**Higher-order derivative of parametric equation**

Suppose that $x$ and $y$ are given by the following parametric equation:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}.$$

One can find the first derivative using the formula

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}. \quad \left( let\ z = h(t) = \frac{g'(t)}{f'(t)} \right)$$

To obtain higher-order derivate $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$, we can repeat the above process:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}z = \frac{dz}{dx} \overset{\substack{z=h(t) \\ x=x(t)}}{\triangleq} \frac{dz/dt}{dx/dt} = \frac{\frac{d}{dt}\left(\frac{g'(t)}{f'(t)}\right)}{f'(t)} = w(t).$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}w(t) = \frac{dw/dt}{dx/dt}.$$

**Example 20**

It is given that $x(t) = 2t - t^2$ and $y(t) = t^3$. Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$.

☺Solution

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{d}{dt}t^3}{\frac{d}{dt}(2t - t^2)} = \frac{3t^2}{2 - 2t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{3t^2}{2 - 2t}\right) \overset{\substack{z=\frac{3t^2}{2-2t} \\ x=x(t)}}{\cong} \frac{\frac{d}{dt}\left(\frac{3t^2}{2 - 2t}\right)}{dx/dt} = \frac{\frac{12t - 6t^2}{(2 - 2t)^2}}{2 - 2t} = \frac{12t - 6t^2}{(2 - 2t)^3}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(\frac{12t - 6t^2}{(2 - 2t)^3}\right) \overset{\substack{z=\frac{12t-6t^2}{(2-2t)^3} \\ x=x(t)}}{\cong} \frac{\frac{d}{dt}\left(\frac{12t - 6t^2}{(2 - 2t)^3}\right)}{dx/dt} = \cdots = \frac{6(4 + 4t - 2t^2)}{(2 - 2t)^5}$$

## Leibnitz' Rule -- A shortcut for calculating the higher order derivative

- In Example 19, we calculate the higher order derivative of the form $\frac{d^n}{dx^n} f(x)g(x)$. Although we can easily calculate the derivatives by using product rule repeatedly, the calculation is lengthy.
- One would like to ask whether there is any shortcut of doing this. Luckily, there is a theorem which provides a general formula of the $\frac{d^n}{dx^n} f(x)g(x)$. This is called Leibnitz' Rule.

**Leibnitz' Rule**

Let $f(x)$ and $g(x)$ be two $n$-times differentiable functions. Then the $n^{th}$ derivative of the product $f(x)g(x)$ is given by

$$\frac{d^n}{dx^n}[f(x)g(x)] = \sum_{r=0}^{n} C_r^n \underbrace{\frac{d^r}{dx^r} f(x)}_{\substack{differentiate \\ r\ times}} \underbrace{\frac{d^{n-r}}{dx^{n-r}} g(x)}_{\substack{differentiate \\ (n-r) times}} = \sum_{r=0}^{n} C_r^n f^{(r)}(x) g^{(n-r)}(x)$$

**Example 21**

Compute

$$\frac{d^4}{dx^4}(1-x^2)e^{2x}$$

☺Solution:

Step 1: Write down the “general” formula using Leibnitz’s rule

We let $f(x) = 1 - x^2$ and $g(x) = e^{2x}$. From Leibnitz’s rule, we have

$$\frac{d^4}{dx^4}(1-x^2)e^{2x} = \frac{d^4}{dx^4}f(x)g(x) \overset{n=4}{=} \sum_{r=0}^{4} C_r^4 f^{(r)} g^{(4-r)}$$

$$= C_0^4 f^{(0)} g^{(4)} + C_1^4 f^{(1)} g^{(3)} + C_2^4 f^{(2)} g^{(2)} + C_3^4 f^{(3)} g^{(1)} + C_4^4 f^{(4)} g^{(0)}.$$

## Step 2: Compute all numbers required in the formula

By direct calculation, we get

| $r$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $C_r^4$ | $C_0^4 = 1$ | $C_1^4 = 4$ | $C_2^4 = 6$ | $C_3^4 = 4$ | $C_4^4 = 1$ |
| $f^{(r)}(x)$ | $1 - x^2$ | $-2x$ | $-2$ | $0$ | $0$ |
| $g^{(r)}(x)$ | $e^{2x}$ | $2e^{2x}$ | $2(2e^{2x})$ $= 4e^{2x}$ | $4(2e^{2x})$ $= 8e^{2x}$ | $8(2e^{2x})$ $= 16e^{2x}$ |

