This is a open-book exam and the exam schedule is October 5th, Tuesday, 9:00 AM - 11:00 AM (two hour). If you need more space, please feel free to attach additional papers. Once you're finished, sign your name and student ID at the top of each page. Also, make sure to sign the following honor pledge.

Honor Pledge

Please review the following honor code, then sign your name and write down the date.

1. I pledge that the answers in this exam are my own and that I will not seek or obtain an unfair advantage in producing these answers. Specifically,
   (a) I will not plagiarize (copy without citation) from any source;
   (b) I will not communicate or attempt to communicate with any other person during the exam;
   (c) neither will I give or attempt to give assistance to another student taking the exam; and
   (d) I will use only approved devices (e.g., calculators) and/or approved device models.
2. I understand that any act of academic dishonesty can lead to disciplinary action.

____
Signature

____
Date

1. (20 points) Sketch the following signals.

   (a) $u(t-5) + u(t-7)$.

   (b) $t^2 [u(t-1) - u(t-2)]$.

   (c) $(t-4)[u(t-2) - u(t-4)]$.

   Consider signals $x_1(t)$ and $x_2(t)$ as plotted in the figures below.

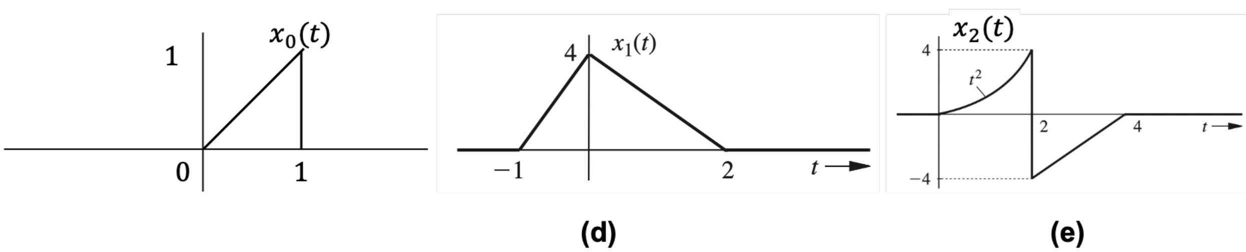


Figure 1:

   Note that $x_0(t)$ can be expressed in terms of unit step function as follows

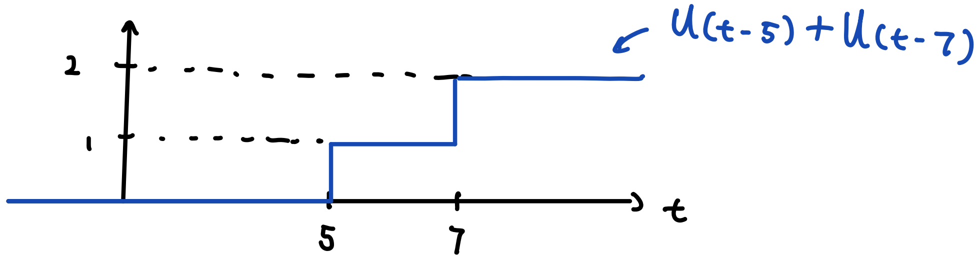
$$x_0(t) = t[u(t) - u(t-1)]. \tag{1}$$

   (d) Express $x_1(t)$ in Fig. 1-(d) in terms of unit step function, similar to (1).

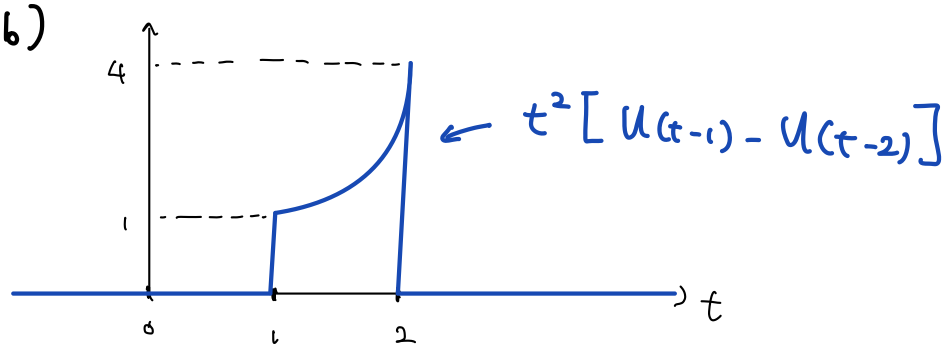
   (e) Express $x_2(t)$ in Fig. 1-(d) in terms of unit step function, similar to (1).

