MA1200 Calculus and Basic Linear Algebra I

Lecture Note 5

Exponential Function and Logarithmic Function

Exponential Function

An exponential function $f(x)$ is a function of the following form:

$$f(x) = a^x$$

where a is a real constant which $a > 0$ and $a \neq 1$.

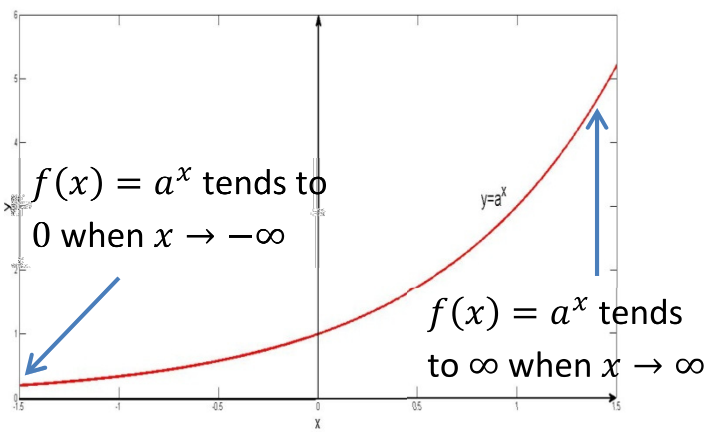
Remark:

- Here, the domain of $f(x)$ is $\mathbb{R}$. (i.e. you can input any real number x into the function $f(x) = a^x$.)
- Here, we do not allow $a \leq 0$ since the function is then undefined for some values of x. For example, one cannot define the values of $0^{-2} = \frac{1}{0^2} = \frac{1}{0}$ and $(-2)^{\frac{1}{2}} = \sqrt{-2}$.
- When $a = 1$, the function simply becomes $f(x) = 1^x = 1$ which is a constant function.

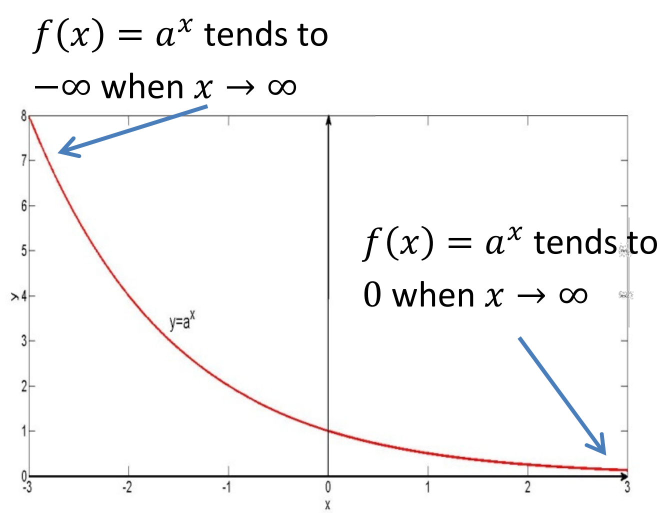
Graph of Exponential Function

From the graphs, we observe that

- The range of the function $f(x)$ is always $(0, \infty)$ (i.e. $0 < f(x) < \infty$).
- For $0 < a < 1$, the function $f(x) = a^x$ is *strictly decreasing*.
  For $a > 1$, the function $f(x) = a^x$ is *strictly increasing*.



$f(x) = a^x$
$a > 1$



$f(x) = a^x$
$0 < a < 1$

Basic Operation of Exponential Functions

(1) $a^x \cdot a^y = a^{x+y}$, (2) $\dfrac{a^x}{a^y} = a^x \cdot a^{-y} = a^{x-y}$.

(3) $(a^x)^y = a^{xy}$

(4) $a^x \cdot b^x = (ab)^x$

Example 1

Compute $\dfrac{3^{2x+1}5^{2x-1}}{15^{2x}}$ and $\dfrac{100^{2x+1}}{25^{2x}}$.

