MA1201 Calculus and Basic Linear Algebra II **Problem Set 2 Techniques of Integration**

Problem 1 (Method of substitution)

Compute the following integral using method of substitution:

(b) $\displaystyle\int x^2 \sec(1-2x^3)\,dx$

let $y = 1-2x^3 \Rightarrow \frac{dy}{dx} = -6x^2 \Rightarrow dx = -\frac{1}{6x^2}dy$

$$\int x^2 \sec(1-2x^3)\,dx = \int x^2 \sec(1-2x^3)\left(-\frac{1}{6x^2}dy\right) = -\frac{1}{6}\int \sec y\,dy$$

$$= -\frac{1}{6}\ln|\sec y + \tan y| + C$$

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$$= -\frac{1}{6}\ln|\sec(1-2x^3) + \tan(1-2x^3)| + C.$$

(d) $\displaystyle\int x\cos^2(x^2)\,dx$

let $y = x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow dx = \frac{1}{2x}dy$

$$\int x\cos^2(x^2)\,dx = \int x\cos^2(x^2)\left(\frac{1}{2x}dy\right) = \frac{1}{2}\int \cos^2 y\,dy$$

$$= \frac{1}{2}\int \frac{1}{2}[\cos(y+y) + \cos(y-y)]\,dy = \frac{1}{4}\int(\cos(2y)+1)\,dy$$

$$= \frac{1}{4}\cdot\left(\frac{1}{2}\sin(2y) + y\right) + C$$

$$= \frac{1}{8}\sin(2x^2) + \frac{1}{4}x^2 + C.$$

(f) $\displaystyle\int \frac{e^{2x}}{(1+e^x)^3}\,dx$

let $y = 1+e^x \Rightarrow \frac{dy}{dx} = e^x \Rightarrow dx = \frac{1}{e^x}dy$

$$\int \frac{e^{2x}}{(1+e^x)^3}\,dx = \int \frac{e^{2x}}{(1+e^x)^3}\cdot\left(\frac{1}{e^x}dy\right) = \int \frac{e^x}{(1+e^x)^3}\,dy = \int \frac{y-1}{y^3}\,dy$$

$$= \int (y^{-2} - y^{-3})\,dy = \frac{y^{-2+1}}{-2+1} - \frac{y^{-3+1}}{-3+1} + C.$$

$$= -\frac{1}{y} + \frac{1}{2y^2} + C = -\frac{1}{1+e^x} + \frac{1}{2(1+e^x)^2} + C.$$

(h) $\displaystyle\int_1^5 \frac{\sin^2(\ln x)}{x}\,dx$

let $y=\ln x \Rightarrow \frac{dy}{dx}=\frac{1}{x} \Rightarrow dx = x\,dy$

$x=5,\ y=\ln 5,\quad x=1,\ y=\ln 1=0.$

$$\int_1^5 \frac{\sin^2(\ln x)}{x}\,dx = \int_0^{\ln 5}\frac{\sin^2(\ln x)}{x}\cdot x\,dy = \int_0^{\ln 5}\sin^2 y\,dy$$

$$= \int_0^{\ln 5} -\frac{1}{2}\left[\cos(y+y)-\cos(y-y)\right]dy$$

$$= \int_0^{\ln 5} -\frac{1}{2}\left[\cos(2y)-1\right]dy = -\frac{1}{2}\cdot\left[\frac{1}{2}\sin(2y)-y\right]\Big|_0^{\ln 5}$$

$$= -\frac{1}{4}\sin(2\ln 5)+\frac{1}{2}\ln 5$$

(j) $\displaystyle\int \frac{2x+1}{x^2-2x+5}\,dx$

let $y=x^2-2x+5 \Rightarrow \frac{dy}{dx}=2x-2 \Rightarrow dx=\frac{1}{2x-2}\,dy$

