MA1200 Solution to Past Exam Paper 1819 A

1. (a) $$4x^2+y^2+24x-4y+24=0$$

$$\Rightarrow 4(x^2+6x)+y^2-4y+24=0$$

$$\Rightarrow 4[(x+3)^2-9]+(y-2)^2-4+24=0$$

$$\Rightarrow 4(x+3)^2+(y-2)^2=16$$

$$\Rightarrow \frac{(x+3)^2}{2^2}+\frac{(y-2)^2}{4^2}=1$$

which is an equation of an ellipse.

(b) Centre: $(-3, 2)$

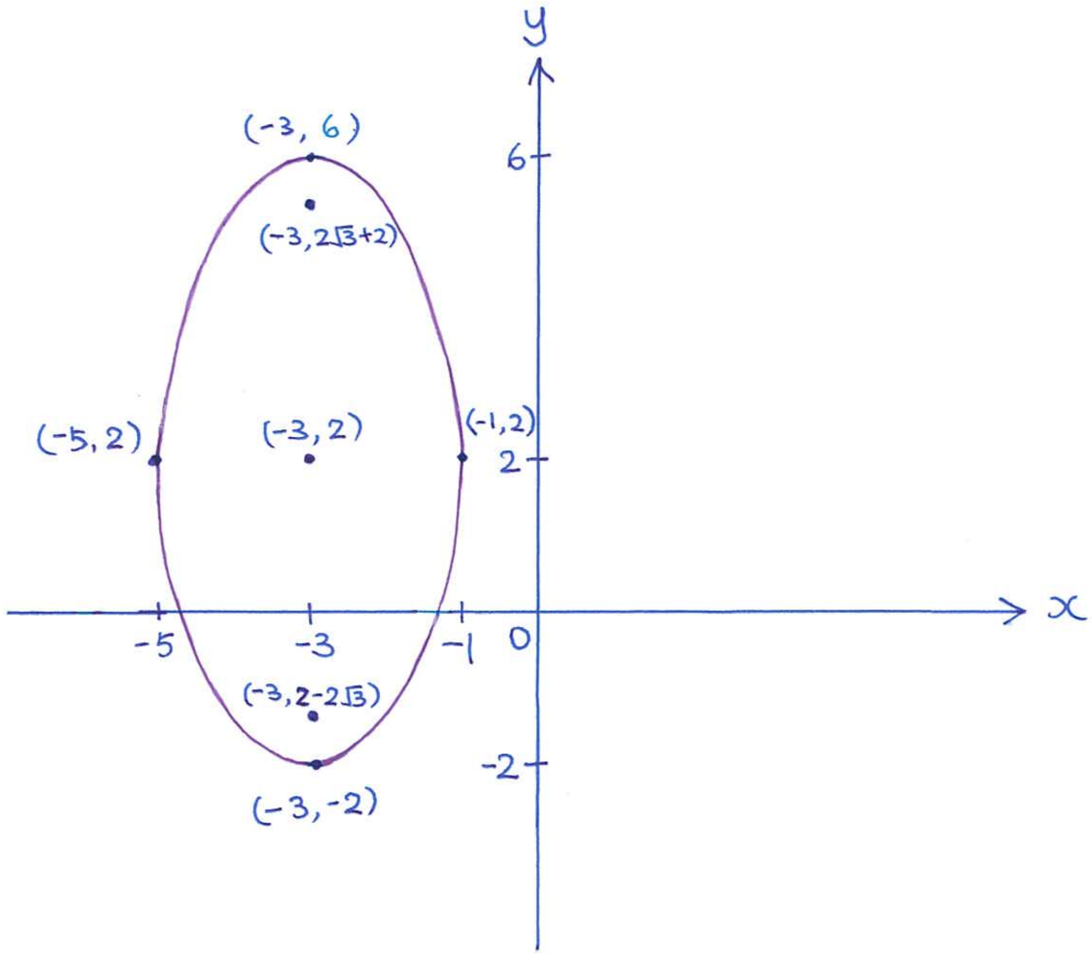
Vertices: $(-2-3, 0+2)$, $(2-3, 0+2)$, $(0-3, -4+2)$, $(0-3, 4+2)$

i.e. $(-5, 2)$, $(-1, 2)$, $(-3, -2)$, $(-3, 6)$

$c=\sqrt{4^2-2^2}=\sqrt{12}=2\sqrt{3}$ (>0)

Foci: $(0-3, 2\sqrt{3}+2)$ & $(0-3, -2\sqrt{3}+2)$, i.e. $(-3, 2\sqrt{3}+2)$ & $(-3, -2\sqrt{3}+2)$.

