MA1201 Calculus and Basic Linear Algebra II **Solution of Problem Set 2**

Problem 1

(a) Let $y = 1 + \frac{1}{x^2} \Rightarrow \frac{dy}{dx} = -\frac{2}{x^3} \Rightarrow dx = -\frac{x^3}{2}dy.$

Then the integral becomes

$$\int \frac{e^{1+\frac{1}{x^2}}}{x^3}dx = \int \frac{e^{1+\frac{1}{x^2}}}{x^3}\left(-\frac{x^3}{2}dy\right) = -\frac{1}{2}\int e^y dy = -\frac{1}{2}e^y + C = -\frac{1}{2}e^{1+\frac{1}{x^2}} + C.$$

(b) Let $y = 1 - 2x^3 \Rightarrow \frac{dy}{dx} = -6x^2 \Rightarrow dx = -\frac{1}{6x^2}dy.$

Then the integral becomes

$$\int x^2 \sec(1-2x^3)\,dx = \int x^2 \sec(1-2x^3)\left(-\frac{1}{6x^2}dy\right) = -\frac{1}{6}\int \sec y\,dy$$
$$= -\frac{1}{6}\ln|\sec y + \tan y| + C = -\frac{1}{6}\ln|\sec(1-2x^3) + \tan(1-2x^3)| + C.$$

(c) Let $y = 1 + x^4 \Rightarrow \frac{dy}{dx} = 4x^3 \Rightarrow dx = \frac{1}{4x^3}dy.$

Then the integral becomes

$$\int x^{11}\sqrt{1+x^4}dx = \int x^{11}\sqrt{1+x^4}\left(\frac{1}{4x^3}dy\right) = \frac{1}{4}\int x^8\sqrt{1+x^4}dy = \frac{1}{4}\int (y-1)^2\sqrt{y}dy$$
$$= \frac{1}{4}\int \left(y^{\frac{5}{2}} - 2y^{\frac{3}{2}} + y^{\frac{1}{2}}\right)dy = \frac{1}{4}\left(\frac{y^{\frac{7}{2}}}{\frac{7}{2}} - 2\frac{y^{\frac{5}{2}}}{\frac{5}{2}} + \frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$
$$= \frac{1}{14}(1+x^4)^{\frac{7}{2}} - \frac{1}{5}(1+x^4)^{\frac{5}{2}} + \frac{1}{6}(1+x^4)^{\frac{3}{2}} + C.$$

(d) Let $y = x^2 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow dx = \frac{1}{2x}dy.$

Then the integral becomes

$$\int x\cos^2(x^2)\,dx = \int x\cos^2(x^2)\left(\frac{1}{2x}dy\right) = \frac{1}{2}\int \cos^2 y\,dy = \frac{1}{2}\int \frac{1}{2}[\cos(y+y) + \cos(y-y)]dy$$
$$= \frac{1}{4}\int (\cos 2y + 1)dy = \frac{1}{4}\left(\frac{1}{2}\sin 2y + y\right) + C = \frac{1}{8}\sin(2x^2) + \frac{x^2}{4} + C.$$

(e) Let $y = \cos x \Rightarrow \frac{dy}{dx} = -\sin x \Rightarrow dx = -\frac{1}{\sin x}dy.$

Together with the fact that $\sin 2x = 2\sin x\cos x$, the integral then becomes

$$\int \sin 2x\sqrt{\cos x}\,dx = \int 2\sin x\cos x\sqrt{\cos x}\left(-\frac{1}{\sin x}dy\right) = -2\int y\sqrt{y}dy = -2\int y^{\frac{3}{2}}dy$$
$$= -2\frac{y^{\frac{5}{2}}}{\frac{5}{2}} + C = -\frac{4}{5}\cos^{\frac{5}{2}}x + C.$$

(f) Let $y = 1 + e^x \Rightarrow \frac{dy}{dx} = e^x \Rightarrow dx = \frac{1}{e^x}dy.$

Then the integral becomes

$$\int \frac{e^{2x}}{(1+e^x)^3}dx = \int \frac{e^{2x}}{(1+e^x)^3}\left(\frac{1}{e^x}dy\right) = \int \frac{e^x}{(1+e^x)^3}dy = \int \frac{y-1}{y^3}dy = \int (y^{-2} - y^{-3})dy$$
$$= -y^{-1} + \frac{1}{2}y^{-2} + C = -\frac{1}{1+e^x} + \frac{1}{2(1+e^x)^2} + C.$$

(g) Let $y = x^2 - 1 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow dx = \frac{1}{2x}dy.$

When $x = 1,\ y = 1^2 - 1 = 0$. When $x = 2,\ y = 2^2 - 1 = 3$.

Then the integral becomes

$$\int_1^2 xe^{x^2-1}dx = \int_0^3 xe^{x^2-1}\left(\frac{1}{2x}dy\right) = \frac{1}{2}\int_0^3 e^y dy = \frac{1}{2}e^y|_0^3 = \frac{e^3}{2} - \frac{1}{2}.$$

(h) Let $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x} \Rightarrow dx = xdy.$

When $x = 5,\ y = \ln 5$. When $x = 1,\ y = \ln 1 = 0$.

Then the integral becomes

$$\int_1^5 \frac{\sin^2(\ln x)}{x}dx = \int_0^{\ln 5}\frac{\sin^2(\ln x)}{x}(xdy) = \int_0^{\ln 5}\sin^2 y\,dy$$
$$= \int_0^{\ln 5} -\frac{1}{2}[\cos(y+y) - \cos(y-y)]dy = -\frac{1}{2}\int_0^{\ln 5}(\cos 2y - 1)dy$$
$$= -\frac{1}{2}\left(\frac{1}{2}\sin 2y - y\right)|_0^{\ln 5} = -\frac{1}{4}\sin(2\ln 5) + \frac{1}{2}\ln 5.$$

(i) Following the idea of Example 7, let $y = x^2 + 4 \Rightarrow \frac{dy}{dx} = 2x \Rightarrow dx = \frac{1}{2x}dy.$

We compute the integral by decomposing it as follows:

$$\int \frac{3x+2}{x^2+4}dx = \int \frac{3x}{x^2+4}dx + \int \frac{2}{x^2+4}dx = \int \frac{3x}{x^2+4}\left(\frac{1}{2x}dy\right) + 2\int \frac{1}{x^2+4}dx$$
$$= \frac{3}{2}\int \frac{1}{y}dy + \frac{2}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}dx = \frac{3}{2}\ln y + \frac{1}{2}\left(\frac{1}{\frac{1}{2}}\tan^{-1}\frac{x}{2}\right) + C = \frac{3}{2}\ln|x^2+4| + \tan^{-1}\frac{x}{2} + C.$$

(j) Let $y = x^2 - 2x + 5 \Rightarrow \frac{dy}{dx} = 2x - 2 \Rightarrow dx = \frac{1}{2x-2}dy.$

Similar to (i), the integral can be computed as

$$\int \frac{2x+1}{x^2-2x+5}dx = \int \frac{2x-2}{x^2-2x+5}dx + \int \frac{3}{x^2-2x+5}dx$$
$$= \int \frac{2x-2}{x^2-2x+5}\left(\frac{1}{2x-2}dy\right) + 3\int \frac{1}{x^2-2x+5}dx$$
$$= \int \frac{1}{y}dy + 3\int \frac{1}{(x-1)^2+4}dx = \int \frac{1}{y}dy + \frac{3}{4}\int \frac{1}{\left(\frac{x-1}{2}\right)^2+1}dx$$
$$= \ln|y| + \frac{3}{4}\left(\frac{1}{\frac{1}{2}}\tan^{-1}\frac{x-1}{2}\right) + C = \ln|x^2-2x+5| + \frac{3}{2}\tan^{-1}\frac{x-1}{2} + C.$$

(k) Let $y = 3x^2 + 6x + 19 \Rightarrow \frac{dy}{dx} = 6x + 6 \Rightarrow dx = \frac{1}{6x+6}dy.$

Then the integral can be computed as

$$\int \frac{4x}{3x^2+6x+19}dx = \int \frac{4x+4}{3x^2+6x+19}dx - \int \frac{4}{3x^2+6x+19}dx$$
$$= \frac{2}{3}\int \frac{1}{y}dy - 4\int \frac{1}{3x^2+6x+19}dx = \frac{2}{3}\int \frac{1}{y}dy - 4\int \frac{1}{3(x+1)^2+16}dx$$
$$= \frac{2}{3}\ln|y| - \frac{4}{16}\int \frac{1}{\left(\frac{\sqrt{3}}{4}x + \frac{\sqrt{3}}{4}\right)^2+1}dx$$

$$= \frac{2}{3}\ln|3x^2+6x+19| - \frac{1}{4}\left(\frac{1}{\frac{\sqrt{3}}{4}}\tan^{-1}\left(\frac{\sqrt{3}}{4}x+\frac{\sqrt{3}}{4}\right)\right)+C$$

$$= \frac{2}{3}\ln|3x^2+6x+19| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}}{4}x+\frac{\sqrt{3}}{4}\right)+C.$$

(l) Let $x=\sin\theta \Rightarrow \dfrac{dx}{d\theta}=\cos\theta \Rightarrow dx=\cos\theta\,d\theta$.

The integral then becomes

$$\int \frac{1}{x^2\sqrt{1-x^2}}dx = \int \frac{1}{\sin^2\theta\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta = \int \frac{1}{\sin^2\theta}d\theta = \int \csc^2\theta\,d\theta$$

$$= -\cot\theta + C \overset{\substack{\sin\theta=x\\ \cos\theta=\sqrt{1-x^2}}}{\cong} -\frac{\sqrt{1-x^2}}{x}+C.$$

(m) Let $x=\dfrac{1}{\sqrt{3}}\tan\theta \Rightarrow \dfrac{dx}{d\theta}=\dfrac{1}{\sqrt{3}}\sec^2\theta \Rightarrow dx=\dfrac{1}{\sqrt{3}}\sec^2\theta\,d\theta$

The integral becomes

$$\int \frac{1}{(1+3x^2)^{\frac{3}{2}}}dx = \int \frac{1}{\left(1+3\left(\frac{1}{3}\tan^2\theta\right)\right)^{\frac{3}{2}}}\left(\frac{1}{\sqrt{3}}\sec^2\theta\,d\theta\right) = \frac{1}{\sqrt{3}}\int\frac{\sec^2\theta}{\sec^3\theta}d\theta = \frac{1}{\sqrt{3}}\int\frac{1}{\sec\theta}d\theta$$

$$= \frac{1}{\sqrt{3}}\int\cos\theta\,d\theta = \frac{1}{\sqrt{3}}\sin\theta + C = \frac{1}{\sqrt{3}}\frac{\sqrt{3}x}{\sqrt{1+3x^2}}+C = \frac{x}{\sqrt{1+3x^2}}+C.$$

(n) Let $y=4x^2+1 \Rightarrow \dfrac{dy}{dx}=8x \Rightarrow dx=\dfrac{1}{8x}dy$.

Then the integral becomes

$$\int\frac{3x}{\sqrt{4x^2+1}}dx = \int\frac{3x}{\sqrt{4x^2+1}}\left(\frac{1}{8x}dy\right) = \frac{3}{8}\int\frac{1}{\sqrt{y}}dy = \frac{3}{8}\frac{y^{\frac{1}{2}}}{\frac{1}{2}}+C = \frac{3}{4}\sqrt{4x^2+1}+C.$$

(o) Let $x=\dfrac{3}{4}\sin\theta \;\Rightarrow\; \dfrac{dx}{d\theta}=\dfrac{3}{4}\cos\theta \;\Rightarrow\; dx=\dfrac{3}{4}\cos\theta\,d\theta$.

