Quadratic Form

(1, 2)

$q(x_1, x_2)$

$x_1 \in \mathbb{R}$

$x_2 \in \mathbb{R}$

$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ✓

$\underbrace{A = A^T}_{\text{Symmetric}}$

Ex1

$$q = 3x_1^2 + 4x_1x_2 - 1x_2^2$$

$$= \underset{1\times 2}{\underline{x}^T}\ \underset{2\times 2}{A}\ \underset{2\times 1}{\underline{x}}$$

$$= [x_1 \quad x_2] \begin{bmatrix} 3 & 4/2 \\ 4/2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2 2

Ex 2.a

$$q = x_1^2 + 2x_2^2 + 7x_3^2 - 2x_1x_2 + 4x_1x_3 - 2x_2x_3$$

$$q = \underset{\underline{x}^T}{[x_1 \quad x_2 \quad x_3]} \underset{A}{\begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & 7 \end{bmatrix}} \underset{\underline{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}$$

$A = A^T$

"General form"

$$q = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i x_j$$

$$= \underline{x}^T \underbrace{A}_{n \times n} \underline{x}$$

$A = [a_{ij}]$

$A = A^T$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$q = \underline{x}^T A \underline{x}$ is +ve definite if $q > 0$ for all $\underline{x} \neq 0$

q is −ve definite if $q < 0$ for all $\underline{x} \neq 0$

q is +ve semi-definite if $q \geq 0$

is −ve semi-definite if $q \leq 0$

q is indefinite if it can be +ve and negative values.

Ex 2b. $q = x_1^2 + 2x_2^2 + 7x_3^2 - 2x_1x_2 + 4x_1x_3 - 2x_2x_3$

$$= (x_1 - x_2 + 2x_3)^2 + 2x_2^2 + 7x_3^2 - 2x_2x_3 - x_2^2 - 4x_3^2 + 4x_2x_3$$

$$= (x_1 - x_2 + 2x_3)^2 + x_2^2 + 3x_3^2 + 2x_2x_3$$

$$= \underbrace{(x_1 - x_2 + 2x_3)^2}_{+ve} + \underbrace{(x_2 + x_3)^2}_{+ve} + \underbrace{2x_3^2}_{+ve}$$

$q > 0$ for any choice of x_1, x_2, x_3

q and $A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & 7 \end{bmatrix}$. are +ve definite

$q = \underline{x}^T A \underline{x} > 0$

for any choice of $\underline{x}$

How do we associate with eigenvalues?

check all $\lambda_i > 0$.

Example

$$q = 3x_1^2 + 4x_1x_2 - x_2^2$$
$$= 3\left(x_1^2 + \frac{4}{3}x_1x_2 - \frac{1}{3}x_2^2\right)$$
$$= 3\left(\underbrace{x_1^2 + 2\left(\frac{2}{3}\right)x_1x_2 + \left(\frac{2}{3}x_2\right)^2} - \left(\frac{2}{3}\right)^2 x_2^2 - \frac{1}{3}x_2^2\right)$$
$$= 3\left(\left(x_1 + \left(\frac{2}{3}x_2\right)\right)^2 - \left(\frac{2}{3}\right)^2 x_2^2 - \frac{1}{3}x_2^2\right)$$
$$= 3\left(\left(x_1 + \frac{2}{3}x_2\right)^2 - \frac{7}{9}x_2^2\right)$$
$$q = \underbrace{3\left(x_1 + 2/3\, x_2\right)^2}_{+ve} - \frac{7}{3}\underbrace{x_2^2}_{+ve}$$

q could be +ve/-ve for different choices of x_1 and x_2

For eg: Choose $(x_1, x_2) = (1, 3)$

$q = 6 > 0$

Choose $(x_1, x_2) = (0, 3)$

$q = -9 < 0$

or

$$q = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$= \underline{x}^T A \underline{x}$$

"q" and "A" are <u>indefinite</u>

Check A has ~~...~~ +ve and -ve eigenvalues

Recall

(A is $n \times n$ matrix)

- "A" is a real symmetric matrix, it is possible to find n linearly independent eigenvectors, $\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_n$
- If $\lambda_1, \lambda_2, \cdots \lambda_n$ are distinct, then these eigenvectors $\underline{x}_1, \underline{x}_2, \ldots \underline{x}_n$ will also be orthogonal. $\boxed{\underline{x}_i^T \underline{x}_j = 0 \quad i \neq j}$
- [Of course, if they are not distinct, it is in fact still possible to obtain n orthogonal eigenvectors via "Gram-Schmidt process"]
(We did not cover this process in our course.)

- If $\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_n$ are normalized, i.e $\underline{x}_1^T \underline{x}_1 = 1, \ \underline{x}_2^T \underline{x}_2 = 1, \ \cdots, \ \underline{x}_n^T \underline{x}_n = 1$

$$P = [\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_n] \ ; \qquad P^T P = I \iff \underbrace{P^T = P^{-1}}$$

P is orthogonal Matrix

Thus, it is possible to diagonalize a symmetric matrix using an orthogonal matrix P

$$P^T A P = D$$

Suppose a quadratic form $\boxed{q = \underline{x}^T A \underline{x}}$

Make the change of variable

$$\boxed{\underline{x} = P \underline{y}}$$

$$P^T \underline{x} = P^T P \underline{y} = \underline{y}$$

$$\begin{aligned} q = \underline{x}^T A \underline{x} &= (P\underline{y})^T A P \underline{y} \\ &= \underline{y}^T P^T A P \underline{y} \\ &= \boxed{\underline{y}^T D \underline{y}} = [y_1, y_2, \dots, y_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \end{aligned}$$

$$\begin{aligned} q &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2 \\ &= \boxed{\sum_{i=1}^{n} \lambda_i y_i^2} \end{aligned}$$

q is written as a sum of squares with the coefficients of the y_i^2 being λ_i of A, Thm

