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$\vec{a}$ $\begin{cases} \text{① magnitude / length / } \cancel{\text{norm}}. \\ \text{② direction.} \end{cases}$

$O = (0,0,0)$.

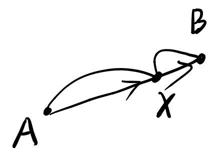
Problem 2

Let $A = (0,1,-1)$ and $B = (1,2,0)$ be two points in a plane. Let X be a point between A and B such that $AX : XB = 2:1$.

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AX}$.

(b) Hence, find the coordinate of X by finding its position vector $\overrightarrow{OX}$. (Hint: $\overrightarrow{AX} = \overrightarrow{OX} - \overrightarrow{OA}$).

(a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$



$= \vec{i} + 2\vec{j} - (\vec{j} - \vec{k})$

$= \vec{i} + \vec{j} + \vec{k}$

$$\overrightarrow{AX} = \underbrace{\frac{2}{3}\cancel{|\overrightarrow{AB}|}}_{\text{magnitude}} \times \underbrace{\frac{\overrightarrow{AB}}{\cancel{|\overrightarrow{AB}|}}}_{\text{direction}} = \frac{2}{3}\overrightarrow{AB} = \frac{2}{3}(\vec{i} + \vec{j} + \vec{k})$$

$= \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$.

(b) $\overrightarrow{AX} = \overrightarrow{OX} - \overrightarrow{OA}$

$\Rightarrow \overrightarrow{OX} = \overrightarrow{AX} + \overrightarrow{OA}$

$= \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k} + \vec{j} - \vec{k}$

$= \frac{2}{3}\vec{i} + \frac{5}{3}\vec{j} - \frac{1}{3}\vec{k}$.

#.

Problem 3

Let $\vec{a} = 2\vec{\imath} - 3\vec{\jmath} + 5\vec{k}$ and $\vec{b} = \vec{\imath} + 3\vec{\jmath}$ be two vectors.

(a) Find $|\vec{a}|$ and $|\vec{a} - 2\vec{b}|$.

(b) Find the unit vector of $\vec{b}$.

(c) Let $\vec{c}$ be another vector with magnitude $|2\vec{a} + \vec{b}|$ and its direction is same as that of $\vec{b}$. Find the vector $\vec{c}$.

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} \qquad (*)$$

(a) $\vec{a} = 2\vec{i} - 3\vec{j} + 5\vec{k}$

$$|\vec{a}| = \sqrt{2^2 + (-3)^2 + 5^2} = \sqrt{38}$$

$$\vec{a} - 2\vec{b} = 2\vec{i} - 3\vec{j} + 5\vec{k} - 2(\vec{i} + 3\vec{j}) = -9\vec{j} + 5\vec{k}.$$

$$|\vec{a} - 2\vec{b}| = \sqrt{9^2 + 5^2} = \sqrt{81 + 25} = \sqrt{106}.$$

(b) $\vec{a}$ $\qquad \hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

$$\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{\vec{i} + 3\vec{j}}{\sqrt{1^2 + 3^2}} = \frac{\vec{i} + 3\vec{j}}{\sqrt{10}} = \frac{1}{\sqrt{10}}\vec{i} + \frac{3}{\sqrt{10}}\vec{j}$$

(c) $2\vec{a} + \vec{b} = 2(2\vec{i} - 3\vec{j} + 5\vec{k}) + \vec{i} + 3\vec{j} = 5\vec{i} - 3\vec{j} + 10\vec{k}$

$$|2\vec{a} + \vec{b}| = \sqrt{5^2 + (-3)^2 + 10^2} = \sqrt{134}$$

$$\vec{c} = \underbrace{\sqrt{134}}_{\text{magnitude}} \times \underbrace{\left(\frac{1}{\sqrt{10}}\vec{i} + \frac{3}{\sqrt{10}}\vec{j}\right)}_{\text{direction (unit vector)}} = \frac{\sqrt{134}}{\sqrt{10}}\vec{i} + \frac{3\sqrt{134}}{\sqrt{10}}\vec{j}.$$

Scalar product / dot product / inner product.

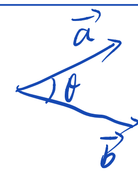
$\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$

$\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$.

$$\boxed{\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (*)}.$$

Theorem:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta \quad (*).$$



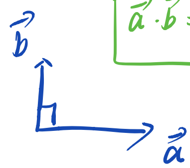
$\Rightarrow$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \quad (**).$$

if $\cos\theta > 0$, $\theta < 90°$

$\cos\theta < 0$, $\theta > 90°$.

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = 0$, $\cos\theta = 0$



Problem 4

Let $\vec{a} = \vec{\imath} + 3\vec{\jmath} - 2\vec{k}$ and $\vec{b} = -2\vec{\imath} + \vec{\jmath} + 3\vec{k}$ be two vectors.

(a) Find $\vec{a}\cdot\vec{b}$.

(b) Find the angle between the vectors $\vec{a}$ and $\vec{b}$.

