ma2001a2 Differential Calculus of Multivariable Functions

1. Let $w = f\left(x,y,z\right) = z\left(x^2+y^2\right)^{-1}$. At $\left(2,1,1\right)$, find the rate of change of w with respect to *y*.
Suppose $x = u+v,\ y = u$ and $z = uv$, at $\left(1,1\right)$, find the rates of change of w with respect to *u* and *v*, respectively. Observe that $\frac{\partial f}{\partial y} \neq \frac{\partial f}{\partial u}$ even though $y = u$. Why is this so?

Solution:

$$w = f\left(x,y,z\right) = z\left(x^2+y^2\right)^{-1} \Rightarrow \frac{\partial w}{\partial y} = -z\left(x^2+y^2\right)^{-2}2y = -\frac{2yz}{\left(x^2+y^2\right)^2}$$

$$\Rightarrow \frac{\partial w}{\partial y}\left(2,1,1\right) = -\frac{2}{25}$$

$$\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} = -\frac{2xz}{\left(x^2+y^2\right)^2} - \frac{2yz}{\left(x^2+y^2\right)^2} + \frac{v}{x^2+y^2}$$

$$\begin{cases} u=1 \\ v=1 \end{cases} \Rightarrow \begin{cases} x=2 \\ y=1 \\ z=1 \end{cases} \Rightarrow \frac{\partial w}{\partial u}\left(1,1\right) = -\frac{2xz}{\left(x^2+y^2\right)^2}\Bigg|_{\substack{x=2\\y=1\\z=1}} - \frac{2yz}{\left(x^2+y^2\right)^2}\Bigg|_{\substack{x=2\\y=1\\z=1}} + \frac{v}{x^2+y^2}\Bigg|_{\substack{x=2\\y=1\\v=1}} = -\frac{4}{25} - \frac{2}{25} + \frac{1}{5} = -\frac{1}{25}$$

$$\frac{\partial w}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v} = -\frac{2xz}{\left(x^2+y^2\right)^2} + \frac{u}{x^2+y^2}$$

$$\frac{\partial w}{\partial v}\left(1,1\right) = -\frac{2xz}{\left(x^2+y^2\right)^2}\Bigg|_{\substack{x=2\\y=1\\z=1}} + \frac{u}{x^2+y^2}\Bigg|_{\substack{x=2\\y=1\\u=1}} = -\frac{4}{25} + \frac{1}{5} = \frac{1}{25}$$

$$\frac{\partial f}{\partial y} - \frac{\partial f}{\partial u} = \frac{2xz}{\left(x^2+y^2\right)^2} - \frac{v}{x^2+y^2} \underset{\text{in most cases}}{\neq} 0$$

$\frac{\partial f}{\partial u}$ is used to measure how *w* changes as *u* changes with *v* kept unchanged. We observe as *u* changes with *v* kept unchanged, all *x*, *y*, *z* will change although *y* will have the same level of change as *u* does and the change of x, y, z all three will affect the change of *w*.

However, $\frac{\partial f}{\partial y}$ is used to measure how *w* changes as *y* changes with *x*, *z* kept unchanged. In that situation only the change of y will affect the change of *w*.

2. Suppose *w* is a function of *u*, *v*, that is, $w = w\left(u,v\right)$. Suppose $u = x+y,\ v = x-y,\ w = xy-z$, therefore, z is a function of *x*, *y*. Transform the following partial differential equation of *z* in *x*, *y*, $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ into a partial differential equation of *w* in *u* and *v*.

Solution:

$$w = xy - z \Rightarrow z = xy - w \Rightarrow \frac{\partial z}{\partial x} = y - \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} - \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} = y - \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}$$

Similarly,

$$w = xy - z \Rightarrow z = xy - w \Rightarrow \frac{\partial z}{\partial y} = x - \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} - \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} = x - \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}.$$

Then

$$\frac{\partial z}{\partial x} = y - \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}$$
$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = -\left(\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v \partial u}\frac{\partial v}{\partial x}\right) - \left(\frac{\partial^2 w}{\partial u \partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial x}\right)$$
$$= -\frac{\partial^2 w}{\partial u^2} - \frac{\partial^2 w}{\partial v \partial u} - \frac{\partial^2 w}{\partial u \partial v} - \frac{\partial^2 w}{\partial v^2} = -\frac{\partial^2 w}{\partial u^2} - 2\frac{\partial^2 w}{\partial u \partial v} - \frac{\partial^2 w}{\partial v^2}$$

$$\frac{\partial z}{\partial y} = x - \frac{\partial w}{\partial u} + \frac{\partial w}{\partial v}$$
$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = -\left(\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v \partial u}\frac{\partial v}{\partial y}\right) + \left(\frac{\partial^2 w}{\partial u \partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial y}\right)$$
$$= -\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v \partial u} + \frac{\partial^2 w}{\partial u \partial v} - \frac{\partial^2 w}{\partial v^2} = -\frac{\partial^2 w}{\partial u^2} + 2\frac{\partial^2 w}{\partial u \partial v} - \frac{\partial^2 w}{\partial v^2}$$

Also

$$\frac{\partial z}{\partial x} = y - \frac{\partial w}{\partial u} - \frac{\partial w}{\partial v}$$
$$\Rightarrow \frac{\partial^2 z}{\partial y \partial x} = 1 - \left(\frac{\partial^2 w}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v \partial u}\frac{\partial v}{\partial y}\right) - \left(\frac{\partial^2 w}{\partial u \partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 w}{\partial v^2}\frac{\partial v}{\partial y}\right) = 1 - \frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v \partial u} - \frac{\partial^2 w}{\partial u \partial v} + \frac{\partial^2 w}{\partial v^2} = 1 - \frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2}$$

Finally

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0 \Rightarrow -\frac{\partial^2 w}{\partial u^2} - 2\frac{\partial^2 w}{\partial u \partial v} - \frac{\partial^2 w}{\partial v^2} + 2\left(1 - \frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2}\right) - \frac{\partial^2 w}{\partial u^2} + 2\frac{\partial^2 w}{\partial u \partial v} - \frac{\partial^2 w}{\partial v^2} = 0$$
$$\Rightarrow -4\frac{\partial^2 w}{\partial u^2} + 2 = 0 \Rightarrow \frac{\partial^2 w}{\partial u^2} = \frac{1}{2}$$

□

3. Let $z = f\left(x, \frac{x}{y}\right)$ and $s = x, t = \frac{x}{y}$. Assume f has continuous second order partial derivatives. Find $\frac{\partial z}{\partial x}$ and show that $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial s^2} + \frac{2}{y}\frac{\partial^2 f}{\partial s \partial t} + \frac{1}{y^2}\frac{\partial^2 f}{\partial t^2}$.

Solution:

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial s}\frac{ds}{dx} + \frac{\partial f}{\partial t}\frac{dt}{dx} = \frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}\frac{1}{y}.$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}\frac{1}{y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial s}\right) + \frac{1}{y}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial t}\right)$$

$$= \frac{\partial^2 f}{\partial s^2}\frac{ds}{dx} + \frac{\partial^2 f}{\partial s \partial t}\frac{\partial t}{\partial x} + \frac{1}{y}\left(\frac{\partial^2 f}{\partial s \partial t}\frac{ds}{dx} + \frac{\partial^2 f}{\partial t^2}\frac{dt}{dx}\right) = \frac{\partial^2 f}{\partial s^2} + \frac{2}{y}\frac{\partial^2 f}{\partial s \partial t} + \frac{1}{y^2}\frac{\partial^2 f}{\partial t^2}$$

4. Suppose $u = u(x, y)$ satisfies the wave equation $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$. Let $y = y(x) = 2x$. Along the line $y = 2x$, we have $u(x, y(x)) = x$ and $\left.\frac{\partial u}{\partial x}\right|_{\substack{x=x\\y=2x}} = x^2$. Find along the line $y = 2x$, $\left.\frac{\partial u}{\partial y}\right|_{\substack{x=x\\y=2x}}$, $\left.\frac{\partial^2 u}{\partial y \partial x}\right|_{\substack{x=x\\y=2x}}$, $\left.\frac{\partial^2 u}{\partial x^2}\right|_{\substack{x=x\\y=2x}}$, $\left.\frac{\partial^2 u}{\partial y^2}\right|_{\substack{x=x\\y=2x}}$.

Solution:

Let $g(x) = u(x, y(x)),\ y = y(x) = 2x$, then $\frac{dg}{dx} = \left.\frac{\partial u}{\partial x}\right|_{\substack{x=x\\y=2x}} + \left.\frac{\partial u}{\partial y}\right|_{\substack{x=x\\y=2x}}\frac{dy}{dx}$.

