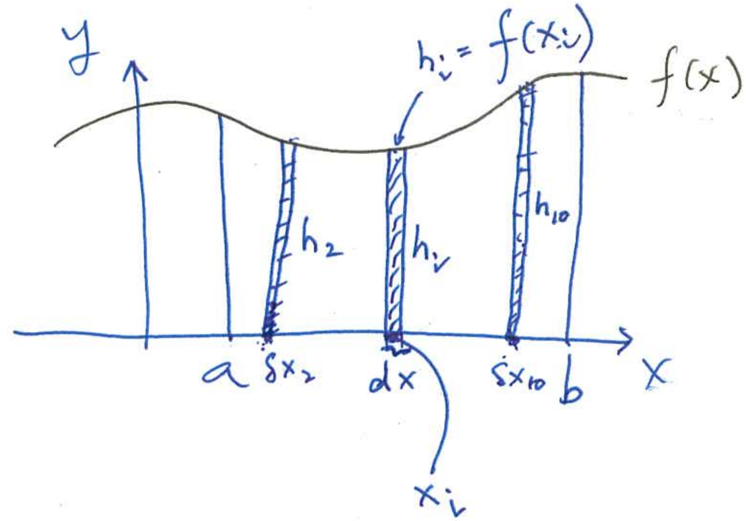
Chapter 3

$$\int_a^b f(x)\,dx$$



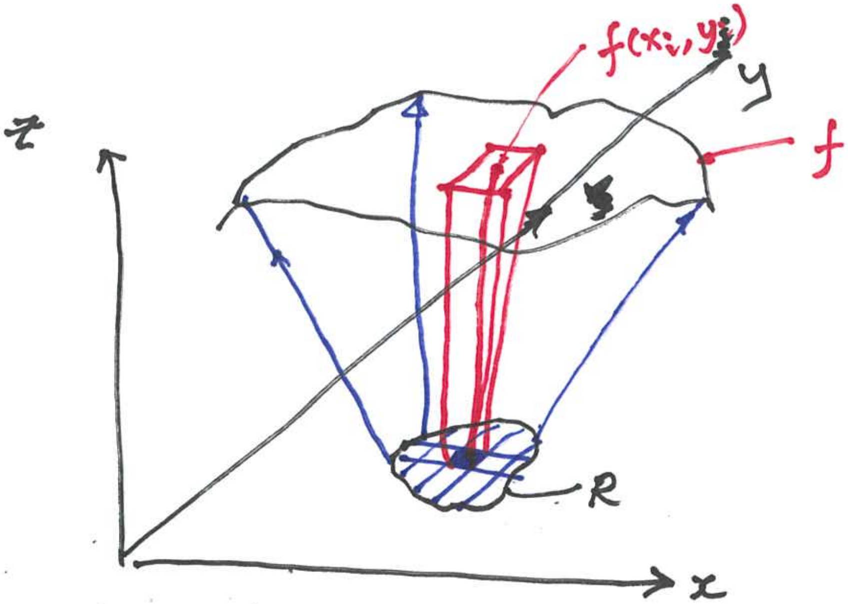
$$\simeq \lim_{|p|\to 0} \sum_{i=1}^{n} \underbrace{f(x_i)}^{h_i} \delta x_i$$

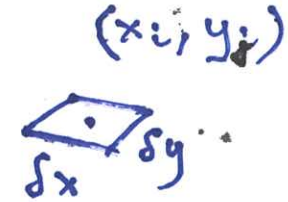
$n \to \infty$ very large

$$|p| = \max_{1\le i\le n} \{ \delta x_i \,,\quad 1 \le i \le n \}$$

$$|p| \to 0 \quad \Rightarrow \quad \delta x_i \to 0$$

$$\iiiint \cdots \qquad ? \int \cdots \iint f(x_1, x_2, \cdots x_n)\, dx_1\, dx_2 \cdots dx_n$$





Small area

$$\delta S_{ij} = \delta x_i \, \delta y_j$$

$$\delta V_i = f(x_i, y_i) \underbrace{\delta x_i \, \delta y_i}_{\delta S_{ij}}$$

$$P = \{ \delta S_{ij} = \delta x_i \, \delta y_j \, , \; 1 \le i \le m , \; 1 \le j \le n . \}$$

$$\iint_R f(x, y) \, dx \, dy$$

$$|P| = \max_{\substack{1 \le i \le m \\ 1 \le j \le n}} \{ \delta S_{ij} \}$$

$|P| \to 0$

$\delta x \to 0$

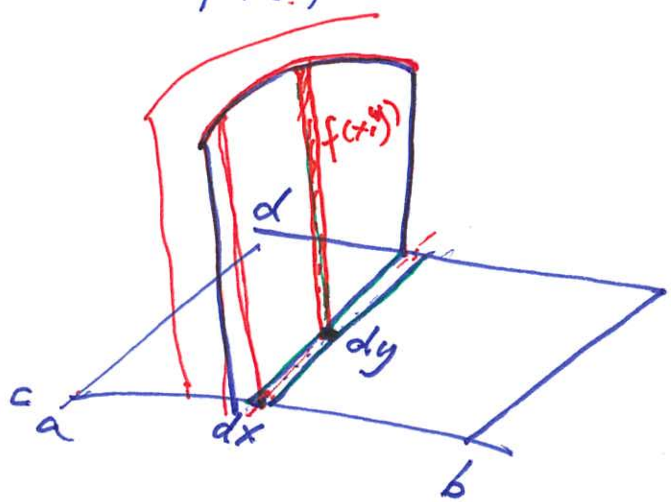
$$\delta y \to 0 \quad \simeq \lim_{\substack{\delta x_i \to 0 \\ \delta y_i \to 0}} \sum_i \sum_j f(x_i, y_j) \, \delta x_i \, \delta y_j$$

