eigenvalue & eigenvector

- Def: $A\vec{x} = \lambda \vec{x}$
- Interpretation of eigenvector & eigenvalue.
- Computation — characteristic polynomial $\det(A-\lambda I)$
- Diagonalizable — Diagonalization thm
  - $A_{n\times n} = P_{n\times n} D P^{-1}$ P invertible D diagonal
- symmetric matrix
  - it must be orthogonally diagonalizable.
    $A = PDP^T$, P orthogonal matrix, D diagonal.
  - P is orthogonal $\Longleftrightarrow$ $P = P^T$ $\Longleftrightarrow$ columns orthogonal + unit
  - Quadratic form $Q(\vec{x}) = \vec{x}^T A \vec{x}$ $\begin{cases} \text{positive} \\ \text{negative.} \end{cases}$ definite.
    $= \vec{x}^T P D P^T \vec{x}$
    $= \vec{y}^T D \vec{y}$

Differentiation

- $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$
- partial derivatives
  - Def: $\partial_x f(x_0,y_0) = \frac{d}{dx} f(x,y_0)\Big|_{x=x_0}$
  - Interpretation: slope of tangent line.
  - Chain Rule
  - Implicit differentiation eg: $x^3 + y^2 + 3xz = 5$ $\quad \cancel{\frac{\partial x}{\partial y}} \quad \frac{\partial x}{\partial z}$
  - Application
    - linear, quadratic Approximation.
    - Max, min, saddle.
- Directional derivative. $D_{\vec{u}} f(P) = \nabla f(P) \cdot \frac{\vec{u}}{\|\vec{u}\|}$.

Vector Field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$

$$\vec{F}(x,y,z) = f_1(x,y,z)\,\vec{i} + f_2(x,y,z)\,\vec{j} + f_3(x,y,z)\,\vec{k}$$

Differentiable operators for vector Field

- gradient: $\nabla \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}\right)$.
  scalar → vector
- divergence: $\operatorname{div}(\vec{F}) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$.
  Field → scalar
- curl: $\operatorname{curl}(\vec{F}) = \nabla \times \vec{F} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ f_1 & f_2 & f_3 \end{pmatrix}$

$\vec{F}$ is called irrotational if $\operatorname{curl}\vec{F} \equiv \vec{0}$

$\vec{F}$ irrotational $\iff$ $\vec{F}$ conservative

$\Updownarrow \qquad\qquad \Updownarrow$

$\operatorname{curl}\vec{F} \equiv \vec{0}$ $\iff$ $\vec{F} = \nabla \varphi$

Integral

Single integral $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j^*)\,\Delta x$

double integral $\iint_D f(x,y)\,dD = \lim_{n\to\infty}\lim_{m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)\,\Delta x\,\Delta y$

triple integral $\iiint_V f(x,y,z)\,dV = \lim_{n\to\infty}\lim_{m\to\infty}\lim_{k\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{s=1}^{k} f(x_i^*, y_j^*, z_s^*)\,\Delta x\,\Delta y\,\Delta z$

line integral

1st kind $\int_C f(x,y,z)\,dC = \lim_{n\to\infty} \sum_{i=1}^{n} f(P_i^*)\,|P_{i-1}P_i| = \int_a^b f(x(t),y(t),z(t))\,|\vec{r}\,'(t)|\,dt$

2nd kind $\int_{\vec{C}} \vec{F}(x,y,z)\cdot d\vec{C} = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}(P_i^*)\cdot \overrightarrow{P_{i-1}P_i} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt.$

Surface integral

1st kind $\iint_S f(x,y,z)\,dS = \lim_{n\to\infty}\lim_{m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(P_{ij}^*)\cdot|S_{ij}| = \iint_D f(\vec{r}(u,v))\cdot|\vec{r}_u\times\vec{r}_v|\,du\,dv$

