MA1200, Sem A, 2014–2015, CA1, CB1, CC1, CD1,
Test 2.

Question 1

(a) It is given that $\sin A = \frac{4}{5}$ and $\cos B = \frac{5}{13}$. Without using a calculator, show that one possible value of $\sin(A+B)$ is $\frac{56}{65}$, and find all the other possible values.
(Hint: $\sin(A+B) = \sin A\cos B + \cos A\sin B$)

(b) (i) Prove that $\tan x + \cot x = 2\operatorname{cosec}2x$.

(ii) Find the general solution, in radians, of the equation $1 + 2\operatorname{cosec}2x = \cot x$. (20 marks)

Question 2

(a) Without the use of De L'Hôpital rule, prove that, for all positive rational number n,
$$\lim_{x\to a}\frac{x^n - a^n}{x-a} = na^{n-1}.$$

(Hint: Consider two cases.
1. For n be a positive integer and $n>1$, we have
$x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^r x^{n-r-1} + \ldots + a^{n-1})$.
2. For $n = \frac{p}{q}$, where p and q are positive integers, and let $y = x^{\frac{1}{q}}$ and $b = a^{\frac{1}{q}}$.)

(b) Evaluate the limit $\lim_{x\to 0}\frac{\sin(2x)}{\sin\left(\frac{x}{2}\right)}$.

(20 marks)

Question 3

(a) Let
$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & \text{if } x \neq 1 \\ c, & \text{if } x = 1 \end{cases}.$$

Find the value of c for which $f(x)$ is continuous at $x=1$. Give your reason.

(b) Let
$$g(x) = |\tan x|, \text{ for } x \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right].$$

Determine whether $g(x)$ is differentiable at $x = 0$. Give your reason. (20 marks)

Question 4

(a) Given that $\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$, prove that
(i) $\cosh^2 x - \sinh^2 x = 1$,
(ii) the inverse of $\sinh x = \sinh^{-1} x = \log_e\left(x + \sqrt{1+x^2}\right)$.

(b) Solve the equation $\cosh^2 x - 2\sinh x = 0$, giving your answer as natural logarithms.

(20 marks)

Question 5

(a) Solve the equation $2\log_{10} x = 1 + \log_{10}\left(\frac{2(2x+5)}{5}\right)$.

(b) The functions F and G are defined by
$F(x) = \log_e(1+x)$, for $x \in \mathbb{R}^+$,
$G(x) = e^{-x}$, for $x \in \mathbb{R}^+$.

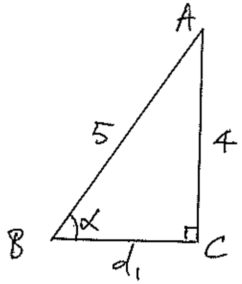
(i) Give the ranges of $F(x)$ and $G(x)$.
(ii) Give definitions of the inverse functions $F^{-1}(x)$ and $G^{-1}(x)$ in a form similar to the above definitions.

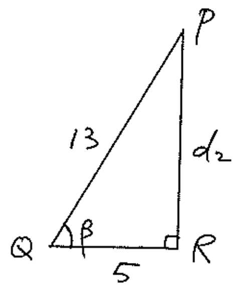
(20 marks)

- END -

MA1200, Sem. A, 2014–2015, CA1, CB1, CC1, CD1, Test 2.

Q.1. (a)





In $\triangle ABC$, $\angle C = 90^\circ$,

$d_1^2 + 4^2 = 5^2$, by Pythagoras' theorem

$d_1 = 3$.

In $\triangle PQR$, $\angle R = 90^\circ$.

$d_2^2 + 5^2 = 13^2$, by Pythagoras' theorem

$d_2 = 12$. 2

Case (i) $0^\circ < A < 90^\circ$, $\sin A = \frac{4}{5}$, $\therefore \cos A = \frac{3}{5}$.

$0^\circ < B < 90^\circ$, $\cos B = \frac{5}{13}$, $\therefore \sin B = \frac{12}{13}$.

$\therefore \sin(A+B) = \sin A \cos B + \cos A \sin B$

$= \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) = \frac{20}{65} + \frac{36}{65} = \frac{56}{65}$ 3

Case (ii) $90^\circ < A < 180^\circ$, $\sin A = \frac{4}{5}$, $\therefore \cos A = -\frac{3}{5}$.

$270^\circ < B < 360^\circ$, $\cos B = \frac{5}{13}$, $\therefore \sin B = -\frac{12}{13}$.

