Higher derivatives

The operation of differentiation takes a differentiable function $f(x)$ and produces a new function $f'(x)$. If $f'(x)$ is also differentiable, we can differentiate $f'(x)$ and produce another function called the second derivative of $f(x)$ and it is denoted by $f''(x)$. We may repeat the process and suppose that $y = f(x)$ is a differentiable function such that $f'(x)$, $f''(x)$, …, up to its $(n-1)^{\text{th}}$ derivative are differentiable. Then the n^{th} derivative of f exists and we denote it by $f^{(n)}(x)$. These are summarized in the following table:

The function $f(x)$	$y = f(x)$
First derivative of $f(x)$ w.r.t. x (i.e. differentiate $f(x)$ once)	$y' = \frac{dy}{dx} = f'(x)$
Second derivative of $f(x)$ w.r.t. x (i.e. differentiate $f(x)$ twice)	$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)$

Third derivative of $f(x)$ w.r.t. x (i.e. differentiate $f(x)$ three times)	$y''' = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = f'''(x)$ (Also denoted as $y^{(3)}$ or $f^{(3)}(x)$.)
$\vdots$	$\vdots$
The n-th derivative of $f(x)$ w.r.t. x (i.e. differentiate $f(x)$ n times, where n is a positive integer)	$y^{(n)} = \frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = f^{(n)}(x)$ ($\frac{d^ny}{dx^n}$ is also denoted by D^ny.)

Note: $\boxed{f^{(n)}(x) \neq f^n(x)}$

$f^{(n)}(x)$ is the n-th derivative of $f(x)$ w.r.t. x, while $f^n(x) = [f(x)]^n$ is $f(x)$ to the power n.

E.g. $\frac{d^2y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2$, $f^{(3)}(x) \neq f^3(x) = [f(x)]^3$, etc.

Example 28

implicit

If $ay^2 + by + c = x$ for any constants $a(\neq 0)$, b and c, show that

$$\underbrace{\frac{d^2y}{dx^2}} + 2a\left(\frac{dy}{dx}\right)^3 = 0.$$

$\because$ the highest derivative is the 2nd derivative

$\therefore$ we need to do differentiation twice

Solution

$$ay^2 + by + c = x$$

Differentiate both sides w.r.t. x:

$$a \cdot 2y\frac{dy}{dx} + b\frac{dy}{dx} + 0 = 1 \quad \Rightarrow \quad (2ay + b)\frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2ay + b} \quad \ldots(*)$$

Differentiate both sides of $(2ay + b)\frac{dy}{dx} = 1$ w.r.t. x:

$$(2ay + b) \cdot \underbrace{\frac{d}{dx}\left(\frac{dy}{dx}\right)}_{=\frac{d^2y}{dx^2}} + 2a\underbrace{\frac{dy}{dx} \cdot \frac{dy}{dx}}_{=\left(\frac{dy}{dx}\right)^2} = 0$$

$$\Rightarrow \quad (2ay+b)\frac{d^2y}{dx^2} + 2a\left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + 2a\underbrace{\left(\frac{1}{2ay+b}\right)}_{=\frac{dy}{dx}\text{ by }(*)}\left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + 2a\left(\frac{dy}{dx}\right)^3 = 0$$

□

For some simple functions like those in the following examples, we may differentiate the function $y = f(x)$ a few times to obtain $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ etc., and then conjecture the general formula for $\frac{d^ny}{dx^n}$, where $n \in \mathbb{N}$.

Example 29

Let $y = x^3$. Find $\frac{d^n y}{dx^n}$, where n is a positive integer.

Solution

$$y = x^3 \leftarrow \text{deg. 3}$$

For $n = 1$: $\frac{dy}{dx} = 3x^2 \leftarrow \text{deg. 2}$

For $n = 2$: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3x^2) = 6x \leftarrow \text{deg. 1}$

For $n = 3$: $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}(6x) = 6 \leftarrow \text{constant}$

For all integers $n \geq 4$: $\frac{d^n y}{dx^n} = 0.$

Hence,
$$\frac{d^n y}{dx^n} = \begin{cases} 3x^2 & , \text{ for } n = 1 \\ 6x & , \text{ for } n = 2 \\ 6 & , \text{ for } n = 3 \\ 0 & , \text{ for } n \geq 4 \end{cases}$$

Example 30

Let $y = x^m$, where m is a positive integer. Find $\frac{d^n y}{dx^n}$, where n is a positive integer.