2nd kind. $\iint_S \vec{F}\cdot d\vec{S} = \lim_{n\to\infty}\lim_{m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} \vec{F}(P_{ij}^*)\cdot\vec{n}_{ij}\,|S_{ij}| = \iint_D \vec{F}(\vec{r}(u,v))\cdot\vec{r}_u\times\vec{r}_v\,du\,dv$

Conservative Thm: If $\vec{F}$ is conservative with potiential function φ, i.e. $\vec{F} = \nabla\varphi$, then for any simple curve C from A to B.

$$\int_{C_{AB}} \vec{F}\cdot d\vec{C} = \varphi(B) - \varphi(A).$$

Divergence Thm: $\oiint_S \vec{F}\,d\vec{S} = \iiint_V \operatorname{div}(\vec{F})\,dV$

Stoke's Thm: $\oint_C \vec{F}\,d\vec{C} = \iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S}$

proof of Conservative thm:

let $\vec{r}(t)$, $t: a \to b$ be a parametric equation for C ($\vec{r}(a) = A$ $\vec{r}(b) = B$). $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

$$\int_{C_{AB}} \vec{F}\, d\vec{C} = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \underbrace{(x'(t), y'(t), z'(t))}_{\vec{r}'(t)}\, dt.$$

$$\overset{\vec{F} = \nabla\varphi}{=} \int_a^b \left(\frac{\partial\varphi}{\partial x}(\vec{r}(t)), \frac{\partial\varphi}{\partial y}(\vec{r}(t)), \frac{\partial\varphi}{\partial z}(\vec{r}(t))\right) \cdot (x'(t), y'(t), z'(t))\, dt.$$

$$= \int_a^b \underbrace{\left(\frac{\partial\varphi}{\partial x}\frac{dx}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dt} + \frac{\partial\varphi}{\partial z}\frac{dz}{dt}\right)(t)}\, dt.$$

$$= \int_a^b \frac{d}{dt}\varphi(x(t), y(t), z(t)).\, dt.$$

$$= \varphi(\underbrace{x(b), y(b), z(b)}) - \varphi(x(a), y(a), z(a)).$$

$$= \varphi(B) - \varphi(A).$$

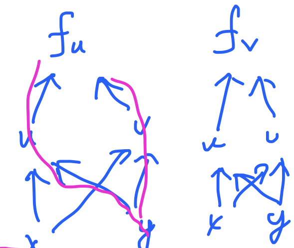
1. **Laplace equations** Show that if $w = f(u, v)$ satisfies the Laplace equation $f_{uu} + f_{vv} = 0$ and if $u = (x^2 - y^2)/2$ and $v = xy$, then w satisfies the Laplace equation

$$w_{xx} + w_{yy} = 0.$$

$$W_{xx} = \frac{\partial w_x}{\partial x}$$

$$W_x = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$

$$= f_u \cdot x + f_v \cdot y$$



$$W_{xx} = \frac{\partial}{\partial x}(f_u \cdot x) + \frac{\partial}{\partial x}(f_v \cdot y).$$

$$= \left(\frac{\partial}{\partial x} f_u\right)\cdot x + f_u \cdot 1 + y \frac{\partial}{\partial x} f_v$$

$$= \left(\frac{\partial f_u}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_u}{\partial v}\frac{\partial v}{\partial x}\right)\cdot x + f_u + y\left(\frac{\partial f_v}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f_v}{\partial v}\frac{\partial v}{\partial x}\right).$$

$$= \left(f_{uu}\cdot x + \cancel{f_{uv}\, y}\right)\cdot x + \cancel{f_u} + y\left(\cancel{f_{vu}\cdot x} + f_{vv}\, y\right).$$

$$W_y = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = f_u \cdot (-y) + f_v \cdot x$$

$$W_{yy} = \frac{\partial}{\partial y}(-f_u \cdot y) + \frac{\partial}{\partial y}(f_v \cdot x)$$