We have $g(x) = u(x, 2x) = x$, $\left.\frac{\partial u}{\partial x}\right|_{\substack{x=x\\y=2x}} = x^2$, so

$$\frac{dg}{dx} = \left.\frac{\partial u}{\partial x}\right|_{\substack{x=x\\y=2x}} + \left.\frac{\partial u}{\partial y}\right|_{\substack{x=x\\y=2x}}\frac{dy}{dx} \Rightarrow 1 = x^2 + 2\left.\frac{\partial u}{\partial y}\right|_{\substack{x=x\\y=2x}} \Rightarrow \left.\frac{\partial u}{\partial y}\right|_{\substack{x=x\\y=2x}} = \frac{1-x^2}{2}$$

Let $h(x) = \left.\frac{\partial u}{\partial y}\right|_{\substack{x=x\\y=2x}} = \frac{1-x^2}{2}$, then $\frac{dh(x)}{dx} = \left.\frac{\partial^2 u}{\partial x \partial y}\right|_{\substack{x=x\\y=2x}} + \left.\frac{\partial^2 u}{\partial y^2}\right|_{\substack{x=x\\y=2x}}\frac{dy}{dx}$.

Again,

$$\frac{dh(x)}{dx} = \left.\frac{\partial^2 u}{\partial x \partial y}\right|_{\substack{x=x\\y=2x}} + \left.\frac{\partial^2 u}{\partial y^2}\right|_{\substack{x=x\\y=2x}}\frac{dy}{dx} \Rightarrow -x = \left.\frac{\partial^2 u}{\partial x \partial y}\right|_{\substack{x=x\\y=2x}} + 2\left.\frac{\partial^2 u}{\partial y^2}\right|_{\substack{x=x\\y=2x}} \cdots\cdots(1)$$

In addition, let $k(x) = \left.\frac{\partial u}{\partial x}\right|_{\substack{x=x\\y=2x}} = x^2$, then

$$2x = \frac{dk(x)}{dx} = \left.\frac{\partial^2 u}{\partial x^2}\right|_{\substack{x=x\\y=2x}} + \left.\frac{\partial^2 u}{\partial x \partial y}\right|_{\substack{x=x\\y=2x}}\frac{dy}{dx} = \left.\frac{\partial^2 u}{\partial x^2}\right|_{\substack{x=x\\y=2x}} + 2\left.\frac{\partial^2 u}{\partial x \partial y}\right|_{\substack{x=x\\y=2x}} \Rightarrow 2x = \left.\frac{\partial^2 u}{\partial x^2}\right|_{\substack{x=x\\y=2x}} + 2\left.\frac{\partial^2 u}{\partial x \partial y}\right|_{\substack{x=x\\y=2x}} \cdots\cdots(2)$$

$(1) - 2\times(2)$ we have

$$-5x = -3\left.\frac{\partial^2 u}{\partial x \partial y}\right|_{\substack{x=x\\y=2x}} \Rightarrow \left.\frac{\partial^2 u}{\partial x \partial y}\right|_{\substack{x=x\\y=2x}} = \frac{5x}{3}.$$

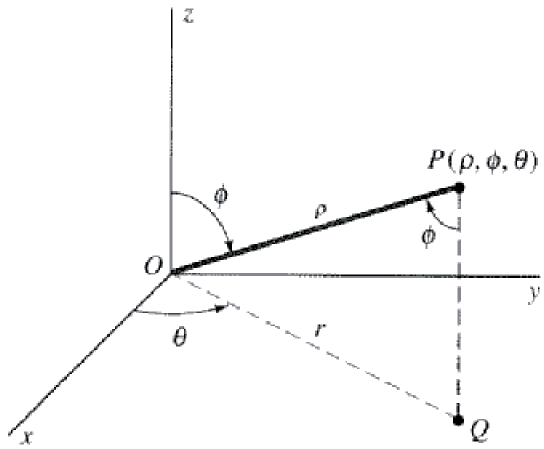
Thus, from (1) $-x = \frac{5x}{3} + 2\left.\frac{\partial^2 u}{\partial x^2}\right|_{\substack{x=x\\y=2x}} \Rightarrow \left.\frac{\partial^2 u}{\partial x^2}\right|_{\substack{x=x\\y=2x}} = \frac{-x - \frac{5x}{3}}{2} = -\frac{4x}{3}$.

Also, we have $\left.\dfrac{\partial^2 u}{\partial x^2}\right|_{\substack{x=x\\y=2x}} = \left.\dfrac{\partial^2 u}{\partial y^2}\right|_{\substack{x=x\\y=2x}} = -\dfrac{4x}{3}$

□

5. One of the most popular alternative coordinate systems to Cartesian coordinates is Spherical polar coordinates. Spherical polar coordinates represent a point $P(x, y, z)$ in space by ordered triples (ρ, θ, φ) in which
   1. ρ is the distance from *P* to the origin.
   2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive *z*-axis.
   3. θ is the angle measured counterclockwise from the positive *x*-axis to $\overrightarrow{OQ}$ which is the projection of $\overrightarrow{OP}$ on *xy*-plane.

   The equations relating spherical polar coordinates to Cartesian coordinates are:
   $x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$, where $\rho \ge 0,\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi$.



(i) Show that

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial V}{\partial \phi}\frac{\partial \phi}{\partial x} = \frac{\partial V}{\partial \rho}\sin\phi\cos\theta - \frac{\partial V}{\partial \theta}\frac{\sin\theta}{\rho\sin\phi} + \frac{\partial V}{\partial \phi}\frac{\cos\theta\cos\phi}{\rho},$$

and find also $\dfrac{\partial V}{\partial y}$, $\dfrac{\partial V}{\partial z}$.

(ii) (optional) Show that in spherical coordinates (ρ, θ, φ) Laplace's equation

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{takes the form}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\cot\phi}{\rho^2}\frac{\partial V}{\partial \phi} = 0.$$

Proof:

$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$. Then we have $x^2 + y^2 + z^2 = \rho^2$, so $\begin{cases} \dfrac{\partial \rho}{\partial x} = \sin\phi\cos\theta \\ \dfrac{\partial \rho}{\partial y} = \sin\phi\sin\theta \\ \dfrac{\partial \rho}{\partial z} = \cos\phi \end{cases}$

$$\frac{y}{x}=\tan\theta \Rightarrow \frac{y}{-x^2}=\sec^2\theta\frac{\partial\theta}{\partial x} \Rightarrow \frac{-\rho\sin\phi\sin\theta\cos^2\theta}{\rho^2\sin^2\phi\cos^2\theta}=\frac{\partial\theta}{\partial x} \Rightarrow \frac{\partial\theta}{\partial x}=-\frac{\sin\theta}{\rho\sin\phi}$$

$$\frac{y}{x}=\tan\theta \Rightarrow \frac{1}{x}=\sec^2\theta\frac{\partial\theta}{\partial y} \Rightarrow \frac{\partial\theta}{\partial y}=\frac{\cos\theta}{\rho\sin\phi} \text{ and } \frac{\partial\theta}{\partial z}=0$$

Thus, $\begin{cases} \dfrac{\partial\theta}{\partial x}=-\dfrac{\sin\theta}{\rho\sin\phi} \\ \dfrac{\partial\theta}{\partial y}=\dfrac{\cos\theta}{\rho\sin\phi} \\ \dfrac{\partial\theta}{\partial z}=0 \end{cases}$,

and

$$z=\rho\cos\phi \Rightarrow 0=-\rho\sin\phi\frac{\partial\phi}{\partial x}+\frac{\partial\rho}{\partial x}\cos\phi \Rightarrow \frac{\partial\phi}{\partial x}=\frac{\frac{\partial\rho}{\partial x}\cos\phi}{\rho\sin\phi}=\frac{\sin\phi\cos\theta\cos\phi}{\rho\sin\phi}=\frac{\cos\theta\cos\phi}{\rho}$$

$$z=\rho\cos\phi \Rightarrow 0=-\rho\sin\phi\frac{\partial\phi}{\partial y}+\frac{\partial\rho}{\partial y}\cos\phi \Rightarrow \frac{\partial\phi}{\partial y}=\frac{\frac{\partial\rho}{\partial y}\cos\phi}{\rho\sin\phi}=\frac{\sin\phi\sin\theta\cos\phi}{\rho\sin\phi}=\frac{\sin\theta\cos\phi}{\rho}$$

$$z=\rho\cos\phi \Rightarrow 1=-\rho\sin\phi\frac{\partial\phi}{\partial z}+\frac{\partial\rho}{\partial z}\cos\phi \Rightarrow \frac{\partial\phi}{\partial z}=\frac{\frac{\partial\rho}{\partial z}\cos\phi-1}{\rho\sin\phi}=\frac{\cos^2\phi-1}{\rho\sin\phi}=-\frac{\sin\phi}{\rho}$$

$$\begin{cases} \dfrac{\partial\phi}{\partial x}=\dfrac{\cos\theta\cos\phi}{\rho} \\ \dfrac{\partial\phi}{\partial y}=\dfrac{\sin\theta\cos\phi}{\rho} \\ \dfrac{\partial\phi}{\partial z}=-\dfrac{\sin\phi}{\rho} \end{cases}$$

