Unit 1

Sets

Albert Sung

The Barber Paradox

- The barber is a man in town who shaves those and only those men who do not shave themselves.

- *Q:* Who shaves the barber?

The Barber Paradox

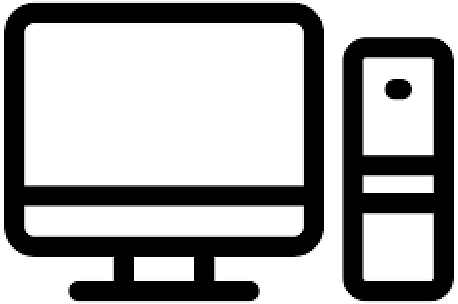
- (1-min video) https://www.youtube.com/watch?v=qQs2ZHV_WBk

The Halting Problem

```
i = 1
while i != 10:
        i += 2
print('Hello world!')
```

```
i = 2
while i != 10:
        i += 2
print('Hello world!')
```

Can you write a program to check whether any given program will halt or not?



Outline of Unit 1

- 1.1 Basic Concepts
- 1.2 Proofs Involving Sets
- 1.3 Functions
- 1.4 Russell's Paradox
- 1.5 The Halting Problem

Unit 1.1

Basic Concepts

Sets

- A set is a collection of objects.
- A is a subset of B, written as $A \subseteq B$, if every member of A is also a member of B.
  - It is a proper subset of B if B contains some elements that are not in A.
    - i.e., A is not the same as B.
- B is then said to be a superset of A.
- The cardinality of a set A is defined as the number of elements in the set.
- It is denoted by $|A|$.
  - If $|A|$ is finite, A is called a finite set.
  - Otherwise, A is called an infinite set.

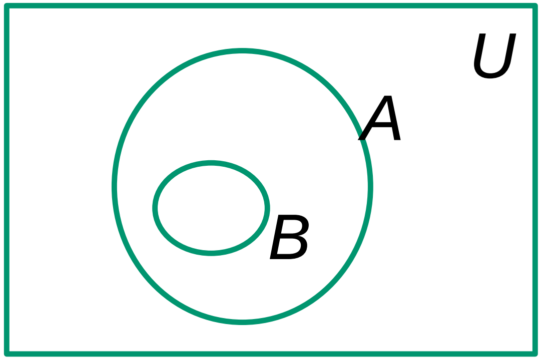
Some Common Sets in Math

Set	Symbols
Natural Numbers*	$\mathbb{N} = \{1, 2, 3, \dots\}$
Whole Numbers	$\mathbb{N} \cup \{0\}$ or $\mathbb{Z}_{\geq 0}$
Integers	$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
Binary Numbers	$\mathbb{B} = \{0, 1\}$
Rational Numbers	$\mathbb{Q}$
Real Numbers	$\mathbb{R}$
Complex Numbers	$\mathbb{C}$

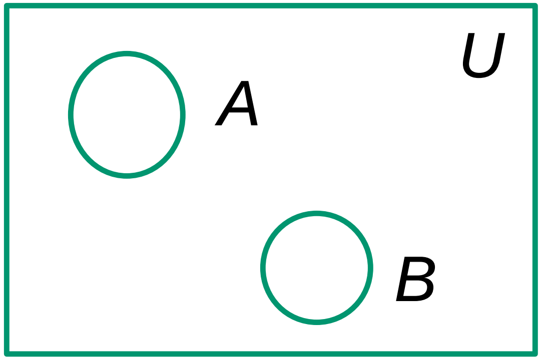
*In some convention, 0 is included in the set of natural numbers.

Relationship between Sets

- A universal set U is a set containing everything that we are considering.
- Venn diagram
  - U is represented by a rectangular box.
  - Subsets of U (e. g. A and B) are represented by circles (more precisely, regions inside closed curves).
- A and B are disjoint if they have no elements in common.

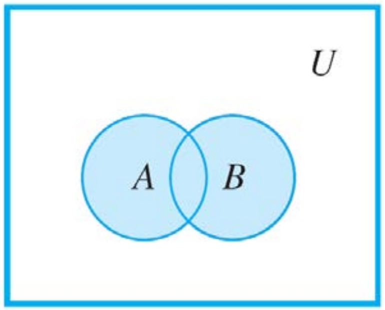


B is a subset of A.

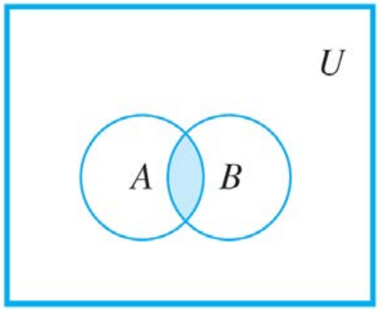


A and B are disjoint.