$\rightarrow (\tan^{-1}x)' = \frac{1}{1+x^2}$

$$\int\frac{2x+1}{x^2-2x+5}\,dx = \int\frac{2x-2}{x^2-2x+5}\,dx + \int\frac{3}{x^2-2x+5}\,dx$$

$$= \int\frac{2x-2}{x^2-2x+5}\left(\frac{1}{2x-2}\,dy\right) + 3\int\frac{1}{(x-1)^2+4}\,dx$$

$$= \int\frac{1}{y}\,dy + \frac{3}{4}\int\frac{1}{1+\left(\frac{x-1}{2}\right)^2}\,dx$$

$$= \ln|y| + C_1 + \frac{3}{4}\cdot\frac{1}{\frac{1}{2}}\tan^{-1}\left(\frac{x-1}{2}\right) + C_2$$

$$= \ln|x^2-2x+5| + \frac{3}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C$$

(l) $\displaystyle\int \frac{1}{x^2\sqrt{1-x^2}}\,dx$ $\qquad$ $\sin^2\theta+\cos^2\theta = 1.$

$$\text{let } x=\sin\theta \Rightarrow \frac{dx}{d\theta}=\cos\theta \Rightarrow dx=\cos\theta\, d\theta.$$

$$\int \frac{1}{x^2\sqrt{1-x^2}}\,dx = \int \frac{1}{\sin^2\theta\cdot\cancel{\cos\theta}}\cdot\cancel{\cos\theta}\, d\theta = \int \frac{1}{\sin^2\theta}\, d\theta.$$

$$\text{let } u=\frac{\cos\theta}{\sin\theta} \Rightarrow \frac{du}{d\theta}=\frac{-\sin\theta\sin\theta-\cos\theta\cos\theta}{\sin^2\theta} = -\frac{1}{\sin^2\theta} \Rightarrow d\theta = -\sin^2\theta\, du.$$

$$= \int \frac{1}{\sin^2\theta}\cdot(-\sin^2\theta\, du) = \int -1\, du = -u + C = -\cot\theta + C$$

$$= -\frac{\sqrt{1-x^2}}{x} + C.$$

$x=\sin\theta$

$\cos\theta = \sqrt{1-x^2}$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{\sqrt{1-x^2}}{x}$

(n) $\displaystyle\int \frac{3x}{\sqrt{4x^2+1}}\,dx$

$$\text{let } y=4x^2+1 \Rightarrow \frac{dy}{dx}=8x \Rightarrow dx=\frac{1}{8x}\,dy$$

$$\int \frac{3x}{\sqrt{4x^2+1}}\,dx = \int \frac{3x}{\sqrt{4x^2+1}}\cdot\left(\frac{1}{8x}\,dy\right) = \frac{3}{8}\int \frac{1}{\sqrt{y}}\,dy = \frac{3}{8}\int y^{-\frac{1}{2}}\,dy = \frac{3}{8}\, 2y^{\frac{1}{2}} + C$$

$$= \frac{3}{4}\sqrt{4x^2+1} + C.$$

(p) $\displaystyle\int \frac{1}{(x^2+6x+10)^{\frac{3}{2}}}\,dx$

$$\text{let } y=x^2+6x+10 \Rightarrow \frac{dy}{dx}=2x+6 \Rightarrow dx=\frac{1}{2x+6}\,dy.\ \times.$$

$$\int \frac{1}{(x^2+6x+10)^{\frac{3}{2}}}\,dx = \int \frac{1}{((x+3)^2+1)^{\frac{3}{2}}}\,dx = \int \frac{1}{(\tan^2\theta+1)^{3/2}}\cdot\sec^2\theta\, d\theta.$$

$$\text{let } x+3=\tan\theta \Rightarrow \frac{dx}{d\theta}=\sec^2\theta \Rightarrow dx=\sec^2\theta\, d\theta.$$