$$\frac{\partial V}{\partial x}=\frac{\partial V}{\partial\rho}\frac{\partial\rho}{\partial x}+\frac{\partial V}{\partial\theta}\frac{\partial\theta}{\partial x}+\frac{\partial V}{\partial\phi}\frac{\partial\phi}{\partial x}=\frac{\partial V}{\partial\rho}\sin\phi\cos\theta-\frac{\partial V}{\partial\theta}\frac{\sin\theta}{\rho\sin\phi}+\frac{\partial V}{\partial\phi}\frac{\cos\theta\cos\phi}{\rho}$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial\left(\frac{\partial V}{\partial x}\right)}{\partial x} = \frac{\partial}{\partial \rho}\left(\frac{\partial V}{\partial x}\right)\sin\phi\cos\theta - \frac{\partial}{\partial \theta}\left(\frac{\partial V}{\partial x}\right)\frac{\sin\theta}{\rho\sin\phi} + \frac{\partial}{\partial \phi}\left(\frac{\partial V}{\partial x}\right)\frac{\cos\theta\cos\phi}{\rho}$$

$$= \frac{\partial}{\partial \rho}\left(\frac{\partial V}{\partial \rho}\sin\phi\cos\theta - \frac{\partial V}{\partial \theta}\frac{\sin\theta}{\rho\sin\phi} + \frac{\partial V}{\partial \phi}\frac{\cos\theta\cos\phi}{\rho}\right)\sin\phi\cos\theta$$

$$- \frac{\partial}{\partial \theta}\left(\frac{\partial V}{\partial \rho}\sin\phi\cos\theta - \frac{\partial V}{\partial \theta}\frac{\sin\theta}{\rho\sin\phi} + \frac{\partial V}{\partial \phi}\frac{\cos\theta\cos\phi}{\rho}\right)\frac{\sin\theta}{\rho\sin\phi}$$

$$+ \frac{\partial}{\partial \phi}\left(\frac{\partial V}{\partial \rho}\sin\phi\cos\theta - \frac{\partial V}{\partial \theta}\frac{\sin\theta}{\rho\sin\phi} + \frac{\partial V}{\partial \phi}\frac{\cos\theta\cos\phi}{\rho}\right)\frac{\cos\theta\cos\phi}{\rho}$$

$$= \frac{\partial^2 V}{\partial \rho^2}\sin^2\phi\cos^2\theta - \frac{\partial^2 V}{\partial \theta\partial \rho}\frac{\sin\theta\cos\theta}{\rho} + \frac{\partial V}{\partial \theta}\frac{\sin\theta\cos\theta}{\rho^2} + \frac{\partial^2 V}{\partial \rho\partial \phi}\frac{\cos^2\theta\cos\phi\sin\phi}{\rho} - \frac{\partial V}{\partial \phi}\frac{\cos^2\theta\sin\phi\cos\phi}{\rho^2}$$

$$- \frac{\partial^2 V}{\partial \theta\partial \rho}\frac{\cos\theta\sin\theta}{\rho} + \frac{\partial V}{\partial \rho}\frac{\sin^2\theta}{\rho} + \frac{\partial^2 V}{\partial \theta^2}\frac{\sin^2\theta}{\rho^2\sin^2\phi} + \frac{\partial V}{\partial \theta}\frac{\cos\theta\sin\theta}{\rho^2\sin^2\phi} - \frac{\partial^2 V}{\partial \theta\partial \phi}\frac{\cos\theta\cos\phi\sin\theta}{\rho^2\sin\phi} + \frac{\partial V}{\partial \phi}\frac{\sin^2\theta\cos\phi}{\rho^2\sin\phi}$$

$$+ \frac{\partial^2 V}{\partial \phi\partial \rho}\frac{\sin\phi\cos^2\theta\cos\phi}{\rho^2} + \frac{\partial V}{\partial \rho}\frac{\cos^2\phi\cos^2\theta}{\rho} - \frac{\partial^2 V}{\partial \phi\partial \theta}\frac{\sin\theta\cos\theta\cos\phi}{\rho^2\sin\phi} + \frac{\partial V}{\partial \theta}\frac{\sin\theta\cos\theta\cos^2\phi}{\rho^2\sin^2\phi}$$

$$+ \frac{\partial^2 V}{\partial \phi^2}\frac{\cos^2\theta\cos^2\phi}{\rho^2} - \frac{\partial V}{\partial \phi}\frac{\cos^2\theta\sin\phi\cos\phi}{\rho^2}$$

$$\frac{\partial V}{\partial y} = \frac{\partial V}{\partial \rho}\frac{\partial \rho}{\partial y} + \frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial V}{\partial \phi}\frac{\partial \phi}{\partial y} = \frac{\partial V}{\partial \rho}\sin\phi\sin\theta + \frac{\partial V}{\partial \theta}\frac{\cos\theta}{\rho\sin\phi} + \frac{\partial V}{\partial \phi}\frac{\sin\theta\cos\phi}{\rho}$$

$$\frac{\partial^2 V}{\partial y^2}=\frac{\partial\left(\frac{\partial V}{\partial y}\right)}{\partial y}=\frac{\partial}{\partial\rho}\left(\frac{\partial V}{\partial y}\right)\sin\phi\sin\theta+\frac{\partial}{\partial\theta}\left(\frac{\partial V}{\partial y}\right)\frac{\cos\theta}{\rho\sin\phi}+\frac{\partial}{\partial\phi}\left(\frac{\partial V}{\partial y}\right)\frac{\sin\theta\cos\phi}{\rho}$$

$$=\frac{\partial}{\partial\rho}\left(\frac{\partial V}{\partial\rho}\sin\phi\sin\theta+\frac{\partial V}{\partial\theta}\frac{\cos\theta}{\rho\sin\phi}+\frac{\partial V}{\partial\phi}\frac{\sin\theta\cos\phi}{\rho}\right)\sin\phi\sin\theta$$

$$+\frac{\partial}{\partial\theta}\left(\frac{\partial V}{\partial\rho}\sin\phi\sin\theta+\frac{\partial V}{\partial\theta}\frac{\cos\theta}{\rho\sin\phi}+\frac{\partial V}{\partial\phi}\frac{\sin\theta\cos\phi}{\rho}\right)\frac{\cos\theta}{\rho\sin\phi}$$

$$+\frac{\partial}{\partial\phi}\left(\frac{\partial V}{\partial\rho}\sin\phi\sin\theta+\frac{\partial V}{\partial\theta}\frac{\cos\theta}{\rho\sin\phi}+\frac{\partial V}{\partial\phi}\frac{\sin\theta\cos\phi}{\rho}\right)\frac{\sin\theta\cos\phi}{\rho}$$

$$=\frac{\partial^2 V}{\partial\rho^2}\sin^2\phi\sin^2\theta+\frac{\partial^2 V}{\partial\rho\partial\theta}\frac{\cos\theta\sin\theta}{\rho}-\frac{\partial V}{\partial\theta}\frac{\cos\theta\sin\theta}{\rho^2}+\frac{\partial^2 V}{\partial\rho\partial\phi}\frac{\sin^2\theta\cos\phi\sin\phi}{\rho}-\frac{\partial V}{\partial\phi}\frac{\sin^2\theta\cos\phi\sin\phi}{\rho^2}$$

$$+\frac{\partial^2 V}{\partial\theta\partial\rho}\frac{\sin\theta\cos\theta}{\rho}+\frac{\partial V}{\partial\rho}\frac{\cos^2\theta}{\rho}+\frac{\partial^2 V}{\partial\theta^2}\frac{\cos^2\theta}{\rho^2\sin^2\phi}-\frac{\partial V}{\partial\theta}\frac{\sin\theta\cos\theta}{\rho^2\sin^2\phi}+\frac{\partial^2 V}{\partial\theta\partial\phi}\frac{\sin\theta\cos\phi\cos\theta}{\rho^2\sin\phi}+\frac{\partial V}{\partial\phi}\frac{\cos^2\theta\cos\phi}{\rho^2\sin\phi}$$

$$+\frac{\partial^2 V}{\partial\phi\partial\rho}\frac{\sin\phi\sin^2\theta\cos\phi}{\rho}+\frac{\partial V}{\partial\rho}\frac{\sin^2\theta\cos^2\phi}{\rho}+\frac{\partial^2 V}{\partial\phi\partial\theta}\frac{\cos\theta\sin\theta\cos\phi}{\rho^2\sin\phi}-\frac{\partial V}{\partial\theta}\frac{\cos^2\phi\cos\theta\sin\theta}{\rho^2\sin^2\phi}$$

$$+\frac{\partial^2 V}{\partial\phi^2}\frac{\sin^2\theta\cos^2\phi}{\rho^2}-\frac{\partial V}{\partial\phi}\frac{\sin^2\theta\sin\phi\cos\phi}{\rho^2}$$