Solution

$$\frac{dy}{dx} = mx^{m-1}$$

2-1

$$\frac{d^2y}{dx^2} = m(m-1)x^{m-2}$$

3-1

$$\frac{d^3y}{dx^3} = m(m-1)(m-2)x^{m-3}$$

$$\vdots$$

$$\frac{d^m y}{dx^m} = m(m-1)(m-2)\cdots\underbrace{[m-(m-1)]}_{=1}\underbrace{x^{m-m}}_{=x^0=1} = m!$$

$$\frac{d^n y}{dx^n} = 0 \quad \text{for } n > m.$$

Hence, $$\frac{d^n y}{dx^n} = \begin{cases} m(m-1)(m-2)\cdots[m-(n-1)]\,x^{m-n} & \text{, if } n \le m \\ 0 & \text{, if } n > m \end{cases}$$

Example 31

Find $\frac{d^n}{dx^n}(e^{ax})$, where a is a non-zero constant and n is a positive integer.

Solution

$$\frac{d}{dx}(e^{ax}) = e^{ax} \cdot \frac{d}{dx}(ax) = ae^{ax}$$

$$\frac{d^2}{dx^2}(e^{ax}) = \frac{d}{dx}(ae^{ax}) = ae^{ax} \cdot a = a^2 e^{ax}$$

$$\frac{d^3}{dx^3}(e^{ax}) = \frac{d}{dx}(a^2 e^{ax}) = a^2 e^{ax} \cdot a = a^3 e^{ax}$$

$\therefore$ By conjecture,

$$\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}, \qquad n \in \mathbb{N}.$$

Example 32

Find $\frac{d^n}{dx^n}\left(\frac{1}{ax+b}\right)$, where $a \neq 0$ and b are constants and n is a positive integer.

Solution

$$\frac{d}{dx}\left(\frac{1}{ax+b}\right) = \frac{d}{dx}[(ax+b)^{-1}] = (-1)\cdot(ax+b)^{-2}\cdot\underbrace{\frac{d}{dx}(ax+b)}_{=a} = (-1)\cdot a\cdot(ax+b)^{-2}$$

$$\frac{d^2}{dx^2}\left(\frac{1}{ax+b}\right) = (-1)\cdot a\cdot\frac{d}{dx}[(ax+b)^{-2}] = \underbrace{(-1)}_{(-1)\cdot 1}\cdot a\cdot\underbrace{(-2)}_{=(-1)\cdot 2}\cdot(ax+b)^{-3}\cdot\underbrace{\frac{d}{dx}(ax+b)}_{=a}$$

$$= (-1)^2\cdot\underbrace{2!}_{=2\cdot 1}\cdot a^2\cdot(ax+b)^{-3} \leftarrow 2+1$$

$$\frac{d^3}{dx^3}\left(\frac{1}{ax+b}\right) = (-1)^2\cdot 2!\cdot a^2\cdot\frac{d}{dx}[(ax+b)^{-3}] = (-1)^2\cdot 2!\cdot a^2\cdot\underbrace{(-3)}_{=(-1)\cdot 3}\cdot(ax+b)^{-4}\cdot a$$

$$= (-1)^3\cdot\underbrace{3!}_{=3\cdot 2\cdot 1}\cdot a^3\cdot(ax+b)^{-4} \leftarrow 3+1$$

$\therefore$ By conjecture, $\frac{d^n}{dx^n}\left(\frac{1}{ax+b}\right) = (-1)^n\cdot n!\cdot a^n\cdot(ax+b)^{-(n+1)} = \frac{(-1)^n\, n!\, a^n}{(ax+b)^{n+1}}$

Example 33

Find $\frac{d^n}{dx^n}[\cos(ax+b)]$, where $a \neq 0$ and b are constants and n is a positive integer.

