EE3210 Signals & Systems
Mid-term Exam 1 (100 points in total) Name: ______________________________

This is a open-book exam. Submission due date is **12 pm, noon, March. 9th, 2021**. Late submission will not be accepted. If you need more space, please feel free to attach additional papers. Once you're finished, scan and upload it to Canvas course website.

**Honor Pledge**

Please review the following honor code, then sign your name and write down the date.

1. I pledge that the answers in this exam are my own and that I will not seek or obtain an unfair advantage in producing these answers. Specifically,
   (a) I will not plagiarize (copy without citation) from any source;
   (b) I will not communicate or attempt to communicate with any other person during the exam;
   (c) neither will I give or attempt to give assistance to another student taking the exam; and
   (d) I will use only approved devices (e.g., calculators) and/or approved device models.
2. I understand that any act of academic dishonesty can lead to disciplinary action.

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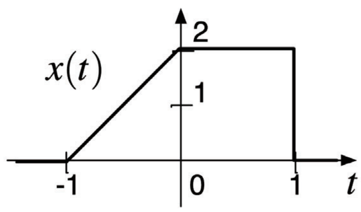
Date

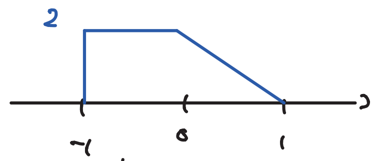
1. (10 points) Consider signal $x(t)$ as plotted in the figure below.



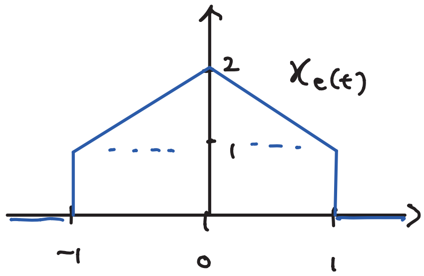

(4pts) (a) Plot the even part $x_e(t)$ and odd part $x_o(t)$ of the given signal $x(t)$.

Solution)

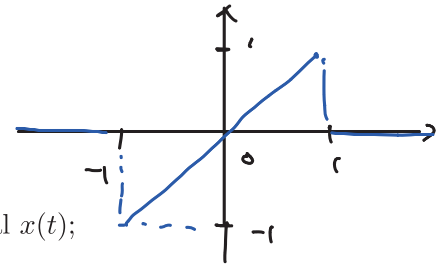
a) Since $x(-t)$ is given by



$x_e(t) = \frac{1}{2}(x(t) + x(-t))$,



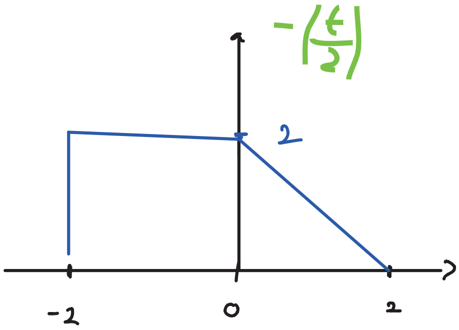

$x_o(t) = \frac{1}{2}(x(t) - x(-t))$



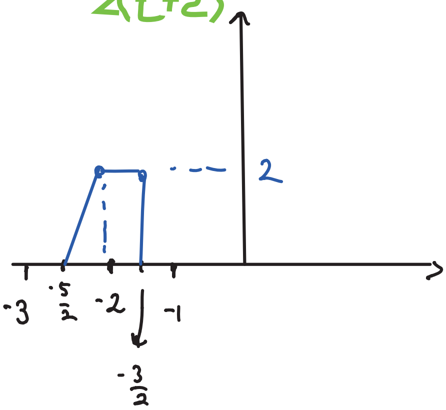
(6pts) (b) Plot the following functions derived from the given signal $x(t)$;

- $x\left(-\frac{t}{2}\right)$
- $x(2t+4)$
- $x(-t+1)$

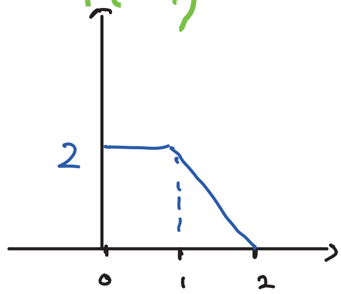
Solution) $x(-\frac{t}{2})$ $-(\frac{t}{2})$