(Answer Page for Question 1)

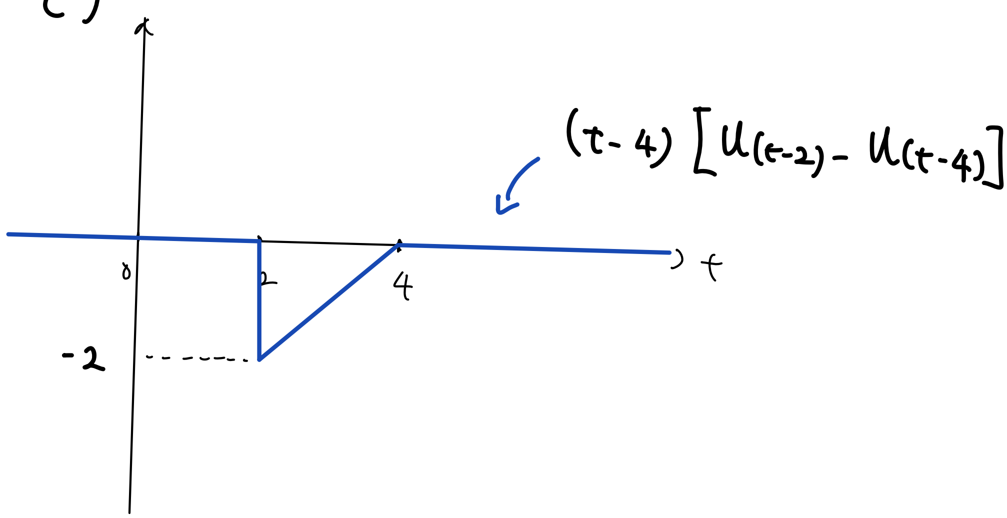
Sol) a)



b)



c)



d) $x_1(t) = 4(t+1)[u(t+1) - u(t)] + (4-2t)[u(t) - u(t-2)]$

$= 4(t+1)\,u(t+1) - 6t\,u(t) - (4-2t)\,u(t-2)$

e) $x_2(t) = t^2[u(t) - u(t-2)] + (2t-8)[u(t-2) - u(t-4)]$

$= t^2\,u(t) - (t^2-2t+8)\,u(t-2) - (2t-8)\,u(t-4)$

2. (20 points) Consider signals $x(t)$ and $y(t)$ as plotted in the figure below.

$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$

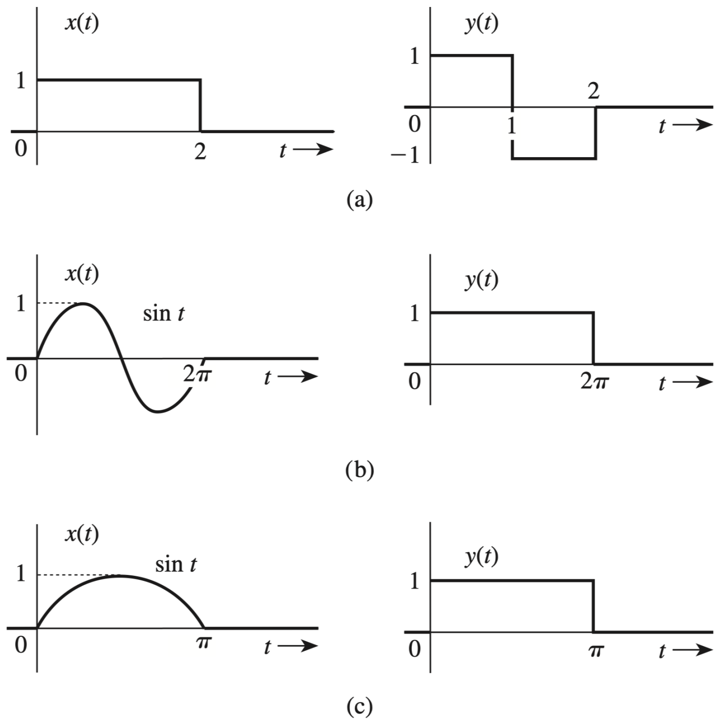


Figure 2:

