Exercise:

(a) If $y = \cos[\ln(1+x)]$, show that $(1+x)^2\, y'' + (1+x)\, y' + y = 0$ — (***).

(b) By applying Leibnitz' rule to equation (***), obtain a relation between $y^{(n)}$, $y^{(n+1)}$ and $y^{(n+2)}$, where $y^{(r)}$ denotes $\frac{d^r y}{dx^r}$.

(c) Hence, or otherwise, find the Maclaurin series expansion for $y = \cos[\ln(1+x)]$ in ascending powers of x as far as the term in x^5.

Solution:

(a) $y = \cos[\ln(1+x)]$

Differentiate both sides w.r.t. x :

$$y' = -\sin[\ln(1+x)] \cdot \frac{1}{1+x}$$

$$\Rightarrow (1+x)\,y' = -\sin[\ln(1+x)]$$

Diff. both sides w.r.t. x :

$$(1+x)\,y'' + y' = -\cos[\ln(1+x)] \cdot \frac{1}{1+x}$$

$$\Rightarrow (1+x)^2 y'' + (1+x)\,y' + \underbrace{\cos[\ln(1+x)]}_{=y} = 0$$

$$\therefore (1+x)^2 y'' + (1+x)\,y' + y = 0 \qquad \text{--- } (***)$$

(b) Differentiating both sides of (***) n times w.r.t. x :

$$\left[(1+x)^2 y''\right]^{(n)} + \left[(1+x)y'\right]^{(n)} + y^{(n)} = 0^{(n)}$$

Apply the Leibnitz rule, we have

$$\left[\sum_{k=0}^{n}\binom{n}{k}\left((1+x)^2\right)^{(k)}(y'')^{(n-k)}\right] + \left[\sum_{k=0}^{n}\binom{n}{k}(1+x)^{(k)}(y')^{(n-k)}\right] + y^{(n)} = 0$$

$$\Rightarrow \left[(1+x)^2(y'')^{(n)} + \binom{n}{1}\cdot 2(1+x)(y'')^{(n-1)} + \binom{n}{2}\cdot 2\cdot(y'')^{(n-2)}\right]$$
$$+ \left[(1+x)(y')^{(n)} + \binom{n}{1}\cdot 1\cdot(y')^{(n-1)}\right] + y^{(n)} = 0$$

$$\Rightarrow \left[(1+x)^2 \, y^{(n+2)} + n\cdot 2(1+x)\cdot y^{(n+1)} + \frac{n(n-1)}{2}\cdot 2\cdot y^{(n)}\right]$$
$$+ \left[(1+x)\cdot y^{(n+1)} + n\cdot y^{(n)}\right] + y^{(n)} = 0$$

$$\Rightarrow (1+x)^2 y^{(n+2)} + (2n+1)(1+x)\,y^{(n+1)} + (n^2+1)\,y^{(n)} = 0 \qquad (\text{for } n \geq 2)$$

Differentiating (***) w.r.t. x :

$$(1+x)^2 y^{(3)} + 2(1+x) y'' + (1+x) y'' + y' + y' = 0$$

$$\Rightarrow (1+x)^2 y^{(3)} + 3(1+x) y'' + 2y' = 0$$ ← ∴ The relation is true for $n=1$.

$$\therefore \quad (1+x)^2 y^{(n+2)} + (2n+1)(1+x) y^{(n+1)} + (n^2+1) y^{(n)} = 0 \quad \text{——} (*)$$

is true for $n \geq 1$.

(c) Putting $x=0$ into (*), we have ∵ Maclaurin series

$$y^{(n+2)}(0) = -(2n+1) y^{(n+1)}(0) - (n^2+1) y^{(n)}(0) \quad \text{for } n \geq 1 \quad \text{——} (**)$$

When $x=0$, we have

$$\underline{y^{(0)}(0)} = y(0) = \cos[\ln(1+0)] = \cos 0 = \underline{1}$$

$$\underline{y^{(1)}(0)} = -\sin[\ln(1+0)] \cdot \frac{1}{1+0} = -\sin 0 = \underline{0}$$

$$\underline{y^{(2)}(0)} = -(1+0) y'(0) - y(0) = -0-1 = \underline{-1}$$, from (***)

Using (**), i.e. $y^{(n+2)}(0) = -(2n+1)\, y^{(n+1)}(0) - (n^2+1)\, y^{(n)}(0)$ for $n \geq 1$:

For $n=1$, $\underline{y^{(3)}(0)} = -3y''(0) - 2y'(0) = -3(-1) - 2(0) = \underline{3}$

For $n=2$, $\underline{y^{(4)}(0)} = -5\,y^{(3)}(0) - 5\,y''(0) = -5(3) - 5(-1) = \underline{-10}$

For $n=3$, $\underline{y^{(5)}(0)} = -7\,y^{(4)}(0) - 10\,y^{(3)}(0) = -7(-10) - 10(3) = \underline{40}$

$\therefore$ The Maclaurin series for $y = \cos(\ln(1+x))$ is

$$y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y^{(3)}(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 + \cdots$$

$$= 1 + \frac{0}{1!}x + \frac{(-1)}{2!}x^2 + \frac{3}{3!}x^3 + \frac{(-10)}{4!}x^4 + \frac{40}{5!}x^5 + \cdots$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{12}x^4 + \frac{1}{3}x^5 + \cdots$$