MA1200 CALCULUS AND BASIC LINEAR ALGEBRA I

LECTURE: CG1

Chapter 5
Exponential and Logarithmic Functions

Exponential Functions

The **exponential function with base b** is defined by

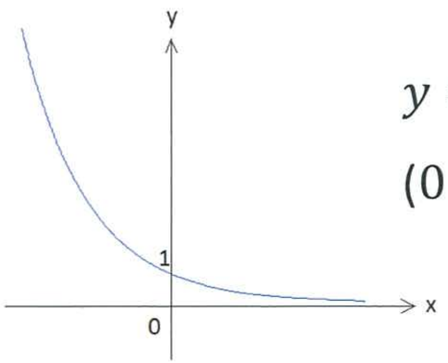
$$f(x) = b^x,$$

where the constant b (with $b > 0$ and $b \neq 1$) is called the **base**, and $x \in \mathbb{R}$ is called the **exponent**.

E.g. $f(x) = 10^x$, $g(x) = \left(\frac{1}{2}\right)^x$, $h(x) = 5^{3x+2}$ are examples of exponential functions.

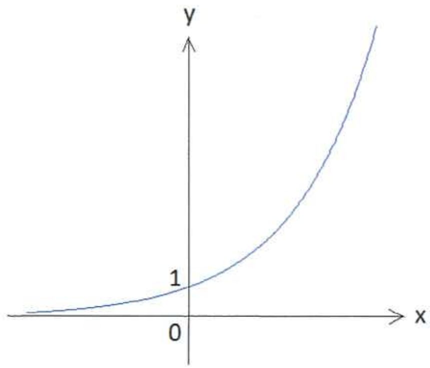
$k(x) = x^{10}$ is NOT an exponential function.

Graphs of exponential functions:



$y = b^x$

$(0 < b < 1)$



$y = b^x$

$(b > 1)$

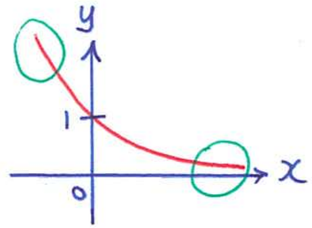
Note that:

1. The largest possible domain of $f(x) = b^x$ is $\boxed{Dom(f) = \mathbb{R}}$.

2. The largest possible range of $f(x) = b^x$ is $\boxed{Ran(f) = (0, \infty)}$.

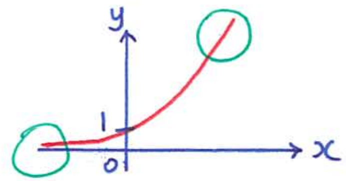
3. For $0 < b < 1$, $f(x) = b^x$ is a **strictly decreasing** function.

