1. (15 points) Evaluate the following integrals.

(a) (5 points) $\displaystyle\int \frac{e^{2x}+2e^{-2x}+3}{e^{-x+1}}\,dx.$

Solution.

$$\int \frac{e^{2x}+2e^{-2x}+3}{e^{-x+1}}\,dx = \int \Big(\frac{e^{2x}}{e^{-x+1}} + \frac{2e^{-2x}}{e^{-x+1}} + \frac{3}{e^{-x+1}}\Big)\,dx$$
$$= \int (e^{3x-1}+2e^{-x-1}+3e^{x-1})\,dx = \frac{1}{3}e^{3x-1} - 2e^{-x-1} + 3e^{x-1} + C.$$

(b) (5 points) $\displaystyle\int \frac{1}{(2x+1)^2}\,dx.$

Solution.

$$\int \frac{1}{(2x+1)^2}\,dx = -\frac{(2x+1)^{-1}}{2} + C = -\frac{1}{2(2x+1)} + C.$$

(c) (5 points) $\displaystyle\int_{-1}^{1} 2x\sin(1-x^2+x^4)\,dx.$

Solution. Since the integral is defined on a symmetric interval $[-1,1]$, and the function being integrated is odd:

$$2(-x)\sin\big[1-(-x)^2+(-x)^4\big] = -2x\sin(1-x^2+x^4),$$

it follows that

$$\int_{-1}^{1} 2x\sin(1-x^2+x^4)\,dx = 0.$$

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2. (20 points) Evaluate the following integrals.

(a) (7 points) $\displaystyle\int x^2\tan^{-1}x\,dx.$

Solution.

$$\int x^2\tan^{-1}x\,dx = \int \tan^{-1}x\,d\Big(\frac{1}{3}x^3\Big) = \tan^{-1}x\cdot\frac{1}{3}x^3 - \int \frac{1}{3}x^3\,d(\tan^{-1}x)$$
$$= \frac{1}{3}x^3\tan^{-1}x - \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx,$$

and the substitution

$$u = 1+x^2, \quad \text{or} \quad x^2 = u-1,$$
$$du = 2x\,dx, \quad \text{or} \quad x\,dx = \frac{1}{2}\,du,$$

yields

$$\int \frac{x^3}{1+x^2}\,dx = \int \frac{x^2}{1+x^2}\cdot x\,dx = \int \frac{u-1}{u}\cdot\left(\frac{1}{2}du\right) = \frac{1}{2}\int\left(1-\frac{1}{u}\right)du$$
$$= \frac{1}{2}(u-\ln|u|)+C = \frac{1}{2}\left[(1+x^2)-\ln(1+x^2)\right]+C.$$

Thus

$$\int x^2\tan^{-1}x\,dx = \frac{1}{3}x^3\tan^{-1}x - \frac{1}{6}(1+x^2)+\frac{1}{6}\ln(1+x^2)+C.$$

(b) (5 points) $\int \frac{\cos^5 x}{\sin^2 x}\,dx$.

Solution.

$$\int \frac{\cos^5 x}{\sin^2 x}\,dx = \int \frac{(\cos^2 x)^2}{\sin^2 x}\cdot(\cos x\,dx) = \int \frac{(1-\sin^2 x)^2}{\sin^2 x}\,d(\sin x)$$
$$\overset{u=\sin x}{=} \int \frac{(1-u^2)^2}{u^2}\,du = \int \frac{1-2u^2+u^4}{u^2}\,du = \int\left(\frac{1}{u^2}-2+u^2\right)du$$
$$= -\frac{1}{u}-2u+\frac{1}{3}u^3+C = -\frac{1}{\sin x}-2\sin x+\frac{1}{3}\sin^3 x+C.$$

(c) (8 points) $\int \frac{5x^2-11x+32}{(x+1)(2x^2-4x+10)}\,dx$.

