4 Continuous Time Fourier Transform

Major References:

- *Chapter 4, Signals and Systems by Alan V. Oppenheim et. al., 2nd edition, Prentice Hall*
- *Chapter 5, Schaum's Outline of Signals and Systems, 2nd Edition, 2010, McGraw-Hill*

Develope Fourier Transform from Fourier Series

Let's consider a nonperiodic signal with finite duration $x(t)$, where $x(t) = 0$ for $|t| > T_1$ and assume that $T_1 < \frac{T_0}{2}$. We introduce a periodic signal $x_{T_0}(t)$ by repeating $x(t)$ with fundamental period T_0 as follows

$$x_{T_0}(t + T_0) = x_{T_0}(t) \quad \text{where} \quad x_{T_0}(t) = x(t) \text{ for } -\frac{T_0}{2} \le t < \frac{T_0}{2}$$

If we let $T_0 \to \infty$, we have the following results by using the complex exponential FS representation

$$\begin{aligned} x(t) &= \lim_{T_0\to\infty} x_{T_0}(t) = \lim_{T_0\to\infty} \sum_{k=-\infty}^{\infty} c_k e^{j2\pi \frac{k}{T_0} t} = \lim_{T_0\to\infty} \sum_{k=-\infty}^{\infty} \left\{ \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_{T_0}(\tau)\, e^{-j2\pi \frac{k}{T_0}\tau} d\tau \right\} e^{j2\pi \frac{k}{T_0} t} \\ &= \lim_{T_0\to\infty} \sum_{k=-\infty}^{\infty} \left\{ \frac{1}{T_0} \int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi \frac{k}{T_0}\tau} d\tau \right\} e^{j2\pi \frac{k}{T_0} t} \end{aligned} \tag{4.1}$$

Since $x_{T_0}(t) = x(t)$ for $-\frac{T_0}{2} \le t < \frac{T_0}{2}$ and $x(t) = 0$ for $|t| > \frac{T_0}{2}$, the last equality of (4.1) follows

$$\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_{T_0}(\tau)\, e^{-jkw_0\tau} d\tau = \int_{-\infty}^{\infty} x(\tau)\, e^{-jkw_0\tau} d\tau$$

Let's denote $f = \frac{k}{T_0}$, then $\lim_{T_0\to\infty} f = \Delta f = \frac{\Delta k}{T_0}$. Since k is an integer, $\Delta k = 1$. By applying $T_0\, \Delta f = 1$ to (4.1),

$$\begin{aligned} x(t) &= \lim_{T_0\to\infty} \sum_{k=-\infty}^{\infty} \left\{ \frac{1}{T_0} \int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi \frac{k}{T_0}\tau} d\tau \right\} e^{j2\pi \frac{k}{T_0} t} \cdot \mathbf{1} = \lim_{T_0\to\infty} \sum_{k=-\infty}^{\infty} \underbrace{\left\{ \int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f\tau} d\tau \right\}}_{\triangleq X(f)} e^{j2\pi f t} \cdot \Delta f \\ &= \lim_{T_0\to\infty} \sum_{T_0 f=-\infty}^{\infty} X(f)\, e^{j2\pi f t} \cdot \Delta f = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t} df, \end{aligned} \tag{4.2}$$

where we denoted $X(f) \triangleq \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t} dt$ and applied the definition of the Riemann Integral.

4.1 Fourier Transform Pair and Properties

Based on (4.1, 4.2), we derived the Fourier Transform pair of an aperiodic continuous time signal $x(t)$ as follows.

1. CT - Fourier Transform Pair

For a non-periodic CT signal $x(t)$, the FT pair in terms of frequence f are given by

- Fourier Transform of $x(t)$; $\mathcal{F}\{x(t)\} = X(f)$
- Inverse Fourier Transform of $X(f)$; $\mathcal{F}^{-1}\{X(f)\} = x(t)$
- Notation; $x(t) \quad \overset{\mathcal{F}}{\underset{\mathcal{F}^{-1}}{\rightleftarrows}} \quad X(f)$

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft} df \quad \overset{\mathcal{F}}{\underset{\mathcal{F}^{-1}}{\rightleftarrows}} \quad X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt \tag{4.3}$$

The FT pair in terms of angular frequence $w = 2\pi f$ are given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\, e^{jwt} dw \quad \overset{\mathcal{F}}{\underset{\mathcal{F}^{-1}}{\rightleftarrows}} \quad X(w) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt} dt \tag{4.4}$$

2. Dirichlet Condition (Sufficient conditions for FT to exist)

1. $x(t)$ is absolutely integrable $\int_{-\infty}^{\infty} |x(t)| dt < \infty$
2. $x(t)$ has a finite number of maxima and minima within any finite interval of t.
3. $x(t)$ has a finite number of discontinuities within any finite interval of t, and each of these discontinuities is finite.

If $x(t)$ satisfies the Dirichlet condition, then the corresponding Fourier transform is convergent.

※ Properties of Fourier Transform

1. **Linear Property**
   If the FT of $x_1(t)$ and $x_2(t)$ are $X_1(f)$ and $X_2(f)$, then the FT of $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ is given by

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \quad \leftrightarrow \quad \alpha_1 X_1(f) + \alpha_2 X_2(f)$$

   Proof) The FT of the linearly combined signals $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ is

$$\begin{aligned}\int_{-\infty}^{\infty} [\alpha_1 x_1(t) + \alpha_2 x_2(t)]\, e^{-j2\pi ft} dt &= \alpha_1 \int_{-\infty}^{\infty} x_1(t) e^{-j2\pi ft} dt + \alpha_2 \int_{-\infty}^{\infty} x_2(t) e^{-j2\pi ft} dt \\ &= \alpha_1 X_1(f) + \alpha_2 X_2(f).\end{aligned}$$

2. **Parameter Shifting**
   If the FT of $x(t)$ is $X(f)$, then the FT of the following mapped signals are given by

$$\mathcal{F}[x(t-t_0)] = e^{-j2\pi f t_0} X(f), \qquad \mathcal{F}\left[x(t)\, e^{\pm j2\pi f_0 t}\right] = X(f \mp f_0)$$

   Proof) The FT of $x(t-t_0)$ and $x(t)\, e^{\pm j2\pi f_0 t}$ are derived as below

$$\begin{aligned}&\int_{-\infty}^{\infty} x(t-t_0)\, e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} x(v)\, e^{-j2\pi f v} dt \cdot e^{-j2\pi f t_0} = X(f)\, e^{-j2\pi f t_0}, \\ &\int_{-\infty}^{\infty} x(t)\, e^{\pm j2\pi f_0 t} e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi(f \mp f_0)t} dt = X(f \mp f_0),\end{aligned}$$

where we used a change of variable ($t - t_0 = v$) in the first equality of the first expression.

3. **Scaling**
If the FT of $x(t)$ is $X(f)$, then the FT of the following scaled signal is given by

$$\mathcal{F}\left[x\left(at\right)\right] = \frac{1}{|a|}X\left(\frac{f}{a}\right), \qquad \mathcal{F}\left[x\left(-t\right)\right] = X\left(-f\right)$$

Proof) The FT of $x\left(at\right)$ is given by

$$\mathcal{F}\left[x\left(at\right)\right] = \int_{-\infty}^{\infty} x\left(at\right)e^{-j2\pi ft}dt = \begin{cases} \frac{1}{a}\int_{-\infty}^{\infty} x\left(\tau\right)e^{-j2\pi f\frac{\tau}{a}}d\tau, & \text{if } a > 0 \\ -\frac{1}{a}\int_{-\infty}^{\infty} x\left(\tau\right)e^{-j2\pi f\frac{\tau}{a}}d\tau, & \text{if } a < 0 \end{cases}$$

4. **Conjugate Property**
If the FT of $x(t)$ is $X(f)$, the FT of $x^*(t)$ for arbitrary signal $x(t)$ is given by

$$\mathcal{F}\left[x^*\left(t\right)\right] = X^*\left(-f\right).$$

If $x(t)$ is a real-valued signal, then $X^*\left(-f\right) = X\left(f\right)$ follows by the conjugate property.
Proof) By taking the complex conjugage of (4.3), we get

$$X^*(f) = \left[\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt\right]^* = \int_{-\infty}^{\infty} x^*(t)e^{j2\pi ft}dt \quad \Rightarrow \quad X^*(-f) = \int_{-\infty}^{\infty} x^*(t)e^{-j2\pi ft}dt$$

5. **Duality**
If the FT of $x(t)$ is $X(f)$, the FT of $X(t)$ can be derived as follows

$$\mathcal{F}\left[X(t)\right] = x(-f)$$

Proof) By using a change of variable ($f' \leftarrow -t$, $t' \leftarrow f$), we get

$$x(t) = \int_{-\infty}^{\infty} X\left(f\right)e^{j2\pi ft}df \quad \Rightarrow \quad x\left(-f'\right) = \int_{-\infty}^{\infty} X\left(t'\right)e^{-j2\pi f't'}dt'$$

