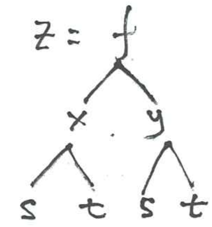
Change of Variables

$z = f(x,y)$, $x = (s,t)$, $y = y(s,t)$



$$\left\{\begin{aligned}
\left(\frac{\partial z}{\partial s}\right)_t &= \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial s}\right)_t + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial s}\right)_t \\
\left(\frac{\partial z}{\partial t}\right)_s &= \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial t}\right)_s + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial t}\right)_s
\end{aligned}\right\} \quad ***$$

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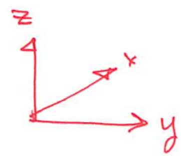


$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$x^2 + y^2 = r^2$

$\boxed{x-y}$ $\boxed{r-\theta}$

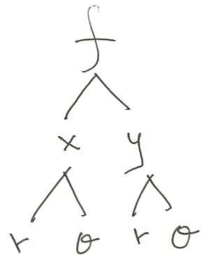
DO NOT !

$$\frac{\partial z}{\partial t} = \cancel{\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial t}\right)_s} + \cancel{\left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial t}\right)_s}$$

Page 17

$$Z = f = e^{2xy}$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$



$$\frac{\partial z}{\partial r} = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial r}\right)_\theta + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial r}\right)_\theta$$

5 marks.

$$= \left(2y e^{2xy}\right)(\cos\theta) + 2x e^{2xy}(\sin\theta)$$

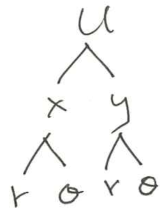
$$= 2r^2 e^{r^2 \sin 2\theta} \sin 2\theta \quad ,,$$

because

$$\boxed{\sin 2\theta = 2\sin\theta\cos\theta}$$

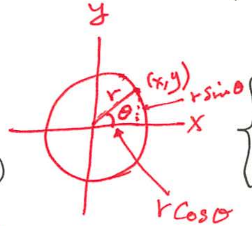
$$u(x,y) = u(r,\theta)$$

$$\begin{cases} \left(\dfrac{\partial u}{\partial r}\right)_\theta = \left(\dfrac{\partial u}{\partial x}\right)_y \left(\dfrac{\partial x}{\partial r}\right)_\theta + \left(\dfrac{\partial u}{\partial y}\right)_x \left(\dfrac{\partial y}{\partial r}\right)_\theta \checkmark \\[2ex] \left(\dfrac{\partial u}{\partial \theta}\right)_r = \left(\dfrac{\partial u}{\partial x}\right)_y \left(\dfrac{\partial x}{\partial \theta}\right)_r + \left(\dfrac{\partial u}{\partial y}\right)_x \left(\dfrac{\partial y}{\partial \theta}\right)_r \checkmark \end{cases}$$



$\boxed{**\ **}$

$$\begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\cos\theta + \dfrac{\partial u}{\partial y}\sin\theta \quad \text{———} (*) \\[2ex] \dfrac{\partial u}{\partial \theta} = \dfrac{\partial u}{\partial x}(-r\sin\theta) + \dfrac{\partial u}{\partial y}(r\cos\theta) \quad \text{———} (**) \end{cases}$$



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

polar form

$\boxed{**\ **}$

$$x^2 + y^2 = r^2$$

$$\boxed{\left[\frac{\partial^2 u}{\partial r^2} \checkmark + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \checkmark + \frac{1}{r}\frac{\partial u}{\partial r} \checkmark\right] = \left[u_{xx} + u_{yy}\right] = 0} \quad *$$

$$\begin{cases} u_r = u_x\cos\theta + u_y\sin\theta \quad \text{———} (1) \\ u_\theta = -r\sin\theta\, u_x + r\cos\theta\, u_y \quad \text{———} (2) \end{cases}$$

$$\begin{cases} \left(\dfrac{\partial y}{\partial r}\right)_\theta = \sin\theta\,; \quad \left(\dfrac{\partial y}{\partial \theta}\right)_r = r\cos\theta \\[2ex] \left(\dfrac{\partial x}{\partial r}\right)_\theta = \cos\theta\,; \quad \left(\dfrac{\partial x}{\partial \theta}\right)_r = -r\sin\theta \end{cases}$$

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from (1)

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\left[u_x\cos\theta + u_y\sin\theta\right]$$

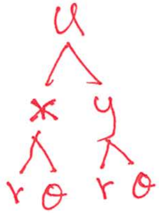
$$= \cos\theta \boxed{\frac{\partial u_x}{\partial r}} + 0 + \sin\theta \boxed{\frac{\partial u_y}{\partial r}} + 0$$

$u_x \cdot 0$ $\quad$ $u_y \cdot 0$

$$\cancel{+\frac{1}{r}\left[\cos\theta\frac{\partial u_y}{\partial \theta} - \sin\theta\, u_y\right]}$$

