MA1200

Practice Exercise for Ch. 4 Trigonometric Functions and Inverse Trigonometric Functions

Solutions

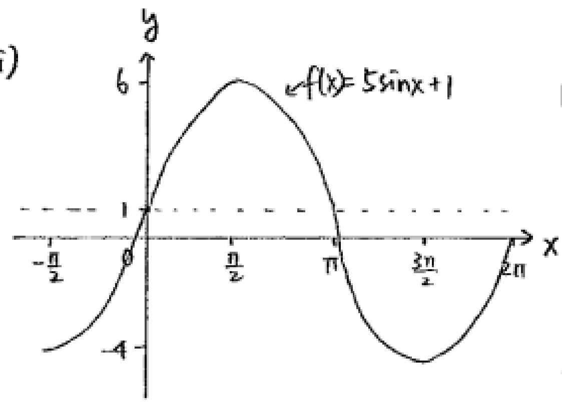
1. (a) Convert the following angles to radians.

(i) $\frac{4\pi}{15}$ rad (ii) $\frac{2\pi}{3}$ rad (iii) $\frac{7\pi}{4}$ rad

(b) Convert the following angles to degree.

(i) 30° (ii) 123° (iii) -72°

(2) (a)(i)



(ii) Domain: $\mathbb{R}$

Range: $[-4,6]$

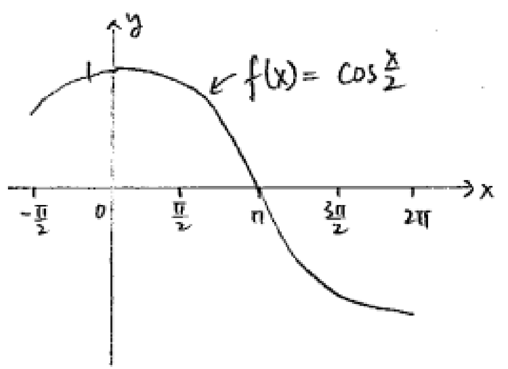
(iii) Since $f(x+2\pi) = 5\sin(x+2\pi)+1$

$= 5\sin x+1$

$= f(x)$

$\therefore f(x)$ is periodic with $T=2\pi$

(b) (i)



(ii) Domain: $\mathbb{R}$

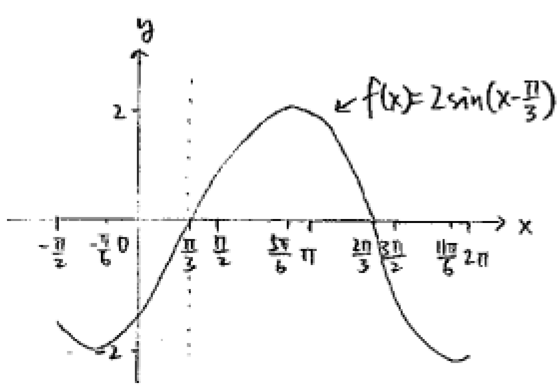
Range: $[-1,1]$

(iii) Since $f(x+4\pi) = \cos\left[\frac{x+4\pi}{2}\right]$

$= \cos\left(\frac{x}{2}+2\pi\right) = \cos\frac{x}{2} = f(x)$

$\therefore f(x)$ is periodic with $T=4\pi$

(c) (i)



(ii) Domain: $\mathbb{R}$

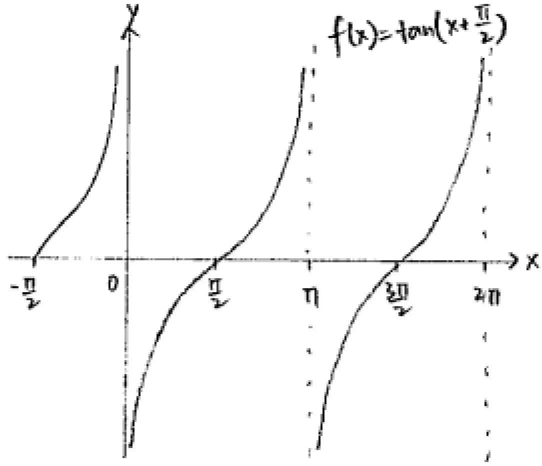
Range: $[-2,2]$

(iii) Since $f(x+2\pi) = 2\sin\left(x+2\pi-\frac{\pi}{3}\right)$

$= 2\sin\left(x-\frac{\pi}{3}\right) = f(x)$

$\therefore f(x)$ is periodic with $T=2\pi$

(d) (i)