Solution

$$\frac{d}{dx}[\cos(ax+b)] = -\sin(ax+b)\cdot\underbrace{\frac{d}{dx}(ax+b)}_{=a}$$

$$= -a\sin(ax+b)$$

Ch.4 (in radians)

$$= a\cos\left(ax+b+\frac{\pi}{2}\right) \quad \because \boxed{\cos\left(\theta+\frac{\pi}{2}\right) = -\sin\theta}$$

Put $\theta = ax+b$

$$\frac{d^2}{dx^2}[\cos(ax+b)] = \frac{d}{dx}\left[a\cos\left(ax+b+\frac{\pi}{2}\right)\right]$$

$$= -a\sin\left(ax+b+\frac{\pi}{2}\right)\cdot\underbrace{\frac{d}{dx}\left(ax+b+\frac{\pi}{2}\right)}_{=a}$$

$$= a^2\cos\left(ax+b+\frac{\pi}{2}+\frac{\pi}{2}\right) \quad \because \boxed{\cos\left(\theta+\frac{\pi}{2}\right) = -\sin\theta}$$

Put $\theta = ax+b+\frac{\pi}{2}$

$$= a^2\cos\left(ax+b+\frac{2\pi}{2}\right)$$

$$\frac{d^3}{dx^3}[\cos(ax+b)] = \frac{d}{dx}\left[a^2\cos\left(ax+b+\frac{2\pi}{2}\right)\right]$$

$$= -a^2\sin\left(ax+b+\frac{2\pi}{2}\right)\cdot\underbrace{\frac{d}{dx}\left(ax+b+\frac{2\pi}{2}\right)}_{=a}$$

$$= a^3\cos\left(ax+b+\frac{2\pi}{2}+\frac{\pi}{2}\right) \quad \because \boxed{\cos\left(\theta+\frac{\pi}{2}\right) = -\sin\theta}$$

Put $\theta = ax+b+\frac{2\pi}{2}$

$$= a^3\cos\left(ax+b+\frac{3\pi}{2}\right)$$

etc.

$$\therefore \quad \frac{d^n}{dx^n}[\cos(ax+b)] = a^n\cos\left(ax+b+\frac{n\pi}{2}\right), \quad \text{where } n \in \mathbb{N}.$$

Homework: Show that $\frac{d^n}{dx^n}[\sin(ax+b)] = a^n\sin\left(ax+b+\frac{n\pi}{2}\right)$.

(Hint: $\boxed{\sin\left(\theta+\frac{\pi}{2}\right) = \cos\theta}$.)

Example 34

Let $y = \ln(2x+3)$. Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$, and then conjecture the formula for $\frac{d^ny}{dx^n}$, where $n \in \mathbb{N}$.

Solution

$$y = \ln(2x+3)$$

$$\frac{dy}{dx} = \frac{1}{2x+3} \cdot \underbrace{\frac{d}{dx}(2x+3)}_{=2} = 2 \cdot (2x+3)^{-1}$$

$$\frac{d^2y}{dx^2} = 2 \cdot (-1)(2x+3)^{-2} \cdot 2 = 2^2 \cdot (-1)(2x+3)^{-2}$$

$$\frac{d^3y}{dx^3} = 2^2 \cdot \underbrace{(-1)}_{=(-1)\cdot 1} \underbrace{(-2)}_{=(-1)\cdot 2} (2x+3)^{-3} \cdot 2 = 2^3 \cdot (-1)^2 \cdot 2!\,(2x+3)^{-3}$$

$$\frac{d^4y}{dx^4} = 2^3 \cdot (-1)^2 \cdot 2!\,\underbrace{(-3)}_{=(-1)\cdot 3}(2x+3)^{-4} \cdot 2 = 2^4 (-1)^3 \cdot 3!\,(2x+3)^{-4}$$

$\therefore$ By conjecture,

$$\frac{d^ny}{dx^n} = 2^n \cdot (-1)^{n-1} \cdot (n-1)!\,(2x+3)^{-n} \quad , \quad n \in \mathbb{N}$$

More on Higher Derivatives

If f and g are n-times differentiable functions of x, then

$$\boxed{\frac{d^n}{dx^n}[\alpha \cdot f(x) \pm \beta \cdot g(x)] = \alpha \cdot \frac{d^n}{dx^n}[f(x)] \pm \beta \cdot \frac{d^n}{dx^n}[g(x)]} \quad \text{......} \ (*)$$

for all constants α and β.