Ex 1

~~Regular~~ Rectangular region

$$\int_{x=a}^{b}\left[\int_{y=c}^{d} f(x,y)\,dy\right]dx$$

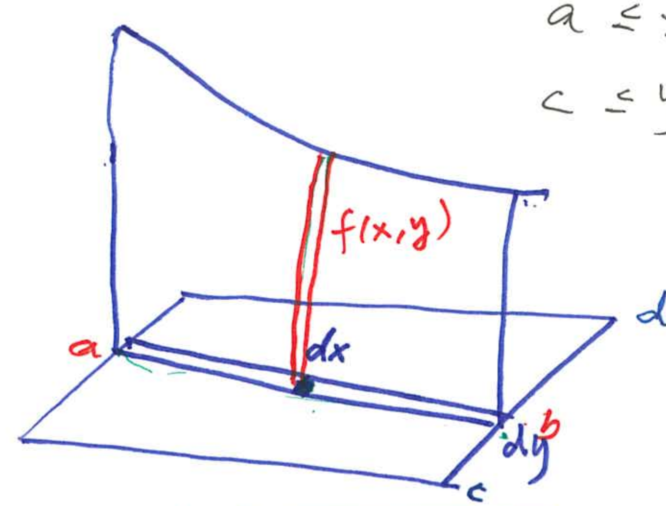
$A(x)$



To obtain the cross-sectional area of A, we hold x fixed; we integrate w.r.t y

$a \le x \le$

$c \le y \le$

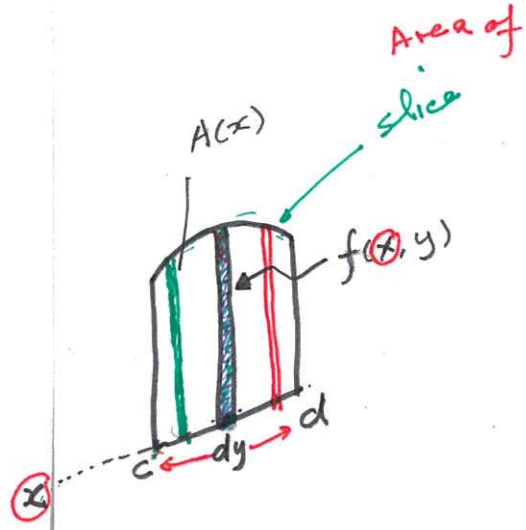


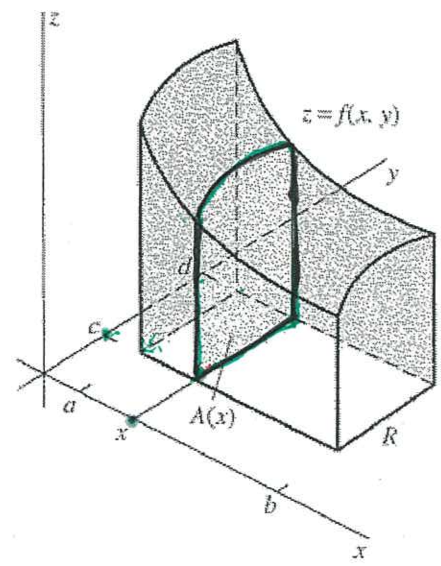
$$\int_{y=c}^{d}\left[\int_{x=a}^{b} f(x,y)\,dx\right]dy$$

