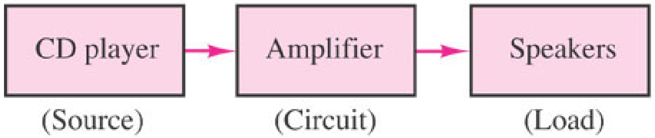
Frequency response

1) Changing the frequency affects the currents and voltages in a circuit
2) This is due to changes in the impedances of the various components in a circuit
3) This affects the working frequency range of a particular device or circuit
4) Hence it is important to find out the frequency response of a circuit
5) The frequency response of a circuit is a measure of the variation of a load-related voltage or current in relation to the input frequency
6) We typically express this in terms of variation in output voltage over the source:

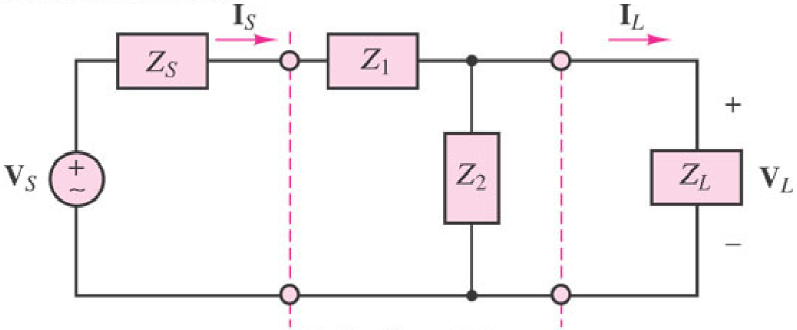
$$H_V(j\omega) = \frac{V_L(j\omega)}{V_S(j\omega)}$$

How does V_L change relative to V_S for different frequencies?
How does V_L change with respect to <u>phase</u> and <u>magnitude</u>?

Voltage divider rule used almost always to derive expression of output/input

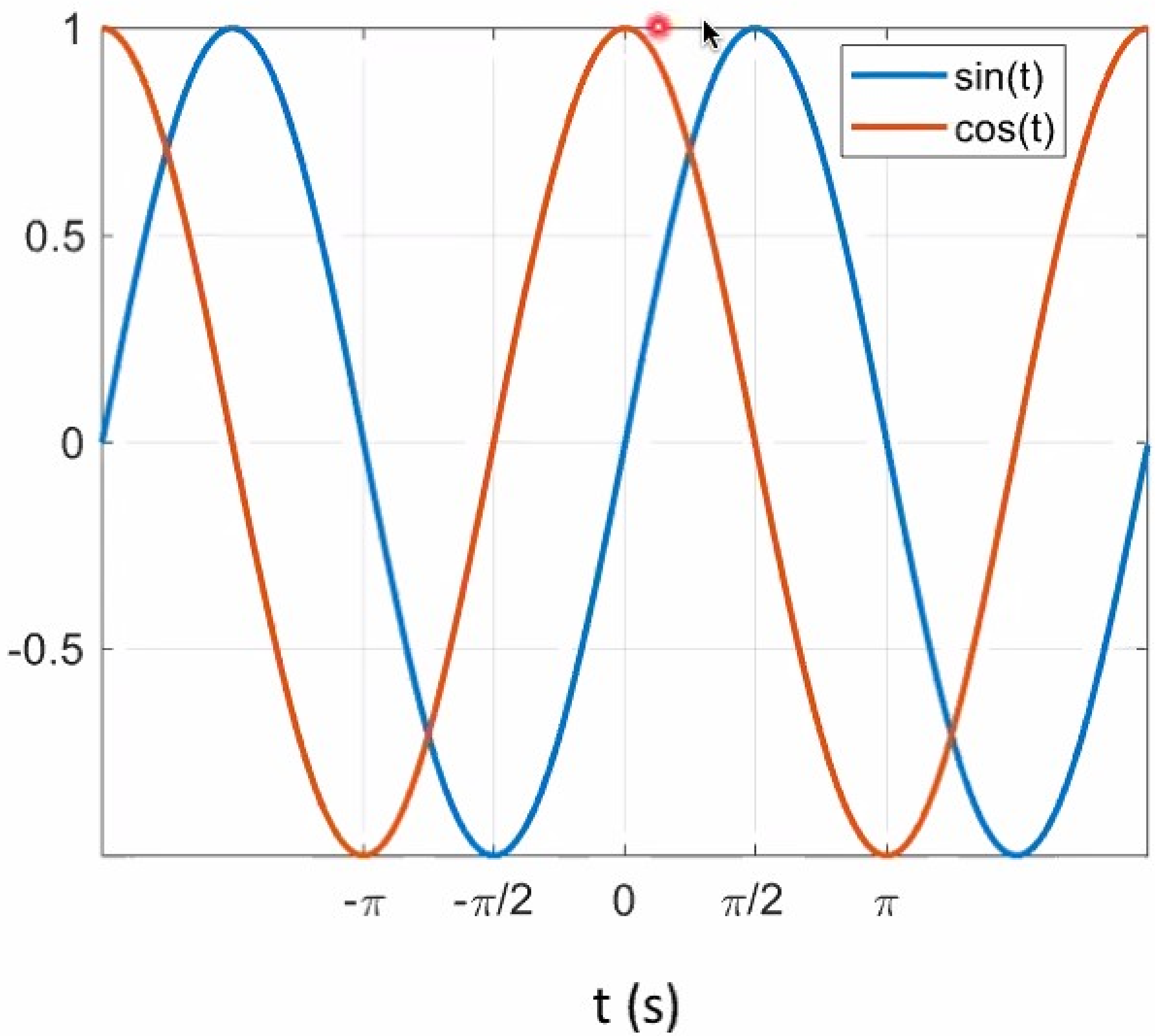


A physical system



A circuit model

Review: Phase



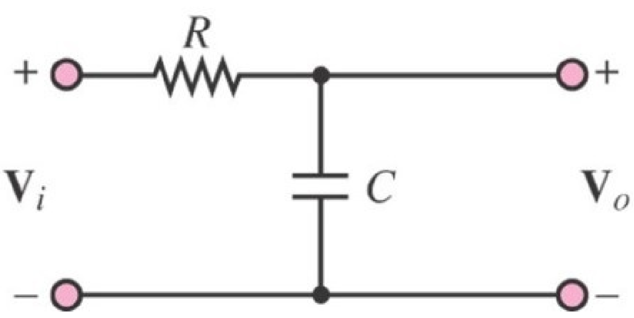
- $\sin\left(\frac{\pi}{2}\right) = \cos(0) = 1$
- $\sin(t) = \cos\left(t - \frac{\pi}{2}\right)$
- $\cos(t) = \sin\left(t + \frac{\pi}{2}\right)$

Low pass filter

Let us consider the response of the output V_o in relation to the input V_i. We keep the amplitude of V_i constant but vary its frequency ω.

By voltage divider rule: $$\frac{V_o}{V_i}(j\omega) = \frac{1/j\omega C}{R + 1/j\omega C}$$

$$= \frac{1}{1 + j\omega CR}$$



Note that CR is a constant based on the circuit values unlike ω, which is a variable.
Re-write CR as a constant with the same unit as angular frequency:

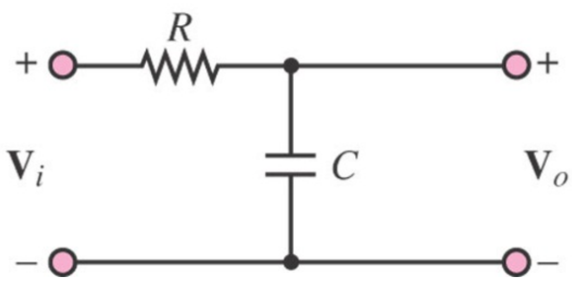
$$\omega_c = \frac{1}{RC}$$

This frequency is called the <u>cutoff</u> radian frequency (ω_c) and is a CONSTANT

Sub back into above equation: $$\frac{V_o}{V_i}(j\omega) = \frac{1}{1 + j(\omega/\omega_c)}$$

Analyze the response of low pass filter

$$\frac{V_o}{V_i}(j\omega) = \frac{1}{1 + j(\omega/\omega_c)}$$



Both the phase and magnitude of V_o/V_i will change when ω is allowed to vary.

