5. Energy and Power Signals

Energy E of a signal $x(t)$ (or $x[n]$) is defined as

$$\mathbf{E} = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad \text{for CT signal,} \quad \mathbf{E} = \sum_{n=-\infty}^{\infty} |x[n]|^2 \quad \text{for DT signal,} \tag{1.6}$$

whereas the Power P of a signal is defined as follows

Periodic signal T = Period.

Non-periodic signal T.

$$\mathbf{P} = \begin{cases} \underline{\displaystyle\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt} & \text{for CT signal,} \\ \displaystyle\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 & \text{for DT signal} \end{cases} \tag{1.7}$$

$= \lim_{T\to\infty} \frac{1}{T} \int_0^T |x(t)|^2 dt$

$\left[-\frac{T}{2}, \frac{T}{2}\right]$

$[0, T]$

- **Energy signal** has finite energy and zero power

$$0 < \mathbf{E} < \infty, \quad \mathbf{P} = 0.$$

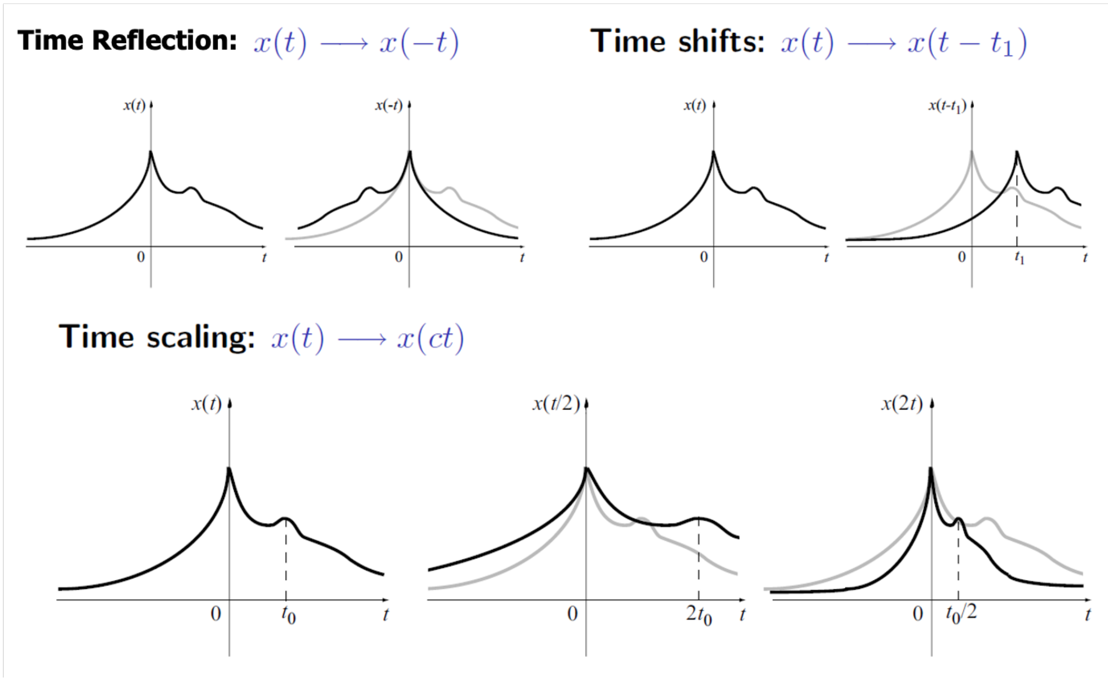
- **Power signal** has finite power and infinite energy

$$0 < \mathbf{P} < \infty, \quad \mathbf{E} = \infty.$$

- Signals that satisfy neither property are neither energy signals nor power signals.

1.2 Basic Signal Operations

- **Time Reversal**: Flip the signal around the vertical axis $\quad x(t) \to x(-t)$
- **Time Shifts**: Shift the signal to left or right $\quad x(t) \to x(t - t_0)$
  - **Right-shift** if $t_0 > 0$, **Left-shift** if $t_0 < 0$.
- **Time Scaling**: Linearly stretch or compress the signal $\quad x(t) \to x(ct)$
  - **Compression** if $|c| > 1$, **Expansion** if $|c| < 1$.
- **Affine Transformation**: $x(t) \to x(\alpha t + \beta) = x(\alpha(t + \beta/\alpha))$ for any real α, β
  - Step 1. **Scale by** α. If $\alpha < 0$, reflection across y-axis
  - Step 2. **Shift by** $-\beta/\alpha$.
    - If α and β have different signs, right-shift.
    - If α and β have same signs, left shift.



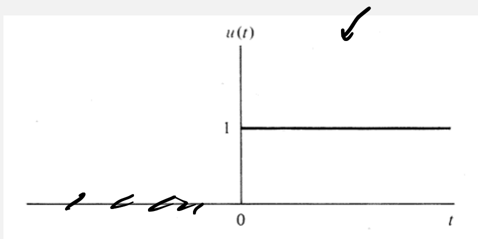
1.3 Example of Important Signals

1. **Unit Step Function** (also referred as *Heaviside unit function*)

- **Definition**

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0, \end{cases} \tag{1.8}$$

- **Properties**
  - Aperiodic signal
  - Power signal $\mathbf{P} = 1/2$
  - Infinite Energy $\mathbf{E} = \infty$



Functions related to the step function $u(t)$

$E = \int_0^\infty u(t)^2 dt = \int_0^\infty 1 \cdot dt = \infty$

a) **Signum Function**

- Definition

$$\boxed{sgn\,(t) = 2u(t) - 1} = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$

$[-\frac{T}{2}, \frac{T}{2}]$

$P = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)^2 dt = \lim_{T\to\infty} \frac{1}{T} \int_0^{T/2} u(t)^2 dt$

$= \lim \frac{1}{T}\left[\cancel{\int_{-\frac{T}{2}}^{0} u(t)^2 dt} + \int_0^{T/2} u(t)^2 dt\right]$

$= \lim_{T\to\infty} \frac{1}{\cancel{T}} \cdot \frac{\cancel{T}}{2} = \frac{1}{2}$

