MA1201 Calculus and Basic Linear Algebra II

Chapter 2

Integration Technique

In this Chapter, we shall discuss some important technique in evaluating integrals:

- Method of Substitution (pp. 3 - 37)
- Integration by parts & Reduction Formula (pp. 38 - 89)
- Integration of rational functions – Method of partial fractions (pp.90 - 110)

Method of substitution

The principle of this method is to "transform" some complicated integrals into some simpler integrals by introducing a new variable y.

$$\int \frac{3x^2}{x^3+1}\,dx \xrightarrow{\ y = x^3+1\ } \int \frac{1}{y}\,dy = \ln|y| + C$$

(Difficult to integrate) (Easy to integrate)

Procedure of transformation

Let $y = x^3 + 1$, then $\frac{dy}{dx} = 3x^2 \Rightarrow dx = \frac{1}{3x^2}dy$.

$$\int \frac{3x^2}{x^3+1}\,dx$$

Change dx to dy

$$\int \frac{3x^2}{x^3+1}\left(\frac{1}{3x^2}dy\right) = \int \frac{1}{x^3+1}\,dy$$

Express the integrand in terms of y

$$\int \frac{1}{y}\,dy$$

Example 1

Compute the integral

$$\int x^2 \cos(x^3 + 1)\,dx$$

☺Solution:

IDEA: Since the function $\cos(x^3 + 1)$ is hard to integrate, it may be easier for us to compute if we "transform" this function into $\cos y$.

Step 1: Choose a substitution

We let $y = x^3 + 1$.

Step 2: Transform the integral $(x \to y)$ and compute the transformed integral

The derivative can be computed as

$$\frac{dy}{dx} = 3x^2 \;\Rightarrow\; dx = \frac{1}{3x^2}\,dy,$$

$$\int x^2 \cos(x^3+1)\,dx = \int x^2 \cos(x^3+1)\left(\frac{1}{3x^2}dy\right)$$

$$= \frac{1}{3}\int \cos(x^3+1)\,dy$$

$$= \frac{1}{3}\int \cos y\,dy = \frac{1}{3}\sin y + C$$

$$\overset{\text{substitue } y=x^3+1}{\cong} \frac{1}{3}\sin(x^3+1) + C.$$

Checking:

$$\frac{d}{dx}\left(\frac{1}{3}\sin(x^3+1)\right) = \frac{1}{3}\frac{d(\sin(x^3+1))}{d(x^3+1)}\frac{d(x^3+1)}{dx} = x^2\cos(x^3+1).$$

☺Reminder:

Remember to transform the answer into function of x after integration!

Example 2

Compute the integral

$$\int x^5\sqrt{x^2+3}\,dx$$

☺Solution:

IDEA: Since the function $\sqrt{x^2+3}$ is hard to integrate, it may be easier for us to compute if we "transform" this function into $\sqrt{y}$.

Step 1: Choose a substitution

We let $y = x^2 + 3$.

Step 2: Transform the integral $(x \to y)$ and compute the transformed integral

$$\frac{dy}{dx} = 2x \;\Rightarrow\; dx = \frac{1}{2x}dy,$$

Then the integral becomes

$$\int x^5\sqrt{x^2+3}dx = \int x^5\sqrt{x^2+3}\left(\frac{1}{2x}dy\right) = \frac{1}{2}\int x^4\sqrt{x^2+3}\,dy$$

$$\overset{\substack{y=x^2+3\\ \Rightarrow x^2=y-3}}{\stackrel{\frown}{=}} \frac{1}{2}\int (y-3)^2\sqrt{y}dy = \frac{1}{2}\int (y^2-6y+9)\left(y^{\frac{1}{2}}\right)dy$$

$$= \frac{1}{2}\left[\int y^{\frac{5}{2}}dy - 6\int y^{\frac{3}{2}}dy + 9\int y^{\frac{1}{2}}dy\right]$$

$$\overset{\int x^a dx = \frac{x^{a+1}}{a+1}+C}{\stackrel{\frown}{=}} \frac{1}{2}\left[\frac{y^{\frac{5}{2}+1}}{\frac{5}{2}+1} - 6\frac{y^{\frac{3}{2}+1}}{\frac{3}{2}+1} + 9\frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right] + C$$

$$\overset{y=x^2+3}{\stackrel{\frown}{=}} \frac{1}{7}(x^2+3)^{\frac{7}{2}} - \frac{6}{5}(x^2+3)^{\frac{5}{2}} + 3(x^2+3)^{\frac{3}{2}} + C.$$

Remark of Example 2 (On the choice of substitution)

One may try to transform the integral by using the substitution $y = \sqrt{x^2+3}$. The computation is more tedious although one can obtain the answer:

Note that $\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+3}}(2x) = \frac{x}{\sqrt{x^2+3}}$, then the integral becomes

$$\int x^5\sqrt{x^2+3}dx = \int x^5\sqrt{x^2+3}\frac{\sqrt{x^2+3}}{x}dy = \int x^4(x^2+3)dy$$

$$= \int (y^2-3)^2y^2dy = \int (y^6-6y^4+9y^2)dy = \frac{y^7}{7} - \frac{6y^5}{5} + 3y^3 + C$$

$$= \frac{1}{7}(x^2+3)^{\frac{7}{2}} - \frac{6}{5}(x^2+3)^{\frac{5}{2}} + 3(x^2+3)^{\frac{3}{2}} + C.$$

Therefore, one should choose some simple (but effective) substitution in order to minimize the computation cost. Remember, “simple is the best”.

Method of substitution for definite integral

The procedure is similar to that of indefinite integral except the determination of upper limits and lower limits of transformed integral since the original range of integration is for the variable x only. One needs to determine the range of integration of new variable y when using the method of substitution.

Example 3 (Definite integral)

Evaluate the integral

$$\int_1^4 \frac{\sin\left(\frac{1}{x}\right)}{x^2}\,dx.$$

☺Solution

Step 1: Choose your substitution

We let $y = \frac{1}{x}$. (so that $\sin\left(\frac{1}{x}\right)$ can turn into $\sin y$)

Step 2: Transform the integral ($x \to y$) and compute the integral

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}x^{-1} = -x^{-2} = -\frac{1}{x^2} \Rightarrow dx = -x^2 dy$$

When $x = 4$, then $y = \frac{1}{4}$; When $x = 1$, then $y = 1$.

Determine the upper limit and lower limit of the new integral

$$\int_1^4 \frac{\sin\left(\frac{1}{x}\right)}{x^2} dx = \int_1^{\frac{1}{4}} \frac{\sin\left(\frac{1}{x}\right)}{x^2} (-x^2 dy)$$

$$= -\int_1^{\frac{1}{4}} \sin y \, dy = -[-\cos y]_1^{\frac{1}{4}} = \cos\frac{1}{4} - \cos 1.$$

Example 4

Evaluate

$$\int_0^{\ln 2} \frac{e^{2x}}{\sqrt{e^x+1}}\,dx$$

Step 1: Choose your substitution

Choose $y = e^x + 1$ (Question: Why not e^x or $\sqrt{e^x+1}$??)

Step 2: Transform the integral ($x \to y$) and compute the integral

$$\frac{dy}{dx} = \frac{d}{dx}(e^x+1) = e^x \;\Rightarrow\; dx = \frac{dy}{e^x}.$$

When $x = \ln 2$, $y = e^{\ln 2} + 1 = 3$; When $x = 0$, $y = e^0 + 1 = 2$.

Then the integral becomes

$$\int_0^{\ln 2} \frac{e^{2x}}{\sqrt{e^x+1}}\,dx = \int_2^3 \frac{e^{2x}}{\sqrt{y}}\frac{dy}{e^x} = \int_2^3 \frac{e^x}{\sqrt{y}}\,dy$$

$$\overset{y=e^x+1}{\widehat{=}} \int_2^3 \frac{y-1}{\sqrt{y}}\,dy$$

$$= \int_2^3 \left(\frac{y}{\sqrt{y}} - \frac{1}{\sqrt{y}}\right) dy = \int_2^3 \left(\sqrt{y} - \frac{1}{\sqrt{y}}\right) dy$$

$$= \int_2^3 \left(y^{\frac{1}{2}} - y^{-\frac{1}{2}}\right) dy \overset{\int x^a dx = \frac{x^{a+1}}{a+1}}{\widehat{=}} \left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}} - \frac{y^{\frac{1}{2}}}{\frac{1}{2}}\right]_2^3 = \frac{2\sqrt{2}}{3}.$$

Example 5

Compute the integral

$$\int \tan x \, dx.$$

☺Solution:

One has to rewrite the integrand as $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx.$

Step 1: Choose your substitution

We choose the substitution $y = \cos x$ (so that the denominator becomes y only).

Step 2: Transform the integral and compute it

Note that $\frac{dy}{dx} = -\sin x,$ then the integral becomes

$$\int \frac{\sin x}{\cos x} dx = \int \frac{\sin x}{\cos x}\left(\frac{1}{-\sin x} dy\right) = -\int \frac{1}{y} dy = -\ln|y| + C$$

$$= -\ln|\cos x| + C \quad (\text{or } \ln|\sec x| + C)$$

Example 6

Evaluate

$$\int \frac{x+1}{x^2+2x+7}\,dx.$$

☺Solution:

Step 1: Choose your substitution: We let $y = x^2 + 2x + 7$.

Step 2: Transform the integral $(x \to y)$ and compute the integral

$$\frac{dy}{dx} = 2x + 2 \Rightarrow dy = (2x+2)dx \Rightarrow dx = \frac{dy}{2x+2}.$$

$$\int \frac{x+1}{x^2+2x+7}\,dx = \int \frac{x+1}{x^2+2x+7}\,\frac{dy}{2x+2} \overset{\substack{2x+2\\=2(x+1)}}{\triangleq} \int \frac{1}{2(x^2+2x+7)}\,dy = \frac{1}{2}\int \frac{1}{y}\,dy$$

$$= \frac{1}{2}\ln|y| + C \overset{y=x^2+2x+7}{\triangleq} \frac{1}{2}\ln|x^2+2x+7| + C.$$

Harder Examples

In some situations, one may not able to transform the integral into the desired form using simple substitution. In this case, one has to either use algebra trick to turn the integral into another form so that the simple substitution still works or use other "more advanced" substitutions.

Example 7 (Related to Example 6)

Compute the integral

$$\int \frac{2x-5}{x^2-6x+10}\,dx.$$

Since this integral looks like the one in Example 6, one may try the substitution $y = x^2-6x+10$ so that $\frac{dy}{dx} = 2x-6$. But we get

$$\int \frac{2x-5}{x^2-6x+10}\,dx = \int \frac{2x-5}{x^2-6x+10}\left(\frac{dy}{2x-6}\right) = \int \frac{2x-5}{(2x-6)y}\,dy.$$

And we cannot move on since there is no cancellation between $(2x-5)$ and $(2x-6)$.

To settle this technical problem, one can modify the integral so that the “cancellation” works.

☺Solution:

$$\int \frac{2x-5}{x^2-6x+10}\,dx = \int \left[\left(\frac{2x-6}{x^2-6x+10}\right) + \overbrace{\frac{1}{x^2-6x+10}}^{\text{extra term}}\right] dx$$

$$= \int \frac{2x-6}{x^2-6x+10}\,dx + \int \frac{1}{x^2-6x+10}\,dx \ldots\ldots\ldots\ldots.(*)$$

For the 1st integral $\int \frac{2x-6}{x^2-6x+10}\,dx,$ we let

$$y = x^2-6x+10 \Rightarrow \frac{dy}{dx} = 2x-6 \Rightarrow dx = \frac{1}{2x-6}\,dy.$$

$$\int \frac{2x-6}{x^2-6x+10}\,dx = \int \frac{2x-6}{x^2-6x+10}\left(\frac{1}{2x-6}\,dy\right) = \int \frac{1}{y}\,dy = \ln|y| + C$$

$$= \ln|x^2-6x+10| + C.$$

Next, we proceed to compute the second integral. One can use similar trick as in Example 15 (p. 27) of Chapter 1 to compute this integral.

For the 2nd integral $\int \frac{1}{x^2-6x+10}dx$, note that

$$\int \frac{1}{x^2-6x+10}dx = \int \frac{1}{(x-3)^2+1}dx = \tan^{-1}(x-3) + C'$$

Note:

The last equality follows from the fact that

$$\int \frac{1}{1+x^2}dx = \tan^{-1}x + C \Rightarrow \int \frac{1}{1+(ax+b)^2}dx = \frac{1}{a}\tan^{-1}(ax+b) + C.$$

Summing up, we have

$$\int \frac{2x-5}{x^2-6x+10}dx = \ln|x^2-6x+10| + \tan^{-1}(x-3) + C.$$

Example 8

Compute the integral

$$\int \frac{3x+11}{2x^2+8x+13}\,dx$$

☺Solution:

We try the substitution $y = 2x^2 + 8x + 13$. Then we have $\frac{dy}{dx} = 4x + 8.$

Following the idea of Example 7, the integral is then rewritten as

$$\int \frac{3x+11}{2x^2+8x+13}\,dx \overset{\substack{3x+11\\=(3x+6)+5}}{\triangleq} \int \frac{3x+6}{2x^2+8x+13}\,dx + \int \frac{5}{2x^2+8x+13}\,dx$$

$$= \overbrace{\int \frac{3x+6}{2x^2+8x+13}\left(\frac{1}{4x+8}\right)dy}^{\text{substitute } y=2x^2+8x+13} + \int \frac{5}{2x^2+8x+13}\,dx$$

$$= \frac{3}{4}\int \frac{1}{y}\,dy + 5\int \frac{1}{2(x+2)^2+5}\,dx$$

$$= \frac{3}{4}\ln|y| + \frac{5}{5}\int \frac{1}{\frac{2}{5}(x+2)^2 + 1}\,dx$$

$$\overset{\frac{2}{5}(x+2)^2 = \left[\sqrt{\frac{2}{5}}(x+2)\right]^2}{\overset{\frown}{=}} \frac{3}{4}\ln|2x^2 + 8x + 13| + \int \frac{1}{\left[\left(\sqrt{\frac{2}{5}}x + \frac{2\sqrt{2}}{\sqrt{5}}\right)\right]^2 + 1}\,dx$$

$$= \frac{3}{4}\ln|2x^2 + 8x + 13| + \left(\frac{1}{\sqrt{\frac{2}{5}}}\tan^{-1}\left(\sqrt{\frac{2}{5}}x + \frac{2\sqrt{2}}{\sqrt{5}}\right)\right) + C$$

$$= \frac{3}{4}\ln|2x^2 + 8x + 13| + \sqrt{\frac{5}{2}}\tan^{-1}\left(\sqrt{\frac{2}{5}}x + \frac{2\sqrt{2}}{\sqrt{5}}\right) + C.$$

Example 9 (Very Tricky, you just need to know the result)

Compute the integral

$$\int \sec x \, dx.$$

☺Solution:

To compute this integral, we multiply both numerator and denominator by a factor of $(\sec x + \tan x)$, i.e.,

$$\int \sec x \, dx = \int \frac{\sec x \, (\sec x + \tan x)}{\sec x + \tan x} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx.$$

We then let $y = \sec x + \tan x$. Then $\frac{dy}{dx} = \sec x \tan x + \sec^2 x$ and the integral becomes

$$\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \left(\frac{1}{\sec x \tan x + \sec^2 x}\right) dy$$

$$= \int \frac{1}{y} dy = \ln|y| + C$$

$$= \ln|\sec x + \tan x| + C.$$

Remark of Example 9

Using similar trick, one can show that

$$\int \csc x \, dx = -\ln|\csc x + \cot x| + C.$$

(Hint: Multiply both numerator and denominator of the integrand by $(\csc x + \cot x)$.)

