value(q2);

71-80. Example CAS commands:

Mathematica: (assigned function and values for a, and b may vary)

For transcendental functions the FindRoot is needed instead of the Solve command.

The Map command executes FindRoot over a set of initial guesses

Initial guesses will vary as the functions vary.

```
Clear[x, f, F]
{a, b}= {0, 2π}; f[x_] = Sin[2x] Cos[x/3]
F[x_] = Integrate[f[t], {t, a, x}]
Plot[{f[x], F[x]},{x, a, b}]
x/.Map[FindRoot[F'[x]==0, {x, #}] &,{2, 3, 5, 6}]
x/.Map[FindRoot[f'[x]==0, {x, #}] &,{1, 2, 4, 5, 6}]
```

Slightly alter above commands for 75 - 80.

```
Clear[x, f, F, u]
a=0;  f[x_] = x^2 − 2x − 3
u[x_] = 1 − x^2
F[x_] = Integrate[f[t], {t, a, u(x)}]
x/.Map[FindRoot[F'[x]==0,{x,#}] &,{1, 2, 3, 4}]
x/.Map[FindRoot[F''[x]==0,{x,#}] &,{1, 2, 3, 4}]
```

After determining an appropriate value for b, the following can be entered

```
b = 4;
Plot[{F[x], {x, a, b}]
```

5.5 INDEFINTE INTEGRALS AND THE SUBSTITUTION RULE

1. Let $u = 3x \Rightarrow du = 3\,dx \Rightarrow \frac{1}{3}\,du = dx$

$\int \sin 3x\,dx = \int \frac{1}{3}\sin u\,du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos 3x + C$

2. Let $u = 2x^2 \Rightarrow du = 4x\,dx \Rightarrow \frac{1}{4}\,du = x\,dx$

$\int x\sin(2x^2)\,dx = \int \frac{1}{4}\sin u\,du = -\frac{1}{4}\cos u + C = -\frac{1}{4}\cos 2x^2 + C$

3. Let $u = 2t \Rightarrow du = 2\,dt \Rightarrow \frac{1}{2}\,du = dt$

$\int \sec 2t\tan 2t\,dt = \int \frac{1}{2}\sec u\tan u\,du = \frac{1}{2}\sec u + C = \frac{1}{2}\sec 2t + C$

4. Let $u = 1 - \cos\frac{t}{2} \Rightarrow du = \frac{1}{2}\sin\frac{t}{2}\,dt \Rightarrow 2\,du = \sin\frac{t}{2}\,dt$

$\int \left(1 - \cos\frac{t}{2}\right)^2\left(\sin\frac{t}{2}\right)dt = \int 2u^2\,du = \frac{2}{3}u^3 + C = \frac{2}{3}\left(1 - \cos\frac{t}{2}\right)^3 + C$

5. Let $u = 7x - 2 \Rightarrow du = 7\,dx \Rightarrow \frac{1}{7}\,du = dx$

$\int 28(7x-2)^{-5}\,dx = \int \frac{1}{7}(28)u^{-5}\,du = \int 4u^{-5}\,du = -u^{-4} + C = -(7x-2)^{-4} + C$

6. Let $u = x^4 - 1 \Rightarrow du = 4x^3\,dx \Rightarrow \frac{1}{4}\,du = x^3\,dx$

$\int x^3\left(x^4 - 1\right)^2dx = \int \frac{1}{4}u^2\,du = \frac{u^3}{12} + C = \frac{1}{12}\left(x^4 - 1\right)^3 + C$

7. Let $u = 1 - r^3 \Rightarrow du = -3r^2\,dr \Rightarrow -3\,du = 9r^2\,dr$
$\int \frac{9r^2\,dr}{\sqrt{1-r^3}} = \int -3u^{-1/2}\,du = -3(2)u^{1/2} + C = -6\left(1 - r^3\right)^{1/2} + C$

8. Let $u = y^4 + 4y^2 + 1 \Rightarrow du = (4y^3 + 8y)\,dy \Rightarrow 3\,du = 12\,(y^3 + 2y)\,dy$
$\int 12\left(y^4 + 4y^2 + 1\right)^2 (y^3 + 2y)\,dy = \int 3u^2\,du = u^3 + C = \left(y^4 + 4y^2 + 1\right)^3 + C$

9. Let $u = x^{3/2} - 1 \Rightarrow du = \frac{3}{2}x^{1/2}\,dx \Rightarrow \frac{2}{3}\,du = \sqrt{x}\,dx$
$\int \sqrt{x}\sin^2\left(x^{3/2} - 1\right)dx = \int \frac{2}{3}\sin^2 u\,du = \frac{2}{3}\left(\frac{u}{2} - \frac{1}{4}\sin 2u\right) + C = \frac{1}{3}\left(x^{3/2} - 1\right) - \frac{1}{6}\sin\left(2x^{3/2} - 2\right) + C$

10. Let $u = -\frac{1}{x} \Rightarrow du = \frac{1}{x^2}\,dx$
$\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx = \int \cos^2(-u)\,du = \int \cos^2(u)\,du = \left(\frac{u}{2} + \frac{1}{4}\sin 2u\right) + C = -\frac{1}{2x} + \frac{1}{4}\sin\left(-\frac{2}{x}\right) + C$
$= -\frac{1}{2x} - \frac{1}{4}\sin\left(\frac{2}{x}\right) + C$

11. (a) Let $u = \cot 2\theta \Rightarrow du = -2\csc^2 2\theta\,d\theta \Rightarrow -\frac{1}{2}\,du = \csc^2 2\theta\,d\theta$
$\int \csc^2 2\theta\cot 2\theta\,d\theta = -\int \frac{1}{2}u\,du = -\frac{1}{2}\left(\frac{u^2}{2}\right) + C = -\frac{u^2}{4} + C = -\frac{1}{4}\cot^2 2\theta + C$
(b) Let $u = \csc 2\theta \Rightarrow du = -2\csc 2\theta\cot 2\theta\,d\theta \Rightarrow -\frac{1}{2}\,du = \csc 2\theta\cot 2\theta\,d\theta$
$\int \csc^2 2\theta\cot 2\theta\,d\theta = \int -\frac{1}{2}u\,du = -\frac{1}{2}\left(\frac{u^2}{2}\right) + C = -\frac{u^2}{4} + C = -\frac{1}{4}\csc^2 2\theta + C$

12. (a) Let $u = 5x + 8 \Rightarrow du = 5\,dx \Rightarrow \frac{1}{5}\,du = dx$
$\int \frac{dx}{\sqrt{5x+8}} = \int \frac{1}{5}\left(\frac{1}{\sqrt{u}}\right)du = \frac{1}{5}\int u^{-1/2}\,du = \frac{1}{5}\left(2u^{1/2}\right) + C = \frac{2}{5}u^{1/2} + C = \frac{2}{5}\sqrt{5x+8} + C$
(b) Let $u = \sqrt{5x+8} \Rightarrow du = \frac{1}{2}(5x+8)^{-1/2}(5)\,dx \Rightarrow \frac{2}{5}\,du = \frac{dx}{\sqrt{5x+8}}$
$\int \frac{dx}{\sqrt{5x+8}} = \int \frac{2}{5}\,du = \frac{2}{5}u + C = \frac{2}{5}\sqrt{5x+8} + C$

13. Let $u = 3 - 2s \Rightarrow du = -2\,ds \Rightarrow -\frac{1}{2}\,du = ds$
$\int \sqrt{3-2s}\,ds = \int \sqrt{u}\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\int u^{1/2}\,du = \left(-\frac{1}{2}\right)\left(\frac{2}{3}u^{3/2}\right) + C = -\frac{1}{3}(3-2s)^{3/2} + C$

14. Let $u = 2x + 1 \Rightarrow du = 2\,dx \Rightarrow \frac{1}{2}\,du = dx$
$\int (2x+1)^3\,dx = \int u^3\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int u^3\,du = \left(\frac{1}{2}\right)\left(\frac{u^4}{4}\right) + C = \frac{1}{8}(2x+1)^4 + C$

15. Let $u = 5s + 4 \Rightarrow du = 5\,ds \Rightarrow \frac{1}{5}\,du = ds$
$\int \frac{1}{\sqrt{5s+4}}\,ds = \int \frac{1}{\sqrt{u}}\left(\frac{1}{5}\,du\right) = \frac{1}{5}\int u^{-1/2}\,du = \left(\frac{1}{5}\right)\left(2u^{1/2}\right) + C = \frac{2}{5}\sqrt{5s+4} + C$

16. Let $u = 2 - x \Rightarrow du = -dx \Rightarrow -du = dx$
$\int \frac{3}{(2-x)^2}\,dx = \int \frac{3(-du)}{u^2} = -3\int u^{-2}\,du = -3\left(\frac{u^{-1}}{-1}\right) + C = \frac{3}{2-x} + C$

17. Let $u = 1 - \theta^2 \Rightarrow du = -2\theta\,d\theta \Rightarrow -\frac{1}{2}\,du = \theta\,d\theta$
$\int \theta\sqrt[4]{1-\theta^2}\,d\theta = \int \sqrt[4]{u}\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\int u^{1/4}\,du = \left(-\frac{1}{2}\right)\left(\frac{4}{5}u^{5/4}\right) + C = -\frac{2}{5}\left(1-\theta^2\right)^{5/4} + C$

18. Let $u = \theta^2 - 1 \Rightarrow du = 2\theta\,d\theta \Rightarrow 4\,du = 8\theta\,d\theta$
$\int 8\theta\sqrt[3]{\theta^2 - 1}\,d\theta = \int \sqrt[3]{u}\,(4\,du) = 4\int u^{1/3}\,du = 4\left(\frac{3}{4}u^{4/3}\right) + C = 3\left(\theta^2 - 1\right)^{4/3} + C$

19. Let $u = 7 - 3y^2 \Rightarrow du = -6y\,dy \Rightarrow -\frac{1}{2}\,du = 3y\,dy$
$\int 3y\sqrt{7-3y^2}\,dy = \int \sqrt{u}\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\int u^{1/2}\,du = \left(-\frac{1}{2}\right)\left(\frac{2}{3}u^{3/2}\right) + C = -\frac{1}{3}\left(7-3y^2\right)^{3/2} + C$

20. Let $u = 2y^2 + 1 \Rightarrow du = 4y\,dy$
$\int \frac{4y\,dy}{\sqrt{2y^2+1}} = \int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{2y^2+1} + C$

21. Let $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2\,du = \frac{1}{\sqrt{x}}\,dx$
$\int \frac{1}{\sqrt{x}\,(1+\sqrt{x})^2}\,dx = \int \frac{2\,du}{u^2} = -\frac{2}{u} + C = \frac{-2}{1+\sqrt{x}} + C$

22. Let $u = 1 + \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\,dx \Rightarrow 2\,du = \frac{1}{\sqrt{x}}\,dx$
$\int \frac{(1+\sqrt{x})^3}{\sqrt{x}}\,dx = \int u^3\,(2\,du) = 2\left(\frac{1}{4}u^4\right) + C = \frac{1}{2}\left(1+\sqrt{x}\right)^4 + C$

23. Let $u = 3z + 4 \Rightarrow du = 3\,dz \Rightarrow \frac{1}{3}\,du = dz$
$\int \cos(3z+4)\,dz = \int (\cos u)\left(\frac{1}{3}\,du\right) = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin(3z+4) + C$

24. Let $u = 8z - 5 \Rightarrow du = 8\,dz \Rightarrow \frac{1}{8}\,du = dz$
$\int \sin(8z-5)\,dz = \int (\sin u)\left(\frac{1}{8}\,du\right) = \frac{1}{8}\int \sin u\,du = \frac{1}{8}(-\cos u) + C = -\frac{1}{8}\cos(8z-5) + C$

25. Let $u = 3x + 2 \Rightarrow du = 3\,dx \Rightarrow \frac{1}{3}\,du = dx$
$\int \sec^2(3x+2)\,dx = \int (\sec^2 u)\left(\frac{1}{3}\,du\right) = \frac{1}{3}\int \sec^2 u\,du = \frac{1}{3}\tan u + C = \frac{1}{3}\tan(3x+2) + C$

26. Let $u = \tan x \Rightarrow du = \sec^2 x\,dx$
$\int \tan^2 x \sec^2 x\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C$

27. Let $u = \sin\left(\frac{x}{3}\right) \Rightarrow du = \frac{1}{3}\cos\left(\frac{x}{3}\right)dx \Rightarrow 3\,du = \cos\left(\frac{x}{3}\right)dx$
$\int \sin^5\left(\frac{x}{3}\right)\cos\left(\frac{x}{3}\right)dx = \int u^5\,(3\,du) = 3\left(\frac{1}{6}u^6\right) + C = \frac{1}{2}\sin^6\left(\frac{x}{3}\right) + C$

28. Let $u = \tan\left(\frac{x}{2}\right) \Rightarrow du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx \Rightarrow 2\,du = \sec^2\left(\frac{x}{2}\right)dx$
$\int \tan^7\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx = \int u^7\,(2\,du) = 2\left(\frac{1}{8}u^8\right) + C = \frac{1}{4}\tan^8\left(\frac{x}{2}\right) + C$

29. Let $u = \frac{r^3}{18} - 1 \Rightarrow du = \frac{r^2}{6}\,dr \Rightarrow 6\,du = r^2\,dr$
$\int r^2\left(\frac{r^3}{18} - 1\right)^5 dr = \int u^5\,(6\,du) = 6\int u^5\,du = 6\left(\frac{u^6}{6}\right) + C = \left(\frac{r^3}{18} - 1\right)^6 + C$

30. Let $u = 7 - \frac{r^5}{10} \Rightarrow du = -\frac{1}{2}r^4\,dr \Rightarrow -2\,du = r^4\,dr$
$\int r^4\left(7 - \frac{r^5}{10}\right)^3 dr = \int u^3\,(-2\,du) = -2\int u^3\,du = -2\left(\frac{u^4}{4}\right) + C = -\frac{1}{2}\left(7 - \frac{r^5}{10}\right)^4 + C$

31. Let $u = x^{3/2} + 1 \Rightarrow du = \frac{3}{2}x^{1/2}\,dx \Rightarrow \frac{2}{3}\,du = x^{1/2}\,dx$
$\int x^{1/2}\sin\left(x^{3/2}+1\right)dx = \int (\sin u)\left(\frac{2}{3}\,du\right) = \frac{2}{3}\int \sin u\,du = \frac{2}{3}(-\cos u) + C = -\frac{2}{3}\cos\left(x^{3/2}+1\right) + C$

32. Let $u = x^{4/3} - 8 \Rightarrow du = \frac{4}{3}x^{1/3}\,dx \Rightarrow \frac{3}{4}\,du = x^{1/3}\,dx$
$\int x^{1/3}\sin\left(x^{4/3}-8\right)dx = \int (\sin u)\left(\frac{3}{4}\,du\right) = \frac{3}{4}\int \sin u\,du = \frac{3}{4}(-\cos u) + C = -\frac{3}{4}\cos\left(x^{4/3}-8\right) + C$

33. Let $u = \sec\left(v+\frac{\pi}{2}\right) \Rightarrow du = \sec\left(v+\frac{\pi}{2}\right)\tan\left(v+\frac{\pi}{2}\right)dv$
$\int \sec\left(v+\frac{\pi}{2}\right)\tan\left(v+\frac{\pi}{2}\right)dv = \int du = u + C = \sec\left(v+\frac{\pi}{2}\right) + C$

34. Let $u = \csc\left(\frac{v-\pi}{2}\right) \Rightarrow du = -\frac{1}{2}\csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv \Rightarrow -2\,du = \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv$
$\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv = \int -2\,du = -2u + C = -2\csc\left(\frac{v-\pi}{2}\right) + C$

35. Let $u = \cos(2t+1) \Rightarrow du = -2\sin(2t+1)\,dt \Rightarrow -\frac{1}{2}\,du = \sin(2t+1)\,dt$
$\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt = \int -\frac{1}{2}\frac{du}{u^2} = \frac{1}{2u} + C = \frac{1}{2\cos(2t+1)} + C$

36. Let $u = 2 + \sin t \Rightarrow du = \cos t\,dt$
$\int \frac{6\cos t}{(2+\sin t)^3}\,dt = \int \frac{6}{u^3}\,du = 6\int u^{-3}\,du = 6\left(\frac{u^{-2}}{-2}\right) + C = -3(2+\sin t)^{-2} + C$

37. Let $u = \cot y \Rightarrow du = -\csc^2 y\,dy \Rightarrow -du = \csc^2 y\,dy$
$\int \sqrt{\cot y}\,\csc^2 y\,dy = \int \sqrt{u}\,(-du) = -\int u^{1/2}\,du = -\frac{2}{3}u^{3/2} + C = -\frac{2}{3}(\cot y)^{3/2} + C = -\frac{2}{3}\left(\cot^3 y\right)^{1/2} + C$

38. Let $u = \sec z \Rightarrow du = \sec z\tan z\,dz$
$\int \frac{\sec z\tan z}{\sqrt{\sec z}}\,dz = \int \frac{1}{\sqrt{u}}\,du = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{\sec z} + C$

39. Let $u = \frac{1}{t} - 1 = t^{-1} - 1 \Rightarrow du = -t^{-2}\,dt \Rightarrow -du = \frac{1}{t^2}\,dt$
$\int \frac{1}{t^2}\cos\left(\frac{1}{t}-1\right)dt = \int (\cos u)(-du) = -\int \cos u\,du = -\sin u + C = -\sin\left(\frac{1}{t}-1\right) + C$

40. Let $u = \sqrt{t} + 3 = t^{1/2} + 3 \Rightarrow du = \frac{1}{2}t^{-1/2}\,dt \Rightarrow 2\,du = \frac{1}{\sqrt{t}}\,dt$
$\int \frac{1}{\sqrt{t}}\cos\left(\sqrt{t}+3\right)dt = \int (\cos u)(2\,du) = 2\int \cos u\,du = 2\sin u + C = 2\sin\left(\sqrt{t}+3\right) + C$

41. Let $u = \sin\frac{1}{\theta} \Rightarrow du = \left(\cos\frac{1}{\theta}\right)\left(-\frac{1}{\theta^2}\right)d\theta \Rightarrow -du = \frac{1}{\theta^2}\cos\frac{1}{\theta}\,d\theta$
$\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta = \int -u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\sin^2\frac{1}{\theta} + C$

42. Let $u = \csc\sqrt{\theta} \Rightarrow du = \left(-\csc\sqrt{\theta}\cot\sqrt{\theta}\right)\left(\frac{1}{2\sqrt{\theta}}\right)d\theta \Rightarrow -2\,du = \frac{1}{\sqrt{\theta}}\cot\sqrt{\theta}\csc\sqrt{\theta}\,d\theta$
$\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta = \int \frac{1}{\sqrt{\theta}}\cot\sqrt{\theta}\csc\sqrt{\theta}\,d\theta = \int -2\,du = -2u + C = -2\csc\sqrt{\theta} + C = -\frac{2}{\sin\sqrt{\theta}} + C$

43. Let $u = s^3 + 2s^2 - 5s + 5 \Rightarrow du = (3s^2 + 4s - 5)\,ds$
$\int (s^3 + 2s^2 - 5s + 5)(3s^2 + 4s - 5)\,ds = \int u\,du = \frac{u^2}{2} + C = \frac{(s^3+2s^2-5s+5)^2}{2} + C$

44. Let $u = \theta^4 - 2\theta^2 + 8\theta - 2 \Rightarrow du = (4\theta^3 - 4\theta + 8)\,d\theta \Rightarrow \frac{1}{4}\,du = (\theta^3 - \theta + 2)\,d\theta$
$\int (\theta^4 - 2\theta^2 + 8\theta - 2)(\theta^3 - \theta + 2)\,d\theta = \int u\left(\frac{1}{4}\,du\right) = \frac{1}{4}\int u\,du = \frac{1}{4}\left(\frac{u^2}{2}\right) + C = \frac{(\theta^4-2\theta^2+8\theta-2)^2}{8} + C$

45. Let $u = 1 + t^4 \Rightarrow du = 4t^3\,dt \Rightarrow \frac{1}{4}\,du = t^3\,dt$

$\int t^3 (1 + t^4)^3\,dt = \int u^3 \left(\frac{1}{4}\,du\right) = \frac{1}{4}\left(\frac{1}{4}u^4\right) + C = \frac{1}{16}(1 + t^4)^4 + C$

46. Let $u = 1 - \frac{1}{x} \Rightarrow du = \frac{1}{x^2}\,dx$

$\int \sqrt{\frac{x-1}{x^5}}\,dx = \int \frac{1}{x^2}\sqrt{\frac{x-1}{x}}\,dx = \int \frac{1}{x^2}\sqrt{1 - \frac{1}{x}}\,dx = \int \sqrt{u}\,du = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}\left(1 - \frac{1}{x}\right)^{3/2} + C$

47. Let $u = x^2 + 1$. Then $du = 2x dx$ and $\frac{1}{2}du = x dx$ and $x^2 = u - 1$. Thus $\int x^3\sqrt{x^2+1}\,dx = \int (u-1)\frac{1}{2}\sqrt{u}\,du$

$= \frac{1}{2}\int\left(u^{3/2} - u^{1/2}\right)du = \frac{1}{2}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] + C = \frac{1}{5}u^{5/2} - \frac{1}{3}u^{3/2} + C = \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C$

48. Let $u = x^3 + 1 \Rightarrow du = 3x^2 dx$ and $x^3 = u - 1$. So $\int 3x^5\sqrt{x^3+1}\,dx = \int (u-1)\sqrt{u}\,du = \int\left(u^{3/2} - u^{1/2}\right)du$

$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{5}(x^3+1)^{5/2} - \frac{2}{3}(x^3+1)^{3/2} + C$

49. (a) Let $u = \tan x \Rightarrow du = \sec^2 x\,dx$; $v = u^3 \Rightarrow dv = 3u^2\,du \Rightarrow 6\,dv = 18u^2\,du$; $w = 2 + v \Rightarrow dw = dv$

$\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx = \int \frac{18u^2}{(2+u^3)^2}\,du = \int \frac{6\,dv}{(2+v)^2} = \int \frac{6\,dw}{w^2} = 6\int w^{-2}\,dw = -6w^{-1} + C = -\frac{6}{2+v} + C$

$= -\frac{6}{2+u^3} + C = -\frac{6}{2+\tan^3 x} + C$

(b) Let $u = \tan^3 x \Rightarrow du = 3\tan^2 x\sec^2 x\,dx \Rightarrow 6\,du = 18\tan^2 x\sec^2 x\,dx$; $v = 2 + u \Rightarrow dv = du$

$\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx = \int \frac{6\,du}{(2+u)^2} = \int \frac{6\,dv}{v^2} = -\frac{6}{v} + C = -\frac{6}{2+u} + C = -\frac{6}{2+\tan^3 x} + C$

(c) Let $u = 2 + \tan^3 x \Rightarrow du = 3\tan^2 x\sec^2 x\,dx \Rightarrow 6\,du = 18\tan^2 x\sec^2 x\,dx$

$\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx = \int \frac{6\,du}{u^2} = -\frac{6}{u} + C = -\frac{6}{2+\tan^3 x} + C$

50. (a) Let $u = x - 1 \Rightarrow du = dx$; $v = \sin u \Rightarrow dv = \cos u\,du$; $w = 1 + v^2 \Rightarrow dw = 2v\,dv \Rightarrow \frac{1}{2}\,dw = v\,dv$

$\int \sqrt{1+\sin^2(x-1)}\sin(x-1)\cos(x-1)\,dx = \int \sqrt{1+\sin^2 u}\sin u\cos u\,du = \int v\sqrt{1+v^2}\,dv$

$= \int \frac{1}{2}\sqrt{w}\,dw = \frac{1}{3}w^{3/2} + C = \frac{1}{3}(1+v^2)^{3/2} + C = \frac{1}{3}(1+\sin^2 u)^{3/2} + C = \frac{1}{3}\left(1+\sin^2(x-1)\right)^{3/2} + C$

(b) Let $u = \sin(x-1) \Rightarrow du = \cos(x-1)\,dx$; $v = 1 + u^2 \Rightarrow dv = 2u\,du \Rightarrow \frac{1}{2}\,dv = u\,du$

$\int \sqrt{1+\sin^2(x-1)}\sin(x-1)\cos(x-1)\,dx = \int u\sqrt{1+u^2}\,du = \int \frac{1}{2}\sqrt{v}\,dv = \int \frac{1}{2}v^{1/2}\,dv$

$= \left(\frac{1}{2}\left(\frac{2}{3}\right)v^{3/2}\right) + C = \frac{1}{3}v^{3/2} + C = \frac{1}{3}(1+u^2)^{3/2} + C = \frac{1}{3}\left(1+\sin^2(x-1)\right)^{3/2} + C$

(c) Let $u = 1 + \sin^2(x-1) \Rightarrow du = 2\sin(x-1)\cos(x-1)\,dx \Rightarrow \frac{1}{2}\,du = \sin(x-1)\cos(x-1)\,dx$

$\int \sqrt{1+\sin^2(x-1)}\sin(x-1)\cos(x-1)\,dx = \int \frac{1}{2}\sqrt{u}\,du = \int \frac{1}{2}u^{1/2}\,du = \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right) + C$

$= \frac{1}{3}\left(1+\sin^2(x-1)\right)^{3/2} + C$

51. Let $u = 3(2r-1)^2 + 6 \Rightarrow du = 6(2r-1)(2)\,dr \Rightarrow \frac{1}{12}\,du = (2r-1)\,dr$; $v = \sqrt{u} \Rightarrow dv = \frac{1}{2\sqrt{u}}\,du \Rightarrow \frac{1}{6}\,dv$

$= \frac{1}{12\sqrt{u}}\,du$

$\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\,dr = \int \left(\frac{\cos\sqrt{u}}{\sqrt{u}}\right)\left(\frac{1}{12}\,du\right) = \int (\cos v)\left(\frac{1}{6}\,dv\right) = \frac{1}{6}\sin v + C = \frac{1}{6}\sin\sqrt{u} + C$

$= \frac{1}{6}\sin\sqrt{3(2r-1)^2+6} + C$

52. Let $u = \cos\sqrt{\theta} \Rightarrow du = \left(-\sin\sqrt{\theta}\right)\left(\frac{1}{2\sqrt{\theta}}\right)d\theta \Rightarrow -2\,du = \frac{\sin\sqrt{\theta}}{\sqrt{\theta}}\,d\theta$

$\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}\,d\theta = \int \frac{\sin\sqrt{\theta}}{\sqrt{\theta}\sqrt{\cos^3\sqrt{\theta}}}\,d\theta = \int \frac{-2\,du}{u^{3/2}} = -2\int u^{-3/2}\,du = -2\left(-2u^{-1/2}\right) + C = \frac{4}{\sqrt{u}} + C$

$= \frac{4}{\sqrt{\cos\sqrt{\theta}}} + C$

53. Let $u = 3t^2 - 1 \Rightarrow du = 6t\,dt \Rightarrow 2\,du = 12t\,dt$
$s = \int 12t\,(3t^2 - 1)^3\,dt = \int u^3\,(2\,du) = 2\left(\frac{1}{4}u^4\right) + C = \frac{1}{2}u^4 + C = \frac{1}{2}(3t^2 - 1)^4 + C;$
$s = 3$ when $t = 1 \Rightarrow 3 = \frac{1}{2}(3 - 1)^4 + C \Rightarrow 3 = 8 + C \Rightarrow C = -5 \Rightarrow s = \frac{1}{2}(3t^2 - 1)^4 - 5$

54. Let $u = x^2 + 8 \Rightarrow du = 2x\,dx \Rightarrow 2\,du = 4x\,dx$
$y = \int 4x\,(x^2 + 8)^{-1/3}\,dx = \int u^{-1/3}\,(2\,du) = 2\left(\frac{3}{2}u^{2/3}\right) + C = 3u^{2/3} + C = 3\,(x^2 + 8)^{2/3} + C;$
$y = 0$ when $x = 0 \Rightarrow 0 = 3(8)^{2/3} + C \Rightarrow C = -12 \Rightarrow y = 3\,(x^2 + 8)^{2/3} - 12$

55. Let $u = t + \frac{\pi}{12} \Rightarrow du = dt$
$s = \int 8\sin^2\left(t + \frac{\pi}{12}\right)dt = \int 8\sin^2 u\,du = 8\left(\frac{u}{2} - \frac{1}{4}\sin 2u\right) + C = 4\left(t + \frac{\pi}{12}\right) - 2\sin\left(2t + \frac{\pi}{6}\right) + C;$
$s = 8$ when $t = 0 \Rightarrow 8 = 4\left(\frac{\pi}{12}\right) - 2\sin\left(\frac{\pi}{6}\right) + C \Rightarrow C = 8 - \frac{\pi}{3} + 1 = 9 - \frac{\pi}{3}$
$\Rightarrow s = 4\left(t + \frac{\pi}{12}\right) - 2\sin\left(2t + \frac{\pi}{6}\right) + 9 - \frac{\pi}{3} = 4t - 2\sin\left(2t + \frac{\pi}{6}\right) + 9$

56. Let $u = \frac{\pi}{4} - \theta \Rightarrow -du = d\theta$
$r = \int 3\cos^2\left(\frac{\pi}{4} - \theta\right)d\theta = -\int 3\cos^2 u\,du = -3\left(\frac{u}{2} + \frac{1}{4}\sin 2u\right) + C = -\frac{3}{2}\left(\frac{\pi}{4} - \theta\right) - \frac{3}{4}\sin\left(\frac{\pi}{2} - 2\theta\right) + C;$
$r = \frac{\pi}{8}$ when $\theta = 0 \Rightarrow \frac{\pi}{8} = -\frac{3\pi}{8} - \frac{3}{4}\sin\frac{\pi}{2} + C \Rightarrow C = \frac{\pi}{2} + \frac{3}{4} \Rightarrow r = -\frac{3}{2}\left(\frac{\pi}{4} - \theta\right) - \frac{3}{4}\sin\left(\frac{\pi}{2} - 2\theta\right) + \frac{\pi}{2} + \frac{3}{4}$
$\Rightarrow r = \frac{3}{2}\theta - \frac{3}{4}\sin\left(\frac{\pi}{2} - 2\theta\right) + \frac{\pi}{8} + \frac{3}{4} \Rightarrow r = \frac{3}{2}\theta - \frac{3}{4}\cos 2\theta + \frac{\pi}{8} + \frac{3}{4}$

57. Let $u = 2t - \frac{\pi}{2} \Rightarrow du = 2\,dt \Rightarrow -2\,du = -4\,dt$
$\frac{ds}{dt} = \int -4\sin\left(2t - \frac{\pi}{2}\right)dt = \int(\sin u)(-2\,du) = 2\cos u + C_1 = 2\cos\left(2t - \frac{\pi}{2}\right) + C_1;$
at $t = 0$ and $\frac{ds}{dt} = 100$ we have $100 = 2\cos\left(-\frac{\pi}{2}\right) + C_1 \Rightarrow C_1 = 100 \Rightarrow \frac{ds}{dt} = 2\cos\left(2t - \frac{\pi}{2}\right) + 100$
$\Rightarrow s = \int\left(2\cos\left(2t - \frac{\pi}{2}\right) + 100\right)dt = \int(\cos u + 50)\,du = \sin u + 50u + C_2 = \sin\left(2t - \frac{\pi}{2}\right) + 50\left(2t - \frac{\pi}{2}\right) + C_2;$
at $t = 0$ and $s = 0$ we have $0 = \sin\left(-\frac{\pi}{2}\right) + 50\left(-\frac{\pi}{2}\right) + C_2 \Rightarrow C_2 = 1 + 25\pi$
$\Rightarrow s = \sin\left(2t - \frac{\pi}{2}\right) + 100t - 25\pi + (1 + 25\pi) \Rightarrow s = \sin\left(2t - \frac{\pi}{2}\right) + 100t + 1$

58. Let $u = \tan 2x \Rightarrow du = 2\sec^2 2x\,dx \Rightarrow 2\,du = 4\sec^2 2x\,dx$; $v = 2x \Rightarrow dv = 2\,dx \Rightarrow \frac{1}{2}\,dv = dx$
$\frac{dy}{dx} = \int 4\sec^2 2x\tan 2x\,dx = \int u(2\,du) = u^2 + C_1 = \tan^2 2x + C_1;$
at $x = 0$ and $\frac{dy}{dx} = 4$ we have $4 = 0 + C_1 \Rightarrow C_1 = 4 \Rightarrow \frac{dy}{dx} = \tan^2 2x + 4 = (\sec^2 2x - 1) + 4 = \sec^2 2x + 3$
$\Rightarrow y = \int(\sec^2 2x + 3)\,dx = \int(\sec^2 v + 3)\left(\frac{1}{2}\,dv\right) = \frac{1}{2}\tan v + \frac{3}{2}v + C_2 = \frac{1}{2}\tan 2x + 3x + C_2;$
at $x = 0$ and $y = -1$ we have $-1 = \frac{1}{2}(0) + 0 + C_2 \Rightarrow C_2 = -1 \Rightarrow y = \frac{1}{2}\tan 2x + 3x - 1$

59. Let $u = 2t \Rightarrow du = 2\,dt \Rightarrow 3\,du = 6\,dt$
$s = \int 6\sin 2t\,dt = \int(\sin u)(3\,du) = -3\cos u + C = -3\cos 2t + C;$
at $t = 0$ and $s = 0$ we have $0 = -3\cos 0 + C \Rightarrow C = 3 \Rightarrow s = 3 - 3\cos 2t \Rightarrow s\left(\frac{\pi}{2}\right) = 3 - 3\cos(\pi) = 6$ m

60. Let $u = \pi t \Rightarrow du = \pi\,dt \Rightarrow \pi\,du = \pi^2\,dt$
$v = \int \pi^2\cos\pi t\,dt = \int(\cos u)(\pi\,du) = \pi\sin u + C_1 = \pi\sin(\pi t) + C_1;$
at $t = 0$ and $v = 8$ we have $8 = \pi(0) + C_1 \Rightarrow C_1 = 8 \Rightarrow v = \frac{ds}{dt} = \pi\sin(\pi t) + 8 \Rightarrow s = \int(\pi\sin(\pi t) + 8)\,dt$
$= \int \sin u\,du + 8t + C_2 = -\cos(\pi t) + 8t + C_2$; at $t = 0$ and $s = 0$ we have $0 = -1 + C_2 \Rightarrow C_2 = 1$

$\Rightarrow\ s = 8t - \cos(\pi t) + 1 \ \Rightarrow\ s(1) = 8 - \cos\pi + 1 = 10$ m

61. All three integrations are correct. In each case, the derivative of the function on the right is the integrand on the left, and each formula has an arbitrary constant for generating the remaining antiderivatives. Moreover, $\sin^2 x + C_1 = 1 - \cos^2 x + C_1 \ \Rightarrow\ C_2 = 1 + C_1$; also $-\cos^2 x + C_2 = -\frac{\cos 2x}{2} - \frac{1}{2} + C_2 \ \Rightarrow\ C_3 = C_2 - \frac{1}{2} = C_1 + \frac{1}{2}$.

