MA1201 Calculus and Basic Linear Algebra II **Problem Set 2 Techniques of Integration**

Problem 1 (Method of substitution)

Compute the following integral using method of substitution:

(b) $\int x^2 \sec(1-2x^3)\, dx$

let $y = 1-2x^3 \Rightarrow \frac{dy}{dx} = -6x^2 \Rightarrow dx = -\frac{1}{6x^2} dy$

$\int x^2 \sec(1-2x^3)\, dx = \int x^2 \sec(1-2x^3) \cdot \left(-\frac{1}{6x^2} dy\right) = \int -\frac{1}{6} \sec y \, dy$

$\overset{P21}{=} -\frac{1}{6} \ln|\sec y + \tan y| + C$

$= -\frac{1}{6} \ln|\sec(1-2x^3) + \tan(1-2x^3)| + C.$

(d) $\int x \cos^2(x^2)\, dx$

let $y = x^2 \Rightarrow \frac{dy}{dx} = 2x. \Rightarrow dx = \frac{1}{2x} dy$

$\int x \cos^2(x^2)\, dx = \int x \cos^2(x^2) \cdot \frac{1}{2x} dy = \frac{1}{2} \int \cos^2 y \, dy$

$= \frac{1}{2} \int \frac{1}{2} [\cos(y+y) + \cos(y-y)]\, dy = \frac{1}{4} \int (\cos(2y) + 1)\, dy$

$= \frac{1}{4} \cdot \left(\frac{1}{2} \sin(2y) + y\right) + C.$

$= \frac{1}{8} \sin(2x^2) + \frac{1}{4} x^2 + C.$

(f) $\int \frac{e^{2x}}{(1+e^x)^3} dx$

let $y = 1+e^x \Rightarrow \frac{dy}{dx} = e^x \Rightarrow dx = \frac{1}{e^x} dy$

$\int \frac{e^{2x}}{(1+e^x)^3} dx = \int \frac{e^{2x}}{(1+e^x)^3} \cdot \frac{1}{e^x} dy = \int \frac{e^x}{(1+e^x)^3} dy = \int \frac{y-1}{y^3} dy$

$= \int (y^{-2} - y^{-3})\, dy = \frac{y^{-2+1}}{-2+1} - \frac{y^{-3+1}}{-3+1} + C = -\frac{1}{y} + \frac{1}{2y^2} + C.$

$= -\frac{1}{1+e^x} + \frac{1}{2(1+e^x)^2} + C$

(h) $\displaystyle\int_1^5 \frac{\sin^2(\ln x)}{x}\,dx$

let $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x} \Rightarrow dx = x\,dy$

$x=1 \quad y = \ln x = \ln 1 = 0 \qquad x = 5\,,\ y = \ln x = \ln 5.$

$$\int_1^5 \frac{\sin^2(\ln x)}{x}\,dx = \int_0^{\ln 5} \frac{\sin^2(\ln x)}{x}\,x\,dy = \int_0^{\ln 5} \sin^2 y\,dy$$

$$= \int_0^{\ln} -\frac{1}{2}\left[\cos(y+y) - \cos(y-y)\right]dy = -\frac{1}{2}\int_0^{\ln 5}(\cos(2y) - 1)\,dy$$

$$= -\frac{1}{2}\left(\frac{1}{2}\sin(2y) - y\right)\Big|_0^{\ln 5}$$

$$= -\frac{1}{4}\sin(2\ln 5) + \frac{1}{2}\ln 5$$

(j) $\displaystyle\int \frac{2x+1}{x^2-2x+5}\,dx$

let $y = x^2 - 2x + 5 \Rightarrow \frac{dy}{dx} = 2x - 2 \Rightarrow dx = \frac{1}{2x-2}\,dy$

$\to \int \frac{1}{1+(\,)^2} \to \tan^{-1}(\cdot)$

$$\int \frac{2x+1}{x^2-2x+5}\,dx = \int\left(\frac{2x-2}{x^2-2x+5} + \frac{3}{x^2-2x+5}\right)dx$$

$$= \int \frac{2x-2}{x^2-2x+5}\cdot\frac{1}{2x-2}\,dy + \int \frac{3}{x^2-2x+5}\,dx$$

$$= \int \frac{1}{y}\,dy + 3\int \frac{1}{(x-1)^2+4}\,dx$$

$$= \ln|y| + C_1 + \frac{3}{4}\int \frac{1}{\left(\frac{x-1}{2}\right)^2+1}\,dx$$

$$= \ln|y| + C_1 + \frac{3}{4}\cdot\frac{1}{\frac{1}{2}}\tan^{-1}\left(\frac{x-1}{2}\right) + C_2$$

$$= \ln|y| + \frac{3}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C$$

$$= \ln|x^2-2x+5| + \frac{3}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C.$$

(l) $\displaystyle\int \frac{1}{x^2\sqrt{1-x^2}}dx$

let $x = \sin\theta \Rightarrow \frac{dx}{d\theta} = \cos\theta \Rightarrow dx = \cos\theta\, d\theta.$

$$\int \frac{1}{\sin^2\theta \cancel{\sqrt{1-\sin^2\theta}}} \cancel{\cos\theta}\, d\theta = \int \frac{1}{\sin^2\theta} d\theta$$

let $u = \frac{\cos\theta}{\sin\theta} \Rightarrow \frac{du}{d\theta} = \frac{-\sin\theta\sin\theta - \cos\theta\cdot\cos\theta}{\sin^2\theta} = -\frac{1}{\sin^2\theta} \Rightarrow d\theta = -\sin^2\theta\, du.$

$$\int \frac{1}{\sin^2\theta} d\theta = \int \frac{1}{\sin^2\theta}\cdot(-\sin^2\theta\, du) = \int -du = -u + C = -\frac{\cos\theta}{\sin\theta} + C$$

$$= -\frac{\sqrt{1-x^2}}{x} + C.$$

(n) $\displaystyle\int \frac{3x}{\sqrt{4x^2+1}}dx$

let $y = 4x^2 + 1 \Rightarrow \frac{dy}{dx} = 8x \Rightarrow dx = \frac{1}{8x} dy$