(a) Find the energies of the signals $x(t)$ and $y(t)$ illustrated in Fig. 2-(a). Then, draw and calculate the energies of signals $x(t) + y(t)$ and $x(t) - y(t)$.

(b) Find the energies of the signals $x(t)$, $y(t)$, $x(t) + y(t)$ and $x(t) - y(t)$ in Fig. 2-(b).

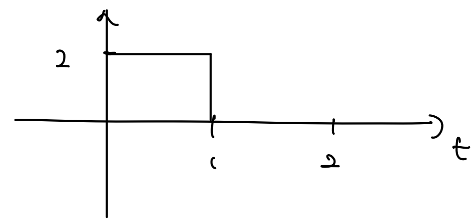
(c) Find the energies of the signals $x(t)$, $y(t)$, $x(t) + y(t)$ and $x(t) - y(t)$ in Fig. 2-(c).

(Answer Page for Question 2)

Sol) a) $E_x = \int_0^2 x(t)^2\, dt = \int_0^2 1 \cdot dt = 2$

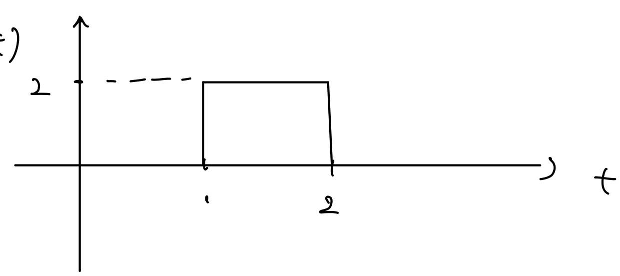
$E_y = \int_0^1 \underbrace{y(t)}_{=1}{}^2\, dt + \int_1^2 \underbrace{y(t)}_{(-1)^2 = 1}{}^2\, dt = 2$

$x(t) + y(t)$



$E_{x+y} = \int_0^1 |x(t) + y(t)|^2 dt = \int_0^1 2^2\, dt = 4$

$x(t) - y(t)$



$E_{x-y} = \int_1^2 |x(t) - y(t)|^2 dt = 4$

b) $E_x = \int_0^{\pi} |\sin(t)|^2 dt + \int_{\pi}^{2\pi} |\sin(t)|^2 dt = 2\int_0^{\pi} \sin^2(t)\, dt$

$= \int_0^{\pi} (1 - \cos 2t)\, dt = \pi - \cancel{\frac{1}{2} \sin 2t \Big|_0^{\pi}}^{\,0} = \pi$

$E_Y = \int_0^{2\pi} 1^2 \cdot dt = 2\pi$

(Answer Page for Question 2)

Continue)

b) $E_{x+y} = \int_0^{2\pi} (1+\sin(t))^2 dt = \int_0^{2\pi} (1 + 2\sin t + \sin^2 t)dt$

$= \int_0^{2\pi} \left(1 + \cancel{2\sin t}^{\,0} + \frac{1}{2}(1 - \cos 2t) \right) dt$

$= \frac{3}{2} \cdot 2\pi - \frac{1}{2} \cancel{\int_0^{2\pi} \cos 2t \cdot dt}^{\,\phi} = 3\pi$

$E_{x-y} = \int_0^{2\pi} |\sin(t) - 1|^2 dt = \int_0^{2\pi} (1 - \sin(t))^2 dt$

$= \int_0^{2\pi} (1 - 2\sin t + \sin^2 t)\, dt = 3\pi.$

c) $E_x = \int_0^{\pi} |\sin t|^2 dt = \frac{1}{2}\int_0^{\pi} (1 - \cos(2t))dt = \frac{\pi}{2} - \cancel{\frac{1}{4} \sin 2t \Big|_0^{\pi}}^{\,\phi}$

