3 Fourier Series

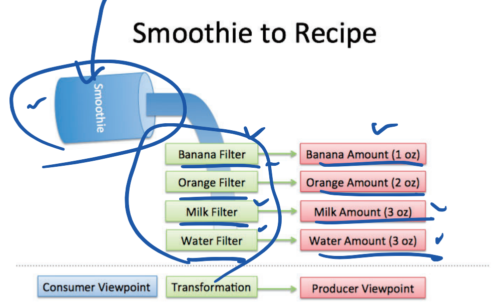
Major References:

- *Chapter 3, Signals and Systems by Alan V. Oppenheim et. al., 2nd edition, Prentice Hall*
- *Chapter 5.2 & 6.2, Schaum's Outline of Signals and Systems, 2nd Edition, 2010, McGraw-Hill*

3.1 Introduction

Signal / or Vector

Jean-Baptiste Joseph Fourier (1768-1830)



1st Metaphor of the Fourier Analysis (Source: https://betterexplained.com)

What is Fourier Analysis?

1. **What does the Fourier Transform do?**
   Given a smoothie, it finds the recipe.
2. **How?**
   Run the smoothie through filters to extract each ingredient.
3. **Why?** Recipes are easier to analyze, compare, and modify than the smoothie itself.
4. **How do we get the smoothie back?**
   Blend the ingredients.

Important Points to consider

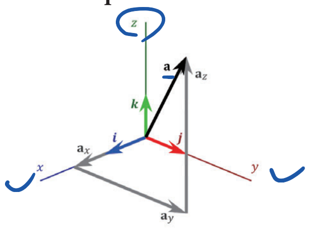
1. **Filters must be independent**
2. **Filters must be complete**
3. **Ingredients must be combineable.**
   The ingredients must make the same result when separated and combined in any order.

Signal → filter

A / base signal A
B / base signal B
C / base signal C
D / base signal D

2nd Metaphor of the Fourier Analysis

$x(t) * y(t)$



- Arbitrary vector can be expressed via unit vectors and the magnitude toward each unit vector.
- Can we break a function into it simple functions (referred to as base functions) just like vector case?
- Can we combine the base functions to represent arbitrary signals?

$$\mathbf{a} = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z$$

$$\Rightarrow \boxed{x(t) = \sum_{n=-\infty}^{\infty} a_n \psi_n(t),}$$ where $\psi_n(t)$ is the base function.

$\boxed{a_n = \text{filtering result to } \psi_n(t)}$

$\vec{e}_x, \vec{e}_y, \vec{e}_z$

$\|\vec{e}_x\| = 1$

$\|\vec{e}_y\| = 1$

$\|\vec{e}_z\| = 1$

$\vec{a} \cdot \vec{e}_x = a_x$

$\vec{a} \cdot \vec{e}_y = a_y$

$\vec{a} \cdot \vec{e}_z = a_z$

similar to filtering

Fourier Analysis

periodic

- Fourier Series (FS) < CT, DT

Aperiodic

- Fourier Transform (FT) < CT, DT

	Periodic	Non-periodic
CT	CT-FS	CT-FT
DT	DT-FS	DT-FT

fundamental Period $T_0 = \frac{1}{f_0}$

fundamental frequency $f_0 = \frac{1}{T_0}$

fundamenal angular frequency $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$

$\rightarrow \boxed{\omega_0 T_0 = 2\pi}$

✓ CT - FS. ($x(t)$ is periodic signal with fundamental period T_0.)

base function $\psi_k(t) = e^{jk\omega_0 t}$

$$= \left(\exp\left(j\frac{2\pi}{T_0}t\right)\right)^k$$

$$\psi_k^*(t) = e^{-jk\omega_0 t}.$$

Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \psi_k(t)$$

$$= \sum_{k=-\infty}^{\infty} \underline{c_k} e^{j2\pi \cdot k f_0 t}.$$

Analysis ($x(t) \longrightarrow \{c_k\}$)

$\underline{[\alpha, \alpha+T_0]}$ α is arbitrary constant.

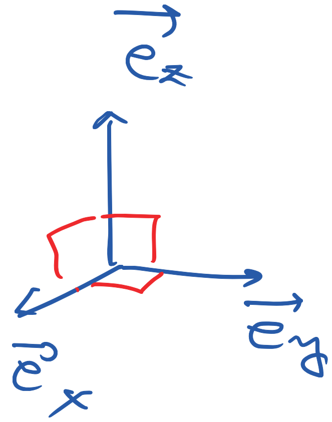
✓ $$c_k = \frac{1}{T_0}\int_{T_0} x(t)\, \psi_k^*(t)\, dt.$$

$$= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)\, e^{-j2\pi k f_0 t}\, dt.$$

$\Leftarrow [0, T_0]$ or $\left[-\frac{T_0}{2}, \frac{T_0}{2}\right]$

$\|\vec{e}_x\| = \|\vec{e}_y\| = \|\vec{e}_z\|$ (Normalized = Size is one.)

