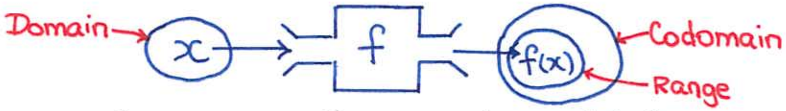
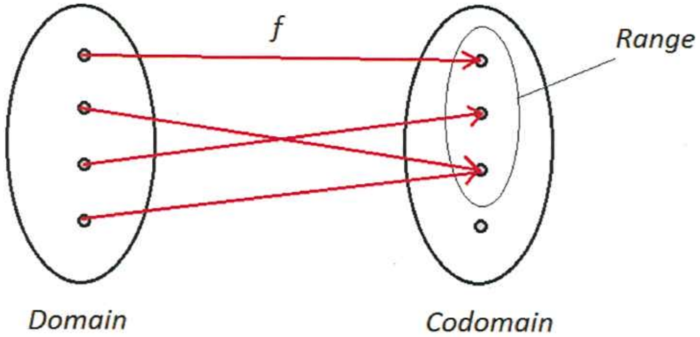
Functions



- A **function** is a rule that assigns a unique value $f(x)$ to any x from a set called the domain.
- The **domain** of a function is the set of all possible input values (i.e. all possible values of x) for which the function is defined.
- The **codomain** of a function is the set which contains all possible output values.
- The **range** is the set of all output values (i.e. all values of y or $f(x)$), which actually result from using the function formula.
- In general, the range of a function is a subset of its codomain but not necessarily the same set.



- Clearly, the range of a function depends on what you put into the function (domain) and the function itself.

- If set A is the domain of f and set B is the codomain of f, we write

$$f : A \to B.$$

For example, we may write the following to define a function:

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^2 + x + 1.$$

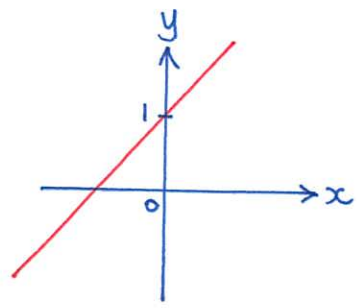
- If $x \in A$ and $y = f(x) \in B$ (for example, $y = x^2 + x + 1$), then x is called the **independent** variable and y is called the **dependent** variable.

- We use the term "**largest possible domain**" to denote the largest possible $\boxed{\textbf{set}}$ of the input values x, not just the largest possible number that x can take.

$$\text{E.g. If } g : \underset{\text{Domain}}{\mathbb{R}} \to \underset{\text{Codomain}}{\mathbb{R}} \text{ \& } g(x) = x+1,$$

then $\mathrm{Dom}(g) = \mathbb{R}$

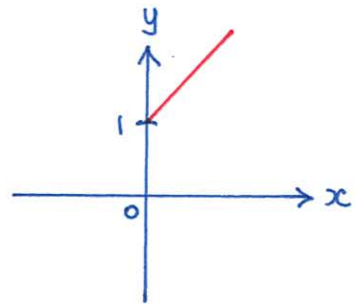
and $\mathrm{Ran}(g) = \mathbb{R}$.



If $h : [0,\infty) \to \mathbb{R}$ & $h(x) = x+1$,

then $\mathrm{Dom}(h) = [0,\infty)$

and $\mathrm{Ran}(h) = [1,\infty)$.



➢ We use the notations $Dom(f)$ and $Ran(f)$ to denote the **largest possible domain** and the **largest possible range** of the function f, respectively. Then $x \in Dom(f)$ and $f(x) \in Ran(f)$.

In this course, we will mainly study those functions whose domains and codomains are subsets of $\mathbb{R}$, i.e. they are real-valued functions.

Summary of the domain, codomain and range of a function:

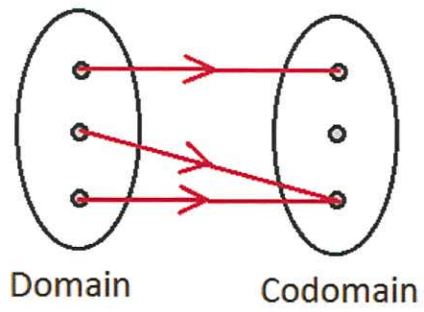
Domain: What can be put into the function?