☺Solution:

$$\frac{3^{2x+1}5^{2x-1}}{15^{2x}} = \frac{3^{2x+1} \cdot 5^{2x-1}}{3^{2x} \cdot 5^{2x}} = 3^{2x+1-2x} \cdot 5^{2x-1-2x} = 3 \cdot 5^{-1} = \frac{3}{5},$$

$$\frac{100^{2x+1}}{25^{2x}} = \frac{100 \cdot 100^{2x}}{25^{2x}} = \frac{100 \cdot (25^{2x} \cdot 4^{2x})}{25^{2x}} = 100 \cdot 4^{2x}.$$

Special exponential function e^x

The number e

Consider the expression $\left(1+\frac{1}{n}\right)^n$ where n is real number, this expression has a special property that $\left(1+\frac{1}{n}\right)^n$ tends to a finite number (≈ 2.71828) when n gets larger ($n \to \infty$):

n	100	1,000	10,000	100,000	1,000,000	10,000,000
$\left(1+\frac{1}{n}\right)^n$	2.70481	2.71692	2.71815	2.71827	2.71828	2.71828

This finite number is denoted by e $\overbrace{\left(=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)}^{value\ of\ \left(1+\frac{1}{n}\right)^n\ when\ n=\infty}$, is called *natural base*. The corresponding exponential function $f(x)=e^x$ is called *natural exponential function* (or simply exponential function).

Importance of the exponential function e^x

- Practically, exponential function in many modeling such as population growth in biology and continuously compound interest in finance.
- In calculus, the exponential function e^x has a special property that

$$\frac{d}{dx}e^x = e^x, \quad \int e^x dx = e^x + C.$$

  This property makes the exponential function useful in theoretical aspect (say in the study of differential equation).

- In algebra, the exponential function e^x is an important element in the Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

  where $i = \sqrt{-1}$. (You will learn more in MA1201)

Logarithmic Functions

Given a real number a (with $a > 0$ and $a \neq 1$), the *logarithmic function with base* a, denoted by $y = f(x) = \log_a x$, is the number such that

$$a^y = x.$$

From the definition, one can see that the function $\log_a x$ is the *inverse function* of a^x. So we expect that

$$\log_a(a^x) = x \quad and \quad a^{\log_a x} = x.$$

Two commonly used logarithmic functions

- When $a = 10$, the function $\log_{10} x$ (or simply $\log x$) is called *common logarithm* which is commonly used in actual computation (say solving equation involving exponential function).
- When $a = e$, the function $\log_e x$ is called *natural logarithm* which is commonly used in calculus (e.g. differentiation and integration). We usually denote this function by $\ln x$.

Example 2

Compute $\log_5 25$, $\log_3 \sqrt[3]{9}$, $\log_2 0.5$ and $\ln 1$ using definition.

☺Solution:

$y = \log_5 25 \Rightarrow 5^y = 25 \Rightarrow 5^y = 5^2 \Rightarrow y = \log_5 25 = 2.$

$y = \log_3 \sqrt[3]{9} \Rightarrow 3^y = \sqrt[3]{9} \Rightarrow 3^y = 9^{\frac{1}{3}} \Rightarrow 3^y = 3^{\frac{2}{3}} \Rightarrow y = \log_3 \sqrt[3]{9} = \frac{2}{3}.$

$y = \log_2 0.5 \Rightarrow 2^y = 0.5 \Rightarrow 2^y = \frac{1}{2} \Rightarrow 2^y = 2^{-1} \Rightarrow y = \log_2 0.5 = -1.$

