MA1200 Calculus and Basic Linear Algebra I
Chapter 3 Polynomials and Rational Functions

Review

A function $f(x)=a_n x^n + a_{n-1}x^{n-1}+\cdots+a_0$ (where the $a'_i\, s$ are real numbers and n is a non-negative integer) is called a *polynomial function.* If $a_n \neq 0$, n is the *degree* of the polynomial function.
In particular,

$$f(x)=ax+b \quad (a\neq 0) \text{ is called a linear function.}$$

$$f(x)=ax^2+bx+c \quad (a\neq 0) \text{ is called a quadratic function.}$$

Quotients of polynomial functions are called *rational functions.* That is, *f* is a rational function if it is of the form

$$f(x)=\frac{a_n x^n + a_{n-1}x^{n-1}+\cdots+a_1 x+a_0}{b_m x^m + b_{m-1}x^{m-1}+\cdots+b_1 x+b_0},$$

where $a'_i\, s$ and $b'_j\, s$ are real numbers and both n and m are non-negative integers.

1 Quadratic Functions

A *quadratic function* is any function of the form

$$f(x)=ax^2+bx+c\,,$$

where a, b and c are real numbers, with $a\neq 0$. The graph of any quadratic function is a parabola.

Using the *method of completing the square*, we can write the quadratic function $f(x)=ax^2+bx+c$ as

$$\begin{aligned} f(x) &= ax^2+bx+c \\ &= a\left(x^2+\frac{b}{a}x\right)+c = a\left[x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2\right]-a\left(\frac{b}{2a}\right)^2+c = a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right) \\ &= a(x-h)^2+k\,, \end{aligned}$$

where $h=-\dfrac{b}{2a}$ and $k=c-\dfrac{b^2}{4a}$.

The form $f(x) = a(x-h)^2+k$ is called the *standard form of a quadratic function*.

Example 1

Express each of the following quadratic functions in its standard form.

(a) $f(x)=3x^2+18x+7$

(b) $g(x)=-2x^2+8x+1$

Solutions

(a) $f(x) = 3x^2+18x+7 = 3(x^2+6x)+7 = 3(x^2+6x+9)-27+7 = 3(x+3)^2-20$
$\quad = 3[x-(-3)]^2+(-20)$

(b) $g(x) = -2x^2+8x+1 = -2(x^2-4x)+1 = -2(x^2-4x+4)+8+1 = -2(x-2)^2+9$

By expressing the quadratic function in its standard form $f(x) = a(x-h)^2 + k$, we can easily observe its properties:

(i) If $a > 0$, the parabola opens upward; if $a < 0$, the parabola opens downward.

(ii) The *vertex* of the parabola is at (h, k), i.e. $\left(-\dfrac{b}{2a}, c-\dfrac{b^2}{4a}\right)$.

If $a > 0$, then function attains minimum value at $x = -\dfrac{b}{2a}$ with the value $f\left(-\dfrac{b}{2a}\right) = c - \dfrac{b^2}{4a}$.

If $a < 0$, then function attains maximum value at $x = -\dfrac{b}{2a}$ with the value $f\left(-\dfrac{b}{2a}\right) = c - \dfrac{b^2}{4a}$.

(iii) The parabola is symmetric about the axis $x = -\dfrac{b}{2a}$.

(iv) To determine the location(s) (if any) where the parabola cuts the x-axis,

solve $a(x-h)^2 + k = 0$, (here $h = -\dfrac{b}{2a}$ and $k = c - \dfrac{b^2}{4a}$.)

$$a\left(x+\frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) = 0$$

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2} \qquad \text{(where } a \neq 0\text{)}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}$$

- it intersects the x-axis at 2 distinct points iff the *discriminant*, $\Delta = b^2 - 4ac > 0$;
- it touches the x-axis at 1 point iff $\Delta = b^2 - 4ac = 0$;
- it does not cut the x-axis iff $\Delta = b^2 - 4ac < 0$.

