# Cartesian coordinates V.S. Polar coordinates in 2D-plane

# Chapter 3. Multiple Integral

# 1 Three-Variable Case (Triple Integral):

## 1.1 Definition

## 1.2 Particular Interpretations:

- (a) If $f(x, y, z) \equiv 1$, then $\iiint 1dxdydz =$ **volume** of the region $V$.
- (b) If the scalar function $\rho(x, y, z)$ gives the density at a point $(x, y, z)$ of the region $V$, then $\iiint_V \rho(x, y, z)dxdydz =$ **mass** of the region $V$.
- (c) If the scalar function $\rho(x, y, z)$ gives the charge density at a point $(x, y, z)$ of the region $V$, then $\iiint_V \rho(x, y, z)dx =$ **total charge** within the region $V$.

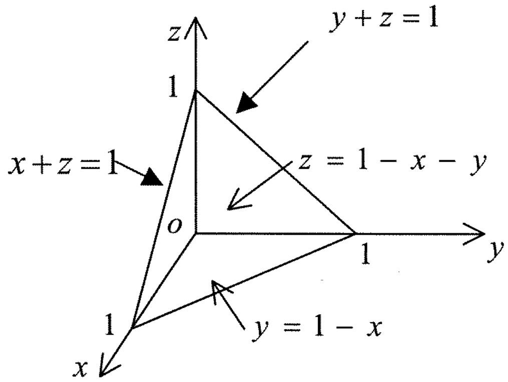
## 1.3 Computation of Double Integrals:

**Case 1:** By iterated integral in some order directly.

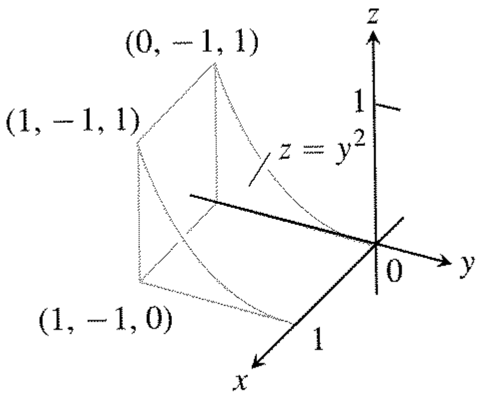
**Example**. The density of a rectangular blocks $V$ bounded by the planes $x = 1, x = 2, y = 0, y = 3, z = -1, z = 0$ is given by the scalar function $\rho(x, y, z) = x(y + 1) - z$. Find the mass of the block.

**Example**. Evaluate $\iiint_V \frac{1}{(x+y+2z+1)^3} dxdydz$ where $V$ is the region enclosed by the planes

$x = 0, y = 0, z = 0, x + y + z = 1.$

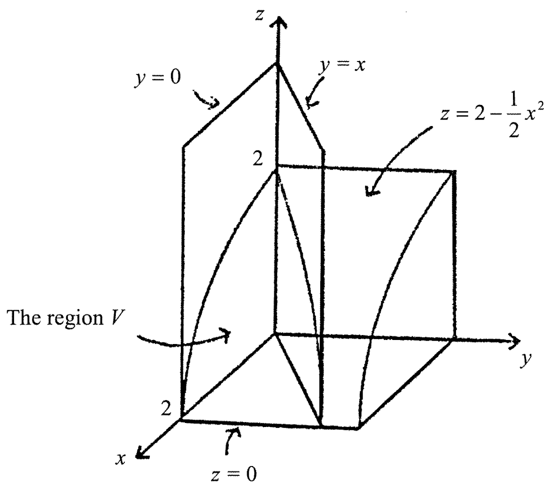


**Example**. With the aid of the following figure, change the order of the iterated integral $\int_0^1 \left[\int_{-1}^0 \left(\int_0^{y^2} f(x,y,z)dz\right) dy\right] dx$ to an equivalent iterated integral with order $dydzdx$.



**Example**. Find the volume of the solid $D$ enclosed by $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

**Example**. Evaluate $\iiint_V 2xyz\ dV$ where $V$ is the region bounded by the parabolic cylinder $z = 2 - \frac{1}{2}x^2$ and the planes $x = 0$, $y = x$ and $y = 0$, $z = 0$.



**Example**. Evaluate $\iiint_V xyz \; dV$, where $V$ is the region enclosed by $x^2 + y^2 + z^2 = 1$ and $x \geq 0, z \geq 0, y \geq 0$ and $x = 0, y = 0, z = 0$.

**Case 2. Substitution needed first.**

For $I = \iiint_V f(x, y, z) dxdydz$, the change of variable $x = x(u, v, w), y = (u, v, w), z = z(u, v, w)$, gives,

$$I = \iiint_V f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dudvdw$$

where $J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$ is the Jacobian of the transformation. $V^*$ is the region in $uvw$-space corresponding to the region $V$ in $xyz$-space injectively (one to one) and $J$ must be of one sign in $V^*$.

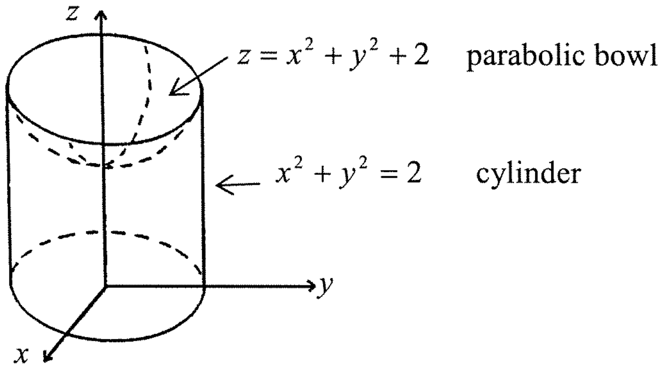
**Example**

$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=\frac{y}{2}+1} \left( \frac{2x-y}{2} + \frac{z}{3} \right) \, dx \, dy \, dz.$$

The most popular alternative coordinate systems to rectangular coordinates are **cylindrical polar coordinates** and **spherical polar coordinates**. They induce the popularly used substitutions: cylindrical substitution and spherical substitution.

**Cylindrical Polar Coordinates v.s. Rectangular Coordinates:**

**Example**. Find the volume $V$ between the surfaces $x^2 + y^2 = 2, z = x^2 + y^2 + 2$ and the plane $z = 0$.



**Example**. $\iiint_V z\, dV$ with $V$ enclosed by $x^2 + y^2 = 4,\ z = x^2 + y^2$ and $xy$-plane.

**Spherical Polar Coordinates:**

**Example**.

Find the volume of the "ice cream cone" D cut from the solid sphere $x^2 + y^2 + z^2 = 1$ by the cone $x^2 + y^2 = 3z^2$ for $z \geq 0$.

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Find an equation for the sphere $x^2 + y^2 + (z-1)^2 = 1$ under spherical coordinates.