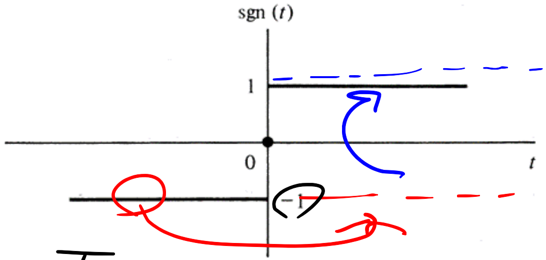
$E=\int_{-\infty}^{\infty} sgn^2(t)\,dt = \int_{-\infty}^{0} 1\cdot dt + \int_{0}^{\infty} 1\cdot dt = \infty$

- Properties

$P=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} sgn^2(t)\,dt$

$=\frac{1}{T}\int_{-T/2}^{0} 1\cdot dt + \frac{1}{T}\int_{0}^{T/2} 1\cdot dt = \frac{1}{T}\cdot T = 1$

  - Aperiodic & odd signal
  - Power signal **P** = 1
  - Infinite Energy **E** = ∞



b) **Ramp Function**

- Definition

$$\boxed{r(t)=\begin{cases} t, & t\geq 0 \\ 0, & t<0 \end{cases}},\qquad \int_{-\infty}^{t} u(\tau)\,d\tau = r(t)$$

- Properties

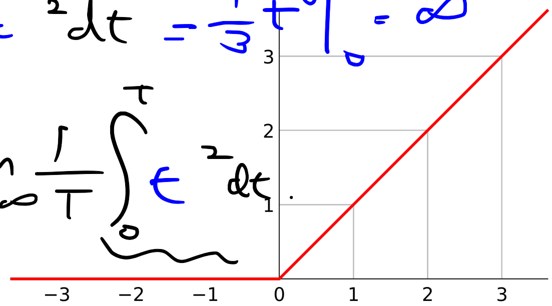
$E=\int_{0}^{\infty} t^2\,dt = \frac{1}{3}t^3\Big|_0^{\infty} = \infty$

[0, T]

[−T/2, T/2]

  - Aperiodic
  - Infinite Power **P** = ∞
  - Infinite Energy **E** = ∞

$P=\lim_{T\to\infty}\frac{1}{T}\int_{0}^{T} t^2\,dt$



$=\lim\frac{1}{3T}\left\{t^3\Big|_0^T\right\} = \lim\frac{T^3}{3T}$

c) **Rectangular Pulse**

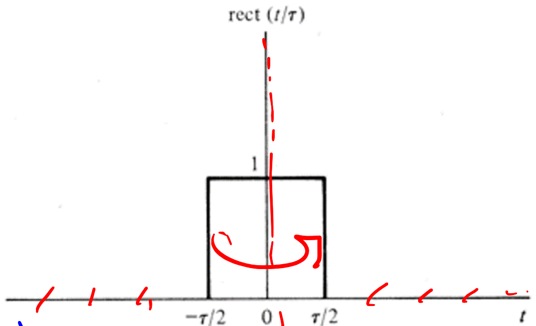
- Definition

$$rect\,(t/\tau)=\begin{cases} 1, & |t|<\frac{\tau}{2} \\ 0, & |t|>\frac{\tau}{2} \end{cases},\qquad rect\,(t/\tau)=u\left(t+\frac{\tau}{2}\right)-u\left(t-\frac{\tau}{2}\right) \quad (1.9)$$

$=\lim_{T\to\infty}\frac{T^2}{3} = \infty$

- Properties

  - Aperiodic & Even signal
  - Zero Power **P** = 0
  - Energy Signal **E** = τ



$E=\int_{-\tau/2}^{\tau/2} 1\cdot dt = \tau$

$P=\lim_{T\to\infty}\frac{1}{T}\int_{-\tau/2}^{\tau/2} 1\cdot dt = \lim_{T\to\infty}\frac{1}{T}\tau = 0$

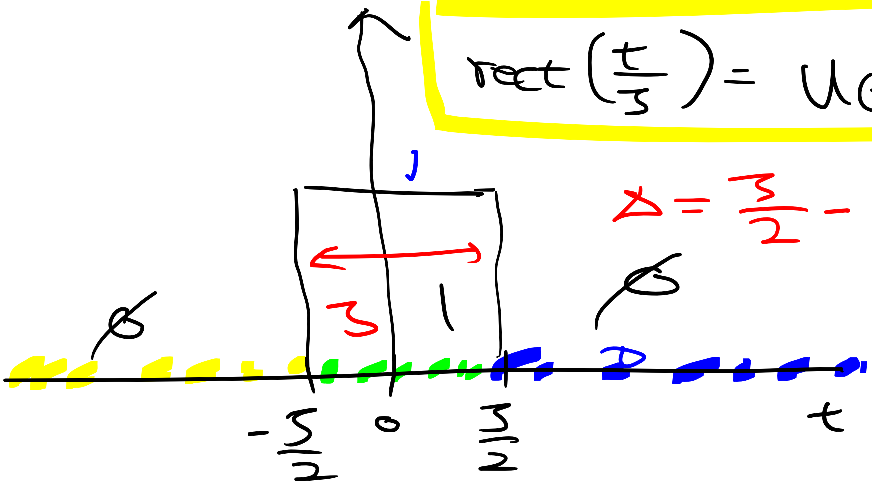
$$\int_{-\infty}^{t} \boxed{u(\tau)}\, d\tau = \begin{cases} \text{if } t>0. & \int_0^t 1\cdot d\tau = t \\ t<0. & \int_{-\infty}^{t} 0\cdot d\tau = \phi \end{cases}$$

$$= \delta(t).$$

$$\text{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1, & |t| < \frac{\tau}{2} \\ 0, & |t| > \frac{\tau}{2} \end{cases}$$

$$\boxed{\text{rect}\left(\frac{t}{\tau}\right) = u\left(t+\frac{\tau}{2}\right) - u\left(t-\frac{\tau}{2}\right)}$$