62. Both integrations are correct. In each case, the derivative of the function on the right is the integrand on the left, and each formula has an arbitrary constant for generating the remaining antiderivatives. Moreover,
$\frac{\tan^2 x}{2} + C = \frac{\sec^2 x - 1}{2} + C = \frac{\sec^2 x}{2} + \underbrace{\left(C - \frac{1}{2}\right)}_{\text{a constant}}$

63. (a) $\left(\frac{1}{\frac{1}{60} - 0}\right)\int_0^{1/60} V_{max}\sin 120\pi t\,dt = 60\left[-V_{max}\left(\frac{1}{120\pi}\right)\cos(120\pi t)\right]_0^{1/60} = -\frac{V_{max}}{2\pi}[\cos 2\pi - \cos 0]$
$= -\frac{V_{max}}{2\pi}[1 - 1] = 0$

(b) $V_{max} = \sqrt{2}\,V_{rms} = \sqrt{2}(240) \approx 339$ volts

(c) $\int_0^{1/60}(V_{max})^2\sin^2 120\pi t\,dt = (V_{max})^2\int_0^{1/60}\left(\frac{1 - \cos 240\pi t}{2}\right)dt = \frac{(V_{max})^2}{2}\int_0^{1/60}(1 - \cos 240\pi t)\,dt$
$= \frac{(V_{max})^2}{2}\left[t - \left(\frac{1}{240\pi}\right)\sin 240\pi t\right]_0^{1/60} = \frac{(V_{max})^2}{2}\left[\left(\frac{1}{60} - \left(\frac{1}{240\pi}\right)\sin(4\pi)\right) - \left(0 - \left(\frac{1}{240\pi}\right)\sin(0)\right)\right] = \frac{(V_{max})^2}{120}$

5.6 SUBSTITUTION AND AREA BETWEEN CURVES

1. (a) Let $u = y + 1 \ \Rightarrow\ du = dy$; $y = 0 \ \Rightarrow\ u = 1$, $y = 3 \ \Rightarrow\ u = 4$
$\int_0^3 \sqrt{y+1}\,dy = \int_1^4 u^{1/2}\,du = \left[\frac{2}{3}u^{3/2}\right]_1^4 = \left(\frac{2}{3}\right)(4)^{3/2} - \left(\frac{2}{3}\right)(1)^{3/2} = \left(\frac{2}{3}\right)(8) - \left(\frac{2}{3}\right)(1) = \frac{14}{3}$

(b) Use the same substitution for u as in part (a); $y = -1 \ \Rightarrow\ u = 0$, $y = 0 \ \Rightarrow\ u = 1$
$\int_{-1}^0 \sqrt{y+1}\,dy = \int_0^1 u^{1/2}\,du = \left[\frac{2}{3}u^{3/2}\right]_0^1 = \left(\frac{2}{3}\right)(1)^{3/2} - 0 = \frac{2}{3}$

2. (a) Let $u = 1 - r^2 \ \Rightarrow\ du = -2r\,dr \ \Rightarrow\ -\frac{1}{2}\,du = r\,dr$; $r = 0 \ \Rightarrow\ u = 1$, $r = 1 \ \Rightarrow\ u = 0$
$\int_0^1 r\sqrt{1 - r^2}\,dr = \int_1^0 -\frac{1}{2}\sqrt{u}\,du = \left[-\frac{1}{3}u^{3/2}\right]_1^0 = 0 - \left(-\frac{1}{3}\right)(1)^{3/2} = \frac{1}{3}$

(b) Use the same substitution for u as in part (a); $r = -1 \ \Rightarrow\ u = 0$, $r = 1 \ \Rightarrow\ u = 0$
$\int_{-1}^1 r\sqrt{1 - r^2}\,dr = \int_0^0 -\frac{1}{2}\sqrt{u}\,du = 0$

3. (a) Let $u = \tan x \ \Rightarrow\ du = \sec^2 x\,dx$; $x = 0 \ \Rightarrow\ u = 0$, $x = \frac{\pi}{4} \ \Rightarrow\ u = 1$
$\int_0^{\pi/4} \tan x \sec^2 x\,dx = \int_0^1 u\,du = \left[\frac{u^2}{2}\right]_0^1 = \frac{1^2}{2} - 0 = \frac{1}{2}$

(b) Use the same substitution as in part (a); $x = -\frac{\pi}{4} \ \Rightarrow\ u = -1$, $x = 0 \ \Rightarrow\ u = 0$
$\int_{-\pi/4}^0 \tan x \sec^2 x\,dx = \int_{-1}^0 u\,du = \left[\frac{u^2}{2}\right]_{-1}^0 = 0 - \frac{1}{2} = -\frac{1}{2}$

4. (a) Let $u = \cos x \ \Rightarrow\ du = -\sin x\,dx \ \Rightarrow\ -du = \sin x\,dx$; $x = 0 \ \Rightarrow\ u = 1$, $x = \pi \ \Rightarrow\ u = -1$
$\int_0^{\pi} 3\cos^2 x \sin x\,dx = \int_1^{-1} -3u^2\,du = \left[-u^3\right]_1^{-1} = -(-1)^3 - \left(-(1)^3\right) = 2$

(b) Use the same substitution as in part (a); $x = 2\pi \ \Rightarrow\ u = 1$, $x = 3\pi \ \Rightarrow\ u = -1$
$\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\,dx = \int_1^{-1} -3u^2\,du = 2$

5. (a) $u = 1 + t^4 \Rightarrow du = 4t^3\,dt \Rightarrow \frac{1}{4}\,du = t^3\,dt$; $t = 0 \Rightarrow u = 1$, $t = 1 \Rightarrow u = 2$

$$\int_0^1 t^3\left(1+t^4\right)^3 dt = \int_1^2 \frac{1}{4}u^3\,du = \left[\frac{u^4}{16}\right]_1^2 = \frac{2^4}{16} - \frac{1^4}{16} = \frac{15}{16}$$

(b) Use the same substitution as in part (a); $t = -1 \Rightarrow u = 2$, $t = 1 \Rightarrow u = 2$

$$\int_{-1}^1 t^3\left(1+t^4\right)^3 dt = \int_2^2 \frac{1}{4}u^3\,du = 0$$

6. (a) Let $u = t^2 + 1 \Rightarrow du = 2t\,dt \Rightarrow \frac{1}{2}\,du = t\,dt$; $t = 0 \Rightarrow u = 1$, $t = \sqrt{7} \Rightarrow u = 8$

$$\int_0^{\sqrt{7}} t\left(t^2+1\right)^{1/3} dt = \int_1^8 \frac{1}{2}u^{1/3}\,du = \left[\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)u^{4/3}\right]_1^8 = \left(\frac{3}{8}\right)(8)^{4/3} - \left(\frac{3}{8}\right)(1)^{4/3} = \frac{45}{8}$$

(b) Use the same substitution as in part (a); $t = -\sqrt{7} \Rightarrow u = 8$, $t = 0 \Rightarrow u = 1$

$$\int_{-\sqrt{7}}^0 t\left(t^2+1\right)^{1/3} dt = \int_8^1 \frac{1}{2}u^{1/3}\,du = -\int_1^8 \frac{1}{2}u^{1/3}\,du = -\frac{45}{8}$$

7. (a) Let $u = 4 + r^2 \Rightarrow du = 2r\,dr \Rightarrow \frac{1}{2}\,du = r\,dr$; $r = -1 \Rightarrow u = 5$, $r = 1 \Rightarrow u = 5$

$$\int_{-1}^1 \frac{5r}{\left(4+r^2\right)^2}\,dr = 5\int_5^5 \frac{1}{2}u^{-2}\,du = 0$$

(b) Use the same substitution as in part (a); $r = 0 \Rightarrow u = 4$, $r = 1 \Rightarrow u = 5$

$$\int_0^1 \frac{5r}{\left(4+r^2\right)^2}\,dr = 5\int_4^5 \frac{1}{2}u^{-2}\,du = 5\left[-\frac{1}{2}u^{-1}\right]_4^5 = 5\left(-\frac{1}{2}(5)^{-1}\right) - 5\left(-\frac{1}{2}(4)^{-1}\right) = \frac{1}{8}$$

8. (a) Let $u = 1 + v^{3/2} \Rightarrow du = \frac{3}{2}v^{1/2}\,dv \Rightarrow \frac{20}{3}\,du = 10\sqrt{v}\,dv$; $v = 0 \Rightarrow u = 1$, $v = 1 \Rightarrow u = 2$

$$\int_0^1 \frac{10\sqrt{v}}{\left(1+v^{3/2}\right)^2}\,dv = \int_1^2 \frac{1}{u^2}\left(\frac{20}{3}\,du\right) = \frac{20}{3}\int_1^2 u^{-2}\,du = -\frac{20}{3}\left[\frac{1}{u}\right]_1^2 = -\frac{20}{3}\left[\frac{1}{2} - \frac{1}{1}\right] = \frac{10}{3}$$

(b) Use the same substitution as in part (a); $v = 1 \Rightarrow u = 2$, $v = 4 \Rightarrow u = 1 + 4^{3/2} = 9$

$$\int_1^4 \frac{10\sqrt{v}}{\left(1+v^{3/2}\right)^2}\,dv = \int_2^9 \frac{1}{u^2}\left(\frac{20}{3}\,du\right) = -\frac{20}{3}\left[\frac{1}{u}\right]_2^9 = -\frac{20}{3}\left(\frac{1}{9} - \frac{1}{2}\right) = -\frac{20}{3}\left(-\frac{7}{18}\right) = \frac{70}{27}$$

9. (a) Let $u = x^2 + 1 \Rightarrow du = 2x\,dx \Rightarrow 2\,du = 4x\,dx$; $x = 0 \Rightarrow u = 1$, $x = \sqrt{3} \Rightarrow u = 4$

$$\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx = \int_1^4 \frac{2}{\sqrt{u}}\,du = \int_1^4 2u^{-1/2}\,du = \left[4u^{1/2}\right]_1^4 = 4(4)^{1/2} - 4(1)^{1/2} = 4$$

(b) Use the same substitution as in part (a); $x = -\sqrt{3} \Rightarrow u = 4$, $x = \sqrt{3} \Rightarrow u = 4$

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx = \int_4^4 \frac{2}{\sqrt{u}}\,du = 0$$

10. (a) Let $u = x^4 + 9 \Rightarrow du = 4x^3\,dx \Rightarrow \frac{1}{4}\,du = x^3\,dx$; $x = 0 \Rightarrow u = 9$, $x = 1 \Rightarrow u = 10$

$$\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx = \int_9^{10} \frac{1}{4}u^{-1/2}\,du = \left[\frac{1}{4}(2)u^{1/2}\right]_9^{10} = \frac{1}{2}(10)^{1/2} - \frac{1}{2}(9)^{1/2} = \frac{\sqrt{10}-3}{2}$$

(b) Use the same substitution as in part (a); $x = -1 \Rightarrow u = 10$, $x = 0 \Rightarrow u = 9$

$$\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}}\,dx = \int_{10}^9 \frac{1}{4}u^{-1/2}\,du = -\int_9^{10} \frac{1}{4}u^{-1/2}\,du = \frac{3-\sqrt{10}}{2}$$

11. (a) Let $u = 1 - \cos 3t \Rightarrow du = 3\sin 3t\,dt \Rightarrow \frac{1}{3}\,du = \sin 3t\,dt$; $t = 0 \Rightarrow u = 0$, $t = \frac{\pi}{6} \Rightarrow u = 1 - \cos\frac{\pi}{2} = 1$

$$\int_0^{\pi/6} (1 - \cos 3t)\sin 3t\,dt = \int_0^1 \frac{1}{3}u\,du = \left[\frac{1}{3}\left(\frac{u^2}{2}\right)\right]_0^1 = \frac{1}{6}(1)^2 - \frac{1}{6}(0)^2 = \frac{1}{6}$$

(b) Use the same substitution as in part (a); $t = \frac{\pi}{6} \Rightarrow u = 1$, $t = \frac{\pi}{3} \Rightarrow u = 1 - \cos\pi = 2$

$$\int_{\pi/6}^{\pi/3} (1 - \cos 3t)\sin 3t\,dt = \int_1^2 \frac{1}{3}u\,du = \left[\frac{1}{3}\left(\frac{u^2}{2}\right)\right]_1^2 = \frac{1}{6}(2)^2 - \frac{1}{6}(1)^2 = \frac{1}{2}$$

12. (a) Let $u = 2 + \tan \frac{t}{2} \Rightarrow du = \frac{1}{2}\sec^2 \frac{t}{2}\,dt \Rightarrow 2\,du = \sec^2 \frac{t}{2}\,dt$; $t = \frac{-\pi}{2} \Rightarrow u = 2 + \tan\left(\frac{-\pi}{4}\right) = 1$, $t = 0 \Rightarrow u = 2$

$$\int_{-\pi/2}^{0}\left(2 + \tan \tfrac{t}{2}\right)\sec^2 \tfrac{t}{2}\,dt = \int_1^2 u\,(2\,du) = \left[u^2\right]_1^2 = 2^2 - 1^2 = 3$$

(b) Use the same substitution as in part (a); $t = \frac{-\pi}{2} \Rightarrow u = 1$, $t = \frac{\pi}{2} \Rightarrow u = 3$

$$\int_{-\pi/2}^{\pi/2}\left(2 + \tan \tfrac{t}{2}\right)\sec^2 \tfrac{t}{2}\,dt = 2\int_1^3 u\,du = \left[u^2\right]_1^3 = 3^2 - 1^2 = 8$$

13. (a) Let $u = 4 + 3\sin z \Rightarrow du = 3\cos z\,dz \Rightarrow \frac{1}{3}\,du = \cos z\,dz$; $z = 0 \Rightarrow u = 4$, $z = 2\pi \Rightarrow u = 4$

$$\int_0^{2\pi}\frac{\cos z}{\sqrt{4 + 3\sin z}}\,dz = \int_4^4 \frac{1}{\sqrt{u}}\left(\tfrac{1}{3}\,du\right) = 0$$

(b) Use the same substitution as in part (a); $z = -\pi \Rightarrow u = 4 + 3\sin(-\pi) = 4$, $z = \pi \Rightarrow u = 4$

$$\int_{-\pi}^{\pi}\frac{\cos z}{\sqrt{4 + 3\sin z}}\,dz = \int_4^4 \frac{1}{\sqrt{u}}\left(\tfrac{1}{3}\,du\right) = 0$$

14. (a) Let $u = 3 + 2\cos w \Rightarrow du = -2\sin w\,dw \Rightarrow -\frac{1}{2}\,du = \sin w\,dw$; $w = -\frac{\pi}{2} \Rightarrow u = 3$, $w = 0 \Rightarrow u = 5$

$$\int_{-\pi/2}^{0}\frac{\sin w}{(3 + 2\cos w)^2}\,dw = \int_3^5 u^{-2}\left(-\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\left[u^{-1}\right]_3^5 = \tfrac{1}{2}\left(\tfrac{1}{5} - \tfrac{1}{3}\right) = -\tfrac{1}{15}$$

(b) Use the same substitution as in part (a); $w = 0 \Rightarrow u = 5$, $w = \frac{\pi}{2} \Rightarrow u = 3$

$$\int_0^{\pi/2}\frac{\sin w}{(3 + 2\cos w)^2}\,dw = \int_5^3 u^{-2}\left(-\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int_3^5 u^{-2}\,du = \tfrac{1}{15}$$

15. Let $u = t^5 + 2t \Rightarrow du = (5t^4 + 2)\,dt$; $t = 0 \Rightarrow u = 0$, $t = 1 \Rightarrow u = 3$

$$\int_0^1 \sqrt{t^5 + 2t}\,(5t^4 + 2)\,dt = \int_0^3 u^{1/2}\,du = \left[\tfrac{2}{3}u^{3/2}\right]_0^3 = \tfrac{2}{3}(3)^{3/2} - \tfrac{2}{3}(0)^{3/2} = 2\sqrt{3}$$

16. Let $u = 1 + \sqrt{y} \Rightarrow du = \frac{dy}{2\sqrt{y}}$; $y = 1 \Rightarrow u = 2$, $y = 4 \Rightarrow u = 3$

$$\int_1^4 \frac{dy}{2\sqrt{y}\,\left(1 + \sqrt{y}\right)^2} = \int_2^3 \tfrac{1}{u^2}\,du = \int_2^3 u^{-2}\,du = \left[-u^{-1}\right]_2^3 = \left(-\tfrac{1}{3}\right) - \left(-\tfrac{1}{2}\right) = \tfrac{1}{6}$$

17. Let $u = \cos 2\theta \Rightarrow du = -2\sin 2\theta\,d\theta \Rightarrow -\frac{1}{2}\,du = \sin 2\theta\,d\theta$; $\theta = 0 \Rightarrow u = 1$, $\theta = \frac{\pi}{6} \Rightarrow u = \cos 2\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$$\int_0^{\pi/6}\cos^{-3} 2\theta\,\sin 2\theta\,d\theta = \int_1^{1/2} u^{-3}\left(-\tfrac{1}{2}\,du\right) = -\tfrac{1}{2}\int_1^{1/2} u^{-3}\,du = \left[-\tfrac{1}{2}\left(\tfrac{u^{-2}}{-2}\right)\right]_1^{1/2} = \tfrac{1}{4\left(\frac{1}{2}\right)^2} - \tfrac{1}{4(1)^2} = \tfrac{3}{4}$$

18. Let $u = \tan\left(\frac{\theta}{6}\right) \Rightarrow du = \frac{1}{6}\sec^2\left(\frac{\theta}{6}\right)d\theta \Rightarrow 6\,du = \sec^2\left(\frac{\theta}{6}\right)d\theta$; $\theta = \pi \Rightarrow u = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, $\theta = \frac{3\pi}{2} \Rightarrow$ $u = \tan\frac{\pi}{4} = 1$

$$\int_\pi^{3\pi/2}\cot^5\left(\tfrac{\theta}{6}\right)\sec^2\left(\tfrac{\theta}{6}\right)d\theta = \int_{1/\sqrt{3}}^{1} u^{-5}\,(6\,du) = \left[6\left(\tfrac{u^{-4}}{-4}\right)\right]_{1/\sqrt{3}}^{1} = \left[-\tfrac{3}{2u^4}\right]_{1/\sqrt{3}}^{1} = -\tfrac{3}{2(1)^4} - \left(-\tfrac{3}{2\left(\frac{1}{\sqrt{3}}\right)^4}\right) = 12$$

19. Let $u = 5 - 4\cos t \Rightarrow du = 4\sin t\,dt \Rightarrow \frac{1}{4}\,du = \sin t\,dt$; $t = 0 \Rightarrow u = 5 - 4\cos 0 = 1$, $t = \pi \Rightarrow$ $u = 5 - 4\cos\pi = 9$

$$\int_0^\pi 5\,(5 - 4\cos t)^{1/4}\sin t\,dt = \int_1^9 5u^{1/4}\left(\tfrac{1}{4}\,du\right) = \tfrac{5}{4}\int_1^9 u^{1/4}\,du = \left[\tfrac{5}{4}\left(\tfrac{4}{5}u^{5/4}\right)\right]_1^9 = 9^{5/4} - 1 = 3^{5/2} - 1$$

20. Let $u = 1 - \sin 2t \Rightarrow du = -2\cos 2t\,dt \Rightarrow -\frac{1}{2}\,du = \cos 2t\,dt$; $t = 0 \Rightarrow u = 1$, $t = \frac{\pi}{4} \Rightarrow u = 0$

$$\int_0^{\pi/4}(1 - \sin 2t)^{3/2}\cos 2t\,dt = \int_1^0 -\tfrac{1}{2}u^{3/2}\,du = \left[-\tfrac{1}{2}\left(\tfrac{2}{5}u^{5/2}\right)\right]_1^0 = \left(-\tfrac{1}{5}(0)^{5/2}\right) - \left(-\tfrac{1}{5}(1)^{5/2}\right) = \tfrac{1}{5}$$

21. Let $u = 4y - y^2 + 4y^3 + 1 \Rightarrow du = (4 - 2y + 12y^2)\,dy$; $y = 0 \Rightarrow u = 1$, $y = 1 \Rightarrow u = 4(1) - (1)^2 + 4(1)^3 + 1 = 8$

$$\int_0^1 \left(4y - y^2 + 4y^3 + 1\right)^{-2/3}\left(12y^2 - 2y + 4\right)dy = \int_1^8 u^{-2/3}\,du = \left[3u^{1/3}\right]_1^8 = 3(8)^{1/3} - 3(1)^{1/3} = 3$$

22. Let $u = y^3 + 6y^2 - 12y + 9 \Rightarrow du = (3y^2 + 12y - 12)\,dy \Rightarrow \frac{1}{3}\,du = (y^2 + 4y - 4)\,dy$; $y = 0 \Rightarrow u = 9$, $y = 1 \Rightarrow u = 4$

$$\int_0^1 (y^3 + 6y^2 - 12y + 9)^{-1/2} (y^2 + 4y - 4)\,dy = \int_9^4 \frac{1}{3} u^{-1/2}\,du = \left[\frac{1}{3}\left(2u^{1/2}\right)\right]_9^4 = \frac{2}{3}(4)^{1/2} - \frac{2}{3}(9)^{1/2} = \frac{2}{3}(2-3) = -\frac{2}{3}$$

23. Let $u = \theta^{3/2} \Rightarrow du = \frac{3}{2}\theta^{1/2}\,d\theta \Rightarrow \frac{2}{3}\,du = \sqrt{\theta}\,d\theta$; $\theta = 0 \Rightarrow u = 0$, $\theta = \sqrt[3]{\pi^2} \Rightarrow u = \pi$

$$\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\cos^2\left(\theta^{3/2}\right) d\theta = \int_0^{\pi} \cos^2 u \left(\frac{2}{3}\,du\right) = \left[\frac{2}{3}\left(\frac{u}{2} + \frac{1}{4}\sin 2u\right)\right]_0^{\pi} = \frac{2}{3}\left(\frac{\pi}{2} + \frac{1}{4}\sin 2\pi\right) - \frac{2}{3}(0) = \frac{\pi}{3}$$

24. Let $u = 1 + \frac{1}{t} \Rightarrow du = -t^{-2}\,dt$; $t = -1 \Rightarrow u = 0$, $t = -\frac{1}{2} \Rightarrow u = -1$

$$\int_{-1}^{-1/2} t^{-2}\sin^2\left(1 + \frac{1}{t}\right) dt = \int_0^{-1} -\sin^2 u\,du = \left[-\left(\frac{u}{2} - \frac{1}{4}\sin 2u\right)\right]_0^{-1} = -\left[\left(-\frac{1}{2} - \frac{1}{4}\sin(-2)\right) - \left(\frac{0}{2} - \frac{1}{4}\sin 0\right)\right]$$
$$= \frac{1}{2} - \frac{1}{4}\sin 2$$

25. Let $u = 4 - x^2 \Rightarrow du = -2x\,dx \Rightarrow -\frac{1}{2}\,du = x\,dx$; $x = -2 \Rightarrow u = 0$, $x = 0 \Rightarrow u = 4$, $x = 2 \Rightarrow u = 0$

$$A = -\int_{-2}^{0} x\sqrt{4 - x^2}\,dx + \int_0^2 x\sqrt{4 - x^2}\,dx = -\int_0^4 -\frac{1}{2}u^{1/2}\,du + \int_4^0 -\frac{1}{2}u^{1/2}\,du = 2\int_0^4 \frac{1}{2}u^{1/2}\,du = \int_0^4 u^{1/2}\,du$$
$$= \left[\frac{2}{3}u^{3/2}\right]_0^4 = \frac{2}{3}(4)^{3/2} - \frac{2}{3}(0)^{3/2} = \frac{16}{3}$$

26. Let $u = 1 - \cos x \Rightarrow du = \sin x\,dx$; $x = 0 \Rightarrow u = 0$, $x = \pi \Rightarrow u = 2$

$$\int_0^{\pi} (1 - \cos x)\sin x\,dx = \int_0^2 u\,du = \left[\frac{u^2}{2}\right]_0^2 = \frac{2^2}{2} - \frac{0^2}{2} = 2$$

27. Let $u = 1 + \cos x \Rightarrow du = -\sin x\,dx \Rightarrow -du = \sin x\,dx$; $x = -\pi \Rightarrow u = 1 + \cos(-\pi) = 0$, $x = 0 \Rightarrow u = 1 + \cos 0 = 2$

$$A = -\int_{-\pi}^{0} 3(\sin x)\sqrt{1 + \cos x}\,dx = -\int_0^2 3u^{1/2}(-du) = 3\int_0^2 u^{1/2}\,du = \left[2u^{3/2}\right]_0^2 = 2(2)^{3/2} - 2(0)^{3/2} = 2^{5/2}$$

28. Let $u = \pi + \pi\sin x \Rightarrow du = \pi\cos x\,dx \Rightarrow \frac{1}{\pi}\,du = \cos x\,dx$; $x = -\frac{\pi}{2} \Rightarrow u = \pi + \pi\sin\left(-\frac{\pi}{2}\right) = 0$, $x = 0 \Rightarrow u = \pi$

Because of symmetry about $x = -\frac{\pi}{2}$, $A = 2\int_{-\pi/2}^{0} \frac{\pi}{2}(\cos x)(\sin(\pi + \pi\sin x))\,dx = 2\int_0^{\pi} \frac{\pi}{2}(\sin u)\left(\frac{1}{\pi}\,du\right)$

$$= \int_0^{\pi} \sin u\,du = [-\cos u]_0^{\pi} = (-\cos\pi) - (-\cos 0) = 2$$

29. For the sketch given, $a = 0$, $b = \pi$; $f(x) - g(x) = 1 - \cos^2 x = \sin^2 x = \frac{1 - \cos 2x}{2}$;

$$A = \int_0^{\pi} \frac{(1 - \cos 2x)}{2}\,dx = \frac{1}{2}\int_0^{\pi} (1 - \cos 2x)\,dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}[(\pi - 0) - (0 - 0)] = \frac{\pi}{2}$$

30. For the sketch given, $a = -\frac{\pi}{3}$, $b = \frac{\pi}{3}$; $f(t) - g(t) = \frac{1}{2}\sec^2 t - (-4\sin^2 t) = \frac{1}{2}\sec^2 t + 4\sin^2 t$;

$$A = \int_{-\pi/3}^{\pi/3}\left(\frac{1}{2}\sec^2 t + 4\sin^2 t\right) dt = \frac{1}{2}\int_{-\pi/3}^{\pi/3}\sec^2 t\,dt + 4\int_{-\pi/3}^{\pi/3}\sin^2 t\,dt = \frac{1}{2}\int_{-\pi/3}^{\pi/3}\sec^2 t\,dt + 4\int_{-\pi/3}^{\pi/3}\frac{(1 - \cos 2t)}{2}\,dt$$
$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\sec^2 t\,dt + 2\int_{-\pi/3}^{\pi/3}(1 - \cos 2t)\,dt = \frac{1}{2}[\tan t]_{-\pi/3}^{\pi/3} + 2\left[t - \frac{\sin 2t}{2}\right]_{-\pi/3}^{\pi/3} = \sqrt{3} + 4\cdot\frac{\pi}{3} - \sqrt{3} = \frac{4\pi}{3}$$

31. For the sketch given, $a = -2$, $b = 2$; $f(x) - g(x) = 2x^2 - (x^4 - 2x^2) = 4x^2 - x^4$;

$$A = \int_{-2}^{2} (4x^2 - x^4)\,dx = \left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_{-2}^{2} = \left(\frac{32}{3} - \frac{32}{5}\right) - \left[-\frac{32}{3} - \left(-\frac{32}{5}\right)\right] = \frac{64}{3} - \frac{64}{5} = \frac{320-192}{15} = \frac{128}{15}$$

32. For the sketch given, $c = 0$, $d = 1$; $f(y) - g(y) = y^2 - y^3$;
$A = \int_0^1 (y^2 - y^3)\,dy = \int_0^1 y^2\,dy - \int_0^1 y^3\,dy = \left[\frac{y^3}{3}\right]_0^1 - \left[\frac{y^4}{4}\right]_0^1 = \frac{(1-0)}{3} - \frac{(1-0)}{4} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

33. For the sketch given, $c = 0$, $d = 1$; $f(y) - g(y) = (12y^2 - 12y^3) - (2y^2 - 2y) = 10y^2 - 12y^3 + 2y$;
$A = \int_0^1 (10y^2 - 12y^3 + 2y)\,dy = \int_0^1 10y^2\,dy - \int_0^1 12y^3\,dy + \int_0^1 2y\,dy = \left[\frac{10}{3}y^3\right]_0^1 - \left[\frac{12}{4}y^4\right]_0^1 + \left[\frac{2}{2}y^2\right]_0^1$
$= \left(\frac{10}{3} - 0\right) - (3 - 0) + (1 - 0) = \frac{4}{3}$

34. For the sketch given, $a = -1$, $b = 1$; $f(x) - g(x) = x^2 - (-2x^4) = x^2 + 2x^4$;
$A = \int_{-1}^1 (x^2 + 2x^4)\,dx = \left[\frac{x^3}{3} + \frac{2x^5}{5}\right]_{-1}^1 = \left(\frac{1}{3} + \frac{2}{5}\right) - \left[-\frac{1}{3} + \left(-\frac{2}{5}\right)\right] = \frac{2}{3} + \frac{4}{5} = \frac{10+12}{15} = \frac{22}{15}$

35. We want the area between the line $y = 1$, $0 \le x \le 2$, and the curve $y = \frac{x^2}{4}$, *minus* the area of a triangle (formed by $y = x$ and $y = 1$) with base 1 and height 1. Thus, $A = \int_0^2 \left(1 - \frac{x^2}{4}\right) dx - \frac{1}{2}(1)(1) = \left[x - \frac{x^3}{12}\right]_0^2 - \frac{1}{2}$
$= \left(2 - \frac{8}{12}\right) - \frac{1}{2} = 2 - \frac{2}{3} - \frac{1}{2} = \frac{5}{6}$

36. We want the area between the x-axis and the curve $y = x^2$, $0 \le x \le 1$ *plus* the area of a triangle (formed by $x = 1$, $x + y = 2$, and the x-axis) with base 1 and height 1. Thus, $A = \int_0^1 x^2\,dx + \frac{1}{2}(1)(1) = \left[\frac{x^3}{3}\right]_0^1 + \frac{1}{2} = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

37. AREA = A1 + A2
A1: For the sketch given, $a = -3$ and we find b by solving the equations $y = x^2 - 4$ and $y = -x^2 - 2x$ simultaneously for x: $x^2 - 4 = -x^2 - 2x \Rightarrow 2x^2 + 2x - 4 = 0 \Rightarrow 2(x+2)(x-1) \Rightarrow x = -2$ or $x = 1$ so $b = -2$: $f(x) - g(x) = (x^2 - 4) - (-x^2 - 2x) = 2x^2 + 2x - 4 \Rightarrow A1 = \int_{-3}^{-2} (2x^2 + 2x - 4)\,dx$
$= \left[\frac{2x^3}{3} + \frac{2x^2}{2} - 4x\right]_{-3}^{-2} = \left(-\frac{16}{3} + 4 + 8\right) - (-18 + 9 + 12) = 9 - \frac{16}{3} = \frac{11}{3}$;
A2: For the sketch given, $a = -2$ and $b = 1$: $f(x) - g(x) = (-x^2 - 2x) - (x^2 - 4) = -2x^2 - 2x + 4$
$\Rightarrow A2 = -\int_{-2}^1 (2x^2 + 2x - 4)\,dx = -\left[\frac{2x^3}{3} + x^2 - 4x\right]_{-2}^1 = -\left(\frac{2}{3} + 1 - 4\right) + \left(-\frac{16}{3} + 4 + 8\right)$
$= -\frac{2}{3} - 1 + 4 - \frac{16}{3} + 4 + 8 = 9$;
Therefore, AREA = A1 + A2 $= \frac{11}{3} + 9 = \frac{38}{3}$

38. AREA = A1 + A2
A1: For the sketch given, $a = -2$ and $b = 0$: $f(x) - g(x) = (2x^3 - x^2 - 5x) - (-x^2 + 3x) = 2x^3 - 8x$
$\Rightarrow A1 = \int_{-2}^0 (2x^3 - 8x)\,dx = \left[\frac{2x^4}{4} - \frac{8x^2}{2}\right]_{-2}^0 = 0 - (8 - 16) = 8$;
A2: For the sketch given, $a = 0$ and $b = 2$: $f(x) - g(x) = (-x^2 + 3x) - (2x^3 - x^2 - 5x) = 8x - 2x^3$
$\Rightarrow A2 = \int_0^2 (8x - 2x^3)\,dx = \left[\frac{8x^2}{2} - \frac{2x^4}{4}\right]_0^2 = (16 - 8) = 8$;
Therefore, AREA = A1 + A2 = 16

39. AREA = A1 + A2 + A3
A1: For the sketch given, $a = -2$ and $b = -1$: $f(x) - g(x) = (-x + 2) - (4 - x^2) = x^2 - x - 2$
$\Rightarrow A1 = \int_{-2}^{-1} (x^2 - x - 2)\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-2}^{-1} = \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(-\frac{8}{3} - \frac{4}{2} + 4\right) = \frac{7}{3} - \frac{1}{2} = \frac{14-3}{6} = \frac{11}{6}$;
A2: For the sketch given, $a = -1$ and $b = 2$: $f(x) - g(x) = (4 - x^2) - (-x + 2) = -(x^2 - x - 2)$
$\Rightarrow A2 = -\int_{-1}^2 (x^2 - x - 2)\,dx = -\left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^2 = -\left(\frac{8}{3} - \frac{4}{2} - 4\right) + \left(-\frac{1}{3} - \frac{1}{2} + 2\right) = -3 + 8 - \frac{1}{2} = \frac{9}{2}$;

A3: For the sketch given, $a = 2$ and $b = 3$: $f(x) - g(x) = (-x + 2) - (4 - x^2) = x^2 - x - 2$

$$\Rightarrow A3 = \int_2^3 (x^2 - x - 2)\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_2^3 = \left(\frac{27}{3} - \frac{9}{2} - 6\right) - \left(\frac{8}{3} - \frac{4}{2} - 4\right) = 9 - \frac{9}{2} - \frac{8}{3};$$

Therefore, AREA $= A1 + A2 + A3 = \frac{11}{6} + \frac{9}{2} + \left(9 - \frac{9}{2} - \frac{8}{3}\right) = 9 - \frac{5}{6} = \frac{49}{6}$

40. AREA $= A1 + A2 + A3$

A1: For the sketch given, $a = -2$ and $b = 0$: $f(x) - g(x) = \left(\frac{x^3}{3} - x\right) - \frac{x}{3} = \frac{x^3}{3} - \frac{4}{3}x = \frac{1}{3}(x^3 - 4x)$

$$\Rightarrow A1 = \frac{1}{3}\int_{-2}^{0} (x^3 - 4x)\,dx = \frac{1}{3}\left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} = 0 - \frac{1}{3}(4 - 8) = \frac{4}{3};$$

A2: For the sketch given, $a = 0$ and we find b by solving the equations $y = \frac{x^3}{3} - x$ and $y = \frac{x}{3}$ simultaneously for x: $\frac{x^3}{3} - x = \frac{x}{3} \Rightarrow \frac{x^3}{3} - \frac{4}{3}x = 0 \Rightarrow \frac{x}{3}(x-2)(x+2) = 0 \Rightarrow x = -2$, $x = 0$, or $x = 2$ so $b = 2$:

$$f(x) - g(x) = \frac{x}{3} - \left(\frac{x^3}{3} - x\right) = -\frac{1}{3}(x^3 - 4x) \Rightarrow A2 = -\frac{1}{3}\int_0^2 (x^3 - 4x)\,dx = \frac{1}{3}\int_0^2 (4x - x^3) = \frac{1}{3}\left[2x^2 - \frac{x^4}{4}\right]_0^2$$

$$= \frac{1}{3}(8 - 4) = \frac{4}{3};$$

A3: For the sketch given, $a = 2$ and $b = 3$: $f(x) - g(x) = \left(\frac{x^3}{3} - x\right) - \frac{x}{3} = \frac{1}{3}(x^3 - 4x)$

$$\Rightarrow A3 = \frac{1}{3}\int_2^3 (x^3 - 4x)\,dx = \frac{1}{3}\left[\frac{x^4}{4} - 2x^2\right]_2^3 = \frac{1}{3}\left[\left(\frac{81}{4} - 2\cdot 9\right) - \left(\frac{16}{4} - 8\right)\right] = \frac{1}{3}\left(\frac{81}{4} - 14\right) = \frac{25}{12};$$

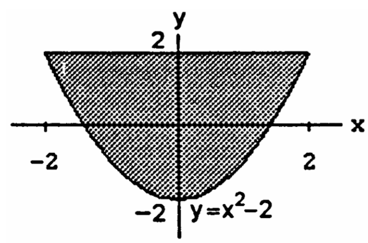
Therefore, AREA $= A1 + A2 + A3 = \frac{4}{3} + \frac{4}{3} + \frac{25}{12} = \frac{32+25}{12} = \frac{19}{4}$

41. $a = -2$, $b = 2$;

$f(x) - g(x) = 2 - (x^2 - 2) = 4 - x^2$

$$\Rightarrow A = \int_{-2}^{2} (4 - x^2)dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$$

$$= 2\cdot\left(\frac{24}{3} - \frac{8}{3}\right) = \frac{32}{3}$$

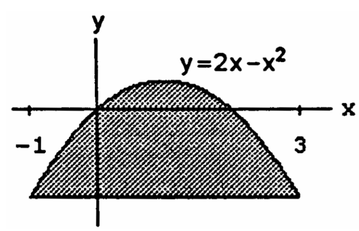


42. $a = -1$, $b = 3$;

$f(x) - g(x) = (2x - x^2) - (-3) = 2x - x^2 + 3$

$$\Rightarrow A = \int_{-1}^{3} (2x - x^2 + 3)\,dx = \left[x^2 - \frac{x^3}{3} + 3x\right]_{-1}^{3}$$

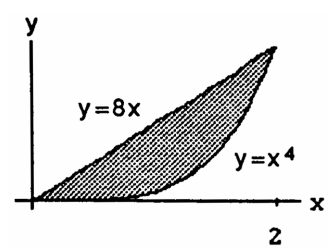
$$= \left(9 - \frac{27}{3} + 9\right) - \left(1 + \frac{1}{3} - 3\right) = 11 - \frac{1}{3} = \frac{32}{3}$$



43. $a = 0$, $b = 2$;

$$f(x) - g(x) = 8x - x^4 \Rightarrow A = \int_0^2 (8x - x^4)\,dx$$

$$= \left[\frac{8x^2}{2} - \frac{x^5}{5}\right]_0^2 = 16 - \frac{32}{5} = \frac{80 - 32}{5} = \frac{48}{5}$$



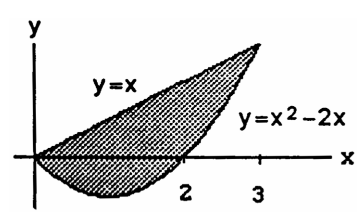
44. Limits of integration: $x^2 - 2x = x \Rightarrow x^2 = 3x$

$\Rightarrow x(x - 3) = 0 \Rightarrow a = 0$ and $b = 3$;

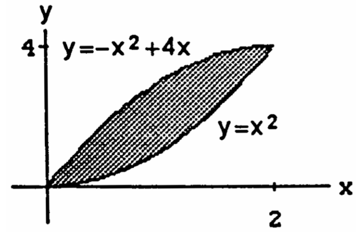
$f(x) - g(x) = x - (x^2 - 2x) = 3x - x^2$

$$\Rightarrow A = \int_0^3 (3x - x^2)\,dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$$

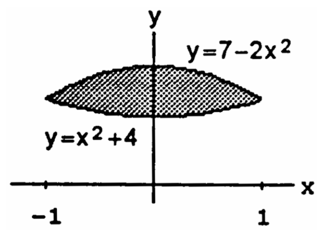
$$= \frac{27}{2} - 9 = \frac{27 - 18}{2} = \frac{9}{2}$$



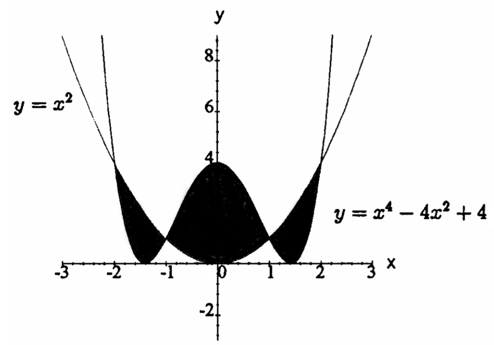
45. Limits of integration: $x^2 = -x^2 + 4x \Rightarrow 2x^2 - 4x = 0$
$\Rightarrow 2x(x-2) = 0 \Rightarrow a = 0$ and $b = 2$;
$f(x) - g(x) = (-x^2 + 4x) - x^2 = -2x^2 + 4x$
$\Rightarrow A = \int_0^2 (-2x^2 + 4x)\,dx = \left[\frac{-2x^3}{3} + \frac{4x^2}{2}\right]_0^2$
$= -\frac{16}{3} + \frac{16}{2} = \frac{-32+48}{6} = \frac{8}{3}$



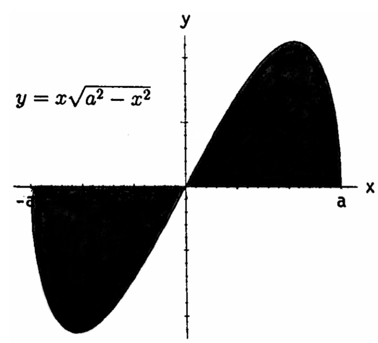
46. Limits of integration: $7 - 2x^2 = x^2 + 4 \Rightarrow 3x^2 - 3 = 0$
$\Rightarrow 3(x-1)(x+1) = 0 \Rightarrow a = -1$ and $b = 1$;
$f(x) - g(x) = (7 - 2x^2) - (x^2 + 4) = 3 - 3x^2$
$\Rightarrow A = \int_{-1}^{1} (3 - 3x^2)\,dx = 3\left[x - \frac{x^3}{3}\right]_{-1}^{1}$
$= 3\left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)\right] = 6\left(\frac{2}{3}\right) = 4$



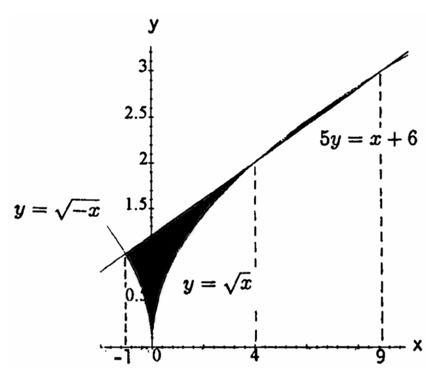
47. Limits of integration: $x^4 - 4x^2 + 4 = x^2$
$\Rightarrow x^4 - 5x^2 + 4 = 0 \Rightarrow (x^2 - 4)(x^2 - 1) = 0$
$\Rightarrow (x+2)(x-2)(x+1)(x-1) = 0 \Rightarrow x = -2, -1, 1, 2$;
$f(x) - g(x) = (x^4 - 4x^2 + 4) - x^2 = x^4 - 5x^2 + 4$ and
$g(x) - f(x) = x^2 - (x^4 - 4x^2 + 4) = -x^4 + 5x^2 - 4$
$\Rightarrow A = \int_{-2}^{-1} (-x^4 + 5x^2 - 4)dx + \int_{-1}^{1} (x^4 - 5x^2 + 4)dx$
$+ \int_1^2 (-x^4 + 5x^2 - 4)dx$
$= \left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x\right]_{-2}^{-1} + \left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right]_{-1}^{1} + \left[\frac{-x^5}{5} + \frac{5x^3}{3} - 4x\right]_1^2$
$= \left(\frac{1}{5} - \frac{5}{3} + 4\right) - \left(\frac{32}{5} - \frac{40}{3} + 8\right) + \left(\frac{1}{5} - \frac{5}{3} + 4\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right) + \left(-\frac{32}{5} + \frac{40}{3} - 8\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right)$
$= -\frac{60}{5} + \frac{60}{3} = \frac{300-180}{15} = 8$



