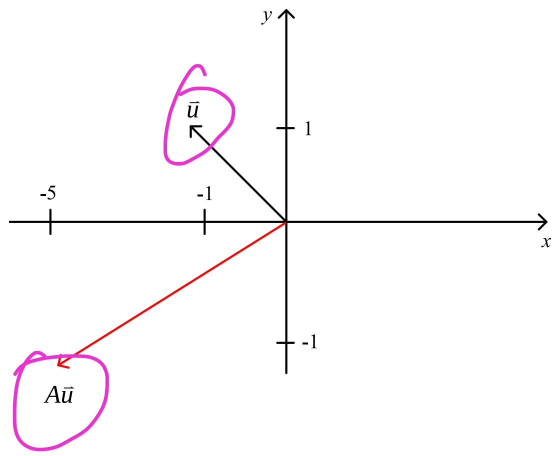
Consider

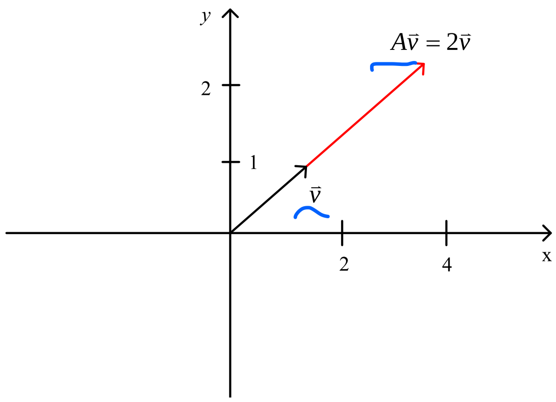
$$A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}, \quad \vec{u} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Then

$$A\vec{u} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$$



$$A\vec{v} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\vec{v}$$



Definition: Given a $n \times n$ matrix A, a nonzero vector $\vec{x}$ is called eigenvector of A if for some scalar λ,

$$A\vec{x} = \lambda\vec{x}.$$

On the other hand, a scalar λ is called eigenvalue of A if there is a nontrivial solution such that

$$A\vec{x} = \lambda\vec{x}$$

and $\vec{x}$ is called eigenvector corresponding to λ.

Remark: Any eigenvector is a nonzero vector.

Example. $A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Since

$$A\vec{v} = 2\vec{v},$$

$\vec{v}$ is an eigenvector of A corresponding to the eigenvalue 2.

Question: Are there other eigenvalues and eigenvectors for A

$\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -2 \\ -1 \end{pmatrix} = 2\begin{pmatrix} -2 \\ -1 \end{pmatrix}$ $\qquad A\, a\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \cdot a\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

all $\left\{ k\begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$ is of A corresponding to $\lambda = 2$

$\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is also an eigenvector of A corr. $\lambda = 1$

$\left\{ k\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ is eigenspace of A corresponding to $\lambda = 1$

Example Show that 7 is an eigenvalue of $A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$.

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is eigenvector cor. to $\lambda = 7$.

Sol.

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 7\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - 7\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow \left[\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} - \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 1 \end{cases}$$

has nontrivial sol.

$\Updownarrow$

Idea: $\det\begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix} = 0 \Longleftrightarrow \begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix}$ not invertible

$= (-6)\cdot(-5) - 5\cdot 6$

$$A\vec{x} = \lambda\vec{x} \Longrightarrow A\vec{x} = \lambda I_n\vec{x}$$
$$\Longrightarrow A\vec{x} - \lambda I_n\vec{x} = \vec{0}$$
$$\Longrightarrow (A - \lambda I_n)\vec{x} = \vec{0}$$

Finding λ to make $A\vec{x} = \lambda\vec{x}$ has nontrivial solution is equal to make

$$\boxed{\det(A - \lambda I_n) = 0.}$$

Definition: For a $n \times n$ matrix A, $\det(A - \lambda I_n)$ is a polynomial in λ, called as characteristic polynomial of A. And, $\det(A - \lambda I_n) = 0$ is called the characteristic equation of A.

$\boxed{\textbf{Question:}}$ Given a square matrix A, how to find its all eigenvalues and eigenvectors.

Example: $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}_{2\times 2}$. Find all eigenvalus and eigenvectors of A.