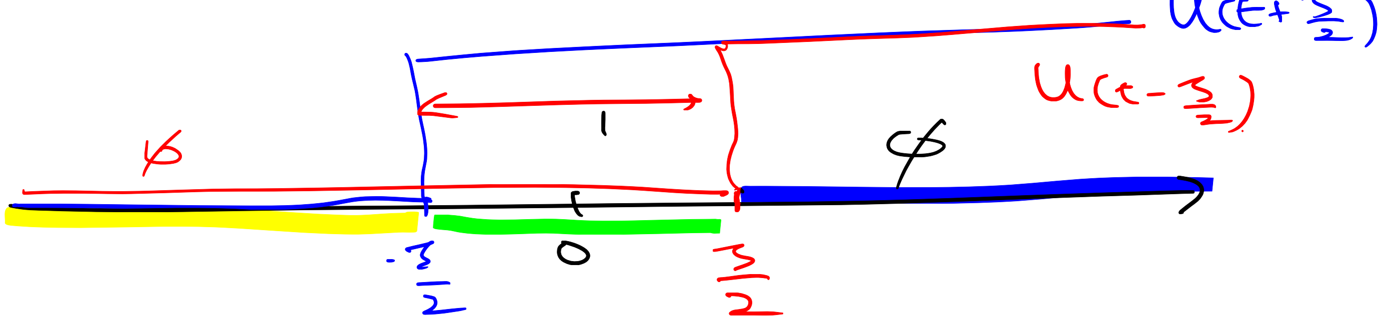
$$\Delta = \frac{\tau}{2} - \left(-\frac{\tau}{2}\right) = \tau$$



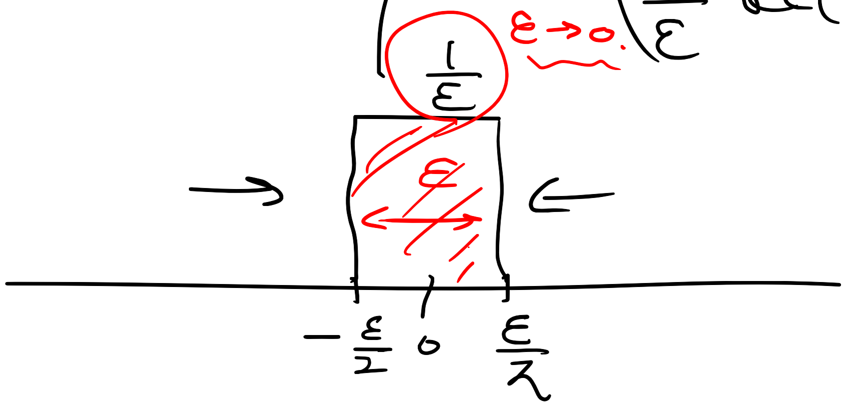
$u(t-t_0)$

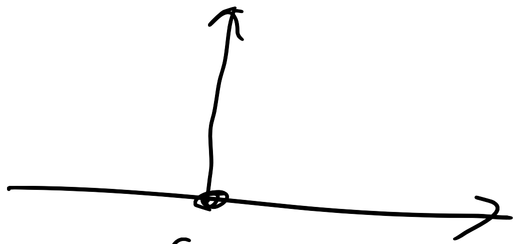
$$\overset{1}{u\left(t+\frac{\tau}{2}\right)} - \overset{\phi}{u\left(t-\frac{\tau}{2}\right)}$$

$t_0 = -\frac{\tau}{2}$ $\quad$ $t_0 = \frac{\tau}{2}$



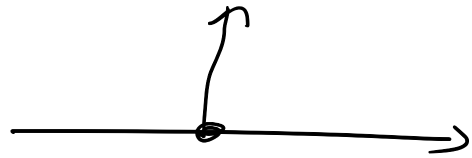
$$\lim_{\varepsilon \to 0}\left(\frac{1}{\varepsilon}\operatorname{rect}\left(\frac{t}{\varepsilon}\right)\right) = \delta(t)$$





$t=0$

$\delta(t-t_0)$



$t=t_0$

$$\frac{1}{\varepsilon}\operatorname{rect}\left(\frac{t}{\varepsilon}\right) = \delta_\varepsilon(t)$$

$$\lim_{\varepsilon \to 0} \boxed{\delta_\varepsilon(t)} = \delta(t)$$

$$\delta_\varepsilon(t)\Big|_{t=0} = \frac{1}{\varepsilon}$$

even function

$$\delta_\varepsilon(t) = 0, \quad |t| > \frac{\varepsilon}{2}$$

$$\lim_{\varepsilon \to 0}\int_{-\infty}^{\infty}\delta_\varepsilon(t)\,dt = 1 \longrightarrow \int_{-\infty}^{\infty}\delta(t)\,dt = 1.$$

