MA1200 Exercise for Chapter 6 Limits, Continuity and Differentiability
Solutions

Limits

1. Evaluate the following limits:

(a) $\lim\limits_{x\to\infty}\dfrac{x^2+1}{2x^3-x}$ (b) $\lim\limits_{x\to\infty}\dfrac{x+\sqrt{x^4-x^2+1}}{2x^2+1+\sqrt{x^4+1}}$ (c) $\lim\limits_{x\to 0}\dfrac{m\sin(mx)-n\sin(nx)}{\tan(mx)+\tan(nx)}$

Solution:

(a)

$$\lim_{x\to\infty}\frac{x^2+1}{2x^3-x}=\lim_{x\to\infty}\frac{\dfrac{x^2+1}{x^3}}{\dfrac{2x^3-x}{x^3}}=\lim_{x\to\infty}\frac{\dfrac{1}{x}+\dfrac{1}{x^3}}{2-\dfrac{1}{x^2}}=0$$

(b)

$$\lim_{x\to\infty}\frac{x+\sqrt{x^4-x^2+1}}{2x^2+1+\sqrt{x^4+1}}=\lim_{x\to\infty}\frac{\dfrac{x+\sqrt{x^4-x^2+1}}{x^2}}{\dfrac{2x^2+1+\sqrt{x^4+1}}{x^2}}=\lim_{x\to\infty}\frac{\dfrac{1}{x}+\sqrt{\dfrac{x^4-x^2+1}{x^4}}}{2+\dfrac{1}{x^2}+\sqrt{\dfrac{x^4+1}{x^4}}}=\frac{1}{3}$$

(c)

$$\lim_{x\to 0}\frac{m\sin(mx)-n\sin(nx)}{\tan(mx)+\tan(nx)}=\lim_{x\to 0}\frac{\dfrac{m\sin(mx)-n\sin(nx)}{mnx}}{\dfrac{\tan(mx)+\tan(nx)}{mnx}}$$

$$=\lim_{x\to 0}\frac{\dfrac{m\sin(mx)}{mnx}-\dfrac{n\sin(nx)}{mnx}}{\dfrac{\sin(mx)}{mnx\cos mx}+\dfrac{\sin(nx)}{mnx\cos nx}}=\frac{\dfrac{m}{n}-\dfrac{n}{m}}{\dfrac{1}{n}+\dfrac{1}{m}}=m-n$$

2.

(a) Divide by highest power, $\lim\limits_{x\to\infty}\dfrac{x+\sqrt{x^4-x^2+1}}{2x^2+1+\sqrt{x^4+1}}=\lim_{x\to\infty}\frac{x+\sqrt{x^4-x^2+1}}{2x^2+1+\sqrt{x^4+1}}\times\frac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$=\lim_{x\to\infty}\frac{1/x+\sqrt{(x^4-x^2+1)/x^4}}{2+1/x^2+\sqrt{(x^4+1)/x^4}}=\lim_{x\to\infty}\frac{1/x+\sqrt{1-1/x^2+1/x^4}}{2+1/x^2+\sqrt{1+1/x^4}}=\frac{\sqrt{1}}{2+\sqrt{1}}=\frac{1}{3}$

*(b) For $x\to 0^+$, $\frac{1}{x}\to+\infty$, $2^{1/x}\to+\infty$. So we can think of it as

$\lim\limits_{y\to\infty}\dfrac{1+y}{3+y}=\lim\limits_{y\to\infty}\dfrac{1+y}{3+y}\times\dfrac{1/y}{1/y}=\dfrac{1}{1}=1$

But for $x\to 0^-$, $\frac{1}{x}\to-\infty$, $2^{1/x}\to 0$. Limit = 1/3. Left Hand Limit$\neq$Right Hand Limit, so limit Does Not Exist.

(c) $\lim_{x\to 0}\frac{\sin ax}{ax}\frac{bx}{\sin bx}\frac{ax}{bx}=1\cdot 1\cdot\frac{a}{b}=\frac{a}{b}$

(d) Numerator $\to 0$, denominator $\to 1$, so limit = 0/1 = 0.

(e) $\lim_{x\to 0}\frac{(\sin 3x)^2}{(3x)^2}\frac{9}{\cos x}=1\cdot\frac{9}{1}=9$

(f) $\lim_{x\to 0}\frac{\sin 2x}{2x}\frac{2}{2x+1}=1\cdot\frac{2}{1}=2$

(g) $\lim_{x\to 3^-}\frac{x-3}{-(x-3)}\frac{x+3}{1}=-6$

3. Evaluate $\lim\limits_{n\to\infty}\frac{n}{2}r^2\sin\frac{2\pi}{n}$, where r is a constant. Interpret this limit geometrically.

