Rational functions

A **rational function** is a quotient of two polynomials. It is of the form

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.

Note that the largest possible domain of $f(x) = \frac{p(x)}{q(x)}$ is $\boxed{\mathbb{R}\backslash\{x \in \mathbb{R} | q(x) = 0\}}$, i.e. the set of all real numbers except the value(s) of x such that $q(x) = 0$.

- $f(x) = \frac{p(x)}{q(x)}$ is called a **proper rational function** if $\boxed{\text{degree of } p(x) < \text{degree of } q(x)}$.

  deg. 2

  For example, $f(x) = \frac{x^2+3}{x^3+2x-4}$ is a proper rational function.

  deg. 3

- $f(x) = \frac{p(x)}{q(x)}$ is called an **improper rational function** if $\boxed{\text{degree of } p(x) \geq \text{degree of } q(x)}$.

  deg. 2 deg.3

  For example, $g(x) = \frac{x^2+3}{x^2+5x-7}$ and $h(x) = \frac{x^3+x-4}{x^2+5x-7}$ are improper rational functions.

  deg. 2 deg. 2

If $f(x) = \frac{p(x)}{q(x)}$ is an **improper** rational function, use **long division** / **synthetic division** to write

$f(x)$ as

$$\boxed{f(x) = \text{a polynomial} + \text{a proper rational function}}.$$

This is because

$$p(x) = \underbrace{s(x)}_{Quotient}\ \underbrace{q(x)}_{Divisor} + \underbrace{r(x)}_{Remainder}$$

where $0 \leq \text{degree of remainder } r(x) < \text{degree of divisor } q(x)$.

$$\begin{array}{r} \boxed{x-5} \\ \boxed{x^2+5x-7}\,\big)\ x^3 \qquad +x-4 \\ -)\ \ x^3+5x^2-7x \\ \hline -5x^2+8x-4 \\ -)\ -5x^2-25x+35 \\ \hline \boxed{33x-39} \end{array}$$

$$\therefore\ f(x) = \frac{p(x)}{q(x)} = \frac{s(x)\,q(x)+r(x)}{q(x)} = \underbrace{s(x)}_{\text{polynomial}} + \underbrace{\frac{r(x)}{q(x)}}_{\text{proper rational function}}$$

For example, $g(x) = \frac{x^2+3}{x^2+5x-7} = 1 + \frac{-5x+10}{x^2+5x-7}$ and $h(x) = \frac{x^3+x-4}{x^2+5x-7} = \underbrace{x-5}_{\text{polynomial}} + \underbrace{\frac{33x-39}{x^2+5x-7}}_{\text{proper rational function}}$.

$$\frac{(x^2+5x-7)-5x+7+3}{x^2+5x-7}$$

Example 9

Find the largest possible domain of the function $f(x) = \frac{x-7}{x^3+2x^2+5x+10}$.

Solution

Let $g(x) = x^3 + 2x^2 + 5x + \enclose{circle}{10}$.

Factors of 10 (the constant term) are $\pm1, \pm2, \pm5, \pm10$.

Use the "trial and error" method to find a root of the equation $g(x) = 0$:

$g(-1) = (-1)^3 + 2(-1)^2 + 5(-1) + 10 = 6\ (\neq 0)$

$g(-2) = (-2)^3 + 2(-2)^2 + 5(-2) + 10 = 0$ ✓ **$\therefore (x-(-2)) = x+2$ is a factor of $g(x)$**

By long division,

$$
\begin{array}{rl}
 & x^2 + 5 \\
x+2 \big) & x^3 + 2x^2 + 5x + 10 \\
 & \underline{x^3 + 2x^2} \\
 & 5x + 10 \\
 & \underline{5x + 10}
\end{array}
$$

$g(x) = x^3 + 2x^2 + 5x + 10 = (x+2)(x^2+5)$

The function $f(x)$ is NOT well-defined when $g(x) = 0$,

i.e. when $(x+2)(x^2+5) = 0 \Rightarrow x+2 = 0$ or $\underbrace{x^2+5=0}_{\textit{has no real solution}} \Rightarrow x = -2$.

$\therefore$ The largest possible domain of $f(x)$ is $\mathbb{R}\backslash\{-2\}$.