Note that $\frac{d^n}{dx^n}[f(x) \cdot g(x)] \neq \left\{\frac{d^n}{dx^n}[f(x)]\right\} \cdot \left\{\frac{d^n}{dx^n}[g(x)]\right\}$.

Question: How to find $\frac{d^n}{dx^n}[f(x) \cdot g(x)]$?

Method 1: For simple functions, decompose $f(x) \cdot g(x)$ into sum or difference of functions in x, then use result (*) and other known results.

Method 2: Use Leibnitz' rule.

Example 35

(a) Resolve $\frac{1}{(x+1)(2x+1)}$ into partial fractions.

(b) Find $\frac{d^n}{dx^n}\left[\frac{1}{(x+1)(2x+1)}\right]$, where $n \in \mathbb{N}$.

Solution

(a) $$\frac{1}{(x+1)(2x+1)} = \frac{A}{x+1} + \frac{B}{2x+1}$$

$$\therefore \quad 1 = A(2x+1) + B(x+1)$$

Put $x = -1$: $1 = -A \Rightarrow A = -1$

Put $x = -\frac{1}{2}$: $1 = \frac{1}{2}B \Rightarrow B = 2$

$$\therefore \quad \frac{1}{(x+1)(2x+1)} = -\frac{1}{x+1} + \frac{2}{2x+1}$$

(b) $\dfrac{d^n}{dx^n}\left[\dfrac{1}{(x+1)(2x+1)}\right]$

$$= -\frac{d^n}{dx^n}\left(\frac{1}{x+1}\right) + 2\cdot\frac{d^n}{dx^n}\left(\frac{1}{2x+1}\right)$$

$a=1, b=1$ $a=2, b=1$

$$= -\left[\frac{(-1)^n\cdot n!\cdot 1^n}{(x+1)^{n+1}}\right] + 2\cdot\left[\frac{(-1)^n\cdot n!\cdot 2^n}{(2x+1)^{n+1}}\right]$$

by using $\dfrac{d^n}{dx^n}\left(\dfrac{1}{ax+b}\right) = \dfrac{(-1)^n\cdot n!\, a^n}{(ax+b)^{n+1}}$ from Ex. 32

$$= (-1)^n\cdot n!\left[\frac{2^{n+1}}{(2x+1)^{n+1}} - \frac{1}{(x+1)^{n+1}}\right]$$

Example 36

Note that $\frac{d^n}{dx^n}(\cos^2 x) \neq \left[\frac{d^n}{dx^n}(\cos x)\right]^2$.

Find $\dfrac{d^n}{dx^n}(\cos^2 x)$, where $n \in \mathbb{N}$.

Solution

$$\frac{d^n}{dx^n}(\cos^2 x) = \frac{d^n}{dx^n}\left[\frac{1}{2}(1+\cos 2x)\right]$$ using Half-angle formula

$$= \frac{1}{2}\left[\underbrace{\frac{d^n}{dx^n}(1)}_{=0 \ \because 1 \text{ is a constant}} + \frac{d^n}{dx^n}(\cos 2x)\right]$$

$$= \frac{1}{2}\left[2^n \cos\left(2x + \frac{n\pi}{2}\right)\right]$$ by using $\frac{d^n}{dx^n}[\cos(ax+b)] = a^n \cos(ax+b+\frac{n\pi}{2})$

from Ex. 33 $(a=2, b=0)$

$$= 2^{n-1}\cos\left(2x + \frac{n\pi}{2}\right), \quad n \in \mathbb{N}$$

Consider differentiation of $f(x) \cdot g(x)$ a few times:

$$(f(x) \cdot g(x))' = f(x) \cdot g'(x) + f'(x) \cdot g(x)$$

$$(f(x) \cdot g(x))'' = f(x) \cdot g''(x) + f'(x) \cdot g'(x) + f'(x) \cdot g'(x) + f''(x) \cdot g(x)$$

$$= f(x) \cdot g''(x) + 2 f'(x) \cdot g'(x) + f''(x) \cdot g(x)$$

$$(f(x) \cdot g(x))''' = f(x) \cdot g'''(x) + f'(x) \cdot g''(x) + 2 [f'(x) \cdot g''(x) + f''(x) \cdot g'(x)]$$

$$+ f''(x) \cdot g'(x) + f'''(x) \cdot g(x)$$

$$= f(x) \cdot g'''(x) + 3 f'(x) \cdot g''(x) + 3 f''(x) \cdot g'(x) + f'''(x) \cdot g(x)$$

etc.