$$\frac{\partial V}{\partial z}=\frac{\partial V}{\partial\rho}\frac{\partial\rho}{\partial z}+\frac{\partial V}{\partial\theta}\frac{\partial\theta}{\partial z}+\frac{\partial V}{\partial\phi}\frac{\partial\phi}{\partial z}=\frac{\partial V}{\partial\rho}\cos\phi-\frac{\partial V}{\partial\phi}\frac{\sin\phi}{\rho}$$

$$\frac{\partial^2 V}{\partial z^2}=\frac{\partial}{\partial\rho}\left(\frac{\partial V}{\partial z}\right)\cos\phi-\frac{\partial}{\partial\phi}\left(\frac{\partial V}{\partial z}\right)\frac{\sin\phi}{\rho}$$

$$=\frac{\partial}{\partial\rho}\left(\frac{\partial V}{\partial\rho}\cos\phi-\frac{\partial V}{\partial\phi}\frac{\sin\phi}{\rho}\right)\cos\phi-\frac{\partial}{\partial\phi}\left(\frac{\partial V}{\partial\rho}\cos\phi-\frac{\partial V}{\partial\phi}\frac{\sin\phi}{\rho}\right)\frac{\sin\phi}{\rho}$$

$$=\frac{\partial^2 V}{\partial\rho^2}\cos^2\phi-\frac{\partial^2 V}{\partial\rho\partial\phi}\frac{\sin\phi\cos\phi}{\rho}+\frac{\partial V}{\partial\phi}\frac{\sin\phi\cos\phi}{\rho^2}-\frac{\partial^2 V}{\partial\phi\partial\rho}\frac{\cos\phi\sin\phi}{\rho}+\frac{\partial V}{\partial\rho}\frac{\sin^2\phi}{\rho}+\frac{\partial^2 V}{\partial\phi^2}\frac{\sin^2\phi}{\rho^2}$$

$$+\frac{\partial V}{\partial\phi}\frac{\cos\phi\sin\phi}{\rho^2}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{\partial^2 V}{\partial \rho^2}\sin^2\phi\cos^2\theta - \frac{\partial^2 V}{\partial\theta\partial\rho}\frac{\sin\theta\cos\theta}{\rho} + \frac{\partial V}{\partial\theta}\frac{\sin\theta\cos\theta}{\rho^2} + \frac{\partial^2 V}{\partial\rho\partial\phi}\frac{\cos^2\theta\cos\phi\sin\phi}{\rho} - \frac{\partial V}{\partial\phi}\frac{\cos^2\theta\sin\phi\cos\phi}{\rho^2}$$

$$- \frac{\partial^2 V}{\partial\theta\partial\rho}\frac{\cos\theta\sin\theta}{\rho} + \frac{\partial V}{\partial\rho}\frac{\sin^2\theta}{\rho} + \frac{\partial^2 V}{\partial\theta^2}\frac{\sin^2\theta}{\rho^2\sin^2\phi} + \frac{\partial V}{\partial\theta}\frac{\cos\theta\sin\theta}{\rho^2\sin^2\phi} - \frac{\partial^2 V}{\partial\theta\partial\phi}\frac{\cos\theta\cos\phi\sin\theta}{\rho^2\sin\phi} + \frac{\partial V}{\partial\phi}\frac{\sin^2\theta\cos\phi}{\rho^2\sin\phi}$$

$$+ \frac{\partial^2 V}{\partial\phi\partial\rho}\frac{\sin\phi\cos^2\theta\cos\phi}{\rho^2} + \frac{\partial V}{\partial\rho}\frac{\cos^2\phi\cos^2\theta}{\rho} - \frac{\partial^2 V}{\partial\phi\partial\theta}\frac{\sin\theta\cos\theta\cos\phi}{\rho^2\sin\phi} + \frac{\partial V}{\partial\theta}\frac{\sin\theta\cos\theta\cos^2\phi}{\rho^2\sin^2\phi}$$

$$+ \frac{\partial^2 V}{\partial\phi^2}\frac{\cos^2\theta\cos^2\phi}{\rho^2} - \frac{\partial V}{\partial\phi}\frac{\cos^2\theta\sin\phi\cos\phi}{\rho^2}$$

$$+ \frac{\partial^2 V}{\partial\rho^2}\sin^2\phi\sin^2\theta + \frac{\partial^2 V}{\partial\rho\partial\theta}\frac{\cos\theta\sin\theta}{\rho} - \frac{\partial V}{\partial\theta}\frac{\cos\theta\sin\theta}{\rho^2} + \frac{\partial^2 V}{\partial\rho\partial\phi}\frac{\sin^2\theta\cos\phi\sin\phi}{\rho} - \frac{\partial V}{\partial\phi}\frac{\sin^2\theta\cos\phi\sin\phi}{\rho^2}$$

$$+ \frac{\partial^2 V}{\partial\theta\partial\rho}\frac{\sin\theta\cos\theta}{\rho} + \frac{\partial V}{\partial\rho}\frac{\cos^2\theta}{\rho} + \frac{\partial^2 V}{\partial\theta^2}\frac{\cos^2\theta}{\rho^2\sin^2\phi} - \frac{\partial V}{\partial\theta}\frac{\sin\theta\cos\theta}{\rho^2\sin^2\phi} + \frac{\partial^2 V}{\partial\theta\partial\phi}\frac{\sin\theta\cos\phi\cos\theta}{\rho^2\sin\phi} + \frac{\partial V}{\partial\phi}\frac{\cos^2\theta\cos\phi}{\rho^2\sin\phi}$$

$$+ \frac{\partial^2 V}{\partial\phi\partial\rho}\frac{\sin\phi\sin^2\theta\cos\phi}{\rho} + \frac{\partial V}{\partial\rho}\frac{\sin^2\theta\cos^2\phi}{\rho} + \frac{\partial^2 V}{\partial\phi\partial\theta}\frac{\cos\theta\sin\theta\cos\phi}{\rho^2\sin\phi} - \frac{\partial V}{\partial\theta}\frac{\cos^2\phi\cos\theta\sin\theta}{\rho^2\sin^2\phi}$$

$$+ \frac{\partial^2 V}{\partial\phi^2}\frac{\sin^2\theta\cos^2\phi}{\rho^2} - \frac{\partial V}{\partial\phi}\frac{\sin^2\theta\sin\phi\cos\phi}{\rho^2}$$

$$+ \frac{\partial^2 V}{\partial\rho^2}\cos^2\phi - \frac{\partial^2 V}{\partial\rho\partial\phi}\frac{\sin\phi\cos\phi}{\rho} + \frac{\partial V}{\partial\phi}\frac{\sin\phi\cos\phi}{\rho^2} - \frac{\partial^2 V}{\partial\phi\partial\rho}\frac{\cos\phi\sin\phi}{\rho} + \frac{\partial V}{\partial\rho}\frac{\sin^2\phi}{\rho} + \frac{\partial^2 V}{\partial\phi^2}\frac{\sin^2\phi}{\rho^2}$$

$$+ \frac{\partial V}{\partial\phi}\frac{\cos\phi\sin\phi}{\rho^2}$$

$$= \left(\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi\right)\frac{\partial^2 V}{\partial\rho^2} + \left(-\frac{\sin\theta\cos\theta}{\rho} - \frac{\cos\theta\sin\theta}{\rho} + \frac{\sin\theta\cos\theta}{\rho} + \frac{\cos\theta\sin\theta}{\rho}\right)\frac{\partial^2 V}{\partial\theta\partial\rho}$$

$$+ \left(\frac{\cos^2\theta\cos\phi\sin\phi}{\rho} + \frac{\sin\phi\cos^2\theta\cos\phi}{\rho} + \frac{\sin^2\theta\cos\phi\sin\phi}{\rho} + \frac{\sin\phi\sin^2\theta\cos\phi}{\rho} - \frac{\sin\phi\cos\phi}{\rho} - \frac{\cos\phi\sin\phi}{\rho}\right)\frac{\partial^2 V}{\partial\phi\partial\rho}$$

$$+ \left(\frac{\sin^2\theta}{\rho^2\sin^2\phi} + \frac{\cos^2\theta}{\rho^2\sin^2\phi}\right)\frac{\partial^2 V}{\partial\theta^2} + \left(-\frac{\cos\theta\cos\phi\sin\theta}{\rho^2\sin\phi} - \frac{\sin\theta\cos\theta\cos\phi}{\rho^2\sin\phi} + \frac{\sin\theta\cos\phi\cos\theta}{\rho^2\sin\phi} + \frac{\cos\theta\sin\theta\cos\phi}{\rho^2\sin\phi}\right)\frac{\partial^2 V}{\partial\phi\partial\theta}$$