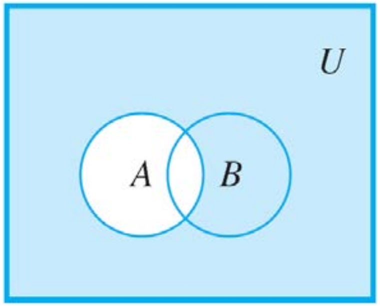
Fundamental Operations



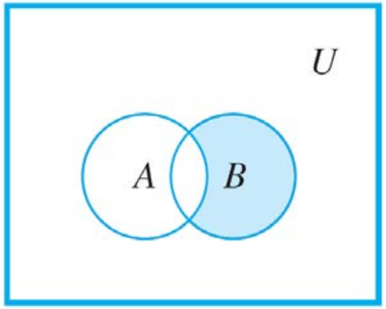
Union: $A \cup B$



Intersection: $A \cap B$



Complement: A^c or $\bar{A}$

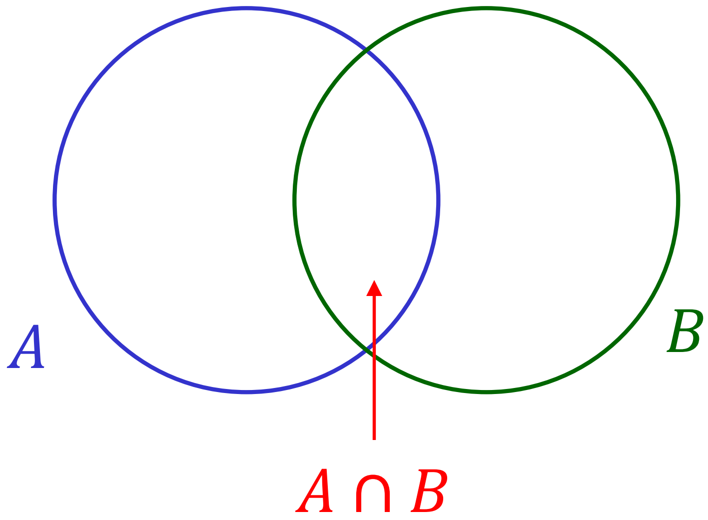


Difference: $B \setminus A$

Inclusion-Exclusion Principle

- For finite sets A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|$$



When adding $|A|$ and $|B|$, $A \cap B$ has been counted twice. That's why we need to subtract it.

Classwork

Consider the numbers 1, 2, …, 100.

How many of them are divisible by 2 or by 3?

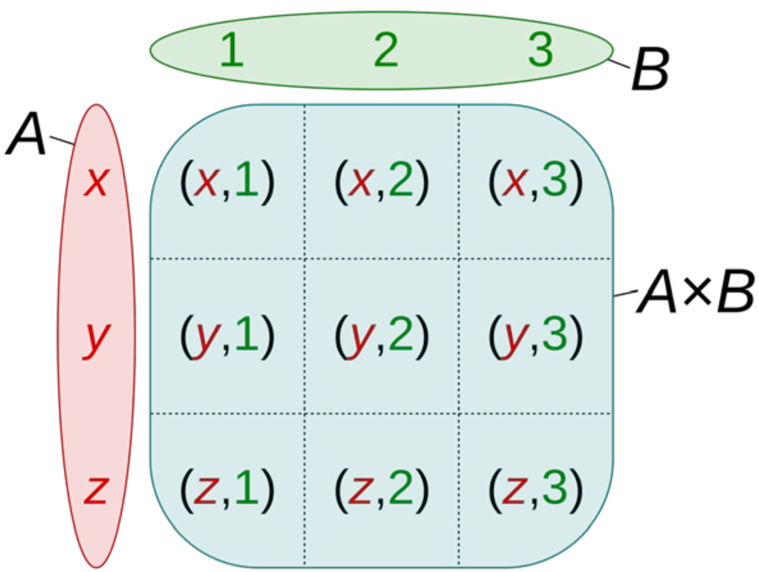
- Solution:

Cartesian Product

- The Cartesian product $A \times B$ of the sets A and B is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.

$$A \times B \triangleq \{(a, b) | a \in A \ \wedge b \in B\}.$$

- Example:



Ordered pair:

- The order is important: $(a, b) \neq (b, a)$

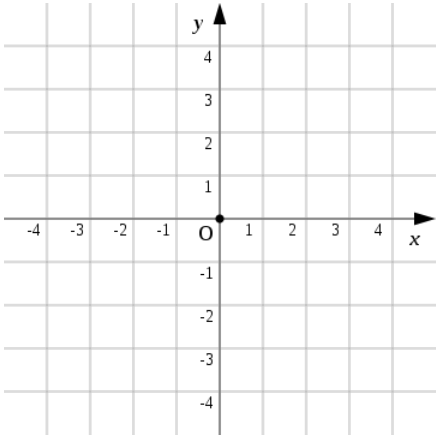
What is $|A \times B|$?

Cartesian Product

- The Cartesian product can be generalized to more than two sets, e.g., $A \times B \times C$.
- If the same set is involved, we write

$$\underbrace{A \times A \times \cdots \times A}_{n} = A^n$$

- For example, the x-y plane is $\mathbb{R}^2$.



