MA1200 CALCULUS AND BASIC LINEAR ALGEBRA I

LECTURE: CG1

Chapter 8
Applications of Differentiation

Applications of Differentiation

In this chapter, we will study the following applications of differentiation:

1. **Equation of tangent / normal to the curve**
2. **Rate of change**
   - rate of change of a variable with respect to time
3. **Local extrema of functions**
   - find the local maximum or local minimum values of the graph of a function
4. **Optimization problem**
   - maximize or minimize a variable
5. **L'Hôpital's rule**
   - find the limits of functions of indeterminate forms
6. **Taylor's series / Maclaurin series**
   - approximate a given function by using higher order polynomials

1. Equation of tangent / normal to the curve

Given the curve of a function $F(x,y)=0$. Suppose that the slope of the function y, i.e. $\frac{dy}{dx}$, at the point (x_0,y_0) exists, and let $m=\left.\frac{dy}{dx}\right|_{(x_0,y_0)}$ be the value of $\frac{dy}{dx}$ of the curve at (x_0,y_0). Then, m is the **slope of the tangent** to the curve at (x_0,y_0).

- The **equation of the tangent to the curve** at the point (x_0,y_0) is $\boxed{\frac{y-y_0}{x-x_0}=m}$.
- The **equation of the normal to the curve** at the point (x_0,y_0) is $\boxed{\frac{y-y_0}{x-x_0}=-\frac{1}{m}}$, provided $m\neq 0$.

Recall: $\boxed{\text{slope of normal}=\frac{-1}{\text{slope of tangent}}}$ (from Chapter 1)

Note that if $m=0$, then

- the equation of the tangent to the curve at (x_0,y_0) is the horizontal line $y=y_0$; and
- the equation of the normal to the curve at (x_0,y_0) is the vertical line $x=x_0$.

Example 1

Given the parametric equations $\begin{cases} x=3\cos t \\ y=2\sin t \end{cases}$, where $0\le t\le 2\pi$.

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(ii) Find the equation of the tangent to the curve at $t=\frac{\pi}{4}$.

(iii) Find the equation of the normal to the curve at $t=\frac{\pi}{4}$.

Solution

(i) $x=3\cos t \Rightarrow \frac{dx}{dt}=3(-\sin t)=-3\sin t$

$y=2\sin t \Rightarrow \frac{dy}{dt}=2\cos t$

Then $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{2\cos t}{-3\sin t}=-\frac{2}{3}\cot t$

and $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}=\frac{\frac{d}{dt}\left(-\frac{2}{3}\cot t\right)}{-3\sin t}=\frac{-\frac{2}{3}(-\operatorname{cosec}^2 t)}{-3\sin t}=-\frac{2}{9}\operatorname{cosec}^3 t$

(ii) At $t=\frac{\pi}{4}$, $y=2\sin\left(\frac{\pi}{4}\right)=2\cdot\frac{\sqrt{2}}{2}=\sqrt{2}$

$x=3\cos\left(\frac{\pi}{4}\right)=3\cdot\frac{\sqrt{2}}{2}=\frac{3\sqrt{2}}{2}$

The slope of the tangent to the curve at $t=\frac{\pi}{4}$ is

$$\frac{dy}{dx}=-\frac{2}{3}\cot\left(\frac{\pi}{4}\right)=-\frac{2}{3}\cdot\frac{1}{\tan\left(\frac{\pi}{4}\right)}=-\frac{2}{3}\cdot\frac{1}{1}=-\frac{2}{3}$$

$\therefore$ The equation of the tangent to the curve at $t=\frac{\pi}{4}$ is given by

$$\frac{y-\sqrt{2}}{x-\frac{3\sqrt{2}}{2}}=-\frac{2}{3}$$

$$\Rightarrow \quad y-\sqrt{2}=-\frac{2}{3}\left(x-\frac{3\sqrt{2}}{2}\right)$$

$$\Rightarrow \quad y=-\frac{2}{3}x+2\sqrt{2}$$

(iii) The slope of the normal to the curve at $t=\frac{\pi}{4}$ is

$$\frac{-1}{\left(-\frac{2}{3}\right)}=\frac{3}{2}.$$

$\therefore$ The equation of the normal to the curve at $t=\frac{\pi}{4}$ is given by

$$\frac{y-\sqrt{2}}{x-\frac{3\sqrt{2}}{2}}=\frac{3}{2}$$

$$\Rightarrow \quad y-\sqrt{2}=\frac{3}{2}\left(x-\frac{3\sqrt{2}}{2}\right)$$

$$\Rightarrow \quad y=\frac{3}{2}x-\frac{5\sqrt{2}}{4}$$

Example 2 **(Explicit function)**

Find the equation of the tangent line to the curve $y = \frac{\sqrt{x}}{x^2+1}$ at the point $\left(1, \frac{1}{2}\right)$.

Solution

$$\frac{dy}{dx} = \frac{(x^2+1) \cdot \frac{d}{dx}\left(x^{\frac{1}{2}}\right) - x^{\frac{1}{2}} \cdot \frac{d}{dx}(x^2+1)}{(x^2+1)^2} = \frac{(x^2+1) \cdot \frac{1}{2}x^{-\frac{1}{2}} - x^{\frac{1}{2}} \cdot (2x)}{(x^2+1)^2}$$

$$= \frac{-3x^2+1}{2\sqrt{x}\,(x^2+1)^2}$$

The slope of the tangent line to the curve at the point $\left(1, \frac{1}{2}\right)$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{-3(1)^2+1}{2\sqrt{1}\,(1^2+1)^2} = -\frac{1}{4}.$$

$\therefore$ The equation of the tangent line to the curve at $\left(1, \frac{1}{2}\right)$ is

$$\frac{y - \frac{1}{2}}{x-1} = -\frac{1}{4} \quad \Rightarrow \quad y = -\frac{1}{4}x + \frac{3}{4} \quad (\text{or written as } 4y + x - 3 = 0).$$

Example 3 **(Implicit function)**

Find the equation of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$.

Solution

Note that $x^2 + y^2 = 25$ is an **implicit** function.

Differentiate both sides of $x^2 + y^2 = 25$ w.r.t. x:

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

$\therefore$ The slope of the tangent line to $x^2 + y^2 = 25$ at the point $(3, 4)$ is

$$\left.\frac{dy}{dx}\right|_{(x,y)=(3,4)} = -\frac{3}{4}.$$

$\therefore$ The equation of the tangent line to the curve at the point $(3, 4)$ is

$$\frac{y-4}{x-3} = -\frac{3}{4} \quad \Rightarrow \quad y = -\frac{3}{4}x + \frac{25}{4}$$

Class Exercise

Given that $x^2 + 2xy^2 - y^3 - 1 = 0$.

Find the equations of the tangent and normal to the curve at the point $P(1, 2)$.

Solution

2. Rate of change

We usually use the notation

$$\frac{d(?)}{dt}$$

to denote "the rate of change of (?) with respect to time t".

E.g. If $h(t)$ denotes the height of an object at time t, then

$\frac{dh}{dt}$ = the rate of change of height with respect to time.

E.g. Suppose an object is moving along a straight line with the **position function**

$x = x(t)$ (i.e. the position of the object at time t).

Then $v(t) = \frac{dx}{dt} =$ **velocity** of the object at time t

(i.e. $v(t)$ is the rate of change of its position x w.r.t. time t),

and $a(t) = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} =$ **acceleration** of the object at time t

(i.e. $a(t)$ is the rate of change of velocity v w.r.t. time t).

Example 4

If the velocity $v(t)$ of a body varies inversely as the square root of the distance $s(t)$, prove that the acceleration $a(t)$ varies as the fourth power of the velocity.

Solution

We have $v(t) = \frac{k}{\sqrt{s(t)}}$, where k is a constant. Then

$$a(t) = \frac{d}{dt}[v(t)] = \frac{d}{dt}\left[\frac{k}{\sqrt{s(t)}}\right] = \frac{d}{dt}\left[k\big(s(t)\big)^{-\frac{1}{2}}\right]$$

$$= k\left(-\frac{1}{2}\right)\big(s(t)\big)^{-\frac{3}{2}} \cdot \frac{d}{dt}[s(t)]$$

$$= -\frac{1}{2}k\big(s(t)\big)^{-\frac{3}{2}} \cdot v(t) = -\frac{1}{2}k\left\{\left[\frac{k}{v(t)}\right]^2\right\}^{-\frac{3}{2}} \cdot v(t)$$

$$= -\frac{1}{2}k\left[\frac{k}{v(t)}\right]^{-3} \cdot v(t)$$

$$= -\frac{[v(t)]^4}{2k^2}$$

Procedure for solving rate of change problems

Step 1: Write an equation which relates relevant variables.

Step 2: Use chain rule to differentiate (implicitly) both sides of the equation w.r.t. time t.

Step 3: Substitute in the values for all known variables and derivatives.

Step 4: Solve for the remaining rate.

Example 5

A ladder which is 5m long is resting against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1m/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3m from the wall?

Solution

Let $x(t)$ be the perpendicular distance from the wall to the bottom of the ladder at time t, and $y(t)$ be the distance from the top of the ladder to the ground at time t.

By Pythagoras' Theorem,

$$x^2 + y^2 = 5^2.$$

Differentiate both sides w.r.t. t:

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0 \quad \Rightarrow \quad \frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$$

Given that $\frac{dx}{dt} = 1 \text{ m/s}$.

When $x = 3 \text{ m}$,

$$y = \sqrt{5^2 - x^2} = \sqrt{5^2 - 3^2} = \sqrt{16} = 4 \text{ m}$$

and

$$\left.\frac{dy}{dt}\right|_{x=3,y=4} = \left.\left(-\frac{x}{y} \cdot \frac{dx}{dt}\right)\right|_{x=3,y=4} = -\frac{3}{4} \cdot 1 = -\frac{3}{4} \text{ m/s}.$$

$\therefore$ The top of the ladder is **sliding down** the wall at a rate of $\mathbf{\frac{3}{4}}$ m/s.

Example 6

A paper cup has the shape of a right circular cone with height 10 cm and diameter 7 cm (at the top). If drinking water is poured into the cup at a rate of 6 cm^3/sec, how fast is the water level rising when the water is 6 cm deep?

(Hint: Volume of right circular cone is $V = \frac{1}{3}\pi r^2 h$.)

Solution

Let V, r and h be the volume of the water, the radius of the surface of water, and the height of water at time t seconds, respectively.

Objective: Find $\left.\frac{dh}{dt}\right|_{h=6\text{ cm}}$

Given information: $\frac{dV}{dt} = 6\text{ cm}^3/\text{sec}$

By similar triangles, $\frac{r}{h} = \frac{3.5}{10} \Rightarrow r = 0.35\,h$

$\therefore\ V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(0.35\,h)^2\,h = \frac{1}{3}\pi(0.35)^2 h^3$ (cm^3)

Differentiate both sides w.r.t. t:

$$\frac{dV}{dt} = \frac{1}{3}\pi(0.35)^2 \cdot \frac{d}{dt}(h^3) = \frac{1}{3}\pi(0.35)^2 \cdot 3h^2\frac{dh}{dt} = \pi(0.35)^2\,h^2\frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{\frac{dV}{dt}}{\pi(0.35)^2\,h^2} = \frac{6}{\pi(0.35)^2\,h^2}$$

When the height of water is $h = 6$ cm,

$$\left.\frac{dh}{dt}\right|_{h=6\text{ cm}} = \frac{6}{\pi(0.35)^2\,6^2} \approx 0.433\text{ cm/sec}$$

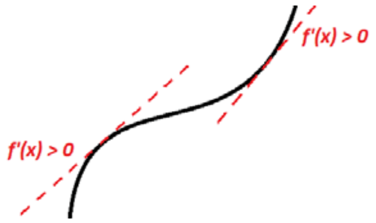
3. Local extrema (local minima / local maxima) of functions

Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b).