   $f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to 0$ as $x \to \infty$



4. For $b > 1$, $f(x) = b^x$ is a **strictly increasing** function.

   $f(x) \to 0$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$



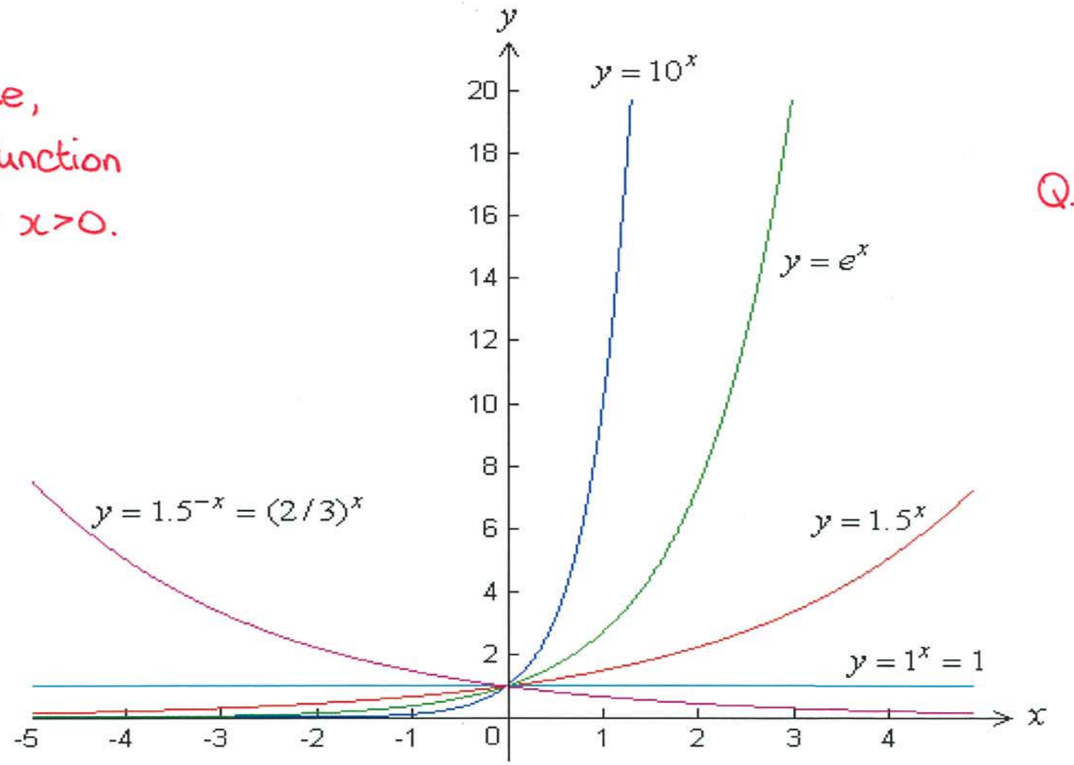
5. For any b (where $b > 0$ and $b \neq 1$), the graph of $f(x) = b^x$ always cuts the y-axis at $y = 1$, since $f(0) = b^0 = 1$ for all $b > 0$. However, it never touches the x-axis, since $f(x) = b^x$ is always positive.

6. Since the exponential function $f(x) = b^x$ is either strictly decreasing (for $0 < b < 1$) or strictly increasing (for $b > 1$), $f(x) = b^x$ is a one-to-one function and its inverse $f^{-1}(x)$ exists.

The graphs of exponential functions with different values of b are shown below.

Note that $y = 1^x$ is not an exponential function, since $y = 1^x = 1$ is a constant function.

Q.1 The larger the base, the faster the function is increasing for $x>0$.



Q.2 The graph of $y=\left(\frac{3}{2}\right)^x$ is a reflection of the graph of $y=\left(\frac{2}{3}\right)^x$ about the y-axis.

$f(x) \xrightarrow[\text{reflect about y-axis}]{} f(-x)$

<u>Question 1</u>: Compare the graphs of $y = \left(\frac{3}{2}\right)^x$ and $y = 10^x$. What do you observe?

<u>Question 2</u>: Compare the graphs of $y = \left(\frac{3}{2}\right)^x$ and $y = \left(\frac{2}{3}\right)^x$. What do you observe?

$= \left(\frac{3}{2}\right)^{-x}$

Laws of indices:

If $a > 0$, $b > 0$, x and y are real numbers, then

(1) $a^0 = 1$ (2) $a^{x+y} = a^x \cdot a^y$ (3) $a^{-x} = \frac{1}{a^x}$

(4) $a^{x-y} = \frac{a^x}{a^y}$ (5) $(a^x)^y = a^{xy}$ (6) $(ab)^x = a^x \cdot b^x$

(7) $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

Natural Base e

A special case, in which we consider $b = e$, where e is defined by the limit of the sequence

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = 2.7182818 \ldots.$$

That is, the value of $\left(1 + \frac{1}{n}\right)^n$ approaches the irrational number $e = 2.7182818 \ldots$ as n gets larger and larger (i.e. as $n \to \infty$). The number e is called the **natural base**. The exponential function with base e, $f(x) = e^x$, is called the **natural exponential function**.

Example 1

For each of the following functions, find its largest possible domain and largest possible range, and then sketch its graph.

