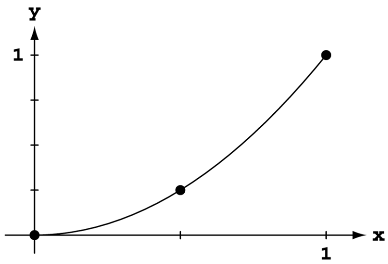
CHAPTER 5 INTEGRATION

5.1 ESTIMATING WITH FINITE SUMS

1. $f(x) = x^2$

Since f is increasing on $[0, 1]$, we use left endpoints to obtain lower sums and right endpoints to obtain upper sums.



(a) $\triangle x = \frac{1-0}{2} = \frac{1}{2}$ and $x_i = i\triangle x = \frac{i}{2} \Rightarrow$ a lower sum is $\sum_{i=0}^{1} \left(\frac{i}{2}\right)^2 \cdot \frac{1}{2} = \frac{1}{2}\left(0^2 + \left(\frac{1}{2}\right)^2\right) = \frac{1}{8}$

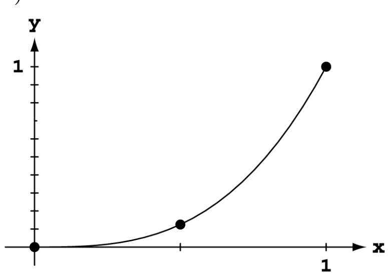
(b) $\triangle x = \frac{1-0}{4} = \frac{1}{4}$ and $x_i = i\triangle x = \frac{i}{4} \Rightarrow$ a lower sum is $\sum_{i=0}^{3} \left(\frac{i}{4}\right)^2 \cdot \frac{1}{4} = \frac{1}{4}\left(0^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2\right) = \frac{1}{4} \cdot \frac{7}{8} = \frac{7}{32}$

(c) $\triangle x = \frac{1-0}{2} = \frac{1}{2}$ and $x_i = i\triangle x = \frac{i}{2} \Rightarrow$ an upper sum is $\sum_{i=1}^{2} \left(\frac{i}{2}\right)^2 \cdot \frac{1}{2} = \frac{1}{2}\left(\left(\frac{1}{2}\right)^2 + 1^2\right) = \frac{5}{8}$

(d) $\triangle x = \frac{1-0}{4} = \frac{1}{4}$ and $x_i = i\triangle x = \frac{i}{4} \Rightarrow$ an upper sum is $\sum_{i=1}^{4} \left(\frac{i}{4}\right)^2 \cdot \frac{1}{4} = \frac{1}{4}\left(\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2 + 1^2\right) = \frac{1}{4} \cdot \left(\frac{30}{16}\right) = \frac{15}{32}$

2. $f(x) = x^3$

Since f is increasing on $[0, 1]$, we use left endpoints to obtain lower sums and right endpoints to obtain upper sums.



(a) $\triangle x = \frac{1-0}{2} = \frac{1}{2}$ and $x_i = i\triangle x = \frac{i}{2} \Rightarrow$ a lower sum is $\sum_{i=0}^{1} \left(\frac{i}{2}\right)^3 \cdot \frac{1}{2} = \frac{1}{2}\left(0^3 + \left(\frac{1}{2}\right)^3\right) = \frac{1}{16}$

(b) $\triangle x = \frac{1-0}{4} = \frac{1}{4}$ and $x_i = i\triangle x = \frac{i}{4} \Rightarrow$ a lower sum is $\sum_{i=0}^{3} \left(\frac{i}{4}\right)^3 \cdot \frac{1}{4} = \frac{1}{4}\left(0^3 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{3}{4}\right)^3\right) = \frac{36}{256} = \frac{9}{64}$

(c) $\triangle x = \frac{1-0}{2} = \frac{1}{2}$ and $x_i = i\triangle x = \frac{i}{2} \Rightarrow$ an upper sum is $\sum_{i=1}^{2} \left(\frac{i}{2}\right)^3 \cdot \frac{1}{2} = \frac{1}{2}\left(\left(\frac{1}{2}\right)^3 + 1^3\right) = \frac{1}{2} \cdot \frac{9}{8} = \frac{9}{16}$

(d) $\triangle x = \frac{1-0}{4} = \frac{1}{4}$ and $x_i = i\triangle x = \frac{i}{4} \Rightarrow$ an upper sum is $\sum_{i=1}^{4} \left(\frac{i}{4}\right)^3 \cdot \frac{1}{4} = \frac{1}{4}\left(\left(\frac{1}{4}\right)^3 + \left(\frac{1}{2}\right)^3 + \left(\frac{3}{4}\right)^3 + 1^3\right) = = \frac{100}{256} = \frac{25}{64}$

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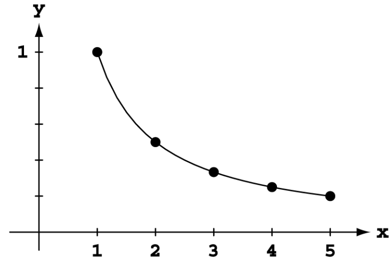
3. $f(x) = \frac{1}{x}$



Since f is decreasing on $[0, 1]$, we use left endpoints to obtain upper sums and right endpoints to obtain lower sums.

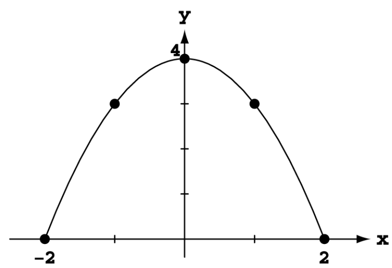
(a) $\triangle x = \frac{5-1}{2} = 2$ and $x_i = 1 + i\triangle x = 1 + 2i \Rightarrow$ a lower sum is $\sum_{i=1}^{2} \frac{1}{x_i} \cdot 2 = 2\left(\frac{1}{3} + \frac{1}{5}\right) = \frac{16}{15}$

(b) $\triangle x = \frac{5-1}{4} = 1$ and $x_i = 1 + i\triangle x = 1 + i \Rightarrow$ a lower sum is $\sum_{i=1}^{4} \frac{1}{x_i} \cdot 1 = 1\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right) = \frac{77}{60}$

(c) $\triangle x = \frac{5-1}{2} = 2$ and $x_i = 1 + i\triangle x = 1 + 2i \Rightarrow$ an upper sum is $\sum_{i=0}^{1} \frac{1}{x_i} \cdot 2 = 2\left(1 + \frac{1}{3}\right) = \frac{8}{3}$

(d) $\triangle x = \frac{5-1}{4} = 1$ and $x_i = 1 + i\triangle x = 1 + i \Rightarrow$ an upper sum is $\sum_{i=0}^{3} \frac{1}{x_i} \cdot 1 = 1\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) = \frac{25}{12}$

4. $f(x) = 4 - x^2$



Since f is increasing on $[-2, 0]$ and decreasing on $[0, 2]$, we use left endpoints on $[-2, 0]$ and right endpoints on $[0, 2]$ to obtain lower sums and use right endpoints on $[-2, 0]$ and left endpoints on $[0, 2]$ to obtain upper sums.

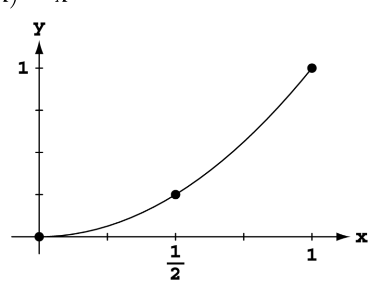
(a) $\triangle x = \frac{2-(-2)}{2} = 2$ and $x_i = -2 + i\triangle x = -2 + 2i \Rightarrow$ a lower sum is $2 \cdot \left(4 - (-2)^2\right) + 2 \cdot (4 - 2^2) = 0$

(b) $\triangle x = \frac{2-(-2)}{4} = 1$ and $x_i = -2 + i\triangle x = -2 + i \Rightarrow$ a lower sum is $\sum_{i=0}^{1} \left(4 - (x_i)^2\right) \cdot 1 + \sum_{i=3}^{4} \left(4 - (x_i)^2\right) \cdot 1$
$= 1\left(\left(4 - (-2)^2\right) + \left(4 - (-1)^2\right) + (4 - 1^2) + (4 - 2^2)\right) = 6$

(c) $\triangle x = \frac{2-(-2)}{2} = 2$ and $x_i = -2 + i\triangle x = -2 + 2i \Rightarrow$ a upper sum is $2 \cdot \left(4 - (0)^2\right) + 2 \cdot (4 - 0^2) = 16$

(d) $\triangle x = \frac{2-(-2)}{4} = 1$ and $x_i = -2 + i\triangle x = -2 + i \Rightarrow$ a upper sum is $\sum_{i=1}^{2} \left(4 - (x_i)^2\right) \cdot 1 + \sum_{i=2}^{3} \left(4 - (x_i)^2\right) \cdot 1$
$= 1\left(\left(4 - (-1)^2\right) + (4 - 0^2) + (4 - 0^2) + (4 - 1^2)\right) = 14$

5. $f(x) = x^2$



Using 2 rectangles $\Rightarrow \triangle x = \frac{1-0}{2} = \frac{1}{2} \Rightarrow \frac{1}{2}\left(f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right)$
$= \frac{1}{2}\left(\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2\right) = \frac{10}{32} = \frac{5}{16}$

Using 4 rectangles $\Rightarrow \triangle x = \frac{1-0}{4} = \frac{1}{4}$
$\Rightarrow \frac{1}{4}\left(f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right)$
$= \frac{1}{4}\left(\left(\frac{1}{8}\right)^2 + \left(\frac{3}{8}\right)^2 + \left(\frac{5}{8}\right)^2 + \left(\frac{7}{8}\right)^2\right) = \frac{21}{64}$

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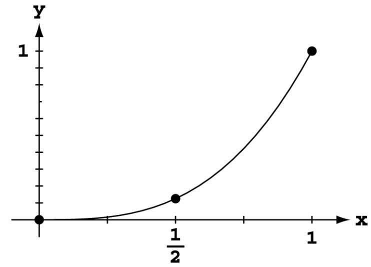
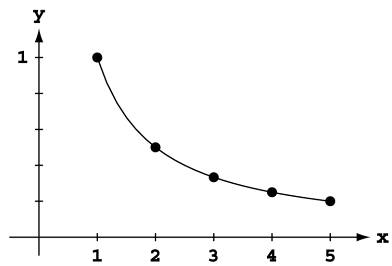
6. $f(x) = x^3$



Using 2 rectangles $\Rightarrow \triangle x = \frac{1-0}{2} = \frac{1}{2} \Rightarrow \frac{1}{2}\left(f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right)$
$= \frac{1}{2}\left(\left(\frac{1}{4}\right)^3 + \left(\frac{3}{4}\right)^3\right) = \frac{28}{2 \cdot 64} = \frac{7}{32}$

Using 4 rectangles $\Rightarrow \triangle x = \frac{1-0}{4} = \frac{1}{4}$
$\Rightarrow \frac{1}{4}\left(f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right)$
$= \frac{1}{4}\left(\frac{1^3+3^3+5^3+7^3}{8^3}\right) = \frac{496}{4 \cdot 8^3} = \frac{124}{8^3} = \frac{31}{128}$

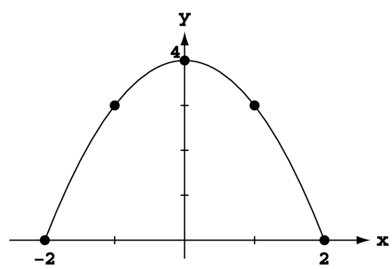
7. $f(x) = \frac{1}{x}$



Using 2 rectangles $\Rightarrow \triangle x = \frac{5-1}{2} = 2 \Rightarrow 2(f(2) + f(4))$
$= 2\left(\frac{1}{2} + \frac{1}{4}\right) = \frac{3}{2}$

Using 4 rectangles $\Rightarrow \triangle x = \frac{5-1}{4} = 1$
$\Rightarrow 1\left(f\left(\frac{3}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{7}{2}\right) + f\left(\frac{9}{2}\right)\right)$
$= 1\left(\frac{2}{3} + \frac{2}{5} + \frac{2}{7} + \frac{2}{9}\right) = \frac{1488}{3 \cdot 5 \cdot 7 \cdot 9} = \frac{496}{5 \cdot 7 \cdot 9} = \frac{496}{315}$

8. $f(x) = 4 - x^2$



Using 2 rectangles $\Rightarrow \triangle x = \frac{2-(-2)}{2} = 2 \Rightarrow 2(f(-1) + f(1))$
$= 2(3 + 3) = 12$

Using 4 rectangles $\Rightarrow \triangle x = \frac{2-(-2)}{4} = 1$
$\Rightarrow 1\left(f\left(-\frac{3}{2}\right) + f\left(-\frac{1}{2}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right)\right)$
$= 1\left(\left(4 - \left(-\frac{3}{2}\right)^2\right) + \left(4 - \left(-\frac{1}{2}\right)^2\right) + \left(4 - \left(\frac{1}{2}\right)^2\right) + \left(4 - \left(\frac{3}{2}\right)^2\right)\right)$
$= 16 - \left(\frac{9}{4} \cdot 2 + \frac{1}{4} \cdot 2\right) = 16 - \frac{10}{2} = 11$

9. (a) $D \approx (0)(1) + (12)(1) + (22)(1) + (10)(1) + (5)(1) + (13)(1) + (11)(1) + (6)(1) + (2)(1) + (6)(1) = 87$ inches
(b) $D \approx (12)(1) + (22)(1) + (10)(1) + (5)(1) + (13)(1) + (11)(1) + (6)(1) + (2)(1) + (6)(1) + (0)(1) = 87$ inches

10. (a) $D \approx (1)(300) + (1.2)(300) + (1.7)(300) + (2.0)(300) + (1.8)(300) + (1.6)(300) + (1.4)(300) + (1.2)(300)$ $+ (1.0)(300) + (1.8)(300) + (1.5)(300) + (1.2)(300) = 5220$ meters (NOTE: 5 minutes = 300 seconds)
(b) $D \approx (1.2)(300) + (1.7)(300) + (2.0)(300) + (1.8)(300) + (1.6)(300) + (1.4)(300) + (1.2)(300) + (1.0)(300)$ $+ (1.8)(300) + (1.5)(300) + (1.2)(300) + (0)(300) = 4920$ meters (NOTE: 5 minutes = 300 seconds)

11. (a) $D \approx (0)(10) + (44)(10) + (15)(10) + (35)(10) + (30)(10) + (44)(10) + (35)(10) + (15)(10) + (22)(10)$ $+ (35)(10) + (44)(10) + (30)(10) = 3490$ feet ≈ 0.66 miles
(b) $D \approx (44)(10) + (15)(10) + (35)(10) + (30)(10) + (44)(10) + (35)(10) + (15)(10) + (22)(10) + (35)(10)$ $+ (44)(10) + (30)(10) + (35)(10) = 3840$ feet ≈ 0.73 miles

12. (a) The distance traveled will be the area under the curve. We will use the approximate velocities at the midpoints of each time interval to approximate this area using rectangles. Thus,
$D \approx (20)(0.001) + (50)(0.001) + (72)(0.001) + (90)(0.001) + (102)(0.001) + (112)(0.001) + (120)(0.001)$ $+ (128)(0.001) + (134)(0.001) + (139)(0.001) \approx 0.967$ miles
(b) Roughly, after 0.0063 hours, the car would have gone 0.484 miles, where 0.0060 hours = 22.7 sec. At 22.7 sec, the velocity was approximately 120 mi/hr.

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13. (a) Because the acceleration is decreasing, an upper estimate is obtained using left end-points in summing acceleration $\cdot \Delta t$. Thus, $\Delta t = 1$ and speed $\approx [32.00 + 19.41 + 11.77 + 7.14 + 4.33](1) = 74.65$ ft/sec
(b) Using right end-points we obtain a lower estimate: speed $\approx [19.41 + 11.77 + 7.14 + 4.33 + 2.63](1)$ $= 45.28$ ft/sec
(c) Upper estimates for the speed at each second are:

t	0	1	2	3	4	5
v	0	32.00	51.41	63.18	70.32	74.65

Thus, the distance fallen when t = 3 seconds is s $\approx [32.00 + 51.41 + 63.18](1) = 146.59$ ft.

14. (a) The speed is a decreasing function of time $\Rightarrow$ right end-points give an lower estimate for the height (distance) attained. Also

t	0	1	2	3	4	5
v	400	368	336	304	272	240

gives the time-velocity table by subtracting the constant g = 32 from the speed at each time increment $\Delta t = 1$ sec. Thus, the speed ≈ 240 ft/sec after 5 seconds.
(b) A lower estimate for height attained is $h \approx [368 + 336 + 304 + 272 + 240](1) = 1520$ ft.

15. Partition $[0, 2]$ into the four subintervals $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, and $[1.5, 2]$. The midpoints of these subintervals are $m_1 = 0.25$, $m_2 = 0.75$, $m_3 = 1.25$, and $m_4 = 1.75$. The heights of the four approximating rectangles are $f(m_1) = (0.25)^3 = \frac{1}{64}$, $f(m_2) = (0.75)^3 = \frac{27}{64}$, $f(m_3) = (1.25)^3 = \frac{125}{64}$, and $f(m_4) = (1.75)^3 = \frac{343}{64}$
Notice that the average value is approximated by $\frac{1}{2}\left[\left(\frac{1}{4}\right)^3\left(\frac{1}{2}\right) + \left(\frac{3}{4}\right)^3\left(\frac{1}{2}\right) + \left(\frac{5}{4}\right)^3\left(\frac{1}{2}\right) + \left(\frac{7}{4}\right)^3\left(\frac{1}{2}\right)\right] = \frac{31}{16}$
$= \frac{1}{\text{length of } [0,2]} \cdot \left[\begin{matrix}\text{approximate area under} \\ \text{curve } f(x) = x^3\end{matrix}\right]$. We use this observation in solving the next several exercises.

16. Partition $[1, 9]$ into the four subintervals $[1, 3]$, $[3, 5]$, $[5, 7]$, and $[7, 9]$. The midpoints of these subintervals are $m_1 = 2$, $m_2 = 4$, $m_3 = 6$, and $m_4 = 8$. The heights of the four approximating rectangles are $f(m_1) = \frac{1}{2}$, $f(m_2) = \frac{1}{4}$, $f(m_3) = \frac{1}{6}$, and $f(m_4) = \frac{1}{8}$. The width of each rectangle is $\Delta x = 2$. Thus,
Area $\approx 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 2\left(\frac{1}{6}\right) + 2\left(\frac{1}{8}\right) = \frac{25}{12} \Rightarrow$ average value $\approx \frac{\text{area}}{\text{length of } [1,9]} = \frac{\left(\frac{25}{12}\right)}{8} = \frac{25}{96}$.

