- The transpose of a sum of matrices is equal to the sum of the transposes, and the transpose of a scalar multiple of a matrix is equal to the scalar multiple of the transpose.
- A matrix is symmetric if and only if it is equal to its transpose. All entries above the main diagonal of a symmetric matrix are reflected into equal entries below the diagonal.
- A matrix is skew-symmetric if and only if it is the opposite of its transpose. All main diagonal entries of a skew-symmetric matrix are zero.
- Every square matrix is the sum in a unique way of a symmetric and a skew-symmetric matrix.

EXERCISES FOR SECTION 1.4

1. Compute the following, if possible, for the matrices

$$\mathbf{A}=\begin{bmatrix}-4 & 2 & 3\\ 0 & 5 & -1\\ 6 & 1 & -2\end{bmatrix}\quad \mathbf{B}=\begin{bmatrix}6 & -1 & 0\\ 2 & 2 & -4\\ 3 & -1 & 1\end{bmatrix}\quad \mathbf{C}=\begin{bmatrix}5 & -1\\ -3 & 4\end{bmatrix}$$

$$\mathbf{D}=\begin{bmatrix}-7 & 1 & -4\\ 3 & -2 & 8\end{bmatrix}\quad \mathbf{E}=\begin{bmatrix}3 & -3 & 5\\ 1 & 0 & -2\\ 6 & 7 & -2\end{bmatrix}\quad \mathbf{F}=\begin{bmatrix}8 & -1\\ 2 & 0\\ 5 & -3\end{bmatrix}.$$

⋆**(a)** $\mathbf{A}+\mathbf{B}$	**(h)** $2\mathbf{D}-3\mathbf{F}$
(b) $\mathbf{C}+\mathbf{D}$	⋆**(i)** $\mathbf{A}^T+\mathbf{E}^T$
⋆**(c)** $4\mathbf{A}$	**(j)** $(\mathbf{A}+\mathbf{E})^T$
(d) $2\mathbf{A}-4\mathbf{B}$	**(k)** $4\mathbf{D}+2\mathbf{F}^T$
⋆**(e)** $\mathbf{C}+3\mathbf{F}-\mathbf{E}$	⋆**(l)** $2\mathbf{C}^T-3\mathbf{F}$
(f) $\mathbf{A}-\mathbf{B}+\mathbf{E}$	**(m)** $5\left(\mathbf{F}^T-\mathbf{D}^T\right)$
⋆**(g)** $2\mathbf{A}-3\mathbf{E}-\mathbf{B}$	⋆**(n)** $\left((\mathbf{B}-\mathbf{A})^T+\mathbf{E}^T\right)^T$

⋆**2.** Indicate which of the following matrices are square, diagonal, upper or lower triangular, symmetric, or skew-symmetric. Calculate the transpose for each matrix.

$$\mathbf{A}=\begin{bmatrix}-1 & 4\\ 0 & 1\\ 6 & 0\end{bmatrix}\quad \mathbf{B}=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\quad \mathbf{C}=\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix}\quad \mathbf{D}=\begin{bmatrix}-1\\ 4\\ 2\end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} 0 & 0 & 6 \\ 0 & -6 & 0 \\ -6 & 0 & 0 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 0 & -1 & 6 & 2 \\ 1 & 0 & -7 & 1 \\ -6 & 7 & 0 & -4 \\ -2 & -1 & 4 & 0 \end{bmatrix} \quad \mathbf{J} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 1 & 5 & 6 \\ -3 & -5 & 1 & 7 \\ -4 & -6 & -7 & 1 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad \mathbf{N} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{Q} = \begin{bmatrix} -2 & 0 & 0 \\ 4 & 0 & 0 \\ -1 & 2 & 3 \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} 6 & 2 \\ 3 & -2 \\ -1 & 0 \end{bmatrix}$$

3. Decompose each of the following as the sum of a symmetric and a skew-symmetric matrix:

★**(a)** $\begin{bmatrix} 3 & -1 & 4 \\ 0 & 2 & 5 \\ 1 & -3 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & -4 \\ 3 & 3 & -1 \\ 4 & -1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 3 & 4 & -1 \\ -3 & 5 & -1 & 2 \\ -4 & 1 & -2 & 0 \\ 1 & -2 & 0 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} -3 & 3 & 5 & -4 \\ 11 & 4 & 5 & -1 \\ -9 & 1 & 5 & -14 \\ 2 & -11 & -2 & -5 \end{bmatrix}$

4. Prove that if $\mathbf{A}^T = \mathbf{B}^T$, then $\mathbf{A} = \mathbf{B}$.

5. **(a)** Prove that any symmetric or skew-symmetric matrix is square.

(b) Prove that every diagonal matrix is symmetric.

(c) Show that $(\mathbf{I}_n)^T = \mathbf{I}_n$. (Hint: Use part (b).)

★**(d)** Describe completely every matrix that is both diagonal and skew-symmetric.

6. Assume that **A** and **B** are square matrices of the same size.

(a) If **A** and **B** are diagonal, prove that $\mathbf{A} + \mathbf{B}$ is diagonal.

(b) If **A** and **B** are symmetric, prove that $\mathbf{A} + \mathbf{B}$ is symmetric.

7. Use induction to prove that, if $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are upper triangular matrices of the same size, then $\Sigma_{i=1}^{n} \mathbf{A}_i$ is upper triangular.

8. **(a)** If $\mathbf{A}$ is a symmetric matrix, show that $\mathbf{A}^T$ and $c\mathbf{A}$ are also symmetric.

(b) If $\mathbf{A}$ is a skew-symmetric matrix, show that $\mathbf{A}^T$ and $c\mathbf{A}$ are also skew-symmetric.

9. The **Kronecker Delta** δ_{ij} is defined as follows: $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$. If $\mathbf{A} = \mathbf{I}_n$, explain why $a_{ij} = \delta_{ij}$.

10. Prove parts (4), (5), and (7) of Theorem 1.11.

►11. Prove parts (1) and (3) of Theorem 1.12.

12. Let $\mathbf{A}$ be an $m \times n$ matrix. Prove that if $c\mathbf{A} = \mathbf{O}_{mn}$, the $m \times n$ zero matrix, then $c = 0$ or $\mathbf{A} = \mathbf{O}_{mn}$.

13. This exercise provides an outline for the proof of Theorem 1.13. Let $\mathbf{A}$ be an $n \times n$ matrix.

(a) Prove that $\frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right)$ is a symmetric matrix.

(b) Prove that $\frac{1}{2}\left(\mathbf{A} - \mathbf{A}^T\right)$ is a skew-symmetric matrix.

(c) Show that $\mathbf{A} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right) + \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^T\right)$.

(d) Suppose that $\mathbf{S}_1$ and $\mathbf{S}_2$ are symmetric matrices and that $\mathbf{V}_1$ and $\mathbf{V}_2$ are skew-symmetric matrices such that $\mathbf{S}_1 + \mathbf{V}_1 = \mathbf{S}_2 + \mathbf{V}_2$. Derive a second equation involving $\mathbf{S}_1, \mathbf{S}_2, \mathbf{V}_1$, and $\mathbf{V}_2$ by taking the transpose of both sides of the equation and simplifying.

(e) Prove that $\mathbf{S}_1 = \mathbf{S}_2$ by adding the two equations from part (d) together.

(f) Use parts (d) and (e) to prove that $\mathbf{V}_1 = \mathbf{V}_2$.

(g) Explain how parts (a) through (f) together prove Theorem 1.13.

14. The **trace** of a square matrix $\mathbf{A}$ is the sum of the elements along the main diagonal.

★(a) Find the trace of each square matrix in Exercise 2.

(b) If $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$ matrices, prove that:

(i) $\text{trace}(\mathbf{A} + \mathbf{B}) = \text{trace}(\mathbf{A}) + \text{trace}(\mathbf{B})$

(ii) $\text{trace}(c\mathbf{A}) = c(\text{trace}(\mathbf{A}))$

(iii) $\text{trace}(\mathbf{A}) = \text{trace}\left(\mathbf{A}^T\right)$

★(c) Suppose that $\text{trace}(\mathbf{A}) = \text{trace}(\mathbf{B})$ for two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$. Does $\mathbf{A} = \mathbf{B}$? Prove your answer.

★15. True or False:

(a) A 5×6 matrix has exactly six entries on its main diagonal.

(b) The transpose of a lower triangular matrix is upper triangular.

(c) No skew-symmetric matrix is diagonal.

(d) If $\mathbf{V}$ is a skew-symmetric matrix, then $-\mathbf{V}^T = \mathbf{V}$.

(e) For all scalars c, and $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, $\left(c\left(\mathbf{A}^T + \mathbf{B}\right)\right)^T = c\mathbf{B}^T + c\mathbf{A}$.

- The kth row of a matrix product is equal to the kth row of the first matrix times the (whole) second matrix, and the lth column of a matrix product is equal to the (whole) first matrix times the lth column of the second matrix.
- The associative and distributive laws hold for matrix multiplication (but *not* the commutative law).
- The cancellation laws do not generally hold for matrix multiplication. That is, $\mathbf{AB} = \mathbf{AC}$ or $\mathbf{BA} = \mathbf{CA}$ do not necessarily imply $\mathbf{B} = \mathbf{C}$.
- Any product of a matrix with a zero matrix is equal to a zero matrix. However, if the product of two matrices is zero, it does not necessarily mean that one of the matrices is zero.
- The usual laws of exponents hold for powers of square matrices, except that a power of a matrix product is usually not equal to the product of the individual powers of the matrices; that is, in general, $(\mathbf{AB})^q \neq \mathbf{A}^q\mathbf{B}^q$. In particular, $\mathbf{ABAB} = (\mathbf{AB})^2 \neq \mathbf{A}^2\mathbf{B}^2 = \mathbf{AABB}$.
- The transpose of a matrix product is found by multiplying the transposes of the matrices in *reverse* order.
- If $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{B}$ is a $1 \times m$ matrix, and $\mathbf{C}$ is an $n \times 1$ matrix, then $\mathbf{BA}$ gives a linear combination of the rows of $\mathbf{A}$, and $\mathbf{AC}$ gives a linear combination of the columns of $\mathbf{A}$.

EXERCISES FOR SECTION 1.5

Note: Exercises 1 through 3 refer to the following matrices:

$$\mathbf{A} = \begin{bmatrix} -2 & 3 \\ 6 & 5 \\ 1 & -4 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -5 & 3 & 6 \\ 3 & 8 & 0 \\ -2 & 0 & 4 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 11 & -2 \\ -4 & -2 \\ 3 & -1 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} -1 & 4 & 3 & 7 \\ 2 & 1 & 7 & 5 \\ 0 & 5 & 5 & -2 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 9 & -3 \\ 5 & -4 \\ 2 & 0 \\ 8 & -3 \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} 5 & 1 & 0 \\ 0 & -2 & -1 \\ 1 & 0 & 3 \end{bmatrix} \quad \mathbf{H} = \begin{bmatrix} 6 & 3 & 1 \\ 1 & -15 & -5 \\ -2 & -1 & 10 \end{bmatrix} \quad \mathbf{J} = \begin{bmatrix} 8 \\ -1 \\ 4 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 2 & 1 & -5 \\ 0 & 2 & 7 \end{bmatrix} \quad \mathbf{L} = \begin{bmatrix} 10 & 9 \\ 8 & 7 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} 7 & -1 \\ 11 & 3 \end{bmatrix}$$

$$\mathbf{N} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{P} = \begin{bmatrix} 3 & -1 \\ 4 & 7 \end{bmatrix} \quad \mathbf{Q} = \begin{bmatrix} 1 & 4 & -1 & 6 \\ 8 & 7 & -3 & 3 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} -3 & 6 & -2 \end{bmatrix} \quad \mathbf{S} = \begin{bmatrix} 6 & -4 & 3 & 2 \end{bmatrix} \quad \mathbf{T} = \begin{bmatrix} 4 & -1 & 7 \end{bmatrix}$$

1. Which of these products are possible? If possible, then calculate the product.

(a) $\mathbf{AB}$
★**(b)** $\mathbf{BA}$
★**(c)** $\mathbf{JM}$
(d) $\mathbf{DF}$
★**(e)** $\mathbf{RJ}$
★**(f)** $\mathbf{JR}$
★**(g)** $\mathbf{RT}$
(h) $\mathbf{SF}$
(i) $\mathbf{KN}$
★**(j)** $\mathbf{F}^2$
(k) $\mathbf{B}^2$
★**(l)** $\mathbf{E}^3$
(m) $(\mathbf{TJ})^3$
★**(n)** $\mathbf{D}(\mathbf{FK})$
(o) $(\mathbf{CL})\mathbf{G}$

2. Determine whether these pairs of matrices commute.

★**(a)** $\mathbf{L}$ and $\mathbf{M}$
(b) $\mathbf{G}$ and $\mathbf{H}$
★**(c)** $\mathbf{A}$ and $\mathbf{K}$
★**(d)** $\mathbf{N}$ and $\mathbf{P}$
(e) $\mathbf{F}$ and $\mathbf{Q}$

3. Find only the indicated row or column of each given matrix product.

★**(a)** The second row of $\mathbf{BG}$
(b) The third column of $\mathbf{DE}$
★**(c)** The first column of $\mathbf{SE}$
(d) The third row of $\mathbf{FQ}$

★**4.** Assuming that all of the following products exist, which of these equations are always valid? If valid, specify which theorems (and parts, if appropriate) apply.

