Two important properties

Property 1

$\underline{x}$ is orthogonal to $\underline{y}$

$$\begin{array}{ccc} \underline{x} & & \underline{y} \\ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} & \cdot & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0 \end{array}$$

$\iff \quad \underline{x} \cdot \underline{y} = 0$

$\iff \quad \underline{x}^T \underline{y} = 0$

$$[-1, 0, 0] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

$\iff \quad \underline{x} \perp \underline{y}$

$\iff \quad x_1 y_1 + x_2 y_2 + \cdots x_n y_n = 0 \iff \sum_i x_i y_i = 0$

Property 2.

If a vector $\hat{x}$ is a normalized vector $\iff$ ($\hat{\underline{x}}$ is a unit vector)

$$\hat{\underline{x}} \cdot \hat{\underline{x}} = 1 \iff \hat{\underline{x}}^T \underline{x} = 1$$

These two properties will be used later

A has $\lambda_1, \lambda_2, \lambda_3$

$$\begin{array}{cccc} 3\times3 & 3\times1 & 3\times1 & 3\times1 \\ P = & [\ \hat{\underline{x}}_1\ , & \hat{\underline{x}}_2\ , & \hat{\underline{x}}_3\ ] \\ & \downarrow & \downarrow & \downarrow \\ & \lambda_1\ , & \lambda_2\ , & \lambda_3 \end{array}$$

(i) Suppose $\lambda_1 \neq \lambda_2 \neq \lambda_3$ are distinct **

(ii) $\hat{\underline{x}}_1, \hat{\underline{x}}_2, \hat{\underline{x}}_3$ are unit vectors

$i=1,2,3$ $\quad \|\hat{\underline{x}}_i\| = 1 \iff \hat{\underline{x}}_i \cdot \hat{\underline{x}}_i = 1$ ✓

$\iff \hat{\underline{x}}_i^T \hat{\underline{x}}_i = 1$ ✓

from (i), it implies "orthogonal"
$$\begin{cases} \hat{\underline{x}}_1 \cdot \underline{x}_2 = 0, & \underline{x}_2 \cdot \underline{x}_3 = 0, & \underline{x}_3 \cdot \underline{x}_1 = 0 \\ \hat{\underline{x}}_1 \cdot \hat{\underline{x}}_3 = 0, & \hat{\underline{x}}_2 \cdot \hat{\underline{x}}_3 = 0, & \hat{\underline{x}}_3 \cdot \hat{\underline{x}}_1 = 0 \end{cases}$$
✓ $\perp$ —

$$\boxed{P^T} P = \begin{array}{c} 3\times3 \\ \begin{array}{c} 1\times3 \\ 1\times3 \\ 1\times3 \end{array}\begin{bmatrix} \boxed{\hat{\underline{x}}_1^T} \\ \hat{\underline{x}}_2^T \\ \hat{\underline{x}}_3^T \end{bmatrix} \end{array} \begin{array}{c} 3\times1 \quad \boxed{3\times3} \\ [\ \hat{\underline{x}}_1\ , \hat{\underline{x}}_2\ , \hat{\underline{x}}_3\ ] \end{array}$$

$$= \begin{bmatrix} \overset{1\times1}{\boxed{\hat{\underline{x}}_1^T \hat{\underline{x}}_1}} \to 1 & \overset{1\times1}{\boxed{\hat{\underline{x}}_1^T \hat{\underline{x}}_2}} \to 0 & \overset{1\times1}{\boxed{\hat{\underline{x}}_1^T \hat{\underline{x}}_3}} \to 0 \\ \boxed{\hat{\underline{x}}_2^T \hat{\underline{x}}_1} \to 0 & \boxed{\hat{\underline{x}}_2^T \hat{\underline{x}}_2} \to 1 & \boxed{\hat{\underline{x}}_2^T \hat{\underline{x}}_3} \to 0 \\ \boxed{\hat{\underline{x}}_3^T \hat{\underline{x}}_1} \to 0 & \boxed{\hat{\underline{x}}_3^T \hat{\underline{x}}_2} \to 0 & \boxed{\hat{\underline{x}}_3^T \hat{\underline{x}}_3} \to 1 \end{bmatrix}$$

$$\overset{3\times3}{I_3}$$

$$= I_3$$

$$\boxed{P^T P = I_3}$$

$\iff \quad P^{-1} = P^T$

"P" is called orthogonal Matrix

Thm If $\overset{n\times n}{A}$ is a real symmetric matrix (ie $A = A^T$)

(i) $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$ are real numbers. ✓

(ii) $\underline{x}_1, \underline{x}_2, \underline{x}_3, \ , \underline{x}_n$ corresponding to distinct eigenvalues are orthogonal.