17. Partition $[0, 2]$ into the four subintervals $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, and $[1.5, 2]$. The midpoints of the subintervals are $m_1 = 0.25$, $m_2 = 0.75$, $m_3 = 1.25$, and $m_4 = 1.75$. The heights of the four approximating rectangles are
$f(m_1) = \frac{1}{2} + \sin^2\frac{\pi}{4} = \frac{1}{2} + \frac{1}{2} = 1$, $f(m_2) = \frac{1}{2} + \sin^2\frac{3\pi}{4} = \frac{1}{2} + \frac{1}{2} = 1$, $f(m_3) = \frac{1}{2} + \sin^2\frac{5\pi}{4} = \frac{1}{2} + \left(-\frac{1}{\sqrt{2}}\right)^2$
$= \frac{1}{2} + \frac{1}{2} = 1$, and $f(m_4) = \frac{1}{2} + \sin^2\frac{7\pi}{4} = \frac{1}{2} + \left(-\frac{1}{\sqrt{2}}\right)^2 = 1$. The width of each rectangle is $\Delta x = \frac{1}{2}$. Thus,
Area $\approx (1 + 1 + 1 + 1)\left(\frac{1}{2}\right) = 2 \Rightarrow$ average value $\approx \frac{\text{area}}{\text{length of } [0,2]} = \frac{2}{2} = 1$.

18. Partition $[0, 4]$ into the four subintervals $[0, 1]$, $[1, 2,]$, $[2, 3]$, and $[3, 4]$. The midpoints of the subintervals are $m_1 = \frac{1}{2}$, $m_2 = \frac{3}{2}$, $m_3 = \frac{5}{2}$, and $m_4 = \frac{7}{2}$. The heights of the four approximating rectangles are
$f(m_1) = 1 - \left(\cos\left(\frac{\pi\left(\frac{1}{2}\right)}{4}\right)\right)^4 = 1 - \left(\cos\left(\frac{\pi}{8}\right)\right)^4 = 0.27145$ (to 5 decimal places),
$f(m_2) = 1 - \left(\cos\left(\frac{\pi\left(\frac{3}{2}\right)}{4}\right)\right)^4 = 1 - \left(\cos\left(\frac{3\pi}{8}\right)\right)^4 = 0.97855$, $f(m_3) = 1 - \left(\cos\left(\frac{\pi\left(\frac{5}{2}\right)}{4}\right)\right)^4 = 1 - \left(\cos\left(\frac{5\pi}{8}\right)\right)^4$
$= 0.97855$, and $f(m_4) = 1 - \left(\cos\left(\frac{\pi\left(\frac{7}{2}\right)}{4}\right)\right)^4 = 1 - \left(\cos\left(\frac{7\pi}{8}\right)\right)^4 = 0.27145$. The width of each rectangle is $\Delta x = 1$. Thus, Area $\approx (0.27145)(1) + (0.97855)(1) + (0.97855)(1) + (0.27145)(1) = 2.5 \Rightarrow$ average value $\approx \frac{\text{area}}{\text{length of } [0,4]} = \frac{2.5}{4} = \frac{5}{8}$.

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19. Since the leakage is increasing, an upper estimate uses right endpoints and a lower estimate uses left endpoints:
 (a) upper estimate $= (70)(1) + (97)(1) + (136)(1) + (190)(1) + (265)(1) = 758$ gal,
 lower estimate $= (50)(1) + (70)(1) + (97)(1) + (136)(1) + (190)(1) = 543$ gal.
 (b) upper estimate $= (70 + 97 + 136 + 190 + 265 + 369 + 516 + 720) = 2363$ gal,
 lower estimate $= (50 + 70 + 97 + 136 + 190 + 265 + 369 + 516) = 1693$ gal.
 (c) worst case: $2363 + 720t = 25{,}000 \Rightarrow t \approx 31.4$ hrs;
 best case: $1693 + 720t = 25{,}000 \Rightarrow t \approx 32.4$ hrs

20. Since the pollutant release increases over time, an upper estimate uses right endpoints and a lower estimate uses left endpoints:
 (a) upper estimate $= (0.2)(30) + (0.25)(30) + (0.27)(30) + (0.34)(30) + (0.45)(30) + (0.52)(30) = 60.9$ tons
 lower estimate $= (0.05)(30) + (0.2)(30) + (0.25)(30) + (0.27)(30) + (0.34)(30) + (0.45)(30) = 46.8$ tons
 (b) Using the lower (best case) estimate: $46.8 + (0.52)(30) + (0.63)(30) + (0.70)(30) + (0.81)(30) = 126.6$ tons, so near the end of September 125 tons of pollutants will have been released.

21. (a) The diagonal of the square has length 2, so the side length is $\sqrt{2}$. Area $= \left(\sqrt{2}\right)^2 = 2$
 (b) Think of the octagon as a collection of 16 right triangles with a hypotenuse of length 1 and an acute angle measuring $\frac{2\pi}{16} = \frac{\pi}{8}$.
 Area $= 16\left(\frac{1}{2}\right)\left(\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right) = 4\sin\frac{\pi}{4} = 2\sqrt{2} \approx 2.828$
 (c) Think of the 16-gon as a collection of 32 right triangles with a hypotenuse of length 1 and an acute angle measuring $\frac{2\pi}{32} = \frac{\pi}{16}$.
 Area $= 32\left(\frac{1}{2}\right)\left(\sin\frac{\pi}{16}\right)\left(\cos\frac{\pi}{16}\right) = 8\sin\frac{\pi}{8} = 2\sqrt{2} \approx 3.061$
 (d) Each area is less than the area of the circle, π. As n increases, the area approaches π.

22. (a) Each of the isosceles triangles is made up of two right triangles having hypotenuse 1 and an acute angle measuring $\frac{2\pi}{2n} = \frac{\pi}{n}$. The area of each isosceles triangle is $A_T = 2\left(\frac{1}{2}\right)\left(\sin\frac{\pi}{n}\right)\left(\cos\frac{\pi}{n}\right) = \frac{1}{2}\sin\frac{2\pi}{n}$.
 (b) The area of the polygon is $A_P = nA_T = \frac{n}{2}\sin\frac{2\pi}{n}$, so $\lim\limits_{n\to\infty}\frac{n}{2}\sin\frac{2\pi}{n} = \lim\limits_{n\to\infty}\pi\cdot\frac{\sin\frac{2\pi}{n}}{\left(\frac{2\pi}{n}\right)} = \pi$
 (c) Multiply each area by r^2.
 $A_T = \frac{1}{2}r^2\sin\frac{2\pi}{n}$
 $A_P = \frac{n}{2}r^2\sin\frac{2\pi}{n}$
 $\lim\limits_{n\to\infty} A_P = \pi r^2$

23-26. Example CAS commands:

Maple:

```
with( Student[Calculus1] );
f := x -> sin(x);
a := 0;
b := Pi;
plot( f(x), x=a..b, title="#23(a) (Section 5.1)" );
N := [ 100, 200, 1000 ];                       # (b)
for n in N do
  Xlist := [ a+1.*(b-a)/n*i $ i=0..n ];
  Ylist := map( f, Xlist );
end do:
for n in N do                                  # (c)
  Avg[n] := evalf(add(y,y=Ylist)/nops(Ylist));
```

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```
end do;
avg := FunctionAverage( f(x), x=a..b, output=value );
evalf( avg );
FunctionAverage(f(x),x=a..b,output=plot);      # (d)
fsolve( f(x)=avg, x=0.5 );
fsolve( f(x)=avg, x=2.5 );
fsolve( f(x)=Avg[1000], x=0.5 );
fsolve( f(x)=Avg[1000], x=2.5 );
```

Mathematica: (assigned function and values for a and b may vary):

Symbols for π, $\rightarrow$, powers, roots, fractions, etc. are available in Palettes (under File).

Never insert a space between the name of a function and its argument.

```
Clear[x]
f[x_]:=x  Sin[1/x]
{a,b}={π/4, π}
Plot[f[x],{x, a, b}]
```

The following code computes the value of the function for each interval midpoint and then finds the average. Each sequence of commands for a different value of n (number of subdivisions) should be placed in a separate cell.

