{x^2-x}{x^2}}}=\lim_{x\to\infty}\frac{4}{\sqrt{1+\frac{3}{x}}+\sqrt{1-\frac{1}{x}}}$$

$$=\frac{4}{\sqrt{1+0}+\sqrt{1-0}}=2$$

Sandwich Theorem (or Squeeze Theorem)

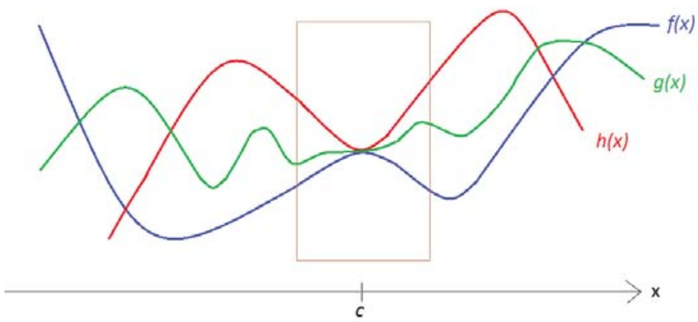
Let I be an interval containing c. Suppose that for every $x \in I$ with $x \neq c$, we have

$$\boxed{f(x) \leq g(x) \leq h(x)}.$$

Furthermore, suppose that $\boxed{\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L}$.

(That is, $f(x)$ and $h(x)$ approach the **same limit** L as x approaches c.)

Then $\boxed{\lim_{x\to c} g(x) = L}$.



The functions $f(x)$ and $h(x)$ are called the **lower** and **upper bounds**, respectively, of $g(x)$.

Example 10

If $2 - x^2 \leq g(x) \leq 2\cos x$ for all $x \in \mathbb{R}$, find $\lim_{x\to 0} g(x)$.

Solution

Limit of lower bound: $\lim_{x\to 0}(2 - x^2) = 2 - 0^2 = 2.$

Limit of upper bound: $\lim_{x\to 0} 2\cos x = 2\underbrace{\cos 0}_{=1} = 2.$

$\because \lim_{x\to 0}(2 - x^2) = \lim_{x\to 0} 2\cos x = 2$

$\therefore$ By the Sandwich (or Squeeze) Theorem,

$$\lim_{x\to 0} g(x) = 2.$$

Example 11

Evaluate the limit $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$.

Solution

First note that the function $x^2 \sin\left(\frac{1}{x}\right)$ is not defined at $x = 0$.

For any $x \neq 0$, we know that $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$.

Multiplying both sides by x^2, we get $-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$.

Limit of lower bound: $\lim_{x \to 0}(-x^2) = -0^2 = 0$

Limit of upper bound: $\lim_{x \to 0} x^2 = 0^2 = 0$

Since $\lim_{x \to 0}(-x^2) = \lim_{x \to 0} x^2 = 0$, by the **Sandwich Theorem**, we have

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

Example 12

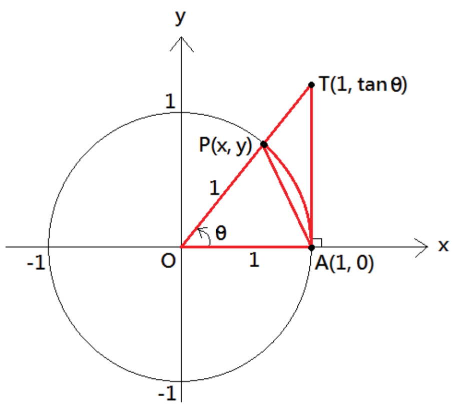
Show that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ (where θ is in radians) by using the Sandwich Theorem.

Solution

First note that the function $\frac{\sin\theta}{\theta}$ is not defined at $\theta = 0$. ($\frac{\sin\theta}{\theta}$ is of the $\frac{0}{0}$ **form** at $\theta = 0$.)

We want to show that $\lim_{\theta \to 0^-} \frac{\sin\theta}{\theta} = \lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$.

To evaluate the right hand limit $\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta}$, we consider a unit circle centered at the origin. Let $P(x, y)$ be a point on the circle in the first quadrant, and θ be the angle (in **radians**) measured from the positive x-axis to the line segment OP. Since $P(x, y)$ lies in the first quadrant, we have $0 < \theta < \frac{\pi}{2}$.



From the diagram, we see that

$$\text{Area of } \Delta\text{OAP} < \text{Area of sector OAP} < \text{Area of } \Delta\text{OAT}.$$

Area of $\Delta\text{OAP} = \frac{1}{2} \cdot (OP) \cdot (OA) \cdot \sin\theta = \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin\theta = \frac{\sin\theta}{2}$

Area of sector OAP $= \pi r^2 \cdot \frac{\theta}{2\pi} = \pi \cdot 1^2 \cdot \frac{\theta}{2\pi} = \frac{\theta}{2}$

Area of $\Delta\text{OAT} = \frac{(OA)\cdot(AT)}{2} = \frac{(1)\cdot(\tan\theta)}{2} = \frac{\tan\theta}{2}$

Therefore, $\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\tan\theta}{2}$, i.e. $\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\sin\theta}{2\cos\theta}$.

Dividing both sides by $\frac{\sin\theta}{2}$, we get

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Taking reciprocal on both sides, we get

$$1 > \frac{\sin\theta}{\theta} > \frac{\cos\theta}{1}, \quad \text{i.e. } \cos\theta < \frac{\sin\theta}{\theta} < 1.$$

Since $\lim_{\theta\to 0^+} \cos\theta = \cos 0 = 1$ and $\lim_{\theta\to 0^+} 1 = 1$, by the **Sandwich Theorem**, we have

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1.$$

To evaluate the left hand limit $\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta}$ (where $\theta < 0$), we let $\theta = -\alpha$ where $\alpha > 0$. Then

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = \lim_{(-\alpha)\to 0^-} \frac{\sin(-\alpha)}{-\alpha} = \lim_{\alpha\to 0^+} \frac{-\sin\alpha}{-\alpha} = \lim_{\alpha\to 0^+} \frac{\sin\alpha}{\alpha} = 1 \text{ (from the above result).}$$

Since $\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = \lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1$, we have $\boxed{\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1}$. □

Useful result:

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x}=1}.$$

It follows that

- $\boxed{\lim_{x\to 0}\frac{x}{\sin x}=1}$

  It is because $\lim_{x\to 0}\frac{x}{\sin x}=\lim_{x\to 0}\frac{1}{\frac{\sin x}{x}}=\frac{1}{\lim_{x\to 0}\frac{\sin x}{x}}=\frac{1}{1}=1.$

- $\boxed{\lim_{x\to 0}\frac{\sin(cx)}{cx}=1}$, where c is a non-zero constant.