Sol:

$$A\vec{x} = \lambda \vec{x} \Leftrightarrow (A - \lambda I_2)\vec{x} = \vec{0}.$$

$$A - \lambda I_2 = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1-\lambda & 4-0 \\ 2-0 & 3-\lambda \end{pmatrix}$$
$$= \begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix}$$

① characteristic poly

$$\det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} =$$
$$= (1-\lambda)(3-\lambda) - 2 \cdot 4$$
$$= \lambda^2 - 4\lambda + 3 - 8$$
$$= \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1).$$
$$= (\lambda - 5)(\lambda + 1) = 0$$

$\checkmark$ $\begin{matrix} 1 & & -5 \\ & \times & \\ 1 & & 1 \end{matrix} = 1 - 5 = -4$

$\begin{matrix} 1 & 5 \\ 1 & -1 \end{matrix} \Rightarrow -1 + 5 = +4$

② solve characteristic equation

$$
\begin{aligned}
&\det(A-\lambda I_2)=0\\
\Longrightarrow&(\lambda-5)(\lambda+1)=0\\
\Longrightarrow&\lambda=5,-1
\end{aligned}
$$

So A has two eigenvalues $5,-1$.

③ Solve $(A-\lambda I_2)\vec{x}=\vec{0}$, for each eigenvalue, $A\vec{x}=\lambda\vec{x}$

when $\lambda=5$.

$$\begin{pmatrix}-4 & 4\\ 2 & -2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}-4 & 4 & 0\\ 2 & -2 & 0\end{pmatrix}\xrightarrow[-\frac{1}{4}R_1]{2R_2+R_1\to R_1}\begin{pmatrix}\overset{\text{basic}}{1} & \overset{\text{free}}{-1} & 0\\ 0 & 0 & 0\end{pmatrix}.$$

reduced echelon.

So $x_1-x_2=0\Rightarrow x_1=x_2$

$$\vec{x}=\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}x_2\\ x_2\end{pmatrix}=x_2\begin{pmatrix}1\\ 1\end{pmatrix}\xlongequal{\text{let } x_2=k}k\begin{pmatrix}1\\ 1\end{pmatrix},\quad k\neq 0.$$

Hence, $\begin{pmatrix}1\\ 1\end{pmatrix}$ is one eigenvector of A corresponding to 5. $\left\{k\begin{pmatrix}1\\ 1\end{pmatrix}: k\neq 0\right\}$ is the set of all eigenvectors of A corresponding to 5.

eigenspace.

When $\lambda = -1$,

$$(A - (-1)I_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \left[\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} 2 & 4 & 0 \\ 2 & 4 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$$

so $x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$.

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} -2 \\ 1 \end{pmatrix} = k\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

So, $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is one eigenvector of A corresponding to -1. $\left\{k\begin{pmatrix} -2 \\ 1 \end{pmatrix} : k \neq 0\right\}$ is eigenspace of A corresponding to -1.

Practice: Find eigenvalues of $A = \begin{pmatrix} 5 & 3 \\ -3 & -1 \end{pmatrix}$ and the corresponding eigenvectors.

$$\det\left[\begin{pmatrix} 5 & 3 \\ -3 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] = \det\begin{pmatrix} 5-\lambda & 3 \\ -3 & -1-\lambda \end{pmatrix}$$

$$= (5-\lambda)(-1-\lambda) + 9 = \lambda^2 - 4\lambda - 5 + 9 = \lambda^2 - 4\lambda + 4$$

$$= (\lambda-2)^2 = 0 \Longrightarrow \lambda = 2$$

multiplicity of $\lambda = 2$ is 2

$$\begin{pmatrix} 5-2 & 3 \\ -3 & -1-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & 3 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left\{ c\begin{pmatrix} 1 \\ -1 \end{pmatrix},\ c \neq 0 \right\}$$

Strategy to find eigenvalues and eigenvectors of $A_{n\times n}$.

① $A - \lambda I_n$

② characteristic polynomial

$$\det(A - \lambda I_n)$$

③ characteristic equation

$$\det(A - \lambda I_n) = 0$$

④ Solve $(A - \lambda I_n)\vec{x} = \vec{0}$ for each eigenvalue.

Example For $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}_{3\times 3}$, find its eigenvalues and eigenvectors.

①

$$A - \lambda I_3 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix}$$

② characteristic poly

$$\det(A - \lambda I_3) = (1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 1-\lambda \end{vmatrix} + 0$$

$$= (1-\lambda)\left[(2-\lambda)(1-\lambda) - 1\right] + (\lambda - 1 - 0) + 0$$

$$= (1-\lambda)\left[\lambda^2 - 3\lambda + 1\right] + (\lambda - 1)$$

$= (\lambda-1)[-\lambda^2+3\lambda] = (\lambda-1)\lambda(3-\lambda).$

$$\det(A-\lambda I_3) = \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} + (-1)^3(-1)\begin{vmatrix} -1 & -1 \\ 0 & 1-\lambda \end{vmatrix} + 0\begin{vmatrix} -1 & 2-\lambda \\ 0 & -1 \end{vmatrix}$$

$$= (1-\lambda)(\lambda^2-3\lambda) = (1-\lambda)\lambda(\lambda-3)$$

③ characteristic equation

$$\det(A-\lambda I_3) = 0$$

$$(1-\lambda)\lambda(\lambda-3) = 0$$

$$\Longrightarrow \lambda = 0, 1, \text{ or } 3.$$

④ Solve $(A-\lambda I_3)\vec{x} = \vec{0}$ for each eigenvalue.