```
n =100; dx = (b − a) /n;
values = Table[N[f[x]], {x, a + dx/2, b, dx}]
average=Sum[values[[i]],{i, 1, Length[values]}] / n
n =200; dx = (b − a) /n;
values = Table[N[f[x]],{x, a + dx/2, b, dx}]
average=Sum[values[[i]],{i, 1, Length[values]}] / n
n =1000; dx = (b − a) /n;
values = Table[N[f[x]],{x, a + dx/2, b, dx}]
average=Sum[values[[i]],{i, 1, Length[values]}] / n
FindRoot[f[x] == average,{x, a}]
```

5.2 SIGMA NOTATION AND LIMITS OF FINITE SUMS

1. $\sum_{k=1}^{2} \frac{6k}{k+1} = \frac{6(1)}{1+1} + \frac{6(2)}{2+1} = \frac{6}{2} + \frac{12}{3} = 7$

2. $\sum_{k=1}^{3} \frac{k-1}{k} = \frac{1-1}{1} + \frac{2-1}{2} + \frac{3-1}{3} = 0 + \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$

3. $\sum_{k=1}^{4} \cos k\pi = \cos(1\pi) + \cos(2\pi) + \cos(3\pi) + \cos(4\pi) = -1 + 1 - 1 + 1 = 0$

4. $\sum_{k=1}^{5} \sin k\pi = \sin(1\pi) + \sin(2\pi) + \sin(3\pi) + \sin(4\pi) + \sin(5\pi) = 0 + 0 + 0 + 0 + 0 = 0$

5. $\sum_{k=1}^{3} (-1)^{k+1} \sin \frac{\pi}{k} = (-1)^{1+1} \sin \frac{\pi}{1} + (-1)^{2+1} \sin \frac{\pi}{2} + (-1)^{3+1} \sin \frac{\pi}{3} = 0 - 1 + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}-2}{2}$

6. $\sum_{k=1}^{4} (-1)^{k} \cos k\pi = (-1)^{1} \cos(1\pi) + (-1)^{2} \cos(2\pi) + (-1)^{3} \cos(3\pi) + (-1)^{4} \cos(4\pi)$

$= -(-1) + 1 - (-1) + 1 = 4$

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7. (a) $\sum_{k=1}^{6} 2^{k-1} = 2^{1-1} + 2^{2-1} + 2^{3-1} + 2^{4-1} + 2^{5-1} + 2^{6-1} = 1 + 2 + 4 + 8 + 16 + 32$

(b) $\sum_{k=0}^{5} 2^{k} = 2^{0} + 2^{1} + 2^{2} + 2^{3} + 2^{4} + 2^{5} = 1 + 2 + 4 + 8 + 16 + 32$

(c) $\sum_{k=-1}^{4} 2^{k+1} = 2^{-1+1} + 2^{0+1} + 2^{1+1} + 2^{2+1} + 2^{3+1} + 2^{4+1} = 1 + 2 + 4 + 8 + 16 + 32$

All of them represent $1 + 2 + 4 + 8 + 16 + 32$

8. (a) $\sum_{k=1}^{6} (-2)^{k-1} = (-2)^{1-1} + (-2)^{2-1} + (-2)^{3-1} + (-2)^{4-1} + (-2)^{5-1} + (-2)^{6-1} = 1 - 2 + 4 - 8 + 16 - 32$

(b) $\sum_{k=0}^{5} (-1)^{k}2^{k} = (-1)^{0}2^{0} + (-1)^{1}2^{1} + (-1)^{2}2^{2} + (-1)^{3}2^{3} + (-1)^{4}2^{4} + (-1)^{5}2^{5} = 1 - 2 + 4 - 8 + 16 - 32$

(c) $\sum_{k=-2}^{3} (-1)^{k+1}2^{k+2} = (-1)^{-2+1}2^{-2+2} + (-1)^{-1+1}2^{-1+2} + (-1)^{0+1}2^{0+2} + (-1)^{1+1}2^{1+2} + (-1)^{2+1}2^{2+2}$
$+ (-1)^{3+1}2^{3+2} = -1 + 2 - 4 + 8 - 16 + 32;$

(a) and (b) represent $1 - 2 + 4 - 8 + 16 - 32$; (c) is not equivalent to the other two

9. (a) $\sum_{k=2}^{4} \frac{(-1)^{k-1}}{k-1} = \frac{(-1)^{2-1}}{2-1} + \frac{(-1)^{3-1}}{3-1} + \frac{(-1)^{4-1}}{4-1} = -1 + \frac{1}{2} - \frac{1}{3}$

(b) $\sum_{k=0}^{2} \frac{(-1)^{k}}{k+1} = \frac{(-1)^{0}}{0+1} + \frac{(-1)^{1}}{1+1} + \frac{(-1)^{2}}{2+1} = 1 - \frac{1}{2} + \frac{1}{3}$

(c) $\sum_{k=-1}^{1} \frac{(-1)^{k}}{k+2} = \frac{(-1)^{-1}}{-1+2} + \frac{(-1)^{0}}{0+2} + \frac{(-1)^{1}}{1+2} = -1 + \frac{1}{2} - \frac{1}{3}$

(a) and (c) are equivalent; (b) is not equivalent to the other two.

10. (a) $\sum_{k=1}^{4} (k-1)^2 = (1-1)^2 + (2-1)^2 + (3-1)^2 + (4-1)^2 = 0 + 1 + 4 + 9$

(b) $\sum_{k=-1}^{3} (k+1)^2 = (-1+1)^2 + (0+1)^2 + (1+1)^2 + (2+1)^2 + (3+1)^2 = 0 + 1 + 4 + 9 + 16$

(c) $\sum_{k=-3}^{-1} k^2 = (-3)^2 + (-2)^2 + (-1)^2 = 9 + 4 + 1$

(a) and (c) are equivalent to each other; (b) is not equivalent to the other two.

11. $\sum_{k=1}^{6} k$

12. $\sum_{k=1}^{4} k^2$

13. $\sum_{k=1}^{4} \frac{1}{2^k}$

14. $\sum_{k=1}^{5} 2k$

15. $\sum_{k=1}^{5} (-1)^{k+1} \frac{1}{k}$

16. $\sum_{k=1}^{5} (-1)^{k} \frac{k}{5}$

17. (a) $\sum_{k=1}^{n} 3a_k = 3 \sum_{k=1}^{n} a_k = 3(-5) = -15$

(b) $\sum_{k=1}^{n} \frac{b_k}{6} = \frac{1}{6} \sum_{k=1}^{n} b_k = \frac{1}{6}(6) = 1$

(c) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = -5 + 6 = 1$

(d) $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k = -5 - 6 = -11$

(e) $\sum_{k=1}^{n} (b_k - 2a_k) = \sum_{k=1}^{n} b_k - 2 \sum_{k=1}^{n} a_k = 6 - 2(-5) = 16$

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18. (a) $\sum_{k=1}^{n} 8a_k = 8\sum_{k=1}^{n} a_k = 8(0) = 0$ (b) $\sum_{k=1}^{n} 250b_k = 250\sum_{k=1}^{n} b_k = 250(1) = 250$

(c) $\sum_{k=1}^{n} (a_k + 1) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} 1 = 0 + n = n$ (d) $\sum_{k=1}^{n} (b_k - 1) = \sum_{k=1}^{n} b_k - \sum_{k=1}^{n} 1 = 1 - n$

19. (a) $\sum_{k=1}^{10} k = \frac{10(10+1)}{2} = 55$ (b) $\sum_{k=1}^{10} k^2 = \frac{10(10+1)(2(10)+1)}{6} = 385$

(c) $\sum_{k=1}^{10} k^3 = \left[\frac{10(10+1)}{2}\right]^2 = 55^2 = 3025$

20. (a) $\sum_{k=1}^{13} k = \frac{13(13+1)}{2} = 91$ (b) $\sum_{k=1}^{13} k^2 = \frac{13(13+1)(2(13)+1)}{6} = 819$

(c) $\sum_{k=1}^{13} k^3 = \left[\frac{13(13+1)}{2}\right]^2 = 91^2 = 8281$

21. $\sum_{k=1}^{7} -2k = -2\sum_{k=1}^{7} k = -2\left(\frac{7(7+1)}{2}\right) = -56$

22. $\sum_{k=1}^{5} \frac{\pi k}{15} = \frac{\pi}{15}\sum_{k=1}^{5} k = \frac{\pi}{15}\left(\frac{5(5+1)}{2}\right) = \pi$

23. $\sum_{k=1}^{6} (3 - k^2) = \sum_{k=1}^{6} 3 - \sum_{k=1}^{6} k^2 = 3(6) - \frac{6(6+1)(2(6)+1)}{6} = -73$

24. $\sum_{k=1}^{6} (k^2 - 5) = \sum_{k=1}^{6} k^2 - \sum_{k=1}^{6} 5 = \frac{6(6+1)(2(6)+1)}{6} - 5(6) = 61$

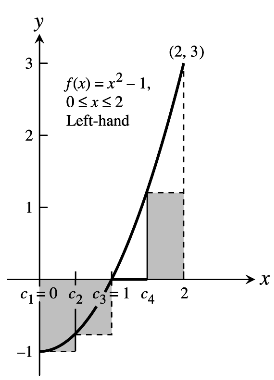
25. $\sum_{k=1}^{5} k(3k+5) = \sum_{k=1}^{5} (3k^2 + 5k) = 3\sum_{k=1}^{5} k^2 + 5\sum_{k=1}^{5} k = 3\left(\frac{5(5+1)(2(5)+1)}{6}\right) + 5\left(\frac{5(5+1)}{2}\right) = 240$

26. $\sum_{k=1}^{7} k(2k+1) = \sum_{k=1}^{7} (2k^2 + k) = 2\sum_{k=1}^{7} k^2 + \sum_{k=1}^{7} k = 2\left(\frac{7(7+1)(2(7)+1)}{6}\right) + \frac{7(7+1)}{2} = 308$

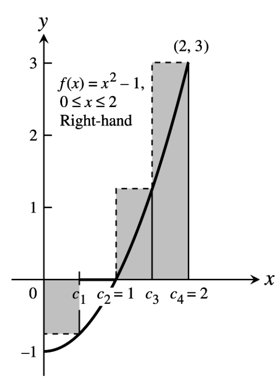
27. $\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3 = \frac{1}{225}\sum_{k=1}^{5} k^3 + \left(\sum_{k=1}^{5} k\right)^3 = \frac{1}{225}\left(\frac{5(5+1)}{2}\right)^2 + \left(\frac{5(5+1)}{2}\right)^3 = 3376$

28. $\left(\sum_{k=1}^{7} k\right)^2 - \sum_{k=1}^{7} \frac{k^3}{4} = \left(\sum_{k=1}^{7} k\right)^2 - \frac{1}{4}\sum_{k=1}^{7} k^3 = \left(\frac{7(7+1)}{2}\right)^2 - \frac{1}{4}\left(\frac{7(7+1)}{2}\right)^2 = 588$

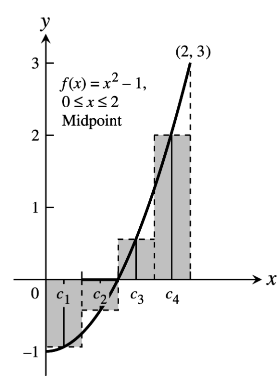
29. (a)



(b)



(c)



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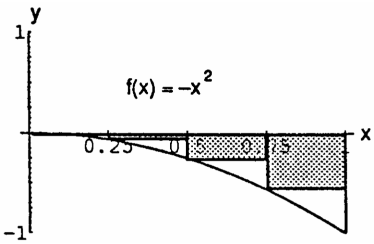
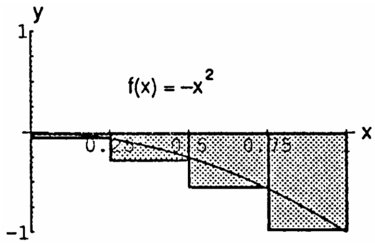
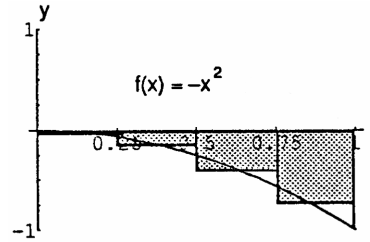
30. (a)



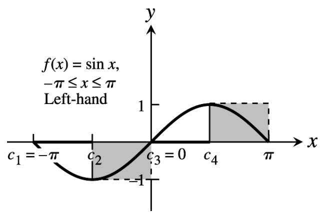
(b)



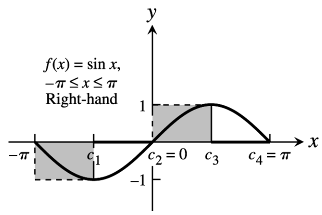
(c)



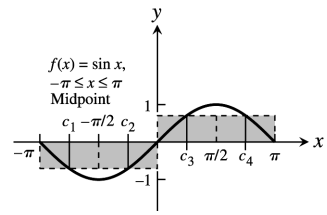
31. (a)



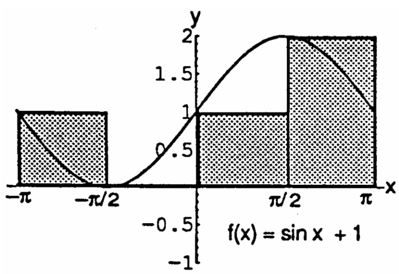
(b)



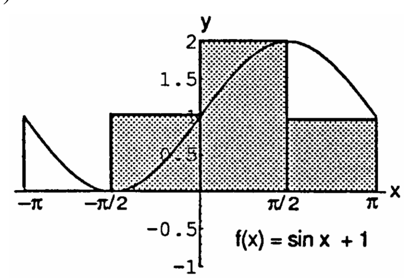
(c)



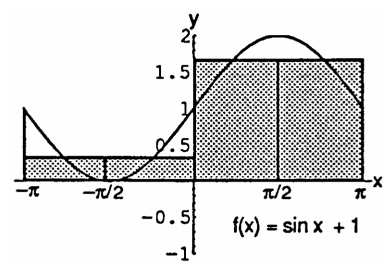
32. (a)



(b)



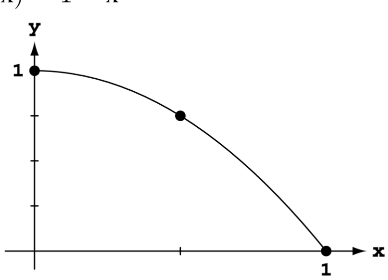
(c)



33. $|x_1 - x_0| = |1.2 - 0| = 1.2$, $|x_2 - x_1| = |1.5 - 1.2| = 0.3$, $|x_3 - x_2| = |2.3 - 1.5| = 0.8$, $|x_4 - x_3| = |2.6 - 2.3| = 0.3$, and $|x_5 - x_4| = |3 - 2.6| = 0.4$; the largest is $\|P\| = 1.2$.

34. $|x_1 - x_0| = |-1.6 - (-2)| = 0.4$, $|x_2 - x_1| = |-0.5 - (-1.6)| = 1.1$, $|x_3 - x_2| = |0 - (-0.5)| = 0.5$, $|x_4 - x_3| = |0.8 - 0| = 0.8$, and $|x_5 - x_4| = |1 - 0.8| = 0.2$; the largest is $\|P\| = 1.1$.

35. $f(x) = 1 - x^2$



Since f is decreasing on $[0, 1]$ we use left endpoints to obtain upper sums. $\triangle x = \frac{1-0}{n} = \frac{1}{n}$ and $x_i = i\triangle x = \frac{i}{n}$. So an upper sum is $\sum_{i=0}^{n-1}(1 - x_i^2)\frac{1}{n} = \frac{1}{n}\sum_{i=0}^{n-1}\left(1 - \left(\frac{i}{n}\right)^2\right) = \frac{1}{n^3}\sum_{i=0}^{n-1}(n^2 - i^2)$

$= \frac{n^3}{n^3} - \frac{1}{n^3}\sum_{i=0}^{n} i^2 = 1 - \frac{(n-1)n(2(n-1)+1)}{6n^3} = 1 - \frac{2n^3 - 3n^2 + n}{6n^3}$

$= 1 - \frac{2 - \frac{3}{n} + \frac{1}{n^2}}{6}$. Thus,

$\lim_{n\to\infty}\sum_{i=0}^{n-1}(1 - x_i^2)\frac{1}{n} = \lim_{n\to\infty}\left(1 - \frac{2 - \frac{3}{n} + \frac{1}{n^2}}{6}\right) = 1 - \frac{1}{3} = \frac{2}{3}$

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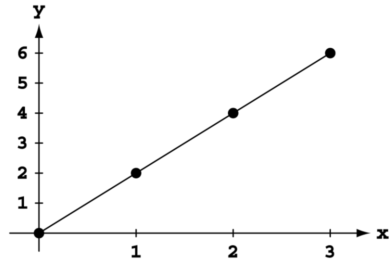
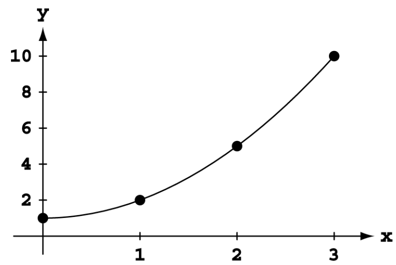
36. $f(x) = 2x$



Since f is increasing on $[0, 3]$ we use right endpoints to obtain upper sums. $\triangle x = \frac{3-0}{n} = \frac{3}{n}$ and $x_i = i\triangle x = \frac{3i}{n}$. So an upper sum is $\sum_{i=1}^{n} 2x_i\left(\frac{3}{n}\right) = \sum_{i=1}^{n} \frac{6i}{n} \cdot \frac{3}{n} = \frac{18}{n^2}\sum_{i=1}^{n} i = \frac{18}{n^2} \cdot \frac{n(n+1)}{2} = \frac{9n^2+9n}{n^2}$

Thus, $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6i}{n} \cdot \frac{3}{n} = \lim_{n\to\infty} \frac{9n^2+9n}{n^2} = \lim_{n\to\infty} \left(9 + \frac{9}{n}\right) = 9.$

37. $f(x) = x^2 + 1$



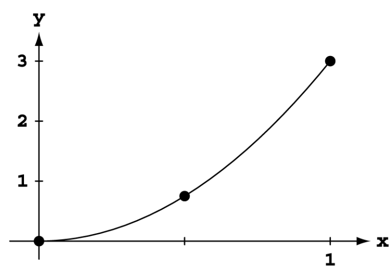
Since f is increasing on $[0, 3]$ we use right endpoints to obtain upper sums. $\triangle x = \frac{3-0}{n} = \frac{3}{n}$ and $x_i = i\triangle x = \frac{3i}{n}$. So an upper sum is $\sum_{i=1}^{n}(x_i^2+1)\frac{3}{n} = \sum_{i=1}^{n}\left(\left(\frac{3i}{n}\right)^2+1\right)\frac{3}{n} = \frac{3}{n}\sum_{i=1}^{n}\left(\frac{9i^2}{n^2}+1\right)$

$= \frac{27}{n}\sum_{i=1}^{n} i^2 + \frac{3}{n} \cdot n = \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + 3$

$= \frac{9(2n^3+3n^2+n)}{2n^3} + 3 = \frac{18+\frac{27}{n}+\frac{9}{n^2}}{2} + 3$. Thus,

$\lim_{n\to\infty} \sum_{i=1}^{n}(x_i^2+1)\frac{3}{n} = \lim_{n\to\infty}\left(\frac{18+\frac{27}{n}+\frac{9}{n^2}}{2} + 3\right) = 9 + 3 = 12.$

38. $f(x) = 3x^2$

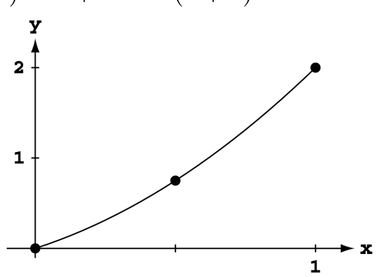


Since f is increasing on $[0, 1]$ we use right endpoints to obtain upper sums. $\triangle x = \frac{1-0}{n} = \frac{1}{n}$ and $x_i = i\triangle x = \frac{i}{n}$. So an upper sum is $\sum_{i=1}^{n} 3x_i^2\left(\frac{1}{n}\right) = \sum_{i=1}^{n} 3\left(\frac{i}{n}\right)^2\left(\frac{1}{n}\right) = \frac{3}{n^3}\sum_{i=1}^{n} i^2 = \frac{3}{n^3} \cdot \left(\frac{n(n+1)(2n+1)}{6}\right)$

$= \frac{2n^3+3n^2+n}{2n^3} = \frac{2+\frac{3}{n}+\frac{1}{n^2}}{2}$. Thus, $\lim_{n\to\infty} \sum_{i=1}^{n} 3x_i^2\left(\frac{1}{n}\right)$

$= \lim_{n\to\infty}\left(\frac{2+\frac{3}{n}+\frac{1}{n^2}}{2}\right) = \frac{2}{2} = 1.$

39. $f(x) = x + x^2 = x(1+x)$



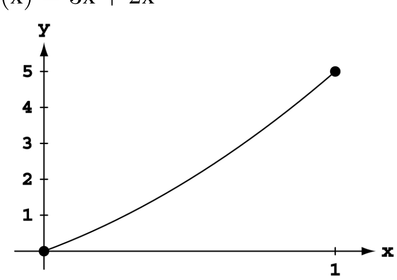
Since f is increasing on $[0, 1]$ we use right endpoints to obtain upper sums. $\triangle x = \frac{1-0}{n} = \frac{1}{n}$ and $x_i = i\triangle x = \frac{i}{n}$. So an upper sum is $\sum_{i=1}^{n}(x_i + x_i^2)\frac{1}{n} = \sum_{i=1}^{n}\left(\frac{i}{n} + \left(\frac{i}{n}\right)^2\right)\frac{1}{n} = \frac{1}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2$

$= \frac{1}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{n^2+n}{2n^2} + \frac{2n^3+3n^2+n}{6n^3}$

$= \frac{1+\frac{1}{n}}{2} + \frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}$. Thus, $\lim_{n\to\infty} \sum_{i=1}^{n}(x_i + x_i^2)\frac{1}{n}$

$= \lim_{n\to\infty}\left[\left(\frac{1+\frac{1}{n}}{2}\right) + \left(\frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}\right)\right] = \frac{1}{2} + \frac{2}{6} = \frac{5}{6}.$

40. $f(x) = 3x + 2x^2$



Since f is increasing on $[0, 1]$ we use right endpoints to obtain upper sums. $\triangle x = \frac{1-0}{n} = \frac{1}{n}$ and $x_i = i\triangle x = \frac{i}{n}$. So an upper sum is $\sum_{i=1}^{n}(3x_i + 2x_i^2)\frac{1}{n} = \sum_{i=1}^{n}\left(\frac{3i}{n} + 2\left(\frac{i}{n}\right)^2\right)\frac{1}{n} = \frac{3}{n^2}\sum_{i=1}^{n} i + \frac{2}{n^3}\sum_{i=1}^{n} i^2$

$= \frac{3}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{2}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{3n^2+3n}{2n^2} + \frac{2n^2+3n+1}{3n^2}$

$= \frac{3+\frac{3}{n}}{2} + \frac{2+\frac{3}{n}+\frac{1}{n^2}}{3}$. Thus, $\lim_{n\to\infty} \sum_{i=1}^{n}(3x_i + 2x_i^2)\frac{1}{n}$

$= \lim_{n\to\infty}\left[\left(\frac{3+\frac{3}{n}}{2}\right) + \left(\frac{2+\frac{3}{n}+\frac{1}{n^2}}{3}\right)\right] = \frac{3}{2} + \frac{2}{3} = \frac{13}{6}.$

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5.3 THE DEFINITE INTEGRAL

1. $\int_0^2 x^2\,dx$

2. $\int_{-1}^0 2x^3\,dx$

3. $\int_{-7}^5 (x^2-3x)\,dx$

4. $\int_1^4 \frac{1}{x}\,dx$

5. $\int_2^3 \frac{1}{1-x}\,dx$

6. $\int_0^1 \sqrt{4-x^2}\,dx$

7. $\int_{-\pi/4}^0 (\sec x)\,dx$

8. $\int_0^{\pi/4} (\tan x)\,dx$

9. (a) $\int_2^2 g(x)\,dx = 0$

(b) $\int_5^1 g(x)\,dx = -\int_1^5 g(x)\,dx = -8$

(c) $\int_1^2 3f(x)\,dx = 3\int_1^2 f(x)\,dx = 3(-4) = -12$

(d) $\int_2^5 f(x)\,dx = \int_1^5 f(x)\,dx - \int_1^2 f(x)\,dx = 6-(-4) = 10$

(e) $\int_1^5 [f(x)-g(x)]\,dx = \int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx = 6-8 = -2$

(f) $\int_1^5 [4f(x)-g(x)]\,dx = 4\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx = 4(6)-8 = 16$

10. (a) $\int_1^9 -2f(x)\,dx = -2\int_1^9 f(x)\,dx = -2(-1) = 2$

(b) $\int_7^9 [f(x)+h(x)]\,dx = \int_7^9 f(x)\,dx + \int_7^9 h(x)\,dx = 5+4 = 9$

(c) $\int_7^9 [2f(x)-3h(x)]\,dx = 2\int_7^9 f(x)\,dx - 3\int_7^9 h(x)\,dx = 2(5)-3(4) = -2$

(d) $\int_9^1 f(x)\,dx = -\int_1^9 f(x)\,dx = -(-1) = 1$

(e) $\int_1^7 f(x)\,dx = \int_1^9 f(x)\,dx - \int_7^9 f(x)\,dx = -1-5 = -6$

(f) $\int_9^7 [h(x)-f(x)]\,dx = \int_7^9 [f(x)-h(x)]\,dx = \int_7^9 f(x)\,dx - \int_7^9 h(x)\,dx = 5-4 = 1$

11. (a) $\int_1^2 f(u)\,du = \int_1^2 f(x)\,dx = 5$

(b) $\int_1^2 \sqrt{3}\,f(z)\,dz = \sqrt{3}\int_1^2 f(z)\,dz = 5\sqrt{3}$

(c) $\int_2^1 f(t)\,dt = -\int_1^2 f(t)\,dt = -5$

(d) $\int_1^2 [-f(x)]\,dx = -\int_1^2 f(x)\,dx = -5$

12. (a) $\int_0^{-3} g(t)\,dt = -\int_{-3}^0 g(t)\,dt = -\sqrt{2}$

(b) $\int_{-3}^0 g(u)\,du = \int_{-3}^0 g(t)\,dt = \sqrt{2}$

(c) $\int_{-3}^0 [-g(x)]\,dx = -\int_{-3}^0 g(x)\,dx = -\sqrt{2}$

(d) $\int_{-3}^0 \frac{g(r)}{\sqrt{2}}\,dr = \frac{1}{\sqrt{2}}\int_{-3}^0 g(t)\,dt = \left(\frac{1}{\sqrt{2}}\right)\left(\sqrt{2}\right) = 1$

13. (a) $\int_3^4 f(z)\,dz = \int_0^4 f(z)\,dz - \int_0^3 f(z)\,dz = 7-3 = 4$

(b) $\int_4^3 f(t)\,dt = -\int_3^4 f(t)\,dt = -4$

14. (a) $\int_1^3 h(r)\,dr = \int_{-1}^3 h(r)\,dr - \int_{-1}^1 h(r)\,dr = 6-0 = 6$

(b) $-\int_3^1 h(u)\,du = -\left(-\int_1^3 h(u)\,du\right) = \int_1^3 h(u)\,du = 6$

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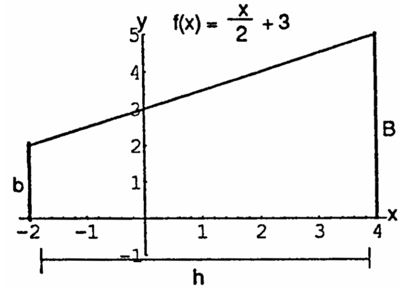
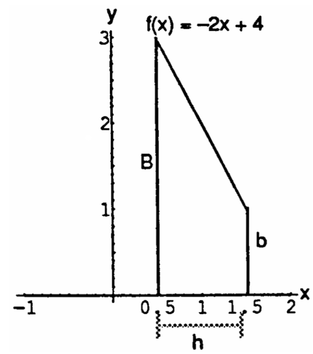
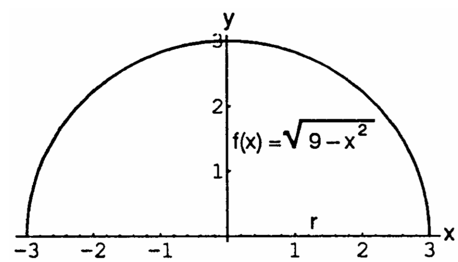
15. The area of the trapezoid is $A = \frac{1}{2}(B + b)h$
$= \frac{1}{2}(5 + 2)(6) = 21 \Rightarrow \int_{-2}^{4} \left(\frac{x}{2} + 3\right) dx$
$= 21$ square units



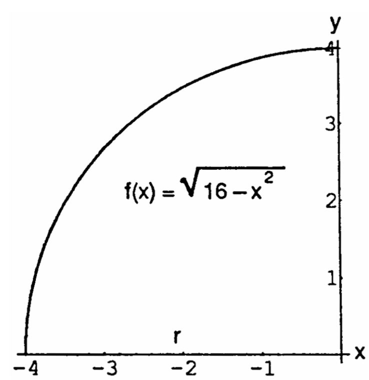
16. The area of the trapezoid is $A = \frac{1}{2}(B + b)h$
$= \frac{1}{2}(3 + 1)(1) = 2 \Rightarrow \int_{1/2}^{3/2} (-2x + 4)\, dx$
$= 2$ square units



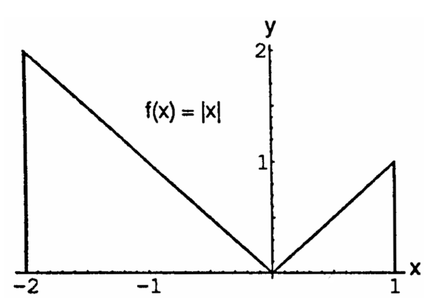
17. The area of the semicircle is $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(3)^2$
$= \frac{9}{2}\pi \Rightarrow \int_{-3}^{3} \sqrt{9 - x^2}\, dx = \frac{9}{2}\pi$ square units



18. The graph of the quarter circle is $A = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(4)^2$
$= 4\pi \Rightarrow \int_{-4}^{0} \sqrt{16 - x^2}\, dx = 4\pi$ square units



19. The area of the triangle on the left is $A = \frac{1}{2}bh = \frac{1}{2}(2)(2)$
$= 2$. The area of the triangle on the right is $A = \frac{1}{2}bh$
$= \frac{1}{2}(1)(1) = \frac{1}{2}$. Then, the total area is 2.5
$\Rightarrow \int_{-2}^{1} |x|\, dx = 2.5$ square units



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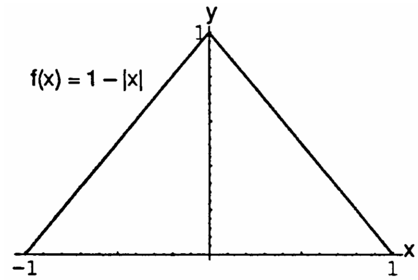
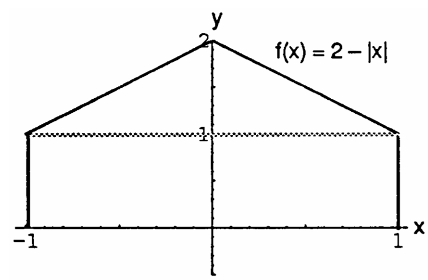
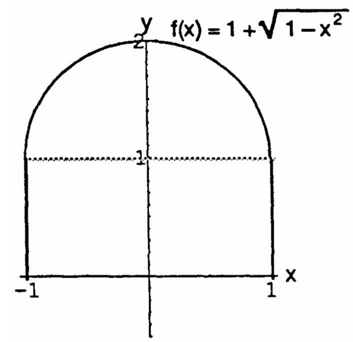
20. The area of the triangle is $A = \frac{1}{2}\,bh = \frac{1}{2}\,(2)(1) = 1$
$\Rightarrow \int_{-1}^{1} (1 - |x|)\,dx = 1$ square unit



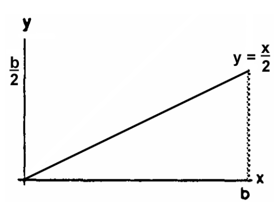
21. The area of the triangular peak is $A = \frac{1}{2}\,bh = \frac{1}{2}\,(2)(1) = 1$.
The area of the rectangular base is $S = \ell w = (2)(1) = 2$.
Then the total area is 3 $\Rightarrow \int_{-1}^{1} (2 - |x|)\,dx = 3$ square units



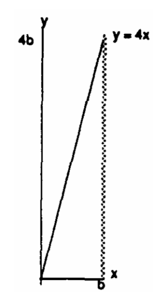
22. $y = 1 + \sqrt{1 - x^2} \Rightarrow y - 1 = \sqrt{1 - x^2}$
$\Rightarrow (y-1)^2 = 1 - x^2 \Rightarrow x^2 + (y-1)^2 = 1$, a circle with center $(0, 1)$ and radius of 1 $\Rightarrow y = 1 + \sqrt{1 - x^2}$ is the upper semicircle. The area of this semicircle is $A = \frac{1}{2}\,\pi r^2 = \frac{1}{2}\,\pi(1)^2 = \frac{\pi}{2}$. The area of the rectangular base is $A = \ell w = (2)(1) = 2$. Then the total area is $2 + \frac{\pi}{2}$
$\Rightarrow \int_{-1}^{1} \left(1 + \sqrt{1 - x^2}\right) dx = 2 + \frac{\pi}{2}$ square units



23. $\int_0^b \frac{x}{2}\,dx = \frac{1}{2}\,(b)(\frac{b}{2}) = \frac{b^2}{4}$



24. $\int_0^b 4x\,dx = \frac{1}{2}\,b(4b) = 2b^2$



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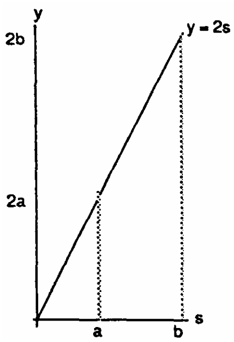
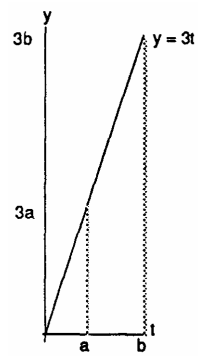
25. $\int_a^b 2s\,ds = \frac{1}{2}b(2b) - \frac{1}{2}a(2a) = b^2 - a^2$



26. $\int_a^b 3t\,dt = \frac{1}{2}b(3b) - \frac{1}{2}a(3a) = \frac{3}{2}(b^2 - a^2)$



27. $\int_1^{\sqrt{2}} x\,dx = \frac{\left(\sqrt{2}\right)^2}{2} - \frac{(1)^2}{2} = \frac{1}{2}$

28. $\int_{0.5}^{2.5} x\,dx = \frac{(2.5)^2}{2} - \frac{(0.5)^2}{2} = 3$

29. $\int_\pi^{2\pi} \theta\,d\theta = \frac{(2\pi)^2}{2} - \frac{\pi^2}{2} = \frac{3\pi^2}{2}$

30. $\int_{\sqrt{2}}^{5\sqrt{2}} r\,dr = \frac{\left(5\sqrt{2}\right)^2}{2} - \frac{\left(\sqrt{2}\right)^2}{2} = 24$

31. $\int_0^{\sqrt[3]{7}} x^2\,dx = \frac{\left(\sqrt[3]{7}\right)^3}{3} = \frac{7}{3}$

32. $\int_0^{0.3} s^2\,ds = \frac{(0.3)^3}{3} = 0.009$

33. $\int_0^{1/2} t^2\,dt = \frac{\left(\frac{1}{2}\right)^3}{3} = \frac{1}{24}$

34. $\int_0^{\pi/2} \theta^2\,d\theta = \frac{\left(\frac{\pi}{2}\right)^3}{3} = \frac{\pi^3}{24}$

35. $\int_a^{2a} x\,dx = \frac{(2a)^2}{2} - \frac{a^2}{2} = \frac{3a^2}{2}$

36. $\int_a^{\sqrt{3}a} x\,dx = \frac{\left(\sqrt{3}a\right)^2}{2} - \frac{a^2}{2} = a^2$

37. $\int_0^{\sqrt[3]{b}} x^2\,dx = \frac{\left(\sqrt[3]{b}\right)^3}{3} = \frac{b}{3}$

38. $\int_0^{3b} x^2\,dx = \frac{(3b)^3}{3} = 9b^3$

39. $\int_3^1 7\,dx = 7(1-3) = -14$

40. $\int_0^{-2} \sqrt{2}\,dx = \sqrt{2}(-2-0) = -2\sqrt{2}$

41. $\int_0^2 5x\,dx = 5\int_0^2 x\,dx = 5\left[\frac{2^2}{2} - \frac{0^2}{2}\right] = 10$

42. $\int_3^5 \frac{x}{8}\,dx = \frac{1}{8}\int_3^5 x\,dx = \frac{1}{8}\left[\frac{5^2}{2} - \frac{3^2}{2}\right] = \frac{16}{16} = 1$

43. $\int_0^2 (2t-3)\,dt = 2\int_1^1 t\,dt - \int_0^2 3\,dt = 2\left[\frac{2^2}{2} - \frac{0^2}{2}\right] - 3(2-0) = 4 - 6 = -2$

44. $\int_0^{\sqrt{2}} \left(t - \sqrt{2}\right) dt = \int_0^{\sqrt{2}} t\,dt - \int_0^{\sqrt{2}} \sqrt{2}\,dt = \left[\frac{\left(\sqrt{2}\right)^2}{2} - \frac{0^2}{2}\right] - \sqrt{2}\left[\sqrt{2} - 0\right] = 1 - 2 = -1$

45. $\int_2^1 \left(1 + \frac{z}{2}\right) dz = \int_2^1 1\,dz + \int_2^1 \frac{z}{2}\,dz = \int_2^1 1\,dz - \frac{1}{2}\int_1^2 z\,dz = 1[1-2] - \frac{1}{2}\left[\frac{2^2}{2} - \frac{1^2}{2}\right] = -1 - \frac{1}{2}\left(\frac{3}{2}\right) = -\frac{7}{4}$

46. $\int_3^0 (2z-3)\,dz = \int_3^0 2z\,dz - \int_3^0 3\,dz = -2\int_0^3 z\,dz - \int_3^0 3\,dz = -2\left[\frac{3^2}{2} - \frac{0^2}{2}\right] - 3[0-3] = -9 + 9 = 0$

47. $\int_1^2 3u^2\,du = 3\int_1^2 u^2\,du = 3\left[\int_0^2 u^2\,du - \int_0^1 u^2\,du\right] = 3\left(\left[\frac{2^3}{3} - \frac{0^3}{3}\right] - \left[\frac{1^3}{3} - \frac{0^3}{3}\right]\right) = 3\left[\frac{2^3}{3} - \frac{1^3}{3}\right] = 3\left(\frac{7}{3}\right) = 7$

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48. $\int_{1/2}^{1} 24u^2\,du = 24\int_{1/2}^{1} u^2\,du = 24\left[\int_0^1 u^2\,du - \int_0^{1/2} u^2\,du\right] = 24\left[\frac{1^3}{3} - \frac{\left(\frac{1}{2}\right)^3}{3}\right] = 24\left[\frac{\left(\frac{7}{8}\right)}{3}\right] = 7$

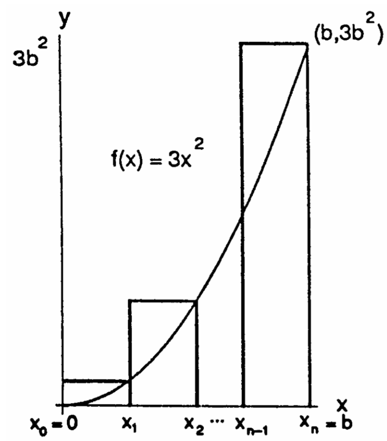
49. $\int_0^2 (3x^2+x-5)\,dx = 3\int_0^2 x^2\,dx + \int_0^2 x\,dx - \int_0^2 5\,dx = 3\left[\frac{2^3}{3} - \frac{0^3}{3}\right] + \left[\frac{2^2}{2} - \frac{0^2}{2}\right] - 5[2-0] = (8+2) - 10 = 0$

50. $\int_1^0 (3x^2+x-5)\,dx = -\int_0^1 (3x^2+x-5)\,dx = -\left[3\int_0^1 x^2\,dx + \int_0^1 x\,dx - \int_0^1 5\,dx\right]$
$= -\left[3\left(\frac{1^3}{3} - \frac{0^3}{3}\right) + \left(\frac{1^2}{2} - \frac{0^2}{2}\right) - 5(1-0)\right] = -\left(\frac{3}{2} - 5\right) = \frac{7}{2}$

51. Let $\Delta x = \frac{b-0}{n} = \frac{b}{n}$ and let $x_0 = 0$, $x_1 = \Delta x$, $x_2 = 2\Delta x, \ldots, x_{n-1} = (n-1)\Delta x$, $x_n = n\Delta x = b$. Let the c_k's be the right end-points of the subintervals $\Rightarrow c_1 = x_1$, $c_2 = x_2$, and so on. The rectangles defined have areas:

$f(c_1)\,\Delta x = f(\Delta x)\,\Delta x = 3(\Delta x)^2\,\Delta x = 3(\Delta x)^3$
$f(c_2)\,\Delta x = f(2\Delta x)\,\Delta x = 3(2\Delta x)^2\,\Delta x = 3(2)^2(\Delta x)^3$
$f(c_3)\,\Delta x = f(3\Delta x)\,\Delta x = 3(3\Delta x)^2\,\Delta x = 3(3)^2(\Delta x)^3$
$\vdots$
$f(c_n)\,\Delta x = f(n\Delta x)\,\Delta x = 3(n\Delta x)^2\,\Delta x = 3(n)^2(\Delta x)^3$

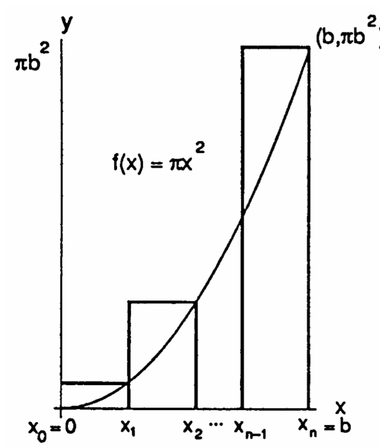
Then $S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x = \sum_{k=1}^{n} 3k^2(\Delta x)^3$