Solution. By partial fractions (note that $2x^2-4x+10$ is irreducible),

$$\frac{5x^2-11x+32}{(x+1)(2x^2-4x+10)} = \frac{A}{x+1}+\frac{Bx+C}{2x^2-4x+10},$$

so that

$$5x^2-11x+32 = A(2x^2-4x+10)+(Bx+C)(x+1).$$

Setting $x=-1$ yields

$$5\cdot(-1)^2-11\cdot(-1)+32 = A\left[2\cdot(-1)^2-4\cdot(-1)+10\right], \qquad \text{or} \qquad A=\frac{48}{16}=3.$$

To find B and C, expand the right side of the equation, combine like powers of x, and then compare coefficients:

$$5x^2-11x+32 = A(2x^2-4x+10)+(Bx+C)(x+1)$$
$$= (2Ax^2-4Ax+10A)+(Bx^2+Bx+Cx+C)$$
$$= (2A+B)x^2+(-4A+B+C)x+(10A+C).$$

This yields the linear system

$$\begin{aligned} 2A+B \quad &= 5 \\ -4A+B+C &= -11\,, \\ 10A \quad +C &= 32 \end{aligned}$$

where the first and third equation implies

$$B = 5-2A = -1, \qquad C = 32-10A = 2.$$

It follows that

$$\frac{5x^2-11x+32}{(x+1)(2x^2-4x+10)} = \frac{3}{x+1} + \frac{-x+2}{2x^2-4x+10},$$

and thus

$$\begin{aligned} \int \frac{5x^2-11x+32}{(x+1)(2x^2-4x+10)}\,dx &= \int \frac{3}{x+1}\,dx + \int \frac{-x+2}{2x^2-4x+10}\,dx \\ &= 3\ln|x+1| + \int \frac{-x+2}{2x^2-4x+10}\,dx. \end{aligned}$$

To evaluate the last integral on the right side, observe that $(2x^2-4x+10)' = 4x-4 = 4(x-1)$ and hence

$$\begin{aligned} \int \frac{-x+2}{2x^2-4x+10}\,dx &= \int \left(\frac{-x+1}{2x^2-4x+10} + \frac{1}{2x^2-4x+10}\right)dx \\ &= -\int \frac{x-1}{2x^2-4x+10}\,dx + \int \frac{1}{2(x-1)^2+8}\,dx \\ &\overset{u=2x^2-4x+10}{=} -\frac{1}{4}\int \frac{1}{u}\,du + \frac{1}{8}\int \frac{1}{(1/4)(x-1)^2+1}\,dx \\ &= -\frac{1}{4}\ln|2x^2-4x+10| + \frac{1}{8}\cdot\frac{\tan^{-1}\left[(1/2)\,(x-1)\right]}{(1/2)} + C. \end{aligned}$$

Consequently,

$$\begin{aligned} \int \frac{5x^2-11x+32}{(x+1)(2x^2-4x+10)}\,dx &= 3\ln|x+1| \\ &\quad -\frac{1}{4}\ln|2x^2-4x+10| + \frac{1}{4}\tan^{-1}\left[(1/2)\,(x-1)\right] + C. \end{aligned}$$

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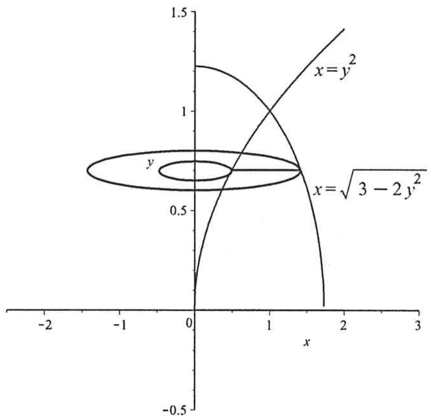
3. (15 points)

(a) (8 points) Find the volume of the solid generated by revolving the region bounded by the upper branch of the parabola $x=y^2$, the upper branch of the ellipse $x^2+2y^2=3$, and the x-axis about the y-axis.



Solution. The slice is chosen to be horizontal to avoid partitioning the interval of integration into multiple subintervals. Since the slice is perpendicular to the axis of rotation, the washer method has to be used. The inner and outer radii $r(y)$ and $R(y)$ of the washer, *expressed as functions of* y, are given by

$$r(y) = y^2, \qquad R(y) = \sqrt{3-2y^2}. \quad 2$$

The bounds of integration, on the other hand, are determined by equating $x = y^2$ to $x = \sqrt{3-2y^2}$ and solving for y, which yields

$$y^2 = \sqrt{3-2y^2} \Longrightarrow y^4 = 3-2y^2 \Longrightarrow y^4+2y^2-3=0 \Longrightarrow (y^2-1)(y^2+3)=0 \quad 2$$
$$\Longrightarrow y^2 = 1,\ y^2=-3 \text{ (discard)} \Longrightarrow y=1,\ y=-1 \text{ (discard)}.$$