Example 10

Evaluate $\dfrac{5}{2x-1}-\dfrac{2}{x+3}$

$$=\frac{5(x+3)-2(2x-1)}{(2x-1)(x+3)}$$

$$=\frac{x+17}{(2x-1)(x+3)}$$

(Combine)

Easy!

(Decompose)

Difficult!

(use **partial fractions**)

☆ Partial Fractions ☆

useful in MA1201 (integration)

Partial fraction is a technique used in writing a complicated fraction as a sum of simpler fractions.

The following table lists some typical cases we shall mostly encounter and how we should resolve them into partial fractions (Note: We assume the expressions are already ***proper*** rational functions.)

If it's improper, use long division first.

Three common types of factors in the denominator:

Type	Expression	Form of Partial Fraction
Distinct Linear Factors	E.g. $\dfrac{f(x)}{(x+a)(x+b)(x+c)}$	$\dfrac{A}{(x+a)}+\dfrac{B}{(x+b)}+\dfrac{C}{(x+c)}$ Constant (deg. 0) linear (deg. 1)
Repeated Linear Factors	E.g. $\dfrac{f(x)}{(x+a)^3}$	$\dfrac{A}{(x+a)}+\dfrac{B}{(x+a)^2}+\dfrac{C}{(x+a)^3}$ Constant (deg. 0) repeated factor is (x+a) which has deg. 1
Quadratic Factors	E.g. $\dfrac{f(x)}{(ax^2+bx+c)(x+d)}$ where $\underset{b^2-4ac<0}{\underline{ax^2+bx+c}}$ cannot be further factorized	$\dfrac{\overset{\text{linear (deg. 1)}}{Ax+B}}{\underset{\text{quadratic (deg. 2)}}{(ax^2+bx+c)}}+\dfrac{\overset{\text{Constant (deg. 0)}}{C}}{\underset{\text{linear (deg. 1)}}{(x+d)}}$

Here, A, B, C are unknown constants to be found.

deg. of numerator is always one less than deg. of denominator (except when factors are repeated).

Note:

In general, if a linear factor $(ax+b)$ is repeated n times, we would have n terms in the decomposition of the form $\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_n}{(ax+b)^n}$. Here, $A_1, A_2, \dots, A_n$ are unknown constants to be found.

Similarly, if a quadratic factor (ax^2+bx+c) is repeated n times, where (ax^2+bx+c) cannot be further factorized, we would have n terms in the decomposition of the form

$$\frac{\boxed{A_1x+B_1}}{(ax^2+bx+c)}+\frac{\boxed{A_2x+B_2}}{(ax^2+bx+c)^2}+\cdots+\frac{\boxed{A_nx+B_n}}{(ax^2+bx+c)^n} \longleftarrow \text{linear} \quad \because \text{repeated factor is quadratic (deg. 2)}$$

Here, $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_n$ are unknown constants to be found.

Procedure for resolving a rational function into partial fractions

Consider the rational function $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials:

Step 1: Check whether $\frac{p(x)}{q(x)}$ is a proper rational function or not. If it is improper, use long division to express $\frac{p(x)}{q(x)}$ as "a polynomial + a proper rational function".

Step 2: For the proper rational function, factorize its denominator.

Step 3: Write down the form of the partial fractions.

Step 4: Find the unknowns.

Note: We may use the results :

$$a^2-b^2 = (a-b)(a+b)$$

$$a^3-b^3 = (a-b)(a^2+ab+b^2)$$

$$a^3+b^3 = (a+b)(a^2-ab+b^2)$$

E.g. $x^2-4 = (x-2)(x+2)$

E.g. $x^3-1 = (x-1)(x^2+x+1)$

E.g. $x^3+8 = (x+2)(x^2-2x+4)$

Example 11 (**Distinct linear factors**)

Express $\frac{x+17}{2x^2+5x-3}$ into partial fractions.

Solution

First note that this is a **proper** rational function. Then notice that the denominator can be factorized as $2x^2+5x-3=(2x-1)(x+3)$. Thus $\frac{x+17}{2x^2+5x-3}=\frac{x+17}{(2x-1)(x+3)}$.

Let $\frac{x+17}{(2x-1)(x+3)}=\frac{\overset{\text{deg.0}}{A}}{\underset{\text{deg.1}}{2x-1}}+\frac{\overset{\text{deg.0}}{B}}{\underset{\text{deg.1}}{x+3}}$, where A and B are constants to be determined.

