CITY UNIVERSITY OF HONG KONG

Course code and title : MA1200 Calculus and Basic Linear Algebra I

Session : Semester A, 2016/2017

Time allowed : Three hours

This paper has **SIX** pages (including this cover page).

A brief table of derivatives is attached on pages 5 and 6.

Instructions to candidates:

1. This paper has **TEN** questions.
2. Attempt **ALL** questions.
3. Each question carries 10 marks.

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*This is a **closed-book** examination.*

Candidates are allowed to use the following materials/aids:

Non-programmable calculators

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorised materials or aids are found on them.

Question 1

The functions $f(x)$ and $g(x)$ are defined by

$$f(x) = 2x - 3 \quad , \text{for} \quad x \in \mathrm{R} ,$$

$$g(x) = \frac{1}{x-2} \quad , \text{for} \quad x \in \mathrm{R} \backslash \{2\} .$$

Find, in a similar form

(a) the inverse function $f^{-1}(x)$, (5 marks)

(b) the composite function $(g \circ f)(x)$. (5 marks)

In each case state the largest possible domain and the range of the function.

Question 2

Evaluate the following limits:

(a) $\lim\limits_{x \to 0} \dfrac{\sinh x}{x}$, (3 marks)

(b) $\lim\limits_{x \to \infty} \cos^{-1}\left(\dfrac{1-x}{1+x}\right)$, (3 marks)

(c) $\lim\limits_{x \to 0^-} \dfrac{2}{3 + e^{\frac{1}{x}}}$ (4 marks)

Question 3

An ellipse has equation $9x^2 + 25y^2 + 36x - 50y - 164 = 0$.

(a) Sketch the ellipse. (2 marks)

(b) Find the coordinates of the foci of the ellipse. (4 marks)

(c) Find the equation of the tangent to the ellipse at the point $P\left(2, \dfrac{14}{5}\right)$ (4 marks)

Question 4

(a) In 263 A.D., a Chinese Mathematician, Liu Hui proposed a method to compute an approximation to the value of π.

Given a unit circle, he calculated the area of inscribed and circumscribed regular hexagon as shown in Figure 1, thus obtaining lower and upper bounds for the area of the unit circle ($= \pi$ sq.unit). Find the areas of hexagon ABCDEF and hexagon PQRSTU, correct your answers to 3 decimal places.

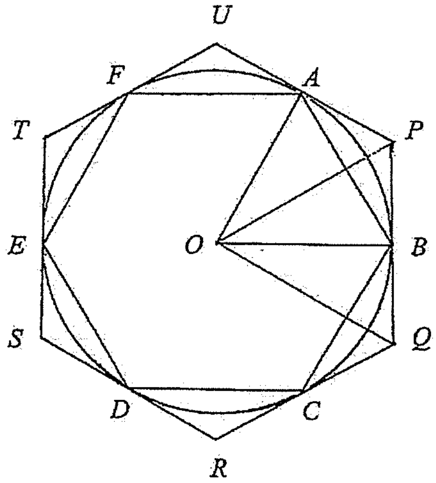

Figure 1

(6 marks)

(b) Find, in radians, the general solution of the trigonometric equation $\sin(5x) = -\dfrac{1}{2}$.

(4 marks)

Question 5

(a) Differentiate with respect to x :

(i) $x^3 e^{-2x}$, (3 marks)

(ii) $\log_e(\cot x + \operatorname{cosec} x)$. (3 marks)

(b) Find the general formula for the nth derivative of the function $F(x) = \dfrac{3x}{x+3}$ with respect to x.

(4 marks)

Question 6

(a) A curve has parametric equations $x = t - t^{-1}$, $y = t + t^{-1}$, for $t \in (0, \infty)$.

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of t . (5 marks)

(b) Differentiate with respect to x :

(i) $(\cos x)^x$, (3 marks)

(ii) $\sqrt{\cos x} + (\cos x)^x$. (2 marks)

Question 7

Given that $\left(x^2+\sqrt{2}x+1\right)$ is a factor of x^4+1 , express $\dfrac{4x^3+8x}{x^4+1}$ as partial fractions.