$y = \ln 1 \Rightarrow e^y = 1 \Rightarrow e^y = e^0 \Rightarrow y = \ln 1 = 0.$

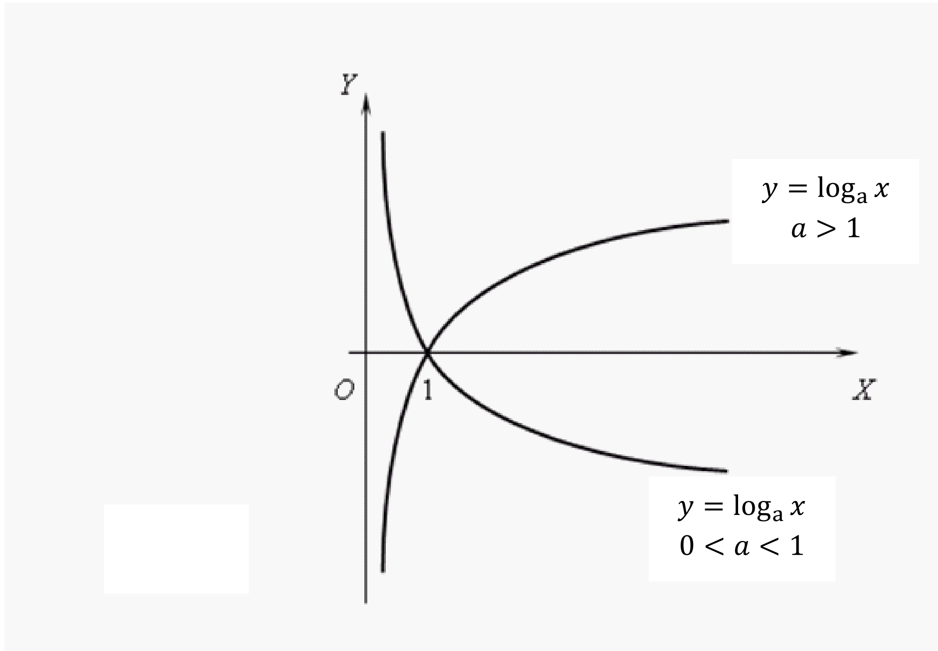
Example 3

Can we find the value of $\log(-2)$?

☺Solution:

$y = \log(-2) \Rightarrow \underbrace{10^y}_{positive} = \underbrace{-2}_{negative}$. So $y = \log(-2)$ does not exist.

Graph of logarithmic functions and its properties



- The domain of $f(x) = \log_a x$ is $(0, \infty)$ (i.e. $0 < x < \infty$).
- The range of $f(x) = \log_a x$ is $\mathbb{R}$.
- For $0 < a < 1$, the function $f(x) = \log_a x$ is decreasing.

  For $a > 1$, the function $f(x) = \log_a x$ is increasing.

Operation of Logarithmic functions

(The proof can be found in Appendix A)

1. $\log_a a^x = x \quad and \quad a^{\log_a x} = x$ (Properties of inverse functions)
2. $\log_a M^n = n \log_a M$
3. $\log_a MN = \log_a M + \log_a N$
4. $\log_a \dfrac{M}{N} = \log_a M - \log_a N$
5. $\log_a M = \dfrac{\log M}{\log a}$.

Example 4

$$2\log 8 + 3\log 5 - 3\log 2 = \log 8^2 + \log 5^3 - \log 2^3$$

$$= \log \frac{8^2 \times 5^3}{2^3} \overset{8=2^3}{\cong} \log \frac{2^6 \times 5^3}{2^3} = \log 10^3 = 3\log 10 \overset{\log 10 = 1}{\cong} 3.$$

Example 5

(a) Solve the equation $3 \cdot 5^{2x-1} + 2 = 23$

(b) Solve the equation $2^{2x+2} - 17 \cdot 2^x + 4 = 0$.

(Hint: Note that $2^{2x+2} = 2^2 \cdot 2^{2x} = 4 \cdot 2^{2x}$)

☺Solution:

(a) $3 \cdot 5^{2x-1} + 2 = 23 \Rightarrow 5^{2x-1} = 7 \Rightarrow (2x-1)\log 5 = \log 7$

$$\Rightarrow x = \frac{1}{2}\left(\frac{\log 7}{\log 5} + 1\right) \approx 1.1045.$$

(b) Let $y = 2^x$, then

$$2^{2x+2} - 17 \cdot 2^x + 4 = 0 \Rightarrow 4 \cdot (2^x)^2 - 17 \cdot 2^x + 4 = 0$$

$$\Rightarrow 4y^2 - 17y + 4 = 0 \Rightarrow (4y-1)(y-4) = 0$$

$$\Rightarrow y = \frac{1}{4} \quad or \quad y = 4$$

$$\Rightarrow 2^x = \frac{1}{4} = 2^{-2} \quad or \quad 2^x = 4 = 2^2$$

$$\Rightarrow x = -2 \quad or \quad x = 2.$$

Example 6

(a) Solve $9^{x+1} = 12^{x-1}$.