$A(y)$

To obtain the cross sectional area of A(y), we hold y f and integrate w.r.t. x

Figure A



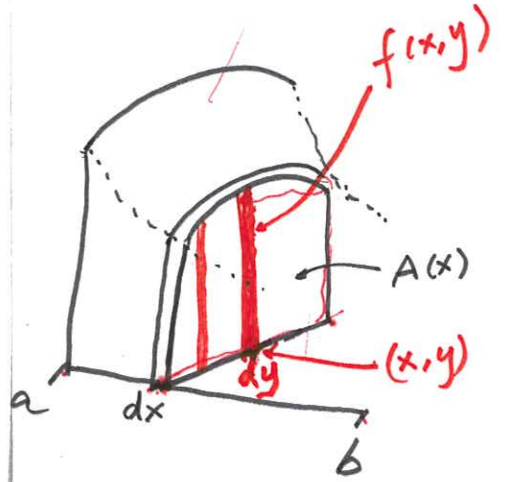


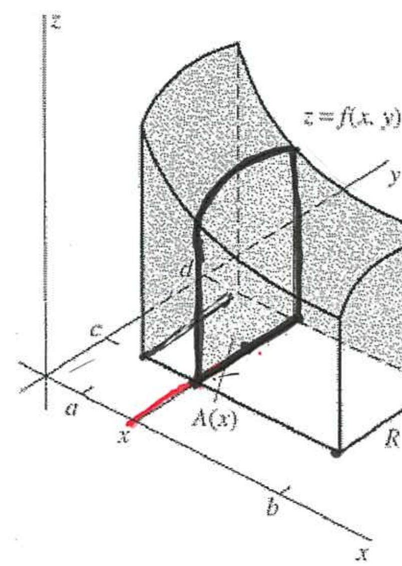
$$A(x) = \int_c^d f(x,y)\,dy$$

$a \le x \le b$

$c \le y \le d$

Figure A



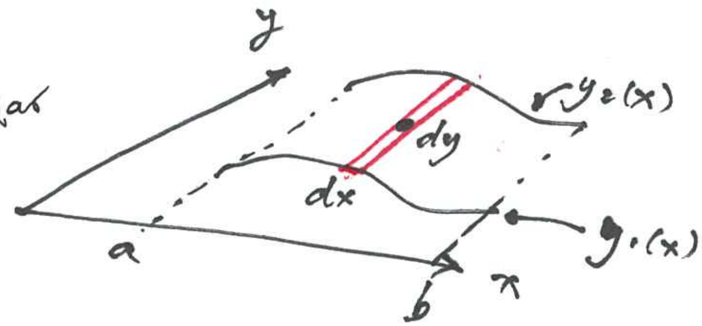


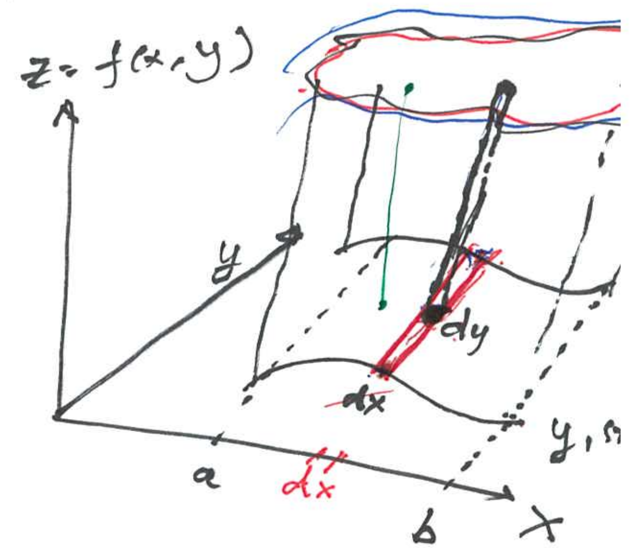
$$V = \int_a^b \underbrace{A(x)}_{\text{obtained from the 1st } \int} dx$$

$$= \underbrace{\int_a^b}_{\text{2nd}} \overbrace{\boxed{\underbrace{\int_{y=c}^{d}}_{\text{1st}} f(x,y)\,dy}}^{A(x)}\, dx$$

Ex2

Non-rectangular regions





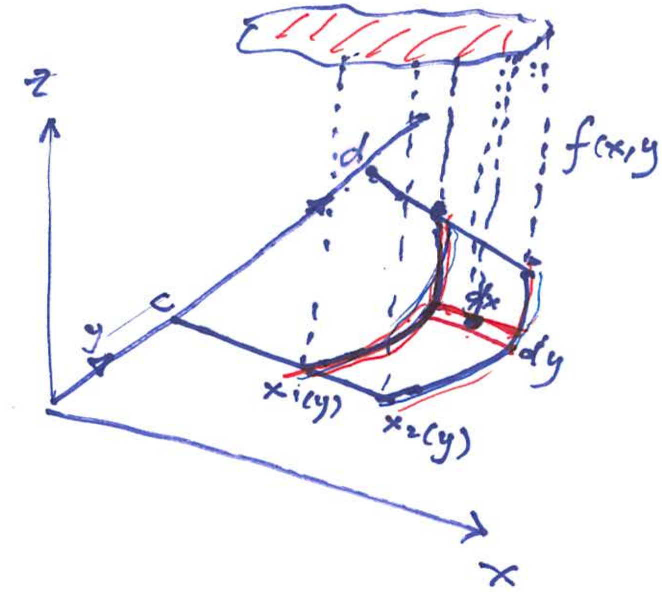
$$a \le x \le b \qquad y_1(x) \le y \le y_2(x)$$

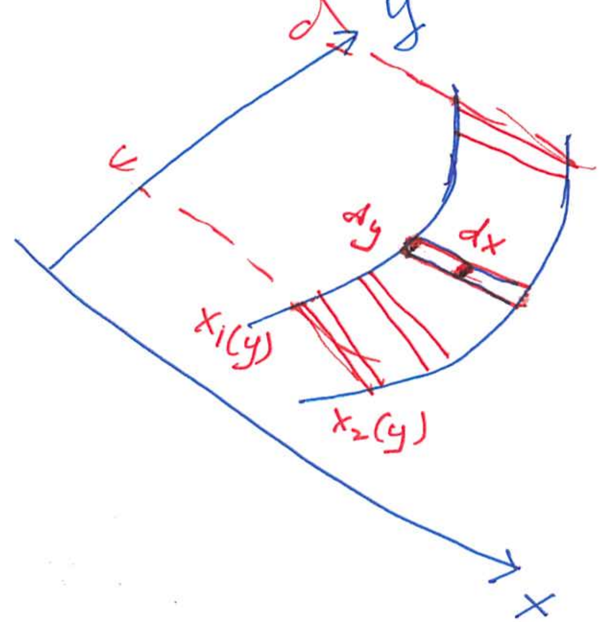
$$\int_{x=a}^{b} \boxed{\int_{y=y_1(x)}^{y=y_2(x)} f(x,y)\, dy}\; dx$$