48. Limits of integration: $x\sqrt{a^2 - x^2} = 0 \Rightarrow x = 0$ or
$\sqrt{a^2 - x^2} = 0 \Rightarrow x = 0$ or $a^2 - x^2 = 0 \Rightarrow x = -a, 0, a$;
$A = \int_{-a}^{0} -x\sqrt{a^2 - x^2}\,dx + \int_0^a x\sqrt{a^2 - x^2}\,dx$
$= \frac{1}{2}\left[\frac{2}{3}(a^2 - x^2)^{3/2}\right]_{-a}^{0} - \frac{1}{2}\left[\frac{2}{3}(a^2 - x^2)^{3/2}\right]_0^a$
$= \frac{1}{3}(a^2)^{3/2} - \left[-\frac{1}{3}(a^2)^{3/2}\right] = \frac{2a^3}{3}$



49. Limits of integration: $y = \sqrt{|x|} = \begin{cases} \sqrt{-x}, & x \le 0 \\ \sqrt{x}, & x \ge 0 \end{cases}$ and
$5y = x + 6$ or $y = \frac{x}{5} + \frac{6}{5}$; for $x \le 0$: $\sqrt{-x} = \frac{x}{5} + \frac{6}{5}$
$\Rightarrow 5\sqrt{-x} = x + 6 \Rightarrow 25(-x) = x^2 + 12x + 36$
$\Rightarrow x^2 + 37x + 36 = 0 \Rightarrow (x+1)(x+36) = 0$
$\Rightarrow x = -1, -36$ (but $x = -36$ is not a solution);
for $x \ge 0$: $5\sqrt{x} = x + 6 \Rightarrow 25x = x^2 + 12x + 36$
$\Rightarrow x^2 - 13x + 36 = 0 \Rightarrow (x-4)(x-9) = 0$
$\Rightarrow x = 4, 9$; there are three intersection points and



$A = \int_{-1}^{0} \left(\frac{x+6}{5} - \sqrt{-x}\right)dx + \int_0^4 \left(\frac{x+6}{5} - \sqrt{x}\right)dx + \int_4^9 \left(\sqrt{x} - \frac{x+6}{5}\right)dx$

$$= \left[\frac{(x+6)^2}{10} + \frac{2}{3}(-x)^{3/2}\right]_{-1}^{0} + \left[\frac{(x+6)^2}{10} - \frac{2}{3}x^{3/2}\right]_{0}^{4} + \left[\frac{2}{3}x^{3/2} - \frac{(x+6)^2}{10}\right]_{4}^{9}$$
$$= \left(\frac{36}{10} - \frac{25}{10} - \frac{2}{3}\right) + \left(\frac{100}{10} - \frac{2}{3}\cdot 4^{3/2} - \frac{36}{10} + 0\right) + \left(\frac{2}{3}\cdot 9^{3/2} - \frac{225}{10} - \frac{2}{3}\cdot 4^{3/2} + \frac{100}{10}\right) = -\frac{50}{10} + \frac{20}{3} = \frac{5}{3}$$

50. Limits of integration:

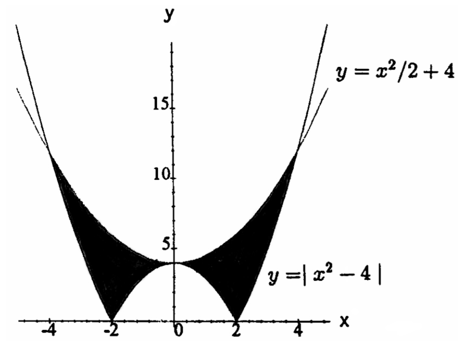
$y = |x^2 - 4| = \begin{cases} x^2 - 4, & x \le -2 \text{ or } x \ge 2 \\ 4 - x^2, & -2 \le x \le 2 \end{cases}$

for $x \le -2$ and $x \ge 2$: $x^2 - 4 = \frac{x^2}{2} + 4$

$\Rightarrow 2x^2 - 8 = x^2 + 8 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$;

for $-2 \le x \le 2$: $4 - x^2 = \frac{x^2}{2} + 4 \Rightarrow 8 - 2x^2 = x^2 + 8$

$\Rightarrow x^2 = 0 \Rightarrow x = 0$; by symmetry of the graph,

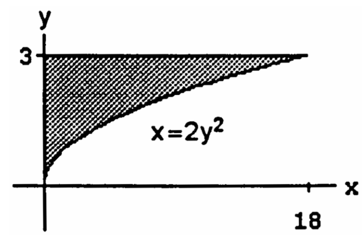


$$A = 2\int_0^2 \left[\left(\frac{x^2}{2} + 4\right) - (4 - x^2)\right]dx + 2\int_2^4 \left[\left(\frac{x^2}{2} + 4\right) - (x^2 - 4)\right]dx = 2\left[\frac{x^3}{2}\right]_0^2 + 2\left[8x - \frac{x^3}{6}\right]_2^4$$
$$= 2\left(\frac{8}{2} - 0\right) + 2\left(32 - \frac{64}{6} - 16 + \frac{8}{6}\right) = 40 - \frac{56}{3} = \frac{64}{3}$$

51. Limits of integration: $c = 0$ and $d = 3$;

$f(y) - g(y) = 2y^2 - 0 = 2y^2$

$\Rightarrow A = \int_0^3 2y^2\,dy = \left[\frac{2y^3}{3}\right]_0^3 = 2 \cdot 9 = 18$

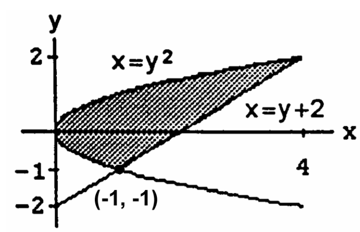


52. Limits of integration: $y^2 = y + 2 \Rightarrow (y+1)(y-2) = 0$

$\Rightarrow c = -1$ and $d = 2$; $f(y) - g(y) = (y+2) - y^2$

$\Rightarrow A = \int_{-1}^{2} (y + 2 - y^2)\,dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2}$

$= \left(\frac{4}{2} + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = 6 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3} = \frac{9}{2}$



53. Limits of integration: $4x = y^2 - 4$ and $4x = 16 + y$

$\Rightarrow y^2 - 4 = 16 + y \Rightarrow y^2 - y - 20 = 0 \Rightarrow$

$(y-5)(y+4) = 0 \Rightarrow c = -4$ and $d = 5$;

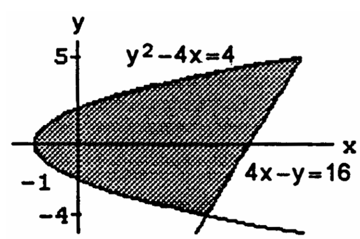
$f(y) - g(y) = \left(\frac{16+y}{4}\right) - \left(\frac{y^2-4}{4}\right) = \frac{-y^2+y+20}{4}$

$\Rightarrow A = \frac{1}{4}\int_{-4}^{5} (-y^2 + y + 20)\,dy$

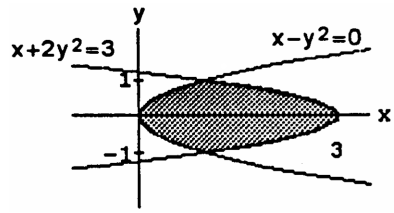
$= \frac{1}{4}\left[-\frac{y^3}{3} + \frac{y^2}{2} + 20y\right]_{-4}^{5}$

$= \frac{1}{4}\left(-\frac{125}{3} + \frac{25}{2} + 100\right) - \frac{1}{4}\left(\frac{64}{3} + \frac{16}{2} - 80\right)$

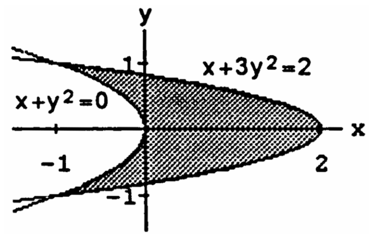
$= \frac{1}{4}\left(-\frac{189}{3} + \frac{9}{2} + 180\right) = \frac{243}{8}$



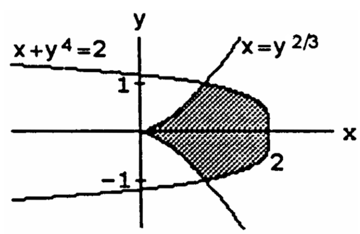
54. Limits of integration: $x = y^2$ and $x = 3 - 2y^2$
$\Rightarrow y^2 = 3 - 2y^2 \Rightarrow 3y^2 = 3 \Rightarrow 3(y-1)(y+1) = 0$
$\Rightarrow c = -1$ and $d = 1$; $f(y) - g(y) = (3 - 2y^2) - y^2$
$= 3 - 3y^2 = 3(1 - y^2) \Rightarrow A = 3\int_{-1}^{1}(1 - y^2)\,dy$
$= 3\left[y - \frac{y^3}{3}\right]_{-1}^{1} = 3\left(1 - \frac{1}{3}\right) - 3\left(-1 + \frac{1}{3}\right)$
$= 3 \cdot 2\left(1 - \frac{1}{3}\right) = 4$



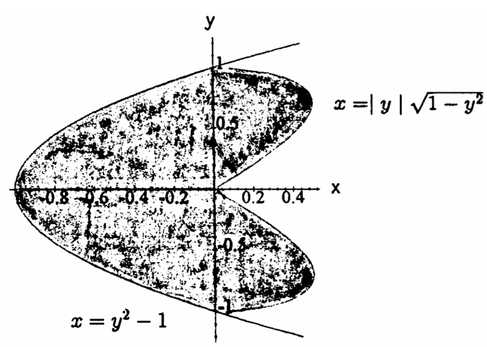
55. Limits of integration: $x = -y^2$ and $x = 2 - 3y^2$
$\Rightarrow -y^2 = 2 - 3y^2 \Rightarrow 2y^2 - 2 = 0$
$\Rightarrow 2(y-1)(y+1) = 0 \Rightarrow c = -1$ and $d = 1$;
$f(y) - g(y) = (2 - 3y^2) - (-y^2) = 2 - 2y^2 = 2(1 - y^2)$
$\Rightarrow A = 2\int_{-1}^{1}(1 - y^2)\,dy = 2\left[y - \frac{y^3}{3}\right]_{-1}^{1}$
$= 2\left(1 - \frac{1}{3}\right) - 2\left(-1 + \frac{1}{3}\right) = 4\left(\frac{2}{3}\right) = \frac{8}{3}$



56. Limits of integration: $x = y^{2/3}$ and $x = 2 - y^4$
$\Rightarrow y^{2/3} = 2 - y^4 \Rightarrow c = -1$ and $d = 1$;
$f(y) - g(y) = (2 - y^4) - y^{2/3}$
$\Rightarrow A = \int_{-1}^{1}\left(2 - y^4 - y^{2/3}\right)dy$
$= \left[2y - \frac{y^5}{5} - \frac{3}{5}y^{5/3}\right]_{-1}^{1}$
$= \left(2 - \frac{1}{5} - \frac{3}{5}\right) - \left(-2 + \frac{1}{5} + \frac{3}{5}\right)$
$= 2\left(2 - \frac{1}{5} - \frac{3}{5}\right) = \frac{12}{5}$

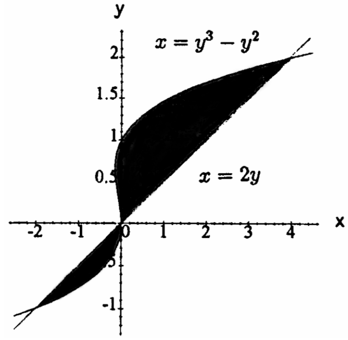


57. Limits of integration: $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$
$\Rightarrow y^2 - 1 = |y|\sqrt{1 - y^2} \Rightarrow y^4 - 2y^2 + 1 = y^2(1 - y^2)$
$\Rightarrow y^4 - 2y^2 + 1 = y^2 - y^4 \Rightarrow 2y^4 - 3y^2 + 1 = 0$
$\Rightarrow (2y^2 - 1)(y^2 - 1) = 0 \Rightarrow 2y^2 - 1 = 0$ or $y^2 - 1 = 0$
$\Rightarrow y^2 = \frac{1}{2}$ or $y^2 = 1 \Rightarrow y = \pm\frac{\sqrt{2}}{2}$ or $y = \pm 1$.
Substitution shows that $\frac{\pm\sqrt{2}}{2}$ are not solutions $\Rightarrow y = \pm 1$;
for $-1 \le y \le 0$, $f(x) - g(x) = -y\sqrt{1 - y^2} - (y^2 - 1)$
$= 1 - y^2 - y(1 - y^2)^{1/2}$, and by symmetry of the graph,
$A = 2\int_{-1}^{0}\left[1 - y^2 - y(1 - y^2)^{1/2}\right]dy$
$= 2\int_{-1}^{0}(1 - y^2)\,dy - 2\int_{-1}^{0}y(1 - y^2)^{1/2}\,dy$
$= 2\left[y - \frac{y^3}{3}\right]_{-1}^{0} + 2\left(\frac{1}{2}\right)\left[\frac{2(1 - y^2)^{3/2}}{3}\right]_{-1}^{0} = 2\left[(0 - 0) - \left(-1 + \frac{1}{3}\right)\right] + \left(\frac{2}{3} - 0\right) = 2$

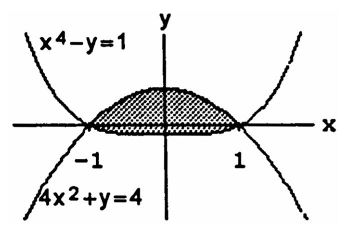


58. AREA = A1 + A2

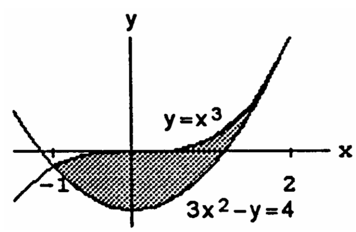
Limits of integration: $x = 2y$ and $x = y^3 - y^2 \Rightarrow$
$y^3 - y^2 = 2y \Rightarrow y(y^2 - y - 2) = y(y+1)(y-2) = 0$
$\Rightarrow y = -1, 0, 2$:
for $-1 \le y \le 0$, $f(y) - g(y) = y^3 - y^2 - 2y$
$\Rightarrow A1 = \int_{-1}^{0} (y^3 - y^2 - 2y)\,dy = \left[\frac{y^4}{4} - \frac{y^3}{3} - y^2\right]_{-1}^{0}$
$= 0 - \left(\frac{1}{4} + \frac{1}{3} - 1\right) = \frac{5}{12}$;
for $0 \le y \le 2$, $f(y) - g(y) = 2y - y^3 + y^2$
$\Rightarrow A2 = \int_{0}^{2} (2y - y^3 + y^2)\,dy = \left[y^2 - \frac{y^4}{4} + \frac{y^3}{3}\right]_{0}^{2}$
$\Rightarrow \left(4 - \frac{16}{4} + \frac{8}{3}\right) - 0 = \frac{8}{3}$;
Therefore, $A1 + A2 = \frac{5}{12} + \frac{8}{3} = \frac{37}{12}$



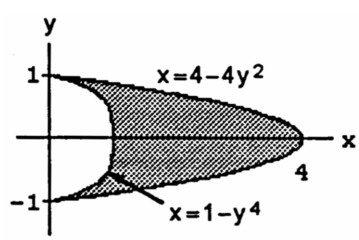
59. Limits of integration: $y = -4x^2 + 4$ and $y = x^4 - 1$
$\Rightarrow x^4 - 1 = -4x^2 + 4 \Rightarrow x^4 + 4x^2 - 5 = 0$
$\Rightarrow (x^2 + 5)(x - 1)(x + 1) = 0 \Rightarrow a = -1$ and $b = 1$;
$f(x) - g(x) = -4x^2 + 4 - x^4 + 1 = -4x^2 - x^4 + 5$
$\Rightarrow A = \int_{-1}^{1} (-4x^2 - x^4 + 5)\,dx = \left[-\frac{4x^3}{3} - \frac{x^5}{5} + 5x\right]_{-1}^{1}$
$= \left(-\frac{4}{3} - \frac{1}{5} + 5\right) - \left(\frac{4}{3} + \frac{1}{5} - 5\right) = 2\left(-\frac{4}{3} - \frac{1}{5} + 5\right) = \frac{104}{15}$



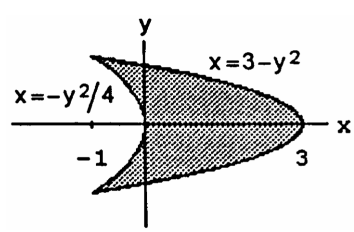
60. Limits of integration: $y = x^3$ and $y = 3x^2 - 4$
$\Rightarrow x^3 - 3x^2 + 4 = 0 \Rightarrow (x^2 - x - 2)(x - 2) = 0$
$\Rightarrow (x + 1)(x - 2)^2 = 0 \Rightarrow a = -1$ and $b = 2$;
$f(x) - g(x) = x^3 - (3x^2 - 4) = x^3 - 3x^2 + 4$
$\Rightarrow A = \int_{-1}^{2} (x^3 - 3x^2 + 4)\,dx = \left[\frac{x^4}{4} - \frac{3x^3}{3} + 4x\right]_{-1}^{2}$
$= \left(\frac{16}{4} - \frac{24}{3} + 8\right) - \left(\frac{1}{4} + 1 - 4\right) = \frac{27}{4}$



61. Limits of integration: $x = 4 - 4y^2$ and $x = 1 - y^4$
$\Rightarrow 4 - 4y^2 = 1 - y^4 \Rightarrow y^4 - 4y^2 + 3 = 0$
$\Rightarrow \left(y - \sqrt{3}\right)\left(y + \sqrt{3}\right)(y - 1)(y + 1) = 0 \Rightarrow c = -1$
and $d = 1$ since $x \ge 0$; $f(y) - g(y) = (4 - 4y^2) - (1 - y^4)$
$= 3 - 4y^2 + y^4 \Rightarrow A = \int_{-1}^{1} (3 - 4y^2 + y^4)\,dy$
$= \left[3y - \frac{4y^3}{3} + \frac{y^5}{5}\right]_{-1}^{1} = 2\left(3 - \frac{4}{3} + \frac{1}{5}\right) = \frac{56}{15}$



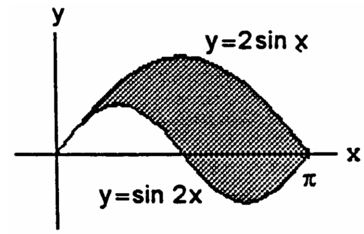
62. Limits of integration: $x = 3 - y^2$ and $x = -\frac{y^2}{4}$
$\Rightarrow 3 - y^2 = -\frac{y^2}{4} \Rightarrow \frac{3y^2}{4} - 3 = 0 \Rightarrow \frac{3}{4}(y - 2)(y + 2) = 0$
$\Rightarrow c = -2$ and $d = 2$; $f(y) - g(y) = (3 - y^2) - \left(\frac{-y^2}{4}\right)$
$= 3\left(1 - \frac{y^2}{4}\right) \Rightarrow A = 3\int_{-2}^{2} \left(1 - \frac{y^2}{4}\right) dy = 3\left[y - \frac{y^3}{12}\right]_{-2}^{2}$
$= 3\left[\left(2 - \frac{8}{12}\right) - \left(-2 + \frac{8}{12}\right)\right] = 3\left(4 - \frac{16}{12}\right) = 12 - 4 = 8$



63. $a = 0, b = \pi$; $f(x) - g(x) = 2\sin x - \sin 2x$

$\Rightarrow A = \int_0^{\pi} (2\sin x - \sin 2x)\,dx = \left[-2\cos x + \frac{\cos 2x}{2}\right]_0^{\pi}$

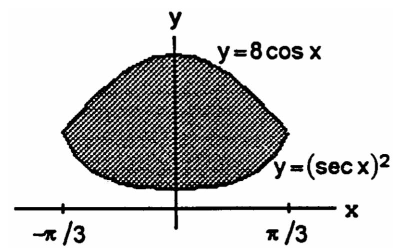
$= \left[-2(-1) + \frac{1}{2}\right] - \left(-2 \cdot 1 + \frac{1}{2}\right) = 4$



64. $a = -\frac{\pi}{3}, b = \frac{\pi}{3}$; $f(x) - g(x) = 8\cos x - \sec^2 x$

$\Rightarrow A = \int_{-\pi/3}^{\pi/3} (8\cos x - \sec^2 x)\,dx = [8\sin x - \tan x]_{-\pi/3}^{\pi/3}$

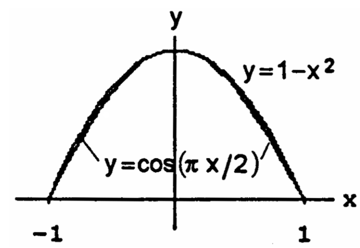
$= \left(8 \cdot \frac{\sqrt{3}}{2} - \sqrt{3}\right) - \left(-8 \cdot \frac{\sqrt{3}}{2} + \sqrt{3}\right) = 6\sqrt{3}$



65. $a = -1, b = 1$; $f(x) - g(x) = (1 - x^2) - \cos\left(\frac{\pi x}{2}\right)$

$\Rightarrow A = \int_{-1}^{1} \left[1 - x^2 - \cos\left(\frac{\pi x}{2}\right)\right] dx = \left[x - \frac{x^3}{3} - \frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_{-1}^{1}$

$= \left(1 - \frac{1}{3} - \frac{2}{\pi}\right) - \left(-1 + \frac{1}{3} + \frac{2}{\pi}\right) = 2\left(\frac{2}{3} - \frac{2}{\pi}\right) = \frac{4}{3} - \frac{4}{\pi}$



66. $A = A1 + A2$

$a_1 = -1, b_1 = 0$ and $a_2 = 0, b_2 = 1$;

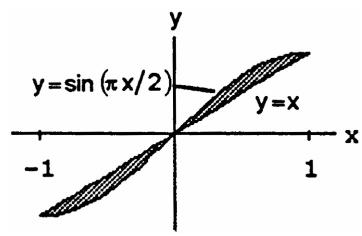
$f_1(x) - g_1(x) = x - \sin\left(\frac{\pi x}{2}\right)$ and $f_2(x) - g_2(x) = \sin\left(\frac{\pi x}{2}\right) - x$

$\Rightarrow$ by symmetry about the origin,

$A_1 + A_2 = 2A_1 \Rightarrow A = 2\int_0^1 \left[\sin\left(\frac{\pi x}{2}\right) - x\right] dx$

$= 2\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right) - \frac{x^2}{2}\right]_0^1 = 2\left[\left(-\frac{2}{\pi} \cdot 0 - \frac{1}{2}\right) - \left(-\frac{2}{\pi} \cdot 1 - 0\right)\right]$

$= 2\left(\frac{2}{\pi} - \frac{1}{2}\right) = 2\left(\frac{4-\pi}{2\pi}\right) = \frac{4-\pi}{\pi}$

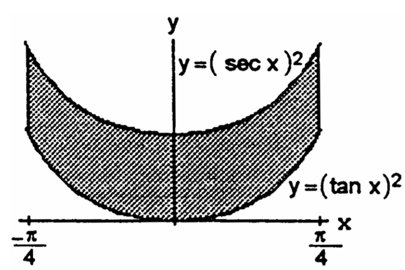


67. $a = -\frac{\pi}{4}, b = \frac{\pi}{4}$; $f(x) - g(x) = \sec^2 x - \tan^2 x$

$\Rightarrow A = \int_{-\pi/4}^{\pi/4} (\sec^2 x - \tan^2 x)\,dx$

$= \int_{-\pi/4}^{\pi/4} [\sec^2 x - (\sec^2 x - 1)]\,dx$

$= \int_{-\pi/4}^{\pi/4} 1 \cdot dx = [x]_{-\pi/4}^{\pi/4} = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$

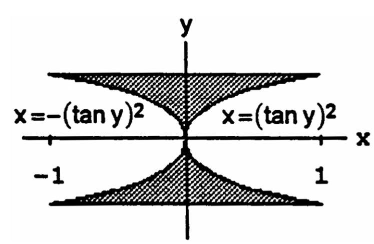


68. $c = -\frac{\pi}{4}, d = \frac{\pi}{4}$; $f(y) - g(y) = \tan^2 y - (-\tan^2 y) = 2\tan^2 y$

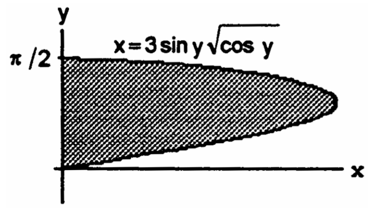
$= 2(\sec^2 y - 1) \Rightarrow A = \int_{-\pi/4}^{\pi/4} 2(\sec^2 y - 1)\,dy$

$= 2[\tan y - y]_{-\pi/4}^{\pi/4} = 2\left[\left(1 - \frac{\pi}{4}\right) - \left(-1 + \frac{\pi}{4}\right)\right]$

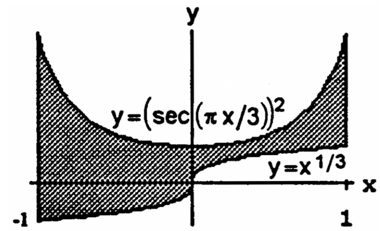
$= 4\left(1 - \frac{\pi}{4}\right) = 4 - \pi$



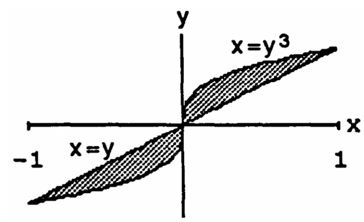
69. $c = 0, d = \frac{\pi}{2}$; $f(y) - g(y) = 3 \sin y \sqrt{\cos y} - 0 = 3 \sin y \sqrt{\cos y}$
$\Rightarrow A = 3\int_0^{\pi/2} \sin y \sqrt{\cos y}\, dy = -3\left[\frac{2}{3}(\cos y)^{3/2}\right]_0^{\pi/2}$
$= -2(0-1) = 2$



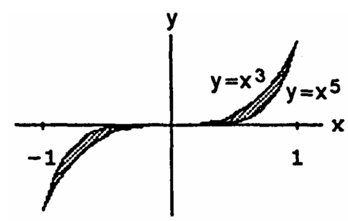
70. $a = -1, b = 1$; $f(x) - g(x) = \sec^2\left(\frac{\pi x}{3}\right) - x^{1/3}$
$\Rightarrow A = \int_{-1}^{1}\left[\sec^2\left(\frac{\pi x}{3}\right) - x^{1/3}\right] dx = \left[\frac{3}{\pi}\tan\left(\frac{\pi x}{3}\right) - \frac{3}{4}x^{4/3}\right]_{-1}^{1}$
$= \left(\frac{3}{\pi}\sqrt{3} - \frac{3}{4}\right) - \left[\frac{3}{\pi}\left(-\sqrt{3}\right) - \frac{3}{4}\right] = \frac{6\sqrt{3}}{\pi}$



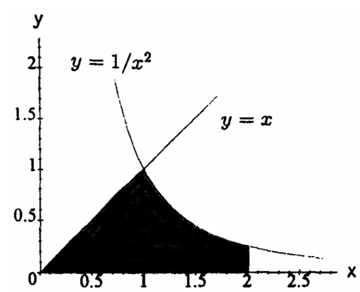
71. $A = A_1 + A_2$
Limits of integration: $x = y^3$ and $x = y \Rightarrow y = y^3$
$\Rightarrow y^3 - y = 0 \Rightarrow y(y-1)(y+1) = 0 \Rightarrow c_1 = -1, d_1 = 0$
and $c_2 = 0, d_2 = 1$; $f_1(y) - g_1(y) = y^3 - y$ and
$f_2(y) - g_2(y) = y - y^3 \Rightarrow$ by symmetry about the origin,
$A_1 + A_2 = 2A_2 \Rightarrow A = 2\int_0^1 (y - y^3)\, dy = 2\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1$
$= 2\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}$



72. $A = A_1 + A_2$
Limits of integration: $y = x^3$ and $y = x^5 \Rightarrow x^3 = x^5$
$\Rightarrow x^5 - x^3 = 0 \Rightarrow x^3(x-1)(x+1) = 0 \Rightarrow a_1 = -1, b_1 = 0$
and $a_2 = 0, b_2 = 1$; $f_1(x) - g_1(x) = x^3 - x^5$ and
$f_2(x) - g_2(x) = x^5 - x^3 \Rightarrow$ by symmetry about the origin,
$A_1 + A_2 = 2A_2 \Rightarrow A = 2\int_0^1 (x^3 - x^5)\, dx = 2\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1$
$= 2\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{1}{6}$



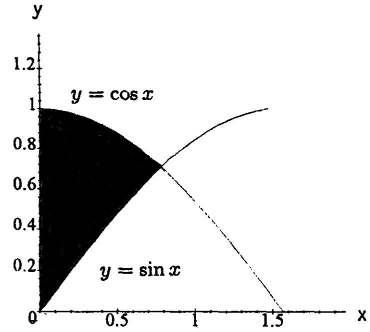
73. $A = A_1 + A_2$
Limits of integration: $y = x$ and $y = \frac{1}{x^2} \Rightarrow x = \frac{1}{x^2}, x \neq 0$
$\Rightarrow x^3 = 1 \Rightarrow x = 1$, $f_1(x) - g_1(x) = x - 0 = x$
$\Rightarrow A_1 = \int_0^1 x\, dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$; $f_2(x) - g_2(x) = \frac{1}{x^2} - 0$
$= x^{-2} \Rightarrow A_2 = \int_1^2 x^{-2}\, dx = \left[\frac{-1}{x}\right]_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$;
$A = A_1 + A_2 = \frac{1}{2} + \frac{1}{2} = 1$



74. Limits of integration: $\sin x = \cos x \Rightarrow x = \frac{\pi}{4} \Rightarrow a = 0$ and $b = \frac{\pi}{4}$; $f(x) - g(x) = \cos x - \sin x$

$\Rightarrow A = \int_0^{\pi/4} (\cos x - \sin x)\, dx = [\sin x + \cos x]_0^{\pi/4}$

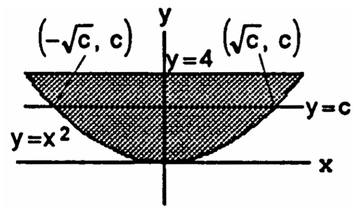
$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1) = \sqrt{2} - 1$



75. (a) The coordinates of the points of intersection of the line and parabola are $c = x^2 \Rightarrow x = \pm\sqrt{c}$ and $y = c$

(b) $f(y) - g(y) = \sqrt{y} - \left(-\sqrt{y}\right) = 2\sqrt{y} \Rightarrow$ the area of the lower section is, $A_L = \int_0^c [f(y) - g(y)]\, dy$

$= 2\int_0^c \sqrt{y}\, dy = 2\left[\frac{2}{3} y^{3/2}\right]_0^c = \frac{4}{3} c^{3/2}$. The area of the entire shaded region can be found by setting $c = 4$: $A = \left(\frac{4}{3}\right) 4^{3/2} = \frac{4 \cdot 8}{3} = \frac{32}{3}$. Since we want c to divide the region into subsections of equal area we have $A = 2A_L \Rightarrow \frac{32}{3} = 2\left(\frac{4}{3} c^{3/2}\right) \Rightarrow c = 4^{2/3}$



(c) $f(x) - g(x) = c - x^2 \Rightarrow A_L = \int_{-\sqrt{c}}^{\sqrt{c}} [f(x) - g(x)]\, dx = \int_{-\sqrt{c}}^{\sqrt{c}} (c - x^2)\, dx = \left[cx - \frac{x^3}{3}\right]_{-\sqrt{c}}^{\sqrt{c}} = 2\left[c^{3/2} - \frac{c^{3/2}}{3}\right]$

$= \frac{4}{3} c^{3/2}$. Again, the area of the whole shaded region can be found by setting $c = 4 \Rightarrow A = \frac{32}{3}$. From the condition $A = 2A_L$, we get $\frac{4}{3} c^{3/2} = \frac{32}{3} \Rightarrow c = 4^{2/3}$ as in part (b).

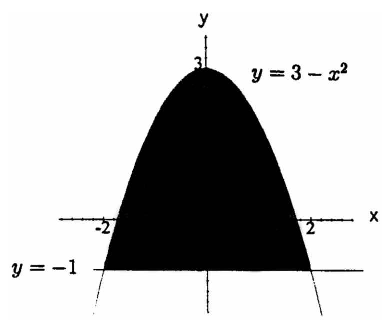
76. (a) Limits of integration: $y = 3 - x^2$ and $y = -1$

$\Rightarrow 3 - x^2 = -1 \Rightarrow x^2 = 4 \Rightarrow a = -2$ and $b = 2$;

$f(x) - g(x) = (3 - x^2) - (-1) = 4 - x^2$

$\Rightarrow A = \int_{-1}^{2} (4 - x^2)\, dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$

$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = 16 - \frac{16}{3} = \frac{32}{3}$



(b) Limits of integration: let $x = 0$ in $y = 3 - x^2$

$\Rightarrow y = 3$; $f(y) - g(y) = \sqrt{3 - y} - \left(-\sqrt{3 - y}\right)$

$= 2(3 - y)^{1/2}$

$\Rightarrow A = 2\int_{-1}^{3} (3 - y)^{1/2}\, dy = -2\int_{-1}^{3} (3 - y)^{1/2}(-1)\, dy = (-2)\left[\frac{2(3-y)^{3/2}}{3}\right]_{-1}^{3} = \left(-\frac{4}{3}\right)\left[0 - (3 + 1)^{3/2}\right]$

$= \left(\frac{4}{3}\right)(8) = \frac{32}{3}$

77. Limits of integration: $y = 1 + \sqrt{x}$ and $y = \frac{2}{\sqrt{x}}$

$\Rightarrow 1 + \sqrt{x} = \frac{2}{\sqrt{x}},\ x \neq 0 \Rightarrow \sqrt{x} + x = 2 \Rightarrow x = (2 - x)^2$

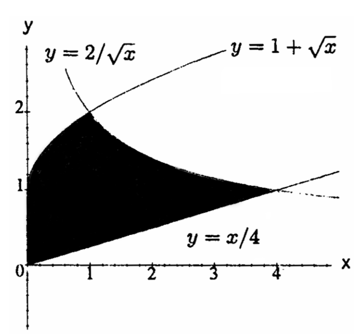
$\Rightarrow x = 4 - 4x + x^2 \Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow (x - 4)(x - 1) = 0 \Rightarrow x = 1, 4$ (but $x = 4$ does not satisfy the equation); $y = \frac{2}{\sqrt{x}}$ and $y = \frac{x}{4} \Rightarrow \frac{2}{\sqrt{x}} = \frac{x}{4}$

$\Rightarrow 8 = x\sqrt{x} \Rightarrow 64 = x^3 \Rightarrow x = 4.$

Therefore, AREA $= A_1 + A_2$: $f_1(x) - g_1(x) = \left(1 + x^{1/2}\right) - \frac{x}{4}$

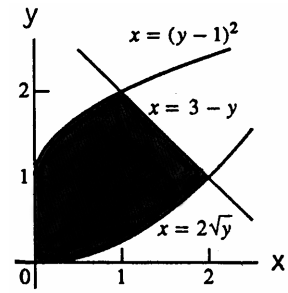
$\Rightarrow A_1 = \int_0^1 \left(1 + x^{1/2} - \frac{x}{4}\right) dx = \left[x + \frac{2}{3} x^{3/2} - \frac{x^2}{8}\right]_0^1$



$= \left(1 + \frac{2}{3} - \frac{1}{8}\right) - 0 = \frac{37}{24}$; $f_2(x) - g_2(x) = 2x^{-1/2} - \frac{x}{4} \Rightarrow A_2 = \int_1^4 \left(2x^{-1/2} - \frac{x}{4}\right) dx = \left[4x^{1/2} - \frac{x^2}{8}\right]_1^4$

$= \left(4 \cdot 2 - \frac{16}{8}\right) - \left(4 - \frac{1}{8}\right) = 4 - \frac{15}{8} = \frac{17}{8}$; Therefore, AREA $= A_1 + A_2 = \frac{37}{24} + \frac{17}{8} = \frac{37+51}{24} = \frac{88}{24} = \frac{11}{3}$

78. Limits of integration: $(y-1)^2 = 3-y \Rightarrow y^2-2y+1 = 3-y \Rightarrow y^2-y-2=0 \Rightarrow (y-2)(y+1)=0 \Rightarrow y=2$ since $y>0$; also, $2\sqrt{y} = 3-y \Rightarrow 4y = 9-6y+y^2 \Rightarrow y^2-10y+9=0 \Rightarrow (y-9)(y-1)=0 \Rightarrow y=1$ since $y=9$ does not satisfy the equation;



AREA $= A_1 + A_2$

$f_1(y) - g_1(y) = 2\sqrt{y} - 0 = 2y^{1/2}$

$\Rightarrow A_1 = 2\int_0^1 y^{1/2}\,dy = 2\left[\frac{2y^{3/2}}{3}\right]_0^1 = \frac{4}{3}$; $f_2(y) - g_2(y) = (3-y)-(y-1)^2$

$\Rightarrow A_2 = \int_1^2 \left[3-y-(y-1)^2\right] dy = \left[3y - \frac{1}{2}y^2 - \frac{1}{3}(y-1)^3\right]_1^2 = \left(6-2-\frac{1}{3}\right) - \left(3-\frac{1}{2}+0\right) = 1-\frac{1}{3}+\frac{1}{2} = \frac{7}{6}$;

Therefore, $A_1 + A_2 = \frac{4}{3} + \frac{7}{6} = \frac{15}{6} = \frac{5}{2}$

79. Area between parabola and $y = a^2$: $A = 2\int_0^a (a^2 - x^2)\,dx = 2\left[a^2x - \frac{1}{3}x^3\right]_0^a = 2\left(a^3 - \frac{a^3}{3}\right) - 0 = \frac{4a^3}{3}$;

Area of triangle AOC: $\frac{1}{2}(2a)(a^2) = a^3$; limit of ratio $= \lim_{a\to 0^+} \frac{a^3}{\left(\frac{4a^3}{3}\right)} = \frac{3}{4}$ which is independent of a.

80. $A = \int_a^b 2f(x)\,dx - \int_a^b f(x)\,dx = 2\int_a^b f(x)\,dx - \int_a^b f(x)\,dx = \int_a^b f(x)\,dx = 4$

81. Neither one; they are both zero. Neither integral takes into account the changes in the formulas for the region's upper and lower bounding curves at $x = 0$. The area of the shaded region is actually
$A = \int_{-1}^0 [-x-(x)]\,dx + \int_0^1 [x-(-x)]\,dx = \int_{-1}^0 -2x\,dx + \int_0^1 2x\,dx = 2.$

82. It is sometimes true. It is true if $f(x) \ge g(x)$ for all x between a and b. Otherwise it is false. If the graph of f lies below the graph of g for a portion of the interval of integration, the integral over that portion will be negative and the integral over $[a, b]$ will be less than the area between the curves (see Exercise 53).

