1. If $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvalue of $A = \begin{bmatrix} a & 2 & -2 \\ 2 & b & 0 \\ -2 & 0 & 7 \end{bmatrix}$, find the value for a, b.

   Solution.

   (a) Eigenvalues: $|A - \lambda I| = -(\lambda - 3)(\lambda - 6)(\lambda - 9) = 0$. Hence eigenvalues are $\lambda_1 = 3, \lambda_2 = 6, \lambda_3 = 9$.

   (b) Eigenvectors: (i) For $\lambda_1 = 3$, we have

$$\left(\begin{array}{ccc|c} 3 & 2 & -2 & 0 \\ 2 & 2 & 0 & 0 \\ -2 & 0 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 3 & 2 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

   By solving

$$3x_1 + 2x_2 - 2x_3 = 0, \quad x_2 + 2x_3 = 0, \quad x_3 = t,$$

   we have eigenvector $t \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, $t \neq 0$.

   (ii) For $\lambda = 6$, we have we have

$$\left(\begin{array}{ccc|c} 0 & 2 & -2 & 0 \\ 2 & -1 & 0 & 0 \\ -2 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

   By solving

$$2x_1 - x_2 = 0, \quad x_2 - x_3 = 0, \quad x_3 = t,$$

   we have eigenvector $t \begin{pmatrix} \frac{1}{2} \\ 1 \\ 1 \end{pmatrix}$, $t \neq 0$.

   (iii) For $\lambda = 9$, we have

$$\left(\begin{array}{ccc|c} -3 & 2 & -2 & 0 \\ 2 & -4 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -3 & 2 & -2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

   By solving

$$-3x_1 + 2x_2 - 2x_3 = 0, \quad 2x_2 + x_3 = 0, \quad x_3 = t,$$

   we have eigenvector $t \begin{pmatrix} -1 \\ -\frac{1}{2} \\ 1 \end{pmatrix}$, $t \neq 0$.

2. It is given the symmetric matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

   (a) find the eigenvalues of A;

   (b) find the eigenvectors corresponding to each of these eigenvalues;

   (c) find an orthogonal matrix P such that $P^\top AP$ gives a diagonal matrix D and calculates P^{-1};

   (d) Determine the eigenvalues of the matrix $B = A^5 + (A^2)^\top$.

Solution. (a) It is easy to show that $|A - \lambda I| = -(\lambda - 1)(\lambda - 2)(\lambda - 3)$. Hence the eigenvalues of A is $1, 2, 3$.

(b) (i) For $\lambda_1 = 1$, we have

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

By solving

$$x_1 + x_3 = 0, \quad x_2 = 0, \quad x_3 = t,$$

we have eigenvector $t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $t \neq 0$. Let $\vec{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

(ii) For $\lambda = 2$, we have we have

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

By solving

$$x_1 = 0, \quad x_2 = t, \quad x_3 = 0,$$

we have eigenvector $t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $t \neq 0$. Let $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

(iii) For $\lambda = 3$, we have

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

By solving

$$x_1 - x_3 = 0, \quad x_2 = 0, \quad x_3 = t,$$

we have eigenvector $t\begin{pmatrix}1\\0\\1\end{pmatrix}$, $t \neq 0$. Let $\vec{v}_3 = \begin{pmatrix}1\\0\\1\end{pmatrix}$.

Finally, by

$$\vec{v}_1 \cdot \vec{v}_2 \times \vec{v}_3 = \begin{vmatrix}-1 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 1\end{vmatrix} = -2 \neq 0,$$

we conclude that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent eigenvectors of A.

(c) Let $\vec{u}_1 = \frac{\vec{v}_1}{|\vec{v}_1}$, $\vec{u}_2 = \frac{\vec{v}_2}{|\vec{v}_2}$, $\vec{u}_3 = \frac{\vec{v}_3}{|\vec{v}_3}$. We obtain an orthonormal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$. Define

$$P = [\vec{u}_1, \vec{u}_2, \vec{u}_3] = \begin{bmatrix}\frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}$$

Note that $PP^\top = P^\top P = I$. Hence P is an orthogonal matrix, i.e., $P^{-1} = P^\top$. Moreover, since $AP = PD$ with $D = \begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$, we have $P^\top AP = D$ is a diagonal matrix.

(d) $B = A^5 + (A^2)^\top = A^5 + A^2 = P(D^5 + D^2)P^\top$. The eigenvalues of A are 1, 2, 3. Hence the eigenvalues of B are given by $1^5 + 1^2 = 2$, $2^5 + 2^2 = 32 + 4 = 36$, and $3^5 + 3^2 = 243 + 9 = 252$.

