MA1201 Calculus and Basic Linear Algebra II

Chapter 1

Basic Concept of Integration

What is integration?

Roughly speaking, integration is the *reverse process* of differentiation: Given a function $f(x)$, we would like to find a (differentiable) function $F(x)$ such that $\frac{d}{dx}F(x) = f(x)$.

$$\begin{tikzcd} F(x) = x^2 \arrow[r, shift left, "\text{Differentiation} \quad \frac{d}{dx}F(x)"] & f(x) = 2x \arrow[l, shift left, "\text{Integration} \quad \int f(x)dx"] \end{tikzcd}$$

Definition (Indefinite Integral)

Given a function $g(x)$, the (indefinite) integral of $g(x)$, denoted by $G(x) = \int g(x)dx$, is a function such that

$$\frac{d}{dx}G(x) = g(x) \ \left(\text{or } G'(x) = g(x)\right).$$

Here, $g(x)$ is called integrand and $G(x)$ is called an *antiderivative* or a *primitive function*.

Integrals of some elementary functions

Example 1

Compute the integral $\int \cos x \, dx$.

☺Solution:

It is equivalent to find a function $F(x)$ $(= \int \cos x \, dx)$ such that

$$F'(x) = \cos x.$$

From the formula in the differentiation, we guess that $F(x) = \sin x$ (so that $\frac{d}{dx} \sin x = \cos x$). Therefore, we conjecture that

$$\int \cos x \, dx = \sin x.$$

Question: Can I try $\boldsymbol{F(x) = \sin x + 2}$?

Answer: Yes. It is because $F'(x) = \frac{d}{dx}(\sin x + 2) = \frac{d}{dx} \sin x + \frac{d}{dx} 2 = \cos x$.

Remark of Example 1

Because of the fact that $\frac{d}{dx}C = 0$ for any constant C, thus for any real constant C, the function $F(x) = \sin x + C$ satisfies

$$F'(x) = \frac{d}{dx}(\sin x + C) = \frac{d}{dx}\sin x + \frac{d}{dx}C = \cos x.$$

In general, the integral $\int \cos x \, dx$ is given by

$$\boxed{\int \cos x \, dx = \sin x + C}$$

where C is arbitrary constant.

Since the final answer is not fixed (C can be any constant), so the integral is called *indefinite integral*.

(Practically, C will be determined based on some *conditions*. You can learn more when you study ordinary differential equation.)

Example 2 (Integral of polynomial)

Using the fact that $\frac{d}{dx}x^a = ax^{a-1}$ for any real number a, compute the integral $\int x^a dx$ for $a \neq -1$.

☺Solution:

Again, we need to find a function $F(x) = \int x^a dx$ such that $F'(x) = x^a$.

1[st] Trial: Try $F(x) = x^{a+1}$

$$\frac{d}{dx}F(x) = \frac{d}{dx}x^{a+1} = \overbrace{(a+1)}^{\text{extra factor!}}\ x^a.$$

2[nd] Trial: Try $F(x) = \frac{x^{a+1}}{a+1}$

Then $\frac{d}{dx}F(x) = \frac{d}{dx}\frac{x^{a+1}}{a+1} = x^a$. Hence we conclude that

$$\int x^a dx = \frac{x^{a+1}}{a+1} + C.$$

Remarks of Example 2

In Example 2, we consider the case when $a \neq -1$. How about the case when $a = -1$? More precisely, what is the integral

$$\int x^{-1} dx = \int \frac{1}{x} dx\,?$$

By direct substitution, one can see that the formula $\int x^a dx = \frac{x^{a+1}}{a+1} + C$ does not work for this case since $\int x^{-1} dx = \frac{x^{-1+1}}{-1+1} + C = \frac{1}{0} + C$ which is undefined.

Note that $\frac{d}{dx} \ln|x| = \frac{1}{x}$.

Absolute sign is needed since x can be positive or negative and $\ln x = \log_e x$ is only defined for $x > 0$.

Therefore, we conclude that

$$\int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C.$$

Example 3

Compute the integrals

$$\int \sqrt{x}dx, \qquad \int \frac{1}{x^3}dx.$$

IDEA: One can first express the functions $\sqrt{x}$ and $\frac{1}{x^3}$ into the form x^a so that we can use the result in Example 2 to compute these two integrals

☺Solution:

$$\int \sqrt{x}dx = \int x^{\frac{1}{2}}dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{2}{3}x^{\frac{3}{2}} + C$$

$$\int \frac{1}{x^3}dx = \int x^{-3}dx = \frac{x^{-3+1}}{-3+1} + C = -\frac{1}{2}x^{-2} + C.$$

Example 4 (Integral of Exponential Function)

Compute the following integral $\int e^x dx$.

☺Solution:

Note that $\frac{d}{dx}e^x = e^x$, thus $\int e^x dx = e^x + C$.

Example 5 (Integral of trigonometric functions)

Compute the following integrals

(a) $\int \sin x \, dx$

(b) $\int \sec^2 x \, dx$

(c) $\int \sec x \tan x \, dx$

☺Solution:

(a) Note that $\frac{d}{dx}\cos x = -\sin x$, so we should try $F(x) = -\cos x$ so that $F'(x) = \frac{d}{dx}(-\cos x) = \sin x$. Thus

$$\int \sin x\, dx = -\cos x + C.$$

(b) Note that $\frac{d}{dx}\tan x = \sec^2 x$, thus

$$\int \sec^2 x\, dx = \tan x + C$$

(c) Note that $\frac{d}{dx}\sec x = \sec x \tan x$, thus

$$\int \sec x \tan x\, dx = \sec x + C$$

Summary of the important integral of some elementary functions

$f(x)$	Antiderivative $F(x)$ $F'(x) = f(x)$	Integral $F(x) = \int f(x)dx$
x^a, a is real	$F(x) = \begin{cases} \dfrac{x^{a+1}}{a+1} & \text{if } a \neq -1 \\ \ln\lvert x\rvert & \text{if } a = -1 \end{cases}$	$\begin{cases} \dfrac{x^{a+1}}{a+1} + C & \text{if } a \neq -1 \\ \ln\lvert x\rvert + C & \text{if } a = -1 \end{cases}$
e^x	$F(x) = e^x$	$e^x + C$
$\sin x$	$F(x) = -\cos x$	$-\cos x + C$
$\cos x$	$F(x) = \sin x$	$\sin x + C$
$\tan x$	$F(x) = -\ln\lvert\cos x\rvert$ or $F(x) = \ln\lvert\sec x\rvert$	$-\ln\lvert\cos x\rvert + C$ or $\ln\lvert\sec x\rvert + C$
$\sec^2 x$	$F(x) = \tan x$	$\tan x + C$
$\dfrac{1}{1+x^2}$	$F(x) = \tan^{-1} x$	$\tan^{-1} x + C$
$\dfrac{1}{\sqrt{1-x^2}}$	$F(x) = \sin^{-1} x$	$\sin^{-1} x + C$

Example 6

Compute the integrals

(a) $\displaystyle\int e^{3x+1}dx$

(b) $\displaystyle\int \cos 2x\, dx$

(c) $\displaystyle\int \frac{1}{1-3x}dx$

☺Solution

(a) If we try $F_1(x) = e^{3x+1}$, then we have $\frac{d}{dx}F_1(x) = \frac{d}{dx}e^{3x+1} = 3e^{3x+1} \neq e^{3x+1}$.