83. Let $u = 2x \Rightarrow du = 2\,dx \Rightarrow \frac{1}{2}du = dx$; $x = 1 \Rightarrow u = 2$, $x = 3 \Rightarrow u = 6$

$\int_1^3 \frac{\sin 2x}{x}\,dx = \int_2^6 \frac{\sin u}{\left(\frac{u}{2}\right)}\left(\frac{1}{2}du\right) = \int_2^6 \frac{\sin u}{u}\,du = [F(u)]_2^6 = F(6) - F(2)$

84. Let $u = 1-x \Rightarrow du = -dx \Rightarrow -du = dx$; $x = 0 \Rightarrow u = 1$, $x = 1 \Rightarrow u = 0$

$\int_0^1 f(1-x)\,dx = \int_1^0 f(u)\,(-du) = -\int_1^0 f(u)\,du = \int_0^1 f(u)\,du = \int_0^1 f(x)\,dx$

85. (a) Let $u = -x \Rightarrow du = -dx$; $x = -1 \Rightarrow u = 1$, $x = 0 \Rightarrow u = 0$

f odd $\Rightarrow f(-x) = -f(x)$. Then $\int_{-1}^0 f(x)\,dx = \int_1^0 f(-u)\,(-du) = \int_1^0 -f(u)\,(-du) = \int_1^0 f(u)\,du = -\int_0^1 f(u)\,du = -3$

(b) Let $u = -x \Rightarrow du = -dx$; $x = -1 \Rightarrow u = 1$, $x = 0 \Rightarrow u = 0$

f even $\Rightarrow f(-x) = f(x)$. Then $\int_{-1}^0 f(x)\,dx = \int_1^0 f(-u)\,(-du) = -\int_1^0 f(u)\,du = \int_0^1 f(u)\,du = 3$

86. (a) Consider $\int_{-a}^0 f(x)\,dx$ when f is odd. Let $u = -x \Rightarrow du = -dx \Rightarrow -du = dx$ and $x = -a \Rightarrow u = a$ and $x = 0 \Rightarrow u = 0$. Thus $\int_{-a}^0 f(x)\,dx = \int_a^0 -f(-u)\,du = \int_a^0 f(u)\,du = -\int_0^a f(u)\,du = -\int_0^a f(x)\,dx$.

Thus $\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx = 0.$

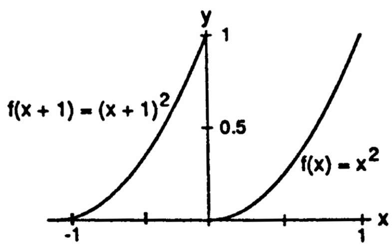
(b) $\int_{-\pi/2}^{\pi/2} \sin x\, dx = [-\cos x]_{-\pi/2}^{\pi/2} = -\cos\left(\frac{\pi}{2}\right) + \cos\left(-\frac{\pi}{2}\right) = 0 + 0 = 0.$

87. Let $u = a - x \Rightarrow du = -dx$; $x = 0 \Rightarrow u = a$, $x = a \Rightarrow u = 0$
$I = \int_0^a \frac{f(x)\,dx}{f(x)+f(a-x)} = \int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}(-du) = \int_0^a \frac{f(a-u)\,du}{f(u)+f(a-u)} = \int_0^a \frac{f(a-x)\,dx}{f(x)+f(a-x)}$
$\Rightarrow I + I = \int_0^a \frac{f(x)\,dx}{f(x)+f(a-x)} + \int_0^a \frac{f(a-x)\,dx}{f(x)+f(a-x)} = \int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)}\,dx = \int_0^a dx = [x]_0^a = a - 0 = a.$
Therefore, $2I = a \Rightarrow I = \frac{a}{2}$.

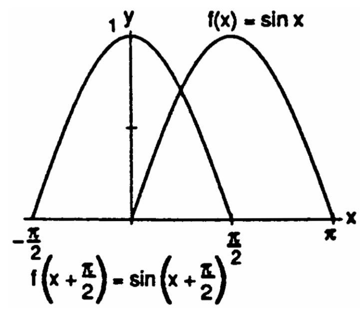
88. Let $u = \frac{xy}{t} \Rightarrow du = -\frac{xy}{t^2}\,dt \Rightarrow -\frac{t}{xy}\,du = \frac{1}{t}\,dt \Rightarrow -\frac{1}{u}\,du = \frac{1}{t}\,dt$; $t = x \Rightarrow u = y$, $t = xy \Rightarrow u = 1$. Therefore,
$\int_x^{xy} \frac{1}{t}\,dt = \int_y^1 -\frac{1}{u}\,du = -\int_y^1 \frac{1}{u}\,du = \int_1^y \frac{1}{u}\,du = \int_1^y \frac{1}{t}\,dt$

89. Let $u = x + c \Rightarrow du = dx$; $x = a - c \Rightarrow u = a$, $x = b - c \Rightarrow u = b$
$\int_{a-c}^{b-c} f(x+c)\,dx = \int_a^b f(u)\,du = \int_a^b f(x)\,dx$

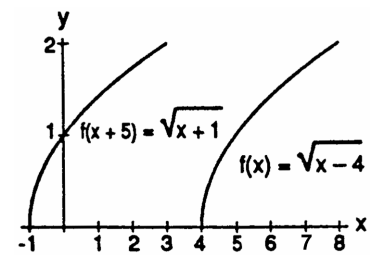
90. (a)



(b)



(c)



91-94. Example CAS commands:

Maple:

```
f := x -> x^3/3-x^2/2-2*x+1/3;
g := x -> x-1;
plot( [f(x),g(x)], x=-5..5, legend=["y = f(x)","y = g(x)"], title="#91(a) (Section 5.6)" );
q1 := [ -5, -2, 1, 4 ];                    # (b)
q2 := [seq( fsolve( f(x)=g(x), x=q1[i]..q1[i+1] ), i=1..nops(q1)-1 )];
for i from 1 to nops(q2)-1 do       # (c)
  area[i] := int( abs(f(x)-g(x)),x=q2[i]..q2[i+1] );
end do;
add( area[i], i=1..nops(q2)-1 );     # (d)
```

Mathematica: (assigned functions may vary)

```
Clear[x, f, g]
f[x_] = x^2 Cos[x]
g[x_] = x^3 − x
Plot[{f[x], g[x]}, {x, −2, 2}]
```

After examining the plots, the initial guesses for FindRoot can be determined.

```
pts = x/.Map[FindRoot[f[x]==g[x],{x, #}]&, {−1, 0, 1}]
i1=NIntegrate[f[x] − g[x], {x, pts[[1]], pts[[2]]}]
i2=NIntegrate[f[x] − g[x], {x, pts[[2]], pts[[3]]}]
i1 + i2
```

CHAPTER 8 TECHNIQUES OF INTEGRATION

8.1 BASIC INTEGRATION FORMULAS

1. $\int \frac{16x\,dx}{\sqrt{8x^2+1}}$; $\begin{bmatrix} u = 8x^2+1 \\ du = 16x\,dx \end{bmatrix}$ $\rightarrow \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{8x^2+1} + C$

2. $\int \frac{3\cos x\,dx}{\sqrt{1+3\sin x}}$; $\begin{bmatrix} u = 1+3\sin x \\ du = 3\cos x\,dx \end{bmatrix}$ $\rightarrow \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{1+3\sin x} + C$

3. $\int 3\sqrt{\sin v}\cos v\,dv$; $\begin{bmatrix} u = \sin v \\ du = \cos v\,dv \end{bmatrix}$ $\rightarrow \int 3\sqrt{u}\,du = 3 \cdot \frac{2}{3}u^{3/2} + C = 2(\sin v)^{3/2} + C$

4. $\int \cot^3 y \csc^2 y\,dy$; $\begin{bmatrix} u = \cot y \\ du = -\csc^2 y\,dy \end{bmatrix}$ $\rightarrow \int u^3(-du) = -\frac{u^4}{4} + C = \frac{-\cot^4 y}{4} + C$

5. $\int_0^1 \frac{16x\,dx}{8x^2+2}$; $\begin{bmatrix} u = 8x^2+2 \\ du = 16x\,dx \\ x = 0 \Rightarrow u = 2,\quad x = 1 \Rightarrow u = 10 \end{bmatrix}$ $\rightarrow \int_2^{10} \frac{du}{u} = [\ln|u|]_2^{10} = \ln 10 - \ln 2 = \ln 5$

6. $\int_{\pi/4}^{\pi/3} \frac{\sec^2 z\,dz}{\tan z}$; $\begin{bmatrix} u = \tan z \\ du = \sec^2 z\,dz \\ z = \frac{\pi}{4} \Rightarrow u = 1, z = \frac{\pi}{3} \Rightarrow u = \sqrt{3} \end{bmatrix}$ $\rightarrow \int_1^{\sqrt{3}} \frac{1}{u}\,du = [\ln|u|]_1^{\sqrt{3}} = \ln\sqrt{3} - \ln 1 = \ln\sqrt{3}$

7. $\int \frac{dx}{\sqrt{x}\left(\sqrt{x}+1\right)}$; $\begin{bmatrix} u = \sqrt{x}+1 \\ du = \frac{1}{2\sqrt{x}}\,dx \\ 2\,du = \frac{dx}{\sqrt{x}} \end{bmatrix}$ $\rightarrow \int \frac{2\,du}{u} = 2\ln|u| + C = 2\ln\left(\sqrt{x}+1\right) + C$

8. $\int \frac{dx}{x-\sqrt{x}} = \int \frac{dx}{\sqrt{x}\left(\sqrt{x}-1\right)}$; $\begin{bmatrix} u = \sqrt{x}-1 \\ du = \frac{1}{2\sqrt{x}}\,dx \\ 2\,du = \frac{dx}{\sqrt{x}} \end{bmatrix}$ $\rightarrow \int \frac{2\,du}{u} = 2\ln|u| + C = 2\ln\left|\sqrt{x}-1\right| + C$

9. $\int \cot(3-7x)\,dx$; $\begin{bmatrix} u = 3-7x \\ du = -7\,dx \end{bmatrix}$ $\rightarrow -\frac{1}{7}\int \cot u\,du = -\frac{1}{7}\ln|\sin u| + C = -\frac{1}{7}\ln|\sin(3-7x)| + C$

10. $\int \csc(\pi x - 1)\,dx$; $\begin{bmatrix} u = \pi x - 1 \\ du = \pi\,dx \end{bmatrix}$ $\rightarrow \int \csc u \cdot \frac{du}{\pi} = \frac{-1}{\pi}\ln|\csc u + \cot u| + C$
$= -\frac{1}{\pi}\ln|\csc(\pi x - 1) + \cot(\pi x - 1)| + C$

11. $\int e^\theta \csc\left(e^\theta + 1\right)\,d\theta$; $\begin{bmatrix} u = e^\theta + 1 \\ du = e^\theta\,d\theta \end{bmatrix}$ $\rightarrow \int \csc u\,du = -\ln|\csc u + \cot u| + C = -\ln\left|\csc\left(e^\theta+1\right) + \cot\left(e^\theta+1\right)\right| + C$

12. $\int \frac{\cot(3+\ln x)}{x}\,dx$; $\begin{bmatrix} u = 3+\ln x \\ du = \frac{dx}{x} \end{bmatrix}$ $\rightarrow \int \cot u\,du = \ln|\sin u| + C = \ln|\sin(3+\ln x)| + C$

13. $\int \sec \frac{t}{3}\, dt;\ \begin{bmatrix} u = \frac{t}{3} \\ du = \frac{dt}{3} \end{bmatrix} \rightarrow \int 3 \sec u\, du = 3 \ln |\sec u + \tan u| + C = 3 \ln \left|\sec \frac{t}{3} + \tan \frac{t}{3}\right| + C$

14. $\int x \sec (x^2 - 5)\, dx;\ \begin{bmatrix} u = x^2 - 5 \\ du = 2x\, dx \end{bmatrix} \rightarrow \int \frac{1}{2} \sec u\, du = \frac{1}{2} \ln |\sec u + \tan u| + C$
$= \frac{1}{2} \ln |\sec (x^2 - 5) + \tan (x^2 - 5)| + C$

15. $\int \csc (s - \pi)\, ds;\ \begin{bmatrix} u = s - \pi \\ du = ds \end{bmatrix} \rightarrow \int \csc u\, du = -\ln |\csc u + \cot u| + C = -\ln |\csc (s - \pi) + \cot (s - \pi)| + C$

16. $\int \frac{1}{\theta^2} \csc \frac{1}{\theta}\, d\theta;\ \begin{bmatrix} u = \frac{1}{\theta} \\ du = \frac{-d\theta}{\theta^2} \end{bmatrix} \rightarrow \int -\csc u\, du = \ln |\csc u + \cot u| + C = \ln \left|\csc \frac{1}{\theta} + \cot \frac{1}{\theta}\right| + C$

17. $\int_0^{\sqrt{\ln 2}} 2xe^{x^2}\, dx;\ \begin{bmatrix} u = x^2 \\ du = 2x\, dx \\ x = 0 \Rightarrow u = 0,\ x = \sqrt{\ln 2} \Rightarrow u = \ln 2 \end{bmatrix} \rightarrow \int_0^{\ln 2} e^u\, du = [e^u]_0^{\ln 2} = e^{\ln 2} - e^0 = 2 - 1 = 1$

18. $\int_{\pi/2}^{\pi} \sin (y)\, e^{\cos y}\, dy;\ \begin{bmatrix} u = \cos y \\ du = -\sin y\, dy \\ y = \frac{\pi}{2} \Rightarrow u = 0,\ y = \pi \Rightarrow u = -1 \end{bmatrix} \rightarrow \int_0^{-1} -e^u\, du = \int_{-1}^{0} e^u\, du = [e^u]_{-1}^{0} = 1 - e^{-1} = \frac{e-1}{e}$

19. $\int e^{\tan v} \sec^2 v\, dv;\ \begin{bmatrix} u = \tan v \\ du = \sec^2 v\, dv \end{bmatrix} \rightarrow \int e^u\, du = e^u + C = e^{\tan v} + C$

20. $\int \frac{e^{\sqrt{t}}\, dt}{\sqrt{t}};\ \begin{bmatrix} u = \sqrt{t} \\ du = \frac{dt}{2\sqrt{t}} \end{bmatrix} \rightarrow \int 2e^u\, du = 2e^u + C = 2e^{\sqrt{t}} + C$

21. $\int 3^{x+1}\, dx;\ \begin{bmatrix} u = x + 1 \\ du = dx \end{bmatrix} \rightarrow \int 3^u\, du = \left(\frac{1}{\ln 3}\right) 3^u + C = \frac{3^{(x+1)}}{\ln 3} + C$

22. $\int \frac{2^{\ln x}}{x}\, dx;\ \begin{bmatrix} u = \ln x \\ du = \frac{dx}{x} \end{bmatrix} \rightarrow \int 2^u\, du = \frac{2^u}{\ln 2} + C = \frac{2^{\ln x}}{\ln 2} + C$

23. $\int \frac{2^{\sqrt{w}}\, dw}{2\sqrt{w}};\ \begin{bmatrix} u = \sqrt{w} \\ du = \frac{dw}{2\sqrt{w}} \end{bmatrix} \rightarrow \int 2^u\, du = \frac{2^u}{\ln 2} + C = \frac{2^{\sqrt{w}}}{\ln 2} + C$

24. $\int 10^{2\theta}\, d\theta;\ \begin{bmatrix} u = 2\theta \\ du = 2\, d\theta \end{bmatrix} \rightarrow \int \frac{1}{2}\, 10^u\, du = \frac{10^u}{2 \ln 10} + C = \frac{1}{2}\left(\frac{10^{2\theta}}{\ln 10}\right) + C$

25. $\int \frac{9\, du}{1 + 9u^2};\ \begin{bmatrix} x = 3u \\ dx = 3\, du \end{bmatrix} \rightarrow \int \frac{3\, dx}{1 + x^2} = 3 \tan^{-1} x + C = 3 \tan^{-1} 3u + C$

26. $\int \frac{4\, dx}{1 + (2x + 1)^2};\ \begin{bmatrix} u = 2x + 1 \\ du = 2\, dx \end{bmatrix} \rightarrow \int \frac{2\, du}{1 + u^2} = 2 \tan^{-1} u + C = 2 \tan^{-1} (2x + 1) + C$

27. $\int_0^{1/6} \frac{dx}{\sqrt{1 - 9x^2}};\ \begin{bmatrix} u = 3x \\ du = 3\, dx \\ x = 0 \Rightarrow u = 0,\ x = \frac{1}{6} \Rightarrow u = \frac{1}{2} \end{bmatrix} \rightarrow \int_0^{1/2} \frac{1}{3} \frac{du}{\sqrt{1 - u^2}} = \left[\frac{1}{3} \sin^{-1} u\right]_0^{1/2} = \frac{1}{3}\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{18}$

28. $\int_0^1 \frac{dt}{\sqrt{4-t^2}} = \left[\sin^{-1} \frac{t}{2}\right]_0^1 = \sin^{-1}\left(\frac{1}{2}\right) - 0 = \frac{\pi}{6}$

29. $\int \frac{2s\,ds}{\sqrt{1-s^4}}$; $\begin{bmatrix} u = s^2 \\ du = 2s\,ds \end{bmatrix} \rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1} s^2 + C$

30. $\int \frac{2\,dx}{x\sqrt{1-4\ln^2 x}}$; $\begin{bmatrix} u = 2\ln x \\ du = \frac{2\,dx}{x} \end{bmatrix} \rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(2\ln x) + C$

31. $\int \frac{6\,dx}{x\sqrt{25x^2-1}} = \int \frac{6\,dx}{5x\sqrt{x^2-\frac{1}{25}}} = \frac{6}{5} \cdot 5\sec^{-1}|5x| + C = 6\sec^{-1}|5x| + C$

32. $\int \frac{dr}{r\sqrt{r^2-9}} = \frac{1}{3}\sec^{-1}\left|\frac{r}{3}\right| + C$

33. $\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x\,dx}{e^{2x}+1}$; $\begin{bmatrix} u = e^x \\ du = e^x\,dx \end{bmatrix} \rightarrow \int \frac{du}{u^2+1} = \tan^{-1} u + C = \tan^{-1} e^x + C$

34. $\int \frac{dy}{\sqrt{e^{2y}-1}} = \int \frac{e^y\,dy}{e^y\sqrt{(e^y)^2-1}}$; $\begin{bmatrix} u = e^y \\ du = e^y\,dy \end{bmatrix} \rightarrow \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C = \sec^{-1} e^y + C$

35. $\int_1^{e^{\pi/3}} \frac{dx}{x\cos(\ln x)}$; $\begin{bmatrix} u = \ln x \\ du = \frac{dx}{x} \\ x = 1 \Rightarrow u = 0,\ x = e^{\pi/3} \Rightarrow u = \frac{\pi}{3} \end{bmatrix} \rightarrow \int_0^{\pi/3} \frac{du}{\cos u} = \int_0^{\pi/3} \sec u\,du = [\ln|\sec u + \tan u|]_0^{\pi/3}$
$= \ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \ln|\sec 0 + \tan 0| = \ln\left(2+\sqrt{3}\right) - \ln(1) = \ln\left(2+\sqrt{3}\right)$

36. $\int \frac{\ln x\,dx}{x + 4x\ln^2 x} = \int \frac{\ln x\,dx}{x(1+4\ln^2 x)}$; $\begin{bmatrix} u = \ln^2 x \\ du = \frac{2}{x}\ln x\,dx \end{bmatrix} \rightarrow \int \frac{1}{2}\,\frac{du}{1+4u} = \frac{1}{8}\ln|1+4u| + C = \frac{1}{8}\ln(1+4\ln^2 x) + C$

37. $\int_1^2 \frac{8\,dx}{x^2-2x+2} = 8\int_1^2 \frac{dx}{1+(x-1)^2}$; $\begin{bmatrix} u = x-1 \\ du = dx \\ x = 1 \Rightarrow u = 0,\ x = 2 \Rightarrow u = 1 \end{bmatrix} \rightarrow 8\int_0^1 \frac{du}{1+u^2} = 8\left[\tan^{-1} u\right]_0^1$
$= 8\left(\tan^{-1} 1 - \tan^{-1} 0\right) = 8\left(\frac{\pi}{4} - 0\right) = 2\pi$

38. $\int_2^4 \frac{2\,dx}{x^2-6x+10} = 2\int_2^4 \frac{dx}{(x-3)^2+1}$; $\begin{bmatrix} u = x-3 \\ du = dx \\ x = 2 \Rightarrow u = -1,\ x = 4 \Rightarrow u = 1 \end{bmatrix} \rightarrow 2\int_{-1}^1 \frac{du}{u^2+1} = 2\left[\tan^{-1} u\right]_{-1}^1$
$= 2\left[\tan^{-1} 1 - \tan^{-1}(-1)\right] = 2\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \pi$

39. $\int \frac{dt}{\sqrt{-t^2+4t-3}} = \int \frac{dt}{\sqrt{1-(t-2)^2}}$; $\begin{bmatrix} u = t-2 \\ du = dt \end{bmatrix} \rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(t-2) + C$

40. $\int \frac{d\theta}{\sqrt{2\theta-\theta^2}} = \int \frac{d\theta}{\sqrt{1-(\theta-1)^2}}$; $\begin{bmatrix} u = \theta-1 \\ du = d\theta \end{bmatrix} \rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(\theta-1) + C$

41. $\int \frac{dx}{(x+1)\sqrt{x^2+2x}} = \int \frac{dx}{(x+1)\sqrt{(x+1)^2-1}}$; $\begin{bmatrix} u = x+1 \\ du = dx \end{bmatrix} \rightarrow \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C = \sec^{-1}|x+1| + C,$
$|u| = |x+1| > 1$

42. $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}} = \int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}}$; $\begin{bmatrix} u = x-2 \\ du = dx \end{bmatrix} \rightarrow \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C$
$= \sec^{-1}|x-2| + C, |u| = |x-2| > 1$

43. $\int (\sec x + \cot x)^2\, dx = \int (\sec^2 x + 2 \sec x \cot x + \cot^2 x)\, dx = \int \sec^2 x\, dx + \int 2 \csc x\, dx + \int (\csc^2 x - 1)\, dx$
$= \tan x - 2 \ln|\csc x + \cot x| - \cot x - x + C$

44. $\int (\csc x - \tan x)^2\, dx = \int (\csc^2 x - 2 \csc x \tan x + \tan^2 x)\, dx = \int \csc^2 x\, dx - \int 2 \sec x\, dx + \int (\sec^2 x - 1)\, dx$
$= -\cot x - 2 \ln|\sec x + \tan x| + \tan x - x + C$

45. $\int \csc x \sin 3x\, dx = \int (\csc x)(\sin 2x \cos x + \sin x \cos 2x)\, dx = \int (\csc x)\,(2 \sin x \cos^2 x + \sin x \cos 2x)\, dx$
$= \int (2 \cos^2 x + \cos 2x)\, dx = \int [(1 + \cos 2x) + \cos 2x]\, dx = \int (1 + 2 \cos 2x)\, dx = x + \sin 2x + C$

46. $\int (\sin 3x \cos 2x - \cos 3x \sin 2x)\, dx = \int \sin(3x - 2x)\, dx = \int \sin x\, dx = -\cos x + C$

47. $\int \frac{x}{x+1}\, dx = \int \left(1 - \frac{1}{x+1}\right) dx = x - \ln|x+1| + C$

48. $\int \frac{x^2}{x^2+1}\, dx = \int \left(1 - \frac{1}{x^2+1}\right) dx = x - \tan^{-1} x + C$

49. $\int_{\sqrt{2}}^{3} \frac{2x^3}{x^2-1}\, dx = \int_{\sqrt{2}}^{3} \left(2x + \frac{2x}{x^2-1}\right) dx = \left[x^2 + \ln|x^2-1|\right]_{\sqrt{2}}^{3} = (9 + \ln 8) - (2 + \ln 1) = 7 + \ln 8$

50. $\int_{-1}^{3} \frac{4x^2-7}{2x+3}\, dx = \int_{-1}^{3} \left[(2x-3) + \frac{2}{2x+3}\right] dx = \left[x^2 - 3x + \ln|2x+3|\right]_{-1}^{3} = (9 - 9 + \ln 9) - (1 + 3 + \ln 1) = \ln 9 - 4$

51. $\int \frac{4t^3 - t^2 + 16t}{t^2+4}\, dt = \int \left[(4t-1) + \frac{4}{t^2+4}\right] dt = 2t^2 - t + 2 \tan^{-1}\left(\frac{t}{2}\right) + C$

52. $\int \frac{2\theta^3 - 7\theta^2 + 7\theta}{2\theta - 5}\, d\theta = \int \left[(\theta^2 - \theta + 1) + \frac{5}{2\theta - 5}\right] d\theta = \frac{\theta^3}{3} - \frac{\theta^2}{2} + \theta + \frac{5}{2} \ln|2\theta - 5| + C$

53. $\int \frac{1-x}{\sqrt{1-x^2}}\, dx = \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x\, dx}{\sqrt{1-x^2}} = \sin^{-1} x + \sqrt{1-x^2} + C$

54. $\int \frac{x + 2\sqrt{x-1}}{2x\sqrt{x-1}}\, dx = \int \frac{dx}{2\sqrt{x-1}} + \int \frac{dx}{x} = (x-1)^{1/2} + \ln|x| + C$

55. $\int_0^{\pi/4} \frac{1 + \sin x}{\cos^2 x}\, dx = \int_0^{\pi/4} (\sec^2 x + \sec x \tan x)\, dx = [\tan x + \sec x]_0^{\pi/4} = \left(1 + \sqrt{2}\right) - (0 + 1) = \sqrt{2}$

56. $\int_0^{1/2} \frac{2 - 8x}{1 + 4x^2}\, dx = \int_0^{1/2} \left(\frac{2}{1+4x^2} - \frac{8x}{1+4x^2}\right) dx = \left[\tan^{-1}(2x) - \ln|1 + 4x^2|\right]_0^{1/2}$
$= (\tan^{-1} 1 - \ln 2) - (\tan^{-1} 0 - \ln 1) = \frac{\pi}{4} - \ln 2$

57. $\int \frac{dx}{1 + \sin x} = \int \frac{(1 - \sin x)}{(1 - \sin^2 x)}\, dx = \int \frac{(1 - \sin x)}{\cos^2 x}\, dx = \int (\sec^2 x - \sec x \tan x)\, dx = \tan x - \sec x + C$

58. $1 + \cos x = 1 + \cos\left(2 \cdot \frac{x}{2}\right) = 2 \cos^2 \frac{x}{2} \Rightarrow \int \frac{dx}{1 + \cos x} = \int \frac{dx}{2 \cos^2\left(\frac{x}{2}\right)} = \frac{1}{2} \int \sec^2\left(\frac{x}{2}\right) dx = \tan \frac{x}{2} + C$

59. $\int \frac{1}{\sec\theta + \tan\theta}\, d\theta = \int d\theta;\ \begin{bmatrix} u = 1 + \sin\theta \\ du = \cos\theta\, d\theta \end{bmatrix} \rightarrow \int \frac{du}{u} = \ln|u| + C = \ln|1 + \sin\theta| + C$

60. $\int \frac{1}{\csc\theta+\cot\theta}\,d\theta = \int \frac{\sin\theta}{1+\cos\theta}\,d\theta;\ \begin{bmatrix} u = 1+\cos\theta \\ du = -\sin\theta\,d\theta \end{bmatrix} \to \int \frac{-du}{u} = -\ln|u| + C = -\ln|1+\cos\theta| + C$

61. $\int \frac{1}{1-\sec x}\,dx = \int \frac{\cos x}{\cos x - 1}\,dx = \int \left(1 + \frac{1}{\cos x - 1}\right) dx = \int \left(1 - \frac{1+\cos x}{\sin^2 x}\right) dx = \int \left(1 - \csc^2 x - \frac{\cos x}{\sin^2 x}\right) dx$
$= \int (1 - \csc^2 x - \csc x \cot x)\,dx = x + \cot x + \csc x + C$

62. $\int \frac{1}{1-\csc x}\,dx = \int \frac{\sin x}{\sin x - 1}\,dx = \int \left(1 + \frac{1}{\sin x - 1}\right) dx = \int \left(1 + \frac{\sin x + 1}{(\sin x - 1)(\sin x + 1)}\right) dx$
$= \int \left(1 - \frac{1+\sin x}{\cos^2 x}\right) dx = \int \left(1 - \sec^2 x - \frac{\sin x}{\cos^2 x}\right) dx = \int (1 - \sec^2 x - \sec x \tan x)\,dx = x - \tan x - \sec x + C$

63. $\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\,dx = \int_0^{2\pi} \left|\sin \frac{x}{2}\right| dx;\ \begin{bmatrix} \sin \frac{x}{2} \ge 0 \\ \text{for } 0 \le \frac{x}{2} \le 2\pi \end{bmatrix} \to \int_0^{2\pi} \sin\left(\frac{x}{2}\right) dx = \left[-2\cos \frac{x}{2}\right]_0^{2\pi} = -2(\cos\pi - \cos 0)$
$= (-2)(-2) = 4$

64. $\int_0^{\pi} \sqrt{1-\cos 2x}\,dx = \int_0^{\pi} \sqrt{2}\,|\sin x|\,dx;\ \begin{bmatrix} \sin x \ge 0 \\ \text{for } 0 \le x \le \pi \end{bmatrix} \to \sqrt{2}\int_0^{\pi} \sin x\,dx = \left[-\sqrt{2}\cos x\right]_0^{\pi}$
$= -\sqrt{2}(\cos\pi - \cos 0) = 2\sqrt{2}$

65. $\int_{\pi/2}^{\pi} \sqrt{1+\cos 2t}\,dt = \int_{\pi/2}^{\pi} \sqrt{2}\,|\cos t|\,dt;\ \begin{bmatrix} \cos t \le 0 \\ \text{for } \frac{\pi}{2} \le t \le \pi \end{bmatrix} \to \int_{\pi/2}^{\pi} -\sqrt{2}\cos t\,dt = \left[-\sqrt{2}\sin t\right]_{\pi/2}^{\pi}$
$= -\sqrt{2}\left(\sin\pi - \sin\frac{\pi}{2}\right) = \sqrt{2}$

66. $\int_{-\pi}^{0} \sqrt{1+\cos t}\,dt = \int_{-\pi}^{0} \sqrt{2}\left|\cos \frac{t}{2}\right| dt;\ \begin{bmatrix} \cos \frac{t}{2} \ge 0 \\ \text{for } -\pi \le t \le 0 \end{bmatrix} \to \int_{-\pi}^{0} \sqrt{2}\cos \frac{t}{2}\,dt = \left[2\sqrt{2}\sin \frac{t}{2}\right]_{-\pi}^{0}$
$= 2\sqrt{2}\left[\sin 0 - \sin\left(-\frac{\pi}{2}\right)\right] = 2\sqrt{2}$

67. $\int_{-\pi}^{0} \sqrt{1-\cos^2\theta}\,d\theta = \int_{-\pi}^{0} |\sin\theta|\,d\theta;\ \begin{bmatrix} \sin\theta \le 0 \\ \text{for } -\pi \le \theta \le 0 \end{bmatrix} \to \int_{-\pi}^{0} -\sin\theta\,d\theta = [\cos\theta]_{-\pi}^{0} = \cos 0 - \cos(-\pi)$
$= 1 - (-1) = 2$

68. $\int_{\pi/2}^{\pi} \sqrt{1-\sin^2\theta}\,d\theta = \int_{\pi/2}^{\pi} |\cos\theta|\,d\theta;\ \begin{bmatrix} \cos\theta \le 0 \\ \text{for } \frac{\pi}{2} \le \theta \le \pi \end{bmatrix} \to \int_{\pi/2}^{\pi} -\cos\theta\,d\theta = [-\sin\theta]_{\pi/2}^{\pi} = -\sin\pi + \sin\frac{\pi}{2} = 1$

69. $\int_{-\pi/4}^{\pi/4} \sqrt{\tan^2 y + 1}\,dy = \int_{-\pi/4}^{\pi/4} |\sec y|\,dy;\ \begin{bmatrix} \sec y \ge 0 \\ \text{for } -\frac{\pi}{4} \le y \le \frac{\pi}{4} \end{bmatrix} \to \int_{-\pi/4}^{\pi/4} \sec y\,dy = [\ln|\sec y + \tan y|]_{-\pi/4}^{\pi/4}$
$= \ln\left|\sqrt{2}+1\right| - \ln\left|\sqrt{2}-1\right|$

70. $\int_{-\pi/4}^{0} \sqrt{\sec^2 y - 1}\,dy = \int_{-\pi/4}^{0} |\tan y|\,dy;\ \begin{bmatrix} \tan y \le 0 \\ \text{for } -\frac{\pi}{4} \le y \le 0 \end{bmatrix} \to \int_{-\pi/4}^{0} -\tan y\,dy = [\ln|\cos y|]_{-\pi/4}^{0} = -\ln\left(\frac{1}{\sqrt{2}}\right)$
$= \ln\sqrt{2}$

71. $\int_{\pi/4}^{3\pi/4} (\csc x - \cot x)^2\,dx = \int_{\pi/4}^{3\pi/4} (\csc^2 x - 2\csc x \cot x + \cot^2 x)\,dx = \int_{\pi/4}^{3\pi/4} (2\csc^2 x - 1 - 2\csc x \cot x)\,dx$
$= [-2\cot x - x + 2\csc x]_{\pi/4}^{3\pi/4} = \left(-2\cot\frac{3\pi}{4} - \frac{3\pi}{4} + 2\csc\frac{3\pi}{4}\right) - \left(-2\cot\frac{\pi}{4} - \frac{\pi}{4} + 2\csc\frac{\pi}{4}\right)$
$= \left[-2(-1) - \frac{3\pi}{4} + 2\left(\sqrt{2}\right)\right] - \left[-2(1) - \frac{\pi}{4} + 2\left(\sqrt{2}\right)\right] = 4 - \frac{\pi}{2}$

72. $\int_0^{\pi/4} (\sec x + 4\cos x)^2\, dx = \int_0^{\pi/4} \left[\sec^2 x + 8 + 16\left(\frac{1+\cos 2x}{2}\right)\right] dx = [\tan x + 16x - 4\sin 2x]_0^{\pi/4}$
$= \left(\tan\frac{\pi}{4} + 4\pi - 4\sin\frac{\pi}{2}\right) - (\tan 0 + 0 - 4\sin 0) = 5 + 4\pi$

73. $\int \cos\theta \csc(\sin\theta)\, d\theta;\ \begin{bmatrix} u = \sin\theta \\ du = \cos\theta\, d\theta \end{bmatrix} \to \int \csc u\, du = -\ln|\csc u + \cot u| + C$
$= -\ln|\csc(\sin\theta) + \cot(\sin\theta)| + C$

74. $\int \left(1 + \frac{1}{x}\right)\cot(x + \ln x)\, dx;\ \begin{bmatrix} u = x + \ln x \\ du = \left(1 + \frac{1}{x}\right) dx \end{bmatrix} \to \int \cot u\, du = \ln|\sin u| + C = \ln|\sin(x + \ln x)| + C$

75. $\int (\csc x - \sec x)(\sin x + \cos x)\, dx = \int (1 + \cot x - \tan x - 1)\, dx = \int \cot x\, dx - \int \tan x\, dx$
$= \ln|\sin x| + \ln|\cos x| + C$

76. $\int 3\sinh\left(\frac{x}{2} + \ln 5\right) dx = \begin{bmatrix} u = \frac{x}{2} + \ln 5 \\ 2\, du = dx \end{bmatrix} = 6\int \sinh u\, du = 6\cosh u + C = 6\cosh\left(\frac{x}{2} + \ln 5\right) + C$

77. $\int \frac{6\, dy}{\sqrt{y}(1+y)};\ \begin{bmatrix} u = \sqrt{y} \\ du = \frac{1}{2\sqrt{y}}\, dy \end{bmatrix} \to \int \frac{12\, du}{1+u^2} = 12\tan^{-1} u + C = 12\tan^{-1}\sqrt{y} + C$

78. $\int \frac{dx}{x\sqrt{4x^2-1}} = \int \frac{2\, dx}{2x\sqrt{(2x)^2-1}};\ \begin{bmatrix} u = 2x \\ du = 2\, dx \end{bmatrix} \to \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C = \sec^{-1}|2x| + C$

79. $\int \frac{7\, dx}{(x-1)\sqrt{x^2-2x-48}} = \int \frac{7\, dx}{(x-1)\sqrt{(x-1)^2-49}};\ \begin{bmatrix} u = x - 1 \\ du = dx \end{bmatrix} \to \int \frac{7\, du}{u\sqrt{u^2-49}} = 7 \cdot \frac{1}{7}\sec^{-1}\left|\frac{u}{7}\right| + C$
$= \sec^{-1}\left|\frac{x-1}{7}\right| + C$

80. $\int \frac{dx}{(2x+1)\sqrt{4x^2+4x}} = \int \frac{dx}{(2x+1)\sqrt{(2x+1)^2-1}};\ \begin{bmatrix} u = 2x + 1 \\ du = 2\, dx \end{bmatrix} \to \int \frac{du}{2u\sqrt{u^2-1}} = \frac{1}{2}\sec^{-1}|u| + C$
$= \frac{1}{2}\sec^{-1}|2x+1| + C$

81. $\int \sec^2 t \tan(\tan t)\, dt;\ \begin{bmatrix} u = \tan t \\ du = \sec^2 t\, dt \end{bmatrix} \to \int \tan u\, du = -\ln|\cos u| + C = \ln|\sec u| + C = \ln|\sec(\tan t)| + C$

82. $\int \frac{dx}{x\sqrt{3+x^2}} = -\frac{1}{3}\operatorname{csch}^{-1}\left|\frac{x}{\sqrt{3}}\right| + C$

83. (a) $\int \cos^3\theta\, d\theta = \int (\cos\theta)(1 - \sin^2\theta)\, d\theta;\ \begin{bmatrix} u = \sin\theta \\ du = \cos\theta\, d\theta \end{bmatrix} \to \int (1 - u^2)\, du = u - \frac{u^3}{3} + C = \sin\theta - \frac{1}{3}\sin^3\theta + C$

(b) $\int \cos^5\theta\, d\theta = \int (\cos\theta)(1 - \sin^2\theta)^2\, d\theta = \int (1 - u^2)^2\, du = \int (1 - 2u^2 + u^4)\, du = u - \frac{2}{3}u^3 + \frac{u^5}{5} + C$
$= \sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta + C$

(c) $\int \cos^9\theta\, d\theta = \int (\cos^8\theta)(\cos\theta)\, d\theta = \int (1 - \sin^2\theta)^4(\cos\theta)\, d\theta$

84. (a) $\int \sin^3\theta\, d\theta = \int (1 - \cos^2\theta)(\sin\theta)\, d\theta;\ \begin{bmatrix} u = \cos\theta \\ du = -\sin\theta\, d\theta \end{bmatrix} \to \int (1 - u^2)(-du) = \frac{u^3}{3} - u + C$
$= -\cos\theta + \frac{1}{3}\cos^3\theta + C$

