MA 1201 Semester B 2018/19

Midterm Exam (100 mins)

Instructions:

- Please show your work. Unsupported answers will receive **NO** credits.
- Make sure you write down the correct lecture session (A/B/C/D/E/F/G/H) you have registered for, together with your full name and student ID on the front page of your answer script.
- Exams submitted to wrong lecture sessions will **NOT** be graded and will receive **0 POINTS**.

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1. (25 points) Let $A(-1,0,2),\ B(3,-2,0),\ C(0,1,-3)$, and $D(1,-2,3)$ be four points in $\mathbb{R}^3$. Using vector method:

(a) (8 points) Find the volume of the parallelepiped with adjacent edges AB, AC, and AD.

Solution. The volume of the parallelepiped is given by

$$V = |\overrightarrow{AB}\cdot(\overrightarrow{AC}\times\overrightarrow{AD})|,$$

where

$$\begin{aligned}\overrightarrow{AB} &= B-A = \langle 3-(-1),-2-0,0-2\rangle = \langle 4,-2,-2\rangle,\\ \overrightarrow{AC} &= C-A = \langle 0-(-1),1-0,-3-2\rangle = \langle 1,1,-5\rangle,\\ \overrightarrow{AD} &= D-A = \langle 1-(-1),-2-0,3-2\rangle = \langle 2,-2,1\rangle.\end{aligned}$$

Since (see page 4 of the review sheet for the determinant formula for computing triple scalar products)

$$\begin{aligned}\overrightarrow{AB}\cdot(\overrightarrow{AC}\times\overrightarrow{AD}) &= \begin{vmatrix}4 & -2 & -2\\ 1 & 1 & -5\\ 2 & -2 & 1\end{vmatrix} = 4\begin{vmatrix}1 & -5\\ -2 & 1\end{vmatrix} - (-2)\begin{vmatrix}1 & -5\\ 2 & 1\end{vmatrix} + (-2)\begin{vmatrix}1 & 1\\ 2 & -2\end{vmatrix}\\ &= 4\big[1\cdot 1-(-2)\cdot(-5)\big]+2\big[1\cdot 1-2\cdot(-5)\big]-2\big[1\cdot(-2)-2\cdot 1\big] = -6,\end{aligned}$$

it follows that

$$V = |-6| = 6.$$

(b) (9 points) Find the equation of the plane that contains A, B, and C. Verify that D does *not* lie on the plane.

Solution. The equation of the plane is given by

$$n_1(x-x_0)+n_2(y-y_0)+n_3(z-z_0)=0,$$

where $\vec{n}=\langle n_1,n_2,n_3\rangle$ is a normal vector of the plane and $P(x_0,y_0,z_0)$ is a point on the plane. Since

$$\begin{aligned}\vec{n} = \overrightarrow{AB}\times\overrightarrow{AC} &= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & -2 & -2\\ 1 & 1 & -5\end{vmatrix} = \vec{i}\begin{vmatrix}-2 & -2\\ 1 & -5\end{vmatrix} - \vec{j}\begin{vmatrix}4 & -2\\ 1 & -5\end{vmatrix} + \vec{k}\begin{vmatrix}4 & -2\\ 1 & 1\end{vmatrix}\\ &= \big[(-2)\cdot(-5)-1\cdot(-2)\big]\vec{i}-\big[4\cdot(-5)-1\cdot(-2)\big]\vec{j}+\big[4\cdot 1-1\cdot(-2)\big]\vec{k} = \langle 12,18,6\rangle = 6\langle 2,3,1\rangle,\end{aligned}$$

and $P(x_0, y_0, z_0) = A(-1, 0, 2)$ is a point on the plane, the equation takes the form

$$2\big[x-(-1)\big] + 3(y-0) + 1(z-2) = 0, \qquad \text{or} \qquad 2x+3y+z=0.$$

Note that $D(1,-2,3)$ does not lie on the plane, since

$$2\cdot 1 + 3\cdot(-2) + 3 = -1 \neq 0.$$

(c) (8 points) Find the distance from D to the plane containing A, B, and C.