Computing integral involving trigonometric functions: $\int \sin^m x \cos^{\mathrm{n}} x\, dx$

Example 10 $(\boldsymbol{m = \mathbf{odd}})$ Compute the integral

$$\int \sin^5 x\, dx.$$

☺IDEA:

Although one can compute this integral by using product-to-sum formula, it would be quite tedious since we need to use this formula for 4 times.

☺Solution:

We let $y = \cos x$. Then $\frac{dy}{dx} = -\sin x$ and the integral becomes

$$\int \sin^5 x\, dx = \int \sin^5 x \left(-\frac{1}{\sin x} dy\right) = -\int \sin^4 x\, dy = -\int (1 - \cos^2 x)^2\, dy$$

$$= -\int (1 - y^2)^2 dy = -\int (1 - 2y^2 + y^4) dy = -y + \frac{2y^3}{3} - \frac{y^5}{5} + C$$

$$= -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5} + C.$$

Example 11 $(\boldsymbol{n} = \mathbf{odd})$ Compute the integral

$$\int \cos^3 x \sin^2 x \, dx$$

☺Solution:

Let $y = \sin x$, then $\frac{dy}{dx} = \cos x$ and the integral becomes

$$\int \cos^3 x \sin^2 x \, dx = \int \cos^3 x \sin^2 x \left(\frac{1}{\cos x} dy\right) = \int \cos^2 x \sin^2 x \, dy$$

$$= \int (1 - \sin^2 x) \sin^2 x \, dy = \int y^2 (1 - y^2) dy = \int (y^2 - y^4) dy$$

$$= \frac{y^3}{3} - \frac{y^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C.$$

Remark: $(\boldsymbol{m}, \boldsymbol{n} = \mathbf{even})$ Use double-angle formulae:

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

Trigonometric substitution

Sometimes, using method of substitution with simple substitution may not be successful. As an example, we consider the following integral

$$\int \frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx\,, \qquad x > 0$$

As a starting point, one may try to use the substitution $y = 1 + x^2$ to simplify the integrand as $\frac{1}{y^{\frac{3}{2}}}$.

Since $\frac{dy}{dx} = 2x \Rightarrow dx = \frac{1}{2x}dy$, then the integral is then transformed into

$$\int \frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx = \int \frac{1}{2x(1+x^2)^{\frac{3}{2}}}\,dy = \frac{1}{2}\int \frac{1}{y^{\frac{3}{2}}\sqrt{y-1}}\,dy.$$

It appears that the integral is still hard to integrate after transformation.

One has to try other substitution that can simplify the integral significantly.

Recall that the main purpose of doing substitution is to transform the denominator (which is the sum of two terms originally) into a single term which is good for algebraic manipulation.

An alternative approach (perhaps the most popular one) is to transform the variable x into some trigonometric functions (say $x = \sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$ etc.,) so that the terms in the denominator can be combined *using trigonometric identities*.

In this example, we take $x = \tan\theta$ $(\Rightarrow \frac{dx}{d\theta} = \sec^2\theta)$ and the integral becomes

$$\int \frac{1}{(1+x^2)^{\frac{3}{2}}}dx = \int \frac{1}{(1+\tan^2\theta)^{\frac{3}{2}}}(\sec^2\theta\, d\theta) \overset{1+\tan^2\theta=\sec^2\theta}{\stackrel{\text{\tiny def}}{=}} \int \frac{\sec^2\theta}{\sec^3\theta}d\theta$$

$$= \int \frac{1}{\sec\theta}d\theta \overset{\sec\theta=\frac{1}{\cos\theta}}{\stackrel{\text{\tiny def}}{=}} \int \cos\theta\, d\theta = \sin\theta + C = \cdots = \frac{x}{\sqrt{1+x^2}} + C.$$

The following identities may be useful for trigonometric substitution:

$$1 - \cos^2\theta = \sin^2\theta$$

$$1 - \sin^2\theta = \cos^2\theta$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\sec^2\theta - 1 = \tan^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\csc^2\theta - 1 = \cot^2\theta$$

Example 12

Compute the integral

$$\int \sqrt{3-x^2}dx\,, \qquad -\sqrt{3} < x < \sqrt{3}$$

Note:

Simple substitution such as $y = 3 - x^2$ does not work in this context. Note that $\frac{dy}{dx} = -2x \;\Rightarrow\; dx = -\frac{1}{2x}dy$ and the integral becomes

$$\int \sqrt{3-x^2}dx = \int \sqrt{3-x^2}\left(-\frac{1}{2x}\right)dy = -\frac{1}{2}\int \frac{1}{\sqrt{3-y}}\sqrt{y}dy$$

☺Solution:

We take $x = \sqrt{3}\sin\theta$ (so that $x^2 = 3\sin^2\theta$), then $\frac{dx}{d\theta} = \sqrt{3}\cos\theta$ and the integral becomes

$$\int \sqrt{3-x^2}dx = \int \sqrt{3-3\sin^2\theta}\left(\sqrt{3}\cos\theta\, d\theta\right)$$

$$= \int 3\cos^2\theta\, d\theta = 3\int (\cos\theta)(\cos\theta)d\theta$$

$$= 3\int \frac{1}{2}[\cos(\theta+\theta)+\cos(\theta-\theta)]d\theta$$

$$= \frac{3}{2}\int \cos 2\theta\, d\theta + \frac{3}{2}\int 1d\theta$$

$$= \frac{3}{2}\left(\frac{1}{2}\sin 2\theta\right) + \frac{3}{2}\theta + C = \frac{3}{4}\sin 2\theta + \frac{3}{2}\theta + C$$

$$\overset{\substack{x=\sqrt{3}\sin\theta \\ \Rightarrow \theta=\sin^{-1}\frac{x}{\sqrt{3}}}}{\triangleq} \frac{3}{4}\sin\left(2\sin^{-1}\frac{x}{\sqrt{3}}\right) + \frac{3}{2}\sin^{-1}\frac{x}{\sqrt{3}} + C.$$

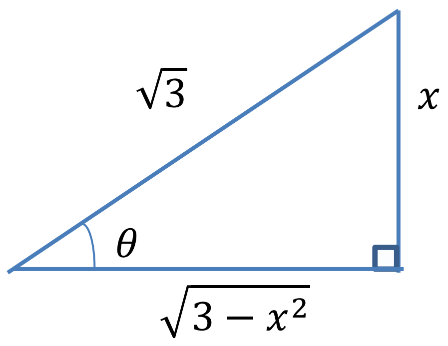
Remark or Example 12:

Alternatively, one can transform the final answer into the function of x *in as follows:*

$$\frac{3}{4}\sin 2\theta + \frac{3}{2}\theta + C \overset{\substack{\sin(A+B) \\ =\sin A\cos B+\cos A\sin B}}{\triangleq} \frac{3}{4}2\sin\theta\cos\theta + \frac{3}{2}\theta + C$$

$$= \frac{3}{2}\left(\frac{x}{\sqrt{3}}\right)\frac{\sqrt{3-x^2}}{\sqrt{3}} + \frac{3}{2}\sin^{-1}\frac{x}{\sqrt{3}} + C$$

$$= \frac{x\sqrt{3-x^2}}{2} + \frac{3}{2}\sin^{-1}\frac{x}{\sqrt{3}} + C$$ (better)



Example 13

Compute the integral

$$\int \frac{1}{x^3\sqrt{x^2-1}}\,dx$$

☺Solution:

Let $x = \sec\theta$ (so that $x^2 - 1 = \sec^2\theta - 1 = \tan^2\theta$), then $\frac{dx}{d\theta} = \sec\theta\tan\theta$ and the integral becomes

$$\int \frac{1}{x^3\sqrt{x^2-1}}\,dx = \int \frac{1}{\sec^3\theta\sqrt{\sec^2\theta-1}}(\sec\theta\tan\theta\,d\theta)$$

$$= \int \frac{1}{\sec^3\theta\tan\theta}(\sec\theta\tan\theta\,d\theta) = \int \frac{1}{\sec^2\theta}\,d\theta$$

$$= \int \cos^2\theta\,d\theta = \int \frac{1}{2}[\cos(\theta+\theta) + \cos(\theta-\theta)]d\theta$$

$$= \frac{1}{2}\int [\cos 2\theta + 1]d\theta$$

$$= \frac{1}{2}\left(\frac{\sin 2\theta}{2} + \theta\right) + C = \frac{1}{4}\sin 2\theta + \frac{1}{2}\theta + C$$

$$= \frac{1}{4}(2\sin\theta\cos\theta) + \frac{1}{2}\theta + C$$

$$= \frac{1}{2}\left(\frac{\sqrt{x^2-1}}{x}\right)\left(\frac{1}{x}\right) + \frac{1}{2}\sec^{-1}x + C$$

$$= \frac{\sqrt{x^2-1}}{2x^2} + \frac{1}{2}\sec^{-1}x + C.$$

Example 14 (A bit tricky)

Compute the integral

$$\int \frac{1}{\sqrt{x^2+4x+7}}\,dx$$

Note:

One has to “combine” the terms x^2 and $4x$ into a single square using *completing square* technique.

☺Solution

Using the completing square technique, we have

$$\int \frac{1}{\sqrt{x^2+4x+7}}\,dx = \int \frac{1}{\sqrt{(x^2+4x+4)+3}}\,dx = \int \frac{1}{\sqrt{(x+2)^2+3}}\,dx$$

We then adopt the trigonometric substitution and let $x + 2 = \sqrt{3}\tan\theta$ (so that $(x+2)^2 + 3 = 3\tan^2\theta + 3 = 3(\tan^2\theta + 1) = 3\sec^2\theta$).

Then $x = \sqrt{3}\tan\theta - 2$ and $\frac{dx}{d\theta} = \sqrt{3}\sec^2\theta$. The integral is transformed into

$$\int \frac{1}{\sqrt{(x+2)^2+3}}\,dx = \int \frac{1}{\sqrt{3\sec^2\theta}}\left(\sqrt{3}\sec^2\theta\,d\theta\right)$$

$$= \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\sqrt{1+\tan^2\theta} + \tan\theta\right| + C$$

$$= \ln\left|\sqrt{1+\left(\frac{x+2}{\sqrt{3}}\right)^2} + \frac{x+2}{\sqrt{3}}\right| + C$$

$$= \ln\left|\sqrt{\frac{x^2+4x+7}{3}} + \frac{x+2}{\sqrt{3}}\right| + C = \ln\left|\sqrt{x^2+4x+7} + x + 2\right| + C'.$$

How to choose the suitable substitution?

So far, we have two different types of substitution (standard one and the trigonometric one). One may ask a question that which substitution we should take when using the method substitution. In fact, there is no clear guideline on the choice the substitution. Here's my suggestion for beginners:

- It is always better to start with a standard, straight-forward substitution (see Example 1 – 6) such that the computation process is easier.

- In the unlucky case when the standard substitution does not work. One can try to use some algebraic trick (see Example 7 – 8) to transform the integral or try to adopt trigonometric substitution (see Example 12-14).

As an illustration, we consider the following two integrals

$$\int \frac{1}{(1+x^2)^2}\,dx\,, \qquad \int \frac{x^5}{(x^2-1)^3}\,dx.$$

Computation of the first integral

For the first integral, one can first try the substitution

$$y = 1 + x^2.$$

Then $\frac{dy}{dx} = 2x \Rightarrow dx = \frac{1}{2x}dy$ and the integral becomes

$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1}{(1+x^2)^2}\left(\frac{1}{2x}dy\right) = \frac{1}{2}\int \frac{1}{y^2\sqrt{y-1}}\,dy$$

which is difficult to integrate.

In this case, one can try other substitutions. Here, we use trigonometric substitution and let $x = \tan\theta$ (so that $1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$).

Then $\frac{dx}{d\theta} = \sec^2\theta$ and the integral becomes

$$\int \frac{1}{(1+x^2)^2}dx = \int \frac{1}{(1+\tan^2\theta)^2}(\sec^2\theta\, d\theta) = \int \frac{1}{\sec^2\theta}d\theta = \int \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\int [\cos(\theta+\theta) + \cos(\theta-\theta)]d\theta = \frac{1}{2}\int (\cos 2\theta + 1)d\theta$$

$$= \frac{1}{4}\sin 2\theta + \frac{\theta}{2} + C = \cdots = \frac{x}{2(1+x^2)} + \frac{\tan^{-1}x}{2} + C.$$

Computation of the second integral

For the second integral, we may try the substitution

$$y = x^2 - 1.$$

Then $\frac{dy}{dx} = 2x \;\Rightarrow\; dx = \frac{1}{2x}dy$ and the integral becomes

$$\int \frac{x^5}{(x^2-1)^3}dx = \int \frac{x^5}{(x^2-1)^3}\left(\frac{1}{2x}dy\right) = \frac{1}{2}\int \frac{x^4}{(x^2-1)^3}dy$$

$$\overset{\substack{y=x^2-1 \\ \Rightarrow x^2=y+1}}{\stackrel{\frown}{=}} \frac{1}{2}\int \frac{(y+1)^2}{y^3}dy = \frac{1}{2}\int \frac{y^2+2y+1}{y^3}dy$$

$$= \frac{1}{2}\int \frac{1}{y}dy + \int y^{-2}dy + \frac{1}{2}\int y^{-3}dy$$

$$= \frac{1}{2}\ln|y| + \frac{y^{-2+1}}{-2+1} + \frac{1}{2}\left(\frac{y^{-3+1}}{-3+1}\right) + C$$

$$= \frac{1}{2}\ln|x^2-1| - \frac{1}{x^2-1} - \frac{1}{4(x^2-1)^2} + C$$

Since the simple substitution is OK for this situation, there is no need to use trigonometric substitution which may complicate the matter.