The volume of the solid is then given by

$$V = \int_0^1 \pi\left[R^2(y)-r^2(y)\right]dy = \int_0^1 \pi\left[\left(\sqrt{3-2y^2}\right)^2 - (y^2)^2\right]dy \quad 1$$
$$= \pi\int_0^1 (3-2y^2-y^4)\,dy \;1\; = \pi\left(3y-2\cdot\frac{1}{3}y^3-\frac{1}{5}y^5\right)\Big|_0^1 = \pi\left(3-\frac{2}{3}-\frac{1}{5}\right) \;2\; = \frac{32}{15}\pi.$$

(b) (7 points) Find the area of the surface generated by revolving the curve $x = \int_0^y \sqrt{4-t^2}\,dt$, $1 \le y \le 2$, about the x-axis.

Solution.

$$S = \int 2\pi\rho\, ds \;1\; = \int_1^2 2\pi y \cdot \sqrt{1+\left[x'(y)\right]^2}\,dy, \quad 1$$

where

$$1+\left[x'(y)\right]^2 = 1+\left[\sqrt{4-y^2}\right]^2 = 1+(4-y^2) = 5-y^2. \quad 1$$

Thus

$$S = 2\pi \int_1^2 y \cdot \sqrt{5-y^2}\, dy \overset{u=5-y^2}{=} 2\pi \int_4^1 u^{1/2} \cdot \left(-\frac{1}{2} du\right)$$
$$= -\pi \int_4^1 u^{1/2}\, du = -\pi \left(\frac{2}{3} u^{3/2}\right)\Big|_4^1 = -\frac{2}{3}\pi(1-4^{3/2}) = \frac{14}{3}\pi.$$

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4. (15 points) Let $A(1,0,-1)$, $B(2,-1,0)$, $C(0,1,-1)$, and $D(1,2,-2)$ be four points in $\mathbb{R}^3$. Using vector method:

(a) (5 points) Find the volume of the parallelepiped with adjacent edges AB, AC, and AD.

Solution. The volume of the parallelepiped is given by

$$V = |\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD})|,$$

where

$$\begin{aligned} \overrightarrow{AB} &= B - A = \langle 2-1, -1-0, 0-(-1)\rangle = \langle 1,-1,1\rangle, \\ \overrightarrow{AC} &= C - A = \langle 0-1, 1-0, -1-(-1)\rangle = \langle -1,1,0\rangle, \\ \overrightarrow{AD} &= D - A = \langle 1-1, 2-0, -2-(-1)\rangle = \langle 0,2,-1\rangle. \end{aligned}$$

Since

$$\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = \begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 2 & -1 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} + 1\begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix}$$
$$= 1[1\cdot(-1) - 2\cdot 0] + 1[(-1)\cdot(-1) - 0\cdot 0] + 1[(-1)\cdot 2 - 0\cdot 1] = -2,$$

it follows that

$$V = |-2| = 2.$$

(b) (5 points) Find the equation of the plane that contains A, B, and C.

Solution. The equation of the plane is given by

$$n_1(x-x_0) + n_2(y-y_0) + n_3(z-z_0) = 0,$$

where $\vec{n} = \langle n_1, n_2, n_3\rangle$ is a normal vector of the plane and $P(x_0, y_0, z_0)$ is a point on the plane. Since

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = \vec{i}\begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$$

$$= \big[(-1)\cdot 0 - 1\cdot 1\big]\vec{i} - \big[1\cdot 0 - (-1)\cdot 1\big]\vec{j} + \big[1\cdot 1 - (-1)\cdot(-1)\big]\vec{k}$$

$$= \langle -1, -1, 0\rangle = -\langle 1, 1, 0\rangle,$$

and $P(x_0, y_0, z_0) = A(1, 0, -1)$ is a point on the plane, the equation takes the form

$$1(x-1) + 1(y-0) + 0\big[z - (-1)\big] = 0, \qquad \text{or} \qquad x + y = 1.$$

(c) (5 points) Find the distance from D to the plane containing A, B, and C.