$x(2t+4)$ $2(t+2)$



$x(-t+1)$ $-(t-1)$



2. (10 points) Determine the Fourier Series (FS) representation of the following signal $x(t)$

$$x(t) = \begin{cases} \cos(w_0 t), & \text{for } -\dfrac{\pi}{2w_0} < t \le \dfrac{\pi}{2w_0}, \\ 0, & \text{for } -\dfrac{\pi}{w_0} < t \le -\dfrac{\pi}{2w_0} \quad \text{or} \quad \dfrac{\pi}{2w_0} < t \le \dfrac{\pi}{w_0}, \end{cases}$$

where $x(t + T_0) = x(t)$, $T_0 = \frac{2\pi}{w_0}$, and $w_0 > 0$ is a real-valued constant number.

(6pts) (a) Derive the complex exponential FS coefficients $c_0$, $c_k$ of the given signal $x(t)$ (*Hint*: Use the integral table provided in the Lecture note at the end of Chapter 3.)

(4pts) (b) Derive the trigonometric FS coefficients $a_0$, $a_k$, and $b_k$ of the signal $x(t)$

**(Answer Page for Question 2)**

Solution ) a)

$$C_0 = \frac{1}{T_0}\int_{-\frac{\pi}{2\omega_0}}^{\frac{\pi}{2\omega_0}} \cos(\omega_0 t)\,dt = \frac{1}{\omega_0 T_0} \sin(\omega_0 t)\Big|_{-\frac{\pi}{2\omega_0}}^{\frac{\pi}{2\omega_0}}$$

$$= \frac{1}{2\pi}\Big(\underbrace{\sin\left(\tfrac{\pi}{2}\right) - \sin\left(-\tfrac{\pi}{2}\right)}_{=1-(-1)=2}\Big) = \frac{1}{\pi}$$

$$C_k = \frac{1}{T_0}\int_{-\frac{\pi}{2\omega_0}}^{\frac{\pi}{2\omega_0}} \cos(\omega_0 t)\, e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\Bigg[\underbrace{\frac{(-jk\omega_0)\cos\omega_0 t + \omega_0 \sin(\omega_0 t)}{(-jk\omega_0)^2 + \omega_0^2}\, e^{-jk\omega_0 t}\Big|_{-\frac{\pi}{2\omega_0}}^{\frac{\pi}{2\omega_0}}}\Bigg]$$

$$\left( = \left(\frac{\omega_0}{\omega_0^2 - (\omega_0 k)^2}\right) e^{-jk\frac{\pi}{2}} - \left(\frac{-\omega_0}{\omega_0^2 - (\omega_0 k)^2}\right) e^{jk\frac{\pi}{2}} \right)$$

$$= \frac{1}{2\pi}\cdot\frac{1}{(1-k^2)}\cdot\left(e^{-jk\frac{\pi}{2}} + e^{jk\frac{\pi}{2}}\right) = \frac{\cos\left(\frac{k\pi}{2}\right)}{\pi(1-k^2)}$$

$$\Rightarrow \begin{cases} \text{even } k = 2m, & \dfrac{(-1)^m}{\pi(1-k^2)} \\ \text{odd } k = 2m+1, & \phi \end{cases}$$

b) $a_0 = 2C_0 = \frac{2}{\pi}$

$$a_k = 2\mathrm{Re}(C_k) = \begin{cases} \text{even } k = 2m, & \dfrac{2(-1)^m}{\pi(1-k^2)} \\ \text{odd } k = 2m+1, & \phi \end{cases} \Rightarrow \text{or } \frac{2\cos\left(\frac{k\pi}{2}\right)}{\pi(1-k^2)}$$

$$b_k = -2\,\mathrm{Im}(C_k) = \phi$$

3. (30 points) Evaluate the convolution integral of the following signals. Provide the

(10pts) (a) $x(t) = \cos(w_0 t)$, $\quad y(t) = \mathrm{e}^{-\alpha|t|}$, where $\alpha$ and $w_0$ are real-valued constants, $\alpha > 0$, $w_0 > 0$.

$$\Rightarrow\ x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

(10pts) (b) $x(t) = \sin(w_1 t)\, U(t)$, $\quad y(t) = \sin(w_2 t)\, U(t)$, where $U(t)$ is a step function, $w_1$ and $w_2$ are real-valued constants, $w_1 > 0$, $w_2 > 0$, and $w_1 \neq w_2$.