The sign of $f'(x)$ tells us whether $f(x)$ is an increasing or decreasing function.

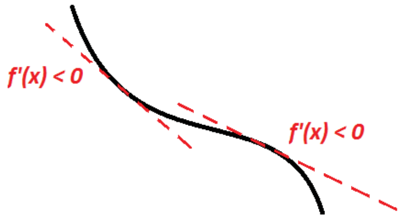
- **Increasing function:**

  If $f'(x) \geq 0$ ($f'(x) > 0$) for every $x \in (a,b)$, then $f(x)$ is **increasing (strictly increasing)** on $[a,b]$.



- **Decreasing function:**

  If $f'(x) \leq 0$ ($f'(x) < 0$) for every $x \in (a,b)$, then $f(x)$ is **decreasing (strictly decreasing)** on $[a,b]$.



Example 7

Identify the increasing and decreasing intervals for $f(x) = x^3 - 3x^2 + 1$.

Solution

$$f'(x) = 3x^2 - 6x$$

Set $f'(x) = 0 \Rightarrow 3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0 \Rightarrow x = 0$ or 2.

	$x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
Sign of x	$-$	0	$+$	$+$	$+$
Sign of $x - 2$	$-$	$-$	$-$	0	$+$
Sign of $f'(x)$	$+$	$\mathbf{0}$	$-$	$\mathbf{0}$	$+$

Observe that $f'(x) > 0$ when $x < 0$ or $x > 2$;

and $f'(x) < 0$ when $0 < x < 2$.

Therefore, f is strictly increasing on $(-\infty, 0]$ and $[2, \infty)$, and strictly decreasing on $[0, 2]$.

Example 8

Show that $\ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ for $x > 0$.

Solution:

Let $f(x) = \ln(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3}$.

Then

$$f'(x) = \frac{1}{1+x} - 1 + x - x^2 = \frac{1-(1+x)+x(1+x)-x^2(1+x)}{1+x} = \frac{-x^3}{1+x} < 0 \text{ for } x > 0.$$

$\therefore$ f is decreasing on $(0, \infty)$.

$\therefore$ $f(x) < f(0)$ for any $x > 0$,

i.e. $\ln(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3} < \ln(1+0) - 0 + 0 - 0 = 0$ for any $x > 0$.

$\Rightarrow$ $\ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ for $x > 0$.

Extreme values

Maximum and **minimum values** of a function are referred to as **extreme values**.

Absolute extreme values

- A function f has an **absolute** (or **global**) **maximum** value $f(x_0)$ at the point $x = x_0$ in $Dom(f)$ if $f(x) \le f(x_0)$ for every $x \in Dom(f)$.

- A function f has an **absolute** (or **global**) **minimum** value $f(x_0)$ at the point $x = x_0$ in $Dom(f)$ if $f(x) \ge f(x_0)$ for every $x \in Dom(f)$.

Local extreme values

- A function f has a **local maximum** value $f(x_0)$ at the point $x = x_0$ in $Dom(f)$ if there is a number $h > 0$ such that $f(x) \leq f(x_0)$ for every x in the open interval $(x_0 - h,\ x_0 + h)$.

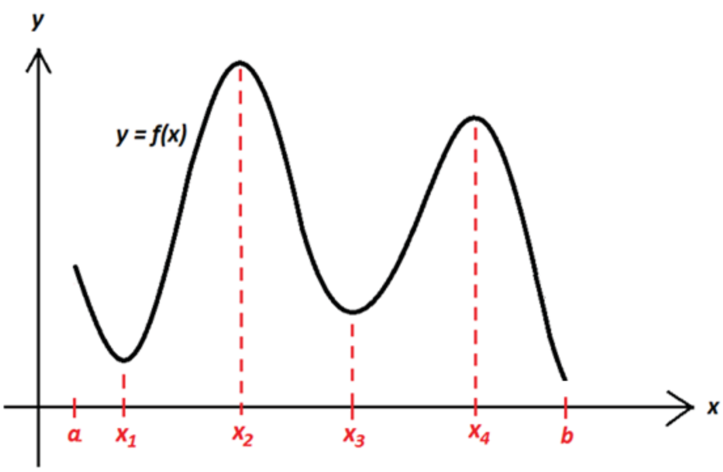
- A function f has a **local minimum** value $f(x_0)$ at the point $x = x_0$ in $Dom(f)$ if there is a number $h > 0$ such that $f(x) \geq f(x_0)$ for every x in the open interval $(x_0 - h,\ x_0 + h)$.

Remarks:

1. In other words, local maximum and minimum values correspond to the points on the graph that are higher or lower than neighboring points.
2. The absolute maximum is the highest of the local maxima (or occurs at boundary point); the absolute minimum is the lowest of the local minima (or occurs at boundary point).

Illustration:

Consider the graph of the function $y = f(x)$ whose domain is $Dom(f) = [a, b]$.



Observe that

- $f(x)$ has local maximum values (or **local maxima**) at x_2 and x_4, and local minimum values (or **local minima**) at x_1 and x_3.
- $f(x)$ has an **absolute maximum** at x_2, and an **absolute minimum** at b.

Local maximum:

Definition:

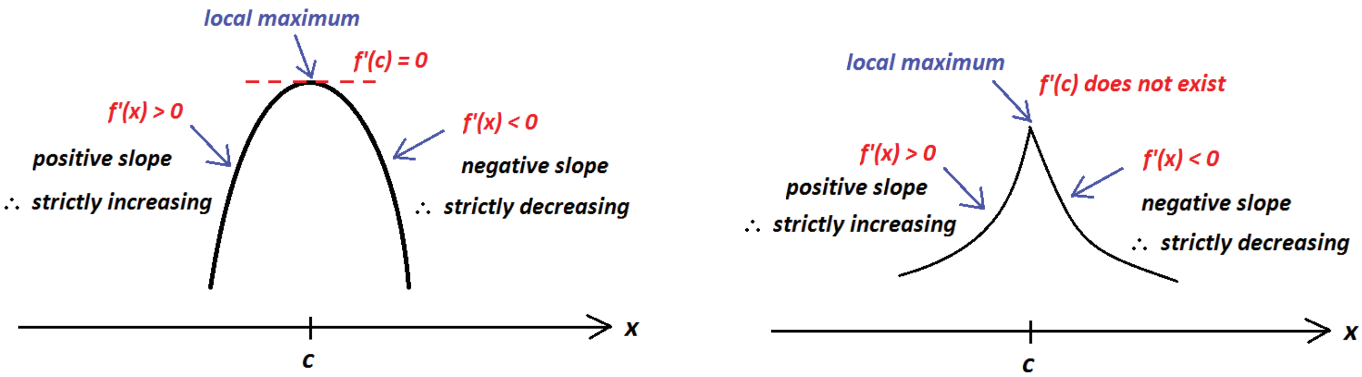
Let f be continuous at $x = c$.

If there exists an open interval (a, b) containing c such that

$$f'(x) > 0 \text{ when } a < x < c$$

and $$f'(x) < 0 \text{ when } c < x < b,$$

then f has a **local maximum** value at $x = c$.



Local minimum:

Definition:

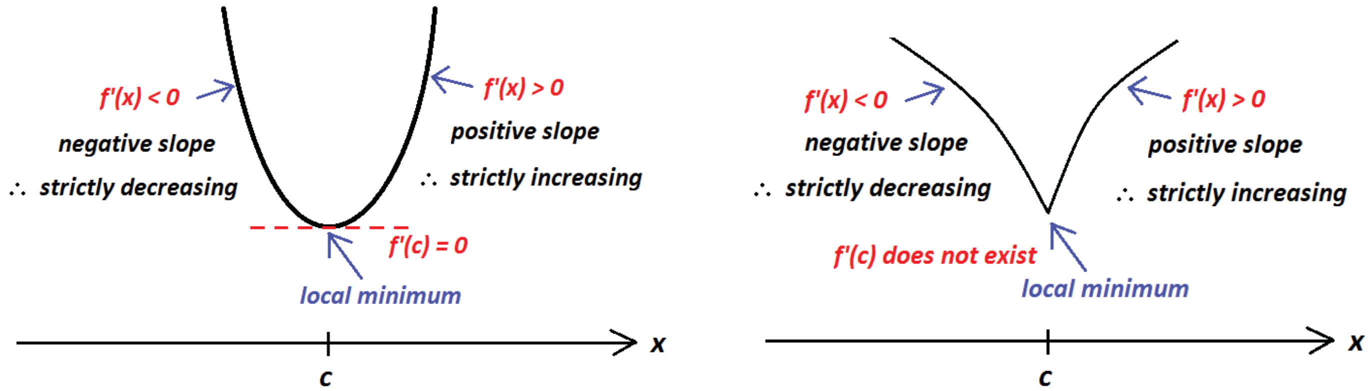
Let f be continuous at $x = c$.

If there exists an open interval (a, b) containing c such that

$$f'(x) < 0 \text{ when } a < x < c$$

and $$f'(x) > 0 \text{ when } c < x < b,$$

then f has a **local minimum** value at $x = c$.



Theorem

Suppose f has a local maximum or local minimum point at $x = c$. **If $f'(c)$ exists**, then

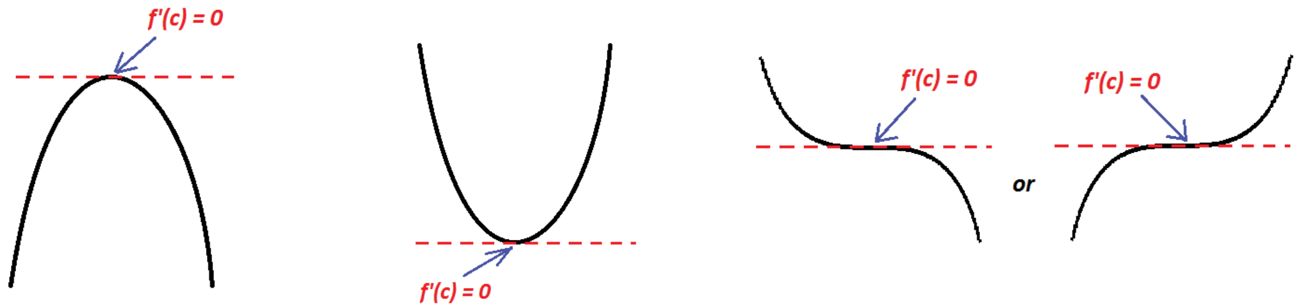
$$\boxed{f'(c) = 0}.$$

Definition

A function f has a **stationary point** at $x = c$ if

$$\boxed{f'(c) = 0}.$$

A stationary point can be a **local maximum**, **local minimum**, or **point of inflection**.



local maximum **local minimum** **point of inflection**

Example 9

Find and classify all stationary points of the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

Solution

$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x+1)(x-2)$.