$$+ \left(\frac{\cos^2\theta\cos^2\phi}{\rho^2} + \frac{\sin^2\theta\cos^2\phi}{\rho^2} + \frac{\sin^2\phi}{\rho^2}\right)\frac{\partial^2 V}{\partial\phi^2} + \left(\frac{\sin^2\theta}{\rho} + \frac{\cos^2\phi\cos^2\theta}{\rho} + \frac{\cos^2\theta}{\rho} + \frac{\sin^2\theta\cos^2\phi}{\rho} + \frac{\sin^2\phi}{\rho}\right)\frac{\partial V}{\partial\rho}$$

$$+ \left(\frac{\sin\theta\cos\theta}{\rho^2} + \frac{\cos\theta\sin\theta}{\rho^2\sin^2\phi} + \frac{\sin\theta\cos\theta\cos^2\phi}{\rho^2\sin^2\phi} - \frac{\cos\theta\sin\theta}{\rho^2} - \frac{\sin\theta\cos\theta}{\rho^2\sin^2\phi} - \frac{\cos^2\phi\cos\theta\sin\theta}{\rho^2\sin^2\phi}\right)\frac{\partial V}{\partial\theta}$$

$$+ \left(-\frac{\cos^2\theta\sin\phi\cos\phi}{\rho^2} + \frac{\sin^2\theta\cos\phi}{\rho^2\sin\phi} - \frac{\cos^2\theta\sin\phi\cos\phi}{\rho^2} - \frac{\sin^2\theta\cos\phi\sin\phi}{\rho^2} + \frac{\cos^2\theta\cos\phi}{\rho^2\sin\phi}\right.$$

$$\left. - \frac{\sin^2\theta\sin\phi\cos\phi}{\rho^2} + \frac{\sin\phi\cos\phi}{\rho^2} + \frac{\cos\phi\sin\phi}{\rho^2}\right)\frac{\partial V}{\partial\phi}$$

$$\Rightarrow \nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2 \sin^2 \phi}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\cot\phi}{\rho^2}\frac{\partial V}{\partial \phi} = 0$$

6. Suppose the equations $\begin{cases} x^2 - y^2 - u^3 + v^2 + 4 = 0 \\ 2xy + y^2 - 2u^2 + 3v^4 + 8 = 0 \end{cases}$ determine functions $u(x,y)$ and $v(x,y)$ near $x = 2$ and $y = -1$ such that $u(2,-1) = 2$ and $v(2,-1) = 1$. Compute $\frac{\partial u}{\partial x}(2,-1)$.

Solution:

$$\begin{cases} x^2 - y^2 - u^3 + v^2 + 4 = 0 \\ 2xy + y^2 - 2u^2 + 3v^4 + 8 = 0 \end{cases} \Rightarrow \begin{cases} 2x - 3u^2\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \\ 2y - 4u\frac{\partial u}{\partial x} + 12v^3\frac{\partial v}{\partial x} = 0 \end{cases}$$

When $x = 2, y = -1$, we have $u = 2, v = 1$

$$\begin{cases} 4 - 12\frac{\partial u}{\partial x} + 2\frac{\partial v}{\partial x} = 0 \\ -2 - 8\frac{\partial u}{\partial x} + 12\frac{\partial v}{\partial x} = 0 \end{cases} \Rightarrow \frac{\partial u}{\partial x}(2,-1) = \frac{13}{32}$$

7. It is given that $\begin{cases} x^2 + y^2 = \frac{1}{2}z^2 \\ x + y + z = 2 \end{cases}$. Find $\frac{dx}{dz}, \frac{dy}{dz}, \frac{d^2x}{dz^2}, \frac{d^2y}{dz^2}$ when $x = 1,\ y = -1,\ z = 2$.

Solution:

$$\begin{cases} x^2 + y^2 = \frac{1}{2}z^2 \\ x + y + z = 2 \end{cases} \Rightarrow \begin{cases} 2x\frac{dx}{dz} + 2y\frac{dy}{dz} = z \cdots (1) \\ \frac{dx}{dz} + \frac{dy}{dz} + 1 = 0 \cdots (2) \end{cases} \Rightarrow \begin{cases} 2\left(\frac{dx}{dz}\right)^2 + 2x\frac{d^2x}{dz^2} + 2\left(\frac{dy}{dz}\right)^2 + 2y\frac{d^2y}{dz^2} = 1 \cdots (3) \\ \frac{d^2x}{dz^2} + \frac{d^2y}{dz^2} = 0 \cdots (4) \end{cases}$$

Put $x = 1, y = -1, z = 2$ in (1) & (2), we have

$$\begin{cases} 2x\frac{dx}{dz} + 2y\frac{dy}{dz} = z \cdots (1) \\ \frac{dx}{dz} + \frac{dy}{dz} + 1 = 0 \cdots (2) \end{cases} \Rightarrow 4\frac{dx}{dz} = 0 \ \&\ 4\frac{dy}{dz} = -4 \Rightarrow \frac{dx}{dz} = 0 \ \&\ \frac{dy}{dz} = -1$$

Put $x = 1, y = -1, z = 2$ & $\frac{dx}{dz} = 0, \frac{dy}{dz} = -1$ in (3) & (4), we have $\frac{d^2x}{dz^2} = -\frac{1}{4}, \frac{d^2y}{dz^2} = \frac{1}{4}$

8. Use Taylor's theorem to expand $f(x,y) = \sin xy$ about the point $\left(1, \frac{\pi}{3}\right)$, neglecting cubic and higher terms. Hence estimate $\sin 0.3\pi$.

Solution:

Expand $f(x,y) = \sin xy$ about $\left(1, \frac{\pi}{3}\right)$.

Then

$$f(x,y)=\sin xy = f\left(1,\frac{\pi}{3}\right)+f_x\left(1,\frac{\pi}{3}\right)(x-1)+f_y\left(1,\frac{\pi}{3}\right)\left(y-\frac{\pi}{3}\right)$$

$$+\frac{1}{2}\left[f_{xx}\left(1,\frac{\pi}{3}\right)(x-1)^2+2f_{xy}\left(1,\frac{\pi}{3}\right)(x-1)\left(y-\frac{\pi}{3}\right)+f_{yy}\left(1,\frac{\pi}{3}\right)\left(y-\frac{\pi}{3}\right)^2\right]+\cdots$$

$$f_x(x,y)=y\cos xy,\ f_y(x,y)=x\cos xy,\ f_{xx}(x,y)=-y^2\sin xy$$

Observe that $\left(0.9,\frac{\pi}{3}\right)$ is close to $\left(1,\frac{\pi}{3}\right)$.

Therefore

$$f\left(0.9,\frac{\pi}{3}\right)=\sin\left(0.9\times\frac{\pi}{3}\right)=\sin 0.3\pi \approx f\left(1,\frac{\pi}{3}\right)+f_x\left(1,\frac{\pi}{3}\right)(0.9-1)+f_y\left(1,\frac{\pi}{3}\right)\left(\frac{\pi}{3}-\frac{\pi}{3}\right)$$

$$+\frac{1}{2}\left[f_{xx}\left(1,\frac{\pi}{3}\right)(0.9-1)^2+2f_{xy}\left(1,\frac{\pi}{3}\right)(0.9-1)\left(\frac{\pi}{3}-\frac{\pi}{3}\right)+f_{yy}\left(1,\frac{\pi}{3}\right)\left(\frac{\pi}{3}-\frac{\pi}{3}\right)^2\right]$$

$$=\sin\frac{\pi}{3}-0.1\times\frac{\pi}{3}\cos\frac{\pi}{3}-(-0.1)^2\times\frac{\pi^2}{9}\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}-0.1\times\frac{\pi}{6}-\frac{1}{2}\times(-0.1)^2\pi^2\frac{\sqrt{3}}{18}=0.80892$$

□

9. In surveying a triangular plot of land, two of its sides were measured as 160m and 210m with maximum possible errors of 0.1m and the included angle was $\frac{\pi}{3}$ exactly. Estimate the maximum error in calculating the length of the third side from the cosine rule $c=\left(a^2+b^2-2ab\ \cos C\right)^{\frac{1}{2}}$.