6. **Convolution and Multiplication**
If the FT of $x(t)$ and $y(t)$ are $X(f)$ and $Y(f)$, respectively, the FT of the following signals are given by

$$\mathcal{F}\left[x(t) * y(t)\right] = X(f)Y(f), \quad \mathcal{F}\left[x(t)y(t)\right] = X(f) * Y(f)$$

Proof) The FT of the convoltion signal can be derived as

$$\begin{aligned} &\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)y\left(t-\tau\right)d\tau\right]e^{-j2\pi ft}dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f\tau}y\left(t-\tau\right)e^{-j2\pi f(t-\tau)}dtd\tau \\ &= \left[\int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f\tau}d\tau\right]\times\left[\int_{-\infty}^{\infty} y\left(l\right)e^{-j2\pi fl}dl\right] = X(f)Y(f) \end{aligned}$$

where we used a change of variable ($t - \tau = l$) in the second equality. Similarly,

$$\begin{aligned} \mathcal{F}^{-1}\left[X(f) * Y(f)\right] &= \int_{-\infty}^{\infty}\left\{X(f) * Y(f)\right\}e^{j2\pi ft}df = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} X\left(l\right)Y\left(f-l\right)dl\right\}e^{j2\pi ft}df \\ &= \left\{\int_{-\infty}^{\infty} X\left(l\right)e^{j2\pi lt}dl\right\}\times\left\{\int_{-\infty}^{\infty} Y\left(f-l\right)e^{j2\pi(f-l)t}df\right\} = x(t)y(t) \end{aligned}$$

where we used a change of variable ($f - l = f'$) in the last equality.

7. **Differentiation**
   If the FT of $x(t)$ is $X(f)$, the FT of the following signals are given by

$$\mathcal{F}\left[\frac{d^n x(t)}{dt^n}\right] = (j2\pi f)^n X(f), \qquad \mathcal{F}\left[(-j2\pi t)^n x(t)\right] = \frac{d^n X(f)}{df^n}$$

   Proof) By differentiating (4.3), we get

$$\begin{aligned} \frac{d^n x(t)}{dt^n} &= \int_{-\infty}^{\infty} (j2\pi f)^n X(f) e^{j2\pi f t} df = \mathcal{F}^{-1}\left[(j2\pi f)^n X(f)\right], \\ \frac{d^n X(f)}{df^n} &= \int_{-\infty}^{\infty} (-j2\pi t)^n x(t) e^{-j2\pi f t} dt = \mathcal{F}\left[(-j2\pi t)^n x(t)\right] \end{aligned}$$

8. **Integration**
   If the FT of $x(t)$ is $X(f)$, the FT of the following signal is given by

$$\mathcal{F}\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = \frac{1}{2} X(0)\,\delta(f) + \frac{1}{j2\pi f} X(f)$$

   Proof) The integral can be expressed in terms of convoltion as follows

$$\int_{-\infty}^{t} x(\tau)\, d\tau = x(t) * u(t)$$

   Then, by using the convoltion property of the FT, we get

$$\mathcal{F}\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = \mathcal{F}\left[x(t) * u(t)\right] = X(f) \times \mathcal{F}\left[u(t)\right]$$

   where the FT pair of the unit step function is derived in the example problem

$$\mathcal{F}\left[u(t)\right] = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$

   Hence, we obtained the FT of integrated signals as follows

$$\mathcal{F}\left[x(t) * u(t)\right] = X(f) \times \mathcal{F}\left[u(t)\right] = \frac{1}{2} X(0)\delta(f) + \frac{1}{j2\pi f} X(f)$$

9. **Parseval's Theorem**
   If the FT of $x(t)$ is $X(f)$, the Parseval's theorem is

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Property	***Signal***	***Fourier Transform***
	$x(t)=\int_{-\infty}^{\infty}X(f)\mathrm{e}^{j2\pi ft}\mathrm{d}f$ $y(t)=\int_{-\infty}^{\infty}Y(f)\mathrm{e}^{j2\pi ft}\mathrm{d}f$	$X(f)=\int_{-\infty}^{\infty}x(t)\mathrm{e}^{-j2\pi ft}\mathrm{d}t$ $Y(f)=\int_{-\infty}^{\infty}y(t)\mathrm{e}^{-j2\pi ft}\mathrm{d}t$
Linearity	$Ax(t)+By(t)$	$AX(f)+BY(f)$
Time Shift	$x(t-t_0)$	$\mathrm{e}^{-j2\pi ft_0}X(f)$
Frequency Shift	$\mathrm{e}^{j2\pi f_0t}x(t)$	$X(f-f_0)$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{f}{a}\right)$
Duality	$X(t)$	$x(-f)$
Complex Conjugate	$x^*(t)$	$X^*(-f)$
Area	$\mathrm{X}(0)=\int_{-\infty}^{\infty}x(t)\mathrm{d}t$	
	$x(0)=\int_{-\infty}^{\infty}X(f)\mathrm{d}f$	
Time Differentiation	$x^{'}(t)$ $\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n}$	$\mathrm{j}2\pi fX(f)$ $(j2\pi f)^nX(f)$
Freq. Differentiation	$-j2\pi tx(t)$ $(-j2\pi t)^nx(t)$	$X^{'}(f)$ $\frac{\mathrm{d}^n X(f)}{\mathrm{d}f^n}$
Time Integration	$\int_{-\infty}^{t}x(\tau)\mathrm{d}\tau$	$\frac{1}{j2\pi f}X(f),\ \text{for } X(0)=0$
Convolution	$\int_{-\infty}^{\infty}x(\tau)y(t-\tau)\mathrm{d}\tau$	$X(f)Y(f)$
Multiplication	$x(t)y(t)$	$\int_{-\infty}^{\infty}X(\nu)Y(f-\nu)\mathrm{d}\nu$
Energy Conservation (Parseval's Theorem)	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2\mathrm{d}t=\int_{-\infty}^{\infty}\lvert X(f)\rvert^2\mathrm{d}f$	

Figure 4.1: Properties of the Fourier Transform

	$g(t)$	$G(f)$	
	$e^{-at}u(t)$	$\dfrac{1}{a+j2\pi f}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j2\pi f}$	$a>0$
3	$e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2+(2\pi f)^2}$	
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j2\pi f)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j2\pi f)^{n+1}}$	
6	$\delta(t)$	1	
7	1	$\delta(f)$	
8	$e^{j2\pi f_0 t}$	$\delta(f-f_0)$	
9	$\cos 2\pi f_0 t$	$0.5\,[\delta(f+f_0)+\delta(f-f_0)]$	
10	$\sin 2\pi f_0 t$	$j0.5\,[\delta(f+f_0)-\delta(f-f_0)]$	
11	$u(t)$	$\frac{1}{2}\delta(f)+\dfrac{1}{j2\pi f}$	
12	$\operatorname{sgn} t$	$\dfrac{2}{j2\pi f}$	
	$\cos 2\pi f_0 t\, u(t)$	$\frac{1}{4}[\delta(f-f_0)+\delta(f+f_0)]+\dfrac{j2\pi f}{(2\pi f_0)^2-(2\pi f)^2}$	
14	$\sin 2\pi f_0 t\, u(t)$	$\frac{1}{4j}[\delta(f-f_0)-\delta(f+f_0)]+\dfrac{2\pi f_0}{(2\pi f_0)^2-(2\pi f)^2}$	
15	$e^{-at}\sin 2\pi f_0 t\, u(t)$	$\dfrac{2\pi f_0}{(a+j2\pi f)^2+4\pi^2 f_0^2}$	
16	$e^{-at}\cos 2\pi f_0 t\, u(t)$	$\dfrac{a+j2\pi f}{(a+j2\pi f)^2+4\pi^2 f_0^2}$	
	$\operatorname{rect}\left(\frac{t}{\tau}\right)$	$\tau\cdot\operatorname{sinc}(f\tau)$	
18	$\operatorname{sinc}(2Bt)$	$\frac{1}{2B}\operatorname{rect}\left(\frac{t}{2B}\right)$	
19	$\tau\operatorname{tri}\left(\frac{t}{\tau}\right)$	$\tau^2\operatorname{sinc}^2(f\tau)$	
20	$\operatorname{sinc}^2(2Bt)$	$\frac{1}{2B}\operatorname{tri}\left(\frac{f}{2B}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT$	$f_0\sum_{n=-\infty}^{\infty}\delta(f-nf_0)$	
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-2(\sigma\pi f)^2}$	

Figure 4.2: Table of the Fourier Transform

[Example 4-1. Functions related to $\delta(t)$] Derive the FT of the following signals

a) $\mathcal{F}\left[\delta\left(t-t_0\right)\right]=$

b) $\mathcal{F}\left[\delta\left(t\right)\right]=$

c) $\mathcal{F}\left[\,1\,\right]=$

d) $\mathcal{F}\left[e^{j2\pi t f_0}\right]=$

Solution) From (4.3), we can directly calculate (a) and (b) as

$$\text{(a) }\mathcal{F}\left[\delta\left(t-t_0\right)\right]=\int_{-\infty}^{\infty}\delta\left(t-t_0\right)e^{-j2\pi f t}dt=e^{-j2\pi f t_0}\xrightarrow{t_0=0}\text{(b) }\mathcal{F}\left[\delta\left(t\right)\right]=1 \tag{4.5}$$

