Unit 2

Functions

Albert Sung

Outline of Unit 2

Natural Numbers vs Even Numbers

- The set of natural numbers

  $\{1, 2, 3, 4, \ldots\}$

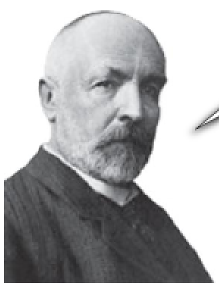
- The set of even numbers

  $\{2, 4, 6, 8\ldots\}$

Which set has a larger size (i.e., more members)?

a) Natural numbers
b) Even numbers
c) They have the same size.
d) Their sizes cannot be compared.

Hilbert's Hotel



I need a room for tonight...
Never mind, I see you're full.

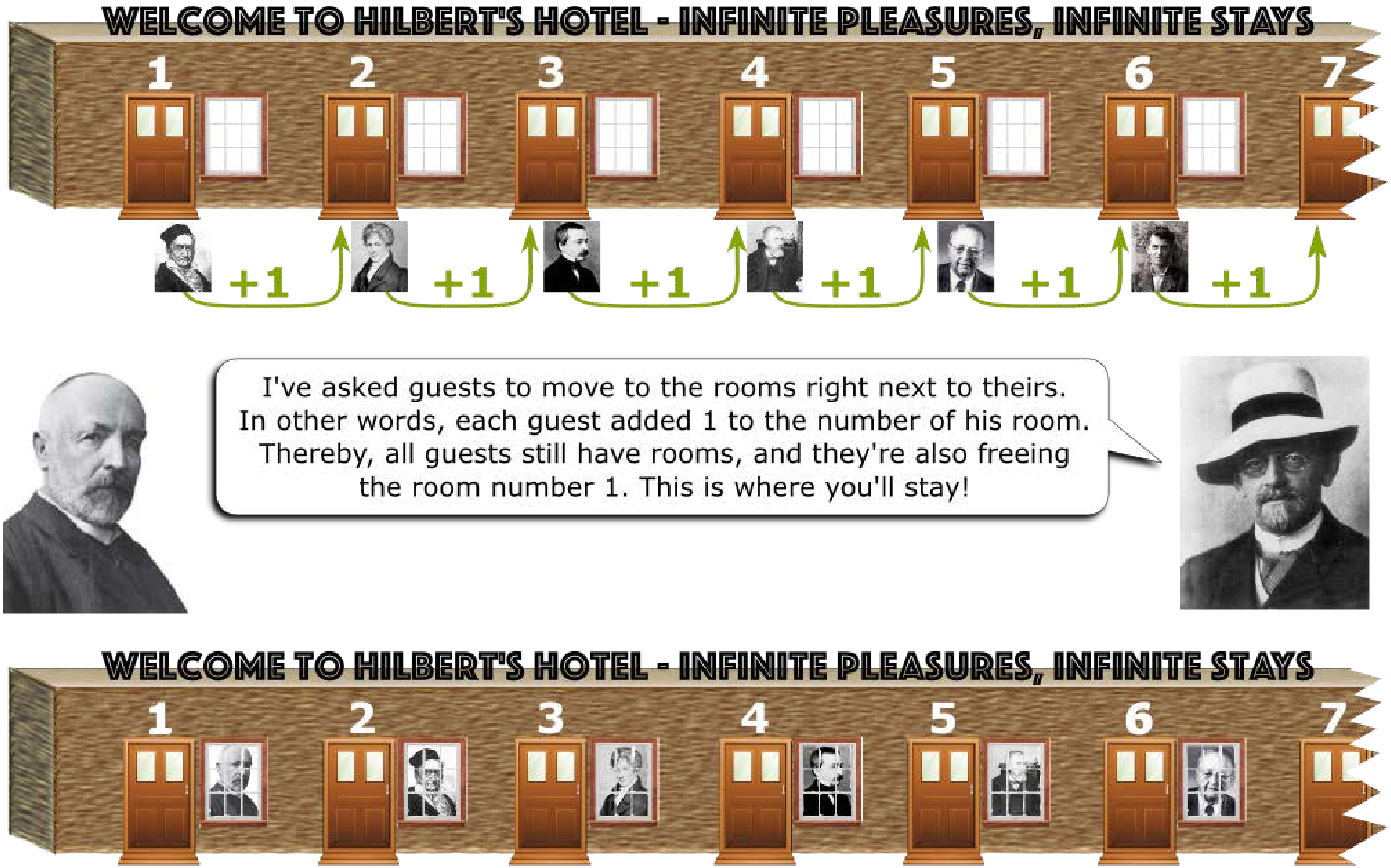
Wait! Wait! We may be full.
But the hotel is *infinite*...

Georg Cantor (1845-1918), a mathematician who has proved that there are different infinities, some are bigger than others.

David Hilbert (1862-1943), a mathematician who has proposed a clever thought experiment to illustrate Cantor's idea on infinities.

Hilbert's Hotel (~1 min video)

https://www.youtube.com/watch?v=faQBrAQ87l4&list=PL73A886F2DD959FF1&index=4



https://www.youtube.com/watch?v=faQBrAQ87l4&list=PL73A886F2DD959FF1&index=4
(~1 min video)