In order to compute the integral, we need to *modify* our choice of $F(x)$ by setting $F_2(x) = \frac{1}{3}e^{3x+1}$ so that $\frac{d}{dx}F_2(x) = \frac{1}{3}\frac{d}{dx}e^{3x+1} = \frac{1}{3}(3e^{3x+1}) = e^{3x+1}$.

Therefore, we get

$$\int e^{3x+1}dx = \frac{1}{3}e^{3x+1} + C.$$

(b) Note that $\frac{d}{dx}\sin 2x = 2\cos 2x$, we should choose our $F(x)$ as $F(x) = \frac{1}{2}\sin 2x$ so that $\frac{d}{dx}F(x) = \frac{1}{2}\frac{d}{dx}\sin 2x = \frac{1}{2}(2\cos 2x) = \cos 2x$. Therefore, we have

$$\int \cos 2x\, dx = \frac{1}{2}\sin 2x + C.$$

(c) Note that $\frac{d}{dx}\ln|1-3x| = -\frac{3}{1-3x}$, we should choose our $F(x)$ as $F(x) = -\frac{1}{3}\ln|1-3x|$

so that

$$\frac{d}{dx}F(x) = -\frac{1}{3}\frac{d}{dx}\ln|1-3x| = -\frac{1}{3}\left(\frac{-3}{1-3x}\right) = \frac{1}{1-3x}.$$

Therefore, we conclude that

$$\int \frac{1}{1-3x}\, dx = -\frac{1}{3}\ln|1-3x| + C.$$

One useful fact in integration

Inspired by the result from Example 6, we can deduce the following useful result

Let $f(x)$ be a function.

$$\text{If } \int f(x)dx = F(x) + C, \quad \text{then } \int f(ax+b)dx = \frac{1}{a}F(ax+b) + C.$$

- One can use this fact to *upgrade* our integration table so that we can compute some more complicated integrals. For example:

$$\int \cos x \, dx = \sin x + C \quad \Rightarrow \quad \int \cos(ax+b) \, dx = \frac{1}{a}\sin(ax+b) + C$$

(original formula) (new formula – more useful)

Put $f(x) = \cos x$ and $F(x) = \sin x$

Example 7

Compute the integral

(a) $\displaystyle\int \frac{1}{(2x-5)^4}\,dx$

(b) $\displaystyle\int \sqrt{3x+1}\,dx$

☺Solution

Recall that $\int x^a dx = \frac{x^{a+1}}{a+1} + C$, one can use the above property and upgrade this formula into $\int (cx+d)^a dx = \frac{1}{c}\left(\frac{(cx+d)^{a+1}}{a+1}\right) + C$. Then, we get

$$\int \frac{1}{(2x-5)^4}\,dx = \int (2x-5)^{-4}dx \overset{\substack{c=2,d=-5\\ a=-4}}{\triangleq} \frac{1}{2}\frac{(2x-5)^{-4+1}}{-4+1} + C = -\frac{1}{6(2x-5)^3} + C.$$

$$\int \sqrt{3x+1}\,dx = \int (3x+1)^{\frac{1}{2}}dx \overset{\substack{c=3,d=1\\ a=1/2}}{\triangleq} \frac{1}{3}\frac{(3x+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{2}{9}(3x+1)^{\frac{3}{2}} + C.$$

More Properties of indefinite integrals

Using the definition of indefinite integral and the properties of differentiation, we can discover the following property of indefinite integral:

Property 1

For any continuous function $f(x)$, we have

$$\frac{d}{dx}\left(\int f(x)dx\right) = f(x) \quad \text{and} \quad \int\left(\frac{d}{dx}f(x)\right)dx = f(x) + C.$$

- The property 1 simply follows from the fact that the integration is the reverse process of differentiation.

Property 2

Let $f(x)$ and $g(x)$ be two continuous functions with $\int f(x)dx = F(x) + C_1$ and $\int g(x)dx = G(x) + C_2$, we have

1. $\displaystyle \int cf(x)dx = c\int f(x)dx = cF(x) + C.$

2. $\displaystyle \int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx = F(x) \pm G(x) + C$

3. $\displaystyle \int f(ax+b)dx = \frac{1}{a}F(ax+b) + C.$

Reasons:

All result can be deduced from the properties of differentiation. (You may refer to Appendix A for more details)

Remark of Property 2

One has to be careful that *there is NO "product rule" or "quotient rule" of integration*:

$$\int f(x)g(x)dx \neq \left(\int f(x)dx\right)\left(\int g(x)dx\right) \text{ and } \int \frac{f(x)}{g(x)}dx \neq \frac{\int f(x)dx}{\int g(x)dx}$$

Take $f(x) = x^2$ and $g(x) = x$ as an example, we can see that

$$\int f(x)g(x)dx = \int x^3 dx = \frac{x^4}{4} + C$$

$$\left(\int f(x)dx\right)\left(\int g(x)dx\right) = \left(\int x^2\,dx\right)\left(\int xdx\right) = \left(\frac{x^3}{3} + C_1\right)\left(\frac{x^2}{2} + C_2\right)$$

$$\neq \int f(x)g(x)dx.$$

Similarly, we have

$$\int \frac{f(x)}{g(x)}dx = \frac{x^2}{2} + C \quad \text{and} \quad \frac{\int f(x)dx}{\int g(x)dx} = \frac{\frac{x^3}{3} + C_1}{\frac{x^2}{2} + C_2} \neq \int \frac{f(x)}{g(x)}dx$$

Example 8

Compute the integral

$$\int \frac{1-2x^2-\frac{1}{\sqrt{x}}}{\sqrt{x}}\,dx.$$

☺Solution:

Using the property of integration, we have

$$\int \frac{1-2x^2-\frac{1}{\sqrt{x}}}{\sqrt{x}}\,dx = \int \frac{1}{\sqrt{x}}\,dx - 2\int \frac{x^2}{\sqrt{x}}\,dx - \int \frac{\frac{1}{\sqrt{x}}}{\sqrt{x}}\,dx$$

$$= \int x^{-\frac{1}{2}}dx - 2\int x^{\frac{3}{2}}\,dx - \int \frac{1}{x}\,dx$$

$$\overset{\substack{\int x^a dx = \frac{x^{a+1}}{a+1} \\ \int \frac{1}{x}dx = \ln|x|}}{=} \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - 2\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \ln|x| + C = 2x^{\frac{1}{2}} - \frac{4}{5}x^{\frac{5}{2}} - \ln|x| + C.$$

Example 9

Compute the integral

$$\int \left(e^{2x+1} + \frac{1}{e^{2x+1}}\right) dx$$

☺Solution:

Using the property of integration, we have

$$\int \left(e^{2x+1} + \frac{1}{e^{2x+1}}\right) dx$$

$$= \int e^{2x+1} dx + \int e^{-2x-1} dx$$

$$\overset{\substack{\int e^x dx = e^x + C \\ \int f(ax+b)dx = \frac{1}{a}F(ax+b)+C}}{=} \frac{1}{2} e^{2x+1} - \frac{1}{2} e^{-2x-1} + C.$$

Example 10

Compute the integral

$$\int \frac{2x+3}{(x-4)^4}\,dx$$

☺IDEA:

We only know how to compute $\int \frac{1}{(x-4)^4}dx$ or $\int \frac{1}{(2x-5)^5}dx$. We have to rewrite the given integral so that we can apply the previous result!