Solution. The distance between D and the plane is given by $|\mathrm{proj}_{\vec{n}}\overrightarrow{AD}|$, the magnitude of the (orthogonal) projection of the vector

$$\overrightarrow{AD} = \langle 0, 2, -1\rangle$$

onto the normal vector $\vec{n} = \langle 1, 1, 0\rangle$:

$$d = |\mathrm{proj}_{\vec{n}}\overrightarrow{AD}| = \left|\frac{\overrightarrow{AD}\cdot\vec{n}}{|\vec{n}|}\right| = \left|\frac{\langle 0, 2, -1\rangle \cdot \langle 1, 1, 0\rangle}{|\langle 1, 1, 0\rangle|}\right|$$

$$= \left|\frac{0\cdot 1 + 2\cdot 1 + (-1)\cdot 0}{\sqrt{1^2 + 1^2 + 0^2}}\right| = \left|\frac{2}{\sqrt{2}}\right| = \sqrt{2}.$$

Note that projecting any one of the vectors $\overrightarrow{AD}$, $\overrightarrow{BD}$, or $\overrightarrow{CD}$ onto $\vec{n}$ yields the same result.

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5. (15 points)

(a) (8 points) Solve the complex equation $(1+\mathrm{i})z^3 - \big[1 + \mathrm{e}^{\mathrm{i}(\pi/3)}\big] = 0$ and list all possible solutions in Euler's form with principal arguments.

Solution. Rearranging the equation yields

$$(1+\mathrm{i})z^3 = 1 + \mathrm{e}^{\mathrm{i}(\pi/3)} = \mathrm{e}^{\mathrm{i}(0)} + \mathrm{e}^{\mathrm{i}(\pi/3)}$$

$$= \mathrm{e}^{\mathrm{i}(\pi/6)}\big[\mathrm{e}^{-\mathrm{i}(\pi/6)} + \mathrm{e}^{\mathrm{i}(\pi/6)}\big] = \mathrm{e}^{\mathrm{i}(\pi/6)}\big[2\cos(\pi/6)\big] = \sqrt{3}\,\mathrm{e}^{\mathrm{i}(\pi/6)},$$

which implies that

$$z^3 = \frac{\sqrt{3}\,\mathrm{e}^{\mathrm{i}(\pi/6)}}{1+\mathrm{i}} = \frac{\sqrt{3}\,\mathrm{e}^{\mathrm{i}(\pi/6)}}{\sqrt{2}\,\mathrm{e}^{\mathrm{i}(\pi/4)}} = \sqrt{3/2}\,\mathrm{e}^{\mathrm{i}(\pi/6 - \pi/4)} = \sqrt{3/2}\,\mathrm{e}^{\mathrm{i}(-\pi/12)}.$$

Thus

$$\begin{aligned} z &= \left[\sqrt{3/2}\,\mathrm{e}^{\mathrm{i}(-\pi/12)}\right]^{1/3} = \left[\sqrt{3/2}\,\mathrm{e}^{\mathrm{i}(-\pi/12+2k\pi)}\right]^{1/3} \\ &= \sqrt[6]{3/2}\,\mathrm{e}^{\mathrm{i}(-\pi/12+2k\pi)/3} = \sqrt[6]{3/2}\,\mathrm{e}^{\mathrm{i}(-\pi/36+2k\pi/3)}, \end{aligned}$$

where

$$\begin{aligned} k=0: &\quad z = \sqrt[6]{3/2}\,\mathrm{e}^{\mathrm{i}(-\pi/36)}, \\ k=1: &\quad z = \sqrt[6]{3/2}\,\mathrm{e}^{\mathrm{i}(-\pi/36+2\pi/3)} = \sqrt[6]{3/2}\,\mathrm{e}^{\mathrm{i}(23\pi/36)}, \\ k=2: &\quad z = \sqrt[6]{3/2}\,\mathrm{e}^{\mathrm{i}(-\pi/36+4\pi/3)} = \sqrt[6]{3/2}\,\mathrm{e}^{\mathrm{i}(47\pi/36)} \\ &\quad\;\; = \sqrt[6]{3/2}\,\mathrm{e}^{\mathrm{i}(47\pi/36-2\pi)} = \sqrt[6]{3/2}\,\mathrm{e}^{\mathrm{i}(-25\pi/36)}. \end{aligned}$$

(b) (7 points) Express the complex number $z = (1-\sin\theta+\mathrm{i}\cos\theta)^{20}$ in Euler's form.