Solution:

$$c=c(a,b)=\left(a^2+b^2-2ab\cos C\right)^{\frac{1}{2}}\Rightarrow\frac{\partial c}{\partial a}=\frac{a-b\cos C}{\sqrt{a^2+b^2-2ab\ \cos C}},\frac{\partial c}{\partial b}=\frac{b-a\cos C}{\sqrt{a^2+b^2-2ab\ \cos C}}.$$

$$\delta c=c(a,b)-c(a_0,b_0)\underset{\substack{\delta a=a-a_0\\ \delta b=b-b_0}}{=}\delta a\frac{\partial c}{\partial a}(a_0+\theta\delta a,b_0+\theta\delta b)+\delta b\frac{\partial c}{\partial b}(a_0+\theta\delta a,b_0+\theta\delta b).$$

Since $\frac{\partial c}{\partial a},\ \frac{\partial c}{\partial b}$ are continuous functions, if $\delta a,\ \delta b$ are quite small, then

$$\frac{\partial c}{\partial a}(a_0+\theta\delta a,b_0+\theta\delta b)\approx\frac{\partial c}{\partial a}(a_0,b_0),\ \frac{\partial c}{\partial b}(a_0+\theta\delta a,b_0+\theta\delta b)\approx\frac{\partial c}{\partial b}(a_0,b_0).$$

Therefore,

$$\delta c=c(a,b)-c(a_0,b_0)\underset{\substack{\delta a=a-a_0\\ \delta b=b-b_0}}{=}\delta a\frac{\partial c}{\partial a}(a_0+\theta\delta a,b_0+\theta\delta b)+\delta b\frac{\partial c}{\partial b}(a_0+\theta\delta a,b_0+\theta\delta b)$$

$$\approx\frac{a_0-b_0\cos C}{\sqrt{a_0{}^2+b_0{}^2-2a_0b_0\ \cos C}}\delta a+\frac{b_0-a_0\cos C}{\sqrt{a_0{}^2+b_0{}^2-2a_0b_0\cos C}}\delta b$$

$$\underset{\sqrt{a_0{}^2+b_0{}^2-2a_0b_0\cos C}=c_0}{=}\frac{a_0-b_0\cos C}{c_0}\delta a+\frac{b_0-a_0\cos C}{c_0}\delta b$$

Observe that $c_0 = \left(a_0{}^2 + b_0{}^2 - 2a_0 b_0 \cos C\right)^{\frac{1}{2}} \underset{\substack{a_0=160 \\ b_0=210 \\ C=\frac{\pi}{3}}}{=} 190$.

$$|\delta c| \le \left|\frac{a_0 - b_0 \cos C}{c_0}\delta a\right| + \left|\frac{b_0 - a_0 \cos C}{c_0}\delta b\right| = \frac{|a_0 - b_0 \cos C|}{|c_0|}|\delta a| + \frac{|b_0 - a_0 \cos C|}{|c_0|}|\delta b|$$

$$\le \frac{|a_0 - b_0 \cos C|}{|c_0|} \times 0.1 + \frac{|b_0 - a_0 \cos C|}{|c_0|} \times 0.1 = \frac{55}{190} \times 0.1 + \frac{130}{190} \times 0.1 = 0.09736$$

□

10. Find and classify the stationary points of
    (a) $f(x, y) = x^3 + 3x^2 - 3y^2 + 6xy$,
    (b) $f(x, y) = x^3 + 3xy - 3x^2 - 3y^2 + 4$,
    (c) $z = f(x, y) = x^4 + y^4 - x^2 - 2xy - y^2$.

Solution:

(a)

$$z = f(x, y) = x^3 + 3x^2 - 3y^2 + 6xy \Rightarrow \begin{cases} \dfrac{\partial z}{\partial x} = 3x^2 + 6x + 6y = 0 \\ \dfrac{\partial z}{\partial y} = -6y + 6x = 0 \end{cases} \Leftrightarrow \begin{cases} 3x^2 + 6x + 6y = 0 \\ x = y \end{cases}$$

$$\Leftrightarrow \begin{cases} 3x^2 + 12x = 0 \\ x = y \end{cases} \Leftrightarrow \begin{cases} x = 0 \text{ or } x = -4 \\ x = y \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \text{ or } \begin{cases} x = -4 \\ y = -4 \end{cases}$$

Also $\dfrac{\partial^2 z}{\partial x^2} = 6x + 6$ & $\dfrac{\partial^2 z}{\partial x \partial y} = 6$ & $\dfrac{\partial^2 z}{\partial y^2} = -6$.

So

$$\Delta(x, y) = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = (6x + 6)(-6) - 36 = -36x - 72$$

And

$$\Delta(0,0) = -72 < 0, \Delta(-4,-4) = 72 > 0, \left.\frac{\partial^2 z}{\partial x^2}\right|_{\substack{x=0 \\ y=0}} = 6 > 0, \left.\frac{\partial^2 z}{\partial x^2}\right|_{\substack{x=-4 \\ y=-4}} = -18 < 0$$

Therefore, $(-4,-4)$ is a local maximum point and $(0,0)$ is a saddle point.

(b)

$$z = f(x, y) = x^3 + 3xy - 3x^2 - 3y^2 + 4 \Rightarrow \begin{cases} \dfrac{\partial z}{\partial x} = 3x^2 + 3y - 6x = 0 \\ \dfrac{\partial z}{\partial y} = 3x - 6y = 0 \end{cases} \Leftrightarrow \begin{cases} 3x^2 + 3y - 6x = 0 \\ x = 2y \end{cases}$$

$$\Leftrightarrow \begin{cases} 12y^2 - 9y = 0 \\ x = 2y \end{cases} \Leftrightarrow \begin{cases} y = 0 \text{ or } y = \dfrac{3}{4} \\ x = 2y \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \text{ or } \begin{cases} x = \dfrac{3}{2} \\ y = \dfrac{3}{4} \end{cases}$$

Also $\dfrac{\partial^2 z}{\partial x^2} = 6x - 6$ & $\dfrac{\partial^2 z}{\partial x \partial y} = 3$ & $\dfrac{\partial^2 z}{\partial y^2} = -6$.

So

$$\Delta(x, y) = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = (6x-6)(-6) - 9 = -36x + 27$$

And

$$\Delta(0,0) = 27 > 0, \Delta\left(\frac{3}{2}, \frac{3}{4}\right) = -27 < 0,\ \left.\frac{\partial^2 z}{\partial x^2}\right|_{\substack{x=0\\y=0}} = -6 < 0, \left.\frac{\partial^2 z}{\partial x^2}\right|_{\substack{x=3/2\\y=3/4}} = 3 > 0$$

Therefore, $(0,0)$ is a local maximum point and $\left(\frac{3}{2}, \frac{3}{4}\right)$ is a saddle point.

(c)

$$z = f(x, y) = x^4 + y^4 - x^2 - 2xy - y^2 \Rightarrow \begin{cases} \dfrac{\partial z}{\partial x} = 4x^3 - 2x - 2y = 0 \\ \dfrac{\partial z}{\partial y} = 4y^3 - 2x - 2y = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial z}{\partial x} = 4x^3 - 2x - 2y = 0 \\ \dfrac{\partial z}{\partial y} = 4y^3 - 2x - 2y = 0 \end{cases} \Rightarrow 4x^3 - 4y^3 = 0$$

And $4x^3 - 4y^3 = 0 \Leftrightarrow 4(x-y)(x^2 + xy + y^2) = 0$

$4(x-y)(x^2 + xy + y^2) = 0 \Leftrightarrow x - y = 0$ or $x^2 + xy + y^2 = 0$

So $\begin{cases} 4x^3 - 2x - 2y = 0 \\ x = y \end{cases} \Leftrightarrow \begin{cases} 4x^3 - 4x = 0 \\ x = y \end{cases} \Leftrightarrow \begin{cases} 4x(x-1)(x+1) = 0 \\ x = y \end{cases}$

So we have the roots for $\begin{cases} \dfrac{\partial z}{\partial x} = 4x^3 - 2x - 2y = 0 \\ \dfrac{\partial z}{\partial y} = 4y^3 - 2x - 2y = 0 \end{cases}$, they are $\begin{cases} x = 0 \\ y = 0 \end{cases}$ or $\begin{cases} x = 1 \\ y = 1 \end{cases}$ or $\begin{cases} x = -1 \\ y = -1 \end{cases}$

Note: Only $(0,0)$ satisfies $x^2 + xy + y^2 = 0$.

Also $\dfrac{\partial z}{\partial x} = 4x^3 - 2x - 2y \Rightarrow \dfrac{\partial^2 z}{\partial x^2} = 12x^2 - 2$ & $\dfrac{\partial^2 z}{\partial x \partial y} = -2$ & $\dfrac{\partial z}{\partial y} = 4y^3 - 2x - 2y \Rightarrow \dfrac{\partial^2 z}{\partial y^2} = 12y^2 - 2$.

So

$$\Delta(x, y) = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = (12x^2 - 2)(12y^2 - 2) - 4$$

And

$$\Delta(1,1) > 0, \Delta(-1,-1) > 0,\ \left.\frac{\partial^2 z}{\partial x^2}\right|_{\substack{x=1\\y=1}} > 0, \left.\frac{\partial^2 z}{\partial x^2}\right|_{\substack{x=-1\\y=-1}} > 0$$

Therefore, $(1,1)$, $(-1,-1)$ are local minimum points.

As for $(0,0)$, $\Delta(0,0)=0$ so the test does not work but we observe that $f(0,0)=0$ and for $x=\dfrac{1}{n}, y=\dfrac{1}{n}$

where *n* is a positive integer . we have $f\left(\dfrac{1}{n},\dfrac{1}{n}\right)-f(0,0)=2\left(\dfrac{1}{n}\right)^4-4\left(\dfrac{1}{n}\right)^2<0$

and $\left(\dfrac{1}{n},\dfrac{1}{n}\right)$ is close to $(0,0)$ when *n* is sufficiently large.