A quadratic form and its associated matrix are positive definite

iff

all eigenvalues of the associated symmetric matrix are positive

i.e λ_i of $A > 0$ for all i

$$\lambda_i > 0$$

Example

$$q = 2x_1^2 - 2x_1x_2 + 2x_2^2 = [x_1\ x_2]\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{x}^T A \underline{x}$$

The eigenvalues and eigenvectors of A are

$\lambda_1 = 1$ $\qquad$ $\lambda_2 = 3$

$\underline{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad$ $\underline{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

normalized vectors

$$\hat{\underline{x}}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \qquad \hat{\underline{x}}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \implies P = \begin{array}{c} \begin{array}{cc} \hat{\underline{x}}_1 & \hat{\underline{x}}_2 \end{array} \\ \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \end{array}$$

$$P^{-1} = P^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} ; \qquad P^T A P = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = D$$

If we make the change of variable

$$\underline{x} = P\underline{y} \quad \text{or} \quad P^{-1}\underline{x} = \underline{y} \quad \text{or} \quad P^T\underline{x} = \underline{y}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} (y_1 - y_2)/\sqrt{2} \\ (y_1 + y_2)/\sqrt{2} \end{bmatrix}$$

$$q = \underline{x}^T A \underline{x} = \underline{y}^T D \underline{y} = \sum_{i=1}^{2} \lambda_i y_i^2 = \lambda_1 y_1^2 + \lambda_2 y_2^2$$

$$\boxed{q = y_1^2 + 3y_2^2}$$

Verification (i) $y = P^T\underline{x} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} (x_1 + x_2)/\sqrt{2} \\ (x_2 - x_1)/\sqrt{2} \end{bmatrix} ; \qquad q = \frac{(x_1 + x_2)^2}{2} + 3\frac{(x_2 - x_1)^2}{2}$$

Verification (ii)

$$q = 2x_1^2 - 2x_1x_2 + 2x_2^2$$

$$= 2(y_1 - y_2)^2/2 \quad - \quad 2\,\frac{(y_1 - y_2)}{\sqrt{2}} \cdot \frac{(y_1 + y_2)}{\sqrt{2}} + 2\,\frac{(y_1 + y_2)^2}{2}$$

$$= y_1^2 + 3y_2^2$$

Verification (iii)

"Completing the Square" does not necessarily give the same factorization.

$$q = 2x_1^2 - 2x_1x_2 + 2x_2^2 = 2(x_1^2 - x_1x_2 + x_2^2)$$

$$= 2\left[\left(x_1 - \frac{x_2}{2}\right)^2 - x_2^2/4 + x_2^2\right]$$

$$= 2(x_1 - x_2/2)^2 + 3x_2^2/2$$

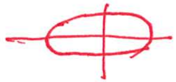
[factorization Different from (i)]

Example

- For any constant $c > 0$, the points (x_1, x_2) satisfying

$$2x_1^2 - 2x_1x_2 + 2x_2^2 = c$$

form a conic in the plane.



- change of variable, we have

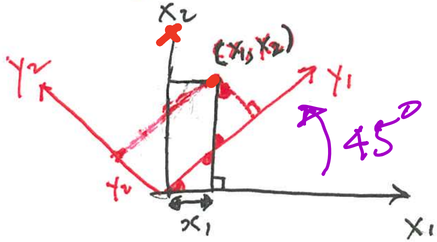
$$y_1^2 + 3y_2^2 = c$$

This is an equation of an ellipse

with transformation

$$\underline{x} = P\underline{y}, \quad \text{where } P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

represents a rotation of the axes through $45°$.



Equivalent definitions

A) If $A_1 = a_{11}$, $A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, $A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \dots$

$A_n = \det(A)$, then A is +ve iff $A_1, A_2, \dots, A_n$ are all positive.

$n \times n$

B) A is +ve iff $A \longrightarrow$ echelon form, all the diagonal elements are positive.

$A \longrightarrow$ eg $\begin{bmatrix} d_{11} & * & * \\ 0 & d_{22} & * \\ 0 & 0 & d_{33} \end{bmatrix}$

$d_{11}, d_{22}, d_{33} > 0$

Example The Matrix

$$A = \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{bmatrix}$$ is +ve definite

Since

$$\det[2] = 2 \; ; \; \det\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 3 \; ; \; \det\begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{bmatrix} = 1$$

Thm: A quadratic form $\underline{x}^T A \underline{x}$ on a symmetric matrix A is

$$\boxed{q = \underline{x}^T A \underline{x}}$$

q is +ve definite iff $\lambda(A) > 0$ — $\lambda_i > 0$

q is semidefinite iff $\lambda(A) \geq 0$ — some $\lambda_i > 0$ & $\lambda_i = 0$

q is -ve definite iff $\lambda(A) < 0$ — $\lambda_i < 0$

q is -ve semidefinite iff $\lambda(A) \leq 0$ — $\lambda_i < 0$ & $\lambda_i = 0$

q is indefinite iff some $\lambda(A) > 0$ and $\lambda(A) < 0$

Eg: Check if $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$ is positive definite.

(Hints: $\lambda = 2, 8$)

Eg: Find a change of variables that reduces the quadratic form $x_1^2 - x_3^2 - 4x_1x_2 + 4x_2x_3$ to a sum of squares and express the quadratic form in terms of the new variables.

Ans: $-3y_2^2 + 3y_3^2$