Compare with binomial expansion:

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

etc.

Leibnitz' Rule

This is used to determine the **n-th derivative of a product of two functions of x**.

Let $y = (fg)(x) = f(x) \cdot g(x)$, where f and g are n-times differentiable functions. Then the n-th derivative of fg is:

$$\boxed{\begin{aligned}(fg)^{(n)}(x) &= \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) \cdot g^{(n-k)}(x) \\ &= \binom{n}{0} f^{(0)}(x)\ g^{(n)}(x) + \binom{n}{1} f^{(1)}(x)\ g^{(n-1)}(x) \\ &\quad + \binom{n}{2} f^{(2)}(x)\ g^{(n-2)}(x) + \cdots + \binom{n}{n} f^{(n)}(x)\ g^{(0)}(x),\end{aligned}}$$

n+1 terms

where $f^{(k)}(x) = \frac{d^k}{dx^k}[f(x)]$, $f^{(0)}(x) = f(x)$,

$g^{(n-k)}(x) = \frac{d^{n-k}}{dx^{n-k}}[g(x)]$, $g^{(0)}(x) = g(x)$, Binomial coefficient, also written as ${}_nC_k$, C_k^n

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)\cdot(n-k)!}{k!(n-k)!} = \frac{\overbrace{n(n-1)(n-2)\cdots(n-k+1)}^{\leftarrow k \text{ factors}}}{k!},$$

$$k! = k(k-1)(k-2)\cdot\cdots\cdot 3\cdot 2\cdot 1 \quad \text{for } k \in \mathbb{N},$$

and $\quad 0! = 1$ (by definition).

k factorial

E.g. $\binom{n}{0} = 1$

$\binom{n}{1} = \frac{n}{1!} = n$

$\binom{n}{2} = \frac{n(n-1)}{2!}$

$\binom{n}{3} = \frac{n(n-1)(n-2)}{3!}$

etc.

Compare the Leibnitz' rule with the Binomial Theorem.

Recall the **Binomial Theorem**:

Also, $\binom{n}{0} = \binom{n}{n}$, $\binom{n}{1} = \binom{n}{n-1}$, $\binom{n}{2} = \binom{n}{n-2}$ etc.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k\, b^{n-k}$$

$$= \binom{n}{0} a^0\, b^n + \binom{n}{1} a^1\, b^{n-1} + \binom{n}{2} a^2\, b^{n-2} + \cdots + \binom{n}{n} a^n\, b^0$$

Note that $f^{(k)}(x) \neq f^k(x)$.

E.g. $f^{(0)}(x) = f(x)$ but $f^0(x) = [f(x)]^0 = 1$, where $f(x)$ is not identically equal to 0.

Example 37

If $y = e^{2x}\underbrace{(x^3 + 5x - 1)}_{\text{deg. } 3}$, find $\dfrac{d^{10}y}{dx^{10}}$.

Solution

By the **Leibnitz' rule**,

$$\frac{d^{10}y}{dx^{10}} = \sum_{k=0}^{10}\binom{10}{k}(x^3+5x-1)^{(k)}\ (e^{2x})^{(10-k)}$$

$$= \binom{10}{0}(x^3+5x-1)^{(0)}\ (e^{2x})^{(10)} + \binom{10}{1}(x^3+5x-1)^{(1)}\ (e^{2x})^{(9)}$$

$$+ \binom{10}{2}(x^3+5x-1)^{(2)}\ (e^{2x})^{(8)} + \binom{10}{3}(x^3+5x-1)^{(3)}\ (e^{2x})^{(7)}$$

$$+ \binom{10}{4}(x^3+5x-1)^{(4)}\ (e^{2x})^{(6)} + \cdots + \binom{10}{10}(x^3+5x-1)^{(10)}\ (e^{2x})^{(0)}$$

$$= 1\cdot\underbrace{(x^3+5x-1)}_{\text{deg. } 3}\cdot 2^{10}e^{2x} + 10\cdot\underbrace{(3x^2+5)}_{\text{deg. } 2}\cdot 2^9 e^{2x} + 45\cdot\underbrace{(6x)}_{\text{deg. } 1}\cdot 2^8 e^{2x}$$

$$+ 120\cdot\underbrace{(6)}_{\text{constant}}\cdot 2^7 e^{2x}$$ **using** $\boxed{\dfrac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}}$ **(Example 31)**