Also $x=\dfrac{1}{n}, y=-\dfrac{1}{n}$, $f\left(\dfrac{1}{n},-\dfrac{1}{n}\right)-f(0,0)=2\left(\dfrac{1}{4}\right)^4>0$ and $\left(\dfrac{1}{n},-\dfrac{1}{n}\right)$ is close to $(0,0)$ when *n* is sufficiently large.

Thus $(0,0)$ is a saddle point.

□

11. Find the minimum distance between the origin and the surface $z^2=x^2y+4$.

Solution:

Let the square of the distance between the origin and a point on the surface $z^2=x^2y+4$ be *w*.

Then $w=x^2+y^2+z^2$ where (x,y,z) is a point on the surface $z^2=x^2y+4$.

Thus $w=x^2+y^2+x^2y+4$

$$w=f(x,y)=x^2+y^2+x^2y+4\Rightarrow\begin{cases}\dfrac{\partial w}{\partial x}=2x+2xy=0\\[2mm]\dfrac{\partial w}{\partial y}=2y+x^2=0\end{cases}\Leftrightarrow\begin{cases}2x-x^3=0\\2y+x^2=0\end{cases}$$

$$\Leftrightarrow\begin{cases}x\left(\sqrt{2}-x\right)\left(\sqrt{2}+x\right)=0\\2y+x^2=0\end{cases}\Leftrightarrow\begin{cases}x=0\text{ or }x=\sqrt{2}\text{ or }x=-\sqrt{2}\\2y+x^2=0\end{cases}\Leftrightarrow\begin{cases}x=0\\y=0\end{cases}\text{or}\begin{cases}x=\sqrt{2}\\y=-1\end{cases}\text{or}\begin{cases}x=-\sqrt{2}\\y=-1\end{cases}$$

$$\Delta(x,y)=\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2=2(2+2y)-4x^2=4+4y-4x^2.$$

$\Delta\left(\pm\sqrt{2},-1\right)=-8$. Neither $\left(\sqrt{2},-1\right)$nor$\left(-\sqrt{2},-1\right)$ yields an extremum.

$\Delta(0,0)=4$, $\left.\dfrac{\partial^2 w}{\partial x^2}\right|_{\substack{x=0\\y=0}}=2>0$. So $(0,0)$ yields the minimum value of *w*.

And $w=f(0,0)=4$.

The minimum distance between the origin and the surface $z^2=x^2y+4$ is 2.

□

12. The equation $x^3-3x+y^2-2y+z^3+z+1=0$ implicitly determines an implicit function $z=z(x,y)$ of *x* and *y* defined in R^2. Find the stationary point(s) of $z=z(x,y)$ and determine the local extreme value(s) of $z=z(x,y)$ that $z=z(x,y)$ will attain there (if any) and state what kind of local extreme value(s) it is or they are.

Solution:

$$x^3-3x+y^2-2y+z^3+z+1=0 \Rightarrow 3x^2-3+\left(3z^2+1\right)\frac{\partial z}{\partial x}=0 \Rightarrow \frac{\partial z}{\partial x}=\frac{3-3x^2}{3z^2+1}$$

$$x^3-3x+y^2-2y+z^3+z+1=0 \Rightarrow 2y-2+\left(3z^2+1\right)\frac{\partial z}{\partial y}=0 \Rightarrow \frac{\partial z}{\partial y}=\frac{2-2y}{3z^2+1}$$

$$\begin{cases} \dfrac{\partial z}{\partial x}=\dfrac{3-3x^2}{3z^2+1}=0 \\ \dfrac{\partial z}{\partial y}=\dfrac{2-2y}{3z^2+1}=0 \end{cases} \Rightarrow \begin{cases} x=1 \text{ or } x=-1 \\ y=1 \end{cases}$$

So the stationary points of $z=z\left(x,y\right)$ are $\left(1,1\right),\left(1,-1\right)$.

Put $\begin{cases} x=1 \\ y=1 \end{cases}$ in $x^3-3x+y^2-2y+z^3+z+1=0$, we have $z=1$

Put $\begin{cases} x=-1 \\ y=1 \end{cases}$ in $x^3-3x+y^2-2y+z^3+z+1=0$, we have $z=-1$

From

$$3x^2-3+\left(3z^2+1\right)\frac{\partial z}{\partial x}=0 \Rightarrow 6x+\left(3z^2+1\right)\frac{\partial^2 z}{\partial x^2}+6z\left(\frac{\partial z}{\partial x}\right)^2=0 \Rightarrow \frac{\partial^2 z}{\partial x^2}=\frac{-6x-6z\left(\dfrac{\partial z}{\partial x}\right)^2}{3z^2+1}$$

From

$$2y-2+\left(3z^2+1\right)\frac{\partial z}{\partial y}=0 \Rightarrow 2+\left(3z^2+1\right)\frac{\partial^2 z}{\partial y^2}+6z\left(\frac{\partial z}{\partial y}\right)^2=0 \Rightarrow \frac{\partial^2 z}{\partial y^2}=\frac{-2-6z\left(\dfrac{dz}{dy}\right)^2}{3z^2+1}$$

Again, from

$$3x^2-3+\left(3z^2+1\right)\frac{\partial z}{\partial x}=0 \Rightarrow \left(3z^2+1\right)\frac{\partial^2 z}{\partial y\partial x}+6z\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}=0 \Rightarrow \frac{\partial^2 z}{\partial y\partial x}=\frac{-6z\dfrac{\partial z}{\partial y}\dfrac{\partial z}{\partial x}}{3z^2+1}$$

$$\left.\frac{\partial^2 z}{\partial x^2}\right|_{\substack{x=1\\y=1\\z=1}}=\left.\frac{-6x-6z\left(\dfrac{\partial z}{\partial x}\right)^2}{3z^2+1}\right|_{\substack{x=1\\y=1\\z=1}}=\frac{-6}{4}<0$$

Note that $\left.\dfrac{\partial z}{\partial x}\right|_{\substack{x=1\\y=1\\z=1}}=0$

.

$$\left.\frac{\partial^2 z}{\partial y^2}\right|_{\substack{x=1\\y=1\\z=1}} = \frac{-2-6z\left(\frac{dz}{dy}\right)^2}{3z^2+1} = \frac{-2}{4} < 0$$

Note that $\left.\frac{\partial z}{\partial y}\right|_{\substack{x=1\\y=1\\z=1}} = 0$

.

$$\left.\frac{\partial^2 z}{\partial y \partial x}\right|_{\substack{x=1\\y=1\\z=1}} = \left.\frac{-6z\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}}{3z^2+1}\right|_{\substack{x=1\\y=1\\z=1}} = 0$$

Therefore,

$$\left.\left[\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial y \partial x}\right)^2\right]\right|_{\substack{x=1\\y=1\\z=1}} = \left.\frac{\partial^2 z}{\partial x^2}\right|_{\substack{x=1\\y=1\\z=1}} \left.\frac{\partial^2 z}{\partial y^2}\right|_{\substack{x=1\\y=1\\z=1}} - \left.\left(\frac{\partial^2 z}{\partial y \partial x}\right)^2\right|_{\substack{x=1\\y=1\\z=1}} = \left(-\frac{6}{4}\right)\left(-\frac{2}{4}\right) - 0 > 0$$

It follows that $z = z(x, y)$ attains the local maximum value $z(1,1) = 1$ at the point $(1,1)$

In addition,

$$\left.\frac{\partial^2 z}{\partial x^2}\right|_{\substack{x=-1\\y=1\\z=-1}} = \left.\frac{-6x-6z\left(\frac{\partial z}{\partial x}\right)^2}{3z^2+1}\right|_{\substack{x=-1\\y=1\\z=-1}} = \frac{6}{4} > 0$$

$$\left.\frac{\partial^2 z}{\partial y^2}\right|_{\substack{x=-1\\y=1\\z=-1}} = \frac{-2-6z\left(\frac{dz}{dy}\right)^2}{3z^2+1} = \frac{-2}{4} < 0$$

$$\left.\frac{\partial^2 z}{\partial y \partial x}\right|_{\substack{x=-1\\y=1\\z=-1}} = \left.\frac{-6z\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}}{3z^2+1}\right|_{\substack{x=-1\\y=1\\z=-1}} = 0$$

Note that $\left.\frac{\partial z}{\partial x}\right|_{\substack{x=-1\\y=1\\z=-1}} = \left.\frac{\partial z}{\partial y}\right|_{\substack{x=-1\\y=1\\z=-1}} = 0$

$$\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial y \partial x}\right)^2\right)\Bigg|_{\substack{x=-1\\y=1\\z=-1}}=\frac{\partial^2 z}{\partial x^2}\Bigg|_{\substack{x=-1\\y=1\\z=-1}}\frac{\partial^2 z}{\partial y^2}\Bigg|_{\substack{x-=1\\y=1\\z=-1}}-\left(\frac{\partial^2 z}{\partial y \partial x}\right)^2\Bigg|_{\substack{x=-1\\y=1\\z=-1}}=\left(\frac{6}{4}\right)\left(-\frac{2}{4}\right)-0<0$$

Therefore, $(1,-1)$ is a saddle point for $z=z(x,y)$, that means $z=z(x,y)$ attains neither local maximum nor local minimum at $(1,-1)$.