(a) $(\mathbf{RG})\mathbf{H} = \mathbf{R}(\mathbf{GH})$
(b) $\mathbf{LP} = \mathbf{PL}$
(c) $\mathbf{E}(\mathbf{FK}) = (\mathbf{EF})\mathbf{K}$
(d) $\mathbf{K}(\mathbf{A} + \mathbf{C}) = \mathbf{KA} + \mathbf{KC}$
(e) $(\mathbf{QF})^T = \mathbf{F}^T\mathbf{Q}^T$
(f) $\mathbf{L}(\mathbf{ML}) = \mathbf{L}^2\mathbf{M}$
(g) $\mathbf{GC} + \mathbf{HC} = (\mathbf{G} + \mathbf{H})\mathbf{C}$
(h) $\mathbf{R}(\mathbf{J} + \mathbf{T}^T) = \mathbf{RJ} + \mathbf{RT}^T$
(i) $(\mathbf{AK})^T = \mathbf{A}^T\mathbf{K}^T$
(j) $(\mathbf{Q} + \mathbf{F}^T)\mathbf{E}^T = \mathbf{QE}^T + (\mathbf{EF})^T$

★5. The following matrices detail the number of employees at four different retail outlets and their wages and benefits (per year). Calculate the total salaries and fringe benefits paid by each outlet per year to its employees.

$$\begin{array}{l}
\\
\text{Outlet 1} \\ \text{Outlet 2} \\ \text{Outlet 3} \\ \text{Outlet 4}
\end{array}
\begin{array}{c}
\begin{array}{ccc} \text{Executives} & \text{Salespersons} & \text{Others} \end{array} \\
\left[\begin{array}{ccc} 3 & 7 & 8 \\ 2 & 4 & 5 \\ 6 & 14 & 18 \\ 3 & 6 & 9 \end{array}\right]
\end{array}$$

$$\begin{array}{r}
\\
\text{Executives} \\ \text{Salespersons} \\ \text{Others}
\end{array}
\begin{array}{c}
\begin{array}{cc} \text{Salary} & \text{Fringe Benefits} \end{array} \\
\left[\begin{array}{cc} \$30000 & \$7500 \\ \$22500 & \$4500 \\ \$15000 & \$3000 \end{array}\right]
\end{array}$$

6. The following matrices detail the typical amount spent on tickets, food, and souvenirs at a Summer Festival by a person from each of four age groups, and the total attendance by these different age groups during each month of the festival. Calculate the total amount spent on tickets, food, and souvenirs each month.

$$\begin{array}{r}
\\
\text{Children} \\ \text{Teens} \\ \text{Adults} \\ \text{Seniors}
\end{array}
\begin{array}{c}
\begin{array}{ccc} \text{Tickets} & \text{Food} & \text{Souvenirs} \end{array} \\
\left[\begin{array}{ccc} \$2 & \$5 & \$8 \\ \$4 & \$12 & \$3 \\ \$6 & \$15 & \$10 \\ \$3 & \$9 & \$12 \end{array}\right]
\end{array}$$

$$\begin{array}{r}
\\
\text{June Attendance} \\ \text{July Attendance} \\ \text{August Attendance}
\end{array}
\begin{array}{c}
\begin{array}{cccc} \text{Children} & \text{Teens} & \text{Adults} & \text{Seniors} \end{array} \\
\left[\begin{array}{cccc} 32500 & 54600 & 121500 & 46300 \\ 37400 & 62800 & 136000 & 52900 \\ 29800 & 48500 & 99200 & 44100 \end{array}\right]
\end{array}$$

★7. Matrix **A** gives the percentage of nitrogen, phosphates, and potash in three fertilizers. Matrix **B** gives the amount (in tons) of each type of fertilizer spread on three different fields. Use matrix operations to find the total amount of nitrogen, phosphates, and potash on each field.

$$\mathbf{A} = \begin{array}{l} \\ \text{Fertilizer 1} \\ \text{Fertilizer 2} \\ \text{Fertilizer 3} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Nitrogen} & \text{Phosphates} & \text{Potash} \end{array} \\ \left[\begin{array}{ccc} 10\% & 10\% & 5\% \\ 25\% & 5\% & 5\% \\ 0\% & 10\% & 20\% \end{array}\right] \end{array}$$

$$\mathbf{B} = \begin{array}{l} \\ \text{Fertilizer 1} \\ \text{Fertilizer 2} \\ \text{Fertilizer 3} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Field 1} & \text{Field 2} & \text{Field 3} \end{array} \\ \left[\begin{array}{ccc} 5 & 2 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 3 \end{array}\right] \end{array}$$

8. Matrix **A** gives the numbers of four different types of computer modules that are needed to assemble various rockets. Matrix **B** gives the amounts of four different types of computer chips that compose each module. Use matrix operations to find the total amount of each type of computer chip needed for each rocket.

$$\mathbf{A} = \begin{array}{l} \\ \text{Rocket 1} \\ \text{Rocket 2} \\ \text{Rocket 3} \\ \text{Rocket 4} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Module A} & \text{Module B} & \text{Module C} & \text{Module D} \end{array} \\ \left[\begin{array}{cccc} 24 & 10 & 5 & 17 \\ 25 & 8 & 6 & 16 \\ 32 & 12 & 8 & 22 \\ 27 & 11 & 7 & 19 \end{array}\right] \end{array}$$

$$\mathbf{B} = \begin{array}{l} \\ \text{Chip 1} \\ \text{Chip 2} \\ \text{Chip 3} \\ \text{Chip 4} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Module A} & \text{Module B} & \text{Module C} & \text{Module D} \end{array} \\ \left[\begin{array}{cccc} 42 & 37 & 52 & 29 \\ 23 & 25 & 48 & 31 \\ 37 & 33 & 29 & 28 \\ 52 & 46 & 35 & 51 \end{array}\right] \end{array}$$

★9. **(a)** Find a nondiagonal matrix **A** such that $\mathbf{A}^2 = \mathbf{I}_2$.

(b) Find a nondiagonal matrix **A** such that $\mathbf{A}^2 = \mathbf{I}_3$. (Hint: Modify your answer to part (a).)

(c) Find a nonidentity matrix **A** such that $\mathbf{A}^3 = \mathbf{I}_3$.

10. Let $\mathbf{A}$ be an $m \times n$ matrix, and let $\mathbf{B}$ be an $n \times m$ matrix, with $m, n \geq 5$. Each of the following sums represents an entry of either $\mathbf{AB}$ or $\mathbf{BA}$. Determine which product is involved and which entry of that product is represented.

★**(a)** $\sum_{k=1}^{n} a_{3k} b_{k4}$

(b) $\sum_{q=1}^{n} a_{4q} b_{q1}$

★**(c)** $\sum_{k=1}^{m} a_{k2} b_{3k}$

(d) $\sum_{q=1}^{m} b_{2q} a_{q5}$

11. Let $\mathbf{A}$ be an $m \times n$ matrix, and let $\mathbf{B}$ be an $n \times m$ matrix, where $m, n \geq 4$. Use sigma (Σ) notation to express the following entries symbolically:

★**(a)** The entry in the third row and second column of $\mathbf{AB}$

(b) The entry in the fourth row and first column of $\mathbf{BA}$

★**12.** For the matrix $\mathbf{A} = \begin{bmatrix} 4 & 7 & -2 \\ -3 & -6 & 5 \\ -9 & 2 & -8 \end{bmatrix}$, use matrix multiplication (as in Example 4) to find the following linear combinations:

(a) $3\mathbf{v}_1 - 2\mathbf{v}_2 + 5\mathbf{v}_3$, where $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are the rows of $\mathbf{A}$

(b) $2\mathbf{w}_1 + 6\mathbf{w}_2 - 3\mathbf{w}_3$, where $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are the columns of $\mathbf{A}$

13. For the matrix $\mathbf{A} = \begin{bmatrix} 7 & -3 & -4 & 1 \\ -5 & 6 & 2 & -3 \\ -1 & 9 & 3 & -8 \end{bmatrix}$, use matrix multiplication (as in Example 4) to find the following linear combinations:

(a) $-5\mathbf{v}_1 + 6\mathbf{v}_2 - 4\mathbf{v}_3$, where $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are the rows of $\mathbf{A}$

(b) $6\mathbf{w}_1 - 4\mathbf{w}_2 + 2\mathbf{w}_3 - 3\mathbf{w}_4$, where $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ are the columns of $\mathbf{A}$

14. **(a)** Consider the unit vectors $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ in $\mathbb{R}^3$. Show that, if $\mathbf{A}$ is an $m \times 3$ matrix, then $\mathbf{Ai}$ = first column of $\mathbf{A}$, $\mathbf{Aj}$ = second column of $\mathbf{A}$, and $\mathbf{Ak}$ = third column of $\mathbf{A}$.

(b) Generalize part (a) to a similar result involving an $m \times n$ matrix $\mathbf{A}$ and the standard unit vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in $\mathbb{R}^n$.

(c) Let $\mathbf{A}$ be an $m \times n$ matrix. Use part (b) to show that, if $\mathbf{Ax} = \mathbf{0}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$, then $\mathbf{A} = \mathbf{O}_{mn}$.

►**15.** Prove parts (2), (3), and (4) of Theorem 1.14.

16. Let $\mathbf{A}$ be an $m \times n$ matrix. Prove $\mathbf{AO}_{np} = \mathbf{O}_{mp}$.

17. Let $\mathbf{A}$ be an $m \times n$ matrix. Prove $\mathbf{AI}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$.

18. **(a)** Prove that the product of two diagonal matrices is diagonal. (Hint: If $\mathbf{C} = \mathbf{AB}$ where $\mathbf{A}$ and $\mathbf{B}$ are diagonal, show that $c_{ij} = 0$ when $i \neq j$.)

(b) Prove that the product of two upper triangular matrices is upper triangular. (Hint: Let $\mathbf{A}$ and $\mathbf{B}$ be upper triangular and $\mathbf{C} = \mathbf{AB}$. Show $c_{ij} = 0$ when $i > j$ by checking that all terms $a_{ik}b_{kj}$ in the formula for c_{ij} have at least one zero factor. Consider the following two cases: $i > k$ and $i \leq k$.)

(c) Prove that the product of two lower triangular matrices is lower triangular. (Hint: Use Theorem 1.16 and part (b) of this exercise.)

19. Show that if $c \in \mathbb{R}$ and $\mathbf{A}$ is a square matrix, then $(c\mathbf{A})^n = c^n\mathbf{A}^n$ for any integer $n \geq 1$. (Hint: Use a proof by induction.)

►**20.** Prove each part of Theorem 1.15 using the method of induction. (Hint: Use induction on t for both parts. Part (1) will be useful in proving part (2).)

21. (a) Show $\mathbf{AB} = \mathbf{BA}$ only if $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size.

(b) Prove two square matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size commute if and only if $(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$.

22. If $\mathbf{A}, \mathbf{B}$, and $\mathbf{C}$ are all square matrices of the same size, show that $\mathbf{AB}$ commutes with $\mathbf{C}$ if $\mathbf{A}$ and $\mathbf{B}$ both commute with $\mathbf{C}$.

23. Show that $\mathbf{A}$ and $\mathbf{B}$ commute if and only if $\mathbf{A}^T$ and $\mathbf{B}^T$ commute.

24. Let $\mathbf{A}$ be any matrix. Show that $\mathbf{AA}^T$ and $\mathbf{A}^T\mathbf{A}$ are both symmetric.

25. Let $\mathbf{A}$ and $\mathbf{B}$ both be $n \times n$ matrices.

(a) Show that $(\mathbf{AB})^T = \mathbf{BA}$ if $\mathbf{A}$ and $\mathbf{B}$ are both symmetric or both skew-symmetric.

(b) If $\mathbf{A}$ and $\mathbf{B}$ are both symmetric, show that $\mathbf{AB}$ is symmetric if and only if $\mathbf{A}$ and $\mathbf{B}$ commute.

26. Recall the definition of the **trace** of a matrix given in Exercise 14 of Section 1.4. If $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$ matrices, show the following:

(a) $\mathrm{Trace}\left(\mathbf{AA}^T\right)$ is the sum of the squares of all entries of $\mathbf{A}$.

(b) If $\mathrm{trace}\left(\mathbf{AA}^T\right) = 0$, then $\mathbf{A} = \mathbf{O}_n$. (Hint: Use part (a) of this exercise.)

(c) $\mathrm{Trace}(\mathbf{AB}) = \mathrm{trace}(\mathbf{BA})$. (Hint: Calculate trace($\mathbf{AB}$) and trace($\mathbf{BA}$) in the 3×3 case to discover how to prove the general $n \times n$ case.)

27. An **idempotent** matrix is a square matrix $\mathbf{A}$ for which $\mathbf{A}^2 = \mathbf{A}$. (Note that if $\mathbf{A}$ is idempotent, then $\mathbf{A}^n = \mathbf{A}$ for every integer $n \geq 1$.)

⋆(a) Find a 2×2 idempotent matrix (besides $\mathbf{I}_n$ and $\mathbf{O}_n$).

(b) Show that $\begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ is idempotent.

(c) If $\mathbf{A}$ is an $n \times n$ idempotent matrix, show that $\mathbf{I}_n - \mathbf{A}$ is also idempotent.

(d) Use parts (b) and (c) to get another example of an idempotent matrix.

(e) Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. Show that $\mathbf{A}$ is idempotent if both $\mathbf{AB} = \mathbf{A}$ and $\mathbf{BA} = \mathbf{B}$.

28. **(a)** Let $\mathbf{A}$ be an $m \times n$ matrix, and let $\mathbf{B}$ be an $n \times p$ matrix. Prove that $\mathbf{AB} = \mathbf{O}_{mp}$ if and only if every (vector) row of $\mathbf{A}$ is orthogonal to each column of $\mathbf{B}$.

★**(b)** Find a 2×3 matrix $\mathbf{A} \neq \mathbf{O}$ and a 3×2 matrix $\mathbf{B} \neq \mathbf{O}$ such that $\mathbf{AB} = \mathbf{O}_2$.

(c) Using your answers from part (b), find a matrix $\mathbf{C} \neq \mathbf{B}$ such that $\mathbf{AB} = \mathbf{AC}$.

★**29.** What form does a 2×2 matrix have if it commutes with every other 2×2 matrix? Prove that your answer is correct.

30. Let $\mathbf{A}$ be an $n \times n$ matrix. Consider the $n \times n$ matrix $\mathbf{\Psi}_{ij}$, which has all entries zero except for an entry of 1 in the (i, j) position.

(a) Show that the jth column of $\mathbf{A\Psi}_{ij}$ equals the ith column of $\mathbf{A}$ and all other columns of $\mathbf{A\Psi}_{ij}$ have only zero entries.

(b) Show that the ith row of $\mathbf{\Psi}_{ij}\mathbf{A}$ equals the jth row of $\mathbf{A}$ and all other rows of $\mathbf{\Psi}_{ij}\mathbf{A}$ have only zero entries.

(c) Use parts (a) and (b) to prove that an $n \times n$ matrix $\mathbf{A}$ commutes with all other $n \times n$ matrices if and only if $\mathbf{A} = c\mathbf{I}_n$, for some $c \in \mathbb{R}$. (Hint: Use $\mathbf{A\Psi}_{kk} = \mathbf{\Psi}_{kk}\mathbf{A}$, for $1 \leq k \leq n$, to prove $a_{ij} = 0$ for $i \neq j$. Then use $\mathbf{A\Psi}_{ij} = \mathbf{\Psi}_{ij}\mathbf{A}$ to show $a_{ii} = a_{jj}$.)

★**31.** True or False:

(a) If $\mathbf{AB}$ is defined, the jth column of $\mathbf{AB} = \mathbf{A}(j$th column of $\mathbf{B})$.

(b) If $\mathbf{A}, \mathbf{B}, \mathbf{D}$ are $n \times n$ matrices, then $\mathbf{D}(\mathbf{A} + \mathbf{B}) = \mathbf{DB} + \mathbf{DA}$.

(c) If t is a scalar, and $\mathbf{D}$ and $\mathbf{E}$ are $n \times n$ matrices, then $(t\mathbf{D})\mathbf{E} = \mathbf{D}(t\mathbf{E})$.

(d) If $\mathbf{D}, \mathbf{E}$ are $n \times n$ matrices, then $(\mathbf{DE})^2 = \mathbf{D}^2\mathbf{E}^2$.

(e) If $\mathbf{D}, \mathbf{E}$ are $n \times n$ matrices, then $(\mathbf{DE})^T = \mathbf{D}^T\mathbf{E}^T$.

