Question 1:

$$D_1 = D \cup \{x_1\}$$
$$D_2 = D \cup \{x_2\}$$

$$\frac{\Pr(y_1 = y|D_1)}{\Pr(y_1 = y|D_2)} = \frac{\Pr\left(y_{1_1}|x_1\right) \cdots \Pr\left(y_{1_i}|x_i\right) \cdots \Pr\left(y_{1_n}|x_n\right)}{\Pr\left(y_{1_1}|x_2\right) \cdots \Pr\left(y_{1_i}|x_i\right) \cdots \Pr\left(y_{1_n}|x_n\right)}$$
$$= \frac{\Pr\left(y_{1_1}|x_1\right)}{\Pr\left(y_{1_1}|x_2\right)}$$

$$\frac{\Pr(y_2 = y|D_2)}{\Pr(y_2 = y|D_1)} = \frac{\Pr\left(y_{2_1}|x_2\right) \cdots \Pr\left(y_{2_i}|x_i\right) \cdots \Pr\left(y_{2_n}|x_n\right)}{\Pr\left(y_{2_1}|x_1\right) \cdots \Pr\left(y_{2_i}|x_i\right) \cdots \Pr\left(y_{2_n}|x_n\right)}$$
$$= \frac{\Pr\left(y_{2_1}|x_2\right)}{\Pr\left(y_{2_1}|x_1\right)}$$

$$\frac{\Pr(y_1 = y|D_1)}{\Pr(y_1 = y|D_2)} \leq \exp(\varepsilon)$$
$$\frac{3/4}{1/4} \leq \exp(\varepsilon)$$
$$\varepsilon \geq \ln(3)$$

$$\alpha = \beta = 1 - \frac{1}{e} - \ln(3)$$
$$= -0.46649$$

$$\frac{\Pr(y_1 = y|D_1)}{\Pr(y_1 = y|D_2)} \times \frac{\Pr(y_2 = y|D_1)}{\Pr(y_2 = y|D_2)} \leq \exp(\varepsilon)\exp(-\varepsilon)$$
$$\frac{\Pr(y_1 = y|D_1)}{\Pr(y_2 = y|D_2)} \leq \frac{\Pr(y_1 = y|D_2)}{\Pr(y_2 = y|D_1)}$$
$$= \frac{\Pr\left(y_{1_1}|x_2\right) \cdots \Pr\left(y_{1_i}|x_i\right) \cdots \Pr\left(y_{1_n}|x_n\right)}{\Pr\left(y_{2_1}|x_1\right) \cdots \Pr\left(y_{2_i}|x_i\right) \cdots \Pr\left(y_{2_n}|x_n\right)}$$
$$= \frac{\Pr\left(y_{1_1}|x_2\right)}{\Pr\left(y_{2_1}|x_1\right)}$$

Question 2:

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \qquad \mathrm{A^T} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\mathrm{A^T}A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$
$$\lambda = 3 + \sqrt{3}, 3 - \sqrt{3}, 0$$

$$u_1 = \begin{bmatrix} \frac{1+\sqrt{3}}{2} \\ \frac{-1+\sqrt{3}}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 0.78867 \\ 0.21132 \\ 0.57735 \end{bmatrix} \qquad u_2 = \begin{bmatrix} -\frac{-1+\sqrt{3}}{2} \\ -\frac{1+\sqrt{3}}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} 0.21132 \\ 0.78867 \\ -0.57735 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.57735 \\ 0.57735 \\ 0.57735 \end{bmatrix}$$

$$\sigma_1 = \sqrt{3+\sqrt{3}} = 2.17533$$
$$\sigma_2 = \sqrt{3-\sqrt{3}} = 1.12603$$
$$\sigma_3 = 0$$

$$v_1 = \frac{1}{\sigma_1} A^T u_1 = \begin{bmatrix} 0.62796 \\ 0.62796 \\ 0 \\ 0.45970 \end{bmatrix} \qquad v_2 = \frac{1}{\sigma_2} A^T u_2 = \begin{bmatrix} 0.32506 \\ 0.32506 \\ 0 \\ -0.88806 \end{bmatrix}$$