$= 1$ $\|\vec{e}_x\| = N$



$$\vec{e}_x \cdot \vec{e}_y = 0 = \vec{e}_y \cdot \vec{e}_z = \vec{e}_x \cdot \vec{e}_z$$

Orthogonal.

(= 90° across different unit vector)

$$\psi_k(t) = e^{jk\omega_0 t}$$

1. Orthogonal → if $k \neq m$

$$(\psi_k(t) \cdot \psi_m(t)) = 0$$

$$\Rightarrow \int_{\alpha}^{\alpha+T_0} \psi_k(t)\,\psi_m^*(t)\,dt = 0$$

$$(x(t) \cdot \psi_m(t)) = \int x(t)\,\psi_m^*(t)\,dt$$

$$\|\psi_k(t)\|^2 = (\psi_k(t) \cdot \psi_k(t))$$

$$= \int_{\alpha}^{\alpha+T_0} \psi_k(t)\,\psi_k^*(t)\,dt$$

$$= T_0$$

$$\left(\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}\right) \quad \left(\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}\right)$$

$$\int_{T_0} \psi_k(t)\,\psi_k^*(t)\,dt = T_0.$$

$$\delta_{k,m} = \begin{cases} 1, & \text{if } k=m \\ 0, & \text{if } k\neq m \end{cases}$$

Dirac delta Notation.

$$\int_{T_0} \psi_k(t)\,\psi_m^*(t)\,dt = 0 \quad \text{if } k \neq m$$

Proof)

$$\int_{T_0} \psi_k(t)\,\psi_m^*(t)\,dt$$

$$= \int_{T_0} e^{jk\omega_0 t} \cdot e^{-jm\omega_0 t}\,dt.$$

$$= \int_{\alpha}^{\alpha+T_0} e^{j(k-m)\omega_0 t}\,dt.$$

Case 1 $(k=m)$

$$\int_{\alpha}^{\alpha+T_0} \underbrace{e^{j\omega_0 t \cdot 0}}_{=1}\,dt$$

$$= \int_{T_0} 1 \cdot dt = T_0.$$

Case 2 $(k \neq m)$

Denote $\boxed{k-m=l}$ → integer.

$\boxed{(\omega_0 T_0 = 2\pi)}$

$$\int_{\alpha}^{\alpha+T_0} e^{jl\cdot\omega_0 t}\,dt$$

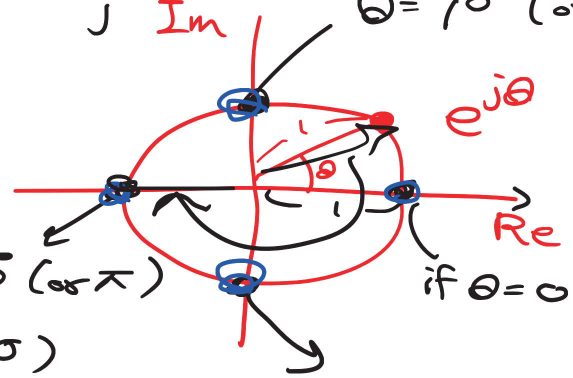
$$= \frac{1}{jl\omega_0}\, e^{jl\cdot\omega_0 t}\Big|_{t=\alpha}^{t=\alpha+T_0}$$

$$= \frac{1}{jl\omega_0}\,\boxed{e^{jl\omega_0\alpha}}\,(\underbrace{e^{jl\omega_0 T_0}}_{=1} - 1)$$

$$= \text{Constant} \cdot \phi = \phi.$$

$$e^{j\ell\omega_0 T_0} = e^{j\ell\cdot 2\pi} = \left(e^{j\cdot 2\pi}\right)^{\ell} = 1^{\ell} = 1$$