(f) If $\mathbf{DE} = \mathbf{O}$, then $\mathbf{D} = \mathbf{O}$ or $\mathbf{E} = \mathbf{O}$.

REVIEW EXERCISES FOR CHAPTER 1

1. Determine whether the quadrilateral $ABCD$ formed by the points $A(6, 4)$, $B(11, 7)$, $C(5, 17)$, $D(0, 14)$ is a rectangle.

★**2.** Find a unit vector $\mathbf{u}$ in the same direction as $\mathbf{x} = \left[\frac{1}{4}, -\frac{3}{5}, \frac{3}{4}\right]$. Is $\mathbf{u}$ shorter or longer than $\mathbf{x}$?

- If the next logical pivot choice is a zero entry, and all entries below this value are zero, then the current column is skipped over.
- At the conclusion of the Gaussian elimination process, if the system is consistent, each nonpivot column represents an (independent) variable that can have any value, and the values of all other (dependent) variables are determined from the independent variables, using back substitution.

EXERCISES FOR SECTION 2.1

1. Use the Gaussian elimination method to solve each of the following systems of linear equations. In each case, indicate whether the system is consistent or inconsistent. Give the complete solution set, and if the solution set is infinite, specify three particular solutions.

★**(a)** $\begin{cases} -5x_1 - 2x_2 + 2x_3 = 14 \\ 3x_1 + x_2 - x_3 = -8 \\ 2x_1 + 2x_2 - x_3 = -3 \end{cases}$

(b) $\begin{cases} 3x_1 - 3x_2 - 2x_3 = 23 \\ -6x_1 + 4x_2 + 3x_3 = -38 \\ -2x_1 + x_2 + x_3 = -11 \end{cases}$

★**(c)** $\begin{cases} 3x_1 - 2x_2 + 4x_3 = -54 \\ -x_1 + x_2 - 2x_3 = 20 \\ 5x_1 - 4x_2 + 8x_3 = -83 \end{cases}$

(d) $\begin{cases} -2x_1 + 3x_2 - 4x_3 + x_4 = -17 \\ 8x_1 - 5x_2 + 2x_3 - 4x_4 = 47 \\ -5x_1 + 9x_2 - 13x_3 + 3x_4 = -44 \\ -4x_1 + 3x_2 - 2x_3 + 2x_4 = -25 \end{cases}$

★**(e)** $\begin{cases} 6x_1 - 12x_2 - 5x_3 + 16x_4 - 2x_5 = -53 \\ -3x_1 + 6x_2 + 3x_3 - 9x_4 + x_5 = 29 \\ -4x_1 + 8x_2 + 3x_3 - 10x_4 + x_5 = 33 \end{cases}$

(f) $\begin{cases} 5x_1 - 5x_2 - 15x_3 - 3x_4 = -34 \\ -2x_1 + 2x_2 + 6x_3 + x_4 = 12 \end{cases}$

★**(g)** $\begin{cases} 4x_1 - 2x_2 - 7x_3 = 5 \\ -6x_1 + 5x_2 + 10x_3 = -11 \\ -2x_1 + 3x_2 + 4x_3 = -3 \\ -3x_1 + 2x_2 + 5x_3 = -5 \end{cases}$

(h) $\begin{cases} 5x_1 - x_2 - 9x_3 - 2x_4 = 26 \\ 4x_1 - x_2 - 7x_3 - 2x_4 = 21 \\ -2x_1 + 4x_3 + x_4 = -12 \\ -3x_1 + 2x_2 + 4x_3 + 2x_4 = -11 \end{cases}$

2. Suppose that each of the following is the final augmented matrix obtained after Gaussian elimination. In each case, give the complete solution set for the corresponding system of linear equations.

★(a) $\left[\begin{array}{ccccc|c} 1 & -5 & 2 & 3 & -2 & -4 \\ 0 & 1 & -1 & -3 & -7 & -2 \\ 0 & 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(b) $\left[\begin{array}{cccccc|c} 1 & -3 & 6 & 0 & -2 & 4 & -3 \\ 0 & 0 & 1 & -2 & 8 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

★(c) $\left[\begin{array}{cccccc|c} 1 & 4 & -8 & -1 & 2 & -3 & -4 \\ 0 & 1 & -7 & 2 & -9 & -1 & -3 \\ 0 & 0 & 0 & 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(d) $\left[\begin{array}{ccccc|c} 1 & -7 & -3 & -2 & -1 & -5 \\ 0 & 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{array}\right]$

★3. Solve the following problem by using a linear system: A certain number of nickels, dimes, and quarters totals \$16.50. There are twice as many dimes as quarters, and the total number of nickels and quarters is 20 more than the number of dimes. Find the correct number of each type of coin.

★4. Find the quadratic equation $y = ax^2 + bx + c$ that goes through the points $(3,18)$, $(2,9)$, and $(-2,13)$.

5. Find the cubic equation $y = ax^3 + bx^2 + cx + d$ that goes through the points $(1,1)$, $(2,-18)$, $(-2,46)$, and $(3,-69)$.

★6. The general equation of a circle is $x^2 + y^2 + ax + by = c$. Find the equation of the circle that goes through the points $(6,8)$, $(8,4)$, and $(3,9)$.

7. Let $\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 1 \\ -2 & 1 & 5 \\ 3 & 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 1 & -5 \\ 2 & 3 & 0 \\ 4 & 1 & 1 \end{bmatrix}$. Compute $R(\mathbf{AB})$ and $(R(\mathbf{A}))\mathbf{B}$ to verify that they are equal, if

★(a) $R: \langle 3\rangle \leftarrow -3\langle 2\rangle + \langle 3\rangle$.

(b) $R: \langle 2\rangle \leftrightarrow \langle 4\rangle$.

8. ▶ **(a)** Prove part (1) of Theorem 2.1 by showing that $R(\mathbf{AB}) = (R(\mathbf{A}))\mathbf{B}$ for each type of row operation ((I), (II), (III)) in turn. (Hint: Use the fact from Section 1.5 that the kth row of $(\mathbf{AB}) = (k\text{th row of } \mathbf{A})\mathbf{B}$.)

(b) Use part (a) and induction to prove part (2) of Theorem 2.1.

9. Explain why the scalar used in a type (I) row operation must be nonzero.

10. Prove that if more than one solution to a system of linear equations exists, then an infinite number of solutions exists. (Hint: Show that if $\mathbf{X}_1$ and $\mathbf{X}_2$ are different solutions to $\mathbf{AX} = \mathbf{B}$, then $\mathbf{X}_1 + c(\mathbf{X}_2 - \mathbf{X}_1)$ is also a solution, for every real number c. Also, show that all these solutions are different.)

★**11.** True or False:

(a) The augmented matrix for a linear system contains all the essential information from the system.

(b) It is possible for a linear system of equations to have exactly three solutions.

(c) A consistent system must have exactly one solution.

(d) type (II) row operations are used to convert nonzero pivot entries to 1.

(e) A type (III) row operation is used to replace a zero pivot entry with a nonzero entry below it.

(f) Multiplying matrices and then performing a row operation on the product has the same effect as performing the row operation on the first matrix and then calculating the product.

2.2 GAUSS-JORDAN ROW REDUCTION AND REDUCED ROW ECHELON FORM

In this section, we introduce the Gauss-Jordan row reduction method, an extension of the Gaussian elimination method. We also examine homogeneous linear systems and their solutions.

Introduction to Gauss-Jordan Row Reduction

In the Gaussian elimination method, we created the augmented matrix for a given linear system and systematically proceeded through the columns from left to right, creating pivots and targeting (zeroing out) entries below the pivots. Although we occasionally skipped over a column, we placed pivots into successive rows, and so the

EXERCISES FOR SECTION 2.2

★1. Which of these matrices are not in reduced row echelon form? Why?

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & -2 & 0 & -2 & 3 \\ 0 & 0 & 1 & 5 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

2. Use the Gauss-Jordan method to convert these matrices to reduced row echelon form, and draw in the correct staircase pattern.

★(a) $\left[\begin{array}{ccc|c} 5 & 20 & -18 & -11 \\ 3 & 12 & -14 & 3 \\ -4 & -16 & 13 & 13 \end{array}\right]$

★(b) $\begin{bmatrix} -2 & 1 & 1 & 15 \\ 6 & -1 & -2 & -36 \\ 1 & -1 & -1 & -11 \\ -5 & -5 & -5 & -14 \end{bmatrix}$

★(c) $\left[\begin{array}{cccc|c} -5 & 10 & -19 & -17 & 20 \\ -3 & 6 & -11 & -11 & 14 \\ -7 & 14 & -26 & -25 & 31 \\ 9 & -18 & 34 & 31 & -37 \end{array}\right]$

(d) $\begin{bmatrix} 2 & -5 & -20 \\ 0 & 2 & 7 \\ 1 & -5 & -19 \\ -5 & 16 & 64 \\ 3 & -9 & -36 \end{bmatrix}$

★(e) $\left[\begin{array}{ccccc|c} -3 & 6 & -1 & -5 & 0 & -5 \\ -1 & 2 & 3 & -5 & 10 & 5 \end{array}\right]$

(f) $\begin{bmatrix} -2 & 1 & -1 & -1 & 3 \\ 3 & 1 & -4 & -2 & -4 \\ 7 & 1 & -6 & -2 & -3 \\ -8 & -1 & 6 & 2 & 3 \\ -3 & 0 & 2 & 1 & 2 \end{bmatrix}$

★3. In parts (a), (e), and (g) of Exercise 1 in Section 2.1, take the final row echelon form matrix that you obtained from Gaussian elimination and convert it to reduced row echelon form. Then check that the reduced row echelon form leads to the same solution set that you obtained using Gaussian elimination.

4. Each of the following homogeneous systems has a nontrivial solution since the number of variables is greater than the number of equations. Use the Gauss-Jordan method to determine the complete solution set for each system, and give one particular nontrivial solution.

★(a) $\begin{cases} -2x_1 - 3x_2 + 2x_3 - 13x_4 = 0 \\ -4x_1 - 7x_2 + 4x_3 - 29x_4 = 0 \\ x_1 + 2x_2 - x_3 + 8x_4 = 0 \end{cases}$

(b) $\begin{cases} 2x_1 + 4x_2 - x_3 + 5x_4 + 2x_5 = 0 \\ 3x_1 + 3x_2 - x_3 + 3x_4 = 0 \\ -5x_1 - 6x_2 + 2x_3 - 6x_4 - x_5 = 0 \end{cases}$

★(c) $\begin{cases} 7x_1 + 28x_2 + 4x_3 - 2x_4 + 10x_5 + 19x_6 = 0 \\ -9x_1 - 36x_2 - 5x_3 + 3x_4 - 15x_5 - 29x_6 = 0 \\ 3x_1 + 12x_2 + 2x_3 + 6x_5 + 11x_6 = 0 \\ 6x_1 + 24x_2 + 3x_3 - 3x_4 + 10x_5 + 20x_6 = 0 \end{cases}$

5. Use the Gauss-Jordan method to find the complete solution set for each of the following homogeneous systems, and express each solution set as linear combinations of particular solutions, as shown after Example 4.

★(a) $\begin{cases} -2x_1 + x_2 + 8x_3 = 0 \\ 7x_1 - 2x_2 - 22x_3 = 0 \\ 3x_1 - x_2 - 10x_3 = 0 \end{cases}$

(b) $\begin{cases} 5x_1 - 2x_3 = 0 \\ -15x_1 - 16x_2 - 9x_3 = 0 \\ 10x_1 + 12x_2 + 7x_3 = 0 \end{cases}$

★(c) $\begin{cases} 2x_1 + 6x_2 + 13x_3 + x_4 = 0 \\ x_1 + 4x_2 + 10x_3 + x_4 = 0 \\ 2x_1 + 8x_2 + 20x_3 + x_4 = 0 \\ 3x_1 + 10x_2 + 21x_3 + 2x_4 = 0 \end{cases}$

(d) $\begin{cases} 2x_1 - 6x_2 + 3x_3 - 21x_4 = 0 \\ 4x_1 - 5x_2 + 2x_3 - 24x_4 = 0 \\ -x_1 + 3x_2 - x_3 + 10x_4 = 0 \\ -2x_1 + 3x_2 - x_3 + 13x_4 = 0 \end{cases}$

6. Use the Gauss-Jordan method to find the minimal integer values for the variables that will balance each of the following chemical equations.[2]

★(a) $aC_6H_6 + bO_2 \rightarrow cCO_2 + dH_2O$

(b) $aC_8H_{18} + bO_2 \rightarrow cCO_2 + dH_2O$

★(c) $aAgNO_3 + bH_2O \rightarrow cAg + dO_2 + eHNO_3$

(d) $aHNO_3 + bHCl + cAu \rightarrow dNOCl + eHAuCl_4 + fH_2O$

[2] The chemical elements used in these equations are silver (Ag), gold (Au), carbon (C), chlorine (Cl), hydrogen (H), nitrogen (N), and oxygen (O). The compounds are water (H_2O), carbon dioxide (CO_2), benzene (C_6H_6), octane (C_8H_{18}), silver nitrate ($AgNO_3$), nitric acid (HNO_3), hydrochloric acid (HCl), nitrous chloride (NOCl), and hydrogen tetrachloroaurate (III) ($HAuCl_4$).

7. Use the Gauss-Jordan method to find the values of A, B, C (and D in part (b)) in the following partial fractions problems:

★**(a)** $$\frac{5x^2+23x-58}{(x-1)(x-3)\left(x+4\right)} = \frac{A}{x-1} + \frac{B}{x-3} + \frac{C}{x+4}$$

(b) $$\frac{-3x^3+29x^2-91x+94}{(x-2)^2(x-3)^2} = \frac{A}{(x-2)^2} + \frac{B}{x-2} + \frac{C}{(x-3)^2} + \frac{D}{x-3}$$

★**8.** Solve the systems $\mathbf{AX} = \mathbf{B}_1$ and $\mathbf{AX} = \mathbf{B}_2$ simultaneously, as illustrated in this section, where

$$\mathbf{A} = \begin{bmatrix} 9 & 2 & 2 \\ 3 & 2 & 4 \\ 27 & 12 & 22 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} -6 \\ 0 \\ 12 \end{bmatrix}, \quad \text{and} \quad \mathbf{B}_2 = \begin{bmatrix} -12 \\ -3 \\ 8 \end{bmatrix}.$$

9. Solve the systems $\mathbf{AX} = \mathbf{B}_1$ and $\mathbf{AX} = \mathbf{B}_2$ simultaneously, as illustrated in this section, where

$$\mathbf{A} = \begin{bmatrix} 12 & 2 & 0 & 3 \\ -24 & -4 & 1 & -6 \\ -4 & -1 & -1 & 0 \\ -30 & -5 & 0 & -6 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 3 \\ 8 \\ -4 \\ 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{B}_2 = \begin{bmatrix} 2 \\ 4 \\ -24 \\ 0 \end{bmatrix}.$$

10. Let $\mathbf{A} = \begin{bmatrix} 0 & 3 & -12 \\ 2 & 4 & -10 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 1 \\ 3 & -3 \\ -4 & 1 \end{bmatrix}$.