Integration by parts

This method is first motivated from the product rule in differentiation: Let $u = u(x)$ and $v = v(x)$ be two differentiable functions in x, then

$$\frac{d}{dx}uv = v\frac{du}{dx} + u\frac{dv}{dx} \dots \dots (*).$$

We integrate the equation $(*)$ with respect to x, we get

$$\int \frac{d(uv)}{dx}\,dx = \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx$$

$$\Rightarrow \int d(uv) = \int vdu + \int udv$$

$$\Rightarrow uv = \int vdu + \int udv$$

Recall that $\int 1dy = y + C$ and we take $y = uv$

$$\Rightarrow \int udv = uv - \int vdu.$$

One can use the similar integral to obtain the integration by parts formula for definite integral. We integrate the equation $(*)$ both sides with respect to x (from a to b):

$$\int_a^b \frac{d(uv)}{dx}\,dx = \int_a^b v\frac{du}{dx}dx + \int_a^b u\frac{dv}{dx}dx$$

$$\Rightarrow \int_a^b d(uv) = \int_a^b vdu + \int_a^b udv$$

$$\Rightarrow \underbrace{u(b)v(b) - u(a)v(a)}_{=uv|_a^b \text{ or } [uv]_a^b} = \int_a^b vdu + \int_a^b udv$$

$$\boxed{\Rightarrow \int_a^b udv = uv|_a^b - \int_a^b vdu.}$$

The two formulae in the boxes are called *integration by parts formulae*.

How do the formulae work?

In order to appreciate this formula, we consider the following integral:

$$\int \ln x \, dx$$

Using the integration by parts formula with $u = \ln x$ and $dv = dx$

(or $v = \int dx = x$), we have

$$\frac{d}{dx}\ln x = \frac{1}{x} \Rightarrow d(\ln x) = \frac{1}{x}dx$$

$$\int \underbrace{\ln x}_{u} \underbrace{dx}_{dv} = x \ln x - \int \underbrace{x}_{v} \; d\,\underbrace{(\ln x)}_{u}$$

$$= x \ln x - \int x \left(\frac{1}{x} dx\right)$$

$$= x \ln x - \int 1 dx$$

$$= x \ln x - x + C.$$

Consider another integral:

$$\int x \sin x \, dx.$$

Using integration by parts with $u = x$ and $dv = \sin x \, dx$

(so that $v = \int \sin x \, dx = -\cos x$), we have

$$\int \underbrace{x}_{u} \underbrace{\sin x \, dx}_{dv} = \int \underbrace{x}_{u} \, d \underbrace{(-\cos x)}_{v} = \underbrace{x(-\cos x)}_{uv} - \int \underbrace{(-\cos x)}_{v} \, d \underbrace{x}_{u}$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + C.$$

Insights about integration by parts

- Similar to the method of substitution, integration by parts aims to transform a given integral into another integral (with some extra terms) in order to simplify the calculation:

$$\underbrace{\int x \sin x \, dx}_{\text{hard to compute}} = -x \cos x + \underbrace{\int \cos x \, dx}_{\text{easier}}$$

- The principle of this method is to "eliminate" some ugly terms (say $\ln x$, x, ... , etc.,) through differentiation (since they will be put in the differential operator after the transform).

$$\int \underbrace{\ln x}_{u} \, d \underbrace{x}_{v} = x \ln x - \int \underbrace{x}_{v} \, d \underbrace{(\ln x)}_{u} = x \ln x - \int x \left(\frac{1}{x} dx \right)$$

$\ln x$ is put inside the operator $d(??)$.

$\ln x$ is being "eliminated" after differentiation

How to use integration by parts properly?

The most crucial part of using integration by parts is how to allocate the functions into $u(x)$ and dv. The following guideline (according to my past experience) suggests one of the ways:

<u>Step 1: Pick $u(x)$</u>

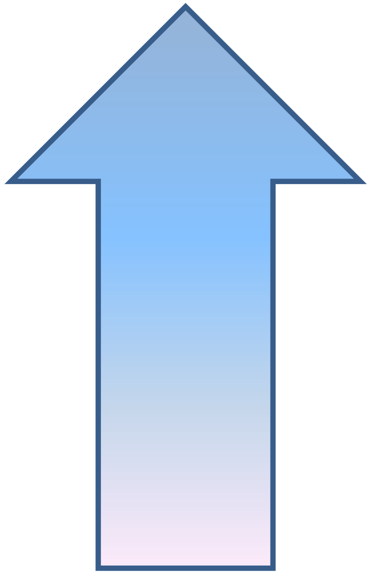
- Recall that the function $u(x)$ will be put inside the differential operator and is then differentiated after the use of the formula.

$$\int u dv = uv - \int v du = uv - \int v \left(\frac{du}{dx}\right) dx$$

- Since we wish to simplify the integral as much as possible, we should choose $u(x)$ in such a way that $u(x)$ can be either "eliminated completely" or "transformed into convenient form" through differentiation during the transform.

The following shows the priority about the choice of $u(x)$

Highest Priority



Lowest Priority

$$f_1(x) = \sin^{-1} x\,,\ \cos^{-1} x \ \text{ or } \ \tan^{-1} x \ \Rightarrow \frac{df_1(x)}{dx} = \underbrace{\frac{1}{\sqrt{1-x^2}}}_{\sin^{-1} x},\ \underbrace{\frac{-1}{\sqrt{1-x^2}}}_{\cos^{-1} x} \ \text{ or } \ \underbrace{\frac{1}{1+x^2}}_{\tan^{-1} x}$$

$$f_2(x) = \ln x \qquad \Rightarrow \frac{df_2(x)}{dx} = \frac{1}{x}$$

$$f_3(x) = x^n, \qquad n = 1,2,3,4, \ldots \qquad \Rightarrow \frac{df_3(x)}{dx} = nx^{n-1}$$

(n must be positive integer!!)

$$f_4(x) = \sin x\,, \cos x \qquad \Rightarrow \frac{df_4(x)}{dx} = \underbrace{\cos x}_{\sin x} \ \text{ or } \ \underbrace{-\sin x}_{\cos x}$$

<u>Step 2: Find $v(x)$</u>

After $u(x)$ is identified, the remaining part in the integrand (say $f(x)dx$) will be dv. The function $v(x)$ can be computed by

$$dv = f(x)dx \Rightarrow \int dv = \int f(x)dx \Rightarrow v = \int f(x)dx.$$

Example 15

Compute the integral

$$\int x^4 \ln x \, dx.$$

IDEA:

One has to eliminate one of the functions x^4 and $\ln x$ using integration by parts. Although both x^4 and $\ln x$ can be "eliminated" through differentiation, we shall choose $u(x) = \ln x$ since it has a higher priority. (It is because we are able to integrate x^4)

☺Solution:

We pick $u = u(x) = \ln x$. Then $dv = x^4 dx \Rightarrow v = \int x^4 dx = \frac{x^5}{5}$.

Using integration by parts, we have

$$\int x^4 \ln x \, dx = \int \underbrace{\ln x}_{u} \underbrace{(x^4 dx)}_{dv} = \underbrace{\int \ln x \, d\left(\frac{x^5}{5}\right)}_{\int u dv}$$

$$= \underbrace{\frac{x^5}{5} \ln x}_{uv} - \underbrace{\int \frac{x^5}{5} d(\ln x)}_{\int v du}$$

$$= \frac{x^5}{5} \ln x - \int \frac{x^5}{5} \left(\frac{1}{x} dx\right)$$

$$= \frac{x^5}{5} \ln x - \frac{1}{5} \int x^4 dx = \frac{x^5}{5} \ln x - \frac{x^5}{25} + C.$$

Example 16

Compute the integral

$$\int x\tan^{-1}x\,dx.$$

IDEA:

Since there is an inverse trigonometric function in the integrand, one has to eliminate $\tan^{-1}x$ first.

☺Solution:

We pick $u = u(x) = \tan^{-1}x$ and $dv = xdx \Rightarrow v = \int xdx = \frac{x^2}{2}$.

From integration by parts, we have

$$\int x\tan^{-1}x\,dx = \int \underbrace{\tan^{-1}x}_{u}\,\underbrace{(xdx)}_{dv} = \underbrace{\int \tan^{-1}x\,d\left(\frac{x^2}{2}\right)}_{\int udv}$$

$$=\underbrace{\frac{x^2}{2}\tan^{-1}x}_{uv}-\underbrace{\int\frac{x^2}{2}d(\tan^{-1}x)}_{\int vdu}$$

$$=\frac{x^2}{2}\tan^{-1}x-\int\frac{x^2}{2}\left(\frac{1}{1+x^2}\right)dx=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{x^2}{1+x^2}dx$$

$$=^{(*)}\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\left[\frac{x^2+1}{x^2+1}-\frac{1}{x^2+1}\right]dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int 1dx+\frac{1}{2}\int\frac{1}{x^2+1}dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{x}{2}+\frac{1}{2}\tan^{-1}x+C.$$

Remark of $(*)$

The purpose of doing such decomposition is to eliminate the x^2 in the numerator.

Example 17

Compute

$$\int x^2 e^{3x} dx$$

☺Solution:

Pick $u = x^2$ and $dv = e^{3x}dx \Rightarrow v = \int e^{3x}dx = \frac{1}{3}e^{3x}$.

From integration by parts, we have

$$\int x^2 e^{3x} dx = \int \underbrace{x^2}_{u} \underbrace{(e^{3x}dx)}_{dv} = \underbrace{\int x^2 d\left(\frac{e^{3x}}{3}\right)}_{\int udv}$$

$$= \underbrace{\frac{x^2 e^{3x}}{3}}_{uv} - \underbrace{\int \frac{e^{3x}}{3} d(x^2)}_{\int vdu} = \frac{x^2 e^{3x}}{3} - \int \frac{e^{3x}}{3}(2xdx) = \frac{x^2 e^{3x}}{3} - \frac{2}{3}\int xe^{3x} dx$$

To continue the computation, we apply integration by parts again on the 2nd integral.

Take $u = x$ and $dv = e^{3x}dx \Rightarrow v = \int e^{3x}dx = \frac{1}{3}e^{3x}$.

$$\int x^2 e^{3x}dx = \frac{x^2 e^{3x}}{3} - \frac{2}{3}\int xe^{3x}dx$$

$$= \frac{x^2 e^{3x}}{3} - \frac{2}{3}\underbrace{\int x\, d\left(\frac{e^{3x}}{3}\right)}_{\int udv} = \frac{x^2 e^{3x}}{3} - \frac{2}{3}\left[\underbrace{\frac{xe^{3x}}{3}}_{uv} - \underbrace{\int \frac{e^{3x}}{3}dx}_{\int vdu}\right]$$

$$= \frac{x^2 e^{3x}}{3} - \frac{2}{9}xe^{3x} + \frac{2}{9}\int e^{3x}dx$$

$$= \frac{x^2 e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2}{27}e^{3x} + C.$$

Example 18 (Integration by parts for definite integral)

Compute the integral

$$\int_0^{\frac{\pi}{2}} x \sin x \, dx$$

☺Solution:

We take $u(x) = x$. Then $dv = \sin x \, dx \Rightarrow v = \int \sin x \, dx = -\cos x$.

From integration by parts, we have

$$\int_0^{\frac{\pi}{2}} x \sin x \, dx = \int_0^{\frac{\pi}{2}} \underbrace{x}_{u} \underbrace{(\sin x \, dx)}_{dv} = \underbrace{\int_0^{\frac{\pi}{2}} x d(-\cos x)}_{\int u dv}$$

$$= \underbrace{[-x \cos x]_0^{\frac{\pi}{2}}}_{=-\frac{\pi}{2}\cos\frac{\pi}{2}-(-0\cos 0)} - \int_0^{\frac{\pi}{2}} (-\cos x) dx = 0 + \underbrace{[\sin x]_0^{\frac{\pi}{2}}}_{\sin\frac{\pi}{2}-\sin 0} = 1.$$

Harder Examples

In this section, we will present some advanced (not straight forward) applications of integration by parts.

Difficult situation #1: The function in the integrand cannot be "eliminated" by differentiation.

Example 19

Compute the integral

$$\int e^{3x} \sin x \, dx$$

IDEA:

Unfortunately, both e^{3x} and $\sin x$ cannot be eliminated through differentiation. However, $\sin x$ can be "transformed" into $\cos x$ after differentiation. Hence, we still use integration by parts to transform the integral into another form.