Power Set

- Given a set A, the set of all its subsets, denoted by $\mathcal{P}(A)$, is called the power set of A.

- Example:
  - Suppose $A = \{1, 2, 3\}$.
  - List all subsets of A:

    $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and $\{1, 2, 3\}$.
  - Hence,

    $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

Cardinality of Power Set

- Suppose $|A| = n$.
- What is $|\mathcal{P}(A)|$?

$$A = \{1,\ 2,\ 3,\ \ldots,\ n\}.$$

Subset: $\{\qquad\qquad\qquad\}$

For each element, there are two possibilities:

- Put it into the subset, or not.

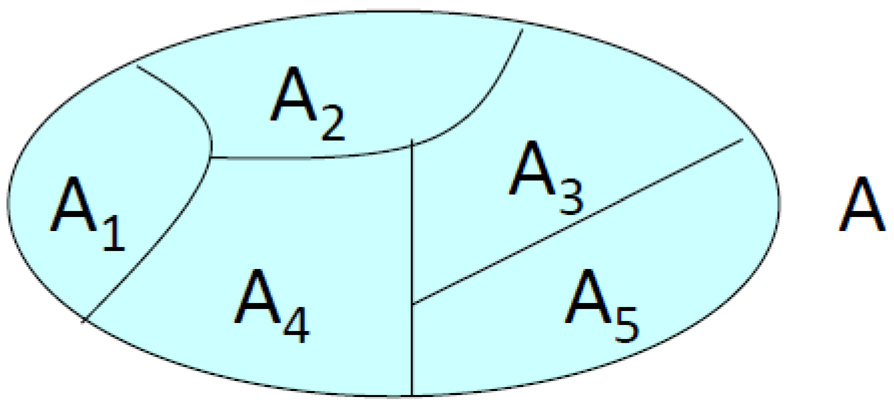
Partition

- A collection of non-empty sets $\{A_1, A_2, \ldots, A_n\}$ is a partition of a set A iff

i. $A = A_1 \cup A_2 \cup \cdots \cup A_n$, and

ii. $A_1, A_2, \ldots, A_n$ are mutually disjoint, i.e.,

$A_i \cap A_j = \emptyset$ for all $i, j = 1, 2, \ldots, n$ and $i \neq j$.



Note: Partition itself is a set.

Bell Numbers

- Consider the set $S_n = \{1, 2, \ldots, n\}$.
- The number of different ways to partition S_n is called the Bell number, denoted by B_n.
  - S_1: $\{\{1\}\}$ is the only partition, so $B_1 = 1$.
  - S_2: $\{\{1\}, \{2\}\}$ and $\{\{1, 2\}\}$ are the partitions, so $B_2 = 2$.
  - S_3: $\{\{1\}, \{2\}, \{3\}\}$, $\{\{1\}, \{2, 3\}\}$, $\{\{2\}, \{1, 3\}\}$, $\{\{3\}, \{1, 2\}\}$, and $\{\{1, 2, 3\}\}$ are the partitions, so $B_3 = 5$.
- How about S_0?
  - S_0 is the empty set $\emptyset$. Its only partition is $\emptyset$, *not* $\{\emptyset\}$.
  - Hence, $B_0 = 1$.

Unit 1.2

Proofs Involving Sets

Subset Relationship

- To prove a subset relationship, the element argument is usually used.

- To prove $A \subseteq B$,
  1) Suppose x is an arbitrarily chosen element of A.
  2) Show that x is also an element of B.
  3) Therefore, $A \subseteq B$.

Example

- Prove that $A \subseteq B$, where

$$A = \{m \in \mathbf{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbf{Z}\}$$
$$B = \{n \in \mathbf{Z} \mid n = 3s \text{ for some } s \in \mathbf{Z}\}.$$

Z denotes the set of integers.

- Proof:
  - Let x be an element of A, so there is an integer r such that
  $$x = 6r + 12 = 3(2r + 4).$$
  - Since $2r + 4$ is an integer, by definition of B, x is an element of B.
  - Therefore, $A \subseteq B$. *Q.E.D.*

Example

- Prove $A \cap B \subseteq A$.

- Proof:
  - Let x be an arbitrary element in $A \cap B$.
  - By definition of intersection, $x \in A$ and $x \in B$.
  - Therefore, $x \in A$ (by simplification rule in propositional logic).

Q.E.D.

Set Equality

- Two sets are the same (or equal) if and only if
  - they contain the same elements, or equivalently,
  - each is a subset of the other.

$$A = B \quad \Leftrightarrow \quad A \subseteq B \text{ and } B \subseteq A.$$

You need to prove both directions: (i) $A \subseteq B$ (ii) $B \subseteq A$

- Exercise:
  - Suppose

$$A = \{m \in \mathbf{Z} \mid m = 2a \text{ for some integer } a\}$$

$$B = \{n \in \mathbf{Z} \mid n = 2b - 2 \text{ for some integer } b\}$$

  - Are they equal?

