Example 20

Find, in radians, the general solution of the equation

$$2\sin^2(2x) - 2\sin x\cos x - 1 = 0,$$

and give all the values of x which lie between 0 and 2π.

Solution

$$2\sin^2(2x) - \underbrace{2\sin x\cos x}_{=\sin 2x} - 1 = 0$$

$\Rightarrow \quad 2\sin^2(2x) - \sin(2x) - 1 = 0$ (by using the **Double angle formula**)

$\Rightarrow \quad [2\sin(2x)+1][\sin(2x)-1] = 0$

$\Rightarrow \quad 2\sin(2x)+1 = 0$ or $\sin(2x) - 1 = 0$

$\Rightarrow \quad \sin(2x) = -\frac{1}{2}$ or $\sin(2x) = 1$

$\therefore$ The general solution of the equation is

$\mathbf{2}x = n\pi + (-1)^n\,\alpha_1$, where $\alpha_1 = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$, for $n \in \mathbb{Z}$,

and $\mathbf{2}x = n\pi + (-1)^n\,\alpha_2$, where $\alpha_2 = \sin^{-1}(1) = \frac{\pi}{2}$, for $n \in \mathbb{Z}$.

That is, $\boxed{x = \frac{n\pi}{2} + (-1)^n \cdot \left(-\frac{\pi}{12}\right)}$ or $\boxed{x = \frac{n\pi}{2} + (-1)^n \cdot \frac{\pi}{4}}$, where $n \in \mathbb{Z}$.

For $x = \frac{n\pi}{2} + (-1)^n \cdot \left(-\frac{\pi}{12}\right)$:

When $n = 1$, $x = \frac{\pi}{2} + \frac{\pi}{12} = \frac{7\pi}{12}$

When $n = 2$, $x = \frac{2\pi}{2} - \frac{\pi}{12} = \frac{11\pi}{12}$

When $n = 3$, $x = \frac{3\pi}{2} + \frac{\pi}{12} = \frac{19\pi}{12}$

When $n = 4$, $x = \frac{4\pi}{2} - \frac{\pi}{12} = \frac{23\pi}{12}$

For $x = \frac{n\pi}{2} + (-1)^n \cdot \frac{\pi}{4}$:

When n=0,

$x = 0 + (-1)^0 \cdot \frac{\pi}{4} = \frac{\pi}{4}$

When $n = 1$, $x = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

When $n = 2$, $x = \frac{2\pi}{2} + \frac{\pi}{4} = \frac{5\pi}{4}$

(When $n = 3$, $x = \frac{3\pi}{2} - \frac{\pi}{4} = \frac{5\pi}{4}$)

Hence, the solutions which lie between 0 and 2π are

$$\boxed{\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{4}}.$$

Example 21

Find the general solution of the equation $\sin x + \cos x = 1$.

Solution

$$\begin{aligned}
\sin x + \cos x = 1 \quad &\Rightarrow \quad \sin x = 1 - \cos x \\
&\Rightarrow \quad 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = 2\sin^2\left(\frac{x}{2}\right) \\
&\Rightarrow \quad 2\sin\left(\frac{x}{2}\right)\left[\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right] = 0 \\
&\Rightarrow \quad 2\sin\left(\frac{x}{2}\right) = 0 \quad \text{or} \quad \cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right) = 0 \\
&\Rightarrow \quad \sin\left(\frac{x}{2}\right) = 0 \quad \text{or} \quad \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = 1 \\
&\Rightarrow \quad \sin\left(\frac{x}{2}\right) = 0 \quad \text{or} \quad \tan\left(\frac{x}{2}\right) = 1
\end{aligned}$$

using Double-angle formula & Half-angle formula

$\therefore$ The general solution of the equation is

$$\frac{x}{2} = n\pi + (-1)^n\alpha_1, \quad \text{where } \alpha_1 = \sin^{-1}(0) = 0, \quad \text{for } n \in \mathbb{Z},$$

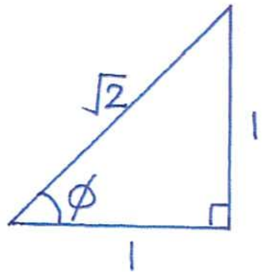
and
$$\frac{x}{2} = n\pi + \alpha_2, \quad \text{where } \alpha_2 = \tan^{-1}(1) = \frac{\pi}{4}, \quad \text{for } n \in \mathbb{Z}.$$

That is, $\boxed{x = 2n\pi}$ or $\boxed{x = 2n\pi + \frac{\pi}{2}}$, for $n \in \mathbb{Z}$.

