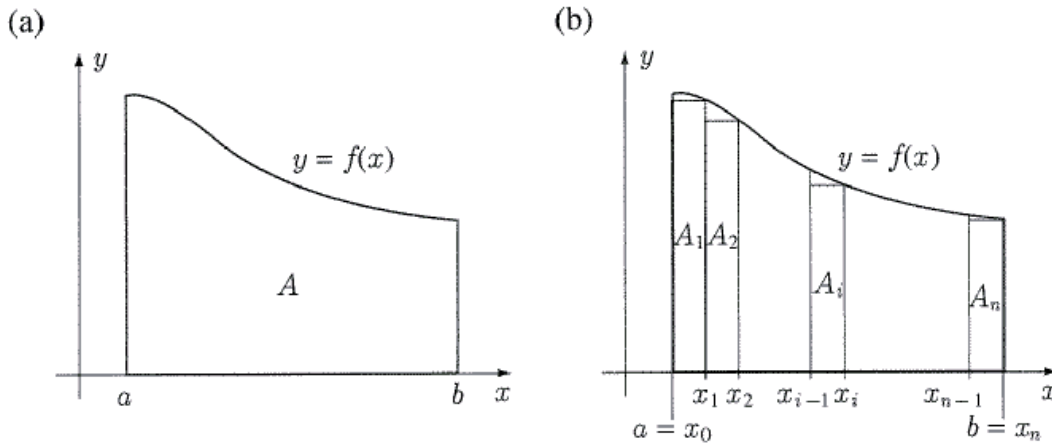


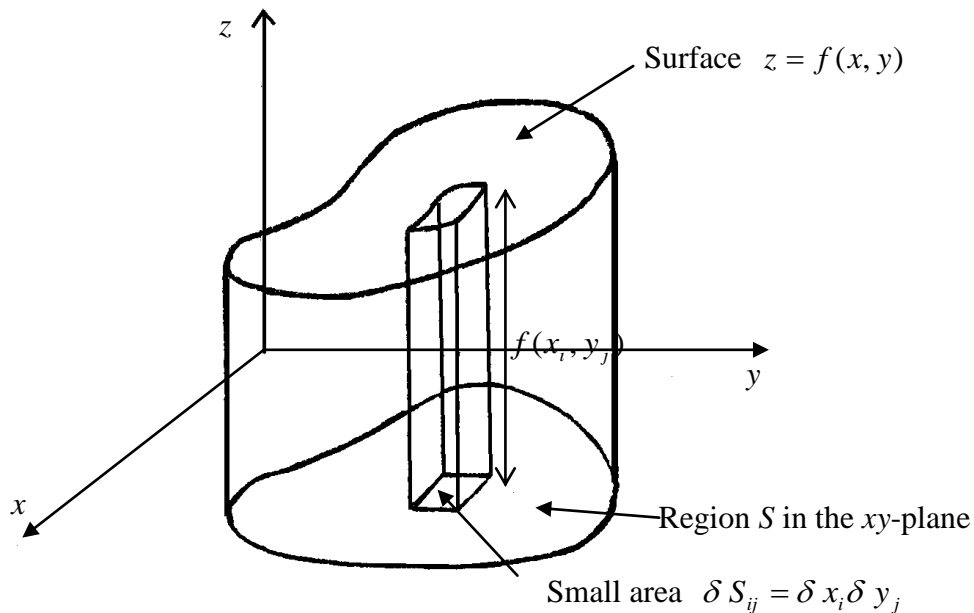
MA2001 Multiple Integrals

1 Double Integrals

For a function $f(x)$ of one variable defined over $a \leq x \leq b$, let $x_0 = a < x_1 < \dots < x_{n-1} < x_n = b$ and $\delta x_i = x_i - x_{i-1}$. Let $P = \{\delta x_i : 1 \leq i \leq n\}$, $|P| = \max_{1 \leq i \leq n} \{\delta x_i : 1 \leq i \leq n\}$. We define the definite integral of f from a to b as $\int_a^b f(x) dx = \lim_{|P| \rightarrow 0} \sum_{i=1}^n f(x_i) \delta x_i$, which corresponds geometrically, if $f(x)$ is positive, to the area between the graph, the x -axis and the lines $x = a$ and $x = b$.



A function $f(x, y)$ defines a surface and, if $f(x, y)$ is positive, we may analogously determine the volume enclosed by this surface and a cylinder erected on a region S of the xy -plane.



Divide S into n small area elements such that the element has area $\delta S_{ij} = \delta x_i \delta y_j$ and contains the point (x_i, y_j) . Then $f(x_i, y_j) \delta x_i \delta y_j$ is the volume of the column with base $\delta S_{ij} = \delta x_i \delta y_j$ and height $f(x_i, y_j)$.

Let $P = \{\delta S_{ij} = \delta x_i \delta y_j : 1 \leq i \leq m, 1 \leq j \leq n\}$, $|P| = \max_{1 \leq i \leq m, 1 \leq j \leq n} \{\delta S_{ij} = \delta x_i \delta y_j : 1 \leq i \leq m, 1 \leq j \leq n\}$.

We define the double integral of $f(x, y)$ over the region S to be

$$\iint_R f(x, y) dx dy = \lim_{\text{all } \delta x_i, \delta y_j \rightarrow 0} \sum_{i=1}^m \sum_{j=1}^n f(x_i, y_j) \delta x_i \delta y_j.$$

The direct evaluation of the limit in the definition of double integral is impractical. We will calculate the double integral by means of the iterated integrals.

Theorem (Rectangular regions)

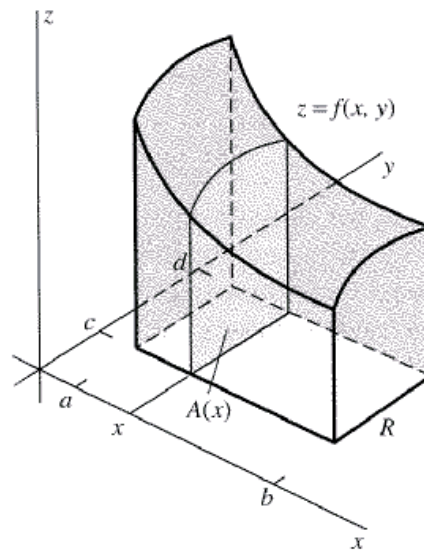
Let $f(x, y)$ be continuous on the region S : $a \leq x \leq b$, $c \leq y \leq d$.

Let $A(x) = \int_c^d f(x, y) dy$, which is the area of the cross section at x (Figure A), then

$$\iint_S f(x, y) dx dy = \int_a^b A(x) dx = \int_a^b \left[\int_c^d f(x, y) dy \right] dx.$$

Similarly, we also have $\iint_S f(x, y) dx dy = \int_c^d \left[\int_a^b f(x, y) dx \right] dy$.

Figure A



Example 1

Evaluate $\iint_S (2xy + y^2) dx dy$ where S is the rectangle $1 \leq x \leq 2$, $0 \leq y \leq 1$.

