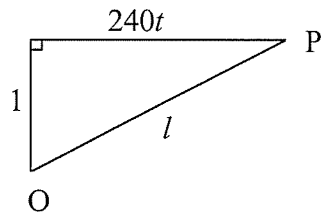
MA1200 Applications of Derivatives
Solutions

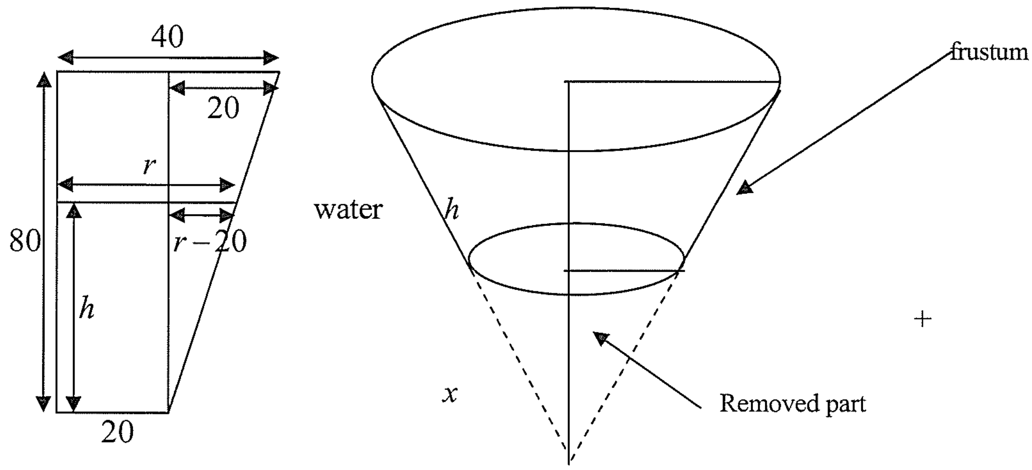
Rate of change

1. Let l be the distance from the observer to the plane. $l^2 = 1^2 + (240t)^2$, then $2l\frac{dl}{dt} = 240^2(2t)$.
   After 30 seconds, $t = 0.5/60$ hr, $l = \sqrt{5}$km, plug in to get
   $\frac{dl}{dt} = \frac{480}{\sqrt{5}}$ km/hr, which is the increasing speed of the distance after 30 seconds.



2. We have $y^2 = x^2 - 4$, $2y\frac{dy}{dt} = 2x\frac{dx}{dt}$. We know $\frac{dx}{dt} = 5$ unit/s and so when $x = 3, y = \sqrt{5}$ and $\frac{dy}{dt} = 3\sqrt{5}$ unit/s.

3.



Suppose the depth of water is h and the radius of the surface of water is d.
Then, using properties of similar triangles, we have $\frac{d}{20} = \frac{h+x}{x}$.
In addition, we know that the tank has altitude 80 centimeters and lower and upper radii of 20 and 40 centimeters, respectively, using properties of similar triangles again we have

$$\frac{40}{20} = \frac{80+x}{x} \Rightarrow 2 = \frac{80}{x} + 1 \Rightarrow x = 80$$

Thus, we have $\frac{d}{20} = \frac{h+80}{80} \Rightarrow d = \frac{h}{4} + 20$.
And the relation between the volume of water V and the depth of water h is :

$V = \frac{1}{3}\pi h\left(d^2 + db + b^2\right)$, where d is the radius of the surface of water and b is lower radius of the frustum..

Since $d = \frac{h}{4} + 20, b = 20$, we have $V = \frac{1}{3}\pi h\left[\left(\frac{h}{4}+20\right)^2 + 20\left(\frac{h}{4}+20\right) + 400\right]$.