Set $f'(x) = 0 \Rightarrow 12x(x+1)(x-2) = 0 \Rightarrow x = -1, 0, 2$.

$\therefore$ f has stationary points at $x = -1, 0, 2$.

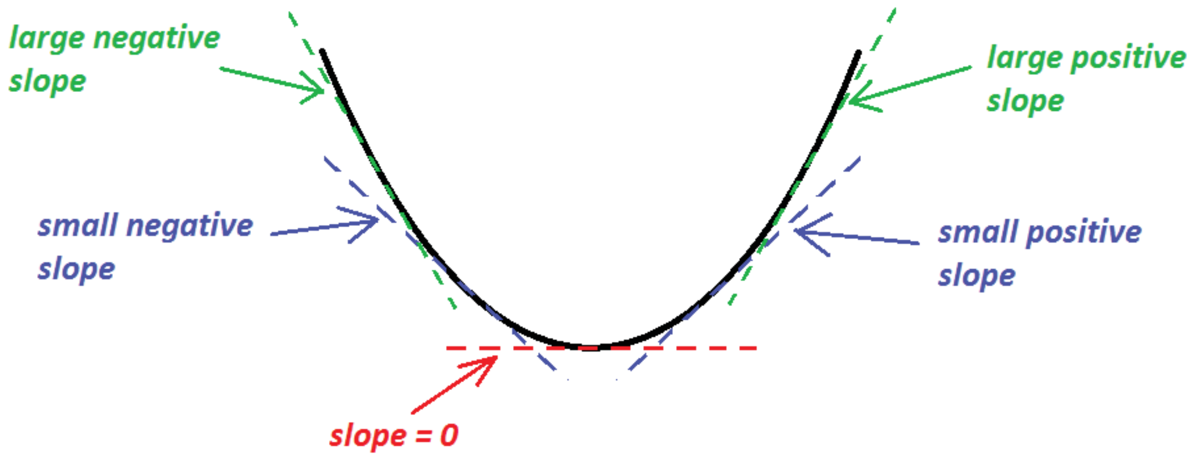
	$x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
Sign of $x + 1$	$-$	0	$+$	$+$	$+$	$+$	$+$
Sign of x	$-$	$-$	$-$	0	$+$	$+$	$+$
Sign of $x - 2$	$-$	$-$	$-$	$-$	$-$	0	$+$
Sign of $f'(x)$	$-$	0	$+$	0	$-$	0	$+$

$\therefore$ f has local maximum at $x = 0$, and the corresponding maximum value is $f(0) = 5$.

f has local minima at $x = -1$ and $x = 2$, and the corresponding local minimum values are $f(-1) = 0$ and $f(2) = -27$, respectively.

Concavity

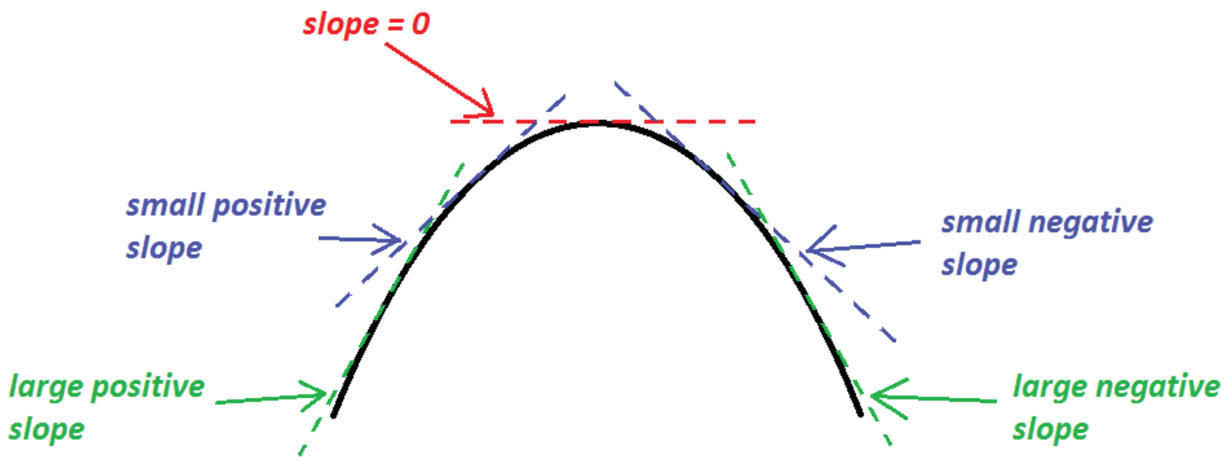
➢ A function f is **concave up** on an interval I if it is differentiable on I and $f'(x)$ is increasing on I, i.e. $f''(x) > 0$ for every $x \in I$.



Concave up: $f'(x)$ increases as x increases.

$\therefore\ f''(x) > 0$

➢ A function f is **concave down** on an interval I if it is differentiable on I and $f'(x)$ is decreasing on I, i.e. $f''(x) < 0$ for every $x \in I$.



Concave down:

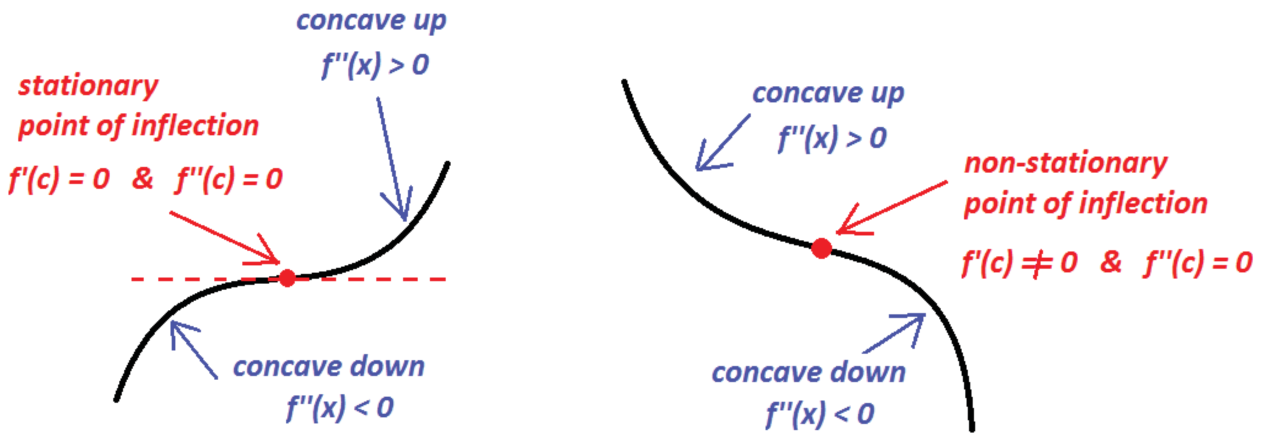
$f'(x)$ decreases as x increases.

$\therefore\ f''(x) < 0$

- Let f be a twice differentiable function. Then f has a **point of inflection** (or **point of inflexion**) at $x = c$ if $\boxed{f''(c) = 0}$ and **f is concave up on one side of c and concave down on the other side of c** (i.e. the sign of $f''(x)$ changes at $x = c$).

  In addition, if $f'(c) = 0$, we say that $(c, f(c))$ is a **stationary point of inflection**.

  If $f'(c) \neq 0$, then $(c, f(c))$ is a **non-stationary point of inflection**.



Summary: Determination of points of inflection of a function

Given a twice differentiable function $y = f(x)$.

To determine the points of inflection of the curve:

Step 1: Find the second derivative $f''(x)$.

Step 2: Set $\boxed{f''(x) = 0}$, and find all the values of c in $Dom(f)$ such that $f''(c) = 0$.

Step 3: If the sign of f'' changes at $x = c$, then f has a point of inflection at $x = c$, i.e. at the point $(c, f(c))$.

Example 10

Determine all (if any) points of inflection for the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

Solution

$f'(x) = 12x^3 - 12x^2 - 24x$

$f''(x) = 36x^2 - 24x - 24 = 12(3x^2 - 2x - 2)$

Set $f''(x) = 0 \Rightarrow 12(3x^2 - 2x - 2) = 0 \Rightarrow 3x^2 - 2x - 2 = 0$

$$\Rightarrow x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-2)}}{2(3)} = \frac{2 \pm \sqrt{28}}{6} = \frac{2 \pm 2\sqrt{7}}{6} = \frac{1 \pm \sqrt{7}}{3}$$

	$x < \frac{1-\sqrt{7}}{3}$	$x = \frac{1-\sqrt{7}}{3}$	$\frac{1-\sqrt{7}}{3} < x < \frac{1+\sqrt{7}}{3}$	$x = \frac{1+\sqrt{7}}{3}$	$x > \frac{1+\sqrt{7}}{3}$
Sign of $f''(x)$	$+$	**0**	$-$	**0**	$+$

Since the sign of f'' changes at $x = \frac{1-\sqrt{7}}{3}$ and $x = \frac{1+\sqrt{7}}{3}$, f has points of inflection at $x = \frac{1-\sqrt{7}}{3}$ and $x = \frac{1+\sqrt{7}}{3}$, i.e. at $(-0.5486, 2.3207)$ and $(1.2153, -13.3578)$.

Classification of the stationary point from the second derivative (Second Derivative Test)

Suppose that the function f has a **stationary point** at $x = c$, i.e. $\boxed{f'(c) = 0}$.

- If $f''(c) > 0$, then f has a **local minimum** at $x = c$.
- If $f''(c) < 0$, then f has a **local maximum** at $x = c$.
- If $f''(c) = 0$, then no conclusion can be drawn. The function f may have a local maximum at $x = c$, a local minimum at $x = c$, or a (stationary) point of inflection at $x = c$.

E.g. 1: Consider $f(x) = x^3$. Then $f'(x) = 3x^2$. Setting $f'(x) = 0$ gives $x = 0$.

$\therefore$ f has a stationary point at $x = 0$, $y = 0$.

$f''(x) = 6x$. Then $f''(0) = 6 \times 0 = 0$, so the Second Derivative Test fails.

Now, $f''(0) = 0$ and let's consider the sign of f'' in the neighborhood of $x = 0$:

	$x < 0$	$x = 0$	$x > 0$
Sign of $f''(x)$	$-$	0	$+$

Since the sign of $f''(x)$ changes at $x = 0$, the function f has a (stationary) point of inflection at $(0, 0)$.

E.g. 2: Consider $f(x) = x^4$. Then $f'(x) = 4x^3$.

Setting $f'(x) = 0$ gives $4x^3 = 0 \Rightarrow x = 0$

$\therefore$ f has a stationary point at $x = 0$, $y = 0$.

$f''(x) = 12x^2$. Then $f''(0) = 12 \times 0^2 = 0$, so the Second Derivative Test fails.

Now, $f''(0) = 0$ and let's consider the sign of f'' in the neighborhood of $x = 0$:

	$x < 0$	$x = 0$	$x > 0$
Sign of $f''(x)$	$+$	0	$+$

Since the sign of $f''(x)$ does not change at $x = 0$, the function f does not have a point of inflection at $(0, 0)$.

