MA 1201 Semester B 2019/20

Midterm Exam (E/F/G/H, 100 mins)

Instructions:

- Please show your work. Unsupported answers will receive **NO** credits.
- Make sure you write down the correct lecture session (E/F/G/H) you have registered for, together with your full name and student ID on the front page of your answer script.
- Exams submitted to wrong lecture sessions will **NOT** be graded and will receive **0 POINTS**.

---

1. (25 points) Let $A(1,2,0)$, $B(-1,3,0)$, $C(-1,2,-1)$, and $D(0,1,1)$ be four points in $\mathbb{R}^3$. Using vector method:

(a) (8 points) Find the area of the triangle $\triangle ABC$.

Solution. The area of the triangle $\triangle ABC$ is given by

$$Area = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|,$$ 2

where

$$\overrightarrow{AB} = B - A = \langle -1-1, 3-2, 0-0 \rangle = \langle -2, 1, 0 \rangle,$$ 1

$$\overrightarrow{AC} = C - A = \langle -1-1, 2-2, -1-0 \rangle = \langle -2, 0, -1 \rangle.$$ 1

Note that

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & 0 \\ -2 & 0 & -1 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} -2 & 0 \\ -2 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} -2 & 1 \\ -2 & 0 \end{vmatrix}$$

$$= -\vec{i} - 2\vec{j} + 2\vec{k} = \langle -1, -2, 2 \rangle.$$ 3

So

$$Area = \frac{1}{2}\sqrt{1+4+4} = \frac{3}{2}.$$ 1

(b) (9 points) Find the equation of the plane that contains A, B, and C.

Solution. The equation of the plane is given by

$$n_1(x-x_0) + n_2(y-y_0) + n_3(z-z_0) = 0,$$ 3

where $\vec{n} = \langle n_1, n_2, n_3 \rangle$ is a normal vector of the plane and $P(x_0, y_0, z_0)$ is a point on the plane. Since $\overrightarrow{n} = (-1,-2,2)$ and $P(x_0,y_0,z_0) = A(1,2,0)$ is a point on the plane, the equation takes the form 2

2 $$-(x-1) - 2(y-2) + 2(z-0) = 0, \qquad \text{or} \qquad -x - 2y + 2z + 5 = 0.$$ 2

(c) (8 points) Find the volume of the parallelepiped with AB, AC, and AD as its adjacent sides.

Solution. Note that

$$\overrightarrow{AD} = D - A = \langle 0-1, 1-2, 1-0 \rangle = \langle -1, -1, 1 \rangle. \quad 2$$

Then the volume of the parallelepiped with AB, AB, and AD as its adjacent sides is given by

$$V = |(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD}| = (1+2+2) = 5. \quad 2$$

4

---

2. (50 points) Evaluate the following integrals.

(a) (7 points) $\int e^{3x+2}\,dx$.

Solution. Let

$$u = 3x+2, \quad 2$$
$$du = (3x+2)'\,dx = 3\,dx, \qquad \text{or} \qquad dx = \frac{1}{3}\,du. \quad 1$$

Then

$$\int e^{3x+2}\,dx = \int e^u \cdot \left(\frac{1}{3}\,du\right) = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{3x+2} + C.$$

2 1 1

(b) (8 points) $\int_1^3 \frac{1}{1+|x-2|}\,dx$.