$$= -f_u \cdot 1 + \frac{\partial}{\partial y}(-f_u)\cdot y + x\frac{\partial}{\partial y} f_v$$

$$= -f_u - y\left(\frac{\partial f_u}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f_u}{\partial v}\frac{\partial v}{\partial y}\right) + x\left(\frac{\partial f_v}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f_v}{\partial v}\frac{\partial v}{\partial y}\right).$$

$$= -f_u - y\left(f_{uu}\cdot(-y) + f_{uv}\, x\right) + x\left(f_{vu}(-y) + f_{vv}\, x\right)$$

$$= \cancel{-f_u} + y^2 f_{uu} - \cancel{xy f_{uv}} - \cancel{xy f_{vu}} + x^2 f_{vv}$$

$$w_{xx} + w_{yy} = (x^2+y^2)(f_{uu} + f_{vv}) = 0$$

2. (a) Find the linearization $L(x, y)$ of the function

$$f(x, y) = (1/2)x^2 + xy + (1/4)y^2 + 3x - 3y + 4 \quad \text{at} \quad P_0(2, 2).$$

$$L(x,y) = f(P_0) + \partial_x f(P_0)(x - x_0) + \partial_y f(P_0)(x - y_0).$$

$$f(P_0) =$$

(b) Then find an upper bound for the magnitude $|E|$ of the error in the approximation $f(x,y) \approx L(x,y)$ over the rectangle

$$R: |x-2| \le 0.1, |y-2| \le 0.1.$$

$$f(x,y) - L(x,y) = \frac{1}{2!}\left[\frac{\partial^2}{\partial x^2} f(x^*, y^*)(x-x_0)^2 + 2\frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x^*, y^*)(x-x_0)(y-y_0) + \frac{\partial^2}{\partial y^2} f(x^*, y^*)(y-y_0)^2\right]$$

x^* is between x_0 and x.

y^* is between y_0 and y.

$$|f - L| \le \frac{M}{2!}\left|\left[(x-x_0)^2 + 2(x-x_0)(y-y_0) + (y-y_0)^2\right]\right|$$

$$\le \frac{M}{2}\left(|x-x_0| + |y-y_0|\right)^2$$

$$\le \frac{M}{2}(0.1 + 0.1)^2$$

$|f_{xx}| \le M \qquad |f_{yy}| \le M \qquad |f_{xy}| \le M.$

3. Find the absolute maxima and minima of the function

$$T(x, y) = x^2 + xy + y^2 - 6x$$

on the rectangular domain: $0 \le x \le 5, \quad -3 \le y \le 3$.

① Find stationary point

$$\left.\begin{aligned} \frac{\partial T}{\partial x} &= 2x + y - 6 = 0 \\ \frac{\partial T}{\partial y} &= x + 2y = 0 \end{aligned}\right\} \Rightarrow \begin{cases} x = 4 \\ y = -2 \end{cases}$$

② $T(4, -2) = \ldots$

③ Find max min on the region $x=0$, $x=5$, $y=-3$, $y=3$.

$T(0, y) = y^2 \quad y \in [-3, 3) \quad \max = 9 \quad \min = 0$

$T(5, y) = 5^2 + 5y + y^2 - 30 \quad y \in [-3, 3] \quad \max = \ldots \quad \ldots$

$T(x, -3) = \quad x \in [0, 5] \quad \max = \quad .$

$T(x, 3) = \quad x \in [0, 5] \quad \max = \quad .$

4.

5. Show that any vector field of the form

$$F(x, y, z) = f(y, z)\mathbf{i} + g(x, z)\mathbf{j} + h(x, y)\mathbf{k}$$

is incompressible (~~conservative~~).

Solenoidal

$$\operatorname{div}(\vec{F}) = \frac{\partial f(y,z)}{\partial x} + \frac{\partial g(x,z)}{\partial y} + \frac{\partial h(x,y)}{\partial z}.$$

$$= 0 + 0 + 0 = 0$$