We can apply duality on (b) to calculate (c), then use parameter shifting (frequency shifting) on (c) to obtain (d).

$$\text{(c) }\mathcal{F}\left[\,1\,\right]=\delta\left(-f\right)=\delta\left(f\right),\quad \text{(d) }\mathcal{F}\left[e^{j2\pi t f_0}\times 1\right]=\delta\left(f-f_0\right) \tag{4.6}$$

[Example 4-2. Functions related to Exponential] Derive the FT of the following signals where $\alpha>0$

a) $\mathcal{F}\left[e^{-\alpha t}u\left(t\right)\right]=$

b) $\mathcal{F}\left[e^{-\alpha t}u\left(t\right)-e^{\alpha t}u\left(-t\right)\right]=$

c) $\mathcal{F}\left[sgn(t)\right]=\mathcal{F}\left[2u(t)-1\right]=$

d) $\mathcal{F}\left[u(t)\right]=$

e) $\mathcal{F}\left[e^{-\alpha|t|}\right]=$

f) $\mathcal{F}\left[\frac{1}{\alpha^2+4(\pi t)^2}\right]=$

Solution) The FT of the one sided decaying exponential function in (a) can be derived as

$$\text{(a) }\mathcal{F}\left[e^{-\alpha t}u\left(t\right)\right]=\int_{0}^{\infty}e^{-\alpha t}e^{-j2\pi f t}dt=\frac{1}{\alpha+j2\pi f} \tag{4.7}$$

The functions in (b) and (e) are referred as *odd two-sided exponential* and *even two-sided exponential signal*, respectively.

$$\begin{aligned}\text{(b) }\mathcal{F}\left[e^{-\alpha t}u\left(t\right)-e^{\alpha t}u\left(-t\right)\right]&=\int_{0}^{\infty}e^{-\alpha t}e^{-j2\pi f t}dt-\int_{-\infty}^{0}e^{\alpha t}e^{-j2\pi f t}dt\\&=\frac{1}{\alpha+j2\pi f}-\frac{1}{\alpha-j2\pi f}=-\frac{j4\pi f}{\alpha^2+4\left(\pi f\right)^2},\\ \text{(e) }\mathcal{F}\left[e^{-\alpha|t|}\right]&=\int_{0}^{\infty}e^{-\alpha t}e^{-j2\pi f t}dt+\int_{-\infty}^{0}e^{\alpha t}e^{-j2\pi f t}dt\\&=\frac{1}{\alpha+j2\pi f}+\frac{1}{\alpha-j2\pi f}=\frac{2\alpha}{\alpha^2+4\left(\pi f\right)^2}\end{aligned} \tag{4.8}$$

The signum function $sgn(t)$ in (c) can be approximated by odd two-sided exponential signal in (b) by $\alpha\longrightarrow 0$ and the unit step function $u(t)$ in (d) can be derived from the signum function as $u(t)=\frac{1}{2}+\frac{1}{2}sgn(t)$.

$$\begin{aligned}\text{(c) }\mathcal{F}\left[sgn(t)\right]&=\lim_{\alpha\to 0}\mathcal{F}\left[e^{-\alpha t}u\left(t\right)-e^{\alpha t}u\left(-t\right)\right]=\lim_{\alpha\to 0}\left[-\frac{j4\pi f}{\alpha^2+4\left(\pi f\right)^2}\right]=\frac{1}{j\pi f},\\ \text{(d) }\mathcal{F}\left[u(t)\right]&=\mathcal{F}\left[\frac{1}{2}+\frac{1}{2}sgn(t)\right]=\frac{1}{2}\mathcal{F}\left[\,1\,\right]+\frac{1}{2}\mathcal{F}\left[sgn(t)\right]=\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}\end{aligned} \tag{4.9}$$

The FT of (f) can be derived by using the duality property with (e).

$$\text{(f) }\mathcal{F}\left[\frac{1}{2\alpha}e^{-\alpha|t|}\right]=\frac{1}{\alpha^2+4\left(\pi f\right)^2}\xrightarrow{Duality}\mathcal{F}\left[\frac{1}{\alpha^2+4\left(\pi t\right)^2}\right]=\frac{1}{2\alpha}e^{-\alpha|f|} \tag{4.10}$$

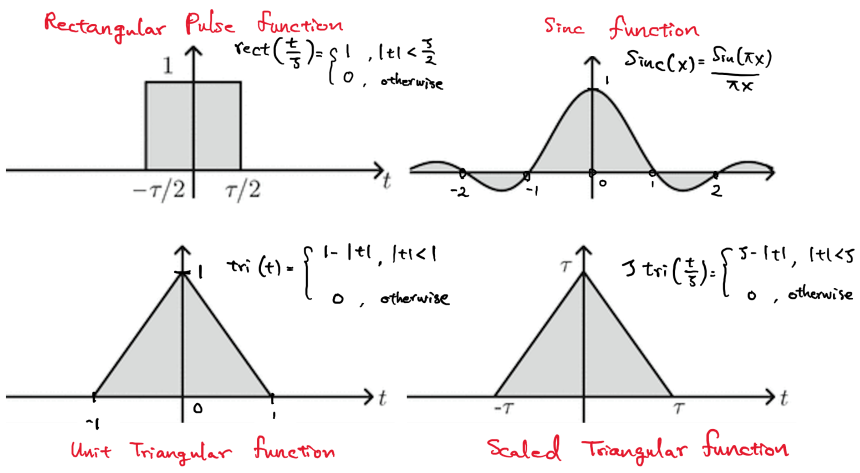
[Example 4-3. Functions related to Rectangular Pulse] Derive the FT of the following signals where $\tau > 0$

a) $\mathcal{F}\left[\mathrm{rect}\left(\frac{t}{\tau}\right)\right] =$ b) $\mathcal{F}\left[\mathrm{sinc}\left(2Bt\right)\right] =$

c) $\mathcal{F}\left[\tau\,\mathrm{tri}\left(\frac{t}{\tau}\right)\right] =$ d) $\mathcal{F}\left[\mathrm{sinc}^2\left(2Bt\right)\right] =$

e) $\mathcal{F}\left[\mathrm{rect}\left(\frac{t-t_0}{\tau}\right)\right] =$ f) $\mathcal{F}\left[\mathrm{sinc}\left(2B\left(t-t_0\right)\right)\right] =$

where the rectangular pulse $rect(t)$, Sinc function $sinc(t)$, and triangular function $tri(t)$ are illustrated below.



Solution) The FT of the rect(t) in (a) can be directly derived using (4.3)

$$\text{(a) } \mathcal{F}\left[\mathrm{rect}\left(\frac{t}{\tau}\right)\right] = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-j2\pi f t}dt = \frac{1}{j2\pi f}e^{-j2\pi f t}\bigg|_{\frac{\tau}{2}}^{-\frac{\tau}{2}} = \frac{2j\sin\left(\pi f\tau\right)}{j2\pi f} = \tau\cdot\mathrm{sinc}\left(f\tau\right).$$

The y-intercept of the Sinc function is 1 and the zero-crossing occurs at every integer t, so τ sinc $(f\tau)$ has a maximum value τ at $f = 0$ with zero-crossing at $f = \frac{n}{\pi}$ for any nonzero integer n.

$$\mathrm{sinc}(t) = \frac{\sin\left(\pi t\right)}{\pi t} \Rightarrow 0 \quad \text{for } t = \text{ arbitrary nonzero integer,} \quad \lim_{t\to 0}\left[\mathrm{sinc}(t)\right] = 1$$

The FT of (b) can be derived by using the duality property with (a) as follows

$$\text{(b) } \frac{1}{\tau}\,\mathrm{rect}\left(\frac{t}{\tau}\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathrm{sinc}\left(f\tau\right), \quad \xRightarrow{\text{Duality}} \quad \mathrm{sinc}\left(2Bt\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} \frac{1}{2B}\,\mathrm{rect}\left(\frac{f}{2B}\right)$$

where we subtituted $\tau = 2B$ and used a change of variable $(t \to -f', f \to t')$. Hence, $\mathcal{F}\left[\mathrm{sinc}\left(2Bt\right)\right] = \frac{1}{2B}\,\mathrm{rect}\left(\frac{f}{2B}\right)$. In Example 2-1. (b) (pp26), we proved that the convoltion of two rectangular pulse signal is a triangluar function

$$\mathrm{rect}\left(\frac{t}{\tau}\right) * \mathrm{rect}\left(\frac{t}{\tau}\right) = \tau\cdot\mathrm{tri}\left(\frac{t}{\tau}\right) = \begin{cases}\tau - |t| & \text{for } |t| \le \tau \\ 0 & \text{for } |t| > \tau\end{cases}$$

Then, the FT of (c) can be obtained by convoltion properties and (d) can be derived based on the duality with (c).