Solution:

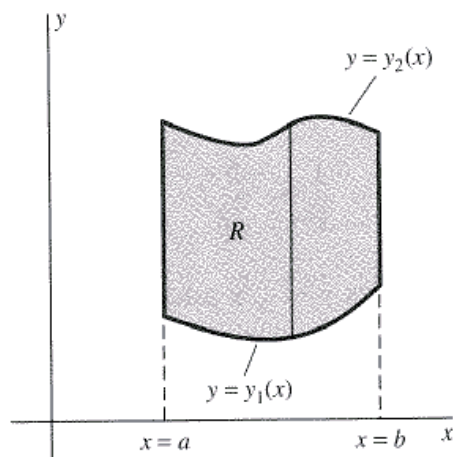
$$\int_0^1 \left[\int_1^2 (2xy + y^2) dx \right] dy = \int_0^1 \left(x^2 y + xy^2 \right) \Big|_1^2 dy = \int_0^1 (4y + 2y^2 - y - y^2) dy = \left(\frac{3y^2}{2} + \frac{y^3}{3} \right) \Big|_0^1 = \frac{11}{6} \quad \text{or}$$

$$\int_1^2 \left[\int_0^1 (2xy + y^2) dy \right] dx = \int_1^2 \left(xy^2 + \frac{y^3}{3} \right) \Big|_0^1 dx = \int_1^2 \left(x + \frac{1}{3} \right) dx = \left(\frac{x^2}{2} + \frac{x}{3} \right) \Big|_1^2 = \frac{4}{2} + \frac{2}{3} - \frac{1}{2} - \frac{1}{3} = \frac{11}{6}.$$

□

Example 2 (Non-rectangular regions)

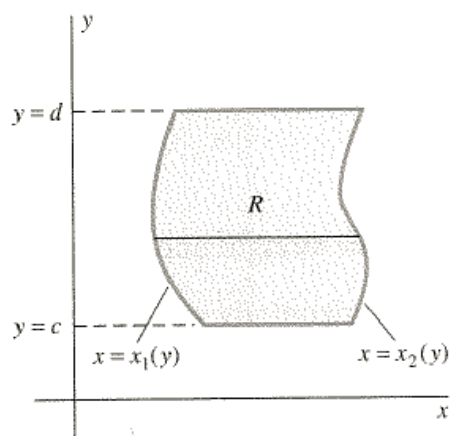
A vertically simple region R



$$a \leq x \leq b, \quad y_1(x) \leq y \leq y_2(x)$$

$$\int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x, y) dy \right] dx$$

A horizontally simple region R



$$c \leq y \leq d, \quad x_1(y) \leq x \leq x_2(y)$$

$$\int_c^d \left[\int_{x_1(y)}^{x_2(y)} f(x, y) dx \right] dy$$

□

Example 3

$$\iint_S 1 dx dy = \text{area of the region } S \text{ if } f(x, y) \equiv 1.$$

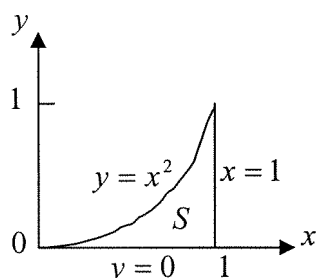
□

Example 4

Evaluate $\iint_S xy^2 dx dy$ where S is the region bounded by $y = 0$, $y = x^2$ and $x = 1$.

Solution:

A sketch of the region S is important, but it is not necessary to be able to visualize the surface $f(x, y)$.



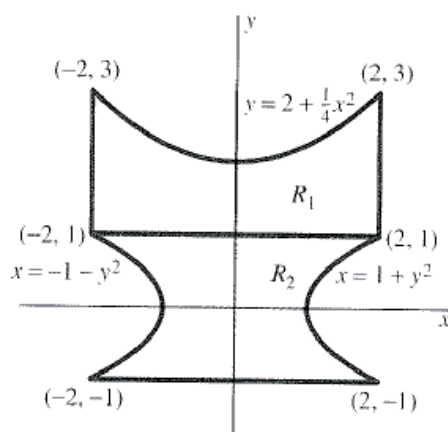
$$\iint_S xy^2 dx dy = \int_0^1 \left(\int_0^{x^2} xy^2 dy \right) dx = \int_0^1 \left(\frac{xy^3}{3} \right) \Big|_0^{x^2} dx = \int_0^1 \frac{x^7}{3} dx = \frac{x^8}{24} \Big|_0^1 = \frac{1}{24} \text{ or}$$

$$\iint_S xy^2 dx dy = \int_0^1 \left(\int_{\sqrt{y}}^1 xy^2 dx \right) dy = \int_0^1 \left(\frac{y^2 x^2}{2} \right) \Big|_{\sqrt{y}}^1 dy = \int_0^1 \left(\frac{y^2}{2} - \frac{y^3}{2} \right) dy = \left(\frac{y^3}{6} - \frac{y^4}{8} \right) \Big|_0^1 = \frac{1}{24}.$$

□

Example 5

The nonsimple region R is the union of the nonoverlapping simple regions R_1 , R_2



Solution:

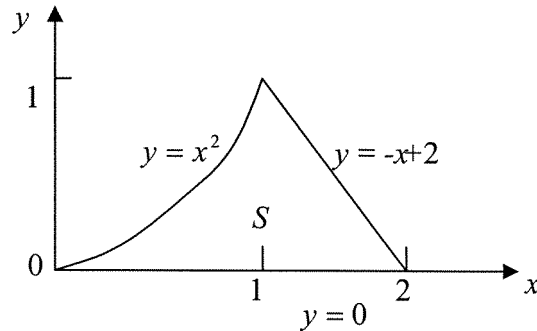
$$\iint_R f(x, y) dx dy = \iint_{R_1} f(x, y) dx dy + \iint_{R_2} f(x, y) dx dy = \int_{-2}^2 \left[\int_1^{2+\frac{x^2}{4}} f(x, y) dy \right] dx + \int_{-1}^1 \left[\int_{-1-y^2}^{1+y^2} f(x, y) dx \right] dy$$

□

In some cases, one order of integration may be easier than the other.

Example 6

Consider an integrable function $f(x, y)$, which is defined on a region of xy - plane shown as follows:



We observe that $\iint_S f(x, y) dx dy = \int_0^1 \left[\int_{\sqrt{y}}^{2-y} f(x, y) dx \right] dy$ looks like easier than

$$\iint_S f(x, y) dx dy = \int_0^1 \left[\int_0^{x^2} f(x, y) dy \right] dx + \int_1^2 \left[\int_0^{2-x} f(x, y) dy \right] dx.$$

□

Example 7 (Optional)

$$\text{Let } f(x, y) = \begin{cases} \frac{1}{y^2} & \text{if } 0 < x < y < 1 \\ -\frac{1}{x^2} & \text{if } 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Show that the double integral $\iint_Q f(x, y) dx dy$, where $Q = [0, 1] \times [0, 1]$ can not exist by pointing out

$$\int_0^1 \left[\int_0^1 f(x, y) dy \right] dx \neq \int_0^1 \left[\int_0^1 f(x, y) dx \right] dy.$$

Solution:

$$\int_0^1 \left[\int_0^1 f(x, y) dy \right] dx = \int_0^1 \left[\int_0^x f(x, y) dy + \int_x^1 f(x, y) dy \right] dx = \int_0^1 \left(-\int_0^x \frac{1}{x^2} dy + \int_x^1 \frac{1}{y^2} dy \right) dx = \int_0^1 -1 dx = -1.$$

$$\int_0^1 \left[\int_0^1 f(x, y) dx \right] dy = \int_0^1 \left[\int_0^y f(x, y) dx + \int_y^1 f(x, y) dx \right] dy = \int_0^1 \left(\int_0^y \frac{1}{y^2} dx - \int_y^1 \frac{1}{x^2} dx \right) dy = \int_0^1 1 dy = 1.$$

Hence,

$$\int_0^1 \left[\int_0^1 f(x, y) dy \right] dx \neq \int_0^1 \left[\int_0^1 f(x, y) dx \right] dy$$

As the two iterated integrals exist but are unequal, the double integral $\iint_Q f(x, y) dx dy$ cannot exist over $Q = [0, 1] \times [0, 1]$.

□

Example 8

Change the order of the integration in $\int_0^1 \left[\int_x^{\sqrt{x}} f(x, y) dy \right] dx$.