To classify the stationary point at $x = 0$, we use the First Derivative Test:

	$x < 0$	$x = 0$	$x > 0$
Sign of $f'(x)$	$-$	0	$+$

$\therefore$ The function has a local minimum at $(0, 0)$.

Summary: Determination of local extrema (local minima / local maxima) of a function

Given a twice differentiable function $y = f(x)$.

To determine the local extrema of the curve:

Step 1: Find the first derivative $f'(x)$.

Step 2: Set $f'(x) = 0$, and find all the values of x_0 in $Dom(f)$ such that $f'(x_0) = 0$.

Step 3: Classify whether x_0 is a local maximum or local minimum by using one of the following tests:

- **The First Derivative Test**:

(a) If there exists an open interval (a, b) containing x_0 such that

$f'(x) > 0$ when $a < x < x_0$

and $f'(x) < 0$ when $x_0 < x < b$,

then f has a **local maximum** value at $x = x_0$, i.e. at the point $(x_0, f(x_0))$.

(b) If there exists an open interval (a, b) containing x_0 such that

$f'(x) < 0$ when $a < x < x_0$

and $f'(x) > 0$ when $x_0 < x < b$,

then f has a **local minimum** value at $x = x_0$, i.e. at the point $(x_0, f(x_0))$.

(c) If the sign of $f'(x)$ does not change at $x = x_0$, then $x = x_0$ is neither a local maximum nor a local minimum.

➢ **The Second Derivative Test**:

(a) If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a **local maximum** value at $x = x_0$, i.e. at the point $(x_0, f(x_0))$.

(b) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a **local minimum** value at $x = x_0$, i.e. at the point $(x_0, f(x_0))$.

(c) If $f'(x_0) = 0$ and $f''(x_0) = 0$, no conclusion can be drawn.

($\rightarrow$ Use the First Derivative Test.)

Example 11

Determine and classify all local extrema of the function $f(x) = x^2 e^x$.

Solution

$f'(x) = x^2 \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(x^2) = x^2 \cdot e^x + e^x \cdot 2x = e^x x(x+2)$.

Set $f'(x) = 0 \quad \Rightarrow \quad e^x x(x+2) = 0$

$\Rightarrow \quad x = 0$ or $x = -2$ (Note: $e^x > 0$ for all $x \in \mathbb{R}$.)

Use either the **First Derivative Test**:

	$x < -2$	$x = -2$	$-2 < x < 0$	$x = 0$	$x > 0$
Sign of x	$-$	$-$	$-$	0	$+$
Sign of $(x+2)$	$-$	0	$+$	$+$	$+$
Sign of $f'(x)$	$+$	0	$-$	0	$+$
		(local max.)		(local min.)	

$\therefore$ f has a **local maximum** at $x = -2$, $y = 4e^{-2}$; and a **local minimum** at $x = 0$, $y = 0$.

Or the **Second Derivative Test**:

$$f''(x) = \frac{d}{dx}[e^x x(x+2)]$$
$$= e^x x \cdot \frac{d(x+2)}{dx} + e^x(x+2) \cdot \frac{d(x)}{dx} + x(x+2) \cdot \frac{d(e^x)}{dx}$$
$$= e^x x \cdot 1 + e^x(x+2) \cdot 1 + x(x+2) \cdot e^x$$
$$= e^x(x^2 + 4x + 2)$$

$$f''(-2) = e^{-2}[(-2)^2 + 4(-2) + 2] = -2e^{-2} < 0$$

$\Rightarrow$ f has a **local maximum** at $x = -2,\ y = 4e^{-2}$.

$$f''(0) = e^0[(0)^2 + 4(0) + 2] = 2 > 0$$

$\Rightarrow$ f a **local minimum** at $x = 0,\ y = 0$.

Example 12

Consider the function $f(x) = 2x^3 - 3x^2 - 12x + 1$, where $Dom(f) = [-2, 4]$. Find all local extrema and the absolute extrema of $f(x)$.

Solution

$f'(x) = 6x^2 - 6x - 12$

Set $f'(x) = 0 \quad \Rightarrow \quad 6x^2 - 6x - 12 = 0 \quad \Rightarrow \quad 6(x^2 - x - 2) = 0$

$\Rightarrow \quad 6(x-2)(x+1) = 0 \quad \Rightarrow \quad x = -1,\ 2 \ \in Dom(f)$

Use the **First Derivative Test:** $f'(x) = 6(x-2)(x+1)$

	$-2 \le x < -1$	$x = -1$	$-1 < x < 2$	$x = 2$	$2 < x \le 4$
Sign of $(x-2)$	$-$	$-$	$-$	0	$+$
Sign of $(x+1)$	$-$	0	$+$	$+$	$+$
Sign of $f'(x)$	$+$	0	$-$	0	$+$
		(local max.)		(local min.)	

$\therefore$ f has a **local maximum** at $(-1, 8)$ and a **local minimum** at $(2, -19)$.

OR use the **Second Derivative Test:**

$$f''(x) = 12x - 6$$

$f''(-1) = 12(-1) - 6 = -18 < 0 \quad \Rightarrow \quad f$ has a **local maximum** at $x = -1,\ y = 8$.

$f''(2) = 12(2) - 6 = 18 > 0 \quad \Rightarrow \quad f$ has a **local minimum** at $x = 2,\ y = -19$.

To find the absolute maximum and minimum of $f(x)$, we compare the local maximum and minimum values with the values of f at the end points of the domain.

$Dom(f) = [-2, 4]$.

$f(-2) = 2(-2)^3 - 3(-2)^2 - 12(-2) + 1 = \mathbf{-3}$

$f(4) = 2(4)^3 - 3(4)^2 - 12(4) + 1 = \mathbf{33}$ **(← absolute maximum)**

Local minimum at $x = 2,\ f(2) = \mathbf{-19}$ **(← absolute minimum)**

Local maximum at $x = -1,\ f(-1) = \mathbf{8}$

$\therefore\ f$ has an **absolute maximum** at $(4, 33)$ and an **absolute minimum** at $(2, -19)$.

4. Optimization problem

Aim: To **maximize** or **minimize** the quantity Q.

E.g. To maximize the volume of a box

E.g. To minimize the surface area of a container, etc.

Procedure for solving optimization problem

Suppose the question asks you to maximize or minimize the quantity Q.

Step 1: Read the question carefully. Draw a diagram if appropriate.

Step 2: Define any variables you wish to use that are not already specified in the question.

Step 3: Use the diagram or the given information from the question to write down one or more constraints which link the variables.

Step 4: Express the quantity Q to be maximized or minimized as a function of one or more variables.

Step 5: If Q depends on more than one variable, use the constraints in Step 3 to express Q as a function of only one variable (say x). Determine the interval in which this variable must lie for the problem to make sense.

Step 6: Differentiate the function Q with respect to the variable x (in Step 5), i.e. find $Q'(x)$.

Step 7: Set $Q'(x) = 0$ and solve for the value(s) of x.

Step 8: Eliminate any values of x obtained in Step 7 that do not make sense.

Step 9: Use either the First Derivative Test or the Second Derivative Test to determine which of the remaining critical point(s) is the one that you are looking for. If there is only one remaining critical point after Step 8, you are still required to check that the extreme value is really a maximum or minimum point.

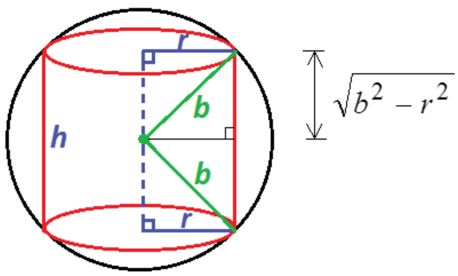
Step 10: Write down the final answer and the optimized value of Q (if necessary).

Example 13

Find the greatest volume of a cylinder that can be inscribed in a sphere of radius b, where b is a given constant.

Solution

Let h and r be the height and radius of the cylinder, respectively.



The height of the cylinder is $h = 2\sqrt{b^2 - r^2}$, where $0 \le h \le 2b$.

Thus, $r^2 = b^2 - \left(\frac{h}{2}\right)^2$.

Volume of the cylinder is $V = \pi r^2 h = \pi\left[b^2 - \left(\frac{h}{2}\right)^2\right]h = \pi\left(b^2 h - \frac{h^3}{4}\right)$.

Differentiate both sides w.r.t. h:

$$V'(h) = \pi\left(b^2 - \frac{3h^2}{4}\right).$$

Set $V'(h) = 0 \quad \Rightarrow \quad \pi\left(b^2 - \frac{3h^2}{4}\right) = 0 \quad \Rightarrow \quad b^2 - \frac{3h^2}{4} = 0 \quad \Rightarrow \quad h^2 = \frac{4b^2}{3}$

$\Rightarrow \quad h = \pm\sqrt{\frac{4b^2}{3}} = \frac{2b}{\sqrt{3}}$ or $-\frac{2b}{\sqrt{3}}$ (rejected since h must be ≥ 0.)

First Derivative Test: $V'(h) = \pi\left(b^2 - \frac{3h^2}{4}\right) = \frac{3}{4}\pi\left(\frac{4b^2}{3} - h^2\right) = \frac{3}{4}\pi\left(\frac{2b}{\sqrt{3}} + h\right)\left(\frac{2b}{\sqrt{3}} - h\right)$

	$0 \leq h < \frac{2b}{\sqrt{3}}$	$h = \frac{2b}{\sqrt{3}}$	$\frac{2b}{\sqrt{3}} < h \leq 2b$
Sign of $V'(h)$	$+$	**0**	$-$

(**local max.**)

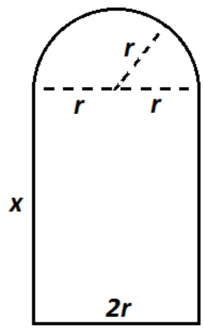
$\therefore$ The volume of the cylinder is maximized at $h = \frac{2b}{\sqrt{3}}$.

$\therefore$ The greatest volume of the cylinder is

$$V\left(\frac{2b}{\sqrt{3}}\right) = \pi\left[b^2\left(\frac{2b}{\sqrt{3}}\right) - \frac{\left(\frac{2b}{\sqrt{3}}\right)^3}{4}\right] = \pi\left(\frac{2}{\sqrt{3}}b^3 - \frac{2}{3\sqrt{3}}b^3\right) = \frac{4\pi}{3\sqrt{3}}b^3 \quad (\text{unit}^3)$$

Example 14

A window is in the shape as shown in the figure on the right. Suppose the perimeter of the window is fixed to be 240 cm. Find the dimensions of the window so that the area of the window is maximized.