Solution. The distance between D and the plane is given by $|\mathrm{proj}_{\vec{n}}\overrightarrow{AD}|$, the magnitude of the (orthogonal) projection of the vector

$$\overrightarrow{AD} = \langle 2, -2, 1\rangle$$

onto the normal vector $\vec{n} = \langle 2, 3, 1\rangle$:

$$d = |\mathrm{proj}_{\vec{n}}\overrightarrow{AD}| = \left|\frac{\overrightarrow{AD}\cdot\vec{n}}{|\vec{n}|}\right| = \left|\frac{\langle 2,-2,1\rangle\cdot\langle 2,3,1\rangle}{|\langle 2,3,1\rangle|}\right| = \left|\frac{2\cdot 2 + (-2)\cdot 3 + 1\cdot 1}{\sqrt{2^2+3^2+1^2}}\right| = \left|\frac{-1}{\sqrt{14}}\right| = \frac{1}{\sqrt{14}}.$$

Note that projecting any one of the vectors $\overrightarrow{AD}$, $\overrightarrow{BD}$, or $\overrightarrow{CD}$ onto $\vec{n}$ yields the same result.

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2. (50 points) Evaluate the following integrals.

(a) (7 points) $\displaystyle\int (2x+5)^{1/3}\,dx$.

Solution. Let

$$\begin{aligned} u &= 2x+5, \\ du &= (2x+5)'\,dx = 2\,dx, \qquad \text{or} \qquad dx = \frac{1}{2}\,du. \end{aligned}$$

Then

$$\int (2x+5)^{1/3}\,dx = \int u^{1/3}\cdot\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int u^{1/3}\,du = \frac{1}{2}\cdot\frac{3}{4}u^{4/3} + C = \frac{3}{8}(2x+5)^{4/3} + C.$$

(b) (8 points) $\displaystyle\int_0^{2\pi} \sin|x-\pi|\,dx$.

Motivation. Since, by definition,

$$|x-\pi| = \begin{cases} x-\pi, & \text{if } x-\pi \geq 0 \\ -(x-\pi), & \text{if } x-\pi < 0 \end{cases} = \begin{cases} x-\pi, & \text{if } x \geq \pi \\ \pi - x, & \text{if } x < \pi \end{cases},$$

it's necessary to partition the interval of integration $[0, 2\pi]$ at $x = \pi$ into two parts, $[0,\pi]$ and $[\pi, 2\pi]$.

Solution.

$$\begin{aligned} \int_0^{2\pi}\sin|x-\pi|\,dx &= \int_0^{\pi}\sin|x-\pi|\,dx + \int_{\pi}^{2\pi}\sin|x-\pi|\,dx = \int_0^{\pi}\sin(\pi-x)\,dx + \int_{\pi}^{2\pi}\sin(x-\pi)\,dx \\ &= -\big[-\cos(\pi-x)\big]\Big|_0^{\pi} + \big[-\cos(x-\pi)\big]\Big|_{\pi}^{2\pi} = \big[\cos 0 - \cos\pi\big] - \big[\cos\pi - \cos 0\big] = 4. \end{aligned}$$

(c) (10 points) $\int \cos(\ln x)\,dx$.

Motivation. Since the integrand contains a composite function $\cos(\ln x)$, which is difficult to work with, it is a good idea to simplify it by making a substitution $w = \ln x$.

Solution. Let

$$w = \ln x, \qquad \text{or} \qquad x = \mathrm{e}^w,$$
$$dw = (\ln x)'\,dx = \frac{1}{x}\,dx, \qquad \text{or} \qquad dx = x\,dw = \mathrm{e}^w\,dw.$$

Then

$$\int \cos(\ln x)\,dx = \int \cos w \cdot (\mathrm{e}^w\,dw) = \int \cos w \cdot \mathrm{e}^w\,dw.$$

To proceed, apply integration by parts with

$$u = \cos w, \qquad dv = \mathrm{e}^w\,dw.$$

Direct integration yields

$$v = \int dv = \int \mathrm{e}^w\,dw = \mathrm{e}^w,$$

so

$$\int \cos w \cdot \mathrm{e}^w\,dw = \int \cos w\,d(\mathrm{e}^w) = \cos w \cdot \mathrm{e}^w - \int \mathrm{e}^w\,d(\cos w) = \cos w \cdot \mathrm{e}^w + \int \sin w \cdot \mathrm{e}^w\,dw.$$

