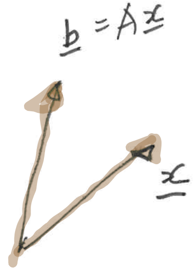
Eigenvalues and Eigenvectors (powerful Maths tool)

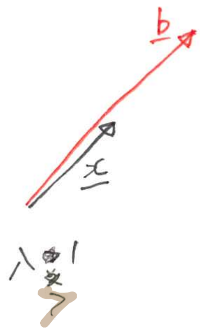
$$\overset{3\times3}{A}\;\overset{3\times1}{\underline{x}} = \overset{3\times1}{\underline{b}}$$

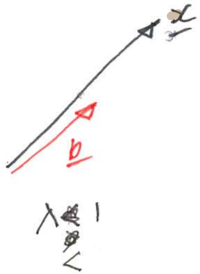


No general relationship between $\underline{x}$ and $\underline{b}$

$\underline{x} \neq 0$

In some special cases, we have





scalar.

$$\underline{b} = \lambda \underline{x}$$

"$A\underline{x} = \lambda \underline{x}$" . corresponding to λ

$\underline{x} \neq 0$

eigenvalue of A.

I can find $\underline{x}$

$\underline{x}$ is known as eigenvector.

Eigenvalue and Eigenvectors

$$\underset{3\times 3}{A}\ \underset{3\times 1}{\underline{x}} = \underbrace{\lambda}_{\text{scalar}}\ \underset{3\times 1}{\underline{x}}$$

special $\underline{x} \neq 0$ eigenvector

special λ eigenvalue

$$\underset{2\times 2}{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}} \underset{2\times 1}{\underline{x}} = \lambda \underset{2\times 1}{\underline{x}}$$

$$\lambda I \underline{x} = \lambda \underline{x}$$

$\cancel{3\times 3}$ 2×2, 3×1 2×1, 2×1

$$A \underline{x} - \lambda \underset{2\times 2}{I} \underset{2\times 1}{\underline{x}} = \underline{0}$$

$$(A - \lambda I)\underline{x} = \underline{0}$$

$$\boxed{B \underline{x} = \underline{0}}$$

homogenous Eqn

If $\boxed{B^{-1}}$ exists

$$\iff B^{-1}B\underline{x} = B^{-1}\underline{0}$$

we want $x \neq 0$,

$$\iff \underline{x} = \underline{0}$$

"trivial sols"

B^{-1} does not exist

$\Downarrow$

$\det B = 0$

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \qquad \det(\underline{A - \lambda I}) = 0$$

$$\Rightarrow \det\begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$$

$$\Rightarrow \lambda^{2} - 4\lambda - 5 = 0$$

2 = dimension of A

$$\Rightarrow (\lambda-5)(\lambda+1) = 0$$

$$\lambda = 5, -1$$

$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ Find the eigenvalues and eigenvectors of A ?

$$(A - \lambda I) = \left(\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8$$

$$= (\lambda^2 - 4\lambda + 3) - 8$$

$$= \boxed{\lambda^2 - 4\lambda - 5}$$

$$= (\lambda - 5)(\lambda + 1)$$

characteristic Polynomial of order 2

$$\det(A - \lambda I) = 0 \iff (\lambda - 5)(\lambda + 1) = 0$$

The eigenvalues of A are $-1, 5$

$\boxed{\text{spectrum}}$

roots of characteristic Eqn

When $\lambda = -1$.

$$A\underline{x} = \lambda \underline{x}$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\left(\underbrace{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

1 Eqn & 2 unknown

1 parameter

$$2x_1 + 4x_2 = 0$$

$$\Rightarrow \quad x_1 = -2x_2$$

choose $x_2 = k$

$$x_1 = -2k$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k\begin{bmatrix} -2 \\ 1 \end{bmatrix} \Rightarrow$$

normalized vector

$$\frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\sqrt{(-2)^2 + 1^2} = \sqrt{5}$$

$k = 1$, $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $k = 2$, $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$

When $\lambda = 5$

$$A\underline{x} = \lambda \underline{x} \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$(A - \lambda I)\underline{x} = 0 \iff \left(\underbrace{\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

rank = 1

$$\iff \left[\begin{array}{cc|c} -4 & 4 & 0 \\ 2 & -2 & 0 \end{array}\right] \iff \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\iff x_1 = x_2 = k \iff \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad k \neq 0$$

One of the eigenvector is $\overset{k=1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ or $\overset{k=\frac{1}{\sqrt{2}}}{\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}}$ ← normalized vector (unit vector) for $\lambda = 5$.

Can we choose eigenvectors / eigenvalues. arbitrary ?

$A\underline{x} = \lambda \underline{x}$?