1. (C)



2. $f(x) = \dfrac{4x+3}{x+2}$

(a) For any $x_1, x_2 \in \mathrm{Dom}(f) = \mathbb{R} \setminus \{-2\}$, where $x_1 \neq x_2$,

$$f(x_1) - f(x_2) = \frac{4x_1+3}{x_1+2} - \frac{4x_2+3}{x_2+2}$$

$$= \frac{(4x_1+3)(x_2+2) - (4x_2+3)(x_1+2)}{(x_1+2)(x_2+2)}$$

$$= \frac{5(x_1 - x_2)}{(x_1+2)(x_2+2)}$$

$$\neq 0 \quad (\because x_1 \neq x_2)$$

$\therefore$ $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$

$\therefore$ $f(x)$ is one-to-one.

2.(b) $-2 = \dfrac{4x+3}{x+2} \Rightarrow -2x-4 = 4x+3$

$\Rightarrow 6x = -7$

$\Rightarrow x = \dfrac{-7}{6}$

$\therefore f^{-1}(-2) = -\dfrac{7}{6}$

<u>OR</u> Find $f^{-1}(x)$ first.

Let $y = \dfrac{4x+3}{x+2}$. Then $(x+2)y = 4x+3$

$\Rightarrow x(4-y) = 2y-3$

$\Rightarrow x = \dfrac{2y-3}{4-y}$

$\therefore f^{-1}(x) = \dfrac{2x-3}{4-x}$

$\therefore f^{-1}(-2) = \dfrac{2(-2)-3}{4-(-2)} = \dfrac{-7}{6}$

2(c) $f^{-1}(x) = \dfrac{2x-3}{4-x}$ (from (b)).

$\text{Dom}(f^{-1}) = \mathbb{R}\setminus\{4\}$

$\text{Ran}(f^{-1}) = \text{Dom}(f) = \mathbb{R}\setminus\{-2\}$ (from (a)).

