MAT201 Section E/F/G/H

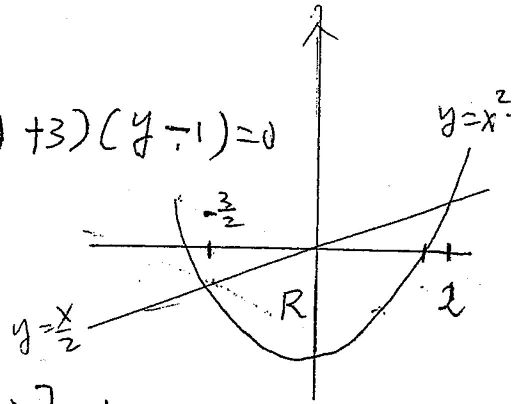
1(a) $R = \begin{cases} y = x^2-3 \\ x = 2y \end{cases}$

[20]

$y = (2y)^2 - 3 = 4y^2 - 3 \Rightarrow 4y^2 - y - 3 = 0 \Rightarrow (4y+3)(y-1) = 0$

$\Rightarrow y = -\frac{3}{4} \quad (x = -\frac{3}{2})$ (3)

or $y = 1 \quad (x = 2)$ (3)



$$A = \int_{-\frac{3}{2}}^{2} [y_{upper} - y_{lower}]\,dx = \int_{-\frac{3}{2}}^{2} \left[\frac{x}{2} - (x^2-3)\right] dx \quad (7)$$

$$= \left[\frac{x^2}{4} - \frac{x^3}{3} + 3x\right]_{-\frac{3}{2}}^{2} \ (3) = \frac{1}{4}\left[2^2 - (-\tfrac{3}{2})^2\right] - \frac{1}{3}\left[2^3 - (-\tfrac{3}{2})^3\right] + 3\left[2 - (-\tfrac{3}{2}\right.$$

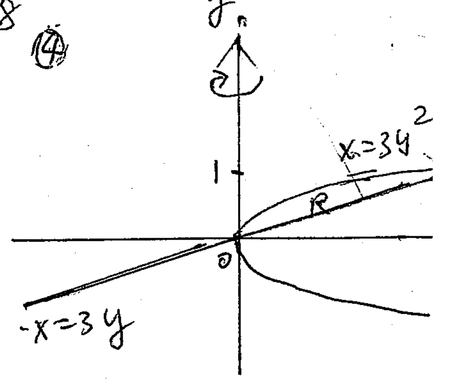
$$= \frac{1}{4}\left(4 - \frac{9}{4}\right) - \frac{1}{3}\left(8 + \frac{27}{8}\right) + 3\left(2 + \frac{3}{2}\right)$$

$$= \frac{1}{4}\left(\frac{16-9}{4}\right) - \frac{1}{3}\left(\frac{64+27}{8}\right) + 3\left(\frac{4+3}{2}\right)$$

$$= \frac{7}{16} - \frac{91}{24} + \frac{21}{2} = \frac{21 - 182 + 504}{48} = \frac{343}{48} \approx 7.1458 \quad (4)$$

1(b) $R = \begin{cases} x = 3y^2 \\ x = 3y \end{cases}$

[20]



$3y^2 = 3y \Rightarrow 3y(y-1) = 0 \Rightarrow y = 0, 1$ (4)

$$V_y = \int_0^1 \pi [x_{outer}^2 - x_{inner}^2]\,dy,$$

$$= \pi \int_0^1 [(3y)^2 - (3y^2)^2]\,dy \quad (9)$$

$$= \pi \int_0^1 [9y^2 - 9y^4]\,dy$$

$$= \pi \left[3y^3 - \frac{9}{5}y^5\right]_0^1 \quad (2)$$

$$= \pi \left\{3(1^3 - 0) - \frac{9}{5}(1^5 - 0)\right\}$$

$$= \pi \left\{3 - \frac{9}{5}\right\} = \pi\,\frac{15-9}{5} = \frac{6\pi}{5} \quad (5)$$

2 (a) [15] $z = -i e^{i\frac{\pi}{6}} = e^{-i\frac{\pi}{2}} e^{i\frac{\pi}{6}}$ (5) $= e^{i(-\frac{\pi}{2}+\frac{\pi}{6})} = e^{-i\frac{\pi}{3}}$ (2)

$= \cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})$ (2)

$z^{10} = (e^{-i\frac{\pi}{3}})^{10} = e^{-i\frac{10\pi}{3}}$ (2) $= e^{i(\frac{-10\pi}{3}+4\pi)} = e^{i(\frac{2\pi}{3})}$ (2)

$= \cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3})$ (2)

b) [15] $z^3 = 2-2i \Rightarrow$ $z^3 = 2-2i = 2\sqrt{2}\, e^{-i\frac{\pi}{4}}$ (5)

$z_k = (2\sqrt{2})^{1/3} e^{i(-\frac{\pi}{4}+2k\pi)/3}$, $k=0,1,2$ (3)

$z_0 = \sqrt{2}\, e^{i(-\frac{\pi}{12})}$ (2)

$z_1 = \sqrt{2}\, e^{i(-\frac{\pi}{4}+2\pi)/3} = \sqrt{2}\, e^{i\frac{7\pi}{12}}$ (2)

$z_2 = \sqrt{2}\, e^{i(-\frac{\pi}{4}+4\pi)/3} = \sqrt{2}\, e^{i\frac{15\pi}{12}} = \sqrt{2}\, e^{i(\frac{5\pi}{4}-2\pi)} = \sqrt{2}\, e^{i(-\frac{3\pi}{4})}$ (3)

3 (a) [15] $(A|b) \overset{R_1 \leftrightarrow R_2}{=} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2 \\ 0 & -1 & c & 1 \\ 2 & c & 1 & -3 \end{array}\right) \xrightarrow{R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & -2 \\ 0 & -1 & c & 1 \\ 0 & c-4 & 3 & 1 \end{array}\right)$ (5)

$\xrightarrow{R_3+(c-4)R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & c & 1 \\ 0 & c-4 & c(c-4)+3 & c-3 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & c & 1 \\ 0 & 0 & (c-1)(c-3) & c-3 \end{array}\right)$ (5)

$\det A = -\det \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & c \\ 0 & 0 & (c-1)(c-3) \end{pmatrix} = -(1)(-1)(c-1)(c-3) = +(c-1)(c-3)$ (3)

A is invertible if $\det A \neq 0$ iff $c \neq 1$, $c \neq 3$ (2)

b) [15] $c=3$ the linear system becomes $\left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ which is consistent (7)

Backward substitution: let $z=t$ (3) be a free variable

R_2: $-y+3z=1 \Rightarrow y=-1+3z=-1+3t$ (2)

R_1: $x+2y-z=2 \Rightarrow x=2-2y+z=2-2(-1+3t)+t$

$=2+2-6t+t = 4-5t$ (2)

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4-5t \\ -1+3t \\ t \end{pmatrix}$ (1)