Magnitude

When $\omega \rightarrow 0$:

$V_o/V_i \rightarrow$ **1**/**1** ⇨ $|V_o/V_i| \rightarrow 1$

When $\omega \rightarrow$ Infinity:

$V_o/V_i \rightarrow$ ________ ⇨ $|V_o/V_i| \rightarrow 0$

Phase

When $\omega \rightarrow 0$:

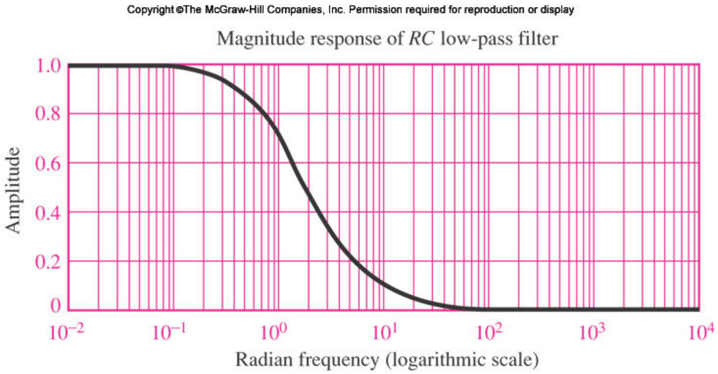
$V_o/V_i \rightarrow$ **1**$\angle 0°$/**1**$\angle 0°$ ⇨ $\angle(V_o/V_i) \rightarrow 0°$

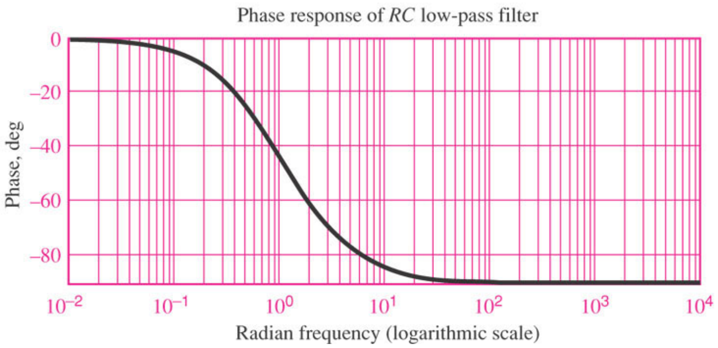
When $\omega \rightarrow$ Infinity:

$V_o/V_i \rightarrow$ ________ ⇨ $\angle(V_o/V_i) \rightarrow -90°$

Sketch the response of low pass filter

Allows lower frequency signals to pass and filters off higher frequency signals





Observations:

When ω approaches zero, magnitude of V_o/V_i approaches 1 and its phase is close to zero

When ω becomes large, magnitude of V_o/V_i approaches zero and its phase is close to $-\pi/2$

Allows lower frequency signal to pass and filters off higher frequency signals

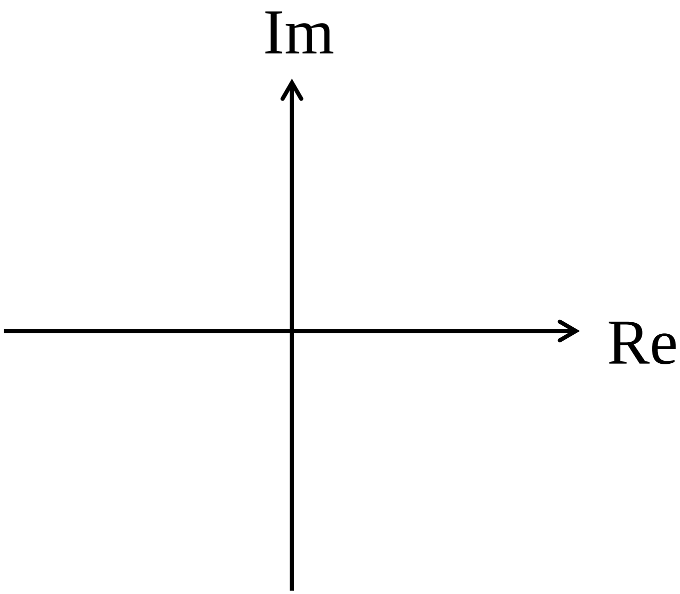
What about in between these two extremes, around ω_c?

The above graphs are presentations of semi-log plots

Semi-log plots: y-axis follows a linear scale, x-axis follows a logarithmic scale

Logarithmic scale (base 10): Between each interval on axis, we increase/decrease by a factor of 10 (it is the power/index that changes)

At the cut off frequency



Plot of denominator for V_o/V_i

$$\frac{V_o}{V_i}(j\omega) = \frac{1}{1 + j(\omega/\omega_c)}$$

At $\omega = \omega_c$:

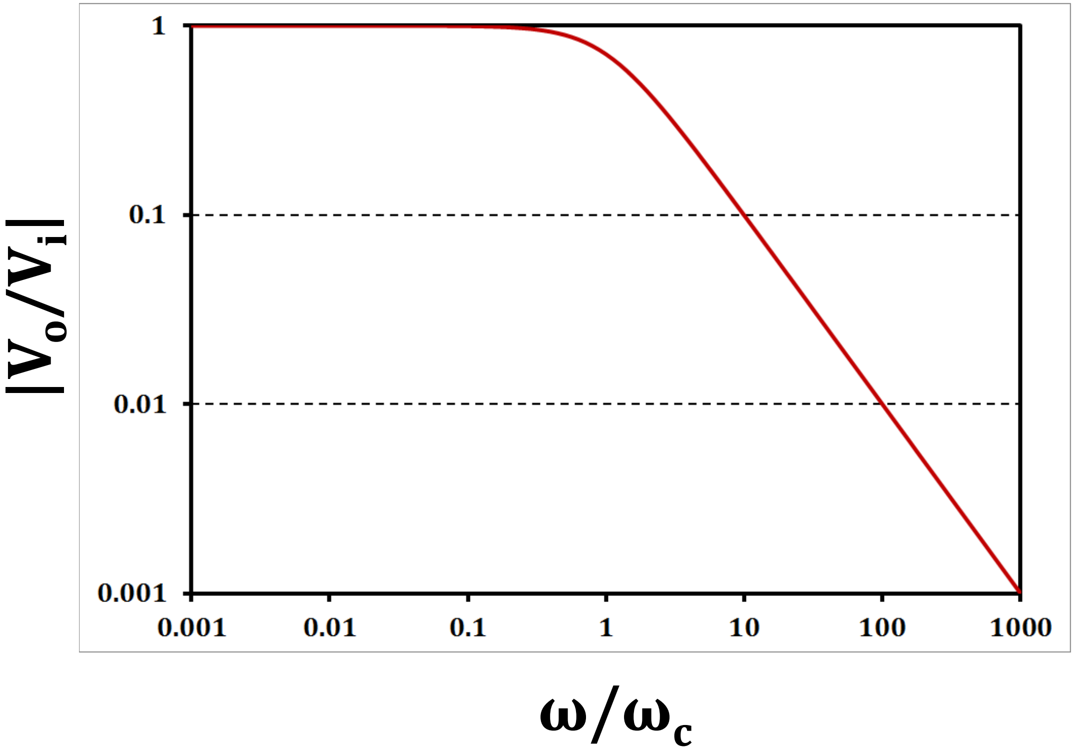
Denominator: **1+ j**

Denominator as a phasor: $\sqrt{2}\angle 45^o$

$$\frac{V_o}{V_i} =$$

At $\omega = \omega_c$, V_o/V_i drops to $1/\sqrt{2}$ of the maximum and has a phase of -45^o

Log-Log Plot for Magnitude (Low Pass)



Two parts of the curve:

When $\omega \ll \omega_c$:
$|V_o/V_i|$ stays flat close to 1

When $\omega \gg \omega_c$:
$|V_o/V_i|$ decreases with ω;
x10 time reduction for every x10 increase in ω

Change of $|V_o/V_i|$ with ω seen as a linear slope on the log-log plot

Change between the 2 parts occurs at ω_c

Log-log plot: **Both** the y-axis and x-axis are on logarithmic scales (base 10). This means moving by 1 interval on either axis, the value increases or decreases by a factor of 10.
Note that on a log scale, one never arrives at zero/infinity.

Bode plot for Low Pass

Bode plot typically comes as a **pair of graphs:**

(1) Log-log plot of magnitude ratio of V_o/V_i vs. frequency

(Log-log plot: Both x and y axes are on logarithmic scales)

(2) Semi-log plot of the phase of V_o/V_i vs. frequency

(Semi-log plot: Linear scale for y-axis (Phase) and log scale for x-axis)

Magnitude

$$\boxed{\frac{V_o}{V_i}(j\omega) = \frac{1}{1 + j\omega/\omega_c}}$$

Phase

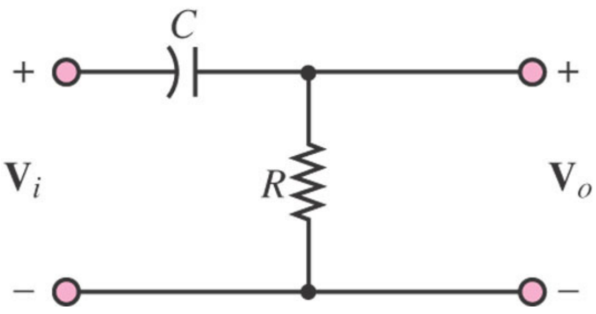
High pass filter

Let us consider the response of the output V_o in relation to the input V_i. We keep the amplitude of V_i constant but vary its frequency ω.