$$\Rightarrow\ x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

(10pts) (c) $x(t)$ and $y(t)$ are periodic signals with the fundamental period $T_0$

$$x(t) = \sin(w_0 t), \quad y(t) = \cos(w_0 t) \text{ for } -\frac{\pi}{w_0} < t \leq \frac{\pi}{w_0},\ w_0 = \frac{2\pi}{T_0},$$
$$x(t) = x(t+T_0), \quad y(t) = y(t+T_0),$$

where $w_0$ is a real-valued constant, $w_0 > 0$. Evaluate the periodic convolution $x(t) \circledast y(t)$.

$$\Rightarrow\ x(t) \circledast y(t) = \int_{-\frac{\pi}{w_0}}^{\frac{\pi}{w_0}} x(\tau)\, y(t-\tau)\, d\tau$$

(*Hint*: Use the integral table provided in the Lecture note at the end of Chapter 3.)

**(Answer Page for Question 3)**

Solution ) a) $x(t) = \cos(\omega_0 t)$ , $y(t) = e^{-\alpha|t|}$

$$\Rightarrow x(t) * y(t) = \int_{-\infty}^{\infty} e^{-\alpha|\tau|} \underbrace{\cos(\omega_0(t-\tau))}_{= \cos(\omega_0 t)\cos(\omega_0 \tau) + \sin(\omega_0 t)\sin(\omega_0 \tau)} d\tau$$

$$= \cos(\omega_0 t) \underbrace{\int_{-\infty}^{\infty} e^{-\alpha|\tau|} \cos(\omega_0 \tau) d\tau}_{①} + \sin(\omega_0 t) \underbrace{\int_{-\infty}^{\infty} e^{-\alpha|\tau|} \sin(\omega_0 \tau) d\tau}_{②}$$

Integral ① $\Rightarrow$ $\int_0^{\infty} e^{-\alpha\tau} \cos(\omega_0 \tau) d\tau + \int_{-\infty}^{0} e^{\alpha\tau} \cos(\omega_0 \tau) d\tau$

$$= \left( \frac{(-\alpha)\cos(\omega_0 \tau) + \omega_0 \sin(\omega_0 \tau)}{(-\alpha)^2 + \omega_0^2} \right) e^{-\alpha\tau} \Big|_0^{\infty} + \left( \frac{\alpha \cos(\omega_0 \tau) + \omega_0 \sin(\omega_0 \tau)}{\alpha^2 + \omega_0^2} \right) e^{\alpha\tau} \Big|_{-\infty}^{0}$$

$$= \frac{\alpha}{\alpha^2 + \omega_0^2} + \frac{\alpha}{\alpha^2 + \omega_0^2} = \frac{2\alpha}{\alpha^2 + \omega_0^2}$$

Integral ② $\Rightarrow$ $\int_0^{\infty} e^{-\alpha\tau} \sin(\omega_0 \tau) d\tau + \int_{-\infty}^{0} e^{\alpha\tau} \sin(\omega_0 \tau) d\tau$

$$= \left( \frac{(-\alpha)\sin(\omega_0 t) - \omega_0 \cos(\omega_0 t)}{(-\alpha)^2 + \omega_0^2} \right) e^{-\alpha t} \Big|_0^{\infty} + \left( \frac{\alpha \sin(\omega_0 \tau) - \omega_0 \cos(\omega_0 \tau)}{\alpha^2 + \omega_0^2} \right) e^{\alpha\tau} \Big|_{-\infty}^{0}$$

$$= \frac{\omega_0}{\alpha^2 + \omega_0^2} - \frac{\omega_0}{\alpha^2 + \omega_0^2} = 0.$$

Therefore, $x(t) * y(t) = \dfrac{2\alpha}{\alpha^2 + \omega_0^2} \cos(\omega_0 t)$

Sol) Q3 - b) $x(t) = \sin(\omega_1 t)\, u(t)$ , $y(t) = \sin(\omega_2 t)\, u(t)$

$$x(t) * y(t) = \int_{-\infty}^{\infty} \sin(\omega_1 \tau)\, u(\tau)\, \sin(\omega_2 (t-\tau))\, u(t-\tau)\, d\tau$$

if $t \geq 0$

$$= \int_0^t \sin(\omega_1 \tau) \underbrace{\sin(\omega_2 (t-\tau))}_{= \sin(\omega_2 t)\cos(\omega_2 \tau) - \cos(\omega_2 t)\sin(\omega_2 \tau)} d\tau$$