## Step 3: Substitute everything into the formula

$$\frac{d^4}{dx^4}(1 - x^2)e^{2x}$$

$$= C_0^4 f^{(0)} g^{(4)} + C_1^4 f^{(1)} g^{(3)} + C_2^4 f^{(2)} g^{(2)} + C_3^4 f^{(3)} g^{(1)} + C_4^4 f^{(4)} g^{(0)}$$

$$= (1 - x^2)(16e^{2x}) + 4(-2x)(8e^{2x}) + 6(-2)(4e^{2x}) + 4(0)(2e^{2x}) + 1(0)(e^{2x})$$

$$= (-16x^2 - 64x - 32)e^{2x}.$$

**Example 22**

Compute

$$\frac{d^3}{dx^3}\cos x \ln x$$

☺Solution:

Step 1: Write down the “general” formula using Leibnitz’s rule

We let $f(x) = \cos x$ and $g(x) = \ln x$. From Leibnitz’s rule, we have

$$\frac{d^3}{dx^3}\cos x \ln x = \frac{d^3}{dx^3} f(x)g(x) \overset{n=3}{\overset{\text{☺}}{=}} \sum_{r=0}^{3} C_r^3 f^{(r)} g^{(3-r)}$$

$$= C_0^3 f^{(0)} g^{(3)} + C_1^3 f^{(1)} g^{(2)} + C_2^3 f^{(2)} g^{(1)} + C_3^3 f^{(3)} g^{(0)}.$$

## Step 2: Compute all numbers required in the formula

By direct calculation, we get

| $r$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|
| $C_r^3$ | $C_0^3 = 1$ | $C_1^3 = 3$ | $C_2^3 = 3$ | $C_3^3 = 1$ |
| $f^{(r)}(x)$ | $\cos x$ | $-\sin x$ | $-\cos x$ | $\sin x$ |
| $g^{(r)}(x)$ | $\ln x$ | $\frac{1}{x} = x^{-1}$ | $-x^{-2}$ | $-(-2x^{-3})$ $= 2x^{-3}$ |

## Step 3: Substitute everything into the formula

$$\frac{d^3}{dx^3}\cos x \ln x = C_0^3 f^{(0)} g^{(3)} + C_1^3 f^{(1)} g^{(2)} + C_2^3 f^{(2)} g^{(1)} + C_3^3 f^{(3)} g^{(0)}$$

$$= \cos x\,(2x^{-3}) + 3(-\sin x)(-x^{-2}) + 3(-\cos x)(x^{-1}) + \sin x\,(\ln x)$$

$$= (2x^{-3} - 3x^{-1})\cos x + (\ln x - 3x^{-2})\sin x.$$

**Example 23**

For any positive integer $n$, compute

$$\frac{d^n}{dx^n} x^2 \cos 3x$$

☺Solution:

Step 1: Write down the "general" formula using Leibnitz's rule

We let $f(x) = x^2$ and $g(x) = \cos 3x$. From Leibnitz's rule, we have

$$\frac{d^n}{dx^n} x^2 \cos 3x = \frac{d^n}{dx^n} f(x)g(x) = \sum_{r=0}^{n} C_r^n f^{(r)} g^{(n-r)}$$

$$= C_0^n f^{(0)} g^{(n)} + C_1^n f^{(1)} g^{(n-1)} + C_2^n f^{(2)} g^{(n-2)} + C_3^n f^{(3)} g^{(n-3)}$$

$$+ C_4^n f^{(4)} g^{(n-4)} + \cdots + C_n^n f^{(n)} g^{(0)}.$$

## Step 2: Compute all numbers required in the formula

Note that for any positive integer $k$, we have

$$g^{(k)}(x) = \frac{d^k}{dx^k}\cos 3x = 3^k \cos\left(\frac{k\pi}{2} + 3x\right).$$

| $r$ | 0 | 1 | 2 | 3 | … | $n$ |
|---|---|---|---|---|---|---|
| $f^{(r)}(x)$ | $x^2$ | $2x$ | 2 | 0 | … | 0 |