Example: De Morgan's Law for Sets

- Prove that $(A \cup B)^c = A^c \cap B^c$.

Idea: The proof should consists of two parts:

1) $(A \cup B)^c \subseteq A^c \cap B^c$
   - Let x be an arbitrary element in $(A \cup B)^c$.
   - Show that $x \in A^c \cap B^c$.
2) $A^c \cap B^c \subseteq (A \cup B)^c$
   - Let x be an arbitrary element in $A^c \cap B^c$.
   - Show that $x \in (A \cup B)^c$.

Example: (Part 1)

1) Prove $(A \cup B)^c \subseteq A^c \cap B^c$.

- Let x be an arbitrary element in $(A \cup B)^c$.
- $x \in (A \cup B)^c \equiv x \notin A \cup B$ (by def. of complement)

$\equiv \sim(x \in A \cup B)$

$\equiv \sim(x \in A \text{ or } x \in B)$ (by def. of union)

$\equiv x \notin A \text{ and } x \notin B$ (by de Morgan's Law in logic)

$\equiv x \in A^c \text{ and } x \in B^c$ (by def. of complement)

$\equiv x \in A^c \cap B^c$ (by def. of intersection)

Classwork: (Part 2)

2) Prove $A^c \cap B^c \subseteq (A \cup B)^c$.

- Let x be an arbitrary element in $A^c \cap B^c$.