Codomain: What may possibly come out of a function?

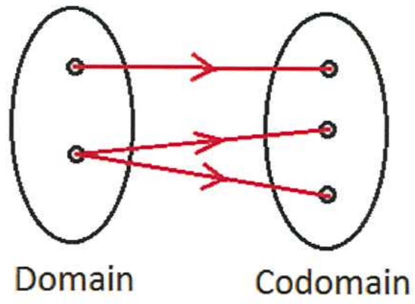
Range: What actually comes out of a function?

Note that every element of the domain A (input) must have exactly one output (in the codomain B).

Consider the following figures:



This is a well-defined function. (Why?)



This is not a well-defined function. (Why?)

Here are some examples of equations which define y as a function of x (where $x \in \mathbb{R}$):

- $y = 3x^2 + 5x + 1, \quad y = 3x - 1$ (These are examples of **polynomials** (Ch.3))
- $y = \sin x, \quad y = \cos x$ (These are examples of **trigonometric functions** (Ch.4))
- $y = e^x, \quad y = 10^x$ (These are examples of **exponential functions** (Ch.5))
- $y = \ln x, \quad y = \log x$ (for $\boldsymbol{x > 0}$) (These are examples of **logarithmic functions** (Ch.5))

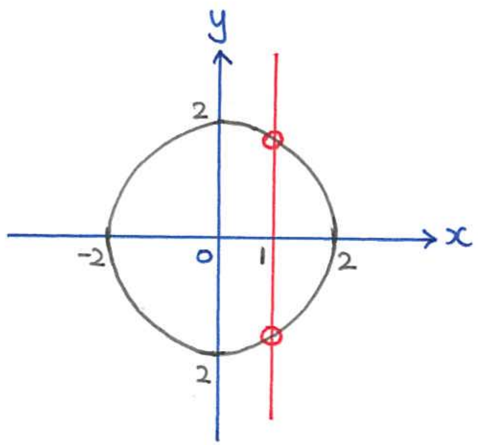
Examples of equations which do not define y as a function of x (where $x \in \mathbb{R}$):

- $x^2 + y^2 = 4$ **(Why?)**

$x^2+y^2=4 \Rightarrow y=\pm\sqrt{4-x^2}$

For every $x \in (-2,2)$, there are two corresponding values of y.

E.g. When $x=1$, $y=\pm\sqrt{4-1^2}=\pm\sqrt{3}$.

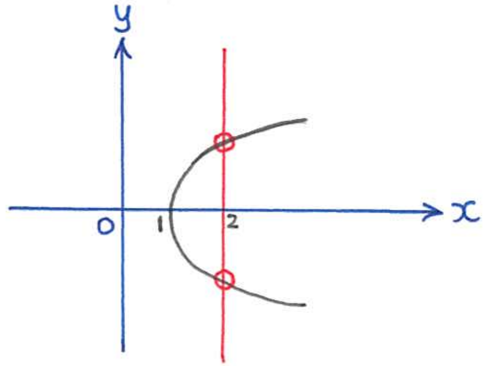


- $x = y^2 + 1$ **(Why?)**

$x = y^2+1 \Rightarrow y=\pm\sqrt{x-1}$.

For every $x>1$, there are two corresponding values of y.

E.g. When $x=2$, $y=\pm\sqrt{2-1}=\pm 1$.



$x=y^2+1 \Rightarrow y^2 = 4\left(\frac{1}{4}\right)(x-1)$

$a=\frac{1}{4}>0$ ∴ opens to the right

Example 7

For each of the following functions, determine the largest possible domain and the largest possible range of f.

(a) $f(x) = x^2 + 1$ (b) $f(x) = 25 - x$

(c) $f(x) = \sqrt{x+4}$ (d) $f(x) = 3 + \frac{1}{x-5}$

(e) $f(x) = 5 + \sin x$

Solution

(a) The function $f(x) = x^2 + 1$ is well-defined for every real number x.