  It is because $\lim_{x\to 0}\frac{\sin(cx)}{cx}=\lim_{(cx)\to 0}\frac{\sin(cx)}{cx}=1.$

- $\boxed{\lim_{x\to 0}\frac{\sin^n x}{x^n}=1}$, where n is an integer.

  It is because $\lim_{x\to 0}\frac{\sin^n x}{x^n}=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^n=1^n=1.$

Example 13

Evaluate the following limits, if they exist.

(a) $\lim_{x\to 0}\frac{\sin 2x}{\sin 5x}$ (b) $\lim_{x\to 0}\frac{\tan^2(3x)}{5x^2}$ (c) $\lim_{x\to\frac{\pi}{2}}\frac{\sin\left(x-\frac{\pi}{2}\right)}{2x-\pi}$

Solution

(a) $$\lim_{x\to 0}\frac{\sin 2x}{\sin 5x}=\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\frac{5x}{\sin 5x}\cdot\frac{2x}{5x}=\frac{2}{5}\underbrace{\left(\lim_{x\to 0}\frac{\sin 2x}{2x}\right)}_{=1}\cdot\underbrace{\left(\lim_{x\to 0}\frac{5x}{\sin 5x}\right)}_{=1}=\frac{2}{5}\cdot 1\cdot 1=\frac{2}{5}$$

(b) $$\lim_{x\to 0}\frac{\tan^2(3x)}{5x^2}=\lim_{x\to 0}\frac{\sin^2(3x)}{\cos^2(3x)}\cdot\frac{1}{5x^2}=\lim_{x\to 0}\frac{\sin^2(3x)}{(3x)^2}\cdot\frac{1}{\cos^2(3x)}\cdot\frac{1}{5x^2}\cdot(3x)^2$$

$$=\frac{9}{5}\cdot\underbrace{\left(\lim_{x\to 0}\frac{\sin 3x}{3x}\right)^2}_{=1^2=1}\cdot\underbrace{\left(\lim_{x\to 0}\frac{1}{\cos^2(3x)}\right)}_{=\frac{1}{\cos^2 0}=\frac{1}{1^2}=1}=\frac{9}{5}\cdot 1^2\cdot 1=\frac{9}{5}$$

(c) $$\lim_{x\to\frac{\pi}{2}}\frac{\sin\left(x-\frac{\pi}{2}\right)}{2x-\pi}=\lim_{x-\frac{\pi}{2}\to 0}\frac{\sin\left(x-\frac{\pi}{2}\right)}{2\left(x-\frac{\pi}{2}\right)}\underbrace{=}_{\text{put }\theta=x-\frac{\pi}{2}}\lim_{\theta\to 0}\frac{\sin\theta}{2\theta}=\frac{1}{2}\left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)=\frac{1}{2}\cdot 1=\frac{1}{2}$$

Example 14

Do the limits (a) $\lim_{x\to 0}\frac{\sin x}{|x|}$ and (b) $\lim_{x\to 0}\frac{|\sin x|}{x}$ exist?

Solution

(a) First note that $\frac{\sin x}{|x|}$ is not defined when $x = 0$. Recall that $|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$.

Then $\frac{\sin x}{|x|} = \begin{cases} \frac{\sin x}{x} & \text{if } x > 0 \\ \frac{\sin x}{-x} & \text{if } x < 0 \end{cases}$.

Since $\frac{\sin x}{|x|}$ has different formulas when x is to the left of 0 and to the right of 0, we have to consider the left-hand and right-hand limits separately.

Left hand limit: $\lim_{x\to 0^-}\frac{\sin x}{|x|} = \lim_{x\to 0^-}\frac{\sin x}{-x} = -\lim_{x\to 0^-}\frac{\sin x}{x} = -1$

Right hand limit: $\lim_{x\to 0^+}\frac{\sin x}{|x|} = \lim_{x\to 0^+}\frac{\sin x}{x} = 1$

Since $\lim_{x\to 0^-}\frac{\sin x}{|x|} \neq \lim_{x\to 0^+}\frac{\sin x}{|x|}$, the limit $\lim_{x\to 0}\frac{\sin x}{|x|}$ does not exist.

(b) First note that $\frac{|\sin x|}{x}$ is not defined when $x = 0$.

Recall that $|\sin x| = \begin{cases} \sin x & \text{if } \sin x \ge 0 \\ -\sin x & \text{if } \sin x < 0 \end{cases}$.

When $0 < x < \frac{\pi}{2}$, $\sin x > 0$.

When $-\frac{\pi}{2} < x < 0$, $\sin x < 0$.

Then $\frac{|\sin x|}{x} = \begin{cases} \frac{\sin x}{x} & \text{if } 0 < x < \frac{\pi}{2} \\ \frac{-\sin x}{x} & \text{if } -\frac{\pi}{2} < x < 0 \end{cases}$.

Since $\frac{|\sin x|}{x}$ has different formulas when x is to the left of 0 and to the right of 0, we have to consider the left-hand and right-hand limits separately.

Left hand limit: $\lim_{x\to 0^-}\frac{|\sin x|}{x} = \lim_{x\to 0^-}\frac{-\sin x}{x} = -\lim_{x\to 0^-}\frac{\sin x}{x} = -1$

Right hand limit: $\lim_{x\to 0^+}\frac{|\sin x|}{x} = \lim_{x\to 0^+}\frac{\sin x}{x} = 1$

Since $\lim_{x\to 0^-}\frac{|\sin x|}{x} \neq \lim_{x\to 0^+}\frac{|\sin x|}{x}$, the limit $\lim_{x\to 0}\frac{|\sin x|}{x}$ does not exist.