(a) Find row operations $R_1, \ldots, R_n$ such that $R_n(R_{n-1}(\cdots(R_2(R_1(\mathbf{A})))\cdots))$ is in reduced row echelon form.

(b) Verify part (2) of Theorem 2.1 using $\mathbf{A}$, $\mathbf{B}$, and the row operations from part (a).

11. Consider the homogeneous system $\mathbf{AX} = \mathbf{O}$ having m equations and n variables.

(a) Prove that, if $\mathbf{X}_1$ and $\mathbf{X}_2$ are both solutions to this system, then $\mathbf{X}_1 + \mathbf{X}_2$ and any scalar multiple $c\mathbf{X}_1$ are also solutions.

★**(b)** Give a counterexample to show that the results of part (a) do not necessarily hold if the system is nonhomogeneous.

(c) Consider a nonhomogeneous system $\mathbf{AX} = \mathbf{B}$ having the same coefficient matrix as the homogeneous system $\mathbf{AX} = \mathbf{O}$. Prove that, if $\mathbf{X}_1$ is a solution of $\mathbf{AX} = \mathbf{B}$ and if $\mathbf{X}_2$ is a solution of $\mathbf{AX} = \mathbf{O}$, then $\mathbf{X}_1 + \mathbf{X}_2$ is also a solution of $\mathbf{AX} = \mathbf{B}$.

(d) Show that if $\mathbf{AX} = \mathbf{B}$ has a unique solution, with $\mathbf{B} \neq \mathbf{O}$, then the corresponding homogeneous system $\mathbf{AX} = \mathbf{O}$ can have only the trivial solution. (Hint: Use part (c).)

12. Prove that the following homogeneous system has a nontrivial solution if and only if $ad - bc = 0$:

$$\begin{cases} ax_1 + bx_2 = 0 \\ cx_1 + dx_2 = 0 \end{cases}.$$

(Hint: First, suppose that $a \neq 0$, and show that under the Gauss-Jordan method, the second column has a nonzero pivot entry if and only if $ad - bc \neq 0$. Then consider the case $a = 0$.)

13. Suppose that $\mathbf{AX} = \mathbf{O}$ is a homogeneous system of n equations in n variables.

(a) If the system $\mathbf{A}^2\mathbf{X} = \mathbf{O}$ has a nontrivial solution, show that $\mathbf{AX} = \mathbf{O}$ also has a nontrivial solution. (Hint: Prove the contrapositive.)

(b) Generalize the result of part (a) to show that, if the system $\mathbf{A}^n\mathbf{X} = \mathbf{O}$ has a nontrivial solution for some positive integer n, then $\mathbf{AX} = \mathbf{O}$ has a nontrivial solution. (Hint: Use a proof by induction.)

⋆14. True or False:

(a) In Gaussian elimination, a descending "staircase" pattern of pivots is created, in which each step starts with 1 and the entries below the staircase are all 0.

(b) Gauss-Jordan row reduction differs from Gaussian elimination by targeting (zeroing out) entries above each nonzero pivot as well as those below the pivot.

(c) In a reduced row echelon form matrix, the nonzero pivot entries are always located in successive rows and columns.

(d) No homogeneous system is inconsistent.

(e) Nontrivial solutions to a homogeneous system are found by setting the dependent (pivot column) variables equal to any real number and then determining the independent (nonpivot column) variables from those choices.

(f) If a homogeneous system has more equations than variables, then the system has a nontrivial solution.

2.3 EQUIVALENT SYSTEMS, RANK, AND ROW SPACE

In this section, we continue discussing the solution sets of linear systems. First we introduce row equivalence of matrices, and use this to prove our assertion in the last two sections that the Gaussian elimination and Gauss-Jordan row reduction methods always produce the complete solution set for a given linear system. We also note that every matrix has a unique corresponding matrix in reduced row echelon form and

- The inverse of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- The inverse of an $n \times n$ matrix $\mathbf{A}$ can be found by row reducing $[\mathbf{A}|\mathbf{I}_n]$ to $\left[\mathbf{I}_n|\mathbf{A}^{-1}\right]$. If this result cannot be obtained, then $\mathbf{A}$ has no inverse (that is, $\mathbf{A}$ is singular).
- A (square) $n \times n$ matrix is nonsingular if and only if its rank is n.
- If $\mathbf{A}$ is nonsingular, then $\mathbf{AX} = \mathbf{B}$ has the unique solution $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$. If $\mathbf{A}$ is singular, then $\mathbf{AX} = \mathbf{B}$ has either no solution or infinitely many solutions.

EXERCISES FOR SECTION 2.4

Note: You should be using a calculator or appropriate computer software to perform nontrivial row reductions.

1. Verify that the following pairs of matrices are inverses:

(a) $\begin{bmatrix} 10 & 41 & -5 \\ -1 & -12 & 1 \\ 3 & 20 & -2 \end{bmatrix}, \begin{bmatrix} 4 & -18 & -19 \\ 1 & -5 & -5 \\ 16 & -77 & -79 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & -1 & 5 \\ -1 & 1 & 0 & -3 \\ 0 & 2 & -3 & 7 \\ 2 & -1 & -2 & 12 \end{bmatrix}, \begin{bmatrix} 1 & -4 & 1 & -2 \\ 4 & 6 & -2 & 1 \\ 5 & 11 & -4 & 3 \\ 1 & 3 & -1 & 1 \end{bmatrix}$

2. Determine whether each of the following matrices is nonsingular by calculating its rank:

★(a) $\begin{bmatrix} 4 & -9 \\ -2 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -1 & 4 \\ 2 & -2 & 1 \\ -1 & 3 & 2 \end{bmatrix}$

★(c) $\begin{bmatrix} -6 & -6 & 1 \\ 2 & 3 & -1 \\ 8 & 6 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} -10 & -3 & 1 & -3 \\ 18 & 5 & -2 & 6 \\ 6 & 2 & -1 & 6 \\ 12 & 3 & -1 & 3 \end{bmatrix}$

★(e) $\begin{bmatrix} 2 & 1 & -7 & 14 \\ -6 & -3 & 19 & -38 \\ 1 & 0 & -3 & 6 \\ 2 & 1 & -6 & 12 \end{bmatrix}$

3. Find the inverse, if it exists, for each of the following 2×2 matrices:

★(a) $\begin{bmatrix} 4 & 2 \\ 9 & -3 \end{bmatrix}$

(b) $\begin{bmatrix} 10 & -5 \\ -4 & 2 \end{bmatrix}$

★(c) $\begin{bmatrix} -3 & 5 \\ -12 & -8 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$

★(e) $\begin{bmatrix} -6 & 12 \\ 4 & -8 \end{bmatrix}$

(f) $\begin{bmatrix} -\frac{1}{2} & \frac{3}{4} \\ \frac{1}{3} & -\frac{1}{2} \end{bmatrix}$

4. Use row reduction to find the inverse, if it exists, for each of the following:

★(a) $\begin{bmatrix} -4 & 7 & 6 \\ 3 & -5 & -4 \\ -2 & 4 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 5 & 7 & -6 \\ 3 & 1 & -2 \\ 1 & -5 & 2 \end{bmatrix}$

★(c) $\begin{bmatrix} 2 & -2 & 3 \\ 8 & -4 & 9 \\ -4 & 6 & -9 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & -2 & -1 \\ -2 & 0 & -1 & 0 \\ -1 & -2 & -1 & -5 \\ 0 & 1 & 1 & 3 \end{bmatrix}$

★(e) $\begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & -2 & 3 & 1 \\ 4 & 1 & 0 & -1 \\ 1 & 3 & -2 & -5 \end{bmatrix}$

(f) $\begin{bmatrix} 3 & 3 & 0 & -2 \\ 14 & 15 & 0 & -11 \\ -3 & 1 & 2 & -5 \\ -2 & 0 & 1 & -2 \end{bmatrix}$

5. Assuming that all main diagonal entries are nonzero, find the inverse of each of the following:

(a) $\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$

(b) $\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

★(c) $\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$

★**6.** The following matrices are useful in computer graphics for rotating vectors (see Section 5.1). Find the inverse of each matrix, and then state what the matrix and its inverse are when $\theta = \frac{\pi}{6}, \frac{\pi}{4}$, and $\frac{\pi}{2}$.

(a) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(b) $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (Hint: Modify your answer from part (a).)

7. In each case, find the inverse of the coefficient matrix and use it to solve the system by matrix multiplication.

★(a) $\begin{cases} 5x_1 - x_2 = 20 \\ -7x_1 + 2x_2 = -31 \end{cases}$

(b) $\begin{cases} -5x_1 + 3x_2 + 6x_3 = 4 \\ 3x_1 - x_2 - 7x_3 = 11 \\ -2x_1 + x_2 + 2x_3 = 2 \end{cases}$

★(c) $\begin{cases} -2x_2 + 5x_3 + x_4 = 25 \\ -7x_1 - 4x_2 + 5x_3 + 22x_4 = -15 \\ 5x_1 + 3x_2 - 4x_3 - 16x_4 = 9 \\ -3x_1 - x_2 + 9x_4 = -16 \end{cases}$

★8. A matrix with the property $\mathbf{A}^2 = \mathbf{I}_n$ is called an **involutory** matrix.

(a) Find an example of a 2×2 involutory matrix other than $\mathbf{I}_2$.

(b) Find an example of a 3×3 involutory matrix other than $\mathbf{I}_3$.

(c) What is $\mathbf{A}^{-1}$ if $\mathbf{A}$ is involutory?

9. (a) Give an example to show that $\mathbf{A} + \mathbf{B}$ can be singular if $\mathbf{A}$ and $\mathbf{B}$ are both nonsingular.

(b) Give an example to show that $\mathbf{A} + \mathbf{B}$ can be nonsingular if $\mathbf{A}$ and $\mathbf{B}$ are both singular.

(c) Give an example to show that even when $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A} + \mathbf{B}$ are all nonsingular, $(\mathbf{A} + \mathbf{B})^{-1}$ is not necessarily equal to $\mathbf{A}^{-1} + \mathbf{B}^{-1}$.

★10. Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be $n \times n$ matrices.

(a) Suppose that $\mathbf{AB} = \mathbf{O}_n$, and $\mathbf{A}$ is nonsingular. What must $\mathbf{B}$ be?

(b) If $\mathbf{AB} = \mathbf{I}_n$, is it possible for $\mathbf{AC}$ to equal $\mathbf{O}_n$ without $\mathbf{C} = \mathbf{O}_n$? Why or why not?

★11. If $\mathbf{A}^4 = \mathbf{I}_n$, but $\mathbf{A} \neq \mathbf{I}_n$, $\mathbf{A}^2 \neq \mathbf{I}_n$, and $\mathbf{A}^3 \neq \mathbf{I}_n$, which powers of $\mathbf{A}$ are equal to $\mathbf{A}^{-1}$?

★12. If the matrix product $\mathbf{A}^{-1}\mathbf{B}$ is known, how could you calculate $\mathbf{B}^{-1}\mathbf{A}$ without necessarily knowing what $\mathbf{A}$ and $\mathbf{B}$ are?

13. Let $\mathbf{A}$ be a symmetric nonsingular matrix. Prove that $\mathbf{A}^{-1}$ is symmetric.

★14. (a) You have already seen in this section that every square matrix containing a row of zeroes must be singular. Why must every square matrix containing a column of zeroes be singular?

(b) Why must every diagonal matrix with at least one zero main diagonal entry be singular?

(c) Why must every upper triangular matrix with no zero entries on the main diagonal be nonsingular?

(d) Use part (c) and the transpose to show that every lower triangular matrix with no zero entries on the main diagonal must be nonsingular.

(e) Prove that if $\mathbf{A}$ is an upper triangular matrix with no zero entries on the main diagonal, then $\mathbf{A}^{-1}$ is upper triangular. (Hint: As $[\mathbf{A}|\mathbf{I}_n]$ is row reduced to $\left[\mathbf{I}_n|\mathbf{A}^{-1}\right]$, consider the effect on the entries in the rightmost columns.)

15. ▶(a) Prove parts (1) and (2) of Theorem 2.11. (Hint: In proving part (2), consider the cases $k \geq 0$ and $k < 0$ separately.)

(b) Use the method of induction to prove the following generalization of part (3) of Theorem 2.11: if $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m$ are nonsingular matrices of the same size, then $(\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_m)^{-1} = \mathbf{A}_m^{-1}\cdots\mathbf{A}_2^{-1}\mathbf{A}_1^{-1}$.

16. If $\mathbf{A}$ is a nonsingular matrix and $c \in \mathbb{R}$ with $c \neq 0$, prove that $(c\mathbf{A})^{-1} = \left(\frac{1}{c}\right)\mathbf{A}^{-1}$.

17. ▶(a) Prove part (1) of Theorem 2.12 if $s < 0$ and $t < 0$.

(b) Prove part (2) of Theorem 2.12 if $s \geq 0$ and $t < 0$.

18. Assume that $\mathbf{A}$ and $\mathbf{B}$ are nonsingular $n \times n$ matrices. Prove that $\mathbf{A}$ and $\mathbf{B}$ commute (that is, $\mathbf{AB} = \mathbf{BA}$) if and only if $(\mathbf{AB})^2 = \mathbf{A}^2\mathbf{B}^2$.

19. Prove that if $\mathbf{A}$ and $\mathbf{B}$ are nonsingular matrices of the same size, then $\mathbf{AB} = \mathbf{BA}$ if and only if $(\mathbf{AB})^q = \mathbf{A}^q\mathbf{B}^q$ for every positive integer $q \geq 2$. (Hint: To prove the "if" part, let $q = 2$. For the "only if" part, first show by induction that if $\mathbf{AB} = \mathbf{BA}$, then $\mathbf{AB}^q = \mathbf{B}^q\mathbf{A}$, for any positive integer $q \geq 2$. Finish the proof with a second induction argument to show $(\mathbf{AB})^q = \mathbf{A}^q\mathbf{B}^q$.)

20. Prove that, if $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{A} - \mathbf{I}_n$ is nonsingular, then for every integer $k \geq 0$,

$$\mathbf{I}_n + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots + \mathbf{A}^k = \left(\mathbf{A}^{k+1} - \mathbf{I}_n\right)(\mathbf{A} - \mathbf{I}_n)^{-1}.$$

★21. Let $\mathbf{A}$ be an $n \times k$ matrix and $\mathbf{B}$ be a $k \times n$ matrix such that $\mathbf{AB} = \mathbf{I}_n$ and $\mathbf{BA} = \mathbf{I}_k$.

(a) Prove that $n \leq k$. (Hint: Assume that $n > k$ and find a contradiction. Show that there is a nontrivial $\mathbf{X}$ such that $\mathbf{BX} = \mathbf{O}$. Then compute $\mathbf{ABX}$ two different ways.)

(b) Prove that $k \leq n$.