3. A quadratic form Q in the components $x_1, \ldots, x_n$ of a vector $\vec{x} = [x_1, \ldots, x_n]^\top$ with symmetric coefficient matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ is defined to be

$$Q(\vec{x}) := \vec{x}^\top A\vec{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j.$$

Determine whether each of the following quadratic forms in two variables is positive or negative definite or semidefinite, or indefinite.

(a) $3x_1^2 + 8x_1x_2 - 3x_2^2$.

(b) $9x_1^2 + 6x_1x_2 + x_2^2$.

(c) $4x_1^2 + 12x_1x_2 + 13x_2^2$.

Solution.

(a) The coefficient matrix is given by

$$A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$$

Its eigenvalues are $5, -5$. So A is indefinite. [Diagonalize A we obtain

$$A = PDP^\top = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

] Let

$$\vec{y} = P^\top \vec{x}.$$

Then

$$Q(x_1, x_2) = 3x_1^2 + 8x_1x_2 - 3x_2^2 = \vec{x}^\top A\vec{x} = \vec{x}^\top PDP^\top \vec{x} = \vec{y}^\top D\vec{y} = 10.$$

Hence its canonical form is given by

$$5y_1^2 - 5y_2^2 = 10 \quad \text{or} \quad y_1^2 - y_2^2 = 2,$$

hyperbola.jpg

which is a hyperbola. See below

(b) The coefficient matrix is given by

$$A = \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}$$

Its eigenvalues are $0, 10 \geq 0$. So A is positive semi-definite.

(c) The coefficient matrix is given by

$$A = \begin{pmatrix} 4 & 6 \\ 6 & 13 \end{pmatrix}$$

Its eigenvalues are $1, 16 > 0$. So A is positive definite. [The 1st LPM of A is $4 > 0$, and the second order is $4 \times 13 - 6^2 = 16 > 0$, so A is positive definite.]

Determine the values of a for which the quadratic form $x^2 + 2xz + y^2 + 2ayz + 2z^2$ is positive definite.

Solution. Its matrix is

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & a \\ 1 & a & 2 \end{pmatrix}.$$

The 1st, 2nd and 3rd order leading principal minors are 2, $\left|\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\right| = 2$, and $\det(A) = 2a^2 - 4 - 1$, respectively. Thus the matrix is positive definite when all of them are positive and thus $a < -\sqrt{5/2}$ or $a > \sqrt{5/2}$.

Find the limit of f as $(x, y) \to (0, 0)$ or show that the limit does not exist.

$$f(x, y) = \frac{2x}{x^2 + x + y^2}$$

Solution: The limit cannot be found by direct substitution, which gives the indeterminate form 0/0. We examine the values of f along different paths that end at (0, 0), which will lead to different results, as we now see. Choose

$$\ell_1 = (x, 0) : x > 0 \qquad \ell_2 = \{(0, y) : y > 0\}.$$

Then

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } \ell_1}} f(x, y) = \lim_{x\to 0} \frac{2x}{x^2 + x} = \lim_{x\to 0} \frac{2}{x + 1} = 2$$

and

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } \ell_2}} f(x, y) = \lim_{y\to 0} \frac{2 \cdot 0}{0^2 + 0 + y} = \lim_{y\to 0} 0 = 0.$$

There is therefore no single number we may call the limit of f as (x, y) approaches the origin. The limit fails to exist, and the function is not continuous.

Let

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

1. Find the limit of f as (x, y) approaches $(0, 0)$ along the line $y = x$.
2. Prove that f is not continuous at the origin.
3. Show that both partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist at the origin.

Solution:

1. Since $f(x,y)$ is constantly zero along the line $y = x$ (except at the origin), we have
$$\lim_{(x,y)\to(0,0)} f(x,y)\Big|_{y=x} = \lim_{(x,y)\to(0,0)} 0 = 0.$$
2. Since $f(0,0) = 1$, the limit in part (a) proves that f is not continuous at $(0,0)$,
3. To find $\partial f/\partial x$ at $(0,0)$, we hold y fixed at $y = 0$. Then $f(x,y) = 1$ for all x, and the graph of is the line L1 in the Figure . The slope of this line at any x is $\partial f/\partial x = 0$. In particular, $\partial f/\partial x = 0$ at (0, 0). Similarly, $\partial f/\partial y$is the slope of line L2 at any y, so $\partial f/\partial y = 0$at (0, 0).

hw2_1.png

It is given that $f(x,y) = x\cos y + ye^x$. Find all the first and second order partial derivatives of f,
$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

Solution: The first step is to calculate both first partial derivatives.
$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x\cos y + ye^x) & \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x\cos y + ye^x)\\ &= \cos y + ye^x & &= -x\sin y + e^x\end{aligned}$$

Now we find both partial derivatives of each first partial:
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x\cos y$$

Suppose $2z^3 - 2yz + x^2 = 1$ determines the function $z = z(x, y)$ as a function of x, y locally at $(x, y, z) = (1, 1, 1)$.