Solution:

$$\lim_{n\to\infty}\frac{n}{2}r^2\sin\frac{2\pi}{n}=\lim_{n\to\infty}\pi r^2\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}=\pi r^2\lim_{n\to\infty}\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}=\pi r^2.$$

This result can be interpreted as the area of a circle being taken as the limit of the area of a regular polygon inscribed in the circle as the number of sides of the polygon increases indefinitely.

*4. Evaluate $\lim\limits_{n\to\infty}\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots\cos\frac{\theta}{2^n}$, where $\theta\neq 0$.

Solution:

Note that

$$\begin{aligned}
&\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots\left(\sin\frac{\theta}{2^n}\cos\frac{\theta}{2^n}\right)\\
&=\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots\cos\frac{\theta}{2^{n-1}}\left(\frac{1}{2}\sin\frac{\theta}{2^{n-1}}\right)\\
&=\frac{1}{2}\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cdots\cos\frac{\theta}{2^{n-2}}\left(\frac{1}{2}\sin\frac{\theta}{2^{n-2}}\right)\\
&=\frac{1}{2^2}\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cdots\cos\frac{\theta}{2^{n-3}}\left(\frac{1}{2}\sin\frac{\theta}{2^{n-3}}\right)\\
&=\cdots\\
&=\frac{1}{2^{n-2}}\cos\frac{\theta}{2}\left(\frac{1}{2}\sin\frac{\theta}{2}\right)\\
&=\frac{1}{2^{n-1}}\left(\frac{1}{2}\sin\theta\right)\\
&=\frac{1}{2^n}\sin\theta.
\end{aligned}$$

$$\Rightarrow\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots\cos\frac{\theta}{2^n}=\frac{\sin\theta}{2^n\sin\frac{\theta}{2^n}}=\frac{\sin\theta}{\theta\frac{\sin\frac{\theta}{2^n}}{\frac{\theta}{2^n}}}=\frac{\sin\theta}{\theta}\frac{1}{\frac{\sin\frac{\theta}{2^n}}{\frac{\theta}{2^n}}}.$$

Therefore, $\lim\limits_{n\to\infty}\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots\cos\frac{\theta}{2^n}=\lim\limits_{n\to\infty}\frac{\sin\theta}{\theta}\frac{1}{\frac{\sin\frac{\theta}{2^n}}{\frac{\theta}{2^n}}}=\frac{\sin\theta}{\theta}\frac{1}{\lim\limits_{n\to\infty}\frac{\sin\frac{\theta}{2^n}}{\frac{\theta}{2^n}}}=\frac{\sin\theta}{\theta}$.

Continuity

5. Discuss the continuity of the following functions at $x=0$:

(a) $f(x)=\dfrac{x^2}{x}$ (b) $h(x)=\begin{cases} |x| & \text{for } x\neq 0 \\ 1 & \text{for } x=0 \end{cases}$ (c) $f(x)=\begin{cases} \dfrac{x^2}{x} & \text{for } x\neq 0 \\ 0 & \text{for } x=0 \end{cases}$

Solution:

(a)

As $f(x)=\dfrac{x^2}{x}$ is not defined at $x=0$, $f(x)=\dfrac{x^2}{x}$ is not continuous at $x=0$.

(b)

$\lim\limits_{x\to 0^+} h(x)=\lim\limits_{x\to 0^+}|x|=\lim\limits_{x\to 0^+} x=0=\lim\limits_{x\to 0^-} h(x)=\lim\limits_{x\to 0^-}|x|=\lim\limits_{x\to 0^-}(-x)\Rightarrow \lim\limits_{x\to 0} h(x)=0$

But $\lim\limits_{x\to 0} h(x)=0\neq h(0)=1$. We conclude that $h(x)=\begin{cases} |x| & \text{for } x\neq 0 \\ 1 & \text{for } x=0 \end{cases}$ is not continuous at $x=0$.

(c)

$\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\dfrac{x^2}{x}=\lim\limits_{x\to 0} x=0=f(0)$. $f(x)$ is continuous at $x=0$.

6. Define $f(0)$ for the following functions such that they are continuous at $x=0$.

(a) $f(x)=\sin x\sin\dfrac{1}{x}$ (b) $f(x)=\dfrac{\tan(2x)}{x}$

Solution

(a)

$0\le\left|\sin x\sin\dfrac{1}{x}\right|=|\sin x|\left|\sin\dfrac{1}{x}\right|\le|\sin x|$. As $\lim\limits_{x\to 0}|\sin x|=0$, we have $\lim\limits_{x\to 0}\left|\sin x\sin\dfrac{1}{x}\right|=0$

Define $f(0)=0$, we then have $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\sin x\sin\dfrac{1}{x}=0=f(0)$. That means $f(x)$ is continuous at $x=0$.