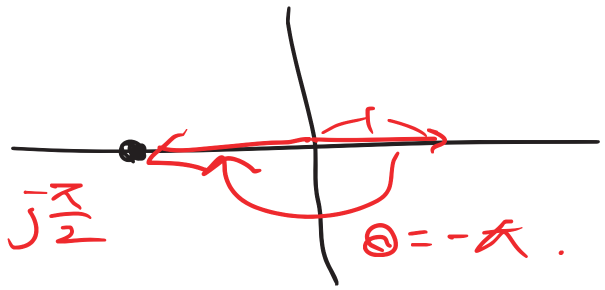
$$= \cos(2\pi\ell) + j\sin(2\pi\ell) = 1$$



$$e^{j\theta} \to e^{j\cdot 0} = e^{j\cdot 2\pi} = 1$$

$$\begin{cases} \boxed{e^{j\frac{\pi}{2}} = j} \\ e^{j\cdot\pi} = -1 = e^{-j\pi} \\ e^{j\frac{3\pi}{2}} = -j = e^{-j\frac{\pi}{2}} \end{cases}$$

$\theta = -\pi$



$$\boxed{\int_{T_0} \psi_k(t)\,\psi_m^*(t)\,dt = T_0\,\delta_{k,m} = \begin{cases} T_0 & \text{if } k = m \\ \phi & \text{if } k \neq m \end{cases}}$$

Assume

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} c_k\,\psi_k(t)}$$ is true. (1)

$$\int_{T_0} x(t)\,\psi_m^*(t)\,dt$$

Substitute ①

$$= \int_{T_0} \left(\sum_{k=-\infty}^{\infty} C_k \psi_k(t) \right) \psi_m^*(t)\, dt$$

$$= \sum_{k=-\infty}^{\infty} C_k \underbrace{\int_{T_0} \psi_k(t) \psi_m^*(t)\, dt}_{= T_0 \delta_{k,m}}$$

$$= T_0 \sum_{k=-\infty}^{\infty} C_k \delta_{k,m}$$

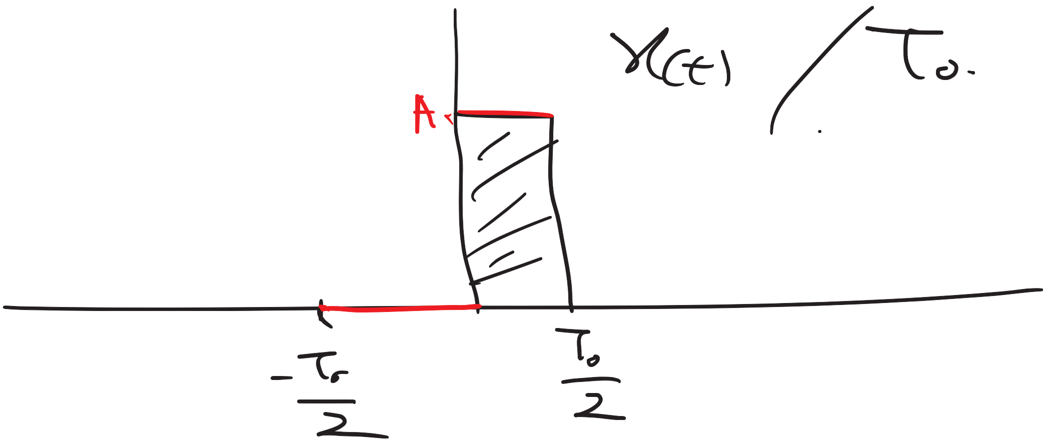
$$\delta_{k,m} = \begin{cases} 1, & \text{only if } k=m \\ 0 & \text{if } k \neq m \end{cases}$$

$$= T_0 C_m.$$

$$\boxed{C_k = \frac{1}{T_0} \int_{T_0} x(t) \psi_k^*(t)\, dt}$$

Ex 3-2)

a)



$$C_k = \frac{1}{T_0}\int_{T_0} x(t)\,\underbrace{e^{-jk\omega_0 t}}_{=1 \text{ if } k=0}\,dt$$

$$\boxed{C_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt} = \frac{1}{T_0}\cdot\frac{A\cdot T_0}{2} = \frac{A}{2}$$

$$C_k = \frac{1}{T_0}\left[\cancel{\int_{-\frac{T_0}{2}}^{0} x(t)e^{-jk\omega_0 t}dt}\ \int_0^{T_0/2} \underbrace{x(t)}_{=A} e^{-jk\omega_0 t}\,dt\right]$$

$\left[-\frac{T_0}{2}, \frac{T_0}{2}\right]$ as the integration interval.