Proof: Read Notes (page 8)

eg: if $\lambda_1 \neq \lambda_2$

$\Rightarrow \quad \underline{x}_1 \perp \underline{x}_2$

$\Rightarrow \quad \underline{x}_1 \cdot \underline{x}_2 = 0$

***

$$\overset{n\times n}{A}\,\underline{x} = \lambda\,\underline{x}$$

A is real symmetric matrix

If A is a real symmetric Matrix

(i) $\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_n$ are distinct ✓

(ii) each eigenvector is normalized ✓
(Unit Vector)

We can find a P

$$P = [\hat{\underline{x}}_1, \hat{\underline{x}}_2, \cdots, \hat{\underline{x}}_n] \iff \boxed{P^{-1} = P^T}$$

① Unit vector

② eigenvector

$$P^{-1}AP = D$$

Diagonalization

$$\Updownarrow$$

$$\boxed{P^TAP = D}$$

"****"

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2, \ldots & \\ 0 & & \lambda_n \end{bmatrix}$$

Example ***

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}, \qquad \overset{\det}{|A - \lambda I|} = \left| \begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix} \right|$$

$$\det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4 = -(\lambda-1)(\lambda+2)^2$$

$$\lambda = 1, \ \underbrace{-2, \ -2} \ \text{(repeated)}$$

When $\lambda = 1$

$$(A-\lambda I)\underline{x} = 0 \Rightarrow \begin{pmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \overset{r_1 \leftrightarrow r_3}{\Longleftrightarrow} \left[\begin{array}{ccc|c} 3 & 3 & 0 & 0 \\ -3 & -6 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 3 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 + x_2 = 0 \\ -x_2 - x_3 = 0 \end{cases}$$

3 unknowns

2 Eqn

1 parameter

$x_3 = k, \quad x_2 = -k, \quad x_1 = k$ ✓

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad k \neq 0 \qquad \text{or} \quad \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix};$$

check

$$A\underline{x} = \lambda \underline{x}$$

When $\lambda = -2$ (repeated) ***

rank = 1   1 eqn

$$(A-\lambda I)\underline{x}=0 \iff \begin{pmatrix}3&3&3\\-3&-3&-3\\3&3&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \iff \left(\begin{array}{ccc|c}3&3&3&0\\0&0&0&0\\0&0&0&0\end{array}\right)$$

# of unknowns
− # of eqns
= # of parameters

$$3(x_1+x_2+x_3)=0$$

$$\begin{cases} n=3 \leftarrow \text{unknowns}\\ n-\boxed{\text{rank}} = 2 \text{ parameters}\\ 3-1 = \underline{2} \end{cases}$$

$$\begin{cases}x_3=t\\x_2=s\end{cases} \Rightarrow \boxed{x_1=-s-t}$$

$\boxed{s^2+t^2\neq 0}$ ✓

$$\begin{cases} s\neq 0, & t=0\\ s=0, & t\neq 0\\ s\neq 0, & t\neq 0\end{cases}$$

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}-s-t\\s\\t\end{pmatrix}= t\begin{pmatrix}-1\\0\\1\end{pmatrix}+s\begin{pmatrix}-1\\1\\0\end{pmatrix}$$

for eg: choose $\underline{s=0}$, $\underline{t=1}$ $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ ✓

$t=0$, $s=1$ $\begin{pmatrix}-1\\1\\0\end{pmatrix}$ ✓

⇒ Two linearly independent eigenvectors

$\lambda=1$ $\lambda=-2$

$$P=\begin{bmatrix}1&-1&-1\\-1&0&1\\1&1&0\end{bmatrix}, \quad \text{"}P^{-1}\text{ exists"} \quad \boxed{P^{-1}AP=D}$$

$$\begin{cases}\text{Check } A\begin{pmatrix}-1\\0\\1\end{pmatrix}=-2\begin{pmatrix}-1\\0\\1\end{pmatrix}\\[2ex] \text{check } A\begin{pmatrix}-1\\1\\0\end{pmatrix}=-2\begin{pmatrix}-1\\1\\0\end{pmatrix}\end{cases}$$

Example : Symmetric Case

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\det(A - \lambda I)$$

$$= \det\left(A - \begin{bmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \lambda \end{bmatrix} \right) = \cdots$$

$$= \cdots$$

$$= (-\lambda^3 + 9\lambda) = -\lambda(\lambda^2 - 9) = -\lambda(\lambda-3)(\lambda+3)$$

Thus, A has eigenvalues , 0, 3, -3

"Is A orthogonally diagonalizable" $\longleftrightarrow$ "$P^T A P = D$" ?

