HW 2-3 Solution

Q1. a)

$$((j2\pi f)+2)\,Y(f) = (1+j2\pi f)\,X(f)$$

freq response: $\Rightarrow H(f) = \dfrac{Y(f)}{X(f)} = 1 - \dfrac{1}{1+j2\pi f}$

impulse response: $h(t) = \delta(t) - e^{-2t}u(t)$

Step response

$$Y(f) = H(f)\,\mathcal{F}(u(t)) = H(f)\left(\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right) + \frac{1}{2}\left(\frac{1}{2+j2\pi f}\right)$$

$$y(t) = \frac{1}{2}u(t) + \frac{1}{2}e^{-2t}u(t)$$

b) $((j2\pi f)^2 + 6(j2\pi f) + 8)\,Y(f) = 2X(f).$

$$\Rightarrow H(f) = \frac{Y(f)}{X(f)} = \frac{1}{2+j2\pi f} - \frac{1}{4+j2\pi f}$$

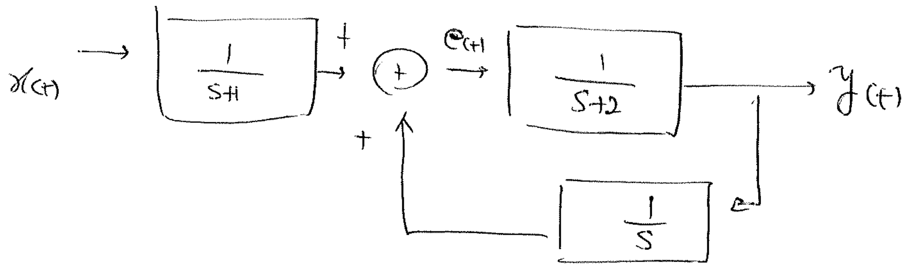
$$h(t) = e^{-2t}u(t) - e^{-4t}u(t)$$

Step response

$$Y(f) = H(f)\,\mathcal{F}(u(t)) = \frac{1}{4}\left(\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right) - \frac{1}{2}\cdot\frac{1}{2+j2\pi f} + \frac{1}{4}\,\frac{1}{4+j2\pi f}$$

$$y(t) = \frac{1}{4}u(t) - \frac{1}{2}e^{-2t}u(t) + \frac{1}{4}e^{-4t}u(t)$$

Q2.



$$\begin{cases} E(s) = \frac{1}{s+1} X(s) + \frac{1}{(s+2)s} E(s) & \text{— ①} \\ Y(s) = \frac{1}{s+2} E(s) & \text{— ②} \end{cases}$$

⇒ From ①, $\left(1 - \frac{1}{s(s+2)}\right) E(s) = \frac{1}{s+1} X(s)$ — ③

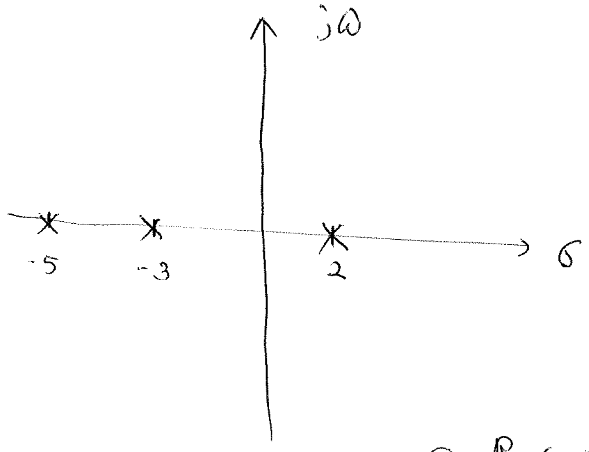
Substitute ③ to ②

$$Y(s) = \frac{1}{(s+2)} \cdot \frac{1}{(s+1)} \cdot \frac{1}{1 - \frac{1}{s(s+2)}} X(s)$$

$$= \frac{s}{s^3+3s^2+s-1} X(s)$$

⇒ ∴ $H(s) = \frac{Y(s)}{X(s)} = \frac{s}{s^3+3s^2+s-1}$

Q3. Three poles at $S=2$, $S=-3$, $S=-5$



a) Possible ROC = $\begin{cases} Re(s) > 2 \\ -3 < Re(s) < 2 \\ -5 < Re(S) < -3 \\ Re(S) < -5 \end{cases}$

b)

