we also going to invent still more systems so as to obtain $\sqrt[4]{-1}$, $\sqrt[6]{-1}$, and so on? But it turns out this is not necessary. These numbers are already expressible in terms of the complex number system $a + ib$. In fact, the Fundamental Theorem of Algebra says that with the introduction of the complex numbers we now have enough numbers to factor every polynomial into a product of linear factors and so enough numbers to solve every possible polynomial equation.

### The Fundamental Theorem of Algebra

Every polynomial equation of the form

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0,$$

in which the coefficients $a_0, a_1, \ldots, a_n$ are any complex numbers, whose degree $n$ is greater than or equal to one, and whose leading coefficient $a_n$ is not zero, has exactly $n$ roots in the complex number system, provided each multiple root of multiplicity $m$ is counted as $m$ roots.

A proof of this theorem can be found in almost any text on the theory of functions of a complex variable.

## EXERCISES A.5

### Operations with Complex Numbers

1. **How computers multiply complex numbers** Find $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$.
   **a.** $(2, 3) \cdot (4, -2)$ **b.** $(2, -1) \cdot (-2, 3)$
   **c.** $(-1, -2) \cdot (2, 1)$
   (This is how complex numbers are multiplied by computers.)
2. Solve the following equations for the real numbers, $x$ and $y$.
   **a.** $(3 + 4i)^2 - 2(x - iy) = x + iy$
   **b.** $\left(\dfrac{1+i}{1-i}\right)^2 + \dfrac{1}{x + iy} = 1 + i$
   **c.** $(3 - 2i)(x + iy) = 2(x - 2iy) + 2i - 1$

### Graphing and Geometry

3. How may the following complex numbers be obtained from $z = x + iy$ geometrically? Sketch.
   **a.** $\bar{z}$ **b.** $\overline{(-z)}$
   **c.** $-z$ **d.** $1/z$
4. Show that the distance between the two points $z_1$ and $z_2$ in an Argand diagram is $|z_1 - z_2|$.

In Exercises 5–10, graph the points $z = x + iy$ that satisfy the given conditions.

5. **a.** $|z| = 2$ **b.** $|z| < 2$ **c.** $|z| > 2$
6. $|z - 1| = 2$
7. $|z + 1| = 1$
8. $|z + 1| = |z - 1|$
9. $|z + i| = |z - 1|$
10. $|z + 1| \geq |z|$

Express the complex numbers in Exercises 11–14 in the form $re^{i\theta}$, with $r \geq 0$ and $-\pi < \theta \leq \pi$. Draw an Argand diagram for each calculation.

11. $\left(1 + \sqrt{-3}\right)^2$
12. $\dfrac{1+i}{1-i}$
13. $\dfrac{1 + i\sqrt{3}}{1 - i\sqrt{3}}$
14. $(2 + 3i)(1 - 2i)$

### Powers and Roots

Use De Moivre's Theorem to express the trigonometric functions in Exercises 15 and 16 in terms of $\cos\theta$ and $\sin\theta$.

15. $\cos 4\theta$
16. $\sin 4\theta$
17. Find the three cube roots of 1.

**18.** Find the two square roots of $i$.

**19.** Find the three cube roots of $-8i$.

**20.** Find the six sixth roots of 64.

**21.** Find the four solutions of the equation $z^4 - 2z^2 + 4 = 0$.

**22.** Find the six solutions of the equation $z^6 + 2z^3 + 2 = 0$.

**23.** Find all solutions of the equation $x^4 + 4x^2 + 16 = 0$.

**24.** Solve the equation $x^4 + 1 = 0$.

## Theory and Examples

**25. Complex numbers and vectors in the plane** Show with an Argand diagram that the law for adding complex numbers is the same as the parallelogram law for adding vectors.

**26. Complex arithmetic with conjugates** Show that the conjugate of the sum (product, or quotient) of two complex numbers, $z_1$ and $z_2$, is the same as the sum (product, or quotient) of their conjugates.

**27. Complex roots of polynomials with real coefficients come in complex-conjugate pairs**

**a.** Extend the results of Exercise 26 to show that $f(\bar{z}) = \overline{f(z)}$ if

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

is a polynomial with real coefficients $a_0, \ldots, a_n$.

**b.** If $z$ is a root of the equation $f(z) = 0$, where $f(z)$ is a polynomial with real coefficients as in part (a), show that the conjugate $\bar{z}$ is also a root of the equation. (*Hint:* Let $f(z) = u + iv = 0$; then both $u$ and $v$ are zero. Use the fact that $f(\bar{z}) = \overline{f(z)} = u - iv$.)

**28. Absolute value of a conjugate** Show that $|\bar{z}| = |z|$.

**29. When $\mathbf{z} = \bar{\mathbf{z}}$** If $z$ and $\bar{z}$ are equal, what can you say about the location of the point $z$ in the complex plane?

**30. Real and imaginary parts** Let Re($z$) denote the real part of $z$ and Im($z$) the imaginary part. Show that the following relations hold for any complex numbers $z$, $z_1$, and $z_2$.

**a.** $z + \bar{z} = 2\text{Re}(z)$ **b.** $z - \bar{z} = 2i\text{Im}(z)$

**c.** $|\text{Re}(z)| \le |z|$

**d.** $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$

**e.** $|z_1 + z_2| \le |z_1| + |z_2|$