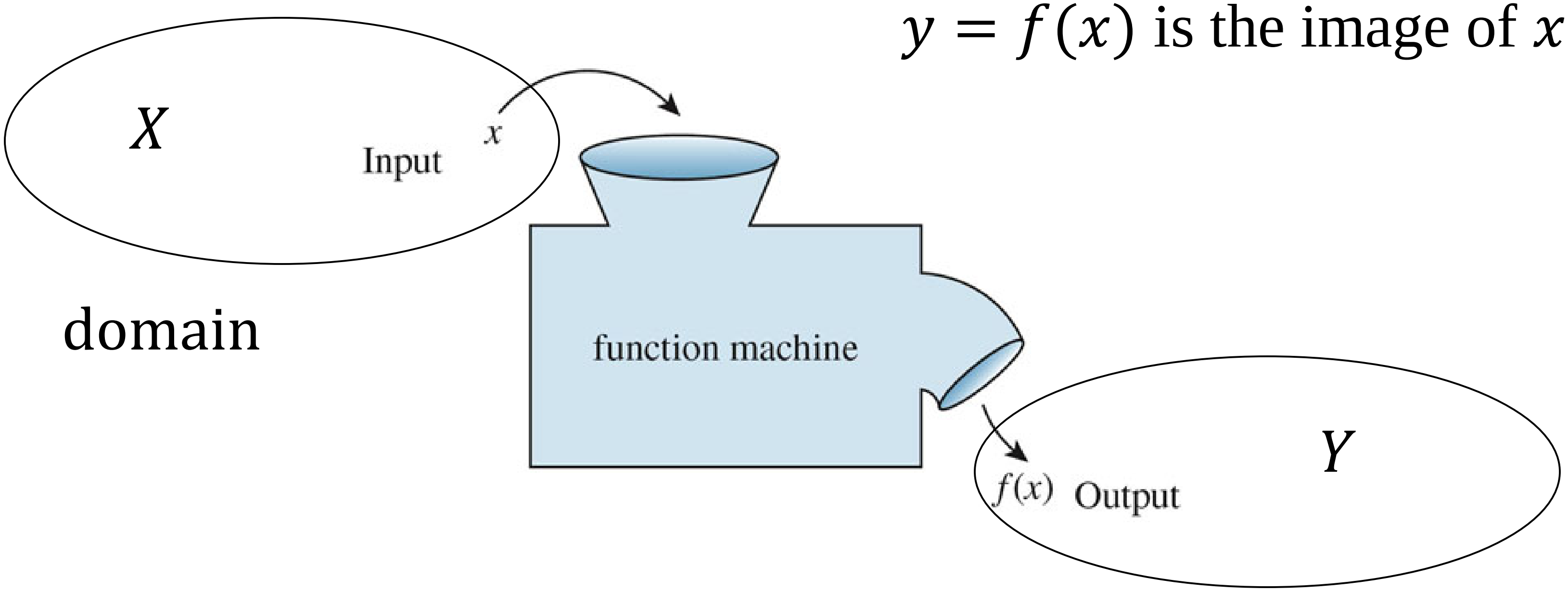
Unit 2.1

Composition of Functions

Functions

- Consider a function $f: X \to Y$.

$y = f(x)$ is the image of x.



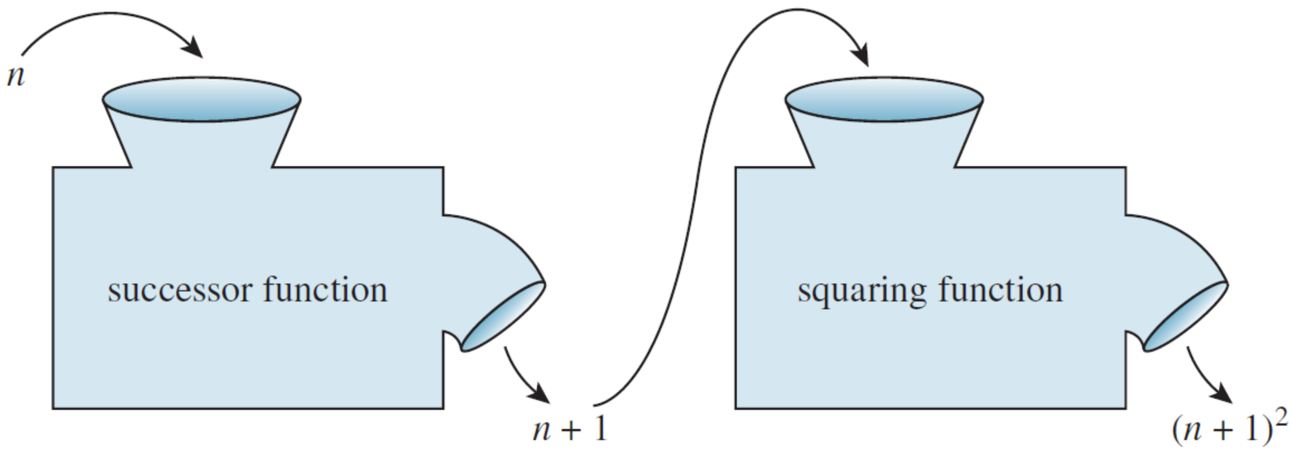
domain

co-domain

$R = \{y \in Y \mid y = f(x)$ for some $x \in X\}$ is the range of f.

Composition of Functions

- If we link two function machines in series as follows, the resultant function is called the composition of them.



What if we change the order of these two machines? Will we get the same output?