$\therefore$ The largest possible domain of f is $Dom(f) = \mathbb{R}$ (the set of all real numbers)

Since $x^2 \geq 0$ for any $x \in Dom(f) = \mathbb{R}$, we have $x^2 + 1 \geq 1$ for any $x \in \mathbb{R}$.

$\therefore$ The largest possible range of f is $Ran(f) = [1, \infty)$ (the set of all real numbers greater than or equal to 1)

(b) The function $f(x) = 25 - x$ is well-defined for every real number x.

$\therefore$ The largest possible domain of f is $Dom(f) = \mathbb{R}$.

For any $x \in Dom(f) = \mathbb{R}$, $25 - x$ can be any real number.

$\therefore$ The largest possible range of f is $Ran(f) = \mathbb{R}$.

(c) The function $f(x) = \sqrt{x+4}$ is well-defined when $x + 4 \geq 0$, i.e. $x \geq -4$.

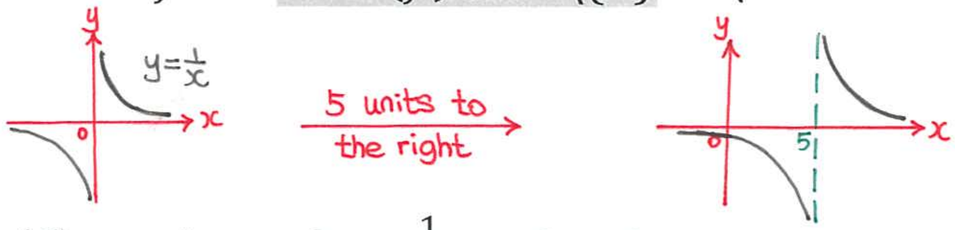
$\therefore$ The largest possible domain of f is $Dom(f) = [-4, \infty)$.

For any $x \in Dom(f) = [-4, \infty)$, we have $x + 4 \geq 0$ and therefore $\underbrace{\sqrt{x+4}}_{=f(x)} \geq 0$.

$\therefore$ The largest possible range of f is $Ran(f) = [0, \infty)$.

(d) The function $f(x) = 3 + \frac{1}{x-5}$ is well-defined when $x - 5 \neq 0$, i.e. $x \neq 5$.

$\therefore$ The largest possible domain of f is $Dom(f) = \mathbb{R}\backslash\{5\}$. (The set of all real numbers except 5)



$y = \frac{1}{x-5}$ can be any real numbers except 0.

Since $\frac{1}{x-5} \neq 0$ for all $x \in Dom(f)$, we have $3 + \frac{1}{x-5} \neq 3 + 0$.

Therefore, $3 + \frac{1}{x-5}$ cannot be equal to 3.

$\therefore$ The largest possible range of f is $Ran(f) = \mathbb{R}\backslash\{3\}$.

(e) The function $f(x) = 5 + \sin x$ is well-defined for all $x \in \mathbb{R}$.

$\therefore$ The largest possible domain of f is $Dom(f) = \mathbb{R}$.

For any $x \in Dom(f)$, $-1 \leq \sin x \leq 1$ and therefore $5 - 1 \leq 5 + \sin x \leq 5 + 1$,

i.e. $4 \leq f(x) \leq 6$.

$\therefore$ The largest possible range of f is $Ran(f) = [4, 6]$.

Ex. 7 (d) Method 2: (To find Ran(f))

Let $y = 3 + \frac{1}{x-5}$.