Solution:

The line $y = x$ and the parabola $y = \sqrt{x}$ cut at $(0, 0)$ and $(1, 1)$. The domain of integration is the area bounded by $y = x$ and $y = \sqrt{x}$ (the same as $x = y^2$).

$$\text{So } \int_0^1 \left[\int_x^{\sqrt{x}} f(x, y) dy \right] dx = \int_0^1 \left[\int_{y^2}^y f(x, y) dx \right] dy.$$

□

Example 9

Change the order of the integration in $\int_0^1 \left[\int_{-y}^y f(x, y) dx \right] dy$

Solution:

Let R be the region of integration. R is given by the inequalities $-y \leq x \leq y$ and $0 \leq y \leq 1$.

Therefore, R is bounded by $y = x$ and $y = -x$ between $y = 0$, $y = 1$.

To find limits for integration in the reverse order, we imagine a vertical line passing from bottom to the top through the region. From $x = -1$ to $x = 0$, it enters at $y = -x$ and leaves at $y = 1$. Then from $x = 0$ to $x = 1$, it enters at $y = x$ and leaves at $y = 1$. So

$$\int_0^1 \left[\int_{-y}^y f(x, y) dx \right] dy = \iint_R f(x, y) dx dy = \int_{-1}^0 \left[\int_{-x}^1 f(x, y) dy \right] dx + \int_0^1 \left[\int_x^1 f(x, y) dy \right] dx$$

□

2 Change of Variable in Double Integrals

For the definite integral $I = \int_a^b f(x) dx$, we know that the change of variable $x = x(u)$ produces

$$I = \int_{\alpha}^{\beta} f[x(u)] \frac{dx}{du} du,$$

where $x(\alpha) = a$, $x(\beta) = b$.

For the double integral $I = \iint_S f(x, y) dx dy$, the change of variable $x = x(u, v)$, $y = y(u, v)$ gives

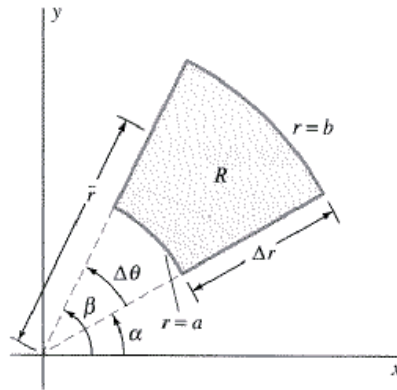
$$I = \iint_{S^*} f[x(u, v), y(u, v)] \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv,$$

where $J = \frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$ is called the Jacobian of the transformation. S^* is the region in the uv -plane corresponding to the region S in the xy -plane injectively (one to one) and J must be of one sign in S^* .

In particular, when using polar coordinates, we have

$$x = r \cos \theta, \quad y = r \sin \theta$$

For polar coordinates, the area of a small polar rectangle R is approximately $r \Delta r \Delta \theta$, (Note $\delta \equiv \Delta$).

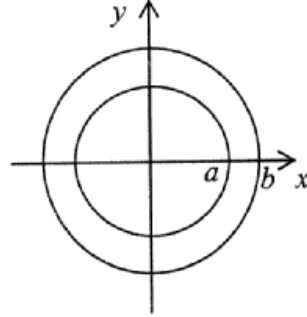


$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \det \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} = r \quad \text{i.e. } dx dy \rightarrow r dr d\theta, \text{ where } r dr d\theta$$

corresponds to an element of area in the $r\theta$ -plane.

Example 10

Find the moment of inertia of a hollow circular cylinder of inner radius a , outer radius b , height h and constant density ρ about the axis of the cylinder. The formula for the moment of inertia of the hollow circular cylinder is given as $MI = \iint_S \rho h (x^2 + y^2) dx dy$.



Solution:

Using polar coordinates: $x = r \cos \theta$, $y = r \sin \theta$.

Under $x = r \cos \theta$, $y = r \sin \theta$, the hollow circular ring S of inner radius a , outer radius b is transformed as a rectangle R in the $r\theta$ -plane, which is bounded by $\theta = 0$, $\theta = 2\pi$, $r = a$, $r = b$.

The region of integration is now a rectangle R in the $r\theta$ -plane, $R: \begin{cases} a \leq r \leq b \\ 0 \leq \theta \leq 2\pi \end{cases}$.

$$J = r \quad |J| = r \quad \text{i.e. } dx dy \rightarrow r dr d\theta$$

$$\begin{aligned} MI &= \iint_S \rho h (x^2 + y^2) dx dy = \iint_R \rho h r^2 |J| dr d\theta = \iint_R \rho h r^2 r dr d\theta \\ &= \int_0^{2\pi} \left[\int_a^b \rho h r^3 dr \right] d\theta = \rho h \int_0^{2\pi} \left[\int_a^b r^3 dr \right] d\theta = \rho h \left(\int_a^b r^3 dr \right) \left(\int_0^{2\pi} 1 d\theta \right) \\ &= \rho h \left(\frac{r^4}{4} \Big|_a^b \right) \left(\theta \Big|_0^{2\pi} \right) = \rho h 2\pi \frac{b^4 - a^4}{4} = \rho h \pi \frac{(a^2 + b^2)(a^2 - b^2)}{2} = \frac{M}{2} (a^2 + b^2) \end{aligned}$$

$$\text{, where mass } = M = \rho h \pi (b^2 - a^2)$$

Example 11

Evaluate the integral $\iint_Q e^{\frac{y-x}{y+x}} dx dy$, where Q is bounded by $x + y = 2$, $x = 0$, $y = 0$.

(Hint: Use the change of variables $u = y - x$ and $v = y + x$.)

Solution:

By using the change of variables $u = y - x$ and $v = y + x$ and solving for x and y ,

$$\text{we find } x = \frac{v-u}{2}, y = \frac{v+u}{2}.$$

Under the transformations $u = y - x$ and $v = y + x$, the lines $x + y = 2, x = 0, y = 0$ map onto the lines $v = 2, u - v = 0, u + v = 0$, respectively. Points inside Q are carried into points inside the triangular region T bounded by $v = 2, u - v = 0, u + v = 0$.

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = -\frac{1}{2} \quad \text{or} \quad J = \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u, v)}{\partial(x, y)}} = \frac{1}{\det \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}} = \frac{1}{-2}$$

$$\text{Therefore, } \iint_Q e^{\frac{y-x}{y+x}} dx dy = \iint_T e^{\frac{u}{v}} \left| -\frac{1}{2} \right| du dv = \frac{1}{2} \int_0^2 \left(\int_{-v}^v e^{\frac{u}{v}} du \right) dv = \frac{1}{2} \int_0^2 v \left(e - \frac{1}{e} \right) dv = e - \frac{1}{e}.$$

□

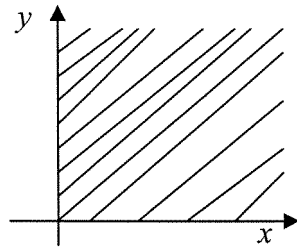
Example 12

By considering $I^2 = \left(\int_0^\infty e^{-x^2} dx \right) \left(\int_0^\infty e^{-y^2} dy \right)$, evaluate $I = \int_0^\infty e^{-x^2} dx$.

Solution:

$$\begin{aligned} I^2 &= \left(\int_0^\infty e^{-x^2} dx \right) \left(\int_0^\infty e^{-y^2} dy \right) = \int_0^\infty \left(\int_0^\infty e^{-x^2} dx \right) e^{-y^2} dy = \int_0^\infty e^{-y^2} \left(\int_0^\infty e^{-x^2} dx \right) dy = \int_0^\infty \left(\int_0^\infty e^{-y^2} e^{-x^2} dx \right) dy \\ &= \int_0^\infty \left[\int_0^\infty e^{-(x^2+y^2)} dx \right] dy \end{aligned}$$

$\int_0^\infty \left[\int_0^\infty e^{-(x^2+y^2)} dx \right] dy$ is equivalent to integrating the function $f(x, y) = e^{-(x^2+y^2)}$ over the first quadrant of xy -plane.



