# MA1201 Calculus and Basic Linear Algebra II   Problem Set 5   Complex Number

**Part A: Basic Concept**

**Problem 1**

Simplify the following expression and express your answer in the form of $a+bi$.

(a) $i^5 - i^7 + i^{10}$  (b) $\dfrac{1-i}{3+2i}$

(c) $(2+i)^2(3-i)$  (d) $(1-3i)^{-1}$

**Problem 2**

Let $z = a + bi$ be a complex numbers satisfying $\frac{1}{z} = 1 + 3i$. Find the value of $z$.

**Problem 3**

Express the following complex numbers in the Polar and Euler form. Write down the modulus and principal argument of each of the complex numbers.

(a) $z_1 = 3 - 3i$  (b) $z_2 = \sqrt{6} + \sqrt{2}i$

(c) $z_3 = -2i$  (d) $z_4 = -4 - \sqrt{48}i$

(e) $z_5 = -1 + 5i$  (f) $z_6 = 5 - 8i$

(g) $z_7 = ie^{\frac{i\pi}{4}}$  (h) $z_8 = -2e^{\frac{i\pi}{3}}$

(i) $z_9 = e^{\frac{i\pi}{5}} + e^{-i\pi}$  (j) $z_{10} = 1 - e^{\frac{i\pi}{4}}$

**Problem 4**

Express the following complex number in the Euler form and polar form. Here, $0 < \theta < \frac{\pi}{2}$.

(a) $-\cos\theta - i\sin\theta$  (b) $-\sin\theta + i\cos\theta$

(c) $1 - \sin\theta + i\cos\theta$  (d) $1 + \cos\theta - i\sin\theta$

**Problem 5**

Simplify the following expression using suitable methods. Express your answer in the Polar form or Euler form.

(a) $z_1 = \dfrac{1+\sqrt{3}i}{2-2i}$  (b) $z_2 = (1+i)^{-5}$

(c) $z_3 = \left(\dfrac{(1-i)(\sqrt{3}+i)}{2i}\right)^{12}$  (d) $z_4 = \sqrt[6]{-\sqrt{48}+4i}$

(e) $z_5 = \sqrt{\dfrac{4+4i}{(-2+\sqrt{12}i)^3}}$  (f) $z_6 = (-2-2i)^{\frac{3}{4}}$

(g) $z_7 = (\sin\theta - i\cos\theta)^5$  (h) $z_8 = \sqrt[4]{1 + e^{\frac{i\pi}{4}}}$

(Hint: For (g) and (h), you need to express the complex number inside the bracket in the polar form first. The techniques used in Problem 3(g)-(j) and Problem 4 will be useful.)

**Problem 6**

Show that for any $0 < \theta < \frac{\pi}{2}$, we have

$$\left(\frac{1 + i\tan\theta}{1 - i\tan\theta}\right)^5 = \frac{1 + i\tan 5\theta}{1 - i\tan 5\theta}.$$

(Hint: Note that $\tan\theta = \frac{\sin\theta}{\cos\theta}$).

**Problem 7 (A bit harder)**

Let $z$ be a complex number with $|z| = 1$ and $z \neq \pm 1$.

(a) Show that the complex number $z_0 = \frac{1+z}{1-z}$ is *purely imaginary*. (i.e. $z_0 = bi$ for some real number $b$.) (Hint: You may consider the polar form of $z$: $z = r(\cos\theta + i\sin\theta)$. What can you say about the value of $r$?)

(b) Using the similar technique, show that if $\theta \neq k\pi$, ($k$ is an integer) then $z_1 = \frac{1+\bar{z}}{1-\bar{z}}$ is also purely imaginary.

**Problem 8**

Let $z = 1 + 3i$ be a complex number, compute the following

(a) $\dfrac{3z}{1 + \bar{z}}$ (b) $(z + 3\bar{z})^2$

(c) $\sqrt[4]{z - \bar{z}}$

**Problem 9**

Using the fact that $z + \bar{z} = 2\mathrm{Re}(z)$ and $z - \bar{z} = 2\mathrm{Im}(z)i$, where $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ represent the real part and imaginary part of $z$ respectively, evaluate

$$\mathrm{Re}\left(\frac{z-1}{z+1}\right) \text{ and } \mathrm{Im}\left(\frac{z-1}{z+1}\right),$$

where $z$ is the complex number with $|z| = 1$. Express your answer in terms of $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ if necessary.

**Problem 10**

Let $z_1$ and $z_2$ be two complex numbers, show that

$$|z_1 + z_2|^2 - |z_1 - z_2|^2 = 4\mathrm{Re}(z_1\bar{z}_2).$$

(☺Hint: $|z|^2 = z\bar{z}$.)

**Application of Complex Number**

**Problem 11**

Solve the following equations

(a) $z^6 = -3 + \sqrt{3}i$

(b) $(1 - z)^7 + (1 + z)^7 = 0$

(c) $z^{10} - 5z^5 - 6 = 0$

(d) $z^8 - 2\sqrt{3}z^4 + 4 = 0$

(e) $\dfrac{z^5}{1 + z^5} = \sqrt{3}i.$

**Problem 12**

Solve the equation $z^4 - 8z^3 + 27z^2 - 50z + 50 = 0$ given that $3 + i$ is one of the root.

**Problem 13**

(a) By considering the expression $(\cos\theta + i\sin\theta)^5$, show that

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

(b) Using similar techniques as in (a), show that

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

**Problem 14**

(a) Let $z = \cos\theta + i\sin\theta$ be a complex number.

By considering the expression $\left(z - \frac{1}{z}\right)^5$ and using the fact that $z^n - \frac{1}{z^n} = 2i\sin n\theta$, show that

$$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta).$$

(b) By considering the expression $\left(z - \frac{1}{z}\right)^3\left(z + \frac{1}{z}\right)^4$ and using the fact that $z^n - \frac{1}{z^n} = 2i\sin n\theta$ and $z^n + \frac{1}{z^n} = 2\cos n\theta$, show that

$$\sin^3\theta\cos^4\theta = -\frac{1}{64}(\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta).$$

(c) Hence, compute the integral $\int_0^{\frac{\pi}{2}} \sin^3\theta\cos^4\theta\, d\theta$.