$= 3(\Delta x)^3 \sum_{k=1}^{n} k^2 = 3\left(\frac{b^3}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right)$
$= \frac{b^3}{2}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \Rightarrow \int_0^b 3x^2\,dx = \lim_{n\to\infty} \frac{b^3}{2}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = b^3.$



52. Let $\Delta x = \frac{b-0}{n} = \frac{b}{n}$ and let $x_0 = 0$, $x_1 = \Delta x$, $x_2 = 2\Delta x, \ldots, x_{n-1} = (n-1)\Delta x$, $x_n = n\Delta x = b$. Let the c_k's be the right end-points of the subintervals $\Rightarrow c_1 = x_1$, $c_2 = x_2$, and so on. The rectangles defined have areas:

$f(c_1)\,\Delta x = f(\Delta x)\,\Delta x = \pi(\Delta x)^2\,\Delta x = \pi(\Delta x)^3$
$f(c_2)\,\Delta x = f(2\Delta x)\,\Delta x = \pi(2\Delta x)^2\,\Delta x = \pi(2)^2(\Delta x)^3$
$f(c_3)\,\Delta x = f(3\Delta x)\,\Delta x = \pi(3\Delta x)^2\,\Delta x = \pi(3)^2(\Delta x)^3$
$\vdots$
$f(c_n)\,\Delta x = f(n\Delta x)\,\Delta x = \pi(n\Delta x)^2\,\Delta x = \pi(n)^2(\Delta x)^3$

Then $S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x = \sum_{k=1}^{n} \pi k^2(\Delta x)^3$
$= \pi(\Delta x)^3 \sum_{k=1}^{n} k^2 = \pi\left(\frac{b^3}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right)$
$= \frac{\pi b^3}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \Rightarrow \int_0^b \pi x^2\,dx = \lim_{n\to\infty} \frac{\pi b^3}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{\pi b^3}{3}.$



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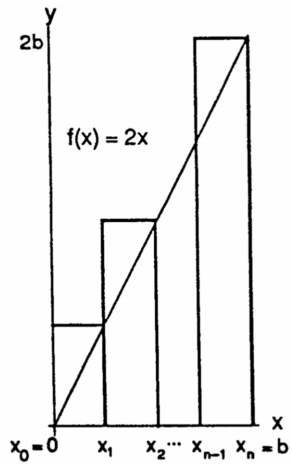
53. Let $\Delta x = \frac{b-0}{n} = \frac{b}{n}$ and let $x_0 = 0$, $x_1 = \Delta x$, $x_2 = 2\Delta x, \ldots, x_{n-1} = (n-1)\Delta x$, $x_n = n\Delta x = b$. Let the c_k's be the right end-points of the subintervals $\Rightarrow$ $c_1 = x_1$, $c_2 = x_2$, and so on. The rectangles defined have areas:

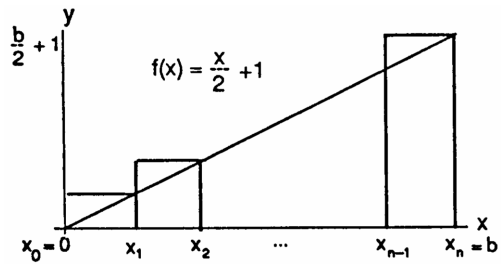
$$f(c_1)\,\Delta x = f(\Delta x)\,\Delta x = 2(\Delta x)(\Delta x) = 2(\Delta x)^2$$
$$f(c_2)\,\Delta x = f(2\Delta x)\,\Delta x = 2(2\Delta x)(\Delta x) = 2(2)(\Delta x)^2$$
$$f(c_3)\,\Delta x = f(3\Delta x)\,\Delta x = 2(3\Delta x)(\Delta x) = 2(3)(\Delta x)^2$$
$$\vdots$$
$$f(c_n)\,\Delta x = f(n\Delta x)\,\Delta x = 2(n\Delta x)(\Delta x) = 2(n)(\Delta x)^2$$

Then $S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x = \sum_{k=1}^{n} 2k(\Delta x)^2$
$= 2(\Delta x)^2 \sum_{k=1}^{n} k = 2\left(\frac{b^2}{n^2}\right)\left(\frac{n(n+1)}{2}\right)$
$= b^2\left(1+\frac{1}{n}\right) \Rightarrow \int_0^b 2x\,dx = \lim_{n\to\infty} b^2\left(1+\frac{1}{n}\right) = b^2.$



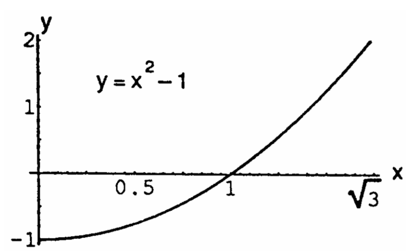
54. Let $\Delta x = \frac{b-0}{n} = \frac{b}{n}$ and let $x_0 = 0$, $x_1 = \Delta x$, $x_2 = 2\Delta x, \ldots, x_{n-1} = (n-1)\Delta x$, $x_n = n\Delta x = b$. Let the c_k's be the right end-points of the subintervals $\Rightarrow$ $c_1 = x_1$, $c_2 = x_2$, and so on. The rectangles defined have areas:

$$f(c_1)\,\Delta x = f(\Delta x)\,\Delta x = \left(\frac{\Delta x}{2}+1\right)(\Delta x) = \frac{1}{2}(\Delta x)^2 + \Delta x$$
$$f(c_2)\,\Delta x = f(2\Delta x)\,\Delta x = \left(\frac{2\Delta x}{2}+1\right)(\Delta x) = \frac{1}{2}(2)(\Delta x)^2 + \Delta x$$
$$f(c_3)\,\Delta x = f(3\Delta x)\,\Delta x = \left(\frac{3\Delta x}{2}+1\right)(\Delta x) = \frac{1}{2}(3)(\Delta x)^2 + \Delta x$$
$$\vdots$$
$$f(c_n)\,\Delta x = f(n\Delta x)\,\Delta x = \left(\frac{n\Delta x}{2}+1\right)(\Delta x) = \frac{1}{2}(n)(\Delta x)^2 + \Delta x$$

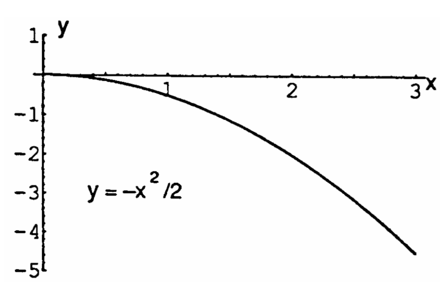


Then $S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x = \sum_{k=1}^{n}\left(\frac{1}{2}k(\Delta x)^2 + \Delta x\right) = \frac{1}{2}(\Delta x)^2\sum_{k=1}^{n} k + \Delta x\sum_{k=1}^{n} 1 = \frac{1}{2}\left(\frac{b^2}{n^2}\right)\left(\frac{n(n+1)}{2}\right) + \left(\frac{b}{n}\right)(n)$
$= \frac{1}{4}b^2\left(1+\frac{1}{n}\right) + b \Rightarrow \int_0^b\left(\frac{x}{2}+1\right)dx = \lim_{n\to\infty}\left(\frac{1}{4}b^2\left(1+\frac{1}{n}\right)+b\right) = \frac{1}{4}b^2 + b.$

55. $\text{av}(f) = \left(\frac{1}{\sqrt{3}-0}\right)\int_0^{\sqrt{3}}(x^2-1)\,dx$
$= \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} x^2\,dx - \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} 1\,dx$
$= \frac{1}{\sqrt{3}}\left(\frac{\left(\sqrt{3}\right)^3}{3}\right) - \frac{1}{\sqrt{3}}\left(\sqrt{3}-0\right) = 1 - 1 = 0.$



56. $\text{av}(f) = \left(\frac{1}{3-0}\right)\int_0^3\left(-\frac{x^2}{2}\right)dx = \frac{1}{3}\left(-\frac{1}{2}\right)\int_0^3 x^2\,dx$
$= -\frac{1}{6}\left(\frac{3^3}{3}\right) = -\frac{3}{2};\ -\frac{x^2}{2} = -\frac{3}{2}.$



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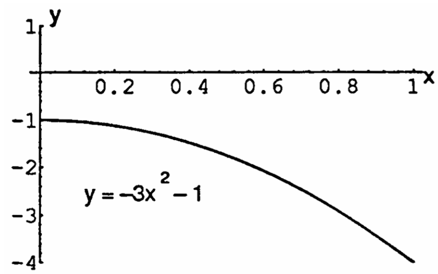
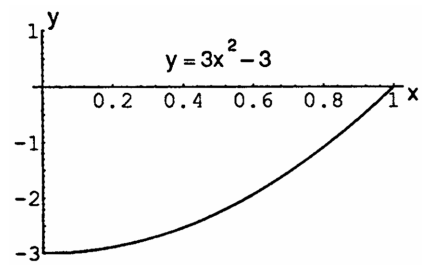
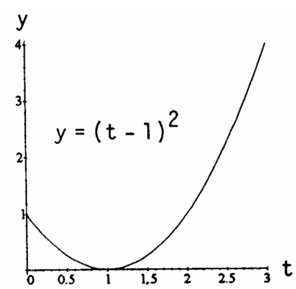
57. $\text{av}(f) = \left(\frac{1}{1-0}\right)\int_0^1 (-3x^2 - 1)\,dx =$
$= -3\int_0^1 x^2\,dx - \int_0^1 1\,dx = -3\left(\frac{1^3}{3}\right) - (1-0)$
$= -2.$



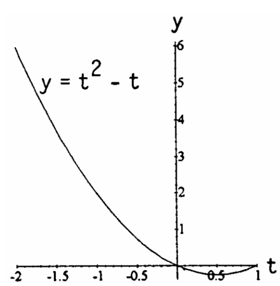
58. $\text{av}(f) = \left(\frac{1}{1-0}\right)\int_0^1 (3x^2 - 3)\,dx =$
$= 3\int_0^1 x^2\,dx - \int_0^1 3\,dx = 3\left(\frac{1^3}{3}\right) - 3(1-0)$
$= -2.$



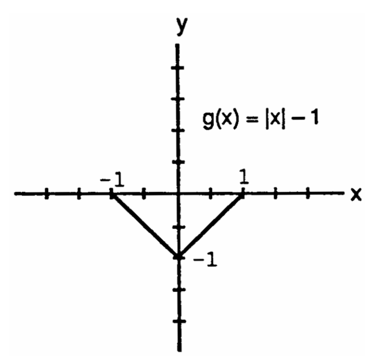
59. $\text{av}(f) = \left(\frac{1}{3-0}\right)\int_0^3 (t-1)^2\,dt$
$= \frac{1}{3}\int_0^3 t^2\,dt - \frac{2}{3}\int_0^3 t\,dt + \frac{1}{3}\int_0^3 1\,dt$
$= \frac{1}{3}\left(\frac{3^3}{3}\right) - \frac{2}{3}\left(\frac{3^2}{2} - \frac{0^2}{2}\right) + \frac{1}{3}(3-0) = 1.$



60. $\text{av}(f) = \left(\frac{1}{1-(-2)}\right)\int_{-2}^1 (t^2 - t)\,dt$
$= \frac{1}{3}\int_{-2}^1 t^2\,dt - \frac{1}{3}\int_{-2}^1 t\,dt$
$= \frac{1}{3}\int_0^1 t^2\,dt - \frac{1}{3}\int_0^{-2} t^2\,dt - \frac{1}{3}\left(\frac{1^2}{2} - \frac{(-2)^2}{2}\right)$
$= \frac{1}{3}\left(\frac{1^3}{3}\right) - \frac{1}{3}\left(\frac{(-2)^3}{3}\right) + \frac{1}{2} = \frac{3}{2}.$



61. (a) $\text{av}(g) = \left(\frac{1}{1-(-1)}\right)\int_{-1}^1 (|x| - 1)\,dx$
$= \frac{1}{2}\int_{-1}^0 (-x-1)\,dx + \frac{1}{2}\int_0^1 (x-1)\,dx$
$= -\frac{1}{2}\int_{-1}^0 x\,dx - \frac{1}{2}\int_{-1}^0 1\,dx + \frac{1}{2}\int_0^1 x\,dx - \frac{1}{2}\int_0^1 1\,dx$
$= -\frac{1}{2}\left(\frac{0^2}{2} - \frac{(-1)^2}{2}\right) - \frac{1}{2}(0-(-1)) + \frac{1}{2}\left(\frac{1^2}{2} - \frac{0^2}{2}\right) - \frac{1}{2}(1-0)$
$= -\frac{1}{2}.$



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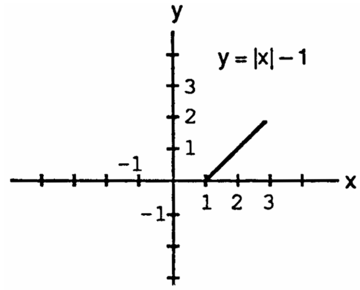
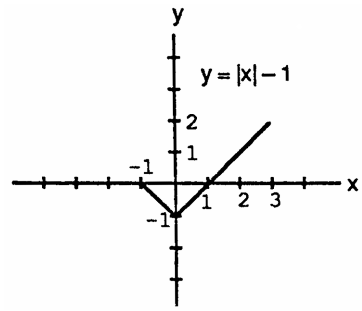
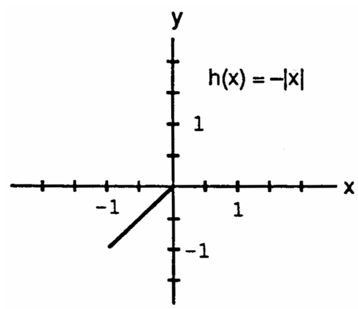
(b) $\text{av}(g) = \left(\frac{1}{3-1}\right)\int_1^3 (|x|-1)\,dx = \frac{1}{2}\int_1^3 (x-1)\,dx$
$= \frac{1}{2}\int_1^3 x\,dx - \frac{1}{2}\int_1^3 1\,dx = \frac{1}{2}\left(\frac{3^2}{2} - \frac{1^2}{2}\right) - \frac{1}{2}(3-1)$
$= 1.$



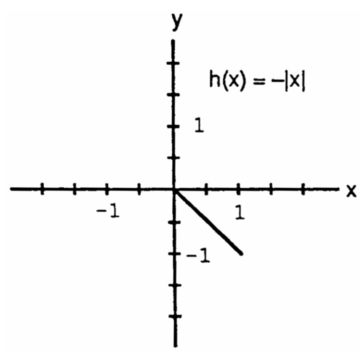
(c) $\text{av}(g) = \left(\frac{1}{3-(-1)}\right)\int_{-1}^3 (|x|-1)\,dx$
$= \frac{1}{4}\int_{-1}^1 (|x|-1)\,dx + \frac{1}{4}\int_1^3 (|x|-1)\,dx$
$= \frac{1}{4}(-1+2) = \frac{1}{4}$ (see parts (a) and (b) above).



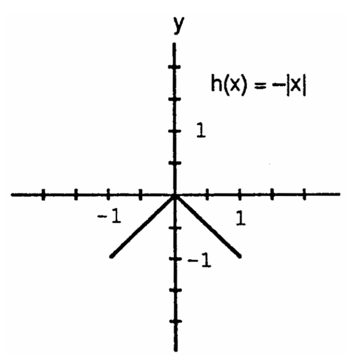
62. (a) $\text{av}(h) = \left(\frac{1}{0-(-1)}\right)\int_{-1}^0 -|x|\,dx = \int_{-1}^0 -(-x)\,dx$
$= \int_{-1}^0 x\,dx = \frac{0^2}{2} - \frac{(-1)^2}{2} = -\frac{1}{2}.$



(b) $\text{av}(h) = \left(\frac{1}{1-0}\right)\int_0^1 -|x|\,dx = -\int_0^1 x\,dx$
$= -\left(\frac{1^2}{2} - \frac{0^2}{2}\right) = -\frac{1}{2}.$



(c) $\text{av}(h) = \left(\frac{1}{1-(-1)}\right)\int_{-1}^1 -|x|\,dx$
$= \frac{1}{2}\left(\int_{-1}^0 -|x|\,dx + \int_0^1 -|x|\,dx\right)$
$= \frac{1}{2}\left(-\frac{1}{2} + \left(-\frac{1}{2}\right)\right) = -\frac{1}{2}$ (see parts (a) and (b) above).



63. To find where $x - x^2 \ge 0$, let $x - x^2 = 0 \Rightarrow x(1-x) = 0 \Rightarrow x = 0$ or $x = 1$. If $0 < x < 1$, then $0 < x - x^2 \Rightarrow a = 0$ and $b = 1$ maximize the integral.

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64. To find where $x^4 - 2x^2 \le 0$, let $x^4 - 2x^2 = 0 \Rightarrow x^2(x^2 - 2) = 0 \Rightarrow x = 0$ or $x = \pm\sqrt{2}$. By the sign graph, $\underset{-\sqrt{2}}{++++++\,0}\;--\;\underset{0}{0}\;--\;\underset{\sqrt{2}}{0}\;+++++++$, we can see that $x^4 - 2x^2 \le 0$ on $\left[-\sqrt{2}, \sqrt{2}\right] \Rightarrow a = -\sqrt{2}$ and $b = \sqrt{2}$ minimize the integral.

65. $f(x) = \frac{1}{1+x^2}$ is decreasing on $[0, 1] \Rightarrow$ maximum value of f occurs at $0 \Rightarrow \max f = f(0) = 1$; minimum value of f occurs at $1 \Rightarrow \min f = f(1) = \frac{1}{1+1^2} = \frac{1}{2}$. Therefore, $(1-0)\min f \le \int_0^1 \frac{1}{1+x^2}\,dx \le (1-0)\max f$ $\Rightarrow \frac{1}{2} \le \int_0^1 \frac{1}{1+x^2}\,dx \le 1$. That is, an upper bound $= 1$ and a lower bound $= \frac{1}{2}$.

66. See Exercise 65 above. On $[0, 0.5]$, $\max f = \frac{1}{1+0^2} = 1$, $\min f = \frac{1}{1+(0.5)^2} = 0.8$. Therefore $(0.5-0)\min f \le \int_0^{0.5} f(x)\,dx \le (0.5-0)\max f \Rightarrow \frac{2}{5} \le \int_0^{0.5} \frac{1}{1+x^2}\,dx \le \frac{1}{2}$. On $[0.5, 1]$, $\max f = \frac{1}{1+(0.5)^2} = 0.8$ and $\min f = \frac{1}{1+1^2} = 0.5$. Therefore $(1-0.5)\min f \le \int_{0.5}^1 \frac{1}{1+x^2}\,dx \le (1-0.5)\max f \Rightarrow \frac{1}{4} \le \int_{0.5}^1 \frac{1}{1+x^2}\,dx \le \frac{2}{5}$. Then $\frac{1}{4} + \frac{2}{5} \le \int_0^{0.5} \frac{1}{1+x^2}\,dx + \int_{0.5}^1 \frac{1}{1+x^2}\,dx \le \frac{1}{2} + \frac{2}{5} \Rightarrow \frac{13}{20} \le \int_0^1 \frac{1}{1+x^2}\,dx \le \frac{9}{10}$.

67. $-1 \le \sin(x^2) \le 1$ for all $x \Rightarrow (1-0)(-1) \le \int_0^1 \sin(x^2)\,dx \le (1-0)(1)$ or $\int_0^1 \sin x^2\,dx \le 1 \Rightarrow \int_0^1 \sin x^2\,dx$ cannot equal 2.

68. $f(x) = \sqrt{x+8}$ is increasing on $[0, 1] \Rightarrow \max f = f(1) = \sqrt{1+8} = 3$ and $\min f = f(0) = \sqrt{0+8} = 2\sqrt{2}$. Therefore, $(1-0)\min f \le \int_0^1 \sqrt{x+8}\,dx \le (1-0)\max f \Rightarrow 2\sqrt{2} \le \int_0^1 \sqrt{x+8}\,dx \le 3$.

69. If $f(x) \ge 0$ on $[a, b]$, then $\min f \ge 0$ and $\max f \ge 0$ on $[a, b]$. Now, $(b-a)\min f \le \int_a^b f(x)\,dx \le (b-a)\max f$. Then $b \ge a \Rightarrow b - a \ge 0 \Rightarrow (b-a)\min f \ge 0 \Rightarrow \int_a^b f(x)\,dx \ge 0$.

70. If $f(x) \le 0$ on $[a, b]$, then $\min f \le 0$ and $\max f \le 0$. Now, $(b-a)\min f \le \int_a^b f(x)\,dx \le (b-a)\max f$. Then $b \ge a \Rightarrow b - a \ge 0 \Rightarrow (b-a)\max f \le 0 \Rightarrow \int_a^b f(x)\,dx \le 0$.

71. $\sin x \le x$ for $x \ge 0 \Rightarrow \sin x - x \le 0$ for $x \ge 0 \Rightarrow \int_0^1 (\sin x - x)\,dx \le 0$ (see Exercise 70) $\Rightarrow \int_0^1 \sin x\,dx - \int_0^1 x\,dx \le 0 \Rightarrow \int_0^1 \sin x\,dx \le \int_0^1 x\,dx \Rightarrow \int_0^1 \sin x\,dx \le \left(\frac{1^2}{2} - \frac{0^2}{2}\right) \Rightarrow \int_0^1 \sin x\,dx \le \frac{1}{2}$. Thus an upper bound is $\frac{1}{2}$.

72. $\sec x \ge 1 + \frac{x^2}{2}$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \sec x - \left(1 + \frac{x^2}{2}\right) \ge 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \int_0^1 \left[\sec x - \left(1 + \frac{x^2}{2}\right)\right] dx \ge 0$ (see Exercise 69) since $[0, 1]$ is contained in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow \int_0^1 \sec x\,dx - \int_0^1 \left(1 + \frac{x^2}{2}\right) dx \ge 0 \Rightarrow \int_0^1 \sec x\,dx$ $\ge \int_0^1 \left(1 + \frac{x^2}{2}\right) dx \Rightarrow \int_0^1 \sec x\,dx \ge \int_0^1 1\,dx + \frac{1}{2}\int_0^1 x^2\,dx \Rightarrow \int_0^1 \sec x\,dx \ge (1-0) + \frac{1}{2}\left(\frac{1^3}{3}\right) \Rightarrow \int_0^1 \sec x\,dx \ge \frac{7}{6}$. Thus a lower bound is $\frac{7}{6}$.

73. Yes, for the following reasons: $\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$ is a constant K. Thus $\int_a^b \text{av}(f)\,dx = \int_a^b K\,dx$ $= K(b-a) \Rightarrow \int_a^b \text{av}(f)\,dx = (b-a)K = (b-a)\cdot\frac{1}{b-a}\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$.

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74. All three rules hold. The reasons: On any interval $[a, b]$ on which f and g are integrable, we have:

(a) $\operatorname{av}(f+g) = \frac{1}{b-a}\int_a^b [f(x)+g(x)]\,dx = \frac{1}{b-a}\left[\int_a^b f(x)\,dx + \int_a^b g(x)\,dx\right] = \frac{1}{b-a}\int_a^b f(x)\,dx + \frac{1}{b-a}\int_a^b g(x)\,dx$
$= \operatorname{av}(f) + \operatorname{av}(g)$

(b) $\operatorname{av}(kf) = \frac{1}{b-a}\int_a^b kf(x)\,dx = \frac{1}{b-a}\left[k\int_a^b f(x)\,dx\right] = k\left[\frac{1}{b-a}\int_a^b f(x)\,dx\right] = k\,\operatorname{av}(f)$

(c) $\operatorname{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{1}{b-a}\int_a^b g(x)\,dx$ since $f(x) \le g(x)$ on $[a, b]$, and $\frac{1}{b-a}\int_a^b g(x)\,dx = \operatorname{av}(g)$.
Therefore, $\operatorname{av}(f) \le \operatorname{av}(g)$.

75. Consider the partition P that subdivides the interval $[a, b]$ into n subintervals of width $\triangle x = \frac{b-a}{n}$ and let c_k be the right endpoint of each subinterval. So the partition is $P = \{a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \ldots, a + \frac{n(b-a)}{n}\}$ and $c_k = a + \frac{k(b-a)}{n}$. We get the Riemann sum $\sum_{k=1}^{n} f(c_k)\triangle x = \sum_{k=1}^{n} c \cdot \frac{b-a}{n} = \frac{c(b-a)}{n}\sum_{k=1}^{n} 1 = \frac{c(b-a)}{n} \cdot n = c(b-a)$. As $n \to \infty$ and $\|P\| \to 0$ this expression remains $c(b-a)$. Thus, $\int_a^b c\,dx = c(b-a)$.

76. Consider the partition P that subdivides the interval $[a, b]$ into n subintervals of width $\triangle x = \frac{b-a}{n}$ and let c_k be the right endpoint of each subinterval. So the partition is $P = \{a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \ldots, a + \frac{n(b-a)}{n}\}$ and $c_k = a + \frac{k(b-a)}{n}$. We get the Riemann sum $\sum_{k=1}^{n} f(c_k)\triangle x = \sum_{k=1}^{n} c_k^2\left(\frac{b-a}{n}\right) = \frac{b-a}{n}\sum_{k=1}^{n}\left(a + \frac{k(b-a)}{n}\right)^2 = \frac{b-a}{n}\sum_{k=1}^{n}\left(a^2 + \frac{2ak(b-a)}{n} + \frac{k^2(b-a)^2}{n^2}\right)$
$= \frac{b-a}{n}\left(\sum_{k=1}^{n} a^2 + \frac{2a(b-a)}{n}\sum_{k=1}^{n} k + \frac{(b-a)^2}{n^2}\sum_{k=1}^{n} k^2\right) = \frac{b-a}{n} \cdot na^2 + \frac{2a(b-a)^2}{n^2} \cdot \frac{n(n+1)}{2} + \frac{(b-a)^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$
$= (b-a)a^2 + a(b-a)^2 \cdot \frac{n+1}{n} + \frac{(b-a)^3}{6} \cdot \frac{(n+1)(2n+1)}{n^2} = (b-a)a^2 + a(b-a)^2 \cdot \frac{1+\frac{1}{n}}{1} + \frac{(b-a)^3}{6} \cdot \frac{2+\frac{3}{n}+\frac{1}{n^2}}{1}$
As $n \to \infty$ and $\|P\| \to 0$ this expression has value $(b-a)a^2 + a(b-a)^2 \cdot 1 + \frac{(b-a)^3}{6} \cdot 2$
$= ba^2 - a^3 + ab^2 - 2a^2b + a^3 + \frac{1}{3}(b^3 - 3b^2a + 3ba^2 - a^3) = \frac{b^3}{3} - \frac{a^3}{3}$. Thus, $\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3}$.

77. (a) $U = \max_1 \Delta x + \max_2 \Delta x + \ldots + \max_n \Delta x$ where $\max_1 = f(x_1)$, $\max_2 = f(x_2)$, $\ldots$, $\max_n = f(x_n)$ since f is increasing on $[a, b]$; $L = \min_1 \Delta x + \min_2 \Delta x + \ldots + \min_n \Delta x$ where $\min_1 = f(x_0)$, $\min_2 = f(x_1)$, $\ldots$, $\min_n = f(x_{n-1})$ since f is increasing on $[a, b]$. Therefore
$U - L = (\max_1 - \min_1)\Delta x + (\max_2 - \min_2)\Delta x + \ldots + (\max_n - \min_n)\Delta x$
$= (f(x_1) - f(x_0))\Delta x + (f(x_2) - f(x_1))\Delta x + \ldots + (f(x_n) - f(x_{n-1}))\Delta x = (f(x_n) - f(x_0))\Delta x = (f(b) - f(a))\Delta x$.

(b) $U = \max_1 \Delta x_1 + \max_2 \Delta x_2 + \ldots + \max_n \Delta x_n$ where $\max_1 = f(x_1)$, $\max_2 = f(x_2)$, $\ldots$, $\max_n = f(x_n)$ since f is increasing on$[a, b]$; $L = \min_1 \Delta x_1 + \min_2 \Delta x_2 + \ldots + \min_n \Delta x_n$ where
$\min_1 = f(x_0)$, $\min_2 = f(x_1)$, $\ldots$, $\min_n = f(x_{n-1})$ since f is increasing on $[a, b]$. Therefore
$U - L = (\max_1 - \min_1)\Delta x_1 + (\max_2 - \min_2)\Delta x_2 + \ldots + (\max_n - \min_n)\Delta x_n$
$= (f(x_1) - f(x_0))\Delta x_1 + (f(x_2) - f(x_1))\Delta x_2 + \ldots + (f(x_n) - f(x_{n-1}))\Delta x_n$
$\le (f(x_1) - f(x_0))\Delta x_{max} + (f(x_2) - f(x_1))\Delta x_{max} + \ldots + (f(x_n) - f(x_{n-1}))\Delta x_{max}$. Then
$U - L \le (f(x_n) - f(x_0))\Delta x_{max} = (f(b) - f(a))\Delta x_{max} = |f(b) - f(a)|\,\Delta x_{max}$ since $f(b) \ge f(a)$. Thus
$\lim_{\|P\| \to 0} (U - L) = \lim_{\|P\| \to 0} (f(b) - f(a))\,\Delta x_{max} = 0$, since $\Delta x_{max} = \|P\|$.

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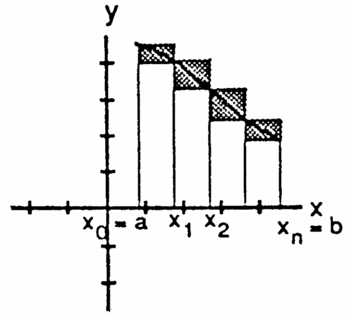

78. (a) $U = \max_1 \Delta x + \max_2 \Delta x + \ldots + \max_n \Delta x$ where $\max_1 = f(x_0), \max_2 = f(x_1), \ldots, \max_n = f(x_{n-1})$ since f is decreasing on $[a, b]$; $L = \min_1 \Delta x + \min_2 \Delta x + \ldots + \min_n \Delta x$ where $\min_1 = f(x_1), \min_2 = f(x_2), \ldots, \min_n = f(x_n)$ since f is decreasing on $[a, b]$. Therefore $U - L = (\max_1 - \min_1)\Delta x + (\max_2 - \min_2)\Delta x + \ldots + (\max_n - \min_n)\Delta x = (f(x_0) - f(x_1))\Delta x + (f(x_1) - f(x_2))\Delta x + \ldots + (f(x_{n-1}) - f(x_n))\Delta x = (f(x_0) - f(x_n))\Delta x = (f(a) - f(b))\Delta x.$

(b) $U = \max_1 \Delta x_1 + \max_2 \Delta x_2 + \ldots + \max_n \Delta x_n$ where $\max_1 = f(x_0), \max_2 = f(x_1), \ldots, \max_n = f(x_{n-1})$ since f is decreasing on$[a, b]$; $L = \min_1 \Delta x_1 + \min_2 \Delta x_2 + \ldots + \min_n \Delta x_n$ where $\min_1 = f(x_1), \min_2 = f(x_2), \ldots, \min_n = f(x_n)$ since f is decreasing on $[a, b]$. Therefore $U - L = (\max_1 - \min_1)\Delta x_1 + (\max_2 - \min_2)\Delta x_2 + \ldots + (\max_n - \min_n)\Delta x_n = (f(x_0) - f(x_1))\Delta x_1 + (f(x_1) - f(x_2))\Delta x_2 + \ldots + (f(x_{n-1}) - f(x_n))\Delta x_n \le (f(x_0) - f(x_n))\Delta x_{max} = (f(a) - f(b)\,\Delta x_{max} = |f(b) - f(a)|\,\Delta x_{max}$ since $f(b) \le f(a)$. Thus $\lim\limits_{\|P\| \to 0} (U - L) = \lim\limits_{\|P\| \to 0} |f(b) - f(a)|\,\Delta x_{max} = 0$, since $\Delta x_{max} = \|P\|$.

79. (a) Partition $\left[0, \frac{\pi}{2}\right]$ into n subintervals, each of length $\Delta x = \frac{\pi}{2n}$ with points $x_0 = 0, x_1 = \Delta x, x_2 = 2\Delta x, \ldots, x_n = n\Delta x = \frac{\pi}{2}$. Since sin x is increasing on $\left[0, \frac{\pi}{2}\right]$, the upper sum U is the sum of the areas of the circumscribed rectangles of areas $f(x_1)\Delta x = (\sin \Delta x)\Delta x, f(x_2)\Delta x = (\sin 2\Delta x)\Delta x, \ldots, f(x_n)\Delta x = (\sin n\Delta x)\Delta x$. Then $U = (\sin \Delta x + \sin 2\Delta x + \ldots + \sin n\Delta x)\Delta x = \left[\frac{\cos \frac{\Delta x}{2} - \cos\left(\left(n + \frac{1}{2}\right)\Delta x\right)}{2 \sin \frac{\Delta x}{2}}\right]\Delta x$

$$= \left[\frac{\cos \frac{\pi}{4n} - \cos\left(\left(n + \frac{1}{2}\right)\frac{\pi}{2n}\right)}{2 \sin \frac{\pi}{4n}}\right]\left(\frac{\pi}{2n}\right) = \frac{\pi\left(\cos \frac{\pi}{4n} - \cos\left(\frac{\pi}{2} + \frac{\pi}{4n}\right)\right)}{4n \sin \frac{\pi}{4n}} = \frac{\cos \frac{\pi}{4n} - \cos\left(\frac{\pi}{2} + \frac{\pi}{4n}\right)}{\left(\frac{\sin \frac{\pi}{4n}}{\frac{\pi}{4n}}\right)}$$

(b) The area is $\int_0^{\pi/2} \sin x\,dx = \lim\limits_{n \to \infty} \frac{\cos \frac{\pi}{4n} - \cos\left(\frac{\pi}{2} + \frac{\pi}{4n}\right)}{\left(\frac{\sin \frac{\pi}{4n}}{\frac{\pi}{4n}}\right)} = \frac{1 - \cos \frac{\pi}{2}}{1} = 1.$

80. (a) The area of the shaded region is $\sum_{i=1}^{n} \triangle x_i \cdot m_i$ which is equal to L.

(b) The area of the shaded region is $\sum_{i=1}^{n} \triangle x_i \cdot M_i$ which is equal to U.

(c) The area of the shaded region is the difference in the areas of the shaded regions shown in the second part of the figure and the first part of the figure. Thus this area is $U - L$.

81. By Exercise 80, $U - L = \sum_{i=1}^{n} \triangle x_i \cdot M_i - \sum_{i=1}^{n} \triangle x_i \cdot m_i$ where $M_i = \max\{f(x)$ on the ith subinterval$\}$ and $m_i = \min\{f(x)$ on the ith subinterval$\}$. Thus $U - L = \sum_{i=1}^{n} (M_i - m_i)\triangle x_i < \sum_{i=1}^{n} \epsilon \cdot \triangle x_i$ provided $\triangle x_i < \delta$ for each $i = 1, \ldots, n$. Since $\sum_{i=1}^{n} \epsilon \cdot \triangle x_i = \epsilon \sum_{i=1}^{n} \triangle x_i = \epsilon(b - a)$ the result, $U - L < \epsilon(b - a)$ follows.

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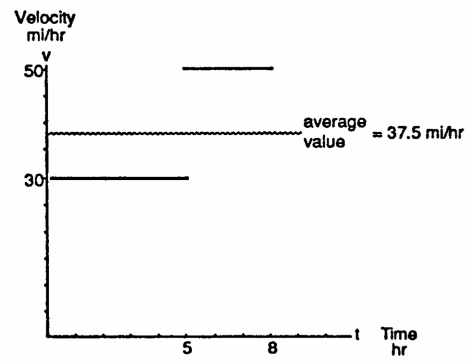
82. The car drove the first 150 miles in 5 hours and the second 150 miles in 3 hours, which means it drove 300 miles in 8 hours, for an average of $\frac{300}{8}$ mi/hr $= 37.5$ mi/hr. In terms of average values of functions, the function whose average value we seek is $v(t) = \begin{cases} 30, & 0 \le t \le 5 \\ 50, & 5 < 1 \le 8 \end{cases}$, and the average value is $\frac{(30)(5)+(50)(3)}{8} = 37.5$.



83-88. Example CAS commands:

Maple:

```
with( plots );
with( Student[Calculus1] );
f := x -> 1-x;
a := 0;
b := 1;
N :=[ 4, 10, 20, 50 ];
P := [seq( RiemannSum( f(x), x=a..b, partition=n, method=random, output=plot ), n=N )]:
display( P, insequence=true );
```

89-92. Example CAS commands:

Maple:

```
with( Student[Calculus1] );
f := x -> sin(x);
a := 0;
b := Pi;
plot( f(x), x=a..b, title="#23(a) (Section 5.1)" );
N := [ 100, 200, 1000 ];                              # (b)
for n in N do
  Xlist := [ a+1.*(b-a)/n*i $ i=0..n ];
  Ylist := map( f, Xlist );
end do:
for n in N do                                         # (c)
  Avg[n] := evalf(add(y,y=Ylist)/nops(Ylist));
end do;
avg := FunctionAverage( f(x), x=a..b, output=value );
evalf( avg );
FunctionAverage(f(x),x=a..b,output=plot);      # (d)
fsolve( f(x)=avg, x=0.5 );
fsolve( f(x)=avg, x=2.5 );
fsolve( f(x)=Avg[1000], x=0.5 );
fsolve( f(x)=Avg[1000], x=2.5 );
```

83-92. Example CAS commands:

Mathematica: (assigned function and values for a, b, and n may vary)

Sums of rectangles evaluated at left-hand endpoints can be represented and evaluated by this set of commands