Motivation. Since, by definition,

$$|x-2| = \begin{cases} x-2, & \text{if } x-2 \geq 0 \\ -(x-2), & \text{if } x-2 < 0 \end{cases} = \begin{cases} x-2, & \text{if } x \geq 2 \\ 2-x, & \text{if } x < 2 \end{cases} \quad 2$$

it's necessary to partition the interval of integration $[1,3]$ at $x = 2$ into two parts, $[1,2]$ and $[2,3]$. 2

Solution.

$$\int_1^3 \frac{1}{1+|x-2|}\,dx = \int_1^2 \frac{1}{1+|x-2|}\,dx + \int_2^3 \frac{1}{1+|x-2|}\,dx = \int_1^2 \frac{1}{1+(2-x)}\,dx + \int_2^3 \frac{1}{1+(x-2)}\,dx$$

1 1

$$= \int_1^2 \frac{1}{3-x}\,dx + \int_2^3 \frac{1}{x-1}\,dx = -\ln|x-3|\,|_1^2 + \ln|x-1|\,|_2^3 = 2\ln 2. \quad 1$$

1

(c) (10 points) $\int \mathrm{e}^{-x}\sin(5x)\,dx$.

Solution. By the integration by part

2 2

$$\int \mathrm{e}^{-x}\sin(5x)\,dx = \int \sin(5x)\,d(-e^{-x}) = -e^{-x}\sin(5x) - \int (-e^{-x})\,d(\sin(5x)) = -e^{-x}\sin(5x) + 5\int e^{-x}\cos(5x)dx$$

2

$$= -e^{-x}\sin(5x) + 5\int \cos(5x)\,d(-e^{-x}) = -e^{-x}\sin(5x) - 5e^{-x}\cos(5x) + 5\int e^{-x}d(\cos(5x))$$

2

$$= -e^{-x}\sin(5x) - 5e^{-x}\cos(5x) - 25\int \mathrm{e}^{-x}\sin(5x)\,dx. \quad 1$$

Thus

$$\int \mathrm{e}^{-x}\sin(5x)\,dx = -\frac{1}{26}(e^{-x}\sin(5x) + 5e^{-x}\cos(5x)) + C. \quad 1$$

(d) (10 points) $\displaystyle\int \frac{dx}{\sin x \cos x}$.

Solution. Note that $\displaystyle\int \frac{dx}{\sin x\cos x} = \int \frac{\cos x dx}{\sin x \cos^2 x} = \int \frac{d(\sin x)}{\sin x(1-\sin^2 x)}$. Let $u = \sin x$. Then 2 2

$$
\begin{aligned}
\int \frac{dx}{\sin x \cos x} &= \int \frac{1}{u(1-u^2)}du = \int \frac{1}{u(1-u)(1+u)}du && 2 \quad 1\\
&= \int \frac{1}{u}du + \frac{1}{2}\int \frac{1}{1-u}du - \frac{1}{2}\int \frac{1}{1+u}du && 1\\
&= \ln|u| - \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + C, && 1\\
&= \frac{1}{2}\ln\left|\frac{\sin^2 x}{(\sin x-1)(\sin x+1)}\right| + C = \ln|\tan x| + C. && 1
\end{aligned}
$$

(e) (15 points) $\displaystyle\int \frac{x^3-3x^2+6x-2}{(x-1)(x^2-2x+2)}dx$.

Solution. Because the rational function is improper, by the long division,

$$\frac{x^3-3x^2+6x-2}{(x-1)(x^2-2x+2)} = 1 + \frac{2x}{(x-1)(x^2-2x+2)} \qquad 2$$

Note that $(x-1)$ and (x^2-2x+2) are the only factors of the denominator, and the partial fraction decomposition of the rational function is given by

$$\frac{2x}{(x-1)(x^2-2x+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2-2x+2}. \qquad 2$$

To find the constants A, B, and C, multiply both sides of the equation by the denominator $(x-1)(x^2-2x+2)$ and rearrange. This yields

$$2x = A(x^2-2x+2) + (Bx+C)(x-1). \qquad 1$$

So

$$A = 2, \qquad B = -2, \qquad C = 4. \qquad 1$$

It follows that

$$\frac{x^3-3x^2+6x-2}{(x-1)(x^2-2x+2)} = 1 + \frac{2}{x-1} + \frac{-2x+4}{x^2-2x+2}, \qquad 1$$

and thus

$$\int \frac{x^3-3x^2+6x-2}{(x-1)(x^2-2x+2)}dx = \int dx + \int \frac{2}{x-1}dx + \int \frac{-2x+4}{x^2-2x+2}dx = x + 2\ln|x-1| + \int \frac{-2x+4}{x^2-2x+2}dx. \qquad 2$$