(c) Use parts (a) and (b) to show that $\mathbf{A}$ and $\mathbf{B}$ are square nonsingular matrices with $\mathbf{A}^{-1} = \mathbf{B}$.

★22. True or False:

(a) Every $n \times n$ matrix $\mathbf{A}$ has a unique inverse.

(b) If $\mathbf{A}, \mathbf{B}$ are $n \times n$ matrices, and $\mathbf{BA} = \mathbf{I}_n$, then $\mathbf{A}$ and $\mathbf{B}$ are inverses.

(c) If $\mathbf{A}, \mathbf{B}$ are nonsingular $n \times n$ matrices, then $((\mathbf{AB})^T)^{-1} = (\mathbf{A}^{-1})^T(\mathbf{B}^{-1})^T$.

(d) $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is singular if and only if $ad - bc \neq 0$.

(e) If $\mathbf{A}$ is an $n \times n$ matrix, then $\mathbf{A}$ is nonsingular if and only if $[\mathbf{A}|\mathbf{I}_n]$ has fewer than n nonzero pivots before the augmentation bar after row reduction.

(f) If $\mathbf{A}$ is an $n \times n$ matrix, then rank($\mathbf{A}$) $= n$ if and only if any system of the form $\mathbf{AX} = \mathbf{B}$ has a unique solution for $\mathbf{X}$.

REVIEW EXERCISES FOR CHAPTER 2

1. For each of the following linear systems,

★**(i)** Use Gaussian elimination to give the complete solution set.

(ii) Use the Gauss-Jordan method to give the complete solution set and the correct staircase pattern for the row reduced echelon form of the augmented matrix for the system.

(a) $$\begin{cases} 2x_1 + 5x_2 - 4x_3 = 48 \\ x_1 - 3x_2 + 2x_3 = -40 \\ -3x_1 + 4x_2 + 7x_3 = 15 \\ -2x_1 + 3x_2 - x_3 = 41 \end{cases}$$

(b) $$\begin{cases} 4x_1 + 3x_2 - 7x_3 + 5x_4 = 31 \\ -2x_1 - 3x_2 + 5x_3 - x_4 = -5 \\ 2x_1 - 6x_2 - 2x_3 + 3x_4 = 52 \\ 6x_1 - 21x_2 - 3x_3 + 12x_4 = 16 \end{cases}$$

(c) $$\begin{cases} 6x_1 - 2x_2 + 2x_3 - x_4 - 6x_5 = -33 \\ -2x_1 + x_2 \qquad\quad + 2x_4 - x_5 = 13 \\ 4x_1 - x_2 + 2x_3 - 3x_4 + x_5 = -24 \end{cases}$$

★**2.** Find the cubic equation that goes through the points $(-3, 120)$, $(-2, 51)$, $(3, -24)$, and $(4, -69)$.

3. Are the following matrices in reduced row echelon form? If not, explain why not.

(a) $$\begin{bmatrix} 1 & -5 & 2 & -4 & -2 \\ 0 & 1 & -3 & 4 & -1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

★**4.** Find minimal integer values for the variables that will satisfy the following chemical equation: $a\,NH_3 + b\,O_2 \longrightarrow c\,NO_2 + d\,H_2O$ (NH_3 = ammonia; NO_2 = nitrogen dioxide).

5. Solve the following linear systems simultaneously:

- The area of a parallelogram or the volume of a parallelepiped is the absolute value of the determinant of the matrix whose rows determine the sides of the figure.

EXERCISES FOR SECTION 3.1

1. Calculate the determinant of each of the following matrices using the quick formulas given at the beginning of this section:

★**(a)** $\begin{bmatrix} -2 & 5 \\ 3 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 5 & -3 \\ 2 & 0 \end{bmatrix}$

★**(c)** $\begin{bmatrix} 6 & -12 \\ -4 & 8 \end{bmatrix}$

(d) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

★**(e)** $\begin{bmatrix} 2 & 0 & 5 \\ -4 & 1 & 7 \\ 0 & 3 & -3 \end{bmatrix}$

(f) $\begin{bmatrix} 3 & -2 & 4 \\ 5 & 1 & -2 \\ -1 & 3 & 6 \end{bmatrix}$

★**(g)** $\begin{bmatrix} 5 & 0 & 0 \\ 3 & -2 & 0 \\ -1 & 8 & 4 \end{bmatrix}$

(h) $\begin{bmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

★**(i)** $\begin{bmatrix} 3 & 1 & -2 \\ -1 & 4 & 5 \\ 3 & 1 & -2 \end{bmatrix}$

★**(j)** $[-3]$

2. Calculate the indicated minors for each given matrix.

★**(a)** $|\mathbf{A}_{21}|$, for $\mathbf{A} = \begin{bmatrix} -2 & 4 & 3 \\ 3 & -1 & 6 \\ 5 & -2 & 4 \end{bmatrix}$

(b) $|\mathbf{B}_{34}|$, for $\mathbf{B} = \begin{bmatrix} 0 & 2 & -3 & 1 \\ 1 & 4 & 2 & -1 \\ 3 & -2 & 4 & 0 \\ 4 & -1 & 1 & 0 \end{bmatrix}$

★**(c)** $|\mathbf{C}_{42}|$, for $\mathbf{C} = \begin{bmatrix} -3 & 3 & 0 & 5 \\ 2 & 1 & -1 & 4 \\ 6 & -3 & 4 & 0 \\ -1 & 5 & 1 & -2 \end{bmatrix}$

3. Calculate the indicated cofactors for each given matrix.

★**(a)** $\mathcal{A}_{22}$, for $\mathbf{A} = \begin{bmatrix} 4 & 1 & -3 \\ 0 & 2 & -2 \\ 9 & 14 & -7 \end{bmatrix}$

(b) $\mathcal{B}_{23}$, for $\mathbf{B} = \begin{bmatrix} -9 & 6 & 7 \\ 2 & -1 & 0 \\ 4 & 3 & -8 \end{bmatrix}$

★**(c)** $\mathcal{C}_{43}$, for $\mathbf{C} = \begin{bmatrix} -5 & 2 & 2 & 13 \\ -8 & 2 & -5 & 22 \\ -6 & -3 & 0 & -16 \\ 4 & -1 & 7 & -8 \end{bmatrix}$

★**(d)** $\mathcal{D}_{12}$, for $\mathbf{D} = \begin{bmatrix} x+1 & x & x-7 \\ x-4 & x+5 & x-3 \\ x-1 & x & x+2 \end{bmatrix}$, where $x \in \mathbb{R}$

4. Calculate the determinant of each of the matrices in Exercise 1 using the formal definition of the determinant.

5. Calculate the determinant of each of the following matrices.

★**(a)** $\begin{bmatrix} 5 & 2 & 1 & 0 \\ -1 & 3 & 5 & 2 \\ 4 & 1 & 0 & 2 \\ 0 & 2 & 3 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 5 & 4 & 0 \\ 4 & 1 & -2 & 7 \\ -1 & 0 & 3 & 0 \\ 0 & 2 & 1 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 1 & 9 & 7 \\ 0 & -1 & 3 & 8 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

★**(d)** $\begin{bmatrix} 0 & 4 & 1 & 3 & -2 \\ 2 & 2 & 3 & -1 & 0 \\ 3 & 1 & 2 & -5 & 1 \\ 1 & 0 & -4 & 0 & 0 \\ 0 & 3 & 0 & 0 & 2 \end{bmatrix}$

6. For a general 4×4 matrix $\mathbf{A}$, write out the formula for $|\mathbf{A}|$ using cofactor expansion along the last row, and simplify as far as possible. (Your final answer should have 24 terms, each being a product of four entries of $\mathbf{A}$.)

★**7.** Give a counterexample to show that for square matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size, it is not always true that $|\mathbf{A}+\mathbf{B}| = |\mathbf{A}| + |\mathbf{B}|$.

8. **(a)** Show that the **cross product** $\mathbf{a} \times \mathbf{b} = [a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1]$ of $\mathbf{a} = [a_1, a_2, a_3]$ and $\mathbf{b} = [b_1, b_2, b_3]$ can be expressed in "determinant notation" as

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

(b) Show that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

9. Calculate the area of the parallelogram in $\mathbb{R}^2$ determined by the following:

★**(a)** $\mathbf{x} = [3, 2]$, $\mathbf{y} = [4, 5]$

(b) $\mathbf{x} = [-4, 3]$, $\mathbf{y} = [-2, 6]$

★(c) $\mathbf{x} = [5, -1]$, $\mathbf{y} = [-3, 3]$

(d) $\mathbf{x} = [-2, 3]$, $\mathbf{y} = [6, -9]$

▶**10.** Prove part (1) of Theorem 3.1. (Hint: See Figure 3.2. The area of the parallelogram is the length of the base $\mathbf{x}$ multiplied by the length of the perpendicular height $\mathbf{h}$. Note that if $\mathbf{p} = \mathbf{proj_x y}$, then $\mathbf{h} = \mathbf{y} - \mathbf{p}$.)

11. Calculate the volume of the parallelepiped in $\mathbb{R}^3$ determined by the following:

★(a) $\mathbf{x} = [-2, 3, 1]$, $\mathbf{y} = [4, 2, 0]$, $\mathbf{z} = [-1, 3, 2]$

(b) $\mathbf{x} = [1, 2, 3]$, $\mathbf{y} = [0, -1, 0]$, $\mathbf{z} = [4, -1, 5]$

★(c) $\mathbf{x} = [-3, 4, 0]$, $\mathbf{y} = [6, -2, 1]$, $\mathbf{z} = [0, -3, 3]$

(d) $\mathbf{x} = [1, 2, 0]$, $\mathbf{y} = [3, 2, -1]$, $\mathbf{z} = [5, -2, -1]$

★**12.** Prove part (2) of Theorem 3.1. (Hint: See Figure 3.3. Let $\mathbf{h}$ be the perpendicular dropped from $\mathbf{z}$ to the plane of the parallelogram. From Exercise 8, $\mathbf{x} \times \mathbf{y}$ is perpendicular to both $\mathbf{x}$ and $\mathbf{y}$, and so $\mathbf{h}$ is actually the projection of $\mathbf{z}$ onto $\mathbf{x} \times \mathbf{y}$. Hence, the volume of the parallelepiped is the area of the parallelogram determined by $\mathbf{x}$ and $\mathbf{y}$ multiplied by the length of $\mathbf{h}$. A calculation similar to that in Exercise 10 shows that the area of the parallelogram is $\sqrt{(x_2y_3 - x_3y_2)^2 + (x_1y_3 - x_3y_1)^2 + (x_1y_2 - x_2y_1)^2}$.)

13. (a) If $\mathbf{A}$ is an $n \times n$ matrix, and c is a scalar, prove that $|c\mathbf{A}| = c^n\,|\mathbf{A}|$. (Hint: Use a proof by induction on n.)

(b) Use part (a) together with part (2) of Theorem 3.1 to explain why, when each side of a parallelepiped is doubled, the volume is multiplied by 8.

14. Show that, for $x \in \mathbb{R}$, $x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ is the determinant of

$$\begin{bmatrix} x & -1 & 0 & 0 \\ 0 & x & -1 & 0 \\ 0 & 0 & x & -1 \\ a_0 & a_1 & a_2 & a_3 + x \end{bmatrix}.$$

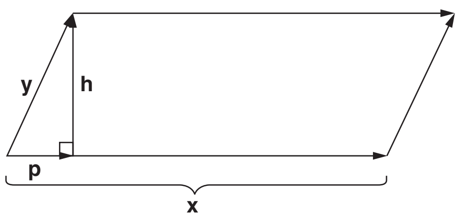

FIGURE 3.2

Parallelogram determined by $\mathbf{x}$ and $\mathbf{y}$.

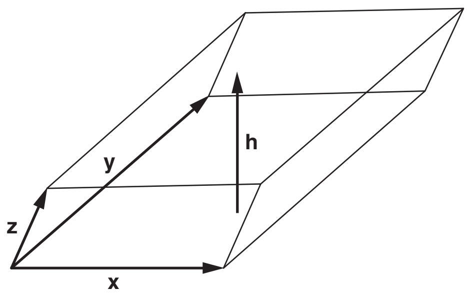

FIGURE 3.3

Parallelepiped determined by $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$.

15. Solve the following determinant equations for $x \in \mathbb{R}$:

★**(a)** $\begin{vmatrix} x & 2 \\ 5 & x+3 \end{vmatrix} = 0$

(b) $\begin{vmatrix} 15 & x-4 \\ x+7 & -2 \end{vmatrix} = 0$

★**(c)** $\begin{vmatrix} x-3 & 5 & -19 \\ 0 & x-1 & 6 \\ 0 & 0 & x-2 \end{vmatrix} = 0$

16. **(a)** Show that the determinant of the 3×3 **Vandermonde matrix**

$$\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$$

is equal to $(a-b)(b-c)(c-a)$.

★**(b)** Using part (a), calculate the determinant of

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -2 \\ 4 & 9 & 4 \end{bmatrix}.$$

17. The purpose of this exercise is to show that it is impossible to have an equilateral triangle whose three vertices all lie on **lattice points** in the plane — that is, points whose coordinates are both integers. Suppose T is such an equilateral triangle. Use the following steps to reach a contradiction:

(a) If s is the length of a side of T, use elementary geometry to find a formula for the area of T in terms of s.

(b) Use your answer for part (a) to show that the area of T is an irrational number. (You may assume $\sqrt{3}$ is irrational.)

(c) Suppose the three vertices of a triangle in the plane are given. Use part (1) of Theorem 3.1 to express the area of the triangle using a determinant.

(d) Use your answer for part (c) to show that the area of T is a rational number, thus contradicting part (b).

⋆**18.** True or False:

(a) The basketweaving technique can be used to find determinants of 3×3 and larger square matrices.

(b) The area of the parallelogram determined by nonparallel vectors $[x_1, x_2]$ and $[y_1, y_2]$ is $|x_1y_2 - x_2y_1|$.

(c) An $n \times n$ matrix has $2n$ associated cofactors.

(d) The cofactor $\mathcal{B}_{23}$ for a square matrix $\mathbf{B}$ equals the minor $|\mathbf{B}_{23}|$.

(e) The determinant of a 4×4 matrix $\mathbf{A}$ is $a_{41}\mathcal{A}_{41} + a_{42}\mathcal{A}_{42} + a_{43}\mathcal{A}_{43} + a_{44}\mathcal{A}_{44}$.

3.2 DETERMINANTS AND ROW REDUCTION

In this section, we provide a method for calculating the determinant of a matrix by using row reduction. For large matrices, this technique is computationally more efficient than cofactor expansion. We will also use the relationship between determinants and row reduction to establish a link between determinants and rank.

Determinants of Upper Triangular Matrices

We begin by proving the following simple formula for the determinant of an upper triangular matrix. Our goal will be to reduce every other determinant computation to this special case using row reduction.

Theorem 3.2 Let $\mathbf{A}$ be an upper triangular $n \times n$ matrix. Then $|\mathbf{A}| = a_{11}a_{22}\cdots a_{nn}$, the product of the entries of $\mathbf{A}$ along the main diagonal.