"$P^T P = I$"

$\lambda = 0$

$$\begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \cdots > \quad \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

unknowns

$3 - 2$ Eqns

$= 1 \leftarrow$ parameter

choose $x_3 = t$, $x_2 = \frac{1}{2} t$, $x_1 = t$

eigenvector $= \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$ $\xrightarrow{\text{unit vector}}$ $\hat{x}_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$ ✩

eg: $t=1$

$$d = \sqrt{1^2 + \tfrac{1}{2}^2 + 1^2}$$

$$= \sqrt{1 + \tfrac{1}{4} + 1}$$

$$= \sqrt{\tfrac{9}{4}} = \tfrac{3}{2}$$

$t=2$ $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ $\longrightarrow$

$$3 = \sqrt{2^2 + 1^2 + 2^2}$$

$\lambda = 3$

$$(A-\lambda I)\underline{x}=0 \iff \begin{bmatrix} -2 & -2 & 0 \\ -2 & -3 & 2 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \cdots \quad \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$x_3 = t \quad , \quad x_1 = -x_2 = -2t \quad , \quad x_2 = 2t$$

$$\underline{x}_2 = \begin{bmatrix} -2t \\ 2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, t=1 \quad \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \quad \text{or} \quad \underline{\hat{x}}_2 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix} \leftarrow \text{unit vector}$$

$$\frac{1}{\sqrt{(-2)^2+2^2+1}}$$

check $\underline{\hat{x}}_1 \cdot \underline{\hat{x}}_2 = 0$

$\lambda = -3$

$$(A-\lambda I)\underline{x}=0 \iff \begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 2 \\ 0 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

3 unknowns

2 Equ

1 parameters

$x_3 = s$

$$\underline{x}_3 = \begin{bmatrix} -\frac{1}{2}s \\ -s \\ s \end{bmatrix} \xrightarrow[\text{unit vector}]{} \underline{\hat{x}}_3 = \begin{bmatrix} -1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

$$\begin{cases} \underline{\hat{x}}_1 \cdot \underline{\hat{x}}_3 = 0 \\ \underline{\hat{x}}_2 \cdot \underline{\hat{x}}_3 = 0 \end{cases}$$

$s = 1$

$$\sqrt{\frac{1}{2}^2 + 1^2 + 1^2}$$

$$P = \begin{bmatrix} 2/3 & -2/3 & -1/3 \\ 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \lambda=0 & \lambda=3 & \lambda=-3 \end{array}$$

check $\begin{cases} P^T P = I \\ P P^T = I \end{cases}$ ✓

$$\boxed{P^T = P^{-1}}$$

Compute

$$\underline{P^T A P} = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

If $A_{(n\times n)}$ has $\lambda_1, \lambda_2, \dots \lambda_n$ eigenvalues.

Find the eigenvalues and their Corresponding eigenvectors

(i) A^k

(ii) A^{-1}

(iii) kA k is a constant.

(iv) A^T

(v) $(A+kI)$ k is a scalar.

If $\overset{3\times 3}{A}$ has eigenvalues, $\boxed{10, 15, 22}$

what about the eigenvalues

$$B = A^5 + 2A^3 + A^{-1} - 10I \quad ?$$

Ans. $\left(\lambda_i^5 + 2\lambda_i^3 + \frac{1}{\lambda_i} - 10\right)$ $\lambda_i = 10, 15, 22$

induction (!) .

$$\boxed{A \underline{x}_i = \lambda_i \underline{x}_i} \qquad i = 1, 2, \ldots, n$$

k = 1

$$A A \underline{x}_i = A(\lambda_i \underline{x}_i)$$

$$A^2 \underline{x}_i = \lambda_i A \underline{x}_i = \lambda_i(\lambda_i \underline{x}_i) = \lambda_i^2 \underline{x}_i$$

k = 2

k = 3

$$= A A^2 \underline{x}_i = A(\lambda_i^2 \underline{x}_i) = \lambda_i^2 A \underline{x}_i = \lambda_i^3 \underline{x}_i$$

$$A^3 \underline{x}_i$$

$\vdots$

k = 3

$\vdots$

k = 4

k = n $\quad \boxed{A^n \underline{x}_i = \lambda_i^n \underline{x}_i}$ is True

k = n+1 $\quad A^{n+1} \underline{x}_i = A A^n \underline{x}_i = \lambda_i^n A \underline{x}_i = \lambda_i^{n+1} \underline{x}_i$