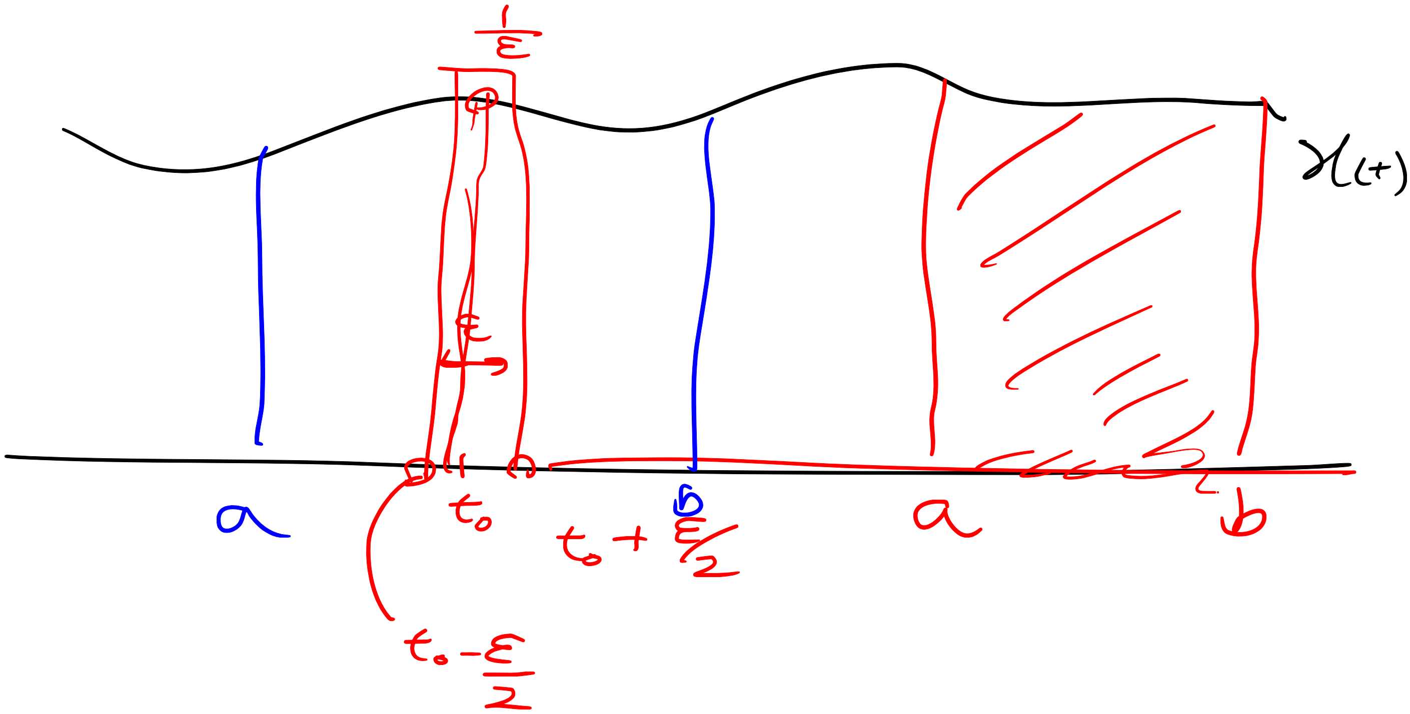
$$\delta(t)\Big|_{t=0} = \infty$$

even function

$$\delta(t) = 0, \quad t \neq 0.$$

$$\int_a^b x(t)\,\underline{\delta(t-t_0)}\,dt = \begin{cases} x(t_0), & t_0 \in [a,b] \\ \boxed{0 \quad , \quad t_0 \notin [a,b]} \end{cases}$$

$$\boxed{\lim_{\varepsilon \to 0}\delta_\varepsilon(t-t_0)}$$



$$a \to -\infty$$

$$b \to \infty$$

$$\boxed{\int_{-\infty}^{\infty} x(\tau)\,\delta(\tau - t)\,d\tau = x(t)}$$

2. **Unit Impulse Function** (also referred as *Direc delta function*)

- **Definition**

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}, \qquad \int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{1.10}$$

- **Properties**
  - **Sampling Property**

$$x(t)\delta(t - t_0) = x(t_0)\,\delta(t - t_0)$$

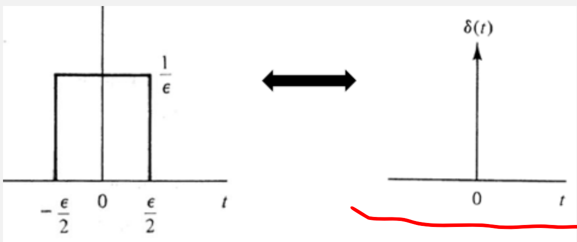
  - **Sifting Property**

$$\int_a^b x(t)\delta(t - t_0)\, dt = \begin{cases} x(t_0), & \text{if } a < t_0 < b \\ 0, & \text{otherwise} \end{cases}$$

  - Impulse function is the *building block of any signal, i.e.*, arbitrary signal can be respresented as an infinite sum of impulse function and signal amplitude.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t - \tau)\, d\tau \tag{1.11}$$

Relationship between Rectangular Pulse and Impulse Function



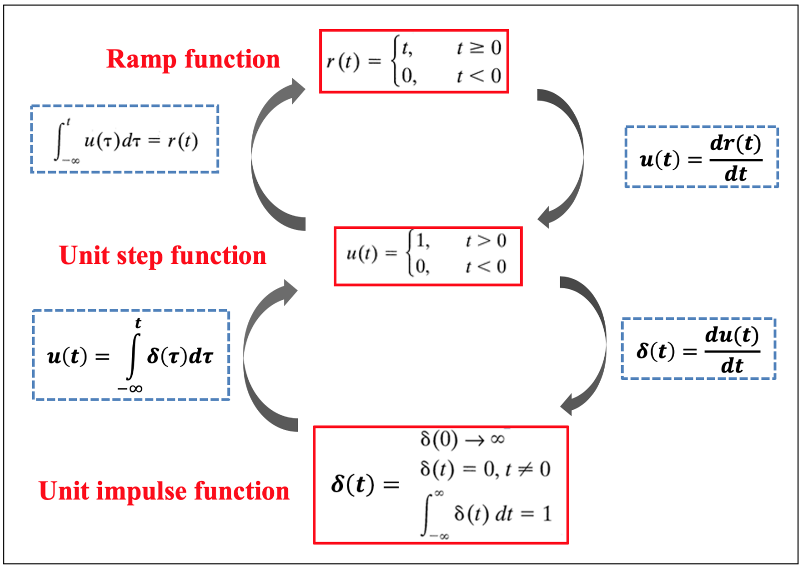
$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$

- $\delta_\epsilon(t) = \frac{1}{\epsilon} rect\left(\frac{t}{\epsilon}\right)$
- $\delta_\epsilon(0) = \frac{1}{\epsilon}$
- $\delta_\epsilon(t) = 0, \ |t| > \frac{\epsilon}{2}$
- $\int_{-\infty}^{\infty} \delta_\epsilon(t)\, dt = 1$

- $\delta(t) = \lim_{\epsilon \to 0} \delta_\epsilon(t)$
- $\delta(0) \to \infty$
- $\delta(t) = 0, t \neq 0$
- $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

Additional Properties of Unit impulse function

- Scaling Property: $\delta(at) = \frac{1}{|a|}\delta(t)$
- Even Function: $\delta(-t) = \delta(t)$
- Derivative and Integral: $u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau, \quad \delta(t) = \frac{du(t)}{dt}$



3. **Complex Exponential Function**

- **Definition**

$$e^{jw_0 t} = \cos(w_0 t) + j\sin(w_0 t)$$

- **Properties**
  - Periodic with $T = \frac{2\pi n}{w_0}$ where n is an integer
  - Fundamental period $T_0 = \frac{2\pi}{|w_0|}$
  - Infinite Energy $\mathbf{E} = \infty$
  - Finite power $\mathbf{P} = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_0^{T_0} |e^{jw_0 t}|^2 dt = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_0^{T_0} 1 \cdot dt = 1$

4. **Sinusoidal Function**

$$A\cos(w_0 t + \theta) \quad \text{or} \quad A\sin(w_0 t + \theta),$$

where A is the *amplitude*, θ is the *phase angle*, w_0 is the *radian frequency* with

$$\text{Fundamental period } T_0 = \frac{2\pi}{w_0} \text{ (sec)}, \quad \text{Fundamental frequency } f_0 = \frac{1}{T_0} \text{ hertz (Hz)}$$

1.4 Classification of System Types

- **[Def]** A *system* is a mathematical model of a physical process that relates the *input signal* to the *output signal* in the form $y = Tx$.