Example 15

Evaluate the limit $\lim\limits_{x\to 0}\frac{2\sin x\cos x}{x}$.

Solution

Method 1:

$$\lim_{x\to 0}\frac{2\sin x\cos x}{x}=\lim_{x\to 0}\frac{\sin 2x}{x}$$ (by using **double angle formula** $\boxed{\sin 2x=2\sin x\cos x}$)

$$=\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot 2=2\cdot\underbrace{\left(\lim_{x\to 0}\frac{\sin 2x}{2x}\right)}_{=1}=2\cdot 1=2$$

Method 2:

$$\lim_{x\to 0}\frac{2\sin x\cos x}{x}=2\underbrace{\left(\lim_{x\to 0}\frac{\sin x}{x}\right)}_{=1}\cdot\underbrace{\left(\lim_{x\to 0}\cos x\right)}_{=\cos 0=1}=2\cdot 1\cdot 1=2$$

Example 16

Evaluate the limit $\lim\limits_{x\to 0}\frac{1-\cos x}{x\sin x}$.

Solution

Note that the function $\frac{1-\cos x}{x\sin x}$ is of the $\frac{\mathbf{0}}{\mathbf{0}}$ **form** when $x=0$.

Method 1:

$$\lim_{x\to 0}\frac{1-\cos x}{x\sin x}=\lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{x\sin x}$$ by the **Half-angle formula** $\boxed{\sin^2\left(\frac{x}{2}\right)=\frac{1}{2}(1-\cos x)}$.

$$=2\lim_{x\to 0}\left[\frac{\sin\left(\frac{x}{2}\right)\cdot\sin\left(\frac{x}{2}\right)}{\frac{x}{2}\cdot\frac{x}{2}}\cdot\frac{\frac{x}{2}\cdot\frac{x}{2}}{x\sin x}\right]=\frac{2}{4}\lim_{x\to 0}\left[\underbrace{\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}}_{\to 1}\cdot\underbrace{\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}}_{\to 1}\cdot\underbrace{\frac{x}{\sin x}}_{\to 1}\right]=\frac{2}{4}\cdot 1\cdot 1\cdot 1=\frac{1}{2}$$

Method 2:

$$\lim_{x\to 0}\frac{1-\cos x}{x\sin x}=\lim_{x\to 0}\left(\frac{1-\cos x}{x\sin x}\cdot\frac{1+\cos x}{1+\cos x}\right)=\lim_{x\to 0}\frac{1-\cos^2 x}{x\sin x(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{x\sin x\,(1+\cos x)}=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{1}{1+\cos x}\right)=1\cdot\frac{1}{1+1}=\frac{1}{2}$$

Example 17

Evaluate the limit $\lim\limits_{x\to 0}\dfrac{(\sin 3x)^2}{x^2\cos x}$.

Solution

$$\lim_{x\to 0}\frac{(\sin 3x)^2}{x^2\cos x}\quad \left(\frac{0}{0}\ \textbf{form}\right)$$

$$=\lim_{x\to 0}\left[\frac{(\sin 3x)^2}{(3x)^2}\cdot\frac{1}{\cos x}\cdot 3^2\right]$$

$$=1^2\cdot\frac{1}{\cos 0}\cdot 3^2$$

$$=9$$

Example 18

Evaluate the limit $\lim\limits_{x\to 0}\dfrac{\cos(3x)-\cos(7x)}{x^2}$.

Solution

$$\lim_{x\to 0}\frac{\cos(3x)-\cos(7x)}{x^2}\quad \left(\frac{0}{0}\ \textbf{form}\right)$$

$$=\lim_{x\to 0}\frac{-2\sin\left(\frac{3x+7x}{2}\right)\sin\left(\frac{3x-7x}{2}\right)}{x^2}$$

by using the **sum-to-product formula**

$$\boxed{\cos A-\cos B=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}$$

$$=\lim_{x\to 0}\frac{-2\sin(5x)\sin(-2x)}{x^2}$$

$$=\lim_{x\to 0}\frac{-2\sin(5x)[-\sin(2x)]}{x^2}$$

since $\sin(2x)$ is an odd function

$$=2\lim_{x\to 0}\underbrace{\frac{\sin(5x)}{5x}}_{\to 1}\cdot\underbrace{\frac{\sin(2x)}{2x}}_{\to 1}\cdot 5\cdot 2$$

$$=2\cdot 1\cdot 1\cdot 5\cdot 2=20$$

Example 19

Evaluate the limit $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$.

Solution

$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}$ $\left(\frac{0}{0}\text{ form}\right)$

$=\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{x^3}$

$=\lim_{x\to 0}\frac{\sin x\left(\frac{1}{\cos x}-1\right)}{x^3}$

$=\lim_{x\to 0}\frac{\sin x\left(\frac{1-\cos x}{\cos x}\right)}{x^3}$

$=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}\cdot\frac{1-\cos x}{x^2}$

$=2\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}\cdot\frac{\sin^2\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^2}\cdot\frac{\left(\frac{x}{2}\right)^2}{x^2}$ by the half angle formula $\sin^2\left(\frac{x}{2}\right)=\frac{1}{2}(1-\cos x)$.

$=2\cdot 1\cdot 1\cdot 1^2\cdot\left(\frac{1}{2}\right)^2$

$=\frac{1}{2}$

Continuity of functions

Definition (Continuity at a point)

Let f be defined on an open interval containing c.