$E_y = \int_0^{\pi} 1^2 dt = \pi$

$E_{x+y} = \int_0^{\pi} (1+\sin t)^2 dt = \int_0^{\pi} \left\{ \frac{3}{2} + 2\sin t - \frac{1}{2}\cos 2t \right\} dt$

$= \frac{3}{2}\pi \underbrace{- 2\cos t \Big|_0^{\pi}}_{-2} \cancel{- \frac{1}{4}\sin 2t \Big|_0^{\pi}}^{\,\phi} = \frac{3}{2}\pi + 4$

$E_{x-y} = \int_0^{\pi} |\sin t - 1|^2 dt = \int_0^{\pi} (1 - \sin(t))^2 dt$

$= \int_0^{\pi} \left\{ \frac{3}{2} - 2\sin t - \frac{1}{2}\cos 2t \right\} dt = \frac{3}{2}\pi - 4$

3. (20 points) Consider the LTI system illustrated in the figure below.

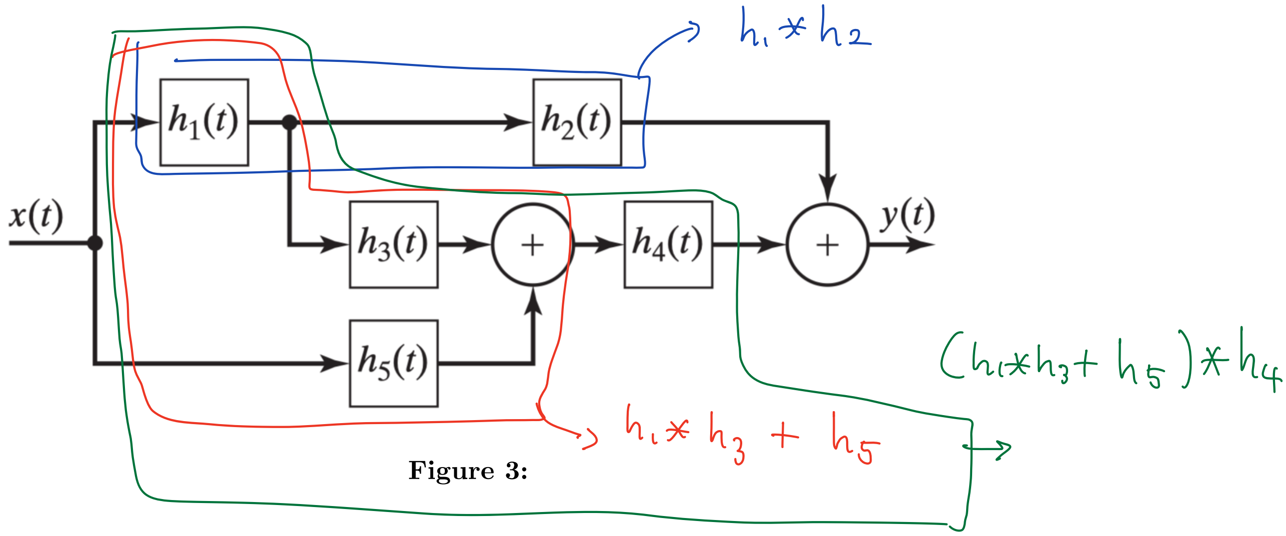


Figure 3:

(a) Express the overall system impulse response $h(t)$ as a function of the impulse responses of the sub-systems $\{h_1(t), h_2(t), h_3(t), h_4(t), h_5(t)\}$.

(b) Assume that $h_1(t) = h_4(t) = u(t)$, $h_2(t) = h_3(t) = 5\delta(t)$, and $h_5(t) = e^{-2t}u(t)$. Find the overall impulse response $h(t)$ of the system.