Multiplying both sides by $(2x-1)(x+3)$, we get

$$x+17=A(x+3)+B(2x-1)$$

Put $x=-3$: $-3+17=A\underbrace{(-3+3)}_{=0}+B[2\cdot(-3)-1] \Rightarrow 14=-7B \Rightarrow B=-2$

Put $x=\frac{1}{2}$: $\frac{1}{2}+17=A\left(\frac{1}{2}+3\right)+B\underbrace{\left[2\cdot\left(\frac{1}{2}\right)-1\right]}_{=0} \Rightarrow \frac{35}{2}=\frac{7}{2}A \Rightarrow A=5$

$\therefore \frac{x+17}{(2x-1)(x+3)}=\frac{5}{2x-1}-\frac{2}{x+3}$.

Example 12 (Three distinct linear factors)

Resolve $\frac{4x^2+12x+18}{x^3-9x}$ into partial fractions.

Solution

First note that this is a **proper** rational function. Also, note that the denominator can be factorized as $x^3-9x=x(x^2-9)=x(x-3)(x+3)$.

Then we have $\frac{4x^2+12x+18}{x^3-9x}=\frac{4x^2+12x+18}{x(x-3)(x+3)}=\frac{A}{x}+\frac{B}{x-3}+\frac{C}{x+3}$.

Multiplying both sides by $x(x-3)(x+3)$, we get

$$4x^2+12x+18=A(x-3)(x+3)+Bx(x+3)+Cx(x-3).$$

Put $x=0$: $18=-9A \Rightarrow A=-2$

Put $x=3$: $90=18B \Rightarrow B=5$

Put $x=-3$: $18=18C \Rightarrow C=1$

$\therefore \frac{4x^2+12x+18}{x^3-9x}=\frac{-2}{x}+\frac{5}{x-3}+\frac{1}{x+3}$

Example 13 (**Improper rational function**)

Express $\frac{2x^3-x^2-9x-10}{x^2-4}$ into partial fractions.

Solution

First note that the degree of the $\overset{\text{(deg. 3)}}{\text{numerator}}$ is greater than the degree of the $\overset{\text{(deg. 2)}}{\text{denominator}}$, i.e. it is an improper rational function.

By long division,

$$\underbrace{\frac{2x^3-x^2-9x-10}{x^2-4}}_{\text{improper rational function}} = \underbrace{\boxed{2x-1}}_{\text{polynomial}} + \underbrace{\frac{\boxed{-x-14}}{\boxed{x^2-4}}}_{\text{proper rational function}}$$

$$\begin{array}{r} \boxed{2x \; - \; 1} \\ \boxed{x^2-4}\,\overline{)\,2x^3-x^2-9x-10} \\ 2x^3 \qquad\quad -8x \quad\;\; \\ \hline -x^2-x-10 \\ -x^2 \qquad\;\; +4 \\ \hline \boxed{-x-14} \end{array}$$

Consider $\frac{-x-14}{x^2-4} = \frac{-x-14}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$.

Multiplying both sides by $(x-2)(x+2)$, we get

$$-x-14 = A(x+2) + B(x-2)$$

Put $x = -2$: $-(-2) - 14 = 0 + B(-4) \Rightarrow -12 = -4B \Rightarrow B = 3$

Put $x = 2$: $-2 - 14 = A(4) + 0 \Rightarrow -16 = 4A \Rightarrow A = -4$

$$\therefore \frac{2x^3 - x^2 - 9x - 10}{x^2 - 4} = 2x - 1 - \frac{4}{x-2} + \frac{3}{x+2}$$

Example 14 (Repeated linear factors)

Express $\frac{-7x^2+11x-3}{x(x-1)^3}$ into partial fractions.