Then f is **continuous** at $x=c$ if and only if

$$\boxed{\lim_{x\to c}f(x)=f(c).}$$

By this definition, there are 3 conditions for continuity of f at $x=c$:

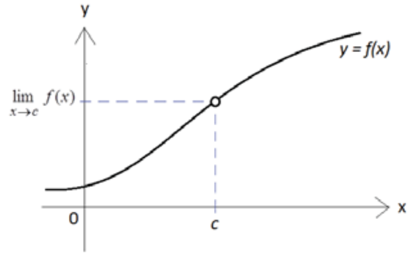
(i) $f(c)$ exists (i.e. c is in the domain of f)

(ii) $\lim_{x\to c}f(x)$ exists

(iii) $\lim_{x\to c}f(x)=f(c)$

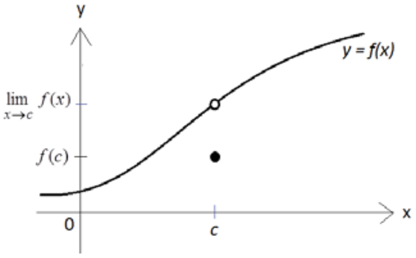
If any one of these three conditions fails, then f is **discontinuous** at $x=c$ (i.e. there is a break on the graph of $y=f(x)$ at $x=c$).

Examples



$f(x)$ is not defined at $x = c$, i.e. $f(c)$ does not exist.

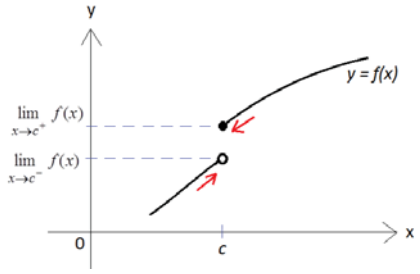
$\therefore$ f is **discontinuous** at $x = c$.



Both $f(c)$ and $\lim_{x\to c} f(x)$ exist.

However, $\lim_{x\to c} f(x) \neq f(c)$

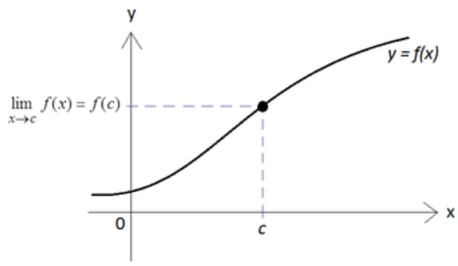
$\therefore$ f is **discontinuous** at $x = c$.



Since $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$,

the limit $\lim_{x\to c} f(x)$ does not exist.

$\therefore$ f is **discontinuous** at $x = c$.



Both $f(c)$ and $\lim_{x\to c} f(x)$ exist.

Moreover, $\lim_{x\to c} f(x) = f(c)$

$\therefore$ f is **continuous** at $x = c$.

Example 20

Is $f(x) = \frac{x^2-1}{x-1}$ continuous at $x = 1$?

Solution

$f(x) = \frac{x^2-1}{x-1}$ is not defined at $x = 1$, i.e. $f(1)$ does not exist.

$\therefore$ $f(x) = \frac{x^2-1}{x-1}$ is not continuous at $x = 1$.

(i.e. f is discontinuous at $x = 1$.)

Example 21

Let $g(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Is g continuous at $x = 0$?

Solution

The function g is defined at $x = 0$, so $g(0)$ exists.

Left hand limit: $\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} \frac{x}{|x|} \underbrace{=}_{\because x<0} \lim_{x \to 0^-} \frac{x}{-x} = \lim_{x \to 0^-} (-1) = -1$

Right hand limit: $\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} \frac{x}{|x|} \underbrace{=}_{\because x>0} \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} (1) = 1$

Since $\lim_{x \to 0^-} g(x) \neq \lim_{x \to 0^+} g(x)$, the limit $\lim_{x \to 0} g(x)$ does not exist.

$\therefore$ $g(x)$ is discontinuous at $x = 0$.

Example 22

Let $h(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$. Is h continuous at $x = 0$?

Solution

$h(x)$ is defined at $x = 0$, so $h(0)$ exists.

$\lim_{x \to 0} h(x) \underbrace{=}_{\because x \neq 0} \lim_{x \to 0} x^2 = 0^2 = 0.$

Since $\lim_{x \to 0} h(x) = 0 \neq 1 = h(0)$, h is discontinuous at $x = 0$.

Example 23

The function $f(x)=\frac{x^2-3x-10}{x-5}$ is undefined at $x=5$, then $f(5)$ doesn't exist and so the function f is not continuous at $x=5$. If we define $g(x)=\begin{cases}\frac{x^2-3x-10}{x-5} & \text{if } x\neq 5\\ c & \text{if } x=5\end{cases}$, where c is a constant, find the value of c such that g is continuous at $x=5$.

Solution

The function g is defined at $x=5$, so $g(5)$ exists.

$$\lim_{x\to 5} g(x)=\lim_{x\to 5}\frac{x^2-3x-10}{x-5}=\lim_{x\to 5}\frac{(x-5)(x+2)}{x-5}=\lim_{x\to 5}(x+2)=5+2=7$$

If we put $g(5)=c=7$, then

$$\lim_{x\to 5} g(x)=7=g(5)$$

and therefore the function g is continuous at $x=5$.

Example 24

Let $f(x)=\begin{cases} k\left|\frac{2+x}{x^2-3}\right| & \text{if } -1\le x<0\\ c & \text{if } x=0\\ \frac{x}{\sqrt{2+3x}-\sqrt{2}} & \text{if } 0<x\le 1\end{cases}$, where k and c are constants.

Find the values of c and k so that $f(x)$ is continuous at $x=0$.

