# Multiple Comparison Tests

# Concepts

Parametric Test

Test hypothesis of the form

$H_0$: $\mu_1 = \mu_2 = \cdots = \mu_m$
$H_1$: at least one of the $\mu_i$ is different from the rest

Tested by ANOVA (similar to the more than two population t-test)

# Non-parametric test

Null hypothesis $F_1 = \cdots = F_m$

Alternative hypothesis At least one of the $F_i$ differs from the others

Tested by Kruskal-Wallis test (similar to the more than two population Mann-Whitney U test)

# One-way analysis of variance (ANOVA)

$m$ independent random samples, each of size $n$

The members of the ith sample - $X_{i1}, X_{i2}, \dots, X_{in}$ are normal random variables with unknown mean $\mu_i$ and unknown variance $\sigma^2$, i.e., $X_{ij} \sim N(\mu_i, \sigma^2)$

$H_0$: $\mu_1 = \mu_2 = \cdots = \mu_m$
$H_1$: at least one of the $\mu_i$ is different from the rest

Note that the test makes the simplifying assumption that the variance $\sigma^2$ is the same for all the $m$ independent samples

Define

$$X_{i.} = \sum_{j=1}^{n} X_{ij}/n \qquad X_{..} = \frac{\sum_{i=1}^{m} X_{i.}}{m}$$

$$SS_W = \sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij} - X_{i.})^2 \qquad SS_b = n\sum_{i=1}^{m}(X_{i.} - X_{..})^2$$

It can be shown that the test statistic

$$TS = \frac{SS_b/(m-1)}{SS_W/(nm-m)}$$

follows the $F$-distribution

## If $TS$ is sufficiently large, then $H_0$ is rejected

reject $H_0$ if $TS > F_{m-1,nm-m,\alpha}$

accept $H_0$ if otherwise

[A more thorough mathematical description can be found in text pg. 448-452.]

## Example

An auto rental firm is using 15 identical motors that are adjusted to run at a fixed speed to test 3 different brands of gasoline. Each brand of gasoline is assigned to exactly 5 of the motors. Each motor runs on 10 gallons of gasoline until it is out of fuel. The following represents the total mileage obtained by the different motors:

Gas 1: 220 251 226 246 260

Gas 2: 244 235 232 242 225

Gas 3: 252 272 250 238 256

Test the hypothesis that the average mileage obtained in not affected by the type of gas used. Use 5% level of significance

$m = 3$ (three groups to test) $n = 5$ (each group has 5 data)

| Gas | Mileage | | | | | $\sum_j X_{ij}$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 31 | 6 | 26 | 40 | 103 |
| 2 | 24 | 15 | 12 | 22 | 5 | 78 |
| 3 | 32 | 52 | 30 | 18 | 36 | 168 |

$$X_{1.} = \frac{103}{5} = 20.6 \qquad X_{2.} = \frac{78}{5} = 15.6 \qquad X_{3.} = \frac{168}{5} = 33.6$$

$$X_{..} = \frac{20.6 + 15.6 + 33.6}{3} = 23.2667$$

$$TS = 2.60$$

From the F-distribution table (Table A4, pg. 651 of Text)

$F_{2,12,0.05} = 3.89$

Since

$TS < F_{2,12,0.05}$

we cannot reject the hypothesis that the gasoline give equal mileage

# Notes on usage

- In general, the ANOVA procedure allows the sample sizes $n_1, \ldots, n_m$ of the $m$ samples to be different
- If the data is paired, then two-way ANOVA should be used

# Kruskal-Wallis test

$m$ independent random samples, of sample sizes $n_1, \ldots, n_m$

The K-W test is a non-parametric test. It does not assume the form of the $m$ distributions (i.e., unlike ANOVA, it does not requires the $m$ distributions to be normal, nor they have the same variance)

Let the $m$ distributions be $F_1, \ldots, F_m$

| | |
|---|---|
| Null hypothesis | $F_1 = \cdots = F_m$ |
| Alternative hypothesis | At least one of the $F_i$ differs from the others |

## In this test, we only consider the ranks of the data

The $n = n_1 + \cdots + n_m$ data are first ranked. Then the test computes a statistic called $H$ statistic

$$H = \frac{12V}{n(n+1)}$$

$$V = \sum_{i=1}^{m} n_i(\bar{R}_i - \bar{R})^2$$

$\bar{R}_i$ is the average rank of the ith sample.