(ii) Domain: $\mathbb{R}\setminus\{x \mid x=n\pi,\ n\in\mathbb{Z}\}$

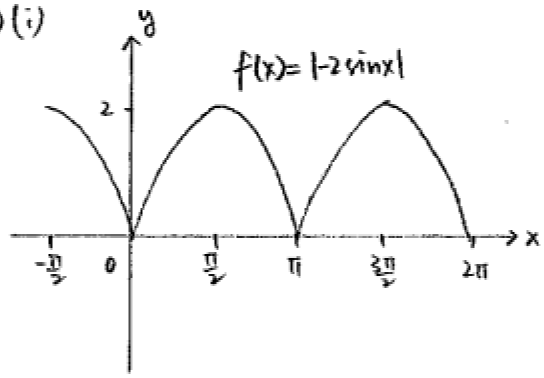
Range: $\mathbb{R}$

(iii) Since $f(x+\pi)=\tan(x+\pi+\frac{\pi}{2})$

$= \tan(x+\frac{\pi}{2}) = f(x)$

$\therefore$ $f(x)$ is periodic with $T=\pi$

(e) (i)



(ii) Domain: $\mathbb{R}$

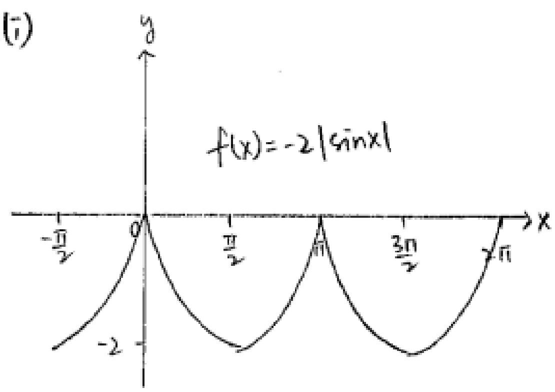
Range: $[0,2]$

(iii) Since $f(x+\pi)=|-2\sin(x+\pi)|$

$=|-2\cdot(-\sin x)| = |-2\sin x| = f(x)$

$\therefore$ $f(x)$ is periodic with $T=\pi$

(f) (i)



(ii) Domain: $\mathbb{R}$

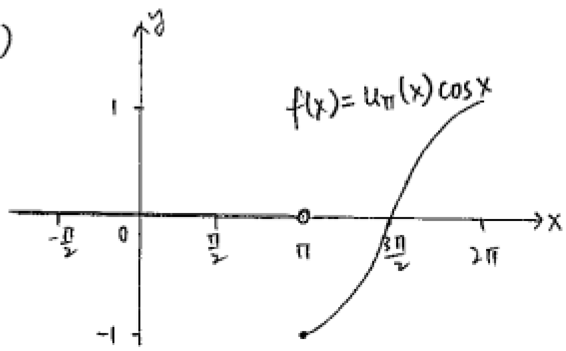
Range: $[-2,0]$

(iii) Since $f(x+\pi)=-2|\sin(x+\pi)|$

$=-2|-\sin x| = -2|\sin x| = f(x)$

$\therefore$ $f(x)$ is periodic with $T=\pi$

(3) (a) (i)