$\therefore \sin(A+B) = \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(-\frac{3}{5}\right)\left(-\frac{12}{13}\right) = \frac{56}{65}$. 1

Case (iii) $0^\circ < A < 90^\circ$, $\sin A = \frac{4}{5}$, $\therefore \cos A = \frac{3}{5}$.

$270^\circ < B < 360^\circ$, $\cos B = \frac{5}{13}$, $\therefore \sin B = -\frac{12}{13}$.

$\therefore \sin(A+B) = \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) = \frac{20}{65} - \frac{36}{65} = -\frac{16}{65}$

3

Case (iv) $90^\circ < A < 180^\circ$, $\sin A = \frac{4}{5}$, $\therefore \cos A = -\frac{3}{5}$.

$0^\circ < B < 90^\circ$, $\cos B = \frac{5}{13}$, $\therefore \sin B = \frac{12}{13}$.

$\therefore \sin(A+B) = \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) + \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right) = \frac{20}{65} - \frac{36}{65} = -\frac{16}{65}$ 1

(b)(i) L.H.S. $= \tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$

$= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x}$

$= \frac{1}{\sin x \cos x}$, since $\sin^2 x + \cos^2 x = 1$

$= \frac{2}{2 \sin x \cos x}$

$= \frac{2}{\sin 2x} = 2 \operatorname{cosec} 2x =$ R.H.S. 3

(ii) $1 + 2 \operatorname{cosec} 2x = \cot x$

$1 + \tan x + \cot x = \cot x$

$\tan x = -1$ 2

$\therefore$ The general solution of the trigonometric equation is

$x = n\pi - \frac{\pi}{4}$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ 5

Q.2.

(a) For $n=1$, $\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}\frac{x-a}{x-a}$

$= 1 = 1\cdot a^{1-1} = n a^{n-1}$. (1)

For n be a positive integer and $n>1$,

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}\frac{(x-a)(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\cdots+a^{n-1})}{x-a}$$

$$=\lim_{x\to a}(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\cdots+a^{n-1})$$

$$=\underbrace{a^{n-1}+a^{n-1}+a^{n-1}+\cdots+a^{n-1}}_{n\text{ terms}}=na^{n-1}. \quad (5)$$

For $n=\frac{p}{q}$, where p and q are positive integers, and let $y=x^{\frac{1}{q}}$ and $b=a^{\frac{1}{q}}$, we have

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{y\to b}\frac{y^p-b^p}{y^q-b^q}$$

$$=\lim_{y\to b}\left(\frac{\frac{y^p-b^p}{y-b}}{\frac{y^q-b^q}{y-b}}\right)$$

$$=\frac{\lim_{y\to b}\left(\frac{y^p-b^p}{y-b}\right)}{\lim_{y\to b}\left(\frac{y^q-b^q}{y-b}\right)}=\frac{pb^{p-1}}{qb^{q-1}}$$

$$=\frac{p}{q}b^{p-1-(q-1)}$$

$$=\frac{p}{q}a^{\frac{1}{q}(p-q)}=\frac{p}{q}a^{\frac{p}{q}-1}$$

$$=na^{n-1}$$

Hence, $\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$, for all positive rational number, n. (6)

(b) $$\lim_{x\to 0}\frac{\sin(2x)}{\sin\left(\frac{x}{2}\right)}=\lim_{x\to 0}\left(\frac{\frac{\sin(2x)}{(2x)}\times 2x}{\frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}\times\frac{x}{2}}\right)$$

$$=4\lim_{x\to 0}\left(\frac{\frac{\sin(2x)}{(2x)}}{\frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}}\right)$$

$$=4\,\frac{\lim_{2x\to 0}\left(\frac{\sin 2x}{2x}\right)}{\lim_{\frac{x}{2}\to 0}\left(\frac{\sin\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)}\right)}$$

$=\underline{\underline{4}}$, since $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$. (8)

Method II.

$$\lim_{x\to 0}\left(\frac{\sin(2x)}{\sin\left(\frac{x}{2}\right)}\right) \quad \left(=\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{(\cos 2x)\,2}{\left(\cos\frac{x}{2}\right)\frac{1}{2}}$$, by De L'Hôpital rule

$$=4\lim_{x\to 0}\left(\frac{\cos 2x}{\cos\frac{x}{2}}\right)$$

$=\underline{\underline{4}}$

(8)

Q.3.