Method 2:

$$\sin x + \cos x = 1$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{\sqrt{2}}$$



$$\Rightarrow \quad \sin\phi \sin x + \cos\phi \cos x = \frac{\sqrt{2}}{2}$$

$$\phi = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \cos\left(x - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$ by using Compound angle formula

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

where $A = x$, $B = \phi = \frac{\pi}{4}$

$$\Rightarrow \quad x - \frac{\pi}{4} = 2n\pi \pm \alpha \quad \text{where } \alpha = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

$$= 2n\pi \pm \frac{\pi}{4}$$

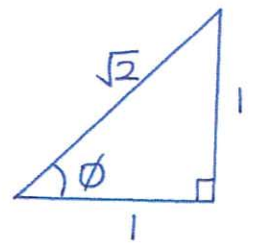
$$\Rightarrow \quad x = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{4}$$

i.e. $x = 2n\pi + \frac{\pi}{4} + \frac{\pi}{4} = 2n\pi + \frac{\pi}{2}$ ✓

or $x = 2n\pi - \frac{\pi}{4} + \frac{\pi}{4} = 2n\pi$ ✓, $n \in \mathbb{Z}$

Method 3:

$$\sin x + \cos x = 1$$



$$\Rightarrow \quad \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos\phi \sin x + \sin\phi \cos x = \frac{\sqrt{2}}{2}$$

$$\phi = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \sin\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

by using compound angle formula

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

where $A = x$, $B = \phi = \frac{\pi}{4}$

$$\Rightarrow \quad x + \frac{\pi}{4} = n\pi + (-1)^n \cdot \alpha \qquad \text{where } \alpha = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

$$= n\pi + (-1)^n \cdot \frac{\pi}{4}$$

$$\Rightarrow \quad x = n\pi + (-1)^n \cdot \frac{\pi}{4} - \frac{\pi}{4}, \qquad n \in \mathbb{Z}$$

Note: When $n = 2m$ is even, $x = 2m\pi + \underbrace{(-1)^{2m}}_{=1} \cdot \frac{\pi}{4} - \frac{\pi}{4} = 2m\pi$. ✓

When $n = 2m+1$ is odd, $x = (2m+1)\pi + \underbrace{(-1)^{2m+1}}_{=-1} \cdot \frac{\pi}{4} - \frac{\pi}{4} = 2m\pi + \frac{\pi}{2}$ ✓, $m \in \mathbb{Z}$.

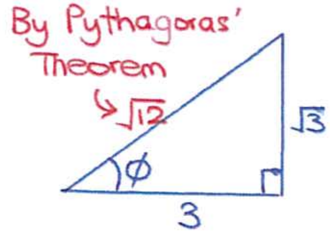
Additional Exercise

(a) Express $3\cos 4x + \sqrt{3}\sin 4x$ in the form $R\cos(4x - \phi)$, where $R > 0$ and $0 < \phi < \frac{\pi}{2}$.

(b) Find the general solution of $\cos 4x + \frac{1}{\sqrt{3}}\sin 4x = 1$.

Solution:

(a) $3\cos 4x + \sqrt{3}\sin 4x$



$$= \sqrt{12}\left(\frac{3}{\sqrt{12}}\cos 4x + \frac{\sqrt{3}}{\sqrt{12}}\sin 4x\right)$$

$$= 2\sqrt{3}\left(\cos\phi\cos 4x + \sin\phi\sin 4x\right)$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$$

$$= 2\sqrt{3}\cos\left(4x - \frac{\pi}{6}\right)$$

by using compound angle formula

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

where $A = 4x$, $B = \phi = \frac{\pi}{6}$

(b) $\cos 4x + \frac{1}{\sqrt{3}} \sin 4x = 1$

$\Rightarrow \quad 3 \cos 4x + \sqrt{3} \sin 4x = 3$

$\Rightarrow \quad 2\sqrt{3} \cos(4x - \frac{\pi}{6}) = 3$ (from (a))

$\Rightarrow \quad \cos(4x - \frac{\pi}{6}) = \frac{3}{2\sqrt{3}}$

$= \frac{\sqrt{3}}{2}$

$\Rightarrow \quad 4x - \frac{\pi}{6} = 2n\pi \pm \alpha$ where $\alpha = \cos^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{6}$

$= 2n\pi \pm \frac{\pi}{6}$

$\Rightarrow \quad x = \dfrac{2n\pi \pm \frac{\pi}{6} + \frac{\pi}{6}}{4}$, $n \in \mathbb{Z}$

i.e. $x = \dfrac{2n\pi + \frac{\pi}{6} + \frac{\pi}{6}}{4} = \frac{n\pi}{2} + \frac{\pi}{12}$

or $x = \dfrac{2n\pi - \frac{\pi}{6} + \frac{\pi}{6}}{4} = \frac{n\pi}{2}$, where $n \in \mathbb{Z}$

Example :

Find the general solutions of the equation :

$$4\sin x - 3\cos x = 5 .$$

Q. Is the following solution correct ?

$$4\sin x - 3\cos x = 5$$

$$\Rightarrow \quad 4\sin x = 5 + 3\cos x$$

$$\Rightarrow \quad 16\sin^2 x = 25 + 30\cos x + 9\cos^2 x$$

$$\Rightarrow \quad 16(1-\cos^2 x) = 25 + 30\cos x + 9\cos^2 x$$

$$\Rightarrow \quad 25\cos^2 x + 30\cos x + 9 = 0$$

$$\Rightarrow \quad (5\cos x + 3)^2 = 0$$

$$\Rightarrow \quad \cos x = -\frac{3}{5}$$

$$\Rightarrow \quad x = 2n\pi \pm \cos^{-1}\left(\frac{-3}{5}\right) , \quad n \in \mathbb{Z}$$

✓ or X ?