Solution

First note that it's a proper rational function.

constants ∵ the repeated factor is linear (deg. 1)

$$\frac{-7x^2+11x-3}{x(x-1)^3}=\frac{A}{x}+\frac{\textcircled{B}}{x-1}+\frac{\textcircled{C}}{(x-1)^2}+\frac{\textcircled{D}}{(x-1)^3}$$

Multiplying both sides by $x(x-1)^3$, we get

$$-7x^2+11x-3=A(x-1)^3+Bx(x-1)^2+Cx(x-1)+Dx$$

Put $x=1$: $-7(1)^2+11(1)-3=0+0+0+D(1) \Rightarrow D=1$

Put $x=0$: $-7(0)^2+11(0)-3=A(0-1)^3+0+0+0 \Rightarrow -3=-A \Rightarrow A=3$

Equating coefficients of x^3: $0=A+B \Rightarrow B=-A=-3$

Put $x=-1$: $-7(-1)^2+11(-1)-3=A(-1-1)^3+B(-1)(-1-1)^2$

$$+C(-1)(-1-1)+D(-1)$$

$$\Rightarrow -21=-8A-4B+2C-D$$

$$\Rightarrow C=\frac{1}{2}(-21+8A+4B+D)=\frac{1}{2}[-21+8(3)+4(-3)+1]=-4$$

$$\therefore \quad \frac{-7x^2+11x-3}{x(x-1)^3}=\frac{3}{x}-\frac{3}{x-1}-\frac{4}{(x-1)^2}+\frac{1}{(x-1)^3}$$

Example 15 (Linear and Quadratic Factors)

Resolve $\frac{9x^2-12x-2}{(2x+1)(x^2-2x+5)}$ into partial fractions.

Solution

First note that $\frac{9x^2-12x-2}{(2x+1)(x^2-2x+5)}$ is a proper rational function.

Also, note that $x^2-2x+5=0$ has no real solution (since the discriminant $b^2-4ac=(-2)^2-4(1)(5)=-16<0$), which means that x^2-2x+5 cannot be further factorized.

Let $\frac{9x^2-12x-2}{(2x+1)(x^2-2x+5)} = \frac{\overset{\text{deg.0}}{A}}{\underset{\text{deg.1}}{2x+1}} + \frac{\overset{\text{deg.1}}{Bx+C}}{x^2-2x+5} \leftarrow \text{deg.2}$

Multiplying both sides by $(2x+1)(x^2-2x+5)$, we get

$$9x^2-12x-2 = A(x^2-2x+5)+(Bx+C)(2x+1)$$

Put $x=-\frac{1}{2}$: $9\left(-\frac{1}{2}\right)^2-12\left(-\frac{1}{2}\right)-2 = A\left[\left(-\frac{1}{2}\right)^2-2\left(-\frac{1}{2}\right)+5\right]+0$

$$\Rightarrow \frac{25}{4}=\frac{25}{4}A \quad \Rightarrow \quad A=1$$

Equating coefficients of x^2: $9 = A + 2B \quad \Rightarrow \quad 9 = 1 + 2B \quad \Rightarrow \quad B = 4$

Equating constant term: $-2 = 5A + C \quad \Rightarrow -2 = 5(1) + C \quad \Rightarrow \quad C = -7$

$$\therefore \; \frac{9x^2-12x-2}{(2x+1)(x^2-2x+5)} = \frac{1}{2x+1} + \frac{4x-7}{x^2-2x+5}$$

Example 16 (**Repeated Quadratic Factors**) – *It's a bit complicated!*

Express $\frac{8x-1}{(x+1)(x^2+2)^2}$ into partial fractions.

Solution

Note that $\frac{8x-1}{(x+1)(x^2+2)^2}$ is a proper rational function, and also x^2+2 cannot be further factorized.

Let

deg.1 ∵ the repeated factor (x^2+2) is of deg. 2

$$\frac{8x-1}{(x+1)(x^2+2)^2} = \frac{\overset{\text{deg.0}}{A}}{\underset{\text{deg.1}}{x+1}} + \frac{\boxed{Bx+C}}{x^2+2} + \frac{\boxed{Dx+E}}{(x^2+2)^2}.$$

Multiplying both sides by $(x+1)(x^2+2)^2$, we get

$$8x-1 = A(x^2+2)^2 + (Bx+C)(x+1)(x^2+2) + (Dx+E)(x+1) \quad \text{......(*)}$$

Put $x=-1$: $\quad -9 = 9A \quad \Rightarrow \quad A=-1$

Equating coefficients of x^4: $\quad 0 = A+B \quad \Rightarrow \quad B=-A=1$

Substitute $A=-1$ and $B=1$ into (*):