The *domain* and the *range* of the quadratic function can also be easily determined, as shown in the following example.

<u>Example 2</u>

It is given that (a) $f(x) = 2x^2 - 4x - 6$ and (b) $g(x) = -3x^2 + 24x - 36$. For each of the functions,

(i) express it in the standard form of quadratic function;
(ii) find the vertex;
(iii) sketch the graph;
(iv) find its largest possible domain and the largest possible range.

Solutions

(a)(i) $f(x) = 2x^2 - 4x - 6 = 2(x^2 - 2x) - 6 = 2(x^2 - 2x + 1) - 2 - 6 = 2(x-1)^2 + (-8)$

(ii) The vertex is at $(1, -8)$

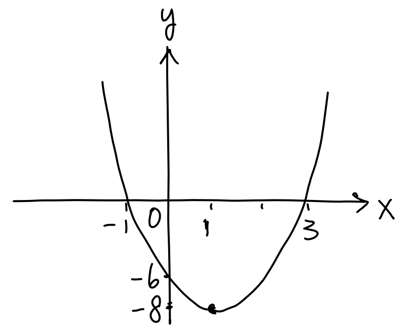
(iii) Since $a = 2 > 0$, The parabola opens upward.

Solving $2x^2 - 4x - 6 = 0$, get

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-6)}}{2(2)} = 3 \text{ or } -1.$$

It cuts the y-axis at $(0,\ f(0))$, i.e. $(0, -6)$.

The parabola is symmetric about $x = 1$.



(iv) The largest possible domain of $f(x)$ is **R**

The largest possible range of $f(x)$ is $[-8, \infty)$.

(b)(i) $g(x) = -3x^2 + 24x - 36 = -3(x^2 - 8x) - 36 = -3(x^2 - 8x + 16) + 48 - 36 = -3(x-4)^2 + 12$

(ii) The vertex is at $(4, 12)$

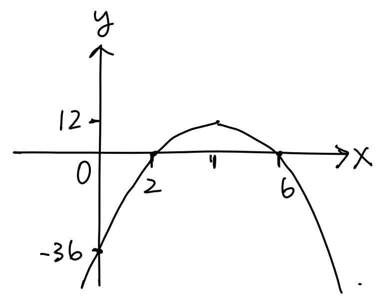
(iii) Since $a = -3 < 0$, The parabola opens downward.

Solving $g(x) = 0$, get

$$x = \frac{-24 \pm \sqrt{24^2 - 4(-3)(-36)}}{2(-3)} = 2 \text{ or } 6.$$

It cuts the y-axis at $(0,\ f(0))$, i.e. $(0, -36)$.

The parabola is symmetric about $x = 4$.



(iv) The largest possible domain of $g(x)$ is **R**

The largest possible range of $g(x)$ is $(-\infty, 12]$.

Questions: Sketch the graph of (a) $f(x) = 4x^2 - 4x + 1$; (b) $g(x) = 3x^2 + 6x + 10$.

2 Polynomial Functions

A *polynomial function of degree n* is a function of the form

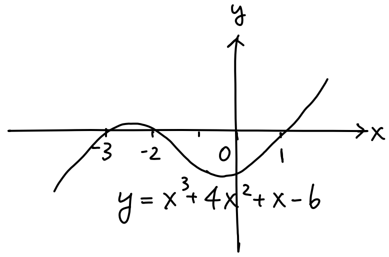
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

where n is a non-negative integer, a_n, a_{n-1}, …, a_1, a_0 are real numbers with $a_n \neq 0$. The number a_n is called the *leading coefficient*.

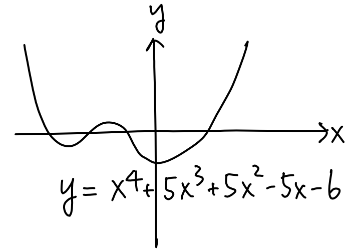
Polynomial functions of degree 2 or above have graphs that are *smooth* (i.e. contain only rounded curves with no sharp corners) and *continuous* (i.e. having no break). By observing the value of the leading coefficient, we can know how the function behaves as x tends to ∞ or $-\infty$.