By voltage divider rule:

$$\frac{V_o}{V_i}(j\omega) = \frac{R}{R + 1/j\omega C}$$

$$= \frac{1}{1 + 1/j\omega CR}$$



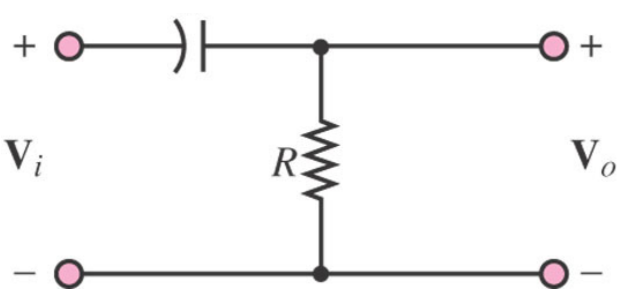
Once again we re-write RC to define the cut off radian frequency, ω_c whereby $\omega_c = 1/RC$:

Sub back into above equation:

$$\frac{V_o}{V_i}(j\omega) = \frac{1}{1 + (\omega_c / j\omega)}$$

Analyze response of high pass filter

$$\frac{V_o}{V_i}(j\omega) = \frac{j(\omega/\omega_c)}{1 + j(\omega/\omega_c)}$$



Both the phase and magnitude of V_o/V_i will change when ω is allowed to vary.

<u>Magnitude</u>

When $\omega \rightarrow 0$:

$V_o/V_i \rightarrow \mathbf{0}/\mathbf{1}$ ⇨ $|V_o/V_i| \rightarrow 0$

When $\omega \rightarrow$ Infinity:

$V_o/V_i \rightarrow$ ____________ ⇨ $|V_o/V_i| \rightarrow 1$

<u>Phase</u>

When $\omega \rightarrow 0$:

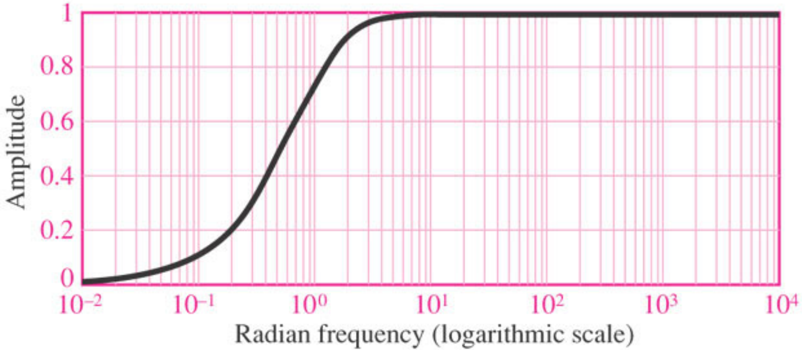
$V_o/V_i \rightarrow \mathbf{(j0)}/\mathbf{(1)} \rightarrow \mathbf{0}\angle 90°/\mathbf{1}\angle 0°$ ⇨ $\angle(V_o/V_i) \rightarrow 90°$

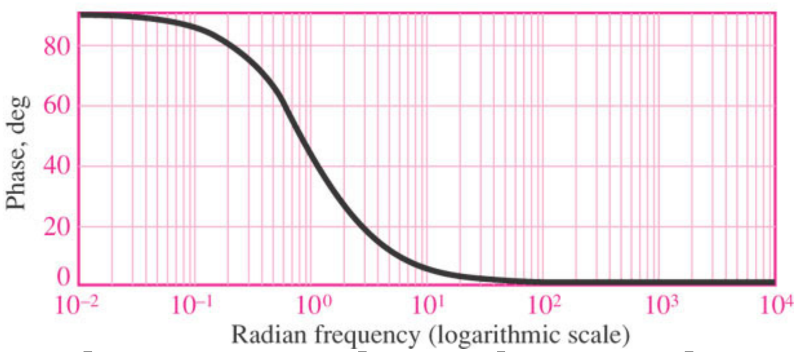
When $\omega \rightarrow$ Infinity:

$V_o/V_i \rightarrow$ ____________ ⇨ $\angle(V_o/V_i) \rightarrow 0°$

Sketch response of high pass filter

Allows higher frequency signals to pass and filters off lower frequency signals





Semi-log plots: y-axis on linear scale, x-axis on logarithmic scale

Observations:

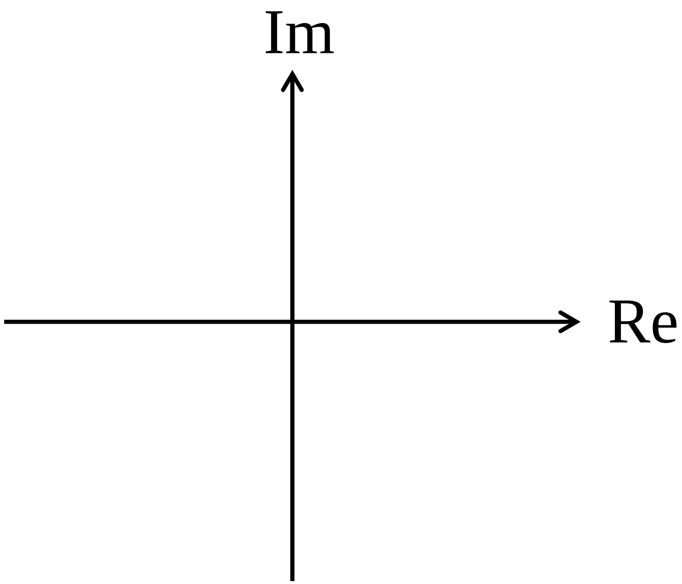
When ω approaches zero, V_o/V_i approaches zero and phase is close to $\pi/2$

When ω becomes large, V_o/V_i approaches 1 and phase is close to 0

Allows higher frequency signals to pass and filters off lower frequency signals

What about in between these two extremes, around ω_c?

At the cut off frequency



Plot of denominator for V_o/V_i

Note that numerator adds a phase shift of 90° from j

$$\frac{V_o}{V_i}(j\omega)=\frac{j(\omega/\omega_c)}{1+j(\omega/\omega_c)}$$

At $\omega = \omega_c$:

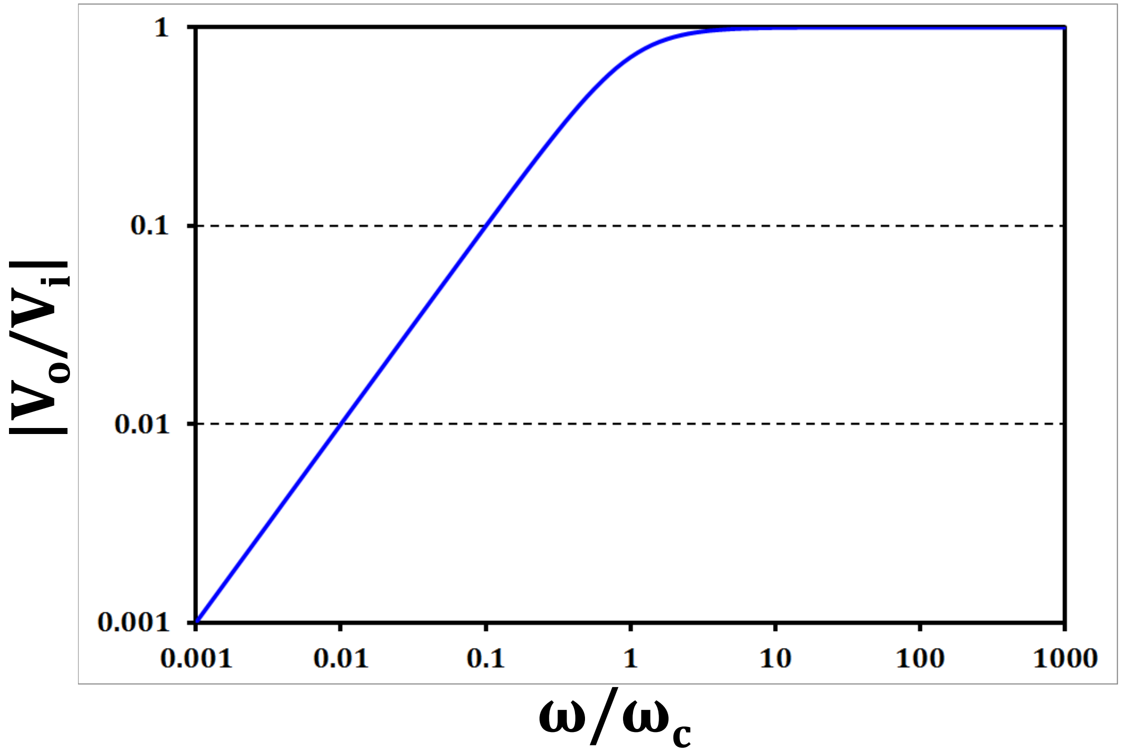
Denominator: **1+ j**

Denominator as a phasor: $\sqrt{2}\angle 45^o$

$$\frac{V_o}{V_i}=$$

At $\omega = \omega_c$, V_o/V_i is once again $1/\sqrt{2}$ of the maximum and has a phase of 45^o

Log-Log Plot for $|V_o/V_i|$ (High Pass)



When $\omega >> \omega_c$:
$|V_o/V_i|$ stays flat close to 1

When $\omega << \omega_c$:
$|V_o/V_i|$ decreases with ω;
x10 time reduction for every x10 reduction in ω

Seen as a linear slope on the log-log plot

Bode plot for High Pass

Bode plot typically comes as a **pair of graphs:**

(1) Log-log plot of magnitude ratio of V_o/V_i vs. frequency

(Log-log plot: Both x and y axes are on logarithmic scales)

(2) Semi-log plot of the phase of V_o/V_i vs. frequency

(Semi-log plot: Linear scale for y-axis (Phase) and log scale for x-axis)

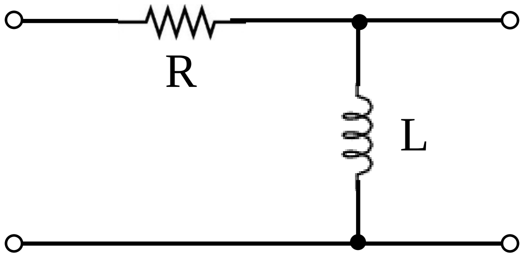
Magnitude

$$\boxed{\frac{V_o}{V_i}(j\omega) = \frac{1}{1+\omega_c / j\omega}}$$

Phase

Other examples on filters 1

Determine the frequency response characteristics (low pass or high pass) for the following filter circuits. Hence draw the bode plot of the filter.