<u>OR</u>

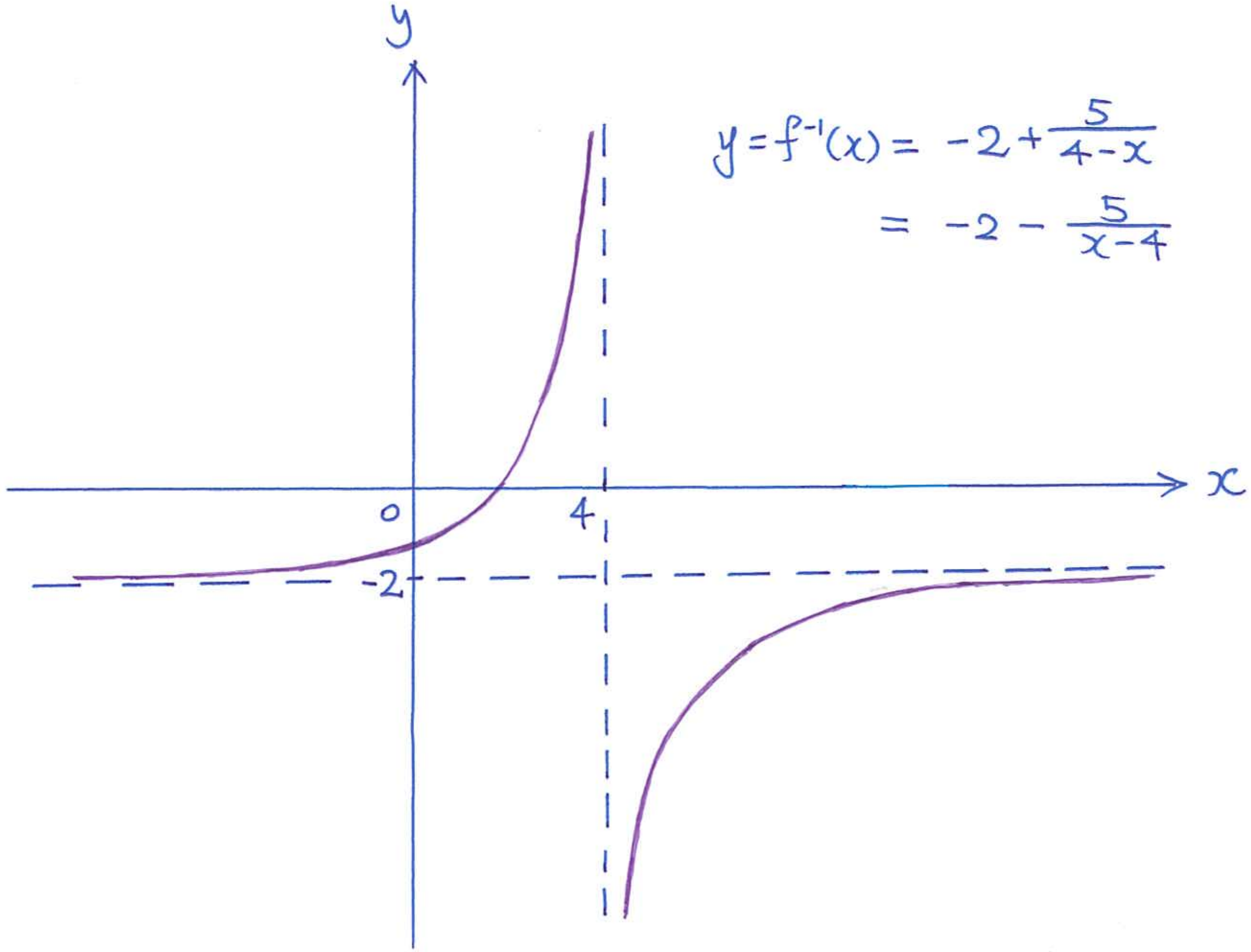
$f^{-1}(x) = \dfrac{-2(4-x)+5}{4-x} = -2 + \dfrac{5}{4-x}$

For any $x \in \text{Dom}(f^{-1})$, $\dfrac{5}{4-x}$ can be any real number except 0.

$\therefore$ $f^{-1}(x) = -2 + \dfrac{5}{4-x}$ can be any real number except $-2+0=-2$.

$\therefore$ $\text{Ran}(f^{-1}) = \mathbb{R}\setminus\{-2\}$.

2(d)



$$y = f^{-1}(x) = -2 + \frac{5}{4-x}$$
$$= -2 - \frac{5}{x-4}$$

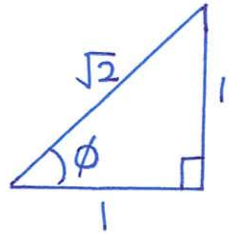
3. (a) $\sin(3x) = \sin(x+2x)$

$= \sin x \cos 2x + \cos x \sin 2x$, using compound angle formula

$= \sin x(\cos^2 x - \sin^2 x) + \cos x \cdot 2\sin x \cos x$, using Double-angle formula

$= 3\sin x \cos^2 x - \sin^3 x$

(b) $\sin(3x) + \cos(3x) + 1 = 0$



$\phi = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$\Rightarrow \sqrt{2}\left(\frac{1}{\sqrt{2}}\sin(3x) + \frac{1}{\sqrt{2}}\cos(3x)\right) = -1$

$\Rightarrow \sqrt{2}\left(\cos\phi \sin(3x) + \sin\phi \cos(3x)\right) = -1$

$\Rightarrow \sqrt{2}\sin(3x+\phi) = -1$

$\Rightarrow \sin(3x+\phi) = -\frac{1}{\sqrt{2}}$

$\Rightarrow 3x + \phi = n\pi + (-1)^n \cdot \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

$= n\pi + (-1)^n \cdot \left(-\frac{\pi}{4}\right)$

$\Rightarrow x = \frac{n\pi}{3} + (-1)^{n+1} \cdot \frac{\pi}{12} - \frac{\pi}{12}$, for $n \in \mathbb{Z}$