☺Solution:

$$\int \frac{2x+3}{(x-4)^4}dx = \int \frac{\color{red}{2x-8}}{(x-4)^4}dx + \int \frac{11}{(x-4)^4}dx$$

$$= 2\int \frac{1}{(x-4)^3}dx + 11\int \frac{1}{(x-4)^4}dx$$

$$= 2\int (x-4)^{-3}dx + 11\int (x-4)^{-4}dx$$

$$= 2\frac{(x-4)^{-3+1}}{-3+1} + 11\frac{(x-4)^{-4+1}}{-4+1} + C$$

$$= -(x-4)^{-2} - \frac{11}{3}(x-4)^{-3} + C.$$

Some useful algebraic tricks in integration

In this section, we will introduce some useful techniques in integration. Some of them involve the materials that you have learnt in MA1200.

Example 11 (Rationalization)

Compute the integral

$$\int \frac{x(x-3)}{\sqrt{x}-\sqrt{3}}\,dx$$

☺Solution:

$$\int \frac{x(x-3)}{\sqrt{x}-\sqrt{3}}\,dx = \int \frac{x(x-3)}{\sqrt{x}-\sqrt{3}}\left(\frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}}\right)dx = \int \frac{x(x-3)(\sqrt{x}+\sqrt{3})}{x-3}\,dx$$

$$= \int x(\sqrt{x}+\sqrt{3})dx = \int x^{\frac{3}{2}}dx + \sqrt{3}\int x dx$$

$$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \sqrt{3}\left(\frac{x^2}{2}\right) + C = \frac{2}{5}x^{\frac{5}{2}} + \frac{\sqrt{3}}{2}x^2 + C.$$

Integration of functions involving trigonometric functions

The *product-to-sum formula* and *compound angle formula* are very useful in computing the integral which the integrand consists of a product of some trigonometric functions. Both formulae aim to express the integrand into the sum of single trigonometric functions so that one can carry the integration term-by-term.

Product to sum formula and Compound angle formula

1. $\cos A \cos B = \dfrac{1}{2}[\cos(A+B) + \cos(A-B)]$
2. $\sin A \sin B = -\dfrac{1}{2}[\cos(A+B) - \cos(A-B)]$
3. $\sin A \cos B = \dfrac{1}{2}[\sin(A+B) + \sin(A-B)]$
4. $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
5. $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

Example 12

Compute the integral

$$\int \sin 8x \sin 3x \, dx$$

☺Solution:

Recall that there is no product rule in integration, one has to rewrite the expression into sum so that the property of integration can be applied.

Using product-to-sum formula, we get

$$\int \sin 8x \sin 3x \, dx = \int -\frac{1}{2}[\cos(8x+3x) - \cos(8x-3x)]dx$$

$$= -\frac{1}{2}\int \cos 11x \, dx + \frac{1}{2}\int \cos 5x \, dx$$

$$\overset{\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b)+C}{\cong} -\frac{1}{22}\sin 11x + \frac{1}{10}\sin 5x + C.$$

Example 13

Compute the integral

$$\int \cos^2 3x\, dx.$$

☺Solution:

Using product to sum formula, we have

$$\int \cos^2 3x\, dx = \int (\cos 3x)(\cos 3x) dx$$

$$= \int \frac{1}{2}[\cos(3x+3x)+\cos(3x-3x)]dx = \frac{1}{2}\int \cos 6x\, dx + \frac{1}{2}\int 1 dx.$$

$$\overset{\int \cos(ax+b)dx=\frac{1}{a}\sin(ax+b)+C}{\cong} \frac{1}{2}\left(\frac{\sin 6x}{6}\right)+\frac{1}{2}x+C = \frac{1}{12}\sin 6x + \frac{1}{2}x + C.$$

Use of Partial Fractions

The method of partial fraction allows us to split the function into sum of simpler rational fractions so that we can do the calculus easily. In MA1200, we have seen how this method works in differentiation. In fact, we can apply the similar method when we encounter the integral involving rational functions so that we can compute the integral easily.

(*We will discuss this method in more detail in later chapter)

Example 14 (Use of Partial Fractions)

(a) Decompose the function $f(x) = \frac{1}{x(2x-1)}$ into partial fractions.

(b) Hence, compute the integral

$$\int \frac{1}{x(2x-1)}\,dx.$$

☺Solution of (a)

Using method of partial fraction, we consider the following decomposition:

$$\frac{1}{x(2x-1)} = \frac{A}{x} + \frac{B}{2x-1} \Rightarrow A(2x-1) + Bx = 1.$$

Substitute $x = \frac{1}{2}$ into the equation on R.H.S., we get $\frac{1}{2}B = 1 \Rightarrow B = 2$.

Substitute $x = 0$ into the same equation, we get $-A = 1 \Rightarrow A = -1$.

Therefore $\frac{1}{x(2x-1)} = -\frac{1}{x} + \frac{2}{2x-1}$.

☺Solution of (b)

Using the result of (a) and the fact that $\int \frac{1}{x} dx = \ln|x| + C$, we have

$$\begin{aligned}\int \frac{1}{x(2x-1)} dx &= -\int \frac{1}{x} dx + 2\int \frac{1}{2x-1} dx = -\ln|x| + \frac{2}{2}\ln|2x-1| + C \\ &= -\ln|x| + \ln|2x-1| + C.\end{aligned}$$

Example 15 (Important)

Using the fact that $\int \frac{1}{1+x^2}dx = \tan^{-1}x + C$ (since $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$), compute the integral

$(a)\ \int \frac{1}{1+16x^2}dx$

$(b)\ \int \frac{3}{4+x^2}dx$

$(c)\ \int \frac{1}{2x^2+4x+4}dx$

☺Solution:

(a) Note that

$$\int \frac{1}{1+16x^2}dx = \int \frac{1}{1+(4x)^2}dx \overset{\int \frac{1}{1+y^2}dy = \tan^{-1}y}{\cong} \frac{\tan^{-1}4x}{4} + C$$

(b) IDEA: One needs to rewrite the integral into the form $\int \frac{C}{(1+y^2)}dy$.

