Problem 9

Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{a}| = 2$ and $|\vec{b}| = 3$ and the angle between these two vectors is $\cos^{-1}\frac{3}{5}$.

(a) Are the vector $\boxed{\vec{a} - 2\vec{b}}$ and $\boxed{-9\vec{a} + 2\vec{b}}$ perpendicular to each other? Explain your answer.

(b) If the angle between the vectors $\boxed{\vec{a}}$ and $\boxed{\vec{a} + k\vec{b}}$ is $60°$, find the value of k.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = 0.$$

$$\Leftrightarrow \vec{a}\cdot\vec{b} = 0.$$

(a). $(\vec{a} - 2\vec{b})\cdot(-9\vec{a} + 2\vec{b}) \overset{?}{=} 0$

$$= -9|\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + 18\,\vec{a}\cdot\vec{b} - 4|\vec{b}|^2$$

$$= -9\times 4 + 20\,|\vec{a}|\,|\vec{b}|\cos\left(\cos^{-1}\frac{3}{5}\right) - 4\times 9$$

$$= -36 + 20\times\boxed{2\times 3\times\frac{3}{5}} - 4\times 9 = 0$$

(b) $$\cos(60^\circ) = \frac{\vec{a}\cdot(\vec{a} + k\vec{b})}{|\vec{a}|\,|\vec{a} + k\vec{b}|} = \frac{1}{2}.$$

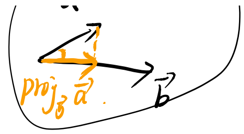
$$\vec{a}\cdot(\vec{a} + k\vec{b}) = |\vec{a}|^2 + k\,\vec{a}\cdot\vec{b} = 4 + \frac{18}{5}k$$

$$|\vec{a} + k\vec{b}| = \sqrt{(\vec{a} + k\vec{b})\cdot(\vec{a} + k\vec{b})} = \sqrt{4 + \frac{36}{5}k + 9k^2}$$

$$\Rightarrow \left(\frac{4 + \frac{18}{5}k}{\sqrt{4 + \frac{36}{5}k + 9k^2}}\right)^2 = \left(\frac{1}{2}\right)^2$$

$$\Rightarrow k_1 = \qquad , k_2 =$$

$\vec{a}$



$proj_{\vec b}\vec a$ $\quad$ $proj_{\vec a}\vec b$

$= |\vec a|\cos\theta\,\frac{\vec b}{|\vec b|}$

Problem 10

Find the projection vector of $\vec a$ onto $\vec b$ $(proj_{\vec b}\vec a)$ for each of the following set of vectors $\vec a$ and $\vec b$.

(a) $\vec a = 3\vec\imath - 4\vec\jmath$ and $\vec b = \vec\imath - 18\vec\jmath$.

$$proj_{\vec b}\vec a = |\vec a|\cos\theta \cdot \frac{\vec b}{|\vec b|}$$

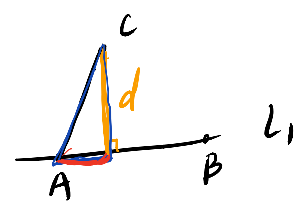
$$= \cancel{|\vec a|}\,\frac{\vec a\cdot\vec b}{\cancel{|\vec a|}\,|\vec b|}\cdot\frac{\vec b}{|\vec b|}$$

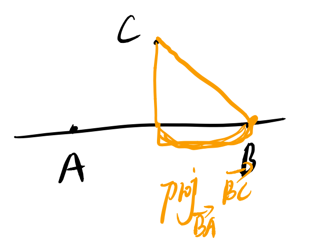
$$= \frac{3\times 1 + (-4)\times(-18)}{\sqrt{1+18^2}}\cdot\frac{\vec i - 18\vec j}{\sqrt{1+18^2}} = \frac{3}{13}\vec i - \frac{54}{13}\vec j.$$

Problem 11

(a) Let L_1 be a line passing through the points $A=(1,1,0)$ and $B=(-1,2,3)$, find the shortest distance between a point $C=(0,1,0)$ and the line L_1.

