IMPORTANT: The answers are provided with the view to deepen your understanding of the topic and to stimulate more in depth discussions. They should not be shared with others or put on the web. The questions and answers may be revised in the next course delivery.

Answers to Tutorial 3

Qn 1

Let F be the event that a plane crash occurs. Let E be the event that a passenger dies.

Find $P(E|F)$ and $P(F)$ from the web.

The probability of dying in a plane crash is

$$P(EF) = P(E|F)P(F)$$

Extended reading: D. Carnegie, How to Stop Worrying and Start Living is a classic. It has a chapter titled “A law that will outlaw many of your worries” which suggests not to over-worry about unlikely events. It is quite fun to read. Both the English and Chinese translation is available in CityU library.

Qn 2

$$P(E) = P\{(2,6), \dots, (6,2)\} = \frac{5}{36}$$

$$P(F) = P\{(2,1), \dots, (2,6)\} = \frac{1}{6}$$

$$P(EF) = P\{(2,6)\} = \frac{1}{36}$$

As $P(EF) \neq P(E)P(F)$, the two events are not independent.

Actually, as long as the sum is 8, the first nor the second dice cannot be 1.

Qn 3

a) Let X_1 and X_2 be the number obtained for the first and second die respectively.

$$E[X_1 + X_2] = E[X_1] + E[X_2] = 3.5 + 3.5 = 7$$

b) Since we may assume that the number obtained for the first and the second die are independent,

$$Var[X_1 + X_2] = Var[X_1] + Var[X_2] = \frac{35}{12} + \frac{35}{12} = 5.833333333$$

Standard deviation = $\sqrt{5.833333333} = 2.415229458 \approx 2.42$

<u>Qn 4</u>

The first and second ranked students are male. The third ranked student is female. The probability is

$$P\{X = 3\} = \left(\frac{44}{52}\right)\left(\frac{43}{51}\right)\left(\frac{8}{50}\right) = 0.114147813 \approx 11.4\%$$

<u>Qn 5</u>

a) The area under the curve is 1. Hence

$$\int_0^\infty \lambda e^{-\frac{x}{9}} dx = 1 \Rightarrow 9\lambda = 1 \Rightarrow \lambda = \frac{1}{9}$$

b) The expected value is

$$E[X] = \int_0^\infty x(\frac{1}{9})e^{-x/9} dx$$

Using integration by part,

$$u = -x \quad dv = e^{-\frac{x}{9}} d\left(-\frac{x}{9}\right) \Rightarrow v = e^{-x/9}$$

$E[X] = -xe^{-\frac{x}{9}}|_0^\infty - \int_0^\infty e^{-\frac{x}{9}} d(-x) = 9$ years

Note: Revision of integration by parts:

Integration by parts stems from the product rule for differentiation. If u and v are functions of x, then

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integration with respect to x gives

$$\int \frac{d}{dx}(uv)dx = \int u\frac{dv}{dx}dx + v\frac{du}{dx}dx$$

Therefore

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

Hence

$$\int udv = uv - \int vdu$$

c)

$$\int_0^6 (\frac{1}{9})e^{-\frac{x}{9}}dx = -\left(e^{-\frac{6}{9}} - 1\right) = 0.486582881 < 0.5$$

There are more than 50% chance that the computer will survive at least 6 years.

d)

The distribution in the question is known as the exponential distribution.

The exponential distribution has the famous property of being memoryless.

A non-negative random variable X is memoryless if

$$P\{X > s + t|X > t\} = P\{X > s\} \quad for\, all\, s,\, t \geq 0$$

It is not fully realistic because the computer components will age. So the computer should be more and more prone to break down given the same period of time s as the computer is getting old.

Also, if the quality control is not good, the chance of a new computer breaking down may also be higher.

Note: It also has the following properties:

1. Probability density function:

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

2. Cumulative distribution:

$$F(x;\lambda) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

3. Expected value and variance

$\mathrm{E}[\mathrm{X}] = \frac{1}{\lambda}$ $\qquad$ $\mathrm{Var}[\mathrm{x}] = \frac{1}{\lambda^2}$

The derivations can be found in pg. 178 of the text.

<u>Qn 6</u>

Geometric Distribution (See A First Course in Probability Pg. 158-159. It is available as a e-book in Course Reserve of EE3001)

8.1 The Geometric Random Variable

Suppose that independent trials, each having a probability $p, 0 < p < 1$, of being a success, are performed until a success occurs. If we let X equal the number of trials required, then

$$P\{X = n\} = (1 - p)^{n-1}p \qquad n = 1, 2, \ldots \tag{8.1}$$

Equation (8.1) follows because, in order for X to equal n, it is necessary and sufficient that the first $n - 1$ trials are failures and the nth trial is a success. Equation (8.1) then follows, since the outcomes of the successive trials are assumed to be independent.

Since

$$\sum_{n=1}^{\infty} P\{X = n\} = p \sum_{n=1}^{\infty} (1 - p)^{n-1} = \frac{p}{1 - (1 - p)} = 1$$

it follows that with probability 1, a success will eventually occur. Any random variable X whose probability mass function is given by Equation (8.1) is said to be a *geometric* random variable with parameter p.

Example 8b Find the expected value of a geometric random variable.

Solution With $q = 1 - p$, we have

$$
\begin{aligned}
E[X] &= \sum_{i=1}^{\infty} iq^{i-1}p \\
&= \sum_{i=1}^{\infty}(i - 1 + 1)q^{i-1}p \\
&= \sum_{i=1}^{\infty}(i - 1)q^{i-1}p + \sum_{i=1}^{\infty} q^{i-1}p \\
&= \sum_{j=0}^{\infty} jq^{j}p + 1 \\
&= q\sum_{j=1}^{\infty} jq^{j-1}p + 1 \\
&= qE[X] + 1
\end{aligned}
$$

Hence,

$$pE[X] = 1$$

yielding the result

$$E[X] = \frac{1}{p}$$

In other words, if independent trials having a common probability p of being successful are performed until the first success occurs, then the expected number of required trials equals $1/p$. For instance, the expected number of rolls of a fair die that it takes to obtain the value 1 is 6. ■

Since $p = 0.4$

The expected number of examinations is $1/p$ = 2.5

It makes the assumption that each attempt is independent. In reality, the attempts are not independent. For example, if the first attempt does not get Distinction, the student has meanwhile gained experience in calmly taking violin examination. Then the second attempt will have a higher probability of getting Distinction, or the student may attempt a violin examination at a higher grade, etc.