NO !

eg:
$$\begin{bmatrix} 1 & 4 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \end{bmatrix} \overset{?}{=} \lambda \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} \overset{?}{=} \lambda \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

we cannot find λ to make L.H.S = R.H.S.

No λ exists for this!

Q1: What are the relationship between a given Matrix A. and its eigenvalue ?

Q2: How can we find λ and $\underline{x}$ for given matrix A ? such that $A\underline{x} = \lambda \underline{x}$?

$$\overset{2\times 2}{A}\,\underline{x} = \lambda \underline{x} \qquad \underline{x} \neq 0$$

When $\lambda = 5$

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \overset{\underline{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = 5 \overset{\underline{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

$$(A - \lambda I)\underline{x} = \underline{0} \Rightarrow \left(\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left(\begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -4 & 4 & 0 \\ 2 & -2 & 0 \end{array}\right] \underset{r_1/2 + r_2}{\overset{(r_1/-4)}{\Longleftrightarrow}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Longleftrightarrow x_1 - x_2 = 0$$

$\boxed{\text{rank} = 1} \Rightarrow$ 1 Eqn, 2 unknowns, infinite solutions

$x_2 = k$

$x_1 = k$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ k \neq 0;$$

magnitude is $\sqrt{2}$

One of the eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ ← unit vector

normalized eigenvector vector for $\lambda = 5$

$$(\lambda_i, \underline{x}_i) = \left(5, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}\right)$$

choose $x_2 = k \Rightarrow x_1 = k$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Verify $\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

L.H.S = R.H.S

normalized Vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ← normalized Eigen vector for $\lambda = 5$

$$A = \begin{pmatrix} 5 & 3 \\ -3 & -1 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 3 \\ -3 & -1-\lambda \end{vmatrix} = (\lambda-2)^2$$

$\lambda = 2$ is the only eigenvalue (repeated)

$$(A - \lambda I)\,\underline{x} = \underline{0}$$

$$\begin{pmatrix} 3 & 3 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad k \neq 0$$

One eigenvalue corresponds to one ~~linearly independent~~ eigenvector.

Note: But sometimes one eigenvalue may ~~correspond~~ lead to more than one linearly independent eigenvectors. (see later)

If P has an inverse, P^{-1} exists

$$APP^{-1} = PDP^{-1} \iff \boxed{A = PDP^{-1}}$$

$$P^{-1}AP = P^{-1}PD \iff \boxed{P^{-1}AP = D}$$ Diagonalizat.

P^{-1} exists $\iff \det(P) \neq 0$

$\iff \underline{x}_1, \underline{x}_2, \underline{x}_3$ Linearly independent

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Q1 Is $\det(P^{-1}AP) = \det(A)$?

$$\det(P^{-1})\det(A)\det(P)$$

$$\frac{1}{\cancel{\det(P)}}\det(A)\cancel{\det(P)} = \det(A)$$

$$\det(P^{-1}AP) = \det(D) = \det\left(\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix}\right) = \lambda_1\lambda_2\lambda_3$$

$$= \det(A)$$

Q2. What happens if $\underline{x}_1, \underline{x}_2$ are interchange in P ?

$$A[\underline{x}_2, \underline{x}_1, \underline{x}_3] = [A\underline{x}_2, A\underline{x}_1, A\underline{x}_3] = [\underline{x}_3, \underline{x}_2, \underline{x}_1]\begin{bmatrix} \lambda_2 & & 0 \\ & \lambda_1 & \\ 0 & & \lambda_3 \end{bmatrix}$$

Q3 How to Compute A^m ?

$$P^{-1}AP = D$$

$$A = PDP^{-1}$$

$$A^m = \underbrace{(PD\cancel{P^{-1}})(\cancel{P}DP^{-1}) \cdots (\cancel{P}DP^{-1})}_{m}$$

$$= P\underbrace{DDD\cdots D}_{m}P^{-1}$$

$$= P D^m P^{-1} = P\begin{bmatrix} \lambda_1^m & & & 0 \\ & \lambda_2^m & & \\ & & \ddots & \\ 0 & & & \lambda_n^m \end{bmatrix}P^{-1}$$

Suppose $\overset{3\times 3}{A}$ has 3 eigenvalues $\lambda_1, \lambda_2, \lambda_3$

with corresponding eigenvectors $\underline{x}_1, \underline{x}_2, \underline{x}_3$

$$A\underline{x}_1 = \lambda_1 \underline{x}_1, \quad A\underline{x}_2 = \lambda_2 \underline{x}_2, \quad A\underline{x}_3 = \lambda_3 \underline{x}_3$$

$$\overset{3\times 3}{P} = [\overset{3\times 1}{\underline{x}_1}, \overset{3\times 1}{\underline{x}_2}, \overset{3\times 1}{\underline{x}_3}] \text{ in a column form}$$