Then the integral becomes

$$\int\sqrt{9-16x^2}dx = \int\sqrt{9-16\left(\frac{3}{4}\sin\theta\right)^2}\left(\frac{3}{4}\cos\theta\,d\theta\right) = \frac{9}{4}\int\cos^2\theta\,d\theta$$

$$= \frac{9}{4}\int\frac{1}{2}[\cos(\theta+\theta)+\cos(\theta-\theta)]d\theta = \frac{9}{8}\int\cos 2\theta\,d\theta + \frac{9}{8}\int 1d\theta$$

$$= \frac{9}{16}\sin 2\theta + \frac{9}{8}\theta + C = \frac{9}{8}\sin\theta\cos\theta + \frac{9}{8}\theta + C$$

$$= \frac{9}{8}\left(\frac{4x}{3}\right)\frac{\sqrt{9-16x^2}}{3} + \frac{9}{8}\sin^{-1}\frac{4}{3}x + C = \frac{1}{2}x\sqrt{9-16x^2} + \frac{9}{8}\sin^{-1}\frac{4}{3}x + C.$$

(p) Using completing square technique, we first rewrite the integral as

$$\int\frac{1}{(x^2+6x+10)^{\frac{3}{2}}}dx = \int\frac{1}{[(x+3)^2+1]^{\frac{3}{2}}}dx.$$

Let $x+3=\tan\theta \Rightarrow \dfrac{dx}{d\theta}=\sec^2\theta \Rightarrow dx=\sec^2\theta\,d\theta$.

Then the integral becomes

$$\int \frac{1}{[(x+3)^2+1]^{\frac{3}{2}}}dx = \int \frac{1}{(\tan^2\theta+1)^{\frac{3}{2}}}\sec^2\theta\, d\theta = \int \frac{1}{\sec\theta}d\theta = \int \cos\theta\, d\theta = \sin\theta + C$$
$$= \frac{x+3}{\sqrt{x^2+6x+10}} + C.$$

(q) Let $y = \cos x \Rightarrow \dfrac{dy}{dx} = -\sin x \Rightarrow dx = -\dfrac{1}{\sin x}dy.$

The integral then becomes

$$\int \sin^7 x\, dx = \int \sin^7 x\left(-\frac{1}{\sin x}dy\right) = -\int \sin^6 x\, dy = -\int (1-\cos^2 x)^3 dy = -\int (1-y^2)^3 dy$$
$$= -\int (1-3y^2+3y^4-y^6)dy = -y + y^3 - \frac{3}{5}y^5 + \frac{y^7}{7} + C$$
$$= -\cos x + \cos^3 x - \frac{3}{5}\cos^5 x + \frac{\cos^7 x}{7} + C.$$

(r) Let $y = \sin x \Rightarrow \dfrac{dy}{dx} = \cos x \Rightarrow dx = \dfrac{1}{\cos x}dy.$

The integral can be rewritten as

$$\int \sin^3 x\cos^5 x\, dx = \int \sin^3 x\cos^5 x\left(\frac{1}{\cos x}dy\right) = \int \sin^3 x\cos^4 x\, dy = \int \sin^3 x\,(1-\sin^2 x)^2 dy$$
$$= \int y^3(1-y^2)^2 dy = \int (y^3 - 2y^5 + y^7)dy = \frac{y^4}{4} - \frac{y^6}{3} + \frac{y^8}{8} + C$$
$$= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{3} + \frac{\sin^8 x}{8} + C.$$

Problem 2

(a) Take $u = x$ and $dv = e^{-3x}dx \Rightarrow v = \int e^{-3x}dx = -\frac{1}{3}e^{-3x}$.

Using integration by parts, we get

$$\int \underbrace{x}_{u}\underbrace{e^{-3x}dx}_{dv} = \underbrace{-\frac{1}{3}xe^{-3x}}_{uv} - \int \underbrace{-\frac{1}{3}e^{-3x}}_{v}\underbrace{dx}_{du} = -\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}dx = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C.$$

(b) Take $u = \ln x$ and $dv = \sqrt{x}dx \Rightarrow v = \int \sqrt{x}dx = \int x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}}$.

Using integration by parts, we get

$$\int_1^e \sqrt{x}\ln x\, dx = \int_1^e \underbrace{\ln x}_{u}\underbrace{\sqrt{x}dx}_{dv} = \underbrace{\frac{2}{3}x^{\frac{3}{2}}\ln x\,|_1^e}_{uv|_1^e} - \int_1^e \underbrace{\frac{2}{3}x^{\frac{3}{2}}}_{v}\underbrace{d(\ln x)}_{du}$$
$$= \frac{2}{3}e^{\frac{3}{2}}\ln e - \frac{2}{3}x^{\frac{3}{2}}\ln 1 - \frac{2}{3}\int_1^e x^{\frac{3}{2}}\left(\frac{1}{x}dx\right) = \frac{2}{3}e^{\frac{3}{2}} - \frac{2}{3}\int_1^e x^{\frac{1}{2}}dx = \frac{2}{3}e^{\frac{3}{2}} - \frac{2}{3}\frac{x^{\frac{3}{2}}}{\frac{3}{2}}|_1^e$$
$$= \frac{2}{3}e^{\frac{3}{2}} - \frac{4}{9}e^{\frac{3}{2}} + \frac{4}{9} = \frac{2}{9}e^{\frac{3}{2}} + \frac{4}{9}.$$

(c) Take $u = x^2$ and $dv = \sin x\, dx \Rightarrow v = \int \sin x\, dx = -\cos x$.

Using integration by parts, we get

$$\int \underbrace{x^2}_{u}\underbrace{\sin x\, dx}_{dv} = \underbrace{-x^2\cos x}_{uv} - \int \underbrace{-\cos x}_{v}\underbrace{d(x^2)}_{du} = -x^2\cos x + 2\int x\cos x\, dx.$$

To compute the second integral, we take $u = x$ and $dv = \cos x\, dx \Rightarrow v = \int \cos x\, dx = \sin x$. Using integration by parts, we have

$$\int \underbrace{x}_{u}\underbrace{\cos x\, dx}_{dv} = \underbrace{x\sin x}_{uv} - \int \underbrace{\sin x}_{v}\underbrace{dx}_{du} = x\sin x + \cos x + C.$$

Therefore, we conclude that

$$\int x^2 \sin x\, dx = -x^2\cos x + 2x\sin x + 2\cos x + C.$$

(d) We first rewrite the integral by using product-to-sum formula:

$$\int x\sin^2 x\, dx = \int x\left[-\frac{1}{2}[\cos(x+x)-\cos(x-x)]\right]dx = -\frac{1}{2}\int \underbrace{x}_{u}\underbrace{\cos 2x\, dx}_{dv} + \frac{1}{2}\int x dx$$

$$\overset{\substack{u=x \\ dv=\cos 2xdx \\ \Rightarrow v=\int \cos 2xdx=\frac{1}{2}\sin 2x}}{\cong} -\frac{1}{2}\left(\underbrace{\frac{1}{2}x\sin 2x}_{uv} - \int \underbrace{\frac{1}{2}\sin 2x}_{v}\underbrace{dx}_{du}\right) + \frac{1}{2}\left(\frac{x^2}{2}\right)$$

$$= -\frac{1}{4}x\sin 2x + \frac{1}{4}\left(-\frac{1}{2}\cos 2x\right) + \frac{x^2}{4} + C = -\frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + \frac{x^2}{4} + C.$$

(e) Take $u = \cos^{-1} x$ and $dv = x^2 dx \Rightarrow v = \int x^2 dx = \frac{x^3}{3}$.

Using integration by parts, we have

$$\int x^2\cos^{-1} x\, dx = \int \underbrace{\cos^{-1} x}_{u}\underbrace{(x^2 dx)}_{dv} = \underbrace{\frac{x^3}{3}\cos^{-1} x}_{uv} - \int \underbrace{\frac{x^3}{3}}_{v}\underbrace{d(\cos^{-1} x)}_{du} = \frac{x^3}{3}\cos^{-1} x + \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}dx$$

$$\overset{\substack{y=1-x^2 \\ \frac{dy}{dx}=-2x}}{\cong} \frac{x^3}{3}\cos^{-1} x + \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\left(-\frac{1}{2x}dy\right) = \frac{x^3}{3}\cos^{-1} x - \frac{1}{6}\int \frac{1-y}{\sqrt{y}}dy$$

$$= \frac{x^3}{3}\cos^{-1} x - \frac{1}{6}\int \left(y^{-\frac{1}{2}} - y^{\frac{1}{2}}\right)dy = \frac{x^3}{3}\cos^{-1} x - \frac{1}{6}\left(\frac{y^{\frac{1}{2}}}{\frac{1}{2}} - \frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{x^3}{3}\cos^{-1} x - \frac{1}{3}\sqrt{1-x^2} + \frac{1}{9}(1-x^2)^{\frac{3}{2}} + C.$$

(f) Take $u = \tan^{-1} x$ and $dv = dx \Rightarrow v = \int dx = x$.

Using integration by parts, we have

$$\int \underbrace{\tan^{-1} x}_{u}\underbrace{dx}_{dv} = \underbrace{x\tan^{-1} x}_{uv} - \int \underbrace{x\, d(\tan^{-1} x)}_{vdu} = x\tan^{-1} x - \int \frac{x}{1+x^2}dx$$

$$\overset{\substack{y=1+x^2 \\ \Rightarrow\frac{dy}{dx}=2x}}{\cong} x\tan^{-1} x - \int \frac{x}{1+x^2}\left(\frac{1}{2x}dy\right) = x\tan^{-1} x - \frac{1}{2}\int \frac{1}{y}dy = x\tan^{-1} x - \frac{1}{2}\ln|y| + C$$

$$= x\tan^{-1} x - \frac{1}{2}\ln|1+x^2| + C.$$

(g) Take $u = \csc x$, $dv = \csc^2 x\, dx \Rightarrow v = \int \csc^2 x\, dx = -\cot x$.

Using integration by parts, we have

$$\int \csc^3 x\, dx = \int \underbrace{\csc x}_{u}\underbrace{(\csc^2 x\, dx)}_{dv} = \underbrace{-\csc x\cot x}_{uv} - \int \underbrace{(-\cot x)}_{v}\underbrace{d(\csc x)}_{du}$$

$$= -\csc x\cot x - \int \csc x\cot^2 x\, dx = -\csc x\cot x - \int \csc x(\csc^2 x - 1)dx$$

$$= -\csc x\cot x - \int \csc^3 x\, dx + \int \csc x\, dx$$

$$= -\csc x\cot x - \ln|\cot x + \csc x| - \int \csc^3 x\, dx.$$

Summing up, we get

$$\int \csc^3 x\,dx = -\csc x\cot x - \ln|\cot x + \csc x| - \int \csc^3 x\,dx$$
$$\Rightarrow 2\int \csc^3 x\,dx = -\csc x\cot x - \ln|\cot x + \csc x|$$
$$\Rightarrow \int \csc^3 x\,dx = -\frac{1}{2}(\csc x\cot x + \ln|\cot x + \csc x|) + C.$$

(h) Take $u = \cos^2 x$, $dv = \cos x\,dx \Rightarrow v = \int \cos x\,dx = \sin x$.
Using integration by parts, we have

$$\int \cos^3 x\,dx = \int \underbrace{\cos^2 x}_{u}\,\underbrace{(\cos x\,dx)}_{dv} = \underbrace{\cos^2 x\sin x}_{uv} - \int \underbrace{(\sin x)}_{v}\,\underbrace{d(\cos^2 x)}_{du}$$
$$= \cos^2 x\sin x - \int \sin x\,(-2\cos x\sin x)dx = \cos^2 x\sin x + 2\int \sin^2 x\cos x\,dx$$
$$= \cos^2 x\sin x + 2\int (1-\cos^2 x)\cos x\,dx$$
$$= \cos^2 x\sin x + 2\int \cos x\,dx - 2\int \cos^3 x\,dx = \cos^2 x\sin x + 2\sin x - 2\int \cos^3 x\,dx.$$

