CITY UNIVERSITY OF HONG KONG

Department of Mathematics

Course Code & Title	:	MA1200 Calculus and Basic Linear Algebra I
Session	:	Semester A, 2018-2019
Time Allowed	:	Three Hours

This paper has **SEVEN** pages. (including this cover page)

A brief table of derivatives is attached on page 6 and 7.

Instructions to candidates:

1. Answer **all** questions.
2. Start each main question on a new page.
3. Show all step.

This is a ***closed-book*** *examination.*

Candidates are allowed to use the following materials/aids:

Non-programmable portable battery operated calculator.

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorized materials or aids are found on them.

Question 1

Consider the conic section described by the equation $4x^2 + y^2 + 24x - 4y + 24 = 0$.

(a) Classify its type. (3 marks)

(b) Find its center, vertices, and foci. (5 marks)

(c) Sketch its graph. (3 marks)

Question 2

Let $f(x) = \dfrac{4x+3}{x+2}$.

(a) Show that f is one-to-one in its domain of definition. (3 marks)

(b) Calculate $f^{-1}(-2)$. (2 marks)

(c) Find the domain and range of $f^{-1}(x)$. (4 marks)

(d) Sketch the graph of the curve $y = f^{-1}(x)$. (2 marks)

Question 3

(a) Show that $\sin(3x) = 3\sin x \cos^2 x - \sin^3 x$. (3 marks)

(b) Find, in radians, the general solutions of the equation (3 marks)

$$\sin(3x) + \cos(3x) + 1 = 0.$$

(Hint: To solve questions (a) and (b), you may use the formulas:

$\sin(A+B) = \sin A \cos B + \cos A \sin B$, $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\sin A + \sin B = 2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$, $\cos A + \cos B = 2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2}$)

(c) Let $f(x) = \dfrac{x^3-3}{x^3-x^2-x+1}$.

(i) Express $f(x)$ in partial fractions. (3 marks)

(ii) Find $f^{(3)}(x)$. (3 marks)

Question 4

Consider the function $f(x) = x^2 \ln x$.

(a) Find its domain of definition and the interval on which $f(x)$ is positive. (2 marks)

(b) Calculate $\lim\limits_{x \to 0+} f(x)$. (3 marks)

(c) Find the inflection point of $f(x)$. (3 marks)

(d) Find the minimum value of $f(x)$. (3 marks)

Question 5

(a) Compute $\lim\limits_{x \to \infty} \left(\dfrac{\ln x}{x} \right)^{1/\ln x}$. (4 marks)

(b) Let $f(x) = \dfrac{\sqrt{|x|}\,\cos(\pi^{1/x^2})}{2 + \sqrt{x^2+3}}$ for $x \neq 0$. How should f be defined at $x = 0$ so that it becomes a continuous function on all $\mathbb{R}$? (4 marks)

(c) Show that the equation $x^5 + x^3 + 2x = 2x^4 + 3x^2 + 4$ has a solution in the open interval $(2, 3)$. (3 marks)

Question 6

Differentiate the following functions about the variable x:

(a) $\dfrac{x^2+2}{x^2-1}$; (2 marks)

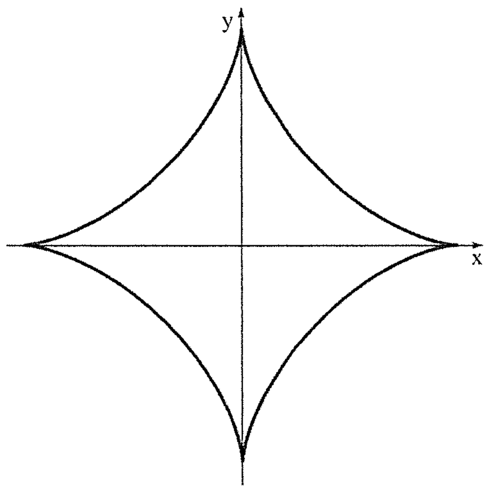
(b) $\sin^{-1}\left(\dfrac{x^2}{3}\right)$; (3 marks)

(c) $\ln \dfrac{(6+\sin^2 x)^{10}}{(7+\cos x)^3}$; (3 marks)

(d) $(\sin x)^{\tan x}$. (3 marks)

Question 7

The graph of the equation $\begin{cases} x = 5\sqrt{5}\,\sin^3 t, \\ y = 5\sqrt{5}\,\cos^3 t, \end{cases}$ for $t \in [0, 2\pi]$ is one of a family of curves called astroids; see the following figure.



