MA1201 (A/B/C/D) Test 2

1. (a) $\begin{cases} x = 3y^2 - 2 \\ x = y^2 \end{cases} \Rightarrow y^2 = 3y^2 - 2 \Rightarrow 2y^2 = 2 \Rightarrow y^2 = 1 \Rightarrow \begin{cases} y = \pm 1 \\ x = 1 \end{cases}$ (4)

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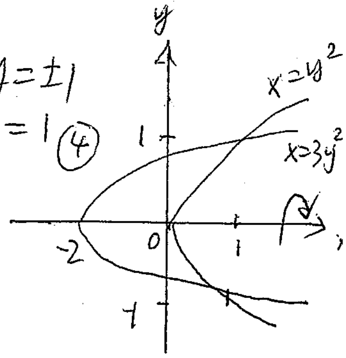


$V_x = \pi \int_{-2}^{1} y_{upper}^2 \, dx - \pi \int_0^1 y_{lower}^2 \, dx$

$= \pi \int_{-2}^{1} \frac{x+2}{3} \, dx - \pi \int_0^1 x \, dx$ (6)

$= \frac{\pi}{3} \left[ \frac{x^2}{2} + 2x \right]_{-2}^{1} - \pi \left[ \frac{x^2}{2} \right]_0^1 = \frac{\pi}{3} \left\{ \frac{1}{2} [1^2 - (-2)^2] + 2[1 - (-2)] \right\} - \frac{\pi}{2} \{ 1^2 - 0^2 \}$

$= \frac{\pi}{3} \left[ -\frac{3}{2} + 6 \right] - \frac{\pi}{2} = \frac{\pi}{3} \left( \frac{9}{2} \right) - \frac{\pi}{2} = \frac{3}{2}\pi - \frac{\pi}{2} = \pi$ (5)

(b) ~~Astroid~~ $x = \cos^3 t$, $y = \sin^3 t$

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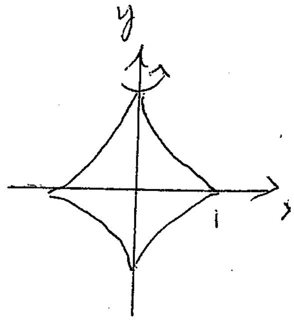


$\frac{dx}{dt} = 3\cos^2 t(-\sin t) = 3\cos^2 t \sin t$ (2); $\frac{dy}{dt} = 3\sin^2 t \cos t$ (2)

$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}$

$= \sqrt{9\sin^2 t \cos^2 t (\cos^2 t + \sin^2 t)} = 3|\sin t \cos t|$ (3)

$S_y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\pi x \, ds = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 t \, |3\sin t \cos t| \, dt$ (5) $= 12\pi \int_0^{\frac{\pi}{2}} \cos^4 t \sin t \, dt$ by symmet

$= -12\pi \int_0^{\frac{\pi}{2}} \cos^4 t \, d(\cos t) = -\frac{12\pi}{5} [\cos^5 t]_0^{\frac{\pi}{2}} = -\frac{12\pi}{5} [\underbrace{\cos^5 \tfrac{\pi}{2}}_{=0} - \underbrace{\cos^5 0}_{1}] = \frac{12\pi}{5}$ (6)

2 (a) $|z_A| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$ (2)

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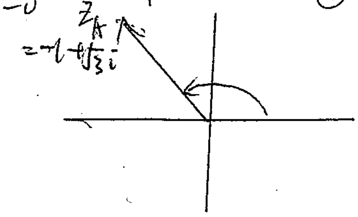


$\arg(z_A) = \pi - \tan^{-1}\left|\frac{\sqrt{3}}{-1}\right| = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ (3) $\therefore z_A = 2e^{i\frac{2\pi}{3}}$ (1)

$75^\circ = 75\frac{\pi}{180} = \frac{5\pi}{12}$ (2)

$z_B = 5e^{i(-\frac{5\pi}{12})} z_A$ (2) $= 5e^{-i\frac{5\pi}{12}} \, 2e^{i\frac{2\pi}{3}} = 10 e^{i(-\frac{5\pi}{12} + \frac{2\pi}{3})} = 10^{i\frac{3\pi}{12}} = 10e^{i\frac{\pi}{4}}$