(b) $\log(x+1) + \log(2x-1) = \log 14$

☺Solution

(a) By taking logarithm on both sides, we have

$$9^{x+1} = 12^{x-1} \Rightarrow \log 9^{x+1} = \log 12^{x-1}$$
$$\Rightarrow (x+1)\log 9 = (x-1)\log 12 \Rightarrow (\log 12 - \log 9)x = \log 9 + \log 12$$
$$\Rightarrow x = \frac{\log 9 + \log 12}{\log 12 - \log 9}.$$

(b) Note that

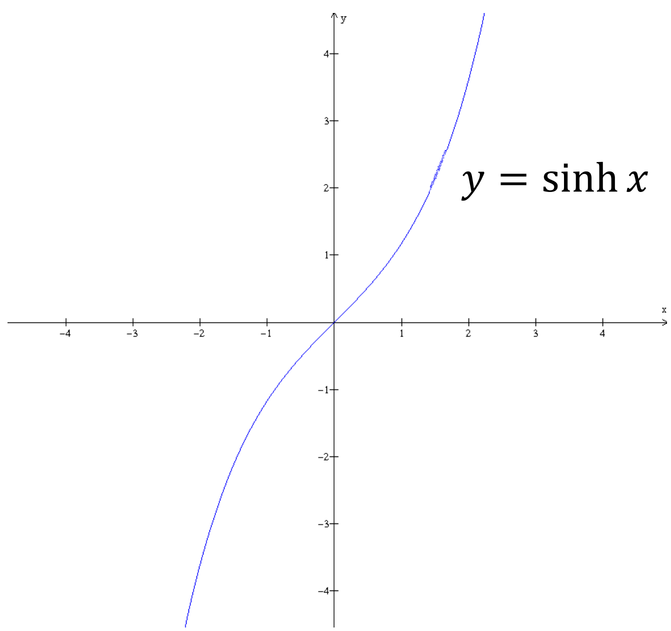
$$\log(x+1) + \log(2x-1) = \log 14 \Rightarrow \log[(x+1)(2x-1)] = \log 14$$
$$\Rightarrow (x+1)(2x-1) = 14 \Rightarrow 2x^2 + x - 15 = 0$$
$$\Rightarrow (x+3)(2x-5) = 0$$
$$\Rightarrow x = -3\ (rejected) \quad or \quad x = \frac{5}{2}.$$

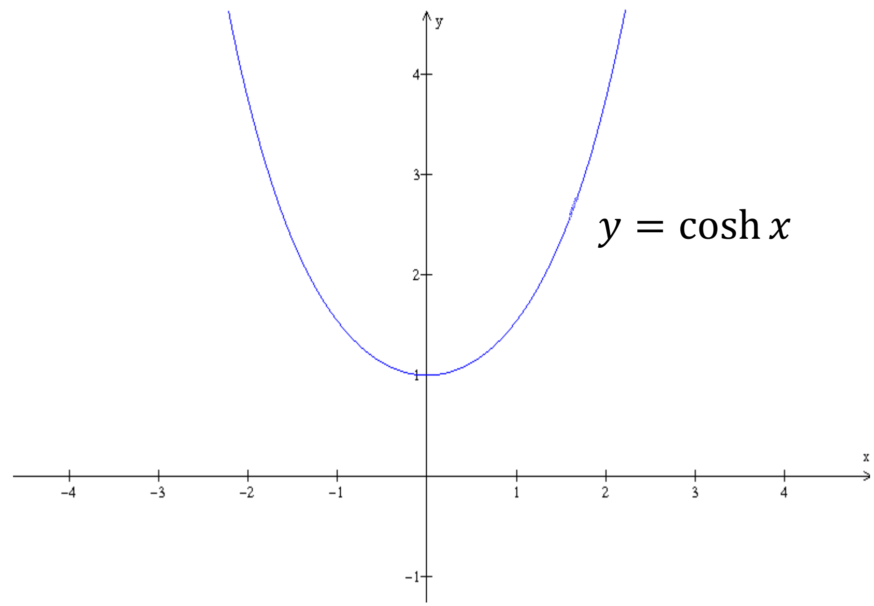
Hyperbolic Sine and Hyperbolic Cosine Functions