Because we have defined the determinant recursively, we prove Theorem 3.2 by induction.

Proof. We use induction on n.

Base Step: $n = 1$. In this case, $\mathbf{A} = [a_{11}]$, and $|\mathbf{A}| = a_{11}$, which verifies the formula in the theorem.

Inductive Step: Let $n > 1$. Assume that for any upper triangular $(n-1) \times (n-1)$ matrix $\mathbf{B}$, $|\mathbf{B}| = b_{11}b_{22}\cdots b_{(n-1)(n-1)}$. We must prove that the formula given in the theorem holds for any $n \times n$ matrix $\mathbf{A}$.

EXERCISES FOR SECTION 3.2

1. Each of the following matrices is obtained from $\mathbf{I}_3$ by performing a single row operation of type (I), (II), or (III). Identify the operation, and use Theorem 3.3 to give the determinant of each matrix.

★(a) $\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

★(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

★(f) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. Calculate the determinant of each of the following matrices by using row reduction to produce an upper triangular form:

★(a) $\begin{bmatrix} 10 & 4 & 21 \\ 0 & -4 & 3 \\ -5 & -1 & -12 \end{bmatrix}$

(b) $\begin{bmatrix} 18 & -9 & -14 \\ 6 & -3 & -5 \\ -3 & 1 & 2 \end{bmatrix}$

★(c) $\begin{bmatrix} 1 & -1 & 5 & 1 \\ -2 & 1 & -7 & 1 \\ -3 & 2 & -12 & -2 \\ 2 & -1 & 9 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} -8 & 4 & -3 & 2 \\ 2 & 1 & -1 & -1 \\ -3 & -5 & 4 & 0 \\ 2 & -4 & 3 & -1 \end{bmatrix}$

★(e) $\begin{bmatrix} 5 & 3 & -8 & 4 \\ \frac{15}{2} & \frac{1}{2} & -1 & -7 \\ -\frac{5}{2} & \frac{3}{2} & -4 & 1 \\ 10 & -3 & 8 & -8 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ 2 & 4 & -3 & 1 & -4 \\ 2 & 6 & 4 & 8 & -4 \\ -3 & -8 & -1 & 1 & 0 \\ 1 & 3 & 3 & 10 & 1 \end{bmatrix}$

3. By calculating the determinant of each matrix, decide whether it is nonsingular.

★(a) $\begin{bmatrix} 5 & 6 \\ -3 & -4 \end{bmatrix}$

(b) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

★(c) $\begin{bmatrix} -12 & 7 & -27 \\ 4 & -1 & 2 \\ 3 & 2 & -8 \end{bmatrix}$ (d) $\begin{bmatrix} 31 & -20 & 106 \\ -11 & 7 & -37 \\ -9 & 6 & -32 \end{bmatrix}$

4. By calculating the determinant of the coefficient matrix, decide whether each of the following homogeneous systems has a nontrivial solution. (You do not need to find the actual solutions.)

★(a) $\begin{cases} -6x + 3y - 22z = 0 \\ -7x + 4y - 31z = 0 \\ 11x - 6y + 46z = 0 \end{cases}$

(b) $\begin{cases} 4x_1 - x_2 + x_3 = 0 \\ -x_1 + x_2 - 2x_3 = 0 \\ -6x_1 + 9x_2 - 19x_3 = 0 \end{cases}$

(c) $\begin{cases} 2x_1 - 2x_2 + x_3 + 4x_4 = 0 \\ 4x_1 + 2x_2 + x_3 = 0 \\ -x_1 - x_2 - x_4 = 0 \\ -12x_1 - 7x_2 - 5x_3 + 2x_4 = 0 \end{cases}$

5. Let $\mathbf{A}$ be an upper triangular matrix. Prove that $|\mathbf{A}| \neq 0$ if and only if all the main diagonal elements of $\mathbf{A}$ are nonzero.

★6. Find the determinant of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & a_{16} \\ 0 & 0 & 0 & 0 & a_{25} & a_{26} \\ 0 & 0 & 0 & a_{34} & a_{35} & a_{36} \\ 0 & 0 & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}.$$

(Hint: Use part (3) of Theorem 3.3 and then Theorem 3.2.)

7. Suppose that $\mathbf{AB} = \mathbf{AC}$ and $|\mathbf{A}| \neq 0$. Show that $\mathbf{B} = \mathbf{C}$.

8. The purpose of this exercise is to outline a proof by induction of part (1) of Theorem 3.3. Let $\mathbf{A}$ be an $n \times n$ matrix, let R be the row operation $\langle i \rangle \leftarrow c\,\langle i \rangle$, and let $\mathbf{B} = R(\mathbf{A})$.

(a) Prove $|\mathbf{B}| = c|\mathbf{A}|$ when $n = 1$. (This is the Base Step.)

(b) State the inductive hypothesis for the Inductive Step.

(c) Complete the Inductive Step for the case in which R is not performed on the last row of $\mathbf{A}$.

(d) Complete the Inductive Step for the case in which R is performed on the last row of $\mathbf{A}$.

9. The purpose of this exercise and the next is to outline a proof by induction of part (2) of Theorem 3.3. This exercise completes the Base Step.

(a) Explain why $n \neq 1$ in this problem.

(b) Prove that applying the row operation $\langle 1 \rangle \leftarrow c\,\langle 2 \rangle + \langle 1 \rangle$ to a 2×2 matrix does not change the determinant.

(c) Repeat part (b) for the row operation $\langle 2 \rangle \leftarrow c\,\langle 1 \rangle + \langle 2 \rangle$.

10. The purpose of this exercise is to outline the Inductive Step in the proof of part (2) of Theorem 3.3. You may assume that part (3) of Theorem 3.3 has already been proved. Let $\mathbf{A}$ be an $n \times n$ matrix, for $n \geq 3$, and let R be the row operation $\langle i \rangle \leftarrow c\langle j \rangle + \langle i \rangle$.

(a) State the inductive hypothesis and the statement to be proved for the Inductive Step. (Assume for size $n-1$, and prove for size n.)

(b) Prove the Inductive Step in the case where $i \neq n$ and $j \neq n$. (Your proof should be similar to that for Case 1 in the proof of part (3) of Theorem 3.3.)

(c) Consider the case $i = n$. Suppose $k \neq j$ and $k \neq n$. Let R_1 be the row operation $\langle k \rangle \leftrightarrow \langle n \rangle$ and R_2 be the row operation $\langle k \rangle \leftarrow c\langle j \rangle + \langle k \rangle$. Prove that $R(\mathbf{A}) = R_1(R_2(R_1(\mathbf{A})))$.

(d) Finish the proof of the Inductive Step for the case $i = n$. (Your proof should be similar to that for Case 3 in the proof of part (3) of Theorem 3.3.)

(e) Finally, consider the case $j = n$. Suppose $k \neq i$ and $k \neq n$. Let R_1 be the row operation $\langle k \rangle \leftrightarrow \langle n \rangle$ and R_3 be the row operation $\langle i \rangle \leftarrow c\,\langle k \rangle + \langle i \rangle$. Prove that $R(\mathbf{A}) = R_1(R_3(R_1(\mathbf{A})))$.

(f) Finish the proof of the Inductive Step for the case $j = n$.

11. Let $\mathbf{A}$ be an $n \times n$ matrix having an entire row of zeroes.

(a) Use part (1) of Theorem 3.3 to prove that $|\mathbf{A}| = 0$.

(b) Use Corollary 3.6 to provide an alternate proof that $|\mathbf{A}| = 0$.

12. Let $\mathbf{A}$ be an $n \times n$ matrix having two identical rows.

(a) Use part (3) of Theorem 3.3 to prove that $|\mathbf{A}| = 0$.

(b) Use Corollary 3.6 to provide an alternate proof that $|\mathbf{A}| = 0$.

13. Let $\mathbf{A}$ be an $n \times n$ matrix.

(a) Show that if the entries of some row of $\mathbf{A}$ are proportional to those in another row, then $|\mathbf{A}| = 0$.

(b) Show that if the entries in every row of $\mathbf{A}$ add up to zero, then $|\mathbf{A}| = 0$. (Hint: Consider the system $\mathbf{AX} = \mathbf{O}$, and note that the $n \times 1$ vector $\mathbf{X}$ having every entry equal to 1 is a nontrivial solution.)

14. **(a)** Use row reduction to show that the determinant of the $n \times n$ matrix symbolically represented by $\begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{O} & \mathbf{B} \end{bmatrix}$ is $|\mathbf{A}|\,|\mathbf{B}|$, where

A is an $m \times m$ submatrix,
B is an $(n-m) \times (n-m)$ submatrix,
C is an $m \times (n-m)$ submatrix, and
O is an $(n-m) \times m$ zero submatrix.

(b) Use part (a) to compute

$$\begin{vmatrix} -2 & 6 & 7 & -1 \\ 3 & -9 & 2 & -2 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & -1 & 5 \end{vmatrix}.$$

15. Suppose that $f\colon \mathcal{M}_{nn} \to \mathbb{R}$ such that $f(\mathbf{I}_n) = 1$, and that whenever a single row operation is performed on $\mathbf{A} \in \mathcal{M}_{nn}$ to create $\mathbf{B}$,

$$f(\mathbf{B}) = \begin{cases} cf(\mathbf{A}) & \text{for a type (I) row operation with } c \neq 0 \\ f(\mathbf{A}) & \text{for a type (II) row operation} \\ -f(\mathbf{A}) & \text{for a type (III) row operation} \end{cases}.$$

Prove that $f(\mathbf{A}) = |\mathbf{A}|$, for all $\mathbf{A} \in \mathcal{M}_{nn}$. (Hint: If **A** is row equivalent to $\mathbf{I}_n$, then the given properties of f guarantee that $f(\mathbf{A}) = |\mathbf{A}|$ (why?). Otherwise, **A** is row equivalent to a matrix with a row of zeroes, and $|\mathbf{A}| = 0$. In this case, apply a type (I) operation with $c = -1$ to obtain $f(\mathbf{A}) = 0$.)

★16. True or False:

(a) The determinant of a square matrix is the product of the main diagonal entries.

(b) Two row operations of type (III) performed in succession have no overall effect on the determinant.

(c) If every row of a 4×4 matrix is multiplied by 3, the determinant is multiplied by 3 also.

(d) If two rows of a square matrix **A** are identical, then $|\mathbf{A}| = 1$.

(e) A square matrix **A** is nonsingular if and only if $|\mathbf{A}| = 0$.

(f) An $n \times n$ matrix **A** has determinant zero if and only if $\text{rank}(\mathbf{A}) < n$.

3.3 FURTHER PROPERTIES OF THE DETERMINANT

In this section, we investigate the determinant of a product and the determinant of a transpose. We also introduce the classical adjoint of a matrix. Finally, we present Cramer's Rule, an alternative technique for solving certain linear systems using determinants.

Theorems 3.9, 3.10, 3.11, and 3.13 are not proven in this section. An interrelated progressive development of these proofs is left as Exercises 23 through 36.

Highlights

- The determinant of a product $\mathbf{AB}$ is the product of the determinants of $\mathbf{A}$ and $\mathbf{B}$.
- The determinant of $\mathbf{A}^{-1}$ is the reciprocal of the determinant of $\mathbf{A}$.
- A matrix and its transpose have the same determinant.
- The determinant of a matrix can be found using cofactor expansion along any row or column.
- The (classical) adjoint $\mathcal{A}$ of a matrix $\mathbf{A}$ is the transpose of the matrix whose (i,j) entry is the (i,j) cofactor of $\mathbf{A}$.
- If $\mathbf{A}$ is nonsingular, then $\mathbf{A}^{-1} = (1/|\mathbf{A}|)\mathcal{A}$.
- A system $\mathbf{AX} = \mathbf{B}$ where $|\mathbf{A}| \neq 0$ can be solved via division of determinants using Cramer's Rule: that is, each $x_i = |\mathbf{A}_i|/|\mathbf{A}|$, where $\mathbf{A}_i = \mathbf{A}$ except that the ith column of $\mathbf{A}_i$ equals $\mathbf{B}$.

EXERCISES FOR SECTION 3.3

1. For a general 4×4 matrix $\mathbf{A}$, write out the formula for $|\mathbf{A}|$ using a cofactor expansion along the indicated row or column.

★**(a)** Third row

(b) First row

★**(c)** Fourth column

(d) First column

2. Find the determinant of each of the following matrices by performing a cofactor expansion along the indicated row or column:

★**(a)** Second row of $\begin{bmatrix} 2 & -1 & 4 \\ 0 & 3 & -2 \\ 5 & -2 & -3 \end{bmatrix}$

(b) First row of $\begin{bmatrix} 10 & -2 & 7 \\ 3 & 2 & -8 \\ 6 & 5 & -2 \end{bmatrix}$

★**(c)** First column of $\begin{bmatrix} 4 & -2 & 3 \\ 5 & -1 & -2 \\ 3 & 3 & 2 \end{bmatrix}$

(d) Third column of $\begin{bmatrix} 4 & -2 & 0 & -1 \\ -1 & 3 & -3 & 2 \\ 2 & 4 & -4 & -3 \\ 3 & 6 & 0 & -2 \end{bmatrix}$

3. Calculate the adjoint matrix for each of the following by finding the associated cofactor for each entry. Then use the adjoint to find the inverse of the original matrix (if it exists).

★**(a)** $\begin{bmatrix} 14 & -1 & -21 \\ 2 & 0 & -3 \\ 20 & -2 & -33 \end{bmatrix}$

(b) $\begin{bmatrix} -15 & -6 & -2 \\ 5 & 3 & 2 \\ 5 & 6 & 5 \end{bmatrix}$

★**(c)** $\begin{bmatrix} -2 & 1 & 0 & -1 \\ 7 & -4 & 1 & 4 \\ -14 & 11 & -2 & -8 \\ -12 & 10 & -2 & -7 \end{bmatrix}$

(d) $\begin{bmatrix} -4 & 0 & 0 \\ -3 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

★**(e)** $\begin{bmatrix} 3 & -1 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

(f) $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

4. Use Cramer's Rule to solve each of the following systems:

★**(a)** $\begin{cases} 3x_1 - x_2 - x_3 = -8 \\ 2x_1 - x_2 - 2x_3 = 3 \\ -9x_1 + x_2 = 39 \end{cases}$

(b) $\begin{cases} -2x_1 + 5x_2 - 4x_3 = -3 \\ 3x_1 - 3x_2 + 4x_3 = 6 \\ 2x_1 - x_2 + 2x_3 = 5 \end{cases}$

(c) $\begin{cases} -5x_1 + 6x_2 + 2x_3 = -16 \\ 3x_1 - 5x_2 - 3x_3 = 13 \\ -3x_1 + 3x_2 + x_3 = -11 \end{cases}$

★**(d)** $\begin{cases} -5x_1 + 2x_2 - 2x_3 + x_4 = -10 \\ 2x_1 - x_2 + 2x_3 - 2x_4 = -9 \\ 5x_1 - 2x_2 + 3x_3 - x_4 = 7 \\ -6x_1 + 2x_2 - 2x_3 + x_4 = -14 \end{cases}$

5. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices.

