1 Directional Derivatives

1.1 Recall Partial Derivatives:

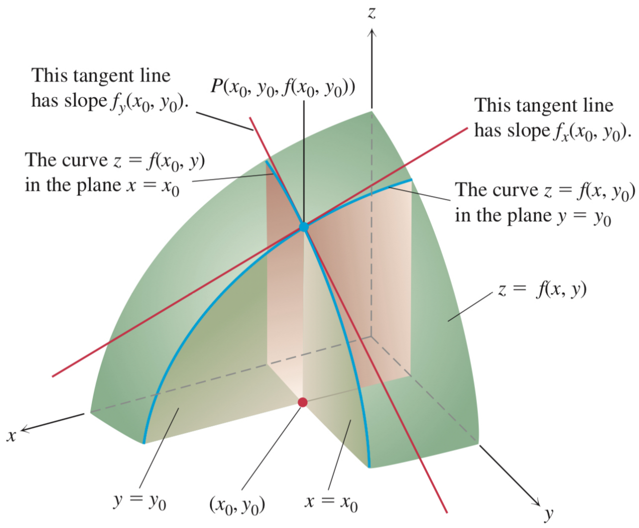


FIGURE 14.18 Figures 14.16 and 14.17 combined. The tangent lines at the point $(x_0, y_0, f(x_0, y_0))$ determine a plane that, in this picture at least, appears to be tangent to the surface.

For $z = f(x, y)$, we have

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}$$

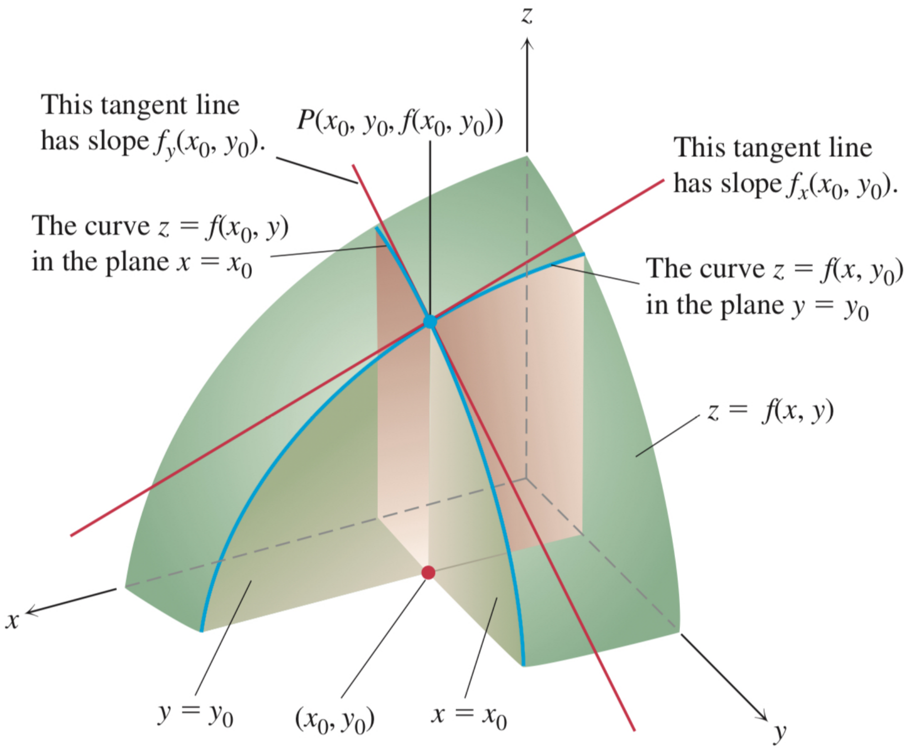


FIGURE 14.18 Figures 14.16 and 14.17 combined. The tangent lines at the point $(x_0, y_0, f(x_0, y_0))$ determine a plane that, in this picture at least, appears to be tangent to the surface.

1.2 Definition of Directional Derivatives:

Definition: For $z = f(x, y)$, given a unit vector $\vec{u} = (u_1, u_2)$,

Remarks:

(1)

$$\frac{\partial f}{\partial x}(a, b) = D_{(1,0)} f(a, b), \qquad \frac{\partial f}{\partial y}(a, b) = D_{(0,1)} f(a, b).$$

(2) For any $\vec{u} \neq \vec{0}$,

$$D_{\vec{u}} f(a, b) = \lim_{s \to 0} \frac{f\left((a, b) + s\left(u_1, u_1\right)\right) - f(a, b)}{s\sqrt{u_1^2 + u_2^2}}.$$

1.3 Geometrical Interpretation of $D_{\vec{u}}f$:

Given $z = f(x, y)$ and a unit vector $\vec{u} \neq \vec{0}$, let

$$g(t) = f\left((x_0, y_0) + t\vec{u}\right) \quad \text{for} \quad t \in [-1, 1],$$

then g forms a curve on the surface of f passing through $(x_0, y_0, f(x_0, y_0))$.