For the integral on the right side, apply integration by parts one more time with

$$u = \sin w, \qquad dv = \mathrm{e}^w\,dw,$$
$$v = \int dv = \int \mathrm{e}^w\,dw = \mathrm{e}^w,$$

to deduce

$$\int \sin w \cdot \mathrm{e}^w\,dw = \int \sin w\,d(\mathrm{e}^w) = \sin w \cdot \mathrm{e}^w - \int \mathrm{e}^w\,d(\sin w) = \sin w \cdot \mathrm{e}^w - \int \cos w \cdot \mathrm{e}^w\,dw.$$

Thus

$$\int \cos w \cdot \mathrm{e}^w\,dw = \cos w \cdot \mathrm{e}^w + \Big(\sin w \cdot \mathrm{e}^w - \int \cos w \cdot \mathrm{e}^w\,dw\Big),$$

from which it follows that

$$2\int \cos w \cdot \mathrm{e}^w\,dw = \cos w \cdot \mathrm{e}^w + \sin w \cdot \mathrm{e}^w,$$

or

$$\int \cos w \cdot \mathrm{e}^w\,dw = \frac{1}{2}(\cos w + \sin w)\mathrm{e}^w + C = \frac{1}{2}\big[\cos(\ln x) + \sin(\ln x)\big]x + C.$$

(d) (10 points) $\int \frac{1}{(4+x^2)^{3/2}}\,dx$.

Motivation. Since the integrand contains a square root function $(4+x^2)^{3/2}$, but no extra factor of x (= a constant multiple of $(4+x^2)'$), the simple substitution $u = 4+x^2$ does not work. Thus it is a good idea to try a trig substitution instead.

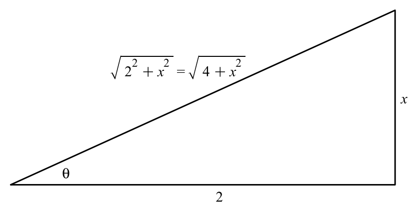
Solution. The Pythagorean identity $1+\tan^2\theta = \sec^2\theta$, or equivalently $4+4\tan^2\theta = 4\sec^2\theta$, motivates the (trig) substitution

$$x^2 = 4\tan^2\theta, \quad \text{or} \quad x = 2\tan\theta, \qquad \text{so that} \qquad (4+x^2)^{3/2} = (4\sec^2\theta)^{3/2} = 8\sec^3\theta,$$
$$dx = (2\tan\theta)'\,d\theta = 2\sec^2\theta\,d\theta.$$

It follows that

$$\int \frac{1}{(4+x^2)^{3/2}}\,dx = \int \frac{1}{8\sec^3\theta}\cdot(2\sec^2\theta\,d\theta) = \frac{1}{4}\int \frac{1}{\sec\theta}\,d\theta$$
$$= \frac{1}{4}\int \cos\theta\,d\theta = \frac{1}{4}\sin\theta + C = \frac{1}{4}\cdot\frac{x}{\sqrt{4+x^2}} + C,$$

where in the last step the following right triangle has been used to find $\sin\theta$. This triangle is con-



structed based on the observation that

$$x = 2\tan\theta, \qquad \text{or} \qquad \tan\theta = \frac{x}{2} = \frac{\text{opp}}{\text{adj}},$$

and its hypothenuse is determined using the Pythagorean theorem.