## Step 3: Substitute everything into the formula

$$\frac{d^n}{dx^n}x^2\cos 3x$$

$$= C_0^n f^{(0)} g^{(n)} + C_1^n f^{(1)} g^{(n-1)} + C_2^n f^{(2)} g^{(n-2)}$$

$$+ \overbrace{C_3^n f^{(3)} g^{(n-3)} + C_4^n f^{(4)} g^{(n-4)} + \cdots + C_n^n f^{(n)} g^{(0)}}^{=0 \text{ since } f^{(k)}(x)=0 \text{ for } k\geq 3}$$

$$= C_0^n x^2 3^n \cos\left(\frac{n\pi}{2} + 3x\right) + C_1^n(2x)\left[3^{n-1}\cos\left(\frac{(n-1)\pi}{2} + 3x\right)\right]$$

$$+ C_2^n(2)\left[3^{n-2}\cos\left(\frac{(n-2)\pi}{2} + 3x\right)\right]$$

Recall that $C_r^n = \frac{n!}{r!(n-r)!}$, so we have

$$C_0^n = 1, \qquad C_1^n = n, \qquad C_2^n = \frac{n(n-1)}{2}.$$

$$\frac{d^n}{dx^n} x^2 \cos 3x$$

$$= x^2 3^n \cos\left(\frac{n\pi}{2} + 3x\right) + n(2x)\left[3^{n-1}\cos\left(\frac{(n-1)\pi}{2} + 3x\right)\right]$$

$$+n(n-1)\left[3^{n-2}\cos\left(\frac{(n-2)\pi}{2} + 3x\right)\right]$$

**Example 24 (Harder Example)**

Let $f(x) = \tan^{-1} x$

(a) Show that $(1+x^2)f''(x) + 2xf'(x) = 0$.

(b) Let $n$ be a positive integer

(i) Using Leibnitz's rule, show that

$$(1+x^2)f^{(n+2)}(x) + 2(n+1)xf^{(n+1)}(x) + n(n+1)f^{(n)}(x) = 0.$$

(ii) Hence, find $f^{(3)}(x), f^{(4)}(x), f^{(5)}(x)$.

☺Solution

(a) Using direct differentiation, we get

$$f'(x) = \frac{1}{1+x^2}, \qquad f''(x) = \frac{d}{dx}\left(\frac{1}{1+x^2}\right) = -\frac{2x}{(1+x^2)^2}$$

Then

$$(1+x^2)f''(x) + 2xf'(x) = (1+x^2)\left(-\frac{2x}{(1+x^2)^2}\right) + 2x\left(\frac{1}{1+x^2}\right) = 0.$$

(b) To obtain the equation (i), one has to differentiate the equation in (a) with respect to $x$ for $n$ times:

$$\frac{d^n}{dx^n}(1+x^2)f''(x)+\frac{d^n}{dx^n}[2xf'(x)]=\frac{d^n}{dx^n}0=0 \ldots\ldots (*)$$

We proceed to compute the two derivatives on L.H.S.

Using Leibnitz's Rule, we get

$$\frac{d^n}{dx^n}(1+x^2)f''(x)=\sum_{r=0}^{n}C_r^n\frac{d^r}{dx^r}(1+x^2)\frac{d^{n-r}}{dx^{n-r}}f''(x)$$

$$=C_0^n\overbrace{\frac{d^0}{dx^0}(1+x^2)}^{1+x^2}\overbrace{\frac{d^n}{dx^n}f''(x)}^{f^{(n+2)}(x)}+C_1^n\overbrace{\frac{d^1}{dx^1}(1+x^2)}^{2x}\overbrace{\frac{d^{n-1}}{dx^{n-1}}f''(x)}^{f^{(n+1)}(x)}$$

$$+C_2^n\overbrace{\frac{d^2}{dx^2}(1+x^2)}^{2}\overbrace{\frac{d^{n-2}}{dx^{n-2}}f''(x)}^{f^{(n)}(x)}+C_3^n\overbrace{\frac{d^3}{dx^3}(1+x^2)}^{0}\overbrace{\frac{d^{n-3}}{dx^{n-3}}f''(x)}^{f^{(n-1)}(x)}+\cdots$$

$$+C_n^n\overbrace{\frac{d^n}{dx^n}(1+x^2)}^{=0}\overbrace{\frac{d^0}{dx^0}f''(x)}^{f^{(2)}(x)}$$

$$= (1+x^2)f^{(n+2)}(x) + n(2x)f^{(n+1)}(x) + \frac{n(n-1)}{2}2\,f^{(n)}(x)$$
$$= (1+x^2)f^{(n+2)}(x) + 2nxf^{(n+1)}(x) + n(n-1)f^{(n)}(x).$$