Under the transformations $x = r \cos \theta, y = r \sin \theta$, the first quadrant R of xy -plane is corresponding to the region \bar{R} , which is bounded by $r = 0, \theta = 0, \theta = \frac{\pi}{2}$ in $r\theta$ -plane. (We observe that under

$x = r \cos \theta, y = r \sin \theta$ the quarter circle $x^2 + y^2 = r_0^2, x, y \geq 0$ to $r = r_0$ corresponds the line segment

$$\begin{cases} r = r_0 \\ 0 \leq \theta \leq \frac{\pi}{2} \end{cases} \text{ in } r\theta\text{-plane). So}$$

$$I^2 = \int_0^\infty \left[\int_0^\infty e^{-(x^2+y^2)} dx \right] dy = \iint_R e^{-(x^2+y^2)} dx dy = \iint_R e^{-r^2} |J| dr d\theta = \iint_R e^{-r^2} r dr d\theta = \int_0^\infty \left(\int_0^{\frac{\pi}{2}} e^{-r^2} r d\theta \right) dr, \text{ where}$$

$$|J| = \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| = r.$$

And

$$\begin{aligned} \int_0^\infty \left(\int_0^{\frac{\pi}{2}} e^{-r^2} r d\theta \right) dr &= \int_0^\infty \frac{\pi}{2} e^{-r^2} r dr \stackrel{\substack{\text{Let } x=r^2 \\ r=0 \rightarrow x=0 \\ r=\infty \rightarrow x=\infty \\ dx=2rdr}}{=} \int_0^\infty \frac{\pi}{2} e^{-x} \frac{dx}{2} = \frac{\pi}{4} \int_0^\infty e^{-x} dx = -\frac{\pi}{4} e^{-x} \Big|_0^\infty = -\frac{\pi}{4} \frac{1}{e^x} \Big|_0^\infty \\ &= -\frac{\pi}{4} \left(\frac{1}{e^\infty} - 1 \right) = -\frac{\pi}{4} (0 - 1) = \frac{\pi}{4} \end{aligned}$$

$$\text{Finally, we have } I^2 = \left(\int_0^\infty e^{-x^2} dx \right) \left(\int_0^\infty e^{-y^2} dy \right) = \frac{\pi}{4} \Rightarrow \int_0^\infty e^{-x^2} dx = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$$

□

3 Triple Integrals

Suppose that a scalar field $f(x, y, z)$ is defined at all points (x_i, y_i, z_i) within a region V of three-dimensional space. Divide V into N subregions such that the i th subregion has volume $\delta V_{ijk} = \delta x_i \delta y_j \delta z_k$ and contains a point (x_i, y_j, z_k) .

$$\begin{aligned} \text{Let } P &= \{ \delta V_{ijk} = \delta x_i \delta y_j \delta z_k : 1 \leq i \leq N_1, 1 \leq j \leq N_2, 1 \leq k \leq N_3 \}, \\ |P| &= \max_{\substack{1 \leq i \leq N_1, \\ 1 \leq j \leq N_2, \\ 1 \leq k \leq N_3}} \{ \delta V_{ijk} = \delta x_i \delta y_j \delta z_k : 1 \leq i \leq N_1, 1 \leq j \leq N_2, 1 \leq k \leq N_3 \}. \end{aligned}$$

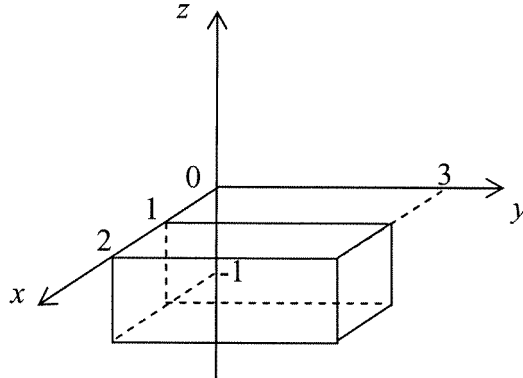
We define the triple integral of $f(x, y, z)$ over the region V to be

$$\iiint_V f(x, y, z) dx dy dz = \lim_{\text{all } \delta x_i, \delta y_j, \delta z_k \rightarrow 0} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \sum_{k=1}^{N_3} f(x_i, y_j, z_k) \delta x_i \delta y_j \delta z_k.$$

As expected, we can evaluate triple integrals by iterated single integration.

Example 13 (Rectangular block)

The density of a rectangular blocks V bounded by the planes $x = 1, x = 2, y = 0, y = 3, z = -1, z = 0$ is given by the scalar function $\rho(x, y, z) = x(y + 1) - z$. Find the mass of the block.



Solution:

$$\begin{aligned} \text{Mass} &= \iiint_V [x(y+1) - z] dx dy dz = \int_1^2 \left[\int_0^3 \left[\int_{-1}^0 [x(y+1) - z] dz \right] dy \right] dx = \int_1^2 \left[\int_0^3 \left[x(y+1)z - \frac{z^2}{2} \right] \Big|_{-1}^0 dy \right] dx \\ &= \int_1^2 \left[\int_0^3 \left[x(y+1) + \frac{1}{2} \right] dy \right] dx = \int_1^2 \left(\frac{xy^2}{2} + xy + \frac{1}{2}y \right) \Big|_0^3 dx = \int_1^2 \left(\frac{9x}{2} + 3x + \frac{3}{2} \right) dx = \left(\frac{15}{4}x^2 + \frac{3}{2}x \right) \Big|_1^2 = \frac{51}{4} \end{aligned}$$

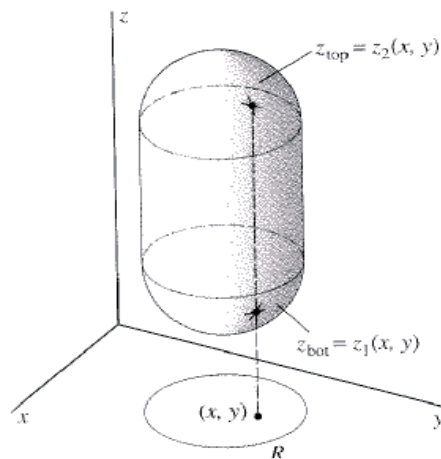
Non-rectangular block:

Suppose that the solid T with piecewise smooth boundary is z -simple: Each line parallel to the z -axis intersects T (if at all) in a single line segment. In effect, this means that T can be described by the inequalities

$z_1(x, y) \leq z \leq z_2(x, y)$ with $(x, y) \in R$, where R is the vertical projection of T onto the xy -plane (Figure

B). Then
$$\iiint_T f(x, y, z) dx dy dz = \iint_R \left[\int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z) dz \right] dx dy.$$

Figure B



Similarly, we may integrate first with respect to either x or y if the space region T is either x -simple or y -simple. Such situations appear in Figure C and Figure D, respectively.