(e) (15 points) $\displaystyle\int \frac{-7x+29}{(x+1)(x^2-4x+13)}\,dx.$

Solution. The idea is to decompose the integrand, a rational function, into partial fractions. To begin with, note that

$$\deg(\text{numerator}) = 1 < \deg(\text{denominator}) = 1+2 = 3.$$

This shows that the rational function is proper, and thus a long division is not needed. Next, note that the denominator is already in a factored form, and the quadratic polynomial $(x^2-4x+13)$ is irreducible ($\Delta = (-4)^2 - 4\cdot 1\cdot 13 < 0$). Thus $(x+1)$ and $(x^2-4x+13)$ are the only factors of the denominator, and the partial fraction decomposition of the rational function is given by

$$\frac{-7x+29}{(x+1)(x^2-4x+13)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-4x+13}.$$

To find the constants A, B, and C, multiply both sides of the equation by the denominator $(x+1)(x^2-4x+13)$ and rearrange. This yields

$$-7x+29 = A(x^2-4x+13) + (Bx+C)(x+1).$$

Setting $x=-1$, the zero of the linear factor $(x+1)$, on both sides of the equation yields

$$-7\cdot(-1)+29 = A\big[(-1)^2 - 4\cdot(-1) + 13\big],$$

which implies

$$36 = 18A, \qquad \text{or} \qquad A = 2.$$

To find B and C, one could substitute other convenient values of x (e.g. $x = 0$ and $x = 1$) into the equation, but it's better here to expand the right side of the equation, combine like powers of x, and then compare coefficients:

$$\begin{aligned} -7x+29 &= A(x^2-4x+13)+(Bx+C)(x+1) \\ &= (Ax^2-4Ax+13A)+(Bx^2+Bx+Cx+C) \\ &= (A+B)x^2+(-4A+B+C)x+(13A+C). \end{aligned}$$

This yields the linear system

$$\begin{aligned} A+B \qquad &= 0 \\ -4A+B+C &= -7 \; , \\ 13A \qquad +C &= 29 \end{aligned}$$

where the first and third equation immediately implies (recall $A = 2$)

$$B = -A = -2, \qquad C = 29-13A = 3.$$

It follows that

$$\frac{-7x+29}{(x+1)(x^2-4x+13)} = \frac{2}{x+1}+\frac{-2x+3}{x^2-4x+13},$$

and thus

$$\int \frac{-7x+29}{(x+1)(x^2-4x+13)}\,dx = \int \frac{2}{x+1}\,dx + \int \frac{-2x+3}{x^2-4x+13}\,dx = 2\ln|x+1| + \int \frac{-2x+3}{x^2-4x+13}\,dx.$$

To evaluate the last integral on the right side, observe that the substitution

$$\begin{aligned} u &= x^2-4x+13, \\ du &= (x^2-4x+13)'\,dx = (2x-4)\,dx = 2(x-2)\,dx, \end{aligned}$$

indicates that an extra factor of $(x-2)$ (= a constant multiple of u') is needed to simplify the integral. This suggests the decomposition of the numerator:

$$-2x+3 = -2(x-2+2)+3 = -2(x-2)-1,$$

and hence

$$\begin{aligned} \int \frac{-2x+3}{x^2-4x+13}\,dx &= \int \left(\frac{-2(x-2)}{x^2-4x+13} - \frac{1}{x^2-4x+13}\right)dx \\ &= -2\int \frac{x-2}{x^2-4x+13}\,dx - \int \frac{1}{x^2-4x+13}\,dx. \end{aligned}$$

The first integral on the right side can be directly solved using the substitution

$$\begin{aligned} u &= x^2-4x+13, \\ du &= 2(x-2)\,dx, \qquad \text{or} \qquad (x-2)\,dx = \frac{1}{2}\,du, \end{aligned}$$

which gives

$$-2\int \frac{x-2}{x^2-4x+13}\,dx = -2\int \frac{1}{x^2-4x+13}\cdot(x-2)\,dx$$
$$= -2\int \frac{1}{u}\cdot\left(\frac{1}{2}\,du\right) = -\int \frac{1}{u}\,du = -\ln|u| + C = -\ln|x^2-4x+13| + C.$$

As for the second integral, the fact that the quadratic polynomial $(x^2-4x+13)$ is irreducible implies that

$$\int \frac{1}{x^2-4x+13}\,dx = \int \frac{1}{(x^2-2\cdot x\cdot 2)+13}\,dx = \int \frac{1}{(x^2-2\cdot x\cdot 2+2^2-2^2)+13}\,dx$$
$$= \int \frac{1}{(x-2)^2+9}\,dx = \frac{1}{9}\int \frac{1}{(1/9)(x-2)^2+1}\,dx = \frac{1}{9}\cdot\frac{\tan^{-1}\left[(1/3)\,(x-2)\right]}{(1/3)} + C.$$