Bode Plots

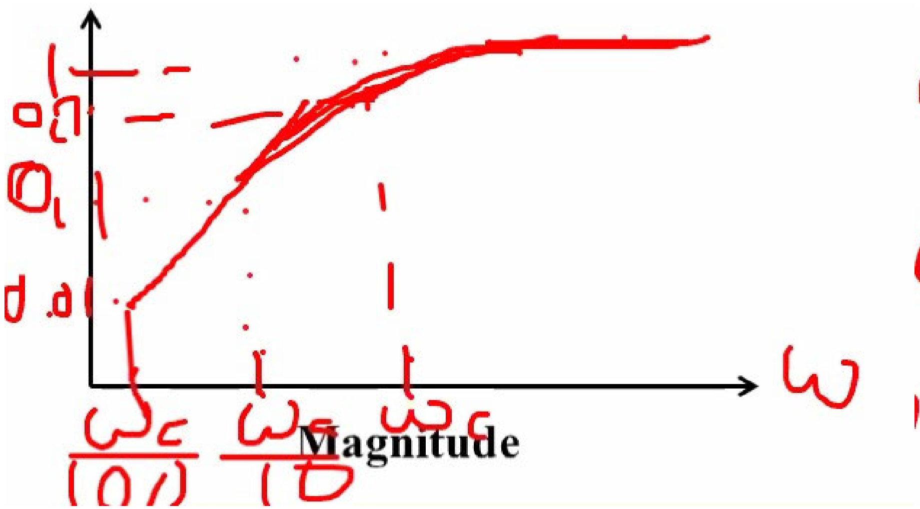
Magnitude

Phase

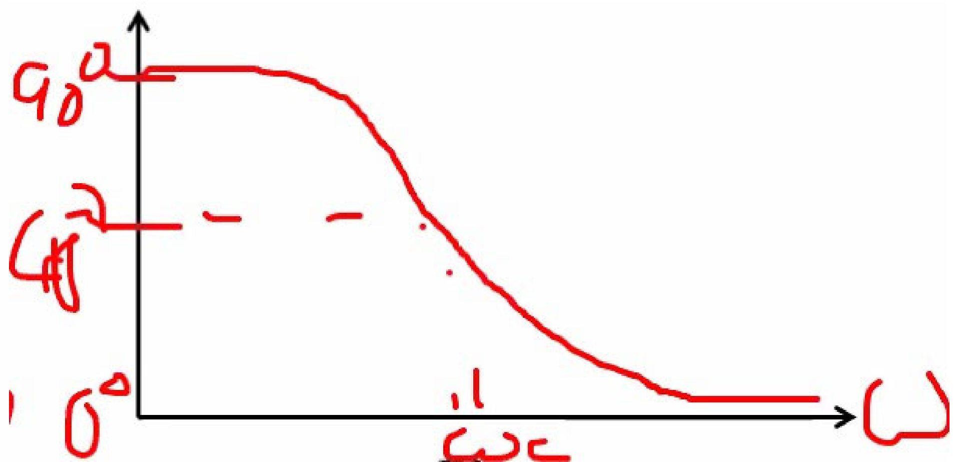
$$\frac{V_0}{V_i} = \frac{j\omega L}{R + j\omega L}$$

$$\omega_c = \frac{R}{L}$$

$$\frac{V_0}{V_i} = \frac{j\omega(\frac{L}{R})}{1 - j\omega(\frac{L}{R})}$$



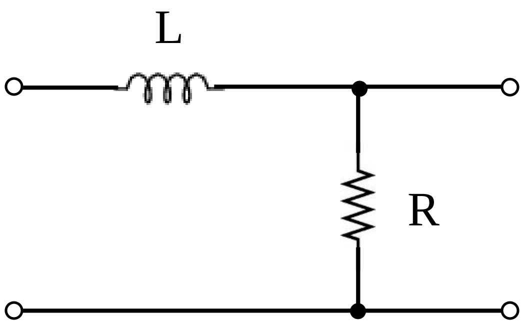
Magnitude



Phase

Other examples on filters 2

Determine the frequency response characteristics (low pass or high pass) for the following filter circuits. Hence draw the bode plot of the filter.



Bode Plots

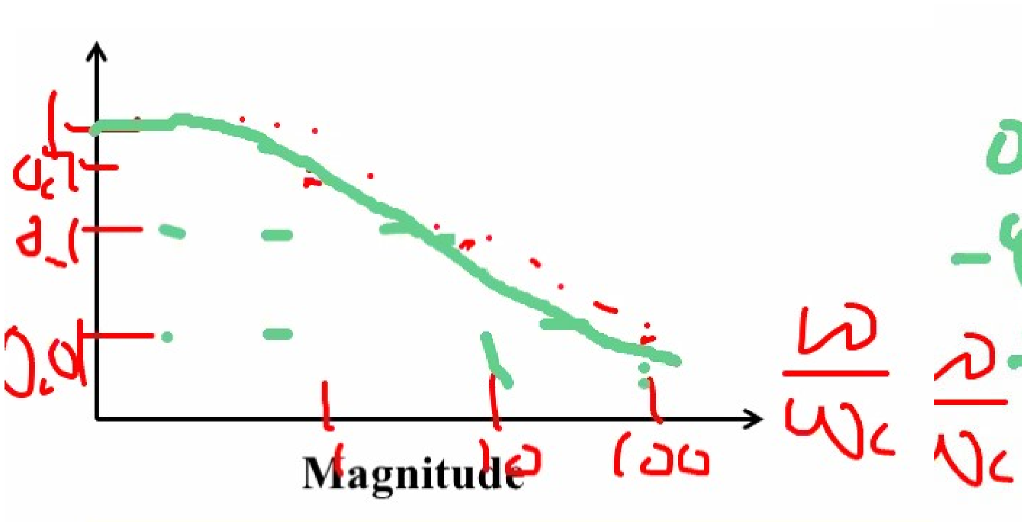
Magnitude

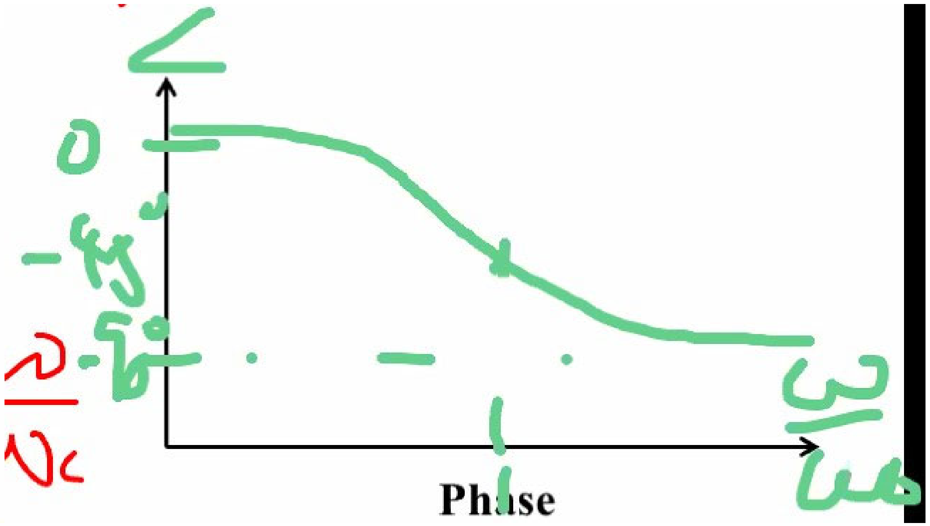
Phase

$$\frac{V_0}{V_i} = \frac{R}{R + j\omega L}$$

$$\omega_c = \frac{R}{L}$$

$$\frac{V_0}{V_i} = \frac{1}{1 + j\omega(\frac{L}{R})}$$

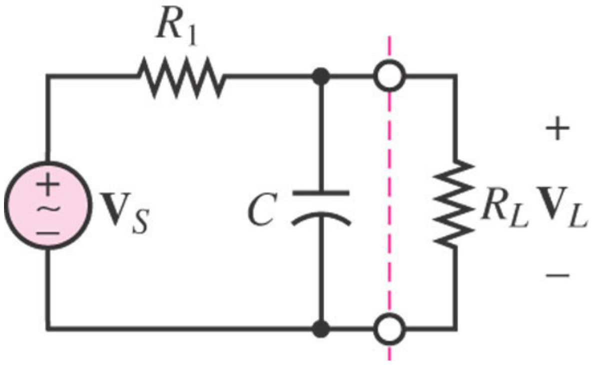




Frequency response example 1

Compute the frequency response of V_L/V_s for the following circuit:

R_1 = 1kΩ; C = 10μF; R_L = 10kΩ



First find the combined impedance of the capacitor C and resistor R_L in parallel:

$$Z_{RC} = \frac{R_L}{(1 + j\omega R_L C)} = \frac{10}{(1 + j0.1\omega)}\ \text{k}\Omega$$

Now apply voltage divider rule:

$$\frac{V_L}{V_S}(j\omega) = \frac{Z_{RC}}{Z_{RC} + R_1} = \frac{10/(1 + j0.1\omega)}{[10/(1 + j0.1\omega)] + 1}$$

$$= \frac{10}{11 + j0.1\omega} = \frac{100}{110 + j\omega}$$

Frequency response example 1

$$\frac{V_L}{V_S}(j\omega) = \frac{100}{110 + j\omega} = \left(\frac{100}{110}\right)\left(\frac{1}{1 + j\omega/110}\right)$$

When $\omega \to 0$:

$V_o/V_i \to 100/110$

$|V_o/V_i| \to 100/110 = 10/11$

$\angle(V_o/V_i) \to 0^o$

When $\omega \to$ Infinity:

$V_o/V_i \to$

$|V_o/V_i| \to 0$

$\angle(V_o/V_i) \to -90^o$

Magnitude

Phase

Frequency response example 1

$$\frac{V_L}{V_S}(j\omega) = \frac{100}{110 + j\omega} = \left(\frac{100}{110}\right)\left(\frac{1}{1 + j\omega/110}\right)$$

$\omega_c = 110\ \text{rad s}^{-1}$

When ω → 0:

$V_o/V_i \rightarrow$ 100/110

$|V_o/V_i| \rightarrow$ 100/110 = 10/11

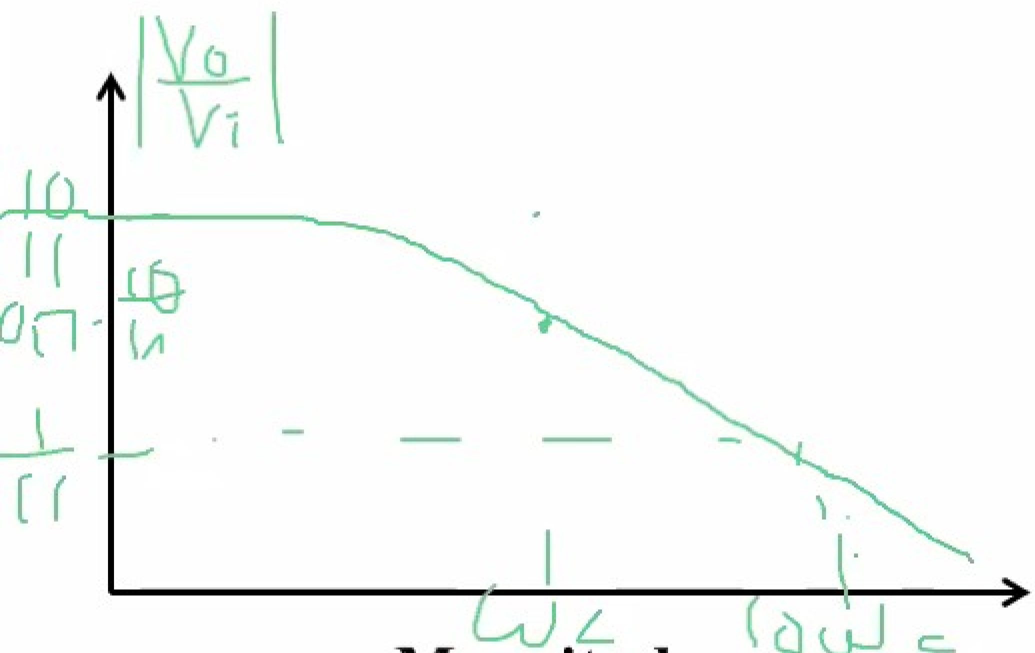
$\angle(V_o/V_i) \rightarrow 0^o$

When ω → Infinity:

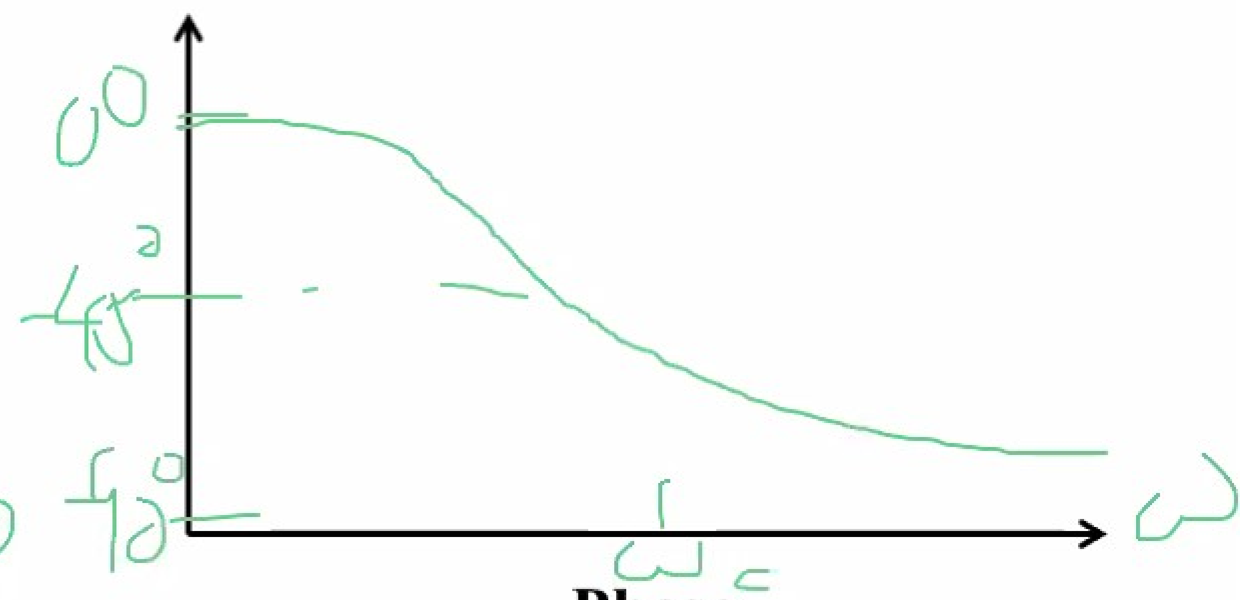
$V_o/V_i \rightarrow$

$|V_o/V_i| \rightarrow$ 0

$\angle(V_o/V_i) \rightarrow -90^o$



Magnitude

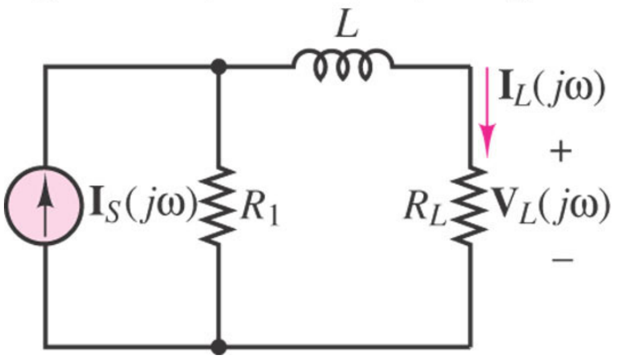


Phase

Frequency response example 2

Compute the frequency response of V_L/I_S for the following circuit:

$R_1 = 1\text{k}\Omega$; $L = 2\text{mH}$; $R_L = 4\text{k}\Omega$



First find the combined impedance of the inductor and resistor in parallel:

$$Z_{RL} = R_L + j\omega L = 4000 + j\left(2\times10^{-3}\right)\omega\ \Omega$$

Now apply current divider rule:

$$\frac{V_L}{I_S}(j\omega) = \left(\frac{I_L}{I_S}\right)R_L = \left(\frac{R_1}{R_1 + Z_{RL}}\right)R_L$$

$$= \frac{(1000)(4000)}{1000 + 4000 + j\left(2\times10^{-3}\right)\omega} = \frac{4\times10^6}{5000 + j\left(2\times10^{-3}\right)\omega}$$

$$= \frac{800}{1 + j\left(4\times10^{-7}\right)\omega}$$

Frequency response example 2

$$\frac{V_L}{I_S}(j\omega) = \frac{800}{1 + j\left(4\times10^{-7}\right)\omega}$$

When $\omega \to 0$:

$V_L/I_s \to 800$

$|V_L/I_s| \to 800$

$\angle(V_L/I_s) \to 0^o$

When $\omega \to$ Infinity:

$V_L/I_S \to$

$|V_L/I_S| \to 0$

$\angle(V_L/I_S) \to -90^o$

Magnitude

Phase

Frequency response example 2

$$\frac{V_L}{I_S}(j\omega)=\frac{800}{1+j\left(4\times10^{-7}\right)\omega}$$

$$\omega=\frac{1}{4\cdot10^{-7}}=0{,}25\cdot10^{7}\ \text{rad/s}$$

When $\omega \to 0$:

$V_L/I_s \to 800$

$|V_L/I_s| \to 800$

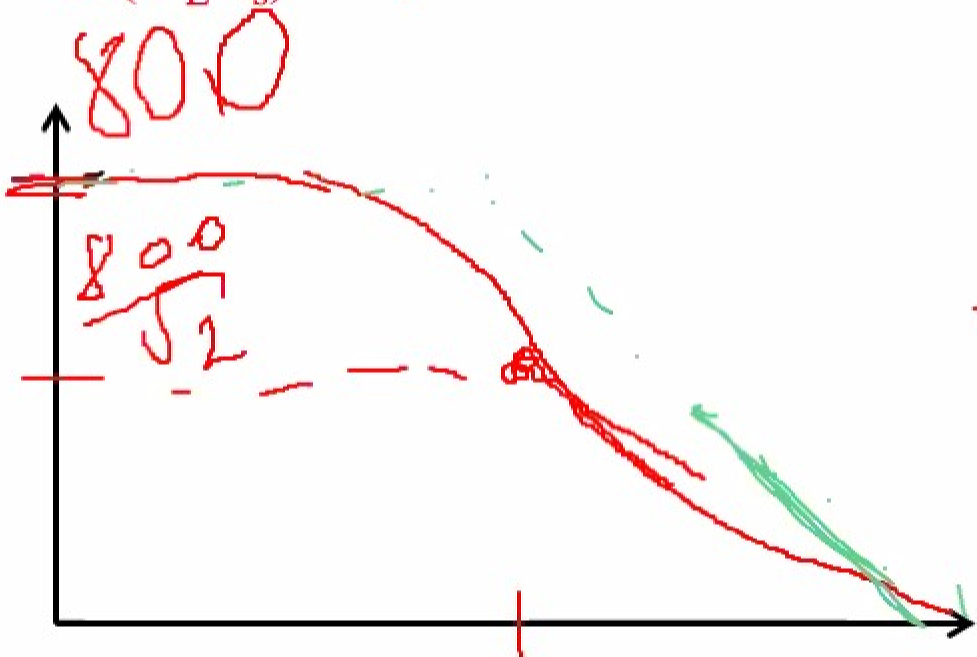
$\angle(V_L/I_s) \to 0^o$

When $\omega \to$ Infinity:

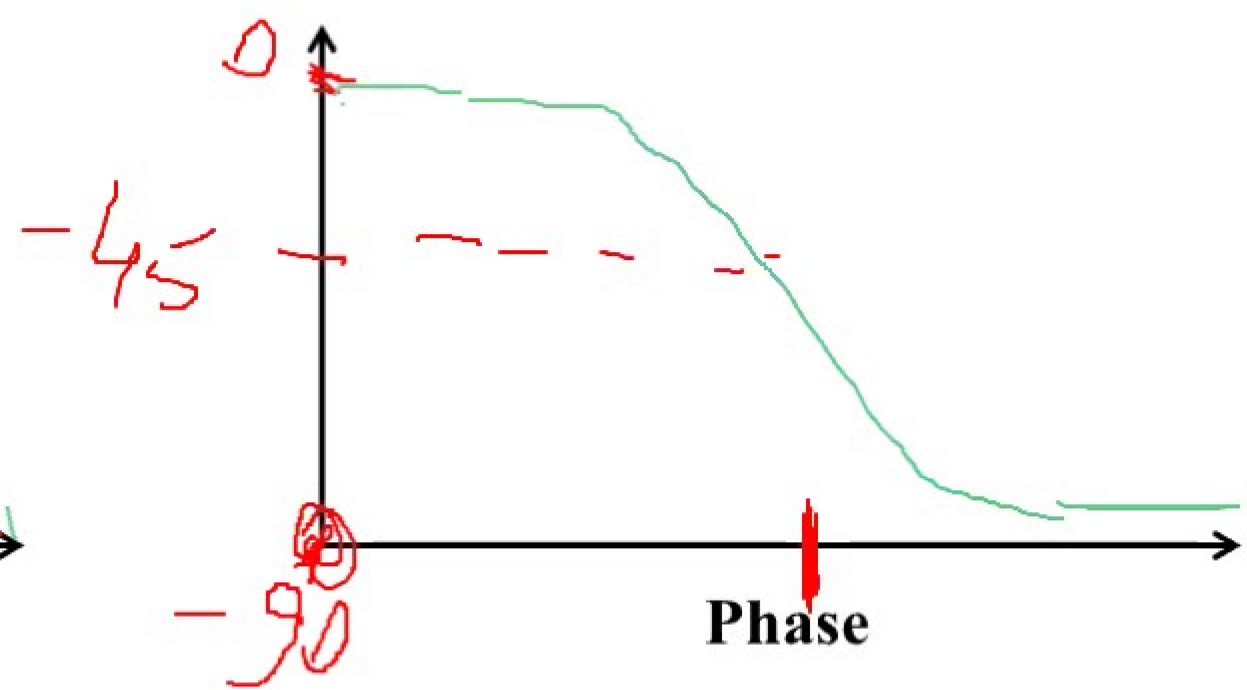
$V_L/I_S \to$

$|V_L/I_S| \to 0$

$\angle(V_L/I_S) \to -90^o$



Magnitude



Phase

General form for low pass filter

$$\frac{V_o}{V_i}(j\omega) = A\left[\frac{1}{1 + j(\omega/\omega_c)}\right]$$

A is a constant and real number

A defines the magnitude of V_o/V_i in the **pass band**

Magnitude

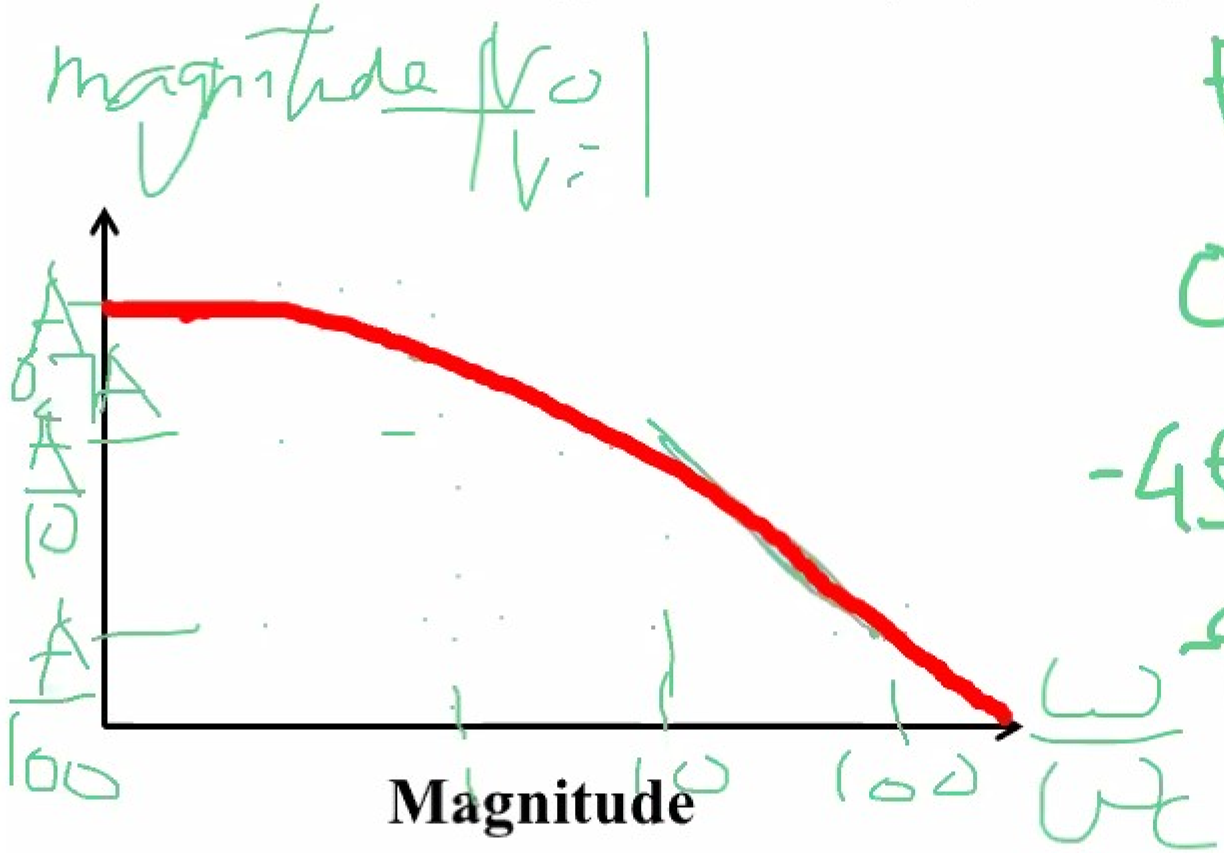
Phase

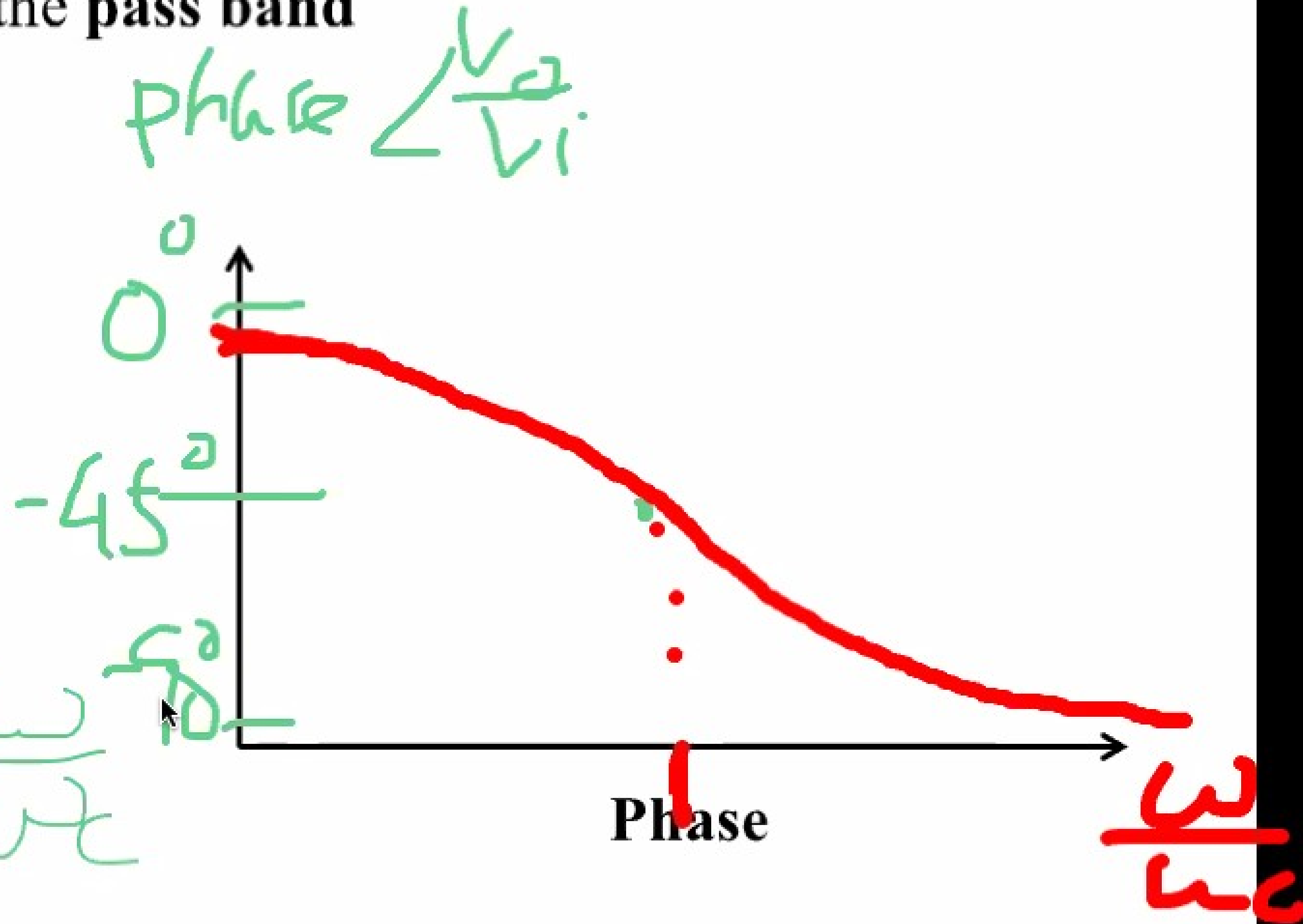
General form for low pass filter

$$\frac{V_o}{V_i}(j\omega) = A\left[\frac{1}{1+j(\omega/\omega_c)}\right]$$

A is a constant and real number

A defines the magnitude of V_o/V_i in the **pass band**





General form for high pass filter

$$\frac{V_o}{V_i}(j\omega) = A\left[\frac{j(\omega/\omega_c)}{1 + j(\omega/\omega_c)}\right]$$

A is a constant and real number

A defines the magnitude of V_o/V_i in the **pass band**

Magnitude

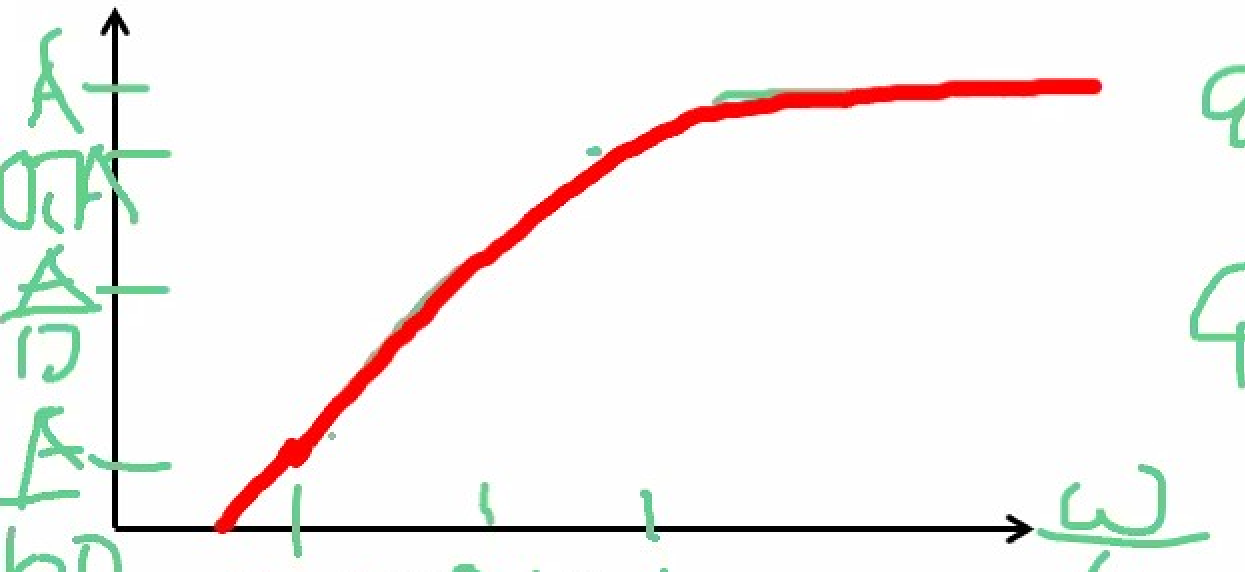
Phase

General form for high pass filter

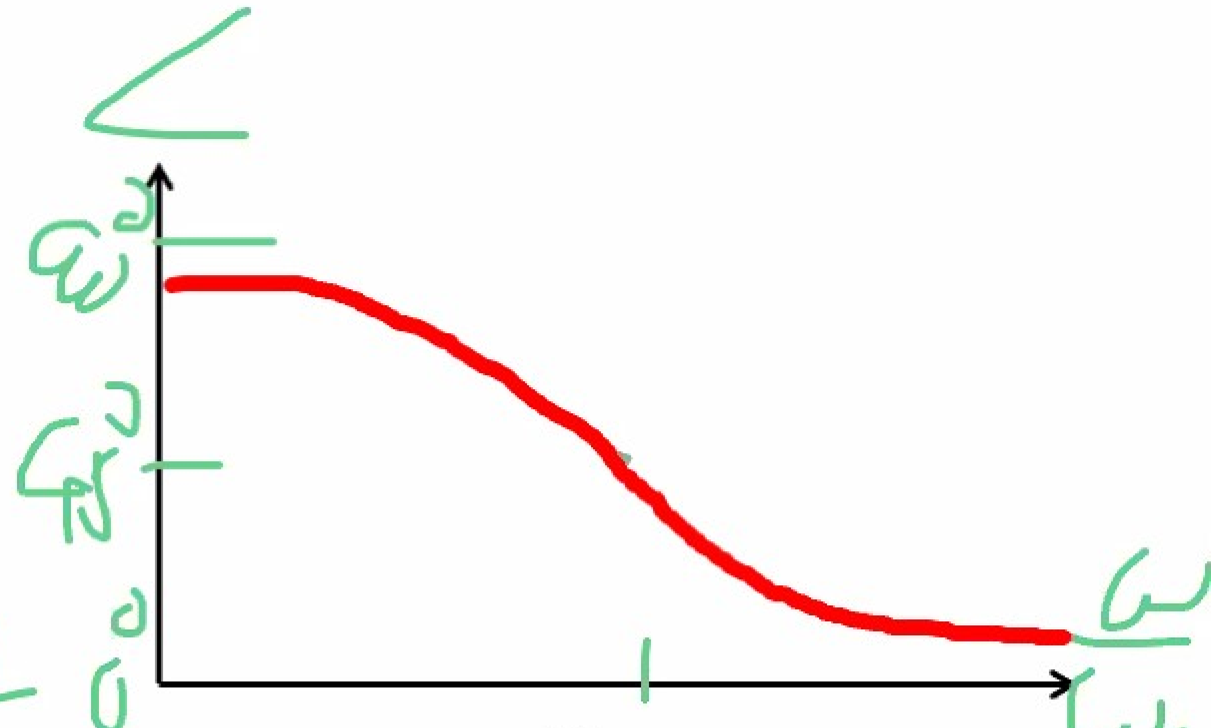
$$\frac{V_o}{V_i}(j\omega) = A\left[\frac{j(\omega/\omega_c)}{1 + j(\omega/\omega_c)}\right]$$

A is a constant and real number

A defines the magnitude of V_o/V_i in the **pass band**



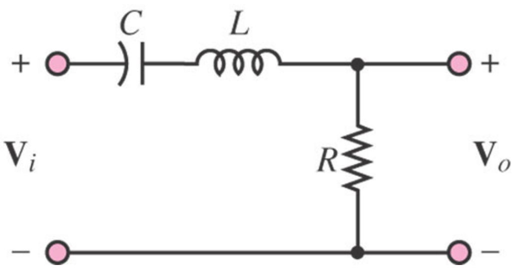
Magnitude



Phase

RLC Series Resonator (Bandpass filter)

RLC bandpass filter. The circuit preserves frequencies within a band.