$\leftarrow$ 3 + 1 = 4 non-zero terms

$$+ \mathbf{210 \cdot (0) \cdot 2^6\, e^{2x}} + \cdots$$ ← **all the remaining terms are 0,**

derivative of constant is 0

since $\frac{d^n}{dx^n}(x^3+5x-1)=0$ **for** $n \geq 4$

$$= 2^7\, e^{2x}[2^3 \cdot (x^3+5x-1) + 10 \cdot (3x^2+5) \cdot 2^2 + 45 \cdot (6x) \cdot 2 + 120 \cdot (6)]$$

$$= 2^7\, e^{2x}\,(8x^3+120x^2+580x+912)$$

$$= 2^9\, e^{2x}\,(2x^3+30x^2+145x+228)$$

Remark:

We usually take polynomial as f(x)

We take $f(x) = x^3+5x-1$ and $g(x) = e^{2x}$ so that the first $1+3=4$ terms are non-zero (i.e. $f^{(k)}(x) \neq 0$ when $k = 0, 1, 2, 3$) and all the remaining terms are zeros.

If we take $f(x) = e^{2x}$ and $g(x) = x^3+5x-1$, then the last 4 terms are non-zero and all the remaining terms are zeros.

Example 38 quadratic (deg. 2) $\therefore$ first 3 terms are non-zero if we take it as f(x).

Given that $y = (2x^2 + 3x - 7)\cos(3x + 2)$. Find $\dfrac{d^n y}{dx^n}$, where $n \in \mathbb{N}$.

Solution

By using the **Leibnitz' rule**,

$$\frac{d^n y}{dx^n} = \sum_{k=0}^{n} \binom{n}{k} (2x^2 + 3x - 7)^{(k)} \, [\cos(3x + 2)]^{(n-k)}$$

$$= \binom{n}{0}(2x^2+3x-7)^{(0)} \cdot [\cos(3x+2)]^{(n)} + \binom{n}{1}(2x^2+3x-7)^{(1)} \cdot [\cos(3x+2)]^{(n-1)}$$

$$+ \binom{n}{2}(2x^2+3x-7)^{(2)} \cdot [\cos(3x+2)]^{(n-2)} + 0$$

$\leftarrow$ remaining terms are 0, since $(2x^2+3x-7)^{(k)} = 0$ for $k > 2$

From Ex. 33

$$= 1 \cdot (2x^2+3x-7) \cdot 3^n \cos\left(3x+2+\frac{n\pi}{2}\right) + n \cdot (4x+3) \cdot 3^{n-1} \cos\left(3x+2+\frac{(n-1)\pi}{2}\right)$$

From Ex. 33

$$+ \frac{n(n-1)}{2} \cdot 4 \cdot 3^{n-2} \cos\left(3x+2+\frac{(n-2)\pi}{2}\right) \quad \boxed{\text{for } n \geq 2} \quad \text{———} \; (*)$$

From Ex. 33

For $n=1$, we have

$$\frac{dy}{dx} = (2x^2+3x-7)\cdot[-\sin(3x+2)\cdot 3] + \cos(3x+2)\cdot(4x+3)$$ by product rule

$$= 3(2x^2+3x-7)\cdot\cos\left(3x+2+\frac{\pi}{2}\right) + (4x+3)\cos(3x+2)$$

which is the same expression as if we put $n=1$ into ⊛.

$\therefore$ ⊛ is true for $n \geq 1$.

Example 39 (This is hard!)

Given that $y = e^{\sin^{-1} x}$.

highest derivative is the second derivative $\therefore$ do differentiation twice

(a) Show that $(1-x^2)\boxed{\dfrac{d^2y}{dx^2}} - x\dfrac{dy}{dx} - y = 0$ …… (*).