3(c)
(i) $f(x) = \dfrac{x^3-3}{x^3-x^2-x+1} = \dfrac{(x^3-x^2-x+1)+x^2+x-4}{x^3-x^2-x+1} = 1+\dfrac{x^2+x-4}{x^3-x^2-x+1}$

improper

$$= 1+\frac{x^2+x-4}{(x+1)(x^2-2x+1)}$$

$$= 1+\frac{x^2+x-4}{(x+1)(x-1)^2}$$

Consider $\dfrac{x^2+x-4}{(x+1)(x-1)^2} = \dfrac{A}{x+1}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$

$\Rightarrow \quad x^2+x-4 = A(x-1)^2+B(x+1)(x-1)+C(x+1)$

Put $x=1$: $\quad -2 = 2C \Rightarrow C=-1$

Put $x=-1$: $\quad -4 = 4A \Rightarrow A=-1$

Compare constant term : $\quad -4 = A-B+C \Rightarrow B = A+C+4 = 2$

$\therefore \quad f(x) = 1-\dfrac{1}{x+1}+\dfrac{2}{x-1}-\dfrac{1}{(x-1)^2}$

3(c) (ii) $f(x) = 1 - (x+1)^{-1} + 2(x-1)^{-1} - (x-1)^{-2}$

$f'(x) = -(-1)(x+1)^{-2} + 2(-1)(x-1)^{-2} - (-2)(x-1)^{-3}$

$f''(x) = (-2)(x+1)^{-3} - 2(-2)(x-1)^{-3} + 2(-3)(x-1)^{-4}$

$\therefore \quad f^{(3)}(x) = (-2)(-3)(x+1)^{-4} + 4(-3)(x-1)^{-4} - 6(-4)(x-1)^{-5}$

$= \dfrac{6}{(x+1)^4} - \dfrac{12}{(x-1)^4} + \dfrac{24}{(x-1)^5}$

4. $f(x) = x^2 \ln x$

(a) $\mathrm{Dom}(f) = (0, \infty)$

$\ln x > 0$ when $x > 1$.

$\therefore$ $f(x)$ is positive when $x \in (1, \infty)$.

(b)
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 \ln x \quad (0 \times (-\infty) \text{ form})$$
$$= \lim_{x\to 0^+} \frac{\ln x}{x^{-2}} \quad \left(\frac{-\infty}{\infty} \text{ form}\right)$$
$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-2x^{-3}} \quad \text{by L'Hôpital's rule}$$
$$= \lim_{x\to 0^+} \frac{x}{-2}$$
$$= 0$$

4c) $f(x) = x^2 \ln x$

$f'(x) = 2x \ln x + x^2 \cdot \frac{1}{x} = 2x \ln x + x$

$f''(x) = 2 \ln x + 2x \cdot \frac{1}{x} + 1 = 2 \ln x + 3$

Set $f''(x) = 0 \Rightarrow 2 \ln x + 3 = 0 \Rightarrow \ln x = \frac{-3}{2} \Rightarrow x = e^{-\frac{3}{2}}$

	$x < e^{-\frac{3}{2}}$	$x = e^{-\frac{3}{2}}$	$x > e^{-\frac{3}{2}}$
sign of $f''(x)$	$-$	0	$+$

$\because$ sign of $f''(x)$ changes at $x = e^{-\frac{3}{2}}$,

$\therefore$ $f(x)$ has an inflection point at $\left(e^{-\frac{3}{2}}, -\frac{3}{2}e^{-3}\right)$.

(d) Set $f'(x) = 0 \Rightarrow 2x \ln x + x = 0 \Rightarrow x(2\ln x + 1) = 0$

$\Rightarrow x = 0$ (rejected $\because 0 \notin Dom(f)$.)

or $x = e^{\frac{-1}{2}}$

$f''(e^{-\frac{1}{2}}) = 2 > 0$

$\therefore$ $f(x)$ has a local minimum at $x = e^{-\frac{1}{2}}$. ≈ 0.6065

$f(e^{-\frac{1}{2}}) = -\frac{1}{2}e^{-1}$ < 0 $\lim_{x \to 0^+} f(x)$

$\because$ $f(x) > 0$ when $x > 1$ (from (a)).