☺Solution:

We try to transform the integral by using integration by parts. Take $u = \sin x$ and $dv = e^{3x}dx \Rightarrow v = \int e^{3x}dx = \frac{1}{3}e^{3x}$, then we have

$$\int e^{3x}\sin x\,dx = \int \underbrace{\sin x}_{u}\,\underbrace{(e^{3x}dx)}_{dv} = \underbrace{\int \sin x\,d\left(\frac{1}{3}e^{3x}\right)}_{\int udv}$$

$$= \frac{1}{3}e^{3x}\sin x - \int \frac{1}{3}e^{3x}d(\sin x)$$

$$= \frac{1}{3}e^{3x}\sin x - \frac{1}{3}\int e^{3x}\cos x\,dx$$

To continue, we use the integration by parts again by taking $u = \cos x$ and $dv = e^{3x}dx \Rightarrow v = \int e^{3x}dx = \frac{1}{3}e^{3x}$, then we have

$$= \frac{1}{3}e^{3x}\sin x - \frac{1}{3}\underbrace{\int \cos x\,d\left(\frac{e^{3x}}{3}\right)}_{\int udv}$$

$$= \frac{1}{3} e^{3x} \sin x - \frac{1}{3}\left[\cos x \left(\frac{e^{3x}}{3}\right) - \int \frac{e^{3x}}{3} d(\cos x)\right]$$

$$= \frac{1}{3} e^{3x} \sin x - \frac{1}{9} e^{3x} \cos x - \frac{1}{9}\int e^{3x} \sin x \, dx$$

Summing up, we have

$$\int e^{3x} \sin x \, dx = \frac{1}{3} e^{3x} \sin x - \frac{1}{9} e^{3x} \cos x - \frac{1}{9}\int e^{3x} \sin x \, dx$$

$$\Rightarrow \frac{10}{9}\int e^{3x} \sin x \, dx = \frac{1}{3} e^{3x} \sin x - \frac{1}{9} e^{3x} \cos x$$

$$\Rightarrow \int e^{3x} \sin x \, dx = \frac{3}{10} e^{3x} \sin x - \frac{1}{10} e^{3x} \cos x + C.$$

Example 20

Compute the integral

$$\int \sec^3 x \, dx$$

IDEA:

Different from the integral $\int \sin^3 x \, dx$ that you have seen in Lecture Note 1 (Review example 3 p. 49), there is no "product-to-sum" formula for the trigonometric function $\sec\theta$. Again, we shall transform the integral using integration by parts.

- However, one should NOT use the integration by parts directly as follows:

$$\int \underbrace{\sec^3 x}_{u} \, \underbrace{dx}_{dv} = x\sec^3 x - \int x d(\sec^3 x) = x\sec^3 x - 3\int \underbrace{x}_{\text{extra!}} \sec^3 x \tan x \, dx$$

To avoid this problem, one has to ensure that both $u(x)$ and $v(x)$ are *trigonometric functions*. Note that $\int \sec^2 x\, dx = \tan x$, one can use integration by parts in the following ways:

Let $u = \sec x$ and $dv = \sec^2 x\, dx \Rightarrow v = \int \sec^2 x\, dx = \tan x$.

$$\int \sec^3 x\, dx = \int \underbrace{\sec x}_{u}\, \underbrace{(\sec^2 x\, dx)}_{dv} = \underbrace{\int \sec x\, d(\tan x)}_{\int u dv}$$

$$= \sec x \tan x - \int \tan x\, d(\sec x)$$

$$\overset{\frac{d}{dx}\sec x = \sec x \tan x}{\triangleq} \sec x \tan x - \int \tan x\, (\sec x \tan x\, dx)$$

$$= \sec x \tan x - \int \sec x \tan^2 x\, dx$$

$$\overset{1+\tan^2 x = \sec^2 x}{\triangleq} \sec x \tan x - \int \sec x\, (\sec^2 x - 1) dx$$

$$= \sec x \tan x + \int \sec x \, dx - \int \sec^3 x \, dx$$

$$\overset{\int \sec x dx = \ln|\sec x + \tan x| + C}{\cong} \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x \, dx$$

Summing up, we get

$$\int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x \, dx$$

$$\Rightarrow 2 \int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x|$$

$$\Rightarrow \int \sec^3 x \, dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C.$$

- Don't forget to add an arbitrary constant C to an indefinite integral.

Difficult situation #2: The integrand involved the complicated function.

Example 21

Compute the integral

$$\int x\ln\left(2\sqrt{x}+1\right)dx$$

☺IDEA:

Due to the existence of logarithmic function $\ln x$, one may prefer to use integration by parts with $u=\ln\left(2\sqrt{x}+1\right)$. However, the process is tedious.

$$\int \underbrace{\ln\left(2\sqrt{x}+1\right)}_{u}\underbrace{(xdx)}_{dv}=\int \ln(2\sqrt{x}+1)\,d\left(\frac{x^2}{2}\right)$$

$$=\frac{x^2}{2}\ln\left(2\sqrt{x}+1\right)-\int\frac{x^2}{2}d\left(\ln\left(2\sqrt{x}+1\right)\right)$$

$$=\frac{x^2}{2}\ln\left(2\sqrt{x}+1\right)-\int\frac{x^2}{2}\left(\frac{1}{\sqrt{x}\left(2\sqrt{x}+1\right)}\right)dx$$

☺Solution:

One has to use the *method of substitution* first to simplify the integral. (Transform the function $\ln\left(2\sqrt{x}+1\right)$ into $\ln y$.

We let $y = 2\sqrt{x}+1$. Then $\frac{dy}{dx} = \frac{d}{dx}\left(2x^{\frac{1}{2}}+1\right) = x^{-\frac{1}{2}}$ and the integral becomes

$$\int x\ln(2\sqrt{x}+1)dx = \int x\ln(2\sqrt{x}+1)\left(\frac{1}{x^{-\frac{1}{2}}}dy\right) = \int x^{\frac{3}{2}}\ln(2\sqrt{x}+1)\,dy$$

$$= \int\left(\frac{y-1}{2}\right)^3 \ln y\,dy$$

$$= \frac{1}{8}\int y^3\ln y\,dy - \frac{3}{8}\int y^2\ln y\,dy + \frac{3}{8}\int y\ln y\,dy - \frac{1}{8}\int \ln y\,dy$$

Each of these four integrals can be computed using the technique used in Example 15. (Left as exercise.) For your reference, the final answer is given by

$$\int x\ln\left(2\sqrt{x}+1\right)dx$$

$$=\frac{1}{8}\left[\frac{y^4}{4}\ln y-\frac{y^4}{16}\right]-\frac{3}{8}\left[\frac{y^3}{3}\ln y-\frac{y^3}{9}\right]+\frac{3}{8}\left[\frac{y^2}{2}\ln y-\frac{y^2}{4}\right]-\frac{1}{8}[y\ln y-y]+C$$

(*Remember to put $y=2\sqrt{x}+1$. I skip this step to save space.)

Example 22

Compute the integral

$$\int_1^2\frac{e^{\frac{1}{x}}}{x^3}dx.$$

☺IDEA:

Similar to Example 21, one needs to simplify the integral using the *method of substitution* to transform the function $e^{\frac{1}{x}}$ into “standard” exponent function e^y.

☺Solution

We let $y=\frac{1}{x}$. Then $\frac{dy}{dx}=-\frac{1}{x^2}$ and the integral becomes

$$\int_1^2 \frac{e^{\frac{1}{x}}}{x^3}dx \overset{\substack{\text{when } x=2,\ y=\frac{1}{2} \\ \text{when } x=1,\ y=1}}{\triangleq} \int_1^{\frac{1}{2}} \frac{e^{\frac{1}{x}}}{x^3}(-x^2dy)=-\int_1^{\frac{1}{2}} e^{\frac{1}{x}}\left(\frac{1}{x}\right)dy$$

$$\overset{\int_a^b f(x)dx=-\int_b^a f(x)dx}{\triangleq} \int_{\frac{1}{2}}^1 \underbrace{y}_{u}\underbrace{e^y dy}_{dv} \overset{\substack{dv=e^y dy \\ \Rightarrow v=\int e^y dy=e^y}}{\triangleq} \int_{\frac{1}{2}}^1 yd(e^y)$$

$$=\underbrace{[ye^y]_{1/2}^1}_{1e^1-\frac{1}{2}e^{\frac{1}{2}}}-\int_{\frac{1}{2}}^1 e^y dy$$

$$=e-\frac{1}{2}e^{\frac{1}{2}}-[e^y]_{\frac{1}{2}}^1=e-\frac{1}{2}e^{\frac{1}{2}}-\left(e-e^{\frac{1}{2}}\right)=\frac{1}{2}e^{\frac{1}{2}}.$$

Method of substitution v.s. integration by parts

Summing up, the method of substitution and integration by parts aim to transform the integral in two different ways:

Method of substitution

Transform the functions (say $e^{\frac{1}{x}}$, $\sin\sqrt{x}$, $\cos(\ln x)$, etc.,) into "standard functions" (say e^y, $\sin y$, $\cos y$) using the substitution (say $y = \frac{1}{x}$, $\sqrt{x}$, $\ln x$)

Integration by parts

Transform the functions (say $\sin^{-1} x$, $\ln x$) into simple function using differentiation (take $u(x) = \sin^{-1} x$, $\ln x$ and $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$, $\frac{1}{x}$).

- This method may not work well for the functions such as $\ln\left(2\sqrt{x}+1\right)$ etc. One needs to transform these functions into standard one $\ln y$ using the method of substitution. (See Example 21 and 22)

Example 23

Compute the integral

$$\int x^5 \cos(3 - x^3)\, dx$$

☺Solution:

We first simplify the integral using the *method of substitution* again.

We let $y = 3 - x^3$. Then $\frac{dy}{dx} = -3x^2$ and the integral becomes

$$\int x^5 \cos(3 - x^3)\, dx = \int x^5 \cos(3 - x^3) \left(\frac{1}{-3x^2} dy\right)$$

$$\overset{x^3=3-y}{\cong} -\frac{1}{3}\int (3 - y) \cos y\, dy = -\frac{1}{3}\int \underbrace{(3 - y)}_{u} \underbrace{(\cos y\, dy)}_{dv}$$

$$\overset{\substack{dv=\cos y dy \\ \Rightarrow v=\int \cos y dy=\sin y}}{\cong} -\frac{1}{3}\int (3 - y) d(\sin y)$$

$$= -\frac{1}{3}\left[(3-y)\sin y - \int \sin y \, d(3-y)\right]$$

$$\overset{\frac{d}{dy}(3-y)=-1}{\cong} \quad -\frac{1}{3}\left[(3-y)\sin y - \int \sin y \, (-dy)\right]$$

$$= -\frac{1}{3}(3-y)\sin y - \frac{1}{3}\int \sin y \, dy$$

$$= -\frac{1}{3}(3-y)\sin y + \frac{1}{3}\cos y + C$$

$$= -\frac{1}{3}x^3\sin(3-x^3) + \frac{1}{3}\cos(3-x^3) + C.$$

There is $\sin^{-1} x$.
Eliminate it using
integration by parts first!

Example 24

Compute the integral

$$\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}}\,dx$$

There is $1-x^2$ in the
denominator. Transform it using
the method of substitution first!

☺Solution

Method 1: Use integration by parts

Let $u(x) = \sin^{-1} x$, then $dv = \frac{x}{\sqrt{1-x^2}}dx \Rightarrow v = \int \frac{x}{\sqrt{1-x^2}}dx \overset{y=1-x^2}{\triangleq} -\sqrt{1-x^2}$.

Then the integral becomes

$$\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}}\,dx = \int \underbrace{\sin^{-1} x}_{u} \underbrace{\left(\frac{x}{\sqrt{1-x^2}}\,dx\right)}_{dv} = \underbrace{\int \sin^{-1} x\, d\left(-\sqrt{1-x^2}\right)}_{\int u dv}$$

$$= -(\sin^{-1} x)\sqrt{1-x^2} - \int -\sqrt{1-x^2}\, d(\sin^{-1} x)$$

$$= -(\sin^{-1} x)\sqrt{1-x^2} + \int \sqrt{1-x^2}\left(\frac{1}{\sqrt{1-x^2}}dx\right)$$

$$= -\sin^{-1} x\sqrt{1-x^2} + \int 1dx = -\sin^{-1} x\sqrt{1-x^2} + x + C.$$

Method 2: Using substitution

We let $x = \sin\theta$ so that $1 - x^2 = 1 - \sin^2\theta = \cos^2\theta$ (NOT $y = 1 - x^2$ or $x = \cos\theta$!!). Then $\frac{dx}{d\theta} = \cos\theta$ and the integral becomes

$$\int \frac{x\sin^{-1} x}{\sqrt{1-x^2}}dx = \int \frac{\sin\theta\,[\sin^{-1}(\sin\theta)]}{\sqrt{1-\sin^2\theta}}(\cos\theta\, d\theta) = \int \theta\sin\theta\, d\theta$$

$$= \int \theta(\sin\theta\, d\theta) = \int \theta d(-\cos\theta) = -\theta\cos\theta - \int(-\cos\theta)d\theta$$

$$= -\theta\cos\theta + \sin\theta + C = -(\sin^{-1} x)\left(\sqrt{1-x^2}\right) + \underbrace{x}_{\sin(\sin^{-1}\theta)} + C.$$

Reduction Formula

How to integrate the following integrals

$$\int x^{10} e^x dx\,, \qquad \int \cos^8 x\, dx\,, \qquad \int (\ln x)^{20} dx\,?$$

For example, we wish to compute the first integral,

$$\int x^{10} e^x dx,$$

- One can use integration by parts to transform the integral into simpler one (Eliminate x^{10} in the integrand through differentiation).

- It cannot be done in one shot (See Example 17)!

One can use integration by parts 10 times to find out the answer, i.e.,

$$\int x^{10}e^x dx = \int x^{10} de^x = x^{10}e^x - \int e^x dx^{10} = x^{10}e^x - 10\int x^9 e^x\, dx$$

$$= x^{10}e^x - 10\int x^9 de^x = x^{10}e^x - 10\left(x^9e^x - \int e^x dx^9\right)$$

$$= x^{10}e^x - 10x^9e^x + 90\int x^8 e^x dx$$

$$= x^{10}e^x - 10x^9e^x + 90\int x^8 de^x = x^{10}e^x - 10x^9e^x + 90\left(x^8e^x - \int e^x dx^8\right)$$

$$= x^{10}e^x - 10x^9e^x + 90x^8e^x - 720\int x^7 e^x dx$$

$= \cdots$ (OK, I give up)

- For these "massive" integrals, it requires repeated use of integration by parts. This causes a huge computational cost.

- Question: Is there any shortcut?

- Answer: Reduction Formula

Reduction Formula is an integration technique which tries to compute the integrals which the expressions involving powers of elementary functions (e.g. $\cos^{\mathrm{n}} x$, $(ax^2+bx+c)^n$, $(\ln x)^n$ etc.).

- Let $I_n = \int \cos^n x \, dx$, $\int (ax^2+bx+c)^n dx$, $\int (\ln x)^n dx$, we first use some technique (usually integration by parts) to derive the equation (called *recurrence relation* or *reduction formula*) governing I_n.
- Then we compute our target integral using that recurrence relation derived in the first step.