(b) Using part (a) and the Leibnitz' rule, show that

$$(1-x^2)\boxed{y^{(n+2)}} - (2n+1)x\,y^{(n+1)} - (n^2+1)\,y^{(n)} = 0\,,$$

where $y^{(k)} = \dfrac{d^k y}{dx^k}$.

highest derivative is the (n+2) th derivative $\therefore$ Differentiate (*) n times

Solution

(a) $y = e^{\sin^{-1} x}$

$$\frac{dy}{dx} = e^{\sin^{-1} x} \cdot \frac{d}{dx}(\sin^{-1} x) = e^{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\,\frac{dy}{dx} = e^{\sin^{-1} x}$$

$$\Rightarrow (1-x^2)^{\frac{1}{2}}\,\frac{dy}{dx} = y \quad \text{…… (**)}$$

product of 2 functions of x (because y depends on x)

Differentiate both sides of $(**)$ w.r.t x:

$$\underbrace{(1-x^2)^{\frac{1}{2}}\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{dy}{dx}\cdot\frac{d}{dx}\left[(1-x^2)^{\frac{1}{2}}\right]}_{by\ product\ rule}=\frac{d}{dx}(y)$$

$$\Rightarrow\ (1-x^2)^{\frac{1}{2}}\cdot\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\underbrace{\frac{1}{2}(1-x^2)^{-\frac{1}{2}}\cdot\frac{d}{dx}(1-x^2)}_{by\ chain\ rule}=\frac{dy}{dx}$$

$$\Rightarrow\ (1-x^2)^{\frac{1}{2}}\cdot\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\frac{1}{2}(1-x^2)^{-\frac{1}{2}}\cdot(-2x)=\frac{dy}{dx}$$

Multiply both sides by $(1-x^2)^{\frac{1}{2}}$:

$$(1-x^2)\cdot\frac{d^2y}{dx^2}-x\,\frac{dy}{dx}=\underbrace{(1-x^2)^{\frac{1}{2}}\,\frac{dy}{dx}}_{=y,\ from\ (**)}$$

$$\therefore\ (1-x^2)\,\frac{d^2y}{dx^2}-x\,\frac{dy}{dx}-y=0\quad \ldots\ldots(*)$$

product of 2 functions of x

(b) Use the Leibnitz' rule to differentiate both sides of $(*)$ n times w.r.t. x:

$$[(1-x^2)\,y'']^{(n)} - (x\,y')^{(n)} - y^{(n)} = (0)^{(n)}$$

$$\Rightarrow \left[\sum_{k=0}^{n} \binom{n}{k} (1-x^2)^{(k)}\ (y'')^{(n-k)}\right] - \left[\sum_{k=0}^{n} \binom{n}{k} (x)^{(k)}\ (y')^{(n-k)}\right] - y^{(n)} = 0$$

$$\Rightarrow \left[\binom{n}{0}(1-x^2)^{(0)}\,(y'')^{(n)} + \binom{n}{1}(1-x^2)^{(1)}\,(y'')^{(n-1)} + \binom{n}{2}(1-x^2)^{(2)}\,(y'')^{(n-2)} + 0\right]$$

$$-\left[\binom{n}{0}(x)^{(0)}\ (y')^{(n)} + \binom{n}{1}(x)^{(1)}\ (y')^{(n-1)} + 0\right] - y^{(n)} = 0$$

$$\Rightarrow \left[1\cdot(1-x^2)\cdot y^{(n+2)} + n\cdot(-2x)\cdot y^{(n+1)} + \frac{n(n-1)}{2}\cdot(-2)\cdot y^{(n)}\right]$$

$$-\left[1\cdot x\cdot y^{(n+1)} + n\cdot(1)\cdot y^{(n)}\right] - y^{(n)} = 0$$

$$\Rightarrow (1-x^2)\ y^{(n+2)} + (-2nx - x)\ y^{(n+1)} + [-n(n-1) - n - 1]\ y^{(n)} = 0$$

Hence,

$$(1-x^2)\,y^{(n+2)} - (2n+1)x\,y^{(n+1)} - (n^2+1)\,y^{(n)} = 0.$$