CITY UNIVERSITY OF HONG KONG

Course code and title	:	MA1200 Calculus and Basic Linear Algebra I Section CA1, CB1, CC1 and CD1 Test 1
Session	:	Semester A, 2017/2018
Time	:	12:30 - 13:30, 20 October 2017 (Friday)
Time allowed	:	1 hour

This paper has **TWO** pages (including this cover page).

Instructions to candidates:

1. This paper has **FIVE** questions.
2. Attempt **ALL** questions.

This is a closed-book test.

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NOT TO BE TAKEN AWAY

Question 1

(a) The straight line $\frac{x}{5}+\frac{y}{12}=1$ meets the x-axis at P and the y-axis at Q.
Find the equation of the perpendicular bisector of PQ. (10 marks)

(b) Find the equation of the circle on PQ as diameter. (10 marks)

Question 2

Express $\frac{2x^2-3x+18}{(x-2)(x^2-x+2)}$ in partial fractions. (20 marks)

Question 3

Let $f(x)$ be a periodic function of x with period 3 and $f(x)=x^2$ for $-1<x\le 2$.

Sketch the graph of the curve $y=f(x)$ in the interval $[-4,5]$. (20 marks)

Question 4

(a) If $t=\tan\frac{x}{2}$, show that $\tan x+\sec x=\frac{1+t}{1-t}$.

(Hint: You may use $\tan x=\frac{2t}{1-t^2}$, $\cos x=\frac{1-t^2}{1+t^2}$, where $t=\tan\frac{x}{2}$.) (10 marks)

(b) Find the general solution of $2\cos x=-1$. (10 marks)

Question 5

Let $F(x)$ and $G(x)$ be two functions defined by

$F(x)=\frac{1}{x}$,

$G(x)=\sqrt{x}$.

(a) Find their largest possible domains and ranges. (12 marks)

(b) Find $(G\circ F)(x)$ and state its largest possible domain. (8 marks)

- END -

MA1200, CA1, CB1, CC1, CD1, 2017/18, Sem A.
Test 1, Suggested solutions.

Question 1

(a) The mid-point of PQ is

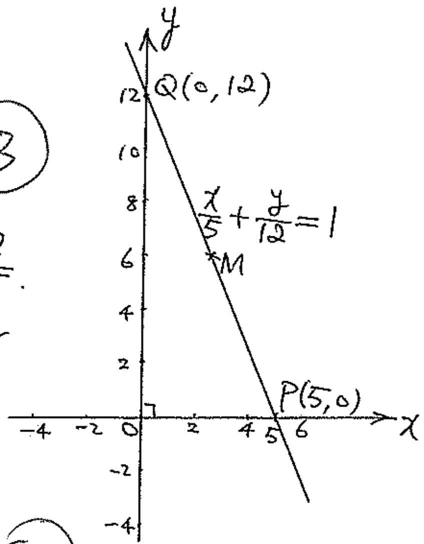
$M\left(\frac{0+5}{2}, \frac{12+0}{2}\right)$. 3

Slope of $PQ = \frac{0-12}{5-0} = -\frac{12}{5}$.

The equation of the perpendicular bisector of PQ is given by

$\frac{y-6}{x-\frac{5}{2}} = \frac{5}{12}$, that is

$10x - 24y + 119 = 0$. // 7



(b) The radius of the required circle is given by

$r = \frac{1}{2}PQ = \frac{1}{2}\sqrt{(5-0)^2+(0-12)^2} = \frac{13}{2}$ units.