(if $t < 0$, then $x(t) * y(t) = 0$)

$$\Rightarrow \sin(\omega_2 t) \int_0^t \underbrace{\sin(\omega_1 \tau)\cos(\omega_2 \tau)}_{= \frac{1}{2}\left(\sin(\omega_1+\omega_2)\tau + \sin(\omega_1-\omega_2)\tau\right)} d\tau - \cos(\omega_2 t) \int_0^t \underbrace{\sin(\omega_1 \tau)\sin(\omega_2 \tau)}_{= \frac{1}{2}\left(\cos(\omega_1-\omega_2)\tau - \cos(\omega_1+\omega_2)\tau\right)} d\tau$$

$$\Rightarrow \frac{1}{2}\sin(\omega_2 t)\left[ \frac{1}{\omega_1+\omega_2} \underbrace{\cos(\omega_1+\omega_2)\tau \Big|_t^0}_{\substack{1-\cos(\omega_1+\omega_2)t \\ = 1-\cos\omega_1 t \cdot \cos\omega_2 t + \sin\omega_1 t \cdot \sin\omega_2 t}} + \frac{1}{\omega_1-\omega_2} \underbrace{\cos(\omega_1-\omega_2)\tau \Big|_t^0}_{1-\cos(\omega_1-\omega_2)t \,=\, 1-\cos\omega_1 t \cos\omega_2 t - \sin\omega_1 t \cdot \sin\omega_2 t} \right]$$

$$- \frac{1}{2}\cos(\omega_2 t)\left[ \frac{1}{\omega_1-\omega_2} \underbrace{\sin(\omega_1-\omega_2)\tau \Big|_0^t}_{\substack{\sin(\omega_1-\omega_2)t \\ = \sin\omega_1 t \cdot \cos\omega_2 t - \sin\omega_2 t \cdot \cos\omega_1 t}} - \frac{1}{\omega_1+\omega_2} \underbrace{\sin(\omega_1+\omega_2)\tau \Big|_0^t}_{\substack{= \sin(\omega_1+\omega_2)t = \sin\omega_1 t \cos\omega_2 t \\ + \cos\omega_1 t \cdot \sin\omega_2 t}} \right]$$

$$\Rightarrow \sin(\omega_2 t)\cdot\left(\frac{\omega_1}{\omega_1^2-\omega_2^2}\right) - \cancel{\cos(\omega_1 t)\sin(\omega_2 t)\cos(\omega_2 t)\left(\frac{\omega_1}{\omega_1^2-\omega_2^2}\right)}$$

$$- \sin(\omega_1 t)\sin^2(\omega_2 t)\left(\frac{\omega_2}{\omega_1^2-\omega_2^2}\right) - \sin(\omega_1 t)\cos^2(\omega_2 t)\left(\frac{\omega_2}{\omega_1^2-\omega_2^2}\right) \leftarrow \left(\sin^2(\omega_2 t) + \cos^2(\omega_2 t)\right) = 1$$

$$+ \cancel{\cos(\omega_1 t)\sin(\omega_2 t)\cos(\omega_2 t)\left(\frac{\omega_1}{\omega_1^2-\omega_2^2}\right)}$$

Therefore

$$\Rightarrow x(t) * y(t) = \left[ \frac{\omega_1}{\omega_1^2-\omega_2^2}\sin(\omega_2 t) - \frac{\omega_2}{\omega_1^2-\omega_2^2}\sin(\omega_1 t) \right] u(t)$$

Sol) Q3-c) $x(t) = \sin(\omega_0 t)$, $y(t) = \cos(\omega_0 t)$

$$x(t) \circledast y(t) = \int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} \sin(\omega_0 \tau) \underbrace{\cos(\omega_0(t-\tau))}_{= \cos(\omega_0 t)\cos(\omega_0 \tau) + \sin(\omega_0 t)\sin(\omega_0 \tau)} d\tau$$

$$\Rightarrow \cos(\omega_0 t) \int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} \underbrace{\sin(\omega_0 \tau)\cos(\omega_0 \tau)}_{= \frac{1}{2}\sin(2\omega_0 \tau)} d\tau + \sin(\omega_0 t) \int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} \underbrace{\sin^2(\omega_0 \tau)}_{\frac{1}{2}(1-\cos(2\omega_0 \tau))} d\tau$$

$$= \frac{1}{2}\cos(\omega_0 t)\left[ \underbrace{\frac{1}{2\omega_0}\cos(2\omega_0 \tau) \Big|_{\frac{\pi}{\omega_0}}^{-\frac{\pi}{\omega_0}}}_{\emptyset} \right] + \frac{1}{2}\sin(\omega_0 t)\left[ \frac{2\pi}{\omega_0} - \underbrace{\frac{1}{2\omega_0}\sin(2\omega_0 \tau) \Big|_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}}}_{\emptyset} \right]$$