$$\int \frac{3}{4+x^2}dx = \int \frac{3}{4}\left(\frac{1}{1+\frac{x^2}{4}}\right)dx = \frac{3}{4}\int \frac{1}{1+\left(\frac{x}{2}\right)^2}dx = \frac{3}{4}\left(\frac{\tan^{-1}\frac{x}{2}}{\frac{1}{2}}\right)+C$$

$$= \frac{3}{2}\tan^{-1}\frac{x}{2}+C.$$

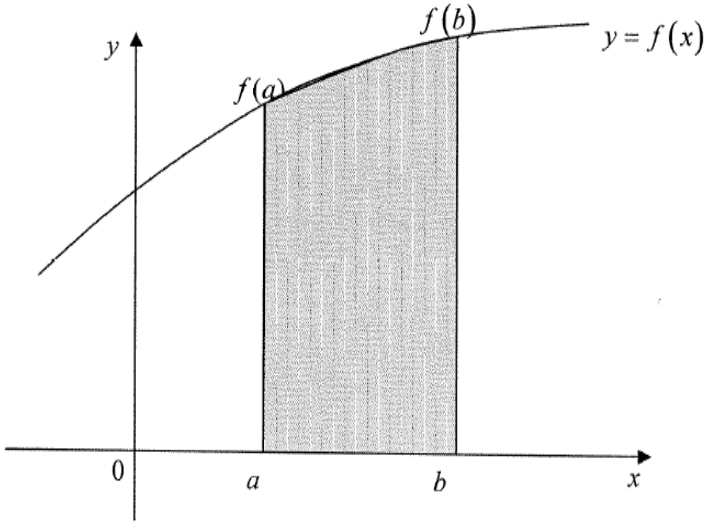
(c) IDEA: One needs to use "completing square" technique to rewrite the integral into the form $\int \frac{C}{(1+y^2)}dy$ in order to apply the given result.

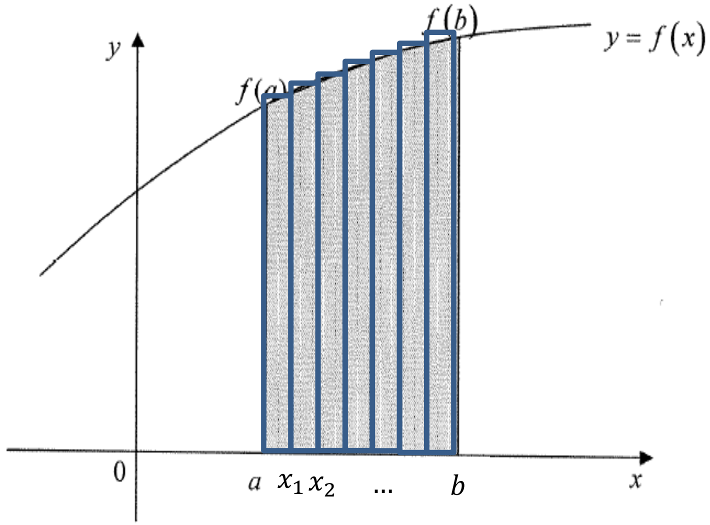
$$\int \frac{1}{2x^2+4x+4}dx = \int \frac{1}{2(x^2+2x)+4}dx = \int \frac{1}{2(x^2+2x+1-1)+4}dx$$

$$= \int \frac{1}{2+2(x+1)^2}dx = \frac{1}{2}\int \frac{1}{1+(x+1)^2}dx = \frac{1}{2}\tan^{-1}(x+1)+C.$$

Definite Integral

Question: How do we measure the area of the shaded region shown below?





We cut the region into n pieces (with equal base length) and each small piece is approximated by a rectangle:

Area of rectangles $= \sum_{i=1}^{n} \underbrace{\frac{b-a}{n}}_{\text{base}} \times \underbrace{f(z_i)}_{\text{height}}$

where z_i is some values between x_{i-1} and x_i.

When n is getting larger (i.e., $n \to \infty$), the total area of the rectangle gives a good approximation to the area of shaded region A. So we expect that

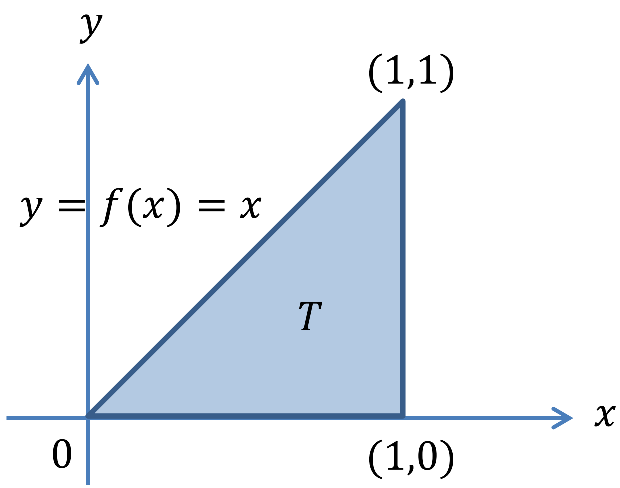
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{b-a}{n} f(z_i) \overset{\text{let } \Delta x = \frac{b-a}{n}}{\triangleq} \lim_{n\to\infty} \sum_{i=1}^{n} f(z_i)\Delta x.$$

We define the *definite integral* of $f(x)$ from $x = a$ to $x = b$ to be the area of shaded region (the area of the region under the graph of $y = f(x)$)

$$A = \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(z_i)\,\Delta x.$$

- Different from the indefinite integral $\int f(x)\,dx$, the $\int_a^b f(x)dx$ returns an unique number instead of a function.
- Conceptually, there is no relationship between indefinite and definite integrals. However, there is a little relationship between the two integrals in terms of computation (will be explored later).

As a verification, we compute the area of the following region using the above approach:



By geometric argument, the area of the triangle T is $\frac{1}{2}(1 \times 1) = \frac{1}{2}$.

We first cut divide the region vertically into n regions with equal width (i.e., width $= \frac{1}{n}$).

Next, we consider the i^{th} piece of the region. We approximate its area by a rectangle with width $\frac{1}{n}$ and the height $f\left(\frac{i-1}{n}\right) = \frac{i-1}{n}$.

Then the approximate area of the region is then given by

$$\text{Area} \approx \sum_{i=1}^{n} \left(\frac{i-1}{n} \times \frac{1}{n}\right) = \frac{1}{n^2} \sum_{i=1}^{n} (i-1) = \frac{1}{n^2} [0 + 1 + 2 + \cdots + (n-1)]$$

$$= \frac{1}{n^2} \frac{\big(0 + (n-1)\big) \times n}{2} = \frac{n^2 - n}{2n^2}.$$

Take $n \to \infty$, the approximate area becomes

$$\text{Area} = \lim_{n\to\infty} \frac{n^2 - n}{2n^2} \overset{\substack{\text{divide both sides} \\ \text{by } n^2}}{\cong} \lim_{n\to\infty} \frac{1 - \frac{1}{n}}{2} = \frac{1}{2}$$

which agrees with the result obtained by geometric argument.