$\bar{R} = (n+1)/2$ is the average rank of the $n$ data

If the null hypothesis is true, then the $\bar{R}_i$ are approximately equal. Hence $V$ will be small

Kruskal and Wallis show that if the $n_i$ values are "large" (rule of thumb is $n_i \geq 5$), $H$ can be approximated by a $\chi^2$ distribution with $m - 1$ d.f.

If $H$ is sufficiently large, then $H_0$ is rejected

reject $H_0$ if $H > \chi^2_{\alpha,m-1}$

accept $H_0$ if otherwise

## Example

In the previous example,

Gas 1: 220 251 226 246 260

Gas 2: 244 235 232 242 225

Gas 3: 252 272 250 238 256

Assume that the form of the distributions is unknown.

Test the hypothesis that the average mileage obtained in not affected by the type of gas used. Use 5% level of significance

| | Gas | Rank |
|---|---|---|
| 220 | 1 | 1 |
| 225 | 2 | 2 |
| 226 | 1 | 3 |
| 232 | 2 | 4 |
| 235 | 2 | 5 |
| 238 | 3 | 6 |
| 242 | 2 | 7 |
| 244 | 2 | 8 |
| 246 | 1 | 9 |
| 250 | 3 | 10 |
| 251 | 1 | 11 |
| 252 | 3 | 12 |
| 256 | 3 | 13 |
| 260 | 1 | 14 |
| 272 | 3 | 15 |

| Gas 1 | | Gas 2 | | Gas 3 | |
|---|---|---|---|---|---|
| Amount | Rank | Amount | Rank | Amount | Rank |
| 220 | 1 | 244 | 8 | 252 | 12 |
| 251 | 11 | 235 | 5 | 272 | 15 |
| 226 | 3 | 232 | 4 | 250 | 10 |
| 246 | 9 | 242 | 7 | 238 | 6 |
| 260 | 14 | 225 | 2 | 256 | 13 |

$$n = 15 \qquad m = 3$$

$$\bar{R}_1 = \frac{1 + 11 + 3 + 9 + 14}{5} = 7.6 \qquad \bar{R}_2 = 5.2 \qquad \bar{R}_3 = 11.2$$

$$\bar{R} = 8$$

$$V = 5(7.6 - 8)^2 + 5(5.2 - 8)^2 + 5(11.2 - 8)^2 = 91.2$$

$$H = 4.56$$

From the Chi-square distribution table (Table A2, pg. 649 of text)

$$\chi^2_{0.05,2} = 5.991$$

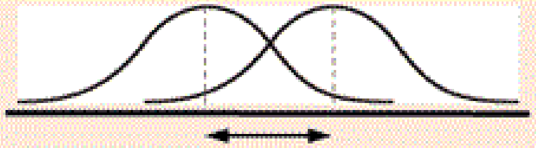
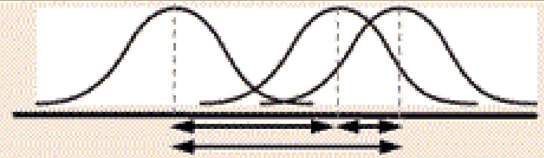
Since

$$H < \chi^2_{0.05,2}$$

we cannot reject the hypothesis that the gasoline give equal mileage

# Notes on usage

- If the data is paired, then Friedman test should be used

| | | 2 groups | $n$ groups ($n > 2$) | |
|---|---|---|---|---|
| data distribution | |  |  | |
| Parametric Test (normality) | unpaired (independent) | ・unpaired $t$-test | ANOVA (Analysis of Variance) | ・one-way ANOVA |
| | paired (related) | ・paired $t$-test | | ・two-way ANOVA |
| Non-parametric Test (no normality) | unpaired (independent) | ・Mann-Whitney $U$-test | one-way data | ・Kruskal-Wallis test |
| | paired (related) | ・sign test<br>・Wilcoxon signed-ranks test | two-way data | ・Friedman test |

http://www.design.kyushu-u.ac.jp/~takagi/
(Downloadable files → Statistical Files)

# References

Text Ch. 10

Kruskal-Wallis test:

Pg. 765-771, D. Wackerly, W. Mendenhall, and R.L. Scheaffer, *Mathematical statistics with applications*, 7th Ed., 2008.