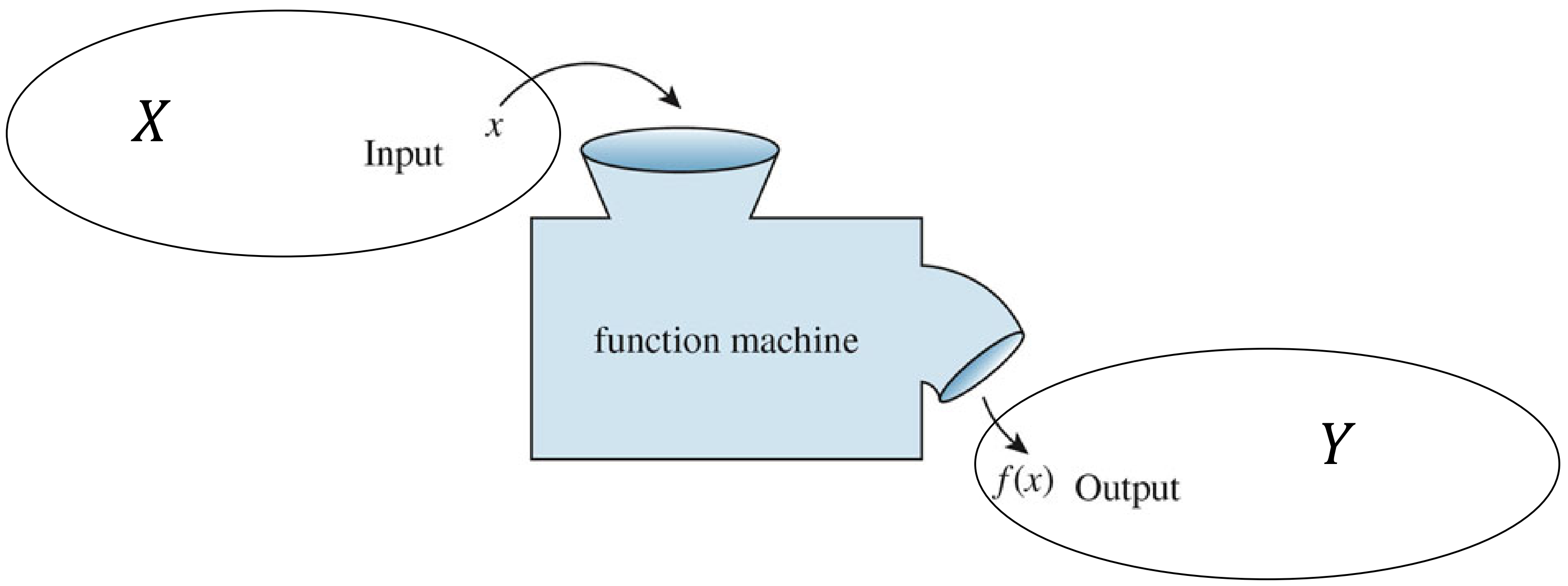
DIY

Unit 1.3

Functions

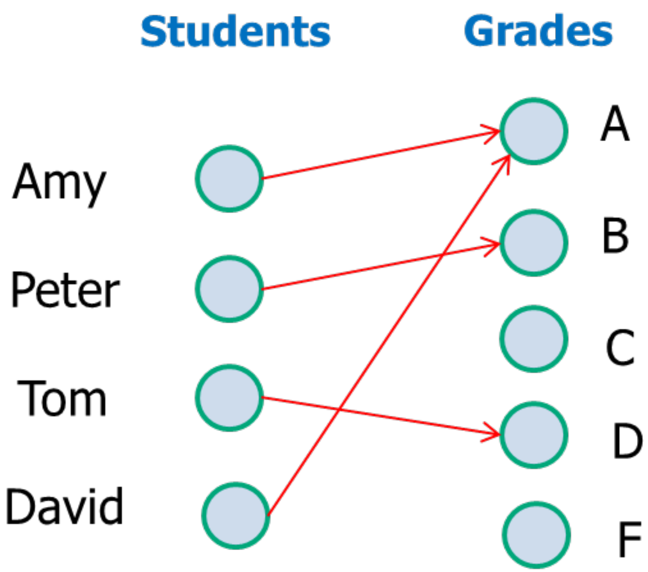
Functions

- A function f from X to Y, denoted by $f : X \to Y$, (or f maps X to Y) is an assignment of each element of X to exactly one element of Y.
  - X and Y are nonempty sets.



Example

- Consider the Grade Assignment Function f which maps a set of students to a set of grades.
  - f assigns each student exactly one grade.



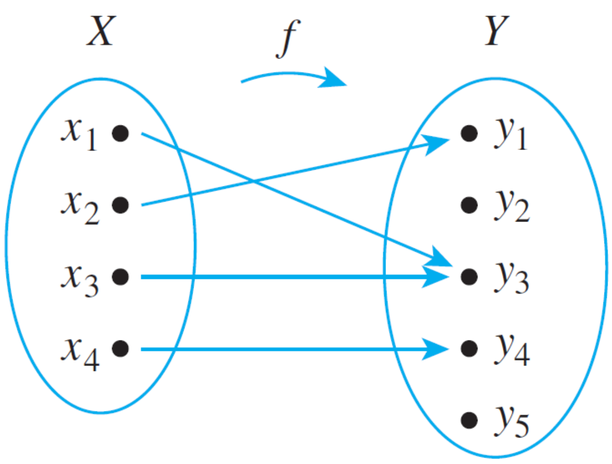
No student is assigned more than one grade.

No student has no grade assigned.

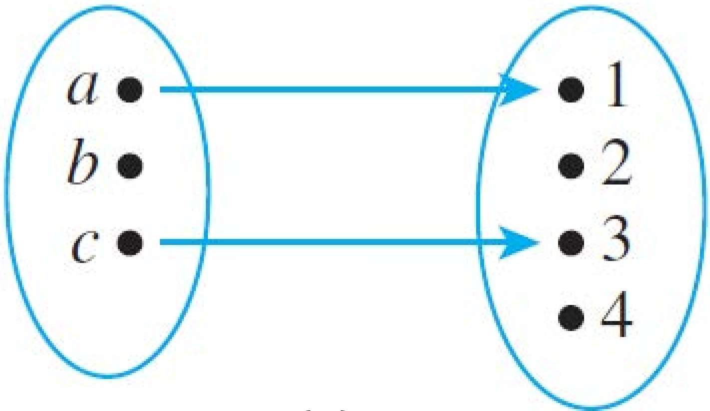
Arrow Diagrams

- A function $f: X \to Y$ can be represented by an arrow diagram.
- An arrow is drawn from each element in X to its corresponding unique element in Y under f.

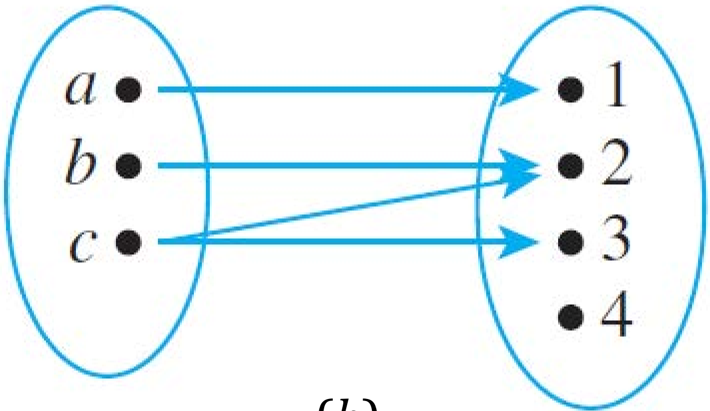
  - Every element in X points to a unique element in Y.
  - No element of X has two arrows coming out of it.



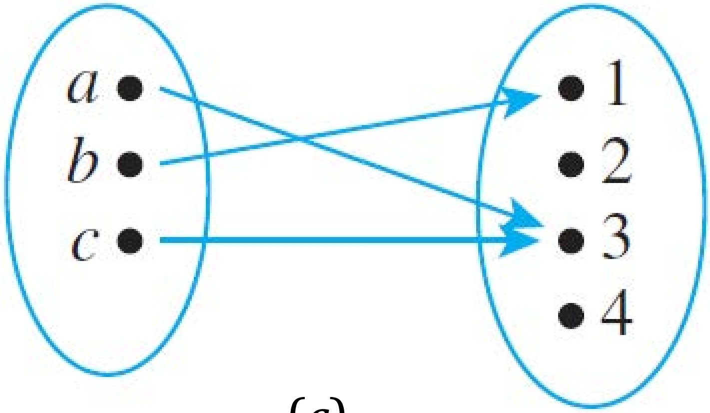
Are They Functions?