Remarks about the definite integral

- The interval $[a, b]$ is called the *range of integration*.
- The numbers a and b are called the *lower* and *upper limits* of integration respectively.
- In fact, the definite integral gives the signed area of region under the curve $y = f(x)$. More precisely,
  - ✓ The integral is *positive* if the entire curve is *above* the x-axis (i.e. $f(x) \geq 0$)
  - ✓ The integral is *negative* if the entire curve is *below* the x-axis (i.e. $f(x) \leq 0$)

  In general, if we wish to find the area of the region under curve $y = f(x)$, we shall use the following modified formula:

$$\text{Area} = \int_a^b |f(x)| dx,$$

  Where $|f(x)|$ is the *absolute value* of $f(x)$.

How to compute the definite integral in general?

In general, it is hard to compute the definite integral using the definition:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(z_i)\Delta x.$$

In practice, people compute the value of definite integral using the following theorem:

Fundamental Theorem of Calculus

If $f(x)$ is continuous on the interval $[a, b]$, then

$$\boxed{\int_a^b f(x)dx = \underbrace{F(b) - F(a)}_{\text{denoted by } F(x)|_a^b \text{ or } [F(x)]_a^b}}$$

where $F(x) = \int f(x)dx$.

The proof of this theorem is presented in Appendix B.

Procedure of computing definite integral

Using this theorem, one can compute the definite integral $\int_a^b f(x)dx$ as follows:

Step 1: Find $F(x)$ by computing the indefinite integral $F(x) = \int f(x)dx$. (You may ignore the arbitrarily constant C since the constant does not affect the result.)

Step 2: Compute the integral $\int_a^b f(x)dx$ using $\int_a^b f(x)dx = F(b) - F(a)$.

Example 16

Compute

$$\int_0^3 x^3 dx \quad \text{and} \quad \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin 3x \, dx.$$

☺Solution:

Step 1: Compute the indefinite integrals

For x^3, we have

$$F(x) = \int x^3 dx \overset{\int x^a dx = \frac{x^{a+1}}{a+1}}{\cong} \frac{x^4}{4}.$$

For $\sin 3x$, we have

$$G(x) = \int \sin 3x \, dx \overset{\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b)}{\cong} -\frac{1}{3}\cos 3x.$$

Step 2: Calculate the integrals

$$\int_0^3 x^3 dx = \frac{x^4}{4}\bigg|_0^3 = \frac{3^4}{4} - \frac{0^4}{4} = \frac{81}{4}, \qquad \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin 3x \, dx = -\frac{1}{3}\cos 3x\bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -\frac{1}{3\sqrt{2}}.$$

Example 17

Compute

$$\int_{-1}^{1} \frac{1}{2+6x^2}\,dx.$$

☺Solution:

Step 1: Compute the indefinite integral

$$F(x) = \int \frac{1}{2+6x^2}\,dx = \frac{1}{2}\int \frac{1}{1+3x^2}\,dx = \frac{1}{2}\int \frac{1}{1+\left(\sqrt{3}x\right)^2}\,dx = \frac{1}{2\sqrt{3}}\tan^{-1}\sqrt{3}x.$$

Step 2: Compute the integral

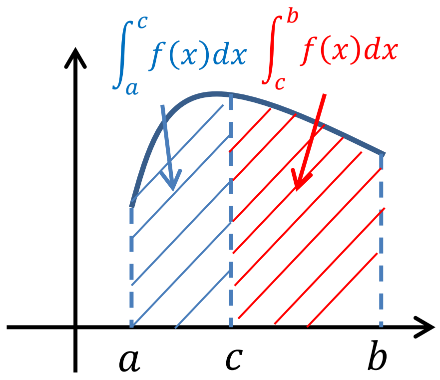
$$\int_{-1}^{1} \frac{1}{2+6x^2}\,dx = \frac{1}{2\sqrt{3}}\tan^{-1}\sqrt{3}x\Big|_{-1}^{1} = \frac{1}{2\sqrt{3}}\left(\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right) = \frac{\pi}{3\sqrt{3}}.$$

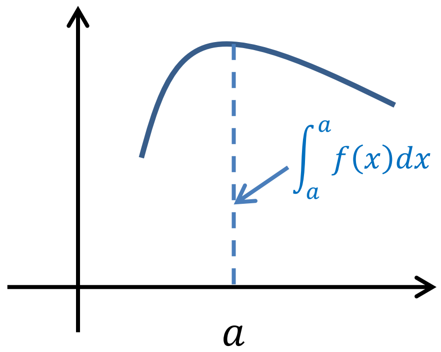
Properties of Definite integral

In this section, we summarize some formulae of definite integral which are useful in computing the definite integrals. All of them can be derived from the fundamental theorem of calculus. Here, $f(x)$ and $g(x)$ are two continuous functions.

1. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$

2. $\int_a^a f(x)dx = 0.$





Example 18

We consider the function defined as $f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ x^2 & \text{if } x > 1 \end{cases}.$

Compute the integrals

$$\int_0^2 f(x)dx.$$

☺Solution:

Using the above property, we can compute the integral as

$$\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx = \int_0^1 \underbrace{x}_{\substack{f(x)=x \\ \text{for } 0 \le x \le 1}} dx + \int_1^2 \underbrace{x^2}_{\substack{f(x)=x^2 \\ \text{for } 1 < x \le 2}} dx$$

$$= \frac{x^2}{2}\bigg|_0^1 + \frac{x^3}{3}\bigg|_1^2 = \frac{17}{6}.$$

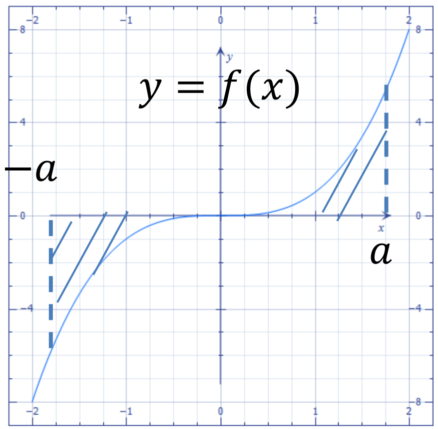
Other properties of definite integrals

3. $\int_b^a f(x)dx = -\int_a^b f(x)dx.$

4. $\int_a^b [c_1 f(x) + c_2 g(x)]dx = c_1 \int_a^b f(x)dx + c_2 \int_a^b g(x)dx.$