Therefore,

$$\Rightarrow x(t) \circledast y(t) = \frac{\pi}{\omega_0}\sin(\omega_0 t)$$

4. (25 points) Calculate the Fourier Series coefficients of a signal $x(t)$, defined as follows

$$\begin{cases} x(t) = t - \frac{1}{\pi^2}\, t^3, & \text{for } 0 < t \leq \pi, \\ x(t+\pi) = x(t). \end{cases} \quad , \quad \omega_0 = \frac{2\pi}{\pi} = 2$$

(20pts) (a) Derive the complex exponential FS coefficients $c_0$, $c_k$ of the given signal $x(t)$

Sol)

a)

$$\frac{1}{\pi}\int_0^{\pi} \underset{v}{t^3}\ \underset{u'}{e^{-j2kt}}\,dt = \frac{1}{\pi}\left[ \frac{t^3 e^{-j2kt}}{(-j2k)}\Big|_0^{\pi} - \frac{3}{(-j2k)}\int_0^{\pi} t^2 e^{-j2kt}\,dt \right]$$

$$= \frac{1}{\pi}\left[ \frac{\pi^3}{(-j2k)} - \frac{3}{(-j2k)}\left\{ \frac{t^2 e^{-j2kt}}{(-j2k)}\Big|_0^{\pi} - \frac{2}{(-j2k)}\int_0^{\pi} t\, e^{-j2kt}\,dt \right\} \right]$$

$$= \frac{1}{\pi}\left[ \frac{\pi^3}{(-j2k)} - \frac{3}{(-j2k)}\left(\frac{\pi^2}{(-j2k)}\right) + \frac{6}{(-j2k)^2}\left\{ \frac{t\,e^{-j2kt}}{(-j2k)}\Big|_0^{\pi} - \cancel{\frac{1}{(-j2k)}\int_0^{\pi} e^{-j2kt}\,dt}^{\ \emptyset} \right\} \right]$$

$$= \frac{1}{\pi}\left[ \frac{\pi^3}{(-j2k)} - \frac{3}{(-j2k)^2}\pi^2 + \frac{6\pi}{(-j2k)^3} \right]$$

$$\frac{1}{\pi}\int_0^{\pi} t\, e^{-j2kt}\,dt = \frac{1}{\pi}\left\{ \frac{t\,e^{-j2kt}}{(-j2k)}\Big|_0^{\pi} - \frac{1}{(-j2k)}\int_0^{\pi} e^{-j2kt}\,dt \right\} = \frac{1}{(-j2k)}$$

(5 pts) (b) Derive the trigonometric FS coefficients $a_0$, $a_k$, and $b_k$ of the signal $x(t)$

a) (Continue)

$$\Rightarrow c_k = \frac{1}{\pi}\int_0^{\pi}\left(t - \frac{1}{\pi^2}t^3\right)e^{-j2kt}\,dt$$

$$= \cancel{\frac{1}{(-j2k)}}^{\ \emptyset} - \left( \cancel{\frac{1}{(-j2k)}} - \frac{3}{\pi}\frac{1}{(-j2k)^2} + \frac{6}{\pi^2}\frac{1}{(-j2k)^3} \right)$$

$$= \frac{3}{\pi}\frac{1}{(-j2k)^2} - \frac{6}{\pi^2}\frac{1}{(-j2k)^3}$$

$$= -\frac{3}{4k^2\pi} + \frac{3}{4k^3\pi^2}j$$

$$c_0 = \frac{1}{\pi}\int_0^{\pi}\left(t - \frac{1}{\pi^2}t^3\right)dt = \frac{1}{\pi}\left[\frac{1}{2}t^2 - \frac{1}{4\pi^2}t^4\right]_0^{\pi} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