$Re(S) > 2$: Unstable Causal

$-3 < Re(S) < 2$: Stable, Noncausal

$-5 < Re(S) < -3$: Unstable, Noncausal

$Re(S) < -5$: Unstable, Noncausal

Q 4

a)

$$X(f) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n}$$

$$= - \sum_{n=-\infty}^{\infty} a^{n} u[-n-1] e^{-j2\pi f n}$$

$$= - \sum_{n=-\infty}^{-1} \left(a e^{-j2\pi f} \right)^{n} = - \sum_{l=1}^{\infty} \left(\frac{1}{a} e^{j2\pi f} \right)^{l}$$ (by using change of variable $n = -l$)

$$= - \frac{\frac{1}{a} e^{j2\pi f}}{1 - \frac{1}{a} e^{j2\pi f}}$$ when $\frac{1}{|a|} < 1 \Leftrightarrow |a| > 1$

$$= \frac{1}{1 - a e^{-j2\pi f}}$$ and $|a| > 1$

b) $$X(f) = \sum_{n=0}^{N-1} e^{-j2\pi f n} = \frac{1 - e^{-j2\pi f N}}{1 - e^{-j2\pi f}} = e^{-j\pi f (N-1)} \cdot \frac{\sin(\pi f N)}{\sin(\pi f)}$$

Q 5.

a)

$$\sum_{n=-\infty}^{\infty} x[n-n_0]e^{-j2\pi f n} = \overset{X(f)}{\overset{\|}{\left\{ \sum_{l=-\infty}^{\infty} x[l]\, e^{-j2\pi f l} \right\}}} \cdot e^{-j2\pi f n_0}$$

use change of variable $n - n_0 = l$

$$\Rightarrow \quad X(f) \cdot e^{-j2\pi f n_0}$$

b)

$$\sum_{n=-\infty}^{\infty} e^{j2\pi f_0 n}\, x[n]\, e^{-j2\pi f n} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j2\pi(f-f_0)n}$$

$$= X(f-f_0)$$

c)

$$\frac{d}{df}\left(X(f) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi f n} \right)$$

$$\Rightarrow \quad \sum_{n=-\infty}^{\infty} (-j2\pi n)\, x[n]\, e^{-j2\pi f n} = \frac{dX(f)}{df}$$

$$\sum_{n=-\infty}^{\infty} (n\, x[n])e^{-j2\pi f n} = \frac{j}{2\pi} \times \frac{dX}{df}.$$

Therefore $\quad n\, x[n] \longleftrightarrow \dfrac{j}{2\pi}\dfrac{dX}{df}$

Q 6.

a) $\mathcal{F}\left(\operatorname{Sinc}^2(4t)\right) = \frac{1}{4}\operatorname{tri}\left(\frac{f}{4}\right).$

b) $\mathcal{F}\left(\operatorname{sgn}(t)\right) = \frac{1}{j\pi f}$

c) $\mathcal{F}\left(e^{-2t}u(t)\right) = \frac{1}{2+j2\pi f}$

d) $\mathcal{F}\left(\operatorname{rect}\left(\frac{t}{3}\right)\right) = 3\operatorname{Sinc}(3f)$

Q7.

a) $X(s) = \frac{1}{s+5}$, $Re(s) > -5$

b) $X(s) = \frac{1}{s^2}$, $Re(s) > 0$.

c) $X_1(s) = \frac{1}{s+3}$, $Re(s) > -3$

$$\mathcal{L}(x_1(t-4)) = e^{-4s} X_1(s) = \frac{e^{-4s}}{s+3} , \quad Re(s) > -3$$

d) $X(s) = \frac{s^2+2s+5}{(s+3)(s+5)^2}$, $Re(s) > -3$

$$= \frac{2}{s+3} - \frac{1}{s+5} - \frac{10}{(s+5)^2}$$

$$x(t) = \left[2e^{-3t} - (1+10t)e^{-5t}\right] u(t)$$

Q 8.