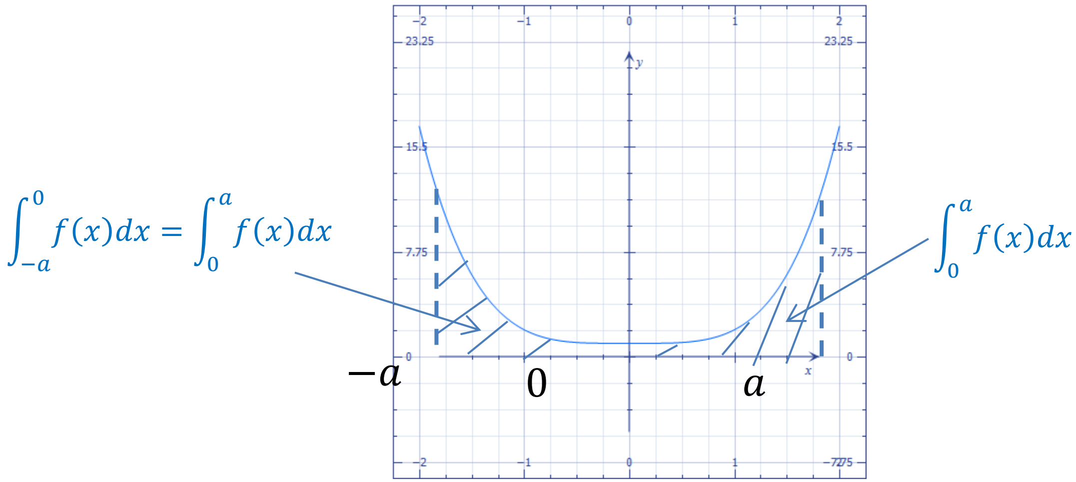
5. If $f(x)$ is an odd function (i.e., $f(-x) = -f(x)$), then

$$\int_{-a}^{a} f(x)dx = 0.$$



6. If $f(x)$ is an even function (i.e., $f(-x) = f(x)$), then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$



7. If $f(x) \leq g(x) \leq h(x)$ for all $x \in [a, b]$, then

$$\int_{a}^{b} f(x)dx \leq \int_{a}^{b} g(x)dx \leq \int_{a}^{b} h(x)dx.$$

Example 19 (Integral of odd functions)

Compute the integral

$$\int_{-\pi}^{\pi} \frac{\sin(x^3 + x)}{x^4 + 3}\, dx.$$

☺Solution:

Let $f(x) = \frac{\sin(x^3+x)}{x^4+3}$. We observe that

$$f(-x) = \frac{\sin[(-x)^3 + (-x)]}{(-x)^4 + 3} = \frac{\sin(-x^3 - x)}{x^4 + 3} \overset{\substack{\sin(-\theta) \\ =-\sin\theta}}{\cong} -\frac{\sin(x^3 + x)}{x^4 + 3} = -f(x).$$

Thus it is an odd function and we can conclude from the property of definite integral that (with $a = \pi$) that

$$\int_{-\pi}^{\pi} \frac{\sin(x^3 + x)}{x^4 + 3}\, dx = 0,$$

Integration of absolute value function

☺IDEA:
We need to remove the absolute value sign $|\cdot|$ in the integral first!!

Example 20

Compute the integral

$$\int_0^4 |x-3|dx, \qquad \int_0^2 e^{1+|x-1|}dx$$

☺Solution:

Recall that $|y| = \begin{cases} y & \text{if } y \geq 0 \\ -y & \text{if } y < 0 \end{cases}$, one can substitute $y = x - 3$ and obtain

$$|x-3| = \begin{cases} x-3 & \text{if } x-3 \geq 0 \\ -(x-3) & \text{if } x-3 < 0 \end{cases} = \begin{cases} x-3 & \text{if } x \geq 3 \\ 3-x & \text{if } x < 3 \end{cases}.$$

Then the integral can be computed as

$$\int_0^4 |x-3|dx = \int_0^3 |x-3|dx + \int_3^4 (|x-3|)dx = \int_0^3 (3-x)dx + \int_3^4 (x-3)dx$$

$$= \left(3x - \frac{x^2}{2}\right)\Big|_0^3 + \left(\frac{x^2}{2} - 3x\right)\Big|_3^4 = \cdots = 5.$$

For the second integral, one can use the similar method and obtain

$|x-1| = \begin{cases} x-1 & \text{if } x \geq 1 \\ -(x-1) = 1-x & \text{if } x < 1 \end{cases}$, so we have

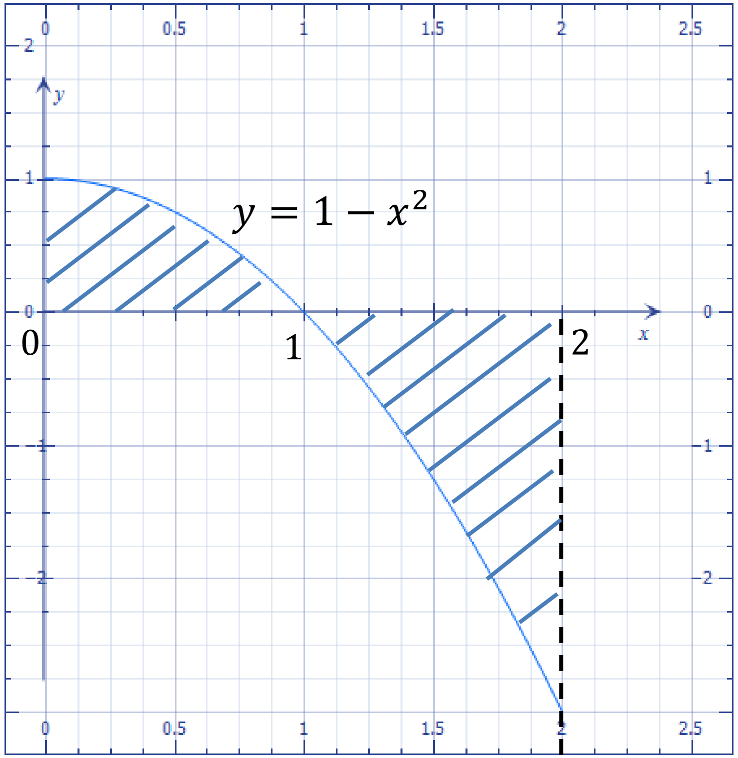
$$\int_0^2 e^{1+|x-1|}dx = \int_0^1 e^{1+|x-1|}dx + \int_1^2 e^{1+|x-1|}dx$$

$$= \int_0^1 e^{1+(1-x)}dx + \int_1^2 e^{1+(x-1)}dx = \int_0^1 e^{2-x}dx + \int_1^2 e^x dx$$

$$= (-e^{2-x})|_0^1 + e^x|_1^2 = 2e^2 - 2e.$$

Example 21

Using definite integral, find the area of the region bounded by the curve $y = f(x) = 1 - x^2$, x-axis, $x = 0$ and $x = 2$.