$$\text{(c) } \mathcal{F}\left[\tau\,\mathrm{tri}\left(\frac{t}{\tau}\right)\right] = \mathcal{F}\left[\mathrm{rect}\left(\frac{t}{\tau}\right) * \mathrm{rect}\left(\frac{t}{\tau}\right)\right] = \mathcal{F}\left[\mathrm{rect}\left(\frac{t}{\tau}\right)\right] \times \mathcal{F}\left[\mathrm{rect}\left(\frac{t}{\tau}\right)\right] = \tau^2\,\mathrm{sinc}^2\left(f\tau\right),$$

$$\text{(d) } \frac{1}{\tau}\,\mathrm{tri}\left(\frac{t}{\tau}\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathrm{sinc}^2\left(f\tau\right), \quad \xRightarrow{\text{Duality}} \quad \mathrm{sinc}^2\left(2Bt\right) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} \frac{1}{2B}\,\mathrm{tri}\left(\frac{f}{2B}\right)$$

where we subtituted $\tau = 2B$ and used a change of variable $(t \to -f', f \to t')$. Hence, $\mathcal{F}\left[\mathrm{sinc}^2(2Bt)\right] = \frac{1}{2B}\,\mathrm{tri}\left(\frac{f}{2B}\right)$. The FT of (e) and (f) can be obtained by using the time-shift property of (a) and frequency-shift of (b), respectively.

$$\text{(e) } \mathcal{F}\left[\mathrm{rect}\left(\frac{t-t_0}{\tau}\right)\right] = \tau e^{-j2\pi f t_0}\mathrm{sinc}(f\tau), \quad \text{(f) } \mathcal{F}\left[\mathrm{sinc}\left(2B\left(t-t_0\right)\right)\right] = \frac{1}{2B}e^{-j2\pi f t_0}\,\mathrm{rect}\left(\frac{t}{2B}\right)$$

[Example 4-4. Functions related to Modulation] Derive the FT of the following signals where $\alpha > 0$

a) $\mathcal{F}\left[\cos(2\pi f_0 t)\right] =$

b) $\mathcal{F}\left[\sin(2\pi f_0 t)\right] =$

c) $\mathcal{F}\left[x(t)\cos(2\pi f_0 t)\right] =$

d) $\mathcal{F}\left[x(t)\sin(2\pi f_0 t)\right] =$

e) $\mathcal{F}\left[\cos(2\pi f_0 t)\,u(t)\right] =$

f) $\mathcal{F}\left[\sin(2\pi f_0 t)\,u(t)\right] =$

g) $\mathcal{F}\left[e^{-\alpha t}\cos(2\pi f_0 t)\,u(t)\right] =$

h) $\mathcal{F}\left[e^{-\alpha t}\sin(2\pi f_0 t)\,u(t)\right] =$

Solution) The FT of (a) and (b) can be derived by using Example 4-1 (d) and linear property as follows

$$\text{(a) } \mathcal{F}\left[\cos(2\pi f_0 t)\right] = \mathcal{F}\left[\frac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)\right] = \frac{1}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right],$$

$$\text{(b) } \mathcal{F}\left[\sin(2\pi f_0 t)\right] = \mathcal{F}\left[\frac{1}{2j}\left(e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right)\right] = -\frac{j}{2}\left[\delta(f-f_0) - \delta(f+f_0)\right]$$

The FT of (c) and (d) can be obtained based on the multiplication property of the FT with (a) and (b).

$$\text{(c) } \mathcal{F}\left[x(t)\cos(2\pi f_0 t)\right] = X(f) * \frac{1}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right] = \frac{1}{2}\left[X(f-f_0) + X(f+f_0)\right],$$

$$\text{(d) } \mathcal{F}\left[x(t)\sin(2\pi f_0 t)\right] = X(f) * \frac{1}{2j}\left[\delta(f-f_0) - \delta(f+f_0)\right] = -\frac{j}{2}\left[X(f-f_0) - X(f+f_0)\right]$$

The FT of (e) and (f) can be obtained by using Example 4-2, *i.e.*, $\mathcal{F}\left[u(t)\right] = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$

$$\begin{aligned}
\text{(e) } \mathcal{F}\left[\cos(2\pi f_0 t)\,u(t)\right] &= \frac{1}{2}\left[\frac{1}{2}\delta(f-f_0) + \frac{1}{j2\pi(f-f_0)} + \frac{1}{2}\delta(f+f_0) + \frac{1}{j2\pi(f+f_0)}\right] \\
&= \frac{1}{4}\left[\delta(f-f_0) + \delta(f+f_0)\right] + \frac{j2\pi f}{(2\pi f_0)^2 - (2\pi f)^2}, \\
\text{(f) } \mathcal{F}\left[\sin(2\pi f_0 t)\,u(t)\right] &= \frac{1}{2j}\left[\frac{1}{2}\delta(f-f_0) + \frac{1}{j2\pi(f-f_0)} - \frac{1}{2}\delta(f+f_0) - \frac{1}{j2\pi(f+f_0)}\right] \\
&= \frac{1}{4j}\left[\delta(f-f_0) - \delta(f+f_0)\right] + \frac{2\pi f_0}{(2\pi f_0)^2 - (2\pi f)^2}
\end{aligned}$$

The FT of (g) and (h) can be obtained by the multiplication property with $\mathcal{F}\left[e^{-\alpha t}u(t)\right] = (\alpha + j2\pi f)^{-1}$

$$\begin{aligned}
\text{(g) } \mathcal{F}\left[e^{-\alpha t}\cos(2\pi f_0 t)\,u(t)\right] &= \frac{1}{\alpha + j2\pi f} * \frac{1}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right] \\
&= \frac{1}{2}\left[\frac{1}{\alpha + j2\pi(f-f_0)} + \frac{1}{\alpha + j2\pi(f+f_0)}\right] = \frac{\alpha + j2\pi f}{(\alpha + j2\pi f)^2 + 4(\pi f_0)^2}, \\
\text{(h) } \mathcal{F}\left[e^{-\alpha t}\sin(2\pi f_0 t)\,u(t)\right] &= \frac{1}{\alpha + j2\pi f} * \frac{1}{2j}\left[\delta(f-f_0) - \delta(f+f_0)\right] \\
&= \frac{1}{2j}\left[\frac{1}{\alpha + j2\pi(f-f_0)} - \frac{1}{\alpha + j2\pi(f+f_0)}\right] = \frac{2\pi f_0}{(\alpha + j2\pi f)^2 + 4(\pi f_0)^2}
\end{aligned}$$

[Example 4-5. etc] Derive the FT of the following signals where $\alpha > 0$

1. FT of a periodic signal $x(t)$ with period T_0
2. FT of the periodic impulse train $\mathcal{F}\left[\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)\right]$
3. $\mathcal{F}\left[t^n e^{-\alpha t} u(t)\right]$

Solution) In (a), we first express a periodic signal $x(t)$ using the FS, then apply $\mathcal{F}\left[e^{j2\pi t f_0}\right] = \delta(f - f_0)$ as follows

$$\text{(a) } \mathcal{F}[x(t)] = \mathcal{F}\left[\sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t}\right] = \sum_{k=-\infty}^{\infty} c_k \mathcal{F}\left[e^{jkw_0 t}\right] = \sum_{k=-\infty}^{\infty} c_k \delta\left(f - \frac{kw_0}{2\pi}\right),$$

which indicates that the Fourier transform of a periodic signal consists of a sequence of equidistant impulses located at the harmonic frequencies. For (b), the FS of the periodic impulse train $\delta_{T_0}(t)$ is derived in Example 3-3, pp42

$$\delta_{T_0}(t) = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} e^{jkw_0 t} \quad \Rightarrow \quad \text{(b) } \mathcal{F}\left[\sum_{k=-\infty}^{\infty} \delta(t - kT_0)\right] = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} \mathcal{F}\left[e^{jkw_0 t}\right] = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} \delta\left(f - \frac{kw_0}{2\pi}\right)$$

The FT of (c) can be derived by using $\mathcal{F}\left[e^{-\alpha t}u(t)\right] = \frac{1}{\alpha + j2\pi f}$ and the higher order differentiation of the FT

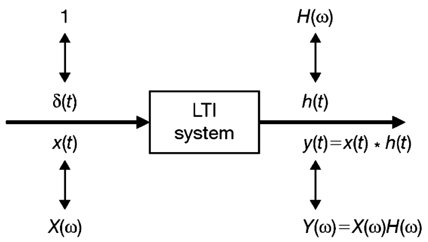
$$\mathcal{F}\left[t^n x(t)\right] = \frac{1}{(-j2\pi)^n}\frac{d^n X(f)}{df^n} \text{ where } x(t) = e^{-\alpha t}u(t),\ X(f) = \frac{1}{\alpha + j2\pi f} \Rightarrow \text{(c) } \mathcal{F}\left[t^n e^{-\alpha t}u(t)\right] = \frac{n!}{(\alpha + j2\pi f)^{n+1}}$$

4.2 Frequency Response

Given the FT of an input $x(t)$ and the impulse response $h(t)$ as $X(f)$ and $H(f)$, the output $y(t)$ of an LTI system is

$$\begin{aligned} y(t) &= h(t) * x(t) = \mathcal{F}^{-1}(Y(f)) = \mathcal{F}^{-1}(X(f)H(f)) \\ &= \int_{-\infty}^{\infty} Y(f)e^{j2\pi f t}df = \int_{-\infty}^{\infty} X(f)H(f)e^{j2\pi f t}df \quad \text{for Aperiodic input} \end{aligned} \tag{4.11}$$

where we refer the FT of the impulse response $H(f)$ as the *Frequency Response*, the magnitude of the frequency response $|H(f)|$ as the *Magnitude Response*, the phase of the frequency response $\theta_H(f)$ as the *Phase response*.