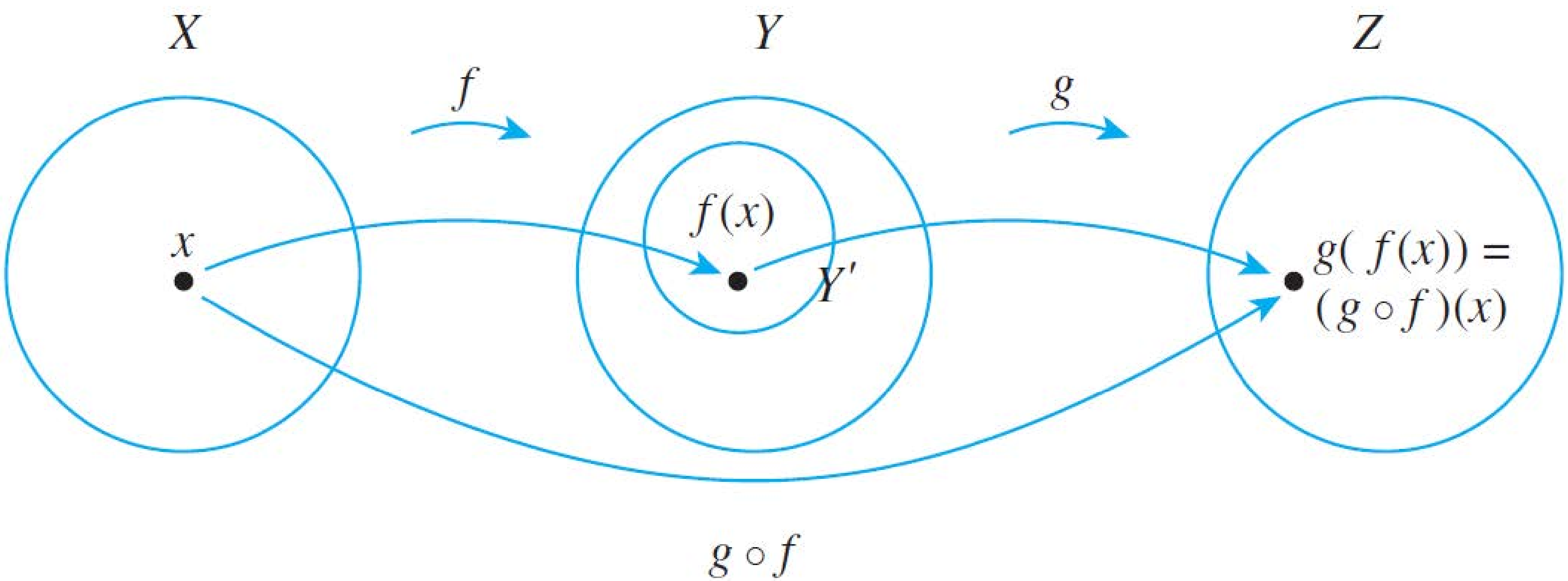
• Definition

Let $f: X \to Y'$ and $g: Y \to Z$ be functions with the property that the range of f is a subset of the domain of g. Define a new function $g \circ f: X \to Z$ as follows:

$$(g \circ f)(x) = g(f(x)) \quad \text{for all } x \in X,$$

where $g \circ f$ is read "g circle f" and $g(f(x))$ is read "g of f of x." The function $g \circ f$ is called the **composition of f and g.**

Note that $Y' \subseteq Y$.



Example

Let $f(n) = n + 1$ and $g(n) = n^2$, where the domains and co-domains of both functions are ***Z***.

a) Find $g \circ f$ and $f \circ g$.

b) Are they equal?

Solution:

a) $(g \circ f)(n) = g\big(f(n)\big) = g(n+1) = (n+1)^2$

$(f \circ g)(n) = f\big(g(n)\big) = f(n^2) = n^2 + 1$

b) No, they are not equal:

$$g \circ f \neq f \circ g.$$

Unit 2.2

One-to-One and Onto

One-to-One Function (Injection)

• Definition

Let F be a function from a set X to a set Y. F is **one-to-one** (or **injective**) if, and only if, for all elements x_1 and x_2 in X,

Useful for proof.

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

or, equivalently,

$$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

(contrapositive)

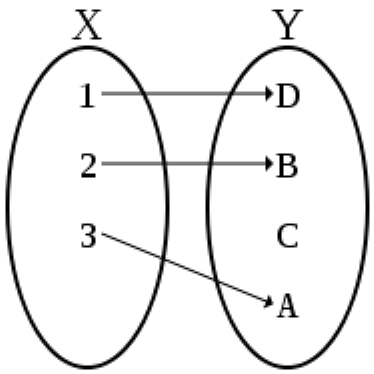
Symbolically,

$$F\colon X \to Y \text{ is one-to-one} \quad \Leftrightarrow \quad \forall x_1, x_2 \in X, \text{if } F(x_1) = F(x_2) \text{ then } x_1 = x_2.$$

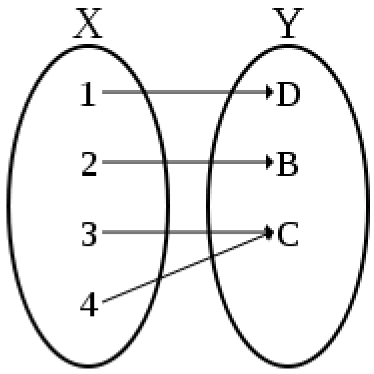
It looks complicated. It's easier (for understanding and for memorization) to use an informal one…

What is an Injection?

- A 1-to-1 function maps distinct elements in its domain to distinct elements in its co-domain.
- Are they injections?



(a)



(b)

Classwork

- Is it injective? Prove or disprove it.
  a) $f: \boldsymbol{R} \rightarrow \boldsymbol{R}$ such that $f(x) = 4x - 1$ for all $x \in R$.

  b) $g: \boldsymbol{Z} \rightarrow \boldsymbol{Z}$ such that $g(n) = n^2$ for all $n \in \boldsymbol{Z}$.

Onto Function (Surjection)

• Definition

Let F be a function from a set X to a set Y. F is **onto** (or **surjective**) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.

Symbolically:

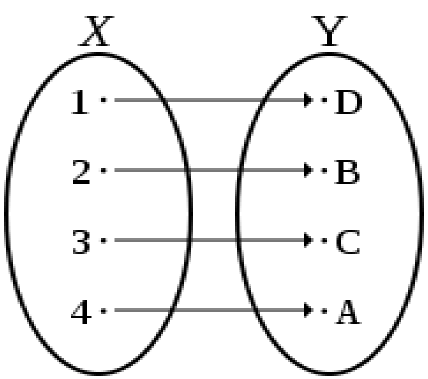
Useful for proof.

$$F: X \to Y \text{ is onto} \quad \Leftrightarrow \quad \forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

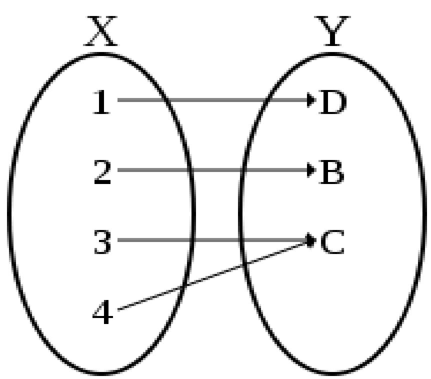
Again we also consider an informal one, in the next slide...

What is a Surjection?