(a) Find the linear approximation of z at $(x, y, z) = (1, 1, 1)$.

(b) Find the quadratic surface approximation of z at $(x, y, z) = (1, 1, 1)$.

Suppose $2z^3 - 2yz + x^2 = 1$ determines the function $z = z(x, y)$ as a function of x, y locally at $(x, y, z) = (1, 1, 1)$.

(a) Find the linear approximation of z at $(x, y, z) = (1, 1, 1)$.

(b) Find the quadratic surface approximation of z at $(x, y, z) = (1, 1, 1)$.

Solution. (a) At $(x, y) = (1, 1)$, we have $z(1, 1) = 1$. By

$$\begin{cases} 6z^2 z_x - 2yz_x + 2x = 0, \\ 6z^2 z_y - 2yz_y - 2z = 0, \end{cases}$$

we have

$$\begin{cases} z_x = \frac{-x}{3z^2-y}, \\ z_y = \frac{z}{3z^2-y}. \end{cases} \tag{1}$$

At $(x, y, z) = (1, 1, 1)$, we get the evaluation of (**??**) as

$$z_x(1, 1) = -\frac{1}{2}, \quad z_y(1, 1) = \frac{1}{2}.$$

Hence, the linear approximation of z at $(x, y, z) = (1, 1, 1)$ is given by

$$z(x, y) \approx z(1, 1) + z_x(1, 1)(x-1) + z_y(1, 1)(y-1) = 1 - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) = 1 - \frac{1}{2}x + \frac{1}{2}y.$$

(b) By (**??**), we have

$$\begin{cases} z_{xx} = \frac{-(3z^2-y)+x(6zz_x)}{(3z^2-y)^2}, \\ z_{xy} = \frac{x(6zz_y-1)}{(3z^2-y)^2}, \\ z_{yx} = \frac{z_x(3z^2-y)-z(6zz_x)}{(3z^2-y)^2}, \\ z_{yy} = \frac{z_y(3z^2-y)-z(6zz_y-1)}{(3z^2-y)^2}. \end{cases}$$

Evaluating at $(x, y, z) = (1, 1, 1)$, we obtain

$$z_{xx}(1, 1) = -\frac{5}{4}, z_{xy}(1, 1) = z_{yx}(1, 1) = \frac{1}{2}, z_{yy}(1, 1) = -\frac{1}{4}.$$

Consequently, the quadratic surface approximation of z at $(x, y, z) = (1, 1, 1)$ is

$$\begin{aligned}
z(x,y) &\approx z(1,1) + z_x(1,1)(x-1) + z_y(1,1)(y-1) \\
&\qquad + \frac{1}{2!}[z_{xx}(1,1)(x-1)^2 + 2z_{xy}(1,1)(x-1)(y-1) + z_{yy}(1,1)(y-1)^2] \\
&= 1 - \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}\left[-\frac{5}{4}(x-1)^2 + (x-1)(y-1) - \frac{1}{4}(y-1)^2\right] \\
&= \frac{3}{4} + \frac{1}{4}x + \frac{1}{4}y - \frac{5}{8}x^2 + \frac{1}{2}xy - \frac{1}{8}y^2.
\end{aligned}$$

It is given that $f(x, y) = e^{2x} \sin 2y$.

(a) Use Taylor's formula to find a linear approximation of $f(x, y)$ at the origin.

(b) Estimate the error in the linear approximation if $|x| \le 0.1$ and $|y| \le 0.1$.

Solution. By Taylor's formula,

$$f(x,y) = e^{2x}\sin(y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + R_2(x,y),$$

where

$$R_2(x,y) = \frac{1}{2!}(x^2 f_{xx}(\theta x, \theta y) + 2xy f_{xy}(\theta x, \theta y) + f_{yy}(\theta x, \theta y)y^2), \quad 0 < \theta < 1.$$

(a) We have

$$f(0,0) = 0, \quad f_x(0,0) = 2e^{2x}\sin(2y)\Big|_{x=0,y=0} = 0, \quad f_y(0,0) = 3e^{2x}\cos(2y)\Big|_{x=0,y=0} = 3.$$

Hence, the linear approximation of f is

$$f(x,y) \approx L(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y = 3y.$$