- Frequency Response:

$$H(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi f t}dt = |H(f)|e^{j\theta_H(f)}$$

- Magnitude Response: $|H(f)|$
- Phase response: $\theta_H(f)$

The output in (4.11) holds for a non-periodic input signal $x(t)$. Given a periodic input $x(t)$,

- If $x(t) = e^{jkw_0 t}$, then $\mathcal{F}(x(t)) = \delta(f - kf_0)$, where $f_0 = \frac{w_0}{2\pi}$, and

$$Y(f) = H(f)X(f) = H(f)\delta(f - kf_0) = H(kf_0)\delta(f - kf_0)$$

- If $x(t)=\sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t}$, then $\mathcal{F}(x(t))=\sum_{k=-\infty}^{\infty} c_k \delta(f-kf_0)$ and

$$Y(f)=H(f)X(f)=\sum_{k=-\infty}^{\infty} c_k H(f)\delta(f-kf_0)=\sum_{k=-\infty}^{\infty} c_k c_k H(kf_0)\delta(f-kf_0) \quad \text{for periodic input}$$

※ LTI Systems Characterized by Differential Equations

If the LTI system is described by differential equations, we can derive the Frequency response using the differentiation property of the Fourier transform as follows

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}=\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \quad \overset{\mathcal{F}}{\Longrightarrow} \quad \sum_{k=0}^{N} a_k \mathcal{F}\left\{\frac{d^k y(t)}{dt^k}\right\}=\sum_{k=0}^{M} b_k \mathcal{F}\left\{\frac{d^k x(t)}{dt^k}\right\}, \quad M\le N$$

$$\Rightarrow \quad H(f)=\frac{Y(f)}{X(f)}=\frac{\sum_{k=0}^{M} b_k (j2\pi f)^k}{\sum_{k=0}^{N} a_k (j2\pi f)^k} \quad \Leftrightarrow \quad h(t)=\mathcal{F}^{-1}(H(f))$$

[Example 4-6.] Consider CT-LTI systems described by the following differential equations. Derive the frequency response $H(f)$ and the corresponding impulse response $h(t)$. Furthermore, derive the system output when the input signal is $x_1(t)=e^{-t}u(t)$, $x_2(t)=te^{-2t}u(t)$, and $x_3(t)=u(t)$, respectively.

a) $\frac{dy(t)}{dt}+2y(t)=x(t)+\frac{dx(t)}{dt} \quad \Rightarrow \quad H(f),\ h(t),\ y_i(t)=\mathbb{T}\{x_i(t)\}$, where $i=\{1,2,3\}$

b) $\frac{dy(t)}{dt}+2y(t)=x(t) \quad \Rightarrow \quad H(f),\ h(t),\ y_i(t)=\mathbb{T}\{x_i(t)\}$, where $i=\{1,2,3\}$

c) $\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+3y(t)=\frac{dx(t)}{dt}+2x(t) \quad \Rightarrow \quad H(f),\ h(t),\ y_i(t)=\mathbb{T}\{x_i(t)\}$, where $i=\{1,2,3\}$

d) $\frac{d^2y(t)}{dt^2}+6\frac{dy(t)}{dt}+8y(t)=2x(t) \quad \Rightarrow \quad H(f),\ h(t),\ y_i(t)=\mathbb{T}\{x_i(t)\}$, where $i=\{1,2,3\}$

Solution) Since $\mathcal{F}\left[\frac{d^n x(t)}{dt^n}\right]=(j2\pi f)^n X(f)$, the FT of each differential equations can be expressed as

$$\text{(a)}\ \mathcal{F}\left\{\frac{dy(t)}{dt}+2y(t)=x(t)+\frac{dx(t)}{dt}\right\} \quad \Leftrightarrow \quad (j2\pi f)Y(f)+2Y(f)=X(f)+(j2\pi f)X(f),$$

$$\Rightarrow \quad H(f)=\frac{Y(f)}{X(f)}=\frac{1+j2\pi f}{2+j2\pi f}=1-\frac{1}{1+j2\pi f} \quad \overset{\mathcal{F}^{-1}}{\Longrightarrow} \quad h(t)=\delta(t)-e^{-2t}u(t),$$

where we used the FT pairs in Table 4.2. The FT of each input signals are given by

$$X_1(f)=\mathcal{F}\left\{e^{-t}u(t)\right\}=\frac{1}{1+j2\pi f}, \quad X_2(f)=\mathcal{F}\left\{te^{-2t}u(t)\right\}=\frac{1}{(2+j2\pi f)^2},$$

$$X_3(f)=\mathcal{F}\{u(t)\}=\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}.$$

Since the FT of the system output is given by $Y_i(f)=\mathcal{F}(x_i(t)*h(t))=X_i(f)H(f)$, the output for each signals are

$$Y_1(f)=\frac{1}{2+j2\pi f} \quad \Rightarrow \quad y_1(t)=e^{-2t}u(t),$$

$$Y_2(f)=\frac{1+j2\pi f}{(2+j2\pi f)^3}=\frac{1}{(2+j2\pi f)^2}-\frac{1}{(2+j2\pi f)^3} \quad \Rightarrow \quad y_2(t)=te^{-2t}u(t)-\frac{1}{2}t^2e^{-2t}u(t),$$

$$Y_3(f)=\frac{1}{4}\delta(f)+\frac{1+j2\pi f}{j2\pi f(2+j2\pi f)}=\frac{1}{2}\left[\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}\right]+\frac{1}{2}\left(\frac{1}{2+j2\pi f}\right) \quad \Rightarrow \quad y_3(t)=\frac{1}{2}u(t)+\frac{1}{2}e^{-2t}u(t),$$

where we applied the Partial Fraction Expansion. *Refer [Oppenheim, Appendix], [*`https://bit.ly/3aND12K`*], [*`https://bit.ly/3cSXtCi`*].*

For system (b), we can derive the Frequency response using similar approach

$$\text{(b) } \mathcal{F}\left\{\frac{dy(t)}{dt} + 2y(t) = x(t)\right\} \quad \Leftrightarrow \quad (j2\pi f)\, Y(f) + 2Y(f) = X(f),$$

$$\Rightarrow \quad H(f) = \frac{Y(f)}{X(f)} = \frac{1}{2 + j2\pi f} \quad \xrightarrow{\mathcal{F}^{-1}} \quad h(t) = e^{-2t}u\,(t)\,,$$

where we used the FT pairs in Table 4.2. The system output for each input signals are derived as

$$\begin{aligned}
Y_1(f) &= \frac{1}{(1+j2\pi f)\,(2+j2\pi f)} = \frac{1}{1+j2\pi f} - \frac{1}{2+j2\pi f} \quad \Rightarrow \quad y_1(t) = e^{-t}u(t) - e^{-2t}u(t),\\
Y_2(f) &= \frac{1}{(2+j2\pi f)^3} \quad \Rightarrow \quad y_2(t) = \frac{1}{2}t^2e^{-2t}u(t),\\
Y_3(f) &= \frac{1}{4}\delta(f) + \frac{1}{j2\pi f\,(2+j2\pi f)} = \frac{1}{2}\left(\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right) - \frac{1}{2}\left(\frac{1}{2+j2\pi f}\right) \quad \Rightarrow \quad y_3(t) = \frac{1}{2}u(t) - \frac{1}{2}e^{-2t}u(t).
\end{aligned}$$