Summing up, we get

$$\int \cos^3 x\,dx = \cos^2 x\sin x + 2\sin x - 2\int \cos^3 x\,dx$$
$$\Rightarrow 3\int \cos^3 x\,dx = \cos^2 x\sin x + 2\sin x \Rightarrow \int \cos^3 x\,dx = \frac{1}{3}(\cos^2 x\sin x + 2\sin x) + C.$$

(i) Take $u = (\ln x)^2$ and $dv = \frac{1}{x^2}dx \Rightarrow v = \int \frac{1}{x^2}dx = -x^{-1} = -\frac{1}{x}$.
Using integration by parts, we have

$$\int_1^e \left(\frac{\ln x}{x}\right)^2 dx = \int_1^e \underbrace{(\ln x)^2}_{u}\,\underbrace{\left(\frac{1}{x^2}dx\right)}_{dv} = \underbrace{-\frac{(\ln x)^2}{x}}_{uv}\Big|_1^e - \int_1^e \underbrace{-\frac{1}{x}}_{v}\,\underbrace{d(\ln x)^2}_{du}$$

$$\overset{\frac{d}{dx}(\ln x)^2 = 2(\ln x)\left(\frac{1}{x}\right)}{\cong} -\frac{(\ln e)^2}{e} + \frac{(\ln 1)^2}{1} + 2\int_1^e \frac{\ln x}{x^2}dx = -\frac{1}{e} + 2\int_1^e \underbrace{(\ln x)}_{u}\,\underbrace{\left(\frac{1}{x^2}dx\right)}_{dv}$$

$$\overset{\substack{u=\ln x\\ dv=\frac{1}{x^2}dx\\ \Rightarrow v=\int\frac{1}{x^2}dx=-\frac{1}{x}}}{\cong} -\frac{1}{e} + 2\left(\underbrace{-\frac{\ln x}{x}}_{uv}\Big|_1^e - \int_1^e \underbrace{-\frac{1}{x}}_{v}\,\underbrace{d(\ln x)}_{du}\right)$$

$$= -\frac{1}{e} + 2\left(-\frac{\ln e}{e} + \frac{\ln 1}{1} + \int_1^e \frac{1}{x^2}dx\right) = -\frac{1}{e} - \frac{2}{e} + 2\left(-\frac{1}{x}\right)\Big|_1^e = 2 - \frac{5}{e}.$$

(j) We take $u = \sin 3x$ and $dv = e^x dx \Rightarrow v = \int e^x dx = e^x$. Using integration by parts, we get

$$\int e^x \sin 3x\,dx = e^x\sin 3x - \int e^x d(\sin 3x) = e^x \sin 3x - 3\int e^x\cos 3x\,dx$$

$$\overset{\substack{u=\cos 3x\\ dv=e^x dx\Rightarrow v=\int e^x dx=e^x}}{\cong} e^x\sin 3x - 3\left(e^x\cos 3x - \int e^x d(\cos 3x)\right)$$

$$= e^x\sin 3x - 3e^x\cos 3x - 9\int e^x\sin 3x\,dx.$$

Summing up, we get

$$\int e^x\sin 3x\,dx = e^x\sin 3x - 3e^x\cos 3x - 9\int e^x\sin 3x\,dx$$
$$\Rightarrow 10\int e^x\sin 3x\,dx = e^x\sin 3x - 3e^x\cos 3x \Rightarrow \int e^x\sin 3x\,dx = \frac{1}{10}e^x\sin 3x - \frac{3}{10}e^x\cos 3x + C.$$

Problem 3

(a) Let $y = 2e^x + 1 \Rightarrow \dfrac{dy}{dx} = 2e^x \Rightarrow dx = \dfrac{1}{2e^x}dy$

The integral then becomes

$$\int e^{2x}\sin(2e^x+1)\,dx = \int e^{2x}\sin(2e^x+1)\left(\frac{1}{2e^x}\right)dy = \frac{1}{2}\int e^x\sin(2e^x+1)\,dy = \frac{1}{2}\int\frac{y-1}{2}\sin y\,dy$$

$$= \frac{1}{4}\int y\sin y\,dy - \frac{1}{4}\int \sin y\,dy = \frac{1}{4}\int \underbrace{y}_{u}\underbrace{\sin y\,dy}_{dv} + \frac{1}{4}\cos y$$

$$\overset{\substack{u=y\\ dv=\sin y dy\\ \Rightarrow v=\int \sin y dy = -\cos y}}{\cong} \frac{1}{4}\left(-y\cos y - \int(-\cos y)dy + \frac{1}{4}\cos y\right) = -\frac{1}{4}y\cos y + \frac{1}{4}\sin y + \frac{1}{4}\cos y$$

$$= -\frac{1}{4}(2e^x+1)\cos(2e^x+1) + \frac{1}{4}\sin(2e^x+1) + \frac{1}{4}\cos(2e^x+1) + C.$$

(b) Let $y = 2\sqrt{x} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}} \Rightarrow dx = \sqrt{x}dy$

When $x = 1,\ y = 2$. When $x = 0,\ y = 0$.

The integral then becomes

$$\int_0^1 \sin(2\sqrt{x})\,dx = \int_0^2 \sin(2\sqrt{x})\,(\sqrt{x}dy) = \int_0^2 \frac{y}{2}\sin y\,dy = \frac{1}{2}\int_0^2 \underbrace{y}_{u}\underbrace{\sin y\,dy}_{dv}$$

$$\overset{\substack{u=y\\ dv=\sin y dy\\ \Rightarrow v=\int \sin y dy = -\cos y}}{\cong} \frac{1}{2}\left(-y\cos y\,|_0^2 - \int_0^2(-\cos y)dy\right) = -\cos 2 + \frac{1}{2}\sin y\,|_0^2 = -\cos 2 + \frac{1}{2}\sin 2.$$

(c) Let $y = 1 + \sqrt[3]{x} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{3x^{\frac{2}{3}}} \Rightarrow dx = 3x^{\frac{2}{3}}dy.$

When $x = 1,\ y = 2$. When $x = 0,\ y = 1$.

$$\int_0^1 \ln(1+\sqrt[3]{x})dx = \int_1^2 3x^{\frac{2}{3}}\ln(1+\sqrt[3]{x})\,dy = 3\int_1^2 (y-1)^2\ln y\,dy.$$

Using integration by parts with $u = \ln y$ and $dv = (y-1)^2dy \Rightarrow v = \int(y-1)^2dy = \frac{(y-1)^3}{3}$. We get

$$\int_1^2 (y-1)^2\ln y\,dy = \frac{(y-1)^3}{3}\ln y\,|_1^2 - \int_1^2\frac{(y-1)^3}{3}d(\ln y) = \frac{1}{3}\ln 2 - \frac{1}{3}\int_1^2\frac{y^3-3y^2+3y-1}{y}dy$$

$$= \frac{1}{3}\ln 2 - \frac{1}{3}\int_1^2\left(y^2-3y+3-\frac{1}{y}\right)dy = \frac{1}{3}\ln 2 - \frac{1}{3}\left(\frac{y^3}{3} - \frac{3y^2}{2} + 3y - \ln y\right)|_1^2 = -\frac{5}{18}.$$

(d) Let $y = \ln x \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x} \Rightarrow dx = xdy.$

Then the integral becomes

$$\int \cos(\ln x)\,dx = \int x\cos(\ln x)\,dy = \int e^y\cos y\,dy.$$

Using similar techniques as in Problem 2(j) (integration by parts twice), we get

$$\int e^y\cos y\,dy = \frac{1}{2}(e^y\cos y + e^y\sin y) + C.$$

Therefore, we conclude that

$$\int \cos(\ln x)\,dx = \frac{1}{2}(e^y\cos y + e^y\sin y) + C = \frac{1}{2}\left(e^{\ln x}\cos(\ln x) + e^{\ln x}\sin(\ln x)\right) + C$$

$$= \frac{1}{2}x\cos(\ln x) + \frac{1}{2}x\sin(\ln x) + C.$$

(e) Let $y = \sin x \Rightarrow \dfrac{dy}{dx} = \cos x \Rightarrow dx = \dfrac{1}{\cos x} dy$

Together with the fact that $\sin 2x = 2 \sin x \cos x$, the integral can be rewritten as

$$\int \sin 2x \ln(\sin x)\, dx = 2 \int \sin x \cos x \ln(\sin x) \left(\frac{1}{\cos x} dy\right) = 2 \int y \ln y\, dy$$

Using integration by parts with $u = \ln y$ and $dv = ydy \Rightarrow v = \int ydy = \frac{y^2}{2}$, the latter integral can be computed as

$$\int y \ln y\, dy = \frac{y^2}{2} \ln y - \int \frac{y^2}{2} d(\ln y) = \frac{y^2}{2} \ln y - \int \frac{y^2}{2}\left(\frac{1}{y} dy\right) = \frac{y^2}{2} \ln y - \frac{1}{2} \int ydy = \frac{y^2}{2} \ln y - \frac{y^2}{4} + C.$$

Therefore, we conclude that

$$\int \sin 2x \ln(\sin x)\, dx = 2\left(\frac{y^2}{2} \ln y - \frac{y^2}{4}\right) + C = \sin^2 x \ln(\sin x) - \frac{\sin^2 x}{2} + C.$$

(f) Let $y = x + 3 \Rightarrow \dfrac{dy}{dx} = 1 \Rightarrow dx = dy.$

The integral becomes

$$\int (x + 1) \ln(x + 3)\, dx = \int (y - 2) \ln y\, dy.$$

Using integration by parts with $u = \ln y$ and $dv = (y - 2)dy \Rightarrow v = \int (y - 2)dy = \frac{(y-2)^2}{2}$. The latter integral can be computed as

$$\int (y - 2) \ln y\, dy = \frac{(y - 2)^2}{2} \ln y - \int \frac{(y - 2)^2}{2} d(\ln y) = \frac{(y - 2)^2}{2} \ln y - \frac{1}{2} \int \frac{y^2 - 4y + 4}{y} dy$$
$$= \frac{(y - 2)^2}{2} \ln y - \frac{1}{2} \int \left(y - 4 + \frac{4}{y}\right) dy = \frac{(y - 2)^2}{2} \ln y - \frac{y^2}{4} + 2y - 2 \ln y + C.$$

Thus we conclude that

$$\int (x + 1) \ln(x + 3)\, dx = \frac{(y - 2)^2}{2} \ln y - \frac{y^2}{4} + 2y - 2 \ln y + C$$
$$= \frac{(x + 1)^2}{2} \ln|x + 3| - \frac{(x + 3)^2}{4} + 2(x + 3) - 2 \ln|x + 3| + C.$$

(g) Let $y = e^x - 1 \Rightarrow \dfrac{dy}{dx} = e^x \Rightarrow dx = \dfrac{1}{e^x} dy$

When $x = 2,\ y = e^2 - 1$. When $x = 1,\ y = e - 1$.