Case 1: $a_n > 0$

- If n is odd, then the graph falls to the left and rises to the right.

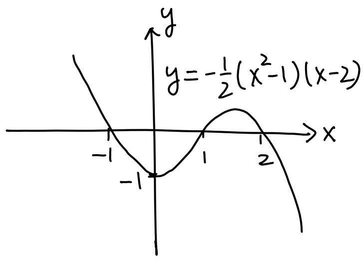


- If n is even, then the graph rises both to the left and to the right.

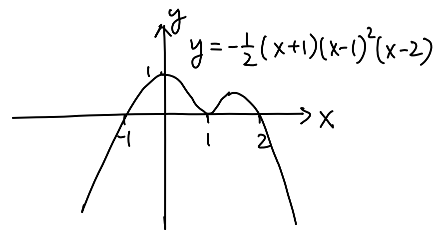


Case 2: $a_n < 0$

- If n is odd, then the graph rises to the left and falls to the right.



- If n is even, then the graph falls both to the left and to the right.



A. Division of Polynomials

Recall that

$$\text{Dividend} = \text{Quotient} \times \text{Divisor} + \text{Remainder}$$

e.g. $51 = 6 \cdot 8 + 3$, $132 = 12 \cdot 11 + 0$, …etc

Similarly, when a polynomial $p(x)$ is divided by another polynomial $d(x)$ of lower degree, we can obtain the quotient $q(x)$ and remainder $r(x)$, i.e.

$$p(x) = q(x)d(x) + r(x)$$

e.g. $2x^3 - 7x^2 + 7x + 1 = (2x^2 - 3x + 1)(x - 2) + 3$

We can obtain the quotient and remainder when $p(x)$ is divided by $d(x)$ using *long division*.
Remark: We can also use synthetic division to divide polynomials if the divisor is of the form $x - c$.

Example 3

Find the quotient and the reminder when the polynomial $p(x) = 2x^3 - 7x^2 + 7x + 1$ is divided by $x - 2$.

Solution

By long division,

$2x^3 - 7x^2 + 7x + 1 = (2x^2 - 3x + 1)(x - 2) + 3$

$\therefore$ The quotient is $2x^2 - 3x + 1$
and the remainder is 3.

$$\begin{array}{r|l}
 & 2x^2 - 3x + 1 \\
\hline
x - 2 & 2x^3 - 7x^2 + 7x + 1 \\
 & \underline{2x^3 - 4x^2} \\
 & \quad -3x^2 + 7x \\
 & \quad \underline{-3x^2 + 6x} \\
 & \qquad\qquad x + 1 \\
 & \qquad\qquad \underline{x - 2} \\
 & \qquad\qquad\quad 3
\end{array}$$

Example 4

Find the quotient and the reminder when $p(x) = 2x^3 - 5x^2 + 5$ is divided by $2x + 3$.

Solution

By long division,

$$\begin{array}{r|l} & x^2 \quad -4x \quad +6 \\ \hline 2x+3 & 2x^3 - 5x^2 \qquad\quad +5 \\ & \underline{2x^3 + 3x^2} \\ & \quad -8x^2 \\ & \quad \underline{-8x^2 - 12x} \\ & \qquad\quad 12x + 5 \\ & \qquad\quad \underline{12x + 18} \\ & \qquad\qquad -13 \end{array}$$

$2x^3 - 5x^2 + 5 = (x^2 - 4x + 6)(2x + 3) - 13$

$\therefore$ The quotient is $x^2 - 4x + 6$
and the remainder is –13.

Question: Find the quotient and the reminder when $p(x) = 5x^3 + 6x + 8$ is divided by $x + 2$.

B. Remainder Theorem

Consider the case when the polynomial $f(x)$ is divided by $ax - b$. The remainder *r* must be a constant.