(a) $f(x) = e^{x+1} - 5$ (b) $g(x) = 3 + 2e^{-x}$ (c) $h(x) = 1 - 3\left(\frac{1}{2}\right)^x$

Solution

(a) Since e^{x+1} is well-defined for all real values of x, the function $f(x) = e^{x+1} - 5$ is also well-defined for all real values of x. $\therefore$ $Dom(f) = \mathbb{R}$

For any $x \in Dom(f) = \mathbb{R}$, e^{x+1} is always greater than 0, and thus $f(x) = e^{x+1} - 5$ is always greater than -5. $\therefore$ $Ran(f) = (-5, \infty)$.

(b) $g(x) = 3 + 2e^{-x}$ is well-defined for all real values of x, so $Dom(g) = \mathbb{R}$.

For any $x \in Dom(g) = \mathbb{R}$, we have $e^{-x} > 0$ and thus $g(x) = 3 + 2e^{-x} > 3$.

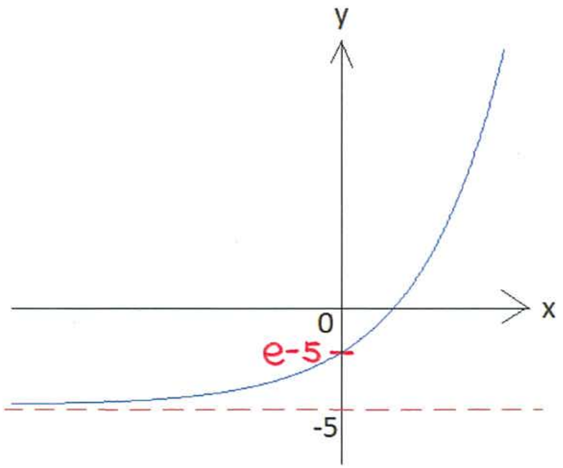
$\therefore$ $Ran(f) = (3, \infty)$.

(c) $h(x) = 1 - 3\left(\frac{1}{2}\right)^x$ is well-defined for all real values of x, so $Dom(h) = \mathbb{R}$.

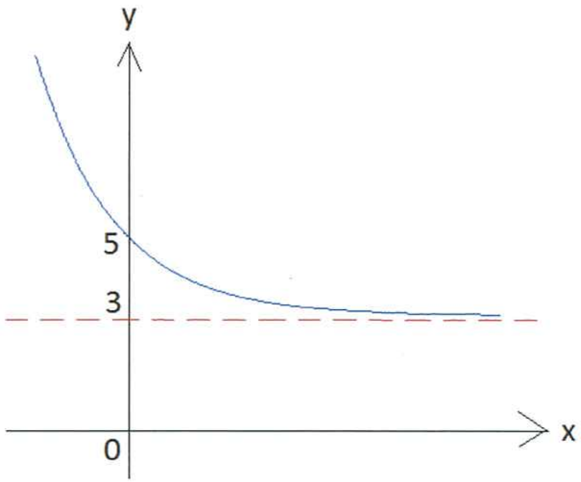
For any $x \in Dom(h) = \mathbb{R}$, we have $\left(\frac{1}{2}\right)^x > 0 \Rightarrow -3\left(\frac{1}{2}\right)^x < 0$

$\Rightarrow h(x) = 1 - 3\left(\frac{1}{2}\right)^x < 1$. Thus, $Ran(f) = (-\infty, 1)$.

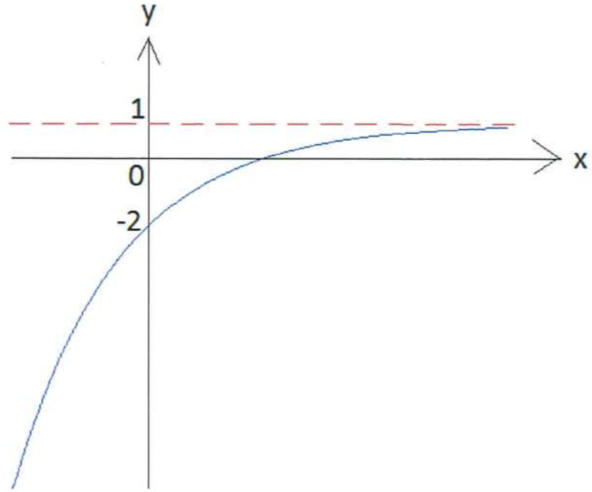
Graphs:



(a) $f(x) = e^{x+1} - 5$



(b) $g(x) = 3 + 2e^{-x}$



(c) $h(x) = 1 - 3\left(\frac{1}{2}\right)^x$

Inverse function of b^x

The exponential function $f(x) = b^x$ (with $b > 0$ and $b \neq 1$) is a one-to-one function and thus it has an inverse. Its inverse is

$$f^{-1}(x) = \log_b x,$$

which is called the logarithmic function with base b.