Also, we observe that $z=z(x,y)$ has neither the global maximum nor the global minimum.

□

13. If $\varphi(x,y,z)=x^2y^2z^2$, find
    (a) the maximum rate of change of φ at the point $(1,1,1)$ and the direction in which this occurs;
    (b) the rate of change of φ at the point $(2,1,1)$ in the direction of $3\vec{i}+4\vec{k}$.

Solution:

(a)

$$\varphi(x,y,z)=x^2y^2z^2 \Rightarrow grad\varphi=\begin{pmatrix}\varphi_x\\ \varphi_y\\ \varphi_z\end{pmatrix}=\begin{pmatrix}2xy^2z^2\\ 2yx^2z^2\\ 2zx^2y^2\end{pmatrix} \& \; grad\varphi\Big|_{\substack{x=1\\y=1\\z=1}}=\begin{pmatrix}2\\2\\2\end{pmatrix}$$

At the point $(1,1,1)$, $\begin{pmatrix}2\\2\\2\end{pmatrix}$ points to the direction in which φ increases most rapidly and the rate of change of φ at the point $(1,1,1)=\left|\begin{pmatrix}2\\2\\2\end{pmatrix}\right|=\sqrt{12}=2\sqrt{3}$.

(b)

$$\varphi(x,y,z)=x^2y^2z^2 \Rightarrow grad\varphi=\begin{pmatrix}\varphi_x\\ \varphi_y\\ \varphi_z\end{pmatrix}=\begin{pmatrix}2xy^2z^2\\ 2yx^2z^2\\ 2zx^2y^2\end{pmatrix} \& \; grad\varphi\Big|_{\substack{x=2\\y=1\\z=1}}=\begin{pmatrix}4\\8\\8\end{pmatrix}.$$

$$\frac{d\varphi}{ds}=D_v\varphi(2,1,1)=grad\varphi(2,1,1)\cdot\frac{3\vec{i}+4\vec{j}}{\left|3\vec{i}+4\vec{j}\right|}=\left(4\vec{i}+8\vec{j}+8\vec{k}\right)\cdot\frac{3\vec{i}+4\vec{j}}{\left|3\vec{i}+4\vec{j}\right|}=\frac{12+32}{\sqrt{9+12}}=\frac{44}{5}$$

□

14. A bomber is carrying a heat seeking missile which has the property that at any point (x,y,z) in space it moves in the direction of maximum temperature increase and the temperature at (x,y,z) is $T=T(x,y,z)=2x^2-xyz$.

    Suppose the bomber has just launched the missile at the point $P(1,2,3)$ and the missile can move at a speed of 50 km/minute in the direction specified.
    (a) In what direction will the missile move?
    (b) How fast is the temperature experienced by the missile changing in degree Celsius per kilometer at that instant when the missile has just left the bomber?

(c) How fast is the temperature experienced by the missile changing in degree Celsius per minute at that instant when the missile has just left the bomber?

(d) Due to the failure of an electronic device, the missile is no longer heat seeking but still can move at a speed of 50 km/minute in the direction specified.
How fast is the temperature experienced by the missile changing in degree Celsius per minute at that instant when the missile has just left the bomber and moved in the direction specified by $\vec{v}=\vec{i}+\vec{j}$?

Solution:

(a)

The temperature gradient vector $\nabla T(x,y,z)=(4x-yz)\vec{i}-xz\vec{j}-xy\vec{k}$.

$$\nabla T(1,2,3)=(4x-yz)\vec{i}-xz\vec{j}-xy\vec{k}\Big|_{\substack{x=1\\y=2\\z=3}}=-2\vec{i}-3\vec{j}-2\vec{k}$$

The missile will move in the direction specific by the vector $-2\vec{i}-3\vec{j}-2\vec{k}$ or $\dfrac{-2\vec{i}-3\vec{j}-2\vec{k}}{\sqrt{17}}$

(b)

The rate of change of temperature <u>with respect to distance</u> experienced by the missile at $P(1,2,3)$ is:

$$\frac{dT}{ds}=\left|-2\vec{i}-3\vec{j}-2\vec{k}\right|=\sqrt{17}\,\frac{{}^{0}C}{\text{km}}.$$

(c)

$$\frac{dw}{dt}=\frac{dw}{ds}\frac{ds}{dt}=\sqrt{17}\,\frac{{}^{0}C}{\text{km}}\times 50\frac{\text{km}}{\text{min}}=50\sqrt{17}\,\frac{{}^{0}C}{\text{mim}}.$$

(d)

The unit vector in the direction of the given vector $\vec{v}=\vec{i}+\vec{j}$ is $\vec{u}=\dfrac{\vec{i}+\vec{j}}{\sqrt{1^2+1^2}}=\dfrac{1}{\sqrt{2}}\vec{i}+\dfrac{1}{\sqrt{2}}\vec{j}$.

The temperature gradient vector $\nabla T(1,2,3)=(4x-yz)\vec{i}-xz\vec{j}-xy\vec{k}\Big|_{\substack{x=1\\y=2\\z=3}}=-2\vec{i}-3\vec{j}-2\vec{k}$.

Therefore the missile's initial rate of change of temperature <u>with respect to distance</u> is:

$$\frac{dT}{ds}\equiv D_{\vec{u}}T(1,2,3)=\nabla T(1,2,3)\cdot\vec{u}=\left(-2\vec{i}-3\vec{j}-2\vec{k}\right)\cdot\left(\frac{1}{\sqrt{2}}\vec{i}+\frac{1}{\sqrt{2}}\vec{j}\right)=\frac{-2-3}{\sqrt{2}}\frac{{}^{0}C}{\text{km}}=-\frac{5}{\sqrt{2}}\frac{{}^{0}C}{\text{km}}$$

Its speed is $\dfrac{ds}{dt}=50\dfrac{\text{km}}{\text{min}}$, so the time rate of change of temperature experienced by the hawk is

$$\frac{dT}{dt}=\frac{dT}{ds}\frac{ds}{dt}=D_{\vec{u}}T\frac{ds}{dt}=\left(-\frac{5}{\sqrt{2}}\frac{{}^{0}C}{\text{km}}\right)\left(50\frac{\text{km}}{\text{min}}\right)=-\frac{250}{\sqrt{2}}\frac{{}^{0}C}{\text{mim}}.$$

□

15. Let $u(x,y)=3x^2+y^2$.

(a) Find the directional derivative of $u(x,y)=3x^2+y^2$ at the point (x,y) in the direction $(1,1)$.

(b) Determine the points (x,y) and directions for which the directional derivative of $u(x,y)=3x^2+y^2$ has its largest value if (x,y) is restricted to lie on the circle $x^2+y^2=1$.

Solution:

(a)

$u(x,y) = 3x^2 + y^2 \Rightarrow gradu = \begin{pmatrix} 6x \\ 2y \end{pmatrix}$. The unit vector with the same direction as $(1,1)$ is $\vec{n} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

Thus, the directional derivative of $u(x,y) = 3x^2 + y^2$ at the point (x,y) in the direction $(1,1)$ is:

$$u_{\vec{n}}(x,y) = \begin{pmatrix} 6x \\ 2y \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{6x+2y}{\sqrt{2}}.$$

(b)

At (x,y), $u(x,y) = 3x^2 + y^2$ has its largest directional derivative in the direction as pointed by $gradu = \begin{pmatrix} 6x \\ 2y \end{pmatrix}$ and the corresponding value is $|gradu| = \left| \begin{pmatrix} 6x \\ 2y \end{pmatrix} \right| = \sqrt{36x^2 + 4y^2}$.

If (x,y) is restricted to lie on the circle $x^2 + y^2 = 1$, then $x = \cos\theta, y = \sin\theta, 0 \le \theta \le 2\pi$.

Then $\left| \begin{pmatrix} 6\cos\theta \\ 2\sin\theta \end{pmatrix} \right| = \sqrt{36\cos\theta^2 + 4\sin\theta^2} = \sqrt{4 + 32\cos^2\theta}$ and $\sqrt{4 + 32\cos^2\theta}$ has the greatest value $\sqrt{36}$ when $\theta = 0, \ \pi$.

For $\theta = 0$, we have $x = 1, y = 0$, for $\theta = \pi$, we have $x = -1, y = 0$.

Answer: at $x = 1, y = 0$ and the direction $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and at $x = -1, y = 0$ and the direction $\begin{pmatrix} -6 \\ 0 \end{pmatrix}$, $u(x,y) = 3x^2 + y^2$ its largest directional derivative 6.

□

-End-