We have $\qquad f(x) = q(x) \cdot (ax - b) + r$

When $x = \dfrac{b}{a}$, $\qquad f\left(\dfrac{b}{a}\right) = q\left(\dfrac{b}{a}\right) \cdot \left(a\left(\dfrac{b}{a}\right) - b\right) + r$

$$f\left(\frac{b}{a}\right) = r$$

Thus we have the *Remainder Theorem*:

> When a polynomial $f(x)$ is divided by $ax - b$, the remainder is $f\left(\dfrac{b}{a}\right)$.

Example 5

Find the remainder when $f(x) = 2x^3 + 3x^2 - 11x + 5$ is divided by $3x - 1$.

Solution

By the Remainder Theorem,

$$\begin{aligned} \text{Remainder} &= f\left(\frac{1}{3}\right) \\ &= 2\left(\frac{1}{3}\right)^3 + 3\left(\frac{1}{3}\right)^2 - 11\left(\frac{1}{3}\right) + 5 \\ &= \frac{47}{27} \end{aligned}$$

C. Zeros of Polynomial Functions and the Factor Theorem

The values of x for which the polynomial function $f(x)$ is equal to 0 are called the *zeros*. These values are the *roots* (*solutions*) of the polynomial equation $f(x)=0$. Each real root appears as an x-intercept of the graph of the polynomial function. The *Factor Theorem* is helpful in finding the real roots of a polynomial function:

$$\boxed{ax-b \text{ is a factor of a polynomial } f(x) \text{ if and only if } f\left(\frac{b}{a}\right)=0.}$$

Example 6

Determine whether $x-3$ is a factor of $f(x)=2x^3-5x^2-4x+3$. Hence factorize $f(x)=2x^3-5x^2-4x+3$.

Solution

Notice that $f(3)=2(3)^3-5(3)^2-4(3)+3=0$.

By the Factor Theorem, $x-3$ is a factor of $f(x)$.

$$\begin{aligned}\therefore\quad f(x) &= 2x^3-5x^2-4x+3\\ &= (x-3)(2x^2+x-1)\\ &= (x-3)(2x-1)(x+1)\end{aligned}$$

$$\begin{array}{r|l} & 2x^2+x-1\\ \hline x-3 & 2x^3-5x^2-4x+3\\ & \underline{2x^3-6x^2}\\ & \quad\ x^2-4x\\ & \quad\ \underline{x^2-3x}\\ & \qquad -x+3\\ & \qquad \underline{-x+3}\end{array}$$

2 Rational Functions

Rational functions are functions of the form $f(x)=\dfrac{h(x)}{g(x)}$, where $h(x)$ and $g(x)$ are polynomials and $g(x)\neq 0$. Examples of rational functions are $\dfrac{x^2+x+2}{x^3-x+6}$, $\dfrac{-x^4-x+1}{x^3-x+1}$ and x^3+2x+7.

If we have a rational function $f(x)=\dfrac{h(x)}{g(x)}$, then $f(x)$ is a *proper rational function* if the degree of $h(x)$ is less than that of $g(x)$. Examples are $\dfrac{2x^2-3x+5}{x^3-x+7}$ and $\dfrac{x^2+4}{(x-1)^2(x+5)}$.

$f(x)=\dfrac{h(x)}{g(x)}$ is an *improper rational function* if the degree of $h(x)$ is greater than or equal to that of $g(x)$. Examples are $\dfrac{2x^4-x+2}{x^3-4x+7}$ and $\dfrac{2x^5+3}{(x+1)^2(x^2+3x-5)}$.

An improper rational function can be written as a sum of a polynomial in x and a proper rational functions.

For example, we can use long division (or synthetic division) to write the improper rational function $f(x)=\dfrac{x^5+2x^3-x+1}{x^3+5x}$ as $f(x)=(x^2-3)+\dfrac{14x+1}{x^3+5x}$.