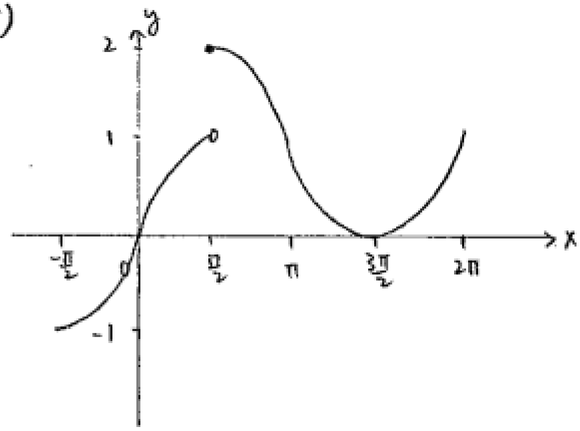


$$f(x)=u_\pi(x)\cos x = \begin{cases} 0 & x<\pi \\ \cos x & x\geq\pi \end{cases}$$

(ii) Domain: $\mathbb{R}$

Range: $[-1,1]$

(b) (i)

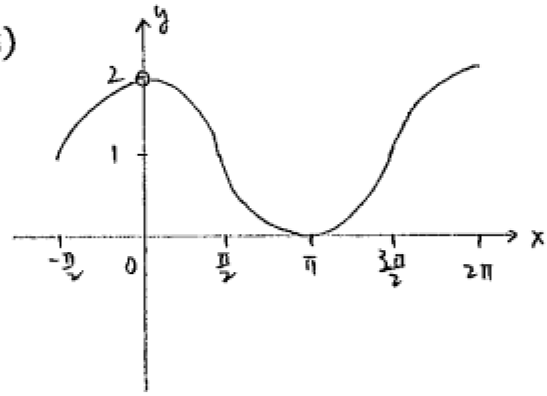


$$f(x) = u_{\frac{\pi}{2}}(x) + \sin x$$
$$= \begin{cases} \sin x & x < \frac{\pi}{2} \\ 1 + \sin x & x \geq \frac{\pi}{2} \end{cases}$$

(ii) Domain: $\mathbb{R}$

Range: $[-1, 2]$

(c) (i)



$$f(x) = \frac{x}{x} + \cos x$$
$$= 1 + \cos x \quad \text{for } x \neq 0$$

$f(x)$ is undefined for $x = 0$

(ii) Domain: $\mathbb{R} \setminus \{0\}$

Range: $[0, 2]$

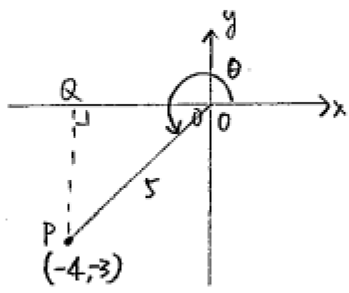
(4). (a)

$$LHS = \frac{\frac{1-\cos^2\theta}{\cos\theta}}{\frac{1-\sin^2\theta}{\sin\theta}} = \frac{\frac{\sin^2\theta}{\cos\theta}}{\frac{\cos^2\theta}{\sin\theta}} = \frac{\sin^2\theta}{\cos\theta} \cdot \frac{\sin\theta}{\cos^2\theta} = \tan^3\theta = RHS$$

(b)

$$LHS = \frac{\csc^2\theta}{1+\tan^2\theta} = \frac{\csc^2\theta}{\sec^2\theta} = \frac{\cos^2\theta}{\sin^2\theta} = \cot^2\theta = \csc^2\theta - 1 = RHS$$

(5).



$\cos\theta = -\frac{4}{5}$ $\quad\theta$ in Quadrant III.

$PQ^2 + OQ^2 = OP^2$

$PQ = \sqrt{5^2 - 4^2} = 3$

$\therefore$ y-coordinate of P is -3.