Consider the integral $\int x^4 e^x dx$ as an example, we compute this integral as follows:

Step 1: Derive the reduction formula

We let $I_n = \int x^n e^x dx$, where n is any non-negative integer. We use the integration by parts (with $u(x) = x^n$ and $dv = e^x dx$) and obtain

$$I_n = \int \underbrace{x^n}_{u} \underbrace{e^x dx}_{dv} \overset{\substack{dv = e^x dx \\ \Rightarrow v = \int e^x dx = e^x}}{\cong} \int \underbrace{x^n}_{u} \underbrace{d(e^x)}_{dv} = x^n e^x - \int e^x d(x^n)$$

$$= x^n e^x - \int e^x (n x^{n-1}) dx = x^n e^x - n \int x^{n-1} e^x dx = x^n e^x - n I_{n-1}.$$

Hence, we obtain a reduction formula for I_n:

$$\boxed{I_n = x^n e^x - n I_{n-1}}$$

Step 2: Use reduction formula to compute the target integral

Note that the target integral is just $I_4 = \int x^4 e^x dx$, using the formula with $n = 4$ obtained in Step 1, we then have

$$I_4 = x^4 e^x - 4I_3$$

Substitute $n = 3$:
$I_3 = x^3 e^x - 3I_2$

$$= x^4 e^x - 4(x^3 e^x - 3I_2)$$

$$= x^4 e^x - 4x^3 e^x + 12I_2$$

Substitute $n = 2$:
$I_2 = x^2 e^x - 2I_1$

$$= x^4 e^x - 4x^3 e^x + 12(x^2 e^x - 2I_1)$$

$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24I_1$$

Substitute $n = 1$:
$I_1 = xe^x - I_0$

$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24(xe^x - I_0)$$

$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24\underbrace{\int x^0 e^x dx}_{\int e^x dx}$$

$$= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24e^x + C.$$

Example 25

Derive the reduction formula of $I_n = \int_1^e (\ln x)^n dx$ and find $\int_1^e (\ln x)^3 dx$.

☺Solution:

Step 1: Derive the reduction formula

Using Integration by parts with $u = (\ln x)^n$ and $dv = dx \Rightarrow v = \int dx = x$, we have

$$\frac{d}{dx}(\ln x)^n = \frac{d(\ln x)^n}{d(\ln x)}\frac{d(\ln x)}{dx}$$

$$= n(\ln x)^{n-1}\left(\frac{1}{x}\right)$$

$$I_n = \int_1^e \underbrace{(\ln x)^n}_{u} \underbrace{dx}_{dv} = [x(\ln x)^n]_1^e - \int_1^e x d(\ln x)^n$$

$$= e - \int_1^e x\left[n(\ln x)^{n-1}\left(\frac{1}{x}\right)\right]dx$$

$$= e - n\int_1^e (\ln x)^{n-1}\,dx = e - nI_{n-1}.$$

Therefore, we obtain the following reduction formula for I_n

$$I_n = e - nI_{n-1}$$

Step 2: Use reduction formula to compute the target integral

Note that $I_3 = \int_1^e (\ln x)^3 dx$, using the reduction formula with $n = 3$, we then have

$$\int_1^e (\ln x)^3 dx = I_3 \overset{n=3}{\cong} e - 3I_2$$

$$\overset{n=2}{\cong} e - 3(e - 2I_1) = -2e + 6I_1$$

$$\overset{n=1}{\cong} -2e + 6(e - 1I_0) = 4e - 6I_0 = 4e - 6\int_1^e (\ln x)^0 dx$$

$$= 4e - 6\int_1^e 1dx = 4e - 6[x]_1^e = 4e - 6e + 6 = 6 - 2e.$$

Remark of reduction formula techniques

- As seen in the above 2 examples, the reduction formula can greatly reduce the computation cost (at least no need to write so many steps). For definite integrals, the reduction formula provides a nice computation formula which is "computer-friendly". Take Example 25 as an example, one can use the recurrent formula obtained in Step 1 and computer software (say excel, Matlab etc.,) to compute the integral likes
$$\int_1^e (\ln x)^{50} dx.$$
- Since the reduction formula is derived by using integration by parts formula, one would like to ask what is the suitable choice of $u(x)$. Recall that the main purpose of reduction is to "reduce" the power n in the integral. Therefore in most of the cases, your $u(x)$ should be chosen to be $(??)^n$ (for example, x^n for $\int x^n e^x dx$ or $(\ln x)^n$ in Example 25.

Example 26

(a) Derive the reduction formula for the integral

$$I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,dx\,, \quad n \geq 1$$

(b) Hence, compute the integral

$$\int_0^1 \frac{1}{(1+x^2)^3}\,dx$$

☺Solution of (a)

Using the integration by parts with $u(x) = \frac{1}{(1+x^2)^n}$ and $dv = dx \;\Rightarrow\; v = \int dx = x,$

we have

$$I_n = \int_0^1 \underbrace{\frac{1}{(1+x^2)^n}}_{u}\underbrace{dx}_{dv} = \left[\frac{x}{(1+x^2)^n}\right]_0^1 - \int_0^1 x\, d\underbrace{\left(\frac{1}{(1+x^2)^n}\right)}_{=(1+x^2)^{-n}}$$

$$= \frac{1}{2^n} - \int_0^1 x\left(-n(1+x^2)^{-(n+1)}(2x)dx\right) = \frac{1}{2^n} + 2n \underbrace{\int_0^1 \frac{x^2}{(1+x^2)^{n+1}} dx}_{\neq I_{n+1} \text{ or } I_n!!!!}$$

$$= \frac{1}{2^n} + 2n\left[\int_0^1 \frac{x^2+1}{(1+x^2)^{n+1}} dx - \int_0^1 \frac{1}{(1+x^2)^{n+1}} dx\right]$$

$$= \frac{1}{2^n} + 2n\left[\int_0^1 \frac{1}{(1+x^2)^{n}} dx - \int_0^1 \frac{1}{(1+x^2)^{n+1}} dx\right]$$

$$= \frac{1}{2^n} + 2nI_n - 2nI_{n+1}$$

After some rearrangement, we finally obtain

$$I_n = \frac{1}{2^n} + 2nI_n - 2nI_{n+1} \Rightarrow 2nI_{n+1} = \frac{1}{2^n} + (2n-1)I_n$$

$$\Rightarrow I_{n+1} \overset{n\neq 0}{\triangleq} \frac{1}{2^{n+1}n} + \frac{2n-1}{2n} I_n.$$

☺Solution of (b)

Using the reduction formula obtained in (a), the integral can be computed as

$$\int_0^1 \frac{1}{(1+x^2)^3}\,dx = I_3$$

$$\overset{n=2}{\cong} \frac{1}{2^{2+1}(2)} + \frac{2(2)-1}{2(2)} I_2 = \frac{1}{16} + \frac{3}{4} I_2$$

$$\overset{n=1}{\cong} \frac{1}{16} + \frac{3}{4}\left(\frac{1}{2^{1+1}(1)} + \frac{2(1)-1}{2(1)} I_1\right) = \frac{1}{4} + \frac{3}{8} I_1$$

$$= \frac{1}{4} + \frac{3}{8}\int_0^1 \frac{1}{1+x^2}\,dx = \frac{1}{4} + \frac{3}{8} \underbrace{[\tan^{-1} x]_0^1}_{\tan^{-1} 1 - \tan^{-1} 0} = \frac{1}{4} + \frac{3\pi}{32}.$$

Reminder:

When computing the numerical value of the inverse trigonometric function, take the value within the principal range.

Some advanced examples about the derivation of reduction formula

Example 27

(a) Derive the reduction formula for the integral

$$I_n = \int x^n e^{x^2+1} dx.$$

(b) Hence, compute the integral

$$I_5 = \int x^5 e^{x^2+1} dx.$$

IDEA:

We will encounter difficulty in deriving (a) if we use integration by parts with $u(x) = x^n$ and $dv = e^{x^2+1} dx$. It is because the function $v = \int e^{x^2+1} dx$ cannot be obtained. Here, we need to make some modification so that the integration by parts can be applied. Note that

$$\int x e^{x^2+1} dx \overset{y=x^2+1}{\cong} \frac{1}{2}\int e^y dy = \frac{1}{2}e^y + C = \frac{1}{2}e^{x^2+1} + C.$$

☺Solution of (a)

Here, we take $u(x) = x^{n-1}$ and $dv = xe^{x^2+1}dx \Rightarrow v = \int xe^{x^2+1}dx = \frac{1}{2}e^{x^2+1}$.

Then we have

$$I_n = \int x^n e^{x^2+1}dx = \int \underbrace{x^{n-1}}_{u} \underbrace{\left(xe^{x^2+1}dx\right)}_{dv} = \int \underbrace{x^{n-1}}_{u} \underbrace{d\left(\frac{1}{2}e^{x^2+1}\right)}_{dv}$$

$$= \frac{1}{2}x^{n-1}e^{x^2+1} - \int \frac{1}{2}e^{x^2+1}d(x^{n-1})$$

$$= \frac{1}{2}x^{n-1}e^{x^2+1} - \frac{n-1}{2}\int x^{n-2}e^{x^2+1}dx$$

$$= \frac{1}{2}x^{n-1}e^{x^2+1} - \frac{n-1}{2}I_{n-2}$$

☺Solution of (b)

Using the formula with $n = 5$, we have

$$\int x^5 e^{x^2+1} dx = I_5 = \frac{1}{2} x^4 e^{x^2+1} - \frac{4}{2} I_3$$

$$\overset{n=3}{\cong} \frac{1}{2} x^4 e^{x^2+1} - 2\left(\frac{1}{2} x^2 e^{x^2+1} - \frac{2}{2} I_1\right)$$

$$= \frac{1}{2} x^4 e^{x^2+1} - x^2 e^{x^2+1} + 2\int x e^{x^2+1} dx$$

$$\overset{\int x e^{x^2+1} dx = \frac{1}{2} e^{x^2+1} + C}{\cong} \frac{1}{2} x^4 e^{x^2+1} - x^2 e^{x^2+1} + e^{x^2+1} + C$$

Example 28

Derive a reduction formula for

$$S_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx,$$

where $n = 2, 3, 4, 5, \dots.$

IDEA:

Here one should not use integration by parts with $u(x) = \sin^n x$ and $dv = dx \Rightarrow v = \int dx = x$ directly since there will be a "undesired function" coming out after the transformation, i.e.

$$\int_0^{\frac{\pi}{2}} \underbrace{\sin^n x}_{u} \underbrace{dx}_{dv} = [x \sin^n x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} x d(\sin^n x)$$

$$= \frac{\pi}{2} - n \int_0^{\frac{\pi}{2}} \underbrace{x}_{\text{extra}} \sin^{n-1} x \cos x \, dx.$$

☺Solution:

Let $S_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$

Similar to Example 20, one should use integration by parts in the following way:

$$S_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx = \int_0^{\frac{\pi}{2}} \underbrace{\sin^{n-1} x}_{u} \underbrace{(\sin x\, dx)}_{dv}$$

$$\overset{\substack{dv=\sin x dx \\ \Rightarrow v=\int \sin x dx=-\cos x}}{\triangleq} \int_0^{\frac{\pi}{2}} \underbrace{\sin^{n-1} x}_{u} \underbrace{d(-\cos x)}_{dv}$$

$$= [-\sin^{n-1} x \cos x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos x\, d(\sin^{n-1} x)$$

$$= 0 + \int_0^{\frac{\pi}{2}} \cos x\, [(n-1) \sin^{n-2} x \cos x\, dx]$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^2 x \sin^{n-2} x \, dx$$

$$\overset{\sin^2 x + \cos^2 x = 1}{\widehat{=}} (n-1)\int_0^{\frac{\pi}{2}} (1 - \sin^2 x) \sin^{n-2} x \, dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x \, dx$$

$$= (n-1)S_{n-2} - (n-1)S_n$$

So we have

$$S_n = (n-1)S_{n-2} - (n-1)S_n$$

$$\Rightarrow S_n = \frac{n-1}{n} S_{n-2}.$$

To compute the exact value of S_n, one can use this formula repeatedly, we get

$$S_n = \frac{n-1}{n} S_{n-2} \overset{\substack{\text{replace } n \\ \text{by } n-2}}{\cong} \frac{n-1}{n}\left(\frac{n-3}{n-2} S_{n-4}\right)$$

$$\overset{\substack{\text{replace } n \\ \text{by } n-4}}{\cong} \frac{n-1}{n}\frac{n-3}{n-2}\left(\frac{n-5}{n-4} S_{n-6}\right)$$

$$= \cdots = \begin{cases} \dfrac{(n-1)\times(n-3)\times \ldots \times 5 \ \times 3 \times 1}{n \times (n-2) \times \ldots \times 6 \ \times 4 \times 2} S_0 & \text{if } n \text{ is even} \\ \dfrac{(n-1)\times(n-3)\times \ldots \times 6 \ \times 4 \times 2}{n \times (n-2) \times \ldots \times 7 \times 5 \ \times 3} S_1 & \text{if } n \text{ is odd} \end{cases}$$

Since

$$S_0 = \int_0^{\frac{\pi}{2}} \sin^0 x \, dx = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}, \qquad S_1 = \int_0^{\frac{\pi}{2}} \sin x \, dx = [-\cos x]_0^{\frac{\pi}{2}} = 1$$

Therefore, we obtain the formula for $S_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$:

$$S_n = \begin{cases} \dfrac{(n-1)\times(n-3)\times \dots \times 5 \ \times 3 \times 1}{n \times (n-2) \times \dots \times 6 \ \times 4 \times 2} \times \dfrac{\pi}{2} & \text{if } n \text{ is even} \\ \dfrac{(n-1)\times(n-3)\times \dots \times 6 \ \times 4 \times 2}{n \times (n-2) \times \dots \times 7 \ \times 5 \times 3} & \text{if } n \text{ is odd} \end{cases}.$$

This formula is called *Wallis's formula*.

Remark:

Using similar method, one can obtain the similar formula for $C_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$.