Solution

Perimeter of the window is $2x + 2r + \pi r = 240$

$\therefore \quad x = \frac{240 - 2r - \pi r}{2} = 120 - r - \frac{\pi}{2}r$

Area of the window: $A(r) = \frac{\pi r^2}{2} + 2rx = \frac{\pi r^2}{2} + 2r\left(120 - r - \frac{\pi}{2}r\right)$

$$= 240r - 2r^2 - \frac{\pi r^2}{2}$$

Differentiate both sides w.r.t. r:

$$A'(r) = 240 - 4r - \pi r$$

Set $A'(r) = 0 \quad \Rightarrow \quad 240 - 4r - \pi r = 0$

$\Rightarrow \quad r = \frac{240}{\pi + 4}$

Using the Second Derivative Test:

$$A''(r) = -4 - \pi$$

$\because\ A''\left(\frac{240}{\pi+4}\right) = -4 - \pi < 0,$

$\therefore\ A(r)$ is maximized when

$$r = \frac{240}{\pi + 4} \approx \boxed{33.6\text{ cm}}$$

and

$$x = 120 - r - \frac{\pi}{2}r = 120 - \frac{240}{\pi + 4} - \frac{\pi}{2}\left(\frac{240}{\pi + 4}\right) = \frac{240}{\pi + 4} \approx \boxed{33.6\text{ cm}}.$$

Example 15

Find the coordinates of a point on the parabola $2y = x^2$ that is closest to the point $(-6, 0)$. Hence find that shortest distance.

Solution

Let (x, y) be a point on the parabola $2y = x^2$.

Then the distance between (x, y) and $(-6, 0)$ is

$$l = \sqrt{[x - (-6)]^2 + (y - 0)^2} = \sqrt{x^2 + 12x + 36 + \left(\frac{1}{2}x^2\right)^2} = \left(\frac{x^4}{4} + x^2 + 12x + 36\right)^{\frac{1}{2}}$$

Differentiating both sides w.r.t. x:

$$\frac{dl}{dx} = \frac{1}{2}\left(\frac{x^4}{4} + x^2 + 12x + 36\right)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left(\frac{x^4}{4} + x^2 + 12x + 36\right)$$

$$= \frac{1}{2} \cdot \frac{x^3 + 2x + 12}{\sqrt{\frac{x^4}{4} + x^2 + 12x + 36}}$$

Set $\dfrac{dl}{dx} = 0 \;\Rightarrow\; \dfrac{1}{2} \cdot \dfrac{x^3 + 2x + 12}{\sqrt{\dfrac{x^4}{4} + x^2 + 12x + 36}} = 0$

$$\Rightarrow\; x^3 + 2x + 12 = 0$$

$$\Rightarrow\; (x+2)(x^2 - 2x + 6) = 0$$

$$\Rightarrow\; x + 2 = 0 \quad \text{or} \quad x^2 - 2x + 6 = 0 \;\; \text{(which has no real solution)}$$

$$\Rightarrow\; x = -2$$

Using the First Derivative Test:

$$\frac{dl}{dx} < 0 \;\text{ when }\; x < -2$$
$$\frac{dl}{dx} > 0 \;\text{ when }\; x > -2$$

$\therefore$ The distance between (x, y) and $(-6, 0)$ is minimized when $x = -2,\; y = 2$.

$\therefore$ The shortest distance is

$$l = \left[\frac{(-2)^4}{4} + (-2)^2 + 12(-2) + 36\right]^{\frac{1}{2}} = \sqrt{20}\;\text{(unit)}$$

5. L'Hôpitals rule

This is used to find **limits of indeterminate forms** such as

$$\boxed{\frac{0}{0}\,,\;\frac{\infty}{\infty}}\;,\;\boxed{0 \times \infty\;,\;\infty - \infty}\;\text{ or }\;\boxed{1^{\infty}\;,\;\infty^0\;,\;0^0}.$$

Type $\frac{0}{0}$: If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, then

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}.$$

Note: a could be any real number, a^-, a^+, $-\infty$ or ∞.

Remark: Always check that you have $\frac{0}{0}$ form **BEFORE** applying the L'Hôpital's rule. DO NOT use the L'Hôpital's rule if any one of the numerator and denominator is non-zero when taking limit.

Example 16

Evaluate the limit $\lim\limits_{x\to 2} \frac{x^2-4}{x^2+x-6}$.

Solution

Method 1:

$$\lim_{x\to 2} \frac{x^2-4}{x^2+x-6} \quad \left(\frac{\mathbf{0}}{\mathbf{0}}\ \textbf{form}\right)$$
$$= \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x+3)} = \lim_{x\to 2} \frac{x+2}{x+3} = \frac{2+2}{2+3} = \frac{4}{5}$$

Method 2:

$$\lim_{x\to 2} \frac{x^2-4}{x^2+x-6} \quad \left(\frac{\mathbf{0}}{\mathbf{0}}\ \textbf{form}\right)$$
$$= \lim_{x\to 2} \frac{2x}{2x+1} \quad \text{by } \mathbf{L'H\hat{o}pital's\ rule}$$
$$= \frac{2(2)}{2(2)+1} = \frac{4}{5}$$

Example 17

Evaluate $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2+x}$.

Solution

Method 1:

$$\lim_{x\to 0} \frac{1-\cos x}{x^2+x} \quad \left(\frac{\mathbf{0}}{\mathbf{0}}\ \textbf{form}\right) = \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{(x^2+x)(1+\cos x)} = \lim_{x\to 0} \frac{1-\cos^2 x}{x(x+1)(1+\cos x)}$$
$$= \lim_{x\to 0} \left(\frac{\sin x}{x} \cdot \frac{\sin x}{(x+1)(1+\cos x)}\right)$$
$$= 1 \cdot \frac{\sin 0}{(0+1)(1+\cos 0)} = \frac{0}{2} = 0$$

Method 2:

$$\lim_{x\to 0} \frac{1-\cos x}{x^2+x} \quad \left(\frac{\mathbf{0}}{\mathbf{0}}\ \textbf{form}\right) = \lim_{x\to 0} \frac{\sin x}{2x+1} \quad \text{by } \mathbf{L'H\hat{o}pital's\ rule}$$
$$= \frac{\sin 0}{2(0)+1} = \frac{0}{1} = 0$$

Example 18

Find $\lim_{x\to 1} \frac{x-1}{\ln x}$.

Solution

$$\lim_{x\to 1} \frac{x-1}{\ln x} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 1} \frac{1}{\frac{1}{x}} \quad \text{by } \mathbf{L'Hôpital's\ rule}$$

$$= \lim_{x\to 1} x$$

$$= 1$$

Example 19

Evaluate $\lim_{x\to 0} \frac{\cos(2x)-\cos x}{x^2}$.

Solution

$$\lim_{x\to 0} \frac{\cos(2x)-\cos x}{x^2} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{-2\sin(2x)+\sin x}{2x} \quad \text{by } \mathbf{L'Hôpital's\ rule} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{-4\cos(2x)+\cos x}{2} \quad \text{by } \mathbf{L'Hôpital's\ rule}$$

$$= \frac{-4\cos 0+\cos 0}{2}$$

$$= \frac{-4+1}{2}$$

$$= -\frac{3}{2}$$

Example 20

Evaluate $\lim_{x\to 0}\frac{\tan x - x}{x - \sin x}$.

Solution

$$\lim_{x\to 0}\frac{\tan x - x}{x-\sin x}\quad \left(\frac{0}{0}\text{ form}\right)$$
$$=\lim_{x\to 0}\frac{\sec^2 x - 1}{1-\cos x}\quad \text{by L'Hôpital's rule}\quad \left(\frac{0}{0}\text{ form}\right)$$
$$=\lim_{x\to 0}\frac{2\sec x\cdot \sec x\tan x}{\sin x}\quad \text{by L'Hôpital's rule}$$
$$=\lim_{x\to 0}\left(\frac{2}{\cos^2 x\cdot \sin x}\cdot\frac{\sin x}{\cos x}\right)$$
$$=\lim_{x\to 0}\frac{2}{\cos^3 x}$$
$$=\frac{2}{\cos^3 0}$$
$$=\frac{2}{1^3}=2$$

Example 21

Evaluate $\lim_{x\to 0}\frac{e^{x^3}-1-x^3}{x^6}$.

Solution

$$\lim_{x\to 0}\frac{e^{x^3}-1-x^3}{x^6}\quad \left(\frac{0}{0}\text{ form}\right)$$
$$=\lim_{x\to 0}\frac{3x^2e^{x^3}-3x^2}{6x^5}\quad \text{by L'Hôpital's rule}$$
$$=\lim_{x\to 0}\frac{e^{x^3}-1}{2x^3}\quad \left(\frac{0}{0}\text{ form}\right)$$
$$=\lim_{x\to 0}\frac{3x^2e^{x^3}}{6x^2}\quad \text{by L'Hôpital's rule}$$
$$=\lim_{x\to 0}\frac{e^{x^3}}{2}$$
$$=\frac{e^0}{2}$$
$$=\frac{1}{2}$$

Example 22

Evaluate $\lim\limits_{x\to 0} \frac{\ln(\cos 5x)}{\ln(\cos 2x)}$.

Solution

$$\lim_{x\to 0} \frac{\ln(\cos 5x)}{\ln(\cos 2x)} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\frac{-5\sin 5x}{\cos 5x}}{\frac{-2\sin 2x}{\cos 2x}} \qquad \text{by } \textbf{L'Hôpital's rule}$$

$$= \frac{5}{2} \lim_{x\to 0} \left(\frac{\sin 5x}{5x} \cdot \frac{2x}{\sin 2x} \cdot \frac{\cos 2x}{\cos 5x} \cdot \frac{5}{2}\right)$$

$$= \frac{5}{2} \cdot 1 \cdot 1 \cdot \frac{1}{1} \cdot \frac{5}{2}$$

$$= \frac{25}{4}$$

Type $\frac{\infty}{\infty}$: If $\lim\limits_{x\to a} |f(x)| = \lim\limits_{x\to a} |g(x)| = \infty$, then

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}.$$

Note: a could be any real number, a^-, a^+, $-\infty$ or ∞.

Remark: Always check that you have $\frac{\infty}{\infty}$ form **BEFORE** applying the L'Hôpital's rule.

Example 23

Find $\lim\limits_{x\to\infty} \frac{x}{e^x}$.

Solution

$$\lim_{x\to\infty} \frac{x}{e^x} \quad \left(\frac{\infty}{\infty} \text{ form}\right) = \lim_{x\to\infty} \frac{1}{e^x} \qquad \text{by } \textbf{L'Hôpital's rule}$$

$$= 0$$

Example 24

Find $\lim_{x\to\infty} \frac{e^x}{x}$, if it exists.

Solution

$$\lim_{x\to\infty} \frac{e^x}{x} \left(\frac{\infty}{\infty} \text{ form}\right) = \lim_{x\to\infty} \frac{e^x}{1} \quad \text{by } \textbf{L'Hôpital's rule}$$

$$= \infty$$

$\therefore$ The limit does not exist.

Example 25

Evaluate $\lim_{x\to 0^+} \frac{\ln x}{\cot x}$.

Solution

$$\lim_{x\to 0^+} \frac{\ln x}{\cot x} \quad \left(\frac{-\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\operatorname{cosec}^2 x} \quad \text{by } \textbf{L'Hôpital's rule}$$

$$= -\lim_{x\to 0^+} \frac{\sin^2 x}{x}$$

$$= -\lim_{x\to 0^+} \left(\frac{\sin x}{x} \cdot \sin x\right)$$

$$= -1 \cdot 0$$

$$= 0$$

Example 26

Evaluate $\lim_{x\to 0^+} \frac{\ln(\sin 2x)}{\ln(\sin 3x)}$.