$= 10(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = 10(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) = 5\sqrt{2} + 5\sqrt{2}i$ (5)

(b) $\frac{1}{i} z^3 = 1 + i \;=\; z^3 = i - 1 = -1 + i$ (2) $= \sqrt{2} e^{i(\pi - \frac{\pi}{4})} = \sqrt{2} e^{i\frac{3\pi}{4}}$ (4)

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$z_k = 2^{\frac{1}{6}} e^{i(\frac{3\pi}{4} + 2k\pi)/3}$, $k = 0, 1, 2$ (3)

$z_0 = 2^{\frac{1}{6}} e^{i\frac{\pi}{4}} = 2^{\frac{1}{6}}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ (2)

$z_1 = 2^{\frac{1}{6}} e^{i(11\pi/12)} = 2^{\frac{1}{6}}(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12})$ (2)

$z_2 = 2^{\frac{1}{6}} e^{i(19\pi/12)} = 2^{1/6} e^{i(\frac{19\pi}{12} - 2\pi)} = 2^{1/6} e^{i(-\frac{5\pi}{12})} = 2^{1/6}\{\cos(-\frac{5\pi}{12}) + i\sin(-\frac{5\pi}{12})$

(4)

3 (a)
34 10

$$\begin{vmatrix} -1 & 2 & 3 \\ 0 & 2 & 0 \\ 2 & -1 & -4 \end{vmatrix} \overset{R_2}{=} 2\begin{vmatrix} -1 & 3 \\ 2 & -4 \end{vmatrix} = 2(4-6) = -4 \quad (3)$$

$$|A^3| + |AA^T| - 2|A^{-1}| = |A|^3 + |A||A^T| - 2|A|^{-1} \;(3) = (-4)^3 + (-4)^2 - \frac{2}{-4}$$

$$= -64 + 16 + \frac{1}{2} = -48 + \frac{1}{2} = -\frac{95}{2} \quad (3)$$

(b) (i)
24 14

$$(A|b) = \left(\begin{array}{cccc|c} 2 & 3 & -1 & 1 & 1 \\ 8 & 12 & -5 & 8 & 3 \\ -2 & -4 & 3 & -4 & -3 \end{array}\right) \xrightarrow[R_3+R_1]{R_2-4R_1} \left(\begin{array}{cccc|c} 2 & 3 & -1 & 1 & 1 \\ 0 & 0 & -1 & 4 & -1 \\ 0 & -1 & 2 & -3 & -2 \end{array}\right) (4) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cccc|c} 2 & 3 & -1 & 1 & 1 \\ 0 & -1 & 2 & -3 & -2 \\ 0 & 0 & -1 & 4 & -1 \end{array}\right) (2)$$

$C_4$ has no pivot $\Rightarrow$ $w = t$ is a free variable (2)

$R_3 : -z + 4w = -1 \Rightarrow z = 1 + 4w = 1 + 4t$ (3)

$R_2 : -y + 2z - 3w = -2 \Rightarrow y = 2 + 2z - 3w = 2 + 2(1+4t) - 3t$
$= 2 + 2 + 8t - 3t = 4 + 5t$ (3)

$R_1 : 2x + 3y - z + w = 1 \Rightarrow 2x = 1 - 3y + z - w$
$= 1 - 3(4+5t) + (1+4t) - t$
$= 1 - 12 - 15t + 1 + 4t - t = -10 - 12t.$

$\therefore x = -5 - 6t$ (3)

$\therefore$ solution $$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -5-6t \\ 4+5t \\ 1+4t \\ t \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -6 \\ 5 \\ 4 \\ 1 \end{pmatrix}$$

(ii) The corresponding homogeneous system
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$$2x + 3y - z + w = 0$$
$$8x + 12y - 5z + 8w = 0$$
$$-2x - 4y + 3z - 4w = 0 \quad (3)$$

A non-trivial solution will be $$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -6 \\ 5 \\ 4 \\ 1 \end{pmatrix} \quad (2)$$