The hyperbolic sine and hyperbolic cosine function are defined as the algebraic combination of some exponential functions:

$$\cosh x = \frac{1}{2}(e^x + e^{-x}), \qquad \sinh x = \frac{1}{2}(e^x - e^{-x}).$$

The graphs of these functions are shown below:





Example 7

Solve the equation

$$\sinh x = 2.$$

☺Solution:

Using the definition of $\sinh x$, we have

$$\sinh x = 2 \Rightarrow \frac{1}{2}(e^x - e^{-x}) = 2 \Rightarrow e^x - e^{-x} = 4$$

$$\Rightarrow e^{2x} - 1 = 4e^x \Rightarrow e^{2x} - 4e^x - 1 = 0.$$

We let $y = e^x$, then the equation becomes

$$y^2 - 4y - 1 = 0 \Rightarrow y = \frac{4 \pm \sqrt{20}}{2} = 1 \pm \sqrt{5}$$

$$\Rightarrow e^x = 1 + \sqrt{5} \ \ or\ 1 - \sqrt{5}\ (rej.)$$

$$\Rightarrow x = \ln(1 + \sqrt{5}).$$

Example 8

(a) Using the definition of $\sinh x$ and $\cosh x$, show that

(i) $\cosh^2 x - \sinh^2 h = 1$

(ii) $\cosh^2 x + \sinh^2 h = \cosh 2x$

(b) Hence, or otherwise, solve the equation $3\sinh x + \cosh 2x = 0$.

☺Solution of (a)

$$\cosh^2 x - \sinh^2 x = \left[\frac{1}{2}(e^x + e^{-x})\right]^2 - \left[\frac{1}{2}(e^x - e^{-x})\right]^2$$

$$= \frac{1}{4}\left[e^{2x} + 2\underbrace{e^x e^{-x}}_{=1} + e^{-2x} - \left(e^{2x} - 2\underbrace{e^x e^{-x}}_{=1} + e^{-2x}\right)\right] = \frac{1}{4}(4) = 1$$

$$\cosh^2 x + \sinh^2 h = \left[\frac{1}{2}(e^x + e^{-x})\right]^2 - \left[\frac{1}{2}(e^x - e^{-x})\right]^2$$

$$= \frac{1}{4}\left[e^{2x} + 2\underbrace{e^x e^{-x}}_{=1} + e^{-2x} + \left(e^{2x} - 2\underbrace{e^x e^{-x}}_{=1} + e^{-2x}\right)\right] = \frac{1}{2}(e^{2x} + e^{-2x})$$

$$= \cosh 2x$$

☺Solution of (b)

Note that

$$3\sinh x + \cosh 2x = 0 \Rightarrow 3\sinh x + (\cosh^2 x + \sinh^2 h) = 0$$

$$\Rightarrow 3\sinh x + (1 + \sinh^2 x) + \sinh^2 x = 0$$

$$\Rightarrow 2\sinh^2 x + 3\sinh x + 1 = 0.$$

Let $y = \sinh x$, then the equation becomes

$$2y^2 + 3y + 1 = 0 \Rightarrow (2y+1)(y+1) = 0 \Rightarrow y = -\frac{1}{2} \quad or \quad y = -1.$$