Now differentiate with respect to t both sides of $V = \frac{1}{3}\pi h\left[\left(\frac{h}{4}+20\right)^2+20\left(\frac{h}{4}+20\right)+400\right]$, we have

$$\frac{dV}{dt}=\frac{1}{3}\pi\frac{dh}{dt}\left[\left(\frac{h}{4}+20\right)^2+20\left(\frac{h}{4}+20\right)+400\right]+\frac{1}{3}\pi h\left[2\left(\frac{h}{4}+20\right)\frac{1}{4}\frac{dh}{dt}+20\left(\frac{1}{4}\right)\frac{dh}{dt}\right]$$

Put $\frac{dV}{dt}=2000$ & $h=30$ into the above identity, we have

$$2000=\frac{1}{3}\pi\frac{dh}{dt}\left[\left(\frac{30}{4}+20\right)^2+20\left(\frac{30}{4}+20\right)+400\right]+\frac{1}{3}\pi 30\left[2\left(\frac{30}{4}+20\right)\frac{1}{4}\frac{dh}{dt}+20\left(\frac{1}{4}\right)\frac{dh}{dt}\right]$$

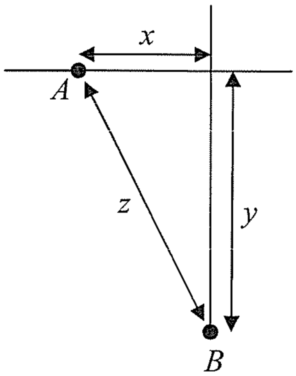
$$\Rightarrow 2000=\frac{1}{3}\pi\frac{dh}{dt}(756.25+550+400)+10\pi\left(13.75\frac{dh}{dt}+5\frac{dh}{dt}\right)=568.75\pi\frac{dh}{dt}+187.5\pi\frac{dh}{dt}$$

$$\Rightarrow\frac{dh}{dt}=\frac{2000}{756.25\pi}\,cm/\text{min}$$

4.

(a) Let x, y be the signed distances from A and B to O respectively (west/south of O means negative, as usual). Let z be the distance between them. Then $z^2 = x^2 + y^2$ and so $z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$. After 1 hr, $x = -5, y = -45, z = 5\sqrt{82}$ and we know $\frac{dx}{dt} = -20m/hr$ and $\frac{dy}{dt} = 15m/hr$, so $\frac{dz}{dt} = -\frac{115}{\sqrt{82}}m/hr$, which means they are approaching at speed $\frac{115}{\sqrt{82}}$ m/hr

(b) $x = x(t) = 15 - 20t$ and $y = y(t) = -60 + 15t$. Plugging into z we get $z = z(t) = \sqrt{625t^2 - 2400t + 3825}$, so minimum at $t = -\frac{b}{2a} = 1.92$ hr



5.

(a)

$$f'(x)=\frac{2}{3}\frac{\left(1+\sqrt{2}x\right)\left(1-\sqrt{2}x\right)}{x^{\frac{1}{3}}\left(x^2+1\right)^2}$$

Note that $\frac{f(x)-f(0)}{x-0}=\frac{1}{x^{\frac{2}{3}}\left(x^2+1\right)}$. Thus, $f'(0)$ does not exist.

(b)

For $x < -\frac{1}{\sqrt{2}}$ and $0 < x < \frac{1}{\sqrt{2}}$, $f'(x) > 0$.

For $-\frac{1}{\sqrt{2}} < x < 0$ and $x > \frac{1}{\sqrt{2}}$, $f'(x) < 0$.

(c)

$f(x) = \frac{\sqrt[3]{x^2}}{x^2+1}$ has relative maxima $y = f(-\frac{1}{\sqrt{2}}) = \frac{2^{\frac{2}{3}}}{3}$ at $x = -\frac{1}{\sqrt{2}}$, and $y = f(\frac{1}{\sqrt{2}}) = \frac{2^{\frac{2}{3}}}{3}$ at $x = \frac{1}{\sqrt{2}}$.

Moreover, $f(x) = \frac{\sqrt[3]{x^2}}{x^2+1}$ has a relative minimum $f(0) = 0$ at $x = 0$.

