MA1200 CALCULUS AND BASIC LINEAR ALGEBRA

LECTURE: CG1

REVIEW EXAMPLES ON CHAPTER 6 TO 8

Example 1 (Exam 1617B)

Evaluate the following limits:

(a) $\displaystyle\lim_{x\to 0}\frac{1-\cos(3x)}{x^2}$

(b) $\displaystyle\lim_{x\to\infty}\frac{1-x^2}{1+x^2}$

(c) $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{3}{3x+2x^2}\right)$

Solutions to Review Examples on Ch. 6-8

Example 1:

(a) **Method 1:**

$$\lim_{x\to 0} \frac{1-\cos(3x)}{x^2} = \lim_{x\to 0} \frac{2\sin^2\left(\frac{3x}{2}\right)}{x^2}$$, using Half-angle formula: $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ where $\theta = \frac{3x}{2}$

$$= \lim_{x\to 0} 2 \cdot \frac{\sin^2\left(\frac{3x}{2}\right)}{\left(\frac{3x}{2}\right)^2} \cdot \left(\frac{3}{2}\right)^2$$

$$= 2 \cdot 1^2 \cdot \frac{9}{4}$$

$$= \frac{9}{2}$$

Method 2:

$$\lim_{x\to 0} \frac{1-\cos(3x)}{x^2} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{3\sin(3x)}{2x}$$, by L'Hôpital's Rule $\left(\tfrac{0}{0} \text{ form}\right)$

$$= \lim_{x\to 0} \frac{3\cdot 3\cos(3x)}{2} \quad \text{, by L'Hopital's rule}$$

$$= \frac{9}{2}\cdot \cos 0$$

$$= \frac{9}{2}$$

(b) Method 1:

$$\lim_{x\to\infty} \frac{1-x^2}{1+x^2} = \lim_{x\to\infty} \frac{\frac{1}{x^2}-1}{\frac{1}{x^2}+1} = \frac{0-1}{0+1} = -1$$

Method 2:

$$\lim_{x\to\infty} \frac{1-x^2}{1+x^2} \quad \left(\frac{-\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to\infty} \frac{-2x}{2x} \quad \text{, by L'Hôpital's rule}$$

$$= \lim_{x\to\infty} (-1)$$

$$= -1$$

(C) $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{3}{3x + 2x^2} \right)$ ($\infty - \infty$ form)

$$= \lim_{x \to 0^+} \frac{(3x + 2x^2) - 3x}{x(3x + 2x^2)}$$

$$= \lim_{x \to 0^+} \frac{2x^2}{x^2(3 + 2x)}$$

$$= \lim_{x \to 0^+} \frac{2}{3 + 2x}$$

$$= \frac{2}{3 + 0}$$

$$= \frac{2}{3}$$

Example 2 (Exam 1314B)

Let $f(x)=\begin{cases} \dfrac{x^3-8}{x-2} & \text{if } x \neq 2 \\ c & \text{if } x=2 \end{cases}$.

Find the value of c for which $f(x)$ is continuous at $x=2$. Give your reason.

Example 2

$$f(x) = \begin{cases} \dfrac{x^3-8}{x-2} & \text{if } x \neq 2 \\ c & \text{if } x = 2 \end{cases}$$

$$\lim_{x\to 2} \frac{x^3-8}{x-2} \qquad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{x-2}$$

$$= \lim_{x\to 2} (x^2+2x+4)$$

$$= 2^2 + 2(2) + 4$$

$$= 12$$

$f(x)$ is continuous at $x = 2$ iff $\lim_{x\to 2} f(x) = f(2)$

$= c$,

i.e. $c = 12$.

Example 3 **(Exam 1213A)**

Let $g(x)=\begin{cases} x^2\sin\dfrac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$. Determine whether $g(x)$ is differentiable at $x=0$, if so, find the value of the first derivative there.

Example 3

$$g(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$g(x)$ is differentiable at $x=0$ iff $\lim\limits_{x\to 0} \dfrac{g(x)-g(0)}{x-0}$ exists.

$$\lim_{x\to 0} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0} \frac{x^2\sin\left(\frac{1}{x}\right) - 0}{x} = \lim_{x\to 0} x\sin\left(\tfrac{1}{x}\right).$$

Since $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ for all $x \neq 0$, we have

$$-|x| \leq x\sin\left(\tfrac{1}{x}\right) \leq |x| \quad \text{for all } x \neq 0.$$

$$\lim_{x\to 0}\left(-|x|\right) = 0 = \lim_{x\to 0} |x|$$

$\therefore$ By the Sandwich Theorem, $\lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$.

$\therefore$ $\lim\limits_{x\to 0} \dfrac{g(x)-g(0)}{x-0}$ exists and $\lim\limits_{x\to 0} \dfrac{g(x)-g(0)}{x-0} = \lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$

$\therefore$ $g(x)$ is differentiable at $x=0$ and $g'(0) = 0$.

<u>Example 4</u> (Exam 1617B)

(a) Prove from first principles that $\frac{d}{dx}\left(x^3\right)=3x^2$.

(b) Let $F(x)=|\cos x|$, for $x\in\mathbf{R}$. Determine whether $F(x)$ is differentiable at $x=0$. Give your reason.

(Hint: You may use $\cos 2\theta = 1-2\sin^2\theta$.)

Example 4

(a) Let $f(x) = x^3$.

From the First Principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$

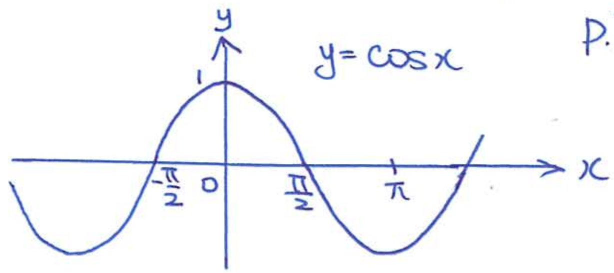
$$= \lim_{h \to 0} (3x^2 + 3xh + h^2)$$

$$= 3x^2 + 3x \cdot (0) + 0^2$$

$$= 3x^2$$

$$\therefore \frac{d}{dx}(x^3) = 3x^2$$

(b) $F(x) = |\cos x| = \begin{cases} \cos x & \text{if } \cos x \geq 0 \\ -\cos x & \text{if } \cos x < 0 \end{cases}$



$$= \cos x \quad \text{if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$\lim_{x\to 0} \frac{F(x)-F(0)}{x-0} = \lim_{x\to 0} \frac{\cos x - |\cos 0|}{x}$$

$$= \lim_{x\to 0} \frac{\cos x - 1}{x}$$

$$= \lim_{x\to 0} \frac{-2\sin^2\left(\frac{x}{2}\right)}{x}$$

$\because \cos 2\theta = 1 - 2\sin^2\theta$

$\Rightarrow \cos 2\theta - 1 = -2\sin^2\theta$

(Put $\theta = \frac{x}{2}$)

$$= \lim_{x\to 0} -\sin\left(\frac{x}{2}\right) \cdot \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}$$

$$= -\sin(0) \cdot 1$$

$$= 0$$

$\therefore \lim_{x\to 0} \frac{F(x)-F(0)}{x-0}$ exists

$\therefore$ $F(x)$ is differentiable at $x=0$.