MA 1201 Semester B 2019/20

Assignment 1 — Due at 5 pm, 27/2/2020 (Thursday) online on Canvas

Instructions:

- Please show your work. Unsupported answers will receive **NO** credits.
- Make sure you write down the correct lecture session (A/B/C/D/E/F/G/H) you have registered for, together with your full name and student ID on the front page of your answer script. Scan your solution into a single pdf file and upload it to Canvas.
- **NO** late homework will be accepted. Homework submitted to wrong tutorial sessions will **NOT** be graded and will receive **0 POINTS**.

Recall that the vector equation for a line L passing through a point $P_0(x_0,y_0,z_0)$ parallel to a vector $\vec{v} = \langle v_1, v_2, v_3\rangle$ is

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}, \qquad -\infty < t < \infty,$$

where $\vec{r}$ is the position vector of a point $P(x,y,z)$ on L and $\vec{r}_0$ is the position vector of $P_0(x_0,y_0,z_0)$. In component form, the vector equation is equivalent to three scalar equations:

$$\begin{cases} x = x_0 + tv_1 \\ y = y_0 + tv_2 \\ z = z_0 + tv_3 \end{cases}.$$

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1. (10 points) Find the volume of the tetrahedron with adjacent vortices $A = (1,2,0)$, $B = (-1,3,4)$, $C = (-1,-2,-3)$ and $D = (0,-1,3)$.

Solution. Note that

$$\vec{AB} = \vec{OB} - \vec{OA} = (-i+3j+4k) - (i+2j) = -2i + j + 4k$$
$$\vec{AC} = \vec{OC} - \vec{OA} = (-i-2j-3k) - (i+2j) = -2i - 4j - 3k$$
$$\vec{AD} = \vec{OD} - \vec{OA} = (-j+3k) - (i+2j) = -i - 3j + 3k$$

Then

$$\vec{AB} \times \vec{AC} = (-2i+j+4k) \times (-2i-4j-3k) = 13i - 14j + 10k$$

and

$$(\vec{AB} \times \vec{AC}) \cdot \vec{AD} = (13i - 14j + 10k) \cdot (-i - 3j + 3k) = 59.$$

So the volume of the tetrahedron is

$$\frac{1}{6}|(\vec{AB} \times \vec{AC}) \cdot \vec{AD}| = \frac{59}{6}.$$

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2. (10 points) Find the equation of the plane passing points $A = (3,1,1)$ and $B = (1,0,-1)$, and parallel with the line

$$\begin{cases} x = -t \\ y = 1 \\ z = 2t + 2. \end{cases}$$

Solution. Note that

$$\vec{AB} = \vec{OB} - \vec{OA} = (i-k) - (3i+j+k) = -2i - j - 2k.$$

Because the direction vector of the line is

$$\vec{v} = -i + 2k,$$

the normal of the plane is

$$\vec{n} = \vec{AB} \times \vec{v} = (-2i - j - 2k) \times (-i + 2k) = -2i + 6j - k.$$

Then for any point $P = (x,y,z)$ in the plane, $\vec{AP} \cdot \vec{n} = 0$. So the equation of the plane is

$$0 = ((x-3)i + (y-1)j + (z-1)k) \cdot (-2i + 6j - k) = -2(x-3) + 6(y-1) - (z-1).$$

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3. (15 points) Find the distance from the line

$$\begin{cases} x = t \\ y = -t + 1 \\ z = 2t + 1 \end{cases}$$

to the line, which is the intersection between plane $x + y - z = 1$ and plane $2x + z = 3$. (Hint: One can find the equation of the intersection line by solving the two equations of the planes together with the substitution $x = t$.)

Solution. Let $x = t$. Then it follows from equations $x + y - z = 1$ and plane $2x + z = 3$ that the equation of the intersecting line is $x = t$, $z = 3 - 2x = -2t + 3$ and $y = 1 - x + z = 1 - t + 3 - 2t = -3t + 4$, *i.e.*,

$$\begin{cases} x = t \\ y = -3t + 4 \\ z = -2t + 3. \end{cases}$$

Select two points A and B on the line

$$\begin{cases} x = t \\ y = -t + 1 \\ z = 2t + 1 \end{cases}$$

by setting $t = 0$ and $t = 1$ respectively, *i.e.*, $A = (0,1,1)$ and $B = (1,0,3)$.
Select two points C and D on the line

$$\begin{cases} x = t \\ y = -3t + 4 \\ z = -2t + 3. \end{cases}$$

by setting $t = 0$ and $t = 1$ respectively, *i.e.*, $C = (0,4,3)$ and $D = (1,1,1)$.
Then the distance between the two lines is $|Proj_{\vec{AB} \times \vec{CD}} \vec{AC}|$.
By the straightforward calculation, $\vec{AB} = i - j + 2k$ and $\vec{CD} = i - 3j - 2k$.

$$\vec{AB} \times \vec{CD} = (i - j + 2k) \times (i - 3j - 2k) = 8i + 4j - 2k.$$

So

$$|Proj_{\vec{AB}\times\vec{CD}}\vec{AC}| = ||\vec{AC}|\cos\theta| = \frac{|\vec{AB}\times\vec{CD}\cdot\vec{AC}|}{|\vec{AB}\times\vec{CD}|} = \frac{|(8i+4j-2k)\cdot(3j+2k)|}{|8i+4j-2k|} = \frac{8}{\sqrt{84}}(\text{or } = \frac{4}{\sqrt{21}}).$$

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4. (30 points) Evaluate the following indefinite integrals.