(c) For the impulse response $h(t)$ derived in Q3-(b), determine whether the LTI system is memoryless, or causal, or stable. Provide explanation to your answer to get a full mark.

(Answer Page for Question 3)

Sol) a)

$$h(t) = h_1(t) * h_2(t) + (h_1(t) * h_3(t) + h_5(t)) * h_4(t)$$

b) $$h_1 * h_2 = u(t) * 5\delta(t) = 5u(t)$$

$$h_1 * h_3 = u(t) * 5\delta(t) = 5u(t)$$

$$(h_1 * h_3 + h_5) * h_4 = (5u(t) + e^{-2t}u(t)) * u(t)$$

$$\Rightarrow 5\, \underbrace{u(t) * u(t)}_{\downarrow} + \underbrace{e^{-2t}u(t) * u(t)}_{\shortparallel}$$

$$= 5t\,u(t) + \int_{-\infty}^{\infty} e^{-2\tau} \underbrace{u(\tau)\,u(t-\tau)}_{1 \text{ if } 0<\tau<t} d\tau$$

$$= 5t\,u(t) + u(t)\int_0^t e^{-2\tau} d\tau$$

$$= 5t\,u(t) + \frac{1}{2}(1-e^{-2t})\,u(t)$$

Therefore, $$h(t) = \left[5 + 5t + \frac{1}{2}(1-e^{-2t})\right] u(t)$$

(Answer Page for Question 3)

C) i) Since $h(t) \neq 0$ for $t \neq 0$, $h(t)$

is a LTI system with memory.

ii) Since $h(t) = 0$ for $t < 0$,

$h(t)$ is a causal system.

iii) $\int_{-\infty}^{\infty} |h(t)| \, dt$

$$= \int_{0}^{\infty} \left[5 + 5t + \frac{1}{2}\left(1 - e^{-2t}\right) \right] dt$$

$\rightarrow \infty$ due to '$5t$' term.

Hence, $h(t)$ is unstable system

$\therefore$ Memory System, Causal System,

Unstable system

4. (20 points) Consider the two signals $x(t)$ and $h(t)$ listed in the following problems. Evaluate the continuous-time convolution $y(t) = x(t) * h(t)$ for each problem using the convolution integral

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau. \tag{2}$$

(a) $x(t) = \delta(t+3) + 3e^{-0.5t}u(t), \quad h(t) = u(t) - u(t-2).$

(b) $x(t) = t^2 u(t), \quad h(t) = \frac{1}{\sqrt{t}} u(t).$

(c) $x(t) = u(t), \quad h(t) = 2e^{-t}u(t) - 2e^{2t}u(-t).$

(d) $x(t) = e^{-2t}\left[u(t) - u(t-2)\right], \quad h(t) = e^{-t}u(t).$

(Answer Page for Question 4)

Sol) a)

$$\left(\delta(t+3) + 3e^{-0.5t}u(t)\right) * \left(u(t) - u(t-2)\right)$$

$$= \delta(t+3) * u(t) + 3e^{-0.5t}u(t) * u(t)$$

$$- \delta(t+3) * u(t-2) - 3e^{-0.5t}u(t) * u(t-2)$$

$$= u(t+3) - u(t+1) + 3u(t)\underbrace{\int_0^t e^{-0.5\tau}\,d\tau}_{=2(1-e^{-\frac{t}{2}})}$$

$$- 3u(t-2)\underbrace{\int_0^{t-2} e^{-0.5\tau}\,d\tau}_{=2(1-e^{-\frac{1}{2}(t-2)})}$$

$$= u(t+3) - u(t+1) + 6\left(1 - e^{-t/2}\right)u(t) - 6\left(1 - e^{-\frac{1}{2}(t-2)}\right)u(t-2)$$

b) $t^2 u(t) * \frac{1}{\sqrt{t}}u(t)$

$$= u(t)\int_0^t \tau^2 \frac{1}{\sqrt{t-\tau}}\,d\tau = u(t)\int_0^t \frac{(t-\tau)^2}{\sqrt{\tau}}\,d\tau$$

$$= u(t)\int_0^t \tau^{-\frac{1}{2}}\left(\tau^2 - 2t\tau + t^2\right)d\tau = u(t)\int_0^t \left(\tau^{3/2} - 2t\,\tau^{\frac{1}{2}} + t^2\tau^{-\frac{1}{2}}\right)d\tau$$

$$= u(t)\left[\frac{2}{5}\tau^{5/2} - \frac{4}{3}t\,\tau^{3/2} + 2t^2\tau^{\frac{1}{2}}\right]_0^t = \frac{16}{15}t^{5/2}\,u(t)$$

