EE3210 Signals & Systems

Due on Midnight, Feb 28, 2020

Homework #1

1. Total mark is 20 points (= 4 points per problem $\times$ 5 problems)
2. Solution will be posted on March 3rd on Canvas website
3. Submission due by Feb 28, 2020, midnight. We will accept late submission until March 2, 2020
4. Late submission penalty; -5 points per day
   - Full mark: 20 points (Feb 28), 15 points (Feb 29), 10 points (March 01), 5 points (March 02), and 0 points for any late submission after March 3rd.
5. Online submission through Canvas
   - Scan or taking a photo of your anwser sheet, then upload to Canvas

Problem 1

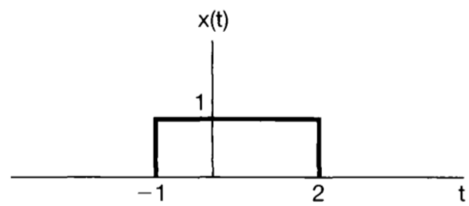
Let's consider an LTI system with intput and output relatex through the equation

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau - 2)\, d\tau \tag{1}$$

a) Find the impulse response $h(t)$ for the given system (1).

b) Is this system causal or not?

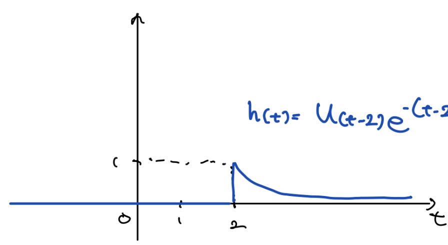
c) Determine the output of the system when the input $x(t)$ is as shown below.



Solution (a) Since the impulse response is the output for an impulse signal input, $h(t)$ is derived as below

$$h(t) = \int_{-\infty}^{t} e^{-(t-\tau)} \delta(\tau - 2)\, d\tau = e^{-(t-2)} u(t-2) \tag{2}$$

(b) As plotted in the figure below, $h(t) = 0$ for $t < 0$. Therefore, the given system is causal.



(c) The input can be expressed in terms of the unit step function as

$$x(t) = u(t+1) - u(t-2)$$

and the output and input of an arbitrary LTI system are related through convolution as follows

$$y(t) = x(t) * h(t) = e^{-(t-2)} u(t-2) * [u(t+1) - u(t-2)]\,. \tag{3}$$

(3) can be evaluated either by using the defintion of the convolution or by applying the following lemma.

Lemma 1

$$e^{-(t+a)} u(t+a) * u(t+b) = \left[1 - e^{-(t+a+b)}\right] u(t+a+b) \tag{4}$$

Proof) Based on the defintion of convolution integration, we can prove that

$$\begin{aligned} e^{-(t+a)} u(t+a) * u(t+b) &= \int_{-\infty}^{\infty} e^{-(\tau+a)} u(\tau+a) * u(t-\tau+b)\, d\tau \\ &= u(t+a+b) \int_{-a}^{t+b} e^{-(\tau+a)} d\tau = \left[1 - e^{-(t+a+b)}\right] u(t+a+b) \end{aligned}$$

Problem 1 continued on next page...

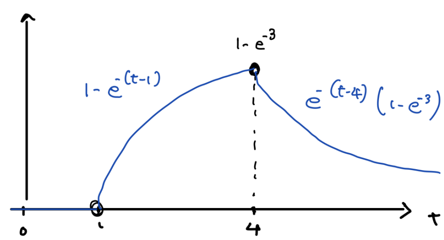
Then, (3) can be evaluated by term-by-term convolution and (4)

$$\begin{aligned} y(t) &= e^{-(t-2)}u(t-2) * u(t+1) - e^{-(t-2)}u(t-2) * u(t-2) \\ &= \left[1 - e^{-(t-1)}\right] u(t-1) - \left[1 - e^{-(t-4)}\right] u(t-4), \end{aligned} \tag{5}$$

where the last expression has three intervals with different value as follows

$$y(t) = \begin{cases} 0, & \text{if } t < 1 \\ 1 - e^{-(t-1)}, & \text{if } 1 \le t < 4 \\ \left[1 - e^{-(t-1)}\right] - \left[1 - e^{-(t-4)}\right] = e^{-(t-4)}\left[1 - e^{-3}\right], & \text{if } 4 \le t \end{cases} \tag{6}$$

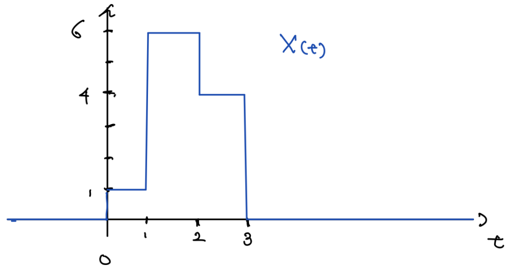
(6) can be plotted as below.

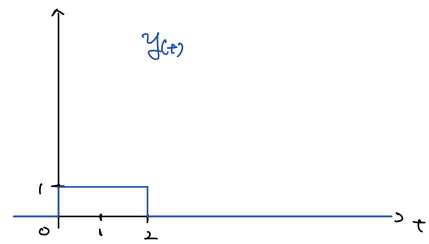


Problem 2

Evaluate the following convolution where $x(t)$ and $y(t)$ are plotted below

$$z(t) = x(t) * y(t)$$





Hint. Express the signals as a linear combination of time-delayed unit step function and apply the lemma

$$u\left(t+a\right) * u\left(t+b\right) = \left(t+a+b\right) u\left(t+a+b\right)$$

Solution $x(t)$ and $y(t)$ can be expressed in terms of the unit step function as follows

$$x(t) = u\left(t\right) + 5u\left(t-1\right) - 2u\left(t-2\right) - 4u\left(t-3\right), \quad y(t) = u\left(t\right) - u\left(t-2\right) \tag{7}$$

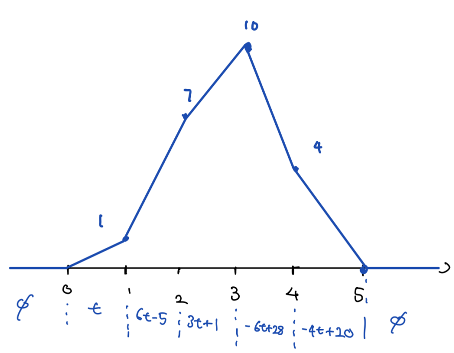
By performing term-by-term convolution and applying the lemma within the hint, $x(t) * y(t)$ is derived as

$$\begin{aligned} x(t) * y(t) &= tu\left(t\right) + 5\left(t-1\right)u\left(t-1\right) - 2\left(t-2\right)u\left(t-2\right) - 4\left(t-3\right)u\left(t-3\right) \\ &\quad - \left(t-2\right)u\left(t-2\right) - 5\left(t-3\right)u\left(t-3\right) + 2\left(t-4\right)u\left(t-4\right) + 4\left(t-5\right)u\left(t-5\right) \\ &= tu\left(t\right) + 5\left(t-1\right)u\left(t-1\right) - 3\left(t-2\right)u\left(t-2\right) - 9\left(t-3\right)u\left(t-3\right) + 2\left(t-4\right)u\left(t-4\right) \end{aligned}$$

The last expression has seven intervals with different value as follows

$$x(t) * y(t) = \begin{cases} 0, & \text{if } t < 0 \\ t, & \text{if } 0 \le t < 1 \\ 6t-5, & \text{if } 1 \le t < 2 \\ 3t+1, & \text{if } 2 \le t < 3 \\ -6t+28, & \text{if } 3 \le t < 4 \\ -4t+20, & \text{if } 4 \le t < 5 \\ 0, & \text{if } 5 \le t \end{cases} \tag{8}$$

(8) can be plotted as below.



Problem 3

Derive the following convolution

$$x(t) * x(t) * x(t) \tag{9}$$

where $x(t) = u(t+1) - u(t-1)$ is a rectangular pulse signal.

Hint. Use the following lemma

$$(t+a)\, u(t+a) * u(t+b) = \frac{1}{2}(t+a+b)^2\, u(t+a+b)$$

Solution Based on the associative property of the convolution, we can rewrite the expression as

$$\{x(t) * x(t)\} * x(t) = y(t) * x(t) \tag{10}$$

where we denote $y(t) = x(t) * x(t)$ and $y(t)$ can be evaluated as follows

$$\begin{aligned} x(t) * x(t) &= (t+2)\, u(t+2) - tu(t) - tu(t) + (t-2)\, u(t-2) \\ &= (t+2)\, u(t+2) - 2tu(t) + (t-2)\, u(t-2) \end{aligned}$$

Then, $x(t) * x(t) * x(t)$ can be expressed as follows

$$y(t) * x(t) = [(t+2)\, u(t+2) - 2tu(t) + (t-2)\, u(t-2)] * (u(t+1) - u(t-1))$$

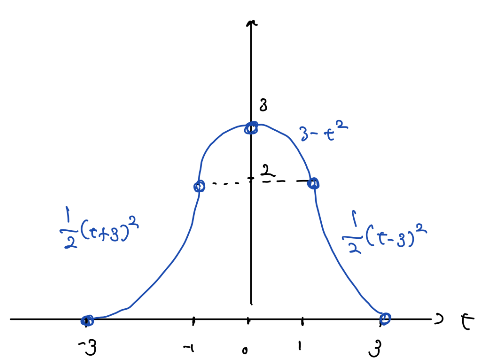
By performing term-by-term convolution and applying the lemma within the hint, $y(t) * x(t)$ is derived as

$$\begin{aligned} y(t) * x(t) = &(t+2)\, u(t+2) * u(t+1) - 2tu(t) * u(t+1) + (t-2)\, u(t-2) * u(t+1) \\ &- (t+2)\, u(t+2) * u(t-1) + 2tu(t) * u(t-1) - (t-2)\, u(t-2) * u(t-1) \\ &= \frac{1}{2}(t+3)^2\, u(t+3) - (t+1)^2\, u(t+1) + \frac{1}{2}(t-1)^2\, u(t-1) \\ &- \frac{1}{2}(t+1)^2\, u(t+1) + (t-1)^2\, u(t-1) - \frac{1}{2}(t-3)^2\, u(t-3) \\ &= \frac{1}{2}(t+3)^2\, u(t+3) - \frac{3}{2}(t+1)^2\, u(t+1) + \frac{3}{2}(t-1)^2\, u(t-1) - \frac{1}{2}(t-3)^2\, u(t-3) \end{aligned}$$

The last expression has five intervals with different value as follows

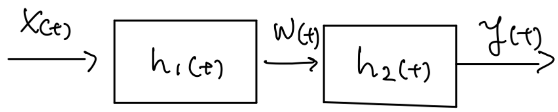
$$x(t) * y(t) = \begin{cases} 0, & \text{if } t < -3 \\ \frac{1}{2}(t+3)^2, & \text{if } -3 \le t < -1 \\ 3 - t^2, & \text{if } -1 \le t < 1 \\ \frac{1}{2}(t-3)^2, & \text{if } 1 \le t < 3 \\ 0, & \text{if } 3 \le t \end{cases} \tag{11}$$

(11) can be plotted as below.



Problem 4

Consider an LTI system with two sub-components connected in a cascaded manner as shown below.



a) Find the overall impulse response $h(t)$ when the impulse response of the each components are given by

$$h_1(t) = \delta(t) - 2e^{-2t}u(t), \quad h_2(t) = e^t u(t)$$

b) Is this system causal or not? Also, is it a stable system or not?

Solution (a) The impulse response for the cascaded connection can be derived by the convolution

$$h_1(t) * h_2(t) = \left[\delta(t) - 2e^{-2t}u(t)\right] * e^t u(t) = e^t u(t) - 2e^{-2t}u(t) * e^t u(t) \tag{12}$$

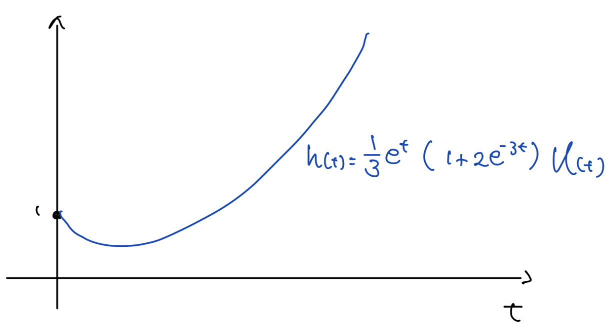
where we used $x(t) * \delta(t) = x(t)$ in the second equality. The convolution term can be evaluated as follows

$$\begin{aligned} 2e^{-2t}u(t) * e^t u(t) &= 2\int_{-\infty}^{\infty} e^{-2\tau}u(\tau)\, e^{t-\tau}u(t-\tau)\, d\tau = 2e^t \int_{-\infty}^{\infty} e^{-3\tau}u(\tau)\, u(t-\tau)\, d\tau \\ &= 2e^t u(t) \int_0^t e^{-3\tau} d\tau = \frac{2}{3}e^t\left[1 - e^{-3t}\right]u(t), \end{aligned} \tag{13}$$

and the overall impulse response is given by

$$h(t) = h_1(t) * h_2(t) = e^t u(t) - \frac{2}{3}e^t\left[1 - e^{-3t}\right]u(t) = \frac{1}{3}e^t\left[1 + 2e^{-3t}\right]u(t) \tag{14}$$

(b) Since $h(t) = 0$ for $t < 0$, it is a causal system. However, as shown in the following figure, the impulse response $h(t)$ of the given system diverge as t increases. Hence, it is not a stable system.



Problem 5

Consider the following systems and answer whether they are linear, causal, or time-invariant.

	Linear	Causal	Time-invariance
a) $y(t) = 2x(t) + 3$			
b) $y(t) = 2x^2(t) + 3x(t)$			
c) $y(t) = Atx(t)$			
d) $y(t) = x(t)x(t-2)$			
e) $y(t) = \exp(x(t))$			
f) $y(t) = \cos(3t)x(t)$			

Solution

	Linear	Causal	Time-invariance
a) $y(t) = 2x(t) + 3$	X	O	O
b) $y(t) = 2x^2(t) + 3x(t)$	X	O	O
c) $y(t) = Atx(t)$	O	O	X
d) $y(t) = x(t)x(t-2)$	X	O	O
e) $y(t) = \exp(x(t))$	X	O	O
f) $y(t) = \cos(3t)x(t)$	O	O	X