Logarithmic Functions

The **logarithmic function with base b** is defined as

$$f(x) = \log_b x$$

for $x > 0$. For $y = \log_b x$, the constant b (with $b > 0$ and $b \neq 1$) is called the **base**, and y is called the **exponent**.

$$\boxed{y = \log_b x \quad \Leftrightarrow \quad x = b^y}$$

Here, $y = \log_b x$ is the **logarithmic form** and $b^y = x$ is the **exponential form**.

Note that exponential function is the inverse function of logarithmic function.

Example 2

Write down each equation in its equivalent exponential form.

(a) $2 = \log_5 x$ (b) $3 = \log_b 64$ (c) $\log_3 7 = y$

Solution

(a) $2 = \log_5 x$ means $5^2 = x$.

(b) $3 = \log_b 64$ means $b^3 = 64$.

(c) $\log_3 7 = y$ means $3^y = 7$.

Example 3

Write down each equation in its equivalent logarithmic form.

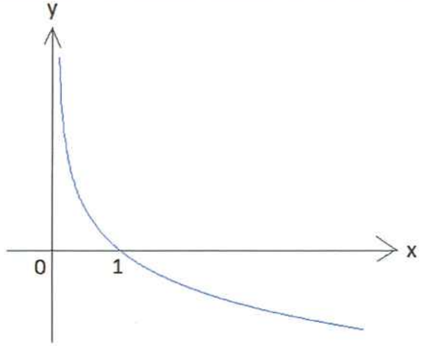
(a) $12^2 = r$ (b) $b^3 = 8$ (c) $e^a = 9$

Solution

(a) $12^2 = r$ means $2 = \log_{12} r$. (b) $b^3 = 8$ means $3 = \log_b 8$.

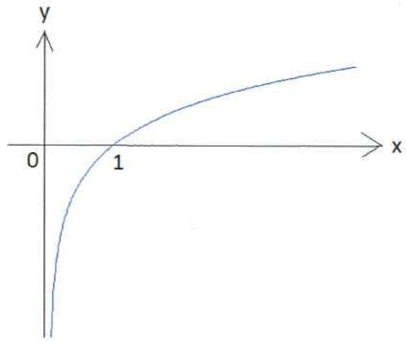
(c) $e^a = 9$ means $a = \log_e 9$.

Graphs of logarithmic functions:



$y = \log_b x$

$(0 < b < 1)$



$y = \log_b x$

$(b > 1)$

Note that:

1. The logarithmic function $f(x) = \log_b x$ is only defined for positive values of x.

   $\therefore$ The largest possible domain of $f(x) = \log_b x$ is $\boxed{Dom(f) = (0, \infty)}$.

2. The largest possible range of $f(x) = \log_b x$ is $\boxed{Ran(f) = \mathbb{R}}$.

3. For $0 < b < 1$, $f(x) = \log_b x$ is a **strictly decreasing** function.

4. For $b > 1$, $f(x) = \log_b x$ is a **strictly increasing** function.

5. For any b (where $b > 0$ and $b \neq 1$), the graph of $f(x) = \log_b x$ always cuts the x-axis at $x = 1$, i.e. $f(1) = \log_b 1 = 0$ for all $b > 0$ and $b \neq 1$. However, it never cuts the y-axis, since $f(x) = \log_b x$ is not defined at zero or negative values of x.

Two commonly used logarithms:

- If the base $b = 10$, $\log_{10} x$ is called the **common logarithm**, usually denoted by $\log x$.
- If the base $b = e$ (the natural number), $\log_e x$ is called the **natural logarithm**, usually denoted by $\ln x$.

The graphs of logarithmic functions with different values of b are shown below.