(b) $\int \sin^5\theta\, d\theta = \int (1 - \cos^2\theta)^2(\sin\theta)\, d\theta = \int (1 - u^2)^2(-du) = \int (-1 + 2u^2 - u^4)\, du$
$= -\cos\theta + \frac{2}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta + C$

(c) $\int \sin^7\theta\, d\theta = \int (1-u^2)^3(-du) = \int (-1+3u^2-3u^4+u^6)\, du = -\cos\theta + \cos^3\theta - \frac{3}{5}\cos^5\theta + \frac{\cos^7\theta}{7} + C$

(d) $\int \sin^{13}\theta\, d\theta = \int (\sin^{12}\theta)(\sin\theta)\, d\theta = \int (1-\cos^2\theta)^6(\sin\theta)\, d\theta$

85. (a) $\int \tan^3\theta\, d\theta = \int (\sec^2\theta - 1)(\tan\theta)\, d\theta = \int \sec^2\theta\tan\theta\, d\theta - \int \tan\theta\, d\theta = \frac{1}{2}\tan^2\theta - \int \tan\theta\, d\theta$
$= \frac{1}{2}\tan^2\theta + \ln|\cos\theta| + C$

(b) $\int \tan^5\theta\, d\theta = \int (\sec^2\theta - 1)(\tan^3\theta)\, d\theta = \int \tan^3\theta\sec^2\theta\, d\theta - \int \tan^3\theta\, d\theta = \frac{1}{4}\tan^4\theta - \int \tan^3\theta\, d\theta$

(c) $\int \tan^7\theta\, d\theta = \int (\sec^2\theta - 1)(\tan^5\theta)\, d\theta = \int \tan^5\theta\sec^2\theta\, d\theta - \int \tan^5\theta\, d\theta = \frac{1}{6}\tan^6\theta - \int \tan^5\theta\, d\theta$

(d) $\int \tan^{2k+1}\theta\, d\theta = \int (\sec^2\theta - 1)(\tan^{2k-1}\theta)\, d\theta = \int \tan^{2k-1}\theta\sec^2\theta\, d\theta - \int \tan^{2k-1}\theta\, d\theta$;

$\begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\, d\theta \end{bmatrix} \rightarrow \int u^{2k-1}\, du - \int \tan^{2k-1}\theta\, d\theta = \frac{1}{2k}u^{2k} - \int \tan^{2k-1}\theta\, d\theta = \frac{1}{2k}\tan^{2k}\theta - \int \tan^{2k-1}\theta\, d\theta$

86. (a) $\int \cot^3\theta\, d\theta = \int (\csc^2\theta - 1)(\cot\theta)\, d\theta = \int \cot\theta\csc^2\theta\, d\theta - \int \cot\theta\, d\theta = -\frac{1}{2}\cot^2\theta - \int \cot\theta\, d\theta$
$= -\frac{1}{2}\cot^2\theta - \ln|\sin\theta| + C$

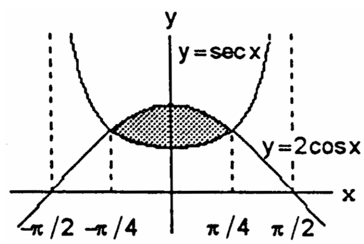
(b) $\int \cot^5\theta\, d\theta = \int (\csc^2\theta - 1)(\cot^3\theta)\, d\theta = \int \cot^3\theta\csc^2\theta\, d\theta - \int \cot^3\theta\, d\theta = -\frac{1}{4}\cot^4\theta - \int \cot^3\theta\, d\theta$

(c) $\int \cot^7\theta\, d\theta = \int (\csc^2\theta - 1)(\cot^5\theta)\, d\theta = \int \cot^5\theta\csc^2\theta\, d\theta - \int \cot^5\theta\, d\theta = -\frac{1}{6}\cot^6\theta - \int \cot^5\theta\, d\theta$

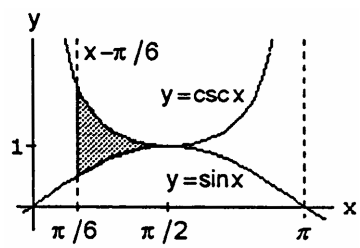
(d) $\int \cot^{2k+1}\theta\, d\theta = \int (\csc^2\theta - 1)(\cot^{2k-1}\theta)\, d\theta = \int \cot^{2k-1}\theta\csc^2\theta\, d\theta - \int \cot^{2k-1}\theta\, d\theta$;

$\begin{bmatrix} u = \cot\theta \\ du = -\csc^2\theta\, d\theta \end{bmatrix} \rightarrow -\int u^{2k-1}\, du - \int \cot^{2k-1}\theta\, d\theta = -\frac{1}{2k}u^{2k} - \int \cot^{2k-1}\theta\, d\theta$
$= -\frac{1}{2k}\cot^{2k}\theta - \int \cot^{2k-1}\theta\, d\theta$

87. $A = \int_{-\pi/4}^{\pi/4} (2\cos x - \sec x)\, dx = [2\sin x - \ln|\sec x + \tan x|]_{-\pi/4}^{\pi/4}$
$= \left[\sqrt{2} - \ln\left(\sqrt{2}+1\right)\right] - \left[-\sqrt{2} - \ln\left(\sqrt{2}-1\right)\right]$
$= 2\sqrt{2} - \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = 2\sqrt{2} - \ln\left(\frac{\left(\sqrt{2}+1\right)^2}{2-1}\right)$
$= 2\sqrt{2} - \ln\left(3 + 2\sqrt{2}\right)$



88. $A = \int_{\pi/6}^{\pi/2} (\csc x - \sin x)\, dx = [-\ln|\csc x + \cot x| + \cos x]_{\pi/6}^{\pi/2}$
$= -\ln|1+0| + \ln\left|2+\sqrt{3}\right| - \frac{\sqrt{3}}{2} = \ln\left(2+\sqrt{3}\right) - \frac{\sqrt{3}}{2}$



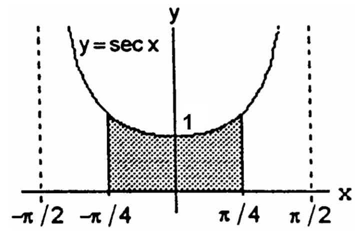
89. $V = \int_{-\pi/4}^{\pi/4} \pi(2\cos x)^2\, dx - \int_{-\pi/4}^{\pi/4} \pi\sec^2 x\, dx = 4\pi\int_{-\pi/4}^{\pi/4} \cos^2 x\, dx - \pi\int_{-\pi/4}^{\pi/4} \sec^2 x\, dx$
$= 2\pi\int_{-\pi/4}^{\pi/4} (1+\cos 2x)\, dx - \pi[\tan x]_{-\pi/4}^{\pi/4} = 2\pi\left[x + \frac{1}{2}\sin 2x\right]_{-\pi/4}^{\pi/4} - \pi[1-(-1)]$
$= 2\pi\left[\left(\frac{\pi}{4}+\frac{1}{2}\right) - \left(-\frac{\pi}{4}-\frac{1}{2}\right)\right] - 2\pi = 2\pi\left(\frac{\pi}{2}+1\right) - 2\pi = \pi^2$

90. $V = \int_{\pi/6}^{\pi/2} \pi\csc^2 x\, dx - \int_{\pi/6}^{\pi/2} \pi\sin^2 x\, dx = \pi\int_{\pi/6}^{\pi/2} \csc^2 x\, dx - \frac{\pi}{2}\int_{\pi/6}^{\pi/2} (1-\cos 2x)\, dx$
$= \pi[-\cot x]_{\pi/6}^{\pi/2} - \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_{\pi/6}^{\pi/2} = \pi\left[0 - \left(-\sqrt{3}\right)\right] - \frac{\pi}{2}\left[\left(\frac{\pi}{2}-0\right) - \left(\frac{\pi}{6} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)\right]$
$= \pi\sqrt{3} - \frac{\pi}{2}\left(\frac{2\pi}{6} + \frac{\sqrt{3}}{4}\right) = \pi\left(\frac{7\sqrt{3}}{8} - \frac{\pi}{6}\right)$

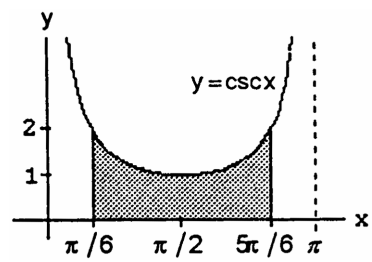
91. $y = \ln(\cos x) \Rightarrow \frac{dy}{dx} = -\frac{\sin x}{\cos x} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \tan^2 x = \sec^2 x - 1;\ L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$
$= \int_0^{\pi/3} \sqrt{1 + (\sec^2 x - 1)}\, dx = \int_0^{\pi/3} \sec x\, dx = [\ln|\sec x + \tan x|]_0^{\pi/3} = \ln\left|2 + \sqrt{3}\right| - \ln|1 + 0| = \ln\left(2 + \sqrt{3}\right)$

92. $y = \ln(\sec x) \Rightarrow \frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \tan^2 x = \sec^2 x - 1;\ L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$
$= \int_0^{\pi/4} \sec x\, dx = [\ln|\sec x + \tan x|]_0^{\pi/4} = \ln\left|\sqrt{2} + 1\right| - \ln|1 + 0| = \ln\left(\sqrt{2} + 1\right)$

93. $M_x = \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2}\sec x\right)(\sec x)\, dx = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \sec^2 x\, dx$
$= \frac{1}{2}[\tan x]_{-\pi/4}^{\pi/4} = \frac{1}{2}[1 - (-1)] = 1;$
$M = \int_{-\pi/4}^{\pi/4} \sec x\, dx = [\ln|\sec x + \tan x|]_{-\pi/4}^{\pi/4}$
$= \ln\left|\sqrt{2} + 1\right| - \ln\left|\sqrt{2} - 1\right| = \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$
$= \ln\left(\frac{\left(\sqrt{2}+1\right)^2}{2-1}\right) = \ln\left(3 + 2\sqrt{2}\right);\ \overline{x} = 0$ by
symmetry of the region, and $\overline{y} = \frac{M_x}{M} = \frac{1}{\ln\left(3+2\sqrt{2}\right)}$



94. $M_x = \int_{\pi/6}^{5\pi/6} \left(\frac{1}{2}\csc x\right)(\csc x)\, dx = \frac{1}{2}\int_{\pi/6}^{5\pi/6} \csc^2 x\, dx$
$= \frac{1}{2}[-\cot x]_{\pi/6}^{5\pi/6} = \frac{1}{2}\left[-\left(-\sqrt{3}\right) - \left(-\sqrt{3}\right)\right] = \sqrt{3};$
$M = \int_{\pi/6}^{5\pi/6} \csc x\, dx = [-\ln|\csc x + \cot x|]_{\pi/6}^{5\pi/6}$
$= -\ln\left|2 - \sqrt{3}\right| - \left(-\ln\left|2 + \sqrt{3}\right|\right) = \ln\left|\frac{2+\sqrt{3}}{2-\sqrt{3}}\right|$
$= \ln\left(\frac{\left(2+\sqrt{3}\right)^2}{4-3}\right) = 2\ln\left(2 + \sqrt{3}\right);\ \overline{x} = \frac{\pi}{2}$ by symmetry
of the region, and $\overline{y} = \frac{M_x}{M} = \frac{\sqrt{3}}{2\ln\left(2+\sqrt{3}\right)}$



95. $\int \csc x\, dx = \int (\csc x)(1)\, dx = \int (\csc x)\left(\frac{\csc x + \cot x}{\csc x + \cot x}\right) dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\, dx;$
$\begin{bmatrix} u = \csc x + \cot x \\ du = (-\csc x \cot x - \csc^2 x)\, dx \end{bmatrix} \to \int \frac{-du}{u} = -\ln|u| + C = -\ln|\csc x + \cot x| + C$

96. $\left[(x^2 - 1)(x + 1)\right]^{-2/3} = \left[(x - 1)(x + 1)^2\right]^{-2/3} = (x - 1)^{-2/3}(x + 1)^{-4/3} = (x + 1)^{-2}\left[(x - 1)^{-2/3}(x + 1)^{2/3}\right]$
$= (x + 1)^{-2}\left(\frac{x-1}{x+1}\right)^{-2/3} = (x + 1)^{-2}\left(1 - \frac{2}{x+1}\right)^{-2/3}$

(a) $\int \left[(x^2 - 1)(x + 1)\right]^{-2/3} dx = \int (x + 1)^{-2}\left(1 - \frac{2}{x+1}\right)^{-2/3} dx;\ \begin{bmatrix} u = \frac{1}{x+1} \\ du = -\frac{1}{(x+1)^2}\, dx \end{bmatrix}$
$\to \int -(1 - 2u)^{-2/3}\, du = \frac{3}{2}(1 - 2u)^{1/3} + C = \frac{3}{2}\left(1 - \frac{2}{x+1}\right)^{1/3} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

(b) $\int \left[(x^2 - 1)(x + 1)\right]^{-2/3} dx = \int (x + 1)^{-2}\left(\frac{x-1}{x+1}\right)^{-2/3} dx;\ u = \left(\frac{x-1}{x+1}\right)^k$
$\Rightarrow du = k\left(\frac{x-1}{x+1}\right)^{k-1} \frac{[(x+1) - (x-1)]}{(x+1)^2}\, dx = 2k\, \frac{(x-1)^{k-1}}{(x+1)^{k+1}}\, dx;\ dx = \frac{(x+1)^2}{2k}\left(\frac{x+1}{x-1}\right)^{k-1} du$
$= \frac{(x+1)^2}{2k}\left(\frac{x-1}{x+1}\right)^{1-k} du;$ then, $\int \left(\frac{x-1}{x+1}\right)^{-2/3} \frac{1}{2k}\left(\frac{x-1}{x+1}\right)^{1-k} du = \frac{1}{2k}\int \left(\frac{x-1}{x+1}\right)^{(1/3-k)} du$
$= \frac{1}{2k}\int \left(\frac{x-1}{x+1}\right)^{k(1/3k-1)} du = \frac{1}{2k}\int u^{(1/3k-1)}\, du = \frac{1}{2k}(3k)\, u^{1/3k} + C = \frac{3}{2}u^{1/3k} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

(c) $\int [(x^2-1)(x+1)]^{-2/3}\,dx = \int (x+1)^{-2}\left(\frac{x-1}{x+1}\right)^{-2/3} dx;$

$\begin{bmatrix} u = \tan^{-1} x \\ x = \tan u \\ dx = \frac{du}{\cos^2 u} \end{bmatrix} \to \int \frac{1}{(\tan u+1)^2}\left(\frac{\tan u - 1}{\tan u + 1}\right)^{-2/3}\left(\frac{du}{\cos^2 u}\right) = \int \frac{1}{(\sin u + \cos u)^2}\left(\frac{\sin u - \cos u}{\sin u + \cos u}\right)^{-2/3} du;$

$\begin{bmatrix} \sin u + \cos u = \sin u + \sin\left(\frac{\pi}{2} - u\right) = 2\sin\frac{\pi}{4}\cos\left(u - \frac{\pi}{4}\right) \\ \sin u - \cos u = \sin u - \sin\left(\frac{\pi}{2} - u\right) = 2\cos\frac{\pi}{4}\sin\left(u - \frac{\pi}{4}\right) \end{bmatrix} \to \int \frac{1}{2\cos^2\left(u - \frac{\pi}{4}\right)}\left[\frac{\sin\left(u-\frac{\pi}{4}\right)}{\cos\left(u - \frac{\pi}{4}\right)}\right]^{-2/3} du$

$= \frac{1}{2}\int \tan^{-2/3}\left(u - \frac{\pi}{4}\right)\sec^2\left(u - \frac{\pi}{4}\right) du = \frac{3}{2}\tan^{1/3}\left(u - \frac{\pi}{4}\right) + C = \frac{3}{2}\left[\frac{\tan u - \tan\frac{\pi}{4}}{1 + \tan u \tan\frac{\pi}{4}}\right]^{1/3} + C$

$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

(d) $u = \tan^{-1}\sqrt{x} \Rightarrow \tan u = \sqrt{x} \Rightarrow \tan^2 u = x \Rightarrow dx = 2\tan u\left(\frac{1}{\cos^2 u}\right) du = \frac{2\sin u}{\cos^3 u}\,du = -\frac{2d(\cos u)}{\cos^3 u};$

$x - 1 = \tan^2 u - 1 = \frac{\sin^2 u - \cos^2 u}{\cos^2 u} = \frac{1 - 2\cos^2 u}{\cos^2 u}$; $x + 1 = \tan^2 u + 1 = \frac{\cos^2 u + \sin^2 u}{\cos^2 u} = \frac{1}{\cos^2 u}$;

$\int (x-1)^{-2/3}(x+1)^{-4/3}\,dx = \int \frac{(1-2\cos^2 u)^{-2/3}}{(\cos^2 u)^{-2/3}} \cdot \frac{1}{(\cos^2 u)^{-4/3}} \cdot \frac{-2d(\cos u)}{\cos^3 u}$

$= \int (1 - 2\cos^2 u)^{-2/3} \cdot (-2) \cdot \cos u \cdot d(\cos u) = \frac{1}{2}\int (1 - 2\cos^2 u)^{-2/3} \cdot d(1 - 2\cos^2 u)$

$= \frac{3}{2}(1 - 2\cos^2 u)^{1/3} + C = \frac{3}{2}\left[\frac{\left(\frac{1 - 2\cos^2 u}{\cos^2 u}\right)}{\left(\frac{1}{\cos^2 u}\right)}\right]^{1/3} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

(e) $u = \tan^{-1}\left(\frac{x-1}{2}\right) \Rightarrow \frac{x-1}{2} = \tan u \Rightarrow x + 1 = 2(\tan u + 1) \Rightarrow dx = \frac{2\,du}{\cos^2 u} = 2d(\tan u);$

$\int (x-1)^{-2/3}(x+1)^{-4/3}\,dx = \int (\tan u)^{-2/3}(\tan u + 1)^{-4/3} \cdot 2^{-2} \cdot 2 \cdot d(\tan u)$

$= \frac{1}{2}\int \left(1 - \frac{1}{\tan u + 1}\right)^{-2/3} d\left(1 - \frac{1}{\tan u + 1}\right) = \frac{3}{2}\left(1 - \frac{1}{\tan u + 1}\right)^{1/3} + C = \frac{3}{2}\left(1 - \frac{2}{x+1}\right)^{1/3} + C$

$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

(f) $\begin{bmatrix} u = \cos^{-1} x \\ x = \cos u \\ dx = -\sin u\,du \end{bmatrix} \to -\int \frac{\sin u\,du}{\sqrt[3]{(\cos^2 u - 1)^2(\cos u + 1)^2}} = -\int \frac{\sin u\,du}{(\sin^{4/3} u)\left(2^{2/3}\cos\frac{u}{2}\right)^{4/3}}$

$= -\int \frac{du}{(\sin u)^{1/3}\left(2^{2/3}\cos\frac{u}{2}\right)^{4/3}} = -\int \frac{du}{2\left(\sin\frac{u}{2}\right)^{1/3}\left(\cos\frac{u}{2}\right)^{5/3}} = -\frac{1}{2}\int \left(\frac{\cos\frac{u}{2}}{\sin\frac{u}{2}}\right)^{1/3}\frac{du}{\left(\cos^2\frac{u}{2}\right)}$

$= -\int \tan^{-1/3}\left(\frac{u}{2}\right) d\left(\tan\frac{u}{2}\right) = -\frac{3}{2}\tan^{2/3}\frac{u}{2} + C = \frac{3}{2}\left(-\tan^2\frac{u}{2}\right)^{1/3} + C = \frac{3}{2}\left(\frac{\cos u - 1}{\cos u + 1}\right)^{1/3} + C$

$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

(g) $\int [(x^2-1)(x+1)]^{-2/3}\,dx;\ \begin{bmatrix} u = \cosh^{-1} x \\ x = \cosh u \\ dx = \sinh u \end{bmatrix} \to \int \frac{\sinh u\,du}{\sqrt[3]{(\cosh^2 u - 1)^2(\cosh u + 1)^2}}$

$= \int \frac{\sinh u\,du}{\sqrt[3]{(\sinh^4 u)(\cosh u + 1)^2}} = \int \frac{du}{\sqrt[3]{(\sinh u)\left(4\cosh^4\frac{u}{2}\right)}} = \frac{1}{2}\int \frac{du}{\sqrt[3]{\sinh\left(\frac{u}{2}\right)\cosh^5\left(\frac{u}{2}\right)}}$

$= \int \left(\tanh\frac{u}{2}\right)^{-1/3} d\left(\tanh\frac{u}{2}\right) = \frac{3}{2}\left(\tanh\frac{u}{2}\right)^{2/3} + C = \frac{3}{2}\left(\frac{\cosh u - 1}{\cosh u + 1}\right)^{1/3} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

8.2 INTEGRATION BY PARTS

1. $u = x$, $du = dx$; $dv = \sin\frac{x}{2}\,dx$, $v = -2\cos\frac{x}{2}$;

$\int x\sin\frac{x}{2}\,dx = -2x\cos\frac{x}{2} - \int\left(-2\cos\frac{x}{2}\right) dx = -2x\cos\left(\frac{x}{2}\right) + 4\sin\left(\frac{x}{2}\right) + C$

2. $u = \theta$, $du = d\theta$; $dv = \cos\pi\theta\,d\theta$, $v = \frac{1}{\pi}\sin\pi\theta$;

$\int \theta\cos\pi\theta\,d\theta = \frac{\theta}{\pi}\sin\pi\theta - \int \frac{1}{\pi}\sin\pi\theta\,d\theta = \frac{\theta}{\pi}\sin\pi\theta + \frac{1}{\pi^2}\cos\pi\theta + C$

3.
$$\begin{array}{lcl} & & \cos t \\ t^2 & \xrightarrow{(+)} & \sin t \\ 2t & \xrightarrow{(-)} & -\cos t \\ 2 & \xrightarrow{(+)} & -\sin t \\ 0 & & \end{array}$$
$$\int t^2 \cos t\, dt = t^2 \sin t + 2t \cos t - 2 \sin t + C$$

4.
$$\begin{array}{lcl} & & \sin x \\ x^2 & \xrightarrow{(+)} & -\cos x \\ 2x & \xrightarrow{(-)} & -\sin x \\ 2 & \xrightarrow{(+)} & \cos x \\ 0 & & \end{array}$$
$$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

5. $u = \ln x,\ du = \frac{dx}{x};\ dv = x\, dx,\ v = \frac{x^2}{2};$
$$\int_1^2 x \ln x\, dx = \left[\frac{x^2}{2} \ln x\right]_1^2 - \int_1^2 \frac{x^2}{2} \frac{dx}{x} = 2 \ln 2 - \left[\frac{x^2}{4}\right]_1^2 = 2 \ln 2 - \frac{3}{4} = \ln 4 - \frac{3}{4}$$

6. $u = \ln x,\ du = \frac{dx}{x};\ dv = x^3\, dx,\ v = \frac{x^4}{4};$
$$\int_1^e x^3 \ln x\, dx = \left[\frac{x^4}{4} \ln x\right]_1^e - \int_1^e \frac{x^4}{4} \frac{dx}{x} = \frac{e^4}{4} - \left[\frac{x^4}{16}\right]_1^e = \frac{3e^4+1}{16}$$

7. $u = \tan^{-1} y,\ du = \frac{dy}{1+y^2};\ dv = dy,\ v = y;$
$$\int \tan^{-1} y\, dy = y \tan^{-1} y - \int \frac{y\, dy}{(1+y^2)} = y \tan^{-1} y - \frac{1}{2} \ln(1+y^2) + C = y \tan^{-1} y - \ln \sqrt{1+y^2} + C$$

8. $u = \sin^{-1} y,\ du = \frac{dy}{\sqrt{1-y^2}};\ dv = dy,\ v = y;$
$$\int \sin^{-1} y\, dy = y \sin^{-1} y - \int \frac{y\, dy}{\sqrt{1-y^2}} = y \sin^{-1} y + \sqrt{1-y^2} + C$$

9. $u = x,\ du = dx;\ dv = \sec^2 x\, dx,\ v = \tan x;$
$$\int x \sec^2 x\, dx = x \tan x - \int \tan x\, dx = x \tan x + \ln|\cos x| + C$$

10. $\int 4x \sec^2 2x\, dx;\ [y = 2x] \to \int y \sec^2 y\, dy = y \tan y - \int \tan y\, dy = y \tan y - \ln|\sec y| + C$
$= 2x \tan 2x - \ln|\sec 2x| + C$

11.
$$\begin{array}{lcl} & & e^x \\ x^3 & \xrightarrow{(+)} & e^x \\ 3x^2 & \xrightarrow{(-)} & e^x \\ 6x & \xrightarrow{(+)} & e^x \\ 6 & \xrightarrow{(-)} & e^x \\ 0 & & \end{array}$$
$$\int x^3 e^x\, dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C = (x^3 - 3x^2 + 6x - 6)\, e^x + C$$

12.

$$\begin{array}{lcl} & & e^{-p} \\ p^4 & \xrightarrow{(+)} & -e^{-p} \\ 4p^3 & \xrightarrow{(-)} & e^{-p} \\ 12p^2 & \xrightarrow{(+)} & -e^{-p} \\ 24p & \xrightarrow{(-)} & e^{-p} \\ 24 & \xrightarrow{(+)} & -e^{-p} \\ 0 & & \end{array}$$

$$\int p^4 e^{-p}\,dp = -p^4 e^{-p} - 4p^3 e^{-p} - 12p^2 e^{-p} - 24pe^{-p} - 24e^{-p} + C$$
$$= \left(-p^4 - 4p^3 - 12p^2 - 24p - 24\right) e^{-p} + C$$

13.

$$\begin{array}{lcl} & & e^{x} \\ x^2 - 5x & \xrightarrow{(+)} & e^{x} \\ 2x - 5 & \xrightarrow{(-)} & e^{x} \\ 2 & \xrightarrow{(+)} & e^{x} \\ 0 & & \end{array}$$

$$\int \left(x^2 - 5x\right) e^x\,dx = \left(x^2 - 5x\right) e^x - (2x - 5)e^x + 2e^x + C = x^2 e^x - 7xe^x + 7e^x + C$$
$$= \left(x^2 - 7x + 7\right) e^x + C$$

14.

$$\begin{array}{lcl} & & e^{r} \\ r^2 + r + 1 & \xrightarrow{(+)} & e^{r} \\ 2r + 1 & \xrightarrow{(-)} & e^{r} \\ 2 & \xrightarrow{(+)} & e^{r} \\ 0 & & \end{array}$$

$$\int \left(r^2 + r + 1\right) e^r\,dr = \left(r^2 + r + 1\right) e^r - (2r + 1)\,e^r + 2e^r + C$$
$$= \left[\left(r^2 + r + 1\right) - (2r + 1) + 2\right] e^r + C = \left(r^2 - r + 2\right) e^r + C$$

15.

$$\begin{array}{lcl} & & e^{x} \\ x^5 & \xrightarrow{(+)} & e^{x} \\ 5x^4 & \xrightarrow{(-)} & e^{x} \\ 20x^3 & \xrightarrow{(+)} & e^{x} \\ 60x^2 & \xrightarrow{(-)} & e^{x} \\ 120x & \xrightarrow{(+)} & e^{x} \\ 120 & \xrightarrow{(-)} & e^{x} \\ 0 & & \end{array}$$

$$\int x^5 e^x\,dx = x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120xe^x - 120e^x + C$$
$$= \left(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120\right) e^x + C$$

16.

$$\begin{array}{lcl} & & e^{4t} \\ t^2 & \xrightarrow{(+)} & \frac{1}{4}e^{4t} \\ 2t & \xrightarrow{(-)} & \frac{1}{16}e^{4t} \\ 2 & \xrightarrow{(+)} & \frac{1}{64}e^{4t} \\ 0 & & \end{array}$$

$$\int t^2 e^{4t}\,dt = \frac{t^2}{4}e^{4t} - \frac{2t}{16}e^{4t} + \frac{2}{64}e^{4t} + C = \frac{t^2}{4}e^{4t} - \frac{t}{8}e^{4t} + \frac{1}{32}e^{4t} + C$$
$$= \left(\frac{t^2}{4} - \frac{t}{8} + \frac{1}{32}\right)e^{4t} + C$$

17.

$$\begin{array}{lcl} & & \sin 2\theta \\ \theta^2 & \xrightarrow{(+)} & -\frac{1}{2}\cos 2\theta \\ 2\theta & \xrightarrow{(-)} & -\frac{1}{4}\sin 2\theta \\ 2 & \xrightarrow{(+)} & \frac{1}{8}\cos 2\theta \\ 0 & & \end{array}$$

$$\int_0^{\pi/2} \theta^2 \sin 2\theta\,d\theta = \left[-\frac{\theta^2}{2}\cos 2\theta + \frac{\theta}{2}\sin 2\theta + \frac{1}{4}\cos 2\theta\right]_0^{\pi/2}$$
$$= \left[-\frac{\pi^2}{8}\cdot(-1) + \frac{\pi}{4}\cdot 0 + \frac{1}{4}\cdot(-1)\right] - \left[0 + 0 + \frac{1}{4}\cdot 1\right] = \frac{\pi^2}{8} - \frac{1}{2} = \frac{\pi^2-4}{8}$$

18.

$$\begin{array}{lcl} & & \cos 2x \\ x^3 & \xrightarrow{(+)} & \frac{1}{2}\sin 2x \\ 3x^2 & \xrightarrow{(-)} & -\frac{1}{4}\cos 2x \\ 6x & \xrightarrow{(+)} & -\frac{1}{8}\sin 2x \\ 6 & \xrightarrow{(-)} & \frac{1}{16}\cos 2x \\ 0 & & \end{array}$$

$$\int_0^{\pi/2} x^3 \cos 2x\,dx = \left[\frac{x^3}{2}\sin 2x + \frac{3x^2}{4}\cos 2x - \frac{3x}{4}\sin 2x - \frac{3}{8}\cos 2x\right]_0^{\pi/2}$$
$$= \left[\frac{\pi^3}{16}\cdot 0 + \frac{3\pi^2}{16}\cdot(-1) - \frac{3\pi}{8}\cdot 0 - \frac{3}{8}\cdot(-1)\right] - \left[0 + 0 - 0 - \frac{3}{8}\cdot 1\right] = -\frac{3\pi^2}{16} + \frac{3}{4} = \frac{3(4-\pi^2)}{16}$$

19. $u = \sec^{-1} t$, $du = \frac{dt}{t\sqrt{t^2-1}}$; $dv = t\,dt$, $v = \frac{t^2}{2}$;

$$\int_{2/\sqrt{3}}^2 t\sec^{-1} t\,dt = \left[\frac{t^2}{2}\sec^{-1} t\right]_{2/\sqrt{3}}^2 - \int_{2/\sqrt{3}}^2 \left(\frac{t^2}{2}\right)\frac{dt}{t\sqrt{t^2-1}} = \left(2\cdot\frac{\pi}{3} - \frac{2}{3}\cdot\frac{\pi}{6}\right) - \int_{2/\sqrt{3}}^2 \frac{t\,dt}{2\sqrt{t^2-1}}$$
$$= \frac{5\pi}{9} - \left[\frac{1}{2}\sqrt{t^2-1}\right]_{2/\sqrt{3}}^2 = \frac{5\pi}{9} - \frac{1}{2}\left(\sqrt{3} - \sqrt{\frac{4}{3}-1}\right) = \frac{5\pi}{9} - \frac{1}{2}\left(\sqrt{3} - \frac{\sqrt{3}}{3}\right) = \frac{5\pi}{9} - \frac{\sqrt{3}}{3} = \frac{5\pi - 3\sqrt{3}}{9}$$

20. $u = \sin^{-1}(x^2)$, $du = \frac{2x\,dx}{\sqrt{1-x^4}}$; $dv = 2x\,dx$, $v = x^2$;

$$\int_0^{1/\sqrt{2}} 2x\sin^{-1}(x^2)\,dx = \left[x^2\sin^{-1}(x^2)\right]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} x^2\cdot\frac{2x\,dx}{\sqrt{1-x^4}} = \left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right) + \int_0^{1/\sqrt{2}} \frac{d(1-x^4)}{2\sqrt{1-x^4}}$$
$$= \frac{\pi}{12} + \left[\sqrt{1-x^4}\right]_0^{1/\sqrt{2}} = \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1 = \frac{\pi + 6\sqrt{3} - 12}{12}$$

21. $I = \int e^\theta \sin\theta\,d\theta$; $[u = \sin\theta,\ du = \cos\theta\,d\theta;\ dv = e^\theta\,d\theta,\ v = e^\theta] \Rightarrow I = e^\theta\sin\theta - \int e^\theta\cos\theta\,d\theta$;
$[u = \cos\theta,\ du = -\sin\theta\,d\theta;\ dv = e^\theta\,d\theta,\ v = e^\theta] \Rightarrow I = e^\theta\sin\theta - \left(e^\theta\cos\theta + \int e^\theta\sin\theta\,d\theta\right)$
$= e^\theta\sin\theta - e^\theta\cos\theta - I + C' \Rightarrow 2I = (e^\theta\sin\theta - e^\theta\cos\theta) + C' \Rightarrow I = \frac{1}{2}(e^\theta\sin\theta - e^\theta\cos\theta) + C$, where $C = \frac{C'}{2}$ is another arbitrary constant

22. $I = \int e^{-y}\cos y\,dy$; $[u = \cos y,\ du = -\sin y\,dy;\ dv = e^{-y}\,dy,\ v = -e^{-y}]$
$\Rightarrow\ I = -e^{-y}\cos y - \int (-e^{-y})(-\sin y)\ dy = -e^{-y}\cos y - \int e^{-y}\sin y\,dy$; $[u = \sin y,\ du = \cos y\,dy;$
$dv = e^{-y}\,dy,\ v = -e^{-y}]\ \Rightarrow\ I = -e^{-y}\cos y - \left(-e^{-y}\sin y - \int (-e^{y})\cos y\,dy\right) = -e^{-y}\cos y + e^{-y}\sin y - I + C'$
$\Rightarrow\ 2I = e^{-y}(\sin y - \cos y) + C'\ \Rightarrow\ I = \frac{1}{2}(e^{-y}\sin y - e^{-y}\cos y) + C$, where $C = \frac{C'}{2}$ is another arbitrary constant

23. $I = \int e^{2x}\cos 3x\,dx$; $\left[u = \cos 3x;\ du = -3\sin 3x\,dx,\ dv = e^{2x}\,dx;\ v = \frac{1}{2}e^{2x}\right]$
$\Rightarrow\ I = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\,dx$; $\left[u = \sin 3x,\ du = 3\cos 3x,\ dv = e^{2x}\,dx;\ v = \frac{1}{2}e^{2x}\right]$
$\Rightarrow\ I = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\left(\frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\,dx\right) = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}I + C'$
$\Rightarrow\ \frac{13}{4}I = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x + C'\ \Rightarrow\ \frac{e^{2x}}{13}(3\sin 3x + 2\cos 3x) + C$, where $C = \frac{4}{13}C'$

24. $\int e^{-2x}\sin 2x\,dx$; $[y = 2x]\ \to\ \frac{1}{2}\int e^{-y}\sin y\,dy = I$; $[u = \sin y,\ du = \cos y\,dy;\ dv = e^{-y}\,dy,\ v = -e^{-y}]$
$\Rightarrow\ I = \frac{1}{2}\left(-e^{-y}\sin y + \int e^{-y}\cos y\,dy\right)$ $[u = \cos y,\ du = -\sin y;\ dv = e^{-y}\,dy,\ v = -e^{-y}]$
$\Rightarrow\ I = -\frac{1}{2}e^{-y}\sin y + \frac{1}{2}\left(-e^{-y}\cos y - \int (-e^{-y})(-\sin y)\,dy\right) = -\frac{1}{2}e^{-y}(\sin y + \cos y) - I + C'$
$\Rightarrow\ 2I = -\frac{1}{2}e^{-y}(\sin y + \cos y) + C'\ \Rightarrow\ I = -\frac{1}{4}e^{-y}(\sin y + \cos y) + C = -\frac{e^{-2x}}{4}(\sin 2x + \cos 2x) + C$, where
$C = \frac{C'}{2}$

25. $\int e^{\sqrt{3s+9}}\,ds$; $\begin{bmatrix} 3s + 9 = x^2 \\ ds = \frac{2}{3}x\,dx \end{bmatrix}\ \to\ \int e^{x}\cdot\frac{2}{3}x\,dx = \frac{2}{3}\int xe^{x}\,dx$; $[u = x,\ du = dx;\ dv = e^{x}\,dx,\ v = e^{x}]$;
$\frac{2}{3}\int xe^{x}\,dx = \frac{2}{3}\left(xe^{x} - \int e^{x}\,dx\right) = \frac{2}{3}(xe^{x} - e^{x}) + C = \frac{2}{3}\left(\sqrt{3s+9}\,e^{\sqrt{3s+9}} - e^{\sqrt{3s+9}}\right) + C$

26. $u = x,\ du = dx;\ dv = \sqrt{1-x}\,dx,\ v = -\frac{2}{3}\sqrt{(1-x)^3}$;
$\int_0^1 x\sqrt{1-x}\,dx = \left[-\frac{2}{3}\sqrt{(1-x)^3}\,x\right]_0^1 + \frac{2}{3}\int_0^1 \sqrt{(1-x)^3}\,dx = \frac{2}{3}\left[-\frac{2}{5}(1-x)^{5/2}\right]_0^1 = \frac{4}{15}$

27. $u = x,\ du = dx;\ dv = \tan^2 x\,dx,\ v = \int \tan^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx = \int \frac{dx}{\cos^2 x} - \int dx$
$= \tan x - x;\int_0^{\pi/3} x\tan^2 x\,dx = [x(\tan x - x)]_0^{\pi/3} - \int_0^{\pi/3}(\tan x - x)\,dx = \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \left[\ln|\cos x| + \frac{x^2}{2}\right]_0^{\pi/3}$
$= \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \ln\frac{1}{2} + \frac{\pi^2}{18} = \frac{\pi\sqrt{3}}{3} - \ln 2 - \frac{\pi^2}{18}$

28. $u = \ln(x + x^2),\ du = \frac{(2x+1)\,dx}{x+x^2}$; $dv = dx,\ v = x$; $\int \ln(x + x^2)\,dx = x\ln(x + x^2) - \int \frac{2x+1}{x(x+1)}\cdot x\,dx$
$= x\ln(x + x^2) - \int \frac{(2x+1)\,dx}{x+1} = x\ln(x + x^2) - \int \frac{2(x+1)-1}{x+1}\,dx = x\ln(x + x^2) - 2x + \ln|x+1| + C$

29. $\int \sin(\ln x)\,dx$; $\begin{bmatrix} u = \ln x \\ du = \frac{1}{x}\,dx \\ dx = e^{u}\,du \end{bmatrix}\ \to\ \int (\sin u)\,e^{u}\,du$. From Exercise 21, $\int (\sin u)\,e^{u}\,du = e^{u}\left(\frac{\sin u - \cos u}{2}\right) + C$
$= \frac{1}{2}[-x\cos(\ln x) + x\sin(\ln x)] + C$