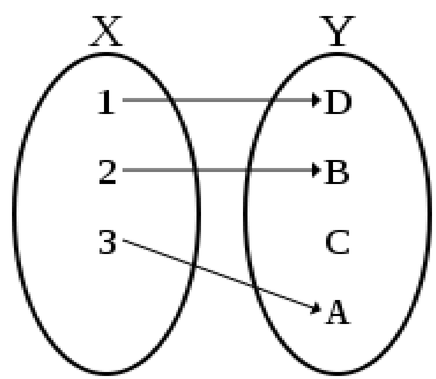
- An onto function has its range equal to its co-domain.
  - i.e. every element in its co-domain has one or more inverse images in its domain.
- Are they surjections?



(a)



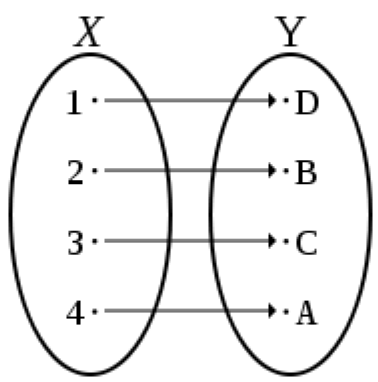
(b)



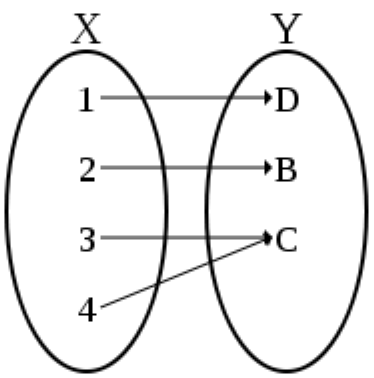
(c)

What is a Bijection?

- A function is a one-to-one correspondence (or bijection) iff it is both 1-to-1 and onto.
- Are they bijections?



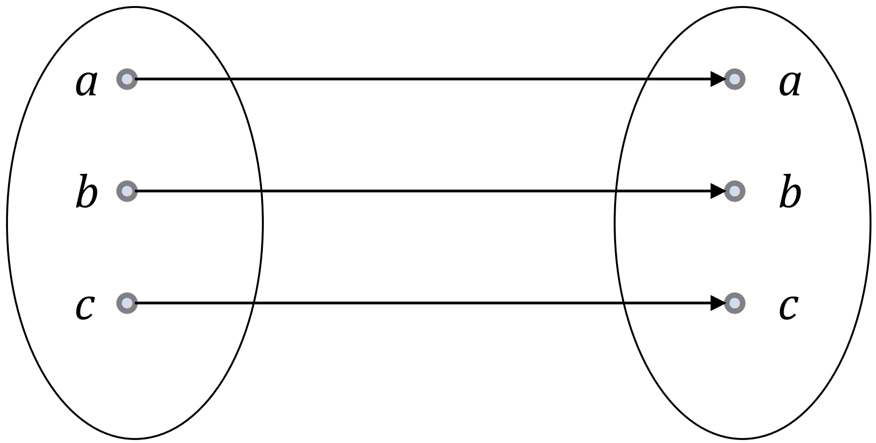
(*a*)



(*b*)

Identity Function

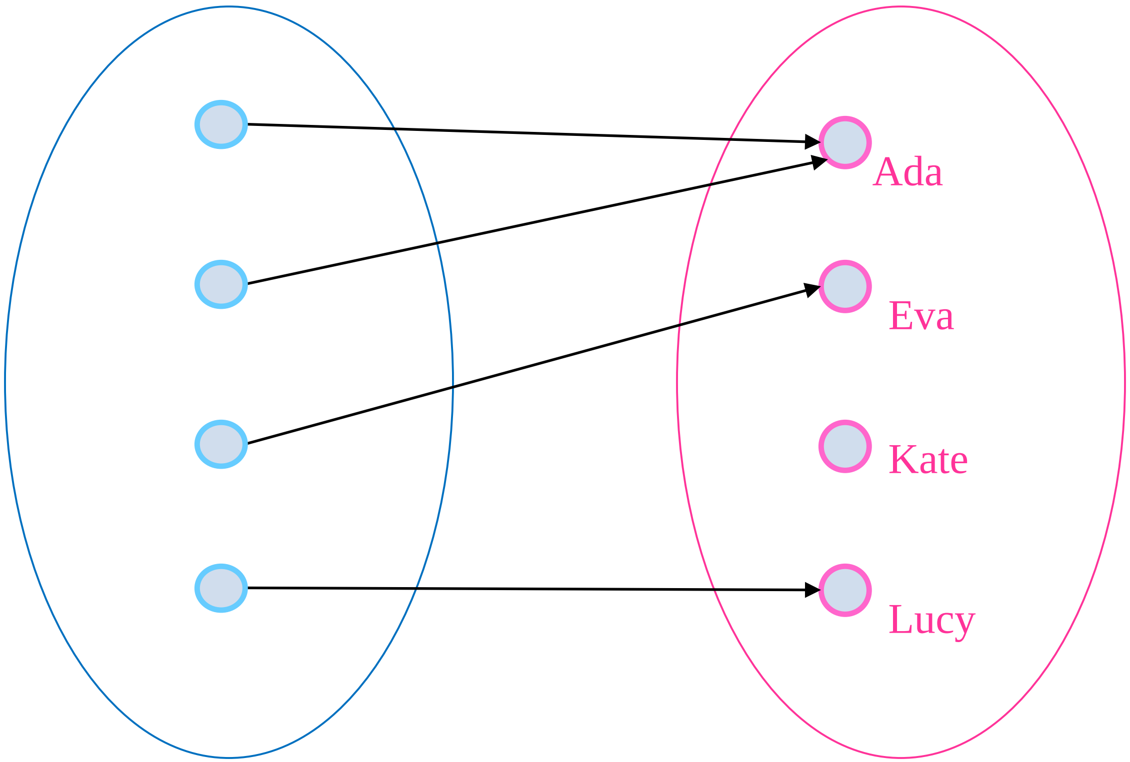
- The identity function I_X on a set X is defined as
$$I_X(x) = x \text{ for all } x \in X.$$



- Any identity function is a bijection.