$$AP = \overset{3\times 3}{A}[\underline{x}_1, \underline{x}_2, \underline{x}_3] = [A\underline{x}_1, A\underline{x}_2, A\underline{x}_3]$$

$$= [\lambda_1 \underline{x}_1, \lambda_2 \underline{x}_2, \lambda_3 \underline{x}_3]$$

$$\parallel$$

$$= [\underline{x}_1, \underline{x}_2, \underline{x}_3] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = PD$$

diagonal matrix with $\lambda_1, \lambda_3, \lambda_3$

** $\boxed{AP = PD}$ **

If P has an inverse, P^{-1} exists

$$APP^{-1} = PDP^{-1} \iff \boxed{A = PDP^{-1}}$$

$$P^{-1}AP = P^{-1}PD \iff \boxed{P^{-1}AP = D}$$

Remember ~~Remember~~ **Diagonalization**

P^{-1} exists $\iff \det(P) \neq 0$

$\iff$ all vector columns in P are linearly independent

$\iff$ $\underline{x}_1, \underline{x}_2, \underline{x}_3$ linearly independent

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Q1 : Is $\det(P^{-1}AP) = \det(A)$?

$$\det(P^{-1})\det(A)\det(P)$$

$$\frac{1}{\cancel{\det(P)}} \times \det(A)\cancel{\det(P)} = \det(A)$$

$$\det(P^{-1}AP) = \det(D) \iff \det(A) = \det(D) = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{bmatrix}$$

$$= \lambda_1\lambda_2\lambda_3$$

Q2: What if we interchange $\underline{x}_1, \underline{x}_2$ in P?

$$A\,\overset{P}{[\underline{x}_2, \underline{x}_1, \underline{x}_3]} = [\lambda_2 \underline{x}_2, \lambda_1 \underline{x}_1, \lambda_3 \underline{x}_3]$$

$$= \underbrace{[\underline{x}_2, \underline{x}_1, \underline{x}_3]}_{P} \begin{bmatrix} \lambda_2 & & 0 \\ & \lambda_1 & \\ 0 & & \lambda_3 \end{bmatrix}$$

Q3: How to compute A^m?

$$A^m = \overbrace{(PD\cancel{P^{-1}})(\cancel{P}DP^{-1})\cdots(\cancel{P}DP^{-1})}^{m}$$

$$= P\underbrace{DDD\cdots D}_{m}P^{-1} = PD^mP^{-1}$$

$$= P \begin{bmatrix} \lambda_1^m & & 0 \\ & \lambda_2^m & \\ 0 & & \lambda_3^m \end{bmatrix} P^{-1}$$

Diagonalization

$$\overset{n\times n}{A}\ \overset{n\times 1}{\underline{x}} = \lambda\, \underline{x}$$

eigenvalue (scalar

unknown eigenvector

Suppose. $\overset{3\times 3}{A}$ has 3 eigenvalues

$$\lambda_1\ ,\ \lambda_2\ ,\ \lambda_3$$

corresponding $\underline{x}_1\ ,\ \underline{x}_2\ ,\ \underline{x}_3$

$$A\,\underline{x}_1 = \lambda_1 \underline{x}_1\ ,\quad A\,\underline{x}_2 = \lambda_2 \underline{x}_2\ ,\quad A\,\underline{x}_3 = \lambda_3 \underline{x}_3$$

$P = [\ \underline{x}_1\ ,\ \underline{x}_2\ ,\ \underline{x}_3\ ]$ in a column form

$$\begin{aligned}
AP &= \overset{3\times 3}{A}\,[\overset{3\times 1}{\underline{x}_1}, \underline{x}_2, \underline{x}_3] = [A\underline{x}_1\ , A\underline{x}_2\ , A\underline{x}_3] \\
&= [\lambda_1 \underline{x}_1\ , \lambda_2 \underline{x}_2\ , \lambda_3 \underline{x}_3] = [\overset{3\times 1}{\underline{x}_1}\ , \overset{3\times 1}{\underline{x}_2}\ , \overset{3\times 1}{\underline{x}_3}]\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix} \\
&= PD
\end{aligned}$$

$D \leftarrow$ diagonal Matrix

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$$\boxed{AP = PD}$$

Compute A^{100} ?

$$AP = PD$$

$$P^{-1}AP = D$$

or

$$A = PDP^{-1}$$

$$A^{100} = (PDP^{-1})^{100}$$

$$= (PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\cancel{(}\cdots\cancel{)}(\cancel{P}DP^{-1})$$

$$= PDD\cdots DP^{-1} = PD^{100}P^{-1}$$

$$= P\begin{bmatrix} \lambda_1^{100} & & 0 \\ & \lambda_2^{100} & \\ 0 & & \ddots \\ & & & \lambda_n^{100} \end{bmatrix} P^{-1}$$

Given A, find P, P^{-1}, D $\quad = A^{100}$