MA1201 Calculus and Basic Linear Algebra II **Problem Set 2 Techniques of Integration**

Problem 1 (Method of substitution)

Compute the following integral using method of substitution:

(b) $\int x^2 \sec(1-2x^3)\,dx$

let $y = 1-2x^3 \Rightarrow \frac{dy}{dx} = -6x^2 \Rightarrow dx = -\frac{1}{6x^2}dy.$

$$\int x^2 \sec(1-2x^3)\,dx = \int x^2 \sec(1-2x^3)\left(-\frac{1}{6x^2}dy\right) = -\frac{1}{6}\int \sec y\,dy$$

$$= -\frac{1}{6}\ln|\sec y + \tan y| + C. = -\frac{1}{6}\ln|\sec(1-2x^3) + \tan(1-2x^3)| + C$$

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(d) $\int x\cos^2(x^2)\,dx$

let $y = x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow dx = \frac{1}{2x}dy$ $\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$$\int x\cos^2(x^2)\,dx = \int x\cos^2(x^2)\left(\frac{1}{2x}dy\right) = \int \frac{1}{2}\cos^2 y\,dy = \frac{1}{2}\int \frac{1}{2}(\cos(2y) + \cos 0)\,dy$$

$$= \frac{1}{4}\int(\cos(2y) + 1)\,dy = \frac{1}{4}\left(\frac{1}{2}\sin(2y) + y\right) + C.$$

$$= \frac{1}{8}\sin(2x^2) + \frac{1}{4}x^2 + C$$

(f) $\int \frac{e^{2x}}{(1+e^x)^3}\,dx$

let $y = 1+e^x \Rightarrow \frac{dy}{dx} = e^x \Rightarrow dx = \frac{1}{e^x}dy.$

$$\int \frac{e^{2x}}{(1+e^x)^3}\,dx = \int \frac{e^{2x}}{(1+e^x)^3}\left(\frac{1}{e^x}dy\right) = \int \frac{e^x}{(1+e^x)^3}\,dy = \int \frac{y-1}{y^3}\,dy$$

$$= \int y^{-2} - y^{-3}\,dy = \frac{y^{-2+1}}{-2+1} - \frac{y^{-3+1}}{-3+1} + C$$

$$= -y^{-1} + \frac{1}{2}y^{-2} + C = -\frac{1}{1+e^x} + \frac{1}{2(1+e^x)^2} + C.$$

(h) $\displaystyle\int_1^5 \frac{\sin^2(\ln x)}{x}\,dx$

let $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x} \Rightarrow dx = x\,dy$.

$x = 5,\ y = \ln 5,\quad x = 1,\ y = \ln 1 = 0$

$$\int_1^5 \frac{\sin^2(\ln x)}{x}\,dx = \int_0^{\ln 5} \frac{\sin^2(\ln x)}{x}\cdot(x\,dy) = \int_0^{\ln 5} \sin^2 y\,dy$$

$$= \int_0^{\ln 5} -\frac{1}{2}\left[\cos(2y) - \cos 0\right] dy = \int_0^{\ln 5} -\frac{1}{2}(\cos 2y - 1)\,dy$$

$$= -\frac{1}{2}\left(\frac{1}{2}\sin 2y - y\right)\Big|_0^{\ln 5} = -\frac{1}{4}\sin(2\ln 5) + \frac{1}{2}\ln 5$$

(j) $\displaystyle\int \frac{2x+1}{x^2-2x+5}\,dx$

let $y = x^2 - 2x + 5 \Rightarrow \frac{dy}{dx} = 2x - 2 \Rightarrow dx = \frac{1}{2x-2}\,dy$

$\frac{1}{1+(\ )^2}$

$$\int \frac{2x+1}{x^2-2x+5}\,dx = \int \frac{2x-2}{x^2-2x+5}\,dx + \int \frac{3}{x^2-2x+5}\,dx.$$

$$= \int \frac{2x-2}{x^2-2x+5}\left(\frac{1}{2x-2}\,dy\right) + 3\int \frac{1}{(x-1)^2+4}\,dx$$

$$= \int \frac{1}{y}\,dy + \frac{3}{4}\int \frac{1}{\left(\frac{x-1}{2}\right)^2+1}\,dx$$

$$= \ln|y| + \frac{3}{4}\cdot\frac{1}{\frac{1}{2}}\tan^{-1}\left(\frac{x-1}{2}\right) + C.$$

$$= \ln|y| + \frac{3}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C.$$

(l) $\displaystyle\int \frac{1}{x^2\sqrt{1-x^2}}dx$

let $x = \sin\theta \Rightarrow \frac{dx}{d\theta} = \cos\theta \Rightarrow dx = \cos\theta\, d\theta$.

$$\int \frac{1}{x^2\sqrt{1-x^2}}\,dx = \int \frac{1}{x^2\sqrt{1-x^2}} \cos\theta\, d\theta = \int \frac{1}{\sin^2\theta \cancel{\cos\theta}} \cancel{\cos\theta}\, d\theta = \int \frac{1}{\sin^2\theta}\, d\theta.$$

$$u = \frac{\cos\theta}{\sin\theta} \Rightarrow \frac{du}{d\theta} = \frac{-\sin\theta\sin\theta - \cos\theta\cos\theta}{\sin^2\theta} = -\frac{1}{\sin^2\theta} \Rightarrow du = -\frac{1}{\sin^2\theta}\, d\theta.$$

$$= \int -1\, du = -u + C = -\cot\theta + C. = -\frac{\sqrt{1-x^2}}{x} + C.$$

$x = \sin\theta$

$\cos\theta = \sqrt{1-x^2}$

$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{1-x^2}}{x}$

(n) $\displaystyle\int \frac{3x}{\sqrt{4x^2+1}}dx$

let $y = 4x^2+1. \Rightarrow \frac{dy}{dx} = 8x \Rightarrow dx = \frac{1}{8x}dy$.