**(Answer Page for Question 4)**

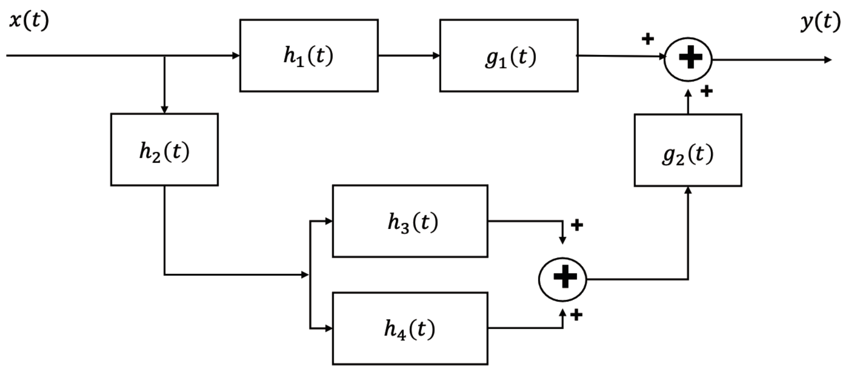
Sol ) Q4 - b)

$$a_0 = 2C_0 = \frac{\pi}{2}$$

$$a_k = 2\mathrm{Re}(C_k) = -\frac{3}{2k^2\pi}$$

$$b_k = -2\mathrm{Im}(C_k) = -\frac{3}{2k^3\pi^2}$$

5. (25 points) Consider the interconnection of six LTI systems, as depicted in the following figure.




The impulse responses of each components are given by

$$h_1(t) = \text{tri}\,(t)\,, \quad h_2(t) = \text{rect}\,(t)\,, \quad h_3(t) = U\,(t+0.5)\,,$$
$$h_4(t) = -U\,(t-0.5)\,, \quad g_1(t) = \delta\,(t+0.5)\,, \quad g_2(t) = \delta\,(t-0.5)\,,$$

where $\text{tri}\,(t) = 1-|t|$ for $|t|<1$ and 0 otherwise, $\text{rect}\,(t)$ is the rectangular pulse signal, $U\,(t)$ is the unit step function, and $\delta(t)$ is the unit impulse function.

(15pts) (a) Find the impulse response $h(t)$ of the overall system.

Sol) a) $h(t) = h_1(t) * g_1(t) + h_2(t) * (h_3(t) + h_4(t)) * g_2(t)$

where $h_1(t) * g_1(t) = \text{tri}(t) * \delta(t+0.5)$

$= \text{tri}(t+0.5)$

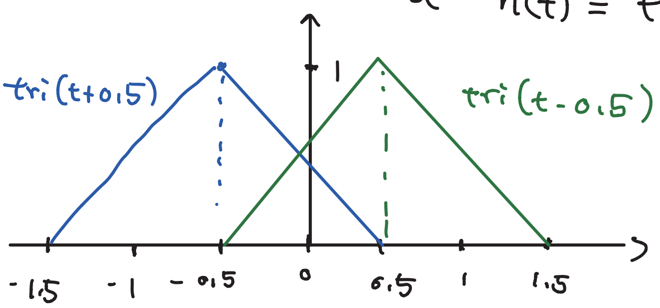
$h_3(t) + h_4(t) = u(t+0.5) - u(t-0.5) = \text{rect}(t)$

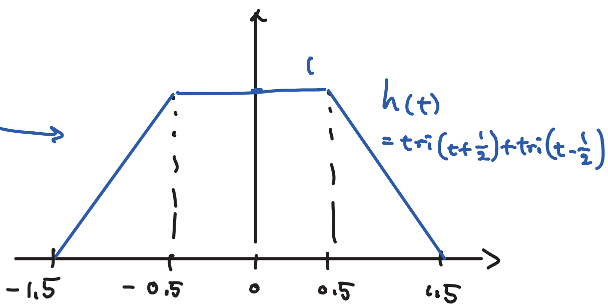
$h_2(t) * (h_3(t) + h_4(t)) = \text{rect}(t) * \text{rect}(t) = \text{tri}(t)$

(10pts) (b) Check if the given system $h(t)$ is memoryless, causal and stable.

Therefore $h_2(t) * (h_3(t) + h_4(t)) * g_2(t) = \text{tri}(t-0.5)$

and $h(t) = \text{tri}(t+0.5) + \text{tri}(t-0.5)$



**(Answer Page for Question 5)**

Solution) (continue)

a)

$$h(t) = \begin{cases} t + 1.5, & \text{for } -1.5 \le t < -0.5 \\ 1, & \text{for } -0.5 \le t < 0.5 \\ 1.5 - t, & \text{for } 0.5 \le t < 1.5 \\ 0, & \text{otherwise} \end{cases}$$

b) memory, non-causal, stable system