$\therefore$ $f(x)$ has an absolute minimum at $x = e^{-\frac{1}{2}}$

and the minimum value of $f(x)$ is $f(e^{-\frac{1}{2}}) = -\frac{1}{2}e^{-1}$.

5. (a) Let $y = \left(\frac{\ln x}{x}\right)^{\frac{1}{\ln x}}$.

Take ln on both sides :

$$\ln y = \ln\left[\left(\frac{\ln x}{x}\right)^{\frac{1}{\ln x}}\right] = \frac{1}{\ln x}\cdot \ln\left(\frac{\ln x}{x}\right) = \frac{\ln(\ln x) - \ln x}{\ln x}$$

$$= \frac{\ln(\ln x)}{\ln x} - 1$$

Take limit on both sides :

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty}\left[\frac{\ln(\ln x)}{\ln x} - 1\right] = \left(\lim_{x\to\infty} \frac{\ln(\ln x)}{\ln x}\right) - 1$$

$\frac{\infty}{\infty}$ form

$$= \left(\lim_{x\to\infty} \frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{\frac{1}{x}}\right) - 1$$

by L'Hopital's rule

$$= \underbrace{\left(\lim_{x\to\infty} \frac{1}{\ln x}\right)}_{=0} - 1$$

$$= -1$$

$$\therefore \lim_{x\to\infty}\left(\frac{\ln x}{x}\right)^{\frac{1}{\ln x}} = e^{-1}.$$

5(b) $\lim_{x\to 0} \dfrac{\sqrt{|x|}\cos(\pi^{\frac{1}{x^2}})}{2+\sqrt{x^2+3}} = \dfrac{\lim_{x\to 0}\sqrt{|x|}\cos(\pi^{\frac{1}{x^2}})}{\lim_{x\to 0}(2+\sqrt{x^2+3})} = \dfrac{\lim_{x\to 0}\sqrt{|x|}\cos(\pi^{\frac{1}{x^2}})}{2+\sqrt{3}}$

Since $-1 \leq \cos(\pi^{\frac{1}{x^2}}) \leq 1$ for all $x \neq 0$, we have

$$-\sqrt{|x|} \leq \sqrt{|x|}\cos(\pi^{\frac{1}{x^2}}) \leq \sqrt{|x|}.$$

$\lim_{x\to 0}(-\sqrt{|x|}) = 0 = \lim_{x\to 0}\sqrt{|x|}$.

$\therefore \lim_{x\to 0}\sqrt{|x|}\cos(\pi^{\frac{1}{x^2}}) = 0$, by the Sandwich Theorem.

$\therefore \lim_{x\to 0} f(x) = \dfrac{\lim_{x\to 0}\sqrt{|x|}\cos(\pi^{\frac{1}{x^2}})}{2+\sqrt{3}} = 0$

For $f(x)$ to be continuous at $x=0$, we define

$$f(0) = \lim_{x\to 0} f(x) = 0.$$

Then $f(x)$ is continuous at all $x \in \mathbb{R}$.

5 (c) $x^5 + x^3 + 2x = 2x^4 + 3x^2 + 4 \Rightarrow x^5 + x^3 + 2x - 2x^4 - 3x^2 - 4 = 0$

Let $f(x) = x^5 + x^3 + 2x - 2x^4 - 3x^2 - 4$.

This is continuous at all $x \in \mathbb{R}$.

$f(2) = -4 < 0$

$f(3) = 83 > 0$

$\therefore$ By the IVT, there is a root $c \in (2,3)$ such that $f(x) = 0$.

$\therefore$ The original equation has solution in $(2, 3)$.