(10 marks)

Question 8

(a) If $y=\left(1+x^2\right)^{-\frac{1}{2}}\log_e\left(x+\sqrt{x^2+1}\right)$, show that $\left(1+x^2\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}+xy=1$. (3 marks)

(b) Deduce that $\left(1+x^2\right)\dfrac{\mathrm{d}^{n+2}y}{\mathrm{d}x^{n+2}}+(2n+3)x\dfrac{\mathrm{d}^{n+1}y}{\mathrm{d}x^{n+1}}+(n+1)^2\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}=0$. (3 marks)

(c) Hence, or otherwise, find the expansion of $\left(1+x^2\right)^{-\frac{1}{2}}\log_e\left(x+\sqrt{x^2+1}\right)$ in ascending powers of x as far as the term in x^5.

(4 marks)

Question 9

The equation of a curve is $y=\dfrac{x+4}{x(x+3)}$, for $x\in \mathrm{R}\backslash\{-3,0\}$.

(a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. (3 marks)

(b) Show that $P\left(-6,-\dfrac{1}{9}\right)$ and $Q(-2,-1)$ are the stationary points of the curve. (4 marks)

(c) Use the second derivative test, determine the nature of the stationary points P and Q .

(3 marks)

Question 10

(a) Prove from first principles that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sin x)=\cos x$. (5 marks)

(Hint: You may use $\sin A-\sin B=2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$.)

(b) Let $F(x)=|\sin x|$, for $x\in\mathrm{R}$.

Determine whether $F(x)$ is differentiable at $x=0$. Give your reason. (5 marks)

Short Table of Derivatives of $y = f(u)$ with respect to x, where u is a function of x

Functions, $y = f(u)$	Derivative of y with respect to x
$y = c$, where c is a constant.	$\frac{dy}{dx} = 0$
$y = cu$, where c is a constant.	$\frac{dy}{dx} = c\frac{du}{dx}$
$y = u^p$, where p is a constant.	$\frac{dy}{dx} = pu^{p-1}\frac{du}{dx}$
$y = u + v$	$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = f(u)$, where u is a function of x.	$\frac{dy}{dx} = \frac{d\,f(u)}{du}\cdot\frac{du}{dx}$, the chain rule
$y = \log_a u$, $a > 0$.	$\frac{dy}{dx} = \frac{1}{u}\log_a e\frac{du}{dx}$
$y = a^u$, $a > 0$.	$\frac{dy}{dx} = a^u \log_e a\frac{du}{dx}$
$y = e^u$	$\frac{dy}{dx} = e^u\frac{du}{dx}$
$y = u^v$	$\frac{dy}{dx} = vu^{v-1}\frac{du}{dx} + u^v \log_e u\frac{dv}{dx}$
$y = \sin u$	$\frac{dy}{dx} = \cos u\frac{du}{dx}$
$y = \cos u$	$\frac{dy}{dx} = -\sin u\frac{du}{dx}$
$y = \tan u$	$\frac{dy}{dx} = \sec^2 u\frac{du}{dx}$
$y = \cot u$	$\frac{dy}{dx} = -\mathrm{cosec}^2 u\frac{du}{dx}$
$y = \sec u$	$\frac{dy}{dx} = \sec u \tan u\frac{du}{dx}$
$y = \mathrm{cosec}\, u$	$\frac{dy}{dx} = -\mathrm{cosec}\, u \cot u\frac{du}{dx}$
$y = \sin^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y = \cos^{-1} u$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y = \tan^{-1} u$	$\frac{dy}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$

Functions, $y = f(u)$	**Derivative of** y **with respect to** x
$y = \cot^{-1} u$	$\frac{dy}{dx} = -\frac{1}{1+u^2}\frac{du}{dx}$
$y = \sec^{-1} u$	$\frac{dy}{dx} = \frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$
$y = \operatorname{cosec}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$
$y = \sinh u$	$\frac{dy}{dx} = \cosh u\frac{du}{dx}$
$y = \cosh u$	$\frac{dy}{dx} = \sinh u\frac{du}{dx}$
$y = \tanh u$	$\frac{dy}{dx} = \operatorname{sech}^2 u\frac{du}{dx}$
$y = \coth u$	$\frac{dy}{dx} = -\operatorname{cosech}^2 u\frac{du}{dx}$
$y = \operatorname{sech} u$	$\frac{dy}{dx} = -\operatorname{sech} u \tanh u\frac{du}{dx}$
$y = \operatorname{cosech} u$	$\frac{dy}{dx} = -\operatorname{cosech} u \coth u\frac{du}{dx}$
$y = \sinh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$
$y = \cosh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$
$y = \tanh^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$
$y = \coth^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$
$y = \operatorname{sech}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$
$y = \operatorname{cosech}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{\lvert u\rvert\sqrt{u^2+1}}\frac{du}{dx}$

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