The integral then becomes

$$\int_1^2 \frac{e^{2x}}{e^x - 1} dx = \int_{e-1}^{e^2-1} \frac{e^{2x}}{e^x - 1}\left(\frac{1}{e^x} dy\right) = \int_{e-1}^{e^2-1} \frac{e^x}{e^x - 1} dy = \int_{e-1}^{e^2-1} \frac{y + 1}{y} dy = \int_{e-1}^{e^2-1} \left(1 + \frac{1}{y}\right) dy$$
$$= (y + \ln y)\Big|_{e-1}^{e^2-1} = e^2 - e + \ln|e^2 - 1| - \ln|e - 1| = e(e - 1) + \ln(e + 1).$$

(h) Let $y = x^2 \Rightarrow \dfrac{dy}{dx} = 2x \Rightarrow dx = \dfrac{1}{2x} dy$

The integral then becomes

$$\int x^3 \cos(3x^2) \sin(x^2)\, dx = \int x^3 \cos(3x^2) \sin(x^2)\left(\frac{1}{2x} dy\right) = \frac{1}{2} \int y \cos 3y \sin y\, dy$$
$$= \frac{1}{2} \int y\left[\frac{1}{2}[\sin(y + 3y) + \sin(y - 3y)]\right] dy = \frac{1}{4} \int \underbrace{y}_{u} \underbrace{\sin 4y\, dy}_{dv} - \frac{1}{4} \int \underbrace{y}_{u} \underbrace{\sin 2y\, dy}_{dv}$$
$$= \frac{1}{4}\left(-\frac{y \cos 4y}{4} - \int \frac{-\cos 4y}{4} dy\right) - \frac{1}{4}\left(-\frac{y \cos 2y}{2} - \int \frac{-\cos 2y}{2} dy\right)$$
$$= -\frac{1}{16} y \cos 4y + \frac{1}{64} \sin 4y + \frac{1}{8} y \cos 2y - \frac{1}{16} \sin 2y + C$$
$$= -\frac{1}{16} x^2 \cos(4x^2) + \frac{1}{64} \sin(4x^2) + \frac{1}{8} x^2 \cos(2x^2) - \frac{1}{16} \sin(2x^2) + C.$$

(i) Let $x = 2\sin\theta \Rightarrow \dfrac{dx}{d\theta} = 2\cos\theta \Rightarrow dx = 2\cos\theta\, d\theta$

Then the integral becomes

$$\int x^2\sqrt{4-x^2}dx = \int 4\sin^2\theta\sqrt{4-4\sin^2\theta}\,(2\cos\theta\, d\theta) = 16\int \sin^2\theta\cos^2\theta\, d\theta$$

$$= 16\int(\sin\theta\cos\theta)^2 d\theta = 16\int\left(\frac{1}{2}\sin 2\theta\right)^2 d\theta = 4\int\sin^2 2\theta\, d\theta$$

$$= 4\int -\frac{1}{2}[\cos(2\theta+2\theta)-\cos(2\theta-2\theta)]d\theta = -2\int\cos 4\theta\, d\theta + 2\int 1 d\theta$$

$$= -\frac{1}{2}\sin 4\theta + 2\theta + C = -\sin 2\theta\cos 2\theta + 2\theta + C$$

$$= -2\sin\theta\cos\theta\,[\cos^2\theta - \sin^2\theta] + 2\theta + C$$

$$= -2\frac{x}{2}\frac{\sqrt{4-x^2}}{2}\left(\frac{4-x^2}{4} - \frac{x^2}{4}\right) + 2\sin^{-1}\frac{x}{2} + C$$

$$= -\frac{1}{2}x\sqrt{4-x^2}\left(1-\frac{x^2}{2}\right) + 2\sin^{-1}\frac{x}{2} + C.$$

(j) Let $y = 4 + x^2 \Rightarrow \dfrac{dy}{dx} = 2x \Rightarrow dx = \dfrac{1}{2x}dy.$

Then the integral becomes

$$\int x^3\sin(4+x^2)\,dx = \int x^3\sin(4+x^2)\left(\frac{1}{2x}dy\right) = \frac{1}{2}\int(y-4)\sin y\, dy.$$

Using integration by parts with $u = y-4$ and $dv = \sin y\, dy \Rightarrow v = \int \sin y\, dy = -\cos y$. The latter integral can be computed as

$$\int(y-4)\sin y\, dy = -(y-4)\cos y - \int(-\cos y)d(y-4) = (4-y)\cos y + \int\cos y\, dy$$

$$= (4-y)\cos y + \sin y + C.$$

Hence, we can conclude that

$$\int x^3\sin(4+x^2)\,dx = \frac{1}{2}[(4-y)\cos y + \sin y] + C = -\frac{x^2}{2}\cos(4+x^2) + \frac{1}{2}\sin(4+x^2) + C.$$

Problem 4

(a) We shall compute $\int e^{2x}\sin 3x\, dx$ using integration by parts.

We take $u = \sin 3x$ and $dv = e^{2x}dx \Rightarrow v = \int e^{2x}dx = \frac{1}{2}e^{2x}$. Using integration by parts, we get

$$\int e^{2x}\sin 3x\, dx = \frac{1}{2}e^{2x}\sin 3x - \int\frac{1}{2}e^{2x}d(\sin 3x) = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\, dx$$

$$\overset{\substack{u=\cos 3x \\ dv=e^{2x}dx\Rightarrow v=\int e^{2x}dx=\frac{1}{2}e^{2x}}}{\cong} \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\left(\frac{1}{2}e^{2x}\cos 3x - \int\frac{1}{2}e^{2x}d(\cos 3x)\right)$$

$$= \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x - \frac{9}{4}\int e^{2x}\sin 3x\, dx.$$

Summing up, we get

$$\int e^{2x}\sin 3x\, dx = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x - \frac{9}{4}\int e^{2x}\sin 3x\, dx$$

$$\Rightarrow \frac{13}{4}\int e^{2x}\sin 3x\, dx = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x$$

$$\Rightarrow \int e^{2x}\sin 3x\, dx = \frac{2}{13}e^{2x}\sin 3x - \frac{3}{13}e^{2x}\cos 3x + C.$$

In a similar fashion, we may obtain

$$\int e^{2x}\cos 3x\, dx = \frac{3}{13}e^{2x}\sin 3x + \frac{2}{13}e^{2x}\cos 3x + C.$$

(b) Comparing with the similar integral in (a), there is an extra term "x" in the integrand. In order to apply the previous result, one has to eliminate "x" from the integrand. This can be done by using integration by parts.

We take $u = x$ and $dv = e^{2x}\cos 3x\,dx \Rightarrow v = \int e^{2x}\cos 3x\,dx = \frac{3}{13}e^{2x}\sin 3x + \frac{2}{13}e^{2x}\cos 3x$.

Using integration by parts, we get

$$\begin{aligned}
\int xe^{2x}\cos 3x\,dx &= x\left(\frac{3}{13}e^{2x}\sin 3x + \frac{2}{13}e^{2x}\cos 3x\right) - \int\left(\frac{3}{13}e^{2x}\sin 3x + \frac{2}{13}e^{2x}\cos 3x\right)dx \\
&= \frac{3}{13}xe^{2x}\sin 3x + \frac{2}{13}xe^{2x}\cos 3x - \frac{3}{13}\int e^{2x}\sin 3x\,dx - \frac{2}{13}\int e^{2x}\cos 3x\,dx \\
&= \frac{3}{13}xe^{2x}\sin 3x + \frac{2}{13}xe^{2x}\cos 3x - \frac{3}{13}\left(\frac{2}{13}e^{2x}\sin 3x - \frac{3}{13}e^{2x}\cos 3x\right) \\
&\quad - \frac{2}{13}\left(\frac{3}{13}e^{2x}\sin 3x + \frac{2}{13}e^{2x}\cos 3x\right) + C \\
&= \frac{3}{13}xe^{2x}\sin 3x + \frac{2}{13}xe^{2x}\cos 3x - \frac{12}{169}e^{2x}\sin 3x + \frac{5}{169}e^{2x}\cos 3x + C.
\end{aligned}$$

Problem 5

The first objective is to eliminate the term "x" in the integrand. This can be done by using integration by parts. Using integration by parts with $u = x$ and $dv = f'(x)dx \Rightarrow v = \int f'(x)dx = f(x)$, we have

$$\int_a^b \underbrace{x}_{u}\underbrace{f'(x)dx}_{dv} = \underbrace{xf(x)}_{uv}|_a^b - \int_a^b \underbrace{f(x)}_{v}\underbrace{dx}_{du} = b\underbrace{f(b)}_{=1} - a\underbrace{f(a)}_{=1} - \underbrace{\int_a^b f(x)dx}_{=0} = b - a.$$

Problem 6

We first simplify the first integral $\int_0^a x^3 f(x^2)dx$ on the left hand side.

Let $y = x^2 \Rightarrow \dfrac{dy}{dx} = 2x \Rightarrow dx = \dfrac{1}{2x}dy$.

When $x = a,\ y = a^2$. When $x = 0,\ y = 0$.

Then the integral $\int_0^a x^3 f(x^2)dx$ can be rewritten as

$$\int_0^a x^3 f(x^2)dx = \int_0^{a^2} x^3 f(x^2)\left(\frac{1}{2x}dy\right) = \frac{1}{2}\int_0^{a^2} yf(y)dy.$$

Since the definite integrals $\int_0^{a^2} yf(y)dy$ and $\int_0^{a^2} xf(x)dx$ give the same values. Hence we conclude that

$$\int_0^a x^3 f(x^2)dx - \frac{1}{2}\int_0^{a^2} xf(x)dx = \frac{1}{2}\int_0^{a^2} yf(y)dy - \frac{1}{2}\int_0^{a^2} xf(x)dx = 0.$$

Problem 7

Using integration by parts with $u = x(1-x)$ and $dv = f''(x)dx \Rightarrow v = \int f''(x)dx = f'(x)$, we have

$$\begin{aligned}
\int_0^1 x(1-x)f''(x)dx &= x(1-x)f'(x)|_0^1 - \int_0^1 f'(x)d[x(1-x)] = 0 - 0 - \int_0^1 f'(x)(1-2x)dx \\
&= \int_0^1 \underbrace{(2x-1)}_{u}\underbrace{f'(x)dx}_{dv} \overset{\substack{u=2x-1 \\ dv=f'(x)dx \\ \Rightarrow v=\int f'(x)dx=f(x)}}{=} (2x-1)f(x)|_0^1 - \int_0^1 f(x)d(2x-1) \\
&= \underbrace{f(1)}_{=1} - \left(-\underbrace{f(0)}_{=1}\right) - 2\underbrace{\int_0^1 f(x)dx}_{=1} = 2 - 2 = 0.
\end{aligned}$$

Problem 8

(a) Using integration by parts with $u = x^n$ and $dv = e^{-3x}dx \Rightarrow v = \int e^{-3x}dx = -\frac{1}{3}e^{-3x}$, we have

$$I_n = \int_0^1 x^n e^{-3x}dx = -\frac{1}{3}x^n e^{-3x}|_0^1 - \int_0^1 -\frac{1}{3}e^{-3x}d(x^n) = -\frac{1}{3}e^{-3} + \frac{n}{3}\int_0^1 x^{n-1}e^{-3x}dx$$
$$= -\frac{1}{3}e^{-3} + \frac{n}{3}I_{n-1}.$$

(b) Using the reduction formula in (a), the integral can be computed as

$$\int_0^1 x^3 e^{-3x}dx = I_3 = -\frac{1}{3}e^{-3} + \frac{3}{3}I_2 = -\frac{1}{3}e^{-3} + I_2 = -\frac{1}{3}e^{-3} + \left(-\frac{1}{3}e^{-3} + \frac{2}{3}I_1\right) = -\frac{2}{3}e^{-3} + \frac{2}{3}I_1$$
$$= -\frac{2}{3}e^{-3} + \frac{2}{3}\left(-\frac{1}{3}e^{-3} + \frac{1}{3}I_0\right) = -\frac{8}{9}e^{-3} + \frac{2}{9}\int_0^1 x^0 e^{-3x}dx$$
$$= -\frac{8}{9}e^{-3} + \frac{2}{9}\int_0^1 e^{-3x}dx = -\frac{8}{9}e^{-3} - \frac{2}{27}e^{-3x}|_0^1 = \frac{2}{27} - \frac{26}{27}e^{-3}.$$

Problem 9

(a) Using integration by parts with $u = (\ln x)^n$ and $dv = x^a dx \Rightarrow v = \int x^a dx = \frac{x^{a+1}}{a+1}$ (since $a \neq -1$),

$$I_n = \int_1^e x^a (\ln x)^n dx = \frac{x^{a+1}}{a+1}(\ln x)^n|_1^e - \int_1^e \frac{x^{a+1}}{a+1}d(\ln x)^n = \frac{e^{a+1}}{a+1} - \int_1^e \frac{x^{a+1}}{a+1}\left(n(\ln x)^{n-1}\frac{1}{x}dx\right)$$
$$= \frac{e^{a+1}}{a+1} - \frac{n}{a+1}\int_1^e x^a (\ln x)^{n-1}dx = \frac{e^{a+1}}{a+1} - \frac{n}{a+1}I_{n-1}.$$