(a) $\sin\theta = \frac{-3}{5}$ $\qquad$ (b) $\tan\theta = \frac{-3}{-4} = \frac{3}{4}$

(c) $\csc\theta = \frac{1}{\sin\theta} = -\frac{5}{3}$

(6) (a) $\dfrac{\sin(\frac{\pi}{2}+\theta)\cos(\frac{3\pi}{2}-\theta)}{\sec(\theta-\pi)} = \dfrac{\cos\theta\,(-\sin\theta)}{\sec[-(\pi-\theta)]} = \dfrac{-\cos\theta\sin\theta}{\sec(\pi-\theta)}$

$= \dfrac{-\cos\theta\sin\theta}{-\sec\theta} = \sin\theta\cos^2\theta$

(b). $\dfrac{\tan(\theta+\frac{3\pi}{2})\cot(\frac{3\pi}{2}+\theta)}{\csc(\theta-\frac{\pi}{2})} = \dfrac{(-\cot\theta)(-\tan\theta)}{\csc[-(\frac{\pi}{2}-\theta)]} = \dfrac{1}{-\csc(\frac{\pi}{2}-\theta)} = -\sin(\frac{\pi}{2}-\theta) = -\cos\theta$

(7) (a) $\sin(\sin^{-1}\frac{2}{5}) = \frac{2}{5}$ (b) $\sin^{-1}(\sin\frac{\pi}{4}) = \frac{\pi}{4}$

(c) $\sin^{-1}(\sin(-\frac{2\pi}{3})) = \sin^{-1}(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$ (d) $\sin^{-1}(\tan(\frac{3\pi}{4})) = \sin^{-1}(-1) = -\frac{\pi}{2}$

(e) $\cos(\cos^{-1}\frac{3}{4}) = \frac{3}{4}$ (f) $\cos^{-1}(\cos\frac{5\pi}{4}) = \cos^{-1}(-\frac{\sqrt{2}}{2}) = \frac{3\pi}{4}$

(g) $\cos^{-1}(\sin(-\frac{\pi}{6})) = \cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$ (h) $\tan^{-1}(\tan\pi) = \tan^{-1}(0) = 0$

(8) (a) $\sin 35^\circ\cos 25^\circ + \sin 25^\circ + \cos 35^\circ$

$= \sin(35^\circ+25^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2}$

(b) $\tan 165^\circ = \tan(135^\circ+30^\circ) = \dfrac{\tan 135^\circ+\tan 30^\circ}{1-\tan 135^\circ\tan 30^\circ}$ (where $\tan 135^\circ$
$= \tan(180^\circ-45^\circ)$
$= -\tan 45^\circ = -1$

$= \dfrac{-1+\frac{1}{\sqrt{3}}}{1-(-1)(\frac{1}{\sqrt{3}})} = \dfrac{-\sqrt{3}+1}{\sqrt{3}+1}$

$= \dfrac{1-\sqrt{3}}{1+\sqrt{3}}$ $\left(= \dfrac{1-\sqrt{3}}{1+\sqrt{3}}\cdot\dfrac{1-\sqrt{3}}{1-\sqrt{3}} = \dfrac{1-2\sqrt{3}+3}{1-3} = \dfrac{4-2\sqrt{3}}{-2} = \sqrt{3}-2\right)$

(9) (a) To prove $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

Method 1

LHS $= \cos(A+B)\cos(A-B)$

$= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$

$= (\cos A\cos B)^2 - (\sin A\sin B)^2$

$= \cos^2 A\,(1-\sin^2 B) - \sin^2 B\,(1-\cos^2 A)$

$= \cos^2 A - \sin^2 B$

$=$ RHS

Method 2

$$LHS = \cos(A+B)\cos(A-B)$$

Take $x = A+B$, $y = A-B$

$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$

$$= \frac{1}{2}[\cos 2A + \cos 2B]$$

$$= \frac{1}{2}[(2\cos^2 A - 1) + (1 - 2\sin^2 B)]$$

$$= \cos^2 A - \sin^2 B$$

$$= RHS$$

(b). To prove $\frac{\sin 2A}{\cos 2A + 1} = \tan A$

$$LHS = \frac{\sin 2A}{\cos 2A + 1} = \frac{2\sin A\cos A}{2\cos^2 A - 1 + 1} = \frac{\sin A}{\cos A} = \tan A = RHS$$