30. $\int z(\ln z)^2\, dz;\ \begin{bmatrix} u = \ln z \\ du = \frac{1}{z}\, dz \\ dz = e^u\, du \end{bmatrix} \rightarrow \int e^u \cdot u^2 \cdot e^u\, du = \int e^{2u} \cdot u^2\, du;$

$$\begin{array}{lcl} & & e^{2u} \\ u^2 & \xrightarrow{(+)} & \frac{1}{2}e^{2u} \\ 2u & \xrightarrow{(-)} & \frac{1}{4}e^{2u} \\ 2 & \xrightarrow{(+)} & \frac{1}{8}e^{2u} \\ 0 & & \end{array}$$

$$\int u^2 e^{2u}\, du = \frac{u^2}{2} e^{2u} - \frac{u}{2} e^{2u} + \frac{1}{4} e^{2u} + C = \frac{e^{2u}}{4}\left[2u^2 - 2u + 1\right] + C$$
$$= \frac{z^2}{4}\left[2(\ln z)^2 - 2\ln z + 1\right] + C$$

31. (a) $u = x,\ du = dx;\ dv = \sin x\, dx,\ v = -\cos x;$

$S_1 = \int_0^{\pi} x \sin x\, dx = \left[-x\cos x\right]_0^{\pi} + \int_0^{\pi} \cos x\, dx = \pi + \left[\sin x\right]_0^{\pi} = \pi$

(b) $S_2 = -\int_{\pi}^{2\pi} x\sin x\, dx = -\left[\left[-x\cos x\right]_{\pi}^{2\pi} + \int_{\pi}^{2\pi} \cos x\, dx\right] = -\left[-3\pi + \left[\sin x\right]_{\pi}^{2\pi}\right] = 3\pi$

(c) $S_3 = \int_{2\pi}^{3\pi} x\sin x\, dx = \left[-x\cos x\right]_{2\pi}^{3\pi} + \int_{2\pi}^{3\pi} \cos x\, dx = 5\pi + \left[\sin x\right]_{2\pi}^{3\pi} = 5\pi$

(d) $S_{n+1} = (-1)^{n+1}\int_{n\pi}^{(n+1)\pi} x\sin x\, dx = (-1)^{n+1}\left[\left[-x\cos x\right]_{n\pi}^{(n+1)\pi} + \left[\sin x\right]_{n\pi}^{(n+1)\pi}\right]$

$= (-1)^{n+1}\left[-(n+1)\pi(-1)^n + n\pi(-1)^{n+1}\right] + 0 = (2n+1)\pi$

32. (a) $u = x,\ du = dx;\ dv = \cos x\, dx,\ v = \sin x;$

$S_1 = -\int_{\pi/2}^{3\pi/2} x\cos x\, dx = -\left[\left[x\sin x\right]_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2} \sin x\, dx\right] = -\left(-\frac{3\pi}{2} - \frac{\pi}{2}\right) - \left[\cos x\right]_{\pi/2}^{3\pi/2} = 2\pi$

(b) $S_2 = \int_{3\pi/2}^{5\pi/2} x\cos x\, dx = \left[x\sin x\right]_{3\pi/2}^{5\pi/2} - \int_{3\pi/2}^{5\pi/2} \sin x\, dx = \left[\frac{5\pi}{2} - \left(-\frac{3\pi}{2}\right)\right] - \left[\cos x\right]_{3\pi/2}^{5\pi/2} = 4\pi$

(c) $S_3 = -\int_{5\pi/2}^{7\pi/2} x\cos x\, dx = -\left[\left[x\sin x\right]_{5\pi/2}^{7\pi/2} - \int_{5\pi/2}^{7\pi/2} \sin x\, dx\right] = -\left(-\frac{7\pi}{2} - \frac{5\pi}{2}\right) - \left[\cos x\right]_{5\pi/2}^{7\pi/2} = 6\pi$

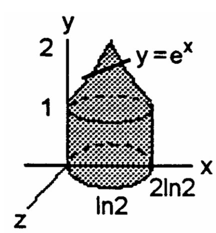
(d) $S_n = (-1)^n \int_{(2n-1)\pi/2}^{(2n+1)\pi/2} x\cos x\, dx = (-1)^n\left[\left[x\sin x\right]_{(2n-1)\pi/2}^{(2n+1)\pi/2} - {}^{n}\int_{(2n-1)\pi/2}^{(2n+1)\pi/2} \sin x\, dx\right]$

$= (-1)^n\left[\frac{(2n+1)\pi}{2}(-1)^n - \frac{(2n-1)\pi}{2}(-1)^{n-1}\right] - \left[\cos x\right]_{(2n-1)\pi/2}^{(2n+1)\pi/2} = \frac{1}{2}(2n\pi + \pi + 2n\pi - \pi) = 2n\pi$

33. $V = \int_0^{\ln 2} 2\pi(\ln 2 - x)\, e^x\, dx = 2\pi\ln 2\int_0^{\ln 2} e^x\, dx - 2\pi\int_0^{\ln 2} xe^x\, dx$

$= (2\pi\ln 2)\left[e^x\right]_0^{\ln 2} - 2\pi\left(\left[xe^x\right]_0^{\ln 2} - \int_0^{\ln 2} e^x\, dx\right)$

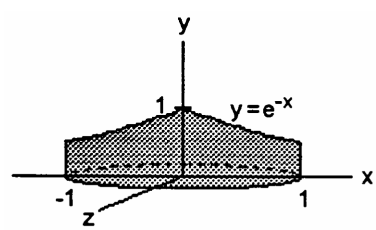
$= 2\pi\ln 2 - 2\pi\left(2\ln 2 - \left[e^x\right]_0^{\ln 2}\right) = -2\pi\ln 2 + 2\pi = 2\pi(1 - \ln 2)$



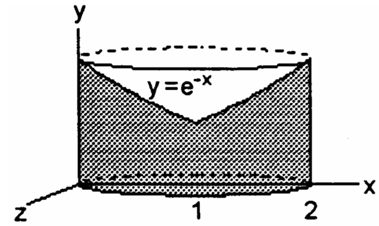
34. (a) $V = \int_0^1 2\pi xe^{-x}\, dx = 2\pi\left(\left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\, dx\right)$

$= 2\pi\left(-\frac{1}{e} + \left[-e^{-x}\right]_0^1\right) = 2\pi\left(-\frac{1}{e} - \frac{1}{e} + 1\right)$

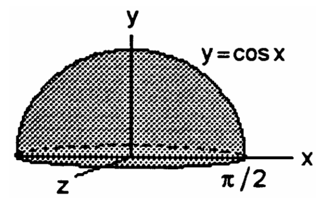
$= 2\pi - \frac{4\pi}{e}$



(b) $V = \int_0^1 2\pi(1-x)e^{-x}\,dx$; $u = 1-x$, $du = -\,dx$; $dv = e^{-x}\,dx$,
$v = -e^{-x}$; $V = 2\pi\left[\left[(1-x)\left(-e^{-x}\right)\right]_0^1 - \int_0^1 e^{-x}\,dx\right]$
$= 2\pi\left[[0-1(-1)] + \left[e^{-x}\right]_0^1\right] = 2\pi\left(1+\frac{1}{e}-1\right) = \frac{2\pi}{e}$



35. (a) $V = \int_0^{\pi/2} 2\pi x\cos x\,dx = 2\pi\left([x\sin x]_0^{\pi/2} - \int_0^{\pi/2}\sin x\,dx\right)$
$= 2\pi\left(\frac{\pi}{2} + [\cos x]_0^{\pi/2}\right) = 2\pi\left(\frac{\pi}{2}+0-1\right) = \pi(\pi-2)$



(b) $V = \int_0^{\pi/2} 2\pi\left(\frac{\pi}{2}-x\right)\cos x\,dx$; $u = \frac{\pi}{2}-x$, $du = -\,dx$; $dv = \cos x\,dx$, $v = \sin x$;
$V = 2\pi\left[\left(\frac{\pi}{2}-x\right)\sin x\right]_0^{\pi/2} + 2\pi\int_0^{\pi/2}\sin x\,dx = 0 + 2\pi[-\cos x]_0^{\pi/2} = 2\pi(0+1) = 2\pi$

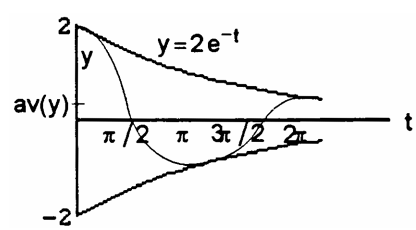
36. (a) $V = \int_0^{\pi} 2\pi x(x\sin x)\,dx$;

$$\begin{array}{lcl} & & \sin x \\ x^2 & \xrightarrow{(+)} & -\cos x \\ 2x & \xrightarrow{(-)} & -\sin x \\ 2 & \xrightarrow{(+)} & \cos x \\ 0 & & \end{array}$$

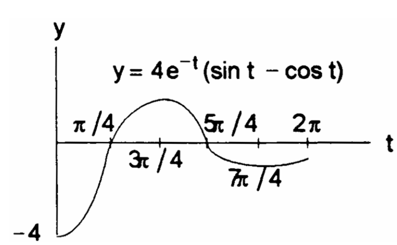
$\Rightarrow V = 2\pi\int_0^{\pi} x^2\sin x\,dx = 2\pi\left[-x^2\cos x + 2x\sin x + 2\cos x\right]_0^{\pi} = 2\pi\left(\pi^2-4\right)$

(b) $V = \int_0^{\pi} 2\pi(\pi-x)x\sin x\,dx = 2\pi^2\int_0^{\pi} x\sin x\,dx - 2\pi\int_0^{\pi} x^2\sin x\,dx = 2\pi^2\left[-x\cos x + \sin x\right]_0^{\pi} - \left(2\pi^3 - 8\pi\right)$
$= 8\pi$

37. $av(y) = \frac{1}{2\pi}\int_0^{2\pi} 2e^{-t}\cos t\,dt$
$= \frac{1}{\pi}\left[e^{-t}\left(\frac{\sin t - \cos t}{2}\right)\right]_0^{2\pi}$
(see Exercise 22) $\Rightarrow$ $av(y) = \frac{1}{2\pi}\left(1-e^{-2\pi}\right)$



38. $av(y) = \frac{1}{2\pi}\int_0^{2\pi} 4e^{-t}(\sin t - \cos t)\,dt$
$= \frac{2}{\pi}\int_0^{2\pi} e^{-t}\sin t\,dt - \frac{2}{\pi}\int_0^{2\pi} e^{-t}\cos t\,dt$
$= \frac{2}{\pi}\left[e^{-t}\left(\frac{-\sin t - \cos t}{2}\right) - e^{-t}\left(\frac{\sin t - \cos t}{2}\right)\right]_0^{2\pi}$
$= \frac{2}{\pi}\left[-e^{-t}\sin t\right]_0^{2\pi} = 0$



39. $I = \int x^n\cos x\,dx$; $[u = x^n,\ du = nx^{n-1}\,dx;\ dv = \cos x\,dx,\ v = \sin x]$
$\Rightarrow I = x^n\sin x - \int nx^{n-1}\sin x\,dx$

40. $I = \int x^n \sin x\, dx$; $[u = x^n,\ du = nx^{n-1}\, dx;\ dv = \sin x\, dx,\ v = -\cos x]$
$\Rightarrow I = -x^n \cos x + \int nx^{n-1} \cos x\, dx$

41. $I = \int x^n e^{ax}\, dx$; $\left[u = x^n,\ du = nx^{n-1}\, dx;\ dv = e^{ax}\, dx,\ v = \frac{1}{a} e^{ax}\right]$
$\Rightarrow I = \frac{x^n e^{ax}}{a} e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax}\, dx,\ a \neq 0$

42. $I = \int (\ln x)^n\, dx$; $\left[u = (\ln x)^n,\ du = \frac{n(\ln x)^{n-1}}{x}\, dx;\ dv = 1\, dx,\ v = x\right]$
$\Rightarrow I = x(\ln x)^n - \int n(\ln x)^{n-1}\, dx$

43. $\int \sin^{-1} x\, dx = x \sin^{-1} x - \int \sin y\, dy = x \sin^{-1} x + \cos y + C = x \sin^{-1} x + \cos(\sin^{-1} x) + C$

44. $\int \tan^{-1} x\, dx = x \tan^{-1} x - \int \tan y\, dy = x \tan^{-1} x + \ln|\cos y| + C = x \tan^{-1} x + \ln|\cos(\tan^{-1} x)| + C$

45. $\int \sec^{-1} x\, dx = x \sec^{-1} x - \int \sec y\, dy = x \sec^{-1} x - \ln|\sec y + \tan y| + C$
$= x \sec^{-1} x - \ln|\sec(\sec^{-1} x) + \tan(\sec^{-1} x)| + C = x \sec^{-1} x - \ln\left|x + \sqrt{x^2 - 1}\right| + C$

46. $\int \log_2 x\, dx = x \log_2 x - \int 2^y\, dy = x \log_2 x - \frac{2^y}{\ln 2} + C = x \log_2 x - \frac{x}{\ln 2} + C$

47. Yes, $\cos^{-1} x$ is the angle whose cosine is x which implies $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$.

48. Yes, $\tan^{-1} x$ is the angle whose tangent is x which implies $\sec(\tan^{-1} x) = \sqrt{1 + x^2}$.

49. (a) $\int \sinh^{-1} x\, dx = x \sinh^{-1} x - \int \sinh y\, dy = x \sinh^{-1} x - \cosh y + C = x \sinh^{-1} x - \cosh(\sinh^{-1} x) + C$;
check: $d[x \sinh^{-1} x - \cosh(\sinh^{-1} x) + C] = \left[\sinh^{-1} x + \frac{x}{\sqrt{1+x^2}} - \sinh(\sinh^{-1} x) \frac{1}{\sqrt{1+x^2}}\right] dx$
$= \sinh^{-1} x\, dx$
(b) $\int \sinh^{-1} x\, dx = x \sinh^{-1} x - \int x \left(\frac{1}{\sqrt{1+x^2}}\right) dx = x \sinh^{-1} x - \frac{1}{2} \int (1 + x^2)^{-1/2} 2x\, dx$
$= x \sinh^{-1} x - (1 + x^2)^{1/2} + C$
check: $d\left[x \sinh^{-1} x - (1 + x^2)^{1/2} + C\right] = \left[\sinh^{-1} x + \frac{x}{\sqrt{1+x^2}} - \frac{x}{\sqrt{1+x^2}}\right] dx = \sinh^{-1} x\, dx$

50. (a) $\int \tanh^{-1} x\, dx = x \tanh^{-1} x - \int \tanh y\, dy = x \tanh^{-1} x - \ln|\cosh y| + C$
$= x \tanh^{-1} x - \ln|\cosh(\tanh^{-1} x)| + C$;
check: $d[x \tanh^{-1} x - \ln|\cosh(\tanh^{-1} x)| + C] = \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{\sinh(\tanh^{-1} x)}{\cosh(\tanh^{-1} x)} \frac{1}{1-x^2}\right] dx$
$= \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{x}{1-x^2}\right] dx = \tanh^{-1} x\, dx$
(b) $\int \tanh^{-1} x\, dx = x \tanh^{-1} x - \int \frac{x}{1-x^2}\, dx = x \tanh^{-1} x - \frac{1}{2} \int \frac{2x}{1-x^2}\, dx = x \tanh^{-1} x + \frac{1}{2} \ln|1 - x^2| + C$
check: $d\left[x \tanh^{-1} x + \frac{1}{2} \ln|1 - x^2| + C\right] = \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{x}{1-x^2}\right] dx = \tanh^{-1} x\, dx$

8.3 INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

1. $\frac{5x - 13}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} \Rightarrow 5x - 13 = A(x - 2) + B(x - 3) = (A + B)x - (2A + 3B)$
$\Rightarrow \left.\begin{array}{r} A + B = 5 \\ 2A + 3B = 13 \end{array}\right\} \Rightarrow -B = (10 - 13) \Rightarrow B = 3 \Rightarrow A = 2$; thus, $\frac{5x - 13}{(x-3)(x-2)} = \frac{2}{x-3} + \frac{3}{x-2}$

2. $\frac{5x-7}{x^2-3x+2} = \frac{5x-7}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} \Rightarrow 5x-7 = A(x-1)+B(x-2) = (A+B)x-(A+2B)$
$\Rightarrow \left.\begin{array}{r} A+B=5 \\ A+2B=7 \end{array}\right\} \Rightarrow B=2 \Rightarrow A=3$; thus, $\frac{5x-7}{x^2-3x+2} = \frac{3}{x-2} + \frac{2}{x-1}$

3. $\frac{x+4}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow x+4 = A(x+1)+B = Ax+(A+B) \Rightarrow \left.\begin{array}{r} A=1 \\ A+B=4 \end{array}\right\} \Rightarrow A=1$ and $B=3$;
thus, $\frac{x+4}{(x+1)^2} = \frac{1}{x+1} + \frac{3}{(x+1)^2}$

4. $\frac{2x+2}{x^2-2x+1} = \frac{2x+2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 2x+2 = A(x-1)+B = Ax+(-A+B) \Rightarrow \left.\begin{array}{r} A=2 \\ -A+B=2 \end{array}\right\}$
$\Rightarrow A=2$ and $B=4$; thus, $\frac{2x+2}{x^2-2x+1} = \frac{2}{x-1} + \frac{4}{(x-1)^2}$

5. $\frac{z+1}{z^2(z-1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1} \Rightarrow z+1 = Az(z-1)+B(z-1)+Cz^2 \Rightarrow z+1 = (A+C)z^2+(-A+B)z-B$
$\Rightarrow \left.\begin{array}{r} A+C=0 \\ -A+B=1 \\ -B=1 \end{array}\right\} \Rightarrow B=-1 \Rightarrow A=-2 \Rightarrow C=2$; thus, $\frac{z+1}{z^2(z-1)} = \frac{-2}{z} + \frac{-1}{z^2} + \frac{2}{z-1}$

6. $\frac{z}{z^3-z^2-6z} = \frac{1}{z^2-z-6} = \frac{1}{(z-3)(z+2)} = \frac{A}{z-3} + \frac{B}{z+2} \Rightarrow 1 = A(z+2)+B(z-3) = (A+B)z+(2A-3B)$
$\Rightarrow \left.\begin{array}{r} A+B=0 \\ 2A-3B=1 \end{array}\right\} \Rightarrow -5B=1 \Rightarrow B=-\frac{1}{5} \Rightarrow A=\frac{1}{5}$; thus, $\frac{z}{z^3-z^2-6z} = \frac{\frac{1}{5}}{z-3} + \frac{-\frac{1}{5}}{z+2}$

7. $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{5t+2}{t^2-5t+6}$ (after long division); $\frac{5t+2}{t^2-5t+6} = \frac{5t+2}{(t-3)(t-2)} = \frac{A}{t-3} + \frac{B}{t-2}$
$\Rightarrow 5t+2 = A(t-2)+B(t-3) = (A+B)t+(-2A-3B) \Rightarrow \left.\begin{array}{r} A+B=5 \\ -2A-3B=2 \end{array}\right\} \Rightarrow -B = (10+2) = 12$
$\Rightarrow B=-12 \Rightarrow A=17$; thus, $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{17}{t-3} + \frac{-12}{t-2}$

8. $\frac{t^4+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^2(t^2+9)}$ (after long division); $\frac{-9t^2+9}{t^2(t^2+9)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+9}$
$\Rightarrow -9t^2+9 = At(t^2+9)+B(t^2+9)+(Ct+D)t^2 = (A+C)t^3+(B+D)t^2+9At+9B$
$\Rightarrow \left.\begin{array}{r} A+C=0 \\ B+D=-9 \\ 9A=0 \\ 9B=9 \end{array}\right\} \Rightarrow A=0 \Rightarrow C=0; B=1 \Rightarrow D=-10$; thus, $\frac{t^4+9}{t^4+9t^2} = 1 + \frac{1}{t^2} + \frac{-10}{t^2+9}$

9. $\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} \Rightarrow 1 = A(1+x)+B(1-x);\ x=1 \Rightarrow A=\frac{1}{2};\ x=-1 \Rightarrow B=\frac{1}{2}$;
$\int \frac{dx}{1-x^2} = \frac{1}{2}\int \frac{dx}{1-x} + \frac{1}{2}\int \frac{dx}{1+x} = \frac{1}{2}\left[\ln|1+x| - \ln|1-x|\right] + C$

10. $\frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2} \Rightarrow 1 = A(x+2)+Bx;\ x=0 \Rightarrow A=\frac{1}{2};\ x=-2 \Rightarrow B=-\frac{1}{2}$;
$\int \frac{dx}{x^2+2x} = \frac{1}{2}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x+2} = \frac{1}{2}\left[\ln|x| - \ln|x+2|\right] + C$

11. $\frac{x+4}{x^2+5x-6} = \frac{A}{x+6} + \frac{B}{x-1} \Rightarrow x+4 = A(x-1)+B(x+6);\ x=1 \Rightarrow B=\frac{5}{7};\ x=-6 \Rightarrow A=\frac{-2}{-7}=\frac{2}{7}$;
$\int \frac{x+4}{x^2+5x-6}\,dx = \frac{2}{7}\int \frac{dx}{x+6} + \frac{5}{7}\int \frac{dx}{x-1} = \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C = \frac{1}{7}\ln\left|(x+6)^2(x-1)^5\right| + C$

12. $\frac{2x+1}{x^2-7x+12} = \frac{A}{x-4} + \frac{B}{x-3} \Rightarrow 2x+1 = A(x-3)+B(x-4);\ x=3 \Rightarrow B=\frac{7}{-1}=-7;\ x=4 \Rightarrow A=\frac{9}{1}=9$;
$\int \frac{2x+1}{x^2-7x+12}\,dx = 9\int \frac{dx}{x-4} - 7\int \frac{dx}{x-3} = 9\ln|x-4| - 7\ln|x-3| + C = \ln\left|\frac{(x-4)^9}{(x-3)^7}\right| + C$

13. $\frac{y}{y^2-2y-3} = \frac{A}{y-3} + \frac{B}{y+1} \Rightarrow y = A(y+1) + B(y-3);\ y = -1 \Rightarrow B = \frac{-1}{-4} = \frac{1}{4};\ y = 3 \Rightarrow A = \frac{3}{4};$
$\int_4^8 \frac{y\,dy}{y^2-2y-3} = \frac{3}{4}\int_4^8 \frac{dy}{y-3} + \frac{1}{4}\int_4^8 \frac{dy}{y+1} = \left[\frac{3}{4}\ln|y-3| + \frac{1}{4}\ln|y+1|\right]_4^8 = \left(\frac{3}{4}\ln 5 + \frac{1}{4}\ln 9\right) - \left(\frac{3}{4}\ln 1 + \frac{1}{4}\ln 5\right)$
$= \frac{1}{2}\ln 5 + \frac{1}{2}\ln 3 = \frac{\ln 15}{2}$

14. $\frac{y+4}{y^2+y} = \frac{A}{y} + \frac{B}{y+1} \Rightarrow y+4 = A(y+1) + By;\ y = 0 \Rightarrow A = 4;\ y = -1 \Rightarrow B = \frac{3}{-1} = -3;$
$\int_{1/2}^1 \frac{y+4}{y^2+y}\,dy = 4\int_{1/2}^1 \frac{dy}{y} - 3\int_{1/2}^1 \frac{dy}{y+1} = [4\ln|y| - 3\ln|y+1|]_{1/2}^1 = (4\ln 1 - 3\ln 2) - \left(4\ln\frac{1}{2} - 3\ln\frac{3}{2}\right)$
$= \ln\frac{1}{8} - \ln\frac{1}{16} + \ln\frac{27}{8} = \ln\left(\frac{27}{8}\cdot\frac{1}{8}\cdot 16\right) = \ln\frac{27}{4}$

15. $\frac{1}{t^3+t^2-2t} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-1} \Rightarrow 1 = A(t+2)(t-1) + Bt(t-1) + Ct(t+2);\ t = 0 \Rightarrow A = -\frac{1}{2};\ t = -2$
$\Rightarrow B = \frac{1}{6};\ t = 1 \Rightarrow C = \frac{1}{3};\ \int \frac{dt}{t^3+t^2-2t} = -\frac{1}{2}\int \frac{dt}{t} + \frac{1}{6}\int \frac{dt}{t+2} + \frac{1}{3}\int \frac{dt}{t-1}$
$= -\frac{1}{2}\ln|t| + \frac{1}{6}\ln|t+2| + \frac{1}{3}\ln|t-1| + C$

16. $\frac{x+3}{2x^3-8x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} \Rightarrow \frac{1}{2}(x+3) = A(x+2)(x-2) + Bx(x-2) + Cx(x+2);\ x = 0 \Rightarrow A = \frac{3}{-8};\ x = -2$
$\Rightarrow B = \frac{1}{16};\ x = 2 \Rightarrow C = \frac{5}{16};\ \int \frac{x+3}{2x^3-8x}\,dx = -\frac{3}{8}\int \frac{dx}{x} + \frac{1}{16}\int \frac{dx}{x+2} + \frac{5}{16}\int \frac{dx}{x-2}$
$= -\frac{3}{8}\ln|x| + \frac{1}{16}\ln|x+2| + \frac{5}{16}\ln|x-2| + C = \frac{1}{16}\ln\left|\frac{(x-2)^5(x+2)}{x^6}\right| + C$

17. $\frac{x^3}{x^2+2x+1} = (x-2) + \frac{3x+2}{(x+1)^2}$ (after long division); $\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow 3x+2 = A(x+1) + B$
$= Ax + (A+B) \Rightarrow A = 3,\ A+B = 2 \Rightarrow A = 3,\ B = -1;\ \int_0^1 \frac{x^3\,dx}{x^2+2x+1}$
$= \int_0^1 (x-2)\,dx + 3\int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{(x+1)^2} = \left[\frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1}\right]_0^1$
$= \left(\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2}\right) - (1) = 3\ln 2 - 2$

18. $\frac{x^3}{x^2-2x+1} = (x+2) + \frac{3x-2}{(x-1)^2}$ (after long division); $\frac{3x-2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 3x-2 = A(x-1) + B$
$= Ax + (-A+B) \Rightarrow A = 3,\ -A+B = -2 \Rightarrow A = 3,\ B = 1;\ \int_{-1}^0 \frac{x^3\,dx}{x^2-2x+1}$
$= \int_{-1}^0 (x+2)\,dx + 3\int_{-1}^0 \frac{dx}{x-1} + \int_{-1}^0 \frac{dx}{(x-1)^2} = \left[\frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1}\right]_{-1}^0$
$= \left(0 + 0 + 3\ln 1 - \frac{1}{(-1)}\right) - \left(\frac{1}{2} - 2 + 3\ln 2 - \frac{1}{(-2)}\right) = 2 - 3\ln 2$

19. $\frac{1}{(x^2-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x+1)^2} + \frac{D}{(x-1)^2} \Rightarrow 1 = A(x+1)(x-1)^2 + B(x-1)(x+1)^2 + C(x-1)^2 + D(x+1)^2;$
$x = -1 \Rightarrow C = \frac{1}{4};\ x = 1 \Rightarrow D = \frac{1}{4};$ coefficient of $x^3 = A + B \Rightarrow A + B = 0;$ constant $= A - B + C + D$
$\Rightarrow A - B + C + D = 1 \Rightarrow A - B = \frac{1}{2};$ thus, $A = \frac{1}{4} \Rightarrow B = -\frac{1}{4};\ \int \frac{dx}{(x^2-1)^2}$
$= \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x-1} + \frac{1}{4}\int \frac{dx}{(x+1)^2} + \frac{1}{4}\int \frac{dx}{(x-1)^2} = \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{x}{2(x^2-1)} + C$

20. $\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \Rightarrow x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1);\ x = -1$
$\Rightarrow C = -\frac{1}{2};\ x = 1 \Rightarrow A = \frac{1}{4};$ coefficient of $x^2 = A + B \Rightarrow A + B = 1 \Rightarrow B = \frac{3}{4};\ \int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$
$= \frac{1}{4}\int \frac{dx}{x-1} + \frac{3}{4}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{(x+1)^2} = \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2(x+1)} + C$
$= \frac{\ln|(x-1)(x+1)^3|}{4} + \frac{1}{2(x+1)} + C$

21. $\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \Rightarrow 1 = A(x^2+1) + (Bx+C)(x+1);\ x = -1 \Rightarrow A = \frac{1}{2};$ coefficient of x^2
$= A + B \Rightarrow A + B = 0 \Rightarrow B = -\frac{1}{2};$ constant $= A + C \Rightarrow A + C = 1 \Rightarrow C = \frac{1}{2};\ \int_0^1 \frac{dx}{(x+1)(x^2+1)}$

$= \frac{1}{2}\int_0^1 \frac{dx}{x+1} + \frac{1}{2}\int_0^1 \frac{(-x+1)}{x^2+1}\,dx = \left[\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x\right]_0^1$
$= \left(\frac{1}{2}\ln 2 - \frac{1}{4}\ln 2 + \frac{1}{2}\tan^{-1}1\right) - \left(\frac{1}{2}\ln 1 - \frac{1}{4}\ln 1 + \frac{1}{2}\tan^{-1}0\right) = \frac{1}{4}\ln 2 + \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{(\pi + 2\ln 2)}{8}$

22. $\frac{3t^2+t+4}{t^3+t} = \frac{A}{t} + \frac{Bt+C}{t^2+1} \Rightarrow 3t^2+t+4 = A(t^2+1) + (Bt+C)t$; $t=0 \Rightarrow A=4$; coefficient of t^2
$= A+B \Rightarrow A+B=3 \Rightarrow B=-1$; coefficient of $t = C \Rightarrow C=1$; $\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+1}\,dt$
$= 4\int_1^{\sqrt{3}} \frac{dt}{t} + \int_1^{\sqrt{3}} \frac{(-t+1)}{t^2+1}\,dt = \left[4\ln|t| - \frac{1}{2}\ln(t^2+1) + \tan^{-1}t\right]_1^{\sqrt{3}}$
$= \left(4\ln\sqrt{3} - \frac{1}{2}\ln 4 + \tan^{-1}\sqrt{3}\right) - \left(4\ln 1 - \frac{1}{2}\ln 2 + \tan^{-1}1\right) = 2\ln 3 - \ln 2 + \frac{\pi}{3} + \frac{1}{2}\ln 2 - \frac{\pi}{4}$
$= 2\ln 3 - \frac{1}{2}\ln 2 + \frac{\pi}{12} = \ln\left(\frac{9}{\sqrt{2}}\right) + \frac{\pi}{12}$

23. $\frac{y^2+2y+1}{(y^2+1)^2} = \frac{Ay+B}{y^2+1} + \frac{Cy+D}{(y^2+1)^2} \Rightarrow y^2+2y+1 = (Ay+B)(y^2+1) + Cy + D$
$= Ay^3 + By^2 + (A+C)y + (B+D) \Rightarrow A=0, B=1; A+C=2 \Rightarrow C=2; B+D=1 \Rightarrow D=0;$
$\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy = \int \frac{1}{y^2+1}\,dy + 2\int \frac{y}{(y^2+1)^2}\,dy = \tan^{-1}y - \frac{1}{y^2+1} + C$

24. $\frac{8x^2+8x+2}{(4x^2+1)^2} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{(4x^2+1)^2} \Rightarrow 8x^2+8x+2 = (Ax+B)(4x^2+1) + Cx + D$
$= 4Ax^3 + 4Bx^2 + (A+C)x + (B+D); A=0, B=2; A+C=8 \Rightarrow C=8; B+D=2 \Rightarrow D=0;$
$\int \frac{8x^2+8x+2}{(4x^2+1)^2}\,dx = 2\int \frac{dx}{4x^2+1} + 8\int \frac{x\,dx}{(4x^2+1)^2} = \tan^{-1}2x - \frac{1}{4x^2+1} + C$

25. $\frac{2s+2}{(s^2+1)(s-1)^3} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{(s-1)^2} + \frac{E}{(s-1)^3} \Rightarrow 2s+2$
$= (As+B)(s-1)^3 + C(s^2+1)(s-1)^2 + D(s^2+1)(s-1) + E(s^2+1)$
$= [As^4 + (-3A+B)s^3 + (3A-3B)s^2 + (-A+3B)s - B] + C(s^4 - 2s^3 + 2s^2 - 2s + 1) + D(s^3 - s^2 + s - 1)$
$\quad + E(s^2+1)$
$= (A+C)s^4 + (-3A+B-2C+D)s^3 + (3A-3B+2C-D+E)s^2 + (-A+3B-2C+D)s + (-B+C-D+E)$

$$\Rightarrow \left.\begin{array}{r} A + C = 0 \\ -3A + B - 2C + D = 0 \\ 3A - 3B + 2C - D + E = 0 \\ -A + 3B - 2C + D = 2 \\ -B + C - D + E = 2 \end{array}\right\}\text{ summing all equations } \Rightarrow 2E = 4 \Rightarrow E = 2;$$

summing eqs (2) and (3) $\Rightarrow -2B+2=0 \Rightarrow B=1$; summing eqs (3) and (4) $\Rightarrow 2A+2=2 \Rightarrow A=0; C=0$ from eq (1); then $-1+0-D+2=2$ from eq (5) $\Rightarrow D=-1$;
$\int \frac{2s+2}{(s^2+1)(s-1)^3}\,ds = \int \frac{ds}{s^2+1} - \int \frac{ds}{(s-1)^2} + 2\int \frac{ds}{(s-1)^3} = -(s-1)^{-2} + (s-1)^{-1} + \tan^{-1}s + C$

26. $\frac{s^4+81}{s(s^2+9)^2} = \frac{A}{s} + \frac{Bs+C}{s^2+9} + \frac{Ds+E}{(s^2+9)^2} \Rightarrow s^4+81 = A(s^2+9)^2 + (Bs+C)s(s^2+9) + (Ds+E)s$
$= A(s^4 + 18s^2 + 81) + (Bs^4 + Cs^3 + 9Bs^2 + 9Cs) + Ds^2 + Es$
$= (A+B)s^4 + Cs^3 + (18A+9B+D)s^2 + (9C+E)s + 81A \Rightarrow 81A = 81$ or $A=1$; $A+B=1 \Rightarrow B=0$;
$C=0$; $9C+E=0 \Rightarrow E=0$; $18A+9B+D=0 \Rightarrow D=-18$; $\int \frac{s^4+81}{s(s^2+9)^2}\,ds = \int \frac{ds}{s} - 18\int \frac{s\,ds}{(s^2+9)^2}$
$= \ln|s| + \frac{9}{(s^2+9)} + C$

27. $\frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2} = \frac{A\theta+B}{\theta^2+2\theta+2} + \frac{C\theta+D}{(\theta^2+2\theta+2)^2} \Rightarrow 2\theta^3+5\theta^2+8\theta+4 = (A\theta+B)(\theta^2+2\theta+2) + C\theta + D$
$= A\theta^3 + (2A+B)\theta^2 + (2A+2B+C)\theta + (2B+D) \Rightarrow A=2; 2A+B=5 \Rightarrow B=1; 2A+2B+C=8 \Rightarrow C=2;$
$2B+D=4 \Rightarrow D=2$; $\int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2}\,d\theta = \int \frac{2\theta+1}{(\theta^2+2\theta+2)}\,d\theta + \int \frac{2\theta+2}{(\theta^2+2\theta+2)^2}\,d\theta$
$= \int \frac{2\theta+2}{\theta^2+2\theta+2}\,d\theta - \int \frac{d\theta}{\theta^2+2\theta+2} + \int \frac{d(\theta^2+2\theta+2)}{(\theta^2+2\theta+2)^2} = \int \frac{d(\theta^2+2\theta+2)}{\theta^2+2\theta+2} - \int \frac{d\theta}{(\theta+1)^2+1} - \frac{1}{\theta^2+2\theta+2}$

$= \frac{-1}{\theta^2+2\theta+2} + \ln(\theta^2+2\theta+2) - \tan^{-1}(\theta+1) + C$

28. $\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3} = \frac{A\theta+B}{\theta^2+1} + \frac{C\theta+D}{(\theta^2+1)^2} + \frac{E\theta+F}{(\theta^2+1)^3} \Rightarrow \theta^4-4\theta^3+2\theta^2-3\theta+1$
$= (A\theta+B)(\theta^2+1)^2 + (C\theta+D)(\theta^2+1) + E\theta+F = (A\theta+B)(\theta^4+2\theta^2+1) + (C\theta^3+D\theta^2+C\theta+D) + E\theta+F$
$= (A\theta^5+B\theta^4+2A\theta^3+2B\theta^2+A\theta+B) + (C\theta^3+D\theta^2+C\theta+D) + E\theta+F$
$= A\theta^5+B\theta^4+(2A+C)\theta^3+(2B+D)\theta^2+(A+C+E)\theta+(B+D+F) \Rightarrow A=0; B=1; 2A+C=-4$
$\Rightarrow C=-4; 2B+D=2 \Rightarrow D=0; A+C+E=-3 \Rightarrow E=1; B+D+F=1 \Rightarrow F=0;$
$\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}\,d\theta = \int \frac{d\theta}{\theta^2+1} - 4\int \frac{\theta\,d\theta}{(\theta^2+1)^2} + \int \frac{\theta\,d\theta}{(\theta^2+1)^3} = \tan^{-1}\theta + 2(\theta^2+1)^{-1} - \frac{1}{4}(\theta^2+1)^{-2} + C$

29. $\frac{2x^3-2x^2+1}{x^2-x} = 2x + \frac{1}{x^2-x} = 2x + \frac{1}{x(x-1)}$; $\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \Rightarrow 1 = A(x-1)+Bx; x=0 \Rightarrow A=-1;$
$x=1 \Rightarrow B=1; \int \frac{2x^3-2x^2+1}{x^2-x} = \int 2x\,dx - \int \frac{dx}{x} + \int \frac{dx}{x-1} = x^2 - \ln|x| + \ln|x-1| + C = x^2 + \ln\left|\frac{x-1}{x}\right| + C$

30. $\frac{x^4}{x^2-1} = (x^2+1) + \frac{1}{x^2-1} = (x^2+1) + \frac{1}{(x+1)(x-1)}$; $\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \Rightarrow 1 = A(x-1)+B(x+1);$
$x=-1 \Rightarrow A=-\frac{1}{2}$; $x=1 \Rightarrow B=\frac{1}{2}$; $\int \frac{x^4}{x^2-1}\,dx = \int (x^2+1)\,dx - \frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{dx}{x-1}$
$= \frac{1}{3}x^3 + x - \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C = \frac{x^3}{3} + x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$

31. $\frac{9x^3-3x+1}{x^3-x^2} = 9 + \frac{9x^2-3x+1}{x^2(x-1)}$ (after long division); $\frac{9x^2-3x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$
$\Rightarrow 9x^2-3x+1 = Ax(x-1)+B(x-1)+Cx^2; x=1 \Rightarrow C=7; x=0 \Rightarrow B=-1; A+C=9 \Rightarrow A=2;$
$\int \frac{9x^3-3x+1}{x^3-x^2}\,dx = \int 9\,dx + 2\int \frac{dx}{x} - \int \frac{dx}{x^2} + 7\int \frac{dx}{x-1} = 9x + 2\ln|x| + \frac{1}{x} + 7\ln|x-1| + C$

32. $\frac{16x^3}{4x^2-4x+1} = (4x+4) + \frac{12x-4}{4x^2-4x+1}$; $\frac{12x-4}{(2x-1)^2} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} \Rightarrow 12x-4 = A(2x-1)+B$
$\Rightarrow A=6; -A+B=-4 \Rightarrow B=2; \int \frac{16x^3}{4x^2-4x+1}\,dx = 4\int (x+1)\,dx + 6\int \frac{dx}{2x-1} + 2\int \frac{dx}{(2x-1)^2}$
$= 2(x+1)^2 + 3\ln|2x-1| - \frac{1}{2x-1} + C_1 = 2x^2 + 4x + 3\ln|2x-1| - (2x-1)^{-1} + C$, where $C = 2 + C_1$