(b) We have

$$f_{xx}(x,y) = 6e^{2x}\sin(2y), \quad f_{xy}(x,y) = 6e^{2x}\cos(2y), \quad f_{yy}(x,y) = -6e^{2x}\sin(2y).$$

Hence, for any $|x| \le 0.1$ and $|y| \le 0.1$,

$$|f_{xx}(x,y)| \le 4e^{0.2}, \ |f_{xy}(x,y)| \le 6e^{0.2} \text{ and } |f_{yy}(x,y)| \le 4e^{0.2}.$$

and then

$$|f(x,y) - L(x,y)| \le \frac{1}{2}6e^{0.2}(0.1 + 0.1)^2 = 0.12e^{0.2}.$$

Find the stationary points of the function $f(x,y) = xye^{-(x^2+2y^2)}$ and determine their nature.

Solution. From,

$$f_x = y(1-2x^2)e^{-x^2-2y^2} = 0, \quad f_y = x(1-4y^2)e^{-x^2-2y^2} = 0.$$

which is equivalent to solving $y(1-2x^2) = 0$ and $x(1-4y^2) = 0$. We get

$$\begin{cases} y = 0 \text{ or } x = \pm\frac{1}{\sqrt{2}}, \\ x = 0 \text{ or } y = \pm\frac{1}{2}. \end{cases}$$

Hence, stationary points are $(0,0), (\frac{1}{\sqrt{2}}, \frac{1}{2}), (\frac{1}{\sqrt{2}}, -\frac{1}{2}), (-\frac{1}{\sqrt{2}}, \frac{1}{2}), (-\frac{1}{\sqrt{2}}, -\frac{1}{2})$.

Note that

$$\begin{aligned} f_{xx} &= 2xy(2x^2-3)e^{-x^2-2y^2}, \\ f_{yy} &= 4xy(4y^2-3)e^{-x^2-2y^2}, \\ f_{xy} &= (1-2x^2)(1-4y^2)e^{-x^2-2y^2}. \end{aligned}$$

Then,

$$D = f_{xx}f_{yy} - f_{xy}^2 = e^{-2x^2-4y^2}[8x^2y^2(2x^2-3)(4y^2-3) - (1-2x^2)^2(1-4y^2)^2].$$

We have Table **??** showing the nature of the stationary points.

point	f_{xx}	f_{yy}	f_{xy}	$D = f_{xx}f_{yy} - f_{xy}^2$	Nature
$(0,0)$	0	0	1	$-1 < 0$	saddle point
$(\frac{1}{\sqrt{2}}, \frac{1}{2})$	$\frac{-\sqrt{2}}{e} < 0$	$\frac{-2\sqrt{2}}{e}$	0	$\frac{4}{e^2}$	local max.
$(\frac{1}{\sqrt{2}}, -\frac{1}{2})$	$\frac{\sqrt{2}}{e} > 0$	$\frac{2\sqrt{2}}{e}$	0	$\frac{4}{e^2}$	local min.
$(-\frac{1}{\sqrt{2}}, \frac{1}{2})$	$\frac{\sqrt{2}}{e} > 0$	$\frac{2\sqrt{2}}{e}$	0	$\frac{4}{e^2}$	local min.
$(-\frac{1}{\sqrt{2}}, -\frac{1}{2})$	$\frac{-\sqrt{2}}{e} < 0$	$\frac{-2\sqrt{2}}{e}$	0	$\frac{4}{e^2}$	local max.

Table 1: Table for Q3

Let $f(x,y) = x^2 - xy + y^2 - y$. Find the directions $\vec{u}$ and the values of $D_{\vec{u}}f(1,-1)$ for which

1. $D_{\vec{u}}f(1,-1)$ is the largest;
2. $D_{\vec{u}}f(1,-1)$ is the smallest;
3. $D_{\vec{u}}f(1,-1) = 0$;
4. $D_{\vec{u}}f(1,-1) = 4$;

5. $D_{\vec{u}} f(1,-1) = -3$.

Solution. Let $\vec{u} = (a,b)^\top$ be a unit vector; i.e., $a^2 + b^2 = 1$. Since

$$\nabla f = (f_x, f_y)^\top = (2x - y, 2y - x - 1)^\top.$$

Hence

$$\nabla f(1,-1) = (3,-4)^\top.$$

Therefore,

$$D_{\vec{u}} f(1,-1) = \nabla f(1,-1) \cdot \vec{u}$$

Note that the gradient ∇f points to the direction where the function changes the most.