1. Invertible and Noninvertible System

A system is said to be **invertible** if distinct inputs lead to distinct outputs. Otherwise, the system is said to be **noninvertible**.

[Examples]

Invertible System

- $y(t) = 2x(t) \leftrightarrow w(t) = \frac{1}{2}y(t)$
- $y[n] = \sum_{k=-\infty}^{n} x[k] \leftrightarrow w[n] = y[n] - y[n-1]$

Noninvertible System

- $y[n] = 0$
- $y(t) = x^2(t)$

2. Memory and Memoryless System

A system is said to be **memoryless** if the output at any time depends only on the input at that same time. Otherwise, the system is said to have **memory**.

[Examples]

Memoryless System

- $y(t) = Rx(t)$
- $y[n] = \left(2x[n] - x^2[n]\right)^2$
-

System with Memory

- $y[n] = \sum_{k=-\infty}^{n} x[k]$
- $y[n] = x[n-1]$
- $y(t) = \frac{1}{c}\int_{-\infty}^{t} x(\tau)d\tau$

3. Causal and Noncausal System

A system is said to be **causal** if its output at the present time depends on only the present and/or past values of the input. If its output at the present time depends on future values of the input, the system is known as **noncausal**.

[Examples]

Causal System

- $y[n] = \sum_{k=-\infty}^{n} x[k]$
- $y(t) = x^2(t)$

Noncausal System

- $y[n] = x[n] + x[n+2]$
- $y[n] = x[-n]$ or $y(t) = x(t+1)$

$*$ **Note)** All memoryless systems are causal, but not vice versa.

4. Linear and Nonlinear System

A system is said to be **linear** if the following superposition property (1.12) holds for a given operator $\mathbf{T}$. If the system does not satisfy (1.12), it is a **nonlinear system**.

$$\mathbf{T}\{\alpha_1 x_1 + \alpha_2 x_2\} = \alpha_1 \mathbf{T}\{x_1\} + \alpha_2 \mathbf{T}\{x_2\} \tag{1.12}$$

[Examples]

Linear System

- $y[n] = \sum_{k=-\infty}^{n} x[k]$
- $y(t) = tx(t)$

Nonlinear System

- $y(t) = x^2(t)$
- $y[n] = 2x[n] + 3$

∗ **Note)** For a linear system, zero input always yields a zero output.

5. Time-invariant and Time-Varying System

A system is **time-invariant** if a time-shift of the input causes a corresponding shift in the output. In other words, the system response is independent of time.

$$\text{If } y(t) = \mathbf{T}\{x(t)\}, \text{ then } y(t - t_0) = \mathbf{T}\{x(t - t_0)\} \tag{1.13}$$

[Examples]

Time invariant System

- $y[n] = \sum_{k=-\infty}^{n} x[k]$
- $y[n] = x[n - n_0]$

Time varying System

- $y(t) = x(2t)$
- $y[n] = nx[n]$

LTI System

Linear time-invariant (LTI) system: A system that is linear and also time-invariant.

6. Stable and Unstable System

A system is **stable** if every bounded input produces a bounded output for all time.

$$\text{If } |x(t)| < A, \text{ then } |y(t)| < B \text{ where } |A| < \infty, |B| < \infty \tag{1.14}$$

[Examples]

Stable System

- $y(t) = x^2(t)$
- $y[n] = x[n] + x[n+2]$

Unstable System

- $y[n] = \frac{1}{x[n]}$
- $y[n] = nx[n]$

1.5 Examples

[Example 1-1] Determine whether or not each of the following signals is periodic. If a signal is periodic, determine its fundamental period.

a) $x(t) = \cos\left(t + \frac{\pi}{4}\right)$

b) $x(t) = \sin\left(\frac{2\pi t}{3}\right)$

c) $x(t) = \cos\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{4}\right)$

d) $x(t) = \cos(t) + \sin\left(\sqrt{2}t\right)$

e) $x(t) = \sin^2(t)$

f) $x(t) = e^{j\left[\frac{\pi}{2}t-1\right]}$

g) $x(t) = \cos\left(2t + \frac{\pi}{4}\right)$

h) $x(t) = \cos^2(t)$

i) $x(t) = (\cos(2\pi t))\, u(t)$

j) $x(t) = e^{j\pi t}$

Solution) To solve this type of problem, try to find the minimum T that satisfy $x(t+T) = x(t)$. For instance, in (a), if the following equality holds with a nonzero constant T, then it is periodic

$$\cos\left(t + \frac{\pi}{4}\right) = \cos\left(t + T + \frac{\pi}{4}\right) \quad \longrightarrow \quad \cos(t') = \cos(t' + T), \tag{1.15}$$

where we used a *change of variable* $t' = t + \frac{\pi}{4}$ in the second equality. Since the minimum T that satisfy (1.15) is 2π, (a) is a periodic signal with period $T = 2\pi$. Similarly, for (b),

$$\sin\left(\frac{2\pi t}{3}\right) = \sin\left(\frac{2\pi t}{3} + \frac{2\pi T}{3}\right) \quad \longrightarrow \quad \frac{2\pi T}{3} = 2\pi, \tag{1.16}$$

and by denoting $t' = \frac{2\pi t}{3}$, the minimum T that satisfy (1.16) is 3.

For (c) and (d), we can use (1.18); The period T_1 for $\cos\left(\frac{\pi t}{3}\right)$ in (c) is $T_1 = 6$ and T_2 for $\sin\left(\frac{\pi t}{4}\right)$ is $T_2 = 8$. Since $T_1/T_2 = 3/4$, (c) is a periodic signal with period $T = 24$. In (d), the period T_1 for $\cos(t)$ is $T_1 = 2\pi$ and T_2 for $\sin\left(\sqrt{2}t\right)$ is $T_2 = \sqrt{2}\pi$. Since $T_1/T_2 = \sqrt{2}$, (d) is aperiodic signal.