(Left as exercise)

$$C_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx = \begin{cases} \dfrac{(n-1)\times(n-3)\times \dots \times 5 \ \times 3 \times 1}{n \times (n-2) \times \dots \times 6 \ \times 4 \times 2} \left(\dfrac{\pi}{2}\right) & \text{if } n \text{ is even} \\ \dfrac{(n-1)\times(n-3)\times \dots \times 6 \ \times 4 \times 2}{n \times (n-2) \times \dots \times 7 \ \times 5 \times 3} & \text{if } n \text{ is odd} \end{cases}$$

Example 29

(a) Derive the reduction formula for the following integral

$$I_n = \int \sec^n x \, dx$$

(b) Hence, compute the following integral

$$\int \frac{1}{(1-x^2)^3} dx$$

☺Solution of (a)

Using similar tactic as in Example 28, we use integration of parts with $u(x) = \sec^{n-2} x$ and $dv = \sec^2 x \, dx \Rightarrow v = \int \sec^2 x \, dx = \tan x$ and obtain

$$I_n = \int \sec^n x \, dx = \int \underbrace{\sec^{n-2} x}_{u} \underbrace{(\sec^2 x \, dx)}_{dv} = \int \underbrace{\sec^{n-2} x}_{u} \underbrace{d(\tan x)}_{dv}$$

$$= \sec^{n-2} x \tan x - \int \tan x \, d(\sec^{n-2} x)$$

$$= \sec^{n-2} x \tan x - \int \tan x \underbrace{[(n-2)\sec^{n-3} x (\sec x \tan x) dx]}_{\frac{d}{dx}(\sec^{n-2} x) = \frac{d(\sec^{n-2} x)}{d(\sec x)} \frac{d(\sec x)}{dx}}$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx$$

$$\stackrel{1+\tan^2 x = \sec^2 x}{\triangleq} \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n} x \, dx + (n-2) \int \sec^{n-2} x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}.$$

Summing up, we obtain

$$I_n = \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$\Rightarrow \; I_n = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2}.$$

☺Solution of (b)

As a starting point, one may try the simple substitution $y = 1 - x^2$ as the first trial. The integral is then becomes

$$\int \frac{1}{(1-x^2)^3}\,dx = -\frac{1}{2}\int \frac{1}{y^3\sqrt{1-y}}\,dy$$

which is complicated to integrate. Hence, we should try trigonometric substitution as an alternative.

We let $x = \sin\theta$. Then $\frac{dx}{d\theta} = \cos\theta$ and the integral becomes

$$\int \frac{1}{(1-x^2)^3}\,dx = \int \frac{1}{\left(\underbrace{1-\sin^2\theta}_{=\cos^2\theta}\right)^3}(\cos\theta\,d\theta) = \int \frac{1}{\cos^5\theta}\,d\theta = \int \sec^5\theta\,d\theta.$$

Using the reduction formula obtained in (a), the above integral is found to be

$$\int \sec^5\theta\,d\theta = I_5 \overset{n=5}{\cong} \frac{1}{5-1}\sec^{5-2}\theta\tan\theta + \frac{5-2}{5-1}I_3$$

$$= \frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{4}I_3$$

$$\overset{n=3}{\cong} \frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{4}\left(\frac{1}{3-1}\sec^{3-2}\theta\tan\theta + \frac{3-2}{3-1}I_1\right)$$

$$= \frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{8}\sec\theta\tan\theta + \frac{3}{8}\int \sec\theta\,d\theta$$

$$= \frac{1}{4}\sec^3\theta\tan\theta + \frac{3}{8}\sec\theta\tan\theta + \frac{3}{8}\ln|\sec\theta + \tan\theta| + C$$

$$\overset{\sec\theta=\frac{1}{\sqrt{1-x^2}},\ \tan\theta=\frac{x}{\sqrt{1-x^2}}}{\cong} \frac{1}{4}\frac{x}{(1-x^2)^2} + \frac{3}{8}\frac{x}{1-x^2} + \frac{3}{8}\ln\left|\frac{1+x}{\sqrt{1-x^2}}\right| + C.$$

Integration of rational function: Method of Partial Fractions

In this section, we shall investigate the integral of the rational function $R(x)$ which is the quotient of two polynomials:

$$\int R(x)dx = \int \frac{P(x)}{Q(x)}dx$$

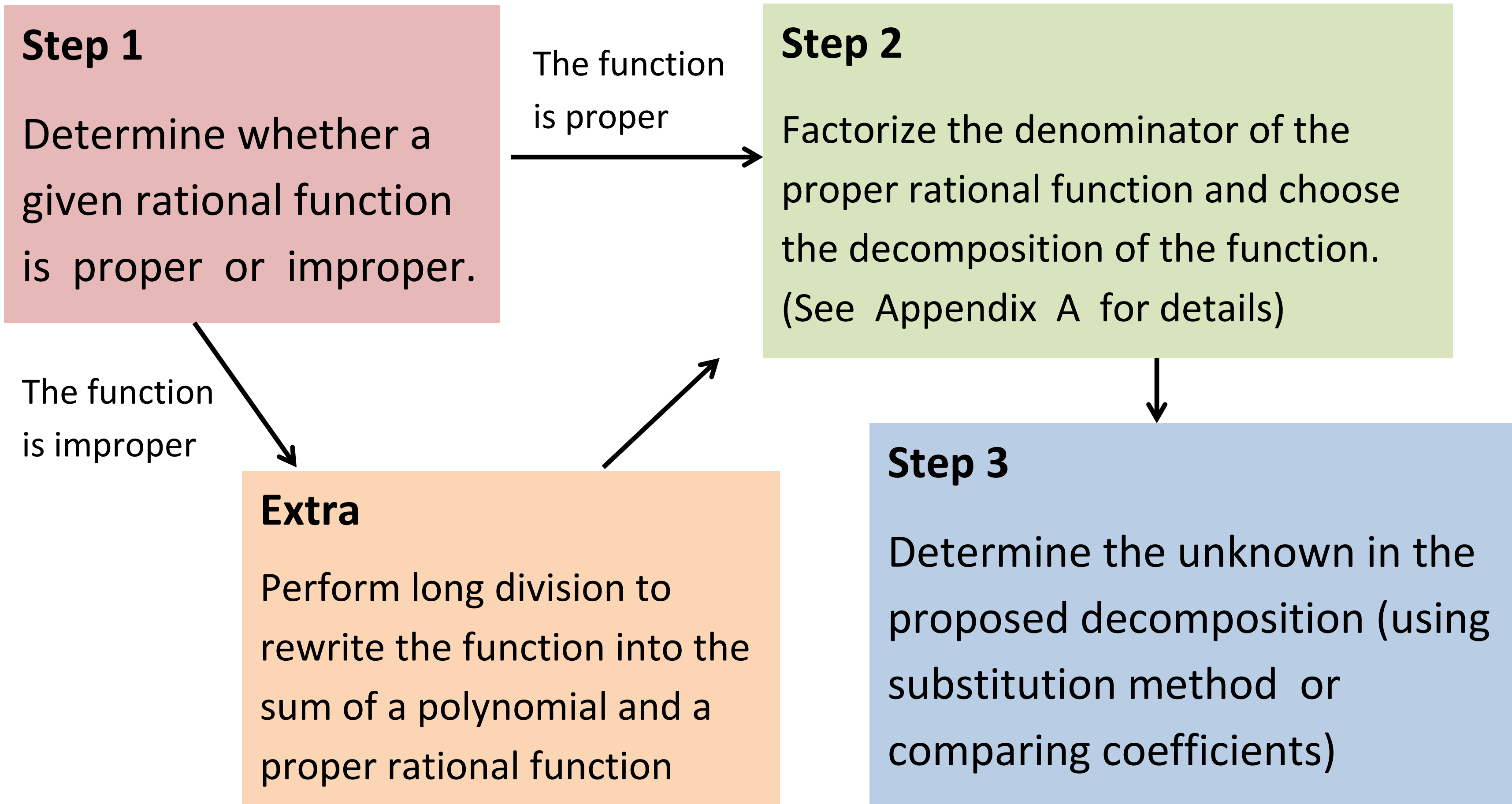
where both $P(x)$ and $Q(x)$ are polynomials.

When we integrate the rational function, it is always convenient for us to first split (decompose) the function into the sum of simpler (rational) functions. This could be done by *method of partial fractions*.

$$\int \frac{x^3 + 5x^2 + 2x - 2}{(x-1)(2x+1)(x^2+1)}dx = \int \frac{1}{x-1}dx + \int \frac{1}{2x+1}dx - \int \frac{x-2}{x^2+1}dx$$

A quick review on the method of partial fraction

The whole process is divided into three steps can be summarized by the following flow chart:



Example 30

Compute the integral

$$\int \frac{1}{(x-3)(2x+1)(x-1)}\,dx$$

☺Solution:

We first decompose the function using the method partial fraction. Note that the function is proper (degree of numerator = 0 < degree of denominator = 3) and the factors in the denominator are linear and distinct, so we propose the following decomposition:

$$\frac{1}{(x-3)(2x+1)(x-1)} = \frac{A}{x-3} + \frac{B}{2x+1} + \frac{C}{x-1}.$$

By taking the common denominator, we obtain the following equation:

$$1 = A(2x+1)(x-1) + B(x-3)(x-1) + C(x-3)(2x+1).$$

Substitute $x = 1$, we obtain $1 = C(-2)(3) \Rightarrow C = -\frac{1}{6}$.

Substitute $x = -\frac{1}{2}$, we obtain $1 = B\left(-\frac{7}{2}\right)\left(-\frac{3}{2}\right) \Rightarrow B = \frac{4}{21}$.

Substitute $x = 3$, we obtain $1 = A(7)(2) \Rightarrow A = \frac{1}{14}$.

Hence, we have

$$\frac{1}{(x-3)(2x+1)(x-1)} = \frac{1}{14(x-3)} + \frac{4}{21(2x+1)} - \frac{1}{6(x-1)}$$

Then we can compute the integral as

$$\int \frac{1}{(x-3)(2x+1)(x-1)} dx$$

$$= \frac{1}{14}\int \frac{1}{x-3} dx + \frac{4}{21}\int \frac{1}{2x+1} dx - \frac{1}{6}\int \frac{1}{x-1} dx$$

$$= \frac{1}{14}\ln|x-3| + \frac{2}{21}\ln|2x+1| - \frac{1}{6}\ln|x-1| + C.$$

Example 31

Compute the integrals

$$\int \frac{1}{x^2 + 4x - 12} dx, \qquad \int \frac{1}{x^2 + 4x + 12} dx$$

☺Solution:

For the first integral, one can rewrite the integral as

$$\int \frac{1}{x^2 + 4x - 12} dx = \int \frac{1}{(x+6)(x-2)} dx$$

Then we decompose the function using the method of partial fractions as follows:

$$\frac{1}{(x+6)(x-2)} = \frac{A}{x+6} + \frac{B}{x-2} \Rightarrow 1 = A(x-2) + B(x+6).$$

Substitute $x = 2$, we get $1 = 8B \Rightarrow B = \frac{1}{8}$.

Substitute $x = -6$, we get $1 = -8A \Rightarrow A = -\frac{1}{8}$.

Thus the integral can be computed as

$$\int \frac{1}{(x+6)(x-2)}\,dx = -\frac{1}{8}\int \frac{1}{x+6}\,dx + \frac{1}{8}\int \frac{1}{x-2}\,dx$$
$$= -\frac{1}{8}\ln|x+6| + \frac{1}{8}\ln|x-2| + C.$$

For the second integral, note that the denominator cannot be factorized, we can compute the integral using the technique in Example 15 of Chapter 1 (pp. 27-28) :

$$\int \frac{1}{x^2+4x+12}\,dx = \int \frac{1}{(x+2)^2+8}\,dx = \frac{1}{8}\int \frac{1}{\frac{1}{8}(x+2)^2+1}\,dx$$
$$= \frac{1}{8}\int \frac{1}{\left(\frac{1}{\sqrt{8}}x+\frac{2}{\sqrt{8}}\right)^2+1}\,dx = \frac{1}{8}\left(\frac{1}{\frac{1}{\sqrt{8}}}\tan^{-1}\left(\frac{1}{\sqrt{8}}x+\frac{2}{\sqrt{8}}\right)\right)+C$$
$$= \frac{1}{\sqrt{8}}\tan^{-1}\left(\frac{1}{\sqrt{8}}x+\frac{2}{\sqrt{8}}\right)+C.$$

Remark of Example 31

- In this example, we see that there are two totally different ways to encounter the integral of the form $\int \frac{A}{ax^2+bx+c} dx$: Method of partial fractions and Technique in Example 15 of Chapter 1.

- This depends on whether the denominator can be further factorized into two linear functions.

- To check whether a given quadratic function $ax^2 + bx + c$ can be factorized or not, one can use the discriminant $\Delta = b^2 - 4ac$:
  - ✓ If $\Delta \geq 0$, the equation $ax^2 + bx + c = 0$ has at least one real root α, β. Then the function can be factorized into $a(x - \alpha)(x - \beta)$.
  - ✓ If $\Delta < 0$, the equation $ax^2 + bx + c = 0$ has no real roots and the given function cannot be factorized.

Example 32

Computed the integral

$$\int \frac{5x^3 - 12x^2 + 5x - 4}{(2x+1)(x-1)^3} dx.$$

☺Solution

We first decompose the rational function using the method of partial fraction. Since the factorization of denominator involve repeated factor, we should propose the following decomposition:

$$\frac{5x^3 - 12x^2 + 5x - 4}{(2x+1)(x-1)^3} = \frac{A}{2x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}.$$

Multiplying the common denominator on both sides, we then obtain

$$A(x-1)^3 + B(x-1)^2(2x+1) + C(x-1)(2x+1) + D(2x+1)$$
$$= 5x^3 - 12x^2 + 5x - 4.$$

Substitute $x = -\frac{1}{2}$, we get $A\left(-\frac{3}{2}\right)^3 = 5\left(-\frac{1}{8}\right) - 12\left(\frac{1}{4}\right) - \frac{5}{2} - 4 \Rightarrow A = 3.$

Substitute $x = 1$, we get $3D = 5 - 12 + 5 - 4 \Rightarrow D = -2$.

Compare the coefficient of x^3: $A + 2B = 5 \Rightarrow B = 1$.

Compare the constant term: $-A + B - C + D = -4 \Rightarrow C = 0$.