Solution

$$\lim_{x\to 0^+} \frac{\ln(\sin 2x)}{\ln(\sin 3x)} \quad \left(\frac{-\infty}{-\infty}\ \textbf{form}\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{2\cos 2x}{\sin 2x}}{\frac{3\cos 3x}{\sin 3x}} \quad \text{by } \textbf{L'Hôpital's rule}$$

$$= \lim_{x\to 0^+} \left(\frac{2\boldsymbol{x}}{\sin 2x} \cdot \frac{\sin 3x}{3\boldsymbol{x}} \cdot \frac{\cos 2x}{\cos 3x}\right)$$

$$= 1 \cdot 1 \cdot \frac{1}{1}$$

$$= 1$$

Type $\mathbf{0 \times \infty}$ and $\infty - \infty$: Rearrange the function so that it is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and then apply L'Hôpital's rule

Example 27

Evaluate $\lim_{x\to 0^+} x^2 \ln x$.

Solution

$$\lim_{x\to 0^+} x^2 \ln x \quad (\mathbf{0} \times (-\infty)\ \textbf{form})$$

$$= \lim_{x\to 0^+} \frac{\ln x}{x^{-2}} \quad \left(\frac{-\infty}{\infty}\ \textbf{form}\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{(-2)x^{-3}} \quad \text{by } \textbf{L'Hôpital's rule}$$

$$= \lim_{x\to 0^+} \left(\frac{1}{-2} \cdot \frac{1}{x} \cdot x^3\right)$$

$$= \lim_{x\to 0^+} \frac{x^2}{-2} = \frac{0}{-2} = 0$$

Example 28

Evaluate $\lim_{x\to 1^+}(x^2-1)\tan\left(\frac{\pi x}{2}\right)$.

Solution

$$\lim_{x\to 1^+}(x^2-1)\tan\left(\frac{\pi x}{2}\right) \quad (\mathbf{0}\times(-\infty)\ \textbf{form})$$

$$=\lim_{x\to 1^+}\frac{x^2-1}{\cot\left(\frac{\pi x}{2}\right)} \quad \left(\frac{\mathbf{0}}{\mathbf{0}}\ \textbf{form}\right)$$

$$=\lim_{x\to 1^+}\frac{2x}{-\csc^2\left(\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}} \quad \text{by } \textbf{L'Hôpital's rule}$$

$$=\frac{2(1)}{-1^2\cdot\frac{\pi}{2}}$$

$$=-\frac{4}{\pi}$$

Example 29

Evaluate $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$.

Solution

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) \quad (\infty-\infty\ \textbf{form})$$

$$=\lim_{x\to 0}\frac{x-\sin x}{x\sin x} \quad \left(\frac{\mathbf{0}}{\mathbf{0}}\ \textbf{form}\right)$$

$$=\lim_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x} \quad \text{by } \textbf{L'Hôpital's rule} \quad \left(\frac{\mathbf{0}}{\mathbf{0}}\ \textbf{form}\right)$$

$$=\lim_{x\to 0}\frac{\sin x}{-x\sin x+\cos x+\cos x} \quad \text{by } \textbf{L'Hôpital's rule}$$

$$=\frac{\sin 0}{-0\cdot\sin 0+\cos 0+\cos 0}$$

$$=\frac{0}{0+1+1}$$

$$=0$$

Example 30

Evaluate $\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{x}{x-1}\right)$.

Solution

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{x}{x-1}\right) \quad (\infty-\infty \text{ form})$$
$$=\lim_{x\to 1}\frac{(x-1)-x\ln x}{(x-1)\ln x} \quad \left(\frac{0}{0}\text{ form}\right)$$
$$=\lim_{x\to 1}\frac{1-\left(x\cdot\frac{1}{x}+\ln x\cdot 1\right)}{(x-1)\cdot\frac{1}{x}+(\ln x)\cdot 1} \quad \text{by L'Hôpital's rule}$$
$$=\lim_{x\to 1}\frac{-\ln x}{\frac{x-1+x\ln x}{x}}$$
$$=\lim_{x\to 1}\frac{-x\ln x}{x-1+x\ln x} \quad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 1}\frac{-\left(x\cdot\frac{1}{x}+\ln x\cdot 1\right)}{1+x\cdot\frac{1}{x}+\ln x} \quad \text{by L'Hôpital's rule}$$
$$=\lim_{x\to 1}\frac{-1-\ln x}{2+\ln x}$$
$$=\frac{-1-\ln 1}{2+\ln 1}$$
$$=\frac{-1-0}{2+0}$$
$$=-\frac{1}{2}$$

$\boxed{\textit{Type}\ \ 0^0\ ,\ \ \infty^0\ ,\ \ 1^\infty}$: Take natural log on both sides first, then take limits on both sides, and finally take exponential.

Example 31

Evaluate $\lim_{x\to 0^+} x^{2x}$.

Solution

$\lim_{x\to 0^+} x^{2x}$ $\left(\mathbf{0^0}\ \textbf{form}\right)$

Let $y = x^{2x}$. Take natural log on both sides:

$$\ln y = \ln(x^{2x}) = 2x\ln x$$

Take limits on both sides:

$$\begin{aligned}\lim_{x\to 0^+}\ln y &= \lim_{x\to 0^+} 2x\ln x \qquad (\mathbf{0}\times(-\infty)\ \textbf{form})\\ &= \lim_{x\to 0^+}\frac{2\ln x}{x^{-1}} \qquad \left(\frac{-\infty}{\infty}\ \textbf{form}\right)\\ &= \lim_{x\to 0^+}\frac{\frac{2}{x}}{(-1)x^{-2}} \qquad \text{by } \textbf{L'Hôpital's rule}\end{aligned}$$

$$= -2\lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{1}{x^2}} = -2\lim_{x\to 0^+} x = -2\cdot 0 = 0$$

$$\therefore \lim_{x\to 0^+} x^{2x} = \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\ln y} = e^{\lim_{x\to 0^+} y} = e^0 = 1$$

Example 32

Evaluate $\lim_{x\to 0^+}(\cot x)^{\frac{1}{\ln x}}$.

Solution

$\lim_{x\to 0^+}(\cot x)^{\frac{1}{\ln x}}$ $\left(\mathbf{\infty^0}\ \textbf{form}\right)$

Let $y = (\cot x)^{\frac{1}{\ln x}}$.

Take natural log on both sides:

$$\ln y = \ln\left[(\cot x)^{\frac{1}{\ln x}}\right] = \frac{1}{\ln x}\cdot\ln(\cot x)$$

Take limits on both sides:

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+}\left[\frac{1}{\ln x}\cdot \ln(\cot x)\right]$$

$$= \lim_{x\to 0^+}\frac{\ln(\cot x)}{\ln x} \qquad \left(\frac{\infty}{-\infty}\ \text{form}\right)$$

$$= \lim_{x\to 0^+}\frac{\dfrac{-\operatorname{cosec}^2 x}{\cot x}}{\dfrac{1}{x}} \qquad \text{by } \mathbf{L'Hôpital's\ rule}$$

$$= \lim_{x\to 0^+}\left(\frac{-x}{\sin^2 x}\cdot\frac{\sin x}{\cos x}\right)$$

$$= -\lim_{x\to 0^+}\left(\frac{x}{\sin x}\cdot\frac{1}{\cos x}\right)$$

$$= -1\cdot\frac{1}{\cos 0}$$

$$= -1$$

$$\therefore \lim_{x\to 0^+}(\cot x)^{\frac{1}{\ln x}} = e^{-1} = \frac{1}{e}$$

Example 33

Evaluate $\lim_{x\to 0^+}(1+\sin 4x)^{\cot x}$.

Solution

$\lim_{x\to 0^+}(1+\sin 4x)^{\cot x}$ $\quad (\mathbf{1^\infty\ form})$

Let $y = (1+\sin 4x)^{\cot x}$. Take natural log on both sides:

$$\ln y = \ln[(1+\sin 4x)^{\cot x}] = \cot x \cdot \ln(1+\sin 4x)$$

Take limits on both sides:

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}[\cot x\cdot\ln(1+\sin 4x)] \quad (\infty\times\mathbf{0\ form})$$

$$= \lim_{x\to 0^+}\frac{\ln(1+\sin 4x)}{\tan x} \quad \left(\frac{\mathbf{0}}{\mathbf{0}}\ \text{form}\right)$$

$$= \lim_{x\to 0^+}\frac{\dfrac{4\cos 4x}{1+\sin 4x}}{\sec^2 x} \quad \text{by } \mathbf{L'Hôpital's\ rule}$$

$$= \lim_{x\to 0^+}\frac{\frac{4}{1+0}}{1} = 4$$

$$\therefore \lim_{x\to 0^+}(1+\sin 4x)^{\cot x} = e^4$$

6. Taylor series/Maclaurin series

It is an approximation of a function $f(x)$ by using high order polynomials, where the terms are calculated from the values of the function's derivatives at a single point $x = a$.

Assume that $f(x)$ can be expressed as the sum of a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + a_4(x-a)^4 + \cdots$$

where a_n's are constants to be determined. Differentiating $f(x)$ a few times w.r.t. x:

$$f'(x) = a_1 + 2a_2(x-a) + 3a_3(x-a)^2 + 4a_4(x-a)^3 + 5a_5(x-a)^4 + \cdots$$

$$f''(x) = 2a_2 + 3 \cdot 2 \cdot a_3(x-a) + 4 \cdot 3 \cdot a_4(x-a)^2 + 5 \cdot 4 \cdot a_5(x-a)^3 + \cdots$$

$$f'''(x) = 3 \cdot 2 \cdot a_3 + 4 \cdot 3 \cdot 2 \cdot a_4(x-a) + 5 \cdot 4 \cdot 3 \cdot a_5(x-a)^2 + \cdots \qquad \text{etc.}$$

Thus, we have

$$f^{(n)}(x) = n!\, a_n + \underbrace{\frac{(n+1)!}{1!} a_{n+1}(x-a) + \frac{(n+2)!}{2!} a_{n+2}(x-a)^2 + \cdots}_{\text{terms with } (x-a) \text{ as factor}}$$

Putting $x = a$, we get

$$f^{(n)}(a) = n!\, a_n \quad \Rightarrow \quad a_n = \frac{f^{(n)}(a)}{n!}.$$

Hence,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

If $f(x)$ is infinitely differentiable on an interval I centred at $x = a$, i.e. $f^{(k)}(a)$ exists for $k = 0, 1, 2, \ldots$, then the series

$$\boxed{\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}}$$

is called the **Taylor series of $f(x)$ about $x = a$**.

If the Taylor series expansion is centred at $x = 0$, i.e. if we put $a = 0$ into the Taylor series, then that series is called a **Maclaurin series of $f(x)$**:

$$\boxed{\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \end{aligned}}$$

Taylor's Theorem

If f is a continuous function of x with continuous derivatives $f'(x)$, $f''(x)$, …, $f^{(n-1)}(x)$ in $[a, b]$, and if $f^{(n)}(x)$ exists in (a, b), then for each x in (a, b), there exists a number $c \in (a, x)$ such that

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n)}(c)}{n!}(x-a)^n$$

is the **remainder term**.

If $\lim_{n\to\infty} R_n(x) = 0$, then $f(x)$ may be represented by the Taylor series expanded about $x = a$, and we say that the Taylor series of f at a converges to f for all x on (a, b).