Quick Summary

- Injection = No girl is loved by more than one boys.
- Surjection = Every girl is loved by some boy.

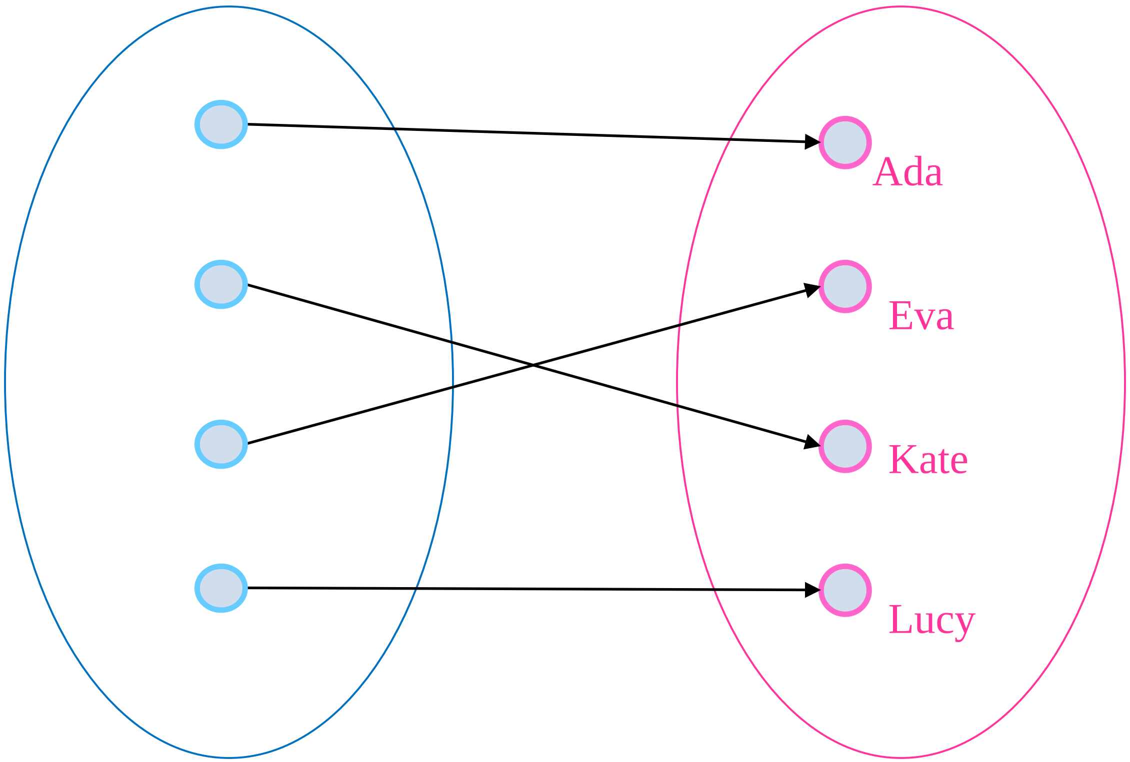


Not 1-to-1 because Ada is torn between two lovers.

Not onto because Kate is loved by none.

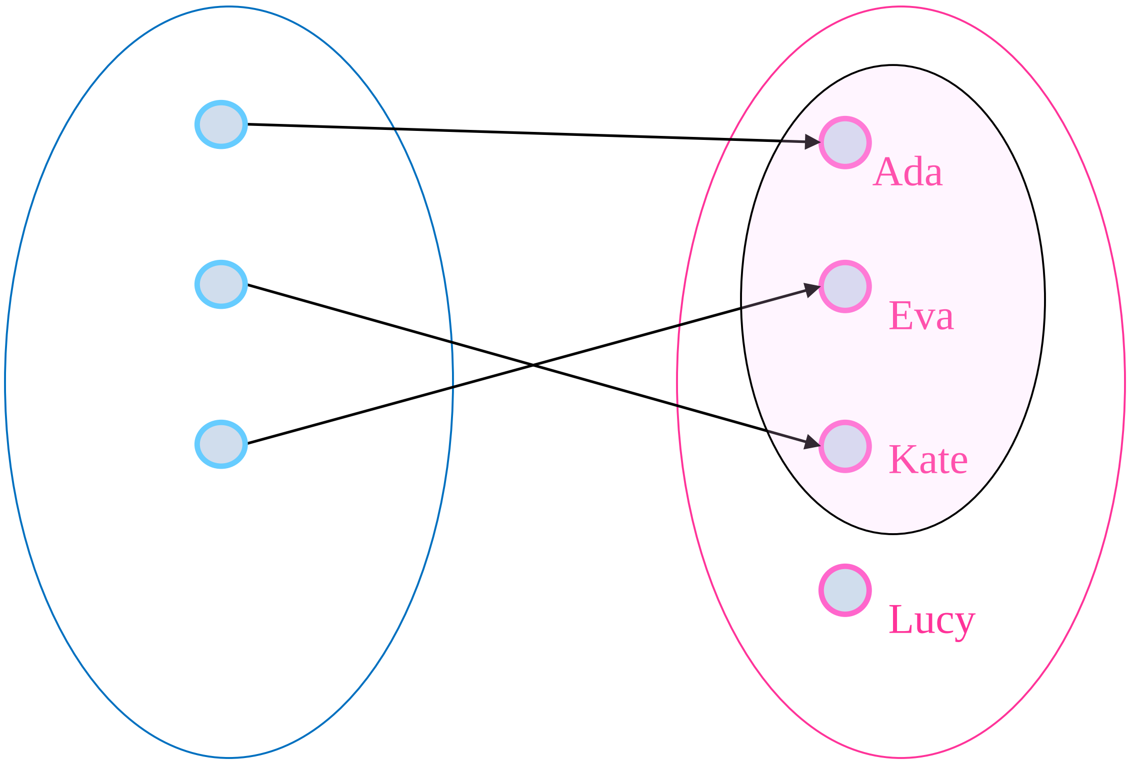
Quick Summary

- Bijection = Perfect Matching!