Then we have

$$\frac{5x^3 - 12x^2 + 5x - 4}{(2x+1)(x-1)^3} = \frac{3}{2x+1} + \frac{1}{x-1} + \frac{0}{(x-1)^2} - \frac{2}{(x-1)^3}$$

The integral can be computed as

$$\int \frac{5x^3 - 12x^2 + 5x - 4}{(2x+1)(x-1)^3}\,dx = 3\int \frac{1}{2x+1}\,dx + \int \frac{1}{x-1}\,dx - 2\underbrace{\int \frac{1}{(x-1)^3}\,dx}_{=\int (x-1)^{-3}\,dx}$$

$$= \frac{3}{2}\ln|2x+1| + \ln|x-1| + (x-1)^{-2} + C.$$

Example 33

Compute the integral

$$\int \frac{6x^2 - 3x + 7}{(4x+1)(x^2+4)}\,dx.$$

☺Solution:

We use the method of partial fractions to decompose the rational function. Since the denominator consists of a quadratic factor $x^2 + 4$, so we shall try the following decomposition:

$$\frac{6x^2 - 3x + 7}{(4x+1)(x^2+4)} = \frac{A}{4x+1} + \frac{Bx + C}{x^2+4}$$

$$\Rightarrow A(x^2+4) + (4x+1)(Bx+C) = 6x^2 - 3x + 7.$$

Substitute $x = -\frac{1}{4}$, we get $\frac{65}{16}A = 6\left(-\frac{1}{4}\right)^2 - 3\left(-\frac{1}{4}\right) + 7 \;\Rightarrow\; A = 2.$

Compare the coefficient of x^2: $A + 4B = 6 \;\Rightarrow\; B = 1.$

Compare the constant term: $4A + C = 7 \;\Rightarrow\; C = -1.$

Thus we obtain

$$\frac{6x^2 - 3x + 7}{(4x+1)(x^2+4)} = \frac{2}{4x+1} + \frac{x-1}{x^2+4}.$$

The integral can be computed as

$$\int \frac{6x^2 - 3x + 7}{(4x+1)(x^2+4)} dx = 2\int \frac{1}{4x+1} dx + \int \frac{x-1}{x^2+4} dx \dots\dots (*)$$

The first term can be computed as

$$\int \frac{1}{4x+1} dx \overset{\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b|+C}{\cong} \frac{1}{4}\ln|4x+1| + C.$$

The second term can be computed as

$$\int \frac{x-1}{x^2+4} dx = \int \frac{x}{x^2+4} dx - \int \frac{1}{x^2+4} dx$$

$$y=x^2+4 \Rightarrow \frac{dy}{dx}=2x$$

$$\overset{\substack{y=x^2+4\\ \Rightarrow\frac{dy}{dx}=2x}}{\cong} \frac{1}{2}\int \frac{1}{y}dy - \frac{1}{4}\int \frac{1}{\frac{x^2}{4}+1}dx$$

$$= \frac{1}{2}\ln|y| - \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}dx$$

$$= \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\frac{x}{2} + C.$$

Combing the result, we have

$$\int \frac{6x^2-3x+7}{(4x+1)(x^2+4)}dx = \frac{1}{2}\ln|4x+1| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\frac{x}{2} + C.$$

Example 34

Compute the integral

$$\int_0^1 \frac{6x^2 + 9x + 19}{(x+1)(x^2+6x+13)}\,dx$$

☺Solution:

We use the method of partial fractions to decompose the rational function. Since the factorization of the denominator consists of a quadratic factor (which cannot be factorized further), we shall propose the following decomposition:

$$\frac{6x^2 + 9x + 19}{(x+1)(x^2+6x+13)} = \frac{A}{x+1} + \frac{Bx + C}{x^2+6x+13}$$

$$\Rightarrow A(x^2 + 6x + 13) + (Bx + C)(x + 1) = 6x^2 + 9x + 19.$$

Substitute $x = -1$, we have $8A = 16 \ \Rightarrow \ A = 2$.

Compare the coefficient of x^2: $A + B = 6 \ \Rightarrow \ B = 4$.

Compare the constant term: $13A + C = 19 \ \Rightarrow \ C = -7$.

Hence, we can compute the integral as

$$\int_0^1 \frac{6x^2+9x+19}{(x+1)(x^2+6x+13)}dx = 2\int_0^1 \frac{1}{x+1}dx + \int_0^1 \frac{4x-7}{x^2+6x+13}dx \ldots\ldots(*)$$

The first term can be computed as

$$\int_0^1 \frac{1}{x+1}dx = [\ln|x+1|]_0^1 = \ln 2.$$

The second term can be computed as

$$\int_0^1 \frac{4x-7}{x^2+6x+13}dx = \int_0^1 \frac{4x+12}{x^2+6x+13}dx - \int_0^1 \frac{19}{x^2+6x+13}dx$$

$$\overset{\substack{y=x^2+6x+13 \\ \Rightarrow \frac{dy}{dx}=2x+6}}{\cong} \int_{13}^{20} \frac{2}{y}dy - 19\int_0^1 \frac{1}{(x+3)^2+4}dx$$

$$= 2[\ln y]_{13}^{20} - \frac{19}{4}\int_0^1 \frac{1}{\frac{(x+3)^2}{4}+1}dx = 2[\ln 20 - \ln 13] - \frac{19}{4}\int_0^1 \frac{1}{\left(\frac{x+3}{2}\right)^2+1}dx$$

$$= 2[\ln 20 - \ln 13] - \left[\frac{19}{2}\tan^{-1}\frac{x+3}{2}\right]_0^1$$

$$= 2[\ln 20 - \ln 13] - \frac{19}{2}\tan^{-1}2 + \frac{19}{2}\tan^{-1}\frac{3}{2}.$$

Combining the result, we finally get

$$\int_0^1 \frac{6x^2+9x+19}{(x+1)(x^2+6x+13)}\,dx$$

$$= 2\ln 2 + 2[\ln 20 - \ln 13] - \frac{19}{2}\tan^{-1}2 + \frac{19}{2}\tan^{-1}\frac{3}{2}$$

$$= \ln\left(\frac{1600}{169}\right) - \frac{19}{2}\tan^{-1}2 + \frac{19}{2}\tan^{-1}\frac{3}{2}.$$

Example 35 (Integral of improper rational function)

Compute the integral

$$\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} dx$$

☺Solution:

Note that the integrand is an improper rational function, one has to decompose the function into a sum of polynomial and a proper rational function before using the method of partial fractions.

We first perform the long division (dividing the numerator by the denominator) and rewrite the integrand as

$$\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} dx = \int \frac{(x^2 + x - 2)(3x^2 + 1) + 1}{x^2 + x - 2} dx$$

$$= \underbrace{\int (3x^2 + 1) dx}_{\text{polynomial}} + \underbrace{\int \frac{1}{x^2 + x - 2} dx}_{\text{proper rational function}} \quad \dots\dots (*)$$

The first integral can be computed as

$$\int (3x^2+1)dx = 3\left(\frac{x^3}{3}\right) + x + C = x^3 + x + C.$$

We proceed to compute the second integral using the method of partial fraction. Note that

$$\int \frac{1}{x^2+x-2}dx = \int \frac{1}{(x+2)(x-1)}dx = \cdots = \frac{1}{3}\int \frac{1}{x-1}dx - \frac{1}{3}\int \frac{1}{x+2}dx$$
$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C.$$

Hence,

$$\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}dx = x^3 + x + \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C.$$

Example 36

Compute the integral

$$\int \frac{9x^3 - 46x^2 + 102x - 9}{(3x+1)^2(x^2 - 6x + 10)}\,dx.$$

☺Solution

Note that the denominator involves *a repeated factor* $(3x+1)$ and *a quadratic factor* $(x^2 - 6x + 10)$, we shall decompose the rational function by using the following decomposition:

$$\frac{9x^3 - 46x^2 + 102x - 9}{(3x+1)^2(x^2 - 6x + 10)} = \frac{A}{3x+1} + \frac{B}{(3x+1)^2} + \frac{Cx + D}{(x^2 - 6x + 10)}.$$

$$\Rightarrow 9x^3 - 46x^2 + 102x - 9$$
$$= A(3x+1)(x^2 - 6x + 10) + B(x^2 - 6x + 10) + (Cx + D)(3x+1)^2.$$

Substitute $x = -\frac{1}{3}$, we get $-\frac{436}{9} = \frac{109}{9}B \;\Rightarrow\; B = -4.$

So $A(3x+1)(x^2 - 6x + 10) + (Cx + D)(3x+1)^2$

$= 9x^3 - 46x^2 + 102x - 9 + 4(x^2 - 6x + 10) = 9x^3 - 42x^2 + 78x + 31$

$= (9x^2 - 45x + 93)\left(x + \frac{1}{3}\right) = (3x^2 - 15x + 31)(3x + 1)$ by long division

Thus $A(x^2 - 6x + 10) + (Cx + D)(3x + 1) = 3x^2 - 15x + 31.$

Substitute $x = -\frac{1}{3}$, we get $\frac{109}{9}A = \frac{109}{3} \Rightarrow A = 3;$

Compare the coefficient of x^2: $A + 3C = 3 \Rightarrow C = 0;$

Compare the constant term: $10A + D = 31 \Rightarrow D = 1.$

$$\int \frac{9x^3 - 46x^2 + 102x - 9}{(3x + 1)^2(x^2 - 6x + 10)}\,dx$$

$$= \int \frac{3}{3x + 1}\,dx - \int \frac{4}{(3x + 1)^2}\,dx + \int \frac{1}{x^2 - 6x + 10}\,dx$$

$$= \ln|3x + 1| + \frac{4}{3(3x + 1)} + \int \frac{1}{(x - 3)^2 + 1}\,dx$$

$$= \ln|3x + 1| + \frac{4}{3(3x + 1)} + \tan^{-1}(x - 3) + C.$$

Appendix A – Various types of decomposition pattern using the method of partial fraction.

The following table summarizes the proposed decomposition under different type of factorization of the denominator:

Scenario	Proposed decomposition
If $Q(x)$ can be factorized into a product of distinct linear factor: $R(x) = \frac{P(x)}{(a_1x+b_1)(a_2x+b_2)\dots(a_nx+b_n)}$	$R(x) = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots \frac{A_n}{a_nx+b_n}.$
If there is some repeated linear factors in the factorization of $Q(x)$: $R(x) = \frac{P(x)}{(ax+b)^n(cx+d)}$	$R(x) = \underbrace{\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}}_{\text{for } (ax+b)^n} + \underbrace{\frac{B}{cx+d}}_{\text{for } (cx+d)}$

If there is quadratic factor (cannot be factorized, e.g. x^2+4) in the factorization of $Q(x)$: $$R(x)=\frac{P(x)}{(ax^2+bx+c)(dx+e)}$$	$$R(x)=\frac{\overbrace{Ax+B}^{\text{degree } 1}}{\underbrace{ax^2+bx+c}_{\text{degree } 2}}+\frac{C}{dx+e}$$
If there is some repeated quadratic factor which cannot be further factorized of $g(x)$: $$R(x)=\frac{P(x)}{(ax^2+bx+c)^n(dx+e)}$$	$$R(x)=\underbrace{\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}}_{\text{for } (ax^2+bx+c)^n}+\frac{C}{dx+e}$$

Appendix B – Brief Table for Integration

1. $\int x^a dx = \begin{cases} \dfrac{x^{a+1}}{a+1} & \text{if } a \neq -1 \\ \ln\lvert x\rvert & \text{if } a = -1 \end{cases}$	2. $\int e^x dx = e^x + C,$ $\int a^x dx = \dfrac{a^x}{\ln a} + C, \qquad a > 0$
3. $\int \sin x \, dx = -\cos x + C$	4. $\int \cos x \, dx = \sin x + C$
5. $\int \tan x \, dx = \ln\lvert\sec x\rvert + C$	6. $\int \cot x \, dx = -\ln\lvert\csc x\rvert + C$
7. $\int \sec^2 x \, dx = \tan x + C$	8. $\int \csc^2 x \, dx = -\cot x + C$
9. $\int \sec x \, dx = \ln\lvert\sec x + \tan x\rvert + C$	10. $\int \csc x \, dx$ $= -\ln\lvert\csc x + \cot x\rvert + C$
11. $\int \sec x \tan x \, dx = \sec x + C$	12. $\int \cot x \csc x \, dx = -\csc x + C$
13. $\int \dfrac{1}{1+x^2} dx = \tan^{-1} x + C$	14. $\int \dfrac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C.$

Review Examples on integration by parts

Review Example 1

Compute the integral

$$\int (2x+3)e^{4x}dx$$

☺Solution:

Using integration by parts formula with $u(x) = 2x + 3$ and $dv = e^{4x}dx$
$\Rightarrow\ v = \int e^{4x}dx = \frac{1}{4}e^{4x}$, we obtain

$$\int \underbrace{(2x+3)}_{u}\,\underbrace{e^{4x}dx}_{dv} = \underbrace{(2x+3)}_{u}\underbrace{\left(\frac{1}{4}e^{4x}\right)}_{v} - \int \underbrace{\frac{1}{4}e^{4x}}_{v}\,\underbrace{d(2x+3)}_{du}$$

$$\overset{\substack{\frac{d}{dx}(2x+3)=2 \\ \Rightarrow d(2x+3)=2dx}}{=} \frac{1}{4}(2x+3)e^{4x} - \frac{1}{2}\int e^{4x}dx = \frac{1}{4}(2x+3)e^{4x} - \frac{1}{8}e^{4x} + C.$$

Review Example 2

Compute the integral

$$\int \sqrt{x}(\ln x)^2 dx$$

☺Solution:

We use the integration by parts with $u = (\ln x)^2$ and $dv = \sqrt{x}dx$

$\Rightarrow\ v = \int \sqrt{x}dx = \int x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}}$. We then get

$$\int \sqrt{x}(\ln x)^2 dx = \int \underbrace{(\ln x)^2}_{u} \underbrace{\left(\sqrt{x}dx\right)}_{dv} = \frac{2}{3}x^{\frac{3}{2}}(\ln x)^2 - \int \frac{2}{3}x^{\frac{3}{2}}d(\ln x)^2$$

$$= \frac{2}{3}x^{\frac{3}{2}}(\ln x)^2 - \frac{2}{3}\int x^{\frac{3}{2}}\left(2(\ln x)\left(\frac{1}{x}\right)dx\right)$$

$$= \frac{2}{3}x^{\frac{3}{2}}(\ln x)^2 - \frac{4}{3}\int x^{\frac{1}{2}}\ln x\, dx$$

We use the integration by parts (with $u(x) = \ln x,\ dv = x^{\frac{1}{2}}dx$

$\Rightarrow\ v = \int x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}}$) again to compute the second integral:

$$\frac{2}{3}x^{\frac{3}{2}}(\ln x)^2 - \frac{4}{3}\int x^{\frac{1}{2}}\ln x\,dx = \frac{2}{3}x^{\frac{3}{2}}(\ln x)^2 - \frac{4}{3}\int \underbrace{(\ln x)}_{u}\underbrace{\left(x^{\frac{1}{2}}dx\right)}_{dv}$$

$$= \frac{2}{3}x^{\frac{3}{2}}(\ln x)^2 - \frac{4}{3}\left[\frac{2}{3}x^{\frac{3}{2}}\ln x - \int \frac{2}{3}x^{\frac{3}{2}}d(\ln x)\right]$$

$$= \frac{2}{3}x^{\frac{3}{2}}(\ln x)^2 - \frac{8}{9}x^{\frac{3}{2}}\ln x + \frac{8}{9}\int x^{\frac{1}{2}}dx$$

$$= \frac{2}{3}x^{\frac{3}{2}}(\ln x)^2 - \frac{8}{9}x^{\frac{3}{2}}\ln x + \frac{16}{27}x^{\frac{3}{2}} + C.$$

Review Example 3

Compute the integral

$$\int x\cos^2(2x)\,dx$$

☺IDEA:

It may be better for us to express the term $\cos^2 x$ into sum of trigonometric substitution using product-to-sum formula to get rid of the "square term" before we use the integration by parts.