Similarly, one can find that

$$\frac{d^n}{dx^n}[2xf'(x)] = 2xf^{(n+1)}(x) + 2n\,f^{(n)}(x).$$

Substitute the formula obtained into the equation $(*)$, we get

$$(1+x^2)f^{(n+2)}(x) + 2nxf^{(n+1)}(x) + n(n-1)f^{(n)}(x) + 2xf^{(n+1)}(x) + 2n\,f^{(n)}(x) = 0.$$
$$\Rightarrow (1+x^2)f^{(n+2)}(x) + (2n+2)xf^{(n+1)}(x) + (n^2+n)f^{(n)}(x) = 0.$$

(b)(ii)

To obtain the derivatives $f^{(3)}(x), f^{(4)}(x), f^{(5)}(x)$, one can use the equation derived in (b)(i).

We first put $n = 1$ into the equation, we get

$$(1+x^2)f^{(3)}(x) + 4x\overbrace{f^{(2)}(x)}^{=\frac{-2x}{(1+x^2)^2}} + 2\overbrace{f^{(1)}(x)}^{=\frac{1}{(1+x^2)}} = 0 \Rightarrow f^{(3)}(x) = \frac{6x^2-2}{(1+x^2)^3}$$

Put $n = 2$ into the equation, we get

$$(1+x^2)f^{(4)}(x) + 6x\overbrace{f^{(3)}(x)}^{=\frac{6x^2-2}{(1+x^2)^3}} + 6\overbrace{f^{(2)}(x)}^{=\frac{-2x}{(1+x^2)^2}} = 0 \Rightarrow f^{(4)}(x) = \frac{24x-24x^3}{(1+x^2)^4}$$

Finally, we put $n = 3$ into the equation, we get

$$(1+x^2)f^{(5)}(x) + 8x\overbrace{f^{(4)}(x)}^{=\frac{24x-24x^3}{(1+x^2)^4}} + 12\overbrace{f^{(3)}(x)}^{=\frac{6x^2-2}{(1+x^2)^3}} = 0 \Rightarrow f^{(5)}(x) = \frac{120x^4-240x^2+24}{(1+x^2)^5}$$

In general, to find the higher-order derivatives of a complicated function, say $f(x) = \sin(\ln(x+1))$, one can do this by following procedure:

Step 1: Compute the first derivative and second derivative $\frac{df}{dx}$ and $\frac{d^2f}{dx^2}$ and use them to derive a differential equation

$$(1+x)^2 f''(x) + (1+x)f'(x) + f(x) = 0$$

Step 2: We obtain the *general equation* by differentiating the equation in Step 1 with respect to $x$ for $n$-times (by Leibnitz's Rule), i.e.

$$\frac{d^n}{dx^n}\left[(1+x)^2 f''(x) + (1+x)f'(x) + f(x)\right] = \frac{d^n}{dx^n}0$$

$$\Rightarrow (1+x)^2 f^{(n+2)}(x) + (2n+1)(1+x)f^{(n+1)}(x) + (n^2+1)f^{(n)}(x) = 0$$

Step 3: Find $f^{(3)}(x)$, $f^{(4)}(x)$, $f^{(5)}(x)$,… by putting $n = 1,2,3,\ldots$. Into the general equation.

**Remark about using Leibnitz's Rule**

- Although the Leibnitz's rule provide a useful formula in finding the higher order derivative, it is only efficient in the case when the function is a product of some elementary functions and the general formula of $n^{th}$ derivative of these elementary functions are available, e.g.

$$f(x) = e^x \sin x\,, \quad g(x) = x^2 \sin x.$$

- In some cases, method of partial fractions or product-to-sum formula may be more useful than Leibnitz's rule when finding the derivatives such as

$$\frac{d}{dx}\sin 3x \cos 4x\,, \frac{d}{dx}\frac{2}{(x-1)(x+3)(2x-1)}.$$