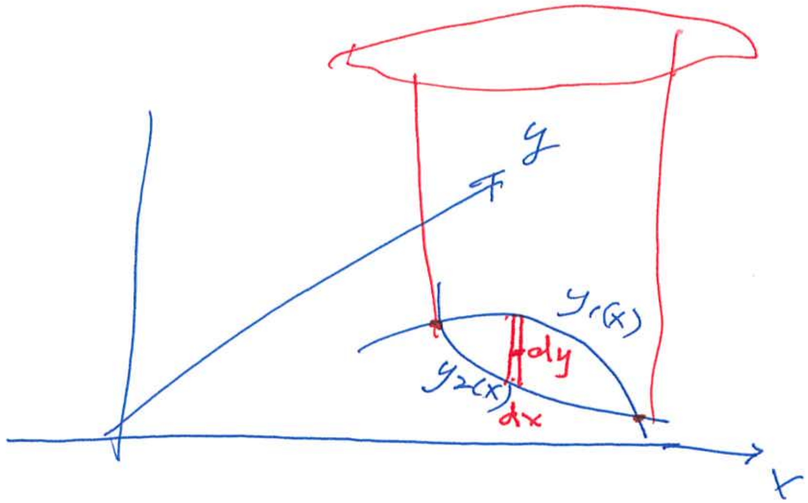
$$x_1(y) \le x \le x_2(y)$$

$$c \le y \le d$$

$$\int_{y=c}^{d} \underbrace{\boxed{\int_{x=x_1(y)}^{x_2(y)} f(x,y)\,dx}}_{A(y)} \, dy$$

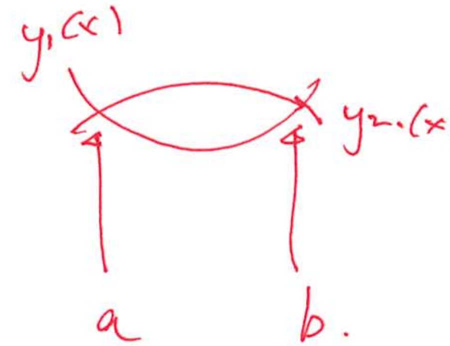






$$y_1(x) \le y \le y_2(x)$$

$$a \le x \le b$$



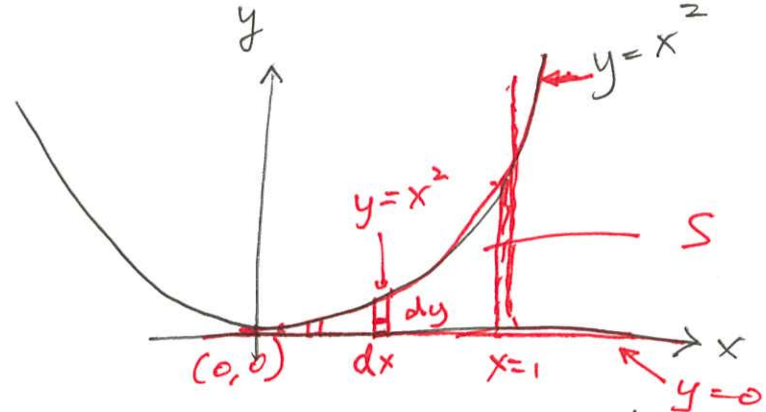
$$\int_{?a}^{?b} \int_{y_1(x)}^{y_2(x)} f(x,y)\, dy \,.\, dx$$

meeting points

Ex 4

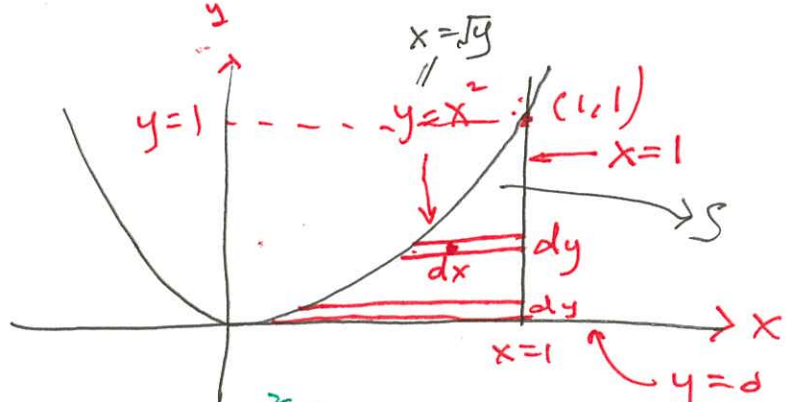
$$\iint_S xy^2\,dx\,dy$$

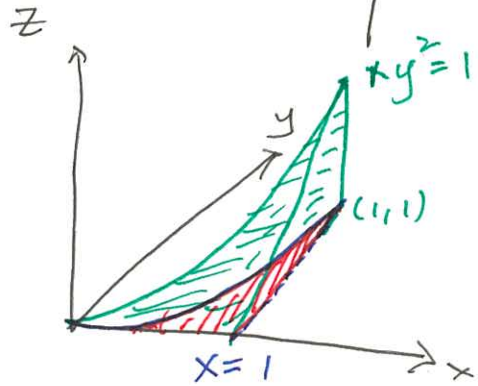
$$= \int_{x=0}^{x=1}\int_{y=0}^{y=x^2} xy^2\,dy\,dx =$$