Figure C

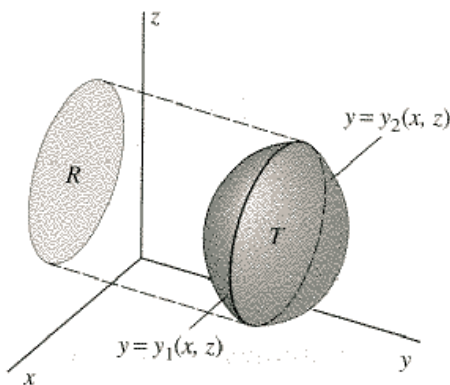
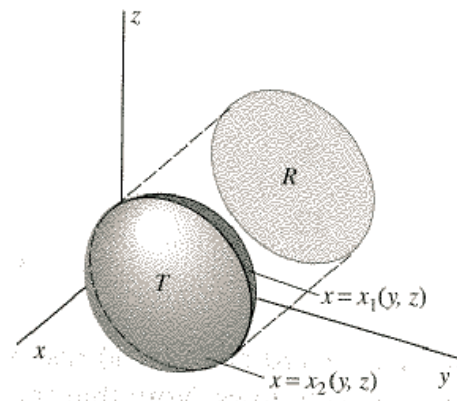


Figure D



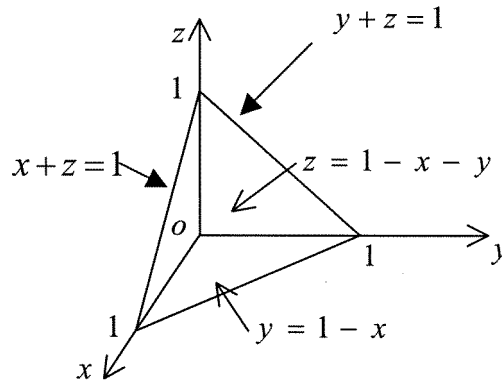
Examples 14

- (a) If $f(x, y, z) \equiv 1$, then $\iiint_V 1 dx dy dz = \mathbf{volume}$ of the region V .
- (b) If the scalar function $\rho(x, y, z)$ gives the density at a point (x, y, z) of the region V , then $\iiint_V \rho(x, y, z) dx dy dz = \mathbf{mass}$ of the region V .
- (c) If the scalar function $\rho(x, y, z)$ gives the charge density at a point (x, y, z) of the region V , then $\iiint_V \rho(x, y, z) dx dy dz = \mathbf{total charge}$ within the region V .

□

Example 15

Evaluate $\iiint_V \frac{1}{(x+y+2z+1)^3} dx dy dz$ where V is the region enclosed by the planes $x=0, y=0, z=0, x+y+z=1$.



Solution:

The projection of V on xy -plane is σ_{xy} which is bounded by $x=0, y=0, y=1-x$.

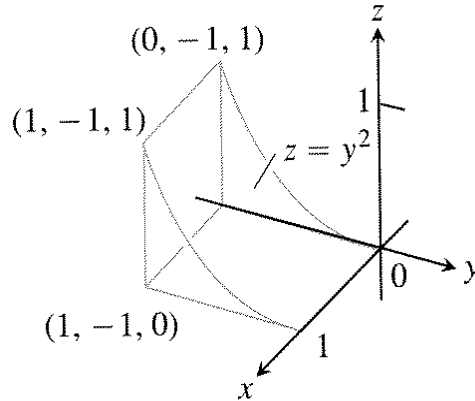
$$\begin{aligned}
 \iiint_{\sigma_{xy}} \left[\int_0^{1-x-y} \frac{1}{(x+y+2z+1)^3} dz \right] dx dy &= \int_0^1 \left[\int_0^{1-x} \left[\int_0^{1-x-y} \frac{1}{(x+y+2z+1)^3} dz \right] dy \right] dx \\
 &= \int_0^1 \left(\int_0^{1-x} \left[\frac{-1}{4(x+y+2z+1)^2} \right]_0^{1-x-y} dy \right) dx = \int_0^1 \left(\int_0^{1-x} \left[\frac{1}{4(x+y+1)^2} - \frac{1}{4(3-x-y)^2} \right] dy \right) dx \\
 &= \int_0^1 \left[\frac{-1}{4(x+y+1)} - \frac{1}{4(3-x-y)} \right]_0^{1-x} dx = \int_0^1 \left[\frac{-1}{8} - \frac{1}{8} + \frac{1}{4(x+1)} + \frac{1}{4(3-x)} \right] dx \\
 &= \left[-\frac{x}{4} + \frac{\log(x+1)}{4} - \frac{\log(3-x)}{4} \right]_0^1 = -\frac{1}{4} + \frac{\log 2}{4} - \frac{\log 2}{4} - \left(-\frac{\log 3}{4} \right) = \frac{\log 3}{4} - \frac{1}{4}
 \end{aligned}$$

□

Example 16

An iterated integral like $\int_0^1 \left[\int_{-1}^0 \left(\int_0^{y^2} f(x, y, z) dz \right) dy \right] dx$ is called an iterated integral with order $dzdydx$.

With the aid of the following figure, change the order of the iterated integral $\int_0^1 \left[\int_{-1}^0 \left(\int_0^{y^2} f(x, y, z) dz \right) dy \right] dx$ to an equivalent iterated integral with order $dydzdx$.



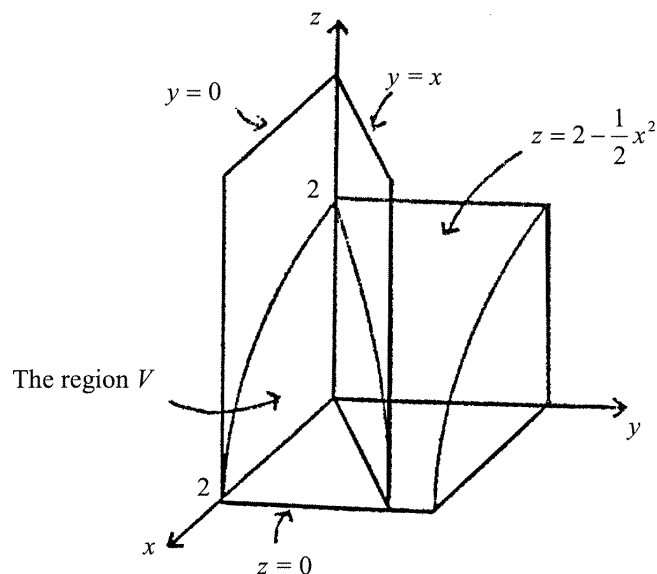
Solution:

$$\int_0^1 \left[\int_{-1}^0 \left(\int_0^{y^2} f(x, y, z) dz \right) dy \right] dx = \int_0^1 \left(\iint_{S_{yz}} f(x, y, z) dy dz \right) dx = \int_0^1 \left[\int_0^1 \left(\int_{-1}^{-\sqrt{z}} f(x, y, z) dy \right) dz \right] dx$$

□

Example 17

Evaluate $\iiint_V 2xyz dx dy dz$ where V is the region bounded by the parabolic cylinder $z = 2 - \frac{1}{2}x^2$ and the planes $x=0$, $y=x$ and $y=0$, $z=0$.



Solution:

Projecting V onto the xz -plane gives σ_{xz} which is bounded by $z = 2 - \frac{1}{2}x^2$, $x = 0$, $z = 0$. Thus,

$$\begin{aligned}\iiint_V 2xyz dy dx dz &= \iint_{\sigma_{xz}} \left[\int_0^x 2xyz dy \right] dx dz = \iint_{\sigma_{xz}} (xy^2 z) \Big|_0^x dx dz = \iint_{\sigma_{xz}} x^3 z dx dz \\ &= \int_0^2 \left(\int_0^{2-\frac{1}{2}x^2} x^3 z dz \right) dx = \int_0^2 \left(\frac{x^3 z^2}{2} \right) \Big|_0^{2-\frac{1}{2}x^2} dx = \int_0^2 \frac{x^3 \left(2 - \frac{1}{2}x^2\right)^2}{2} dx = \frac{4}{3}\end{aligned}$$

Alternatively, projecting V onto the xy -plane gives σ_{xy} which is bounded, $y = 0$, $y = x$, $x = 2$. Thus,

$$\begin{aligned}\iiint_V 2xyz dy dx dz &= \iint_{\sigma_{xy}} \left(\int_0^{2-\frac{1}{2}x^2} 2xyz dz \right) dx dy = \int_0^2 \left[\int_0^x \left(\int_0^{2-\frac{1}{2}x^2} 2xyz dz \right) dy \right] dx = \int_0^2 \left(\int_0^x xyz^2 \Big|_{z=0}^{z=2-\frac{1}{2}x^2} dy \right) dx = \\ &= \int_0^2 \int_0^x xy \left(2 - \frac{1}{2}x^2 \right)^2 dy dx = \int_0^2 \frac{xy^2 \left(2 - \frac{1}{2}x^2 \right)^2}{2} \Big|_{y=0}^{y=x} dx = \frac{1}{2} \int_0^2 x^3 \left(2 - \frac{1}{2}x^2 \right)^2 dx = \frac{4}{3}.\end{aligned}$$

□

Example 18

Evaluate $\iiint_V xyz dx dy dz$, where V is the region enclosed by $x^2 + y^2 + z^2 = 1$ and $x \geq 0$, $z \geq 0$, $y \geq 0$ and

$x = 0$, $y = 0$, $z = 0$.