(c) Let $\vec{c} = 3\vec{\imath} + x\vec{\jmath} - 2\vec{k}$ be a vector which is perpendicular to $\vec{b}$, find the value of x.

(d) Let $\vec{d} = y\vec{a} + 3\vec{b}$ be a vector which is perpendicular to $\vec{a} - \vec{b}$ find the value of y.

(a) $\vec{a} = 1\vec{i} + 3\vec{j} - 2\vec{k}$, $\vec{b} = -2\vec{i} + 1\vec{j} + 3\vec{k}$

$\vec{a}\cdot\vec{b} = 1\times(-2) + 3\times1 + (-2)\times3 = -2 + 3 - 6 = -5$

(b) $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$|\vec{a}| = \sqrt{1^2+3^2+(-2)^2} = \sqrt{14}$

$|\vec{b}| = \sqrt{(-2)^2+1^2+3^2} = \sqrt{14}$

$\cos\theta = \dfrac{-5}{\sqrt{14}\cdot\sqrt{14}} = -\dfrac{5}{14} < 0$, $\theta > 90^\circ$

$\theta \approx 110.92^\circ$

(c) $\vec{c}\cdot\vec{b} = 0$

$(3\vec{i} + x\vec{j} - 2\vec{k})\cdot(-2\vec{i} + \vec{j} + 3\vec{k}) = 3\times(-2) + x - 2\times3 = 0$

$\Rightarrow x = 12$

(d)

$\vec{d}\cdot(\vec{a}-\vec{b}) = 0$

$(y\vec{a} + 3\vec{b})\cdot(\vec{a}-\vec{b}) = 0$

$y\,\vec{a}\cdot\vec{a} - y\,\vec{a}\cdot\vec{b} + 3\vec{a}\cdot\vec{b} - 3\vec{b}\cdot\vec{b} = 0$

$y|\vec{a}|^2 + (3-y)\vec{a}\cdot\vec{b} - 3|\vec{b}|^2 = 0$

$14y - 5(3-y) - 3\times14 = 0$

$\Rightarrow y = 3$

Problem 8

Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{a}| = 1$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = 1$.

(a) Find the angle between the vectors $\vec{a}$ and $\vec{b}$.

(b) Find the value of $(3\vec{a} - 2\vec{b}) \cdot (\vec{a} + 3\vec{b})$ and $|\vec{a} - 2\vec{b}|$.

(c) Find the angle between two vectors $\vec{a} - 2\vec{b}$ and $2\vec{a} + 3\vec{b}$.

(a) $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{1}{1\times 2} = \dfrac{1}{2} > 0$.

$\theta = 60^\circ$.

(b) $(3\vec{a} - 2\vec{b}) \cdot (\vec{a} + 3\vec{b})$

$= 3\vec{a}\cdot\vec{a} + 9\vec{a}\cdot\vec{b} - 2\vec{b}\cdot\vec{a} - 2\vec{b}\cdot 3\vec{b}$

$= 3|\vec{a}|^2 + 7\vec{a}\cdot\vec{b} - 6|\vec{b}|^2 = 3 + 7 - 6\times 2^2 = -14$

$|\vec{a} - 2\vec{b}|^2 = (\vec{a} - 2\vec{b})\cdot(\vec{a} - 2\vec{b})$.

$|\vec{a} - 2\vec{b}| = \sqrt{(\vec{a} - 2\vec{b})\cdot(\vec{a} - 2\vec{b})}$

$= \sqrt{\underbrace{\vec{a}\cdot\vec{a}}_{|\vec{a}|^2} + 4\underbrace{\vec{b}\cdot\vec{b}}_{|\vec{b}|^2} - 4\vec{a}\cdot\vec{b}}$

$= \sqrt{13}$.

(c) $\cos\theta = \dfrac{(\vec{a} - 2\vec{b})\cdot(2\vec{a} + 3\vec{b})}{|\vec{a} - 2\vec{b}|\,|2\vec{a} + 3\vec{b}|}$.

$(\vec{a} - 2\vec{b})\cdot(2\vec{a} + 3\vec{b}) = 2\vec{a}\cdot\vec{a} + 3\vec{a}\cdot\vec{b} - 2\vec{b}\cdot 2\vec{a} - 2\vec{b}\cdot 3\vec{b}$

$= -23$.

$$|\vec{a}-2\vec{b}| = \sqrt{(\vec{a}-2\vec{b})\cdot(\vec{a}-2\vec{b})} = \sqrt{13}.$$

$$|2\vec{a}+3\vec{b}| = \sqrt{(2\vec{a}+3\vec{b})\cdot(2\vec{a}+3\vec{b})} = \sqrt{4|\vec{a}|^2+12\vec{a}\cdot\vec{b}+9|\vec{b}|^2}.$$

$$= \sqrt{52}$$

$$\cos\theta = \frac{-23}{\sqrt{13}\cdot\sqrt{52}} = -\frac{23}{26} < 0. \quad \Rightarrow \theta > 90^\circ.$$

$$\theta \approx 152.2^\circ$$