Test 2. Solution)

Q1. a) $\begin{pmatrix} \text{Since} \\ T_0 = 1 \\ \omega_0 = 2\pi \end{pmatrix}$, $\left(c_0 = 1 = c_k \right.$ ← (No derivation required)

$a_0 = 2$, $a_k = 2Re(c_k) = 2$, $b_k = -2Im(c_k) = 0$

$$\Rightarrow \delta_1(t) = \sum_{k=-\infty}^{\infty} \delta(t-k) = \sum_{k=-\infty}^{\infty} e^{j2\pi kt} = 1 + 2\sum_{k=1}^{\infty} \cos(2\pi kt)$$

b) $$c_0 = \frac{A}{2\pi}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} dt = \frac{A}{2}, \quad \omega_0 = \frac{2\pi}{2\pi} = 1$$

The given signal $x(t)$ is time-shifted version of Ex 3-2) a) in the lecture note.

$x(t) = x_a(t + \frac{\pi}{4})$ where FS d_k of $x_a(t)$ is derived as

$$d_k = \begin{cases} 0 & \text{for even } k = 2m \\ \frac{A}{j\pi(2m+1)} & \text{for odd } k = 2m+1 \end{cases}$$

Hence, the FS coefficient c_k of $x(t)$ is given by

$$c_k = e^{jk\frac{\pi}{4}} d_k = \frac{A e^{jk\frac{\pi}{4}}}{j2\pi k}\left[1 - e^{-jk\pi}\right] = \frac{A e^{jk\frac{\pi}{4}}}{j2\pi k}\left(1 - (-1)^k\right)$$

$$= \begin{cases} \emptyset & \text{for even } k = 2m \\ \frac{A e^{jk\frac{\pi}{4}}}{j\pi(2m+1)} & \text{for odd } k = 2m+1 \end{cases}, \quad c_0 = \frac{A}{2}$$

$c_k = \frac{A}{j2\pi k}\left[e^{jk\frac{\pi}{4}} - e^{-jk\frac{3\pi}{4}}\right]$ or $\frac{A e^{-jk\frac{\pi}{4}}}{2} \operatorname{Sinc}\left(\frac{k}{2}\right)$ are all ok.

Test 2. Solution)

Q1. c) $\mathcal{F}(\text{sinc}^2(2Bt)) = \frac{1}{2B}\,\text{tri}\left(\frac{f}{2B}\right)$

$$\mathcal{F}(\cos(2\pi f_0 t)\,\text{sinc}^2(2Bt)) = \frac{1}{4B}\left[\text{tri}\left(\frac{f-f_0}{2B}\right) + \text{tri}\left(\frac{f+f_0}{2B}\right)\right]$$

Hence $\mathcal{F}\left[4B\,\text{sinc}^2(2Bt)\cos(2\pi f_0 t)\right]$

$$= \left[\text{tri}\left(\frac{f-f_0}{2B}\right) + \text{tri}\left(\frac{f+f_0}{2B}\right)\right]$$

Q2. a) $T_0 = 2.\quad \omega_0 = \frac{2\pi}{T_0} = \pi$

$$C_0 = \frac{1}{2}\int_0^2 \frac{3t}{2}\,dt = \frac{3}{4}\cdot\frac{t^2}{2}\Big|_0^2 = \frac{3}{2}$$

$$C_k = \frac{1}{2}\int_0^2 \frac{3t}{2}\,e^{-jk\pi t}\,dt = \frac{3}{4}\int_0^2 t\,e^{-jk\pi t}\,dt$$

$$= \frac{3}{4}\left[\left(\frac{t}{-jk\pi} - \frac{1}{(-jk\pi)^2}\right)e^{-jk\pi t}\Big|_0^2\right]$$

$$= \frac{3}{4}\left[\left(\frac{2}{-jk\pi} - \cancel{\frac{1}{(-jk\pi)^2}}\right)e^{-j2\pi k} - \left(-\cancel{\frac{1}{(-jk\pi)^2}}\right)\right] \qquad \text{where } e^{-j2\pi k} = 1$$

$$= \frac{3j}{2k\pi} \Rightarrow \begin{array}{l} a_0 = 2C_0 = 3 \\ a_k = 2\,\text{Re}(C_k) = 0 \end{array}, \quad b_k = -2\,\text{Im}(C_k) = -\frac{3}{k\pi}$$

Test 2. Solution)

Q 2. a)

$$x(t) = \frac{3}{2} + \left(\frac{3j}{2\pi}\right) \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{k} e^{jk\pi t}$$

$$= \frac{3}{2} + \left(-\frac{3}{\pi}\right) \sum_{k=1}^{\infty} \frac{1}{k} \sin(k \cdot \pi t) \quad \text{for} \quad 0 < t < 2$$

b) Based on the previous result,

$$\frac{3t}{2} - \frac{3}{2} = -\frac{3}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \sin(k\pi t) \quad \text{— (1)}$$

let's assume $t = \frac{1}{2}$, then (1) becomes

$$\sum_{k=1}^{\infty} \frac{1}{k} \sin\left(\frac{k\pi}{2}\right) = \frac{\pi}{4}$$

$$\Rightarrow 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

Hence, the sum of the infinite series is $\frac{\pi}{4}$.

c) $$P = \frac{1}{2} \int_0^2 |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |C_k|^2$$

(left-hand side) $\Rightarrow \frac{1}{2} \cdot \frac{9}{4} \int_0^2 t^2 dt = \frac{9}{8} \cdot \frac{1}{3} t^3 \Big|_0^2 = 3$.