☺Solution:

The area of the shaded region is given by

$$|1-x^2| = \begin{cases} 1-x^2 & \text{if } 1-x^2 \geq 0 \\ -(1-x^2) & \text{if } 1-x^2 < 0 \end{cases}$$

$$= \begin{cases} 1-x^2 & \text{if } 0 \leq x \leq 1 \\ x^2-1 & \text{if } 1 < x \leq 2 \end{cases}.$$

$$\text{Area} = \int_0^2 |1-x^2|dx$$

$$= \int_0^1 (1-x^2)dx + \int_1^2 (x^2-1)dx$$

$$= \left[x-\frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3}-x\right]_1^2$$

$$= 1-\frac{1}{3}+\left(\frac{2^3}{3}-2-\frac{1^3}{3}+1\right)$$

$$= 2.$$

Example 22 (Harder Example)

Compute the derivative

$$\frac{d}{dx}\left(\int_{1}^{2x} e^{3y^2-1}dy\right)$$

☺Solution

Let $F(y)=\int e^{3y^2-1}dy$, then by fundamental theorem of calculus, we have

$$\int_{1}^{2x} e^{3y^2-1}dy = F(y)|_{1}^{2x} = F(2x)-F(1).$$

Using the properties of differentiation and the fact that $\frac{d}{dy}F(y)=\frac{d}{dy}\int e^{3y^2-1}dy=e^{3y^2-1}$, we finally get

$$\frac{d}{dx}\left(\int_{1}^{2x} e^{3y^2-1}dy\right)=\frac{d}{dx}F(2x)-\frac{d}{dx}F(1)=\frac{dF(2x)}{d(2x)}\frac{d(2x)}{dx}-0=e^{3(2x)^2-1}(2)$$

$$=2e^{12x^2-1}.$$

Review Example 1

Compute the integrals

$$\int \frac{1}{(3x-5)^5}\,dx\,, \qquad \int \frac{e^{5x+3}-e^{x-1}}{e^{x+1}}\,dx\,.$$

☺Solution:

$$\int \frac{1}{(3x-5)^5}\,dx = \int (3x-5)^{-5}dx = \frac{1}{3}\frac{(3x-5)^{-5+1}}{-5+1} + C = -\frac{(3x-5)^{-4}}{12} + C$$

$$\int \frac{e^{5x+3}-e^{x-1}}{e^{x+1}}\,dx = \int \frac{e^{5x+3}}{e^{x+1}}\,dx - \int \frac{e^{x-1}}{e^{x+1}}\,dx = \int e^{4x+2}dx - \int e^{-2}dx$$

$$= \int e^{4x+2}dx - e^{-2}\int 1dx = \frac{1}{4}e^{4x+2} - e^{-2}x + C$$

Review Example 2

Compute the integral

$$\int \sin^3 2x\, dx.$$

☺Solution:

Using the product-to-sum formula, we have

$\sin^3 2x = (\sin 2x)(\sin 2x)(\sin 2x)$

$$\overset{\substack{\sin A \sin B \\ = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]}}{=} -\frac{1}{2}\left[\cos(2x+2x) - \overbrace{\cos(2x-2x)}^{\cos 0 = 1}\right](\sin 2x)$$

$$= -\frac{1}{2}(\cos 4x - 1)(\sin 2x) = -\frac{1}{2}\cos 4x \sin 2x + \frac{1}{2}\sin 2x$$

$$\overset{\substack{\sin A \cos B \\ = \frac{1}{2}[\sin(A+B) + \sin(A-B)]}}{=} -\frac{1}{4}[\sin(2x+4x) + \sin(2x-4x)] + \frac{1}{2}\sin 2x$$

$$= -\frac{1}{4}\sin 6x - \frac{1}{4}\sin(-2x) + \frac{1}{2}\sin(2x) \overset{\sin(-\theta)=-\sin\theta}{\cong} -\frac{1}{4}\sin 6x + \frac{3}{4}\sin 2x.$$

Thus we have

$$\int \sin^3 2x\, dx = -\frac{1}{4}\int \sin 6x\, dx + \frac{3}{4}\int \sin 2x\, dx$$

$$= -\frac{1}{4}\left(\frac{-\cos 6x}{6}\right) + \frac{3}{4}\left(\frac{-\cos 2x}{2}\right) + C$$

$$= \frac{\cos 6x}{24} - \frac{3\cos 2x}{8} + C.$$

Review Example 3

Compute the integrals

$$\int_0^3 e^{2x+1}\,dx, \qquad \int_0^1 \frac{4x+5}{\sqrt{2x+1}}\,dx$$

☺Solution

For the first integral, we have

$$\int_0^3 e^{2x+1}\,dx \overset{\int e^{2x+1}dx = \frac{1}{2}e^{2x+1}}{\cong} \left[\frac{1}{2}e^{2x+1}\right]_0^3 = \frac{1}{2}e^{2(3)+1} - \frac{1}{2}e^{2(0)+1} = \frac{1}{2}(e^7 - e).$$

For the second integral, we need to split the integral as

$$\int_0^1 \frac{4x+5}{\sqrt{2x+1}}\,dx = \int_0^1 \frac{4x+2}{\sqrt{2x+1}}\,dx + \int_0^1 \frac{3}{\sqrt{2x+1}}\,dx$$

$$= 2\int_0^1 \sqrt{2x+1}\,dx + 3\int_0^1 \frac{1}{\sqrt{2x+1}}\,dx$$

$$= 2\int_0^1 (2x+1)^{\frac{1}{2}}dx + 3\int_0^1 (2x+1)^{-\frac{1}{2}}dx$$

$$= 2\left[\frac{1}{2}\frac{(2x+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^1 + 3\left[\frac{1}{2}\frac{(2x+1)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_0^1$$

$$= 5\sqrt{3} - \frac{11}{3}.$$

Review Example 4

Compute the integral

$$\int_0^{2\pi} |\sin x| dx$$

☺Solution:

Using the graph of $y = \sin x$ for $0 \le x \le 2\pi$, we get

$$|\sin x| = \begin{cases} \sin x & \text{if } \sin x \ge 0 \\ -\sin x & \text{if } \sin x < 0 \end{cases} = \begin{cases} \sin x & \text{if } 0 \le x \le \pi \\ -\sin x & \text{if } \pi < x < 2\pi \end{cases}.$$

Hence, we calculate the integral as

$$\int_0^{2\pi} |\sin x| dx = \underbrace{\int_0^{\pi} \sin x \, dx}_{|\sin x| = \sin x} + \underbrace{\int_{\pi}^{2\pi} (-\sin x) dx}_{|\sin x| = -\sin x}$$

$$\overset{\int \sin x dx = -\cos x}{\widehat{=}} \quad [-\cos x]_0^{\pi} - [-\cos x]_{\pi}^{2\pi} = 4.$$

Review Example 5 (A bit harder problem)

Let $f(x)$ be a continuous function

(a) Compute the derivative

$$\frac{d}{dx}\int_{2x}^{x^2+1} f(y)dy$$

(b) Show that

$$(b-a)\int_0^1 f[a+(b-a)x]dx = \int_a^b f(x)dx$$

☺Solution of (a)

Let $F(y) = \int f(y)dy,$ then by fundamental theorem of calculus we get

$$\int_{2x}^{x^2+1} f(y)dy = F(x^2+1) - F(2x).$$

Using the chain rule and the fact that $\frac{d}{dy}F(y) = \frac{d}{dy}\int f(y)dy = f(y),$ we have

$$\frac{d}{dx}\int_{2x}^{x^2+1} f(y)dy = \frac{d}{dx}F(x^2+1) - \frac{d}{dx}F(2x)$$

$$= \frac{dF(x^2+1)}{d(x^2+1)}\frac{d(x^2+1)}{dx} - \frac{dF(2x)}{d(2x)}\frac{d(2x)}{dx}$$

$$\overset{\text{Take } y=x^2+1 \text{ or } y=2x}{=} \quad f(x^2+1)\,(2x) - f(2x)(2)$$