S is the region bounded by $y=0$, $y=x^2$, $x=1$

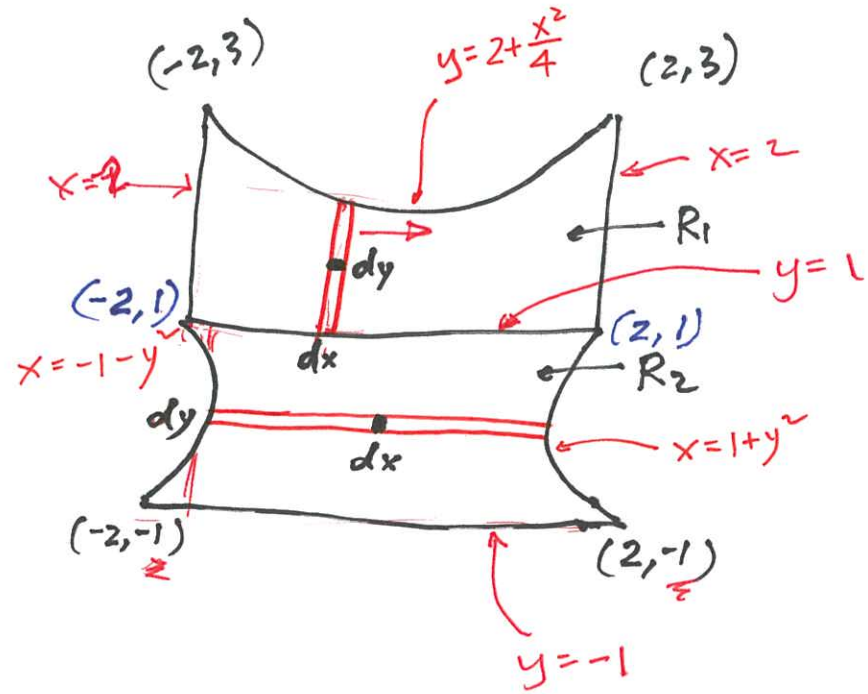
$$= \int_{y=0}^{y=1}\int_{x=\sqrt{y}}^{x=1} \underbrace{(xy^2)}_{f(x,y)}\,dx\,dy =$$





Ex 5

$$\iint_R f(x,y)\,dx\,dy$$



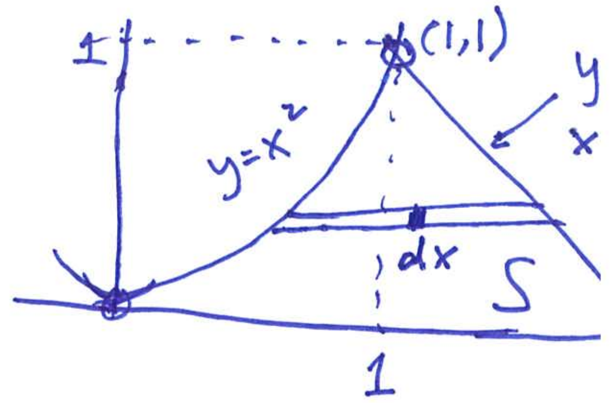
$$= \iint_{R_1} f(x,y)\,dx\,dy + \iint_{R_2} f(x,y)\,dx\,dy$$

$$= \int_{x=-2}^{x=2}\int_{y=1}^{\boxed{y=2+\frac{x^2}{4}}} f(x,y)\,dy\,dx \quad + \quad \int_{y=-1}^{y=1}\int_{x=-1-y^2}^{x=1+y^2} f(x,y)\,dx\,dy$$

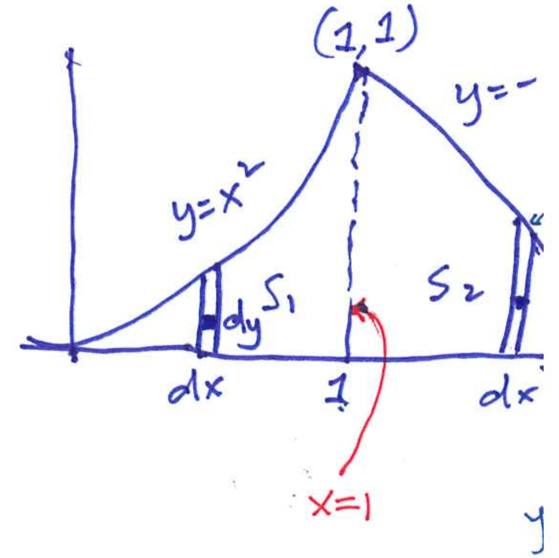
R_1

Ex 6

$$\iint_S f(x,y)\,dx\,dy = \int_{y=0}^{y=1}\int_{x=\sqrt{y}}^{x=2-y} f(x,y)\,dx\,dy$$



$$\iint_{S_1} f(x,y)\,dy\,dx + \iint_{S_2} f(x,y)\,dy\,dx$$



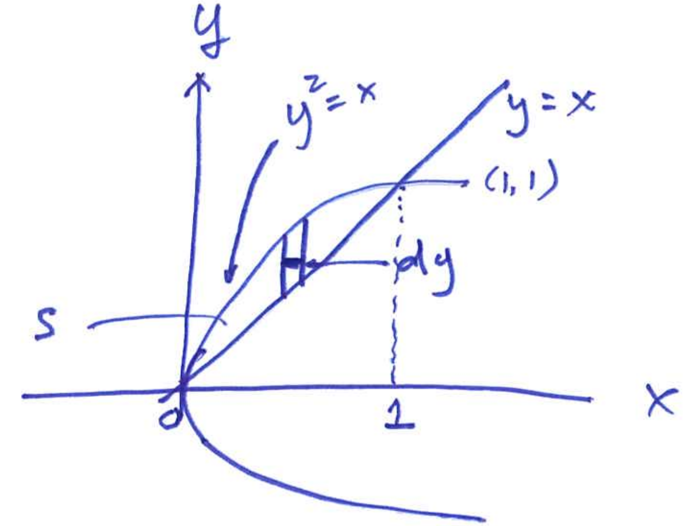
$$\int_{x=0}^{x=1}\int_{y=0}^{y=x^2} f(x,y)\,dy\,dx + \int_{x=1}^{x=2}\int_{y=0}^{y=-x+2} f(x,y)\,dy\,dx$$