(a)



(b)



(c)

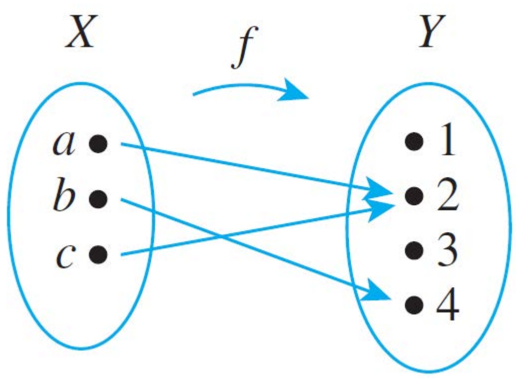
Terminologies

Consider a function $f: X \rightarrow Y$.

- X is called the domain of f while Y is called the co-domain of f.
- Given $x \in X$ and $f(x) = y \in Y$, y is called the image of x under f.
- The range of f is the set of images of all elements in X.
  - Note: range $\subseteq$ co-domain.
- Given $y \in Y$, the inverse image of y is the set of all elements $x \in X$ such that $f(x) = y$.

Classwork

a) What are the domain, co-domain and range of f?



b) What is the image of a under f?

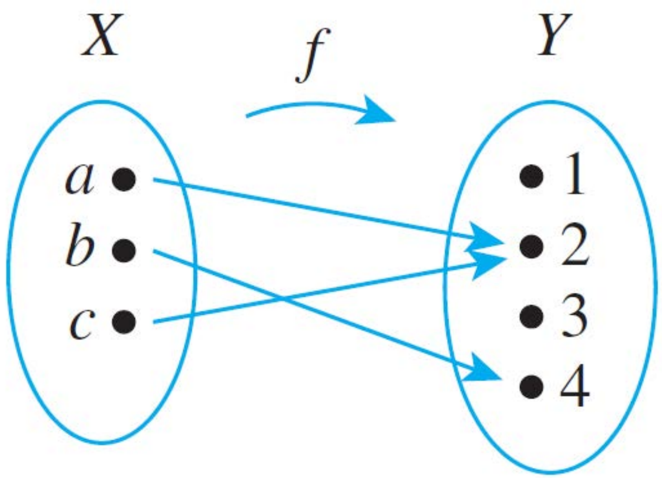
c) What is the inverse image of 2 under f?

d) What is the inverse image of 3 under f?

Function as Subset of Cartesian Product

- A function $f : X \rightarrow Y$ is a subset of the Cartesian product between X and Y.

- $X \times Y =$
  $\{(a,1),(a,2),(a,3),(a,4),$
  $(b,1),(b,2),(b,3),(b,4),$
  $(c,1),(c,2),(c,3),(c,4)\}$



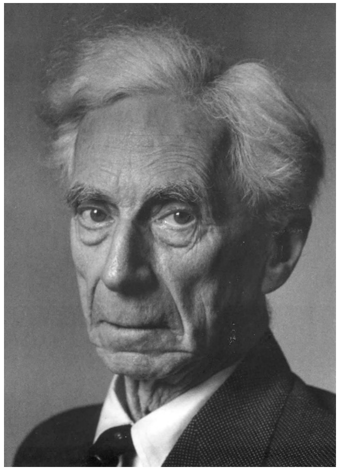
- $f = \{(a,2),(b,4),(c,2)\}$

Unit 1.4

Russell's Paradox

Naïve Set Theory

- In naïve set theory, a set is simply a *collection of objects*.
- Given any *property*, there is a set which contains all objects that have the property.
  - For example, students enrolled in EE2302 this semester form a set.
  - The set is empty if no object has the property.
- Russell found that a paradox arises!



Bertrand Russell (1872-1970), a British philosopher, logician, and writer.

Can $X \in X$?

- Russell's paradox is based on this construction:

$$\boxed{S = \{X \mid X \notin X\}}$$

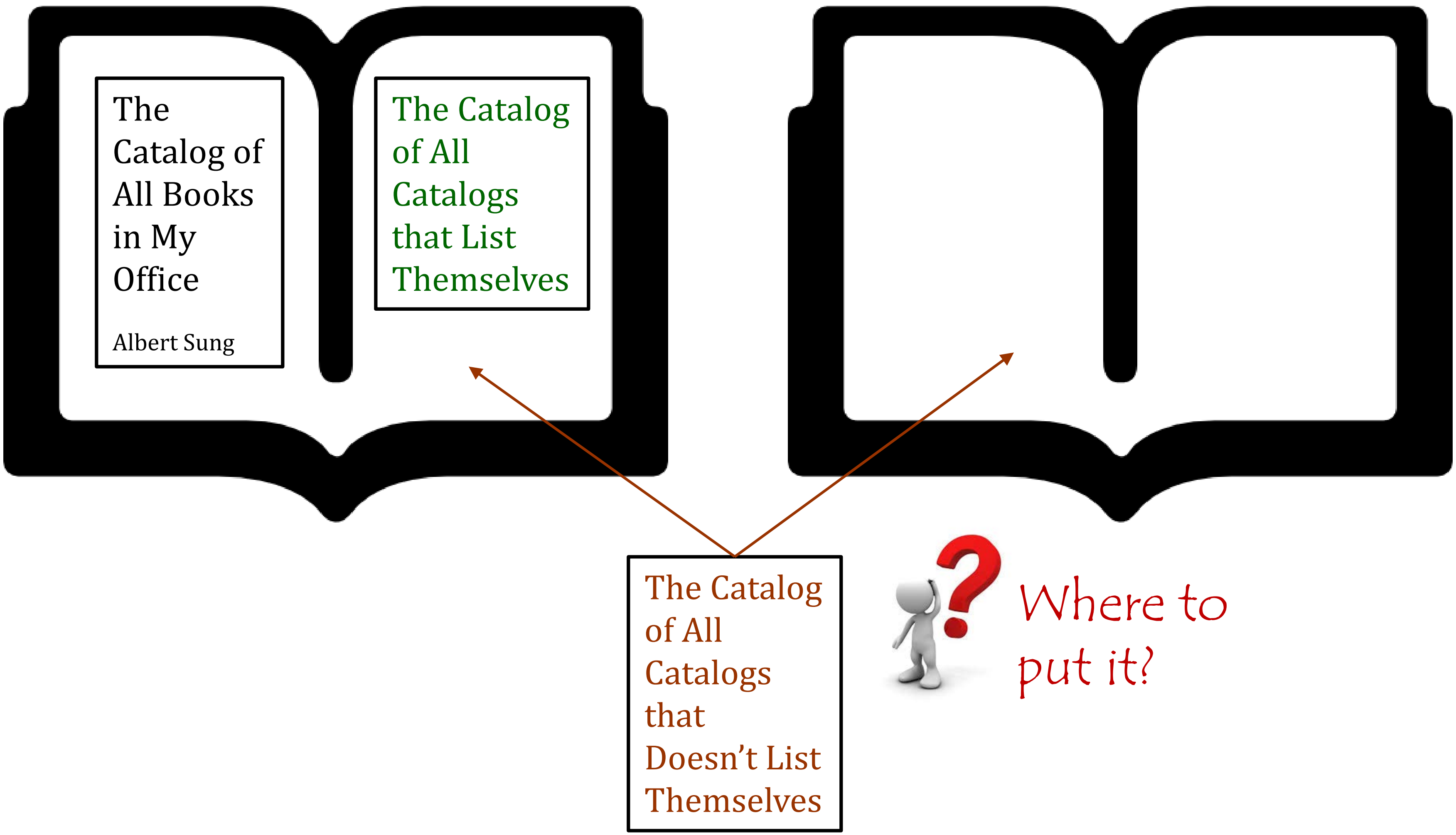
  - Can a set be a member of itself?
  - Or equivalently, can $X \in X$?

Suppose I create a catalog of all books in my office.

This is the catalog.

The Catalog of All Catalogs that List Themselves

The Catalog of All Catalogs that Doesn't List Themselves



Russell's Paradox

- Let S be the set of all sets that are not members of themselves:

$$S = \{X \mid X \notin X\}.$$

Note: X is a set.

- *Q:* Is S an element of itself?
  - i.e., is $S \in S$?
- *A:* Neither yes nor no, because either way leads to a contradiction:
  - Suppose $S \in S$. By the defining property of S, $S \notin S$.
  - Suppose $S \notin S$. By the defining property of S, $S \in S$.

Remarks

- The barber's paradox is a popular version of Russell's paradox.
  - Russell uses it to explain the paradox to layman.
- To resolve Russell's paradox, a set has to be *properly* defined.
- Russell's paradox facilitates the development of axiomatic set theory.
  - There are different ways to do it... (details omitted)

Unit 1.5

Halting Problem

The Halting Problem

(8 min): https://www.youtube.com/watch?v=92WHN-pAFCs

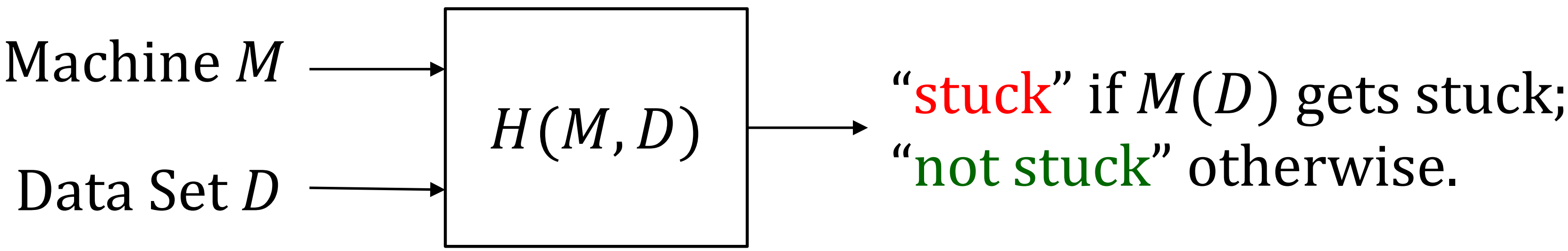
The Halting Theorem

Theorem: There is no computer algorithm that

i. accepts any algorithm M and data set D as input, and then
ii. correctly outputs “stuck” or “not stuck” to indicate whether or not M terminates in a finite number of steps when M is run with data set D.

