MA 1201 Semester B 2019/20

Assignment 3 — Due at 5 pm, 30/4/2020 (Thursday) online on Canvas

Instructions:

- Please show your work. Unsupported answers will receive **NO** credits.
- Make sure you write down the correct lecture session (A/B/C/D/E/F/G/H) you have registered for, together with your full name and student ID on the front page of your answer script. Scan your solution into a single pdf file and upload it to Canvas.
- **NO** late homework will be accepted. Homework submitted to wrong tutorial sessions will **NOT** be graded and will receive **0 POINTS**.

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1. (20 points) Find the area of the surface generated by revolving the curve $x = t - \sin t$ and $y = 1 - \cos t$ with $t \in [0, 2\pi]$, about the line $y = 2$.

Solution. The surface area formula is

$$S = \int 2\pi(2-y)\,ds,$$

where

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2}\,dt,$$

where

$$\left[x'(t)\right]^2 + \left[y'(t)\right]^2 = (1-\cos t)^2 + (\sin t)^2 = 2 - 2\cos t.$$

It then follows that

$$S = \int_0^{2\pi} 2\pi(1+\cos t)\sqrt{2-2\cos t}\,dt = 8\pi \int_0^{2\pi} \cos^2\frac{t}{2}\sin\frac{t}{2}\,dt = -16\pi\int_0^{2\pi}\cos^2\frac{t}{2}d(\cos\frac{t}{2}) = -\frac{16\pi}{3}\cos^3\frac{t}{3}\Big|_0^{2\pi} = \frac{32\pi}{3}.$$

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2. (15 points) Suppose a complex number z satisfies the equation

$$(1+z)^4 = e^{i\theta}(1-z)^4,$$

for some $\theta \in (\pi, 2\pi)$. Find the complex number z and express the result in Euler's form.

Solution. It follows from

$$\left(\frac{1+z}{1-z}\right)^4 = e^{i\theta} = e^{i\theta + i(2k\pi)},$$

that

$$\frac{1+z}{1-z} = w_k = e^{i\frac{\theta+2k\pi}{4}} \qquad \text{for } k = 0,\,1,\,2,\,3.$$

Then

$$1 + z = w_k(1-z) \Longrightarrow z = \frac{w_k - 1}{w_k + 1} \Longrightarrow z = \frac{e^{i\frac{\theta+2k\pi}{4}} - 1}{e^{i\frac{\theta+2k\pi}{4}} + 1} = \frac{e^{i\frac{\theta+2k\pi}{4}} + e^{i\pi}}{e^{i\frac{\theta+2k\pi}{4}} + e^{i0}}.$$

To express z in Euler's form, we write $1 = \mathrm{e}^{\mathrm{i}0}$ and $-1 = \mathrm{e}^{\mathrm{i}\pi}$ above. Then

$$z = \frac{\mathrm{e}^{\mathrm{i}(\frac{\theta+2k\pi}{8}+\frac{\pi}{2})}(\mathrm{e}^{\mathrm{i}(\frac{\theta+2k\pi}{8}-\frac{\pi}{2})}+\mathrm{e}^{-\mathrm{i}(\frac{\theta+2k\pi}{8}-\frac{\pi}{2})})}{\mathrm{e}^{\mathrm{i}\frac{\theta+2k\pi}{8}}(\mathrm{e}^{\mathrm{i}\frac{\theta+2k\pi}{8}}+\mathrm{e}^{-\mathrm{i}\frac{\theta+2k\pi}{8}})} = \frac{\mathrm{e}^{\mathrm{i}(\frac{\theta+2k\pi}{8}+\frac{\pi}{2})}(2\cos(\frac{\theta+2k\pi}{8}-\frac{\pi}{2}))}{\mathrm{e}^{\mathrm{i}\frac{\theta+2k\pi}{8}}(2\cos(\frac{\theta+2k\pi}{8}))} = \tan(\frac{\theta+2k\pi}{8})\mathrm{e}^{\mathrm{i}\frac{\pi}{2}},$$

where $k = 0,1,2,3$. Since $\theta \in (\pi, 2\pi)$, the angle $(\frac{\theta+2k\pi}{8})$ lies in quadrant I when $k = 0,1$, and lies in quadrant II when $k = 2,3$. It implies that when $k = 0,1$,

$$z = \tan(\frac{\theta+2k\pi}{8})\mathrm{e}^{\mathrm{i}\frac{\pi}{2}},$$

and when $k = 2,3$,

$$z = -\tan(\frac{\theta+2k\pi}{8})\mathrm{e}^{-\mathrm{i}\pi}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}} = -\tan(\frac{\theta+2k\pi}{8})\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}.$$

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3. (15 points) Solve the equation $x^3 - 3x^2 + 4x - 2 = 0$ given that $1+i$ is one of the roots.
Solution. By the fundamental theorem, $1-i$ is also a root. So $(x-(1+i))(x-(1-i)) = x^2-2x+2$ is a factor of x^3-3x^2+4x-2. By the long division, $x^3-3x^2+4x-2 = (x^2-2x+2)(x-1)$. So the roots of equation $x^3-3x^2+4x-2 = 0$ are $x = 1+i$, $x = 1-i$, and $x = 1$.