Matrix A^k has eigenvalues $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$

eigenvectors $\underline{x}_1, \underline{x}_2, \ldots \underline{x}_n$

(same as those of eigenvector of A)

(ii) A^{-1} ?

exists. ✓

$$A\underline{x}_i = \lambda_i \underline{x}_i$$

$$A^{-1}A\underline{x}_i = A^{-1}(\lambda_i \underline{x}_i)$$

$$\underline{x}_i = \lambda_i A^{-1}\underline{x}_i$$

$$\frac{1}{\lambda_i}\underline{x}_i = A^{-1}\underline{x}_i$$

$$\boxed{A^{-1}\underline{x}_i = \frac{1}{\lambda_i}\underline{x}_i}$$

If A^{-1} exists, it has

$$\begin{array}{lcccccc} \text{eigenvalues of } A^{-1} & \Big(\frac{1}{\lambda_1}, & \frac{1}{\lambda_2}, & \dots, & \frac{1}{\lambda_n} \Big) & & \lambda_i \neq 0 \\ & \downarrow & \downarrow & & \downarrow & & \\ & \underline{x}_1 & \underline{x}_2 & & \underline{x}_n & & \end{array}$$

(same eigenvectors as A)

(iii) (kA)

$$A\underline{x}_i = \lambda_i \underline{x}_i$$

$$kA\underline{x}_i = k(\lambda_i \underline{x}_i)$$

$$(kA)\underline{x}_i = \boxed{(k\lambda_i)}\,\underline{x}_i$$

kA has eigenvalues

$$\begin{tikzcd}
(k\lambda_1 \arrow[d] & k\lambda_2 \arrow[d] & \cdots & k\lambda_n) \arrow[d] \\
\underline{x}_1 & \underline{x}_2 & & \underline{x}_n
\end{tikzcd}$$

(iv) Hints. $\overset{n\times n}{B}$ $\det(B) = \det(B^T)$ ✓✓

$$A\underline{x} = \lambda \underline{x}$$

$$\underline{x}^T A^T = \lambda \underline{x}^T$$

(X) x^T cannot be eigenvector of A^T **

$$(A - \lambda I)\underline{x} = 0$$

$$\det(A - \lambda I) = 0 \longrightarrow (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)=0$$

$$\det((A-\lambda I)^T) = 0 \longrightarrow \det(A^T - \lambda I) = 0$$

$$\hookrightarrow (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)=0$$

A has the same set of eigenvalues as that of A^T

$\underline{x}_i$ is not the eigenvector of A^T in general

(unless $\boxed{A^T = A}$, $\underline{x}_i$ will be the same eigenvector for A^T)

(v) $(A+kI)$ $\quad$ k is a scalar.

$$
\begin{aligned}
\boxed{(A+kI)}\,\underline{x}_i &= A\underline{x}_i + k\underline{x}_i \\
&= \lambda_i\underline{x}_i + k\underline{x}_i \\
&= \boxed{(\lambda_i+k)}\,\underline{x}_i
\end{aligned}
$$

$(A+kI)$ has eigenvalues

$$
\begin{array}{ccccccc}
(& \lambda_1+k, & \lambda_2+k & , & \cdots, & \lambda_n+k &) \\
 & \downarrow & \downarrow & & & \downarrow & \\
 & \underline{x}_1 & \underline{x}_2 & & & \underline{x}_n &
\end{array}
$$

Ex 1

Q: If $\overset{3\times3}{A}$ has eigenvalues , 10, 15, 22

What about the eigenvalues of

$$B = \underline{A^5 + 2A^3 + A^{-1} - 10I} \quad ?$$

Ans $\left(\lambda_i^5 + 2\lambda_i^3 + \frac{1}{\lambda_i} - 10\right)$.

Ex Book page 20-21

* 1, 5, 9, 11, 12, 15, 29, 30 . *

Book: Maths for Engineering and Science
CityU, Prentice Hall.
or other Engineering Maths book.

Ex. 2

Q 1 $A = \begin{pmatrix} -5 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

a). Find the eigenvalues and the corresponding eigenvectors of A.

b). Is the matrix A diagonalizable, briefly explain.

c). Let B be a square matrix, if $B = PDP^{-1}$, D is a diagonal matrix,

show that $\boxed{B^3 = PD^3P^{-1}}$.

Try it at Home!!