(b) Using the reduction formula in (a) (set $a = 2$), the integral can be compute as

$$\int_1^e x^2 (\ln x)^3 dx = I_3 \overset{\substack{n=3\\a=2}}{\cong} \frac{e^3}{3} - \frac{3}{3}I_2 = \frac{e^3}{3} - \left(\frac{e^3}{3} - \frac{2}{3}I_1\right) = \frac{2}{3}I_1 = \frac{2}{3}\left(\frac{e^3}{3} - \frac{1}{3}I_0\right)$$
$$= \frac{2}{9}e^3 - \frac{2}{9}\int_1^e x^2 (\ln x)^0 dx = \frac{2}{9}e^3 - \frac{2}{9}\int_1^e x^2 dx = \frac{2}{9}e^3 - \frac{2}{27}x^3|_1^e = \frac{4}{27}e^3 + \frac{2}{27}.$$

(c) When $a = -1$, the integral becomes $I_n = \int_1^e \frac{(\ln x)^n}{x}dx$ and the above reduction formula cannot be applied. One can compute this integral using method of substitution.
Let $y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x} \Rightarrow dx = xdy$. When $x = e$, $y = \ln e = 1$. When $x = 1$, $y = \ln 1 = 0$.
The integral then becomes

$$I_n = \int_1^e \frac{(\ln x)^n}{x}dx = \int_0^1 y^n dy = \frac{y^{n+1}}{n+1}|_0^1 = \frac{1}{n+1}.$$

Problem 10

(a) Using integration by parts with $u = (x^2 + a^2)^n$ and $dv = dx \Rightarrow v = \int dx = x$, the integral can be expressed as

$$I_n = \int (x^2 + a^2)^n dx = x(x^2 + a^2)^n - \int xd(x^2 + a^2)^n$$

$$= x(x^2+a^2)^n - \int x(n(x^2+a^2)^{n-1}(2x)dx) = x(x^2+a^2)^n - 2n\int x^2(x^2+a^2)^{n-1}dx$$

$$= x(x^2+a^2)^n - 2n\int (x^2+a^2-a^2)(x^2+a^2)^{n-1}dx$$

$$= x(x^2+a^2)^n - 2n\int (x^2+a^2)^n dx + 2na^2\int (x^2+a^2)^{n-1}dx$$

$$= x(x^2+a^2)^n - 2nI_n + 2na^2 I_{n-1}.$$

Summing up, we obtain

$$I_n = x(x^2+a^2)^n - 2nI_n + 2na^2I_{n-1} \Rightarrow (2n+1)I_n = x(x^2+a^2)^n + 2na^2I_{n-1}$$

$$\Rightarrow I_n = \frac{1}{2n+1}x(x^2+a^2)^n + \frac{2n}{2n+1}a^2 I_{n-1}.$$

(b) Using the reduction formula in (a), we can compute the integral as

$$\int (x^2+a^2)^4 dx = I_4 = \frac{1}{9}x(x^2+a^2)^4 + \frac{8}{9}a^2 I_3 = \frac{1}{9}x(x^2+a^2)^4 + \frac{8}{9}a^2\left(\frac{1}{7}x(x^2+a^2)^3 + \frac{6}{7}a^2 I_2\right)$$

$$= \frac{1}{9}x(x^2+a^2)^4 + \frac{8}{63}a^2x(x^2+a^2)^3 + \frac{16}{21}a^4 I_2$$

$$= \frac{1}{9}x(x^2+a^2)^4 + \frac{8}{63}a^2x(x^2+a^2)^3 + \frac{16}{21}a^4\left(\frac{1}{5}x(x^2+a^2)^2 + \frac{4}{5}a^2 I_1\right)$$

$$= \frac{1}{9}x(x^2+a^2)^4 + \frac{8}{63}a^2x(x^2+a^2)^3 + \frac{16}{105}a^4x(x^2+a^2)^2 + \frac{64}{105}a^6\left(\frac{1}{3}x(x^2+a^2) + \frac{2}{3}a^2 I_0\right)$$

$$= \frac{1}{9}x(x^2+a^2)^4 + \frac{8}{63}a^2x(x^2+a^2)^3 + \frac{16}{105}a^4x(x^2+a^2)^2 + \frac{64}{315}a^6x(x^2+a^2) + \frac{128}{315}a^8\int 1dx$$

$$= \frac{1}{9}x(x^2+a^2)^4 + \frac{8}{63}a^2x(x^2+a^2)^3 + \frac{16}{105}a^4x(x^2+a^2)^2 + \frac{64}{315}a^6x(x^2+a^2) + \frac{128}{315}a^8x + C.$$

Problem 11

(a) Using integration by parts with $u = x^n$ and $dv = \cos 3x\,dx \Rightarrow v = \int \cos 3x\,dx = \frac{1}{3}\sin 3x$, we have

$$I_n = \int x^n \cos 3x\,dx = \frac{1}{3}x^n \sin 3x - \int \frac{1}{3}\sin 3x\,d(x^n) = \frac{1}{3}x^n\sin 3x - \frac{n}{3}\int x^{n-1}\sin 3x\,dx$$

$$\overset{\substack{u=x^{n-1}\\ dv=\sin 3x dx\\ \Rightarrow v=\int \sin 3x dx=-\frac{1}{3}\cos 3x}}{=} \frac{1}{3}x^n\sin 3x - \frac{n}{3}\left(-\frac{1}{3}x^{n-1}\cos 3x - \int -\frac{1}{3}\cos 3x\,d(x^{n-1})\right)$$

$$= \frac{1}{3}x^n\sin 3x + \frac{n}{9}x^{n-1}\cos 3x - \frac{n(n-1)}{9}\int x^{n-2}\cos 3x\,dx$$

$$= \frac{1}{3}x^n\sin 3x + \frac{n}{9}x^{n-1}\cos 3x - \frac{n(n-1)}{9}I_{n-2}.$$

(b) Using the reduction formula in (a), the integral can be computed as

$$\begin{aligned}\int x^4\cos 3x\,dx &= \frac{1}{3}x^4\sin 3x+\frac{4}{9}x^3\cos 3x-\frac{4}{3}I_2\\ &= \frac{1}{3}x^4\sin 3x+\frac{4}{9}x^3\cos 3x-\frac{4}{3}\left(\frac{1}{3}x^2\sin 3x+\frac{2}{9}x\cos 3x-\frac{2}{9}I_0\right)\\ &= \frac{1}{3}x^4\sin 3x+\frac{4}{9}x^3\cos 3x-\frac{4}{9}x^2\sin 3x-\frac{8}{27}x\cos 3x+\frac{8}{27}\int\cos 3x\,dx\\ &= \frac{1}{3}x^4\sin 3x+\frac{4}{9}x^3\cos 3x-\frac{4}{9}x^2\sin 3x-\frac{8}{27}x\cos 3x+\frac{8}{81}\sin 3x+C.\end{aligned}$$

Problem 12

(a) Using similar method as in Example 28 of Chapter 2, we use integration by parts with $u=\cos^{n-1}x$ and $dv=\cos x\,dx\Rightarrow v=\int\cos x\,dx=\sin x$ and obtain

$$I_n=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^n x\,dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\underbrace{\cos^{n-1}x}_{u}\underbrace{(\cos x\,dx)}_{dv}=\cos^{n-1}x\sin x\,\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin x\,d(\cos^{n-1}x)$$

$$=0-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin x\left((n-1)\cos^{n-2}x\,(-\sin x\,dx)\right)=(n-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2 x\cos^{n-2}x\,dx$$

$$=(n-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\cos^2 x)\cos^{n-2}x\,dx=(n-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^{n-2}x\,dx-(n-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^n x\,dx$$

$$=(n-1)I_{n-2}-(n-1)I_n.$$

Summing up, we have

$$I_n=(n-1)I_{n-2}-(n-1)I_n\Rightarrow nI_n=(n-1)I_{n-2}\Rightarrow I_n=\frac{n-1}{n}I_{n-2}.$$

(b) Using the reduction formula obtained in (a), the first integral can be computed as

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^5 x\,dx=I_5\overset{n=5}{\triangleq}\frac{4}{5}I_3\overset{n=3}{\triangleq}\frac{4}{5}\left(\frac{2}{3}I_1\right)=\frac{8}{15}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos x\,dx=\frac{8}{15}\sin x\,\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{16}{15}.$$

Using the same reduction formula, the second integral can be computed as

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2 x\cos^4 x\,dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\cos^2 x)\cos^4 x\,dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4 x\,dx-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^6 x\,dx=I_4-I_6$$

$$=\frac{3}{4}I_2-\frac{5}{6}I_4=\frac{3}{4}\left(\frac{1}{2}I_0\right)-\frac{5}{6}\left(\frac{3}{4}I_2\right)=\frac{3}{8}I_0-\frac{15}{24}\left(\frac{1}{2}I_0\right)=\frac{3}{48}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos x)^0dx$$

$$=\frac{1}{16}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}1dx=\frac{1}{16}x\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{1}{16}\pi.$$

(c) Let $x=\sin\theta\Rightarrow\dfrac{dx}{d\theta}=\cos\theta\Rightarrow dx=\cos\theta\,d\theta.$

When $x=1$, $\theta=\sin^{-1}1=\frac{\pi}{2}$. When $x=-1$, $\theta=\sin^{-1}(-1)=-\frac{\pi}{2}$. The integral becomes

$$\int_{-1}^{1}(1-x^2)^{\frac{5}{2}}dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\sin^2\theta)^{\frac{5}{2}}(\cos\theta\,d\theta)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos^2\theta)^{\frac{5}{2}}(\cos\theta\,d\theta)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^6\theta\,d\theta$$

$$=I_6\overset{\text{from }(b)}{\triangleq}\frac{15}{48}I_0=\frac{15}{48}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos x)^0dx\overset{\text{from }(b)}{\triangleq}\frac{5}{16}\pi.$$

Problem 13

(a) Using integration of parts with $u = \sec^{n-2}x$ and $dv = \sec^2 x\,dx \Rightarrow v = \int \sec^2 x\,dx = \tan x$,

$$I_n = \int_0^{\frac{\pi}{4}} \sec^n x\,dx = \int_0^{\frac{\pi}{4}} \underbrace{\sec^{n-2}x}_{u}\,\underbrace{(\sec^2 x\,dx)}_{dv} = \sec^{n-2}x\tan x\,\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x\,d(\sec^{n-2}x)$$

$$\overset{\sec\frac{\pi}{4}=\frac{1}{\cos\frac{\pi}{4}}=\frac{1}{\frac{1}{\sqrt{2}}}=\sqrt{2}}{\cong} \left(\sqrt{2}\right)^{n-2} - \int_0^{\frac{\pi}{4}} \tan x\,\underbrace{[(n-2)\sec^{n-3}x\,(\sec x\tan x)dx]}_{\frac{d}{dx}(\sec^{n-2}x)=\frac{d(\sec^{n-2}x)}{d(\sec x)}\frac{d(\sec x)}{dx}}$$

$$= \left(\sqrt{2}\right)^{n-2} - (n-2)\int_0^{\frac{\pi}{4}} \sec^{n-2}x\tan^2 x\,dx$$

$$\overset{1+\tan^2 x=\sec^2 x}{\cong} \left(\sqrt{2}\right)^{n-2} - (n-2)\int \sec^{n-2}x\,(\sec^2 x - 1)dx$$

$$= \left(\sqrt{2}\right)^{n-2} - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2}x\,dx = \left(\sqrt{2}\right)^{n-2} - (n-2)I_n + (n-2)I_{n-2}.$$

Summing up, we obtain

$$I_n = \left(\sqrt{2}\right)^{n-2} - (n-2)I_n + (n-2)I_{n-2} \Rightarrow I_n = \frac{\left(\sqrt{2}\right)^{n-2}}{n-1} + \frac{n-2}{n-1}I_{n-2}.$$

(b) <u>First Integral</u>

Let $x = \tan\theta \Rightarrow \dfrac{dx}{d\theta} = \sec^2\theta \Rightarrow dx = \sec^2\theta\,d\theta$.