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4. (15 points) Compute

$$\frac{-i+\cos\theta+i\sin\theta}{\sin\theta+i\cos\theta},$$

where $\theta \in (\frac{\pi}{2}, \pi)$.
Solution. It follows from $\cos\theta = \frac{\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}}{2}$ and $\sin\theta = \frac{\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta}}{2i}$ that,

$$-i+\cos\theta+i\sin\theta = -i+\frac{\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}}{2}+\frac{\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta}}{2} = \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}+\mathrm{e}^{\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}(\frac{\theta}{2}-\frac{\pi}{4})}(\mathrm{e}^{-\mathrm{i}(\frac{\theta}{2}+\frac{\pi}{4})}+\mathrm{e}^{\mathrm{i}(\frac{\theta}{2}+\frac{\pi}{4})}) = 2\cos(\frac{\theta}{2}+\frac{\pi}{4})\mathrm{e}^{\mathrm{i}(\frac{\theta}{2}-\frac{\pi}{4})}$$

and

$$\sin\theta+i\cos\theta = \frac{\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta}}{2i}+i\frac{\mathrm{e}^{\mathrm{i}\theta}+\mathrm{e}^{-\mathrm{i}\theta}}{2} = \mathrm{e}^{\mathrm{i}(\frac{\pi}{2}-\theta)}.$$

Then

$$\frac{-i+\cos\theta+i\sin\theta}{\sin\theta+i\cos\theta} = \frac{2\cos(\frac{\theta}{2}+\frac{\pi}{4})\mathrm{e}^{\mathrm{i}(\frac{\theta}{2}-\frac{\pi}{4})}}{\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}-\theta)}} = 2\cos(\frac{\theta}{2}+\frac{\pi}{4})\mathrm{e}^{\mathrm{i}(\frac{\theta}{2}-\frac{\pi}{4}-\frac{\pi}{2}+\theta)} = 2\cos(\frac{\theta}{2}+\frac{\pi}{4})\mathrm{e}^{\mathrm{i}(\frac{3\theta}{2}-\frac{3\pi}{4})}.$$

Because $\theta \in (\frac{\pi}{2}, \pi)$, $\frac{\theta}{2}+\frac{\pi}{4} \in (\frac{\pi}{2}, \frac{3}{4}\pi)$. So $\cos(\frac{\theta}{2}+\frac{\pi}{4}) < 0$. Then

$$\frac{-i+\cos\theta+i\sin\theta}{\sin\theta+i\cos\theta} = -2\cos(\frac{\theta}{2}+\frac{\pi}{4})\mathrm{e}^{\mathrm{i}\pi}\mathrm{e}^{\mathrm{i}(\frac{3\theta}{2}-\frac{3\pi}{4})} = -2\cos(\frac{\theta}{2}+\frac{\pi}{4})\mathrm{e}^{\mathrm{i}(\frac{3\theta}{2}+\frac{\pi}{4})}.$$

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5. (20 points) Let

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}.$$

(a) (10 points) Evaluate the determinant of A by the cofactor expansion.

Solution. Expand it along the third row,

$$|A| = (2)(-1)^{3+1}\begin{vmatrix}1 & -1\\ 2 & -1\end{vmatrix} + (2)(-1)^{3+2}\begin{vmatrix}3 & -1\\ 2 & -1\end{vmatrix} = (2)(-1+2)-(2)(-3+2) = 4.$$

(b) (10 points) Find all values of λ such that $\det(A-\lambda I_3)=0$, where I_3 is the 3×3 identity matrix.

Solution. Note that

$$\det(A-\lambda I_3) = \begin{vmatrix}3-\lambda & 1 & -1\\ 2 & 2-\lambda & -1\\ 2 & 2 & -\lambda\end{vmatrix} = (3-\lambda)\begin{vmatrix}2-\lambda & -1\\ 2 & -\lambda\end{vmatrix} - \begin{vmatrix}2 & -1\\ 2 & -\lambda\end{vmatrix} - \begin{vmatrix}2 & 2-\lambda\\ 2 & 2\end{vmatrix} = -\lambda^3+5\lambda^2-8\lambda+4.$$

So the values of λ, that satisfies

$$0 = \det(A-\lambda I_3) = -\lambda^3+5\lambda^2-8\lambda+4 = -(\lambda-1)(\lambda-2)^2,$$

are $\lambda = 1,2,2$.

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6. (15 points) Let

$$A = \begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}.$$

(a) (8 points) Show that A is invertible and find A^{-1}.

Solution. Because $|A| = (1)(5)-(2)(2) = 1 \neq 0$, A is invertible. $A^{-1} = \frac{1}{|A|}\begin{pmatrix}5 & -2\\ -2 & 1\end{pmatrix}^T = \begin{pmatrix}5 & -2\\ -2 & 1\end{pmatrix}$.

(b) (7 points) Compute $\det(A^{-2})$.

Solution. By the property of the determinant, $|A^{-2}| = \frac{1}{|A|^2} = 1$.

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