$y^{-\frac{1}{2}}$

$$\int \frac{3x}{\sqrt{4x^2+1}} dx = \int \frac{3x}{\sqrt{4x^2+1}} \cdot \frac{1}{8x} dy = \frac{3}{8}\int \frac{1}{\sqrt{y}} dy = \frac{3}{8}\cdot \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= \frac{3}{4}\sqrt{y} + C = \frac{3}{4}\sqrt{4x^2+1} + C$$

(p) $\displaystyle\int \frac{1}{(x^2+6x+10)^{\frac{3}{2}}}dx$

let $y = x^2 + 6x + 10 \Rightarrow \frac{dy}{dx} = 2x + 6 \Rightarrow dx = \frac{1}{2x+6} dy$

$$\int \frac{1}{(x^2+6x+10)^{\frac{3}{2}}} dx = \int \frac{1}{[(x+3)^2+1]^{\frac{3}{2}}} dx$$

$\tan^2\theta + 1 = \sec^2\theta$

$(\tan\theta)' = \sec^2\theta$

let $x + 3 = \tan\theta \Rightarrow \frac{dx}{d\theta} = \sec^2\theta \Rightarrow dx = \sec^2\theta\, d\theta.$

$$\int \frac{1}{[(x+3)^2+1]^{\frac{3}{2}}} dx = \int \frac{1}{(\tan^2\theta+1)^{\frac{3}{2}}} \sec^2\theta\, d\theta = \int \frac{\sec^2\theta}{\sec^3\theta} d\theta = \int \frac{1}{\sec\theta} d\theta$$

$$= \int \cos\theta\, d\theta = \sin\theta + C = \frac{x+3}{\sqrt{x^2+6x+10}} + C$$

$\tan\theta = x+3$

$\sec\theta = \sqrt{x^2+6x+10}$

$\sin\theta = \frac{\tan\theta}{\sec\theta}$

(r) $\int \sin^3 x \cos^5 x \, dx$

$$\text{let } y = \sin x \Rightarrow \frac{dy}{dx} = \cos x \Rightarrow dx = \frac{1}{\cos x} dy \qquad (\cos^2 x)^2 = (1 - \sin^2 x)^2 = (1 - y^2)^2$$

$$\int \sin^3 x \cos^5 x \, dx = \int \sin^3 x \cos^5 x \cdot \frac{1}{\cos x} dy = \int \sin^3 x \cos^4 x \, dy$$

$$= \int y^3 (1 - y^2)^2 dy = \int y^3 (1 - 2y^2 + y^4) dy$$

$$= \int (y^3 - 2y^5 + y^7) dy = \frac{y^4}{4} - \frac{2}{6} y^6 + \frac{y^8}{8} + C$$

$$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{3} + \frac{\sin^8 x}{8} + C$$

Problem 2 (Integration by parts)

Compute the following integrals using integration by parts:

(b) $\int_1^e \sqrt{x} \ln x \, dx$

$$\text{let } u = \ln x, \; dv = \sqrt{x} dx \Rightarrow v = \int dv = \int \sqrt{x} dx = \frac{2}{3} x^{\frac{3}{2}}$$

$$\int_1^e \sqrt{x} \ln x \, dx = \int_1^e \underbrace{\ln x}_{u} \underbrace{\sqrt{x} dx}_{dv} = \underbrace{\frac{2}{3} x^{\frac{3}{2}}}_{v} \underbrace{\ln x}_{u} \Big|_1^e - \int_1^e \underbrace{\frac{2}{3} x^{\frac{3}{2}}}_{v} \underbrace{d(\ln x)}_{du}$$

$$= \frac{2}{3} e^{\frac{3}{2}} \ln e - \frac{2}{3} \ln 1 - \int_1^e \frac{2}{3} x^{\frac{3}{2}} \cdot \frac{1}{x} dx$$

$$= \frac{2}{3} e^{\frac{3}{2}} - \frac{2}{3} \int_1^e x^{\frac{1}{2}} dx$$

$$= \frac{2}{3} e^{\frac{3}{2}} - \frac{2}{3} \cdot \frac{2}{3} x^{\frac{3}{2}} \Big|_1^e = \frac{2}{3} e^{\frac{3}{2}} - \frac{4}{9} e^{\frac{3}{2}} + \frac{4}{9}$$

$$= \frac{2}{9} e^{\frac{3}{2}} + \frac{4}{9}$$

(d) $\int x \sin^2 x \, dx$

$$\int x \sin^2 x \, dx = \int x \cdot -\frac{1}{2} [\cos(x+x) - \cos(x-x)] dx = -\frac{1}{2} \int (x \cos 2x - x) dx$$

$$= -\frac{1}{2} \int x \cos 2x \, dx + \frac{1}{2} \int x \, dx$$

$u = x$

$dv = \cos 2x \, dx$

$V = \int dv = \frac{1}{2} \sin 2x$

$$= -\frac{1}{2} \left(\frac{1}{2} x \sin 2x - \int \frac{1}{2} \sin 2x \, dx \right) + \frac{1}{4} x^2$$

$$= -\frac{1}{4} x \sin 2x + \frac{1}{4} \int \sin 2x \, dx + \frac{1}{4} x^2$$

$$= -\frac{1}{4} x \sin 2x - \frac{1}{8} \cos 2x + \frac{1}{4} x^2 + C$$