```
Clear[x, f, a, b, n]
```

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```
{a, b}={0, π}; n =10; dx = (b − a)/n;
f = Sin[x]^2;
xvals =Table[N[x], {x, a, b − dx, dx}];
yvals = f /.x → xvals;
boxes = MapThread[Line[{{#1,0},{#1, #3},{#2, #3},{#2, 0}]&,{xvals, xvals + dx, yvals}];
Plot[f, {x, a, b}, Epilog → boxes];
Sum[yvals[[i]] dx, {i, 1, Length[yvals]}]//N
```

Sums of rectangles evaluated at right-hand endpoints can be represented and evaluated by this set of commands.

```
Clear[x, f, a, b, n]
{a, b}={0, π}; n =10; dx = (b − a)/n;
f = Sin[x]^2;
xvals =Table[N[x], {x, a + dx, b, dx}];
yvals = f /.x → xvals;
boxes = MapThread[Line[{{#1,0},{#1, #3},{#2, #3},{#2, 0}]&,{xvals − dx,xvals, yvals}];
Plot[f, {x, a, b}, Epilog → boxes];
Sum[yvals[[i]] dx, {i, 1,Length[yvals]}]//N
```

Sums of rectangles evaluated at midpoints can be represented and evaluated by this set of commands.

```
Clear[x, f, a, b, n]
{a, b}={0, π}; n =10; dx = (b − a)/n;
f = Sin[x]^2;
xvals =Table[N[x], {x, a + dx/2, b − dx/2, dx}];
yvals = f /.x → xvals;
boxes = MapThread[Line[{{#1,0},{#1, #3},{#2, #3},{#2, 0}]&,{xvals − dx/2, xvals + dx/2, yvals}];
Plot[f, {x, a, b},Epilog → boxes];
Sum[yvals[[i]] dx, {i, 1, Length[yvals]}]//N
```

5.4 THE FUNDAMENTAL THEOREM OF CALCULUS

1. $\int_{-2}^{0}(2x+5)\,dx = \left[x^2+5x\right]_{-2}^{0} = \left(0^2+5(0)\right) - \left((-2)^2+5(-2)\right) = 6$

2. $\int_{-3}^{4}\left(5-\frac{x}{2}\right)dx = \left[5x-\frac{x^2}{4}\right]_{-3}^{4} = \left(5(4)-\frac{4^2}{4}\right) - \left(5(-3)-\frac{(-3)^2}{4}\right) = \frac{133}{4}$

3. $\int_{0}^{4}\left(3x-\frac{x^3}{4}\right)dx = \left[\frac{3x^2}{2}-\frac{x^4}{16}\right]_{0}^{4} = \left(\frac{3(4)^2}{2}-\frac{4^4}{16}\right) - \left(\frac{3(0)^2}{2}-\frac{(0)^4}{16}\right) = 8$

4. $\int_{-2}^{2}(x^3-2x+3)\,dx = \left[\frac{x^4}{4}-x^2+3x\right]_{-2}^{2} = \left(\frac{2^4}{4}-2^2+3(2)\right) - \left(\frac{(-2)^4}{4}-(-2)^2+3(-2)\right) = 12$

5. $\int_{0}^{1}\left(x^2+\sqrt{x}\right)dx = \left[\frac{x^3}{3}+\frac{2}{3}x^{3/2}\right]_{0}^{1} = \left(\frac{1}{3}+\frac{2}{3}\right) - 0 = 1$

6. $\int_{0}^{5} x^{3/2}\,dx = \left[\frac{2}{5}x^{5/2}\right]_{0}^{5} = \frac{2}{5}(5)^{5/2} - 0 = 2(5)^{3/2} = 10\sqrt{5}$

7. $\int_{1}^{32} x^{-6/5}\,dx = \left[-5x^{-1/5}\right]_{1}^{32} = \left(-\frac{5}{2}\right) - (-5) = \frac{5}{2}$

8. $\int_{-2}^{-1}\frac{2}{x^2}\,dx = \int_{-2}^{-1}2x^{-2}\,dx = \left[-2x^{-1}\right]_{-2}^{-1} = \left(\frac{-2}{-1}\right) - \left(\frac{-2}{-2}\right) = 1$

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9. $\int_0^{\pi} \sin x\, dx = [-\cos x]_0^{\pi} = (-\cos \pi) - (-\cos 0) = -(-1) - (-1) = 2$

10. $\int_0^{\pi} (1 + \cos x)\, dx = [x + \sin x]_0^{\pi} = (\pi + \sin \pi) - (0 + \sin 0) = \pi$

11. $\int_0^{\pi/3} 2 \sec^2 x\, dx = [2 \tan x]_0^{\pi/3} = \left(2 \tan\left(\frac{\pi}{3}\right)\right) - (2 \tan 0) = 2\sqrt{3} - 0 = 2\sqrt{3}$

12. $\int_{\pi/6}^{5\pi/6} \csc^2 x\, dx = [-\cot x]_{\pi/6}^{5\pi/6} = \left(-\cot\left(\frac{5\pi}{6}\right)\right) - \left(-\cot\left(\frac{\pi}{6}\right)\right) = -\left(-\sqrt{3}\right) - \left(-\sqrt{3}\right) = 2\sqrt{3}$

13. $\int_{\pi/4}^{3\pi/4} \csc\theta \cot\theta\, d\theta = [-\csc\theta]_{\pi/4}^{3\pi/4} = \left(-\csc\left(\frac{3\pi}{4}\right)\right) - \left(-\csc\left(\frac{\pi}{4}\right)\right) = -\sqrt{2} - \left(-\sqrt{2}\right) = 0$

14. $\int_0^{\pi/3} 4 \sec u \tan u\, du = [4 \sec u]_0^{\pi/3} = 4 \sec\left(\frac{\pi}{3}\right) - 4 \sec 0 = 4(2) - 4(1) = 4$

15. $\int_{\pi/2}^{0} \frac{1+\cos 2t}{2}\, dt = \int_{\pi/2}^{0} \left(\frac{1}{2} + \frac{1}{2}\cos 2t\right) dt = \left[\frac{1}{2}t + \frac{1}{4}\sin 2t\right]_{\pi/2}^{0} = \left(\frac{1}{2}(0) + \frac{1}{4}\sin 2(0)\right) - \left(\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin 2\left(\frac{\pi}{2}\right)\right)$
$= -\frac{\pi}{4}$

16. $\int_{-\pi/3}^{\pi/3} \frac{1-\cos 2t}{2}\, dt = \int_{-\pi/3}^{\pi/3} \left(\frac{1}{2} - \frac{1}{2}\cos 2t\right) dt = \left[\frac{1}{2}t - \frac{1}{4}\sin 2t\right]_{-\pi/3}^{\pi/3}$
$= \left(\frac{1}{2}\left(\frac{\pi}{3}\right) - \frac{1}{4}\sin 2\left(\frac{\pi}{3}\right)\right) - \left(\frac{1}{2}\left(-\frac{\pi}{3}\right) - \frac{1}{4}\sin 2\left(-\frac{\pi}{3}\right)\right) = \frac{\pi}{6} - \frac{1}{4}\sin\frac{2\pi}{3} + \frac{\pi}{6} + \frac{1}{4}\sin\left(\frac{-2\pi}{3}\right) = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$