In conclusion,

$$\int \frac{-7x+29}{(x+1)(x^2-4x+13)}\,dx = 2\ln|x+1| - \ln|x^2-4x+13| - \frac{1}{3}\tan^{-1}\left[(1/3)\,(x-2)\right] + C.$$

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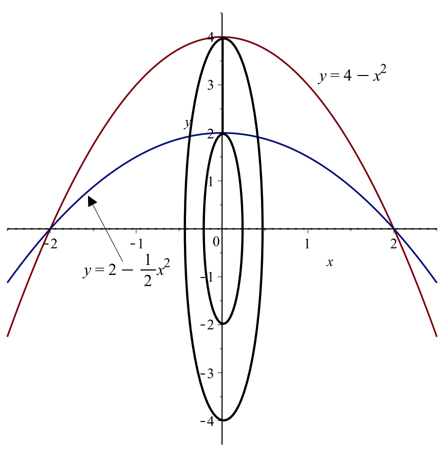
3. (25 points)

(a) (15 points) Find the volume of the solid generated by revolving the region bounded from above by the parabola $y = 4 - x^2$ and from below by the parabola $y = 2 - \frac{1}{2}x^2$ about the x-axis.

Solution. As shown in the following figure, the slice is chosen to be vertical (and hence the integra-



tion has to be carried out in x), since this avoids partitioning the interval of integration into multiple subintervals. Since the slice is perpendicular to the axis of rotation, the revolution of the slice generates a washer, which then suggests that the washer method has to be used. Now the inner and outer radii $r(x)$ and $R(x)$ of the washer, *expressed as functions of* x, are given by

$$r(x) = 2 - \frac{1}{2}x^2, \qquad R(x) = 4 - x^2.$$

The bounds of integration, on the other hand, are determined by equating $y = 4 - x^2$ to $y = 2 - \frac{1}{2}x^2$ and solving for x, which yields

$$4 - x^2 = 2 - \frac{1}{2}x^2 \Longrightarrow \frac{1}{2}x^2 = 2 \Longrightarrow x^2 = 4 \Longrightarrow x = \pm 2.$$

The volume of the solid is then given by

$$\begin{aligned} V &= \int_{-2}^{2} \pi \left[R^2(x) - r^2(x) \right] dx = \int_{-2}^{2} \pi \left\{ (4 - x^2)^2 - \left(2 - \frac{1}{2}x^2 \right)^2 \right\} dx = \pi \int_{-2}^{2} \left(\frac{3}{4}x^4 - 6x^2 + 12 \right) dx \\ &= \pi \cdot \left(\frac{3}{4} \cdot \frac{1}{5}x^5 - 6 \cdot \frac{1}{3}x^3 + 12x \right) \Big|_{-2}^{2} = \pi \cdot \left(\frac{48}{5} - 32 + 48 \right) = \frac{128}{5}\pi. \end{aligned}$$

(b) (10 points) Find the length of the curve $y = \ln(\cos x),\ 0 \le x \le \pi/3$.

Solution. Since the curve is described by a function of x, the curve length should be expressed as an integral of x:

$$L = \int ds = \int_0^{\pi/3} \sqrt{1 + \left[y'(x) \right]^2}\, dx,$$

where

$$1 + \left[y'(x) \right]^2 = 1 + \left(\frac{1}{\cos x} \cdot (-\sin x) \right)^2 = 1 + \tan^2 x = \sec^2 x.$$

It then follows that (note $\sec x > 0$ for $0 \le x \le \pi/3$)

$$L = \int_0^{\pi/3} |\sec x|\, dx = \int_0^{\pi/3} \sec x\, dx = \ln|\sec x + \tan x| \Big|_0^{\pi/3} = \ln|2 + \sqrt{3}| - \ln|1 + 0| = \ln(2 + \sqrt{3}).$$

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