33. $\frac{y^4+y^2-1}{y^3+y} = y - \frac{1}{y(y^2+1)}$; $\frac{1}{y(y^2+1)} = \frac{A}{y} + \frac{By+C}{y^2+1} \Rightarrow 1 = A(y^2+1)+(By+C)y = (A+B)y^2+Cy+A$
$\Rightarrow A=1; A+B=0 \Rightarrow B=-1; C=0; \int \frac{y^4+y^2-1}{y^3+y}\,dy = \int y\,dy - \int \frac{dy}{y} + \int \frac{y\,dy}{y^2+1}$
$= \frac{y^2}{2} - \ln|y| + \frac{1}{2}\ln(1+y^2) + C$

34. $\frac{2y^4}{y^3-y^2+y-1} = 2y + 2 + \frac{2}{y^3-y^2+y-1}$; $\frac{2}{y^3-y^2+y-1} = \frac{2}{(y^2+1)(y-1)} = \frac{A}{y-1} + \frac{By+C}{y^2+1}$
$\Rightarrow 2 = A(y^2+1)+(By+C)(y-1) = (Ay^2+A)+(By^2+Cy-By-C) = (A+B)y^2+(-B+C)y+(A-C)$
$\Rightarrow A+B=0, -B+C=0$ or $C=B, A-C=A-B=2 \Rightarrow A=1, B=-1, C=-1;$
$\int \frac{2y^4}{y^3-y^2+y-1}\,dy = 2\int (y+1)\,dy + \int \frac{dy}{y-1} - \int \frac{y}{y^2+1}\,dy - \int \frac{dy}{y^2+1}$
$= (y+1)^2 + \ln|y-1| - \frac{1}{2}\ln(y^2+1) - \tan^{-1}y + C_1 = y^2 + 2y + \ln|y-1| - \frac{1}{2}\ln(y^2+1) - \tan^{-1}y + C,$
where $C = C_1 + 1$

35. $\int \frac{e^t\,dt}{e^{2t}+3e^t+2} = [e^t = y]\int \frac{dy}{y^2+3y+2} = \int \frac{dy}{y+1} - \int \frac{dy}{y+2} = \ln\left|\frac{y+1}{y+2}\right| + C = \ln\left(\frac{e^t+1}{e^t+2}\right) + C$

36. $\int \frac{e^{4t}+2e^{2t}-e^t}{e^{2t}+1}\,dt = \int \frac{e^{3t}+2e^t-1}{e^{2t}+1}e^t dt;\ \begin{bmatrix} y = e^t \\ dy = e^t\,dt \end{bmatrix} \to \int \frac{y^3+2y-1}{y^2+1}\,dy = \int \left(y + \frac{y-1}{y^2+1}\right)dy = \frac{y^2}{2} + \int \frac{y}{y^2+1}\,dy - \int \frac{dy}{y^2+1}$
$= \frac{y^2}{2} + \frac{1}{2}\ln(y^2+1) - \tan^{-1}y + C = \frac{1}{2}e^{2t} + \frac{1}{2}\ln(e^{2t}+1) - \tan^{-1}(e^t) + C$

37. $\int \frac{\cos y\, dy}{\sin^2 y + \sin y - 6}$; $[\sin y = t, \cos y\, dy = dt] \rightarrow \int \frac{dy}{t^2 + t - 6} = \frac{1}{5} \int \left(\frac{1}{t-2} - \frac{1}{t+3} \right) dt = \frac{1}{5} \ln \left| \frac{t-2}{t+3} \right| + C$
$= \frac{1}{5} \ln \left| \frac{\sin y - 2}{\sin y + 3} \right| + C$

38. $\int \frac{\sin \theta\, d\theta}{\cos^2 \theta + \cos \theta - 2}$; $[\cos \theta = y] \rightarrow -\int \frac{dy}{y^2 + y - 2} = \frac{1}{3} \int \frac{dy}{y+2} - \frac{1}{3} \int \frac{dy}{y-1} = \frac{1}{3} \ln \left| \frac{y+2}{y-1} \right| + C = \frac{1}{3} \ln \left| \frac{\cos \theta + 2}{\cos \theta - 1} \right| + C$
$= \frac{1}{3} \ln \left| \frac{2 + \cos \theta}{1 - \cos \theta} \right| + C = -\frac{1}{3} \ln \left| \frac{\cos \theta - 1}{\cos \theta + 2} \right| + C$

39. $\int \frac{(x-2)^2 \tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2}\, dx = \int \frac{\tan^{-1}(2x)}{4x^2+1}\, dx - 3 \int \frac{x}{(x-2)^2}\, dx$
$= \frac{1}{2} \int \tan^{-1}(2x)\, d\left(\tan^{-1}(2x)\right) - 3 \int \frac{dx}{x-2} - 6 \int \frac{dx}{(x-2)^2} = \frac{\left(\tan^{-1} 2x\right)^2}{4} - 3 \ln |x-2| + \frac{6}{x-2} + C$

40. $\int \frac{(x+1)^2 \tan^{-1}(3x) + 9x^3 + x}{(9x^2+1)(x+1)^2}\, dx = \int \frac{\tan^{-1}(3x)}{9x^2+1}\, dx + \int \frac{x}{(x+1)^2}\, dx$
$= \frac{1}{3} \int \tan^{-1}(3x)\, d\left(\tan^{-1}(3x)\right) + \int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2} = \frac{\left(\tan^{-1} 3x\right)^2}{6} + \ln |x+1| + \frac{1}{x+1} + C$

41. $(t^2 - 3t + 2) \frac{dx}{dt} = 1$; $x = \int \frac{dt}{t^2 - 3t + 2} = \int \frac{dt}{t-2} - \int \frac{dt}{t-1} = \ln \left| \frac{t-2}{t-1} \right| + C$; $\frac{t-2}{t-1} = Ce^x$; $t = 3$ and $x = 0$
$\Rightarrow \frac{1}{2} = C \Rightarrow \frac{t-2}{t-1} = \frac{1}{2} e^x \Rightarrow x = \ln \left| 2 \left(\frac{t-2}{t-1} \right) \right| = \ln |t-2| - \ln |t-1| + \ln 2$

42. $(3t^4 + 4t^2 + 1) \frac{dx}{dt} = 2\sqrt{3}$; $x = 2\sqrt{3} \int \frac{dt}{3t^4 + 4t^2 + 1} = \sqrt{3} \int \frac{dt}{t^2 + \frac{1}{3}} - \sqrt{3} \int \frac{dt}{t^2 + 1}$
$= 3 \tan^{-1}\left(\sqrt{3}t\right) - \sqrt{3} \tan^{-1} t + C$; $t = 1$ and $x = \frac{-\pi\sqrt{3}}{4} \Rightarrow -\frac{\sqrt{3}\pi}{4} = \pi - \frac{\sqrt{3}}{4}\pi + C \Rightarrow C = -\pi$
$\Rightarrow x = 3 \tan^{-1}\left(\sqrt{3}t\right) - \sqrt{3} \tan^{-1} t - \pi$

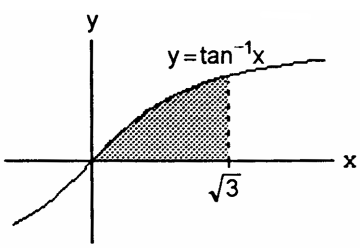
43. $(t^2 + 2t) \frac{dx}{dt} = 2x + 2$; $\frac{1}{2} \int \frac{dx}{x+1} = \int \frac{dt}{t^2 + 2t} \Rightarrow \frac{1}{2} \ln |x+1| = \frac{1}{2} \int \frac{dt}{t} - \frac{1}{2} \int \frac{dt}{t+2} \Rightarrow \ln |x+1| = \ln \left| \frac{t}{t+2} \right| + C$;
$t = 1$ and $x = 1 \Rightarrow \ln 2 = \ln \frac{1}{3} + C \Rightarrow C = \ln 2 + \ln 3 = \ln 6 \Rightarrow \ln |x+1| = \ln 6 \left| \frac{t}{t+2} \right| \Rightarrow x + 1 = \frac{6t}{t+2}$
$\Rightarrow x = \frac{6t}{t+2} - 1, t > 0$

44. $(t+1) \frac{dx}{dt} = x^2 + 1 \Rightarrow \int \frac{dx}{x^2+1} = \int \frac{dt}{t+1} \Rightarrow \tan^{-1} x = \ln |t+1| + C$; $t = 0$ and $x = \frac{\pi}{4} \Rightarrow \tan^{-1} \frac{\pi}{4} = \ln |1| + C$
$\Rightarrow C = \tan^{-1} \frac{\pi}{4} = 1 \Rightarrow \tan^{-1} x = \ln |t+1| + 1 \Rightarrow x = \tan(\ln(t+1) + 1), t > -1$

45. $V = \pi \int_{0.5}^{2.5} y^2\, dx = \pi \int_{0.5}^{2.5} \frac{9}{3x - x^2}\, dx = 3\pi \left(\int_{0.5}^{2.5} \left(-\frac{1}{x-3} + \frac{1}{x} \right) \right) dx = \left[3\pi \ln \left| \frac{x}{x-3} \right| \right]_{0.5}^{2.5} = 3\pi \ln 25$

46. $V = 2\pi \int_0^1 xy\, dx = 2\pi \int_0^1 \frac{2x}{(x+1)(2-x)}\, dx = 4\pi \int_0^1 \left(-\frac{1}{3} \left(\frac{1}{x+1} \right) + \frac{2}{3} \left(\frac{1}{2-x} \right) \right) dx$
$= \left[-\frac{4\pi}{3} \left(\ln |x+1| + 2 \ln |2-x| \right) \right]_0^1 = \frac{4\pi}{3} (\ln 2)$

47. $A = \int_0^{\sqrt{3}} \tan^{-1} x\, dx = \left[x \tan^{-1} x \right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x}{1+x^2}\, dx$
$= \frac{\pi\sqrt{3}}{3} - \left[\frac{1}{2} \ln (x^2 + 1) \right]_0^{\sqrt{3}} = \frac{\pi\sqrt{3}}{3} - \ln 2$;
$\overline{x} = \frac{1}{A} \int_0^{\sqrt{3}} x \tan^{-1} x\, dx$
$= \frac{1}{A} \left(\left[\frac{1}{2} x^2 \tan^{-1} x \right]_0^{\sqrt{3}} - \frac{1}{2} \int_0^{\sqrt{3}} \frac{x^2}{1+x^2}\, dx \right)$
$= \frac{1}{A} \left[\frac{\pi}{2} - \left[\frac{1}{2} (x - \tan^{-1} x) \right]_0^{\sqrt{3}} \right]$
$= \frac{1}{A} \left(\frac{\pi}{2} - \frac{\sqrt{3}}{2} + \frac{\pi}{6} \right) = \frac{1}{A} \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right) \cong 1.10$

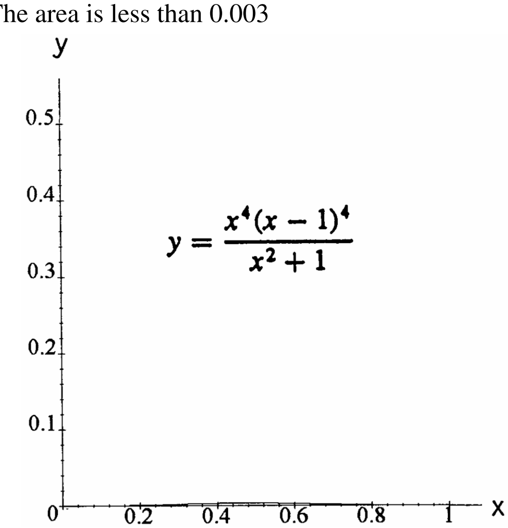


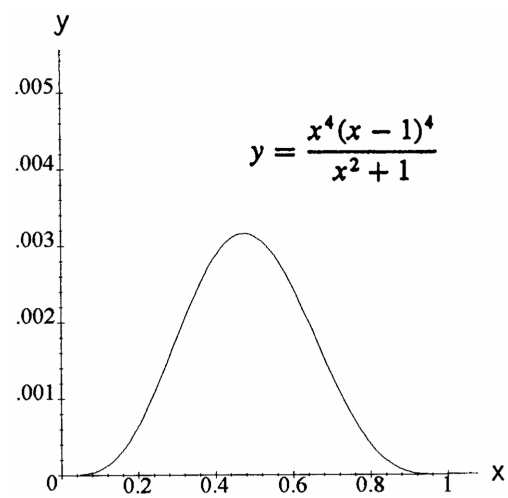
48. $A = \int_3^5 \frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx = 3\int_3^5 \frac{dx}{x} - \int_3^5 \frac{dx}{x+3} + 2\int_3^5 \frac{dx}{x-1} = \left[3\ln|x| - \ln|x+3| + 2\ln|x-1|\right]_3^5 = \ln\frac{125}{9}$;
$\overline{x} = \frac{1}{A}\int_3^5 \frac{x(4x^2+13x-9)}{x^3+2x^2-3x}\,dx = \frac{1}{A}\left([4x]_3^5 + 3\int_3^5 \frac{dx}{x+3} + 2\int_3^5 \frac{dx}{x-1}\right) = \frac{1}{A}(8 + 11\ln 2 - 3\ln 6) \cong 3.90$

49. (a) $\frac{dx}{dt} = kx(N-x) \Rightarrow \int \frac{dx}{x(N-x)} = \int k\,dt \Rightarrow \frac{1}{N}\int \frac{dx}{x} + \frac{1}{N}\int \frac{dx}{N-x} = \int k\,dt \Rightarrow \frac{1}{N}\ln\left|\frac{x}{N-x}\right| = kt + C$;
$k = \frac{1}{250}$, $N = 1000$, $t = 0$ and $x = 2 \Rightarrow \frac{1}{1000}\ln\left|\frac{2}{998}\right| = C \Rightarrow \frac{1}{1000}\ln\left|\frac{x}{1000-x}\right| = \frac{t}{250} + \frac{1}{1000}\ln\left(\frac{1}{499}\right)$
$\Rightarrow \ln\left|\frac{499x}{1000-x}\right| = 4t \Rightarrow \frac{499x}{1000-x} = e^{4t} \Rightarrow 499x = e^{4t}(1000-x) \Rightarrow (499+e^{4t})\,x = 1000e^{4t} \Rightarrow x = \frac{1000e^{4t}}{499+e^{4t}}$
(b) $x = \frac{1}{2}N = 500 \Rightarrow 500 = \frac{1000e^{4t}}{499+e^{4t}} \Rightarrow 500 \cdot 499 + 500e^{4t} = 1000e^{4t} \Rightarrow e^{4t} = 499 \Rightarrow t = \frac{1}{4}\ln 499 \approx 1.55$ days

50. $\frac{dx}{dt} = k(a-x)(b-x) \Rightarrow \frac{dx}{(a-x)(b-x)} = k\,dt$
(a) $a = b$: $\int \frac{dx}{(a-x)^2} = \int k\,dt \Rightarrow \frac{1}{a-x} = kt + C$; $t = 0$ and $x = 0 \Rightarrow \frac{1}{a} = C \Rightarrow \frac{1}{a-x} = kt + \frac{1}{a}$
$\Rightarrow \frac{1}{a-x} = \frac{akt+1}{a} \Rightarrow a - x = \frac{a}{akt+1} \Rightarrow x = a - \frac{a}{akt+1} = \frac{a^2kt}{akt+1}$
(b) $a \ne b$: $\int \frac{dx}{(a-x)(b-x)} = \int k\,dt \Rightarrow \frac{1}{b-a}\int \frac{dx}{a-x} - \frac{1}{b-a}\int \frac{dx}{b-x} = \int k\,dt \Rightarrow \frac{1}{b-a}\ln\left|\frac{b-x}{a-x}\right| = kt + C$;
$t = 0$ and $x = 0 \Rightarrow \frac{1}{b-a}\ln\frac{b}{a} = C \Rightarrow \ln\left|\frac{b-x}{a-x}\right| = (b-a)kt + \ln\left(\frac{b}{a}\right) \Rightarrow \frac{b-x}{a-x} = \frac{b}{a}e^{(b-a)kt}$
$\Rightarrow x = \frac{ab\left[1-e^{(b-a)kt}\right]}{a-be^{(b-a)kt}}$

51. (a) $\int_0^1 \frac{x^4(x-1)^4}{x^2+1}\,dx = \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{x^2+1}\right) dx = \frac{22}{7} - \pi$
(b) $\frac{\frac{22}{7}-\pi}{\pi} \cdot 100\% \cong 0.04\%$
(c) The area is less than 0.003





52. $P(x) = ax^2 + bx + c$, $P(0) = c = 1$ and $P'(0) = 0 \Rightarrow b = 0 \Rightarrow P(x) = ax^2 + 1$. Next,
$\frac{ax^2+1}{x^3(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2}$; for the integral to be a rational function, we must have $A = 0$ and $D = 0$. Thus, $ax^2 + 1 = Bx(x-1)^2 + C(x-1)^2 + Ex^3 = (B+E)x^3 + (C-2B)x^2 + (B-2C)x + C$
$\Rightarrow \left.\begin{array}{r} B + E = 0 \\ C - 2B = a \\ C = 1 \end{array}\right\} \Rightarrow E = -B$; $x = 1 \Rightarrow a + 1 = E$; therefore, $1 - 2B = a \Rightarrow 1 + 2E = a \Rightarrow 1 + 2(a+1) = a$
$\Rightarrow a = -3$

8.4 TRIGONOMETRIC INTEGRALS

1. $\int_0^{\pi/2} \sin^5 x\,dx = \int_0^{\pi/2} (\sin^2 x)^2 \sin x\,dx = \int_0^{\pi/2} (1-\cos^2 x)^2 \sin x\,dx = \int_0^{\pi/2} (1-2\cos^2 x+\cos^4 x)\sin x\,dx$
$= \int_0^{\pi/2} \sin x\,dx - \int_0^{\pi/2} 2\cos^2 x \sin x\,dx + \int_0^{\pi/2} \cos^4 x \sin x\,dx = \left[-\cos x + 2\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5}\right]_0^{\pi/2}$
$= (0) - \left(-1+\frac{2}{3}-\frac{1}{5}\right) = \frac{8}{15}$

2. $\int_0^{\pi} \sin^5\left(\frac{x}{2}\right)dx$ (using Exercise 1) $= \int_0^{\pi} \sin\left(\frac{x}{2}\right)dx - \int_0^{\pi} 2\cos^2\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)dx + \int_0^{\pi} \cos^4\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)dx$
$= \left[-2\cos\left(\frac{x}{2}\right) + \frac{4}{3}\cos^3\left(\frac{x}{2}\right) - \frac{2}{5}\cos^5\left(\frac{x}{2}\right)\right]_0^{\pi} = (0) - \left(-2+\frac{4}{3}-\frac{2}{5}\right) = \frac{16}{15}$

3. $\int_{-\pi/2}^{\pi/2} \cos^3 x\,dx = \int_{-\pi/2}^{\pi/2} (\cos^2 x)\cos x\,dx = \int_{-\pi/2}^{\pi/2} (1-\sin^2 x)\cos x\,dx = \int_{-\pi/2}^{\pi/2} \cos x\,dx - \int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\,dx$
$= \left[\sin x - \frac{\sin^3 x}{3}\right]_{-\pi/2}^{\pi/2} = \left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right) = \frac{4}{3}$

4. $\int_0^{\pi/6} 3\cos^5 3x\,dx = \int_0^{\pi/6} (\cos^2 3x)^2 \cos 3x \cdot 3dx = \int_0^{\pi/6} (1-\sin^2 3x)^2 \cos 3x \cdot 3dx = \int_0^{\pi/6} (1-2\sin^2 3x+\sin^4 3x)\cos 3x \cdot 3dx$
$= \int_0^{\pi/6} \cos 3x \cdot 3dx - 2\int_0^{\pi/6} \sin^2 3x \cos 3x \cdot 3dx + \int_0^{\pi/6} \sin^4 3x \cos 3x \cdot 3dx = \left[\sin 3x - 2\frac{\sin^3 3x}{3} + \frac{\sin^5 3x}{5}\right]_0^{\pi/6}$
$= \left(1-\frac{2}{3}+\frac{1}{5}\right) - (0) = \frac{8}{15}$

5. $\int_0^{\pi/2} \sin^7 y\,dy = \int_0^{\pi/2} \sin^6 y \sin y\,dy = \int_0^{\pi/2} (1-\cos^2 y)^3 \sin y\,dy = \int_0^{\pi/2} \sin y\,dy - 3\int_0^{\pi/2} \cos^2 y \sin y\,dy$
$+ 3\int_0^{\pi/2} \cos^4 y \sin y\,dy - \int_0^{\pi/2} \cos^6 y \sin y\,dy = \left[-\cos y + 3\frac{\cos^3 y}{3} - 3\frac{\cos^5 y}{5} + \frac{\cos^7 y}{7}\right]_0^{\pi/2} = (0) - \left(-1+1-\frac{3}{5}+\frac{1}{7}\right) = \frac{16}{35}$

6. $\int_0^{\pi/2} 7\cos^7 t\,dt$ (using Exercise 5) $= 7\left[\int_0^{\pi/2} \cos t\,dt - 3\int_0^{\pi/2} \sin^2 t \cos t\,dt + 3\int_0^{\pi/2} \sin^4 t \cos t\,dt - \int_0^{\pi/2} \sin^6 t \cos t\,dt\right]$
$= 7\left[\sin t - 3\frac{\sin^3 t}{3} + 3\frac{\sin^5 t}{5} - \frac{\sin^7 t}{7}\right]_0^{\pi/2} = 7\left(1-1+\frac{3}{5}-\frac{1}{7}\right) - 7(0) = \frac{16}{5}$

7. $\int_0^{\pi} 8\sin^4 x\,dx = 8\int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right)^2 dx = 2\int_0^{\pi} (1-2\cos 2x+\cos^2 2x)dx = 2\int_0^{\pi} dx - 2\int_0^{\pi} \cos 2x \cdot 2dx + 2\int_0^{\pi} \frac{1+\cos 4x}{2}\,dx$
$= [2x - 2\sin 2x]_0^{\pi} + \int_0^{\pi} dx + \int_0^{\pi} \cos 4x\,dx = 2\pi + \left[x + \frac{1}{2}\sin 4x\right]_0^{\pi} = 2\pi + \pi = 3\pi$

8. $\int_0^1 8\cos^4 2\pi x\,dx = 8\int_0^1 \left(\frac{1+\cos 4\pi x}{2}\right)^2 dx = 2\int_0^1 (1+2\cos 4\pi x+\cos^2 4\pi x)dx = 2\int_0^1 dx + 4\int_0^1 \cos 4\pi x\,dx + 2\int_0^1 \frac{1+\cos 8\pi x}{2}\,dx$
$= \left[2x + \frac{1}{\pi}\sin 4\pi x\right]_0^1 + \int_0^1 dx + \int_0^1 \cos 8\pi x\,dx = 2 + \left[x + \frac{1}{8\pi}\sin 8\pi x\right]_0^1 = 2 + 1 = 3$

9. $\int_{-\pi/4}^{\pi/4} 16\sin^2 x \cos^2 x\,dx = 16\int_{-\pi/4}^{\pi/4} \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)dx = 4\int_{-\pi/4}^{\pi/4} (1-\cos^2 2x)dx = 4\int_{-\pi/4}^{\pi/4} dx - 4\int_{-\pi/4}^{\pi/4} \left(\frac{1+\cos 4x}{2}\right)dx$
$= [4x]_{-\pi/4}^{\pi/4} - 2\int_{-\pi/4}^{\pi/4} dx - 2\int_{-\pi/4}^{\pi/4} \cos 4x\,dx = \pi + \pi - \left[2x + \frac{\sin 4x}{2}\right]_{-\pi/4}^{\pi/4} = 2\pi - \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \pi$

10. $\int_0^{\pi} 8\sin^4 y \cos^2 y\,dy = 8\int_0^{\pi} \left(\frac{1-\cos 2y}{2}\right)^2\left(\frac{1+\cos 2y}{2}\right)dy = \int_0^{\pi} dy - \int_0^{\pi} \cos 2y\,dy - \int_0^{\pi} \cos^2 2y\,dy + \int_0^{\pi} \cos^3 2y\,dy$
$= \left[y - \frac{1}{2}\sin 2y\right]_0^{\pi} - \int_0^{\pi} \left(\frac{1+\cos 4y}{2}\right)dy + \int_0^{\pi} (1-\sin^2 2y)\cos 2y\,dy = \pi - \frac{1}{2}\int_0^{\pi} dy - \frac{1}{2}\int_0^{\pi} \cos 4y\,dy + \int_0^{\pi} \cos 2y\,dy$
$- \int_0^{\pi} \sin^2 2y \cos 2y\,dy = \pi + \left[-\frac{1}{2}y - \frac{1}{8}\sin 4y + \frac{1}{2}\sin 2y - \frac{1}{2}\cdot\frac{\sin^3 2y}{3}\right]_0^{\pi} = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

11. $\int_0^{\pi/2} 35\sin^4 x\cos^3 x\,dx = \int_0^{\pi/2} 35\sin^4 x\,(1-\sin^2 x)\cos x\,dx = 35\int_0^{\pi/2}\sin^4 x\cos x\,dx - 35\int_0^{\pi/2}\sin^6 x\cos x\,dx$
$= \left[35\frac{\sin^5 x}{5} - 35\frac{\sin^7 x}{7}\right]_0^{\pi/2} = (7-5)-(0) = 2$

12. $\int_0^{\pi}\cos^2 2x\sin 2x\,dx = \left[-\frac{1}{2}\frac{\cos^3 2x}{3}\right]_0^{\pi} = -\frac{1}{6}+\frac{1}{6} = 0$

13. $\int_0^{\pi/4} 8\cos^3 2\theta\sin 2\theta\,d\theta = \left[8\left(-\frac{1}{2}\right)\frac{\cos^4 2\theta}{4}\right]_0^{\pi/4} = \left[-\cos^4 2\theta\right]_0^{\pi/4} = (0)-(-1) = 1$

14. $\int_0^{\pi/2}\sin^2 2\theta\cos^3 2\theta\,d\theta = \int_0^{\pi/2}\sin^2 2\theta(1-\sin^2 2\theta)\cos 2\theta\,d\theta = \int_0^{\pi/2}\sin^2 2\theta\cos 2\theta\,d\theta - \int_0^{\pi/2}\sin^4 2\theta\cos 2\theta\,d\theta$
$= \left[\frac{1}{2}\cdot\frac{\sin^3 2\theta}{3} - \frac{1}{2}\cdot\frac{\sin^5 2\theta}{5}\right]_0^{\pi/2} = 0$

15. $\int_0^{2\pi}\sqrt{\frac{1-\cos x}{2}}\,dx = \int_0^{2\pi}\left|\sin\frac{x}{2}\right|dx = \int_0^{2\pi}\sin\frac{x}{2}\,dx = \left[-2\cos\frac{x}{2}\right]_0^{2\pi} = 2+2 = 4$

16. $\int_0^{\pi}\sqrt{1-\cos 2x}\,dx = \int_0^{\pi}\sqrt{2}\,|\sin 2x|\,dx = \int_0^{\pi}\sqrt{2}\sin 2x\,dx = \left[-\sqrt{2}\cos 2x\right]_0^{\pi} = \sqrt{2}+\sqrt{2} = 2\sqrt{2}$

17. $\int_0^{\pi}\sqrt{1-\sin^2 t}\,dt = \int_0^{\pi}|\cos t|\,dt = \int_0^{\pi/2}\cos t\,dt - \int_{\pi/2}^{\pi}\cos t\,dt = [\sin t]_0^{\pi/2} - [\sin t]_{\pi/2}^{\pi} = 1-0-0+1 = 2$

18. $\int_0^{\pi}\sqrt{1-\cos^2\theta}\,d\theta = \int_0^{\pi}|\sin\theta|\,d\theta = \int_0^{\pi}\sin\theta\,d\theta = [-\cos\theta]_0^{\pi} = 1+1 = 2$

19. $\int_{-\pi/4}^{\pi/4}\sqrt{1+\tan^2 x}\,dx = \int_{-\pi/4}^{\pi/4}|\sec x|\,dx = \int_{-\pi/4}^{\pi/4}\sec x\,dx = [\ln|\sec x+\tan x|]_{-\pi/4}^{\pi/4} = \ln\left(\sqrt{2}+1\right) - \ln\left(\sqrt{2}-1\right)$
$= \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = 2\ln\left(1+\sqrt{2}\right)$

20. $\int_{-\pi/4}^{\pi/4}\sqrt{\sec^2 x-1}\,dx = \int_{-\pi/4}^{\pi/4}|\tan x|\,dx = -\int_{-\pi/4}^{0}\tan x\,dx + \int_0^{\pi/4}\tan x\,dx = [-\ln|\sec x|]_{-\pi/4}^{0} + [-\ln|\sec x|]_0^{\pi/4}$
$= -\ln(1) + \ln\sqrt{2} + \ln\sqrt{2} - \ln(1) = 2\ln\sqrt{2} = \ln 2$

21. $\int_0^{\pi/2}\theta\sqrt{1-\cos 2\theta}\,d\theta = \int_0^{\pi/2}\theta\sqrt{2}\,|\sin\theta|\,d\theta = \sqrt{2}\int_0^{\pi/2}\theta\sin\theta\,d\theta = \sqrt{2}\,[-\theta\cos\theta+\sin\theta]_0^{\pi/2} = \sqrt{2}(1) = \sqrt{2}$

22. $\int_{-\pi}^{\pi}(1-\cos^2 t)^{3/2}\,dt = \int_{-\pi}^{\pi}(\sin^2 t)^{3/2}\,dt = \int_{-\pi}^{\pi}\left|\sin^3 t\right|\,dt = -\int_{-\pi}^{0}\sin^3 t\,dt + \int_0^{\pi}\sin^3 t\,dt = -\int_{-\pi}^{0}(1-\cos^2 t)\sin t\,dt$
$+ \int_0^{\pi}(1-\cos^2 t)\sin t\,dt = -\int_{-\pi}^{0}\sin t\,dt + \int_{-\pi}^{0}\cos^2 t\sin t\,dt + \int_0^{\pi}\sin t\,dt - \int_0^{\pi}\cos^2 t\sin t\,dt = \left[\cos t - \frac{\cos^3 t}{3}\right]_{-\pi}^{0}$
$+ \left[-\cos t + \frac{\cos^3 t}{3}\right]_0^{\pi} = \left(1-\frac{1}{3}+1-\frac{1}{3}\right) + \left(1-\frac{1}{3}+1-\frac{1}{3}\right) = \frac{8}{3}$

23. $\int_{-\pi/3}^{0} 2\sec^3 x\,dx$; $u = \sec x$, $du = \sec x\tan x\,dx$, $dv = \sec^2 x\,dx$, $v = \tan x$;
$\int_{-\pi/3}^{0} 2\sec^3 x\,dx = [2\sec x\tan x]_{-\pi/3}^{0} - 2\int_{-\pi/3}^{0}\sec x\tan^2 x\,dx = 2\cdot 1\cdot 0 - 2\cdot 2\cdot\sqrt{3} - 2\int_{-\pi/3}^{0}\sec x\,(\sec^2 x-1)\,dx$
$= 4\sqrt{3} - 2\int_{-\pi/3}^{0}\sec^3 x\,dx + 2\int_{-\pi/3}^{0}\sec x\,dx$; $2\int_{-\pi/3}^{0} 2\sec^3 x\,dx = 4\sqrt{3} + [2\ln|\sec x+\tan x|]_{-\pi/3}^{0}$
$2\int_{-\pi/3}^{0} 2\sec^3 x\,dx = 4\sqrt{3} + 2\ln|1+0| - 2\ln|2-\sqrt{3}| = 4\sqrt{3} - 2\ln\left(2-\sqrt{3}\right)$
$\int_{-\pi/3}^{0} 2\sec^3 x\,dx = 2\sqrt{3} - \ln\left(2-\sqrt{3}\right)$

24. $\int e^x \sec^3(e^x)dx$; $u = \sec(e^x)$, $du = \sec(e^x)\tan(e^x)e^x dx$, $dv = \sec^2(e^x)e^x dx$, $v = \tan(e^x)$.

$$\begin{aligned}\int e^x \sec^3(e^x)\,dx &= \sec(e^x)\tan(e^x) - \int \sec(e^x)\tan^2(e^x)e^x dx \\ &= \sec(e^x)\tan(e^x) - \int \sec(e^x)(\sec^2(e^x) - 1)e^x dx \\ &= \sec(e^x)\tan(e^x) - \int \sec^3(e^x)e^x dx + \int \sec(e^x)e^x dx\end{aligned}$$

$$2\int e^x \sec^3(e^x)\,dx = \sec(e^x)\tan(e^x) + \ln\left|\sec(e^x) + \tan(e^x)\right| + C$$

$$\int e^x \sec^3(e^x)\,dx = \tfrac{1}{2}\left(\sec(e^x)\tan(e^x) + \ln\left|\sec(e^x) + \tan(e^x)\right|\right) + C$$

25. $\int_0^{\pi/4} \sec^4\theta\, d\theta = \int_0^{\pi/4} (1 + \tan^2\theta)\sec^2\theta\, d\theta = \int_0^{\pi/4} \sec^2\theta\, d\theta + \int_0^{\pi/4} \tan^2\theta \sec^2\theta\, d\theta = \left[\tan\theta + \frac{\tan^3\theta}{3}\right]_0^{\pi/4}$
$= \left(1 + \frac{1}{3}\right) - (0) = \frac{4}{3}$

26. $\int_0^{\pi/12} 3\sec^4(3x)\,dx = \int_0^{\pi/12} (1 + \tan^2(3x))\sec^2(3x)3dx = \int_0^{\pi/} \sec^2(3x)3dx + \int_0^{\pi/12} \tan^2(3x)\sec^2(3x)3dx$
$= \left[\tan(3x) + \frac{\tan^3(3x)}{3}\right]_0^{\pi/12} = \left(1 + \frac{1}{3}\right) - (0) = \frac{4}{3}$

27. $\int_{\pi/4}^{\pi/2} \csc^4\theta\, d\theta = \int_{\pi/4}^{\pi/2} (1 + \cot^2\theta)\csc^2\theta\, d\theta = \int_{\pi/4}^{\pi/2} \csc^2\theta\, d\theta + \int_{\pi/4}^{\pi/2} \cot^2\theta \csc^2\theta\, d\theta = \left[-\cot\theta - \frac{\cot^3\theta}{3}\right]_{\pi/4}^{\pi/2}$
$= (0) - \left(-1 - \frac{1}{3}\right) = \frac{4}{3}$

28. $\int_{\pi/2}^{\pi} 3\csc^4\frac{\theta}{2}\, d\theta = 3\int_{\pi/2}^{\pi} \left(1 + \cot^2\frac{\theta}{2}\right)\csc^2\frac{\theta}{2}\, d\theta = 3\int_{\pi/2}^{\pi} \csc^2\frac{\theta}{2}\, d\theta + 3\int_{\pi/2}^{\pi} \cot^2\frac{\theta}{2}\csc^2\frac{\theta}{2}\, d\theta = \left[-6\cot\frac{\theta}{2} - 6\frac{\cot^3\frac{\theta}{2}}{3}\right]_{\pi/2}^{\pi}$
$= (-6\cdot 0 - 2\cdot 0) - (-6\cdot 1 - 2\cdot 1) = 8$

29. $\int_0^{\pi/4} 4\tan^3 x\, dx = 4\int_0^{\pi/4} (\sec^2 x - 1)\tan x\, dx = 4\int_0^{\pi/4} \sec^2 x \tan x\, dx - 4\int_0^{\pi/4} \tan x\, dx = \left[4\frac{\tan^2 x}{2} - 4\ln|\sec x|\right]_0^{\pi/4}$
$= 2(1) - 4\ln\sqrt{2} - 2\cdot 0 + 4\ln 1 = 2 - 2\ln 2$

30. $\int_{-\pi/4}^{\pi/4} 6\tan^4 x\, dx = 6\int_{-\pi/4}^{\pi/4} (\sec^2 x - 1)\tan^2 x\, dx = 6\int_{-\pi/4}^{\pi/4} \sec^2 x \tan^2 x\, dx - 6\int_{-\pi/4}^{\pi/4} \tan^2 x\, dx$
$= 6\int_{-\pi/4}^{\pi/4} \sec^2 x\tan^2 x\, dx - 6\int_{-\pi/4}^{\pi/4} (\sec^2 x - 1)dx = \left[6\frac{\tan^3 x}{3}\right]_{-\pi/4}^{\pi/4} - 6\int_{-\pi/4}^{\pi/4} \sec^2 x\, dx + 6\int_{-\pi/4}^{\pi/4} dx$
$= 2(1 - (-1)) - [6\tan x]_{-\pi/4}^{\pi/4} + [6x]_{-\pi/4}^{\pi/4} = 4 - 6(1 - (-1)) + \frac{3\pi}{2} + \frac{3\pi}{2} = 3\pi - 8$

31. $\int_{\pi/6}^{\pi/3} \cot^3 x\, dx = \int_{\pi/6}^{\pi/3} (\csc^2 x - 1)\cot x\, dx = \int_{\pi/6}^{\pi/3} \csc^2 x \cot x\, dx - \int_{\pi/6}^{\pi/3} \cot x\, dx = \left[-\frac{\cot^2 x}{2} + \ln|\csc x|\right]_{\pi/6}^{\pi/3}$
$= -\frac{1}{2}\left(\frac{1}{3} - 3\right) + \left(\ln\frac{2}{\sqrt{3}} - \ln 2\right) = \frac{4}{3} - \ln\sqrt{3}$

32. $\int_{\pi/4}^{\pi/2} 8\cot^4 t\, dt = 8\int_{\pi/4}^{\pi/2} (\csc^2 t - 1)\cot^2 t\, dt = 8\int_{\pi/4}^{\pi/2} \csc^2 t\cot^2 t\, dt - 8\int_{\pi/4}^{\pi/2} \cot^2 t\, dt$
$= -8\left[-\frac{\cot^3 t}{3}\right]_{\pi/4}^{\pi/2} - 8\int_{\pi/4}^{\pi/2} (\csc^2 t - 1)dt = -\frac{8}{3}(0 - 1) + [8\cot t]_{\pi/4}^{\pi/2} + [8t]_{\pi/4}^{\pi/2} = \frac{8}{3} + 8(0 - 1) + 4\pi - 2\pi = 2\pi - \frac{16}{3}$

33. $\int_{-\pi}^{0} \sin 3x\cos 2x\, dx = \frac{1}{2}\int_{-\pi}^{0} (\sin x + \sin 5x)\, dx = \frac{1}{2}\left[-\cos x - \frac{1}{5}\cos 5x\right]_{-\pi}^{0} = \frac{1}{2}\left(-1 - \frac{1}{5} - 1 - \frac{1}{5}\right) = -\frac{6}{5}$