Proof by Contradiction

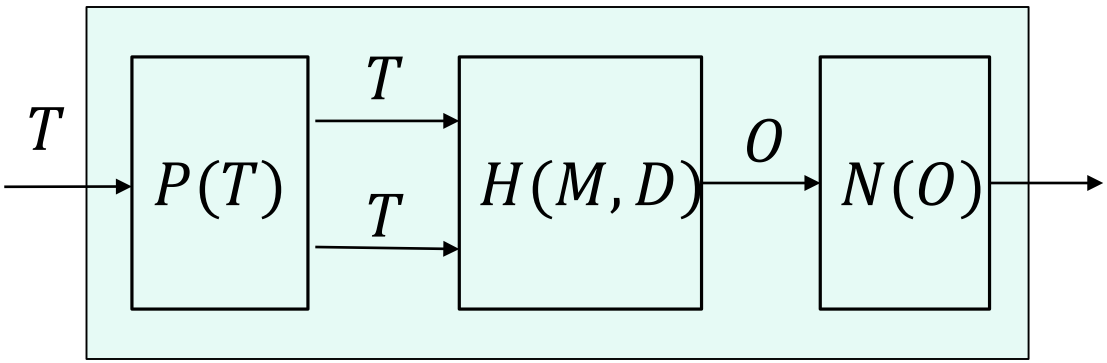
- Assume there is a halting machine:



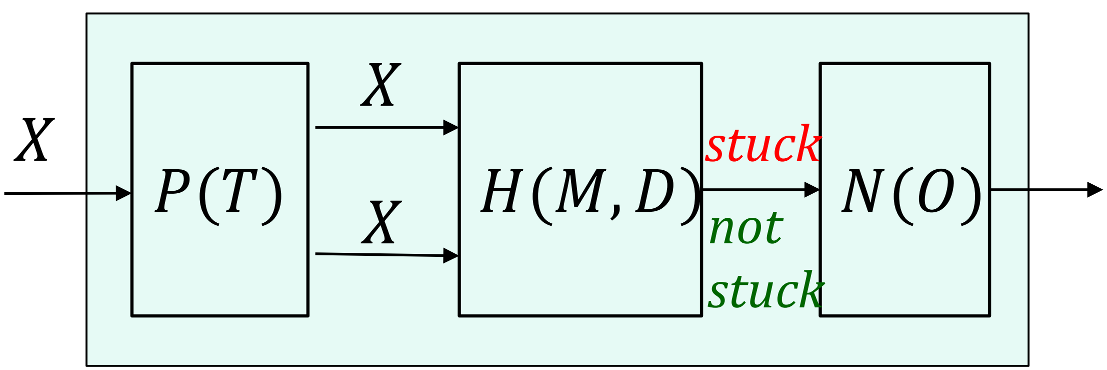
- Try to show that the existence of H leads to a contradiction.

Contradiction Arises…

- Construct machine X as follows:



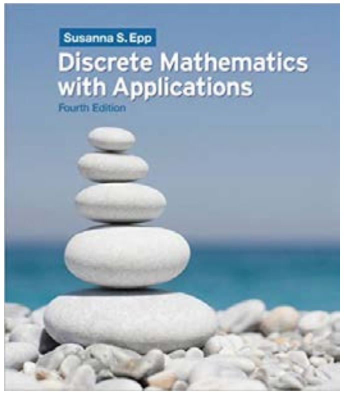
"smile" if O = "stuck";
loop forever otherwise.



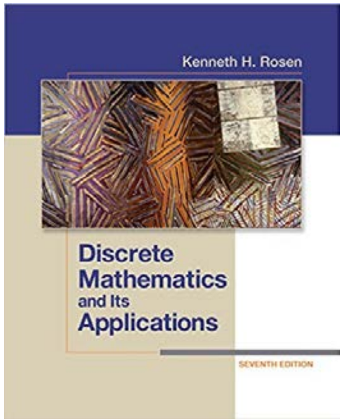
"smile"
loop forever

?!

Recommended Reading



- Chapter 6 and Section 7.1, S. S. Epp, *Discrete Mathematics with Applications*, 4th ed., Brooks Cole, 2010.



- Sections 2.1 and 2.2, K. H. Rosen, *Discrete Mathematics and its Applications*, 7th ed., McGraw-Hill Education, 2011.