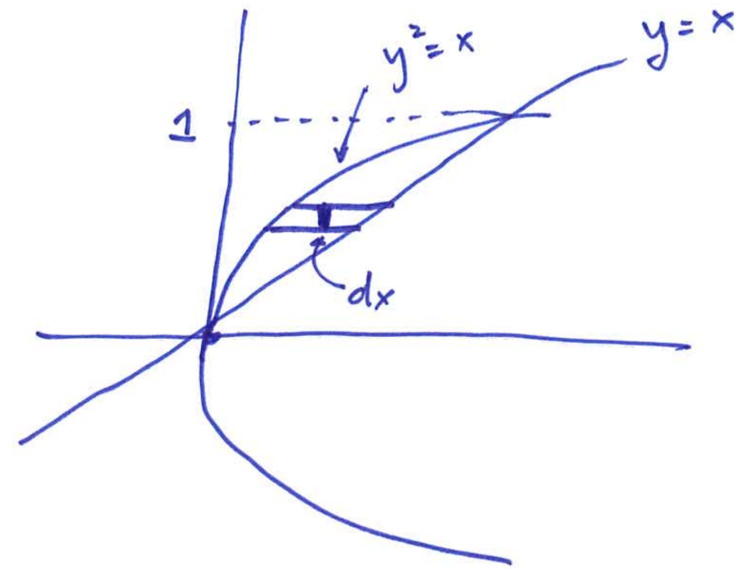
"Change" of Order

8/.

$$\int_{x=0}^{1}\int_{y=x}^{y=\sqrt{x}} f(x,y)\,dy\,dx$$

$y^2 = x$





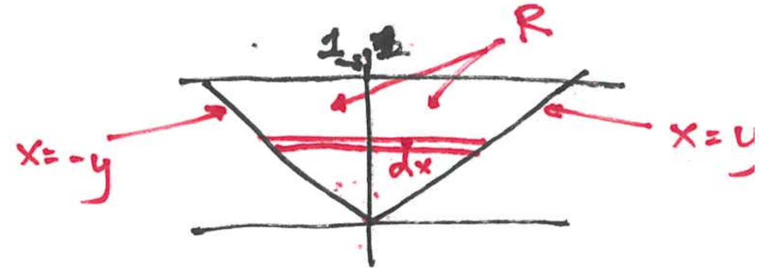
$$\int_{y=0}^{1}\int_{x=y^2}^{x=y} f(x,y)\,dx\,dy$$

Ex 9

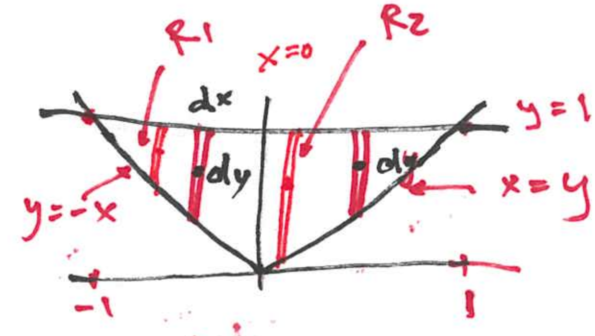
$$\int_0^1 \int_{-y}^{y} f(x,y)\,dx\,dy$$

$y \to x = y$

$-y \to x = -y$



$$= \iint_{R_1} f(x,y)\,dy\,dx + \iint_{R_2} f(x,y)\,dy\,dx$$



$$= \int_{x=-1}^{x=0} \int_{y=-x}^{y=1} f(x,y)\,dy\,dx + \int_{x=0}^{1} \int_{y=x}^{y=1} f(x,y)\,dy\,dx$$

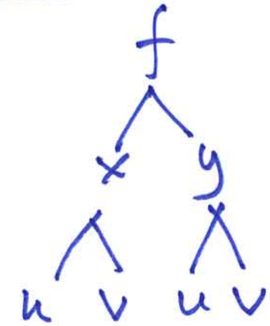
=

Change of Variables

$$I = \int_a^b f(x)\,dx \qquad x = x(u) \qquad x \longrightarrow u$$

$$= \int_\alpha^\beta f(x(u)) \left(\frac{dx}{du}\right) du \qquad x(\alpha) = a \qquad x(\beta) = b$$

