Chapter 5. Line and Surface Integrals

1 Mathematical Representation of Surfaces

Question 1: What is a surface in 3-dimensional (2-dimensional) space?

Question 2: How to represent it in mathematics?

1.1 Parameterization of Surfaces

Example Parameterize the following surfaces:

1. Spherical band: the portion of the sphere $x^2 + y^2 + z^2 = 36$ between the plane $z = -3$ and $z = 3\sqrt{3}$.

2. The surface $z = x + y^2, 0 \leq x \leq 1,\ 0 \leq y \leq 2$.

3. Find a parametrization of the cone $z = \sqrt{x^2 + y^2}$, $1 \leq z \leq 2$.

4. The plane with norm vector $(-1, 2, 3)$ passes through $(2, 3, 1)$.

5. S is the surface whose sides S_1 is given by $x^2 + y^2 = 1$, bottom S_2 $x^2 + y^2 \leq 1$ in the plane $z = 0$, and top S_3 $z = 1 + x$ lies above S_2.

6. Find a parametrization of the cylinder

$$x^2 + (y-3)^2 = 9, \quad 0 \leq z \leq 5$$

1.2 Tangent Plane of Curves

Definition Given a surface S with a parameterization $\vec{r}(u,v)$, the tangent plane of S at point $P=\vec{r}(u_0,v_0)$ is the one with equation

$$[(x,y,z)-P]\cdot\left[\partial_u\vec{r}(u_0,v_0)\times\partial_v\vec{r}(u_0,v_0)\right]=0,$$

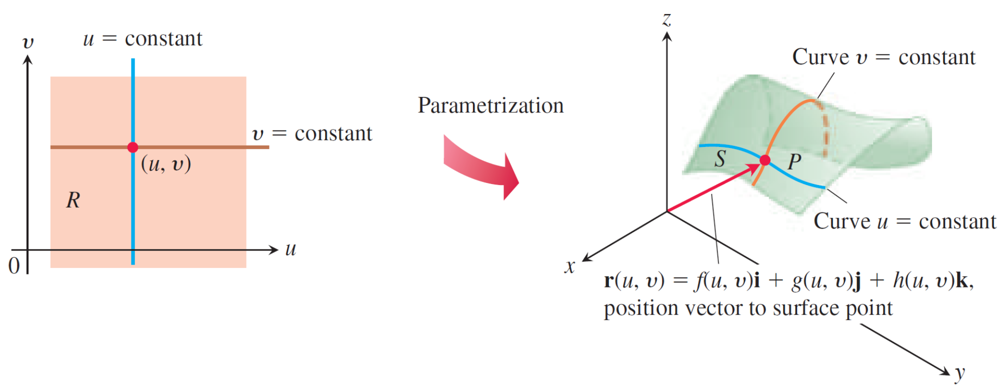
where

$$\partial_u\vec{r}(u_0,v_0)\times\partial_v\vec{r}(u_0,v_0)$$

is the norm vector of the tangent plane at $P=\vec{r}(u_0,v_0)$.

Particularly, $|\vec{r_u}\,(u_0,v_0)\times\vec{r_v}\,(u_0,v_0)\,|$ represents the area on the tangent plane over unit square on uv-plane.

surface.png



Example Let S as the surface of the cone $z = 1 + \sqrt{x^2 + y^2}$, for $0 \leq z \leq 8$. Find its tangent plane and norm vector at $P(3, 4, 6)$.

2 Surface Integral of 1st kind

Definition:

Remark: If S is a region D on xy-plan, $\int_S f dS$ is same as the double integral $\int_D f(x, y, 0)dxdy$.

Physical Interpretation Let $f(x, y, z)$ be the point density of a thin sheet shaped of S. Then

- $\int_S f(x, y, z)dS$ is the mass of the sheet.
- $(\int_S xf(x, y, z)dS, \int_S yf(x, y, z)dS, \int_S zf(x, y, z)dS)$ is the center of mass of the wire.
- $\int_S 1\ dS$ is the area of the surface S.

Example Compute the surface integral $\iint_S x^2 dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

Example Evaluate $\iint_s y\,dS$ where S is the surface $z = x + y^2, 0 \leq x \leq 1, 0 \leq y \leq 2$.

Example Evaluate $\iint_s z\,dS$ where S is the surface whose sides S_1 are given by the cylinder $x^2+y^2=1$, whose bottom S_2 is the disk $x^2+y^2\leq 1$ in the plane $z=0$, and whose top S_3 is the part of the plane $z=1+x$ that lies above S_2.

Example Find the area of tilted plane $x - y + 2z = 2$ inside cylinder

a.) Inside the cylinder $x^2 + z^2 = 3$

b.) Inside the cylinder $y^2 + z^2 = 2$.

3 Surface integral of 2nd kind

Definition: