Summary of Lecture 1

- Euler Theorem
- Algorithm for Euler Circuit.
- Basic techniques for designing algorithms:
  greedy, divide-and-conquer, dynamic programming

Week 2: Greedy Algorithms

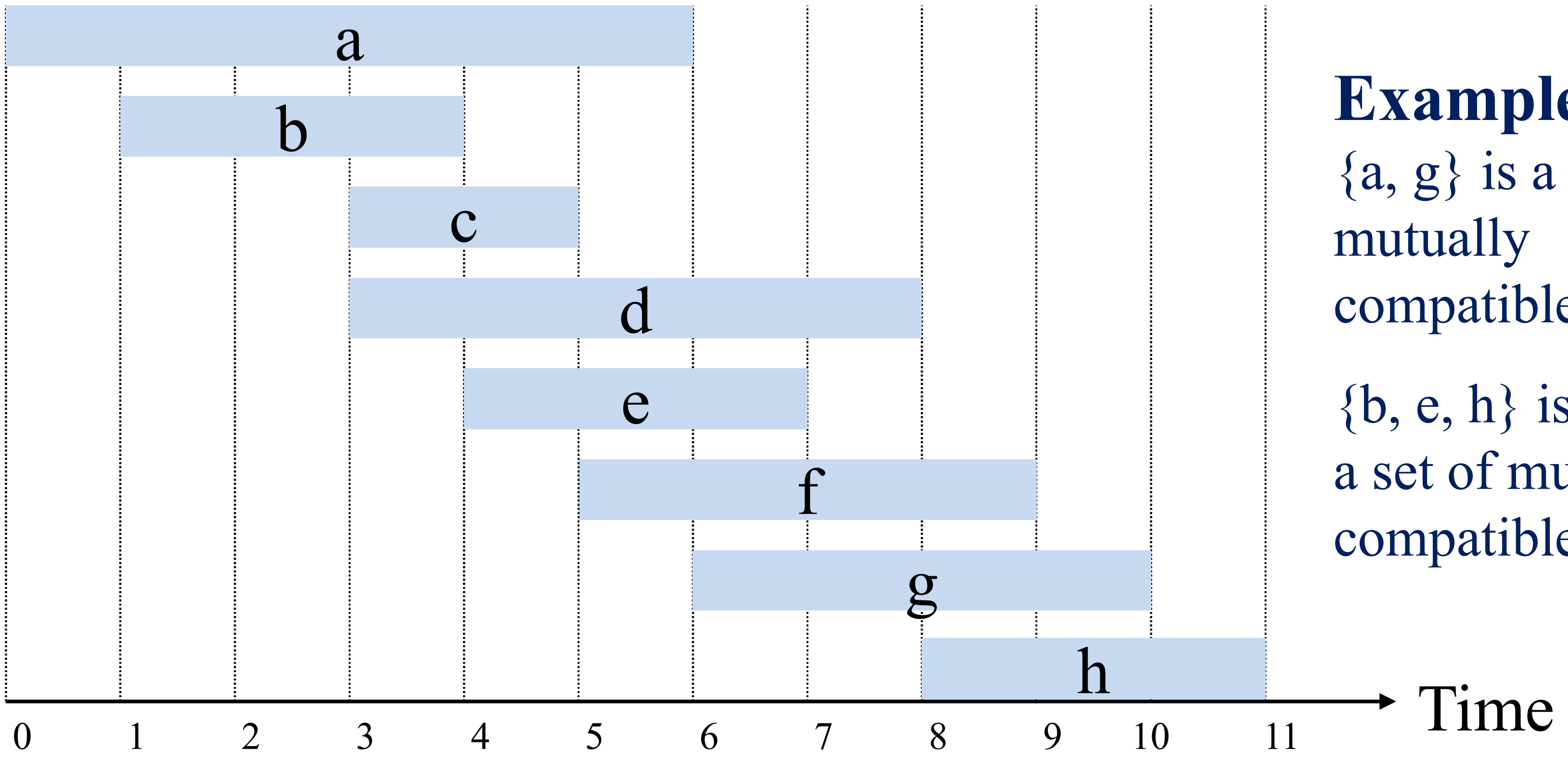
Greedy Algorithm

- A technique to solve problems:
  - always makes the locally best choice at the moment (local optimal).
  - Hopefully, a series of locally best choices will lead to a globally best solution.
- Greedy algorithms yield optimal solutions for many (but not all) problems.

Interval Scheduling

Interval Scheduling

- Interval scheduling.
  - Job j starts at s_j and finishes at f_j.
  - Two jobs compatible if they don't overlap.
  - Goal: find maximum subset of mutually compatible jobs.



Examples:

{a, g} is a set of mutually compatible jobs.

{b, e, h} is also a set of mutually compatible jobs.

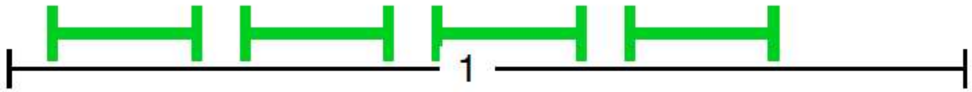
Ideas for Interval Scheduling

Greedy Algorithm: sort all the jobs in a list using a ‘greedy’ principle, and then choose it one by one

- What are possible rules for greedy sorting?
  - Choose the interval that starts earliest.
    - Rationale: start using the resource as soon as possible.
  - Choose the smallest interval.
    - Rationale: try to have lots of small jobs.
  - Choose the interval that overlaps (conflicts) with the fewest remaining intervals.
    - Rationale: keep our options open and eliminate as few intervals as possible.

Rules That Don't Work

1. Earliest start time



2. Shortest job

3. Fewest conflicts

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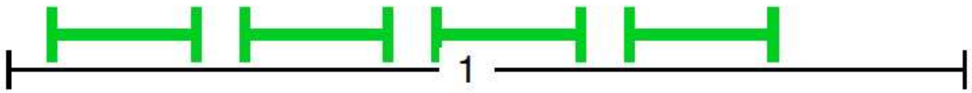
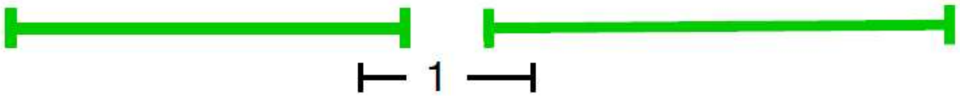


2. Shortest job



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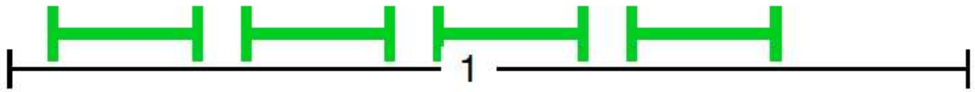
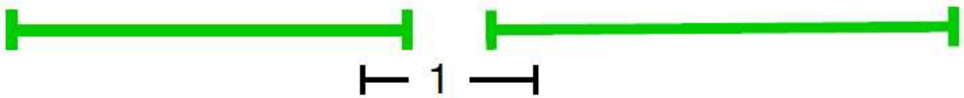
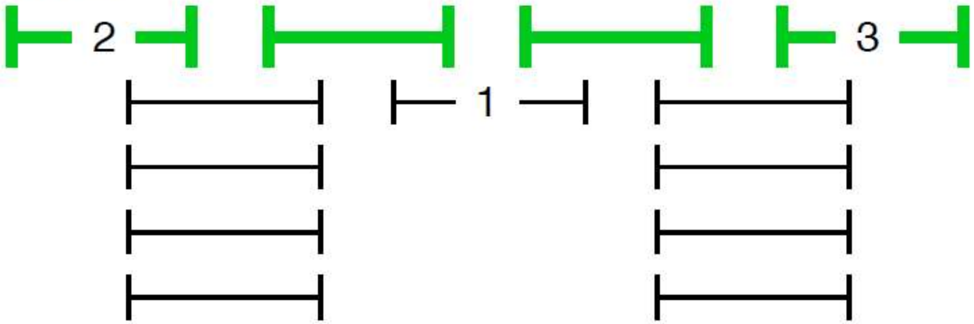


2. Shortest job



3. Fewest conflicts



Example: sort based on finish time

- Jobs (s, f): (0, 10), (3, 4), (2, 8), (1, 5), (4, 5), (4, 8), (5, 6) (7,9).
- Sorting based on f_i:
  - (3, 4) (1, 5), (4, 5) (5, 6) (4,8) (2,8) (7, 9)(0,10).
- Selecting jobs:
  - (3, 4),
  - (4, 5),
  - (5, 6),
  - (7, 9),

Interval Scheduling: Greedy Algorithm

- Greedy algorithm. Consider jobs in increasing order of finishing time. Take each job provided it's compatible with the ones already taken.

```
Sort jobs by finish times so that f₁ ≤ f₂ ≤ ... ≤ fₙ.
     jobs selected
A ← φ
for j = 1 to n {
   if (job j compatible with A)
      A ← A ∪ {j}
}
return A
```

- Implementation. O(n log n).
  - Remember job j* that was added last to A.
  - Job j is compatible with A if $s_j \geq f_{j^*}$.

how to decide if job j is compatible with A

How to Prove Optimality

- How can we prove the schedule returned is optimal?
  - Let A be the schedule returned by this algorithm.
  - Let OPT be some optimal solution (there may be many optimal solutions!).

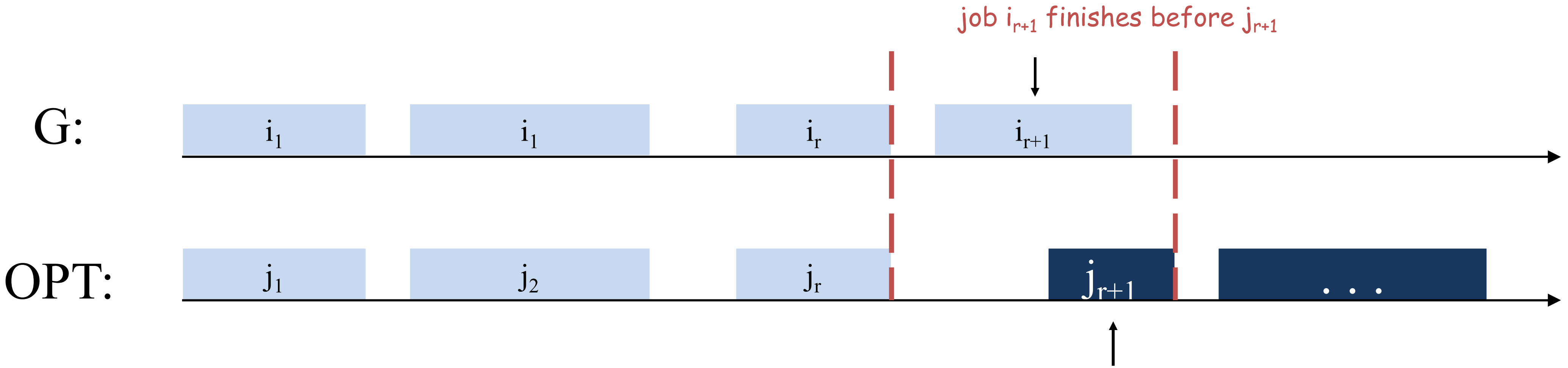
- Might be hard to show that A = OPT, instead we need only to show that |A| = |OPT| or equivalently A is one of the optimal solutions.

- **Note the distinction**: instead of proving directly that a choice of intervals A is the same as an optimal choice, we prove that it has the same number of intervals as an optimal. Therefore, it is optimal.

Interval Scheduling: Analysis

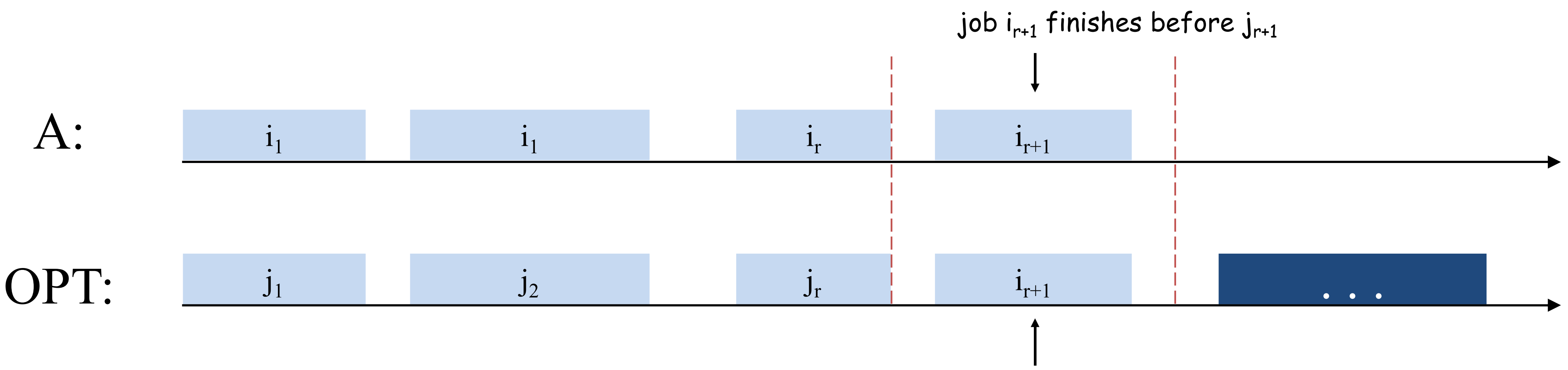
- Theorem. Greedy algorithm is optimal.
- Proof:
  - We compare the solution obtained from greedy algorithm with an optimal solution.
  - Let $G=i_1, i_2, \ldots, i_k$ denote the set of jobs selected by greedy.
  - Let Opt=$j_1, j_2, \ldots, j_n$ denote the set of jobs in the optimal solution.
    - The set of jobs are mutually compatible and the number of jobs is the largest.
  - Without loss of generality, we assume that

    $i_1=j_1, i_2=j_2, \ldots, i_r=j_r$ and $i_{r+1}\neq j_{r+1}$, where r could be 0, 1, 2, ….
  - Job i_{r+1} finishes before (or at the same time of) j_{r+1} due to our greedy algorithm.



Interval Scheduling: Analysis

- Theorem. Greedy algorithm is optimal.
- Pf.
  - Let A=i_1, i_2, ... i_k denote set of jobs selected by greedy.
  - Let Opt=j_1, j_2, ... j_m denote set of jobs in the optimal solution with $i_1 = j_1$, $i_2 = j_2$, ..., $i_r = j_r$ and $i_{r+1} \neq j_{r+1}$ (Note that r can be 0, 1, 2, ...)



Another Proof for Interval Scheduling:

Let: $G = i_1 i_2 \dots . i_m$ be the solution return by greedy

Let : $Opt = j_1 j_2 \dots . j_k$ be an optimal solution

Both are in the order of finish time.

1. If $i_1 \neq j_1$, since i_1 finishes the earliest, and thus not later than j_1, we can replace j_1 by i_1 in Opt. After this replacement, Opt is still optimal. Now

$$Opt = i_1 j_2 \dots j_k$$

2. If $i_2 \neq j_2$, since i_2 finishes earlier than j_2, we can replace j_2 by i_2 in Opt. After this replacement , Opt is still optimal. Now

$$Opt = i_1 i_2 j_3 \dots j_k$$

…….

After a number of replacements:

$$Opt = i_1 i_2 \dots i_m$$

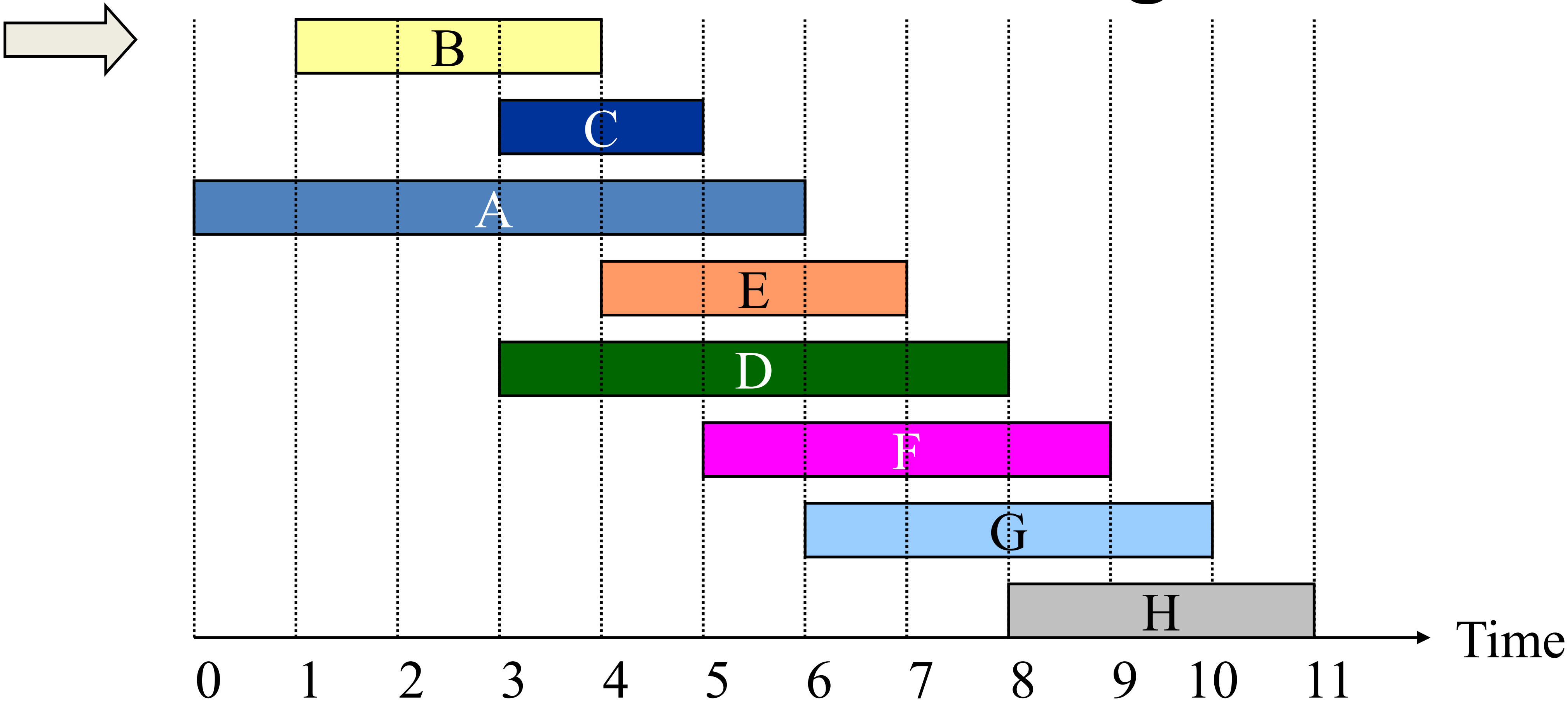
Or

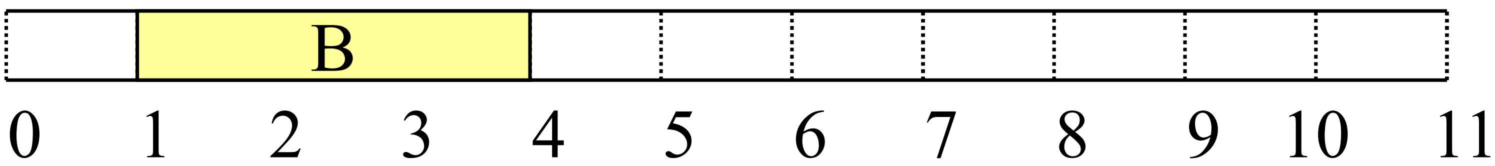
$$Opt = i_1 i_2 \dots i_m j_{m+1} \dots . j_k$$

The last case is impossible, since no job starts later than finish time of i_m.

Therefore, G is optimal (it have the same number of jobs as Opt)

Interval Scheduling





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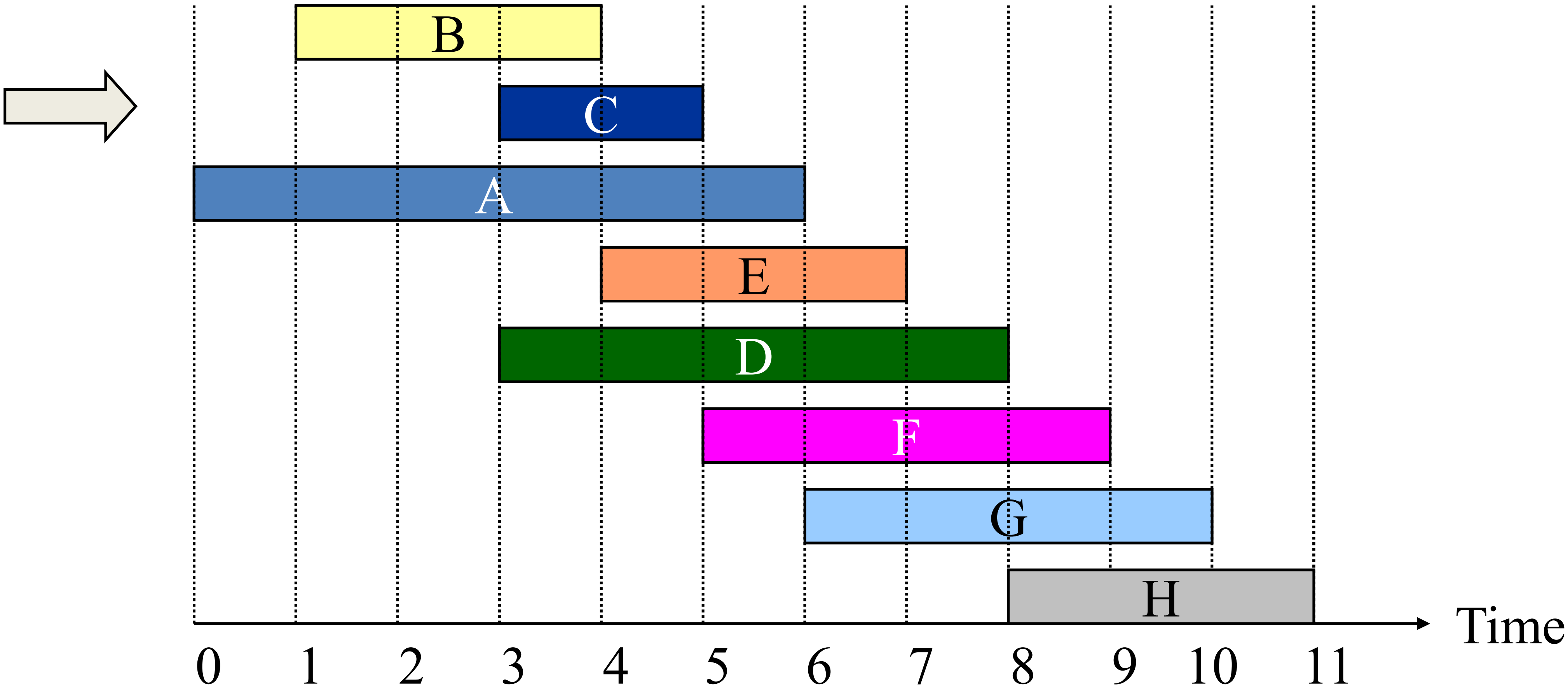
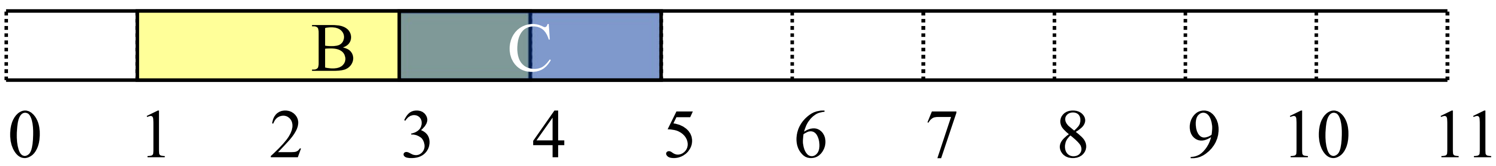
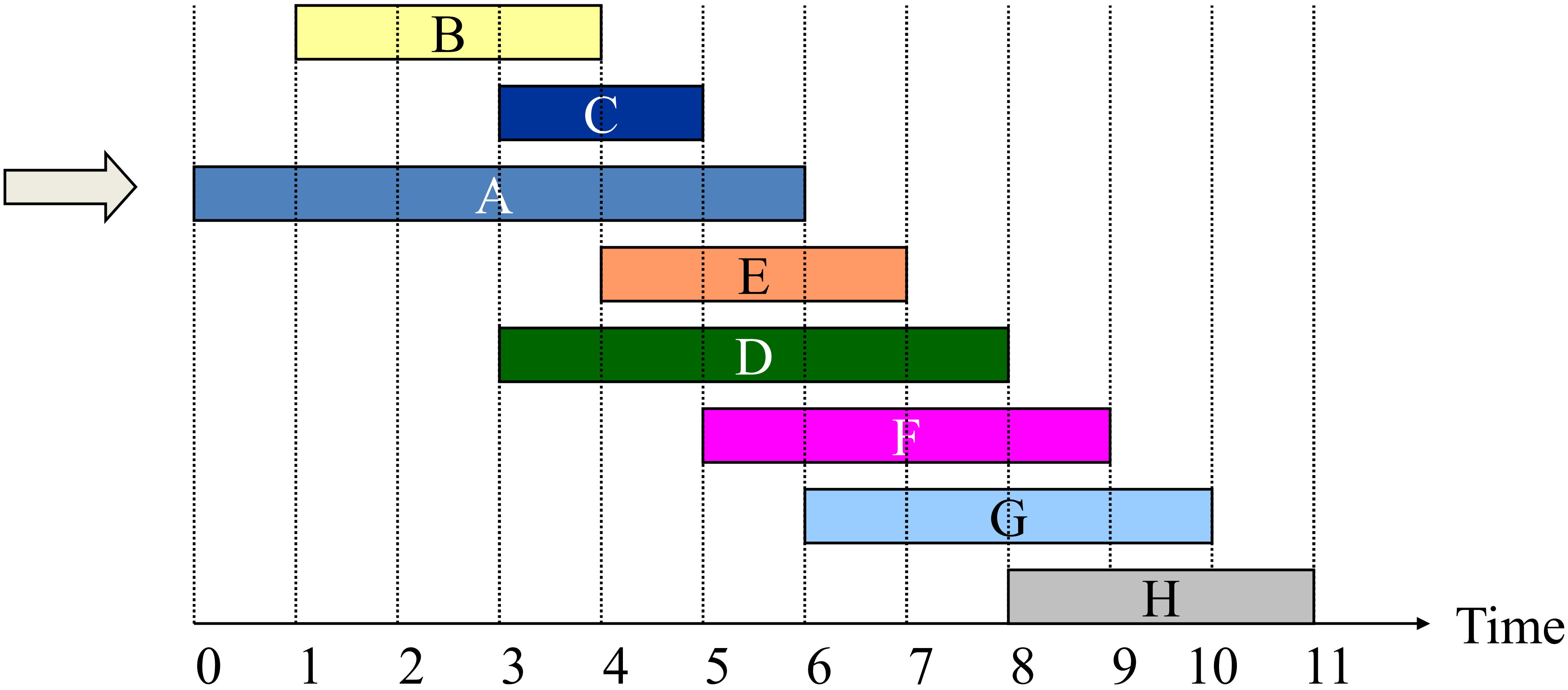
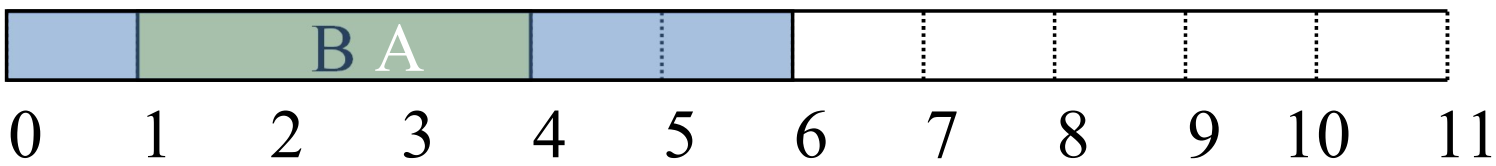




Interval Scheduling





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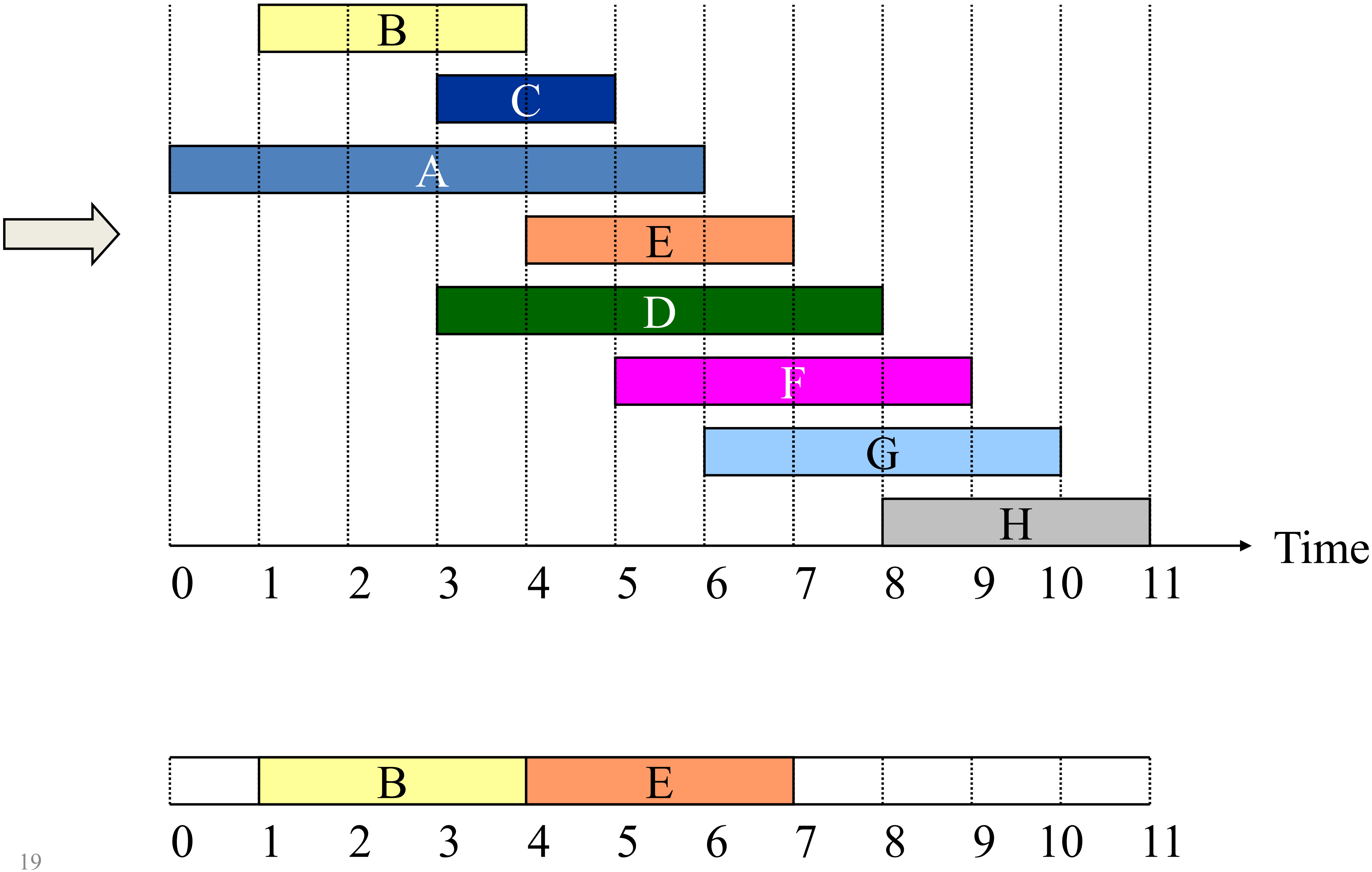

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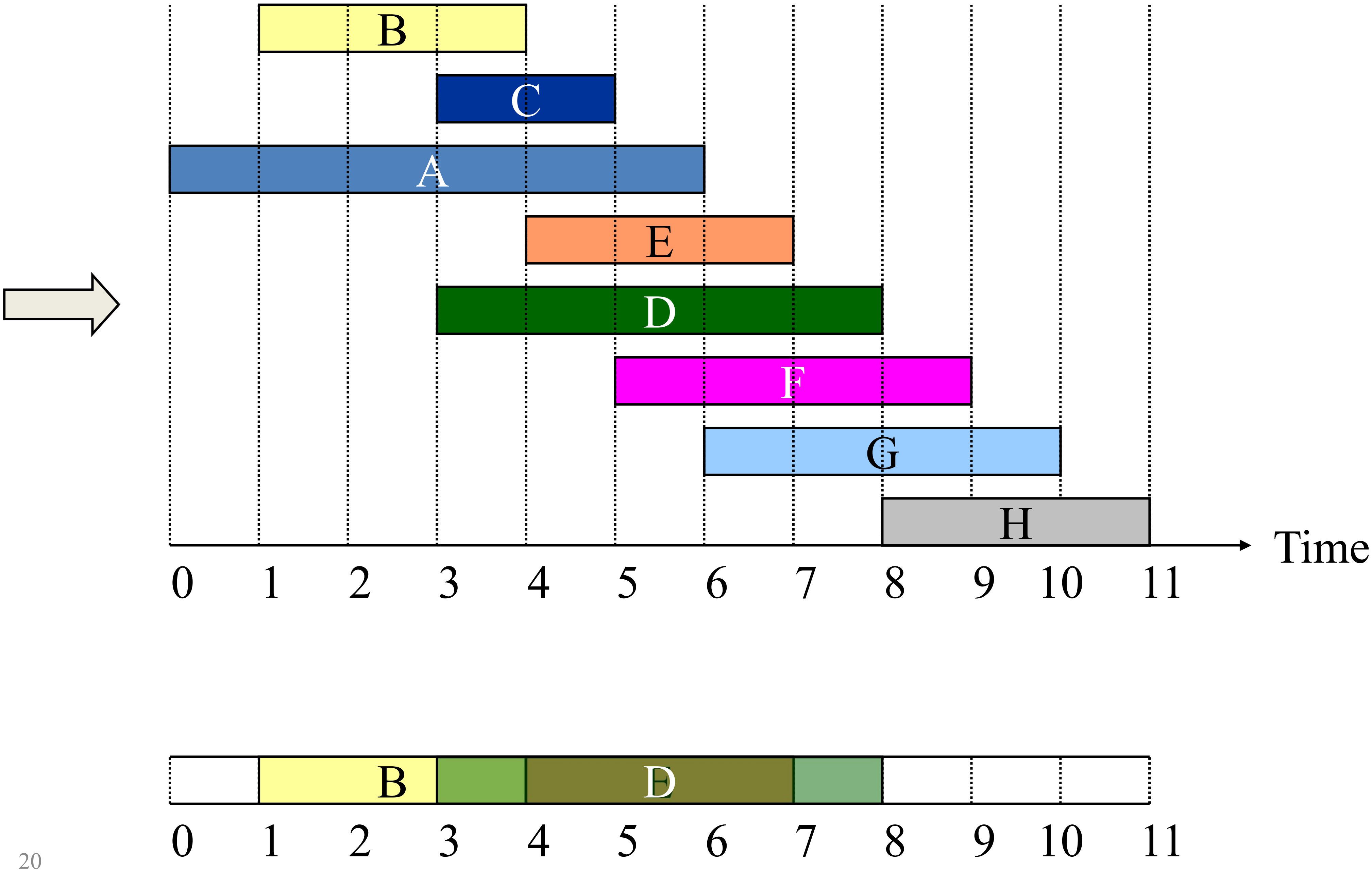

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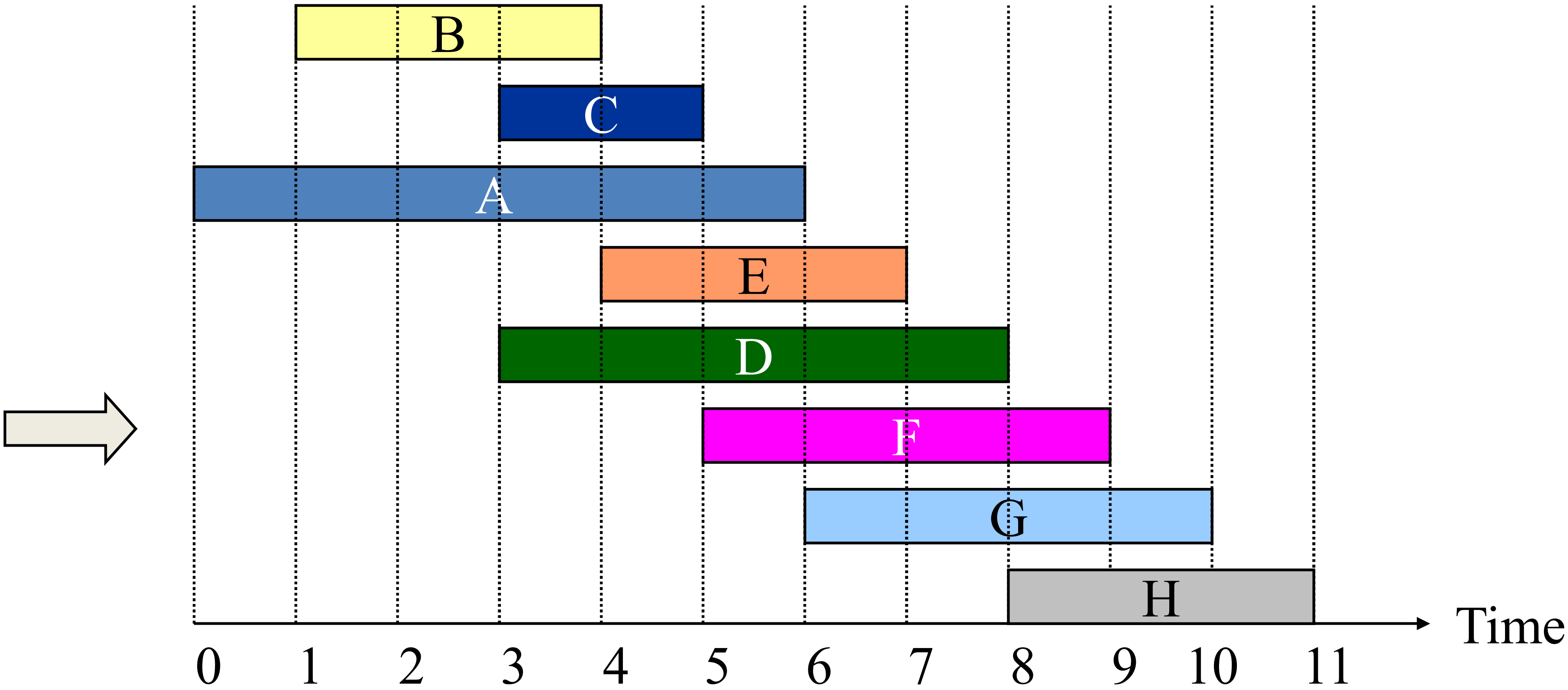

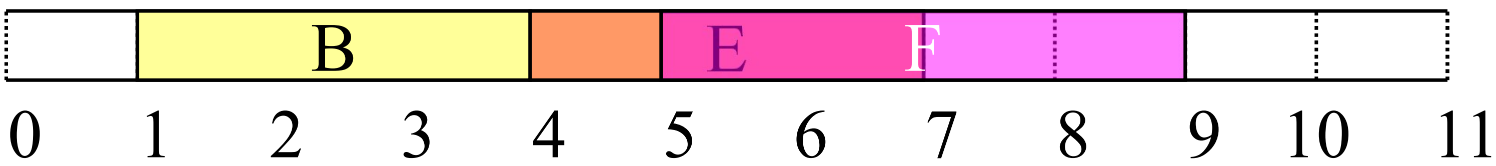

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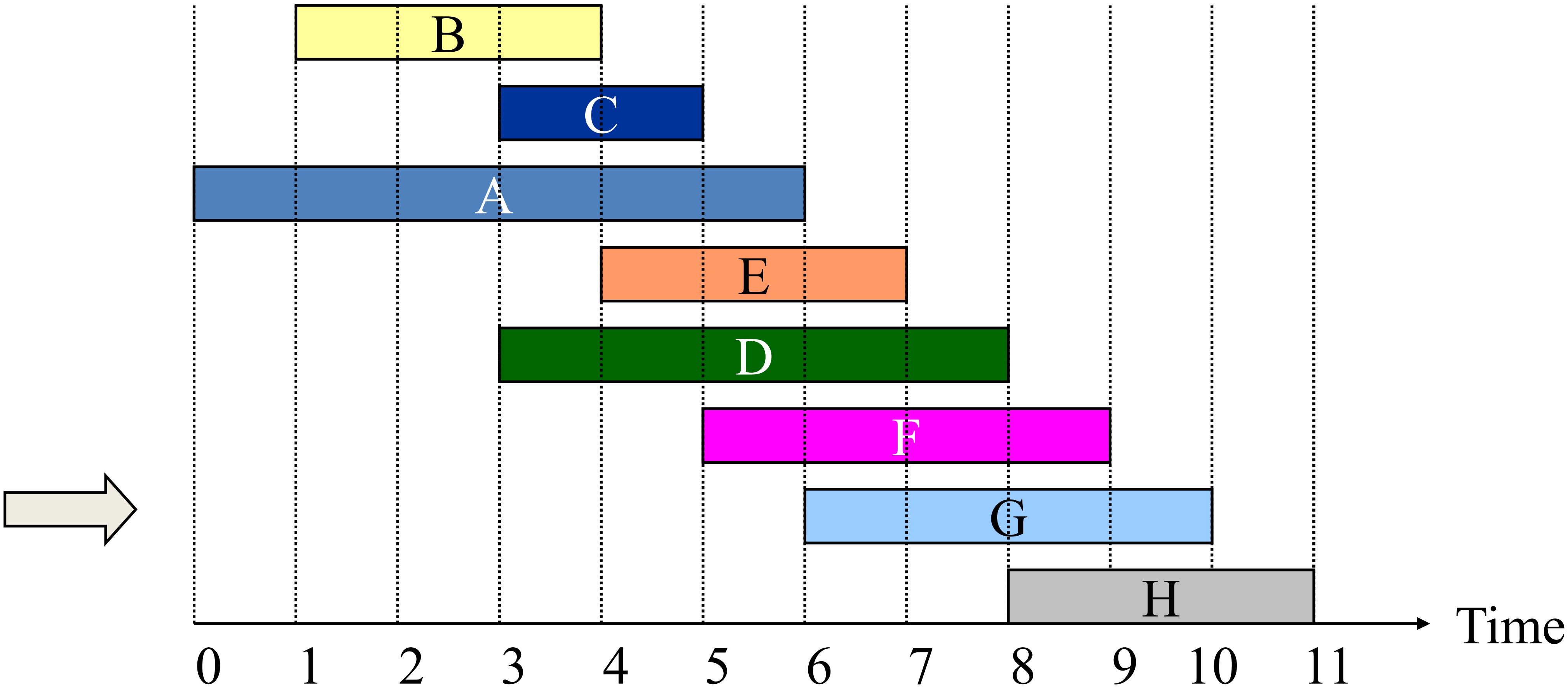

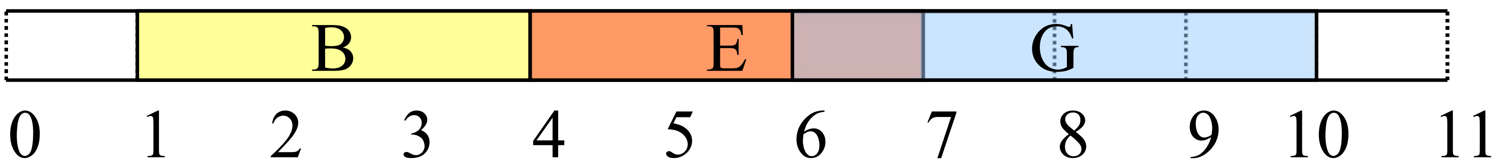

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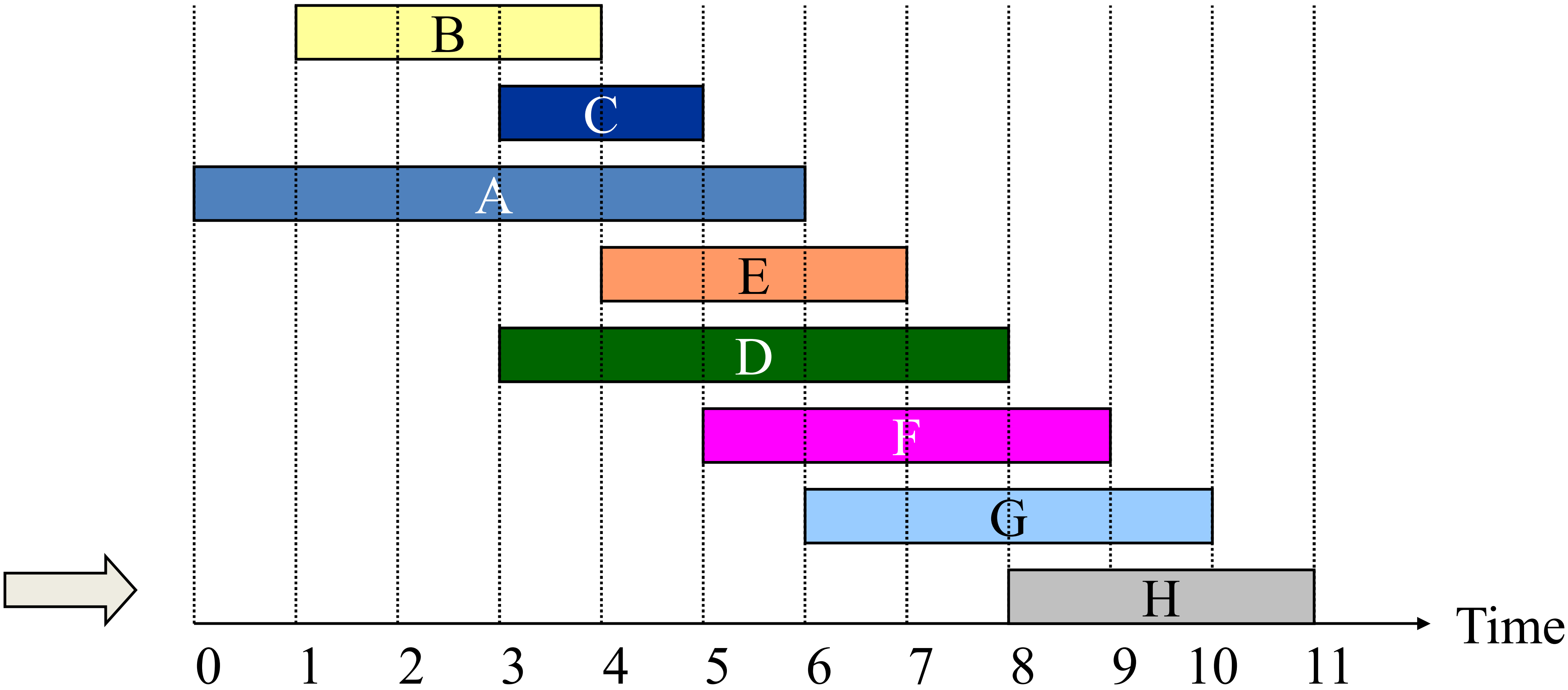
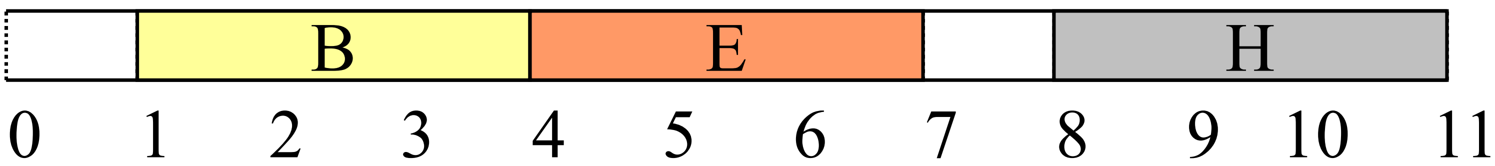


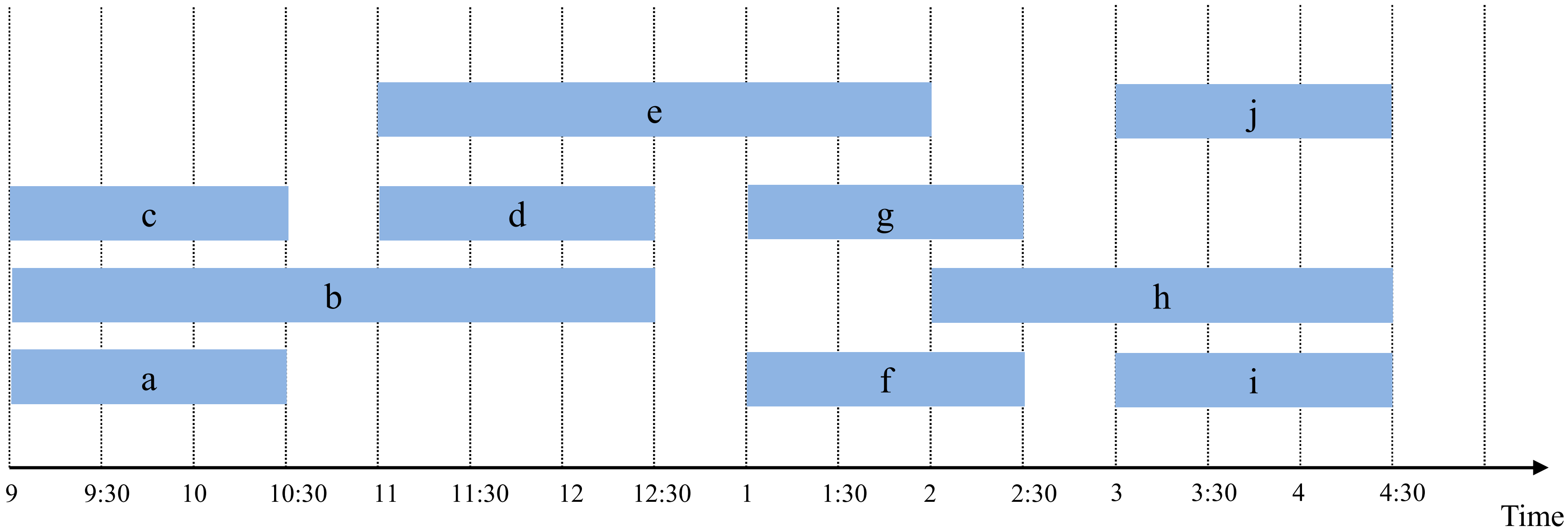


Interval Partitioning

Interval Partitioning

- Interval partitioning.
  - Lecture j starts at s_j and finishes at f_j.
  - Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

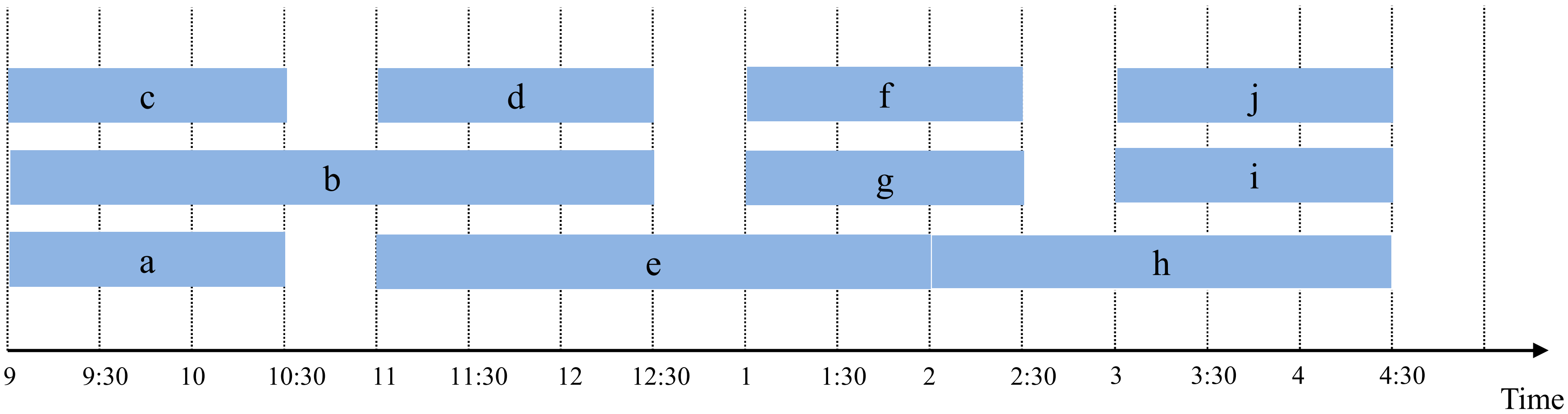
- Ex: This schedule uses 4 classrooms to schedule 10 lectures.



Interval Partitioning

- Interval partitioning.
  - Lecture j starts at s_j and finishes at f_j.
  - Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

- Ex: This schedule uses only 3.

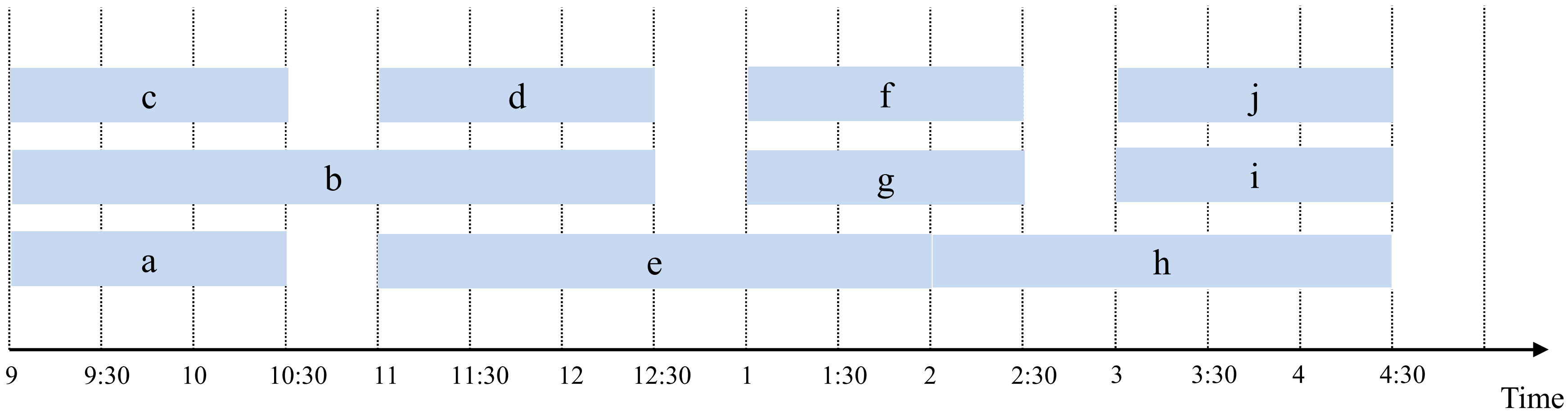


Interval Partitioning: Lower Bound on Optimal Solution

- Def. The depth of a set of open intervals is the maximum number of overlapped lectures during the whole period.
- **Key observation**. Number of classrooms needed $\geq$ depth.
- **Ex**: Depth of schedule below = 3 $\Rightarrow$ schedule below is optimal.

a, b, c all contain 9:30

- **Q**. Does there always exist a schedule equal to depth of intervals?



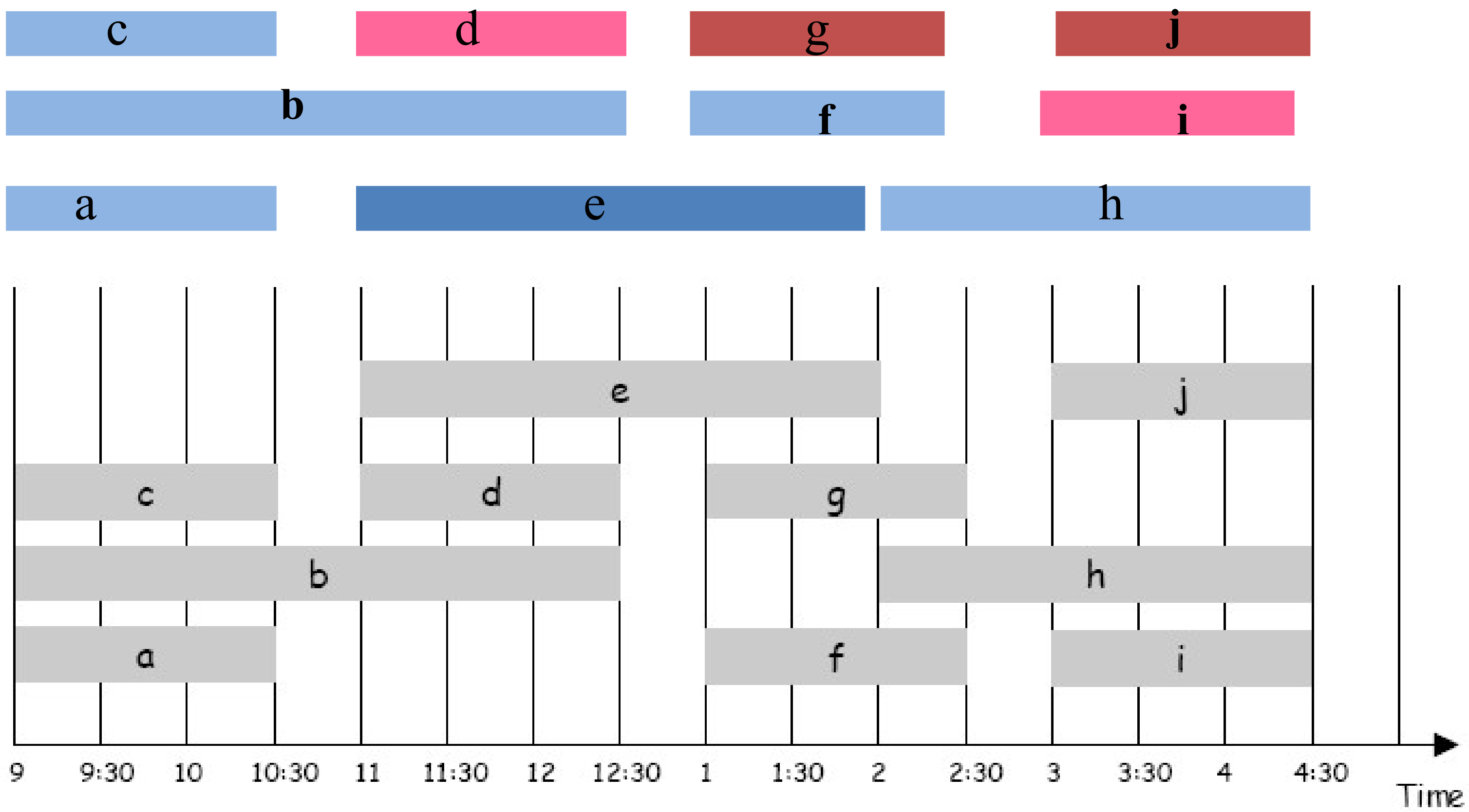
Interval Partitioning: Greedy Algorithm

- Greedy algorithm. Consider lectures in increasing order of start time: assign lecture to any compatible classroom (Don't open any new classroom unless necessary).

```
Sort intervals by starting time so that s1 ≤ s2 ≤ ... ≤ sn.
d ← 0  ← number of allocated classrooms
for j = 1 to n {
   if (lecture j is compatible with some classroom k)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d ← d + 1
}
```

- Implementation. O(n log n).
  - For each classroom k, maintain the finish time of the last job added.
  - Keep the classrooms in a priority queue.

Greedy Algorithm:



Interval Partitioning: Greedy Analysis

- **Observation**. Greedy algorithm never schedules two incompatible lectures in the same classroom (its solution is always feasible).
- **Theorem**. Greedy algorithm is optimal.
- Pf.
  - Let d = number of classrooms that the greedy algorithm allocates.
  - Classroom d is opened because we needed to schedule a job, say j, that is incompatible with all d-1 other classrooms.
  - Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than s_j.
  - Thus, we have d lectures overlapping at time s_j .
  - $d \leq depth$
  - Key observation $\Rightarrow$ all schedules need to use $\geq$ depth classrooms.

Greedy Analysis Strategies

- **Exchange argument**. Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.

- **Structural**. Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.