When $x = 1$, $\theta = \tan^{-1}1 = \frac{\pi}{4}$. When $x = 0$, $\theta = \tan^{-1}0 = 0$. The integral then becomes

$$\int_0^1 (1+x^2)^{\frac{3}{2}}dx = \int_0^{\frac{\pi}{4}} (1+\tan^2\theta)^{\frac{3}{2}}\sec^2\theta\,d\theta = \int_0^{\frac{\pi}{4}} \sec^5\theta\,d\theta = I_5.$$

Using the reduction formula in (a), we have

$$I_5 = \frac{\left(\sqrt{2}\right)^3}{4} + \frac{3}{4}I_3 = \frac{\sqrt{2}}{2} + \frac{3}{4}\left(\frac{\sqrt{2}}{2} + \frac{1}{2}I_1\right) = \frac{7\sqrt{2}}{8} + \frac{3}{8}\int_0^{\frac{\pi}{4}} \sec\theta\,d\theta = \frac{7\sqrt{2}}{8} + \frac{3}{8}\ln|\sec\theta + \tan\theta|_0^{\frac{\pi}{4}}$$

$$= \frac{7\sqrt{2}}{8} + \frac{3}{8}\ln\left|\sqrt{2}+1\right|.$$

Hence, we can conclude that

$$\int_0^1 (1+x^2)^{\frac{3}{2}}dx = I_5 = \frac{7\sqrt{2}}{8} + \frac{3}{8}\ln\left|\sqrt{2}+1\right|.$$

<u>Second Integral</u>

Let $x = 2\sec\theta \Rightarrow \dfrac{dx}{d\theta} = 2\sec\theta\tan\theta \Rightarrow dx = 2\sec\theta\tan\theta\,d\theta$.

When $x = 2\sqrt{2}$, $\sqrt{2} = \sec\theta \Rightarrow \theta = \cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$. When $x = 2$, $1 = \sec\theta \Rightarrow \theta = \cos^{-1}1 = 0$.

The integral then becomes

$$\int_2^{2\sqrt{2}} x^2\sqrt{x^2-4}dx = \int_0^{\frac{\pi}{4}} 4\sec^2\theta\sqrt{4\sec^2\theta-4}(2\sec\theta\tan\theta\, d\theta) = 16\int_0^{\frac{\pi}{4}}\sec^3\theta\tan^2\theta\, d\theta$$

$$= 16\int_0^{\frac{\pi}{4}}\sec^3\theta(\sec^2\theta-1)d\theta = 16\int_0^{\frac{\pi}{4}}\sec^5\theta\, d\theta - 16\int_0^{\frac{\pi}{4}}\sec^3\theta\, d\theta$$

$$= 16I_5 - 16I_3 = 16\left(\frac{7\sqrt{2}}{8}+\frac{3}{8}\ln\left|\sqrt{2}+1\right|\right) - 16\left(\frac{\sqrt{2}}{2}+\frac{1}{2}\ln\left|\sqrt{2}+1\right|\right)$$

$$= 6\sqrt{2} - 2\ln\left|\sqrt{2}+1\right|.$$

Problem 14

(a) Using the integration by parts with $u = x^{n-1}$ and $dv = x\sqrt{a^2-x^2}dx \Rightarrow$

$v = \int x\sqrt{a^2-x^2}dx \overset{y=a^2-x^2}{\cong} -\frac{1}{2}\int\sqrt{y}dy = -\frac{1}{3}y^{\frac{3}{2}} = -\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}$, we have

$$I_n = \int_0^a x^n\sqrt{a^2-x^2}dx = -\frac{1}{3}x^{n-1}(a^2-x^2)^{\frac{3}{2}}|_0^a - \int_0^a -\frac{1}{3}(a^2-x^2)^{\frac{3}{2}}d(x^{n-1})$$

$$= 0 - 0 + \frac{n-1}{3}\int_0^a x^{n-2}(a^2-x^2)^{\frac{3}{2}}dx = \frac{n-1}{3}\int_0^a x^{n-2}(a^2-x^2)\sqrt{a^2-x^2}dx$$

$$= \frac{n-1}{3}a^2\int_0^a x^{n-2}\sqrt{a^2-x^2}dx - \frac{n-1}{3}\int_0^a x^n\sqrt{a^2-x^2}dx = \frac{n-1}{3}a^2 I_{n-2} - \frac{n-1}{3}I_n.$$

Summing up, we have

$$I_n = \frac{n-1}{3}a^2 I_{n-2} - \frac{n-1}{3}I_n \Rightarrow I_n = \left(\frac{n-1}{n+2}\right)a^2 I_{n-2}.$$

Remark of (a)

It is not a good idea to use integration by parts with $u = x^n$ and $dv = \sqrt{a^2-x^2}dx$. It is because the function v becomes $v = \int\sqrt{a^2-x^2}dx = \cdots = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$ and the integral becomes

$$\int_0^a x^n\sqrt{a^2-x^2}dx = x^n\left(\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)|_0^a - n\int_0^a x^{n-1}\left(\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right)dx.$$

The computation will be tedious. Hence, one has to modify the choice of v (or dv) so that the integral will not be too complicated after using integration by parts.

(b) Using the reduction formula in (a), we have

$$\int_0^a x^2(a^2-x^2)^{\frac{3}{2}}dx = \int_0^a x^2(a^2-x^2)\sqrt{a^2-x^2}dx = a^2\int_0^a x^2\sqrt{a^2-x^2}\,dx - \int_0^a x^4\sqrt{a^2-x^2}\,dx$$

$$= a^2 I_2 - I_4 = a^2\left(\frac{1}{4}a^2 I_0\right) - \frac{3}{6}a^2 I_2 = \frac{1}{4}a^4 I_0 - \frac{1}{2}a^2\left(\frac{1}{4}a^2 I_0\right) = \frac{a^4}{8}I_0 = \frac{a^4}{8}\int_0^a\sqrt{a^2-x^2}dx$$

$$\overset{\substack{x=a\sin\theta\\ \Rightarrow\frac{dx}{d\theta}=a\cos\theta}}{\cong} \frac{a^4}{8}\int_0^{\frac{\pi}{2}} a^2\cos^2\theta\, d\theta = \frac{a^6}{8}\int_0^1\frac{1}{2}(\cos 2\theta+1)d\theta = \frac{a^6}{16}\left(\frac{\sin 2\theta}{2}+\theta\right)\Big|_0^{\frac{\pi}{2}} = \frac{a^6}{16}\left(\frac{\pi}{2}\right) = \frac{a^6\pi}{32}.$$

Problem 15

(a) Using integration by parts with $u = (\sin^{-1}x)^n$ and $dv = dx \Rightarrow v = \int dx = x$, we have

$$I_n = \int (\sin^{-1} x)^n dx = x\,(\sin^{-1} x)^n - \int x\, d(\sin^{-1} x)^n = x\,(\sin^{-1} x)^n - n \int \frac{x(\sin^{-1} x)^{n-1}}{\sqrt{1-x^2}}\, dx$$

$$= x\,(\sin^{-1} x)^n - n \int \underbrace{(\sin^{-1} x)^{n-1}}_{u} \underbrace{\frac{x}{\sqrt{1-x^2}}\, dx}_{dv}$$

$$\overset{\substack{u=(\sin^{-1} x)^{n-1} \\ dv=\frac{x}{\sqrt{1-x^2}}dx \\ \Rightarrow v=\int \frac{x}{\sqrt{1-x^2}}dx=-\sqrt{1-x^2}}}{\hat{=}} \quad x\,(\sin^{-1} x)^n - n\left(-\sqrt{1-x^2}(\sin^{-1} x)^{n-1} - \int -\sqrt{1-x^2}\, d(\sin^{-1} x)^{n-1}\right)$$

$$= x\,(\sin^{-1} x)^n + n\sqrt{1-x^2}(\sin^{-1} x)^{n-1} - n(n-1)\int \frac{\sqrt{1-x^2}(\sin^{-1} x)^{n-2}}{\sqrt{1-x^2}}\, dx$$

$$= x\,(\sin^{-1} x)^n + n\sqrt{1-x^2}(\sin^{-1} x)^{n-1} - n(n-1)\int (\sin^{-1} x)^{n-2}\, dx$$

$$= x\,(\sin^{-1} x)^n + n\sqrt{1-x^2}(\sin^{-1} x)^{n-1} - n(n-1) I_{n-2}.$$

(b) Using the reduction formula in (a), the first integral can be computed as

$$\int (\sin^{-1} x)^3 dx = I_3 = x\,(\sin^{-1} x)^3 + 3\sqrt{1-x^2}(\sin^{-1} x)^2 - 6I_1$$

$$= x\,(\sin^{-1} x)^3 + 3\sqrt{1-x^2}(\sin^{-1} x)^2 - 6\int \sin^{-1} x\, dx \ldots (*)$$

To compute the last integral, we use integration by parts with $u = \sin^{-1} x$ and $dv = dx$ $\Rightarrow v = \int dx = x$ and obtain

$$\int \sin^{-1} x\, dx = x \sin^{-1} x - \int x\, d(\sin^{-1} x) = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\, dx$$

$$\overset{\substack{y=1-x^2 \\ \frac{dy}{dx}=-2x}}{\hat{=}} \quad x \sin^{-1} x + \frac{1}{2}\int \frac{1}{\sqrt{y}}\, dy = x \sin^{-1} x + \sqrt{y} + C = x \sin^{-1} x + \sqrt{1-x^2} + C.$$

We conclude from $(*)$ that

$$\int (\sin^{-1} x)^3 dx = x\,(\sin^{-1} x)^3 + 3\sqrt{1-x^2}(\sin^{-1} x)^2 - 6x \sin^{-1} x - 6\sqrt{1-x^2} + C.$$

Consider the second integral, we use integration by parts with $u = (\sin^{-1} x)^5$ and $dv = \frac{x}{\sqrt{1-x^2}}\, dx$ $\Rightarrow v = -\sqrt{1-x^2}$ (from (a)), we have

$$\int \frac{x(\sin^{-1} x)^5}{\sqrt{1-x^2}}\, dx = \int \underbrace{(\sin^{-1} x)^5}_{u} \underbrace{\left(\frac{x}{\sqrt{1-x^2}}\, dx\right)}_{dv} = -(\sin^{-1} x)^5\sqrt{1-x^2} - \int -\sqrt{1-x^2}\, d(\sin^{-1} x)^5$$

$$= -(\sin^{-1} x)^5\sqrt{1-x^2} + \int 5\frac{\sqrt{1-x^2}(\sin^{-1} x)^4}{\sqrt{1-x^2}}\, dx = -(\sin^{-1} x)^5\sqrt{1-x^2} + 5\int (\sin^{-1} x)^4 dx$$

$$= -(\sin^{-1} x)^5\sqrt{1-x^2} + 5I_4 \ldots\ldots (**)$$

Note that I_4 can be computed as

$$I_4 = x(\sin^{-1} x)^4 + 4(\sin^{-1} x)^3\sqrt{1-x^2} - 12I_2$$

$$= x(\sin^{-1} x)^4 + 4(\sin^{-1} x)^3\sqrt{1-x^2} - 12\left(x(\sin^{-1} x)^2 + 2\sin^{-1} x\sqrt{1-x^2} - 2I_0\right)$$

$$= x(\sin^{-1} x)^4 + 4(\sin^{-1} x)^3\sqrt{1-x^2} - 12x(\sin^{-1} x)^2 - 24\sin^{-1} x\sqrt{1-x^2} + 24\int 1 dx$$

$$= x(\sin^{-1} x)^4 + 4(\sin^{-1} x)^3\sqrt{1-x^2} - 12x(\sin^{-1} x)^2 - 24\sin^{-1} x\sqrt{1-x^2} + 24x + C.$$

Substitute this into $(**)$, we obtain

$$\int \frac{x(\sin^{-1} x)^5}{\sqrt{1-x^2}}\,dx = -(\sin^{-1} x)^5\sqrt{1-x^2} + 5x(\sin^{-1} x)^4 + 20(\sin^{-1} x)^3\sqrt{1-x^2}$$
$$-60x(\sin^{-1} x)^2 - 120\sin^{-1} x\sqrt{1-x^2} + 120x + C.$$

Problem 16

(a) Note that $3x - x^2 = x(3-x)$, we may try the following decomposition:

$$\frac{1}{3x-x^2} = \frac{1}{x(3-x)} = \frac{A}{x} + \frac{B}{3-x} \Rightarrow 1 = A(3-x) + Bx.$$

- Substitute $x = 3$, we have $1 = 3B \Rightarrow B = \frac{1}{3}$.
- Substitute $x = 0$, we have $1 = 3A \Rightarrow A = \frac{1}{3}$.