34. $\int_0^{\pi/2} \sin 2x\cos 3x\, dx = \frac{1}{2}\int_0^{\pi/2} (\sin(-x) + \sin 5x)\, dx = \frac{1}{2}\left[\cos(-x) - \frac{1}{5}\cos 5x\right]_0^{\pi/2} = \frac{1}{2}(0) - \frac{1}{2}\left(1 - \frac{1}{5}\right) = -\frac{2}{5}$

35. $\int_{-\pi}^{\pi} \sin 3x \sin 3x\,dx = \frac{1}{2}\int_{-\pi}^{\pi} (\cos 0 - \cos 6x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos 6x\,dx = \frac{1}{2}\left[x - \frac{1}{12}\sin 6x\right]_{-\pi}^{\pi} = \frac{\pi}{2} + \frac{\pi}{2} - 0 = \pi$

36. $\int_{0}^{\pi/2} \sin x \cos x\,dx = \frac{1}{2}\int_{0}^{\pi/2} (\sin 0 + \sin 2x)\,dx = \frac{1}{2}\int_{0}^{\pi/2} \sin 2x\,dx = -\frac{1}{4}[\cos 2x]_{0}^{\pi/2} = -\frac{1}{4}(-1-1) = \frac{1}{2}$

37. $\int_{0}^{\pi} \cos 3x \cos 4x\,dx = \frac{1}{2}\int_{0}^{\pi} (\cos(-x) + \cos 7x)\,dx = \frac{1}{2}\left[-\sin(-x) + \frac{1}{7}\sin 7x\right]_{0}^{\pi} = \frac{1}{2}(0) = 0$

38. $\int_{-\pi/2}^{\pi/2} \cos 7x \cos x\,dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (\cos 6x + \cos 8x)\,dx = \frac{1}{2}\left[\frac{1}{6}\sin 6x + \frac{1}{8}\sin 8x\right]_{-\pi/2}^{\pi/2} = 0$

39. $x = t^{2/3} \Rightarrow t^2 = x^3;\ y = \frac{t^2}{2} \Rightarrow y = \frac{x^3}{2};\ 0 \le t \le 2 \Rightarrow 0 \le x \le 2^{2/3};$
$A = \int_{0}^{2^{2/3}} 2\pi\left(\frac{x^3}{2}\right)\sqrt{1 + \frac{9}{4}x^4}\,dx;\ \begin{bmatrix} u = \frac{9}{4}x^4 \\ du = 9x^3dx \end{bmatrix} \to \frac{\pi}{9}\int_{0}^{9(2^{2/3})} \sqrt{1+u}\,du = \left[\frac{\pi}{9} \cdot \frac{2}{3}(1+u)^{3/2}\right]_{0}^{9(2^{2/3})}$
$= \frac{2\pi}{27}\left[\left(1 + 9\left(2^{2/3}\right)\right)^{3/2} - 1\right]$

40. $y = \ln(\cos x);\ y' = \frac{-\sin x}{\cos x} = -\tan x;\ (y')^2 = \tan^2 x;\ \int_{0}^{\pi/3} \sqrt{1 + \tan^2 x}\,dx = \int_{0}^{\pi/3} |\sec x|\,dx = [\ln|\sec x + \tan x|]_{0}^{\pi/3}$
$= \ln\left(2 + \sqrt{3}\right) - \ln(1+0) = \ln\left(2 + \sqrt{3}\right)$

41. $y = \ln(\sec x);\ y' = \frac{\sec x \tan x}{\sec x} = \tan x; (y')^2 = \tan^2 x;\ \int_{0}^{\pi/4} \sqrt{1 + \tan^2 x}\,dx = \int_{0}^{\pi/4} |\sec x|\,dx = [\ln|\sec x + \tan x|]_{0}^{\pi/4}$
$= \ln\left(\sqrt{2} + 1\right) - \ln(0+1) = \ln\left(\sqrt{2} + 1\right)$

42. $M = \int_{-\pi/4}^{\pi/4} \sec x\,dx = [\ln|\sec x + \tan x|]_{-\pi/4}^{\pi/4} = \ln\left(\sqrt{2} + 1\right) - \ln\left|\sqrt{2} - 1\right| = \ln\frac{\sqrt{2}+1}{\sqrt{2}-1}$
$\overline{y} = \frac{1}{\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}\int_{-\pi/4}^{\pi/4} \frac{\sec^2 x}{2}\,dx = \frac{1}{2\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}[\tan x]_{-\pi/4}^{\pi/4} = \frac{1}{2\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}(1 - (-1)) = \frac{1}{\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}$
$\Rightarrow (\overline{x}, \overline{y}) = \left(0, \left(\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)^{-1}\right)$

43. $V = \pi\int_{0}^{\pi} \sin^2 x\,dx = \pi\int_{0}^{\pi} \frac{1 - \cos 2x}{2}\,dx = \frac{\pi}{2}\int_{0}^{\pi} dx - \frac{\pi}{2}\int_{0}^{\pi} \cos 2x\,dx = \frac{\pi}{2}[x]_{0}^{\pi} - \frac{\pi}{4}[\sin 2x]_{0}^{\pi} = \frac{\pi}{2}(\pi - 0) - \frac{\pi}{4}(0-0) = \frac{\pi^2}{2}$

44. $A = \int_{0}^{\pi} \sqrt{1 + \cos 4x}\,dx = \int_{0}^{\pi} \sqrt{2}\,|\cos 2x|dx = \sqrt{2}\int_{0}^{\pi/4} \cos 2x\,dx - \sqrt{2}\int_{\pi/4}^{3\pi/4} \cos 2x\,dx + \sqrt{2}\int_{3\pi/4}^{\pi} \cos 2x\,dx$
$= \frac{\sqrt{2}}{2}[\sin 2x]_{0}^{\pi/4} - \frac{\sqrt{2}}{2}[\sin 2x]_{\pi/4}^{3\pi/4} + \frac{\sqrt{2}}{2}[\sin 2x]_{3\pi/4}^{\pi} = \frac{\sqrt{2}}{2}(1-0) - \frac{\sqrt{2}}{2}(-1-1) + \frac{\sqrt{2}}{2}(0+1) = \sqrt{2} + \sqrt{2} = 2\sqrt{2}$

45. (a) $m^2 \ne n^2 \Rightarrow m + n \ne 0$ and $m - n \ne 0 \Rightarrow \int_{k}^{k+2\pi} \sin mx \sin nx\,dx = \frac{1}{2}\int_{k}^{k+2\pi} [\cos(m-n)x - \cos(m+n)x]dx$
$= \frac{1}{2}\left[\frac{1}{m-n}\sin(m-n)x - \frac{1}{m+n}\sin(m+n)x\right]_{k}^{k+2\pi}$
$= \frac{1}{2}\left(\frac{1}{m-n}\sin((m-n)(k+2\pi)) - \frac{1}{m+n}\sin((m+n)(k+2\pi))\right) - \frac{1}{2}\left(\frac{1}{m-n}\sin((m-n)k) - \frac{1}{m+n}\sin((m+n)k)\right)$
$= \frac{1}{2(m-n)}\sin((m-n)k) - \frac{1}{2(m+n)}\sin((m+n)k) - \frac{1}{2(m-n)}\sin((m-n)k) + \frac{1}{2(m+n)}\sin((m+n)k) = 0$
$\Rightarrow$ sin mx and sin nx are orthogonal.

(b) Same as part since $\frac{1}{2}\int_{k}^{k+2\pi} \cos 0\,dx = \pi$. $m^2 \ne n^2 \Rightarrow m + n \ne 0$ and $m - n \ne 0 \Rightarrow \int_{k}^{k+2\pi} \cos mx \cos nx\,dx$
$= \frac{1}{2}\int_{k}^{k+2\pi} [\cos(m-n)x + \cos(m+n)x]dx = \frac{1}{2}\left[\frac{1}{m-n}\sin(m-n)x + \frac{1}{m+n}\sin(m+n)x\right]_{k}^{k+2\pi}$
$= \frac{1}{2(m-n)}\sin((m-n)(k+2\pi)) + \frac{1}{2(m+n)}\sin((m+n)(k+2\pi)) - \frac{1}{2(m-n)}\sin((m-n)k) - \frac{1}{2(m+n)}\sin((m+n)k)$
$= \frac{1}{2(m-n)}\sin((m-n)k) + \frac{1}{2(m+n)}\sin((m+n)k) - \frac{1}{2(m-n)}\sin((m-n)k) - \frac{1}{2(m+n)}\sin((m+n)k) = 0$

$\Rightarrow \cos mx$ and $\cos nx$ are orthogonal.

(c) Let $m = n \Rightarrow \sin mx \cos nx = \frac{1}{2}(\sin 0 + \sin((m+n)x))$ and $\frac{1}{2}\int_k^{k+2\pi} \sin 0\, dx = 0$ and $\frac{1}{2}\int_k^{k+2\pi} \sin((m+n)x)\, dx = 0$

$\Rightarrow \sin mx$ and $\cos nx$ are orthogonal if $m = n$.

Let $m \neq n$.

$\int_k^{k+2\pi} \sin mx \cos nx\, dx = \frac{1}{2}\int_k^{k+2\pi} [\sin(m-n)x + \sin(m+n)x] dx = \frac{1}{2}\left[-\frac{1}{m-n}\cos(m-n)x - \frac{1}{m+n}\cos(m+n)x\right]_k^{k+2\pi}$

$= -\frac{1}{2(m-n)}\cos((m-n)(k+2\pi)) - \frac{1}{2(m+n)}\cos((m+n)(k+2\pi)) + \frac{1}{2(m-n)}\cos((m-n)k) + \frac{1}{2(m+n)}\cos((m+n)k)$

$= -\frac{1}{2(m-n)}\cos((m-n)k) - \frac{1}{2(m+n)}\cos((m+n)k) + \frac{1}{2(m-n)}\cos((m-n)k) + \frac{1}{2(m+n)}\cos((m+n)k) = 0$

$\Rightarrow \sin mx$ and $\cos nx$ are orthogonal.

46. $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\, dx = \sum_{n=1}^{N} \frac{a_n}{\pi}\int_{-\pi}^{\pi} \sin nx \sin mx\, dx$. Since $\frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \sin mx\, dx = \begin{cases} 0 & \text{for } m \neq n \\ 1 & \text{for } m = n \end{cases}$,
the sum on the right has only one nonzero term, namely $\frac{a_m}{\pi}\int_{-\pi}^{\pi} \sin mx \sin mx\, dx = a_m$.

8.5 TRIGONOMETRIC SUBSTITUTIONS

1. $y = 3\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dy = \frac{3\,d\theta}{\cos^2\theta},\ 9 + y^2 = 9(1 + \tan^2\theta) = \frac{9}{\cos^2\theta} \Rightarrow \frac{1}{\sqrt{9+y^2}} = \frac{|\cos\theta|}{3} = \frac{\cos\theta}{3}$
(because $\cos\theta > 0$ when $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$);
$\int \frac{dy}{\sqrt{9+y^2}} = 3\int \frac{\cos\theta\, d\theta}{3\cos^2\theta} = \int \frac{d\theta}{\cos\theta} = \ln|\sec\theta + \tan\theta| + C' = \ln\left|\frac{\sqrt{9+y^2}}{3} + \frac{y}{3}\right| + C' = \ln\left|\sqrt{9+y^2} + y\right| + C$

2. $\int \frac{3\,dy}{\sqrt{1+9y^2}}$; $[3y = x]$ $\rightarrow$ $\int \frac{dx}{\sqrt{1+x^2}}$; $x = \tan t,\ -\frac{\pi}{2} < t < \frac{\pi}{2},\ dx = \frac{dt}{\cos^2 t},\ \sqrt{1+x^2} = \frac{1}{\cos t}$;
$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{dt}{\cos^2 t\left(\frac{1}{\cos t}\right)} = \ln|\sec t + \tan t| + C = \ln\left|\sqrt{x^2+1} + x\right| + C = \ln\left|\sqrt{1+9y^2} + 3y\right| + C$

3. $\int_{-2}^{2} \frac{dx}{4+x^2} = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{-2}^{2} = \frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}(-1) = \left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) - \left(\frac{1}{2}\right)\left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$

4. $\int_0^2 \frac{dx}{8+2x^2} = \frac{1}{2}\int_0^2 \frac{dx}{4+x^2} = \frac{1}{2}\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2 = \frac{1}{2}\left(\frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}0\right) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) - 0 = \frac{\pi}{16}$

5. $\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_0^{3/2} = \sin^{-1}\frac{1}{2} - \sin^{-1}0 = \frac{\pi}{6} - 0 = \frac{\pi}{6}$

6. $\int_0^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$; $[t = 2x]$ $\rightarrow$ $\int_0^{1/2\sqrt{2}} \frac{dt}{\sqrt{1-t^2}} = \left[\sin^{-1} t\right]_0^{1/\sqrt{2}} = \sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

7. $t = 5\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = 5\cos\theta\, d\theta,\ \sqrt{25-t^2} = 5\cos\theta$;
$\int \sqrt{25-t^2}\, dt = \int (5\cos\theta)(5\cos\theta)\, d\theta = 25\int \cos^2\theta\, d\theta = 25\int \frac{1+\cos 2\theta}{2}\, d\theta = 25\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) + C$
$= \frac{25}{2}(\theta + \sin\theta\cos\theta) + C = \frac{25}{2}\left[\sin^{-1}\left(\frac{t}{5}\right) + \left(\frac{t}{5}\right)\left(\frac{\sqrt{25-t^2}}{5}\right)\right] + C = \frac{25}{2}\sin^{-1}\left(\frac{t}{5}\right) + \frac{t\sqrt{25-t^2}}{2} + C$

8. $t = \frac{1}{3}\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = \frac{1}{3}\cos\theta\, d\theta,\ \sqrt{1-9t^2} = \cos\theta$;
$\int \sqrt{1-9t^2}\, dt = \frac{1}{3}\int (\cos\theta)(\cos\theta)\, d\theta = \frac{1}{3}\int \cos^2\theta\, d\theta = \frac{1}{6}(\theta + \sin\theta\cos\theta) + C = \frac{1}{6}\left[\sin^{-1}(3t) + 3t\sqrt{1-9t^2}\right] + C$

9. $x = \frac{7}{2}\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \frac{7}{2}\sec\theta\tan\theta\, d\theta,\ \sqrt{4x^2-49} = \sqrt{49\sec^2\theta - 49} = 7\tan\theta$;
$\int \frac{dx}{\sqrt{4x^2-49}} = \int \frac{\left(\frac{7}{2}\sec\theta\tan\theta\right) d\theta}{7\tan\theta} = \frac{1}{2}\int \sec\theta\, d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \frac{1}{2}\ln\left|\frac{2x}{7} + \frac{\sqrt{4x^2-49}}{7}\right| + C$

10. $x = \frac{3}{5}\sec\theta$, $0 < \theta < \frac{\pi}{2}$, $dx = \frac{3}{5}\sec\theta\tan\theta\, d\theta$, $\sqrt{25x^2 - 9} = \sqrt{9\sec^2\theta - 9} = 3\tan\theta$;
$\int \frac{5\, dx}{\sqrt{25x^2-9}} = \int \frac{5\left(\frac{3}{5}\sec\theta\tan\theta\right) d\theta}{3\tan\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{5x}{3} + \frac{\sqrt{25x^2-9}}{3}\right| + C$

11. $y = 7\sec\theta$, $0 < \theta < \frac{\pi}{2}$, $dy = 7\sec\theta\tan\theta\, d\theta$, $\sqrt{y^2 - 49} = 7\tan\theta$;
$\int \frac{\sqrt{y^2-49}}{y}\, dy = \int \frac{(7\tan\theta)(7\sec\theta\tan\theta)\, d\theta}{7\sec\theta} = 7\int \tan^2\theta\, d\theta = 7\int (\sec^2\theta - 1)\, d\theta = 7(\tan\theta - \theta) + C$
$= 7\left[\frac{\sqrt{y^2-49}}{7} - \sec^{-1}\left(\frac{y}{7}\right)\right] + C$

12. $y = 5\sec\theta$, $0 < \theta < \frac{\pi}{2}$, $dy = 5\sec\theta\tan\theta\, d\theta$, $\sqrt{y^2 - 25} = 5\tan\theta$;
$\int \frac{\sqrt{y^2-25}}{y^3}\, dy = \int \frac{(5\tan\theta)(5\sec\theta\tan\theta)\, d\theta}{125\sec^3\theta} = \frac{1}{5}\int \tan^2\theta\cos^2\theta\, d\theta = \frac{1}{5}\int \sin^2\theta\, d\theta = \frac{1}{10}\int (1 - \cos 2\theta)\, d\theta$
$= \frac{1}{10}(\theta - \sin\theta\cos\theta) + C = \frac{1}{10}\left[\sec^{-1}\left(\frac{y}{5}\right) - \left(\frac{\sqrt{y^2-25}}{y}\right)\left(\frac{5}{y}\right)\right] + C = \left[\frac{\sec^{-1}\left(\frac{y}{5}\right)}{10} - \frac{\sqrt{y^2-25}}{2y^2}\right] + C$

13. $x = \sec\theta$, $0 < \theta < \frac{\pi}{2}$, $dx = \sec\theta\tan\theta\, d\theta$, $\sqrt{x^2 - 1} = \tan\theta$;
$\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\, d\theta}{\sec^2\theta\tan\theta} = \int \frac{d\theta}{\sec\theta} = \sin\theta + C = \frac{\sqrt{x^2-1}}{x} + C$

14. $x = \sec\theta$, $0 < \theta < \frac{\pi}{2}$, $dx = \sec\theta\tan\theta\, d\theta$, $\sqrt{x^2 - 1} = \tan\theta$;
$\int \frac{2\, dx}{x^3\sqrt{x^2-1}} = \int \frac{2\tan\theta\sec\theta\, d\theta}{\sec^3\theta\tan\theta} = 2\int \cos^2\theta\, d\theta = 2\int \left(\frac{1+\cos 2\theta}{2}\right) d\theta = \theta + \sin\theta\cos\theta + C$
$= \theta + \tan\theta\cos^2\theta + C = \sec^{-1}x + \sqrt{x^2-1}\left(\frac{1}{x}\right)^2 + C = \sec^{-1}x + \frac{\sqrt{x^2-1}}{x^2} + C$

15. $x = 2\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $dx = \frac{2\, d\theta}{\cos^2\theta}$, $\sqrt{x^2 + 4} = \frac{2}{\cos\theta}$;
$\int \frac{x^3\, dx}{\sqrt{x^2+4}} = \int \frac{(8\tan^3\theta)(\cos\theta)\, d\theta}{\cos^2\theta} = 8\int \frac{\sin^3\theta\, d\theta}{\cos^4\theta} = 8\int \frac{(\cos^2\theta - 1)(-\sin\theta)\, d\theta}{\cos^4\theta}$;
$[t = \cos\theta] \rightarrow 8\int \frac{t^2-1}{t^4}\, dt = 8\int \left(\frac{1}{t^2} - \frac{1}{t^4}\right) dt = 8\left(-\frac{1}{t} + \frac{1}{3t^3}\right) + C = 8\left(-\sec\theta + \frac{\sec^3\theta}{3}\right) + C$
$= 8\left(-\frac{\sqrt{x^2+4}}{2} + \frac{(x^2+4)^{3/2}}{8\cdot 3}\right) + C = \frac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4} + C$

16. $x = \tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $dx = \sec^2\theta\, d\theta$, $\sqrt{x^2 + 1} = \sec\theta$;
$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\sec\theta} = \int \frac{\cos\theta\, d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = \frac{-\sqrt{x^2+1}}{x} + C$

17. $w = 2\sin\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $dw = 2\cos\theta\, d\theta$, $\sqrt{4 - w^2} = 2\cos\theta$;
$\int \frac{8\, dw}{w^2\sqrt{4-w^2}} = \int \frac{8\cdot 2\cos\theta\, d\theta}{4\sin^2\theta\cdot 2\cos\theta} = 2\int \frac{d\theta}{\sin^2\theta} = -2\cot\theta + C = \frac{-2\sqrt{4-w^2}}{w} + C$

18. $w = 3\sin\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $dw = 3\cos\theta\, d\theta$, $\sqrt{9 - w^2} = 3\cos\theta$;
$\int \frac{\sqrt{9-w^2}}{w^2}\, dw = \int \frac{3\cos\theta\cdot 3\cos\theta\, d\theta}{9\sin^2\theta} = \int \cot^2\theta\, d\theta = \int \left(\frac{1-\sin^2\theta}{\sin^2\theta}\right) d\theta = \int (\csc^2\theta - 1)\, d\theta$
$= -\cot\theta - \theta + C = -\frac{\sqrt{9-w^2}}{w} - \sin^{-1}\left(\frac{w}{3}\right) + C$

19. $x = \sin\theta$, $0 \le \theta \le \frac{\pi}{3}$, $dx = \cos\theta\, d\theta$, $(1 - x^2)^{3/2} = \cos^3\theta$;
$\int_0^{\sqrt{3}/2} \frac{4x^2\, dx}{(1-x^2)^{3/2}} = \int_0^{\pi/3} \frac{4\sin^2\theta\cos\theta\, d\theta}{\cos^3\theta} = 4\int_0^{\pi/3} \left(\frac{1-\cos^2\theta}{\cos^2\theta}\right) d\theta = 4\int_0^{\pi/3} (\sec^2\theta - 1)\, d\theta$
$= 4[\tan\theta - \theta]_0^{\pi/3} = 4\sqrt{3} - \frac{4\pi}{3}$

20. $x = 2\sin\theta,\ 0 \le \theta \le \frac{\pi}{6},\ dx = 2\cos\theta\, d\theta,\ (4 - x^2)^{3/2} = 8\cos^3\theta;$
$$\int_0^1 \frac{dx}{(4-x^2)^{3/2}} = \int_0^{\pi/6} \frac{2\cos\theta\, d\theta}{8\cos^3\theta} = \frac{1}{4}\int_0^{\pi/6} \frac{d\theta}{\cos^2\theta} = \frac{1}{4}\left[\tan\theta\right]_0^{\pi/6} = \frac{\sqrt{3}}{12} = \frac{1}{4\sqrt{3}}$$

21. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\, d\theta,\ (x^2 - 1)^{3/2} = \tan^3\theta;$
$$\int \frac{dx}{(x^2-1)^{3/2}} = \int \frac{\sec\theta\tan\theta\, d\theta}{\tan^3\theta} = \int \frac{\cos\theta\, d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = -\frac{x}{\sqrt{x^2-1}} + C$$

22. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\, d\theta,\ (x^2 - 1)^{5/2} = \tan^5\theta;$
$$\int \frac{x^2\, dx}{(x^2-1)^{5/2}} = \int \frac{\sec^2\theta\cdot\sec\theta\tan\theta\, d\theta}{\tan^5\theta} = \int \frac{\cos\theta}{\sin^4\theta}\, d\theta = -\frac{1}{3\sin^3\theta} + C = -\frac{x^3}{3(x^2-1)^{3/2}} + C$$

23. $x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\, d\theta,\ (1 - x^2)^{3/2} = \cos^3\theta;$
$$\int \frac{(1-x^2)^{3/2}\, dx}{x^6} = \int \frac{\cos^3\theta\cdot\cos\theta\, d\theta}{\sin^6\theta} = \int \cot^4\theta\csc^2\theta\, d\theta = -\frac{\cot^5\theta}{5} + C = -\frac{1}{5}\left(\frac{\sqrt{1-x^2}}{x}\right)^5 + C$$

24. $x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\, d\theta,\ (1 - x^2)^{1/2} = \cos\theta;$
$$\int \frac{(1-x^2)^{1/2}\, dx}{x^4} = \int \frac{\cos\theta\cdot\cos\theta\, d\theta}{\sin^4\theta} = \int \cot^2\theta\csc^2\theta\, d\theta = -\frac{\cot^3\theta}{3} + C = -\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C$$

25. $x = \frac{1}{2}\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \frac{1}{2}\sec^2\theta\, d\theta,\ (4x^2 + 1)^2 = \sec^4\theta;$
$$\int \frac{8\, dx}{(4x^2+1)^2} = \int \frac{8\left(\frac{1}{2}\sec^2\theta\right) d\theta}{\sec^4\theta} = 4\int \cos^2\theta\, d\theta = 2(\theta + \sin\theta\cos\theta) + C = 2\tan^{-1}2x + \frac{4x}{(4x^2+1)} + C$$

26. $t = \frac{1}{3}\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = \frac{1}{3}\sec^2\theta\, d\theta,\ 9t^2 + 1 = \sec^2\theta;$
$$\int \frac{6\, dt}{(9t^2+1)^2} = \int \frac{6\left(\frac{1}{3}\sec^2\theta\right) d\theta}{\sec^4\theta} = 2\int \cos^2\theta\, d\theta = \theta + \sin\theta\cos\theta + C = \tan^{-1}3t + \frac{3t}{(9t^2+1)} + C$$

27. $v = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dv = \cos\theta\, d\theta,\ (1 - v^2)^{5/2} = \cos^5\theta;$
$$\int \frac{v^2\, dv}{(1-v^2)^{5/2}} = \int \frac{\sin^2\theta\cos\theta\, d\theta}{\cos^5\theta} = \int \tan^2\theta\sec^2\theta\, d\theta = \frac{\tan^3\theta}{3} + C = \frac{1}{3}\left(\frac{v}{\sqrt{1-v^2}}\right)^3 + C$$

28. $r = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2};$
$$\int \frac{(1-r^2)^{5/2}\, dr}{r^8} = \int \frac{\cos^5\theta\cdot\cos\theta\, d\theta}{\sin^8\theta} = \int \cot^6\theta\csc^2\theta\, d\theta = -\frac{\cot^7\theta}{7} + C = -\frac{1}{7}\left[\frac{\sqrt{1-r^2}}{r}\right]^7 + C$$

29. Let $e^t = 3\tan\theta,\ t = \ln(3\tan\theta),\ \tan^{-1}\left(\frac{1}{3}\right) \le \theta \le \tan^{-1}\left(\frac{4}{3}\right),\ dt = \frac{\sec^2\theta}{\tan\theta}\, d\theta,\ \sqrt{e^{2t} + 9} = \sqrt{9\tan^2\theta + 9} = 3\sec\theta;$
$$\int_0^{\ln 4} \frac{e^t\, dt}{\sqrt{e^{2t}+9}} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \frac{3\tan\theta\cdot\sec^2\theta\, d\theta}{\tan\theta\cdot 3\sec\theta} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \sec\theta\, d\theta = \left[\ln|\sec\theta + \tan\theta|\right]_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)}$$
$$= \ln\left(\frac{5}{3} + \frac{4}{3}\right) - \ln\left(\frac{\sqrt{10}}{3} + \frac{1}{3}\right) = \ln 9 - \ln\left(1 + \sqrt{10}\right)$$

30. Let $e^t = \tan\theta,\ t = \ln(\tan\theta),\ \tan^{-1}\left(\frac{3}{4}\right) \le \theta \le \tan^{-1}\left(\frac{4}{3}\right),\ dt = \frac{\sec^2\theta}{\tan\theta}\, d\theta,\ 1 + e^{2t} = 1 + \tan^2\theta = \sec^2\theta;$
$$\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\, dt}{(1+e^{2t})^{3/2}} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{(\tan\theta)\left(\frac{\sec^2\theta}{\tan\theta}\right) d\theta}{\sec^3\theta} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \cos\theta\, d\theta = \left[\sin\theta\right]_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$$

31. $\int_{1/12}^{1/4} \frac{2\, dt}{\sqrt{t} + 4t\sqrt{t}};\ \left[u = 2\sqrt{t},\ du = \frac{1}{\sqrt{t}}\, dt\right] \rightarrow \int_{1/\sqrt{3}}^{1} \frac{2\, du}{1+u^2};\ u = \tan\theta,\ \frac{\pi}{6} \le \theta \le \frac{\pi}{4},\ du = \sec^2\theta\, d\theta,\ 1 + u^2 = \sec^2\theta;$
$$\int_{1/\sqrt{3}}^{1} \frac{2\, du}{1+u^2} = \int_{\pi/6}^{\pi/4} \frac{2\sec^2\theta\, d\theta}{\sec^2\theta} = \left[2\theta\right]_{\pi/6}^{\pi/4} = 2\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\pi}{6}$$

32. $y = e^{\tan\theta}$, $0 \le \theta \le \frac{\pi}{4}$, $dy = e^{\tan\theta}\sec^2\theta\, d\theta$, $\sqrt{1+(\ln y)^2} = \sqrt{1+\tan^2\theta} = \sec\theta$;
$\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_0^{\pi/4} \frac{e^{\tan\theta}\sec^2\theta}{e^{\tan\theta}\sec\theta}\, d\theta = \int_0^{\pi/4} \sec\theta\, d\theta = [\ln|\sec\theta + \tan\theta|]_0^{\pi/4} = \ln\left(1+\sqrt{2}\right)$

33. $x = \sec\theta$, $0 < \theta < \frac{\pi}{2}$, $dx = \sec\theta\tan\theta\, d\theta$, $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$;
$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\, d\theta}{\sec\theta\tan\theta} = \theta + C = \sec^{-1} x + C$

34. $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, $1 + x^2 = \sec^2\theta$;
$\int \frac{dx}{x^2+1} = \int \frac{\sec^2\theta\, d\theta}{\sec^2\theta} = \theta + C = \tan^{-1} x + C$

35. $x = \sec\theta$, $dx = \sec\theta\tan\theta\, d\theta$, $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$;
$\int \frac{x\, dx}{\sqrt{x^2-1}} = \int \frac{\sec\theta \cdot \sec\theta\tan\theta\, d\theta}{\tan\theta} = \int \sec^2\theta\, d\theta = \tan\theta + C = \sqrt{x^2-1} + C$

36. $x = \sin\theta$, $dx = \cos\theta\, d\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$;
$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta\, d\theta}{\cos\theta} = \theta + C = \sin^{-1} x + C$

37. $x\frac{dy}{dx} = \sqrt{x^2-4}$; $dy = \sqrt{x^2-4}\,\frac{dx}{x}$; $y = \int \frac{\sqrt{x^2-4}}{x}\, dx$; $\left[\begin{array}{c} x = 2\sec\theta,\ 0 < \theta < \frac{\pi}{2} \\ dx = 2\sec\theta\tan\theta\, d\theta \\ \sqrt{x^2-4} = 2\tan\theta \end{array}\right]$
$\rightarrow y = \int \frac{(2\tan\theta)(2\sec\theta\tan\theta)\, d\theta}{2\sec\theta} = 2\int \tan^2\theta\, d\theta = 2\int (\sec^2\theta - 1)\, d\theta = 2(\tan\theta - \theta) + C$
$= 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\left(\frac{x}{2}\right)\right] + C$; $x = 2$ and $y = 0 \Rightarrow 0 = 0 + C \Rightarrow C = 0 \Rightarrow y = 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\frac{x}{2}\right]$

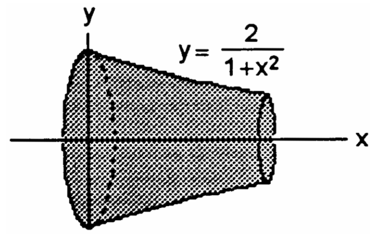
38. $\sqrt{x^2-9}\,\frac{dy}{dx} = 1$, $dy = \frac{dx}{\sqrt{x^2-9}}$; $y = \int \frac{dx}{\sqrt{x^2-9}}$; $\left[\begin{array}{c} x = 3\sec\theta,\ 0 < \theta < \frac{\pi}{2} \\ dx = 3\sec\theta\tan\theta\, d\theta \\ \sqrt{x^2-9} = 3\tan\theta \end{array}\right] \rightarrow y = \int \frac{3\sec\theta\tan\theta\, d\theta}{3\tan\theta}$
$= \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C$; $x = 5$ and $y = \ln 3 \Rightarrow \ln 3 = \ln 3 + C \Rightarrow C = 0$
$\Rightarrow y = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|$

39. $(x^2+4)\frac{dy}{dx} = 3$, $dy = \frac{3\, dx}{x^2+4}$; $y = 3\int \frac{dx}{x^2+4} = \frac{3}{2}\tan^{-1}\frac{x}{2} + C$; $x = 2$ and $y = 0 \Rightarrow 0 = \frac{3}{2}\tan^{-1} 1 + C$
$\Rightarrow C = -\frac{3\pi}{8} \Rightarrow y = \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{3\pi}{8}$

40. $(x^2+1)^2\frac{dy}{dx} = \sqrt{x^2+1}$, $dy = \frac{dx}{(x^2+1)^{3/2}}$; $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, $(x^2+1)^{3/2} = \sec^3\theta$;
$y = \int \frac{\sec^2\theta\, d\theta}{\sec^3\theta} = \int \cos\theta\, d\theta = \sin\theta + C = \tan\theta\cos\theta + C = \frac{\tan\theta}{\sec\theta} + C = \frac{x}{\sqrt{x^2+1}} + C$; $x = 0$ and $y = 1$
$\Rightarrow 1 = 0 + C \Rightarrow y = \frac{x}{\sqrt{x^2+1}} + 1$

41. $A = \int_0^3 \frac{\sqrt{9-x^2}}{3}\, dx$; $x = 3\sin\theta$, $0 \le \theta \le \frac{\pi}{2}$, $dx = 3\cos\theta\, d\theta$, $\sqrt{9-x^2} = \sqrt{9 - 9\sin^2\theta} = 3\cos\theta$;
$A = \int_0^{\pi/2} \frac{3\cos\theta \cdot 3\cos\theta\, d\theta}{3} = 3\int_0^{\pi/2} \cos^2\theta\, d\theta = \frac{3}{2}[\theta + \sin\theta\cos\theta]_0^{\pi/2} = \frac{3\pi}{4}$

42. $V = \int_0^1 \pi \left(\frac{2}{1+x^2}\right)^2 dx = 4\pi \int_0^1 \frac{dx}{(x^2+1)^2}$;
$x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, $x^2+1 = \sec^2\theta$;
$V = 4\pi \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = 4\pi \int_0^{\pi/4} \cos^2\theta\, d\theta$
$= 2\pi \int_0^{\pi/4} (1+\cos 2\theta)\, d\theta = 2\pi \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4} = \pi\left(\frac{\pi}{2}+1\right)$



43. $\int \frac{dx}{1-\sin x} = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{1-\left(\frac{2z}{1+z^2}\right)} = \int \frac{2\,dz}{(1-z)^2} = \frac{2}{1-z} + C = \frac{2}{1-\tan\left(\frac{x}{2}\right)} + C$

44. $\int \frac{dx}{1+\sin x+\cos x} = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{1+\left(\frac{2z}{1+z^2}+\frac{1-z^2}{1+z^2}\right)} = \int \frac{2\,dz}{1+z^2+2z+1-z^2} = \int \frac{dz}{1+z} = \ln|1+z| + C$
$= \ln\left|\tan\left(\frac{x}{2}\right)+1\right| + C$

45. $\int_0^{\pi/2} \frac{dx}{1+\sin x} = \int_0^1 \frac{\left(\frac{2\,dz}{1+z^2}\right)}{1+\left(\frac{2z}{1+z^2}\right)} = \int_0^1 \frac{2\,dz}{(1+z)^2} = -\left[\frac{2}{1+z}\right]_0^1 = -(1-2) = 1$

46. $\int_{\pi/3}^{\pi/2} \frac{dx}{1-\cos x} = \int_{1/\sqrt{3}}^1 \frac{\left(\frac{2\,dz}{1+z^2}\right)}{1-\left(\frac{1-z^2}{1+z^2}\right)} = \int_{1/\sqrt{3}}^1 \frac{dz}{z^2} = \left[-\frac{1}{z}\right]_{1/\sqrt{3}}^1 = \sqrt{3}-1$

47. $\int_0^{\pi/2} \frac{d\theta}{2+\cos\theta} = \int_0^1 \frac{\left(\frac{2\,dz}{1+z^2}\right)}{2+\left(\frac{1-z^2}{1+z^2}\right)} = \int_0^1 \frac{2\,dz}{2+2z^2+1-z^2} = \int_0^1 \frac{2\,dz}{z^2+3} = \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{z}{\sqrt{3}}\right]_0^1 = \frac{2}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}$
$= \frac{\pi}{3\sqrt{3}} = \frac{\sqrt{3}\pi}{9}$

48. $\int_{\pi/2}^{2\pi/3} \frac{\cos\theta\, d\theta}{\sin\theta\cos\theta+\sin\theta} = \int_1^{\sqrt{3}} \frac{\left(\frac{1-z^2}{1+z^2}\right)\left(\frac{2\,dz}{1+z^2}\right)}{\left[\frac{2z(1-z^2)}{(1+z^2)^2}+\left(\frac{2z}{1+z^2}\right)\right]} = \int_1^{\sqrt{3}} \frac{2(1-z^2)\,dz}{2z-2z^3+2z+2z^3} = \int_1^{\sqrt{3}} \frac{1-z^2}{2z}\,dz$
$= \left[\frac{1}{2}\ln z - \frac{z^2}{4}\right]_1^{\sqrt{3}} = \left(\frac{1}{2}\ln\sqrt{3}-\frac{3}{4}\right) - \left(0-\frac{1}{4}\right) = \frac{\ln 3}{4} - \frac{1}{2} = \frac{1}{4}(\ln 3 - 2) = \frac{1}{2}\left(\ln\sqrt{3}-1\right)$

49. $\int \frac{dt}{\sin t-\cos t} = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{\left(\frac{2z}{1+z^2}-\frac{1-z^2}{1+z^2}\right)} = \int \frac{2\,dz}{2z-1+z^2} = \int \frac{2\,dz}{(z+1)^2-2} = \frac{1}{\sqrt{2}}\ln\left|\frac{z+1-\sqrt{2}}{z+1+\sqrt{2}}\right| + C$
$= \frac{1}{\sqrt{2}}\ln\left|\frac{\tan\left(\frac{t}{2}\right)+1-\sqrt{2}}{\tan\left(\frac{t}{2}\right)+1+\sqrt{2}}\right| + C$

50. $\int \frac{\cos t\, dt}{1-\cos t} = \int \frac{\left(\frac{1-z^2}{1+z^2}\right)\left(\frac{2\,dz}{1+z^2}\right)}{1-\left(\frac{1-z^2}{1+z^2}\right)} = \int \frac{2(1-z^2)\,dz}{(1+z^2)^2-(1+z^2)(1-z^2)} = \int \frac{2(1-z^2)\,dz}{(1+z^2)(1+z^2-1+z^2)}$
$= \int \frac{(1-z^2)\,dz}{(1+z^2)z^2} = \int \frac{dz}{z^2(1+z^2)} - \int \frac{dz}{1+z^2} = \int \frac{dz}{z^2} - 2\int \frac{dz}{z^2+1} = -\frac{1}{z} - 2\tan^{-1}z + C = -\cot\left(\frac{t}{2}\right) - t + C$

51. $\int \sec\theta\, d\theta = \int \frac{d\theta}{\cos\theta} = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{\left(\frac{1-z^2}{1+z^2}\right)} = \int \frac{2\,dz}{1-z^2} = \int \frac{2\,dz}{(1+z)(1-z)} = \int \frac{dz}{1+z} + \int \frac{dz}{1-z}$
$= \ln|1+z| - \ln|1-z| + C = \ln\left|\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right| + C$

52. $\int \csc\theta\, d\theta = \int \frac{d\theta}{\sin\theta} = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{\left(\frac{2z}{1+z^2}\right)} = \int \frac{dz}{z} = \ln|z| + C = \ln\left|\tan\frac{\theta}{2}\right| + C$