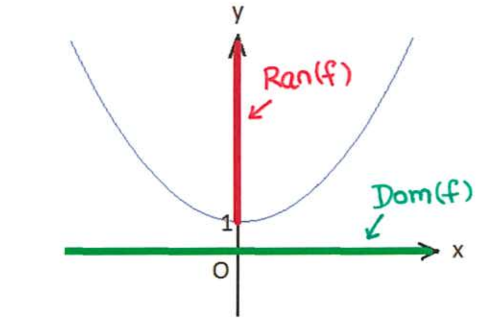
Then $y - 3 = \frac{1}{x-5} \Rightarrow x - 5 = \frac{1}{y-3}$

$\Rightarrow x = 5 + \frac{1}{y-3}$

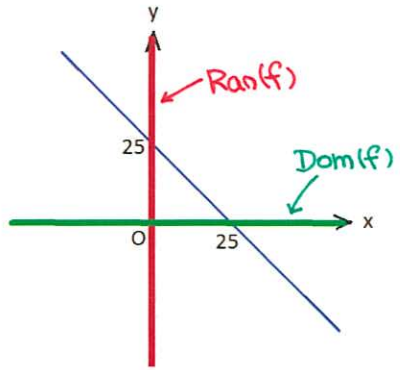
y can be any real number except 3.

$\therefore$ $\mathrm{Ran}(f) = \mathbb{R} \setminus \{3\}$.

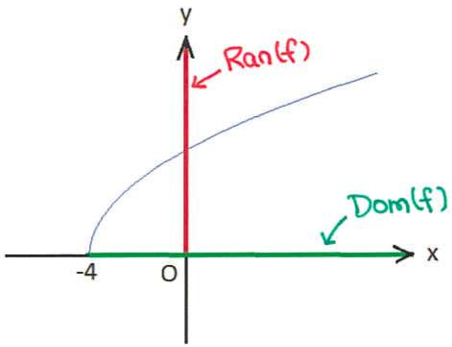
An alternative way to find the domain and range of a function is to sketch its graph first and then determine its domain and range from the graph. For example, the graphs of the first 4 functions in Example 7 are shown below (with **domain** highlighted in green and **range** highlighted in red):



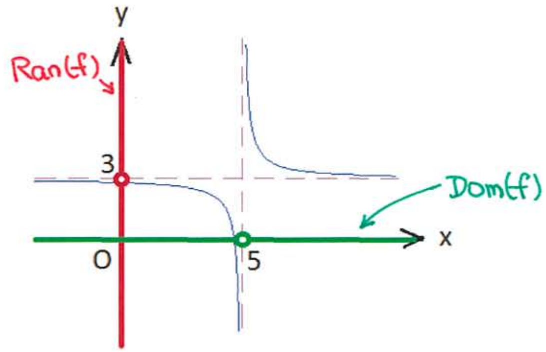
(a) $f(x) = x^2 + 1$



(b) $f(x) = 25 - x$



(c) $f(x) = \sqrt{x + 4}$



(d) $f(x) = 3 + \frac{1}{x-5}$

Example 8 (A bit harder examples)

Find the largest possible domain and largest possible range for each of the following functions:

(a) $f(x) = \frac{3x+1}{x-1}$ (b) $f(x) = 3 + \sqrt{x^2 - 16}$

(c) $f(x) = 3 + \sqrt{x^2 + 16}$ (d) $f(x) = 1 + 2x - x^2$

Solution

(a) $f(x) = \frac{3x+1}{x-1}$ is well-defined only when $x - 1 \neq 0$, i.e. $x \neq 1$.

$\therefore$ The largest possible domain of f is $Dom(f) = \mathbb{R} \setminus \{1\}$

$$f(x) = \frac{3x+1}{x-1} = \frac{3(x-1+1)+1}{x-1} = \frac{3(x-1)+4}{x-1} = 3 + \frac{4}{x-1}$$

Since $\frac{4}{x-1} \neq 0$ for any $x \in Dom(f)$, it follows that $f(x) = 3 + \frac{4}{x-1}$ cannot be equal to 3. (Similar to Ex. 7d)

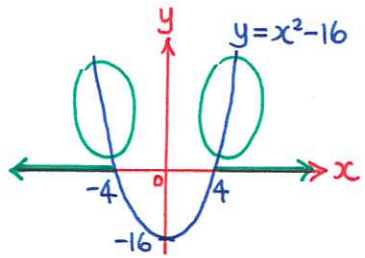
$\therefore$ The largest possible range of f is $Ran(f) = \mathbb{R} \setminus \{3\}$

Alternative method to find its range:

Let $y = \frac{3x+1}{x-1}$. Then express x in terms of y:

$$y = \frac{3x+1}{x-1} \Rightarrow y(x-1) = 3x+1 \Rightarrow x(y-3) = 1+y \Rightarrow x = \frac{1+y}{y-3}.$$

From this expression, y can be any real number except 3. Hence, $Ran(f) = \mathbb{R}\backslash\{3\}$.