Also Note

$$\begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\cos\theta + \dfrac{\partial u}{\partial y}\sin\theta \\[2ex] \dfrac{\partial u}{\partial \theta} = \dfrac{\partial u}{\partial x}(-r\sin\theta) + \dfrac{\partial u}{\partial y}(r\cos\theta) \end{cases}$$



Write in Matrix form

rθ plane

$$\begin{bmatrix} \dfrac{\partial u}{\partial r} \\[2ex] \dfrac{\partial u}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial x} \\[2ex] \dfrac{\partial u}{\partial y} \end{bmatrix}$$

x-y plane

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} \\[2ex] \dfrac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{\partial u}{\partial r} \\[2ex] \dfrac{\partial u}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\dfrac{\sin\theta}{r} \\[2ex] \sin\theta & \dfrac{\cos\theta}{r} \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial r} \\[2ex] \dfrac{\partial u}{\partial \theta} \end{bmatrix}$$

$$\begin{cases} \dfrac{\partial u}{\partial x} = \cos\theta\dfrac{\partial u}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial u}{\partial \theta} \\[2ex] \dfrac{\partial u}{\partial y} = \sin\theta\dfrac{\partial u}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial u}{\partial \theta} \end{cases}$$

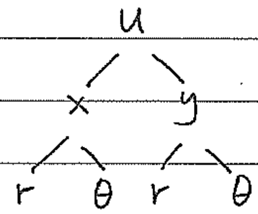
☆

Ch.2 Method 2 for the example in P.18 (easier)   ※※ must Re-do

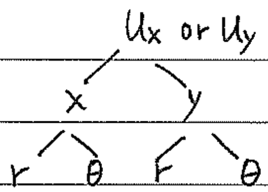
MA2001 $u(r,\theta)$ $u(x,y)$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \qquad U_r = U_x\cos\theta + U_y\sin\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) \qquad U_\theta = -r\sin\theta\, U_x + r\cos\theta\, U_y$$



$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial}{\partial r}\left[U_x\cos\theta + U_y\sin\theta\right]$$
$$= \cos\theta\,\frac{\partial U_x}{\partial r} + \sin\theta\,\frac{\partial U_y}{\partial r}$$



$$\frac{\partial U_x}{\partial r} = \frac{\partial U_x}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial U_x}{\partial y}\cdot\frac{\partial y}{\partial r}$$
$$= U_{xx}\cos\theta + U_{xy}\sin\theta$$
$$\frac{\partial U_y}{\partial r} = \frac{\partial U_y}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial U_y}{\partial y}\cdot\frac{\partial y}{\partial r}$$
$$= U_{yx}\cos\theta + U_{yy}\sin\theta$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial}{\partial\theta}\left[-r\sin\theta\, U_x + r\cos\theta\, U_y\right]$$
$$= -r\left[\sin\theta\,\frac{\partial U_x}{\partial\theta} + \cos\theta\, U_x\right]$$
$$+ r\left[\cos\theta\,\frac{\partial U_y}{\partial\theta} - \sin\theta\, U_y\right]$$
$$= -r\left[\sin\theta\,(-U_{xx}\, r\sin\theta + U_{xy}\, r\cos\theta) + \cos\theta\, U_x\right]$$
$$+ r\left[\cos\theta(-U_{yx}\, r\sin\theta + U_{yy}\, r\cos\theta) - \sin\theta\, U_y\right]$$

$$\therefore \quad \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}$$
$$= \cos\theta\left[\cos\theta\, U_{xx} + \sin\theta\, U_{xy}\right] + \sin\theta\left[\cos\theta\, U_{yx} + \sin\theta\, U_{yy}\right]$$
$$+ \sin^2\theta\, U_{xx} - \sin\theta\cos\theta\, U_{xy} - \frac{1}{r}\cos\theta\, U_x - \sin\theta\cos\theta\, U_{yx}$$
$$+ \cos^2\theta\, U_{yy} - \frac{1}{r}\sin\theta\, U_y + \frac{1}{r}\left[U_x\cos\theta\right] + \frac{1}{r}U_y\sin\theta$$
$$= U_{xx} + U_{yy} = 0$$

Laplace Equation

Example Page 22

Implicit Functions

$$z = e^x \cos y$$

$$\begin{array}{ccc} & z & \\ x & & y \\ & t & \end{array}$$

$$① \begin{cases} x^3 + e^x - t^2 - t = 1 \\ yt^2 + y^2 t - t + y = 0 \end{cases}$$

} we cannot find explicit form $x = ?$, $y = ?$, Only know $x = x(t)$, $y = y(t)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \quad \text{———} \quad *$$

Diff ① on both sides

$$\begin{cases} 3x^2\frac{dx}{dt} + e^x\frac{dx}{dt} - 2t - 1 = 0 \\ 2yt + t^2\frac{dy}{dt} + 2yt\frac{dy}{dt} + y^2 - 1 + \frac{dy}{dt} = 0 \end{cases}$$

$$\Rightarrow \frac{dx}{dt} = \frac{2t+1}{(3x^2+e^x)_{\neq 0}}$$

$$\Rightarrow \frac{dy}{dt} = \frac{1 - y^2 - 2yt}{1 + t^2 + 2yt}$$

Also $\begin{cases} \text{if } t=0, & x=0 \\ \text{if } t=0, & y=0 \end{cases}$

$$\left.\frac{dx}{dt}\right|_{t=0} = \left.\frac{2t+1}{3x^2+e^x}\right|_{\substack{t=0\\x=0}} = 1$$

$$\left.\frac{dy}{dt}\right|_{t=0} = \left.\frac{1-y^2-2yt}{1+t^2+2yt}\right|_{\substack{t=0\\y=0}} = 1$$

from (*)