For system (c), the Frequency response and impulse response are derived as follows

$$\text{(c) } \mathcal{F}\left\{\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)\right\} \quad \Leftrightarrow \quad \left\{(j2\pi f)^2 + 4\,(j2\pi f) + 3\right\} Y(f) = \{j2\pi f + 2\}\, X(f),$$

$$\Rightarrow \quad H(f) = \frac{Y(f)}{X(f)} = \frac{2+j2\pi f}{(j2\pi f)^2 + 4\,(j2\pi f) + 3} = \frac{1/2}{3+j2\pi f} + \frac{1/2}{1+j2\pi f} \xrightarrow{\mathcal{F}^{-1}} h(t) = \frac{1}{2}e^{-3t}u(t) + \frac{1}{2}e^{-t}u(t),$$

where we used the FT pairs in Table 4.2. The system output for each input signals are derived as

$$\begin{aligned}
& Y_1(f) = \frac{2+j2\pi f}{(1+j2\pi f)^2\,(3+j2\pi f)} = \frac{1/4}{1+j2\pi f} + \frac{1/2}{(1+j2\pi f)^2} + \frac{-1/4}{3+j2\pi f}\\
\xrightarrow{\mathcal{F}^{-1}} \quad & y_1(t) = \frac{1}{4}e^{-t}u(t) + \frac{1}{2}te^{-t}u(t) - \frac{1}{4}e^{-3t}u(t),\\
& Y_2(f) = \frac{1}{(1+j2\pi f)\,(2+j2\pi f)\,(3+j2\pi f)} = \frac{1/2}{1+j2\pi f} + \frac{-1}{2+j2\pi f} + \frac{1/2}{3+j2\pi f}\\
\xrightarrow{\mathcal{F}^{-1}} \quad & y_2(t) = \frac{1}{2}e^{-t}u(t) - e^{-2t}u(t) + \frac{1}{2}e^{-3t}u(t),\\
& Y_3(f) = \frac{1}{3}\delta(f) + \frac{2+j2\pi f}{j2\pi f\,(1+j2\pi f)\,(3+j2\pi f)} = \frac{1}{3}\delta(f) + \frac{2/3}{j2\pi f} + \frac{-1/2}{1+j2\pi f} + \frac{-1/6}{3+j2\pi f}\\
& = \frac{2}{3}\left(\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right) + \frac{-1/2}{1+j2\pi f} + \frac{-1/6}{3+j2\pi f}\\
\xrightarrow{\mathcal{F}^{-1}} \quad & y_3(t) = \frac{2}{3}u(t) - \frac{1}{2}e^{-t}u(t) - \frac{1}{6}e^{-3t}u(t).
\end{aligned}$$

For system (d), we can derive the Frequency response using similar approach

$$\text{(d) } \mathcal{F}\left\{\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)\right\} \quad \Leftrightarrow \quad \left\{(j2\pi f)^2 + 6\,(j2\pi f) + 8\right\} Y(f) = 2X(f),$$

$$\Rightarrow \quad H(f) = \frac{Y(f)}{X(f)} = \frac{2}{(j2\pi f)^2 + 6\,(j2\pi f) + 8} = \frac{2}{(4+j2\pi f)\,(2+j2\pi f)} = \frac{1}{2+j2\pi f} + \frac{-1}{4+j2\pi f}$$

$$\xrightarrow{\mathcal{F}^{-1}} h(t) = e^{-2t}u(t) - e^{-4t}u(t)$$

where we used the FT pairs in Table 4.2. The system output for each input signals are derived as

$$
\begin{aligned}
&Y_1(f) = \frac{2}{(1+j2\pi f)(2+j2\pi f)(4+j2\pi f)} = \frac{2/3}{1+j2\pi f} + \frac{-1}{2+j2\pi f} + \frac{1/3}{4+j2\pi f} \\
\xrightarrow{\mathcal{F}^{-1}} \quad &y_1(t) = \frac{2}{3}e^{-t}u(t) - e^{-2t}u(t) + \frac{1}{3}e^{-4t}u(t), \\
&Y_2(f) = \frac{2}{(2+j2\pi f)^3(4+j2\pi f)} = \frac{1/4}{2+j2\pi f} + \frac{-1/2}{(2+j2\pi f)^2} + \frac{1}{(2+j2\pi f)^3} + \frac{-1/4}{4+j2\pi f} \\
\xrightarrow{\mathcal{F}^{-1}} \quad &y_2(t) = \frac{1}{4}e^{-2t}u(t) - \frac{1}{2}te^{-2t}u(t) + \frac{1}{2}t^2e^{-2t}u(t) - \frac{1}{4}e^{-4t}u(t), \\
&Y_3(f) = \frac{1}{8}\delta(f) + \frac{2}{j2\pi f(2+j2\pi f)(4+j2\pi f)} = \frac{1}{4}\left(\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right) + \frac{-1/2}{2+j2\pi f} + \frac{1/4}{4+j2\pi f} \\
\xrightarrow{\mathcal{F}^{-1}} \quad &y_3(t) = \frac{1}{4}u(t) - \frac{1}{2}e^{-2t}u(t) + \frac{1}{4}e^{-4t}u(t).
\end{aligned}
$$

4.3 Nyquist Sampling

Given a CT signal $x(t)$, if we measure its values every T time units, then we get the following samples

$$\cdots, x(-2T), x(-T), x(0), x(T), x(2T), \cdots$$

This process is called Sampling.

$$\text{Sampling period: } T, \quad \text{Sampling frequency: } f_s = \frac{1}{T}$$

In general, some information is lost in this process if the signal is not bandlimited (the bandwidth is not finite).

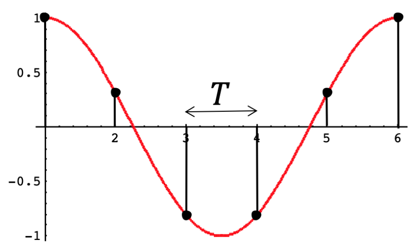
Sampling Theorem

Let $x(t)$ be a band-limited signal with $X(f) = 0$ for $|f| > f_M$. Then $x(t)$ is uniquely determined by its samples $\{x(-kT), \text{ arbitrary integer} k\}$ if the following condition is satisfied

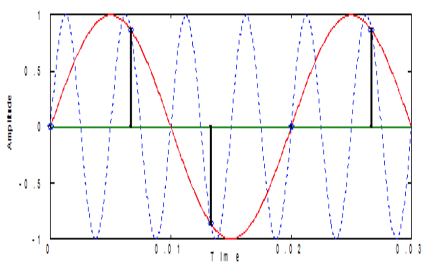
$$f_s \geq 2f_M, \tag{4.12}$$

where f_M, f_s, $2f_M$ are the signal bandwidth, sampling frequency, *Nyquist rate* (minimum f_s), respectively.

※ **Intuitive Explanation:** Sampling theorem in (4.12) indicates that the sampling rate should be higher than the frequency content of the signal. Otherwise, the recovered signal after sampling will suffer significant distortion.



Sampling rate is high enough.



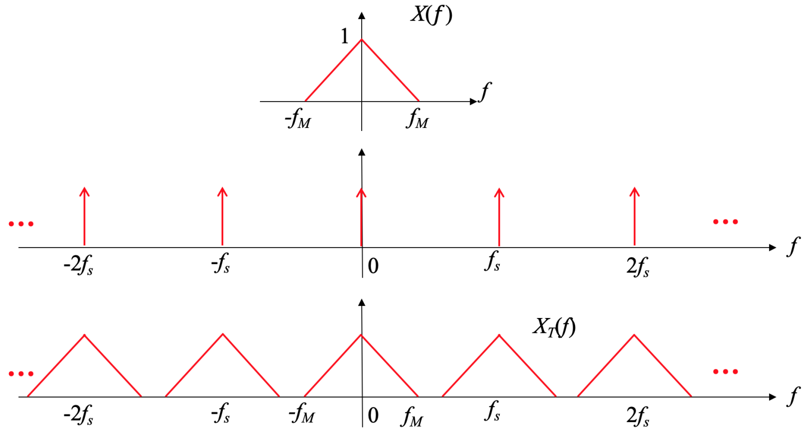
Sampling rate is too low!

※ **Formal Explanation:** Analytically, Sampling process is multiplication of the bandlimited signal $x(t)$ and the periodic impulse train $\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$, which we will denote as $x_T(t)$

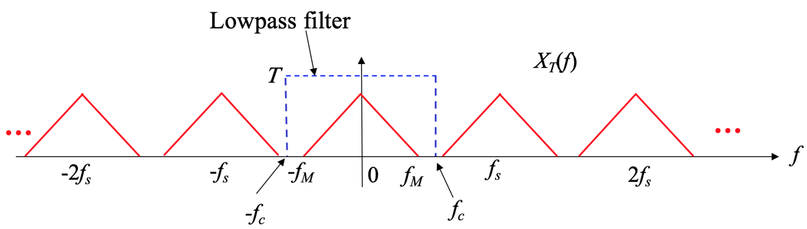
$$x_T(t) = x(t)\delta_T(t) = \sum_{k=-\infty}^{\infty} x(kT)\,\delta(t-kT)\,.$$

By using the multiplication property, the FT of the sampled signal $x_T(t)$ is obtained as follows

$$X_T(f) = \mathcal{F}(x(t)) * \mathcal{F}(\delta_T(t)) = X(f) * \frac{1}{T}\sum_{k=-\infty}^{\infty} \delta(f-kf_s) = f_s \sum_{k=-\infty}^{\infty} X(f-kf_s)\,, \quad f_s \triangleq \frac{1}{T}$$



The shifted replicas do not overlap if and only if $f_s \geq 2f_M$. If (4.12) is satisfied, then $x(t)$ can be exactly recovered from $x_T(t)$ by using an ideal lowpass filter with amplitude $\frac{1}{f_s}$ and cutoff frequency f_c where $f_M \leq f_c \leq f_s - f_M$. In other words, ideal lowpass filter (or equivalently, sinc interpolation) can exactly reproduce the original signal.