$$\Rightarrow \frac{A}{T_0}\int_0^{T_0/2} e^{-jk\omega_0 t}\,dt$$

$$\boxed{(-t \rightarrow t')}\ (dt \rightarrow -dt')$$

$$\Rightarrow \frac{A}{T_0}\int_{-\frac{T_0}{2}}^{0} e^{jk\omega_0 t'}\,dt'$$

$$\Rightarrow \frac{A}{T_0}\cdot\frac{1}{jk\omega_0}\cdot e^{jk\omega_0 t'}\Big|_{-T_0/2}^{0}$$

$$\Rightarrow \frac{A}{jk\boxed{\omega_0 T_0}}\left(1 - e^{-jk\boxed{\omega_0 T_0}/2}\right)$$

$$\omega_0 T_0 = 2\pi$$

$$\Rightarrow \frac{A}{j2\pi k}\left(1 - \underbrace{e^{-jk\pi}}_{\substack{\parallel \\ (e^{-j\pi})^{k}}}\right)$$

$$\Rightarrow \frac{A}{j\pi k}\left(1 - (-1)^{k}\right)$$

$$1-(-1) = 2$$

if k = even number. $= 2m$

$\Rightarrow C_k = 0.$

if k = odd number $= 2m+1$

$$\Rightarrow C_k = \frac{A}{j\pi k} = \frac{A}{j\pi(2m+1)}$$

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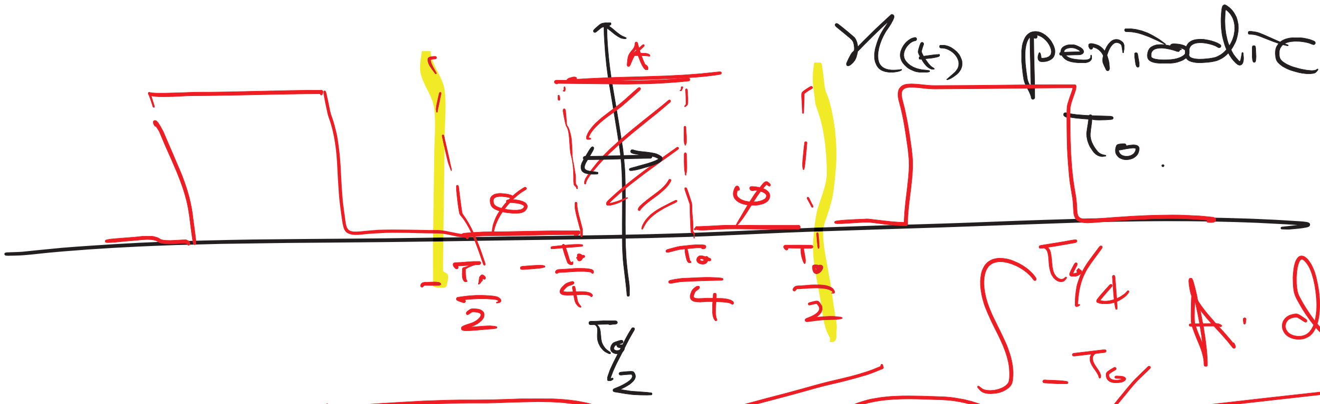
b) Case (b)



$$\int_{-T_0/4}^{T_0/4} A\cdot dt$$

$$c_0 = \frac{1}{T_0}\boxed{\int_{T_0} x(t)\,dt} = \frac{1}{T_0}\frac{A T_0}{2} = \frac{A}{2}$$

$$c_k = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} \underbrace{x(t)}_{A}\, e^{-jk\omega_0 t}\,dt$$

$$= \frac{A}{T_0}\frac{1}{(jk\omega_0)} e^{-jk\omega_0 t}\Big|_{t\,T_0/4}^{T_0/4}$$

$$\Rightarrow \frac{A}{jk\cdot 2\pi}\left(e^{jk\frac{\pi}{2}} - e^{-jk\frac{\pi}{2}}\right)$$

$$\underbrace{\left(e^{j\frac{\pi}{2}}\right)^k}_{j^k} \qquad \underbrace{\left(e^{-j\frac{\pi}{2}}\right)^k}_{(-j)^k}$$

$$\Rightarrow \frac{A}{jk\cdot 2\pi}\left(j^k - (-j)^k\right)$$

$$c_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt$$

$$= \frac{1}{T_0} \left[\cancel{\int_{-T_0/2}^{-T_0/4} \underbrace{x(t)}_{0} dt} + \int_{-T_0/4}^{T_0/4} \underbrace{x(t)}_{A} dt + \cancel{\int_{T_0/4}^{T_0/2} \underbrace{x(t)}_{\phi} dt} \right]$$

$$= \frac{A}{T_0} \int_{-T_0/4}^{T_0/4} 1 \cdot dt$$

$$\frac{A}{T_0} \cdot \frac{T_0}{2}$$

1. Fourier analysis is the study of the way general functions may be represented or approximated by sums of simpler trigonometric functions.
   - Analysis: breaking up a signal into simpler constituent parts
   - Synthesis: reassembling a signal from its constituent parts

