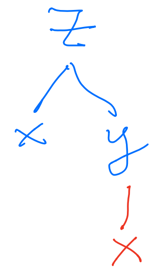
$Z = F(x) = x^3 + y^2 - 3xy = 0$



"y is a function of x" $y = f(x)$

but not in an explicit form

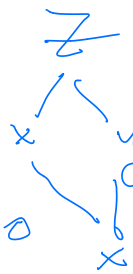
Eg: $y = x^3 + 3x + \sin x$

$Z = f(x, y) = 0$

The implicit function implies that this equation implicitly defines y as a function of x.

By Chain Rule

$$\frac{dz}{dx} = \left(\frac{\partial F}{\partial x}\right)_y \overbrace{\frac{dx}{dx}}^{1} + \left(\frac{\partial F}{\partial y}\right)_x \frac{dy}{dx} = 0$$



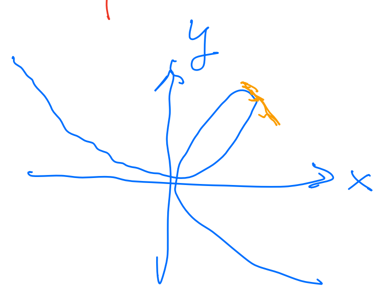
$$= \boxed{(3x^2 - 3y) \cdot 1 + (3y^2 - 3x)\frac{dy}{dx} = 0}$$

We can solve for the slope

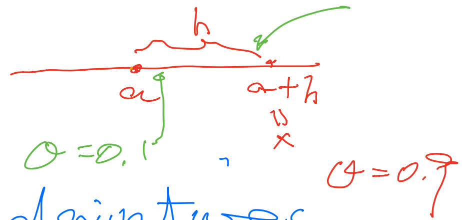
$$\frac{dy}{dx} = \frac{-(x^2 - y)}{(y^2 - x)}$$

Line tangent "slope"

Line tangent to the curve



Taylor Series



$f(x)$ with continuous derivatives

$f'(x),\ f''(x),\ \cdots\ f^{N-1}(x)$

$a \le x \le b$

$x = a+h$

$$f(x) = f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a)$$

$$+ \cdots + \frac{h^{N-1}}{(N-1)!} f^{(N-1)}(a) + R_N(h)$$

$10! = 10 \times 9 \times 8 \times \cdots \times 1$

$7! = 7 \times 6 \times 5 \cdots \times 1$

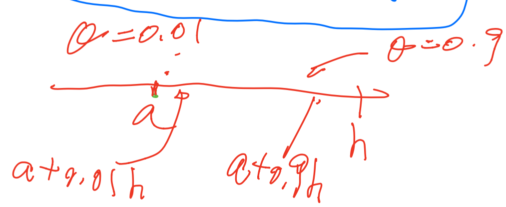
L.H.S. "Approximation" → R.H.S.

polynomial in h

$$\boxed{R_N(h) = \frac{h^N}{N!} f^{(N)}(a+\theta h) \qquad 0 < \theta < 1}$$

h^N → small

small h $\qquad h \ll 1$



$f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$\vdots$

$$\sin x = \sin(a+h) = \sin(30^\circ) + \frac{h}{1}\cos(30^\circ) + \frac{h^2}{2!}\left(-\sin(30^\circ)\right) + \cdots + R_N$$

$a = 30^\circ = \frac{\pi}{6}$

** Two Variables $f(x,y)$ at a point (a,b)

$$f(x,y) = f(a+h,\ b+k) = f(a,b) \longleftarrow r=0$$

$$\begin{cases} x = a+h \\ y = b+k \end{cases}$$

Linear Approx

$$+\frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right] \longleftarrow r=1$$

Quadratic

$$+\frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right] \longleftarrow r=2$$

$$+\frac{1}{3!}\left[1h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + 1k^3 f_{yyy}(a,b)\right] \longleftarrow r=3$$

$$+ \cdots + R_N(h,k)$$

$$= \sum_{r=0}^{N-1} \frac{1}{r!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^r f(a,b) + R_N(h,k)$$

1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$\sqrt{2\cdot(2.02)^3 + (2.97)^2}$$

$$\begin{cases} a = 2 \\ b = 3 \end{cases}$$

Let $f(x,y) = \sqrt{2x^3 + y^2}$

$$f(2,3) = \sqrt{2\times 8 + 9} = \sqrt{25} = 5$$

$$\frac{\partial f}{\partial x} = \frac{3x^2}{\sqrt{2x^3+y^2}} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{2x^3+y^2}}$$

$$f_x(2,3) = 12/5 \qquad f_y = 3/5$$

$h = \Delta x = 0.02$

$$\sqrt{2(2.02)^3 + (2.97)^2}$$

$= 5.0305$

$k = -0.03 = \Delta y$

$f(\overset{x}{2.02}, \overset{y}{2.97})$

$= f(2,3) + [f_x(2,3)(\overset{\Delta x}{0.02}) + f_y(2,3)(\overset{\Delta y}{-0.03})]$

$= 5 + \frac{12}{5}(0.02) + (\frac{3}{5})(-0.03)$

$= 5.03$

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Q: $z^3 - 2xz + y = 0$

Evaluate $z(x, y)$ at $(1,1)$?

z cannot explicitly in terms of x, y

$$\begin{cases} 3z^2 \frac{\partial z}{\partial x} - 2z - 2x\frac{\partial z}{\partial x} + 0 = 0 \\ 3z^2 \frac{\partial z}{\partial y} - 0 \cdot z - 2x\frac{\partial z}{\partial y} + 1 = 0 \end{cases}$$

At $\begin{cases} x=1 \\ y=1 \end{cases}$ we have $z=1$

$(1, 1, 1)$

$x = a$, $y = b$, z

$$\Rightarrow \begin{cases} \frac{\partial z}{\partial x} = \frac{2z}{3z^2 - 2x} \Big|_{\substack{x=1 \\ y=1}} = 2 \\ \frac{\partial z}{\partial y} = \frac{-1}{3z^2 - 2x} \Big|_{\substack{x=1 \\ y=1}} = -1 \end{cases}$$

$$Z(x,y) = Z(1,1) + \left[h \frac{\partial z}{\partial x}\Big|_{\substack{x=1\\y=1}} + k \frac{\partial z}{\partial y}\Big|_{\substack{x=1\\y=1}} \right]$$

$h = (x-a)$ 1

$k = (y-b)$ 1

$$= Z(1,1) + \left[(x-1)\frac{\partial z}{\partial x}\Big|_{\substack{x=1\\y=1}} + (y-1)\frac{\partial z}{\partial y}\Big|_{\substack{x=1\\y=1}} \right]$$

$$= 1 + 2(x-1) - 1(y-1) + \cdots$$