$\boxed{\begin{array}{l} 1^\circ.\ \text{the magnitude of } proj_{\vec{AB}}\vec{AC} \\ 2^\circ\ |\vec{AC}| \\ 3^\circ.\ d=\sqrt{|AC|^2-|proj_{\vec{AB}}\vec{AC}|^2} \end{array}}$





$1^\circ.$ $\vec{AC} = \vec{OC} - \vec{OA} = -\vec i$

$\vec{AB} = \vec{OB} - \vec{OA} = -2\vec i + \vec j + 3\vec k$.

$$|\vec{AC}|\,\underbrace{\frac{\vec{AC}\cdot\vec{AB}}{|\vec{AC}|\,|\vec{AB}|}}_{\cos\theta} = \frac{2}{\sqrt{14}}.$$

$2^\circ.$ $|\vec{AC}| = 1$

$3^\circ.$ $d = \sqrt{1-\left(\frac{2}{\sqrt{14}}\right)^2} = \sqrt{\frac{10}{14}}.$

$$\vec{a} \times \vec{b} = \underbrace{|\vec{a}||\vec{b}|\sin\theta}_{\text{magnitude}} \cdot \vec{n}$$

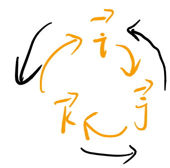
Part C: Vector Product and Scalar Triple Product

Problem 12

Find the value of $\vec{a} \times \vec{b}$ for each of following set of the vectors $\vec{a}$ and $\vec{b}$.

(a) $\vec{a} = \vec{\imath} + 3\vec{\jmath}$ and $\vec{b} = -2\vec{\jmath} + 5\vec{k}$.

(b) $\vec{a} = \vec{\imath} + \vec{\jmath} - 2\vec{k}$ and $\vec{b} = -3\vec{\imath} + 2\vec{\jmath} + 5\vec{k}$



$\vec{i} \times \vec{j} = \vec{k}$ | $\vec{j} \times \vec{i} = -\vec{k}$

$\vec{j} \times \vec{k} = \vec{i}$ | $\vec{k} \times \vec{j} = -\vec{i}$.

$\vec{k} \times \vec{i} = \vec{j}$ | $\vec{i} \times \vec{k} = -\vec{j}$

(a). $\vec{a} = \vec{i} + 3\vec{j}$, $\vec{b} = -2\vec{j} + 5\vec{k}$

$$\vec{a} \times \vec{b} = (\vec{i} + 3\vec{j}) \times (-2\vec{j} + 5\vec{k})$$

$$= -2\vec{k} - 5\vec{j} + 0 + 15\vec{i}$$

$$= 15\vec{i} - 5\vec{j} - 2\vec{k}.$$

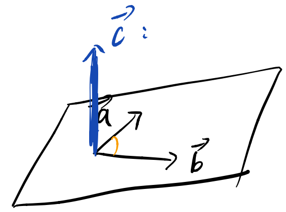
(b).

$$9\vec{i} + \vec{j} + 5\vec{k}.$$

Problem 13

Let $\vec{a}$ and $\vec{b}$ be two vectors in a plane, what is the value of $\vec{a}\cdot(\vec{a}\times\vec{b})$? (Hint: Think about the relationship between the vector $\vec{a}$ and $\vec{a}\times\vec{b}$.) $=0$.

$$\vec{a}\cdot(\vec{a}\times\vec{b})$$
$$=\vec{a}\cdot\vec{c}$$
$$=0$$



$|\vec{a}||\vec{b}|\sin\theta$.

Problem 15

Let $\vec{a}=2\vec{\imath}-\vec{\jmath}+2\vec{k}$ and $\vec{b}=4\vec{\imath}-4\vec{\jmath}+3\vec{k}$ be two vectors.

(a) Find a vector $\vec{c}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$.

(b) Find the area of the triangle with $\vec{a}$ and $\vec{b}$ as its adjacent sides.

(c) Find the equation of the plane passing through a point $(1,1,1)$ and containing the vectors $\vec{a}$ and $\vec{b}$. (Hint: See the remark of Example 12 of Chapter 4.)

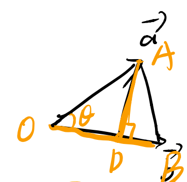
(d) Let $\vec{d}=\vec{\imath}+2\vec{k}$ be a vector. Determine whether the vectors $\vec{a}$, $\vec{b}$ and $\vec{d}$ are coplanar by finding the volume of parallelepiped with $\vec{a}$, $\vec{b}$ and $\vec{d}$ as adjacent sides.