$\Rightarrow \sinh x = -\frac{1}{2}$ or $\sinh x = -1$.

$$\Rightarrow \frac{1}{2}(e^x - e^{-x}) = -\frac{1}{2} \quad and \quad \frac{1}{2}(e^x - e^{-x}) = -1.$$

Using similar techniques as in Example 7, we get

$$x = \ln\frac{-1+\sqrt{5}}{2} \quad or \quad x = \ln\left(-1+\sqrt{2}\right).$$

Summary of Chapter 3, 4 and 5

In these 3 chapters, we learn a lot about the properties of some elementary functions such as polynomials, rational functions, trigonometric functions, exponential and logarithmic functions. These properties will e useful when we study the calculus.

Things you need to know

Chapter 3 -- Polynomial and rational functions

- Factorize the polynomial using remainder theorem and factor theorem.
- Method of partial functions – "Decompose" a complicated rational functions into a sum of simpler fractions:

$$\frac{5x+3}{x^3-2x^2-3x} = -\frac{1}{x} - \frac{1}{2(x+1)} + \frac{3}{2(x-3)}.$$

Chapter 4 – Trigonometric Functions

- Properties of trigonometric functions
- Compound angle formula $\cos(A+B)$, $\sin(A+B)$ and $\tan(A+B)$
- Sum-to-product formula (useful in computing limits) and Product-to-sum (useful in differentiation/ integration)

$$\underbrace{\frac{\sin 5\theta + \sin 3\theta}{\cos 5\theta - \cos 3\theta}}_{\substack{difficult\ to \\ calculate}} \overset{\substack{sum-to-product \\ formula}}{\triangleq} \underbrace{-\frac{\sin 4\theta \cos \theta}{\sin 4\theta \sin \theta}}_{\substack{easy\ to \\ calculate}}$$

$$\underbrace{\sin^2\theta \cos^3\theta}_{difficult\ for\ calculus} \overset{\substack{product-to-sum \\ formula}}{\triangleq} \underbrace{\frac{1}{8}\cos\theta - \frac{1}{16}\cos 5\theta - \frac{1}{16}\cos 3\theta}_{easy\ for\ calculus}$$

Chapter 5 – Exponential and Logarithmic Function

- Basic properties of exponential and logarithmic function
- The natural exponential function $f(x) = e^x$ and natural logarithmic function $g(x) = \ln x$.

Appendix A – Proof of the "Operation of Logarithmic Function"

Property 1

It follows from the fact that $\log_a x$ is the inverse function of a^x.

Property 2

We let $A = \log_a M \Leftrightarrow M = a^A$.

Let $y = \log_a M^n$, then by the definition of logarithmic function, we have

$a^y = M^n \Rightarrow a^y = a^{An} \Rightarrow y = nA \Rightarrow \log_a M^n = n \log_a M$.

Property 3 & 4

We let $A = \log_a M \Leftrightarrow M = a^A$ and $B = \log_a N \Leftrightarrow N = a^B$.

Let $y = \log_a MN$, then by the definition of logarithmic function, we have

$a^y = MN \Rightarrow a^y = a^A a^B \Rightarrow a^y = a^{A+B} \Rightarrow y = A + B$

$\Rightarrow \log_a MN = \log_a M + \log_a N$.

Let $z = \log_a \frac{M}{N}$, then we have

$$a^z = \frac{M}{N} \Rightarrow a^z = \frac{a^A}{a^B} \Rightarrow a^z = a^{A-B} \Rightarrow z = A - B$$

$$\Rightarrow \log_a \frac{M}{N} = \log_a M - \log_a N.$$

<u>Property 5</u>

We let $A = \log M$ and $B = \log a$ and we let $y = \log_a M$, then we have

$$a^y = M \Rightarrow a^y = 10^A \Rightarrow \log a^y = \log 10^A$$

$$\overset{\substack{From \\ Property\ 2}}{\Rightarrow} y \log a = A \Rightarrow y = \frac{A}{\log a} \Rightarrow \log_a M = \frac{A}{B}.$$