Take Home Assignment MA2001 #2

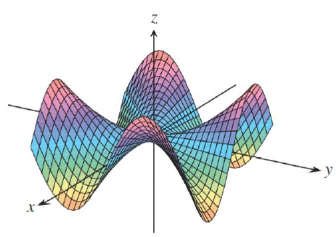
For each of the following questions, write down your solution with details of steps. Marks will not be given if only final answers are provided.

1. Find and sketch the level curves $f(x, y) = c$ on the same set of coordinate axes for the given values of c. We refer to these level curves as a contour map.

   (a) $f(x, y) = x^2 + y^2, c = 0, 1, 4, 9, 16, 25$

   (b) $f(x, y) = xy, c = -9, -4, -1, 0, 1, 4, 9$

2. Let $f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & \text{if } (x, y) \neq 0, \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$ The graph of f is shown as below.



$z = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}$

   a. Show that $\dfrac{\partial f}{\partial y}(x, 0) = x$ for all x, and $\dfrac{\partial f}{\partial x}(0, y) = -y$ for all y.

   b. Show that $\dfrac{\partial^2 f}{\partial y \partial x}(0, 0) \neq \dfrac{\partial^2 f}{\partial x \partial y}(0, 0)$.

3. Let $w = f(r, s, t)$ with $r = g(x, y), s = h(x, y), t = k(x, y)$. Draw a branch diagram and write a Chain Rule formula for partial derivatives $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$.

4. **Laplace equations** Show that if $w = f(u, v)$ satisfies the Laplace equation $f_{uu} + f_{vv} = 0$ and if $u = (x^2 - y^2)/2$ and $v = xy$, then w satisfies the Laplace equation

$$w_{xx} + w_{yy} = 0.$$

5. Assuming the equation defines y as a differentiable function of x, use implicit differentiation find the value of dy/dx at the given point.

$$xe^y + \sin(xy) + y - \ln 2 = 0, \quad (0, \ln 2)$$

6. (a) Find the linearization $L(x, y)$ of the function

$$f(x, y) = (1/2)x^2 + xy + (1/4)y^2 + 3x - 3y + 4 \quad \text{at} \quad P_0(2, 2).$$

   (b) Then find an upper bound for the magnitude $|E|$ of the error in the approximation $f(x, y) \approx L(x, y)$ over the rectangle

$$R : |x - 2| \leq 0.1, |y - 2| \leq 0.1.$$

7. Find all the local maxima, local minima, and saddle points of the functions.

   (a) $f(x, y) = x^2 - 2xy + 2y^2 - 2x + 2y + 1$

   (b) $f(x, y) = e^y - ye^x$

8. Find the absolute maxima and minima of the function

$$T(x, y) = x^2 + xy + y^2 - 6x$$

on the rectangular domain: $0 \leq x \leq 5, \quad -3 \leq y \leq 3$.

9. Let $f(x, y) = \dfrac{(x - y)}{(x + y)}$. Find $D_{\mathbf{u}}f(-\frac{1}{2}, \frac{3}{2})$ if $\mathbf{u} = (1, 2)$, and the directions $\mathbf{u}$ and the values of $D_{\mathbf{u}}f(-\frac{1}{2}, \frac{3}{2})$ for which

   (a) $D_{\mathbf{u}}f(-\frac{1}{2}, \frac{3}{2})$ is largest

   (b) $D_{\mathbf{u}}f(-\frac{1}{2}, \frac{3}{2})$ is smallest

   (c) $D_{\mathbf{u}}f(-\frac{1}{2}, \frac{3}{2}) = 0$

   (d) $D_{\mathbf{u}}f(-\frac{1}{2}, \frac{3}{2}) = -2$

10. **Discovery Question**. Given the function $f(x, y)$ and the positive number ϵ. Show that there exists a $\delta > 0$ such that for all (x,y),

$$\sqrt{x^2 + y^2} < \delta \Rightarrow |f(x, y) - f(0, 0)| < \epsilon$$

(i) $f(x,y) = y/(x^2+1), \quad \epsilon = 0.05$

(ii) $f(x,y) = \dfrac{x^3+y^4}{x^2+y^2}$ and $f(0,0) = 0, \quad \epsilon = 0.02$

11. **Discovery Question**. **Changing voltage in a circuit** The voltage V in a circuit that satisfies the law $V = IR$ is slowly dropping as the battery wears out. At the same time, the resistance R is increasing as the resistor heats up. Use the equation

$$\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt}$$

to find how the current is changing at the instant when $R = 600$ ohms, $I = 0.04$ amp, $dR/dt = 0.5$ ohm/sec, and $dV/dt = -0.01$ volt/sec.