Solution

Left-hand limit:

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} k\left|\frac{2+x}{x^2-3}\right|=k\left|\frac{2+0}{0^2-3}\right|=k\left|-\frac{2}{3}\right|=\frac{2}{3}k$$

Right-hand limit:

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{x}{\sqrt{2+3x}-\sqrt{2}}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0^+}\frac{x\left(\sqrt{2+3x}+\sqrt{2}\right)}{\left(\sqrt{2+3x}-\sqrt{2}\right)\left(\sqrt{2+3x}+\sqrt{2}\right)}$$

$$= \lim_{x \to 0^+} \frac{x\left(\sqrt{2+3x}+\sqrt{2}\right)}{(2+3x)-2}$$

$$= \lim_{x \to 0^+} \frac{x\left(\sqrt{2+3x}+\sqrt{2}\right)}{3x}$$

$$= \lim_{x \to 0^+} \frac{\sqrt{2+3x}+\sqrt{2}}{3}$$

$$= \frac{\sqrt{2+0}+\sqrt{2}}{3}$$

$$= \frac{2\sqrt{2}}{3}$$

$\lim_{x \to 0} f(x)$ exists iff $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$, i.e.

$$\frac{2}{3}k = \frac{2\sqrt{2}}{3} \Rightarrow k = \sqrt{2}.$$

$f(x)$ is continuous at $x = 0$ iff $\lim_{x \to 0} f(x) = f(0)$, i.e.

$$c = \frac{2\sqrt{2}}{3}.$$

Examples of continuous functions

- All polynomials, $\sin x$, $\cos x$ and $|x|$ are continuous at every $x = c$ where $c \in \mathbb{R}$.
- A rational function $\frac{f(x)}{g(x)}$ is continuous at every $x = c$ where $c \in \mathbb{R}$, provided $g(c) \neq 0$, and $f(x)$ and $g(x)$ are both continuous at $x = c$.
- e^x is continuous at every $x = c$ where $c \in \mathbb{R}$.
- $\ln x$ is continuous at every $x = c$ where $c > 0$.

Theorems on continuity

1. If f and g are continuous at c, then so are
   - kf (where k is any real number),
   - $f+g$, $f-g$, fg, $\frac{f}{g}$ (where $g(c) \neq 0$), and
   - f^n (where n is a positive integer).

2. If $\lim_{x\to c} g(x) = l$ and if f is continuous at l, then

$$\boxed{\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(l)}.$$

E.g. $\lim_{x\to c}\left[e^{f(x)}\right] = e^{\lim_{x\to c} f(x)}$.

3. If g is continuous at c and f is continuous at $g(c)$, then the composite function $f \circ g$ is continuous at c.

$$\boxed{\lim_{x\to c}(f\circ g)(x) = \lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f\big(g(c)\big)}$$

E.g. $\cos x$ is continuous at every $x \in \mathbb{R}$, and e^x is continuous at every $x \in [-1, 1]$.

$\therefore$ $e^{\cos x}$ is continuous at every $x \in \mathbb{R}$.

Example 25

Find $\lim_{x\to 1} \sin\frac{(x^2-1)\pi}{x-1}$.

Solution

$$\begin{aligned}\lim_{x\to 1} \sin\frac{(x^2-1)\pi}{x-1} &= \sin\left[\lim_{x\to 1}\frac{(x^2-1)\pi}{x-1}\right], \quad \text{since sin function is continuous everywhere}\\ &= \sin\left[\lim_{x\to 1}\frac{(x-1)(x+1)\pi}{x-1}\right]\\ &= \sin\left[\lim_{x\to 1}(x+1)\pi\right]\\ &= \sin(2\pi)\\ &= 0\end{aligned}$$

Example 26

Is $f(x) = \begin{cases} \frac{1-\cos x}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ continuous everywhere?

Solution

Both $1-\cos x$ and x^2 are continuous at every $x \neq 0$, so $\frac{1-\cos x}{x^2}$ is continuous at every $x \neq 0$. Now we determine whether $f(x)$ is continuous at $x = 0$ or not.

$$\lim_{x\to 0} f(x) \underbrace{=}_{\because x\neq 0} \lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0}\left[\frac{(1-\cos x)}{x^2}\cdot\frac{(1+\cos x)}{(1+\cos x)}\right]$$

$$= \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x^2(1+\cos x)}$$

$$= \lim_{x\to 0}\left[\frac{\sin^2 x}{x^2}\cdot\frac{1}{(1+\cos x)}\right] = 1^2\cdot\frac{1}{(1+\cos 0)} = 1^2\cdot\frac{1}{(1+1)} = \frac{1}{2}$$

Since $\lim_{x\to 0} f(x) = \frac{1}{2} \neq 1 = f(0)$, f is discontinuous at $x = 0$.

$\therefore$ f is continuous everywhere except at $x = 0$.

(Note: If we define $f(0) = \frac{1}{2}$, then f is continuous at $x = 0$.)

Example 27

Let $f(x) = \begin{cases} \cos\left(\frac{\pi x}{2}\right) & \text{if } |x| \leq 1 \\ |x-1| & \text{if } |x| > 1 \end{cases}$. Determine the values of x at which f is continuous.

Solution

Rewrite $f(x)$ as $f(x) = \begin{cases} -(x-1) & \text{if } x < -1 \\ \cos\left(\frac{\pi x}{2}\right) & \text{if } -1 \leq x \leq 1 \\ x-1 & \text{if } x > 1 \end{cases}$.

$f(x)$ is continuous at every $x \neq \pm 1$, since $\cos\left(\frac{\pi x}{2}\right)$ and $|x-1|$ are continuous at every $x \neq \pm 1$.

Is $f(x)$ continuous at $x = -1$? Check whether $\lim_{x\to -1} f(x) = f(-1)$.

$$\lim_{x\to -1^-} f(x) \underbrace{=}_{x<-1} \lim_{x\to -1^-} -(x-1) = -(-1-1) = 2$$

$$\lim_{x\to -1^+} f(x) \underbrace{=}_{x>-1} \lim_{x\to -1^+} \cos\left(\frac{\pi x}{2}\right) = \cos\left(\frac{-\pi}{2}\right) = 0$$

Since $\lim_{x \to -1^-} f(x) \neq \lim_{x \to -1^+} f(x)$, the limit $\lim_{x \to -1} f(x)$ does not exist. Hence $f(x)$ is not continuous at $x = -1$.

Is $f(x)$ continuous at $x = 1$? Check whether $\lim_{x \to 1} f(x) = f(1)$.

$$\lim_{x \to 1^-} f(x) \underbrace{=}_{x<1} \lim_{x \to 1^-} \cos\left(\frac{\pi x}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$\lim_{x \to 1^+} f(x) \underbrace{=}_{x>1} \lim_{x \to 1^+} (x-1) = 1 - 1 = 0$$

Since $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 0$, the limit $\lim_{x \to 1} f(x)$ exists and $\lim_{x \to 1} f(x) = 0$.