(a) (15 points) $\int \frac{2x-1}{x^2-2x+2}\,dx$.

Solution. Let $u = x^2-2x+2$. Then $du = (2x-2)dx$. So

$$\int \frac{2x-1}{x^2-2x+2}\,dx = \int \frac{2x-2}{x^2-2x+2}\,dx + \int \frac{1}{x^2-2x+2}\,dx = \int \frac{du}{u} + \int \frac{dx}{(x-1)^2+1}.$$

Then

$$\int \frac{2x-1}{x^2-2x+2}\,dx = \ln|u| + \tan^{-1}(x-1) + C = \ln|x^2-2x+2| + \tan^{-1}(x-1) + C.$$

(b) (15 points) $\int \frac{4x}{\sqrt{2x+4}}\,dx$.

Solution.

$$\int \frac{4x}{\sqrt{2x+4}}\,dx = \int \frac{2(2x+4)-8}{\sqrt{2x+4}}\,dx = 2\int (2x+4)^{1/2}\,dx - 8\int (2x+4)^{-1/2}\,dx$$

So

$$\int \frac{4x}{\sqrt{2x+4}}\,dx = \frac{2}{3}(2x+4)^{3/2} - 8(2x+4)^{1/2} + C.$$

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5. (25 points) Evaluate the following definite integrals.

(a) (10 points) $\int_0^{\pi} \cos(|x-\frac{\pi}{2}|+\frac{\pi}{2})\,dx$.

Solution.

$$|x-\frac{\pi}{2}|+\frac{\pi}{2} = \begin{cases} x-\frac{\pi}{2}+\frac{\pi}{2} = x & \text{when } x \geq \frac{\pi}{2} \\ \frac{\pi}{2}-x+\frac{\pi}{2} = \pi - x & \text{when } x < \frac{\pi}{2} \end{cases}$$

So

$$\int_0^{\pi} \cos(|x-\frac{\pi}{2}|+\frac{\pi}{2})\,dx = \int_0^{\pi/2} \cos(\pi-x)\,dx + \int_{\pi/2}^{\pi} \cos x\,dx = -\sin(\pi-x)\Big|_0^{\pi/2} + \sin x\Big|_{\pi/2}^{\pi} = -2.$$

(b) (15 points) $\int_{-\pi/3}^{\pi/3} (|x|+\sin x)^2\,dx$. (Hint: Expand the square, split the integral into three pieces, and exploit the even/odd symmetry properties of the integrand. In particular, no integration by parts is needed.)

Solution.

$$\int_{-\pi/3}^{\pi/3}(|x|+\sin x)^2\,dx = \int_{-\pi/3}^{\pi/3}x^2\,dx + \int_{-\pi/3}^{\pi/3}2|x|\sin x\,dx + \int_{-\pi/3}^{\pi/3}\sin^2 x\,dx.$$

Because $|x|\sin x$ is an odd function, $\int_{-\pi/3}^{\pi/3}|x|\sin x\,dx = 0$. So

$$\int_{-\pi/3}^{\pi/3}(|x|+\sin x)^2\,dx = \int_{-\pi/3}^{\pi/3}x^2\,dx + \int_{-\pi/3}^{\pi/3}\sin^2 x\,dx = \int_{-\pi/3}^{\pi/3}x^2\,dx + \int_{-\pi/3}^{\pi/3}\frac{1-\cos 2x}{2}\,dx.$$

Then

$$\int_{-\pi/3}^{\pi/3}(|x|+\sin x)^2\,dx = \frac{1}{3}x^3\Big|_{-\pi/3}^{\pi/3} + \left(\frac{1}{2}x - \frac{1}{4}\sin(2x)\right)\Big|_{-\pi/3}^{\pi/3} = \frac{2\pi^3}{81} + \frac{\pi}{3} - \frac{\sqrt{3}}{4}.$$

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6. (10 points) Compute the derivative $\dfrac{d}{dx}\left(\displaystyle\int_{x^2}^{3x}\frac{y^3}{\sqrt{2y+1}}\,dy\right)$.

Solution. Let $F(y) = \int \frac{y^3}{\sqrt{2y+1}}\,dy$. Then $\frac{dF(y)}{dy} = \frac{y^3}{\sqrt{2y+1}}$. So

$$\frac{d}{dx}\left(\int_{x^2}^{3x}\frac{y^3}{\sqrt{2y+1}}\,dy\right) = \frac{d}{dx}(F(3x)-F(x^2)) = \frac{dF(3x)}{d(3x)}\frac{d(3x)}{dx} - \frac{dF(x^2)}{d(x^2)}\frac{d(x^2)}{dx}.$$

Then

$$\frac{d}{dx}\left(\int_{x^2}^{3x}\frac{y^3}{\sqrt{2y+1}}\,dy\right) = \frac{81x^3}{\sqrt{6x+1}} - \frac{2x^7}{\sqrt{2x^2+1}}.$$

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