$$8x-1=-(x^2+2)^2+(x+C)(x+1)(x^2+2)+(Dx+E)(x+1)$$

$$=\underline{-(x^2+2)^2+x(x+1)(x^2+2)}+C(x+1)(x^2+2)+(Dx+E)(x+1)$$

$$\Rightarrow \underbrace{8x-1+(x^2+2)^2-x(x+1)(x^2+2)}=C\underline{(x+1)}(x^2+2)+(Dx+E)\underline{(x+1)}$$

$$=8x-1+(x^2+2)[x^2+2-x(x+1)]$$

$$=8x-1+(x^2+2)[x^2+2-x^2-x]$$

$$=8x-1+(x^2+2)(2-x)$$

$$=8x-1+(2x^2-x^3+4-2x)$$

$$=-x^3+2x^2+6x+3$$

Since $(x+1)$ is a factor of the RHS, it must be a factor of the LHS as well. Using long division to divide the LHS $(-x^3+2x^2+6x+3)$ by $(x+1)$, we get

$$\begin{array}{r} -x^2+3x+3 \\ x+1\overline{)-x^3+2x^2+6x+3} \\ \underline{-x^3-x^2} \\ 3x^2+6x \\ \underline{3x^2+3x} \\ 3x+3 \\ \underline{3x+3} \end{array}$$

remainder must be 0 →

$$LHS=-x^3+2x^2+6x+3=(x+1)(-x^2+3x+3)$$

Hence, $\underline{(x+1)}(-x^2+3x+3)=C\underline{(x+1)}(x^2+2)+(Dx+E)\underline{(x+1)}$

$$\Rightarrow -x^2+3x+3=C(x^2+2)+(Dx+E)$$

Equating coefficients of x^2: $-1 = C \quad \Rightarrow \quad C = -1$

Equating coefficients of x: $3 = D \quad \Rightarrow \quad D = 3$

Equating constant term: $3 = 2C + E \quad \Rightarrow \quad E = 3 - 2C = 3 - 2(-1) = 5$

Hence, $\frac{8x-1}{(x+1)(x^2+2)^2} = \frac{-1}{x+1} + \frac{x-1}{x^2+2} + \frac{3x+5}{(x^2+2)^2}$

Class Exercise

Express $\frac{5x^3-16x^2+14x-11}{(x-1)^2\,(x^2+3)}$ in partial fractions.

Solution :

proper

$$\text{Let} \quad \frac{5x^3-16x^2+14x-11}{(x-1)^2\underbrace{(x^2+3)}_{\text{cannot be factorized}}} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+3}.$$

Multiply both sides by $(x-1)^2(x^2+3)$:

$$5x^3-16x^2+14x-11 = A(x-1)(x^2+3) + B(x^2+3) + (Cx+D)(x-1)^2 \quad \text{———} \quad \textcircled{**}$$

Put $x=1$: $-8 = 4B \Rightarrow B=-2$

Substitute $B=-2$ into $\textcircled{**}$:

$$5x^3-16x^2+14x-11 = A(x-1)(x^2+3) \underline{-2(x^2+3)} + (Cx+D)(x-1)^2$$

$$\Rightarrow \underline{5x^3-16x^2+14x-11+2(x^2+3)} = A(x-1)(x^2+3) + (Cx+D)(x-1)^2$$

$$= 5x^3-14x^2+14x-5$$

$$= (x-1)(\underline{5x^2-9x+5})$$

$$\begin{array}{r} \boxed{5x^2-9x+5} \\ x-1\overline{\big)\,5x^3-14x^2+14x-5} \\ \underline{5x^3-5x^2} \\ -9x^2+14x \\ \underline{-9x^2+9x} \\ 5x-5 \\ \underline{5x-5} \end{array}$$

$\therefore \quad 5x^2 - 9x + 5 = A(x^2+3) + (Cx+D)(x-1)$

Put $x = 1$: $\quad 1 = 4A \Rightarrow A = \frac{1}{4}$

Compare coeff. of x^2 : $\quad 5 = A + C \Rightarrow C = 5 - \frac{1}{4} = \frac{19}{4}$

Compare constant term : $\quad 5 = 3A - D \Rightarrow D = 3\cdot\left(\frac{1}{4}\right) - 5 = \frac{-17}{4}$

$$\therefore \quad \frac{5x^3 - 16x^2 + 14x - 11}{(x-1)^2\ (x^2+3)} = \frac{1}{4(x-1)} - \frac{2}{(x-1)^2} + \frac{19x-17}{4(x^2+3)}$$