$$X(z) = \sum_{n=0}^{\infty} \sin(\omega_0 n) \, (r z^{-1})^n$$

$$= \frac{1}{2j} \sum_{n=0}^{\infty} \left[e^{j\omega_0 n} - e^{-j\omega_0 n} \right] (r z^{-1})^n$$

$$= \frac{1}{2j} \left[\sum_{n=0}^{\infty} \left(e^{j\omega_0} r z^{-1} \right)^n - \sum_{n=0}^{\infty} \left(e^{-j\omega_0} r z^{-1} \right)^n \right]$$

$$= \frac{1}{2j} \left[\frac{1}{1 - e^{j\omega_0} r z^{-1}} - \frac{1}{1 - e^{-j\omega_0} r z^{-1}} \right] \quad \& \quad |r z^{-1}| < 1$$

$$\Rightarrow \text{ROC}: |z| > |r|$$

$$= \frac{2j \sin(\omega_0)(r z^{-1})}{2j \left(1 - 2\cos\omega_0 \, r z^{-1} + (r z^{-1})^2 \right)}$$

$$\therefore \; X(z) = \frac{\sin(\omega_0) \cdot r z^{-1}}{1 - 2\cos\omega_0 \cdot r z^{-1} + (r z^{-1})^2}$$ and ROC $|z| > |r|$

Q 9.

a) $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = 5z^{2} + 3z^{1} + 4 + z^{-1} - 2z^{-3}$

$\text{ROC} = \{0 < |z| < \infty\}$

b) $x[n] = \frac{1}{3} \cdot \frac{1}{4^n} u[n] - \frac{2}{2^n} u[n] + \frac{8}{3} u[n]$

$$X(z) = \frac{1}{3} \underbrace{\frac{z}{z - \frac{1}{4}}} - 2 \cdot \underbrace{\frac{z}{z-2}} + \frac{8}{3} \underbrace{\frac{z}{z-1}}$$

ROC: $|z| > \frac{1}{4}$ ROC: $|z| > 2$ ROC: $|z| > 1$

both forms are ok.

$$= \frac{z\left(z^{2} - \frac{9}{2}z + \frac{3}{2}\right)}{\left(z - \frac{1}{4}\right)(z-2)(z-1)}$$

with $\text{ROC} = \{|z| > 2\}$

c) $X(z) = \log\left(\frac{1}{1-z}\right)$ $|z| < 1$

$$= -\log(1-z) = \sum_{n=1}^{\infty} \frac{1}{n} z^{n} \xrightarrow{\text{change of variable } n = -l} = -\sum_{l=-\infty}^{-1} \frac{1}{l} z^{-l}$$

Since $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$, the inverse z-transform is

$x[n] = -\frac{1}{n} u[-n-1]$

d) $X(z) = \frac{3}{z-4}$, $|z| > 4$

$$= 3z^{-1} \cdot \frac{z}{z-4} \longrightarrow Z^{-1}(X(z)) = x[n] = 3 \cdot 4^{n-1} u[n-1]$$

Q 10.

$$Y(z) - 5\left(\frac{Y(z)}{z} + y[-1]\right) + 6\left(\frac{Y(z)}{z^2} + \frac{y[-1]}{z} + y[-2]\right)$$

$$= 3\left(\frac{X(z)}{z} + x[-1]\right) + 5\left(\frac{X(z)}{z^2} + \frac{x[-1]}{z} + x[-2]\right)$$

→ by using the initial condition

$$Y(z) - 5\left(\frac{Y(z)}{z} + \frac{11}{6}\right) + 6\left(\frac{Y(z)}{z^2} + \frac{11}{6z} + \frac{37}{36}\right) = \frac{3X(z)}{z} + \frac{5X(z)}{z^2} \quad -①$$

where the input is

$$X(z) = \frac{z}{z-0.5} \quad -②$$

→ Substitute ② into ① and by using partial fraction

$$Y(z) = \frac{26}{15}\cdot\frac{z}{z-0.5} - \frac{7}{3}\,\frac{z}{z-2} + \frac{18}{5}\,\frac{z}{z-3}$$

z^{-1}

$$\to y[n] = \frac{26}{15}(0.5)^n u[n] - \frac{7}{3}2^n u[n] + \frac{18}{5}3^n u[n]$$