Solution:

The projection of V onto xy -plane is σ_{xy} which is bounded by $x^2 + y^2 = 1$ where $x \geq 0$, $y \geq 0$ and

$$x = 0, y = 0. \text{ So } \iiint_V xyz dx dy dz = \iint_{\sigma_{xy}} \left(\int_0^{\sqrt{1-x^2-y^2}} xyz dz \right) dx dy = \int_0^1 \left[\int_0^{\sqrt{1-x^2}} \left(\int_0^{\sqrt{1-x^2-y^2}} xyz dz \right) dy \right] dx$$

Then

$$\begin{aligned}
\iiint_V xyz dx dy dz &= \int_0^1 \left[\int_0^{\sqrt{1-x^2}} \left(\int_0^{\sqrt{1-x^2-y^2}} xyz dz \right) dy \right] dx = \int_0^1 \left(\int_0^{\sqrt{1-x^2}} xy \frac{z^2}{2} \Big|_0^{\sqrt{1-x^2-y^2}} dy \right) dx \\
&= \int_0^1 \left[\int_0^{\sqrt{1-x^2}} \left(xy \frac{1-x^2-y^2}{2} \right) dy \right] dx = \int_0^1 \left[\int_0^{\sqrt{1-x^2}} \left(\frac{xy - x^3y - xy^3}{2} \right) dy \right] dx \\
&= \int_0^1 \left[\frac{(x-x^3)y^2}{4} - \frac{xy^4}{8} \right] \Big|_0^{\sqrt{1-x^2}} dx = \int_0^1 \frac{(x-x^3)(1-x^2)}{4} - \frac{x(1-x^2)^2}{8} dx \\
&= \int_0^1 \frac{(1-x^2)}{4} \left[x - x^3 - \frac{x(1-x^2)}{2} \right] dx = \int_0^1 \frac{(1-x^2)}{4} \left(\frac{x-x^3}{2} \right) dx = \int_0^1 \frac{x-2x^3+x^5}{8} dx \\
&= \left(\frac{x^2}{16} - \frac{x^4}{16} + \frac{x^6}{48} \right) \Big|_0^1 = \frac{1}{48}
\end{aligned}$$

□

Illustration

Find the volume V of the region R bounded by the parabolic cylinder $z = 4 - x^2$ and the planes $x = 0, y = 0, y = 6, z = 0$.

Solution:

$$V = \iiint_R 1 dx dy dz = \int_0^2 \left[\int_0^6 \left(\int_0^{4-x^2} 1 dz \right) dy \right] dx = \int_0^2 \left[\int_0^6 (4-x^2) dy \right] dx = \int_0^2 (4-x^2) y \Big|_0^6 dx = \int_0^2 (24-6x^2) dx = 32.$$

□

4 Change of Variable in Triple Integrals

If $I = \iiint_V f(x, y, z) dx dy dz$, the change of variable $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$, gives

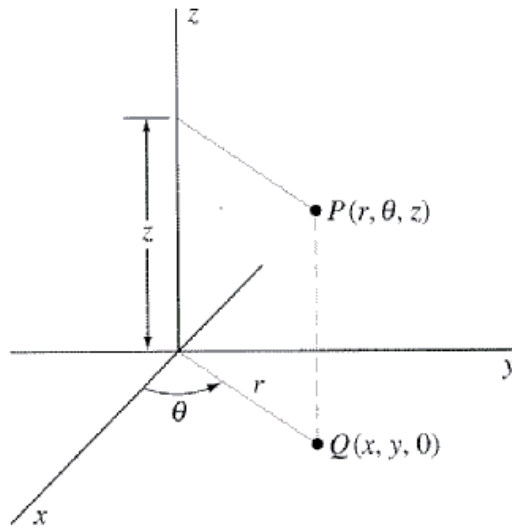
$$I = \iiint_{V^*} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw,$$

where $J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$ is the Jacobian of the transformation. V^* is the region in

uvw -space corresponding to the region V in xyz -space injectively (one to one) and J must be of one sign in V^* .

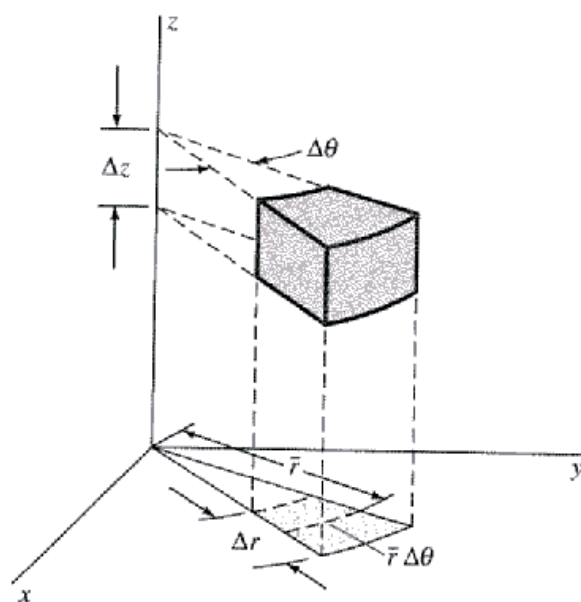
The most popular alternative coordinate systems to Cartesian coordinates are cylindrical polar coordinates and spherical polar coordinates.

Cylindrical Polar Coordinates:



$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z, \quad r \geq 0, \quad 0 \leq \theta < 2\pi, \quad -\infty < z < \infty$$

$r = \text{constant}$ – cylinder, $\theta = \text{constant}$ – plane, $z = \text{constant}$ – plane



The volume of the cylindrical block is approximately $\bar{r}\Delta z\Delta r\Delta\theta$. (Note $\delta \equiv \Delta$)

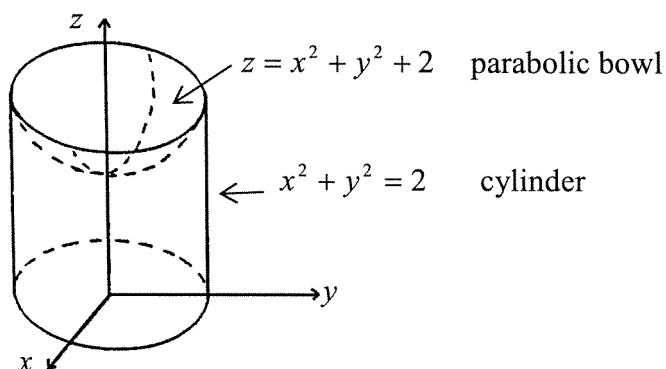
$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \det \begin{pmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r \cos^2 \theta + r \sin^2 \theta = r$$

$$dx dy dz \rightarrow r dr d\theta dz,$$

where $r dr d\theta dz$ corresponds to an element of volume in cylindrical polar coordinates.

Example 19

Find the volume V between the surfaces $x^2 + y^2 = 2$, $z = x^2 + y^2 + 2$ and the plane $z = 0$.