$f(1) = \cos\left(\frac{\pi}{2}\right) = 0$

Since $\lim_{x \to 1} f(x) = 0 = f(1)$, $f(x)$ is continuous at $x = 1$.

Hence, $f(x)$ is continuous at every $x \in \mathbb{R}\backslash\{-1\}$.

Continuity on an interval

- A function f is **continuous on the open interval** (a, b) if it is continuous at every point inside the interval.

- A function f is **continuous on the closed interval** $[a, b]$ if it is

  (1) **continuous** on the open interval (a, b);

  (2) **right continuous** at the left endpoint a (i.e. $\lim_{x \to a^+} f(x) = f(a)$); and

  (3) **left continuous** at the right endpoint b (i.e. $\lim_{x \to b^-} f(x) = f(b)$).

  If any one of the above conditions fails, f is not continuous on $[a, b]$.

Remark:

A function f is continuous at c if and only if it is both left continuous and right continuous at c, i.e.

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$$

and therefore

$$\lim_{x \to c} f(x) = f(c).$$

Examples

- The function $f(x) = \sqrt{x}$ is continuous on the interval $[0, \infty)$, since f is continuous at every x in the open interval $(0, \infty)$, and also it is right continuous at the left endpoint $x = 0$, i.e.

$$\lim_{x \to 0^+} f(x) = 0 = f(0).$$

- The function $g(x) = \ln x$ is not continuous on the interval $[0, 1]$, since g is not defined at $x = 0$ (i.e. 0 is not in the domain of g).

Intermediate Value Theorem (IVT)

Suppose f is continuous on $[a, b]$, and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Example 28

Show that there is a root of the equation $4x^3 - 6x^2 = -3x + 2$ between 1 and 2.

Solution

$$4x^3 - 6x^2 = -3x + 2 \Rightarrow 4x^3 - 6x^2 + 3x - 2 = \mathbf{0}$$

Let $f(x) = 4x^3 - 6x^2 + 3x - 2$.

Then $f(\mathbf{1}) = -1 < \mathbf{0}$ and $f(\mathbf{2}) = 12 > \mathbf{0}$.

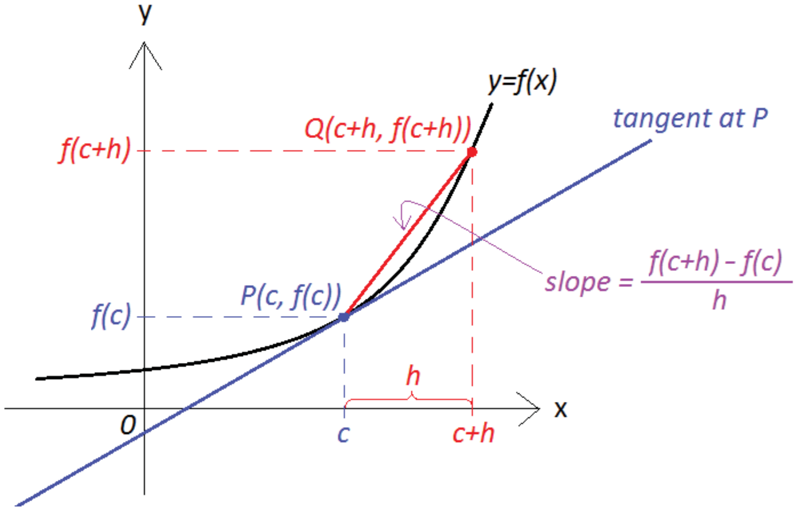
Note that $f(x)$ is continuous everywhere.

By the IVT, there is a number $\boldsymbol{c} \in (\mathbf{1}, \mathbf{2})$ such that $f(\boldsymbol{c}) = \mathbf{0}$,

i.e. there is a root of the equation $4x^3 - 6x^2 = -3x + 2$ between $\mathbf{1}$ and $\mathbf{2}$.

Differentiability of functions

Consider the graph of the function $y = f(x)$.



- A function f is **differentiable at $x = c$** if the limit

$$\boxed{\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}} \quad \text{(or equivalently, } \boxed{\lim_{x\to c}\frac{f(x)-f(c)}{x-c}}\text{)}$$

exists.

- The **derivative of $f(x)$ at $x = c$** (i.e. the slope of the tangent to the curve of $y = f(x)$ at $P(c, f(c))$) is given by

$$\boxed{f'(c) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}},$$

or equivalently,

$$\boxed{f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}},$$

if the limit exists.

(Note: $f'(c)$ is also denoted by $\left.\frac{dy}{dx}\right|_{x=c}$ if $y = f(x)$.)

Now, if we consider all those points x at which f is differentiable, then we can establish a function f' which gives the value of the limit at each x. This function is called the **derivative of f with respect to x**, or the **first derivative of f with respect to x**, and is denoted by $f'(x)$ or $\frac{df(x)}{dx}$.

- From the **First Principle**, the **derivative of $f(x)$** is given by

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}.$$

This gives the slope of the tangent to the curve of $y = f(x)$ at the point $P\big(x, f(x)\big)$ for every x.

(Note: $f'(x)$ is also denoted by $\frac{dy}{dx}$ or y' if $y = f(x)$.)

Example 29

Is $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ differentiable at $x = 0$?

Solution

$f(x)$ is differentiable at $x = 0$ if $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ exists.

$$\lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0} \frac{x^{\frac{1}{3}} - 0^{\frac{1}{3}}}{x-0} = \lim_{x \to 0} x^{\frac{1}{3}-1} = \lim_{x \to 0} x^{-\frac{2}{3}} = \lim_{x \to 0} \frac{1}{x^{\frac{2}{3}}} = \infty$$

[Or using $\lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0} \frac{h^{\frac{1}{3}} - 0^{\frac{1}{3}}}{h} = \lim_{h \to 0} h^{-\frac{2}{3}} = \lim_{h \to 0} \frac{1}{h^{\frac{2}{3}}} = \infty$]

$\therefore$ The limit does not exist.