To evaluate the last integral on the right side, observe that the substitution

$$
\begin{aligned}
u &= x^2-2x+2, && 1\\
du &= (x^2-2x+2)'\,dx = (2x-2)\,dx = 2(x-1)\,dx,
\end{aligned}
$$

indicates that an extra factor of $(x-1)$ ($=$ a constant multiple of u') is needed to simplify the integral. This suggests the decomposition of the numerator:

$$-2x+4 = -2(x-1)+2,$$

1

and hence

$$\begin{aligned}\int \frac{-2x+4}{x^2-2x+2}\,dx &= \int \Big(\frac{-2(x-1)}{x^2-2x+2} + \frac{2}{x^2-2x+2}\Big)\,dx \\ &= \int \frac{-2(x-1)}{x^2-2x+2}\,dx + 2\int \frac{1}{x^2-2x+2}\,dx.\end{aligned}$$

1

The first integral on the right side can be directly solved using the substitution

$$\begin{aligned} u &= x^2-2x+2, \\ du &= 2(x-1)\,dx,\end{aligned}$$

which gives

$$\int \frac{-2(x-1)}{x^2-2x+2}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C = -\ln|x^2-2x+2| + C.$$

1

As for the second integral, the fact that the quadratic polynomial (x^2-2x+2) is irreducible implies that

$$\int \frac{1}{x^2-2x+2}\,dx = \int \frac{1}{(x-1)^2+1}\,dx = \tan^{-1}(x-1)+C.$$

1

In conclusion,

$$\int \frac{x^3-3x^2+6x-2}{(x-1)(x^2-2x+2)}\,dx = x + 2\ln|x-1| - \ln|x^2-2x+2| + 2\tan^{-1}(x-1) + C.$$

1

---

3. (25 points)

(a) (15 points) Find the volume of the solid generated by revolving the region in the first quadrant bounded from above by $y = a(1-\cos x)$ for $0 \le x \le 2\pi$, from below by the x-axis, about the y-axis.

Solution. By the shell method, the volume of the solid is then given by

$$\begin{aligned} V &= \int_0^{2\pi} 2\pi x y(x)\,dx = 2a\pi \int_0^{2\pi} x(1-\cos x)\,dx = 2a\pi \int_0^{2\pi} x\,dx - 2a\pi \int_0^{2\pi} x\cos x\,dx \\ &= a\pi x^2\big|_0^{2\pi} - 2a\pi \int_0^{2\pi} x\,d(\sin x) = 4a\pi^3 - 2a\pi x \sin x\big|_0^{2\pi} + 2a\pi \int_0^{2\pi} \sin x\,dx = 4a\pi^3.\end{aligned}$$

4 4 2 2 2 1

(b) (10 points) Find the length of the curve $y = \frac{x^2}{4} - \frac{\ln x}{2}$, $1 \le x \le e$.

Solution. Since the curve is described by a function of x, the curve length should be expressed as an integral of x:

$$L = \int ds = \int_1^e \sqrt{1+\left[y'(x)\right]^2}\,dx,$$

2

where

$$1+\left[y'(x)\right]^2 = 1+(\frac{1}{2}x-\frac{1}{2x})^2 = (\frac{x}{2}+\frac{1}{2x})^2.$$

2

This shows that

$$L = \int_1^e (\frac{x}{2}+\frac{1}{2x})\,dx = \frac{x^2}{4}|_1^e + \frac{1}{2}\ln|x||_1^e = \frac{e^2}{4} - \frac{1}{4} + \frac{1}{2} = \frac{e^2+1}{4}.$$

2 2 1 1

— THE END —