(a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at the point $(-1, 8)$. (6 marks)

(b) Find the tangent line at the point $(-1, 8)$. (3 marks)

Question 8

A farmer has 100 pigs each weighing 300 pounds. It costs \$10 a day to keep one pig. The pigs gain weight at 10 pounds a day. They sell today for \$15 a pound, but the price is falling by \$0.2 a day. How many days should the farmer wait to sell his pigs in order to maximize his profit? (11 marks)

Question 9

Let $f(x) = \sin(\sinh^{-1} x)$.

(a) Show that (4 marks)

$$(1 + x^2)f''(x) + xf'(x) + f(x) = 0.$$

(b) Let n be a positive integer, show that (5 marks)

$$(1 + x^2)f^{(n+2)}(x) + (2n + 1)xf^{(n+1)}(x) + (n^2 + 1)f^{(n)}(x) = 0.$$

(c) Hence, or otherwise, find the Maclaurin series of $\sin(\sinh^{-1} x)$ as far as the terms in x^5. (4 marks)

Short Table of Derivatives of $y = f(u)$ with respect to x, where u is a function of x

Functions, $y = f(u)$	**Derivative of y with respect to x**
$y = c$, where c is a constant.	$\frac{dy}{dx} = 0$
$y = cu$, where c is a constant.	$\frac{dy}{dx} = c\frac{du}{dx}$
$y = u^p$, where p is a constant.	$\frac{dy}{dx} = pu^{p-1}\frac{du}{dx}$
$y = u + v$	$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = f(u)$, where u is a function of x.	$\frac{dy}{dx} = \frac{d f(u)}{du} \cdot \frac{du}{dx}$, the chain rule
$y = \log_a u$, $a > 0$.	$\frac{dy}{dx} = \frac{1}{u}\log_a e\frac{du}{dx}$
$y = a^u$, $a > 0$.	$\frac{dy}{dx} = a^u \log_e a\frac{du}{dx}$
$y = e^u$	$\frac{dy}{dx} = e^u\frac{du}{dx}$
$y = u^v$	$\frac{dy}{dx} = vu^{v-1}\frac{du}{dx} + u^v \log_e u\frac{dv}{dx}$
$y = \sin u$	$\frac{dy}{dx} = \cos u\frac{du}{dx}$
$y = \cos u$	$\frac{dy}{dx} = -\sin u\frac{du}{dx}$
$y = \tan u$	$\frac{dy}{dx} = \sec^2 u\frac{du}{dx}$
$y = \cot u$	$\frac{dy}{dx} = -\mathrm{cosec}^2 u\frac{du}{dx}$
$y = \sec u$	$\frac{dy}{dx} = \sec u \tan u\frac{du}{dx}$
$y = \mathrm{cosec}\, u$	$\frac{dy}{dx} = -\mathrm{cosec}\, u \cot u\frac{du}{dx}$
$y = \sin^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y = \cos^{-1} u$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y = \tan^{-1} u$	$\frac{dy}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$

Functions, $y=f(u)$	**Derivative of** y **with respect to** x
$y=\cot^{-1}u$	$\frac{dy}{dx}=-\frac{1}{1+u^2}\frac{du}{dx}$
$y=\sec^{-1}u$	$\frac{dy}{dx}=\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$
$y=\operatorname{cosec}^{-1}u$	$\frac{dy}{dx}=-\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$
$y=\sinh u$	$\frac{dy}{dx}=\cosh u\frac{du}{dx}$
$y=\cosh u$	$\frac{dy}{dx}=\sinh u\frac{du}{dx}$
$y=\tanh u$	$\frac{dy}{dx}=\operatorname{sech}^2 u\frac{du}{dx}$
$y=\coth u$	$\frac{dy}{dx}=-\operatorname{cosech}^2 u\frac{du}{dx}$
$y=\operatorname{sech} u$	$\frac{dy}{dx}=-\operatorname{sech} u\tanh u\frac{du}{dx}$
$y=\operatorname{cosech} u$	$\frac{dy}{dx}=-\operatorname{cosech} u\coth u\frac{du}{dx}$
$y=\sinh^{-1}u$	$\frac{dy}{dx}=\frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$
$y=\cosh^{-1}u$	$\frac{dy}{dx}=\frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$
$y=\tanh^{-1}u$	$\frac{dy}{dx}=\frac{1}{1-u^2}\frac{du}{dx}$
$y=\coth^{-1}u$	$\frac{dy}{dx}=\frac{1}{1-u^2}\frac{du}{dx}$
$y=\operatorname{sech}^{-1}u$	$\frac{dy}{dx}=-\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$
$y=\operatorname{cosech}^{-1}u$	$\frac{dy}{dx}=-\frac{1}{\lvert u\rvert\sqrt{u^2+1}}\frac{du}{dx}$

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