(b) $f(x) = 3 + \sqrt{x^2 - 16}$ is well-defined only when $x^2 - 16 \geq 0$

$\Rightarrow x^2 \geq 16 \Rightarrow x \geq 4$ or $x \leq -4$.

$\therefore$ The largest possible domain of f is $Dom(f) = (-\infty, -4] \cup [4, \infty)$

For any $x \in Dom(f)$, $x^2 - 16 \geq 0 \Rightarrow \sqrt{x^2-16} \geq 0 \Rightarrow 3 + \sqrt{x^2-16} \geq 3 + 0$,

i.e. $f(x) \geq 3$.

$\therefore$ The largest possible range of f is $Ran(f) = [3, \infty)$

(c) $f(x) = 3 + \sqrt{x^2+16}$ is well-defined only when $x^2 + 16 \geq 0$.

Clearly, $x^2 + 16 \geq 16 > 0$ for any real number x, thus the largest possible domain of f is $Dom(f) = \mathbb{R}$.

Since $x^2 + 16 \geq 16$ for all $x \in Dom(f)$, we have $\sqrt{x^2+16} \geq \sqrt{16} = 4$ and thus $f(x) = 3 + \sqrt{x^2+16} \geq 3 + 4 = 7$.

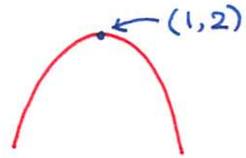
$\therefore$ The largest possible range of f is $Ran(f) = [7, \infty)$.

Coeff. of $x^2 < 0$

$\therefore$ opens downward

(d) $f(x) = 1 + 2x - x^2$ is well-defined for all $x \in \mathbb{R}$.

$\therefore$ The largest possible domain of f is $Dom(f) = \mathbb{R}$.



By completing the square,

$$f(x) = 1 + 2x - x^2 = -(x^2 - 2x) + 1 = -[(x-1)^2 - 1^2] + 1 = 2 - (x-1)^2.$$

For any $x \in Dom(f)$, $(x-1)^2 \geq 0 \Rightarrow -(x-1)^2 \leq 0 \Rightarrow 2 - (x-1)^2 \leq 2 + 0$, i.e. $f(x) \leq 2$. Hence the largest possible range of f is $Ran(f) = (-\infty, 2]$.

Example 9 **(More harder examples)**

Find the largest possible domain for each of the following functions:

(a) $f(x) = \sqrt{x^2 - 3x + 2}$ (b) $f(x) = \sqrt{3 + 2x - x^2}$

(c) $f(x) = \frac{9}{x^2+4x-5}$ (d) $f(x) = \sqrt{\frac{x+1}{x+2}}$

Solution

Two important things to remember when determining the largest possible domain of a function which involves square root or quotient:

1. We cannot take square root of a negative number. Solve inequality $\cdots \geq 0$.
2. We cannot divide by zero. Solve equation $\cdots = 0$, then exclude these x values from $\mathbb{R}$.

(a) The function $f(x) = \sqrt{x^2 - 3x + 2}$ is well-defined only when $x^2 - 3x + 2 \geq 0$, i.e. $(x-1)(x-2) \geq 0$. We want to find all those values of x which satisfy the inequality $(x-1)(x-2) \geq 0$.

One way is to draw a table like the one shown below:

Method 1:

	$x < 1$	$x = 1$	$1 < x < 2$	$x = 2$	$x > 2$
Sign of $(x-1)$	$-$	0	+	+	+
Sign of $(x-2)$	$-$	$-$	$-$	0	+
Sign of $(x-1)(x-2)$	**+**	**0**	$-$	**0**	**+**

i.e. we get $(x-1)(x-2) \geq 0$ only when $x \leq 1$ or $x \geq 2$.