1. When $\vec{u}$ has the same direction as ∇f, $D_{\vec{u}} f$ is the largest. Hence

$$\vec{u} = \frac{\nabla f(1,-1)}{|\nabla f(1,-1)|} = (\tfrac{3}{5}, -\tfrac{4}{5})^\top$$

and

$$D_{\vec{u}} f(1,-1) = |\nabla f(1,1)| = 5.$$

2. When $\vec{u}$ has the opposite direction as ∇f, $D_{\vec{u}} f$ is the smallest. Hence

$$\vec{u} = -\frac{\nabla f(1,-1)}{|\nabla f(1,-1)|} = (-\tfrac{3}{5}, \tfrac{4}{5})^\top.$$

and

$$D_{\vec{u}} f(1,-1) = -|\nabla f(1,1)| = -5.$$

3. When $\vec{u}$ is orthogonal to ∇f, $D_{\vec{u}} f = 0$. Hence

$$\vec{u} \perp \nabla f(1,-1) \Longrightarrow \vec{u} = \pm(\tfrac{4}{5}, \tfrac{3}{5})^\top.$$

4. $D_{\vec{u}} f = 4 = \nabla f(1,-1) \cdot \vec{u}$ implies $3a - 4b = 4$. Together with $a^2 + b^2 = 1$, we get

$$\vec{u} = (0,-1)^\top \text{ or } \vec{u} = (\frac{24}{25}, -\frac{7}{25})^\top.$$

5. $D_{\vec{u}} f = -3 = \nabla f(1,-1) \cdot \vec{u}$ implies $3a - 4b = -3$. Together with $a^2 + b^2 = 1$, we get

$$\vec{u} = (-1,0)^\top \text{ or } \vec{u} = (\frac{7}{25}, \frac{24}{25})^\top.$$

Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 13 & 5 & 2 \\ 2 & 7 & -8 \\ 5 & 4 & 7 \end{bmatrix}$.

Solution.

(a) Eigenvalues: $|A - \lambda I| = -(\lambda - 9)^3$. Hence eigenvalues are $\lambda = 9$(algebraic multiplicity 3).

(b) Eigenvectors: For $\lambda = 9$, we have

$$\left(\begin{array}{ccc|c} 4 & 5 & 2 & 0 \\ 2 & -2 & -8 & 0 \\ 5 & 4 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 4 & 5 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

By solving

$$4x_1 + 5x_2 + 2x_3 = 0, \quad x_2 + 2x_3 = 0, \quad x_3 = t,$$

we have eigenvector $t \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, $t \neq 0$. Hence, the geometric multiplicity of $\lambda = 9$ is 1.

If $\begin{pmatrix} 3 \\ 4 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvalue of $A = \begin{bmatrix} -1 & 0 & 12 & 0 \\ 0 & c & 0 & 12 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -4 & -1 \end{bmatrix}$. Determine the value for c and find eigenvalues and eigenvectors of A.

Solution.

(a) Eigenvalues: $|A - \lambda I| = (\lambda + 1)^2(\lambda + 5)(\lambda - 3) = 0$. Hence eigenvalues are $\lambda_1 = -1, \lambda_2 = 3, \lambda_3 = -5$.

(b) Eigenvectors: (i) For $\lambda_1 = -1$, we have

$$\left(\begin{array}{cccc|c} 0 & 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & -4 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right).$$

By solving

$$x_1 = t, \quad x_2 = s, \quad x_3 = 0, \quad x_4 = 0,$$

we have eigenvector $t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $t^2 + s^2 \neq 0$.

(ii) For $\lambda = 3$, we have

$$\left(\begin{array}{cccc|c} -4 & 0 & 12 & 0 & 0 \\ 0 & -4 & 0 & 12 & 0 \\ 0 & 0 & -4 & -4 & 0 \\ 0 & 0 & -4 & -4 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

By solving

$$x_1 - 3x_3 = 0, \quad x_2 - 3x_4 = 0, \quad x_3 + x_4 = 0, \quad x_4 = t,$$

we have eigenvector $t\begin{pmatrix} -3 \\ 3 \\ -1 \\ 1 \end{pmatrix}$, $t \neq 0$.

(iii) For $\lambda = -5$, we have

$$\left(\begin{array}{cccc|c} 4 & 0 & 12 & 0 & 0 \\ 0 & 4 & 0 & 12 & 0 \\ 0 & 0 & 4 & -4 & 0 \\ 0 & 0 & -4 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

By solving

$$x_1 + 3x_3 = 0, \quad x_2 + 3x_4 = 0, \quad x_3 - x_4 = 0, \quad x_4 = t,$$

we have eigenvector $t\begin{pmatrix} -3 \\ -3 \\ 1 \\ 1 \end{pmatrix}$, $t \neq 0$.