(d)

$$f'(x) = \frac{2}{3}\frac{\left(1+\sqrt{2}x\right)(1-\sqrt{2}x)}{x^{\frac{1}{3}}\left(x^2+1\right)^2} = \frac{2}{3}\frac{1-2x^2}{x^{\frac{1}{3}}\left(x^2+1\right)^2}$$

$$\Rightarrow f''(x) = \frac{2}{9}\frac{\left(x^2 - \frac{23+\sqrt{585}}{28}\right)\left(x^2 + \frac{\sqrt{585}-23}{28}\right)}{x^{\frac{4}{3}}\left(x^2+1\right)^3}$$

Then $f''(x) = 0$ when $x = \pm x_0$, where $x_0 = \sqrt{\frac{23+\sqrt{585}}{28}}$ (≈ 1.298). From the following table

x	$(-\infty, -x_0)$	$-x_0$	$(-x_0, 0) \cup (0, x_0)$	x_0	(x_0, ∞)
f''	$+$	0	$-$	0	$+$

$(-x_0, f(-x_0))$ and $(x_0, f(x_0))$ are the points of inflexion of $y = f(x)$.

Optimization

6. Let the tangent touches the parabola at $(a, 1-a^2)$, where $0 \le a \le 1$ so that the triangle is in quadrant I.
   Slope of tangent $= -2a$ and equation is $y - (1-a^2) = -2a(x-a)$
   x-intercept $= \left(\frac{a^2+1}{2a}, 0\right)$, so base of triangle $= \frac{a^2+1}{2a}$
   y-intercept $= (0, a^2+1)$, so height of triangle $= a^2+1$
   Area of triangle $A = \frac{1}{4}\frac{(a^2+1)^2}{a} = \frac{1}{4}(a^3 + 2a + \frac{1}{a})$
   $\frac{dA}{da} = \frac{1}{4}(3a^2 + 2 - \frac{1}{a^2})$. *Set* $\frac{dA}{da} = 0 = 3b^2 + 2b - 1 = (3b-1)(b+1)$, where $b = a^2$. So $a = \sqrt{\frac{1}{3}}$
   and it is a local minimum. So at $\left(\frac{1}{\sqrt{3}}, \frac{2}{3}\right)$.

7.

$$P(t) = \frac{t^2}{5\left(1+t^2\right)^2}$$

$$\Rightarrow P'(t) = \frac{5\left(1+t^2\right)^2 2t - t^2 10\left(1+t^2\right)2t}{25\left(1+t^2\right)^4} = \frac{10t\left(1+t^2\right)\left[1+t^2-2t^2\right]}{25\left(1+t^2\right)^4} = \frac{2t\left[1-t^2\right]}{5\left(1+t^2\right)^3} = \frac{2t(1-t)(1+t)}{5\left(1+t^2\right)^3}$$

Since domain of $P(t)=\dfrac{t^2}{5(1+t^2)^2}$ is $[0,\infty)$, for $0<t<1$ $P'(t)>0$ and for $t>1$ $P'(t)<0$,

also $P(t)=\dfrac{t^2}{5(1+t^2)^2}$ is a continuous function on $[0,\infty)$, $P(t)=\dfrac{t^2}{5(1+t^2)^2}$ attains the absolute maximum at $t=1$ and $P(1)=\dfrac{1}{20}$.

8. Let y be the length and width of the box and x be the height.
Then $A=4xy+y^2$, where $xy^2=60$ & $x,y>0$.

We have $A=\dfrac{240}{y}+y^2,\ y>0$.

$$A=\frac{240}{y}+y^2 \Rightarrow \frac{dA}{dy}=-\frac{240}{y^2}+2y=\frac{2y^3-240}{y^2}=\frac{2(y^3-120)}{y^2}=\frac{2\left(y-2\times15^{\frac{1}{3}}\right)\left(y^2+2\times15^{\frac{1}{3}}y+4\times15^{\frac{2}{3}}\right)}{y^2}.$$

$\dfrac{dA}{dy}=0 \Rightarrow y=2\times15^{\frac{1}{3}}$.

For $y>0$ note that $y^2+2\times15^{\frac{1}{3}}y+4\times15^{\frac{2}{3}}>0$.
Also $\dfrac{dA}{dy}<0$ for $0<y<2\times15^{\frac{1}{3}}$ & $\dfrac{dA}{dx}>0$ for $y>2\times15^{\frac{1}{3}}$.