Solution. Observe that

$$\begin{aligned} 1-\sin\theta+\mathrm{i}\cos\theta &= 1+\mathrm{i}(\cos\theta+\mathrm{i}\sin\theta) = 1+\mathrm{e}^{\mathrm{i}(\pi/2)}\mathrm{e}^{\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}(0)}+\mathrm{e}^{\mathrm{i}(\pi/2+\theta)} \\ &= \mathrm{e}^{\mathrm{i}(\pi/4+\theta/2)}\left[\mathrm{e}^{-\mathrm{i}(\pi/4+\theta/2)}+\mathrm{e}^{\mathrm{i}(\pi/4+\theta/2)}\right] = \mathrm{e}^{\mathrm{i}(\pi/4+\theta/2)}\left[2\cos(\pi/4+\theta/2)\right]. \end{aligned}$$

Thus

$$\begin{aligned} z &= \left[2\cos(\pi/4+\theta/2)\,\mathrm{e}^{\mathrm{i}(\pi/4+\theta/2)}\right]^{20} = \left[2\cos(\pi/4+\theta/2)\right]^{20}\left[\mathrm{e}^{\mathrm{i}(\pi/4+\theta/2)}\right]^{20} \\ &= \left[2\cos(\pi/4+\theta/2)\right]^{20}\mathrm{e}^{\mathrm{i}(5\pi+10\theta)} = \left[2\cos(\pi/4+\theta/2)\right]^{20}\mathrm{e}^{\mathrm{i}(\pi+10\theta)}. \end{aligned}$$

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6. (20 points)

(a) (5 points) Let A and B be two 4×4 matrices such that $\det(A)=3$ and $\det(B)=2$. Using the properties of determinant, find $\det(2A^TB^{-1})$.

Solution.

$$\det(2A^TB^{-1}) = 2^4\det(A^T)\det(B^{-1}) = 2^4\det(A)\left[\det(B)\right]^{-1} = 16\cdot 3\cdot\frac{1}{2} = 24.$$

(b) (10 points) Solve the linear system

$$\begin{aligned} -x+3y-2z+\;\,w &= 3 \\ 2x-\;\,y+2z+\;\,w &= -1 \\ 3x+\;\,y+2z+3w &= 1 \end{aligned}$$

by Gaussian elimination and express the general solution in vector form.

Solution. Write the system as $Au = b$ and apply Gaussian elimination to the augmented matrix $[A|b]$. The result is

$$[A|b] = \left(\begin{array}{cccc|c} -1 & 3 & -2 & 1 & 3 \\ 2 & -1 & 2 & 1 & -1 \\ 3 & 1 & 2 & 3 & 1 \end{array}\right) \xrightarrow[R_2+2R_1]{R_3+3R_1} \left(\begin{array}{cccc|c} -1 & 3 & -2 & 1 & 3 \\ 0 & 5 & -2 & 3 & 5 \\ 0 & 10 & -4 & 6 & 10 \end{array}\right)$$ 5

$$\xrightarrow{R_3-2R_2} \left(\begin{array}{cccc|c} -1 & 3 & -2 & 1 & 3 \\ 0 & 5 & -2 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$ 2

1

Note that the system is consistent, and the reduced system takes the form

$$\begin{aligned} -x + 3y - 2z + \ \ w &= 3 \\ 5y - 2z + 3w &= 5 \end{aligned}.$$ 1

Solving for the pivot variables x, y in terms of the free variables z, w from the above equations then yields

$$y = 1 + \frac{2}{5}z - \frac{3}{5}w,$$ 1

$$x = -3 + 3y - 2z + w = -3 + 3\left(1 + \frac{2}{5}z - \frac{3}{5}w\right) - 2z + w = -\frac{4}{5}z - \frac{4}{5}w,$$ 1

and the general solution of the system is given by

$$u = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -(4/5)z - (4/5)w \\ 1 + (2/5)z - (3/5)w \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{z}{5}\begin{pmatrix} -4 \\ 2 \\ 5 \\ 0 \end{pmatrix} + \frac{w}{5}\begin{pmatrix} -4 \\ -3 \\ 0 \\ 5 \end{pmatrix}.$$ 1 1

(c) (5 points) Write down the corresponding homogenous system explicitly and determine all non-trivial solutions from (b) without resolving the system.

Solution. The homogenous system associated with the linear system $Au = b$ is

$$\begin{aligned} -x + 3y - 2z + \ \ w &= 0 \\ 2x - \ \ y + 2z + \ \ w &= 0 \\ 3x + \ \ y + 2z + 3w &= 0 \end{aligned}.$$ 2

Its general solution is given by

$$u = \frac{z}{5}\begin{pmatrix} -4 \\ 2 \\ 5 \\ 0 \end{pmatrix} + \frac{w}{5}\begin{pmatrix} -4 \\ -3 \\ 0 \\ 5 \end{pmatrix}.$$ 3