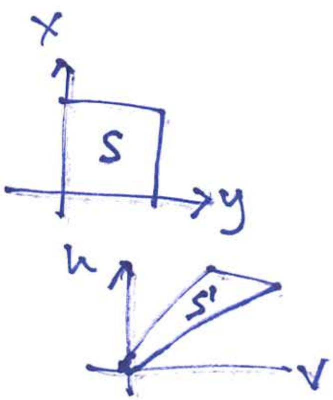
$$x = x(u,v)$$
$$y = y(u,v)$$
$$(x,y) \longrightarrow (u,v)$$



$$I = \iint_S f(x,y)\,dx\,dy$$

$$= \iint_{S'} f(x(u,v), y(u,v)) \, |J| \, du\,dv$$

absolute

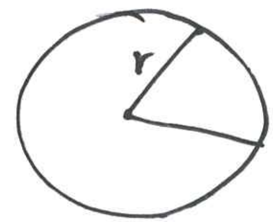


$$J = \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

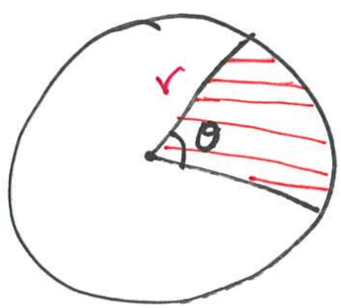
Jacobian

Area of a Circle

$= \pi r^2$



Circumerance of a Circle $= 2\pi r$

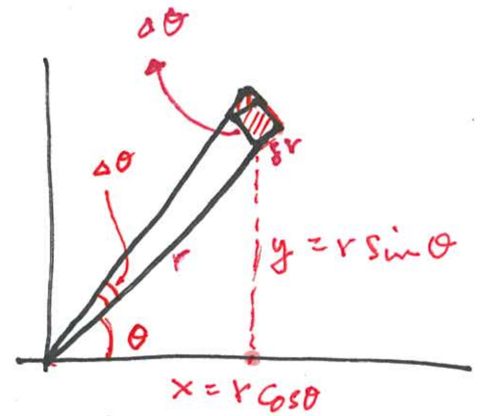


Arc length $= r\theta$

Area of segment $= \frac{r^2}{2}\theta$

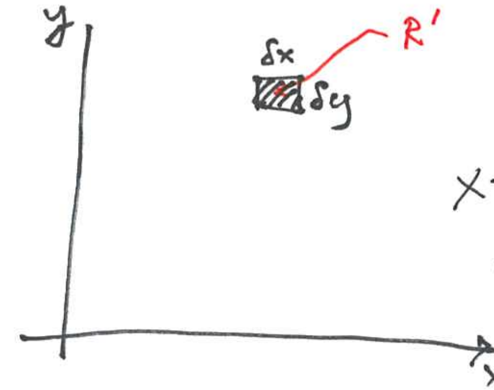
P.7 Change of Variables in Double integral

r-θ plane

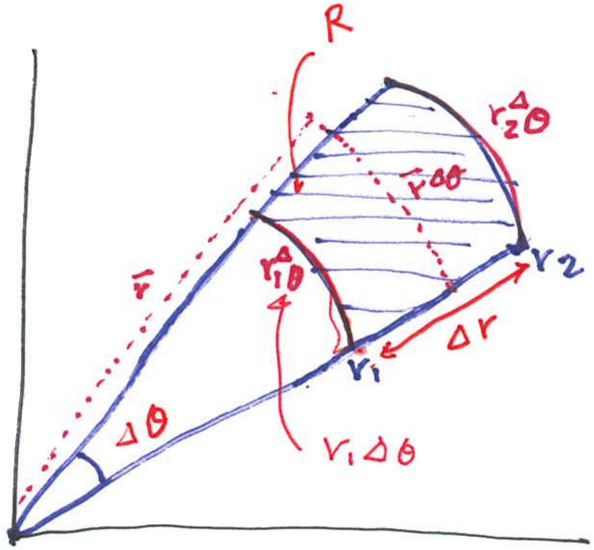


$$\Longleftarrow \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\delta x\, \delta y = \delta A$$



x-y pla



Area of

Area of R ≃ rectangle

$$\simeq \Delta r(\bar{r}\Delta\theta)$$

$$\boxed{\text{Let } \frac{r_1+r_2}{2} = \bar{r}}$$

$$\therefore \quad \delta A = \delta x\,\delta y \simeq \Delta r(\bar{r}\Delta\theta$$

$\delta x \to 0$

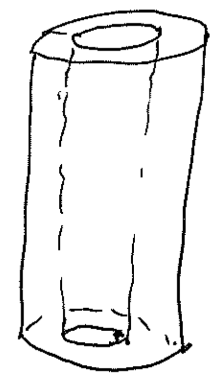
$\delta y \to 0$

$$dx\,dy \simeq \bar{r}\,dr\,d\theta$$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r$$

Example 10

$$MI = \iint_S \rho h \,(x^2+y^2)\,dx\,dy$$

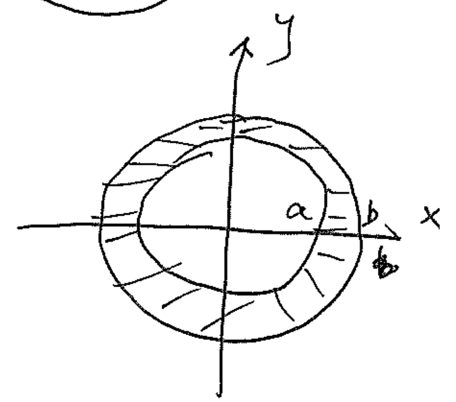


$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$J = r$$
$$|J| = r$$

$$R \begin{cases} 0 \le \theta \le 2\pi \\ a \le r \le b \end{cases}$$

$$dx\,dy \longrightarrow r\,dr\,d\theta$$



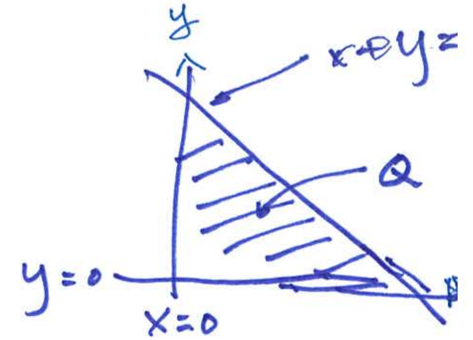
$$MI = \iint_S \rho h r^3 \; r\,dr\,d\theta$$