Quick Summary

- Injection = Bijection to a subset of the co-domain

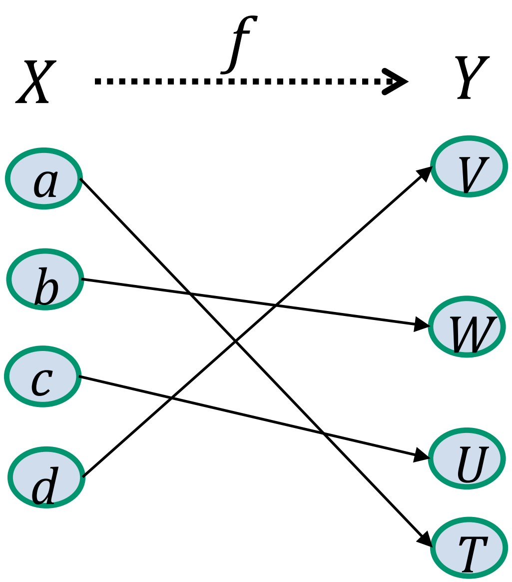


Unit 2.3

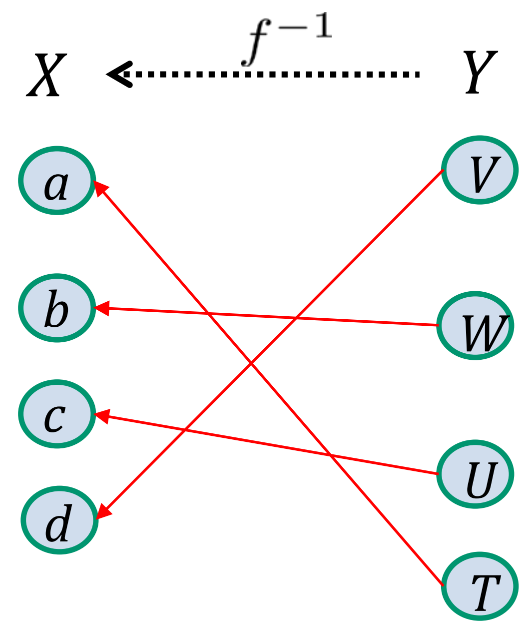
Some Properties

Inverse Functions

- Given a bijection f, we can "undo" the action of f by defining an inverse function f^{-1}.



Undo the original assignment



f^{-1} is also a bijection.

Example

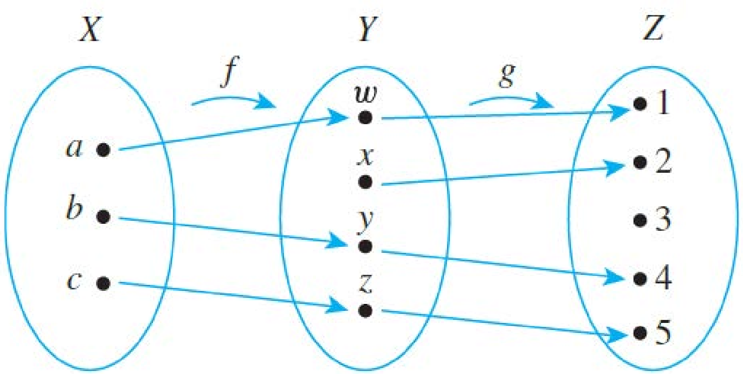
- Find the inverse function of $f(x) = 4x - 1$.
  - Note: the inverse function exists because f is a bijection.
- Solution:

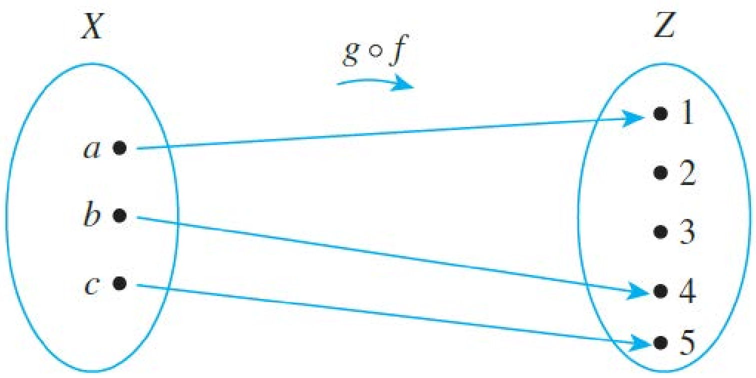
$$
\begin{aligned}
f(x) &= y \\
4x - 1 &= y && \text{by definition of } f \\
x &= \frac{y+1}{4} && \text{by algebra.} \\
f^{-1}(y) &= \frac{y+1}{4}
\end{aligned}
$$

Composition of Injections

Theorem: If $f: X \to Y$ and $g: Y \to Z$ are both injections, then $g \circ f$ is an injection.

Proof Idea:

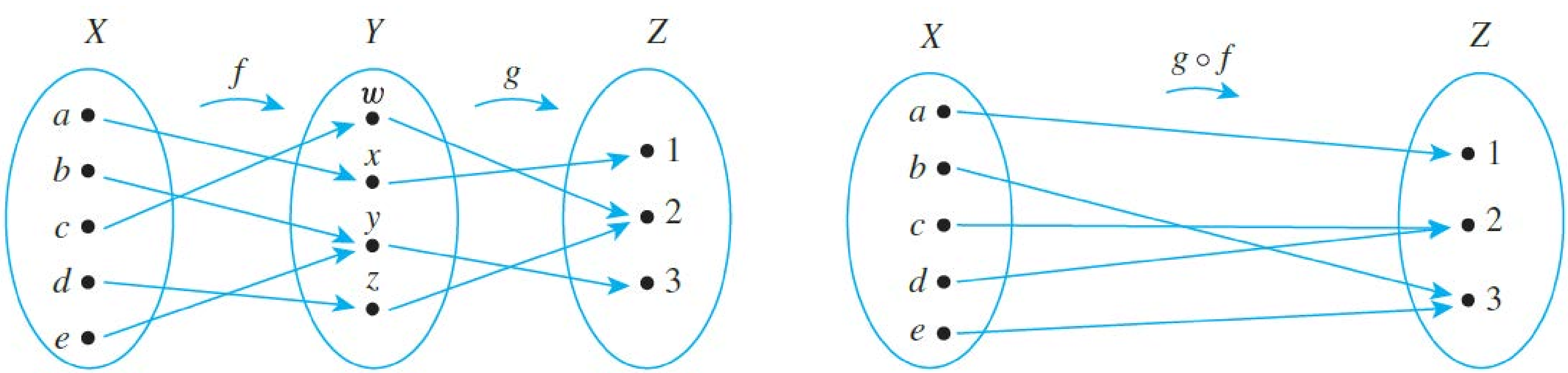




Composition of Surjections

Theorem: If $f: X \to Y$ and $g: Y \to Z$ are both surjections, then $g \circ f$ is a surjection.