6 (a) $\frac{d}{dx}\left(\frac{x^2+2}{x^2-1}\right) = \frac{d}{dx}\left(\frac{(x^2-1)+3}{x^2-1}\right) = \frac{d}{dx}\left(1+3(x^2-1)^{-1}\right)$

$$= 0 + 3(-1)(x^2-1)^{-2}\cdot 2x$$

$$= \frac{-6x}{(x^2-1)^2}$$

(b) $\frac{d}{dx}\left[\sin^{-1}\left(\frac{x^2}{3}\right)\right] = \frac{1}{\sqrt{1-\left(\frac{x^2}{3}\right)^2}}\cdot\frac{2x}{3} = \frac{2x}{\sqrt{9-x^4}}$

(c) $\frac{d}{dx}\left[\ln\frac{(6+\sin^2 x)^{10}}{(7+\cos x)^3}\right] = \frac{d}{dx}\left[10\ln(6+\sin^2 x) - 3\ln(7+\cos x)\right]$

$$= 10\cdot\frac{2\sin x\cdot\cos x}{6+\sin^2 x} - 3\cdot\frac{-\sin x}{7+\cos x}$$

$$= \frac{20\sin x\cos x}{6+\sin^2 x} + \frac{3\sin x}{7+\cos x}$$

(d) Let $y=(\sin x)^{\tan x}$. Then $\ln y = \tan x\ln(\sin x)$

Diff. both sides w.r.t. x:

$$\frac{1}{y}\frac{dy}{dx} = \sec^2 x\cdot\ln(\sin x) + \tan x\cdot\frac{\cos x}{\sin x}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^{\tan x}\left[1+\sec^2 x\cdot\ln(\sin x)\right]$$

7. $\begin{cases} x = 5\sqrt{5}\,\sin^3 t \\ y = 5\sqrt{5}\,\cos^3 t \end{cases}$, $t \in [0, 2\pi]$

$$\frac{dx}{dt} = 5\sqrt{5} \cdot 3\sin^2 t\cos t$$

$$\frac{dy}{dt} = 5\sqrt{5} \cdot 3\cos^2 t \cdot (-\sin t)$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-15\sqrt{5}\cos^2 t\sin t}{15\sqrt{5}\sin^2 t\cos t} = -\frac{\cos t}{\sin t} = -\cot x$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(-\cot x)}{15\sqrt{5}\sin^2 t\cos t} = \frac{\csc^2 t}{15\sqrt{5}\sin^2 t\cos t} = \frac{1}{15\sqrt{5}\sin^4 t\cos t}$$

At $(-1, 8)$, $5\sqrt{5}\sin^3 t = -1 \Rightarrow \sin^3 t = -\frac{1}{5\sqrt{5}} \Rightarrow \sin t = -\frac{1}{\sqrt{5}}$

& $5\sqrt{5}\cos^3 t = 8 \Rightarrow \cos^3 t = \frac{8}{5\sqrt{5}} \Rightarrow \cos t = \frac{2}{\sqrt{5}}$

$$\therefore \left.\frac{dy}{dx}\right|_{(-1,8)} = -\left.\frac{\cos t}{\sin t}\right|_{\substack{(x=-1, \\ y=8)}} = -\frac{\frac{2}{\sqrt{5}}}{\left(-\frac{1}{\sqrt{5}}\right)} = 2$$

$$\& \left.\frac{d^2y}{dx^2}\right|_{(-1,8)} = \frac{1}{15\sqrt{5}\cdot\left(-\frac{1}{\sqrt{5}}\right)^4\cdot\left(\frac{2}{\sqrt{5}}\right)} = \frac{5}{6}$$

7(b) The tangent line to the curve at $(-1, 8)$ is

$$\frac{y-8}{x-(-1)} = 2$$

$$\Rightarrow \quad y = 2x + 10$$

8. Assume that the farmer sells the pigs on day t.

The weight of a pig on day t is $300+10t$.

The cost per pig on day t is $10t$.

The price of a pig per pound on day t is $15-0.2t$.

$\therefore$ The profit per pig is

$$P(t) = (300+10t)(15-0.2t) - 10t$$

$$= -2t^2 + 80t + 4500$$

$P'(t) = -4t + 80$

Set $P'(t) = 0 \Rightarrow t = 20$

When $0 < t < 20$, $P'(t) > 0$.

When $t > 20$, $P'(t) < 0$.

$\therefore$ $P(t)$ is maximized at $t = 20$.

$\therefore$ The farmer should sell the pigs on day 20.