(a) Show that $\mathbf{A}$ is nonsingular if and only if $\mathbf{A}^T$ is nonsingular.

(b) Show that $|\mathbf{AB}| = |\mathbf{BA}|$. (Remember that, in general, $\mathbf{AB} \neq \mathbf{BA}$.)

6. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices.

(a) Show that $|\mathbf{AB}| = 0$ if and only if $|\mathbf{A}| = 0$ or $|\mathbf{B}| = 0$.

(b) Show that if $\mathbf{AB} = -\mathbf{BA}$ and n is odd, then $\mathbf{A}$ or $\mathbf{B}$ is singular.

7. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices.

(a) Show that $\left|\mathbf{AA}^T\right| \geq 0$.

(b) Show that $\left|\mathbf{AB}^T\right| = \left|\mathbf{A}^T\right| |\mathbf{B}|$.

8. Let $\mathbf{A}$ be an $n \times n$ skew-symmetric matrix.

(a) If n is odd, show that $|\mathbf{A}| = 0$.

★**(b)** If n is even, give an example where $|\mathbf{A}| \neq 0$.

9. An **orthogonal** matrix is a (square) matrix $\mathbf{A}$ with $\mathbf{A}^T = \mathbf{A}^{-1}$.

(a) Why is $\mathbf{I}_n$ orthogonal?

★**(b)** Find a 3×3 orthogonal matrix other than $\mathbf{I}_3$.

(c) Show that $|\mathbf{A}| = \pm 1$ if $\mathbf{A}$ is orthogonal.

10. Show that there is no matrix $\mathbf{A}$ such that

$$\mathbf{A}^2 = \begin{bmatrix} 9 & 0 & -3 \\ 3 & 2 & -1 \\ -6 & 0 & 1 \end{bmatrix}.$$

11. Give a proof by induction in each case.

(a) **General form of Theorem 3.7:** Assuming Theorem 3.7, prove $|\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_k| = |\mathbf{A}_1|\,|\mathbf{A}_2| \cdots |\mathbf{A}_k|$ for any $n \times n$ matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k$.

(b) Prove $\left|\mathbf{A}^k\right| = |\mathbf{A}|^k$ for any $n \times n$ matrix $\mathbf{A}$ and any integer $k \geq 1$.

(c) Let $\mathbf{A}$ be an $n \times n$ matrix. Show that if $\mathbf{A}^k = \mathbf{O}_n$, for some integer $k \geq 1$, then $|\mathbf{A}| = 0$.

12. Suppose that $|\mathbf{A}|$ is an integer.

(a) Prove that $|\mathbf{A}^n|$ is not prime, for $n \geq 2$. (Recall that a **prime** number is an integer > 1 with no positive integer divisors except itself and 1.)

(b) Prove that if $\mathbf{A}^n = \mathbf{I}$, for some $n \geq 1$, n odd, then $|\mathbf{A}| = 1$.

13. We say that a matrix $\mathbf{B}$ is **similar** to a matrix $\mathbf{A}$ if there exists some (nonsingular) matrix $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{B}$.

(a) Show that if $\mathbf{B}$ is similar to $\mathbf{A}$, then they are both square matrices of the same size.

★**(b)** Find two different matrices $\mathbf{B}$ similar to $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

(c) Show that every square matrix $\mathbf{A}$ is similar to itself.

(d) Show that if $\mathbf{B}$ is similar to $\mathbf{A}$, then $\mathbf{A}$ is similar to $\mathbf{B}$.

(e) Prove that if $\mathbf{A}$ is similar to $\mathbf{B}$ and $\mathbf{B}$ is similar to $\mathbf{C}$, then $\mathbf{A}$ is similar to $\mathbf{C}$.

(f) Prove that if $\mathbf{A}$ is similar to $\mathbf{I}_n$, then $\mathbf{A} = \mathbf{I}_n$.

(g) Show that if $\mathbf{A}$ and $\mathbf{B}$ are similar, then $|\mathbf{A}| = |\mathbf{B}|$.

★**14.** Let $\mathbf{A}$ and $\mathbf{B}$ be nonsingular matrices of the same size, with adjoints $\mathcal{A}$ and $\mathcal{B}$. Express $(\mathbf{AB})^{-1}$ in terms of $\mathcal{A}$, $\mathcal{B}$, $|\mathbf{A}|$, and $|\mathbf{B}|$.

15. If all entries of a (square) matrix $\mathbf{A}$ are integers and $|\mathbf{A}| = \pm 1$, show that all entries of $\mathbf{A}^{-1}$ are integers.

16. If $\mathbf{A}$ is an $n \times n$ matrix with adjoint $\mathcal{A}$, show that $\mathbf{A}\mathcal{A} = \mathbf{O}_n$ if and only if $\mathbf{A}$ is singular.

17. Let $\mathbf{A}$ be an $n \times n$ matrix with adjoint $\mathcal{A}$.

(a) Show that the adjoint of $\mathbf{A}^T$ is $\mathcal{A}^T$.

(b) Show that the adjoint of $k\mathbf{A}$ is $k^{n-1}\mathcal{A}$, for any scalar k.

18. **(a)** Prove that if $\mathbf{A}$ is symmetric with adjoint matrix $\mathcal{A}$, then $\mathcal{A}$ is symmetric. (Hint: Show that the cofactors $\mathcal{A}_{ij}$ and $\mathcal{A}_{ji}$ of $\mathbf{A}$ are equal.)

★**(b)** Give an example to show that part (a) is not necessarily true when "symmetric" is replaced by "skew-symmetric."

19. Use Corollary 3.12 to prove that if $\mathbf{A}$ is nonsingular and upper triangular, then $\mathbf{A}^{-1}$ is also upper triangular.

20. Let $\mathbf{A}$ be a matrix with adjoint $\mathcal{A}$.

(a) Prove that if $\mathbf{A}$ is singular, then $\mathcal{A}$ is singular. (Hint: Use Exercise 16 and a proof by contradiction.)

(b) Prove that $|\mathcal{A}| = |\mathbf{A}|^{n-1}$. (Hint: Consider the cases $|\mathbf{A}| = 0$ and $|\mathbf{A}| \neq 0$.)

21. Recall the 3×3 Vandermonde matrix from Exercise 16 of Section 3.1. For $n \geq 3$, the **general** $n \times n$ **Vandermonde matrix** is

$$\mathbf{V}_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}.$$

If $x_1, x_2, \ldots, x_n$ are distinct real numbers, show that

$$|\mathbf{V}_n| = (-1)^{n+1}(x_1 - x_n)(x_2 - x_n)\cdots(x_{n-1} - x_n)|\mathbf{V}_{n-1}|.$$

(Hint: Subtract the last column from every other column, and use cofactor expansion along the first row to show that $|\mathbf{V}_n|$ is equal or opposite to the determinant of a matrix $\mathbf{W}$ of size $(n-1) \times (n-1)$. Next, divide each column of $\mathbf{W}$ by the first element of that column, using the "column" version of part (1) of Theorem 3.3 to pull out the factors $x_1 - x_n$, $x_2 - x_n, \ldots, x_{n-1} - x_n$. (Note that $\left(x_1^k - x_n^k\right)/(x_1 - x_n) = x_1^{k-1} + x_1^{k-2}x_n + x_1^{k-3}x_n^2 + \cdots + x_1x_n^{k-2} + x_n^{k-1}$.) Finally, create $|\mathbf{V}_{n-1}|$ from the resulting matrix by going through each row from 2 to n in *reverse order* and adding $-x_n$ times the previous row to it.)

★22. True or False:

(a) If $\mathbf{A}$ is a nonsingular matrix, then $|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}^T|}$.

(b) If $\mathbf{A}$ is a 5×5 matrix, a cofactor expansion along the second row gives the same result as a cofactor expansion along the third column.

(c) If $\mathbf{B}$ is obtained from a type (III) column operation on a square matrix $\mathbf{A}$, then $|\mathbf{B}| = |\mathbf{A}|$.

(d) The (i,j) entry of the adjoint of $\mathbf{A}$ is $(-1)^{i+j}|\mathbf{A}_{ji}|$.

(e) For every nonsingular matrix $\mathbf{A}$, we have $\mathbf{A}\mathcal{A} = \mathbf{I}$.

(f) For the system $\begin{cases} 4x_1 - 2x_2 - x_3 = -6 \\ \quad -3x_2 + 4x_3 = 5 \\ \qquad\qquad x_3 = 3 \end{cases}$, $x_2 = -\frac{1}{12}\begin{vmatrix} 4 & -6 & -1 \\ 0 & 5 & 4 \\ 0 & 3 & 1 \end{vmatrix}$.

Taken together, the remaining exercises outline the proofs of Theorems 3.9, 3.10, 3.11, and 3.13 but not in the order in which these theorems were stated. Almost every exercise in this group is dependent on those which precede it.

★23. This exercise will prove part (1) of Theorem 3.10.

(a) Show that if part (1) of Theorem 3.10 is true for some $i = k$ with $2 \le k \le n$, then it is also true for $i = k - 1$. (Hint: Let $\mathbf{B} = R(\mathbf{A})$, where R is the row operation $\langle k \rangle \leftrightarrow \langle k-1 \rangle$. Show that $|\mathbf{B}_{kj}| = |\mathbf{A}_{(k-1)j}|$ for each j. Then apply part (1) of Theorem 3.10 along the kth row of $\mathbf{B}$.)

(b) Use part (a) to complete the proof of part (1) of Theorem 3.10.

►24. Let $\mathbf{A}$ be an $n \times n$ matrix. Prove that if $\mathbf{A}$ has two identical rows, then $|\mathbf{A}| = 0$. (This was also proven in Exercise 12 in Section 3.2.)

►25. Let $\mathbf{A}$ be an $n \times n$ matrix. Prove that $a_{i1}\mathcal{A}_{j1} + a_{i2}\mathcal{A}_{j2} + \cdots + a_{in}\mathcal{A}_{jn} = 0$, for $i \ne j$, $1 \le i, j \le n$. (Hint: Form a new matrix $\mathbf{B}$, which has all entries equal to $\mathbf{A}$, except that both the ith and jth rows of $\mathbf{B}$ equal the ith row of $\mathbf{A}$. Show that the cofactor expansion along the jth row of $\mathbf{B}$ equals $a_{i1}\mathcal{A}_{j1} + a_{i2}\mathcal{A}_{j2} + \cdots + a_{in}\mathcal{A}_{jn}$. Then apply Exercises 23 and 24.)

►26. Let $\mathbf{A}$ be an $n \times n$ matrix. Prove that $\mathbf{A}\mathcal{A} = (|\mathbf{A}|)\,\mathbf{I}_n$. (Hint: Use Exercises 23 and 25.)

►27. Let $\mathbf{A}$ be a nonsingular $n \times n$ matrix. Prove that $\mathcal{A}\mathbf{A} = (|\mathbf{A}|)\,\mathbf{I}_n$. (Hint: Use Exercise 26 and Theorem 2.9.)

►28. Prove part (2) of Theorem 3.10 if $\mathbf{A}$ is nonsingular. (Hint: Use Exercise 27.)

►29. Let $\mathbf{A}$ be a singular $n \times n$ matrix. Prove that $|\mathbf{A}| = |\mathbf{A}^T|$. (Hint: Use a proof by contradiction to show $\mathbf{A}^T$ is also singular, and then use Theorem 3.5.)

►30. Let $\mathbf{A}$ be an $n \times n$ matrix. Show that $(\mathbf{A}_{jm})^T = (\mathbf{A}^T)_{mj}$, for $1 \le j, m \le n$, where $(\mathbf{A}^T)_{mj}$ refers to the (m,j) submatrix of $\mathbf{A}^T$.

▶**31.** Let $\mathbf{A}$ be a nonsingular $n \times n$ matrix. Prove that $|\mathbf{A}| = |\mathbf{A}^T|$. (Hint: Note that $\mathbf{A}^T$ is also nonsingular by part (4) of Theorem 2.11. Use induction on n. The Base Step ($n = 1$) is straightforward. For the Inductive Step, show that a cofactor expansion along the last column of $\mathbf{A}$ equals a cofactor expansion along the last row of $\mathbf{A}^T$. (Use Exercise 30 to obtain that each minor $|(\mathbf{A}^T)_{ni}| = |(\mathbf{A}_{in})^T|$, and then use either the inductive hypothesis or Exercise 29 to show $|(\mathbf{A}_{in})^T| = |\mathbf{A}_{in}|$.) Finally, note that a cofactor expansion along the last column of $\mathbf{A}$ equals $|\mathbf{A}|$ by Exercise 28.) (This exercise completes the proof of Theorem 3.9.)

▶**32.** Prove part (2) of Theorem 3.10 if $\mathbf{A}$ is singular. (Hint: Show that a cofactor expansion along the jth column of $\mathbf{A}$ is equal to a cofactor expansion along the jth row of $\mathbf{A}^T$. (Note that each $|\mathbf{A}_{kj}| = |(\mathbf{A}_{kj})^T|$ (from Exercises 29 and 31) $= |(\mathbf{A}^T)_{jk}|$ (by Exercise 30). Next, apply Exercise 23 to $\mathbf{A}^T$. Finally, use Exercise 29.) (This exercise completes the proof of Theorem 3.10.)

▶**33.** Let $\mathbf{A}$ be an $n \times n$ matrix. Prove that if $\mathbf{A}$ has two identical columns, then $|\mathbf{A}| = 0$. (Hint: Use Exercises 29 and 31 together with Exercise 24.)

▶**34.** Let $\mathbf{A}$ be an $n \times n$ matrix. Prove that $a_{1i}\mathcal{A}_{1j} + a_{2i}\mathcal{A}_{2j} + \cdots + a_{ni}\mathcal{A}_{nj} = 0$, for $i \neq j, 1 \leq i, j \leq n$. (Hint: Use an argument similar to that in Exercise 25, but with columns instead of rows. Use Exercises 28 and 32 together with Exercise 33.)

▶**35.** Let $\mathbf{A}$ be a singular $n \times n$ matrix. Prove that $\mathcal{A}\mathbf{A} = (|\mathbf{A}|)\,\mathbf{I}_n$. (Hint: Use Exercises 32 and 34.) (This exercise completes the proof of Theorem 3.11.)

▶**36.** This exercise outlines the proof that Cramer's Rule (Theorem 3.13) is valid. We want to solve $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A}$ is an $n \times n$ matrix with $|\mathbf{A}| \neq 0$. Assume $n \geq 2$ (since the case $n = 1$ is trivial).

(a) Show that $\mathbf{X} = (1/|\mathbf{A}|)(\mathcal{A}\mathbf{B})$.

(b) Prove that the kth entry of $\mathbf{X}$ is $(1/|\mathbf{A}|)(b_1\mathcal{A}_{1k} + \cdots + b_n\mathcal{A}_{nk})$.