☺Solution

Using the product-to-sum formula, we have

$$\int x\cos^2(2x)\,dx \overset{\substack{\cos^2 2x \\ =(\cos 2x)(\cos 2x)}}{\triangleq} \int x\left[\frac{1}{2}(\cos(2x+2x)+\cos(2x-2x))\right]dx$$

$$=\frac{1}{2}\int x\cos 4x\,dx+\frac{1}{2}\int xdx=\frac{1}{2}\int x\cos 4x\,dx+\frac{x^2}{4}\ldots.(*)$$

It remains to compute the first integral. Using integration by parts with $u(x) = x$ and $dv = \cos 4x\, dx \ \Rightarrow\ v = \int \cos 4x\, dx = \frac{1}{4}\sin 4x$, we have

$$\int \underbrace{x}_{u} \underbrace{\cos 4x\, dx}_{dv} = \underbrace{x}_{u} \underbrace{\left(\frac{1}{4}\sin 4x\right)}_{v} - \int \underbrace{\frac{1}{4}\sin 4x}_{v} \underbrace{dx}_{du}$$

$$\overset{\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b)+C}{\cong} \quad \frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x + C.$$

From $(*)$, we conclude that

$$\int x\cos^2(2x)\, dx = \frac{1}{8}x\sin 4x + \frac{1}{32}\cos 4x + \frac{x^2}{4} + C.$$

Review Example 4

We let $f(x) = \int_1^{\sqrt{x}} e^{-t^2} dt$.

(a) Find $f'(x)$.

(b) Hence, compute the integral

$$\int_0^1 \sqrt{x} f(x) dx.$$

☺Solution of (a)

Let $G(t) = \int e^{-t^2} dt$, then by the fundamental theorem of calculus, we have

$$f(x) = G(\sqrt{x}) - G(1).$$

Using the fact that $\frac{d}{dt} G(t) = \frac{d}{dt} \int e^{-t^2} dt = e^{-t^2}$, we have

$$f'(x) = \frac{d}{dx} G(\sqrt{x}) - \frac{d}{dx} G(1) = \frac{dG(\sqrt{x})}{d\sqrt{x}} \frac{d\sqrt{x}}{dx} - 0 \overset{\text{take } t=\sqrt{x}}{\cong} e^{-(\sqrt{x})^2} \frac{1}{2} x^{-\frac{1}{2}}$$

$$= \frac{1}{2} x^{-\frac{1}{2}} e^{-x}.$$

☺Solution of (b)

IDEA:

One cannot compute the integral directly since the expression of $f(x)$ is unavailable. However since $f'(x)$ is known, one can use integration by parts formula to transform $f(x)$ into $f'(x)$.

Using integration by parts with $u(x) = f(x)$ and $dv = \sqrt{x}dx$

$\Rightarrow v = \int \sqrt{x}dx = \frac{2}{3}x^{\frac{3}{2}}$, we have

$$\int_0^1 \sqrt{x}f(x)dx = \int_0^1 \underbrace{f(x)}_{u} \underbrace{\left(\sqrt{x}dx\right)}_{dv} = \underbrace{\frac{2}{3}x^{\frac{3}{2}}f(x)|_0^1}_{uv|_0^1} - \int_0^1 \underbrace{\frac{2}{3}x^{\frac{3}{2}}}_{v} \underbrace{df(x)}_{du}$$

$$= \frac{2}{3}(1) \underbrace{f(1)}_{=\int_1^1 e^{-t^2}dt=0} - \underbrace{\frac{2}{3}(0)f(0)}_{=0} - \frac{2}{3}\int_0^1 x^{\frac{3}{2}}f'(x)dx$$

$$= -\frac{2}{3}\int_0^1 x^{\frac{3}{2}}\left(\frac{1}{2}x^{-\frac{1}{2}}e^{-x}\right)dx \quad (\text{from } (a))$$

$$= -\frac{1}{3}\int_0^1 xe^{-x}dx = -\frac{1}{3}\int_0^1 \underbrace{x}_{u}\underbrace{(e^{-x}dx)}_{dv}$$

$$\overset{\substack{u=x \\ dv=e^{-x}dx \\ \Rightarrow v=\int e^{-x}dx=-e^{-x}}}{\cong} -\frac{1}{3}\left[\underbrace{(-xe^{-x})|_0^1}_{uv|_0^1} - \int_0^1 \underbrace{(-e^{-x})}_{v}\underbrace{dx}_{du}\right]$$

$$= -\frac{1}{3}[-e^{-1} + (-e^{-x})|_0^1]$$

$$= \frac{2}{3}e^{-1} - \frac{1}{3}.$$

Review Examples on Integration Technique

Review Example 5

Compute the integral

$$\int \frac{\tan^{-1}(e^x)}{e^x}\,dx$$

☺Solution:

We let $y = e^x$. Then we have $\frac{dy}{dx} = e^x \Rightarrow dx = \frac{1}{e^x}dy$. The integral becomes

$$\int \frac{\tan^{-1}(e^x)}{e^x}\,dx = \int \frac{\tan^{-1}(e^x)}{e^x}\left(\frac{1}{e^x}dy\right) = \int \frac{\tan^{-1} y}{y^2}\,dy.$$

Using integration by parts, we have

$$\int \frac{\tan^{-1} y}{y^2}\,dy = \int \underbrace{\tan^{-1} y}_{u}\,\underbrace{(y^{-2}dy)}_{dv} \overset{\substack{dv=y^{-2}dy \\ v=\int y^{-2}dy=-y^{-1}}}{\cong} \underbrace{-y^{-1}\tan^{-1} y}_{uv} - \underbrace{\int -y^{-1}d(\tan^{-1} y)}_{\int vdu}$$

$$= -y^{-1}\tan^{-1} y + \int \frac{1}{y(1+y^2)}\,dy \ldots\ldots(*)$$

Since there is a quadratic factor in the denominator, we propose the following decomposition:

$$\frac{1}{y(1+y^2)} = \frac{A}{y} + \frac{By+C}{y^2+1} \Rightarrow 1 = A(y^2+1) + y(By+C).$$

Substitute $y = 0$, we get $A = 1$;

so $y(By+C) = -y^2 \Rightarrow By + C = -y \Rightarrow B = -1,\ C = 0.$

Hence, we have

$$\int \frac{1}{y(1+y^2)}\,dy = \int \frac{1}{y}\,dy - \int \frac{y}{1+y^2}\,dy$$

$$\overset{u=1+y^2 \Rightarrow \frac{du}{dy}=2y}{=} \ln|y| - \frac{1}{2}\int \frac{1}{u}\,du = \ln|y| - \frac{1}{2}\ln|u| + C = \ln|y| - \frac{1}{2}\ln|1+y^2| + C$$

Hence from (*), we conclude that

$$\int \frac{\tan^{-1}(e^x)}{e^x}\,dx = \int \frac{\tan^{-1} y}{y^2}\,dy = -y^{-1}\tan^{-1} y + \ln|y| - \frac{1}{2}\ln|1+y^2| + C$$

$$= -\frac{\tan^{-1}(e^x)}{e^x} + \ln|e^x| - \frac{1}{2}\ln|1+e^{2x}| + C = -\frac{\tan^{-1}(e^x)}{e^x} + x - \frac{1}{2}\ln|1+e^{2x}| + C.$$

Review Example 6

Compute the integral

$$\int_0^1 \frac{x+4}{\sqrt{x^2+2x+2}}\,dx$$

☺Solution:

We let $y = x^2 + 2x + 2$, then $\frac{dy}{dx} = 2x + 2 \Rightarrow dx = \frac{1}{2x+2}dy$.

When $x = 1$, $y = 1^2 + 2(1) + 2 = 5$; When $x = 0$, $y = 2$.

Then we split the integral as

$$\int_0^1 \frac{x+4}{\sqrt{x^2+2x+2}}\,dx = \int_0^1 \frac{x+1}{\sqrt{x^2+2x+2}}\,dx + \int_0^1 \frac{3}{\sqrt{x^2+2x+2}}\,dx \;\ldots (*)$$

For the first integral, we use the proposed substitution and get

$$\int_0^1 \frac{x+1}{\sqrt{x^2+2x+2}}\,dx = \int_2^5 \frac{x+1}{\sqrt{x^2+2x+2}}\left(\frac{1}{2x+2}dy\right) = \frac{1}{2}\int_2^5 \frac{1}{\sqrt{y}}\,dy = \left[y^{\frac{1}{2}}\right]_2^5 = \sqrt{5} - \sqrt{2}.$$

For the second integral, we need to propose another substitution (probably trigonometric substitution) since the original substitution does not work. The second integral becomes

$$\int_0^1 \frac{3}{\sqrt{x^2+2x+2}}dx = 3\int_0^1 \frac{1}{\sqrt{(x+1)^2+1}}dx$$

We let $x+1=\tan\theta \Rightarrow x=\tan\theta-1$. Then $\frac{dx}{d\theta}=\sec^2\theta$.

When $x=1$, $\theta=\tan^{-1}2$; When $x=0$, $\theta=\tan^{-1}1=\frac{\pi}{4}$. Then

$$3\int_0^1 \frac{1}{\sqrt{(x+1)^2+1}}dx = 3\int_{\frac{\pi}{4}}^{\tan^{-1}2} \frac{1}{\sqrt{\tan^2\theta+1}}\sec^2\theta\, d\theta = 3\int_{\frac{\pi}{4}}^{\tan^{-1}2} \sec\theta\, d\theta$$

$$= 3\ln|\sec\theta+\tan\theta|\,\Big|_{\frac{\pi}{4}}^{\tan^{-1}2} = 3\ln|\sec(\tan^{-1}2)+\tan(\tan^{-1}2)| - 3\ln\left|\sec\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{4}\right)\right|$$

$$\overset{\sec\theta=\sqrt{1+\tan^2\theta}}{\triangleq} 3\ln\left|\sqrt{5}+2\right| - 3\ln\left|\sqrt{2}+1\right|.$$

Thus, we can conclude from (*) that

$$\int_0^1 \frac{x+4}{\sqrt{x^2+2x+2}}dx = \sqrt{5}-\sqrt{2}+3\ln\left|\sqrt{5}+2\right|-3\ln\left|\sqrt{2}+1\right|.$$

Review Example 7

Compute the integral

$$\int \frac{\ln(x^2+2x+10)}{(2x+1)^2}\,dx.$$

☺Solution: Using integration by parts, we have

$$\int \frac{\ln(x^2+2x+10)}{(2x+1)^2}\,dx = \int \underbrace{\ln(x^2+2x+10)}_{u}\,\underbrace{[(2x+1)^{-2}dx]}_{dv}$$

$$\overset{v=-\frac{1}{2}(2x+1)^{-1}}{\cong} \quad -\frac{1}{2}(2x+1)^{-1}\ln(x^2+2x+10) - \int -\frac{1}{2}(2x+1)^{-1}d\ln(x^2+2x+10)$$

$$= -\frac{1}{2}(2x+1)^{-1}\ln(x^2+2x+10) + \frac{1}{2}\int \frac{2x+2}{(2x+1)(x^2+2x+10)}\,dx \;\ldots (*)$$

Since the integrand is a proper rational function and the denominator contains a quadratic factor, we shall propose the following decomposition:

$$\frac{2x+2}{(2x+1)(x^2+2x+10)} = \frac{A}{2x+1} + \frac{Bx+C}{x^2+2x+10}.$$

$$\Rightarrow\ 2x+2=A(x^2+2x+10)+(2x+1)(Bx+C)$$

Substitute $x=-\frac{1}{2}$, we get $1=A\left(\frac{37}{4}\right)\ \Rightarrow\ A=\frac{4}{37}$.

Compare the coefficient of x^2: $0=A+2B\ \Rightarrow\ B=-\frac{2}{37}$;

Compare the constant term: $2=10A+C\ \Rightarrow C\ =\frac{34}{37}$.

Hence, we can decompose the integral as

$$\int\frac{2x+2}{(2x+1)(x^2+2x+10)}dx=\frac{4}{37}\int\frac{1}{2x+1}dx+\frac{1}{37}\int\frac{-2x+34}{x^2+2x+10}dx$$

$$=\frac{4}{37}\int\frac{1}{2x+1}dx-\frac{1}{37}\int\frac{2x+2}{x^2+2x+10}dx+\frac{1}{37}\int\frac{36}{x^2+2x+10}dx.$$

$$\overset{y=x^2+2x+10}{\triangleq}\ \frac{4}{37}\left(\frac{1}{2}\ln|2x+1|\right)-\frac{1}{37}\int\frac{1}{y}dy+\frac{36}{37}\int\frac{1}{(x+1)^2+9}dx$$

$$=\frac{2}{37}\ln|2x+1|-\frac{1}{37}\ln|y|+\frac{4}{37}\int\frac{1}{\left(\frac{x+1}{3}\right)^2+1}dx$$

$$=\frac{2}{37}\ln|2x+1|-\frac{1}{37}\ln|x^2+2x+10|+\frac{12}{37}\tan^{-1}\left(\frac{x+1}{3}\right)+C$$