For (e) and (h), convert $x(t)$ as follows, then apply similar approach as (a).

$$\cos^2(t) = \frac{1}{2}(1 + \cos(2t)), \quad \sin^2(t) = \frac{1}{2}(1 - \cos(2t)), \tag{1.17}$$

and the remaining can be solved using similar method. The solutions are summarized below.

a) Periodic with $T = 2\pi$

b) Periodic with $T = 3$

c) Periodic with $T = 24$

d) Aperiodic

e) Periodic with $T = \pi$

f) Periodic with $T = 4$

g) Periodic with $T = \pi$

h) Periodic with $T = \pi$

i) Aperiodic

j) Periodic with $T = 2$

Sum of Periodic Signals

- Let $x_1(t)$ and $x_2(t)$ be periodic signals with fundamental periods T_1 and T_2, respectively. The sum $x(t) = x_1(t) + x_2(t)$ is periodic if and only if the following condition holds
$$\frac{T_1}{T_2} = \frac{k}{m} = \text{rational number} \tag{1.18}$$
where the fundamental period T is the least common multiple of T_1 and T_2.

- Let $x_1[n]$ and $x_2[n]$ be periodic sequence with fundamental periods N_1 and N_2, respectively. The sum $x[n] = x_1[n]+x_2[n]$ is periodic given the following condition
$$mN_1 = kN_2 = N \tag{1.19}$$
where the fundamental period N is the least common multiple of N_1 and N_2.
Refer [Schaum's text, Problem 1.14 & 1.15]

[Example 1-2] Determine whether the following signals are energy signals, power signals, or neither.

a) $x(t) = e^{-at}u(t), a > 0$

b) $x(t) = A\cos(\omega_0 t + \theta)$

Solution) To solve this type of problem, **(Step 1.)** you need to calculate the energy **E** first. If **E** is finite, the signal is a Energy signal. Otherwise, **(Step 2.)** if **E** is infinite, you need to calculate the power **P** as well. If **P** is finite, the signal is a Power signal. Otherwise, if **P** is infinite, then it is neither a energy nor a power signal. For example, in (a),

$$\mathbf{E} = \int_{-\infty}^{\infty} e^{-2at}u(t)dt = \int_0^{\infty} e^{-2at}dt = \frac{1}{2a}, \tag{1.20}$$

where we used the definition of the step function in the second equality. Since $\frac{1}{2a}$ is finite, $x(t)$ in (a) is a energy signal. For a periodic signal, the integration interval T in (1.7) is equal to the period. In (b), the period is $T = \frac{2\pi}{\omega_0}$ and the signal power can be calculated as follows

$$\begin{aligned}\mathbf{P} &= \lim_{T\to\infty} \frac{1}{T}\int_0^T A^2\cos^2(\omega_0 t + \theta)\,dt = \lim_{T\to\infty}\frac{A^2}{2\pi}\int_\theta^{2\pi+\theta}\cos^2(l)\,dl \\ &= \lim_{T\to\infty}\frac{A^2}{4\pi}\int_\theta^{2\pi+\theta}[1+\cos(2l)]\,dl = \frac{A^2}{2},\end{aligned} \tag{1.21}$$

where we used $T = \frac{2\pi}{\omega_0}$ and *a change of variable*, $l = \omega_0 t + \theta$ or $\omega_0 dt = dl$, in the second equality, then applied the Cosine rule $\cos^2(t) = \frac{1}{2}(1+\cos(2t))$ in the third equality. Since $\frac{A^2}{2}$ is finite, $x(t)$ in (b) is a power signal. In summary, the solutions are

a) Energy signal

b) Power signal

Definition of Energy and Power Signals

- **Energy signal** has finite energy and zero power, i.e., $0 < \mathbf{E} < \infty,\ \mathbf{P} = 0$
- **Power signal** has finite power and infinite energy, i.e., $0 < \mathbf{P} < \infty,\ \mathbf{E} = \infty$, where

$$\mathbf{E} = \int_{-\infty}^{\infty} |x(t)|^2 dt, \quad \mathbf{P} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

Properties of Periodic Signals

The following equalities hold for a periodic signal $x\,(t+T) = x(t)$

$$\int_{\alpha}^{\beta} x(t)dt = \int_{\alpha+T}^{\beta+T} x(t)dt, \quad \int_{0}^{T} x(t)dt = \int_{a}^{a+T} x(t)dt,$$
$$\mathbf{P} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T_0} \int_{0}^{T_0} |x(t)|^2 dt,$$

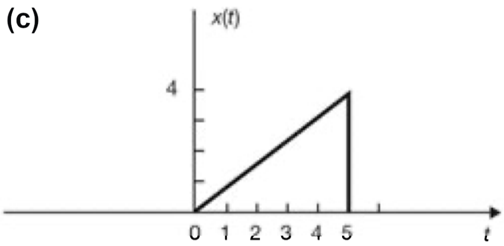
where T_0 is the fundamental period and α, β, a are arbitrary real valued constants.
Refer [Schaum's text, Problem 1.17 & 1.18]

[Example 1-3] Determine the even and odd component of the following signals

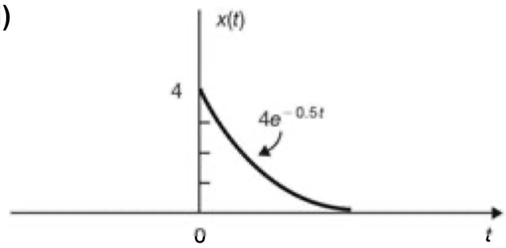
a) $x(t) = u(t)$

b) $x(t) = \sin\left(\omega_0 t + \frac{\pi}{4}\right)$

(c)



(d)



Solution) To solve this type of problem, you need to apply (1.22). In (a), $x(-t) = u(-t) = 1$ for $t < 0$ and $u(-t) = 0$ for $t > 0$. Then, the following results can be derived