(c) Prove that $|\mathbf{A}_k| = b_1\mathcal{A}_{1k} + \cdots + b_n\mathcal{A}_{nk}$, where $\mathbf{A}_k$ is defined as in the statement of Theorem 3.13. (Hint: Perform a cofactor expansion along the kth column of $\mathbf{A}_k$, and use part (2) of Theorem 3.10.)

(d) Explain how parts (b) and (c) together prove Theorem 3.13.

3.4 EIGENVALUES AND DIAGONALIZATION

In this section, we define eigenvalues and eigenvectors in the context of matrices, in order to find, when possible, a diagonal form for a square matrix. Some of the theoretical details involved cannot be discussed fully until we have introduced vector spaces and linear transformations, which are covered in Chapters 4 and 5. Thus, we will take a more comprehensive look at eigenvalues and eigenvectors at the end of Chapter 5, as well as in Chapters 6 and 7.

- An infinite set of vectors is linearly dependent if some finite subset is linearly dependent.
- An infinite set of vectors is linearly independent if every finite subset is linearly independent.
- A set S of vectors is linearly independent if and only if every vector in span(S) is produced by a unique linear combination of the vectors in S.

EXERCISES FOR SECTION 4.4

★**1.** In each part, determine by quick inspection whether the given set of vectors is linearly independent. State a reason for your conclusion.

(a) $\{[0,1,1]\}$

(b) $\{[1,2,-1],[3,1,-1]\}$

(c) $\{[1,2,-5],[-2,-4,10]\}$

(d) $\{[4,2,1],[-1,3,7],[0,0,0]\}$

(e) $\{[2,-5,1],[1,1,-1],[0,2,-3],[2,2,6]\}$

2. Use the Independence Test Method to determine which of the following sets of vectors are linearly independent:

★**(a)** $\{[1,9,-2],[3,4,5],[-2,5,-7]\}$

★**(b)** $\{[2,-1,3],[4,-1,6],[-2,0,2]\}$

(c) $\{[-2,4,2][-1,5,2],[3,5,1]\}$

(d) $\{[5,-2,3],[-4,1,-7],[7,-4,-5]\}$

★**(e)** $\{[2,5,-1,6],[4,3,1,4],[1,-1,1,-1]\}$

(f) $\{[1,3,-2,4],[3,11,-2,-2],[2,8,3,-9],[3,11,-8,5]\}$

3. Use the Independence Test Method to determine which of the following subsets of $\mathcal{P}_2$ are linearly independent:

★**(a)** $\left\{x^2+x+1,\ x^2-1,\ x^2+1\right\}$

(b) $\{x^2-x+3, 2x^2-3x-1, 5x^2-9x-7\}$

★**(c)** $\{2x-6, 7x+2, 12x-7\}$

(d) $\left\{x^2+ax+b \,\middle|\, |a|=|b|=1\right\}$

4. Determine which of the following subsets of $\mathcal{P}$ are linearly independent:

★**(a)** $\{x^2-1, x^2+1, x^2+x\}$

(b) $\{1+x^2-x^3, 2x-1, x+x^3\}$

★**(c)** $\{4x^2+2, x^2+x-1, x, x^2-5x-3\}$

(d) $\{3x^3+2x+1, x^3+x, x-5, x^3+x-10\}$

★(e) $\{1, x, x^2, x^3, \ldots\}$

(f) $\{1, 1+2x, 1+2x+3x^2, 1+2x+3x^2+4x^3, \ldots\}$

5. Show that the following is a linearly dependent subset of $\mathcal{M}_{22}$:

$$\left\{ \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ -6 & 1 \end{bmatrix}, \begin{bmatrix} 4 & -1 \\ -5 & 2 \end{bmatrix}, \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix} \right\}.$$

6. Prove that the following is linearly independent in $\mathcal{M}_{32}$:

$$\left\{ \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ -6 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 7 \\ 5 & 2 \\ -1 & 6 \end{bmatrix} \right\}.$$

7. Let $S = \{[1,1,0], [-2,0,1]\}$.

(a) Show that S is a linearly independent subset of $\mathbb{R}^3$.

★(b) Find a vector $\mathbf{v}$ in $\mathbb{R}^3$ such that $S \cup \{\mathbf{v}\}$ is also linearly independent.

★(c) Is the vector $\mathbf{v}$ from part (b) unique, or could some other choice for $\mathbf{v}$ have been made? Why or why not?

★(d) Find a nonzero vector $\mathbf{u}$ in $\mathbb{R}^3$ such that $S \cup \{\mathbf{u}\}$ is linearly dependent.

8. Suppose that S is the subset $\{[2,-1,0,5], [1,-1,2,0], [-1,0,1,1]\}$ of $\mathbb{R}^4$.

(a) Show that S is linearly independent.

(b) Find a linear combination of vectors in S that produces $[-2,0,3,-4]$ (an element of span(S)).

(c) Is there a different linear combination of the elements of S that yields $[-2,0,3,-4]$? If so, find one. If not, why not?

9. Consider $S = \{2x^3 - x + 3, 3x^3 + 2x - 2, x^3 - 4x + 8, 4x^3 + 5x - 7\} \subseteq \mathcal{P}_3$.

(a) Show that S is linearly dependent.

(b) Show that every three-element subset of S is linearly dependent.

(c) Explain why every subset of S containing exactly two vectors is linearly independent. (Note: There are six possible two-element subsets.)

10. Let $\mathbf{u} = [u_1, u_2, u_3], \mathbf{v} = [v_1, v_2, v_3], \mathbf{w} = [w_1, w_2, w_3]$ be three vectors in $\mathbb{R}^3$. Show that $S = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent if and only if

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \neq 0.$$

(Hint: Consider the transpose and use the Independence Test Method.) (Compare this exercise with Exercise 19 in Section 4.3.)

11. For each of the following vector spaces, find a linearly independent subset S containing exactly four elements:

★**(a)** $\mathbb{R}^4$

(b) $\mathbb{R}^5$

★**(c)** $\mathcal{P}_3$

(d) $\mathcal{M}_{23}$

★**(e)** $\mathcal{V}$ = set of all symmetric matrices in $\mathcal{M}_{33}$.

12. Let S be a (possibly infinite) subset of a vector space $\mathcal{V}$. Prove that S is linearly dependent if and only if there is a vector $\mathbf{v} \in S$ such that $\text{span}(S - \{\mathbf{v}\}) = \text{span}(S)$. (We say that such a vector $\mathbf{v}$ is **redundant** in S because the same set of linear combinations is obtained after $\mathbf{v}$ is removed from S; that is, $\mathbf{v}$ is not needed.)

13. Find a redundant vector in each given linearly dependent set, and show that it satisfies the definition of a redundant vector given in Exercise 12.

(a) $\{[4,-2,6,1],[1,0,-1,2],[0,0,0,0],[6,-2,5,5]\}$

★**(b)** $\{[1,1,0,0],[1,1,1,0],[0,0,-6,0]\}$

(c) $\left\{[x_1,x_2,x_3,x_4] \in \mathbb{R}^4 \,\middle|\, x_i = \pm 1, \text{ for each } i\right\}$

14. Verify that the Diagonalization Method of Section 3.4 produces the fundamental eigenvectors given in the text for the matrix $\mathbf{A}$ of Example 13.

15. Let $S_1 = \{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ be a subset of a vector space $\mathcal{V}$, let c be a nonzero real number, and let $S_2 = \{c\mathbf{v}_1,\ldots,c\mathbf{v}_n\}$. Show that S_1 is linearly independent if and only if S_2 is linearly independent.

▶**16.** Prove Theorem 4.7. (Hint: Use the definition of linear dependence. Construct an appropriate homogeneous system of linear equations, and show that the system has a nontrivial solution.)

17. Let $\mathbf{f}$ be a polynomial with at least two nonzero terms having different degrees. Prove that the set $\{\mathbf{f}(x), x\mathbf{f}'(x)\}$ (where $\mathbf{f}'$ is the derivative of $\mathbf{f}$) is linearly independent in $\mathcal{P}$.

18. Let $\mathcal{V}$ be a vector space, $\mathcal{W}$ a subspace of $\mathcal{V}$, S a linearly independent subset of $\mathcal{W}$, and $\mathbf{v} \in \mathcal{V} - \mathcal{W}$. Prove that $S \cup \{\mathbf{v}\}$ is linearly independent.

19. Let $\mathbf{A}$ be an $n \times m$ matrix, let $S = \{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ be a finite subset of $\mathbb{R}^m$, and let $T = \{\mathbf{A}\mathbf{v}_1,\ldots,\mathbf{A}\mathbf{v}_k\}$, a subset of $\mathbb{R}^n$.

(a) Prove that if T is a linearly independent subset of $\mathbb{R}^n$ containing k distinct vectors, then S is a linearly independent subset of $\mathbb{R}^m$.

★**(b)** Find a matrix $\mathbf{A}$ for which the converse to part (a) is false.

(c) Show that the converse to part (a) is true if $\mathbf{A}$ is square and nonsingular.

20. Prove that every subset of a linearly independent set is linearly independent.

21. Let S be a subset of a vector space $\mathcal{V}$. If $S = \{\mathbf{a}\}$ or $S = \{\ \}$, prove that S is linearly independent if and only if there is no vector $\mathbf{v} \in S$ such that $\mathbf{v} \in \mathrm{span}(S - \{\mathbf{v}\})$.

22. Suppose $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a finite subset of a vector space $\mathcal{V}$. Prove that S is linearly independent if and only if $\mathbf{v}_1 \neq \mathbf{0}$ and, for each k with $2 \leq k \leq n$, $\mathbf{v}_k \notin \mathrm{span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\})$. (Hint: Half of the proof is done by contrapositive. For this half, assume that S is linearly dependent, and use an argument similar to the first half of the proof of Theorem 4.8 to show some $\mathbf{v}_k$ is in $\mathrm{span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\})$. For the other half, assume S is linearly independent and show $\mathbf{v}_1 \neq \mathbf{0}$ and each $\mathbf{v}_k \notin \mathrm{span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\})$.)

23. Let $\mathbf{f}$ be an nth-degree polynomial in $\mathcal{P}$, and let $\mathbf{f}^{(i)}$ be the ith derivative of $\mathbf{f}$. Show that $\left\{\mathbf{f}, \mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \ldots, \mathbf{f}^{(n)}\right\}$ is a linearly independent subset of $\mathcal{P}$. (Hint: Reverse the order of the elements, and use Exercise 22.)

24. Let S be a nonempty (possibly infinite) subset of a vector space $\mathcal{V}$.

(a) Prove that S is linearly independent if and only if *some* vector $\mathbf{v}$ in $\mathrm{span}(S)$ has a unique expression as a linear combination of the vectors in S (ignoring zero coefficients).

(b) The contrapositive of both halves of the "if and only if" statement in part (a), when combined, gives a necessary and sufficient condition for S to be linearly dependent. What is this condition?

25. Suppose $\mathbf{A}$ is an $n \times n$ matrix and that λ is an eigenvalue for $\mathbf{A}$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a set of fundamental eigenvectors for $\mathbf{A}$ corresponding to λ. Prove that S is linearly independent. (Hint: Consider that each $\mathbf{v}_i$ has a 1 in a coordinate in which all the other vectors in S have a 0.)

26. Suppose T is a linearly independent subset of a vector space $\mathcal{V}$ and that $\mathbf{v} \in \mathcal{V}$.

(a) Prove that if $T \cup \{\mathbf{v}\}$ is linearly dependent, then $\mathbf{v} \in \mathrm{span}(T)$.

(b) Prove that if $\mathbf{v} \in \mathrm{span}(T)$, then $T \cup \{\mathbf{v}\}$ is linearly independent. (Compare this to Exercise 18.)

►27. Prove Theorem 4.10. (Hint: Generalize the proof of Theorem 4.9. In the first half of the proof, suppose that $\mathbf{v} \in \mathrm{span}(S)$ and that $\mathbf{v}$ can be expressed as both $a_1\mathbf{u}_1 + \cdots + a_k\mathbf{u}_k$ and $b_1\mathbf{v}_1 + \cdots + b_l\mathbf{v}_l$ for distinct $\mathbf{u}_1, \ldots, \mathbf{u}_k$ and distinct $\mathbf{v}_1, \ldots, \mathbf{v}_l$ in S. Consider the union $W = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \cup \{\mathbf{v}_1, \ldots, \mathbf{v}_l\}$, and label the distinct vectors in the union as $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$. Then use the given linear combinations to express $\mathbf{v}$ in two ways as a linear combination of the vectors in W. Finally, use the fact that W is a linearly independent set.)

★28. True or False:

(a) The set $\{[2, -3, 1], [-8, 12, -4]\}$ is a linearly independent subset of $\mathbb{R}^3$.

(b) A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in a vector space $\mathcal{V}$ is linearly dependent if $\mathbf{v}_2$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_3$.

(c) A subset $S = \{\mathbf{v}\}$ of a vector space $\mathcal{V}$ is linearly dependent if $\mathbf{v} = \mathbf{0}$.

(d) A subset S of a vector space $\mathcal{V}$ is linearly independent if there is a vector $\mathbf{v} \in S$ such that $\mathbf{v} \in \text{span}(S - \{\mathbf{v}\})$.

(e) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set of vectors in a vector space $\mathcal{V}$, and $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = \mathbf{0}$, then $a_1 = a_2 = \cdots = a_n = 0$.

(f) If S is a subset of $\mathbb{R}^4$ containing six vectors, then S is linearly dependent.

(g) Let S be a finite nonempty set of vectors in $\mathbb{R}^n$. If the matrix $\mathbf{A}$ whose rows are the vectors in S has n pivots after row reduction, then S is linearly independent.

(h) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of a vector space $\mathcal{V}$, then no vector in span(S) can be expressed as two different linear combinations of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$.

(i) If $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a subset of a vector space $\mathcal{V}$, and $\mathbf{v}_3 = 3\mathbf{v}_1 - 2\mathbf{v}_2$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

4.5 BASIS AND DIMENSION

Suppose that S is a subset of a vector space $\mathcal{V}$ and that $\mathbf{v}$ is some vector in $\mathcal{V}$. We can ask the following two fundamental questions about S and $\mathbf{v}$:

Existence: Is there a linear combination of vectors in S equal to $\mathbf{v}$?

Uniqueness: If so, is this the only such linear combination?

The interplay between existence and uniqueness questions is a pervasive theme throughout mathematics. Answering the existence question is equivalent to determining whether $\mathbf{v} \in \text{span}(S)$. Answering the uniqueness question is equivalent (by Theorem 4.10) to determining whether S is linearly independent.

We are most interested in cases where both existence and uniqueness occur. In this section, we tie together these concepts by examining those subsets of vector spaces that simultaneously span and are linearly independent. Such a subset is called a **basis**.

Definition of Basis

Definition Let $\mathcal{V}$ be a vector space, and let B be a subset of $\mathcal{V}$. Then B is a **basis** for $\mathcal{V}$ if and only if both of the following are true:

(1) B spans $\mathcal{V}$.

(2) B is linearly independent.