$\tan^2\theta+1=\sec^2\theta$

$$= \int \frac{1}{(\sec^2\theta)^{\frac{3}{2}}}\cdot\sec^2\theta\, d\theta = \int \frac{1}{\sec\theta}\,d\theta = \int \cos\theta\, d\theta = \sin\theta + C$$

$$= \frac{x+3}{\sqrt{x^2+6x+10}} + C.$$

$$\left.\begin{aligned}\tan\theta &= x+3\\ \sec\theta &= \sqrt{x^2+6x+10}\end{aligned}\right\} \Rightarrow \sin\theta = \frac{\tan\theta}{\sec\theta}$$

(r) $\int \sin^3 x \cos^5 x \, dx$

let $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x \Rightarrow dx = \frac{1}{\cos x} dy$. $\Rightarrow \cos^2 x = 1 - \sin^2 x$.

$\int \sin^3 x \cos^5 x \, dx = \int \sin^3 x \cos^5 x \cdot \left(\frac{1}{\cos x} dy\right) = \int \sin^3 x \cos^4 x \, dy$.

$= \int y^3 (1-y^2)^2 dy = \int y^3 (1 - 2y^2 + y^4) \, dy = \int (y^3 - 2y^5 + y^7) dy$

$= \frac{y^4}{4} - \frac{2}{6} y^6 + \frac{1}{8} y^8 + C$.

$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{3} + \frac{\sin^8 x}{8} + C$.

Problem 2 (Integration by parts)

Compute the following integrals using integration by parts:

(b) $\int_1^e \sqrt{x} \ln x \, dx$

$u = \ln x$, $dv = \sqrt{x} \, dx \Rightarrow v = \int dv = \int \sqrt{x} \, dx = \frac{2}{3} x^{\frac{3}{2}}$

$\int_1^e \sqrt{x} \ln x \, dx = \int_1^e \ln x \sqrt{x} \, dx = \frac{2}{3} x^{\frac{3}{2}} \ln x \Big|_1^e - \int_1^e \frac{2}{3} x^{\frac{3}{2}} d(\ln x)$

$= \frac{2}{3} e^{\frac{3}{2}} \ln e - \frac{2}{3} \ln 1 - \int_1^e \frac{2}{3} x^{\frac{3}{2}} \frac{1}{x} dx$

$= \frac{2}{3} e^{\frac{3}{2}} - \int_1^e \frac{2}{3} x^{\frac{1}{2}} dx = \frac{2}{3} e^{\frac{3}{2}} - \frac{2}{3} \cdot \frac{2}{3} x^{\frac{3}{2}} \Big|_1^e$

$= \frac{2}{3} e^{\frac{3}{2}} - \frac{4}{9} e^{\frac{3}{2}} + \frac{4}{9} \cdot 1 = \frac{2}{9} e^{\frac{3}{2}} + \frac{4}{9}$.

(d) $\int x \sin^2 x \, dx = \int x \left[-\frac{1}{2} [\cos(x+x) - \cos(x-x)]\right] dx$

$= -\frac{1}{2} \int (x \cos(2x) - x) dx = -\frac{1}{2} \int x \cos 2x \, dx + \frac{1}{2} \int x \, dx$

$u = x$

$dv = \cos 2x \, dx$

$v = \int dv = \int \cos 2x \, dx$

$= \frac{1}{2} \sin 2x$

$= -\frac{1}{2} \left[\frac{1}{2} x \sin 2x - \int \frac{1}{2} \sin 2x \cdot dx \right] + \frac{1}{2} \int x \, dx$

$= -\frac{1}{4} x \sin 2x + \frac{1}{2} \cdot \frac{1}{2} \cdot \left(-\frac{1}{2} \cos 2x\right) + C_1 + \frac{1}{2} \cdot \frac{1}{2} x^2 + C_2$

$= -\frac{1}{4} x \sin 2x - \frac{1}{8} \cos 2x + \frac{1}{4} x^2 + C$.