(Right-hand side) $\Rightarrow \frac{9}{4} + 2 \times \frac{9}{4\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2}$

Negative k and Positive k are symmetric

Test 2. Solution

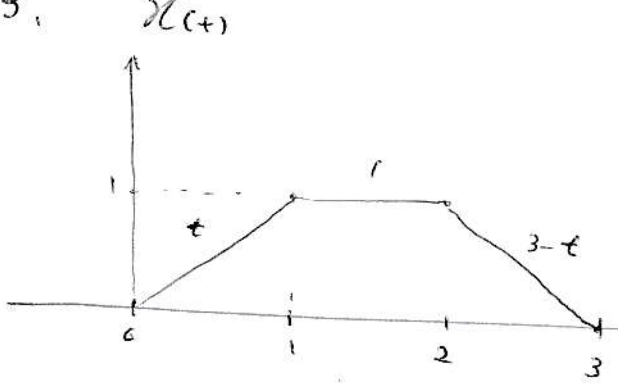
Q2. c) Due to the Parseval's theorem.

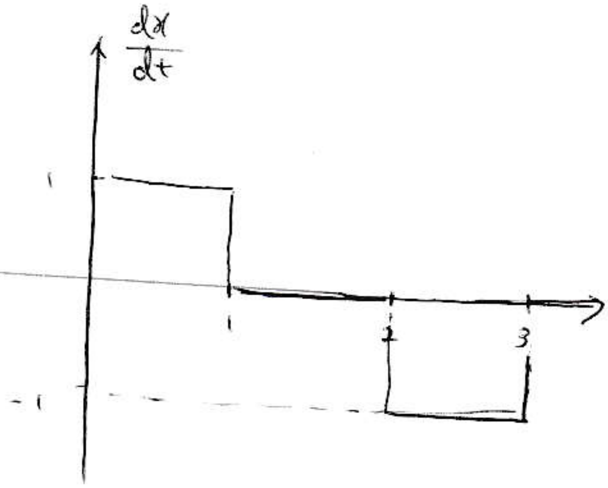
$$3 = \frac{9}{4}\left[1 + \frac{2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\right]$$

$\Rightarrow$ Hence, the sum of the infinite series is

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$$

Q3.





$$\mathcal{F}\left(\frac{dx}{dt}\right) = \int_0^1 e^{-j2\pi f t}\,dt - \int_2^3 e^{-j2\pi f t}\,dt$$

$$= \frac{1}{j2\pi f}\left(1 - e^{-j2\pi f}\right) + \frac{1}{j2\pi f}\left(e^{-j6\pi f} - e^{-j4\pi f}\right)$$

Hence

$$\mathcal{F}(x(t)) = \frac{1}{4(\pi f)^2}\left[e^{-j2\pi f} - 1 - e^{-j6\pi f} + e^{-j4\pi f}\right]$$

Test 2. Solution)

Q4. a) Since the BW of $\cos(150\pi t)$, $\sin(300\pi t)$, $\cos(600\pi t)$

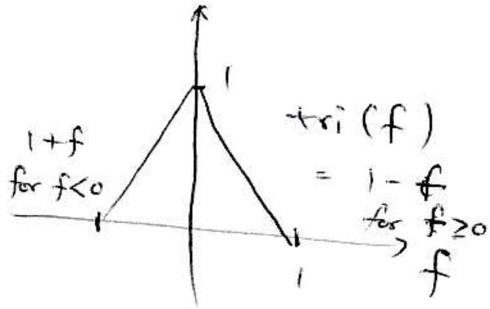
are $f_1 = 75$, $f_2 = 150$, $f_3 = 300$, respectively,

the BW of signal $x(t)$ is $f_M = 300$ Hz.

Hence, the Nyquist sampling rate is

$2f_M = 600$ Hz.

b) $\mathcal{F}(\mathrm{Sinc}^2(t)) = \mathrm{tri}(f)$



3dB BW: $|H(0)| = 1$ and $|H(f_{3dB})| = \dfrac{|H(0)|}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

$= 1 - |f|$

$f_{3dB} = 1 - \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}-1}{\sqrt{2}}$

Equivalent BW: $f_{eq} = \displaystyle\int_0^1 |H(f)|^2 df = \int_0^1 f^2 df = \frac{1}{3}$

Hence $\begin{cases} f_{3dB} = 1 - \dfrac{1}{\sqrt{2}} \text{ or } \dfrac{\sqrt{2}-1}{\sqrt{2}} \\ f_{eq} = \dfrac{1}{3} \end{cases}$

Test 2. Solution)

Q5. a)

$$\{(j2\pi f)^2 + 7(j2\pi f) + 12\} Y(f) = (j2\pi f + 2) X(f)$$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{(j2\pi f + 2)}{(j2\pi f)^2 + 7(j2\pi f) + 12} = \frac{(j2\pi f + 2)}{(j2\pi f + 4)(j2\pi f + 3)}$$

$$= \frac{\alpha_1}{j2\pi f + 4} + \frac{\alpha_2}{j2\pi f + 3}$$

and after Partial Frational Expansion

$\alpha_1 = 2$, $\alpha_2 = -1$

$\mathcal{F}^{-1}$

$$h(t) = \left(2e^{-4t} - e^{-3t}\right) u(t)$$

b) If $x(t) = e^{-2t} u(t)$, then $X(f) = \frac{1}{j2\pi f + 2}$ and

$$Y(f) = H(f)\, X(f)$$

$$= \frac{j2\pi f + 2}{(j2\pi f + 4)(j2\pi f + 3)} \times \frac{1}{j2\pi f + 2}$$

$$= \frac{-1}{j2\pi f + 4} + \frac{1}{j2\pi f + 3}$$

$\mathcal{F}^{-1}$

$$y(t) = \left(-e^{-4t} + e^{-3t}\right) u(t)$$