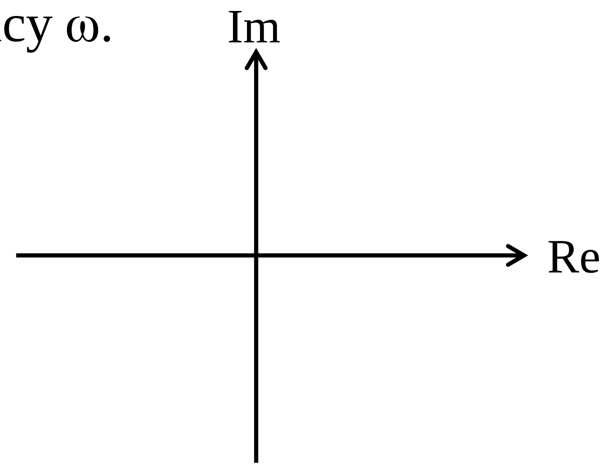


Let us consider the response of the output V_o in relation to the input V_i. We keep the amplitude of V_i constant but vary its frequency ω.

By voltage divider rule:

$$\frac{V_o}{V_i}(j\omega) = \frac{R}{R + j\omega L + 1/j\omega C}$$

$$= \frac{1}{1 + j\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)}$$



Plot of denominator for V_o/V_i

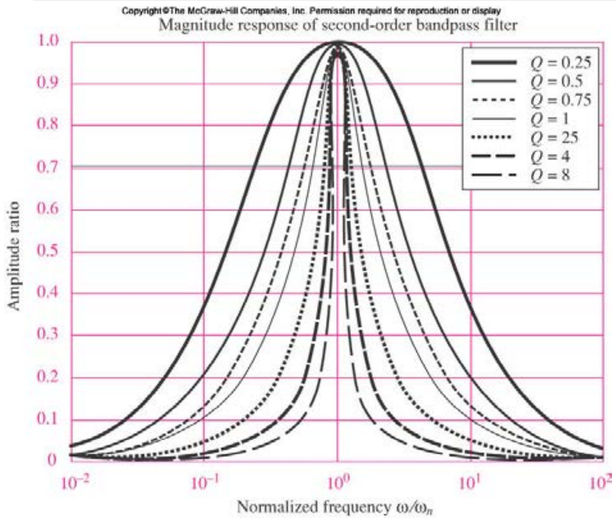
$|V_o/V_i|$ is max when denominator is minimized

We can see that the output will be at its maximum when the imaginary part of the denominator is zero:

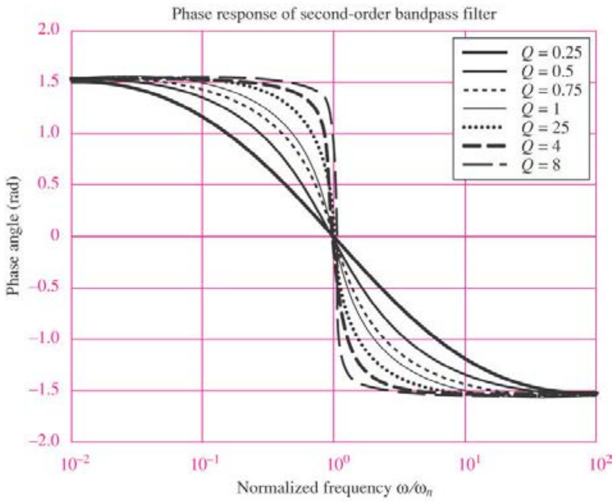
$(\omega L/R) - [1/(\omega RC)] = 0 \rightarrow \omega^2 = 1/(LC)$

When this happens, the impedances of the capacitor and inductor are equal and opposite. This is known as resonance. Max value of V_o/V_i for all frequencies is 1 in this case.

Quality factor



(a)



(b)

(1) There is no "flat" part in the frequency response curve
(2) Response peaks at one frequency: $\omega = 1/\sqrt{(LC)}$, this is known at the resonance frequency, ω_0
(3) For frequencies move further away from ω_0 (whether higher or lower), $|V_o/V_i|$ gets increasingly smaller

$$\frac{V_o}{V_i}(j\omega) = \frac{j\omega CR}{1-\omega^2 LC + j\omega CR}$$

Make the subst. using: $\omega_0 = \dfrac{1}{\sqrt{LC}}$

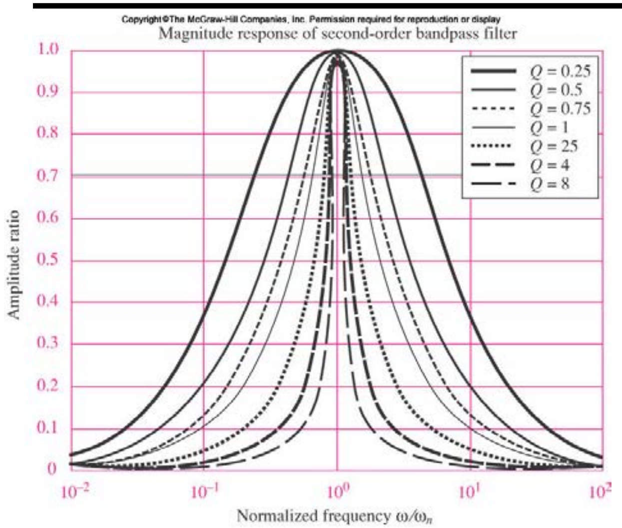
$$\frac{V_o}{V_i}(j\omega) = \frac{j\left(\dfrac{\omega}{\omega_0}\right)\dfrac{R}{\sqrt{L/C}}}{1-\left(\dfrac{\omega}{\omega_0}\right)^2 + j\left(\dfrac{\omega}{\omega_0}\right)\dfrac{R}{\sqrt{L/C}}}$$

If we define: $Q = \dfrac{\sqrt{L/C}}{R}$

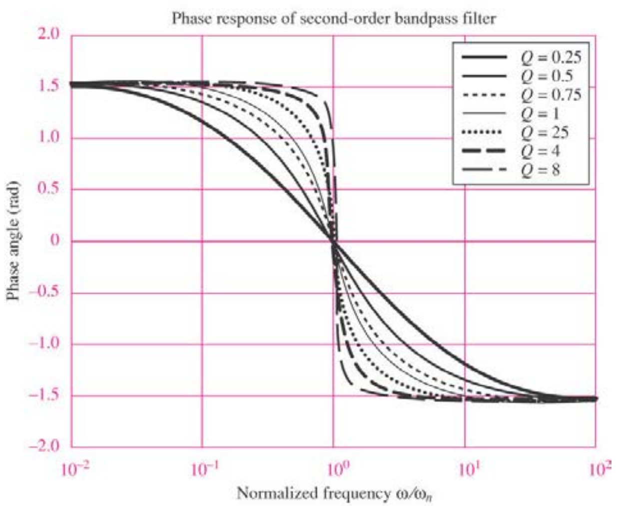
$$\frac{V_o}{V_i}(j\omega) = \frac{j(\omega/\omega_0)(1/Q)}{1-(\omega/\omega_0)^2 + j(\omega/\omega_0)(1/Q)}$$

Q is known as the **Quality Factor** and it describes the sharpness or width of the peak relative to ω_0

Frequency response



(a)



(b)

As Q increases, the resonance peak becomes sharper

In this sense, the resonator only responds at the resonance frequency, ω_0

Frequencies outside ω_0 are filtered out

The resonator works like frequency selector

$$\frac{V_o}{V_i}(j\omega) = \frac{j(\omega/\omega_0)(1/Q)}{1-(\omega/\omega_0)^2 + j(\omega/\omega_0)(1/Q)} \qquad \frac{V_o}{V_i}(j\omega) = \frac{R}{R + j\omega L + 1/j\omega C}$$

<u>Magnitude</u>

When $\omega \rightarrow 0$ or $\rightarrow$ Infinity: $|V_o/V_i| \rightarrow 0$

When $\omega = \omega_0$: $|V_o/V_i| = 1$

<u>Phase</u>

When $\omega \rightarrow 0$: $\angle(V_o/V_i) = 90°$ since $V_o/V_i \rightarrow j\omega CR$

When $\omega \rightarrow$ Infinity: $\angle(V_o/V_i) = -90°$ since $V_o/V_i \rightarrow R/j\omega L$

When $\omega = \omega_0$: $\angle(V_o/V_i) = 0$ since $V_o/V_i \rightarrow 1$