(a) $$\lim_{x\to 1} f(x) = \lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x\to 1}(x+1)$$

$$= 2 \quad 3$$

If $\lim_{x\to 1} f(x) = f(1) = c$, then $f(x)$ is continuous at $x=1$. 3

$\therefore$ The value of c is 2. // 2

(b) $$\lim_{x\to 0^-}\frac{g(x)-g(0)}{x-0} = \lim_{x\to 0^-}\frac{|\tan x| - |\tan 0|}{x}$$

$$= \lim_{x\to 0^-}\left(\frac{-\tan x}{x}\right) = \lim_{x\to 0^-}\left(-\frac{\sin x}{x}\cdot\frac{1}{\cos x}\right)$$

$$= (-1)\left(\lim_{x\to 0^-}\frac{\sin x}{x}\right)\left(\lim_{x\to 0^-}\frac{1}{\cos x}\right)$$

$$= -1, \text{ since } \lim_{x\to 0^-}\left(\frac{\sin x}{x}\right) = 1. \quad 4$$

$$\lim_{x\to 0^+}\frac{g(x)-g(0)}{x-0} = \lim_{x\to 0^+}\frac{|\tan x| - |\tan 0|}{x}$$

$$= \lim_{x\to 0^+}\left(\frac{\tan x}{x}\right)$$

$$= \left(\lim_{x\to 0^+}\frac{\sin x}{x}\right)\left(\lim_{x\to 0^+}\frac{1}{\cos x}\right)$$

$$= 1, \text{ since } \lim_{x\to 0^+}\frac{\sin x}{x} = 1$$

$$\neq \lim_{x\to 0^-}\frac{g(x)-g(0)}{x-0}. \quad 4$$

$\therefore \lim_{x\to 0}\frac{g(x)-g(0)}{x-0}$ does not exist.

$\therefore g(x)$ is not differentiable at $x=0$. // 4

Q.4. (a)(i) $\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\sinh x = \frac{1}{2}(e^x - e^{-x})$, by definition.

$\therefore \cosh x + \sinh x = e^x$ — ①

$\cosh x - \sinh x = e^{-x}$ — ②

From ① and ②, we have

$$(\cosh x + \sinh x)(\cosh x - \sinh x) = (e^x)(e^{-x})$$

$$\therefore \cosh^2 x - \sinh^2 x = e^0 = 1. \quad // \quad 4$$

(ii) Let $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$.

Then $2y(e^x) = (e^x)^2 - 1$.

$(e^x)^2 - 2y(e^x) - 1 = 0$, which is a quadratic equation in e^x.

$$e^x = \frac{2y \pm \sqrt{(-2y)^2 - 4(1)(-1)}}{2(1)}$$

$$= y \pm \sqrt{y^2+1} \quad 4$$

But $e^x \neq y - \sqrt{y^2+1} < 0$, (rejected)

since $e^x > 0$ for all values of x. 2

$\therefore \quad e^x = y + \sqrt{y^2+1}$

$x = \log_e(y + \sqrt{y^2+1})$

$\therefore$ The inverse of $\sinh x$ is

$\sinh^{-1} x = \log_e(x + \sqrt{x^2+1})$. // 3

(b) $\cosh^2 x - 2\sinh x = 0$

$1 + \sinh^2 x - 2\sinh x = 0$, from Q.4(a)(i).

$(\sinh x - 1)^2 = 0$

$\therefore \sinh x = 1$ 4

$\therefore x = \sinh^{-1}(1) = \log_e(1+\sqrt{2})$ // from Q.4 (a)(ii). 3

Q5.(a) $2\log_{10} x = 1 + \log_{10}\left(\frac{2(2x+5)}{5}\right)$

$\log_{10}(x^2) = \log_{10}\left[10 \times \frac{2(2x+5)}{5}\right]$

$x^2 = 4(2x+5)$

$x^2 - 8x - 20 = 0$

$(x+2)(x-10) = 0$

$\therefore x = -2$, or $x = 10$ 4

$x \neq -2$ (rejected), since the domain of $\log x$ is $(0, \infty)$. 2

$\therefore x = 10$. // 2

(b) $F(x) = \log_e(1+x)$, for $x \in \mathbb{R}^+$,

$G(x) = e^{-x}$, for $x \in \mathbb{R}^+$.

(i) $Ran(F) = \mathbb{R}^+$ // 2

$Ran(G) = (0, 1)$. // 2

(ii) Let $y = F(x) = \log_e(1+x)$.

Then $e^y = 1+x$

$x = e^y - 1$

$\therefore F^{-1}(x) = e^x - 1$, for $x \in \mathbb{R}^+$. // 4

Let $y = G(x) = e^{-x}$.

Then $\log_e y = -x$

$x = -\log_e y$

$\therefore G^{-1}(x) = -\log_e x$, for $x \in (0, 1)$. // 4