Solution:

Cartesian coordinates:

We observe that the projection of V onto xy -plane is $x^2 + y^2 \leq 2$. So

$$\iiint_V 1 dx dy dz = \iint_{x^2+y^2 \leq 2} \left(\int_0^{x^2+y^2+2} 1 dz \right) dx dy = \int_{-\sqrt{2}}^{\sqrt{2}} \left[\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \left(\int_0^{x^2+y^2+2} 1 dz \right) dy \right] dx$$

Cylindrical polar coordinates:

Under the cylindrical transformations $x = r \cos \theta$, $y = r \sin \theta$, $z = z$, the volume V is transformed into

the solid V^* : $0 \leq \theta \leq 2\pi$, $0 \leq r \leq \sqrt{2}$, $0 \leq z \leq x^2 + y^2 + 2$ in $r\theta z$ -space. The projection of V^* onto

$r\theta$ -plane is $0 \leq \theta \leq 2\pi$, $0 \leq r \leq \sqrt{2}$. Thus

$$\begin{aligned} \iiint_V 1 dx dy dz &= \iiint_{V^*} r dr d\theta dz = \iint_{0 \leq \theta \leq 2\pi, 0 \leq r \leq \sqrt{2}} \left(\int_0^{r^2+2} r dz \right) dr d\theta = \int_0^{\sqrt{2}} \left[\int_0^{2\pi} \left(\int_0^{r^2+2} r dz \right) d\theta \right] dr = \int_0^{\sqrt{2}} \left[\int_0^{2\pi} (r^3 + 2r) d\theta \right] dr \\ &= \int_0^{\sqrt{2}} 2\pi (r^3 + 2r) dr = 6\pi \end{aligned}$$

□

Example 20

Using cylindrical polar coordinates, evaluate $\iiint_V (x^2 + y^2) dx dy dz$, where V is the solid enclosed by

$$x^2 + y^2 = 2z, \quad z = 2$$

Solution:

Under the cylindrical transformation $x = r \cos \theta$, $y = r \sin \theta$, $z = z$, the solid V is transformed into V^* :

$0 \leq \theta \leq 2\pi$, $0 \leq r \leq 2$, $\frac{x^2 + y^2}{2} = \frac{r^2}{2} \leq z \leq 2$. The projection of V^* onto $r\theta$ -plane is

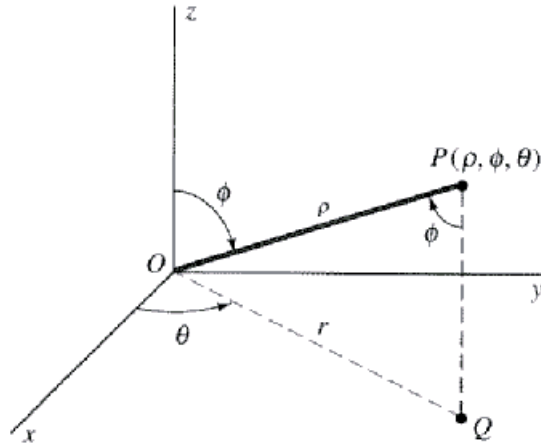
$0 \leq \theta \leq 2\pi$, $0 \leq r \leq 2$.

So

$$\begin{aligned} \iiint_V (x^2 + y^2) dx dy dz &= \iiint_{V^*} r^2 r dr d\theta dz = \iint_{0 \leq \theta \leq 2\pi, 0 \leq r \leq 2} \left(\int_{\frac{r^2}{2}}^2 r^3 dz \right) dr d\theta = \int_0^2 \left[\int_0^{2\pi} \left(\int_{\frac{r^2}{2}}^2 r^3 dz \right) d\theta \right] dr \\ &= \int_0^2 \left[\int_0^{2\pi} r^3 \left(2 - \frac{r^2}{2} \right) d\theta \right] dr = \int_0^2 2\pi r^3 \left(2 - \frac{r^2}{2} \right) dr = \left(\pi r^4 - \frac{\pi r^6}{6} \right) \Big|_0^2 = 16\pi - \frac{64\pi}{6} = \frac{32\pi}{6} = \frac{16\pi}{3} \end{aligned}$$

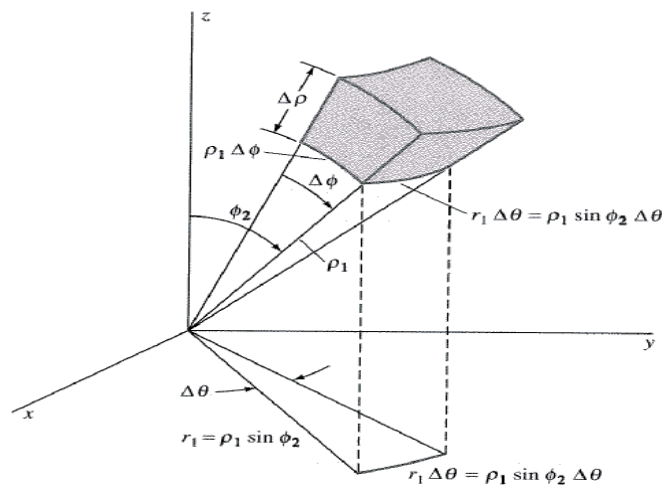
□

Spherical Polar Coordinates:



$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi, \quad \rho \geq 0, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi$$

In particular: $\rho = \text{constant}$ – sphere, $\phi = \text{constant}$ – cone, $\theta = \text{constant}$ – plane



The volume of the spherical block is approximately $\rho^2 \sin \phi \Delta \rho \Delta \phi \Delta \theta$.

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \det \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix} = \det \begin{pmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{pmatrix} = \rho^2 \sin \phi$$

$$dx dy dz \rightarrow \rho^2 \sin \phi d\rho d\phi d\theta$$

$\rho^2 \sin \phi d\rho d\phi d\theta$ corresponds to an element of volume in spherical polar coordinates.

Example 21

Find an equation for the sphere $x^2 + y^2 + (z-1)^2 = 1$ in spherical coordinate.

Solution:

$$\begin{aligned} x^2 + y^2 + (z-1)^2 = 1 &\Rightarrow \begin{matrix} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{matrix} \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta + (\rho \cos \phi - 1)^2 = 1 \\ &\Rightarrow \rho^2 \sin^2 \phi + \rho^2 \cos^2 \phi - 2\rho \cos \phi = 0 \Rightarrow \rho^2 = 2\rho \cos \phi \Rightarrow \rho = 2 \cos \phi \end{aligned}$$

□

Example 22

Find the volume of the “ice cream cone” D cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \frac{\pi}{3}$.

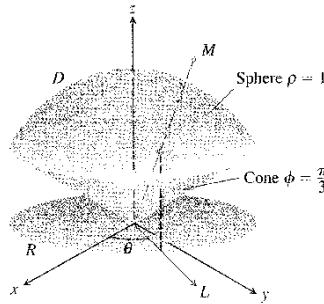
Solution:

The volume is $V = \iiint_D \rho^2 \sin \phi d\rho d\phi d\theta$.

To find the limits of integration for evaluating the integral, we take the following steps.

Step 1: *A sketch.*

We sketch D and its projection R on the xy -plane (Figure A).



Step 2: *The ρ -limits of integration.*

We draw a ray M from the origin through D making an angle ϕ with the positive z -axis. We also draw L , the projection of M on the xy -plane, along with the angle θ that L makes with the positive x -axis. Ray M enters D at $\rho = 0$ and leaves at $\rho = 1$.

Step 3: *The ϕ -limits of integration.*

The cone $\phi = \frac{\pi}{3}$ makes an angle of $\frac{\pi}{3}$ with the positive z -axis. For any given θ , the angle ϕ can run from $\phi = 0$ to $\phi = \frac{\pi}{3}$.