The integral can be computed as

$$\int \frac{1}{3x-x^2}\,dx = \frac{1}{3}\int \frac{1}{x}\,dx + \frac{1}{3}\int \frac{1}{3-x}\,dx = \frac{1}{3}\ln|x| - \frac{1}{3}\ln|3-x| + C.$$

(b) Note that the denominator consists of a repeated factor $(x-1)^2$, we should try the decomposition:

$$\frac{2x^2-5x+5}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$$
$$\Rightarrow 2x^2-5x+5 = A(x-1)(x-2) + B(x-2) + C(x-1)^2.$$

- Put $x = 1$, we have $2 = -B \Rightarrow B = -2$.
- Put $x = 2$, we have $3 = C$.
- Compare the coefficient of x^2 : $2 = A + C \Rightarrow A = 2 - C = -1$.

The integral can be computed as

$$\int \frac{2x^2-5x+5}{(x-1)^2(x-2)}\,dx = -\int \frac{1}{x-1}\,dx - 2\int \frac{1}{(x-1)^2}\,dx + 3\int \frac{1}{x-2}\,dx$$
$$= -\ln|x-1| + \frac{2}{x-1} + 3\ln|x-2| + C.$$

(c) Note that the rational function is an improper one, we first use *long division* (numerator divided by denominator) to decompose the integrand into the sum of a polynomial and a proper rational function.

$$\int \frac{3x^4-5x^3+x^2+2x+1}{3x^3-2x^2-x}\,dx = \int \frac{(3x^3-2x^2-x)(x-1)+x+1}{3x^3-2x^2-x}\,dx$$
$$= \int (x-1)dx + \int \frac{x+1}{3x^3-2x^2-x}\,dx = \frac{x^2}{2} - x + \int \frac{x+1}{3x^3-2x^2-x}\,dx \;\ldots\ldots (*)$$

To compute the second integral, we first note that $3x^3 - 2x^2 - x = x(3x+1)(x-1)$. We then decompose the integrand as follows:

$$\frac{x+1}{3x^3-2x^2-x} = \frac{x+1}{x(3x+1)(x-1)} = \frac{A}{x} + \frac{B}{3x+1} + \frac{C}{x-1}$$
$$\Rightarrow x+1 = A(3x+1)(x-1) + Bx(x-1) + Cx(3x+1).$$

- Put $x = 0$, $1 = -A \Rightarrow A = -1$.
- Put $x = 1$, $2 = 4C \Rightarrow C = \frac{1}{2}$.
- Put $x = -\frac{1}{3}$, $\frac{2}{3} = \frac{4B}{9} \Rightarrow B = \frac{3}{2}$.

Then the second integral can be computed as

$$\int \frac{x+1}{3x^3-2x^2-x}dx = -\int \frac{1}{x}dx + \frac{3}{2}\int \frac{1}{3x+1}dx + \frac{1}{2}\int \frac{1}{x-1}dx$$
$$= -\ln|x| + \frac{1}{2}\ln|3x+1| + \frac{1}{2}\ln|x-1| + C.$$

From (*), we conclude that

$$\int \frac{3x^4-5x^3+x^2+2x+1}{3x^3-2x^2-x}dx = \frac{x^2}{2} - x - \ln|x| + \frac{1}{2}\ln|3x+1| + \frac{1}{2}\ln|x-1| + C.$$

(d) Since the denominator consists of a quadratic factor, we shall decompose the integrand as follows:

$$\frac{3x^2+10}{(x+3)(x^2-6x+10)} = \frac{Ax+B}{(x^2-6x+10)} + \frac{C}{x+3}$$
$$\Rightarrow 3x^2+10 = (Ax+B)(x+3) + C(x^2-6x+10)$$

- Put $x=-3,\ 37 = 37C \Rightarrow C = 1.$
- Compare the coefficient of x^2 : $3 = A + C \Rightarrow A = 3 - C = 2.$
- Compare the constant term : $10 = 3B + 10C \Rightarrow B = 0.$

Thus the integral can be computed as

$$\int \frac{3x^2+10}{(x+3)(x^2-6x+10)}dx = \int \frac{2x}{x^2-6x+10}dx + \int \frac{1}{x+3}dx$$
$$= \int \frac{2x-6}{x^2-6x+10}dx + \int \frac{6}{x^2-6x+10}dx + \ln|x+3| + C$$

Let $y=x^2-6x+10$

$\frac{dy}{dx}=2x-6$

$$\stackrel{\text{def}}{=} \int \frac{1}{y}dy + 6\int \frac{1}{(x-3)^2+1}dx + \ln|x+3| + C$$
$$= \ln|y| + 6\tan^{-1}(x-3) + \ln|x+3| + C = \ln|x^2-6x+10| + 6\tan^{-1}(x-3) + \ln|x+3| + C.$$

(e) Since the denominator contains a quadratic factor, we shall decompose the integrand as follows:

$$\frac{-7x+19}{(2x+1)(x^2-4x+9)} = \frac{A}{2x+1} + \frac{Bx+C}{x^2-4x+9}$$
$$\Rightarrow -7x+19 = A(x^2-4x+9) + (Bx+C)(2x+1).$$

- Put $x = -\frac{1}{2}$, we have $\frac{45}{2} = \frac{45}{4}A \Rightarrow A = 2.$
- Compare the coefficient of x^2 : $0 = A + 2B \Rightarrow B = -1.$
- Compare the constant term : $19 = 9A + C \Rightarrow C = 1.$

Then the integral becomes

$$\int \frac{-7x+19}{(2x+1)(x^2-4x+9)}dx = \int \frac{2}{2x+1}dx + \int \frac{-x+1}{x^2-4x+9}dx$$
$$= 2\int \frac{1}{2x+1}dx - \int \frac{x-1}{x^2-4x+9}dx = 2\int \frac{1}{2x+1}dx - \int \frac{x-2}{x^2-4x+9}dx - \int \frac{1}{x^2-4x+9}dx$$

$$\overset{\substack{\text{Let } y=x^2-4x+9 \\ \frac{dy}{dx}=2x-4}}{\cong} \quad \ln|2x+1| - \frac{1}{2}\int \frac{1}{y}\,dy - \int \frac{1}{(x-2)^2+5}\,dx$$

$$= \ln|2x+1| - \frac{1}{2}\ln|y| - \frac{1}{5}\int \frac{1}{\left(\frac{x-2}{\sqrt{5}}\right)^2+1}\,dx$$

$$= \ln|2x+1| - \frac{1}{2}\ln|x^2-4x+9| - \frac{1}{5}\left(\frac{1}{\frac{1}{\sqrt{5}}}\tan^{-1}\frac{x-2}{\sqrt{5}}\right) + C$$

$$= \ln|2x+1| - \frac{1}{2}\ln|x^2-4x+9| - \frac{\sqrt{5}}{5}\tan^{-1}\frac{x-2}{\sqrt{5}} + C.$$

(f) Note that by synthetic division $4x^3-3x+1=(x+1)(2x-1)^2$, we shall decompose the integrand as follows:

$$\frac{8x^2-3x-2}{4x^3-3x+1} = \frac{8x^2-3x-2}{(x+1)(2x-1)^2} = \frac{A}{x+1} + \frac{B}{2x-1} + \frac{C}{(2x-1)^2}$$
$$\Rightarrow 8x^2-3x-2 = A(2x-1)^2 + B(x+1)(2x-1) + C(x+1)$$

- Put $x=\frac{1}{2}$, $-\frac{3}{2}=\frac{3}{2}C \Rightarrow C=-1$.
- Put $x=-1$, $9=9A \Rightarrow A=1$.
- Compare the coefficient of x^2 : $8=4A+2B \Rightarrow B=2$.

Then the integral can be computed as

$$\int \frac{8x^2-3x-2}{4x^3-3x+1}\,dx = \int \frac{1}{x+1}\,dx + 2\int \frac{1}{2x-1}\,dx - \int \frac{1}{(2x-1)^2}\,dx$$
$$= \ln|x+1| + \ln|2x-1| + \frac{1}{2(2x-1)} + C.$$

(g) Note that the denominator contains a quadratic factor, we decompose the integrand as follows:

$$\frac{x^2-5x-5}{(x-2)(x^2+2x+3)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+3}$$
$$\Rightarrow x^2-5x-5 = A(x^2+2x+3) + (Bx+C)(x-2).$$

- Put $x=2$, $-11=11A \Rightarrow A=-1$.
- Compare the coefficient of x^2 : $1=A+B \Rightarrow B=1-A=2$.
- Compare the constant term : $-5=3A-2C \Rightarrow C=1$.

Thus the integral can be computed as

$$\int \frac{x^2-5x-5}{(x-2)(x^2+2x+3)}\,dx = -\int \frac{1}{x-2}\,dx + \int \frac{2x+1}{x^2+2x+3}\,dx$$
$$= -\ln|x-2| + \int \frac{2x+2}{x^2+2x+3}\,dx - \int \frac{1}{x^2+2x+3}\,dx + C$$

$$\overset{\substack{y=x^2+2x+3 \\ \frac{dy}{dx}=2x+2}}{\cong} \quad -\ln|x-2| + \int \frac{1}{y}\,dy - \int \frac{1}{x^2+2x+3}\,dx + C$$

$$= -\ln|x-2| + \ln|y| - \int \frac{1}{(x+1)^2+2}\,dx + C$$

$$= -\ln|x-2| + \ln|x^2+2x+3| - \frac{1}{2}\int \frac{1}{\left(\frac{x+1}{\sqrt{2}}\right)^2+1}\,dx + C$$

$$= -\ln|x-2| + \ln|x^2+2x+3| - \frac{\sqrt{2}}{2}\tan^{-1}\frac{x+1}{\sqrt{2}} + C.$$

(h) Similar to (c), the integrand is again an improper rational function, one has to use *long division* and rewrite the integrand as the sum of a polynomial and a proper rational function:

$$\int \frac{x^5+2x^4-x+2}{x^3+2x^2-x-2}\,dx = \int \frac{(x^2+1)(x^3+2x^2-x-2)+4}{x^3+2x^2-x-2}\,dx$$

$$= \int (x^2+1)dx + \int \frac{4}{x^3+2x^2-x-2}\,dx = \frac{x^3}{3} + x + \int \frac{4}{x^3+2x^2-x-2}\,dx \ldots (*)$$

Note that by synthetic division $x^3+2x^2-x-2 = (x-1)(x+1)(x+2)$, we can decompose the integrand as follows:

$$\frac{4}{x^3+2x^2-x-2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2}$$

$$\Rightarrow 4 = A(x+1)(x+2) + B(x-1)(x+2) + C(x-1)(x+1).$$

- Put $x=-1$, $4 = B(-2)(1) \Rightarrow B = -2$.
- Put $x=-2$, $4 = C(-3)(-1) \Rightarrow C = \frac{4}{3}$.
- Put $x=1$, $4 = 6A \Rightarrow A = \frac{2}{3}$.