$= 2xf(x^2+1) - 2f(2x).$

☺Solution of (b)

Note that $\int_a^b f(x)dx = F(b) - F(a),$ so we just need to show

$$(b-a)\int_0^1 f[a+(b-a)x]dx = F(b) - F(a).$$

To see this, we use the fact that $\int f(cx+d)dx = \frac{1}{c}F(cx+d)$ for any constant c and $d,$ we have

$$(b-a)\int_0^1 f\left[a+\underbrace{(b-a)}_{=c}x\right]dx$$

$$\overset{\substack{c=b-a\\ d=a\\ \frown}}{\equiv}\ (b-a)\left[\frac{1}{b-a}F(a+(b-a)x)\right]_0^1$$

$$=(b-a)\left[\frac{1}{b-a}F(a+(b-a)(1))-\frac{1}{b-a}F\big(a+(b-a)(0)\big)\right]$$

$$=F(b)-F(a)$$

$$=\int_a^b f(x)dx.$$

Appendix A – Properties of derivatives

Let $f(x)$ and $g(x)$ be two differentiable functions, then we have

1. $\dfrac{d}{dx}kf(x) = k\dfrac{d}{dx}f(x)$

2. $\dfrac{d}{dx}[f(x) \pm g(x)] = \dfrac{d}{dx}f(x) \pm \dfrac{d}{dx}g(x).$

3. $\dfrac{d}{dx}[f(x)g(x)] = g(x)\dfrac{d}{dx}f(x) + f(x)\dfrac{d}{dx}g(x).$

4. $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)\dfrac{d}{dx}f(x) - f(x)\dfrac{d}{dx}g(x)}{[g(x)]^2}.$

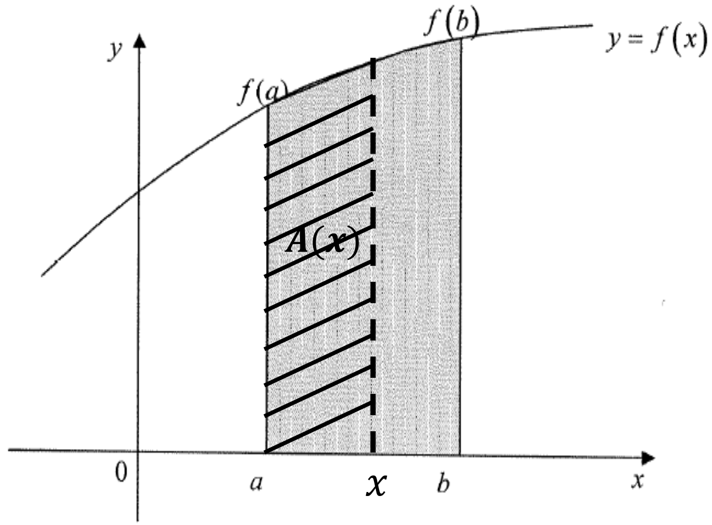
5. If $f(x)$ and $u(x)$ be two differentiable functions, then

$$\frac{d}{dx}f\big(u(x)\big) = \frac{df(u(x))}{du(x)}\frac{du(x)}{dx} \ \left(\text{or } \frac{df}{du}\frac{du}{dx}\right)$$

Appendix B – Proof of Fundamental Theorem of Calculus

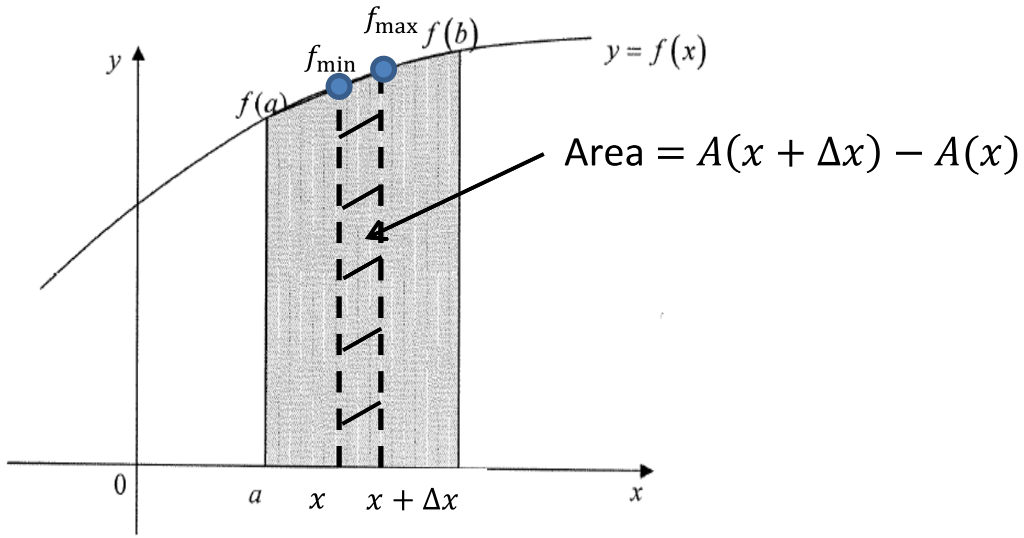
To start with, we define the function $A: [a, b] \to [0, \infty)$ as

$$A(x) = \text{Area under the curve } y = f(x) \text{ between } a \text{ and } x$$



- $A(b)$ is the area of the entire shaded region.

Next, we consider the area difference $A(x + \Delta x) - A(x)$. Using the following graph we observed that



$$f_{\min}\Delta x \leq A(x + \Delta x) - A(x) \leq f_{\max}\Delta x$$

$$\Rightarrow f_{\min} \leq \frac{A(x + \Delta x) - A(x)}{\Delta x} \leq f_{\max}.$$

where $f_{\min}$ and $f_{\max}$ are the minimum and maximum values of the function within x and $x + \Delta x$ respectively.

Taking $\Delta x \to 0$, we get

$$f(x) = \lim_{\Delta x \to 0} f_{\min} \le \underbrace{\lim_{\Delta x \to 0} \frac{A(x+\Delta x) - A(x)}{\Delta x}}_{=\frac{dA}{dx}} \le \lim_{\Delta x \to 0} f_{\max} = f(x)$$

By sandwich theorem, we get $\frac{dA}{dx} = f(x)$.

Next, we integrate both side with respect to x, we have

$$\int \frac{dA}{dx} dx = \underbrace{\int f(x)dx}_{\text{call this } F(x)} \Rightarrow A(x) + C = F(x) \Rightarrow A(x) = F(x) - C.$$

To find the value of C, note that $A(a) = 0$, we then get $A(a) = F(a) - C \Rightarrow C = F(a)$. Therefore, we have $A(x) = F(x) - F(a)$ and

$$\int_a^b f(x)dx \overset{\text{by definition}}{\triangleq} A(b) = F(b) - F(a) \quad \text{where } F(x) = \int f(x)dx.$$