Problem 1 (Method of substitution)

Compute the following integral using method of substitution:

(a) $$\int \frac{e^{1+\frac{1}{x^2}}}{x^3}dx$$

(c) $$\int x^{11}\sqrt{1+x^4}dx$$

(e) $$\int \sin 2x \sqrt{\cos x}\, dx$$

(g) $$\int_1^2 xe^{x^2-1}dx$$

(i) $$\int \frac{3x+2}{x^2+4}dx$$

(k) $$\int \frac{4x}{3x^2+6x+19}dx$$

(a). 1°. $y = 1+\frac{1}{x^2} \Rightarrow \frac{dy}{dx} = -\frac{2}{x^3} \Rightarrow \underline{dx = -\frac{x^3}{2}dy}$.

2°. $\int \frac{e^{1+\frac{1}{x^2}}}{\cancel{x^3}}\left(-\frac{\cancel{x^3}}{2}dy\right)$

$= -\frac{1}{2}\int e^y\, dy = -\frac{1}{2}e^y + C = -\frac{1}{2}e^{1+\frac{1}{x^2}} + C$.

(c) 1°. $y = 1+x^4 \Rightarrow \frac{dy}{dx} = 4x^3 \Rightarrow dx = \frac{1}{4x^3}dy$.

$x^4 = y-1$

2°. $\int x^{\cancel{11}}\sqrt{1+x^4}\left(\frac{1}{4\cancel{x^3}}dy\right) = \frac{1}{4}\int \underline{x^8\sqrt{1+x^4}}\, dy$

$= \frac{1}{4}\int \boxed{(y-1)^2}\, y^{\frac{1}{2}}dy$ $\quad (y^2-2y+1)\cdot y^{\frac{1}{2}}$

$= \frac{1}{4}\int \underline{y^{\frac{5}{2}} - 2y^{\frac{3}{2}} + y^{\frac{1}{2}}}\, dy$

$= \frac{1}{14}(1+x^4)^{\frac{7}{2}} - \frac{1}{5}(1+x^4)^{\frac{5}{2}} + \frac{1}{6}(1+x^4)^{\frac{3}{2}} + C.$

(e) $$\int \sin 2x \sqrt{\cos x}\, dx$$

1°. $y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x \Rightarrow dx = -\frac{1}{\sin x}dy$.

$\sin 2x = 2\sin x\cos x$.

2° $\int \sin 2x\sqrt{\cos x}\left(-\frac{1}{\sin x}dy\right) = -2\int \underline{\cos x\sqrt{\cos x}}\, dy$.

$= -2\int y^{\frac{3}{2}}dy = -2\cdot\frac{1}{\frac{5}{2}}y^{\frac{5}{2}} + C$.

$= -\frac{4}{5}\cos^{\frac{5}{2}}x + C.$

(g) $\int_{1}^{2} x e^{x^2-1} dx$

$1^\circ.$ $y = x^2 - 1 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow dx = \frac{1}{2x} dy.$

$2^\circ.$ when $x=1$, $y=0$; $x=2$, $y=3$.

$$\int_0^3 \cancel{x} e^{x^2-1} \left(\frac{1}{2\cancel{x}} dy\right) = \frac{1}{2}\int_0^3 e^y dy$$

$$= \frac{1}{2} e^y \Big|_0^3$$

$$= \frac{1}{2}(e^3 - 1).$$

(i) $\int \frac{3x+2}{x^2+4} dx$

$1^\circ.$ $y = x^2 + 4 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow dx = \frac{1}{2x} dy.$

$2^\circ.$ $\int \frac{3x+2}{x^2+4}\left(\frac{1}{2x} dy\right) = \int \frac{3\cancel{x}}{x^2+4}\left(\frac{1}{2\cancel{x}} dy\right) + \int \frac{2}{x^2+4} dx.$

$$\boxed{\frac{1}{1+x^2} \to \tan^{-1} x.}$$

$$= \frac{3}{2}\int \frac{1}{y} dy + 2\int \frac{1}{x^2+4} dx$$

$$= \frac{3}{2}\ln|y| + \frac{2}{4}\int \frac{1}{1+\left(\frac{x}{2}\right)^2} dx.$$

$$= \frac{3}{2}\ln|x^2+4| + \frac{2}{4}\cdot\frac{1}{\frac{1}{2}}\tan^{-1}\left(\frac{x}{2}\right) + C$$

$$= \frac{3}{2}\ln|x^2+4| + \tan^{-1}\left(\frac{x}{2}\right) + C.$$

(k) $\displaystyle\int \frac{4x}{3x^2+6x+19}\,dx$

1°. $y = 3x^2+6x+19 \Rightarrow \frac{dy}{dx} = 6x+6 \Rightarrow dx = \frac{1}{6x+6}\,dy$