The domain of a rational function $f(x)=\dfrac{h(x)}{g(x)}$ is the set **R** except the value(s) of x such that $g(x)=0$.

<u>Example 7</u>

Find the largest possible domain of each of the following rational functions.

(a) $r(x)=\dfrac{2x^2-x+1}{x^3+2x^2+5x+10}$ (b) $m(x)=\dfrac{x^4+2x^3-x+1}{x^3+4x^2+x-6}$ (c) $k(x)=\dfrac{x^3+x+2}{x^2+1}$

Solutions

(a) $r(x)=\dfrac{2x^2-x+1}{x^3+2x^2+5x+10}=\dfrac{2x^2-x+1}{(x+2)(x^2+5)}$

the root of $(x+2)(x^2+5)=0$ is $x=-2$.

$\therefore$ The largest possible domain of $r(x)$ is $\mathbf{R}\setminus\{-2\}$.

(b) $m(x)=\dfrac{x^4+2x^3-x+1}{x^3+4x^2+x-6}=\dfrac{x^4+2x^3-x+1}{(x-1)(x+2)(x+3)}$

the root of $(x-1)(x+2)(x+3)=0$ are $x=1$, $x=-2$ and $x=-3$.

$\therefore$ The largest possible domain of $r(x)$ is $\mathbf{R}\setminus\{1,-2,-3\}$.

(c) For $k(x)=\dfrac{x^3+x+2}{x^2+1}$, there is no real root for $x^2+1=0$.

$\therefore$ The largest possible domain of $k(x)$ is **R**.

3 Partial Fractions

If the denominator of a rational function can be factorized into 2 or more linear or quadratic functions, we can decompose this function by a process called *partial fractions*. The following examples demonstrate how to decompose *proper* rational functions. (Note: We assume the expressions are already *proper* rational functions.):

Type	Expression	Form of Partial Fraction
Distinct Linear Factors	e.g. $\dfrac{f(x)}{(x+a)(x+b)(x+c)}$	$\dfrac{A}{(x+a)}+\dfrac{B}{(x+b)}+\dfrac{C}{(x+c)}$
Repeated Linear Factors	e.g. $\dfrac{f(x)}{(x+a)^3}$	$\dfrac{A}{(x+a)}+\dfrac{B}{(x+a)^2}+\dfrac{C}{(x+a)^3}$
Quadratic Factors	e.g. $\dfrac{f(x)}{(ax^2+bx+c)(x+d)}$ where ax^2+bx+c cannot be further factorized	$\dfrac{Ax+B}{(ax^2+bx+c)}+\dfrac{C}{(x+d)}$
⋮		

Remark: Notice that the degree of the trial polynomial in the numerator is *one less than* the degree of the polynomial in the denominator.

Type A: Distinct Linear Factors

Example 8

Resolve $\dfrac{3x-11}{x^2+2x-3}$ into partial fractions.

Solution

$$\frac{3x-11}{x^2+2x-3} \equiv \frac{3x-11}{(x-1)(x+3)} = \frac{A}{(x-1)}+\frac{B}{(x+3)}$$

$$\frac{3x-11}{(x-1)(x+3)} = \frac{A(x+3)+B(x-1)}{(x-1)(x+3)}$$

$$3x - 11 = A(x + 3) + B(x - 1)$$

Put $x = 1$, $-8 = A(1 + 3)$, get $A = -2$;

Put $x = -3$, $-20 = B(-3 - 1)$, get $B = 5$.

Thus, $$\frac{3x-11}{x^2+2x-3} \equiv \frac{3x-11}{(x-1)(x+3)} = -\frac{2}{(x-1)}+\frac{5}{(x+3)}$$

$$= \frac{5}{(x+3)}-\frac{2}{(x-1)}$$

Example 9

Resolve $\dfrac{5x+3}{x^3-2x^2-3x}$ into partial fractions.