$\therefore$ The equation of the required circle is

$\left(x-\frac{5}{2}\right)^2 + (y-6)^2 = \left(\frac{13}{2}\right)^2$, that is

$x^2+y^2-5x-12y=0$. // 10

Question 2 Method I

Let $\frac{2x^2-3x+18}{(x-2)(x^2-x+2)} = \frac{A}{x-2} + \frac{Bx+C}{x^2-x+2}$. 5

Then $2x^2-3x+18 = A(x^2-x+2) + (Bx+C)(x-2)$ 5

Put $x=2$, then $2(2^2)-3(2)+18 = A(2^2-2+2)+0$,

$\therefore A = 5$

So, $2x^2-3x+18-5(x^2-x+2) = (Bx+C)(x-2)$

$= -3x^2+2x+8$

$= (-3x-4)(x-2)$, $\therefore Bx+C = -3x-4$

$\therefore \frac{2x^2-3x+18}{(x-2)(x^2-x+2)} = \frac{5}{x-2} - \frac{3x+4}{x^2-x+2}$. // 10

Method II (Method of undetermined coefficients)

Let $\frac{2x^2-3x+18}{(x-2)(x^2-x+2)} = \frac{A}{x-2} + \frac{Bx+C}{x^2-x+2}$. 5

Then $2x^2-3x+18 = A(x^2-x+2) + (Bx+C)(x-2)$ 5

$= (A+B)x^2 + (-A-2B+C)x + 2A-2C$

Comparing the corresponding coefficients, we have

$$\left.\begin{aligned} A+B &= 2 \\ -A-2B+C &= -3 \\ 2A-2C &= 18 \end{aligned}\right\} (*)$$

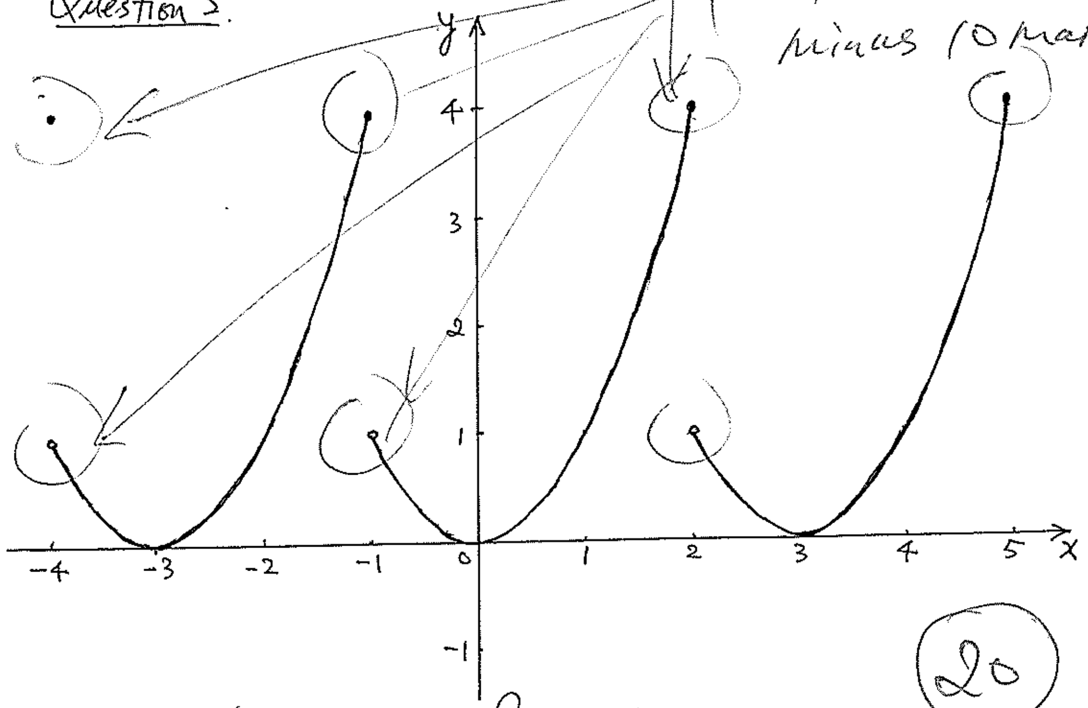
Solving the system (*) for A, B and C, we get

$A=5$, $B=-3$, $C=-4$.

Therefore, $\frac{2x^2-3x+18}{(x-2)(x^2-x+2)} = \frac{5}{x-2} - \frac{3x+4}{x^2-x+2}$. // 10

Question 3.

(Without these: o minus 10 marks)



20

Graph of $y = f(x)$, for $x \in [-4, 5]$

Question 4.