[Example 4-7] Find the Nyquist sampling rate for the following signals

a) $x(t) = 2\cos(200\pi t) + \sin(500\pi t)$ b) $x(t) = 2\cos^2(200\pi t)$

Solution) In (a), the signal bandwidth for $\cos(200\pi t)$ and $\sin(500\pi t)$ are respectively given by

$$f_1 = \frac{200\pi}{2\pi} = 100, \quad f_2 = \frac{500\pi}{2\pi} = 250 \Rightarrow f_M = 250$$

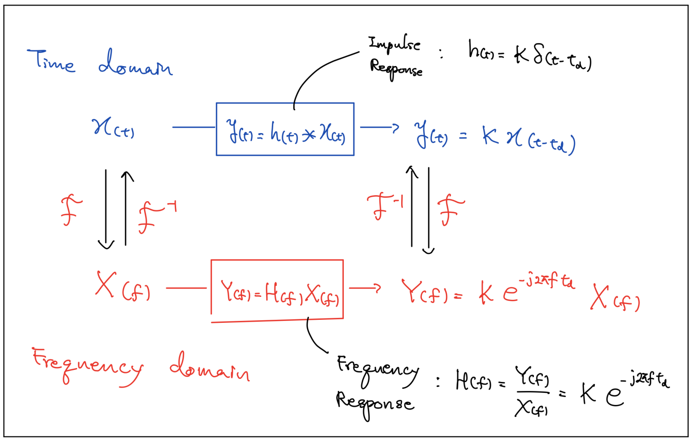
and the bandwidth of the $x(t)$ is $f_M = 250$. Hence the Nyquist sampling rate for (a) is $2f_M = 500$. In (b), $2\cos^2(200\pi t) = 1 + \cos(400\pi t)$ whose signal bandwidth is $f_M = \frac{400\pi}{2\pi} = 200$. Hence the Nyquist sampling rate for (b) is $2f_M = 400$.

4.4 Filtering

※ **Distortionless Transmission:** In order to have a distortionless transmission, the output of an LTI system needs to have an identical shape to the input signal, although its amplitude may be different and it may be delayed in time.

$$\text{Input signal } x(t) \quad \Rightarrow \quad \text{Output Signal } y(t) = Kx\,(t - t_d)\,,$$

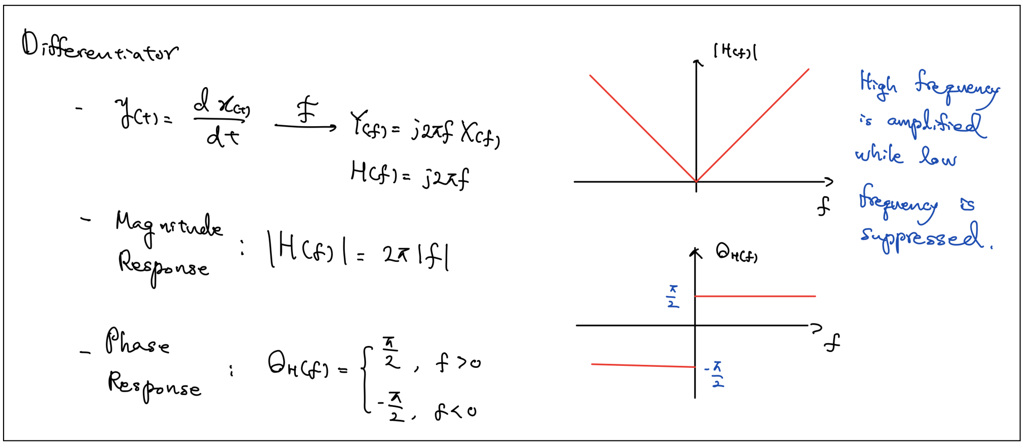
where t_d is the time delay, K is a gain constant. By using the FT, we can obtain the Frequency response as follows



Hence, the systems with distortionless transmission should have Frequency response $H(f) = Ke^{-j2\pi f t_d}$, constant magnitude response $|H(f)| = K$, and linear phase response $\theta_H(f) = -j2\pi f t_d$ over the entire frequency range.

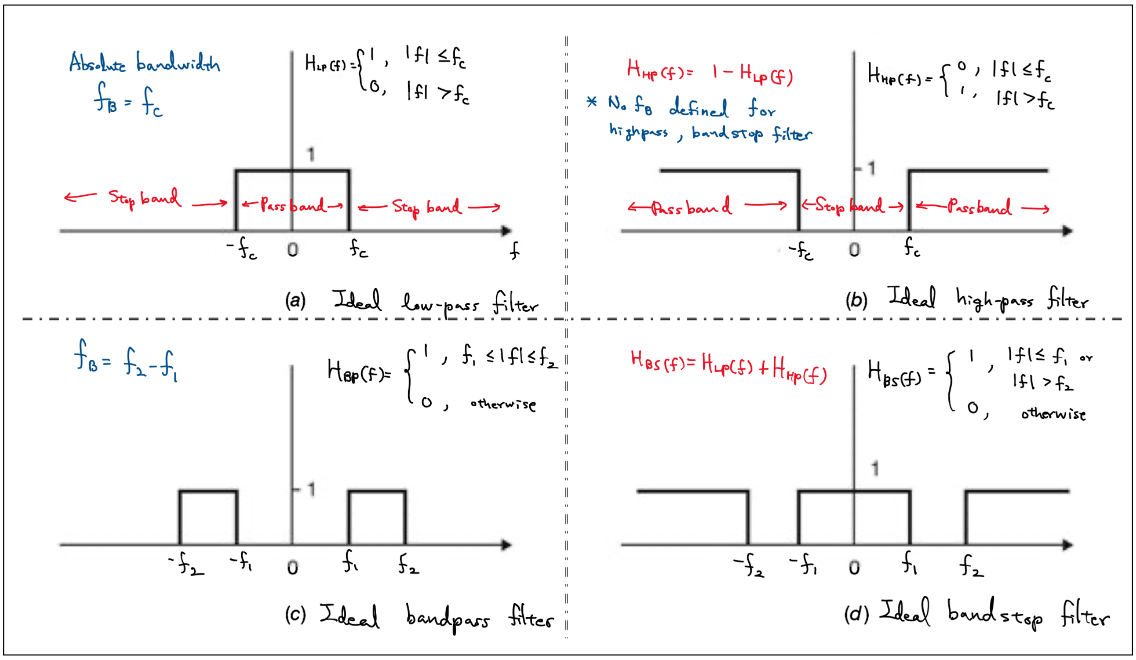
※ **Filtering** is a process that changes the amplitude (or phase) of some frequency components of an input signal.

1. Frequency-Shaping Filter
   - amplify some frequency components while suppress some other frequency components
   - e.g. differntiator, equalizer in a Hi-Fi system



2. Frequency-Selective Filter
   - select some bands of frequencies and reject others.
   - e.g. low-pass filter, high-pass filter, band-pass filter, band-stop filter

※ **Ideal Frequency Selective Filter:** An *ideal* frequency selective filter is one that passes signals at one set of frequencies (referred as the *pass band*) and completely rejects the rest (referred as the *stop band*).



a) **Ideal Low-Pass Filter**

$$H(f) = \begin{cases} 1, & |f| \le f_c \\ 0, & |f| > f_c \end{cases}$$

where w_c is the *cutoff frequency*.

b) **Ideal High-Pass Filter**

$$H(f) = \begin{cases} 0, & |f| \le f_c \\ 1, & |f| > f_c \end{cases}$$

c) **Ideal Bandpass Filter**

$$H(f) = \begin{cases} 1, & f_1 < |f| < f_2 \\ 0, & \text{otherwise} \end{cases}$$

d) **Ideal Bandstop Filter**

$$H(f) = \begin{cases} 0, & f_1 < |f| < f_2 \\ 1, & \text{otherwise} \end{cases}$$

※ **Bandwidth:** There are many different definitions of filter bandwidth.