Example 11

$$\iint_Q e^{\frac{y-x}{y+x}}\,dx\,dy$$

$$\begin{cases} u = y-x \\ v = y+x \end{cases}$$

$$x = \frac{v-u}{2}, \quad y = \frac{v+u}{2}$$

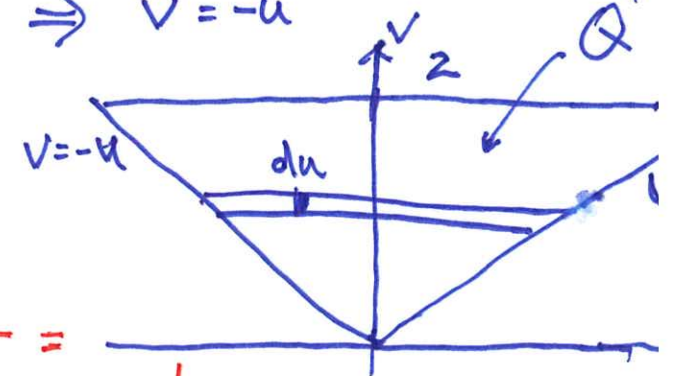


$$J = \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \det\begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= -\frac{1}{2}$$

$$\begin{cases} x+y=2 & \Rightarrow & v=2 \\ x=0 & \Rightarrow & v=u \\ y=0 & \Rightarrow & v=-u \end{cases}$$



$$J = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\det\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}} = \frac{1}{\det\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}} = \dots = -\frac{1}{2}$$

$$\iint_Q e^{\frac{y-x}{y+x}}\,dx\,dy = \int_{v=0}^{2}\int_{u=-v}^{u=v} e^{\frac{u}{v}}\,|J|\,du\,dv = \frac{1}{2}\iint_{Q'} e^{\frac{u}{v}}\,du\,dv$$

Example 17

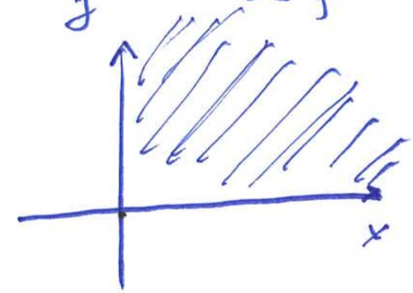
$$\frac{1}{2}\int_{v=0}^{2} v \left(e - \frac{1}{e}\right) dv = e - \frac{1}{e}$$

Example 12

x-y

$0 < x < \infty$

$0 < y < \infty$



$$I = \int_0^\infty e^{-x^2}\,dx$$

$$I^2 = I \cdot I = \int_0^\infty e^{-x^2}\,dx \cdot \int_0^\infty e^{-y^2}\,dy$$

$$I^2 = \int_0^\infty \left(\int_0^\infty e^{-x^2}\,dx \right) e^{-y^2}\,dy = \int_0^\infty e^{-y^2} \left(\int_0^\infty e^{-x^2}\,dx \right) dy$$

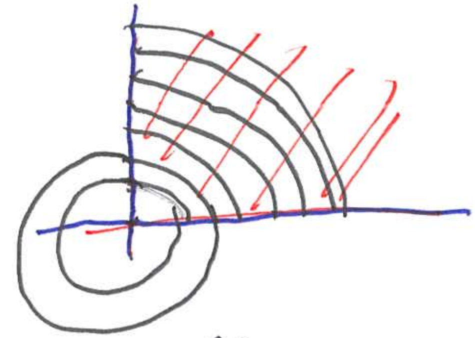
$f(x, y)$

$$= \int_0^\infty \left(\int_0^\infty e^{-y^2} \cdot e^{-x^2}\,dx \right) dy = \int_0^\infty \left(\int_0^\infty e^{-(y^2+x^2)}\,dx \right) dy$$

$x = r\cos\theta$, $y = r\sin\theta$ $\qquad x^2 + y^2 = r^2$ $\qquad 0 \le r \le \infty$, $0 \le \theta \le \frac{\pi}{2}$

12

$$I^2 = \int_{r=0}^{\infty}\int_{\theta=0}^{\frac{\pi}{2}} e^{-r^2}\,\overset{\to r}{|J|}\,d\theta\,dr.$$



$$= \int_{r=0}^{\infty}\boxed{\int_{\theta=0}^{\frac{\pi}{2}} e^{-r^2}\,r\,d\theta}\,dr \;=\; \int_{r=0}^{\infty}\frac{\pi}{2}e^{-r^2}r\;dr \;=\; \frac{\pi}{2}\int_{0}^{\infty}e^{-r^2}\,d(-r^2)$$

$$= -\frac{\pi}{4}\left(\frac{1}{e^{\infty^2}}-1\right) = \frac{\pi}{4}$$

$$I^2 = \frac{\pi}{4} \Rightarrow \quad I = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$$