Hence, $f(x)$ is not differentiable at $x = 0$.

Remark: We say that f is differentiable in an open interval I if it is differentiable at every point of I. For example, $f(x) = \sqrt[3]{x}$ is differentiable at every real number x except at $x = 0$, i.e. it is differentiable in the open intervals $(\infty, 0)$ and $(0, \infty)$.

Example 30

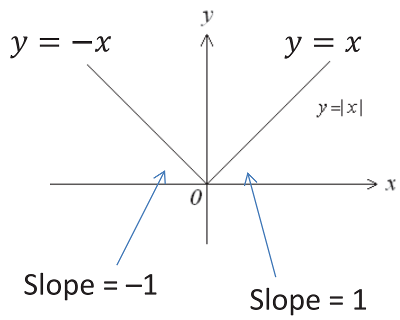
Is $f(x) = |x|$ differentiable at $x = 0$?

Solution

$f(x)$ is differentiable at $x = 0$ if the limit $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$, or equivalently $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$, exists.

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{|0+h|-|0|}{h} = \lim_{h\to 0}\frac{|h|}{h}$$

The above limit involves the absolute value function, so we consider the left hand limit and right hand limit separately.



Right hand limit: $\lim_{h\to 0^+}\frac{|h|}{h} = \lim_{h\to 0^+}\frac{h}{h} = \lim_{h\to 0^+} 1 = 1$

Left hand limit: $\lim_{h\to 0^-}\frac{|h|}{h} = \lim_{h\to 0^-}\frac{-h}{h} = \lim_{h\to 0^-}(-1) = -1$

Since $\lim_{h\to 0^+}\frac{|h|}{h} \neq \lim_{h\to 0^-}\frac{|h|}{h}$, the limit $\lim_{h\to 0}\frac{|h|}{h}$ does not exist.

Hence, $f(x) = |x|$ is not differentiable at $x = 0$.

Summary:

- $f(x)$ is continuous at $x = c$ iff $\lim_{x\to c} f(x) = f(c)$.
- $f(x)$ is differentiable at $x = c$ iff $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ (or $\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$) exists.

Theorem

If f is differentiable at $x = c$, then f is continuous at $x = c$.

Proof: (For your reference)

If f is differentiable at $x = c$, then $f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ exists.

Consider $f(x) - f(c) = \frac{f(x)-f(c)}{x-c}\cdot(x-c)$ for $x \neq c$. Take limits on both sides:

$$\lim_{x\to c}\big(f(x)-f(c)\big) = \lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\cdot(x-c)\right] = \left(\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\right)\left(\lim_{x\to c}(x-c)\right) = f'(c)\cdot 0 = 0$$

$$\Rightarrow \lim_{x\to c} f(x) = f(c)$$

$\therefore$ f is continuous at $x = c$. $\square$

The above theorem says that if a function f is differentiable at $x = c$, then f is continuous at $x = c$. However, the converse is not true. That is, if a function f is continuous at $x = c$, then f is not necessarily differentiable at $x = c$.

For example, $f(x) = |x|$ is continuous at $x = 0$ but it is not differentiable at $x = 0$ (see Example 30).

Differentiability of $f(x)$ at $x = c$ $\Rightarrow$ Continuity of $f(x)$ at $x = c$

$\not\Leftarrow$

Example 31

Is $f(x) = |x|^3$ differentiable at $x = 0$?

Solution

$f(x)$ is differentiable at $x = \mathbf{0}$ if the limit $\lim_{h\to 0}\frac{f(\mathbf{0}+h)-f(\mathbf{0})}{h}$, or equivalently $\lim_{x\to \mathbf{0}}\frac{f(x)-f(\mathbf{0})}{x-\mathbf{0}}$, exists.

$$\begin{aligned}\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{|0+h|^3-|0|^3}{h} = \lim_{h\to 0}\frac{|h|^3}{h} &= \lim_{h\to 0}\frac{|h|^2\cdot|h|}{h}\\ &= \lim_{h\to 0}\frac{h^2\cdot|h|}{h} \quad (\text{since } |h|^2 = h^2)\\ &= \lim_{h\to 0} h\cdot|h|\\ &= 0\cdot|0|\\ &= \mathbf{0}\end{aligned}$$

Since the limit $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ exists, $f(x) = |x|^3$ is differentiable at $x = \mathbf{0}$ and $f'(\mathbf{0}) = \mathbf{0}$.

Example 32

Is $f(x) = |\sin x|$ differentiable at $x = 0$?

Solution

$f(x) = |\sin x|$ is differentiable at $x = 0$ if the limit $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ exists.

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{|\sin x|-|\sin 0|}{x-0} = \lim_{x\to 0} \frac{|\sin x|-0}{x-0} = \lim_{x\to 0} \frac{|\sin x|}{x}$$

Recall that $|\sin x| = \begin{cases} \sin x & \text{if } 0 \le x \le \frac{\pi}{2} \\ -\sin x & \text{if } -\frac{\pi}{2} \le x < 0 \end{cases}.$

Left-hand limit: $\lim_{x\to 0^-} \frac{|\sin x|}{x} = \lim_{x\to 0^-} \frac{-\sin x}{x} = -\lim_{x\to 0^-} \frac{\sin x}{x} = -1.$

Right-hand limit: $\lim_{x\to 0^+} \frac{|\sin x|}{x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1.$

Since $\lim_{x\to 0^-} \frac{|\sin x|}{x} \neq \lim_{x\to 0^+} \frac{|\sin x|}{x}$, the limit $\lim_{x\to 0} \frac{|\sin x|}{x}$ does not exist.

Thus, $f(x) = |\sin x|$ is not differentiable at $x = 0$.

Example 33

Let $f(x) = \begin{cases} \sqrt{x} & \text{if } x \ge 1 \\ x^2 & \text{if } x < 1 \end{cases}.$

(a) Is f continuous at $x = 1$?

(b) Is f differentiable at $x = 1$?