Solution

Note that $x^3-2x^2-3x = x(x+1)(x-3)$.

$$\frac{5x+3}{x^3-2x^2-3x} \equiv \frac{5x+3}{x(x+1)(x-3)} = \frac{A}{x}+\frac{B}{(x+1)}+\frac{C}{(x-3)}$$

$$\frac{5x+3}{x(x+1)(x-3)} = \frac{A(x+1)(x-3)+Bx(x-3)+Cx(x+1)}{x(x+1)(x-3)}$$

$$5x + 3 = A(x + 1)(x - 3) + Bx(x - 3) + Cx(x + 1)$$

Put $x = 0$, $3 = -3A$, get $A = -1$;

Put $x = -1$, $-2 = 4B$, get $B = -\dfrac{1}{2}$;

Put $x = 3$, $18 = 12C$, get $C = \dfrac{3}{2}$.

Thus, $$\frac{5x+3}{x^3-2x^2-3x} \equiv \frac{5x+3}{x(x+1)(x-3)} = -\frac{1}{x}-\frac{1}{2(x+1)}+\frac{3}{2(x-3)}$$

$$= \frac{3}{2(x-3)}-\frac{1}{x}-\frac{1}{2(x+1)}$$

Type B: Repeated Linear Factors

Example 10 Resolve $\dfrac{3x^2-8x+13}{(x+3)(x-1)^2}$ into partial fractions.

Solution

$$\frac{3x^2-8x+13}{(x+3)(x-1)^2} = \frac{A}{(x+3)}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}$$

$$\frac{3x^2-8x+13}{(x+3)(x-1)^2} = \frac{A(x-1)^2+B(x+3)(x-1)+C(x+3)}{(x+3)(x-1)^2}$$

$$3x^2 - 8x + 13 = A(x-1)^2 + B(x+3)(x-1) + C(x+3)$$

Put $x=1$, $8 = 4C$, get $C = 2$;
Put $x = -3$, $64 = 16A$, get $A = 4$;
Put $x = 0$, $13 = A - 3B + 3C = 10 - 3B$, get $B = -1$.

Thus, $$\frac{3x^2-8x+13}{(x+3)(x-1)^2} = \frac{4}{(x+3)}-\frac{1}{(x-1)}+\frac{2}{(x-1)^2}$$

Remark: In general, if a factor ($ax+b$) is repeated n times, we would have n terms in the decomposition of the forms $\dfrac{A_1}{(ax+b)}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_n}{(ax+b)^n}$.

Type C: Irreducible Quadratic Factors

In factoring the denominator of a fraction, we may get some quadratic terms, which cannot be further factorized into real linear factors.

Example 11 Resolve $\dfrac{6x^2-3x+1}{(4x+1)(x^2+1)}$ into partial fractions.

Solution $$\frac{6x^2-3x+1}{(4x+1)(x^2+1)} = \frac{A}{(4x+1)}+\frac{Bx+C}{(x^2+1)}$$

$$6x^2 - 3x + 1 = A(x^2 + 1) + (Bx + C)(4x + 1)$$

Put $x = -\dfrac{1}{4}$,

$$6\left(-\frac{1}{4}\right)^2-3\left(-\frac{1}{4}\right)+1 = A\left[\left(-\frac{1}{4}\right)^2+1\right], \qquad \frac{17}{8}=\frac{17A}{16}\text{, get } A = 2$$

The equation becomes

$$6x^2 - 3x + 1 = 2(x^2 + 1) + (Bx + C)(4x + 1)$$
$$6x^2 - 3x + 1 = 2x^2 + 2 + (Bx + C)(4x + 1)$$
$$4x^2 - 3x - 1 = (Bx + C)(4x + 1)$$
$$4x^2 - 3x - 1 = (Bx + C)(4x + 1)$$
$$(4x + 1)(x - 1) = (Bx + C)(4x + 1)\text{, get } B = 1 \text{ and } C = -1$$

Thus, $$\frac{6x^2-3x+1}{(4x+1)(x^2+1)} = \frac{2}{(4x+1)}+\frac{x-1}{(x^2+1)}.$$