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \left(e^x \cos y \frac{dx}{dt} - e^x \sin y \frac{dy}{dt}\right)$$

$$\frac{dz}{dt}\Big|_{t=0} = \Big(\underbrace{e^x \cos y\Big|_{\substack{x=0\\y=0}}}_{\to 1}\Big)\Big(\underbrace{\frac{dx}{dt}\Big|_{t=0}}_{\to 1}\Big)$$

$$-\Big(\Big(\underbrace{e^x \sin y\Big|_{\substack{x=0\\y=0}}}_{\to 0}\Big)\Big(\underbrace{\frac{dy}{dt}\Big|_{t=0}}_{\to 1}\Big)\Big)$$

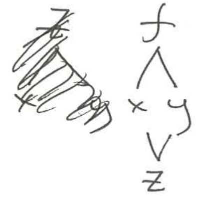
$$= 1 - 0$$

$$= 1$$

Example

$$\begin{cases} x^2+y^2=\frac{1}{2}z^2 \\ x+y+z=2 \end{cases} \qquad \text{For } \begin{cases} x=1 \\ y=-1 \\ z=2 \end{cases}$$

Page 23



Find $\frac{dx}{dz}$, $\frac{dy}{dz}$, $\frac{d^2x}{dz^2}$, $\frac{d^2y}{dz^2}$

$$\begin{cases} x^2+y^2=\frac{1}{2}z^2 \\ x+y+z=2 \end{cases} \Longrightarrow \begin{cases} 2x\frac{dx}{dz}+2y\frac{dy}{dz}=z & (1) \\ \frac{dx}{dz}+\frac{dy}{dz}+1=0 & (2) \end{cases}$$

$$\Rightarrow \begin{cases} 2\left(\frac{dx}{dz}\right)^2+2x\frac{d^2x}{dz^2}+2\left(\frac{dy}{dz}\right)^2+2y\frac{d^2y}{dz^2}=1 & (3) \\ \frac{d^2x}{dz^2}+\frac{d^2y}{dz^2}=0 & (4) \end{cases}$$

Put $x=1$, $y=-1$, $z=2$ in (1) & (2)

$$\begin{cases} 2x\frac{dx}{dz}+2y\frac{dy}{dz}=z \\ \frac{dx}{dz}+\frac{dy}{dz}+1=0 \end{cases} \Longrightarrow 4\frac{dx}{dz}=0,\ 4\frac{dy}{dz}=-4 \Rightarrow \frac{dx}{dz}=0,\ \frac{dy}{dz}=-1$$

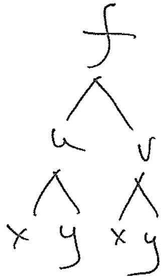
Put $x=1$, $y=-1$, $z=2$, $\frac{dx}{dz}=0$, $\frac{dy}{dz}=-1$ into (3) & (4)

we have $\left\{ \frac{d^2x}{dz^2} = -\frac{1}{4}, \quad \frac{d^2y}{d^2z} = \frac{1}{4} \right.$

Example. (page 24)

Solve ① $\begin{cases} xu + yvu^2 = 2 \\ xu^3 + y^2v^4 = 2 \end{cases}$ for u and v as $u(x,y)$, $v(x,y)$

near $(x,y,u,v) = (1,1,1,1)$. Compute $\left.\dfrac{\partial u}{\partial x}\right|_{\substack{x=1 \\ y=1}}$.



Differentiate ① w.r.t. x

$$\begin{cases} u + x\dfrac{\partial u}{\partial x} + yv2u\dfrac{\partial u}{\partial x} + yu^2\dfrac{\partial v}{\partial x} = 0 \\ u^3 + x3u^2\dfrac{\partial u}{\partial x} + y^2 4v^3\dfrac{\partial v}{\partial x} = 0 \end{cases}$$

$x=1$, $y=1$, $u=1$, $v=1$, we have

$$\begin{cases} 1 + 3\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial x} = 0 \\ 1 + 3\dfrac{\partial u}{\partial x} + 4\dfrac{\partial v}{\partial x} = 0 \end{cases} \Rightarrow \begin{cases} 4 + 12\dfrac{\partial u}{\partial x} + 4\dfrac{\partial v}{\partial x} = 0 \\ 1 + 3\dfrac{\partial u}{\partial x} + 4\dfrac{\partial v}{\partial x} = 0 \end{cases} \Rightarrow 9\dfrac{\partial u}{\partial x} = -3 \Rightarrow \dfrac{\partial u}{\partial x} = -\dfrac{1}{3}$$

at $x=1$, $y=1$