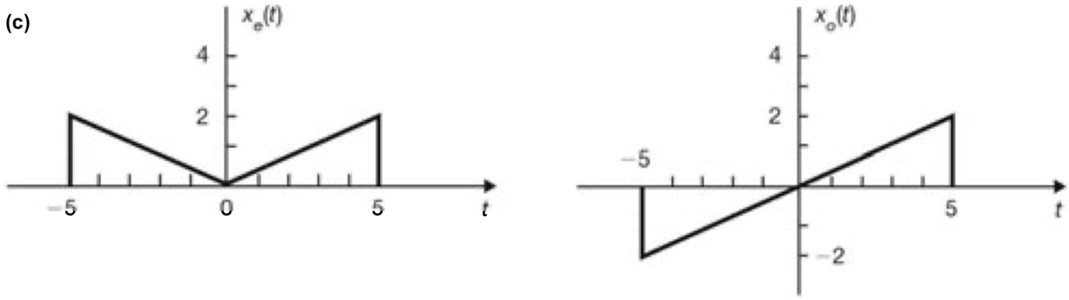
$$x_e\,(t) = \frac{1}{2}\,[u\,(t) + u\,(-t)] = \frac{1}{2},$$
$$x_o\,(t) = \frac{1}{2}\,[u\,(t) - u\,(-t)] = \frac{1}{2} sgn\,(t) = \begin{cases} 0.5, & t > 0, \\ -0.5, & t < 0 \end{cases}$$

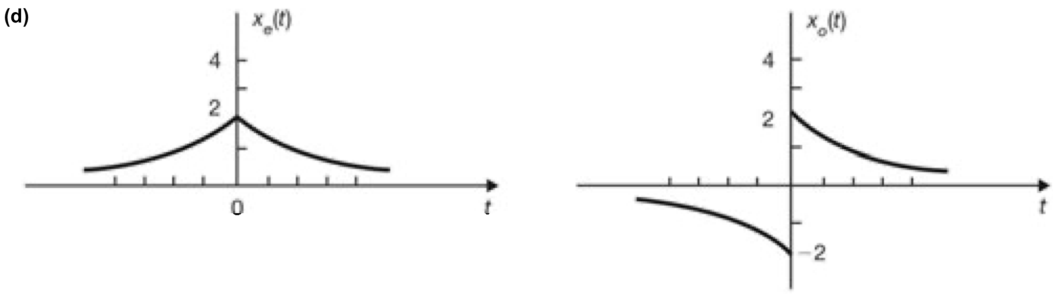
In (b), we first use Sine rule to expand the Sine function, then the following reults can be derived.

$$\sin\left(\omega_0 t + \frac{\pi}{4}\right) = \sin\left(\omega_0 t\right)\cos\left(\frac{\pi}{4}\right) + \cos\left(\omega_0 t\right)\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\left(\sin\left(\omega_0 t\right) + \cos\left(\omega_0 t\right)\right).$$

$$x_e\left(t\right) = \frac{1}{\sqrt{2}}\cos\left(\omega_0 t\right), \quad x_o\left(t\right) = \frac{1}{\sqrt{2}}\sin\left(\omega_0 t\right).$$

Similarly, the even and odd component of (c) and (d) can be found as follows





Even and Odd Component

Any signal $x(t)$ can be expressed as a sum of two signals

$$x\left(t\right) = x_e\left(t\right) + x_o\left(t\right),$$

where $x_e\left(t\right)$ and $x_o\left(t\right)$ are related to the original signal $x(t)$ as follows

$$x_e\left(t\right) = \frac{1}{2}\left[x\left(t\right) + x\left(-t\right)\right], \quad x_o\left(t\right) = \frac{1}{2}\left[x\left(t\right) - x\left(-t\right)\right]. \tag{1.22}$$

[Example 1-4] **[Part 1]** Sketch the following signals.

a) $x_1(t) = u(t) + 5u\left(t-1\right) - 2u\left(t-2\right)$

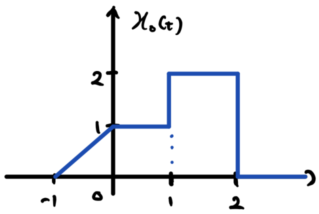
b) $x_2(t) = r(t) - r\left(t-1\right) - u\left(t-2\right)$

c) $x_3(t) = u(t)u\left(a-t\right), a > 0$

d) $x_4(t) = x_0(t)u(1-t)$

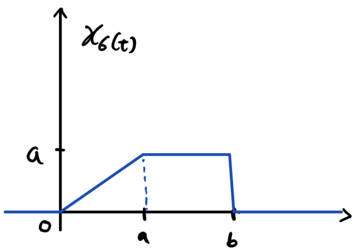
e) $x_5(t) = x_0(t)\,[u(t) - u(t-1)]$

where the signal $x_0(t)$ is plotted below.

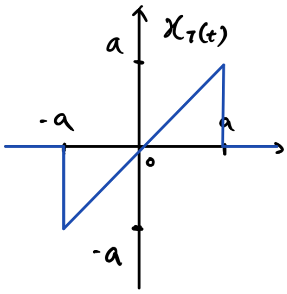


[Part 2] For each of the signals plotted below, write an expression in terms of unit step and unit ramp functions.

(f)

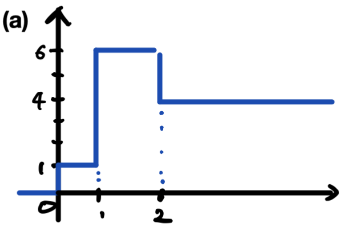


(g)

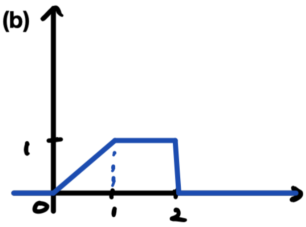


Solution)[Part 1]

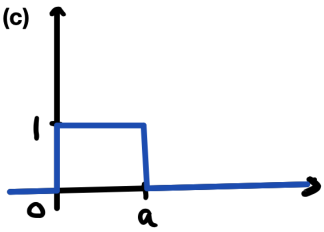
(a)



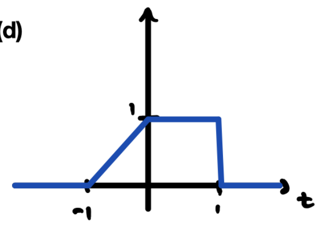
(b)



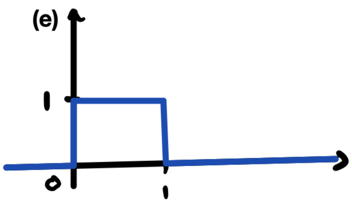
(c)



(d)



(e)



[Part 2]

(f) $x_6(t) = r(t) - r(t-a) - au(t-b)$, (g) $x_7(t) = (r(t) - r(-t))\,(u(t+a) - u(t-a))$

- **Time Reversal**: Flip the signal around the vertical axis $x(t) \longrightarrow x(-t)$
- **Time Shifts**: Shift the signal to left or right $x(t) \longrightarrow x(t - t_0)$
  - **Right-shift** if $t_0 > 0$, **Left-shift** if $t_0 < 0$.
- **Time Scaling**: Linearly stretch or compress the signal $x(t) \longrightarrow x(ct)$
  - **Compression** if $|c| > 1$, **Expansion** if $|c| < 1$.