Proof Idea:



Composition of Bijections

Theorem: If $f: X \to Y$ and $g: Y \to Z$ are both bijections, then $g \circ f$ is a bijection.

Proof: A direct consequence of the two previous results.

Q.E.D.

Unit 2.4

Cardinality of an Infinite Set

Finite and Infinite Sets

- A set S is said to be finite if there exists a bijection
$$f: S \rightarrow \{1, 2, \ldots, n\}$$
for some natural number n.
- The number n is called the cardinality of S, denoted as $|S|$.
  - i.e., $|S|$ represents the number of elements in S.
- The empty set, $\emptyset$, is considered finite, with cardinality 0.
- A set S is said to be infinite if it is not finite.

Comparison of Cardinalities

- **Definition 1**
  - The sets A and B have the same cardinality (denoted by $|A| = |B|$) iff there is a bijection from A to B.

- **Definition 2**
  - $|A| \leq |B|$ if there is an injection from A to B.
  - $|A| < |B|$ if $|A| \leq |B|$ and $|A| \neq |B|$.

Countable Sets

- A set S is countable if
  - it is finite, or
  - it can be placed in a one-to-one correspondence (i.e. bijection) with the set of natural numbers, {1, 2, 3, …}.
- The cardinality of the set of natural numbers is denoted by $\aleph_0$ (read as aleph-null)
  - ℵ is the first letter of the Hebrew alphabet.

Example: Even Numbers

Show that the set of even numbers is countable.

Solution: True because of the bijection $f(n) = 2n$.

$$\{1,\ 2,\ 3,\ 4,\ 5, \ldots,\ n, \ldots\}$$

$$\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \quad \downarrow$$

$$\{2,\ 4,\ 6,\ 8, 10, \ldots, 2n, \ldots\}$$

Q.E.D.

Example: All Integers

- Show that the set of all integers is countable.

Solution: True because of the following bijection:

$$\{1,\ 2,\ 3,\ 4,\ 5, \ldots\}$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$\{0,\ 1,\ -1,\ 2, -2, \ldots\}$$

Q.E.D.

$$f(n) = \begin{cases} -\frac{n-1}{2} & n \text{ is odd} \\ \frac{n}{2} & n \text{ is even} \end{cases}$$

Note: To prove a set is countable, it is not needed to write down the bijection explicitly. We only need to list its members.

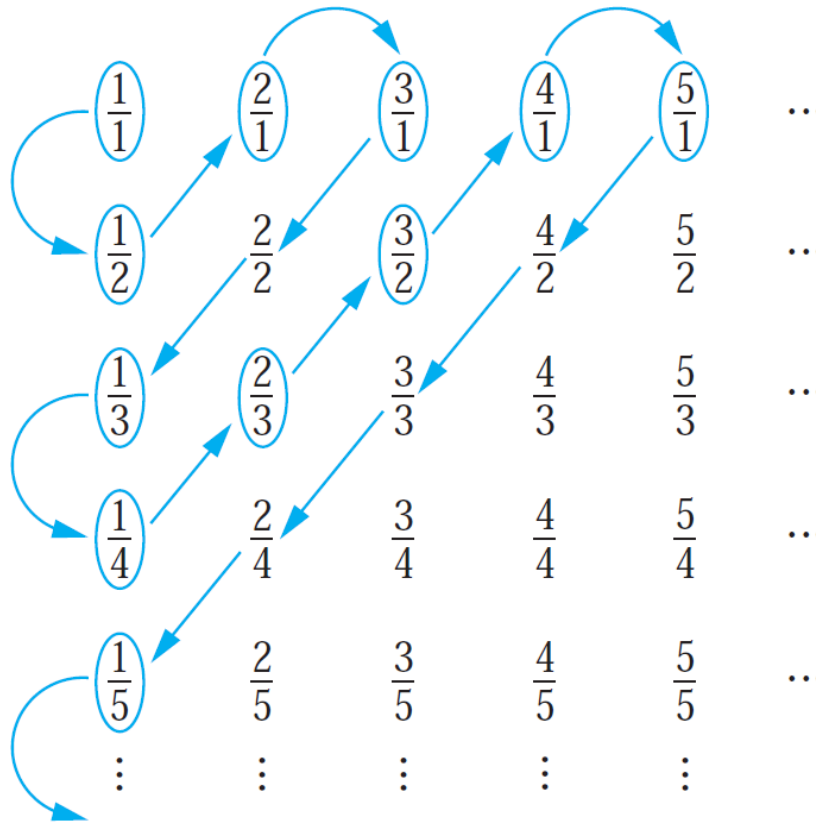
Example: Positive Rationals

Show that the set of positive rational numbers is countable.

Solution:

- By definition, a rational number can be written as p/q, for integers p and $q \neq 0$.
- We can list all rational numbers in the way shown in the next slide.

Terms not circled are not listed because they repeat previously listed terms



There is no need to specify an explicit formula for the bijection.

Stating a rule to pair up the numbers is sufficient.

$$
\begin{array}{cccccccccc}
\{1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, & \ldots\} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\Big\{1, & \frac{1}{2}, & 2, & 3, & \frac{1}{3}, & \frac{1}{4}, & \frac{2}{3}, & 4, & 5, & \ldots\Big\}
\end{array}
$$

Q.E.D.