Step 4: *The θ -limits of integration.*

The ray L sweeps over R as θ runs from 0 to 2π .

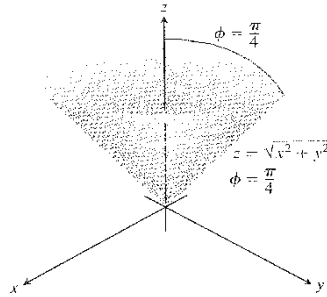
The volume is

$$\begin{aligned} V &= \iiint_D \rho^2 \sin \phi d\rho d\phi d\theta = \int_0^{2\pi} \left[\int_0^{\frac{\pi}{3}} \left(\int_0^1 \rho^2 \sin \phi d\rho \right) d\phi \right] d\theta = \int_0^{2\pi} \left[\int_0^{\frac{\pi}{3}} \left(\frac{\rho^3 \sin \phi}{3} \right) \Big|_0^1 d\phi \right] d\theta \\ &= \int_0^{2\pi} \left(\int_0^{\frac{\pi}{3}} \frac{\sin \phi}{3} d\phi \right) d\theta = \int_0^{2\pi} \left(-\frac{\cos \phi}{3} \right) \Big|_0^{\frac{\pi}{3}} d\theta = \int_0^{2\pi} \left(-\frac{1}{6} + \frac{1}{3} \right) d\theta = \frac{2\pi}{6} = \frac{\pi}{3} \end{aligned}$$

□

Example 23

With the aid of the following, convert $\int_{-1}^1 \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\int_{\sqrt{x^2+y^2}}^1 1 dz \right) dy \right] dx$ to an integral using spherical coordinates with order $d\theta d\rho d\phi$ and then evaluate the obtained integral.



Solution:

For $\int_{-1}^1 \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\int_{\sqrt{x^2+y^2}}^1 1 dz \right) dy \right] dx$, the integration is done in the solid V which is bound above by $z=1$ and bound below by $z = \sqrt{x^2 + y^2}$.

Under the spherical transformations $\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases}$, V is transformed into $V^* = \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \frac{\pi}{4} \\ 0 \leq \rho \leq \sec \phi \end{cases}$.

$$\begin{aligned}
& \int_{-1}^1 \left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\int_{\sqrt{x^2+y^2}}^1 1 dz \right) dy \right] dx = \int_0^{\frac{\pi}{4}} \int_0^{\sec \phi} \left(\int_0^{2\pi} \rho^2 \sin \phi d\theta \right) d\rho d\phi = \int_0^{\frac{\pi}{4}} \int_0^{\sec \phi} \left(\int_0^{2\pi} \rho^2 \sin \phi d\theta \right) d\rho d\phi \\
& = \int_0^{\frac{\pi}{4}} \left(\int_0^{\sec \phi} 2\pi \rho^2 \sin \phi d\rho \right) d\phi = \int_0^{\frac{\pi}{4}} \left(\frac{2\pi \rho^3 \sin \phi}{3} \right) \Big|_0^{\sec \phi} d\phi = \int_0^{\frac{\pi}{4}} \frac{2\pi \sec^3 \phi \sin \phi}{3} d\phi = \frac{2\pi}{3} \int_0^{\frac{\pi}{4}} \frac{\sin \phi}{\cos^3 \phi} d\phi \\
& = -\frac{2\pi}{3} \int_0^{\frac{\pi}{4}} \cos^{-3} \phi d(\cos \phi) = -\frac{2\pi \cos^{-2} \phi}{-6} \Big|_0^{\frac{\pi}{4}} = \frac{\pi}{3 \cos^2 \phi} \Big|_0^{\frac{\pi}{4}} = \frac{\pi}{3} \left(\frac{1}{\cos^2 \frac{\pi}{4}} - \frac{1}{\cos^2 0} \right) = \frac{\pi}{3} \left(\frac{1}{\frac{1}{2}} - \frac{1}{1} \right) = \frac{\pi}{3}
\end{aligned}$$

□

Example 24

In a sample model of the charge distribution around the positively charged (Q) nucleus of the hydrogen atom the charge density in the electron cloud at a distance ρ from the atom is $f(\rho) = \frac{-Q}{\pi a^3} e^{-\frac{2\rho}{a}}$ where a is the Bohr radius. Determine the total charge in the electron cloud.

Solution:

Under the spherical transformation $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, $z = \rho \cos \phi$, R^3 is transformed into V^* : $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$, $0 \leq \rho < \infty$. The total charge q is given by

$$\begin{aligned}
q &= \iiint_{R^3} \left(\frac{-Q}{\pi a^3} e^{-\frac{2\sqrt{x^2+y^2+z^2}}{a}} \right) dx dy dz = \iiint_{V^*} \frac{-Q}{\pi a^3} e^{-\frac{2\rho}{a}} \rho^2 \sin \phi d\theta d\phi d\rho \\
&= \iint_{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi} \left(\int_0^{\infty} \frac{-Q}{\pi a^3} e^{-\frac{2\rho}{a}} \rho^2 \sin \phi d\rho \right) d\theta d\phi = \iint_{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi} \left[\int_0^{\infty} \frac{Q}{2\pi a^2} \sin \phi \rho^2 d\left(e^{-\frac{2\rho}{a}}\right) \right] d\theta d\phi \\
&= \iint_{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi} \frac{Q}{2\pi a^2} \sin \phi \left[\rho^2 e^{-\frac{2\rho}{a}} \Big|_0^{\infty} - \int_0^{\infty} e^{-\frac{2\rho}{a}} d(\rho^2) \right] d\theta d\phi \\
&= \iint_{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi} \frac{Q}{2\pi a^2} \sin \phi \left(-\int_0^{\infty} 2e^{-\frac{2\rho}{a}} \rho d\rho \right) d\theta d\phi = \iint_{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi} \frac{Q}{2\pi a} \sin \phi \left[\int_0^{\infty} \rho d\left(e^{-\frac{2\rho}{a}}\right) \right] d\theta d\phi \\
&= \iint_{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi} \frac{Q}{2\pi a} \sin \phi \left(\rho e^{-\frac{2\rho}{a}} \Big|_0^{\infty} - \int_0^{\infty} e^{-\frac{2\rho}{a}} d\rho \right) d\theta d\phi = \iint_{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi} \frac{Q}{2\pi a} \sin \phi \left(-\int_0^{\infty} e^{-\frac{2\rho}{a}} d\rho \right) d\theta d\phi \\
&= \iint_{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi} \frac{Q}{4\pi} \sin \phi \left(e^{-\frac{2\rho}{a}} \Big|_0^{\infty} \right) d\theta d\phi = \iint_{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi} -\frac{Q}{4\pi} \sin \phi d\theta d\phi = \int_0^{\pi} \left(\int_0^{2\pi} \frac{-Q}{4\pi} \sin \phi d\theta \right) d\phi \\
&= \int_0^{\pi} \frac{-Q}{2} \sin \phi d\phi = \frac{-Q}{2} \cos \phi \Big|_0^{\pi} = -Q
\end{aligned}$$

Note: To evaluate $\rho^2 e^{-\frac{2\rho}{a}} \Big|_{\rho=0}^{\rho=\infty}$, we apply l'Hôpital's rule twice, i.e.,

$$\rho^2 e^{-\frac{2\rho}{a}} \Big|_{\rho=0}^{\rho=\infty} = \lim_{\rho \rightarrow \infty} \frac{\rho^2}{e^{\frac{2\rho}{a}}} = \lim_{\rho \rightarrow \infty} \frac{2\rho}{\frac{2}{a} e^{\frac{2\rho}{a}}} = \lim_{\rho \rightarrow \infty} \frac{2}{\frac{4}{a^2} e^{\frac{2\rho}{a}}} = 0.$$

□