17. $\int_{-\pi/2}^{\pi/2} (8y^2 + \sin y)\, dy = \left[\frac{8y^3}{3} - \cos y\right]_{-\pi/2}^{\pi/2} = \left(\frac{8\left(\frac{\pi}{2}\right)^3}{3} - \cos\frac{\pi}{2}\right) - \left(\frac{8\left(-\frac{\pi}{2}\right)^3}{3} - \cos\left(-\frac{\pi}{2}\right)\right) = \frac{2\pi^3}{3}$

18. $\int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \frac{\pi}{t^2}\right) dt = \int_{-\pi/3}^{-\pi/4} \left(4\sec^2 t + \pi t^{-2}\right) dt = \left[4\tan t - \frac{\pi}{t}\right]_{-\pi/3}^{-\pi/4}$
$= \left(4\tan\left(-\frac{\pi}{4}\right) - \frac{\pi}{\left(-\frac{\pi}{4}\right)}\right) - \left(4\tan\left(\frac{\pi}{3}\right) - \frac{\pi}{\left(-\frac{\pi}{3}\right)}\right) = (4(-1) + 4) - \left(4\left(-\sqrt{3}\right) + 3\right) = 4\sqrt{3} - 3$

19. $\int_1^{-1} (r+1)^2\, dr = \int_1^{-1} (r^2 + 2r + 1)\, dr = \left[\frac{r^3}{3} + r^2 + r\right]_1^{-1} = \left(\frac{(-1)^3}{3} + (-1)^2 + (-1)\right) - \left(\frac{1^3}{3} + 1^2 + 1\right) = -\frac{8}{3}$

20. $\int_{-\sqrt{3}}^{\sqrt{3}} (t+1)(t^2+4)\, dt = \int_{-\sqrt{3}}^{\sqrt{3}} (t^3 + t^2 + 4t + 4)\, dt = \left[\frac{t^4}{4} + \frac{t^3}{3} + 2t^2 + 4t\right]_{-\sqrt{3}}^{\sqrt{3}}$
$= \left(\frac{\left(\sqrt{3}\right)^4}{4} + \frac{\left(\sqrt{3}\right)^3}{3} + 2\left(\sqrt{3}\right)^2 + 4\sqrt{3}\right) - \left(\frac{\left(-\sqrt{3}\right)^4}{4} + \frac{\left(-\sqrt{3}\right)^3}{3} + 2\left(-\sqrt{3}\right)^2 + 4\left(-\sqrt{3}\right)\right) = 10\sqrt{3}$

21. $\int_{\sqrt{2}}^{1} \left(\frac{u^7}{2} - \frac{1}{u^5}\right) du = \int_{\sqrt{2}}^{1} \left(\frac{u^7}{2} - u^{-5}\right) du = \left[\frac{u^8}{16} + \frac{1}{4u^4}\right]_{\sqrt{2}}^{1} = \left(\frac{1^8}{16} + \frac{1}{4(1)^4}\right) - \left(\frac{\left(\sqrt{2}\right)^8}{16} + \frac{1}{4\left(\sqrt{2}\right)^4}\right) = -\frac{3}{4}$

22. $\int_{1/2}^{1} \left(\frac{1}{v^3} - \frac{1}{v^4}\right) dv = \int_{1/2}^{1} (v^{-3} - v^{-4})\, dv = \left[\frac{-1}{2v^2} + \frac{1}{3v^3}\right]_{1/2}^{1} = \left(\frac{-1}{2(1)^2} + \frac{1}{3(1)^3}\right) - \left(\frac{-1}{2\left(\frac{1}{2}\right)^2} + \frac{1}{3\left(\frac{1}{2}\right)^3}\right) = -\frac{5}{6}$

23. $\int_1^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2}\, ds = \int_1^{\sqrt{2}} \left(1 + s^{-3/2}\right) ds = \left[s - \frac{2}{\sqrt{s}}\right]_1^{\sqrt{2}} = \left(\sqrt{2} - \frac{2}{\sqrt{\sqrt{2}}}\right) - \left(1 - \frac{2}{\sqrt{1}}\right) = \sqrt{2} - 2^{3/4} + 1$
$= \sqrt{2} - \sqrt[4]{8} + 1$

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24. $\int_9^4 \frac{1-\sqrt{u}}{\sqrt{u}}\,du = \int_9^4 \left(u^{-1/2} - 1\right) du = \left[2\sqrt{u} - u\right]_9^4 = \left(2\sqrt{4} - 4\right) - \left(2\sqrt{9} - 9\right) = 3$

25. $\int_{-4}^4 |x|\,dx = \int_{-4}^0 |x|\,dx + \int_0^4 |x|\,dx = -\int_{-4}^0 x\,dx + \int_0^4 x\,dx = \left[-\frac{x^2}{2}\right]_{-4}^0 + \left[\frac{x^2}{2}\right]_0^4 = \left(-\frac{0^2}{2} + \frac{(-4)^2}{2}\right) + \left(\frac{4^2}{2} - \frac{0^2}{2}\right)$
$= 16$

26. $\int_0^\pi \frac{1}{2}\left(\cos x + |\cos x|\right) dx = \int_0^{\pi/2} \frac{1}{2}(\cos x + \cos x)\,dx + \int_{\pi/2}^\pi \frac{1}{2}(\cos x - \cos x)\,dx = \int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2}$
$= \sin\frac{\pi}{2} - \sin 0 = 1$

27. (a) $\int_0^{\sqrt{x}} \cos t\,dt = [\sin t]_0^{\sqrt{x}} = \sin\sqrt{x} - \sin 0 = \sin\sqrt{x} \Rightarrow \frac{d}{dx}\left(\int_0^{\sqrt{x}} \cos t\,dt\right) = \frac{d}{dx}\left(\sin\sqrt{x}\right) = \cos\sqrt{x}\left(\frac{1}{2}x^{-1/2}\right)$
$= \frac{\cos\sqrt{x}}{2\sqrt{x}}$

(b) $\frac{d}{dx}\left(\int_0^{\sqrt{x}} \cos t\,dt\right) = \left(\cos\sqrt{x}\right)\left(\frac{d}{dx}\left(\sqrt{x}\right)\right) = \left(\cos\sqrt{x}\right)\left(\frac{1}{2}x^{-1/2}\right) = \frac{\cos\sqrt{x}}{2\sqrt{x}}$

28. (a) $\int_1^{\sin x} 3t^2\,dt = \left[t^3\right]_1^{\sin x} = \sin^3 x - 1 \Rightarrow \frac{d}{dx}\left(\int_1^{\sin x} 3t^2\,dt\right) = \frac{d}{dx}\left(\sin^3 x - 1\right) = 3\sin^2 x \cos x$

(b) $\frac{d}{dx}\left(\int_1^{\sin x} 3t^2\,dt\right) = \left(3\sin^2 x\right)\left(\frac{d}{dx}(\sin x)\right) = 3\sin^2 x \cos x$

29. (a) $\int_0^{t^4} \sqrt{u}\,du = \int_0^{t^4} u^{1/2}\,du = \left[\frac{2}{3}u^{3/2}\right]_0^{t^4} = \frac{2}{3}\left(t^4\right)^{3/2} - 0 = \frac{2}{3}t^6 \Rightarrow \frac{d}{dt}\left(\int_0^{t^4} \sqrt{u}\,du\right) = \frac{d}{dt}\left(\frac{2}{3}t^6\right) = 4t^5$

(b) $\frac{d}{dt}\left(\int_0^{t^4} \sqrt{u}\,du\right) = \sqrt{t^4}\left(\frac{d}{dt}\left(t^4\right)\right) = t^2\left(4t^3\right) = 4t^5$

30. (a) $\int_0^{\tan\theta} \sec^2 y\,dy = [\tan y]_0^{\tan\theta} = \tan(\tan\theta) - 0 = \tan(\tan\theta) \Rightarrow \frac{d}{d\theta}\left(\int_0^{\tan\theta} \sec^2 y\,dy\right) = \frac{d}{d\theta}(\tan(\tan\theta))$
$= \left(\sec^2(\tan\theta)\right)\sec^2\theta$

(b) $\frac{d}{d\theta}\left(\int_0^{\tan\theta} \sec^2 y\,dy\right) = \left(\sec^2(\tan\theta)\right)\left(\frac{d}{d\theta}(\tan\theta)\right) = \left(\sec^2(\tan\theta)\right)\sec^2\theta$

31. $y = \int_0^x \sqrt{1+t^2}\,dt \Rightarrow \frac{dy}{dx} = \sqrt{1+x^2}$

32. $y = \int_1^x \frac{1}{t}\,dt \Rightarrow \frac{dy}{dx} = \frac{1}{x}, x > 0$

33. $y = \int_{\sqrt{x}}^0 \sin t^2\,dt = -\int_0^{\sqrt{x}} \sin t^2\,dt \Rightarrow \frac{dy}{dx} = -\left(\sin\left(\sqrt{x}\right)^2\right)\left(\frac{d}{dx}\left(\sqrt{x}\right)\right) = -(\sin x)\left(\frac{1}{2}x^{-1/2}\right) = -\frac{\sin x}{2\sqrt{x}}$

34. $y = \int_0^{x^2} \cos\sqrt{t}\,dt \Rightarrow \frac{dy}{dx} = \left(\cos\sqrt{x^2}\right)\left(\frac{d}{dx}\left(x^2\right)\right) = 2x\cos|x|$

35. $y = \int_0^{\sin x} \frac{dt}{\sqrt{1-t^2}}, |x| < \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 x}}\left(\frac{d}{dx}(\sin x)\right) = \frac{1}{\sqrt{\cos^2 x}}(\cos x) = \frac{\cos x}{|\cos x|} = \frac{\cos x}{\cos x} = 1$ since $|x| < \frac{\pi}{2}$

36. $y = \int_0^{\tan x} \frac{dt}{1+t^2} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{1+\tan^2 x}\right)\left(\frac{d}{dx}(\tan x)\right) = \left(\frac{1}{\sec^2 x}\right)\left(\sec^2 x\right) = 1$

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37. $-x^2 - 2x = 0 \Rightarrow -x(x+2) = 0 \Rightarrow x = 0$ or $x = -2$; Area

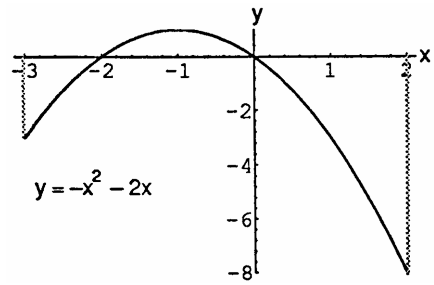
$$= -\int_{-3}^{-2}(-x^2-2x)dx + \int_{-2}^{0}(-x^2-2x)dx - \int_{0}^{2}(-x^2-2x)dx$$

$$= -\left[-\frac{x^3}{3} - x^2\right]_{-3}^{-2} + \left[-\frac{x^3}{3} - x^2\right]_{-2}^{0} - \left[-\frac{x^3}{3} - x^2\right]_{0}^{2}$$

$$= -\left(\left(-\frac{(-2)^3}{3} - (-2)^2\right) - \left(-\frac{(-3)^3}{3} - (-3)^2\right)\right)$$

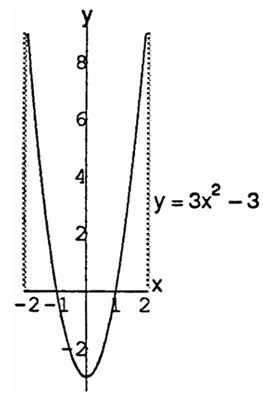
$$+ \left(\left(-\frac{0^3}{3} - 0^2\right) - \left(-\frac{(-2)^3}{3} - (-2)^2\right)\right)$$

$$- \left(\left(-\frac{2^3}{3} - 2^2\right) - \left(-\frac{0^3}{3} - 0^2\right)\right) = \frac{28}{3}$$



38. $3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$; because of symmetry about the y-axis, Area $= 2\left(-\int_{0}^{1}(3x^2-3)dx + \int_{1}^{2}(3x^2-3)dx\right)$

$2\left(-\left[x^3 - 3x\right]_0^1 + \left[x^3 - 3x\right]_1^2\right) = 2\left[-\left((1^3 - 3(1)) - (0^3 - 3(0))\right)\right.$
$\left. + \left((2^3 - 3(2)) - (1^3 - 3(1))\right)\right] = 2(6) = 12$



39. $x^3 - 3x^2 + 2x = 0 \Rightarrow x(x^2 - 3x + 2) = 0$

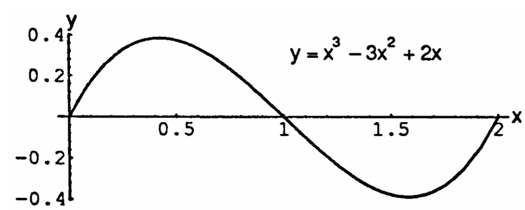
$\Rightarrow x(x-2)(x-1) = 0 \Rightarrow x = 0, 1,$ or 2;

Area $= \int_{0}^{1}(x^3 - 3x^2 + 2x)dx - \int_{1}^{2}(x^3 - 3x^2 + 2x)dx$

$$= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 - \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2$$

$$= \left(\frac{1^4}{4} - 1^3 + 1^2\right) - \left(\frac{0^4}{4} - 0^3 + 0^2\right)$$

$$- \left[\left(\frac{2^4}{4} - 2^3 + 2^2\right) - \left(\frac{1^4}{4} - 1^3 + 1^2\right)\right] = \frac{1}{2}$$

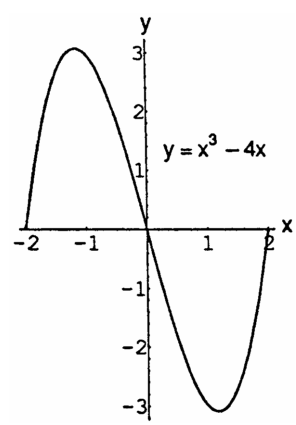


40. $x^3 - 4x = 0 \Rightarrow x(x^2 - 4) = 0 \Rightarrow x(x-2)(x+2) = 0$

$\Rightarrow x = 0, 2,$ or -2. Area $= \int_{-2}^{0}(x^3 - 4x)dx - \int_{0}^{2}(x^3 - 4x)dx$

$$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} - \left[\frac{x^4}{4} - 2x^2\right]_{0}^{2} = \left(\frac{0^4}{4} - 2(0)^2\right)$$

$$- \left(\frac{(-2)^4}{4} - 2(-2)^2\right) - \left[\left(\frac{2^4}{4} - 2(2)^2\right) - \left(\frac{0^4}{4} - 2(0)^2\right)\right] = 8$$

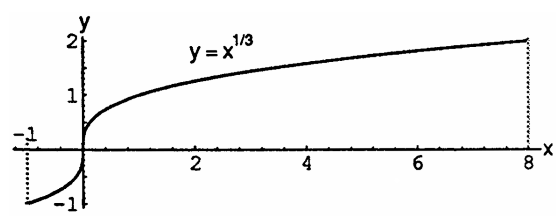


41. $x^{1/3} = 0 \Rightarrow x = 0$; Area $= -\int_{-1}^{0} x^{1/3}\,dx + \int_{0}^{8} x^{1/3}\,dx$

$$= \left[-\frac{3}{4}x^{4/3}\right]_{-1}^{0} + \left[\frac{3}{4}x^{4/3}\right]_{0}^{8}$$

$$= \left(-\frac{3}{4}(0)^{4/3}\right) - \left(-\frac{3}{4}(-1)^{4/3}\right) + \left(\frac{3}{4}(8)^{4/3}\right) - \left(\frac{3}{4}(0)^{4/3}\right)$$

$$= \frac{51}{4}$$



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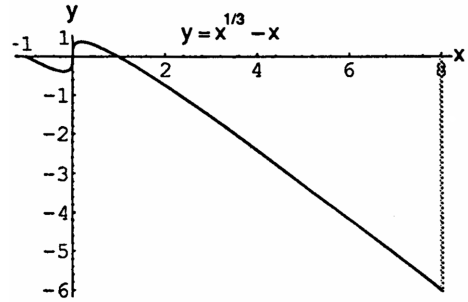
42. $x^{1/3} - x = 0 \Rightarrow x^{1/3}\left(1 - x^{2/3}\right) = 0 \Rightarrow x^{1/3} = 0$ or $1 - x^{2/3} = 0 \Rightarrow x = 0$ or $1 = x^{2/3} \Rightarrow x = 0$ or $1 = x^2 \Rightarrow x = 0$ or ± 1;



$\text{Area} = -\int_{-1}^{0}\left(x^{1/3} - x\right)dx + \int_{0}^{1}\left(x^{1/3} - x\right)dx - \int_{1}^{8}\left(x^{1/3} - x\right)dx$

$= -\left[\frac{3}{4}x^{4/3} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{3}{4}x^{4/3} - \frac{x^2}{2}\right]_{0}^{1} - \left[\frac{3}{4}x^{4/3} - \frac{x^2}{2}\right]_{1}^{8}$

$= -\left[\left(\frac{3}{4}(0)^{4/3} - \frac{0^2}{2}\right) - \left(\frac{3}{4}(-1)^{4/3} - \frac{(-1)^2}{2}\right)\right]$

$+ \left[\left(\frac{3}{4}(1)^{4/3} - \frac{1^2}{2}\right) - \left(\frac{3}{4}(0)^{4/3} - \frac{0^2}{2}\right)\right]$

$- \left[\left(\frac{3}{4}(8)^{4/3} - \frac{8^2}{2}\right) - \left(\frac{3}{4}(1)^{4/3} - \frac{1^2}{2}\right)\right]$

$= \frac{1}{4} + \frac{1}{4} - \left(-20 - \frac{3}{4} + \frac{1}{2}\right) = \frac{83}{4}$

43. The area of the rectangle bounded by the lines $y = 2$, $y = 0$, $x = \pi$, and $x = 0$ is 2π. The area under the curve $y = 1 + \cos x$ on $[0, \pi]$ is $\int_0^{\pi} (1 + \cos x)\, dx = [x + \sin x]_0^{\pi} = (\pi + \sin \pi) - (0 + \sin 0) = \pi$. Therefore the area of the shaded region is $2\pi - \pi = \pi$.

44. The area of the rectangle bounded by the lines $x = \frac{\pi}{6}$, $x = \frac{5\pi}{6}$, $y = \sin \frac{\pi}{6} = \frac{1}{2} = \sin \frac{5\pi}{6}$, and $y = 0$ is $\frac{1}{2}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) = \frac{\pi}{3}$. The area under the curve $y = \sin x$ on $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$ is $\int_{\pi/6}^{5\pi/6} \sin x\, dx = [-\cos x]_{\pi/6}^{5\pi/6}$ $= \left(-\cos \frac{5\pi}{6}\right) - \left(-\cos \frac{\pi}{6}\right) = -\left(-\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2} = \sqrt{3}$. Therefore the area of the shaded region is $\sqrt{3} - \frac{\pi}{3}$.