(c) To prove $(\sin A - \cos A)^2 = 1 - \sin 2A$

$$LHS = (\sin A - \cos A)^2$$

$$= \sin^2 A - 2\sin A\cos A + \cos^2 A = 1 - 2\sin A\cos A = 1 - \sin 2A = RHS$$

(10).
$$\frac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ} = \frac{2\cos\frac{75^\circ+15^\circ}{2}\sin\frac{75^\circ-15^\circ}{2}}{2\cos\frac{75^\circ+15^\circ}{2}\cos\frac{75^\circ-15^\circ}{2}} = \frac{2\cos 45^\circ \sin 30^\circ}{2\cos 45^\circ \cos 30^\circ}$$

$$= \tan 30^\circ = \frac{1}{\sqrt{3}}.$$

(11). To prove $4\cos A\cos\left(\frac{2\pi}{3} + A\right)\cos\left(\frac{2\pi}{3} - A\right) = \cos 3A$

$$LHS = 4\cos A\cos\left(\frac{2\pi}{3} + A\right)\cos\left(\frac{2\pi}{3} - A\right)$$

$$= 4\cos A\left[\frac{1}{2}\left(\cos\left(\frac{4\pi}{3}\right) + \cos 2A\right)\right]$$

$$= 2\cos A\left[-\frac{1}{2} + \cos 2A\right]$$

$$= -\cos A + 2\cos A\cos 2A = -\cos A + \cos(A+2A) + \cos(A-2A)$$

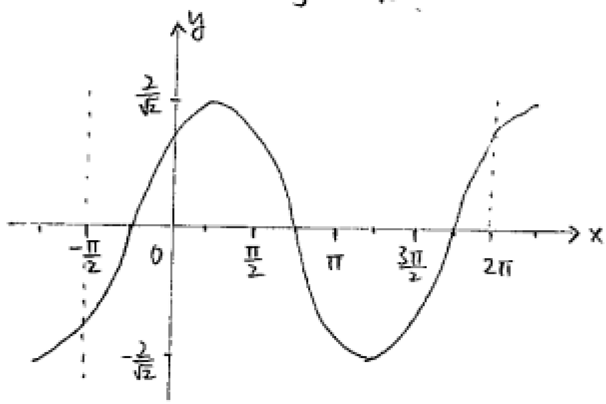
$$= -\cos A + \cos 3A + \cos A = \cos 3A = RHS$$

(12). (a) $\sin(x+45°) = \sin x \cos 45° + \cos x \sin 45°$

$$= \frac{\sqrt{2}}{2}\sin x + \frac{\sqrt{2}}{2}\cos x = \frac{\sqrt{2}}{2}(\sin x + \cos x)$$

(b) $y = \cos x + \sin x$

$$= \frac{2}{\sqrt{2}}\left[\frac{\sqrt{2}}{2}(\sin x + \cos x)\right] = \frac{2}{\sqrt{2}}\sin(x+45°)$$



(13) (a) Notice that $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$, so that

$\frac{x}{2} = 2n\pi \pm \frac{\pi}{6}$ $\therefore$ $x = 4n\pi \pm \frac{\pi}{3}$, where n is any integer.

(b)

$$2\sin^2 x + \sin x - 1 = 0$$
$$(2\sin x - 1)(\sin x + 1) = 0$$
$$2\sin x - 1 = 0 \quad \text{or} \quad \sin x + 1 = 0$$
$$\sin x = \frac{1}{2} \quad \text{or} \quad \sin x = -1$$
$$x = n\pi + (-1)^n \frac{\pi}{6} \quad \text{or} \quad x = n\pi + (-1)^n\left(-\frac{\pi}{2}\right)$$

$\therefore$ The solution is $x = n\pi + (-1)^n \frac{\pi}{6}$ or $x = n\pi - (-1)^n\left(\frac{\pi}{2}\right)$, where n is any integer.