Answer: $x=15^{\frac{1}{3}}$, $y=2\times15^{\frac{1}{3}}$

9.
$P=\rho\left[a\left(1-e^{-kx}\right)+b\right]-c_0-cx=\rho a\left(1-e^{-kx}\right)+\rho b-c_0-cx$, where $x\geq 0$.
Then
$P=\rho a\left(1-e^{-kx}\right)+\rho b-c_0-cx \Rightarrow \dfrac{dP}{dx}=\rho ake^{-kx}-c$.
$\dfrac{dP}{dx}=\rho ake^{-kx}-c=0 \Rightarrow e^{-kx}=\dfrac{c}{\rho ak} \Rightarrow \ln e^{-kx}=\ln\dfrac{c}{\rho ak} \Rightarrow -kx=\ln\dfrac{c}{\rho ak} \Rightarrow x=-\dfrac{1}{k}\ln\dfrac{c}{\rho ak}$.

Suppose $\dfrac{c}{\rho ak}<1$.

For $0<x<-\dfrac{1}{k}\left[\ln\dfrac{c}{\rho ak}\right]$, $\dfrac{dP}{dx}=\rho ake^{-kx}-c>0$.

For $-\dfrac{1}{k}\left[\ln\dfrac{c}{\rho ak}\right]<x$, $\dfrac{dP}{dx}=\rho ake^{-kx}-c<0$.
Since $P=\rho a\left(1-e^{-kx}\right)+\rho b-c_0-cx$ where $x\geq 0$ is a continuous function on $x\geq 0$,

$P=\rho a\left(1-e^{-kx}\right)+\rho b-c_0-cx$ where $x\geq 0$ attains the absolute maximum at $x=-\dfrac{1}{k}\left[\ln\dfrac{c}{\rho ak}\right]$ pounds.

Taylor/Maclaurin Series

10.

(a) The nth derivative is $\dfrac{(n+1)!}{(1-x)^{n+2}}$. So the Taylor series at $x=0.5$ is

$$\frac{1}{(1-x)^2}=\sum_{n=0}^{\infty}\frac{(n+1)!}{(1-0.5)^{n+2}}\frac{1}{n!}(x-0.5)^n=\sum_{n=0}^{\infty}(n+1)2^{n+2}(x-0.5)^n$$

b) The nth derivative is $\dfrac{(-1)^{n-1}1\cdot 3\cdot\ldots\cdot(2n-3)}{2^n}(1+x)^{-\frac{2n-1}{2}}$ for $n\geq 2$. So the Maclaurin series is

$$\sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4+\ldots$$

Plug in $x=0.1$, we get $\sqrt{1.1}\approx 1+\dfrac{1}{20}-\dfrac{1}{800}+\dfrac{1}{16000}\approx 1.04881$ (to 5 d.p.)

c) The nth derivative is $(-1)^{n-1}(n-1)!(1+x)^{-n}$. So the Maclaurin series is

$$\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(n-1)!}{(1+0)^n n!}x^n=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\ldots$$

Plug in $x=1$ we get $\ln 2=1-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{4}+\dfrac{1}{5}-\ldots$

Remark: So far there is no reason why we <u>can</u> plug in $x=1$. But it can be proved that it is legitimate to plug in $x=1$.

d) $f'=2\sin x\cos x=\sin 2x$. Then using formula in notes, the nth derivative is $2^{n-1}\sin\left(2x+\frac{(n-1)\pi}{2}\right)$. So the Taylor series at $x=\pi/4$ is

$$\sin^2 x=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{2^{n-1}\sin\left(\frac{\pi}{2}+\frac{(n-1)\pi}{2}\right)}{n!}\left(x-\frac{\pi}{4}\right)^n$$

$$=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{2^{n-1}\sin\left(\frac{n\pi}{2}\right)}{n!}\left(x-\frac{\pi}{4}\right)^n$$

$$=\frac{1}{2}+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}2^{2k-2}}{(2k-1)!}\left(x-\frac{\pi}{4}\right)^{2k-1}$$

$$=\frac{1}{2}+\left(x-\frac{\pi}{4}\right)-\frac{4}{6}\left(x-\frac{\pi}{4}\right)^3+\frac{16}{120}\left(x-\frac{\pi}{4}\right)^5-\cdots$$

because $\sin\left(\frac{n\pi}{2}\right)=\begin{cases}(-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd,}\\ 0 & \text{if } n \text{ is even.}\end{cases}$