[Example 1-5] Evaluate the following integrals.

a) $\int_{-\infty}^{t} \cos(\tau)\, u(\tau) d\tau$

b) $\int_{-\infty}^{t} \cos(\tau)\, \delta(\tau) d\tau$

c) $\int_{-\infty}^{\infty} \cos(t) u(t-1)\, \delta(t) dt$

d) $\int_{0}^{2\pi} t \sin\left(\frac{t}{2}\right) \delta(t-\pi)$

e) $\int_{-\infty}^{\infty} \left(\frac{2}{3}t - \frac{3}{2}\right) \delta(t-1)\, dt$

f) $\int_{-3}^{2} \left[\exp(1-t) + \sin\left(\frac{2\pi t}{3}\right)\right] \delta\left(t - \frac{3}{2}\right) dt$

Solution)

(a)

$$\int_{-\infty}^{t} \cos(\tau)\, u(\tau) d\tau = \begin{cases} \text{If } t > 0, \int_{0}^{t} \cos(\tau)\, d\tau = \sin(t) \\ \text{If } t < 0, 0 \end{cases} = u(t)\sin(t)$$

(b)

$$\int_{-\infty}^{t} \cos(\tau)\, \delta(\tau) d\tau = \begin{cases} \text{If } t > 0, \cos 0 = 1 \\ \text{If } t < 0, 0 \end{cases} = u(t)$$

(c)

$$\int_{-\infty}^{\infty} \cos(t) u(t-1)\, \delta(t) dt = \cos(0) u(-1) = 0$$

(d)

$$\int_{0}^{2\pi} t \sin\left(\frac{t}{2}\right) \delta(t-\pi) = \pi \sin\left(\frac{\pi}{2}\right) = \pi$$

(e)

$$\int_{-\infty}^{\infty} \left(\frac{2}{3}t - \frac{3}{2}\right) \delta(t-1)\, dt = \frac{2}{3} - \frac{3}{2} = -\frac{5}{6}$$

(f)

$$\int_{-3}^{2} \left[\exp(1-t) + \sin\left(\frac{2\pi t}{3}\right)\right] \delta\left(t - \frac{3}{2}\right) dt = \exp\left(-\frac{1}{2}\right) + \sin(\pi) = \exp(-0.5)$$

Properties of Unit impulse function

- $\int_a^b x(t)\delta(t-t_0)\,dt = \begin{cases} x(t_0), & \text{if } a < t_0 < b \\ 0, & \text{otherwise} \end{cases}$
- $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$
- $\delta(at) = \frac{1}{|a|}\delta(t), \quad u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau, \quad \delta(t) = \frac{du(t)}{dt}$

[Example 1-6] Determine whether the following system is (i) memoryless, (ii) causal, (iii) linear, (iv) time-invariant, or (v) stable. *Refer [Schaum's text, Problem 1.33, 1.34, 1.36, 1.38]*

a) $y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$

b) $y(t) = x(t)\cos(\omega_0 t)$

c) $y[n] = x[n-1]$

d) $y[n] = nx[n]$

Solution) In (a), the output depends on the past input, so it is not memoryless system. The output depends on the present and past values of the input, so it is a Causal system. To test linearity, substitute $x(t) \leftarrow \alpha_1 x_1(t) + \alpha_2 x_2(t)$ as the input, where $y_1(t)$ and $y_2(t)$ is the corresponding output of $x_1(t)$ and $x_2(t)$, respectively. Then,

$$\begin{aligned} y(t) &= \frac{1}{C}\int_{-\infty}^{t} [\alpha_1 x_1(\tau) + \alpha_2 x_2(\tau)]\,d\tau \\ &= \alpha_1 \left[\frac{1}{C}\int_{-\infty}^{t} x_1(\tau)\,d\tau\right] + \alpha_2 \left[\frac{1}{C}\int_{-\infty}^{t} x_2(\tau)\,d\tau\right] = \alpha_1 y_1(t) + \alpha_2 y_2(t), \end{aligned}$$

so the superposition property holds, which indicates a linear system. To test time-invariance, input time shifted signal $x(t-t_0)$. If the corresponding output is $y(t-t_0)$, then it is a time invariant system.

$$\frac{1}{C}\int_{-\infty}^{t} x(\tau - t_0)\,d\tau = \frac{1}{C}\int_{-\infty}^{t-t_0} x(l)\,dl = y(t-t_0),$$

by using *a change of variable* $l = \tau - t_0$ in the first equality. Hence, it is a time-invariant system. For stability, (a) can be easily proved to be a unstable by substituting a unit step function $x(t) = u(t)$ as the input, which achieves unbounded $y(t) = \frac{tu(t)}{C}$. The remaining can be proved using similar method. The solutions are summarized below.

a) memory, causal, linear, time-invariant, unstable

b) memoryless, causal, linear, time-variant, stable

c) memory, causal, linear, time-invariant, stable

d) memoryless, causal, linear, time-variant, unstable.

System Characterization

1. **Memoryless System**; output at any time depends only on the input at that same time
2. **Causal System**; output at the present time depends only on the present and/or past input values
3. **Linear System**; the superposition property holds, i.e., $\mathbf{T}\{\alpha_1 x_1 + \alpha_2 x_2\} = \alpha_1 \mathbf{T}\{x_1\} + \alpha_2 \mathbf{T}\{x_2\}$
4. **Time-invariant System**; time-shift of the input causes a same amount of shifting in the output
5. **Stable System**; If $|x(t)| < A$, then $|y(t)| < B$ where $|A| < \infty, |B| < \infty$
6. **LTI System**; A system that is linear and also time-invariant