Problem 9

For any real number $a \neq -1$ and non-negative integer n, we define the integral as

$$I_n = \int_1^e x^a (\ln x)^n dx.$$

(a) Deduce the following reduction formula for I_n:

$$I_n = \frac{e^{a+1}}{a+1} - \frac{n}{a+1} I_{n-1}, \quad n \geq 1.$$

(b) Using the reduction formula in (a), find the value of

$$\int_1^e x^2 (\ln x)^3 dx.$$

(c) What is the value of I_n when $a = -1$?

(a). $I_n = \int_1^e x^a (\ln x)^n dx.$

$u = (\ln x)^n, \quad dv = x^a dx \Rightarrow v = \int x^a dx = \frac{x^{a+1}}{a+1}.$

$$I_n = \frac{x^{a+1}}{a+1} (\ln x)^n \Big|_1^e - \int_1^e \frac{x^{a+1}}{a+1} \cdot \frac{1}{x} \boxed{d (\ln x)^n} = \frac{d(\ln x)^n}{d\ln x} \cdot \frac{d\ln x}{dx}.$$

$$= \boxed{n (\ln x)^{n-1} \cdot \frac{1}{x}.}$$

$$= \frac{e^{a+1}}{a+1} - \frac{n}{a+1} \underbrace{\int_1^e x^a (\ln x)^{n-1} dx}_{I_{n-1}}. \quad n \geq 1.$$

(b). $a=2, \ n=3. \quad I_3 = \frac{e^3}{3} - I_2.$

$a=2, \ n=2. \quad = \frac{e^3}{3} - \left[\frac{e^3}{3} - \frac{2}{3} I_1\right].$

$= \frac{2}{3} I_1$

$a=2, \ n=1 \quad = \frac{2}{3}\left[\frac{e^3}{3} - \frac{1}{3} I_0\right]$

$I_0 = \int_1^e x^2 dx = \frac{x^3}{3}\Big|_1^e = \frac{e^3}{3} - \frac{1}{3}.$

$$I_n = \int_1^e x^a (\ln x)^n dx.$$

(c). $a = -1$. $I_n = \int_1^e \frac{(\ln x)^n}{x} dx$.

"substitution" let $y = \ln x$. $\Rightarrow \frac{dy}{dx} = \frac{1}{x} \Rightarrow dx = x\,dy$.

when $x = 1$, $y = 0$; $x = e$, $y = 1$.

$$I_n = \int_0^1 \frac{(\overbrace{\ln x}^{=y})^n}{\cancel{x}} \cdot \cancel{x}\,dy = \int_0^1 y^n dy = \left.\frac{y^{n+1}}{n+1}\right|_0^1 = \frac{1}{n+1}.$$

Problem 16 (Method of Partial Fractions)

(e) $\int \frac{-7x+19}{(x^2-4x+9)(2x+1)}dx$

$$\frac{-7x+19}{(x^2-4x+9)(2x+1)} \begin{matrix} 1 \\ 3 \end{matrix} = \frac{Ax+B}{x^2-4x+9} + \frac{C}{2x+1}.$$

$$\Rightarrow -7x+19 = (Ax+B)(2x+1) + C(x^2-4x+9)$$

let $x=-\frac{1}{2}$, $C=2$.

compare the coefficient of x^2. $0 = 2A + C \Rightarrow A = -1$

constant . $19 = B + 9C \Rightarrow B = 1$

$$\boxed{\int \frac{-x+1}{x^2-4x+9}dx + \int \frac{2}{2x+1}dx}$$

(i) $\int \frac{2x^2-x+1}{x^3(x-1)}dx$

$\frac{Ax^2+Bx+C}{x^3}$

$$\frac{2x^2-x+1}{x^3(x-1)} \begin{matrix} 2 \\ 4 \end{matrix} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1}$$

$$\Rightarrow 2x^2-x+1 = Ax^2(x-1) + Bx(x-1) + C(x-1) + Dx^3.$$

Bx^2 $-Bx$

Let $x=0$, $C=-1$

$x=1$, $D=2$.

compare the coefficient of x^3: $0 = A+D \Rightarrow A=-2$

$x: \quad -1 = -B + C \Rightarrow B = 0.$

$$\int \frac{-2}{x}\,dx + \int \frac{-1}{x^3}\,dx + \int \frac{2}{x-1}\,dx.$$

*(k) $\displaystyle\int \frac{6x^3 - 27x^2 + 5x - 1}{(x-2)^2(4x^2+1)}\,dx$

$$\frac{6x^3-27x^2+5x-1}{(x-2)^2(4x^2+1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{4x^2+1}$$

$$\Rightarrow 6x^3-27x^2+5x-1 = A(x-2)(4x^2+1) + B(4x^2+1) + (Cx+D)(x-2)^2.$$

let $x=2$, $B=-3$.

$$6x^3-27x^2+5x-1+3(4x^2+1) = A(x-2)(4x^2+1) + (Cx+D)(x-2)^2.$$

$$= 6x^3-15x^2+5x+2.$$

synthetic division

$$6x^2-3x-1 = A(4x^2+1) + (Cx+D)(x-2).$$

let $x=2$, $A=1$

Compare the coefficient of x^2: $6 = 4A + C \Rightarrow C = 2.$

constant : $-1 = A - 2D \Rightarrow D = 1$

$$\begin{array}{r|l} & 6x^2-3x-1 \\ x-2 & 6x^3-15x^2+5x+2 \\ & 6x^3-12x^2 \\ \hline & -3x^2+5x \\ & -3x^2+6x \\ \hline & -x+2 \end{array}$$

$$\int \frac{1}{x-2}\,dx - 3\int \frac{1}{(x-2)^2}\,dx + \boxed{\int \frac{2x+1}{4x^2+1}\,dx}.$$

$$\boxed{\frac{1}{1+x^2}} \rightarrow$$

$$\int \frac{2x}{4x^2+1}\,dx + \int \frac{1}{4x^2+1}\,dx$$

$$y = 4x^2+1 \Rightarrow \frac{dy}{dx} = 8x \Rightarrow dx = \frac{1}{8x}\,dy.$$

$$\int \frac{2x}{4x^2+1}\,\frac{1}{8x}\,dy = \frac{1}{4}\int \frac{1}{y}\,dy \ldots\ldots$$