*11.

a) If $y=\left(1+x^2\right)^{-\frac{1}{2}}$, show that $\left(1+x^2\right)\dfrac{dy}{dx}+xy=0$ … (*)

$$\frac{dy}{dx}=-\frac{1}{2}\left(1+x^2\right)^{-\frac{3}{2}}(2x)=-x\cdot\frac{1}{1+x^2}\cdot y$$

$$\Rightarrow\left(1+x^2\right)\frac{dy}{dx}+xy=0\ \ldots\ (*).$$

b) Deduce that $\left(1+x^2\right)\dfrac{d^{n+1}y}{dx^{n+1}}+(2n+1)x\dfrac{d^n y}{dx^n}+n^2\dfrac{d^{n-1}y}{dx^{n-1}}=0$.

Apply Leibnitz's rule on (*):

$$\overset{f\quad g}{\left(1+x^2\right)y'} \xrightarrow{\quad} \begin{matrix}\text{differentiate}\\ n\text{ times}\end{matrix} \longrightarrow g^{(n)}f+\binom{n}{1}g^{(n-1)}f^{(1)}+\binom{n}{2}g^{(n-2)}f^{(2)}+0$$

$$=\left(1+x^2\right)y^{(n+1)}+ny^{(n)}2x+\frac{n(n-1)}{2}y^{(n-1)}2$$

$$\overset{f\,g}{xy} \xrightarrow{\quad} \begin{matrix}\text{differentiate}\\ n\text{ times}\end{matrix} \longrightarrow g^{(n)}f+\binom{n}{1}g^{(n-1)}f^{(1)}+0$$

$$=xy^{(n)}+ny^{(n-1)}$$

$\therefore$ Differentiating (*) n times we get

$$\left(1+x^2\right)y^{(n+1)}+ny^{(n)}2x+n(n-1)y^{(n-1)}+xy^{(n)}+ny^{(n-1)}=0$$

$$\left(1+x^2\right)\frac{d^{n+1}y}{dx^{n+1}}+(2n+1)x\frac{d^ny}{dx^n}+n^2\frac{d^{n-1}y}{dx^{n-1}}=0\ldots(**)$$

c) Find Taylor series at $x=1$:

$$y(1)=\frac{1}{\sqrt{2}}$$

Plug in $x=1$ in (*), $y'(1)=-\frac{1}{2\sqrt{2}}$

Plug in $x=1$ in (**), $y^{(n+1)}(1)=\frac{-(2n+1)y^{(n)}(1)-n^2y^{(n-1)}(1)}{2}\ \ldots(***)$

Let $n=1,2$ in (***) we get

$$y^{(2)}(1)=\frac{1}{4\sqrt{2}}$$

$$y^{(3)}(1)=\frac{3}{8\sqrt{2}}$$

$$\therefore\frac{1}{\sqrt{1+x^2}}=\frac{1}{\sqrt{2}}-\frac{1}{2\sqrt{2}}\cdot\frac{1}{1!}(x-1)+\frac{1}{4\sqrt{2}}\cdot\frac{1}{2!}(x-1)^2+\frac{3}{8\sqrt{2}}\cdot\frac{1}{3!}(x-1)^3+\ldots$$

$$=\frac{1}{\sqrt{2}}\left[1-\frac{1}{2}(x-1)+\frac{1}{8}(x-1)^2+\frac{1}{16}(x-1)^3\right]+\ldots$$