(a). $\vec{c}=\vec{a}\times\vec{b}=(2\vec{i}-\vec{j}+2\vec{k})\times(4\vec{i}-4\vec{j}+3\vec{k})=5\vec{i}+2\vec{j}-4\vec{k}$

(b). $\boxed{A=\frac{1}{2}|\vec{a}\times\vec{b}|}$

$$|\vec{a}\times\vec{b}|=\sqrt{5^2+2^2+(-4)^2}=\sqrt{45}.$$

$$A=\frac{\sqrt{45}}{2}.$$

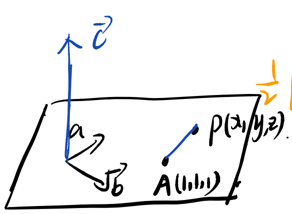


$\frac{1}{2}\times AD\times OB$.

$\frac{1}{2}|\vec{a}\times\vec{b}|=\frac{1}{2}|\vec{a}||\vec{b}|\sin\theta$

$=\frac{1}{2}AD\cdot OB$.

(c) $\vec{c}\cdot\vec{AP}=0$.



$$\vec{AP}=\vec{OP}-\vec{OA}=(x-1)\vec{i}+(y-1)\vec{j}+(z-1)\vec{k}.$$

$$5(x-1)+2(y-1)-4(z-1)=0.$$

$\Leftrightarrow$ $5x+2y-4z=3$. this is the equation for the plane.

(d) $V = \left| \vec{d} \cdot (\vec{a} \times \vec{b}) \right|$ ← absolute value.

$$= \left| (\vec{i} + 2\vec{k}) \cdot (5\vec{i} + 2\vec{j} - 4\vec{k}) \right|$$

$$= \left| 1 \times 5 + 2 \times (-4) \right| = |-3| = 3 \neq 0.$$

Problem 19

(a) Let π_1 be a plane containing the points $A = (3, -2, 0)$, $B = (2, 0, 3)$ and $C = (1, -1, 1)$, find the shortest distance between the point $D = (1, 0, -1)$ and the plane π_1.

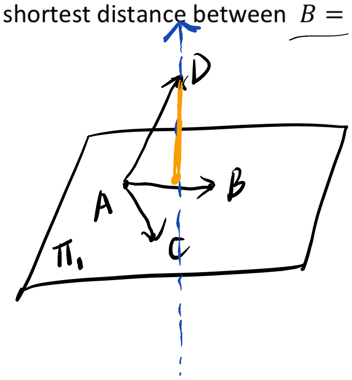
(b) Let π_2 be a plane passing through a point $A = (2, 1, -6)$. It is also given that the vector $\vec{n} = -\vec{\imath} - \vec{\jmath} - \vec{k}$ is perpendicular to the plane π_2. Find the shortest distance between $B = (1, -1, 1)$ and the plane π_2.

(a).

① $\overrightarrow{AB}$, $\overrightarrow{AC}$.

$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$

② $\overrightarrow{AD}$. $\quad proj_{\vec{n}} \overrightarrow{AD}$



1°. $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -\vec{i} + 2\vec{j} + 3\vec{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -2\vec{i} + \vec{j} + \vec{k}$.

$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \cdots = -\vec{i} - 5\vec{j} + 3\vec{k}$.

2°. $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = -2\vec{i} + 2\vec{j} - \vec{k}$.

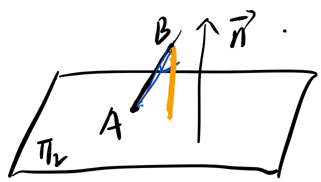
magnitude of $proj_{\vec{n}} \overrightarrow{AD} = \left| |\overrightarrow{AD}| \cdot \dfrac{\overrightarrow{AD} \cdot \vec{n}}{|\overrightarrow{AD}| \cdot |\vec{n}|} \right| = \cdots = \dfrac{11}{\sqrt{35}}$.

← absolute value.

(b).

$\overrightarrow{AB}$

$\vec{n}$. ✓



$\overrightarrow{AB}$

the magnitude $proj_{\vec{n}} \overrightarrow{AB} = \left| |\overrightarrow{AB}| \cdot \dfrac{\overrightarrow{AB} \cdot \vec{n}}{|\overrightarrow{AB}| |\vec{n}|} \right|$

$= \dfrac{4}{\sqrt{3}}$