(b)

As $f(x)=\dfrac{\tan(2x)}{x}=\dfrac{\sin(2x)}{2x}\dfrac{2}{\cos(2x)}$, $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\dfrac{\tan(2x)}{x}=\lim\limits_{x\to 0}\dfrac{\sin(2x)}{2x}\dfrac{2}{\cos(2x)}=2$.

Define $f(0)=2$, we then have $\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\dfrac{\tan(2x)}{x}=2=f(0)$. That means $f(x)$ is continuous at $x=0$.

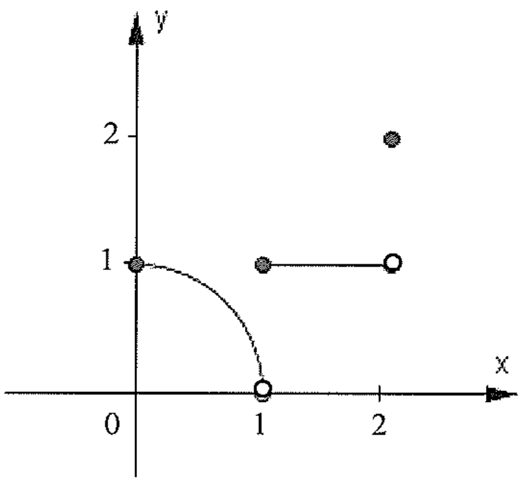
7. Sketch the graph of the following function on $[0,2]$

$$f(x)=\begin{cases}\sqrt{1-x^2} & 0\le x<1\\ 1 \quad \text{for} & 1\le x<2\\ 2 & x=2\end{cases}$$

(a) For what values of c in the domain does $\lim_{x\to c} f(x)$ exist?

(b) At what points does only the left-hand limit exist?

(c) At what points does only the right-hand limit exist?



(a) $(0,1)\cup(1,2)$

(b) 2

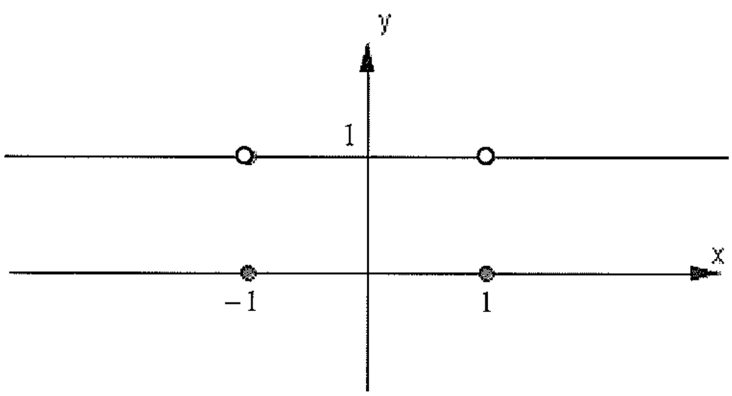
(c) 0

8. Given the function $y=f(x)$ defined as follows:

$$f(x)=\begin{cases}0, & x^2=1\\ 1, & \text{otherwise}\end{cases}$$

Sketch the function. At what points is the function discontinuous? Explain.

The function looks like this:



It is discontinuous at $x=-1,1$, because $\lim_{x \to -1,1} f(x) = 1 \neq 0 = f(1) = f(-1)$

Differentiability

9. Given $f(x) = \begin{cases} \frac{1}{2}(x^2-4) & 0 < x < 2, \\ 0 & x = 2, \\ \frac{2}{x^2}(x^2-4) & x > 2. \end{cases}$ Show that f is differentiable at $x = 2$.

Solution

$$\lim_{h \to 0^+} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h \to 0^+} \frac{\frac{2}{(2+h)^2}\left((2+h)^2-4\right)-0}{h} = \lim_{h \to 0^+} \frac{2(2+h)^2-8}{h(2+h)^2} = \lim_{h \to 0^+} \frac{8h+2h^2}{h(2+h)^2} = \lim_{h \to 0^+} \frac{8+2h}{(2+h)^2} = 2$$

$$\lim_{h \to 0^-} \frac{f(2+h)-f(2)}{h} = \lim_{h \to 0^-} \frac{\frac{1}{2}\left((2+h)^2-4\right)-0}{h} = \lim_{h \to 0^-} \frac{4h+h^2}{2h} = \lim_{h \to 0^-} \frac{4+h}{2} = 2$$

$\therefore$ f is differentiable at $x = 2$.

-End-