MA1200 CALCULUS AND BASIC LINEAR ALGEBRA

LECTURE: CG1

REVIEW EXAMPLES ON CHAPTER 6 TO 8

<u>Example 1</u> **(Exam 1617B)**

Evaluate the following limits:

(a) $\lim_{x\to 0}\frac{1-\cos(3x)}{x^2}$

(b) $\lim_{x\to\infty}\frac{1-x^2}{1+x^2}$

(c) $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{3}{3x+2x^2}\right)$

Example 2 (Exam 1314B)

Let $f(x) = \begin{cases} \dfrac{x^3 - 8}{x - 2} & \text{if} \quad x \neq 2 \\ c & \text{if} \quad x = 2 \end{cases}$.

Find the value of c for which $f(x)$ is continuous at $x = 2$. Give your reason.

Example 3 **(Exam 1213A)**

Let $g(x)=\begin{cases} x^2\sin\dfrac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$. Determine whether $g(x)$ is differentiable at $x=0$, if so, find the value of the first derivative there.

Example 4 **(Exam 1617B)**

(a) Prove from first principles that $\frac{d}{dx}\left(x^3\right)=3x^2$.

(b) Let $F(x)=\left|\cos x\right|$, for $x\in\mathbf{R}$. Determine whether $F(x)$ is differentiable at $x=0$. Give your reason.

(Hint: You may use $\cos 2\theta=1-2\sin^2\theta$.)

Example 5

Find $\dfrac{dy}{dx}$ for each of the following:

(a) $y = \tan^{-1}\left[5\cosh(x^3+2)\right]$

(b) $y = \left(\sqrt{x} + \dfrac{1}{x}\right)^{x^2}$

(c) $xy^2 + e^{-xy} = 1$

(d) $y = \cos^3(4x) \cdot \ln\left[\sinh(2x)\right]$

(e) $y = \dfrac{(2x+3)^6}{(x^2+1)\,e^{4x}}$

Example 6 **(Exam 1415A)**

A curve has parametric equations

$$x = 2t - \log_e(2t),$$

$$y = t^2 - \log_e(t^2),$$

where t is the parameter and $t > 0$.

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of t.

Example 7 **(Exam 1415A)**

Given that $y = e^{\sqrt{3}x}\cos x$, express $\frac{dy}{dx}$ in the form $re^{\sqrt{3}x}\cos(x+\phi)$, where $r > 0$ and $0 < \phi < \frac{\pi}{2}$, and state the numerical value of r and ϕ. Express $\frac{d^2y}{dx^2}$ in similar form. (Hint: $\cos(A+B) = \cos A\cos B - \sin A\sin B$.)

Example 8 **(Exam 1415B)**

(a) Express $\dfrac{7x+17}{(x-4)(2x+1)}$ in partial fractions.

(b) If $y=(\alpha x+\beta)^{-1}$, where α and β are non-zero constants, find the general formula for the nth derivative of y with respect to x.

(c) Using the result in parts (a) and (b), or otherwise, find the sixth derivative of $\dfrac{7x+17}{(x-4)(2x+1)}$ with respect to x. You need not simplify your answer.

<u>Example 9</u> **(Exam 1617B)**

Find the seventh derivative of $\left(x^2 - x + 3\right)e^{-2x}$ with respect to x.

Example 10 **(Exam 1314B)**

$P\left(2, \dfrac{2+\sqrt{3}}{2}\right)$ is a point on the curve $x^2 + 4y^2 - 6x - 8y + 9 = 0$.

(a) Find the slope of the tangent to the curve at P.

(b) Find the equation of the normal to the curve at P.

Example 11

Evaluate $\lim_{x \to 0} \dfrac{1+x-e^{\sin x}}{x-\ln(1+x)}$.

Example 12 **(Exam 1415B)**

(a) Starting from the formula $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$, show that

$\tan 4\theta=\dfrac{4t(1-t^2)}{1-6t^2+t^4}$, where $t=\tan\theta$. Deduce that, when $\theta=\tan^{-1}\left(\dfrac{1}{5}\right)$,

$\tan\left(4\theta-\dfrac{\pi}{4}\right)=\dfrac{1}{239}$.

(b) If $y=\tan^{-1}x$, prove that $(1+x^2)\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}=0$. By repeated differentiation of this result and use the Maclaurin theorem, or otherwise, prove that the first three non-zero terms in the series expansion of $\tan^{-1}x$ are $x-\dfrac{x^3}{3}+\dfrac{x^5}{5}+\ldots$.

(c) Using the results in parts (a) and (b), find an approximation to the value of π, giving 5 decimal places in your answer.

Example 13

Let $P(0,-2)$ be a turning point of the curve $y=\dfrac{x^2+px+q}{x+1}$.

(a) Find the values of p and q.

(b) Find all local extrema of the curve.

(c) Find the largest possible domain and largest possible range of the function

$f(x)=\dfrac{x^2+px+q}{x+1}$, where p and q are the values you found in (a).

Example 14

Let $y = \sinh^{-1} x$. Use implicit differentiation to show that $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+x^2}}$.

Hence, or otherwise show that

$$(1+x^2)y^{(2)} + xy^{(1)} = 0,$$

and that, for $n \geq 0$,

$$(1+x^2)y^{(n+2)} + (2n+1)xy^{(n+1)} + n^2 y^{(n)} = 0,$$

where $y^{(r)}$ denotes $\dfrac{d^r y}{dx^r}$.

Hence, or otherwise, find the expansion of $\sinh^{-1} x$ in ascending powers of x as far as the term in x^5.