MA1200 Exercise for Chapter 6 Limits, Continuity and Differentiability

Limits

1. Evaluate the following limits:

(a) $\lim_{x\to\infty}\dfrac{x^2+1}{2x^3-x}$

(b) $\lim_{x\to\infty}\dfrac{x+\sqrt{x^4-x^2+1}}{2x^2+1+\sqrt{x^4+1}}$

(c) $\lim_{x\to 0}\dfrac{m\sin(mx)-n\sin(nx)}{\tan(mx)+\tan(nx)}$ $(m\neq -n)$

2. Evaluate the following limits:

(a) $\lim_{x\to\infty}\dfrac{x+\sqrt{x^4-x^2+1}}{2x^2+1+\sqrt{x^4+1}}$

*(b) $\lim_{x\to 0}\dfrac{1+2^{1/x}}{3+2^{1/x}}$

(Hint: For $x\to 0^+$, $\frac{1}{x}\to+\infty$, $2^{\frac{1}{x}}\to+\infty$. We can consider $\lim_{x\to 0^+}\dfrac{1+2^{1/x}}{3+2^{1/x}}$ as $\lim_{y\to\infty}\dfrac{1+y}{3+y}$)

(c) $\lim_{x\to 0}\dfrac{\sin ax}{\sin bx}$

(d) $\lim_{x\to 0}\dfrac{|x|}{|x|+1}$

(e) $\lim_{x\to 0}\dfrac{(\sin 3x)^2}{x^2\cos x}$

(f) $\lim_{x\to 0}\dfrac{\sin 2x}{2x^2+x}$

(g) $\lim_{x\to 3^-}\dfrac{x^2-9}{|x-3|}$

3. Evaluate $\lim_{n\to\infty}\dfrac{n}{2}r^2\sin\dfrac{2\pi}{n}$, where r is a constant. Interpret this limit geometrically.

*4. Evaluate $\lim_{n\to\infty}\cos\dfrac{\theta}{2}\cos\dfrac{\theta}{4}\cos\dfrac{\theta}{8}\cdots\cos\dfrac{\theta}{2^n}$, where $\theta\neq 0$.

Continuity

5. Discuss the continuity of the following functions at $x=0$:

(a) $f(x)=\dfrac{x^2}{x}$

(b) $h(x)=\begin{cases}|x| & \text{for } x\neq 0\\ 1 & \text{for } x=0\end{cases}$

(c) $f(x)=\begin{cases}\dfrac{x^2}{x} & \text{for } x\neq 0\\ 0 & \text{for } x=0\end{cases}$

6. Define $f(0)$ for the following functions such that they are continuous at $x=0$.

(a) $f(x)=\sin x\sin\dfrac{1}{x}$

(b) $f(x)=\dfrac{\tan(2x)}{x}$

7. Sketch the graph of the following function on $[0,2]$

$$f(x)=\begin{cases}\sqrt{1-x^2} & 0\le x<1\\ 1 & \text{for}\quad 1\le x<2\\ 2 & x=2\end{cases}$$

(a) For what values of c in the domain does $\lim_{x\to c} f(x)$ exist?

(b) At what points does only the left-hand limit exist?

(c) At what points does only the right-hand limit exist?

8. Given the function $y=f(x)$ defined as follows:

$$f(x)=\begin{cases}0, & x^2=1\\ 1, & \text{otherwise}\end{cases}$$

Sketch the function. At what points is the function discontinuous? Explain.

Differentiability

9. Given $f(x)=\begin{cases}\frac{1}{2}\left(x^2-4\right) & 0<x<2,\\ 0 & x=2,\\ \frac{2}{x^2}\left(x^2-4\right) & x>2.\end{cases}$ Show that f is differentiable at $x = 2$.

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