Solution

(a) f is continuous at $x = 1$ iff $\lim_{x\to 1} f(x) = f(1)$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1^2 = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \sqrt{x} = \sqrt{1} = 1$$

$$f(1) = \sqrt{1} = 1$$

$\because \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = 1$

$\therefore$ f is continuous at $x = 1$.

(b) f is differentiable at $x = 1$ iff $\lim_{x\to1}\frac{f(x)-f(1)}{x-1}$ exists.

$$\lim_{x\to1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to1^-}\frac{x^2-\sqrt{1}}{x-1} = \lim_{x\to1^-}\frac{(x-1)(x+1)}{x-1} = \lim_{x\to1^-}(x+1) = 1+1 = 2$$

$$\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1} = \lim_{x\to1^+}\frac{\sqrt{x}-\sqrt{1}}{x-1} = \lim_{x\to1^+}\frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x-1)(\sqrt{x}+1)} = \lim_{x\to1^+}\frac{1}{\sqrt{x}+1}$$

$$= \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

$$\because \lim_{x\to1^-}\frac{f(x)-f(1)}{x-1} \neq \lim_{x\to1^+}\frac{f(x)-f(1)}{x-1}$$

$$\therefore \lim_{x\to1}\frac{f(x)-f(1)}{x-1} \text{ does not exist.}$$

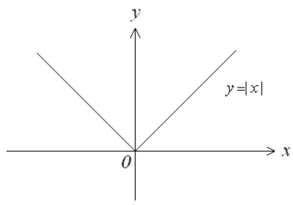
$\therefore$ f is not differentiable at $x = 1$.

Function f is **not differentiable** at $x = c$ if one of the following situations is true:

(i) **f has a sharp corner at c**

E.g. $f(x) = |x|$ has a sharp corner at $x = 0$.

$\therefore$ f is not differentiable at $x = 0$.

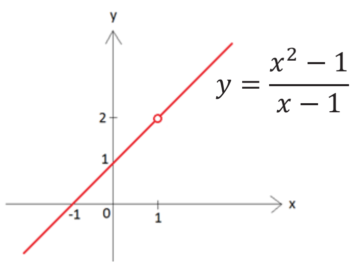


(ii) **f is discontinuous at c**

(i.e. f is not defined at c, or $\lim_{x\to c} f(x)$ does not exist, or $\lim_{x\to c} f(x) \neq f(c)$).

E.g. $f(x) = \frac{x^2-1}{x-1}$ is not defined at $x = 1$, so it is discontinuous at $x = 1$.

$\therefore$ f is not differentiable at $x = 1$.



(iii) **f has a vertical tangent line at c** (i.e. $\lim_{x\to c}|f'(x)| = \infty$).

E.g. $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ has a vertical tangent line at $x = 0$.

$\therefore$ f is not differentiable at $x = 0$.

Differentiation from the First Principle

From the **First Principle**, the **derivative of $f(x)$** is given by

$$\boxed{f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}.$$

Example 34

Let $f(x) = \frac{1}{x}$. Find $f'(x)$ from the **First Principle**.

Solution

From the First Principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0}\frac{x-(x+h)}{x(x+h)h}$$

$$= \lim_{h\to 0}\frac{-h}{x(x+h)h} = \lim_{h\to 0}\frac{-1}{x(x+h)} = \frac{-1}{x(x+0)} = \frac{-1}{x^2}$$

Note: $f(x) = \frac{1}{x}$ is differentiable at every real number x except at $x = 0$.

Example 35

Let $f(x) = x^n$, where n is a positive integer. Find $f'(x)$ from the **First Principle**.

Solution

From the **First Principle**,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{\left[x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n\right] - x^n}{h}$$

$$= \lim_{h\to 0}\left[\binom{n}{1}x^{n-1} + \underbrace{\binom{n}{2}x^{n-2}h}_{\substack{\to 0\\ \text{as } h\to 0}} + \cdots + \underbrace{\binom{n}{n-1}xh^{n-2}}_{\substack{\to 0\\ \text{as } h\to 0}} + \underbrace{h^{n-1}}_{\substack{\to 0\\ \text{as } h\to 0}}\right]$$

$$= \binom{n}{1}x^{n-1}$$

$$= nx^{n-1}$$

<u>Note:</u> In the above calculation, we have used the **Binomial Theorem** to expand $(x+h)^n$.

Binomial Theorem: For all positive integers n,

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}\, b + \binom{n}{2} a^{n-2}\, b^2 + \cdots + \binom{n}{n-1} a\, b^{n-1} + b^n$$

$$= \sum_{r=0}^{n} \binom{n}{r} a^{n-r}\, b^r$$

where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ (called the **binomial coefficient**),

$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$ and

$0! = 1$ (by definition).

Example 36

Let $g(x) = \sin x$. Find $g'(x)$ from the **First Principle**.

Solution

From the **First Principle**,

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\left[\frac{(x+h)+x}{2}\right]\sin\left[\frac{(x+h)-x}{2}\right]}{h}$$

(using the **sum-to-product formula:** $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$)

$$= \lim_{h\to 0} \frac{\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} = \lim_{h\to 0}\left[\underbrace{\cos\left(x+\frac{h}{2}\right)}_{\substack{\to\cos(x)\\ \text{as } h\to 0}} \cdot \underbrace{\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}_{\to 1 \text{ as } h\to 0}\right]$$

$$= \cos x$$

Similarly, it can be shown from the First Principle that if $f(x) = \cos x$, then $f'(x) = -\sin x$.

Example 37

Let $f(x) = \sqrt{x^2+1}$. Find $f'(x)$ from the **First Principle**.

Solution

From the First Principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h}$$

$$= \lim_{h\to 0}\left(\frac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h}\cdot\frac{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}\right)$$

$$= \lim_{h\to 0}\frac{[(x+h)^2+1]-(x^2+1)}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)} = \lim_{h\to 0}\frac{(x^2+2xh+h^2+1)-(x^2+1)}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}$$

$$= \lim_{h\to 0}\frac{2xh+h^2}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)} = \lim_{h\to 0}\frac{2x+h}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}$$

$$= \frac{2x+0}{\sqrt{(x+0)^2+1}+\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$$