$$\begin{aligned} D_{\vec{u}}(f)\,(x_0, y_0) &= \lim_{t \to 0} \frac{f\left((x_0, y_0) + t\vec{u}\right) - f\left(x_0, y_0\right)}{t} \\ &= \lim_{t \to 0} \frac{g(t) - g(0)}{t} \\ &= g'(0), \end{aligned}$$

Therefore, $D_{\vec{u}}(f)\,(x_0, y_0)$ is the slope of the tangent line of the curve $f\left((x_0, y_0) + t\vec{u}\right)$ at $(x_0, y_0, f(x_0, y_0))$.

Example. Given $f(x, y) = x^2 + y^2$, $\vec{u} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Find $D_{\vec{u}} f(1, 1)$.

Example. Given $f(x, y) = x^2 - y^2$, $\vec{u} = (3, 4)$. Find $D_{\vec{u}} f(1, 1)$.

Computation Remark: If f is partial differentiable at (a, b), then for any vector $\vec{v} = (v_1, v_2)$, usually

$$D_{\vec{u}} f(a, b) = (f_x(a, b), f_y(a, b)) \cdot \frac{\vec{v}}{\|\vec{v}\|}.$$

Particularly, if $\vec{v}$ is a unit vector, i.e. $\|\vec{v}\|$=1, then

$$\begin{aligned} D_{\vec{v}} f(a, b) &= (f_x(a, b), f_y(a, b)) \cdot (v_1, v_2) \\ &= v_1 f_x(a, b) + v_2 f_y(a, b) \\ &= \nabla f(a, b) \cdot \vec{v}, \end{aligned}$$

where we define ∇f (or denoted by grad f)

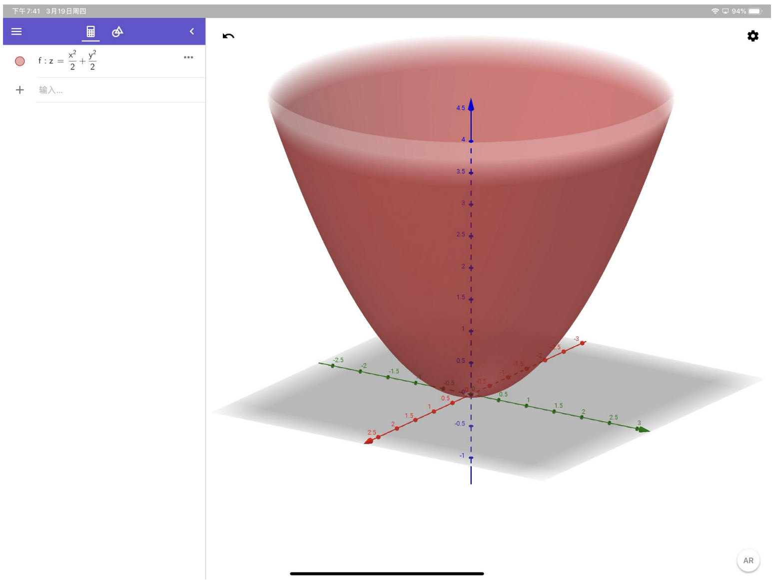
$$\nabla f = (f_x, f_y)$$

to be the gradient vector of f.

Example. Given $f(x, y) = x^2 + y^2$, $\vec{u} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Find $D_{\vec{u}} f(1, 1)$.

Example $f(x, y) = xe^y + \cos(xy)$. Find directional derivative of f at $P(2, 0)$ along $\vec{u} = 3\vec{i} - 4\vec{j}$.

1.4 Directions along which f changes most rapidly or unchanges



Example Find the direction in which $f(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$

(1) increases most rapidly at $(2, 1)$.

(2) decreases most rapidly at $(2, 1)$.

(3) what are the direction of zera change of f at $(1, 1)$?

Example Find the direction in which $f(x,y) = xe^y + \cos(xy)$

(1) increases most rapidly at $(2,0)$.

(2) decreases most rapidly at $(2,0)$.

(3) what are the direction of zera change of f at $(2,0)$?

2 Differentiable

Definition $z = f(x, y)$ is differentiable at point (x_0, y_0) if

Remarks:

- If f is differentiable, then f is directional differentiable along any direction.
- If f is differentiable at P, then f is continuous at P.
- Directional differentiability can not guarantee continuity.

Example Let

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(1) Show f is directional differentiable at $(0,0)$ along any direction. Say $\vec{u} = (a,b)$ with $a^2 + b^2 = 1$.

(2) Show f is not differentiable.

(3) Check $\nabla f(0,0) \cdot \vec{u}$ and $D_{\vec{u}}(0,0)$.