Example 34

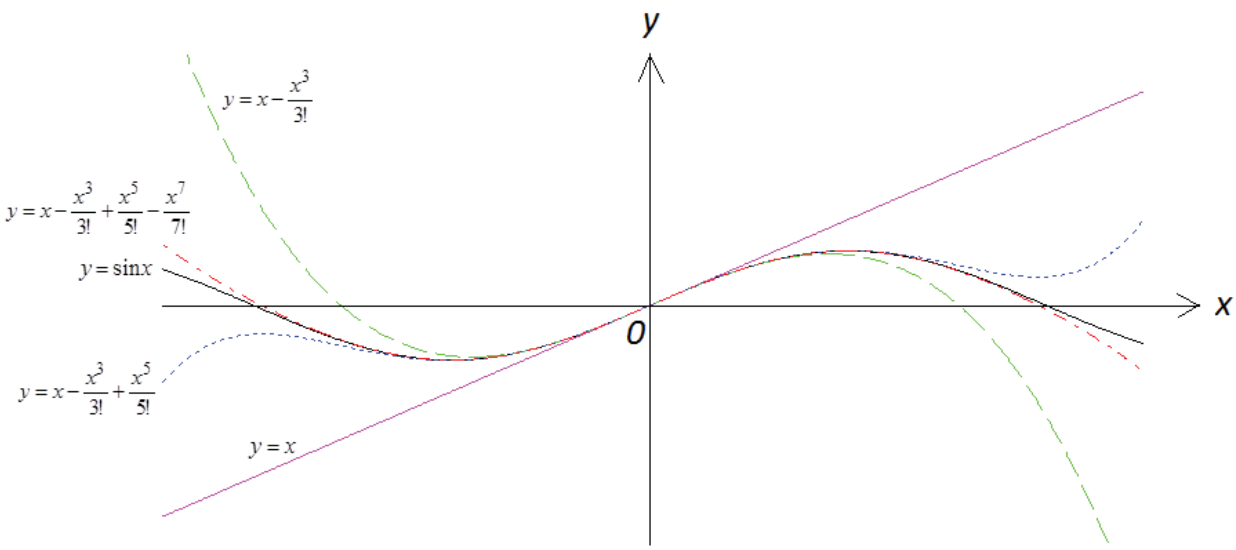
Find the Maclaurin series for $f(x) = \sin x$ as far as the term in x^7.

Solution

$f(x) = \sin x$	$f(0) = \sin 0 = 0$
$f'(x) = \cos x$	$f'(0) = \cos 0 = 1$
$f''(x) = -\sin x$	$f''(0) = -\sin 0 = 0$
$f'''(x) = -\cos x$	$f'''(0) = -\cos 0 = -1$
$f^{(4)}(x) = \sin x$	$f^{(4)}(0) = \sin 0 = 0$
$f^{(5)}(x) = \cos x$	$f^{(5)}(0) = \cos 0 = 1$
$f^{(6)}(x) = -\sin x$	$f^{(6)}(0) = -\sin 0 = 0$
$f^{(7)}(x) = -\cos x$	$f^{(7)}(0) = -\cos 0 = -1$

$\therefore$ The Maclaurin series for $f(x) = \sin x$ is:

$$\begin{aligned} f(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \\ &= 0 + \frac{1}{1!}x + 0 + \frac{(-1)}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0 + \frac{(-1)}{7!}x^7 + \cdots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \end{aligned}$$



Remark:

The remainder term is $R_n(x) = \frac{f^{(n)}(c)}{n!}x^n$, for some c between x and 0.

Since the n^{th} derivative of $\sin x$ is either $\pm\sin x$ or $\pm\cos x$, then

$$|R_n(x)| = \left|\frac{f^{(n)}(c)}{n!}x^n\right| \leq \frac{|x|^n}{n!}.$$

Since $\lim_{n\to\infty}\frac{|x|^n}{n!} = 0$ **for any x** (we omit the proof here), we have $\lim_{n\to\infty}|R_n(x)| = 0$.

Since $-|R_n(x)| \leq R_n(x) \leq |R_n(x)|$ for any x, we have $\lim_{n\to\infty} R_n(x) = 0$, by the Sandwich Theorem.

Therefore, the Maclaurin series for $\sin x$ **converges** to $\sin x$ **for all x**.

Example 35

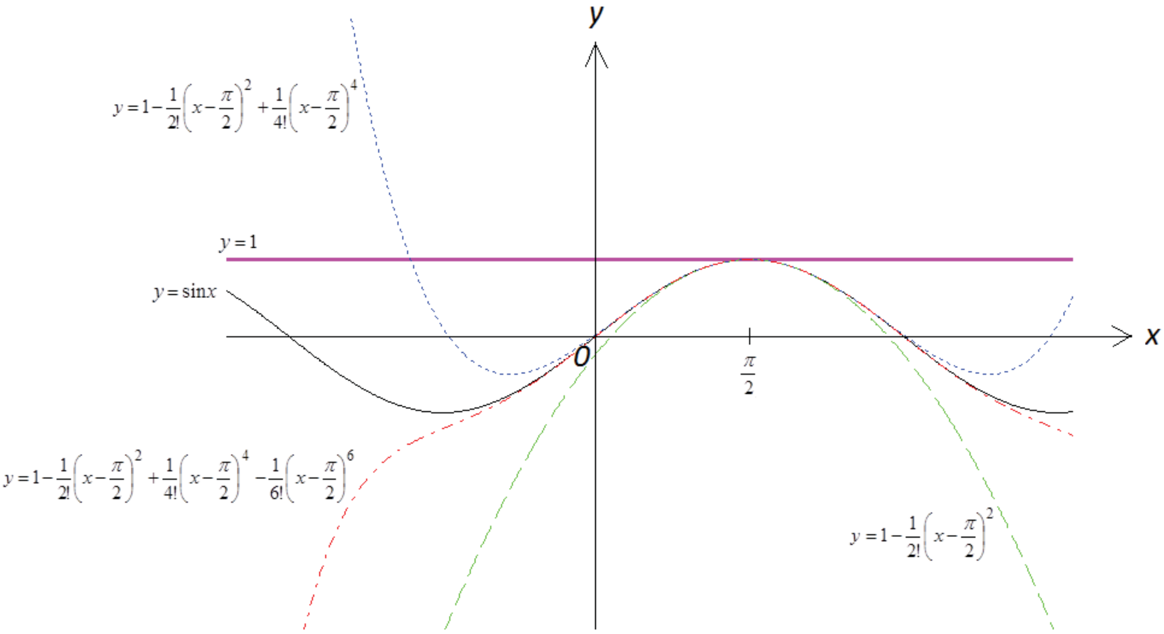
Find the Taylor series for $f(x) = \sin x$ about $x = \frac{\pi}{2}$ for at least four non-zero terms.

Solution

$f(x) = \sin x$	$f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$
$f'(x) = \cos x$	$f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$
$f''(x) = -\sin x$	$f''\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$
$f'''(x) = -\cos x$	$f'''\left(\frac{\pi}{2}\right) = -\cos\frac{\pi}{2} = 0$
$f^{(4)}(x) = \sin x$	$f^{(4)}\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$
$f^{(5)}(x) = \cos x$	$f^{(5)}\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$
$f^{(6)}(x) = -\sin x$	$f^{(6)}\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1$

$\therefore$ The Taylor series for $f(x) = \sin x$ about $x = \frac{\pi}{2}$ is:

$$\begin{aligned} f(x) &= f\left(\frac{\pi}{2}\right) + \frac{f'\left(\frac{\pi}{2}\right)}{1!}\left(x-\frac{\pi}{2}\right) + \frac{f''\left(\frac{\pi}{2}\right)}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{f'''\left(\frac{\pi}{2}\right)}{3!}\left(x-\frac{\pi}{2}\right)^3 + \cdots \\ &= 1 + 0 + \frac{(-1)}{2!}\left(x-\frac{\pi}{2}\right)^2 + 0 + \frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4 + 0 + \frac{(-1)}{6!}\left(x-\frac{\pi}{2}\right)^6 + \cdots \\ &= 1 - \frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x-\frac{\pi}{2}\right)^6 + \cdots \end{aligned}$$



Homework: Show that the Maclaurin series for $f(x)=\cos x$ is given by

$$f(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots.$$

Example 36

(a) Find the Maclaurin series of $f(x)=e^x$ as far as the term in x^5.

(b) Use your answer in part (a) to find an approximation to the value of $\sqrt{e}$.

Solution

(a) $f(x)=e^x$ $\qquad f(0)=e^0=1$

$f'(x)=e^x$ $\qquad f'(0)=e^0=1$

$\vdots \qquad \vdots$

$f^{(n)}(x)=e^x$ for all $n\in\mathbb{N}$. $\qquad f^{(n)}(0)=e^0=1$ for all $n\in\mathbb{N}$.

$\therefore$ The Maclaurin series of $f(x)=e^x$ is

$$\begin{aligned} f(x) &= f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(0)}{5!}x^5+\cdots \\ &= 1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+\cdots \\ &= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots \end{aligned}$$

(b) Put $x = \frac{1}{2}$ into the Maclaurin series obtained in part (a):

$$\sqrt{e} = e^{\frac{1}{2}} = f\left(\frac{1}{2}\right)$$

$$= 1 + \left(\frac{1}{2}\right) + \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^3}{3!} + \frac{\left(\frac{1}{2}\right)^4}{4!} + \frac{\left(\frac{1}{2}\right)^5}{5!} + \cdots$$

$$\approx 1.648698 \quad (\text{to 6 d.p.})$$

Remarks:

1. The actual answer is $\sqrt{e} = 1.648721 \ldots$
2. If we just take the first 4 terms, we have $1 + \left(\frac{1}{2}\right) + \frac{\left(\frac{1}{2}\right)^2}{2!} + \frac{\left(\frac{1}{2}\right)^3}{3!} \approx 1.645833.$
3. The more terms you take, the better the approximation.

Example 37

(a) Find the Maclaurin series for $f(x) = \ln(1 + x)$ as far as the term in x^4.

(b) Use your answer in part (a) to approximate the value of $\ln(1.1)$.

Solution

(a)

$f(x) = \ln(1 + x)$ $\qquad$ $f(0) = \ln(1) = 0$

$f'(x) = \frac{1}{1+x} = (1 + x)^{-1}$ $\qquad$ $f'(0) = 1$

$f''(x) = (-1)(1 + x)^{-2}$ $\qquad$ $f''(0) = -1$

$f'''(x) = (-1)(-2)(1 + x)^{-3}$ $\qquad$ $f'''(0) = 2$

$f^{(4)}(x) = (-1)(-2)(-3)(1 + x)^{-4}$ $\qquad$ $f^{(4)}(0) = -3!$

$\therefore$ The Maclaurin series of $f(x) = \ln(1 + x)$ is

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$$= 0 + \frac{1}{1!}x + \frac{(-1)}{2!}x^2 + \frac{2}{3!}x^3 + \frac{-3!}{4!}x^4 + \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

(b) Put $x = 0.1$ into the Maclaurin series obtained in part (a):

$$\ln(1.1) = f(0.1) = 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \cdots \approx 0.095308 \quad (\text{to 6 d.p.})$$

Note that the exact value is $\ln(1.1) = 0.095310179\ldots$.