when $\lambda = 0$

$$A\vec{x} = \vec{0} \Longrightarrow \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1+R_2 \to R_1} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_3 free

$x_1 = x_3$
$x_2 = x_3$

$$\Longrightarrow \begin{cases} x_1 - x_2 = 0 \\ x_2 - x_3 = 0 \end{cases} \Longrightarrow \begin{cases} x_1 = x_2 = x_3 \\ x_2 = x_3 \end{cases}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \boxed{K \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}, K \neq 0$$

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to $\lambda = 0$.

$\det A = 0$

If A is not invertible, then 0 must be an eigenvalue.

when $\lambda = 1$ ..

when $\lambda = 3$ ..

Example $A = \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}_{2\times 2}$

①

$$A - \lambda I_2 = \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$
$$= \begin{pmatrix} 1-\lambda & -1 \\ 4 & 1-\lambda \end{pmatrix}$$

②

$$\det(A - \lambda I_2) = \begin{vmatrix} 1-\lambda & -1 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4$$
$$= \lambda^2 - 2\lambda + 1 + 4$$
$$= \lambda^2 - 2\lambda + 5. = (\lambda - 1)^2 + 4 = 0$$

$i^2 = -1$

$\Rightarrow (\lambda - 1)^2 = -4$

$\Rightarrow \lambda - 1 = \pm\sqrt{-4} = \pm 2i$

$\Rightarrow \lambda = 1 \pm 2i.$

③ $\lambda^2 - 2\lambda + 5 = 0.$

Recall: $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1, b = -2, c = 5$

$$\begin{aligned}
\lambda &= \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} \\
&= \frac{2 \pm \sqrt{-16}}{2} \overset{i^2=-1}{=\!=\!=} \frac{2 \pm 4i}{2} \\
&= 1 \pm 2i
\end{aligned}$$

So the eigenvalues of A are $1 + 2i$ and $1 - 2i$.

④ Solve $(A - \lambda I_2)\vec{x} = \vec{0}$

when $\lambda = 1 - 2i$.

$$\begin{pmatrix} 2i & -1 \\ 4 & 2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2i & -1 & 0 \\ 4 & 2i & 0 \end{pmatrix} \xrightarrow{\frac{1}{2i} r_1} \begin{pmatrix} 1 & \frac{-1}{2i} & 0 \\ 4 & 2i & 0 \end{pmatrix} \xrightarrow[\frac{1}{i}=-i]{-4\times r_1 + r_2} \begin{pmatrix} 1 & \frac{-1}{2i} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned}
x_1 - \frac{1}{2i}x_2 &= 0 \\
\Longrightarrow x_1 + \frac{1}{2}ix_2 &= 0 \\
\Longrightarrow x_1 &= \left(-\frac{1}{2}i\right) x_2
\end{aligned}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}ix_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -\frac{1}{2}i \\ 1 \end{pmatrix}$$

$\begin{pmatrix} -\frac{1}{2}i \\ 1 \end{pmatrix}$ is one eigenvector of A corresponding to $\lambda = 1 - 2i$.

when $\lambda = 1 + 2i$.

Recall: Given $A_{n\times n}$. eigenvalue λ, eigenvector $\vec{x} \neq 0$: such that

$$A\vec{x} = \lambda\vec{x}.$$

Key idea

$$A\vec{x} = \lambda\vec{x} \iff (A - \lambda I_n)\vec{x} = \vec{0}.$$

characteristic poly in λ: $\det(A - \lambda I_n)$

characteristic equation: $\det(A - \lambda I_n) = 0$

Remarks:

① eigenvalue and eigenvector are only defined for square matrix.

② eigenvalue could be complex scalar.

$P_n(\lambda) = 0$ must have.

n solutions in account multiplic[ity] and complex.

eg. $A = \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}$, $\lambda = 1 \pm 2i$.

③ If $A_{n\times n}$ is a singular matrix ($\det A = 0$), then 0 must be one eigenvalue of A.

not invertible

④ For a $n \times n$ matrix A, it has at most n distinct eigenvalues, at most n linearly independent eigenvectors.

⑤ For one eigenvalue, it has infinite many eigenvectors, but only finitely many linearly independent eigenvectors.

eg. $A = \begin{pmatrix} 3 & 3 & 3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

A has two eigenvalues $\lambda_1 = 0$, $\lambda_2 = 4$.