Union of Countable Sets

Theorem: If A and B are countable, then $A \cup B$ is countable.

Proof:

- $A = \{a_1, a_2, a_3, \ldots\}$
- $B = \{b_1, b_2, b_3, \ldots\}$
- $A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3 \ldots\}$
  - Common elements of A and B, if any, are listed only once.

Q.E.D.

Example: Real Numbers in (0,1)

Show that the set of real numbers in the interval (0, 1) is uncountable.

Solution: We prove by contradiction.
Suppose they are *countable* then we can create a list like

1	$\leftrightarrow$	$x_1 = 0.256173...$
2	$\leftrightarrow$	$x_2 = 0.654321...$
3	$\leftrightarrow$	$x_3 = 0.876241...$
4	$\leftrightarrow$	$x_4 = 0.600002...$
5	$\leftrightarrow$	$x_5 = 0.676783...$
6	$\leftrightarrow$	$x_6 = 0.387514...$
.	.	. .
.	.	. .
n	$\leftrightarrow$	$x_n = 0.a_1a_2a_3a_4a_5 \ldots a_n \ldots$
.	.	. .
.	.	. .

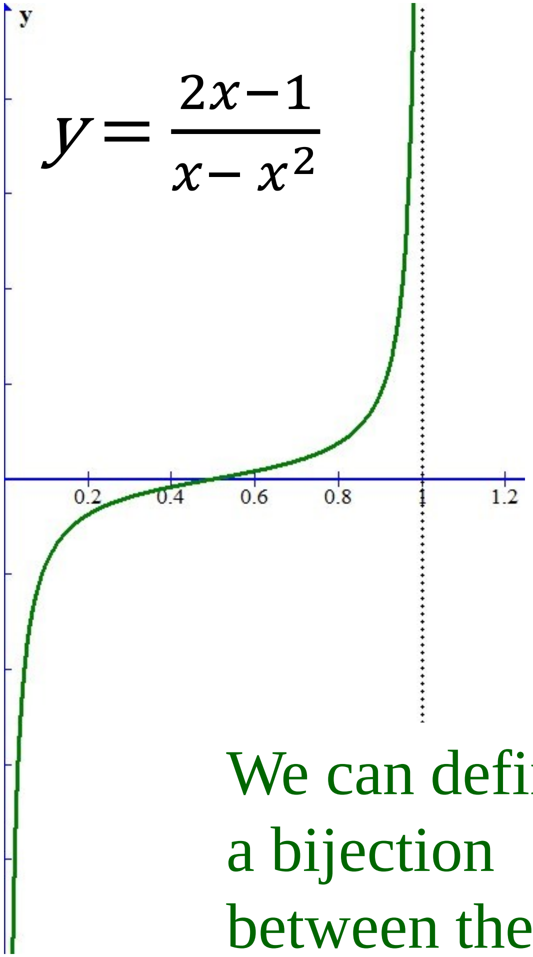
1	$\leftrightarrow$	$x_1 = 0.256173\ldots$	Construct the number
2	$\leftrightarrow$	$x_2 = 0.654321\ldots$	$b = 0.b_1b_2b_3b_4b_5\ \ldots$
3	$\leftrightarrow$	$x_3 = 0.876241\ldots$	
4	$\leftrightarrow$	$x_4 = 0.600002\ldots$	Choose
5	$\leftrightarrow$	$x_5 = 0.676783\ldots$	b_1 not equal to 2 say is 4
6	$\leftrightarrow$	$x_6 = 0.387514\ldots$	b_2 not equal to 5 say is 7
.	.	. .	b_3 not equal to 6 say is 8
.	.	. .	b_4 not equal to 0 say is 3
n	$\leftrightarrow$	$x_n = 0.a_1a_2a_3a_4a_5\ \ldots a_n$	b_5 not equal to 8 say is 7
…			
.	.	. .	b_n not equal to a_n
.	.	. .	

Then $b = 0.b_1b_2b_3b_4b_5\ \ldots = 0.47837\ldots$ is NOT in the list.

The set of real numbers is uncountable! *Q.E.D.*

Example: All Real Numbers

- The set of real numbers has the same cardinality as the set of real numbers in $(0,1)$.   *Why?*



We can define a bijection between them.

- The cardinality is often denoted by *c*.
  - i.e., the continuum of real numbers.

Summary: Cardinality of Some Sets

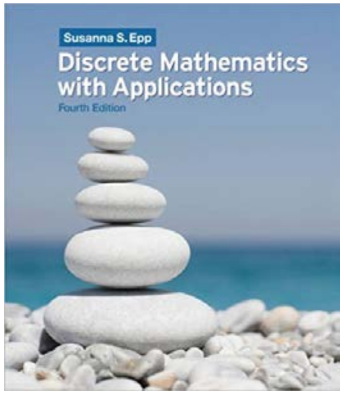
Set	Description	Cardinality
Natural numbers	1, 2, 3, 4, 5, …	$\aleph_0$
Integers	…, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, …	$\aleph_0$
Rational numbers or fractions	All the decimals which terminate or repeat	$\aleph_0$
Irrational numbers	All the decimals which do not terminate or repeat	c
Real numbers	All decimals	c

Is there any cardinality between $\aleph_0$ and c?

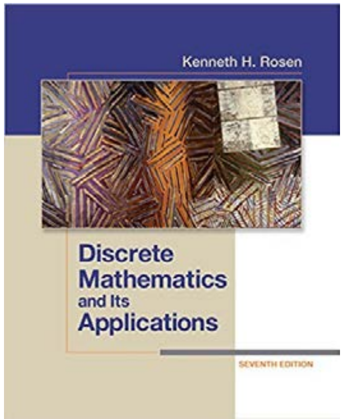
A Hierarchy of Infinities (8 min video)

https://www.youtube.com/watch?v=i7c2qz7sO0I

Recommended Reading



- Chapter 7, S. S. Epp, *Discrete Mathematics with Applications*, 4th ed., Brooks Cole, 2010.



- Sections 2.3 and 2.5, K. H. Rosen, *Discrete Mathematics and its Applications*, 7th ed., McGraw-Hill Education, 2011.