Therefore the second integral of $(*)$ can be computed as

$$\int \frac{4}{x^3+2x^2-x-2}\,dx = \frac{2}{3}\int \frac{1}{x-1}\,dx - 2\int \frac{1}{x+1}\,dx + \frac{4}{3}\int \frac{1}{x+2}\,dx$$

$$= \frac{2}{3}\ln|x-1| - 2\ln|x+1| + \frac{4}{3}\ln|x+2| + C.$$

From $(*)$, we conclude that

$$\int \frac{x^5+2x^4-x+2}{x^3+2x^2-x-2}\,dx = \frac{x^3}{3} + x + \frac{2}{3}\ln|x-1| - 2\ln|x+1| + \frac{4}{3}\ln|x+2| + C.$$

(i) Since there is a repeated factor x^3 in the denominator, we shall decompose the integrand as follows:

$$\frac{2x^2-x+1}{x^3(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1}$$

$$\Rightarrow 2x^2-x+1 = Ax^2(x-1) + Bx(x-1) + C(x-1) + Dx^3.$$

- Put $x=1$, $2 = D$.
- Put $x=0$, $1 = -C \Rightarrow C = -1$.

$$\Rightarrow Ax^2(x-1) + Bx(x-1) = 2x^2-x+1+(x-1)-2x^3 = -2x^3+2x^2 = -2x^2(x-1)$$

$$\Rightarrow Ax + B = -2x \Rightarrow A = -2, \qquad B = 0$$

Hence, the integral can be computed as

$$\int \frac{2x-x+1}{x^3(x-1)}\,dx = -2\int \frac{1}{x}\,dx - \int \frac{1}{x^3}\,dx + 2\int \frac{1}{x-1}\,dx = -2\ln|x| + \frac{1}{2x^2} + 2\ln|x-1| + C.$$

(j) There are two quadratic factors in the denominator, we decompose the integrand as

$$\frac{3x^3-2x-20}{(x^2+3)(2x^2-6x+5)} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{2x^2-6x+5}$$

Method I (Compare coefficients):

$\Rightarrow 3x^3-2x-20=(Ax+B)(\,2x^2-6x+5)+(Cx+D)(x^2+3) \qquad (*)$

$\qquad\qquad = (2A+C)x^3+(-6A+2B+D)x^2+(5A-6B+3C)x+5B+3D$

- Compare the coefficient of x^3 : $2A+C=3 \cdots\cdots\cdots (1)$
- Compare the coefficient of x^2 : $-6A+2B+D=0 \cdots\cdots\cdots (2)$
- Compare the coefficient of x : $5A-6B+3C=-2 \cdots\cdots\cdots (3)$
- Compare the constant term : $5B+3D=-20 \cdots\cdots\cdots (4)$

3(2)--(4) : $-18A+B=20 \cdots\cdots\cdots (5)$

3(1)—(3) : $A+6B=11 \cdots\cdots\cdots (6)$

(6)—6(5) : $109A=-109 \Rightarrow A=-1 \Rightarrow B=2$ from (5).

From (1) $C=5$ while from (4) $D=-10$.

Method II (Substitution using complex numbers):

- Put $x=\sqrt{3}i$ into (*), we have $-9\sqrt{3}i-2\sqrt{3}i-20=(A\sqrt{3}i+B)(-6-6\sqrt{3}i+5)$

  $\Rightarrow -20-11\sqrt{3}i=\left(A\sqrt{3}i+B\right)\left(-1-6\sqrt{3}i\right)=(18A-B)-(A+6B)\sqrt{3}i$

  By comparison, we have: $18A-B=-20 \ \ldots\ldots (7)$ and $A+6B=11 \ \ldots\ldots (8)$

  $6(7)+(8)$: $109A=-109 \ \Rightarrow\ A=-1 \ \Rightarrow\ B=2$ from (7)

- Put $x=\frac{3}{2}+\frac{1}{2}i$ into (*), we have

$$3\left(\frac{9}{4}+\frac{13}{4}i\right)-2\left(\frac{3}{2}+\frac{1}{2}i\right)-20=\left[\left(\frac{3}{2}+\frac{1}{2}i\right)C+D\right]\left[\left(2+\frac{3}{2}i\right)+3\right]=\left[\left(\frac{3}{2}C+D\right)+\frac{1}{2}Ci\right]\left(5+\frac{3}{2}i\right)$$

$$\Rightarrow -\frac{65}{4}+\frac{35}{4}i=\left(\frac{15}{2}C+5D-\frac{3}{4}C\right)+\left(\frac{9}{4}C+\frac{3}{2}D+\frac{5}{2}C\right)i=\left(\frac{27}{4}C+5D\right)+\left(\frac{19}{4}C+\frac{3}{2}D\right)i$$

By comparison, we have $\frac{27}{4}C+5D=-\frac{65}{4} \ \ldots\ldots (9)$ and $\frac{19}{4}C+\frac{3}{2}D=\frac{35}{4} \ \ldots\ldots (10)$

$\frac{3}{2}(9)-5(10)$: $-\frac{109}{8}C=-\frac{545}{8} \ \Rightarrow\ C=5$. Thus from (9), $D=\frac{-\frac{65}{4}-\frac{135}{4}}{5}=-\frac{200}{20}=-10.$

Then the integral can be computed as

$$\int\frac{3x^3-2x-20}{(x^2+3)(2x^2-6x+5)}dx=\int\frac{-x+2}{x^2+3}dx+\int\frac{5x-10}{2x^2-6x+5}dx$$

$$=-\int\frac{x}{x^2+3}dx+2\int\frac{1}{x^2+3}dx+\int\frac{5x-\frac{15}{2}}{2x^2-6x+5}dx-\frac{5}{2}\int\frac{1}{2x^2-6x+5}dx$$

$$\overset{\substack{y=x^2+3\Rightarrow\frac{dy}{dx}=2x \\ z=2x^2-6x+5\Rightarrow\frac{dz}{dx}=4x-6}}{=} -\frac{1}{2}\int\frac{1}{y}dy+\frac{2}{3}\int\frac{1}{\left(\frac{x}{\sqrt{3}}\right)^2+1}dx+\frac{5}{4}\int\frac{1}{z}dz-5\int\frac{1}{(2x-3)^2+1}dx$$

$$=-\frac{1}{2}\ln|y|+\frac{2\sqrt{3}}{3}\tan^{-1}\frac{x}{\sqrt{3}}+\frac{5}{4}\ln|z|-\frac{5}{2}\tan^{-1}(2x-3)+C$$

$$=-\frac{1}{2}\ln|x^2+3|+\frac{2\sqrt{3}}{3}\tan^{-1}\frac{x}{\sqrt{3}}+\frac{5}{4}\ln|2x^2-6x+5|-\frac{5}{2}\tan^{-1}(2x-3)+C.$$

(k) Note that there are one repeated factor and a quadratic factor in the denominator, we shall decompose the integrand as

$$\frac{6x^3-27x^2+5x-1}{(x-2)^2(4x^2+1)}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{Cx+D}{4x^2+1}$$

$$\Rightarrow 6x^3-27x^2+5x-1=A(x-2)(4x^2+1)+B(4x^2+1)+(Cx+D)(x-2)^2.$$

- Put $x=2,\ -51=17B \Rightarrow B=-3.$

$$\Rightarrow\ A(x-2)(4x^2+1)+(Cx+D)(x-2)^2=6x^3-27x^2+5x-1+3(4x^2+1)$$
$$=6x^3-15x^2+5x+2=(x-2)(6x^2-3x-1)\ \text{by synthetic division}$$
$$\Rightarrow A(4x^2+1)+(Cx+D)(x-2)=6x^2-3x-1$$

- Put $x=2,\ 17A=24-6-1=17 \Rightarrow A=1.$
- Compare the coefficient of x^2 : $4A+C=6 \Rightarrow C=6-4A=2.$
- Compare the constant term : $A-2D=-1 \Rightarrow D=1.$

Then the integral can be computed as

$$\int \frac{6x^3-27x^2+5x-1}{(x-2)^2(4x^2+1)}dx=\int \frac{1}{x-2}dx-3\int \frac{1}{(x-2)^2}dx+\int \frac{2x+1}{4x^2+1}dx$$
$$=\ln|x-2|+\frac{3}{x-2}+\int \frac{2x}{4x^2+1}dx+\int \frac{1}{4x^2+1}dx+C$$
$$\overset{\substack{y=4x^2+1\\ \Rightarrow\frac{dy}{dx}=8x}}{\cong}\ \ln|x-2|+\frac{3}{x-2}+\frac{1}{4}\int \frac{1}{y}dy+\int \frac{1}{(2x)^2+1}dx+C$$
$$=\ln|x-2|+\frac{3}{x-2}+\frac{1}{4}\ln|y|+\frac{1}{2}\tan^{-1}2x+C$$
$$=\ln|x-2|+\frac{3}{x-2}+\frac{1}{4}\ln|4x^2+1|+\frac{1}{2}\tan^{-1}2x+C.$$

(l) Note that there is a repeated quadratic factor in the denominator, one shall decompose the integrand

$$\frac{x^2+2x+4}{x(x^2+2)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+2}+\frac{Dx+E}{(x^2+2)^2}$$
$$\Rightarrow x^2+2x+4=A(x^2+2)^2+x(x^2+2)(Bx+C)+x(Dx+E).$$

- Put $x=0,\ 4A=4 \Rightarrow A=1.$

$$\Rightarrow x(x^2+2)(Bx+C)+x(Dx+E)=x^2+2x+4-(x^2+2)^2=-x^4-3x^2+2x$$
$$=-x(x^3+3x-2)$$
$$\Rightarrow (x^2+2)(Bx+C)+Dx+E=-(x^3+3x-2)=-x(x^2+2)-(x-2)\ \text{by long division}$$

By comparing the coefficients, we get $B=-1,\ C=0,\ D=-1$ and $E=2.$

Then the integral can be computed as

$$\int \frac{x^2+2x+4}{x(x^2+2)^2}dx=\int \frac{1}{x}dx-\int \frac{x}{x^2+2}dx-\int \frac{x-2}{(x^2+2)^2}dx$$
$$=\ln|x|-\int \frac{x}{x^2+2}dx-\int \frac{x}{(x^2+2)^2}dx+2\int \frac{1}{(x^2+2)^2}dx+C$$
$$\overset{\substack{y=x^2+2\ \text{(for 1st and 2nd integrals)}\\ x=\sqrt{2}\tan\theta\ \text{(for 3rd integral)}}}{\cong}\ \ln|x|-\frac{1}{2}\int \frac{1}{y}dy-\frac{1}{2}\int \frac{1}{y^2}dy+\frac{\sqrt{2}}{2}\int \frac{1}{\sec^2\theta}d\theta$$

$$= \ln|x| - \frac{1}{2}\ln|y| + \frac{1}{2y} + \frac{\sqrt{2}}{2}\int \cos^2\theta\, d\theta = \ln|x| - \frac{1}{2}\ln|x^2+2| + \frac{1}{2(x^2+2)} + \frac{\sqrt{2}}{4}\int (\cos 2\theta + 1) d\theta$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+2| + \frac{1}{2(x^2+2)} + \frac{\sqrt{2}}{8}\sin 2\theta + \frac{\sqrt{2}}{4}\theta + C$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+2| + \frac{1}{2(x^2+2)} + \frac{\sqrt{2}}{4}\sin\theta\cos\theta + \frac{\sqrt{2}}{4}\tan^{-1}\frac{x}{\sqrt{2}} +$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+2| + \frac{1}{2(x^2+2)} + \frac{\sqrt{2}}{4}\frac{x}{\sqrt{x^2+2}}\frac{\sqrt{2}}{\sqrt{x^2+2}} + \frac{\sqrt{2}}{4}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+2| + \frac{x+1}{2(x^2+2)} + \frac{\sqrt{2}}{4}\tan^{-1}\frac{x}{\sqrt{2}} + C$$