1. Absolute Bandwidth f_B
   - absolute BW (Bandwidth) of an ideal low-pass filter: $f_B = f_c$
   - absolute BW of an ideal bandpass filter: $f_B = f_2 - f_1$
   - absolute BS of an ideal high-pass or bandstop filter are not defined.
2. 3-dB Bandwidth (*Half-Power Bandwidth*) $f_{3\text{ dB}}$
   - $f_{3\text{ dB}}$ is the frequency where the peak magnitude spectrum $|H(0)|$ drops to $|H(0)|/\sqrt{2}$. Since the power is proportional to the square of $|H(f)|$, *i.e.*, $P \propto |H(f)|$, then the 3-dB bandwidth represent the frequency where the peak power reduces to the half.

$$10\log_{10}\left(\frac{P(f_{3\text{ dB}})}{P(f_{\max})}\right)\ dB = 10\log_{10}\left(\frac{|H(f_{3\text{ dB}})|^2}{|H(0)|^2}\right)\ dB \xrightarrow{|H(f_{3\text{ dB}})| = \frac{|H(0)|}{\sqrt{2}}} 10\log_{10}(1/2)\ dB$$
$$= -10\log_{10}(2)\ dB \simeq -3dB$$

3. Equivalent Bandwidth f_{eq}. *Refer [Schaum's text, Problem 5.55]*
   - f_{eq} is defined as the bandwidth of an ideal filter where the power of the ideal filter is equal to that of the real filter given the same input signals as follows

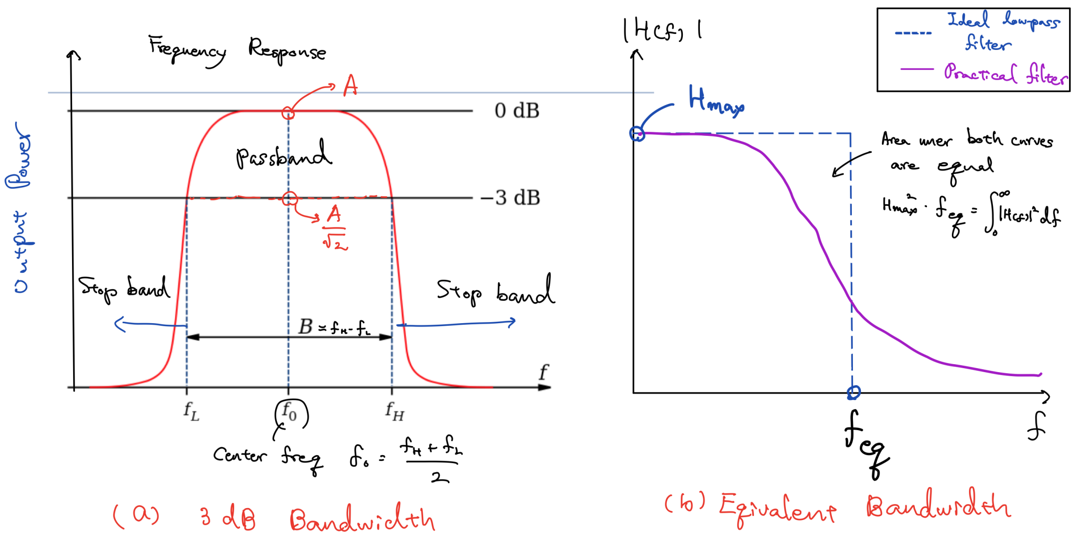
$$f_{eq} = \int_0^\infty \left|\frac{H(f)}{H_{max}}\right|^2 df$$

   where H_{max} denotes the maximum value of the magnitude spectrum.

4. x percent Energy containment Bandwidth $f_{x\%}$. (e.g., 90 percent energy containment BW $f_{90\%}$)

$$\int_{-f_{x\%}}^{f_{x\%}} |X(f)|^2 df = (x\ \%)\, E_x \quad \text{where} \quad E_x = \int_{-\infty}^{\infty} |X(f)|^2 df.$$

   Refer [Schaum's text, Problem 5.57]



[Example 4-8] Consider the following filters with impulse response $h(t)$. For each filter, derive the 3-dB bandwidth $f_{3\text{ dB}}$, equivalent bandwidth f_{eq}, and 90 percent energy containment bandwidth $f_{90\%}$, respectively.

a) $h(t) = w_0\, e^{-w_0 t} u(t)$ where $w_0 = 2\pi f_0$ b) $h(t) = \frac{1}{\alpha^2 + (2\pi t)^2}$ where $\alpha > 0$

Solution) For (a), the frequency response $H(f)$ is obtained as

$$\text{(a) } h(t) = w_0\, e^{-w_0 t} u(t) \quad \Leftrightarrow \quad H(f) = \frac{w_0}{w_0 + j2\pi f} = \frac{1}{1 + j(f/f_0)} \quad \text{where } f_0 = \frac{w_0}{2\pi}$$

Since the maximum magnitude spectrum is $|H(0)| = 1$, the 3-dB bandwidth $f_{3\text{ dB}}$, the equivalent bandwidth f_{eq}, and the 90 percent energy containment bandwidth $f_{90\%}$ occurs when the following conditions are satisfied.

$$\begin{cases} f_{3\text{ dB}}; \quad |H(f_{3\text{ dB}})| = \dfrac{|H(0)|}{\sqrt{2}}, \\ f_{eq}; \quad f_{eq} = \displaystyle\int_0^\infty \frac{|H(f)|^2}{|H(0)|^2} df, \\ f_{90\%}; \quad \displaystyle\int_{-f_{90\%}}^{f_{90\%}} |H(f)|^2 df = 0.9 \int_{-\infty}^{\infty} |H(f)|^2 df \end{cases}, \quad \text{where } |H(f)| = \frac{1}{\sqrt{1 + (f/f_0)^2}} \tag{4.13}$$

Since $|H(f_{3\text{ dB}})| = \frac{1}{\sqrt{1+(f_{3\text{ dB}}/f_0)^2}} = \frac{1}{\sqrt{2}}$ when $f_{3\text{ dB}} = f_0$, the 3-dB bandwidth is equal to f_0. By using (4.14), we obtain

$$
\begin{aligned}
&f_{\text{eq}} = \int_0^\infty \frac{1}{1+(f/f_0)^2} df = f_0 \int_0^\infty \frac{1}{1+x^2} dx = \frac{f_0 \pi}{2},\\
&\text{LHS of } f_{90\%} \text{ in (1.13) } \int_{-f_{90\%}}^{f_{90\%}} |H(f)|^2 df = 2\int_0^{f_{90\%}} \frac{1}{1+(f/f_0)^2} df = 2f_0 \tan^{-1}(x)\Big|_0^{f_{90\%}/f_0} = 2f_0 \tan^{-1}\left(\frac{f_{90\%}}{f_0}\right),\\
&\text{RHS of } f_{90\%} \text{ in (1.13) } 0.9 \int_{-\infty}^{\infty} \frac{1}{1+(f/f_0)^2} df = 0.9 * 2 \int_0^\infty \frac{1}{1+(f/f_0)^2} df = 0.9 f_0 \pi \;\Rightarrow\; f_{90\%} = f_0 \tan(0.45\pi)
\end{aligned}
$$

Hence, the three BWs $f_{3\text{ dB}}$, f_{eq}, and $f_{90\%}$ are given by

$$
\text{(a) } f_{3\text{ dB}} = f_0, \quad f_{\text{eq}} = \frac{f_0\pi}{2} \simeq 1.57 f_0, \quad f_{90\%} = f_0 \tan(0.45\pi) \simeq 6.31 f_0, \quad \Leftrightarrow \quad f_{3\text{ dB}} < f_{\text{eq}} < f_{90\%}
$$

Similarly, for (b), the frequency response $H(f)$ is obtained as

$$
\text{(b) } h(t) = \frac{1}{\alpha^2 + (2\pi t)^2} \quad \Leftrightarrow \quad H(f) = \frac{1}{2\alpha} e^{-\alpha|f|}
$$

and the maximum magnitude spectrum is $|H(0)| = \frac{1}{2\alpha}$. Then, based on (4.13), we obtain

$$
\begin{aligned}
&|H(f_{3\text{ dB}})| = \frac{|H(0)|}{\sqrt{2}} \quad \Rightarrow \frac{1}{2\alpha} e^{-\alpha|f_{3\text{ dB}}|} = \frac{1}{2\alpha\sqrt{2}} \quad \Rightarrow f_{3\text{ dB}} = \frac{1}{\alpha}\ln\sqrt{2},\\
&f_{\text{eq}} = \int_0^\infty e^{-2\alpha f} df = \frac{1}{2\alpha},\\
&\text{LHS of } f_{90\%} \text{ in (1.13) } \int_{-f_{90\%}}^{f_{90\%}} |H(f)|^2 df = 2\int_0^{f_{90\%}} e^{-2\alpha f} df = \frac{1}{\alpha}\left[1 - e^{-2\alpha f_{90\%}}\right],\\
&\text{RHS of } f_{90\%} \text{ in (1.13) } 0.9 \int_{-\infty}^{\infty} e^{-2\alpha f} df = 0.9 * 2 \int_0^\infty e^{-2\alpha f} df = \frac{0.9}{\alpha} \;\Rightarrow\; f_{90\%} = \frac{\ln(10)}{2\alpha}
\end{aligned}
$$

Hence, the BWs $f_{3\text{ dB}}$, f_{eq}, and $f_{90\%}$ are given by

$$
\text{(b) } f_{3\text{ dB}} = \frac{1}{\alpha}\ln\sqrt{2} \simeq \frac{0.35}{\alpha}, \quad f_{\text{eq}} = \frac{1}{2\alpha} = \frac{0.5}{\alpha}, \quad f_{90\%} = \frac{\ln(10)}{2\alpha} \simeq \frac{1.15}{\alpha}, \quad \Leftrightarrow \quad f_{3\text{ dB}} < f_{\text{eq}} < f_{90\%}
$$

Useful Equality

Refer Wolfram Alpha [https://bit.ly/3d3RZoA]

$$
\int \frac{1}{a^2+x^2} dx = \frac{\tan^{-1}\left(\frac{x}{a}\right)}{a}, \quad \int_0^\infty \frac{1}{a^2+x^2} dx = \frac{\pi}{2|a|} \quad \text{for real-valued } a \tag{4.14}
$$