$\therefore$ The largest possible domain of f is $Dom(f) =$ $(-\infty, 1] \cup [2, \infty)$

or written as $\mathbb{R} \setminus (1, 2)$.

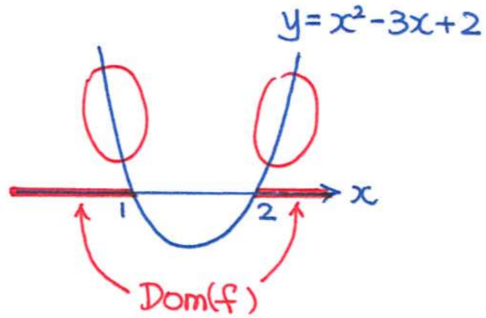
Method 2: $f(x) = \sqrt{x^2-3x+2}$ is defined when $x^2-3x+2 \geq 0$.

Consider $x^2-3x+2 = (x-1)(x-2)$.

Coeff. of x^2 is $1 > 0$

$\therefore$ parabola opens upward

$\therefore$ $\text{Dom}(f) = (-\infty, 1] \cup [2, \infty)$.



Remark: To find the range of $f(x) = \sqrt{x^2-3x+2}$:

For any $x \in \text{Dom}(f) = (-\infty, 1] \cup [2, \infty)$,

observe that $x^2-3x+2 \geq 0$

$$\Rightarrow \underbrace{\sqrt{x^2-3x+2}}_{=f(x)} \geq \sqrt{0} = 0$$

$\therefore$ $\text{Ran}(f) = [0, \infty)$.

(b) $f(x) = \sqrt{3 + 2x - x^2}$ is well-defined only when $3 + 2x - x^2 \geq 0$,

i.e. $(3 - x)(1 + x) \geq 0$. To solve this inequality, we draw the following table:

Don't put $\cancel{(x-3)(x+1) \geq 0}$

It should be $-(x-3)(x+1) \geq 0$

i.e. $(3-x)(x+1) \geq 0$

	$x < -1$	$x = -1$	$-1 < x < 3$	$x = 3$	$x > 3$
Sign of $(3 - x)$	+	+	+	0	−
Sign of $(1 + x)$	−	0	+	+	+
Sign of $(3 - x)(1 + x)$	−	**0**	**+**	**0**	−

i.e. we get $(3 - x)(1 + x) \geq 0$ only when $-1 \leq x \leq 3$.

$\therefore$ The largest possible domain of f is $Dom(f) = [-1, 3]$.

Remark:

To find the range of $f(x) = \sqrt{3+2x-x^2}$:

Consider $3+2x-x^2 = (3-x)(1+x)$.

$\therefore$ Parabola passes through x-axis at $x=-1$ and $x=3$.

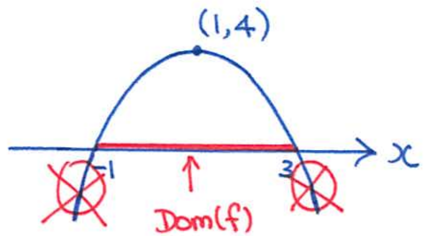
Coeff. of x^2 is $-1 < 0$

$\therefore$ Parabola opens downward.

By completing the square,

$$\begin{aligned} 3+2x-x^2 &= -(x^2-2x)+3 \\ &= -[(x-1)^2-1^2]+3 \\ &= \boxed{4}-(x-\boxed{1})^2 \end{aligned}$$

$\therefore$ Vertex at $(1,4)$.



For any $x \in \mathrm{Dom}(f) = [-1, 3]$,

$$0 \leq 3+2x-x^2 \leq 4$$

$$\Rightarrow \sqrt{0} \leq \sqrt{3+2x-x^2} \leq \sqrt{4}$$

i.e. $0 \leq f(x) \leq 2$

$\therefore \mathrm{Ran}(f) = [0, 2]$

Note: (Ch.3)

Quadratic function of the form

$$y = a(x-\boxed{h})^2 + \boxed{k}$$

vertex at (h, k)