If we take more terms in the series, i.e. $f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$, and then put $x = 0.1$, we get

$$\ln(1.1) = f(0.1) \approx 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \frac{(0.1)^5}{5} - \frac{(0.1)^6}{6}$$

$= 0.095310166$ (which is more closer to the exact value)

Example 38

Given the Maclaurin series of $\tan^{-1} x$,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

By putting the $x = \frac{1}{2}$ and $x = \frac{1}{3}$ into the above result, find an approximation to the value of π to the nearest 4 decimal places.

Solution

$$\tan^{-1}\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{5}\left(\frac{1}{2}\right)^5 - \frac{1}{7}\left(\frac{1}{2}\right)^7 + \cdots = \frac{1}{2} - \frac{1}{3 \times 8} + \frac{1}{5 \times 32} - \frac{1}{7 \times 128} + \cdots$$

$$\tan^{-1}\left(\frac{1}{3}\right) = \frac{1}{3} - \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{5}\left(\frac{1}{3}\right)^5 - \frac{1}{7}\left(\frac{1}{3}\right)^7 + \cdots = \frac{1}{3} - \frac{1}{3 \times 27} + \frac{1}{5 \times 243} - \frac{1}{7 \times 2187} + \cdots$$

By putting $A = \tan^{-1}\left(\frac{1}{2}\right)$ and $B = \tan^{-1}\left(\frac{1}{3}\right)$ into the compound angle formula

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

and then taking $\tan^{-1}$ on both sides, we have

$$\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\tan^{-1}\left[\frac{\tan\left(\tan^{-1}\left(\frac{1}{2}\right)\right)+\tan\left(\tan^{-1}\left(\frac{1}{3}\right)\right)}{1-\tan\left(\tan^{-1}\left(\frac{1}{2}\right)\right)\cdot\tan\left(\tan^{-1}\left(\frac{1}{3}\right)\right)}\right]$$

$$=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right)$$

$$=\tan^{-1}(1)$$

$$=\frac{\pi}{4}$$

$$\therefore\ \ \pi\approx 4\left[\left(\frac{1}{2}-\frac{1}{3\times 8}+\frac{1}{5\times 32}-\frac{1}{7\times 128}\right)+\left(\frac{1}{3}-\frac{1}{3\times 27}+\frac{1}{5\times 243}-\frac{1}{7\times 2187}\right)\right]$$

≈ 3.1409 (to 4 d.p.)

Example 39

Find the Taylor series expansion of $\cos x$ about $x=\frac{\pi}{3}$. Hence find an approximation to $\cos 61°$ to the nearest 5 decimal places.

Solution

Let $f(x)=\cos x$. Then $f\left(\frac{\pi}{3}\right)=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$.

$f'(x)=-\sin x$, $\quad f'\left(\frac{\pi}{3}\right)=-\sin\left(\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}$

$f''(x)=-\cos x$, $\quad f''\left(\frac{\pi}{3}\right)=-\cos\left(\frac{\pi}{3}\right)=-\frac{1}{2}$

$f'''(x)=\sin x$, $\quad f'''\left(\frac{\pi}{3}\right)=\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$, etc.

The Taylor series expansion of $\cos x$ about $x=\frac{\pi}{3}$ is

$$\cos x=f\left(\frac{\pi}{3}\right)+\frac{f'\left(\frac{\pi}{3}\right)}{1!}\left(x-\frac{\pi}{3}\right)+\frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x-\frac{\pi}{3}\right)^2+\frac{f'''\left(\frac{\pi}{3}\right)}{3!}\left(x-\frac{\pi}{3}\right)^3+\cdots$$

$$=\frac{1}{2}-\frac{\sqrt{3}}{2}\left(x-\frac{\pi}{3}\right)-\frac{1}{2(2!)}\left(x-\frac{\pi}{3}\right)^2+\frac{\sqrt{3}}{2(3!)}\left(x-\frac{\pi}{3}\right)^3+\cdots$$

Note that $61° = \frac{61\pi}{180}\ rad$.

$$\begin{aligned}\cos 61° &= \cos\left(\frac{61\pi}{180}\right)\\ &= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{61\pi}{180} - \frac{\pi}{3}\right) - \frac{1}{2(2!)}\left(\frac{61\pi}{180} - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{2(3!)}\left(\frac{61\pi}{180} - \frac{\pi}{3}\right)^3 + \cdots\\ &= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{180}\right) - \frac{1}{2(2!)}\left(\frac{\pi}{180}\right)^2 + \frac{\sqrt{3}}{2(3!)}\left(\frac{\pi}{180}\right)^3 + \cdots\\ &\approx 0.5 - 0.015115 - 0.000076 + 0.000001\\ &\approx 0.48481 \text{ (to 5 d.p.)}\end{aligned}$$

Example 40

(a) If $y = (1-x^2)^{-\frac{1}{2}} \sin^{-1} x$, show that $(1-x^2)\, y' - xy = 1$. —(*)

(b) By applying the Leibnitz' rule to equation (*), deduce that

$(1-x^2)\, y^{(n+1)} - (2n+1)\, x\, y^{(n)} - n^2\, y^{(n-1)} = 0$, for any integer $n \geq 1$.

(c) Hence, or otherwise, find the Maclaurin series expansion for $(1-x^2)^{-\frac{1}{2}} \sin^{-1} x$ as far as the term in x^7.

Solution

(a) $y = (1-x^2)^{-\frac{1}{2}} \sin^{-1} x$

$$\begin{aligned}y' &= (1-x^2)^{-\frac{1}{2}} \cdot \frac{d}{dx}(\sin^{-1} x) + \sin^{-1} x \cdot \frac{d}{dx}\left[(1-x^2)^{-\frac{1}{2}}\right]\\ &= (1-x^2)^{-\frac{1}{2}} \cdot \frac{1}{\sqrt{1-x^2}} + \sin^{-1} x \cdot \left[-\frac{1}{2}(1-x^2)^{-\frac{3}{2}} \cdot (-2x)\right]\\ &= \frac{1}{1-x^2} + \frac{x}{1-x^2} \cdot \underbrace{\frac{\sin^{-1} x}{\sqrt{1-x^2}}}_{=y} = \frac{1}{1-x^2} + \frac{xy}{1-x^2} = \frac{1+xy}{1-x^2}\end{aligned}$$

$$\Rightarrow \quad (1-x^2)\,y' = 1+xy \qquad \therefore \quad (1-x^2)\,y' - xy = 1 \quad —(*)$$

(b) Differentiating both sides of (*) n times with respect to x, we have

$$[(1-x^2)\,y']^{(n)} - [xy]^{(n)} = [1]^{(n)}.$$

By Leibnitz' rule, we have

$$\left[\sum_{k=0}^{n}\binom{n}{k}(1-x^2)^{(k)}\,(y')^{(n-k)}\right] - \left[\sum_{k=0}^{n}\binom{n}{k}x^{(k)}\,y^{(n-k)}\right] = 0$$

$$\Rightarrow \left[\binom{n}{0}(1-x^2)^{(0)}\,(y')^{(n)} + \binom{n}{1}(1-x^2)^{(1)}\,(y')^{(n-1)} + \binom{n}{2}(1-x^2)^{(2)}\,(y')^{(n-2)}\right]$$

$$-\left[\binom{n}{0}x^{(0)}\,y^{(n)} + \binom{n}{1}x^{(1)}\,y^{(n-1)}\right] = 0$$

$$\Rightarrow \left[1\cdot(1-x^2)\cdot\, y^{(n+1)} + n\cdot(-2x)\cdot\, y^{(n)} + \frac{n(n-1)}{2}\cdot(-2)\cdot\, y^{(n-1)}\right]$$

$$-\left[1\cdot x\cdot\, y^{(n)} + n\cdot(1)\cdot\, y^{(n-1)}\right] = 0$$

$$\Rightarrow (1-x^2)\,y^{(n+1)} - (2n+1)\,x\,y^{(n)} - n^2\,y^{(n-1)} = 0 \quad (\text{for } n \ge 2) \quad —(**)$$

Differentiating both sides of (*) w.r.t. x:

$$(1-x^2)\,y'' - 2xy' - (xy'+y) = 0 \quad \Rightarrow \quad (1-x^2)y'' - 3xy' - y = 0$$

$\therefore$ (**) is also true for $n = 1$, i.e. it is true for $n \ge 1$.

(c) Putting $x = 0$ into $y = (1-x^2)^{-\frac{1}{2}}\sin^{-1}x$, we have

$$\boldsymbol{y(0)} = (1-0^2)^{-\frac{1}{2}}\cdot\sin^{-1}0 = \boldsymbol{0}.$$

Since $y' = \frac{1+xy}{1-x^2}$ from (*), we get

$$\boldsymbol{y'(0)} = \frac{1+0}{1-0} = \boldsymbol{1}.$$

Putting $x = 0$ into (**), we have

$$(1-0^2)\cdot y^{(n+1)}(0) - (2n+1)\cdot\, 0\,\cdot y^{(n)}(0) - n^2\cdot y^{(n-1)}(0) = 0,$$

i.e. $\boxed{\boldsymbol{y^{(n+1)}(0) = n^2\cdot y^{(n-1)}(0)}}$.

For $n = 1$, $\boldsymbol{y^{(2)}(0)} = 1^2\cdot y^{(0)}(0) = y(0) = \boldsymbol{0}$.

For $n = 2$, $\boldsymbol{y^{(3)}(0)} = 2^2\cdot y^{(1)}(0) = 4\times 1 = \boldsymbol{4}$.

For $n = 3$, $\mathbf{y^{(4)}(0)} = 3^2 \cdot y^{(2)}(0) = 9 \times 0 = \mathbf{0}$.

For $n = 4$, $\mathbf{y^{(5)}(0)} = 4^2 \cdot y^{(3)}(0) = 16 \times 4 = \mathbf{64}$.

For $n = 5$, $\mathbf{y^{(6)}(0)} = 5^2 \cdot y^{(4)}(0) = 25 \times 0 = \mathbf{0}$.

For $n = 6$, $\mathbf{y^{(7)}(0)} = 6^2 \cdot y^{(5)}(0) = 36 \times 64 = \mathbf{2304}$.

$\therefore$ The Maclaurin series expansion for $y = (1-x^2)^{-\frac{1}{2}} \sin^{-1} x$ is

$$(1-x^2)^{-\frac{1}{2}} \sin^{-1} x = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \cdots$$

$$= 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{4}{3!}x^3 + \frac{0}{4!}x^4 + \frac{64}{5!}x^5 + \frac{0}{6!}x^6 + \frac{2304}{7!}x^7 + \cdots$$

$$= x + \frac{2}{3}x^3 + \frac{8}{15}x^5 + \frac{16}{35}x^7 + \cdots$$

Exercise:

(a) If $y = \cos[\ln(1+x)]$, show that $(1+x)^2 y'' + (1+x)\, y' + y = 0$ — (***).

(b) By applying Leibnitz' rule to equation (***), obtain a relation between $y^{(n)}$, $y^{(n+1)}$ and $y^{(n+2)}$, where $y^{(r)}$ denotes $\frac{d^r y}{dx^r}$.

(c) Hence, or otherwise, find the Maclaurin series expansion for $y = \cos[\ln(1+x)]$ in ascending powers of x as far as the term in x^5.