   $\Rightarrow$ **Fourier analysis is all about breaking & reassembling a function**
2. Fourier Series: fourier analysis for periodic signals
3. Fourier Transform: fourier analysis for non-periodic signals

	Periodic signal	Aperiodic Signal
Continuous Time	Fourier Series (FS)	Fourier Transform (FT)
Discrete Time	Discrete time FS	Discrete time FT

3.2 Continuous Time Fourier Series

For a periodic signal $x(t)$ with fundamental period T_0, we adopt sinusoidal signals as the base function

$$\left\{\begin{array}{l} \text{fundamental period:} T_0, \quad \text{fundamental frequency:} f_0 = \frac{1}{T_0}, \\ \text{fundamental angular frequency:} w_0 = 2\pi f_0 = \frac{2\pi}{T_0} \end{array}\right\}$$

Then the CT-Fourier series can be expressed into the following two representations. All of the proof for Chapter 3.2 are summarized at the end of the section.

1. Fourier Series (Complex Exponential Series Form)

The base function for this form is $\boxed{\psi_k(t) = e^{jkw_0 t} = e^{j2\pi k f_0 t}}$

C_k

1. Synthesis:

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} c_k \psi_k(t)} = \sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t} \tag{3.1}$$

2. Analysis

$x(t) \to C_k$.

$$\boxed{c_k = \frac{1}{T_0} \int_{\mathbf{T}_0} x(t) e^{-jkw_0 t} dt = \frac{1}{T_0} \int_{\mathbf{T}_0} x(t) \psi_k^*(t) dt,} \tag{3.2}$$

where the integration interval $\mathbf{T}_0$ is any period with length T_0, *e.g.*, $[0, T_0]$ or $\left[-\frac{T_0}{2}, \frac{T_0}{2}\right]$

[Properties]

1. The set of base functions $\{\psi_k(t)\}$ is orthogonal on any interval over a period T_0, $(\alpha, \alpha + T_0)$

$$\int_{\alpha}^{\alpha+T_0} \psi_m(t) \psi_k^*(t) dt = \begin{cases} 0, & m \neq k \\ T_0, & m = k \end{cases} \tag{3.3}$$

2. If $x(t)$ is a real funcction, then $c_{-k} = c_k^*$

By using the Euler's Formula, $e^{jkw_0 t} = \cos(kw_0 t) + \sin(kw_0 t)$, the Fourier series in the complex exponential series form can be converted to a trigonometric series form as follows.

2. Fourier Series (Trigonometric Series Form)

1. Synthesis

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(kw_0 t) + b_k \sin(kw_0 t)\right) \tag{3.4}$$

2. Analysis

$$a_k = \frac{2}{T_0} \int_{\mathrm{T}_0} x(t) \cos(kw_0 t)\, dt, \quad b_k = \frac{2}{T_0} \int_{\mathrm{T}_0} x(t) \sin(kw_0 t)\, dt \tag{3.5}$$

[Properties]

1. The convertion between two representations

$$\begin{cases} \dfrac{a_0}{2} = c_0, \\ a_k = c_k + c_{-k}, \quad b_k = j\left(c_k - c_{-k}\right) \end{cases} \Leftrightarrow \begin{cases} c_k = \dfrac{1}{2}\left(a_k - jb_k\right), \\ c_{-k} = \dfrac{1}{2}\left(a_k + jb_k\right), \end{cases} \tag{3.6}$$

2. If $x(t)$ is a real funcction, then $a_k = 2\,\mathrm{Re}\,[c_k]$, $b_k = -2\,\mathrm{Im}\,[c_k]$.

A periodic signal $x(t)$ has a Fourier series representation if it satisfies the Dirichlet conditions. In other words, Dirichlet conditions are the sufficient conditions (but not necessary condition) for the Fourier series to converge.

Peter Gustav Lejeune Dirichlet (1805-1859)

3. Dirichlet Condition (Sufficient conditions for FS to exist)

1. $x(t)$ is absolutely integrable over any period $\int_{\mathrm{T}_0} |x(t)| dt < \infty$
2. $x(t)$ has a finite number of maxima and minima within any finite interval of t.
3. $x(t)$ has a finite number of discontinuities within any finite interval of t, and each of these discontinuities is finite.

If $x(t)$ satisfies the Dirichlet condition, then the corresponding Fourier series is convergent and its sum is $x(t)$, except at any point t_0 at which $x(t)$ is discontinuous.

$$x(t_0) = \frac{1}{2}\left[x\left(t_0^+\right) + x\left(t_0^-\right)\right]$$