For $\boxed{\lambda_1 = 0,}$

$$(A - 0I_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\implies \begin{pmatrix} 3 & 3 & 3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 3 & 3 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{array}{c} \begin{array}{cccc} \text{basic} & \text{free} & \text{free} & \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix}$$

$$x_1 + x_2 + x_3 = 0 \implies x_1 = -x_2 - x_3.$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix} = \boxed{x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}.$$

So for $\lambda = 0$, $\underset{\vec{v}_1}{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$, $\underset{\vec{v}_2}{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$ are two linearly independent eigenvectors.

$x_2 = 1$, $x_3 = 0$ $\qquad$ $x_2 = 0$, $x_3 = 1$

$$\left\{ a\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},\ \begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

eigenspace of A for $\lambda = 0$

Properties of eigenvalue and eigenvectors: Let A be a $n \times n$ matrix, λ is an eigenvalue of A, $\vec{v}$ is an eigenvector corresponding for λ. Then:

① A^k has eigenvalue λ^k with eigenvector $\vec{v}$. k integer.

② kA has eigenvalue $k\lambda$ with eigenvector $\vec{v}$. for any $k \in \mathbb{R}$.

③ A^{-1}(if A is invertible $\Rightarrow \lambda \neq 0$) has eigenvalue $\frac{1}{\lambda}$, with eigenvector $\vec{v}$.

④ A^T has eigenvalue λ with different eigenvector $\vec{v}$.

⑤ $A + kI_n$ has eigenvalue $\lambda + k$ with eigenvector $\vec{v}$.

Proof: ① take $k=3$ as example.

$$A\vec{v} = \lambda\vec{v}$$

$$\xRightarrow{\text{multiply by } A} A(A\vec{v}) = A\cdot(\lambda\vec{v})$$

$$\Longrightarrow A^2\vec{v} = \lambda A\vec{v} = \lambda\cdot(\lambda\vec{v}) = \lambda^2\vec{v}$$

$$\xRightarrow{\text{multiply by } A} A^3\vec{v} = \lambda^2 A\vec{v} = \lambda^3\vec{v}$$

③

$$A\vec{v} = \lambda\vec{v}$$

$$\xRightarrow{\text{multiply by } A^{-1}} A^{-1}A\vec{v} = \lambda A^{-1}\vec{v}$$

$$\Longrightarrow I_n\vec{v} = \vec{v} = \lambda A^{-1}\vec{v}$$

$$\Longrightarrow \frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$$

$\frac{1}{\lambda}$ is eigenvalue of A^{-1}

 $(\lambda+1)(\lambda-5)$ $(\lambda-1)(\lambda-5)$.

eg. $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$

Definition: An $n \times n$ matrix A is said to be diagonalizable if there is a non-singular(invertible) $n \times n$ matrix P such that $A = PDP^{-1}$,where D is a diagonal matrix.

$\Leftrightarrow \det(P) \neq 0$

$\Leftrightarrow$ Columns of P is linear. indep.

Remarks:

① A and $P^{-1}AP = D$ have the same characteristic polynomial thus the same eigenvalues.

$\det(A - \lambda I_n) = \det(PDP^{-1} - \lambda I_n) = \det(PDP^{-1} - \lambda PP^{-1})$
$= \det P[D - \lambda I]P^{-1}$
$= \det P \cdot \det(D - \lambda I_n) \det P^{-1}$
$= \det(D - \lambda I_n)$

② The eigenvalues of D are just the main diagonal entries.

③ Not every square matrix is diagnolizable.

④ $A = PDP^{-1}$ equivalent $AP = PD$ equivalent $D = P^{-1}AP$.

Example: Show that $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagnolizable.

if A is diagolizable

$\det\left[A - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] = \det\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2$ $\quad \lambda = 1$

So D has to be $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

But for any non-singular P, $\quad PDP^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq A$

So A can not be diagonalizable.

Example: Show that $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ is diagonalizable.

$A = \underbrace{\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}}_{P}\begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$

$D = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$.

$AP = PD$.

$[A\vec{v}_1, A\vec{v}_2] \qquad [\lambda_1\vec{v}_1, \lambda_2\vec{v}_2]$

Proof:

For $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$, we have eigenvalues $\lambda_1 = 5, \lambda_2 = -1$(they are distinct).

For $\lambda_1 = 5$, one of the corresponding eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = -1$, one of the corresponding eigenvector is $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Let $P = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$, as the eigenvectors of A are linearly independent,
$P^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}^{-1}$ exists and $P^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}$.

Also $P^{-1}AP = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}$.

It concludes that $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ is diagonalizable.