9 (a) $f(x) = \sin(\sinh^{-1}x)$

$$f'(x) = \cos(\sinh^{-1}x) \cdot \frac{1}{\sqrt{1+x^2}}$$

$$f''(x) = -\sin(\sinh^{-1}x) \cdot \frac{1}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}} + \cos(\sinh^{-1}x) \cdot \left(-\frac{1}{2}\right)(1+x^2)^{-\frac{3}{2}} \cdot 2x$$

$$= -\sin(\sinh^{-1}x) \cdot \frac{1}{1+x^2} - \cos(\sinh^{-1}x) \cdot \frac{x}{(1+x^2)^{3/2}}$$

Then $(1+x^2) f''(x) + x f'(x) + f(x)$

$$= -\sin(\sinh^{-1}x) - \cos(\sinh^{-1}x) \cdot \frac{x}{\sqrt{1+x^2}} + \cos(\sinh^{-1}x) \cdot \frac{x}{\sqrt{1+x^2}} + \sin(\sinh^{-1}x)$$

$$= 0$$

$\therefore \quad (1+x^2) f''(x) + x f'(x) + f(x) = 0 \quad \text{———} \; (*)$

(b) Diff. both sides of (*) n times w.r.t. x (using Leibnitz' rule):

$$\left[(1+x^2)f''(x)\right]^{(n)} + \left[x f'(x)\right]^{(n)} + \left[f(x)\right]^{(n)} = 0^{(n)}$$

$$\Rightarrow \left[\sum_{k=0}^{n}\binom{n}{k}(1+x^2)^{(k)}\left(f''(x)\right)^{(n-k)}\right] + \left[\sum_{k=0}^{n}\binom{n}{k}x^{(k)}\left(f'(x)\right)^{(n-k)}\right] + f^{(n)}(x) = 0$$

$$\Rightarrow \left[1\cdot(1+x^2)\cdot f^{(n+2)}(x) + n\cdot(2x)\cdot f^{(n+1)}(x) + \frac{n(n-1)}{2}\cdot 2\cdot f^{(n)}(x)\right]$$

$$+\left[1\cdot x\cdot f^{(n+1)}(x) + n\cdot 1\cdot f^{(n)}(x)\right] + f^{(n)}(x) = 0$$

$$\Rightarrow (1+x^2)\cdot f^{(n+2)}(x) + (2n+1)x\cdot f^{(n+1)}(x) + (n^2+1)\cdot f^{(n)}(x) = 0 \qquad (**)$$

for $n \geq 2$.

Now check that (**) is also true for $n=1$:

Diff. both sides of (*) w.r.t. x:

$$(1+x^2)f^{(3)}(x) + 3x f''(x) + 2f'(x) = 0$$

$\therefore$ (**) is true for $n=1$ and hence it is true for $n \geq 1$.

9.(c) $f(0) = \sin(\underbrace{\sinh^{-1} 0}_{=0}) = \sin 0 = 0$

$=0 \because \sinh 0 = \frac{1}{2}(e^0 - e^{-0}) = 0$

$$f'(0) = \cos(\underbrace{\sinh^{-1} 0}_{=0}) \cdot \frac{1}{\sqrt{1+0^2}} = \cos 0 = 1$$

$$f''(0) = -\sin(\underbrace{\sinh^{-1} 0}_{=0}) \cdot \frac{1}{1+0^2} - \cos(\sinh^{-1} 0) \cdot \frac{0}{(1+0^2)^{3/2}} = -\sin 0 = 0$$

Put $x=0$ into (**) :

$$f^{(n+2)}(0) = -(n^2+1) f^{(n)}(0)$$

When $n=1$: $f^{(3)}(0) = -2 f'(0) = -2(1) = -2$

When $n=2$: $f^{(4)}(0) = -5 f^{(2)}(0) = -5 \cdot (0) = 0$

When $n=3$: $f^{(5)}(0) = -10 f^{(3)}(0) = -10 \cdot (-2) = 20$

$\therefore$ The Maclaurin series of $\sin(\sinh^{-1}x)$ is

$$f(x) = \sin(\sinh^{-1}x)$$

$$= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$= 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-2}{3!}x^3 + \frac{0}{4!}x^4 + \frac{20}{5!}x^5 + \cdots$$

$$= x - \frac{1}{3}x^3 + \frac{1}{6}x^5 + \cdots$$