45. On $\left[-\frac{\pi}{4}, 0\right]$: The area of the rectangle bounded by the lines $y = \sqrt{2}$, $y = 0$, $\theta = 0$, and $\theta = -\frac{\pi}{4}$ is $\sqrt{2}\left(\frac{\pi}{4}\right)$ $= \frac{\pi\sqrt{2}}{4}$. The area between the curve $y = \sec \theta \tan \theta$ and $y = 0$ is $-\int_{-\pi/4}^{0} \sec \theta \tan \theta\, d\theta = [-\sec \theta]_{-\pi/4}^{0}$ $= (-\sec 0) - \left(-\sec\left(-\frac{\pi}{4}\right)\right) = \sqrt{2} - 1$. Therefore the area of the shaded region on $\left[-\frac{\pi}{4}, 0\right]$ is $\frac{\pi\sqrt{2}}{4} + \left(\sqrt{2} - 1\right)$. On $\left[0, \frac{\pi}{4}\right]$: The area of the rectangle bounded by $\theta = \frac{\pi}{4}$, $\theta = 0$, $y = \sqrt{2}$, and $y = 0$ is $\sqrt{2}\left(\frac{\pi}{4}\right) = \frac{\pi\sqrt{2}}{4}$. The area under the curve $y = \sec \theta \tan \theta$ is $\int_0^{\pi/4} \sec \theta \tan \theta\, d\theta = [\sec \theta]_0^{\pi/4} = \sec \frac{\pi}{4} - \sec 0 = \sqrt{2} - 1$. Therefore the area of the shaded region on $\left[0, \frac{\pi}{4}\right]$ is $\frac{\pi\sqrt{2}}{4} - \left(\sqrt{2} - 1\right)$. Thus, the area of the total shaded region is $\left(\frac{\pi\sqrt{2}}{4} + \sqrt{2} - 1\right) + \left(\frac{\pi\sqrt{2}}{4} - \sqrt{2} + 1\right) = \frac{\pi\sqrt{2}}{2}$.

46. The area of the rectangle bounded by the lines $y = 2$, $y = 0$, $t = -\frac{\pi}{4}$, and $t = 1$ is $2\left(1 - \left(-\frac{\pi}{4}\right)\right) = 2 + \frac{\pi}{2}$. The area under the curve $y = \sec^2 t$ on $\left[-\frac{\pi}{4}, 0\right]$ is $\int_{-\pi/4}^{0} \sec^2 t\, dt = [\tan t]_{-\pi/4}^{0} = \tan 0 - \tan\left(-\frac{\pi}{4}\right) = 1$. The area under the curve $y = 1 - t^2$ on $[0, 1]$ is $\int_0^1 \left(1 - t^2\right) dt = \left[t - \frac{t^3}{3}\right]_0^1 = \left(1 - \frac{1^3}{3}\right) - \left(0 - \frac{0^3}{3}\right) = \frac{2}{3}$. Thus, the total area under the curves on $\left[-\frac{\pi}{4}, 1\right]$ is $1 + \frac{2}{3} = \frac{5}{3}$. Therefore the area of the shaded region is $\left(2 + \frac{\pi}{2}\right) - \frac{5}{3} = \frac{1}{3} + \frac{\pi}{2}$.

47. $y = \int_{\pi}^{x} \frac{1}{t}\, dt - 3 \Rightarrow \frac{dy}{dx} = \frac{1}{x}$ and $y(\pi) = \int_{\pi}^{\pi} \frac{1}{t}\, dt - 3 = 0 - 3 = -3 \Rightarrow$ (d) is a solution to this problem.

48. $y = \int_{-1}^{x} \sec t\, dt + 4 \Rightarrow \frac{dy}{dx} = \sec x$ and $y(-1) = \int_{-1}^{-1} \sec t\, dt + 4 = 0 + 4 = 4 \Rightarrow$ (c) is a solution to this problem.

49. $y = \int_{0}^{x} \sec t\, dt + 4 \Rightarrow \frac{dy}{dx} = \sec x$ and $y(0) = \int_{0}^{0} \sec t\, dt + 4 = 0 + 4 = 4 \Rightarrow$ (b) is a solution to this problem.

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50. $y = \int_1^x \frac{1}{t}\,dt - 3 \Rightarrow \frac{dy}{dx} = \frac{1}{x}$ and $y(1) = \int_1^1 \frac{1}{t}\,dt - 3 = 0 - 3 = -3 \Rightarrow$ (a) is a solution to this problem.

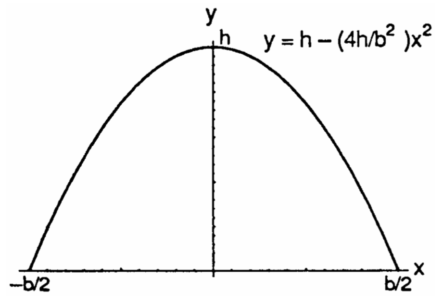
51. $y = \int_2^x \sec t\,dt + 3$

52. $y = \int_1^x \sqrt{1+t^2}\,dt - 2$

53. $s = \int_{t_0}^t f(x)\,dx + s_0$

54. $v = \int_{t_0}^t g(x)\,dx + v_0$

55. Area $= \int_{-b/2}^{b/2} \left(h - \left(\frac{4h}{b^2}\right)x^2\right) dx = \left[hx - \frac{4hx^3}{3b^2}\right]_{-b/2}^{b/2}$
$= \left(h\left(\frac{b}{2}\right) - \frac{4h\left(\frac{b}{2}\right)^3}{3b^2}\right) - \left(h\left(-\frac{b}{2}\right) - \frac{4h\left(-\frac{b}{2}\right)^3}{3b^2}\right)$
$= \left(\frac{bh}{2} - \frac{bh}{6}\right) - \left(-\frac{bh}{2} + \frac{bh}{6}\right) = bh - \frac{bh}{3} = \frac{2}{3}bh$



56. $r = \int_0^3 \left(2 - \frac{2}{(x+1)^2}\right) dx = 2\int_0^3 \left(1 - \frac{1}{(x+1)^2}\right) dx = 2\left[x - \left(\frac{-1}{x+1}\right)\right]_0^3 = 2\left[\left(3 + \frac{1}{(3+1)}\right) - \left(0 + \frac{1}{(0+1)}\right)\right]$
$= 2\left[3\,\frac{1}{4} - 1\right] = 2\left(2\,\frac{1}{4}\right) = 4.5$ or \$4500

57. $\frac{dc}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2} \Rightarrow c = \int_0^x \frac{1}{2}t^{-1/2}dt = \left[t^{1/2}\right]_0^x = \sqrt{x}$
$c(100) - c(1) = \sqrt{100} - \sqrt{1} = \$9.00$

58. By Exercise 57, $c(400) - c(100) = \sqrt{400} - \sqrt{100} = 20 - 10 = \$10.00$

59. (a) $v = \frac{ds}{dt} = \frac{d}{dt}\int_0^t f(x)\,dx = f(t) \Rightarrow v(5) = f(5) = 2$ m/sec
(b) $a = \frac{df}{dt}$ is negative since the slope of the tangent line at $t = 5$ is negative
(c) $s = \int_0^3 f(x)\,dx = \frac{1}{2}(3)(3) = \frac{9}{2}$ m since the integral is the area of the triangle formed by $y = f(x)$, the x-axis, and $x = 3$
(d) $t = 6$ since from $t = 6$ to $t = 9$, the region lies below the x-axis
(e) At $t = 4$ and $t = 7$, since there are horizontal tangents there
(f) Toward the origin between $t = 6$ and $t = 9$ since the velocity is negative on this interval. Away from the origin between $t = 0$ and $t = 6$ since the velocity is positive there.
(g) Right or positive side, because the integral of f from 0 to 9 is positive, there being more area above the x-axis than below it.

60. (a) $v = \frac{dg}{dt} = \frac{d}{dt}\int_0^t g(x)\,dx = g(t) \Rightarrow v(3) = g(3) = 0$ m/sec.
(b) $a = \frac{df}{dt}$ is positive, since the slope of the tangent line at $t = 3$ is positive
(c) At $t = 3$, the particle's position is $\int_0^3 g(x)\,dx = \frac{1}{2}(3)(-6) = -9$
(d) The particle passes through the origin at $t = 6$ because $s(6) = \int_0^6 g(x)\,dx = 0$
(e) At $t = 7$, since there is a horizontal tangent there
(f) The particle starts at the origin and moves away to the left for $0 < t < 3$. It moves back toward the origin for $3 < t < 6$, passes through the origin at $t = 6$, and moves away to the right for $t > 6$.
(g) Right side, since its position at $t = 9$ is positive, there being more area above the x-axis than below it at $t = 9$.

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61. $k > 0 \Rightarrow$ one arch of $y = \sin kx$ will occur over the interval $\left[0, \frac{\pi}{k}\right] \Rightarrow$ the area $= \int_0^{\pi/k} \sin kx\, dx = \left[-\frac{1}{k}\cos kx\right]_0^{\pi/k}$
$= -\frac{1}{k}\cos\left(k\left(\frac{\pi}{k}\right)\right) - \left(-\frac{1}{k}\cos(0)\right) = \frac{2}{k}$

62. $\lim\limits_{x\to 0} \frac{1}{x^3}\int_0^x \frac{t^2}{t^4+1}\,dt = \lim\limits_{x\to 0} \frac{\int_0^x \frac{t^2}{t^4+1}dt}{x^3} = \lim\limits_{x\to 0} \frac{\frac{x^2}{x^4+1}}{3x^2} = \lim\limits_{x\to 0} \frac{1}{3(x^4+1)} = \infty.$

63. $\int_1^x f(t)\,dt = x^2 - 2x + 1 \Rightarrow f(x) = \frac{d}{dx}\int_1^x f(t)\,dt = \frac{d}{dx}(x^2 - 2x + 1) = 2x - 2$

64. $\int_0^x f(t)\,dt = x\cos\pi x \Rightarrow f(x) = \frac{d}{dx}\int_0^x f(t)\,dt = \cos\pi x - \pi x\sin\pi x \Rightarrow f(4) = \cos\pi(4) - \pi(4)\sin\pi(4) = 1$

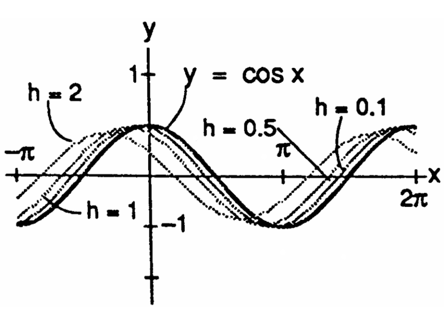
65. $f(x) = 2 - \int_2^{x+1} \frac{9}{1+t}\,dt \Rightarrow f'(x) = -\frac{9}{1+(x+1)} = \frac{-9}{x+2} \Rightarrow f'(1) = -3;\ f(1) = 2 - \int_2^{1+1} \frac{9}{1+t}\,dt = 2 - 0 = 2;$
$L(x) = -3(x-1) + f(1) = -3(x-1) + 2 = -3x + 5$

66. $g(x) = 3 + \int_1^{x^2} \sec(t-1)\,dt \Rightarrow g'(x) = (\sec(x^2-1))(2x) = 2x\sec(x^2-1) \Rightarrow g'(-1) = 2(-1)\sec((-1)^2-1)$
$= -2;\ g(-1) = 3 + \int_1^{(-1)^2} \sec(t-1)\,dt = 3 + \int_1^1 \sec(t-1)\,dt = 3 + 0 = 3;\ L(x) = -2(x-(-1)) + g(-1)$
$= -2(x+1) + 3 = -2x + 1$

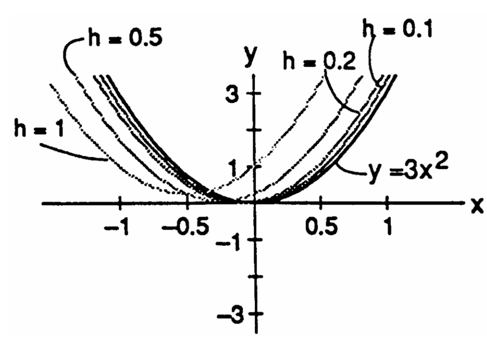
67. (a) True: since f is continuous, g is differentiable by Part 1 of the Fundamental Theorem of Calculus.
(b) True: g is continuous because it is differentiable.
(c) True, since $g'(1) = f(1) = 0$.
(d) False, since $g''(1) = f'(1) > 0$.
(e) True, since $g'(1) = 0$ and $g''(1) = f'(1) > 0$.
(f) False: $g''(x) = f'(x) > 0$, so g'' never changes sign.
(g) True, since $g'(1) = f(1) = 0$ and $g'(x) = f(x)$ is an increasing function of x (because $f'(x) > 0$).

68. (a) True: by Part 1 of the Fundamental Theorem of Calculus, $h'(x) = f(x)$. Since f is differentiable for all x, h has a second derivative for all x.
(b) True: they are continuous because they are differentiable.
(c) True, since $h'(1) = f(1) = 0$.
(d) True, since $h'(1) = 0$ and $h''(1) = f'(1) < 0$.
(e) False, since $h''(1) = f'(1) < 0$.
(f) False, since $h''(x) = f'(x) < 0$ never changes sign.
(g) True, since $h'(1) = f(1) = 0$ and $h'(x) = f(x)$ is a decreasing function of x (because $f'(x) < 0$).

69.



70. The limit is $3x^2$



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71-74. Example CAS commands:

Maple:

```
with( plots );
f := x -> x^3-4*x^2+3*x;
a := 0;
b := 4;
F := unapply( int(f(t),t=a..x), x );                    # (a)
p1 := plot( [f(x),F(x)], x=a..b, legend=["y = f(x)","y = F(x)"], title="#71(a) (Section 5.4)" ):
p1;
dF := D(F);                                             # (b)
q1 := solve( dF(x)=0, x );
pts1 := [ seq( [x,f(x)], x=remove(has,evalf([q1]),I) ) ];
p2 := plot( pts1, style=point, color=blue, symbolsize=18, symbol=diamond, legend="(x,f(x)) where F '(x)=0" ):
display( [p1,p2], title="71(b) (Section 5.4)" );
incr := solve( dF(x)>0, x );                            # (c)
decr := solve( dF(x)<0, x );
df := D(f);                                             # (d)
p3 := plot( [df(x),F(x)], x=a..b, legend=["y = f '(x)","y = F(x)"], title="#71(d) (Section 5.4)" ):
p3;
q2 := solve( df(x)=0, x );
pts2 := [ seq( [x,F(x)], x=remove(has,evalf([q2]),I) ) ];
p4 := plot( pts2, style=point, color=blue, symbolsize=18, symbol=diamond, legend="(x,f(x)) where f '(x)=0" ):
display( [p3,p4], title="71(d) (Section 5.4)" );
```

75-78. Example CAS commands:

Maple:

```
a := 1;
u := x -> x^2;
f := x -> sqrt(1-x^2);
F := unapply( int( f(t), t=a..u(x) ), x );
dF := D(F);                  # (b)
cp := solve( dF(x)=0, x );
solve( dF(x)>0, x );
solve( dF(x)<0, x );
d2F := D(dF);                # (c)
solve( d2F(x)=0, x );
plot( F(x), x=-1..1, title="#75(d) (Section 5.4)" );
```

79. Example CAS commands:

Maple:

```
f := `f`;
q1 := Diff( Int( f(t), t=a..u(x) ), x );
d1 := value( q1 );
```

80. Example CAS commands:

Maple:

```
f := `f`;
q2 := Diff( Int( f(t), t=a..u(x) ), x,x );
```

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value(q2);

71-80. Example CAS commands:

Mathematica: (assigned function and values for a, and b may vary)

For transcendental functions the FindRoot is needed instead of the Solve command.

The Map command executes FindRoot over a set of initial guesses

Initial guesses will vary as the functions vary.

```
Clear[x, f, F]
{a, b}= {0, 2π}; f[x_] = Sin[2x] Cos[x/3]
F[x_] = Integrate[f[t], {t, a, x}]
Plot[{f[x], F[x]},{x, a, b}]
x/.Map[FindRoot[F'[x]==0, {x, #}] &,{2, 3, 5, 6}]
x/.Map[FindRoot[f'[x]==0, {x, #}] &,{1, 2, 4, 5, 6}]
```

Slightly alter above commands for 75 - 80.

```
Clear[x, f, F, u]
a=0; f[x_] = x^2 − 2x − 3
u[x_] = 1 − x^2
F[x_] = Integrate[f[t], {t, a, u(x)}]
x/.Map[FindRoot[F'[x]==0,{x, #}] &,{1, 2, 3, 4}]
x/.Map[FindRoot[F''[x]==0,{x,#}] &,{1, 2, 3, 4}]
```

After determining an appropriate value for b, the following can be entered

```
b = 4;
Plot[{F[x], {x, a, b}]
```

5.5 INDEFINTE INTEGRALS AND THE SUBSTITUTION RULE

1. Let $u = 3x \Rightarrow du = 3\,dx \Rightarrow \frac{1}{3}\,du = dx$

$\int \sin 3x\,dx = \int \frac{1}{3}\sin u\,du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos 3x + C$

2. Let $u = 2x^2 \Rightarrow du = 4x\,dx \Rightarrow \frac{1}{4}\,du = x\,dx$

$\int x\sin(2x^2)\,dx = \int \frac{1}{4}\sin u\,du = -\frac{1}{4}\cos u + C = -\frac{1}{4}\cos 2x^2 + C$

3. Let $u = 2t \Rightarrow du = 2\,dt \Rightarrow \frac{1}{2}\,du = dt$

$\int \sec 2t\tan 2t\,dt = \int \frac{1}{2}\sec u\tan u\,du = \frac{1}{2}\sec u + C = \frac{1}{2}\sec 2t + C$

4. Let $u = 1 - \cos\frac{t}{2} \Rightarrow du = \frac{1}{2}\sin\frac{t}{2}\,dt \Rightarrow 2\,du = \sin\frac{t}{2}\,dt$

$\int \left(1 - \cos\frac{t}{2}\right)^2\left(\sin\frac{t}{2}\right)dt = \int 2u^2\,du = \frac{2}{3}u^3 + C = \frac{2}{3}\left(1 - \cos\frac{t}{2}\right)^3 + C$

5. Let $u = 7x - 2 \Rightarrow du = 7\,dx \Rightarrow \frac{1}{7}\,du = dx$

$\int 28(7x-2)^{-5}\,dx = \int \frac{1}{7}(28)u^{-5}\,du = \int 4u^{-5}\,du = -u^{-4} + C = -(7x-2)^{-4} + C$

6. Let $u = x^4 - 1 \Rightarrow du = 4x^3\,dx \Rightarrow \frac{1}{4}\,du = x^3\,dx$

$\int x^3\left(x^4 - 1\right)^2 dx = \int \frac{1}{4}u^2\,du = \frac{u^3}{12} + C = \frac{1}{12}\left(x^4 - 1\right)^3 + C$

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