$$null([v_1, v_2]) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

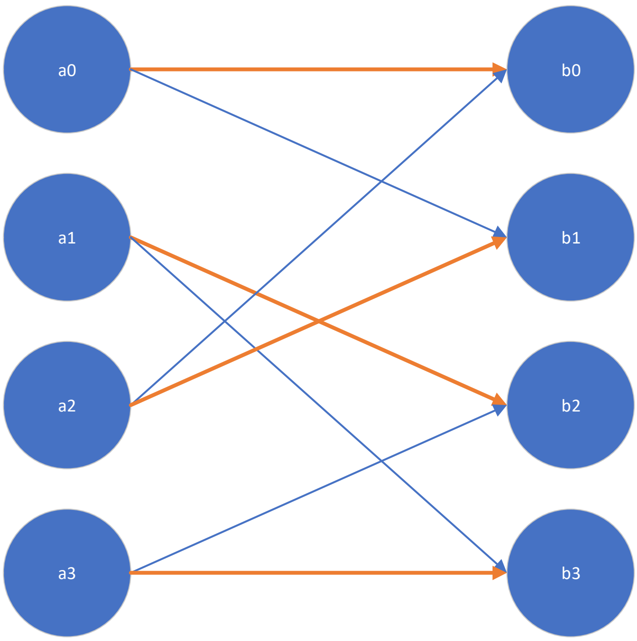
$$v_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.70711 \\ 0.70711 \\ 0 \\ 0 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

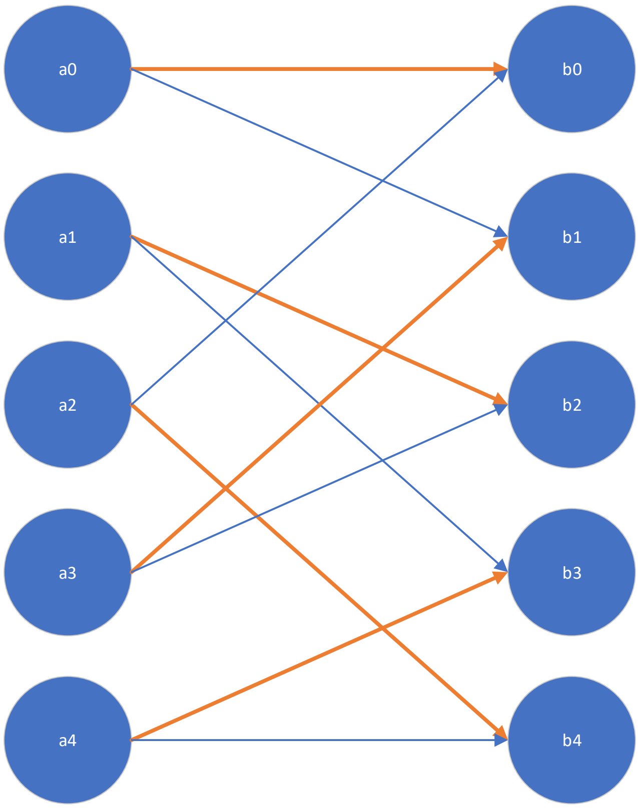
$$D = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \end{bmatrix} = \begin{bmatrix} 2.17533 & 0 & 0 & 0 \\ 0 & 1.12603 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$U = [u_1, u_2, u_3] = \begin{bmatrix} 0.78867 & 0.21132 & -0.57735 \\ 0.21132 & 0.78867 & 0.57735 \\ 0.57735 & -0.57735 & 0.57735 \end{bmatrix}$$

$$V = [v_1, v_2, v_3, v_4] = \begin{bmatrix} 0.62796 & 0.32506 & -0.70711 & 0 \\ 0.62796 & 0.32506 & 0.70711 & 0 \\ 0 & 0 & 0 & 1 \\ 0.45970 & -0.88806 & 0 & 0 \end{bmatrix}$$

Question 3:





Perfect matching for G_4:

$$M = \{(a_0, b_0), (a_1, b_2), (a_2, b_1), (a_3, b_3)\}$$

Perfect matching for G_5:

$$M = \{(a_0, b_0), (a_1, b_2), (a_2, b_4), (a_3, b_1), (a_4, b_3)\}$$

Question 4a:

The method I implemented in the code with the library use the SVD algorithm, with the prediction of rating is set as below.

$$r_{ui} = \mu + b_u + b_i + q_i^T p_u$$

Which r_{ui} is the predicted rating, u is the user, i is the item.

If the user u is unknown, then the bias b_u and the factors p_u are assumed to be zero. Same as item i with b_i and q_i.