(Answer Page for Question 4)

Continue)

c) $u(t) * \left[2e^{-t} u(t) - 2e^{2t} u(-t) \right]$

$$y(t) = 2\int_0^t e^{-\tau} d\tau \cdot u(t) - \boxed{2\int_{-\infty}^{\infty} e^{2\tau} \underbrace{u(-\tau)\, u(t-\tau)}_{1 \text{ if } \tau<0,\ \tau<t} d\tau}$$

(let's denote this as I)

$$= 2(1-e^{-t})u(t) - I$$

if $t>0$, $I = 2\int_{-\infty}^{0} e^{2\tau} d\tau = 1$

if $t<0$, $I = 2\int_{-\infty}^{t} e^{2\tau} d\tau = e^{2t}$

$$I = u(t) + e^{2t} u(-t)$$

$$y(t) = 2(1-e^{-t})u(t) - u(t) - e^{2t} u(-t)$$

$$= (1-2e^{-t})u(t) - e^{2t} u(-t)$$

d) $e^{-2t}\left[u(t) - u(t-2) \right] * e^{-t} u(t)$

$$= \int_{-\infty}^{\infty} e^{-2\tau} \left(u(\tau) - u(\tau-2) \right) e^{-(t-\tau)} u(t-\tau)\, d\tau$$

Continue) d)

$$\Rightarrow e^{-t}\int_{-\infty}^{\infty} e^{-\tau} \underbrace{u(\tau)\, u(t-\tau)}_{=1 \text{ if } 0<\tau<t} d\tau$$

$$- e^{-t}\int_{-\infty}^{\infty} e^{-\tau} \underbrace{u(\tau-2)\, u(t-\tau)}_{=1 \text{ if } 2<\tau<t} d\tau$$

$$\Rightarrow u(t)\, e^{-t} \underbrace{\int_{0}^{t} e^{-\tau} d\tau}_{=1-e^{-t}} - u(t-2)\, e^{-t} \underbrace{\int_{2}^{t} e^{-\tau} d\tau}_{e^{-2}-e^{-t}}$$

$$= \left[e^{-t} - e^{-2t}\right] u(t) - \left[e^{-(t+2)} - e^{-2t}\right] u(t-2)$$

5. (20 points) Consider the following signals.

(a) Evaluate the following integral

$$\int_{-\infty}^{\infty}\left[e^{-2t^2}\delta(t-1)+t^2\delta(t-6)\right]dt.$$

Sol) $e^{-2} + 6^2 = e^{-2} + 36$

(b) Find if the following signals are periodic and if so, find the fundamental period T_0

$$x_1(t) = |\cos(4\pi t)|, \qquad x_2(t) = 2t + \cos(4\pi t).$$

Sol) $x_1 \Rightarrow$ Periodic with $T_0 = \frac{1}{4}$

$x_2 \Rightarrow$ Nonperiodic

(c) Find if the following signal is causal or not. Provide explanation to get a full mark

$$x(t) = e^t\left[u(t+4) - u(t-3)\right].$$

Sol) Non-causal as it depends on future values of the input

(d) Find if the following signal is an energy signal or a power signal

$$x(t) = u(t) - u(t-7).$$

Sol) Energy signal with $E = \int_0^7 x(t)^2\,dt = 7$

(e) Consider the following system described by

$$y(t) = \cos\left[x(t-1)\right]$$

and answer whether it is

- memoryless,
- invertible,
- causal,
- stable,
- time invariant,
- linear.

Sol) Memory, Stable, Non-invertible, Time invariant, Non-causal, and Non-linear