Let's put $x = 2n\pi \pm \cos^{-1}(-\frac{3}{5})$ back into the equation for some integers n.

When $n=0$: $x = \cos^{-1}(-\frac{3}{5})$.

LHS of equation $= 4\sin x - 3\cos x = 5$ ✓ (= RHS of equation)

OR $x = -\cos^{-1}(-\frac{3}{5})$. LHS of equation $= -1.4$ X ($\neq$ RHS of equation)

When $n=1$: $x = 2\pi + \cos^{-1}(-\frac{3}{5})$

LHS of equation $= 5$ ✓ (= RHS of equation)

OR $x = 2\pi - \cos^{-1}(-\frac{3}{5})$

LHS of equation $= -1.4$ X ($\neq$ RHS of equation)

When $n=2$: $x = 4\pi + \cos^{-1}(-\frac{3}{5})$

LHS of equation $= 5$ ✓

OR $x = 4\pi - \cos^{-1}(-\frac{3}{5})$

LHS of equation $= -1.4$ X

Some of the solutions are correct, some are wrong. Why?

Reason:

When we square both sides

$$(4\sin x)^2 = (5+3\cos x)^2,$$

its solutions consist of solutions to

$$4\sin x = 5 + 3\cos x \leftarrow 4\sin x - 3\cos x = 5$$

(our original equation)

as well as solutions to

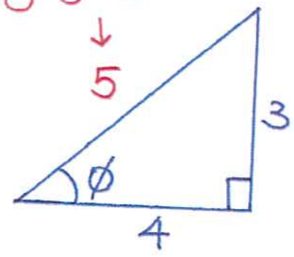
$$4\sin x = -(5+3\cos x). \leftarrow 4\sin x + 3\cos x = -5$$

(Not our original equation)

Correct solution:

$$4 \sin x - 3 \cos x = 5$$

By Pythagora's Theorem



$$\Rightarrow \quad \frac{4}{5} \sin x - \frac{3}{5} \cos x = 1$$

$$\Rightarrow \quad \cos \phi \, \sin x - \sin \phi \, \cos x = 1$$

$$\phi = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\Rightarrow \quad \sin(x - \phi) = 1$$

by compound angle formula

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

where $A = x$, $B = \phi = \tan^{-1}\left(\frac{3}{4}\right)$.

$$\Rightarrow \quad x - \phi = n\pi + (-1)^n \cdot \alpha, \quad \text{where } \alpha = \sin^{-1}(1) = \frac{\pi}{2}$$

$$= n\pi + (-1)^n \cdot \frac{\pi}{2}$$

$\therefore$ The general solution is

$$x = n\pi + (-1)^n \cdot \frac{\pi}{2} + \tan^{-1}\left(\frac{3}{4}\right), \quad \text{where } n \in \mathbb{Z}.$$

Method 2 :

$$4 \sin x - 3 \cos x = 5$$

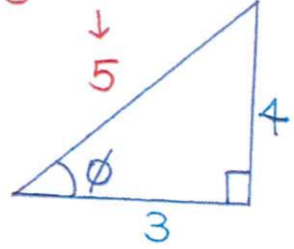
$$\Rightarrow \quad \frac{4}{5} \sin x - \frac{3}{5} \cos x = 1$$

$$\Rightarrow \quad \sin\phi \sin x - \cos\phi \cos x = 1$$

$$\Rightarrow \quad \cos x \cos\phi - \sin x \sin\phi = -1$$

$$\Rightarrow \quad \cos(x + \phi) = -1$$

By Pythagora's Theorem



$$\phi = \tan^{-1}\left(\frac{4}{3}\right)$$

by compound angle formula

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

where $A = x$, $B = \phi = \tan^{-1}\left(\frac{4}{3}\right)$.

$$\Rightarrow \quad x + \phi = 2n\pi \pm \alpha \text{ ,} \qquad \text{where } \alpha = \cos^{-1}(-1) = \pi$$

$$= 2n\pi \pm \pi$$

$\therefore$ The general solution is

$$x = 2n\pi \pm \pi - \tan^{-1}\left(\frac{4}{3}\right) \text{ ,} \qquad \text{where } n \in \mathbb{Z}.$$