L'Hopital's rule

12. a) $\lim_{x\to 0}\frac{1-\cos x}{x^2+3x}=\lim_{x\to 0}\frac{\sin x}{2x+3}=\frac{0}{3}=0$

b) $y=\ln(x^x)=x\ln x$ and

$\lim_{x\to 0^+}y=\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{\ln x}{1/x}=\lim_{x\to 0^+}\frac{1/x}{-1/x^2}=\lim_{x\to 0^+}-x=0$, so

$\lim_{x\to 0^+}x^x=\lim_{x\to 0^+}e^{\ln(x^x)}=e^{\lim\limits_{x\to 0^+}y}=e^0=1$

c) $\lim_{x\to\infty}\frac{ax^{a-1}}{1/x}=\lim_{x\to\infty}ax^a=$DNE ($\infty$)

d) Let $y=\ln(x+1)^{\cot x}=\cot x\ln(x+1)$.

Then $\lim_{x\to 0^+}y=\lim_{x\to 0^+}\frac{\ln(x+1)}{\tan x}=\lim_{x\to 0^+}\frac{1/(x+1)}{\sec^2 x}=1$ and so

$\lim_{x\to 0^+}(x+1)^{\cot x}=\lim_{x\to 0^+}e^{\ln(x+1)^{\cot x}}=e^{\lim\limits_{x\to 0^+}y}=e$

e) $\lim_{x\to\pi/2^+}\frac{\ln(\sin x)}{\cot x}=\lim_{x\to\pi/2^+}\frac{\frac{\cos x}{\sin x}}{-\csc^2 x}=\lim_{x\to\pi/2^+}-\sin x\cos x=0$

f) $\lim_{x\to 0}\frac{\frac{1}{2\sqrt{a+x}}-\frac{-1}{2\sqrt{a-x}}}{1}=\lim_{x\to 0}\frac{1}{2\sqrt{a+x}}+\frac{1}{2\sqrt{a-x}}=\frac{1}{\sqrt{a}}$

g) $\lim_{x\to 0}\frac{2/\sqrt{1-4x^2}}{1}=2$

h) $\lim_{x\to 0^+}\frac{\cos x-1}{\sin x}=\lim_{x\to 0^+}\frac{-\sin x}{\cos x}=0$

i) Let $y=\ln(\sqrt{x}-1)^{1/\sqrt{x}}=\frac{1}{\sqrt{x}}\ln(\sqrt{x}-1)$. Then

$\lim_{x\to\infty}y=\lim_{x\to\infty}\frac{\ln(\sqrt{x}-1)}{\sqrt{x}}=\lim_{x\to\infty}\frac{1/[(\sqrt{x}-1)2\sqrt{x}]}{1/2\sqrt{x}}=\lim_{x\to\infty}\frac{1}{\sqrt{x}-1}=0$ and so

$\lim_{x\to\infty}(\sqrt{x}-1)^{1/\sqrt{x}}=e^{\lim\limits_{x\to\infty}y}=1$

j) Let $y=\ln(x^3+1)^{1/\ln x}=\frac{1}{\ln x}\ln(x^3+1)$. Then

$\lim_{x\to\infty}y=\lim_{x\to\infty}\frac{3x^2/(x^3+1)}{1/x}=\lim_{x\to\infty}\frac{3x^3}{x^3+1}=\lim_{x\to\infty}\frac{3}{1+\frac{1}{x^3}}=3$ and so

$\lim_{x\to\infty}(x^3+1)^{1/\ln x}=e^{\lim\limits_{x\to\infty}y}=e^3$

k) $\lim_{x\to 1/2}\frac{\cos^2\pi x}{e^{2x}-2ex}=\lim_{x\to 1/2}\frac{2\cos\pi x(-\sin\pi x)\pi}{2e^{2x}-2e}=\lim_{x\to 1/2}\frac{-\pi\sin(2\pi x)}{2e^{2x}-2e}=\lim_{x\to 1/2}\frac{-2\pi^2\cos(2\pi x)}{4e^{2x}}=\frac{\pi^2}{2e}$

l) $\lim_{x\to a^+}\frac{\cot(x-a)}{\sec^2(x-a)/\tan(x-a)}=\lim_{x\to a^+}\cos^2(x-a)=1$

-End-