$$\int \frac{3x}{\sqrt{4x^2+1}}\,dx = \int \frac{3x}{\sqrt{4x^2+1}}\left(\frac{1}{8x}dy\right) = \frac{3}{8}\int \frac{1}{\sqrt{y}}\,dy = \frac{3}{8}\cdot \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C.$$

$y^{-\frac{1}{2}}$

$$= \frac{3}{8}\cdot 2\cdot y^{\frac{1}{2}} + C = \frac{3}{4}\sqrt{4x^2+1} + C.$$

(p) $\displaystyle\int \frac{1}{(x^2+6x+10)^{\frac{3}{2}}}dx = \int \frac{1}{[(x+3)^2+1]^{\frac{3}{2}}}\,dx$

let $x+3 = \tan\theta \Rightarrow \frac{dx}{d\theta} = \sec^2\theta \Rightarrow dx = \sec^2\theta\, d\theta$.

$$= \int \frac{1}{(\tan^2\theta+1)^{\frac{3}{2}}}(\sec^2\theta\, d\theta) = \int \frac{1}{\sec^3\theta}\cdot \sec^2\theta\, d\theta = \int \frac{1}{\sec\theta}\, d\theta = \int \cos\theta\, d\theta$$

$\tan^2\theta + 1 = \sec^2\theta$.

$$= \sin\theta + C. = \frac{x+3}{\sqrt{x^2+6x+10}} + C$$

$\frac{\tan\theta}{\sec\theta} = \sin\theta$.

(r) $\int \sin^3 x \cos^5 x \, dx$

let $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x \Rightarrow dx = \frac{1}{\cos x} dy$. $\quad \cos^2 x = 1 - \sin^2 x$

$$\int \sin^3 x \cos^5 x \, dx = \int \sin^3 x \cos^5 x \left(\frac{1}{\cos x} dy\right) = \int \sin^3 x \cos^4 x \, dy$$

$$= \int y^3 (1-y^2)^2 dy = \int y^3 (1 - 2y^2 + y^4) \, dy$$

$$= \int y^3 - 2y^5 + y^7 \, dy = \frac{1}{4} y^4 - \frac{2}{6} y^6 + \frac{1}{8} y^8 + C.$$

$$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{3} + \frac{\sin^8 x}{8} + C.$$

(b) $\int_1^e \sqrt{x} \ln x \, dx$

let $u = \ln x$ and $dv = \sqrt{x} \, dx \Rightarrow v = \int \sqrt{x} \, dx = \frac{2}{3} x^{\frac{3}{2}}$

$$\int_1^e \sqrt{x} \ln x \, dx = \int_1^e \underbrace{\ln x}_{u} \underbrace{\sqrt{x} \, dx}_{dv} = \underbrace{\frac{2}{3} x^{\frac{3}{2}}}_{v} \cdot \ln x \Big|_1^e - \int_1^e \underbrace{\frac{2}{3} x^{\frac{3}{2}}}_{v} \underbrace{d(\ln x)}_{du.}$$

$$= \frac{2}{3} e^{\frac{3}{2}} \ln e - \frac{2}{3} 1^{\frac{3}{2}} \ln 1 - \int_1^e \frac{2}{3} x^{\frac{3}{2}} \cdot \frac{1}{x} dx.$$

$$= \frac{2}{3} e^{\frac{3}{2}} - \int_1^e \frac{2}{3} x^{\frac{1}{2}} dx = \frac{2}{3} e^{\frac{3}{2}} - \frac{2}{3} \cdot \frac{2}{3} x^{\frac{3}{2}} \Big|_1^e$$

$$= \frac{2}{3} e^{\frac{3}{2}} - \frac{4}{9} e^{\frac{3}{2}} + \frac{4}{9} = \frac{2}{9} e^{\frac{3}{2}} + \frac{4}{9}.$$

(d) $\int x \sin^2 x \, dx = \int x \left(-\frac{1}{2} [\cos(2x) - \cos 0]\right) dx$

$$= -\frac{1}{2} \int \underbrace{x}_{u} \underbrace{\cos(2x) dx}_{dv} + \frac{1}{2} \int x \, dx$$

$dv = \cos(2x) \, dx$

$v = \int dv = \int \cos(2x) \, dx = \frac{1}{2} \sin 2x$

$$= -\frac{1}{2} \left(\overbrace{x \cdot \frac{1}{2} \sin 2x}^{uv} - \int \overbrace{\frac{1}{2} \sin 2x \, dx}^{v\,du.} \right) + \frac{1}{2} \int x \, dx.$$

$$= -\frac{1}{4} x \sin 2x - \frac{1}{2} \left(-\frac{1}{2} \cdot \frac{1}{2} \cos 2x\right) + C_1 + \frac{1}{2} \cdot \frac{1}{2} x^2 + C_2.$$

$$= -\frac{1}{4} x \sin 2x + \frac{1}{8} \cos(2x) + \frac{1}{4} x^2 + C.$$