2°. $\displaystyle\int \frac{4x}{3x^2+6x+19}\left(\frac{1}{6x+6}\,dy\right)$

$$= \int \frac{\cancel{4x+4}^{\,2}}{3x^2+6x+19}\left(\frac{1}{\cancel{6x+6}_{\,3}}\,dy\right) - \int \frac{4}{3x^2+6x+19}\,dx.$$

$$= \frac{2}{3}\int \frac{1}{y}\,dy - 4\int \frac{1}{3(x+1)^2+16}\,dx.$$

$$\boxed{\frac{1}{1+x^2} \leftarrow \tan^{-1} x}$$

$$= \frac{2}{3}\ln|y| - \frac{4}{16}\int \frac{1}{1+\frac{3}{16}(x+1)^2}\,dx$$

$$= \frac{2}{3}\ln|y| - \frac{1}{4}\int \frac{1}{1+\left[\frac{\sqrt{3}}{4}(x+1)\right]^2}\,dx$$

$$= \frac{2}{3}\ln|3x^2+6x+19| - \frac{1}{4}\,\frac{1}{\frac{\sqrt{3}}{4}}\tan^{-1}\left(\frac{\sqrt{3}}{4}(x+1)\right)$$

$$= \frac{2}{3}\ln|3x^2+6x+19| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}}{4}x+\frac{\sqrt{3}}{4}\right) + C.$$

$\int u\,dv = uv - \int v\,du$

Problem 2 (Integration by parts)

Compute the following integrals using integration by parts:

(a) $\int xe^{-3x}dx$ (b) $\int_1^e \sqrt{x}\ln x\,dx$

(c) $\int x^2 \sin x\,dx$ (d) $\int x\sin^2 x\,dx$

(a). let $u = x$, $dv = e^{-3x}dx \Rightarrow v = \int e^{-3x}dx = -\frac{1}{3}e^{-3x}$.

$$\int xe^{-3x}dx = -\frac{1}{3}xe^{-3x} - \int -\frac{1}{3}e^{-3x}dx$$

$$= -\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}dx$$

$$= -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C.$$

(b). $u = \ln x$, $dv = \sqrt{x}\,dx \Rightarrow v = \int \sqrt{x}\,dx = \frac{2}{3}x^{\frac{3}{2}}$

$$\int_1^e \sqrt{x}\ln x\,dx = \frac{2}{3}\ln x \cdot x^{\frac{3}{2}}\Big|_1^e - \int_1^e \frac{2}{3}x^{\frac{3}{2}}\,d\ln x.$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln x\Big|_1^e - \frac{2}{3}\int_1^e x^{\frac{1}{2}}dx$$

$$= \frac{2}{3}x^{\frac{3}{2}}\ln x\Big|_1^e - \frac{2}{3}\cdot\frac{2}{3}x^{\frac{3}{2}}\Big|_1^e$$

$$= \frac{2}{9}e^{\frac{3}{2}} + \frac{4}{9}.$$

(c) $\int x^2 \sin x\,dx$

Let $u = x^2$. $dv = \sin x\,dx \Rightarrow v = \int \sin x\,dx = -\cos x$.

$$\int x^2 \sin x\,dx = -x^2\cos x - \left(\int -\cos x\,dx^2\right)$$

$$= -x^2\cos x + 2\int x\cos x\,dx$$

$u = x$

$dv = \cos x\,dx$, $v = \sin x$

$$-x^2\cos x + 2\left[x\sin x - \int \sin x\,dx\right].$$

$$-\cos x$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + C.$$

(d) $\int x\sin^2 x\,dx$

$$\boxed{\sin^2 x = -\frac{1}{2}[\cos 2x - 1]}$$

$$= -\frac{1}{2}\int x[\cos 2x - 1]\,dx$$

$$= -\frac{1}{2}\int x\cos 2x\,dx + \frac{1}{2}\int x\,dx.$$

$u = x$

$dv = \cos 2x\,dx,\ v = \frac{1}{2}\sin 2x$

$$-\frac{1}{2}\Big[\frac{1}{2}x\sin 2x - \int\frac{1}{2}\sin 2x\,dx. + \frac{1}{2}\int x\,dx$$

$$= -\frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + \frac{1}{4}x^2 + C$$