(a)

$$\tan x + \sec x = \frac{2t}{1-t^2} + \frac{1+t^2}{1-t^2}, \quad \text{since } \sec x = \frac{1}{\cos x}.$$

$$= \frac{2t + 1 + t^2}{1-t^2}$$

$$= \frac{(t+1)^2}{(1-t)(1+t)} = \frac{t+1}{1-t},$$

10

where $t = \tan\frac{x}{2}$. //

(b)

$$2\cos x = -1$$

$$\cos x = -\frac{1}{2}$$

The general solution of the trigonometric equation is $x = 2n\pi \pm \alpha$,

10

where $\alpha = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$, $n = 0, \pm 1, \pm 2, \dots$

Question 5.

(a)

$$F(x) = \frac{1}{x}, \quad G(x) = \sqrt{x}$$

$\mathrm{Dom}(F) = \mathbb{R} \setminus \{0\}$ = the set of all real numbers except 0. 3

$\mathrm{Ran}(F) = \mathbb{R} \setminus \{0\}$. 3

$\mathrm{Dom}(G) = [0, \infty)$. 3

$\mathrm{Ran}(G) = [0, \infty)$. 3

(b)

$$(G \circ F)(x) = G(F(x))$$

$$= G\left(\frac{1}{x}\right)$$

$$= \sqrt{\frac{1}{x}}$$

4

$$\left.\begin{array}{l} \frac{1}{x} \geq 0 \\ x \neq 0 \end{array}\right\} \Rightarrow x > 0, \quad \therefore \mathrm{Dom}(G \circ F) = (0, \infty)$$

4

//

CITY UNIVERSITY OF HONG KONG

Course code and title	:	MA1200 Calculus and Basic Linear Algebra I Section CE1, CF1, CG1 and CH1 Test 1
Session	:	Semester A, 2017/2018
Time	:	17:30 - 18:30, 20 October 2017 (Friday)
Time allowed	:	1 hour

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NOT TO BE TAKEN AWAY

Question 1

Show that the equation $25x^2+50x+169y^2-676y=3524$ represents an ellipse, and find the coordinates of its centre, vertices and foci.

(Hint: You may use the method of completing the squares.) (20 marks)

Question 2

(a) It is given that $\cos A=\frac{3}{5}$, where $270°<A<360°$, and that $\sin B=\frac{5}{13}$, where $90°<B<180°$. Without using a calculator, find the values of

(i) $\cos(A-B)$,

(ii) $\cos\frac{A}{2}$.

(Hint: You may use $\cos(A-B)=\cos A\cos B+\sin A\sin B$
$\cos(2x)=2\cos^2 x-1$.) (10 marks)

(b) Find the general solution of $2\sin x=-1$. (10 marks)

Question 3

Let $f(x)=\sqrt{(1+x)(3-x)}$.
Find the largest possible domain and the largest possible range of $f(x)$. (20 marks)

Question 4

Express $\frac{6x^2+30x+52}{(x-3)(2x+1)^2}$ in partial fractions. (20 marks)

Question 5

Let $F(x)$ and $G(x)$ be two functions defined by
$F(x)=2x-3$ for $x\in[-1,\infty)$,
$G(x)=x^3$ for $x\in\mathrm{R}$.

(a) Find the inverse function for each and state its largest possible domain. (12 marks)

(b) Find $(\frac{G}{F})(x)$ and state its largest possible domain. (8 marks)

- END -

MA1200, CE1, CF1, CG1, CH1, 2017/18, Sem. A
Test 1, suggested solutions

Question 1.

$$25x^2+50x+169y^2-676y=3524$$

$$25(x^2+2x+1)+169(y^2-4y+4)=3524+25\times1+169\times4$$

$$25(x+1)^2+169(y-2)^2=4225$$

$$\frac{(x+1)^2}{13^2}+\frac{(y-2)^2}{5^2}=1,$$

5

which is the equation of an ellipse with centre at $(-1,2)$.
Comparing the equation with the standard form of the equation of an ellipse,

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1\quad,\ a>b>0,$$

we have $h=-1$, $k=2$,

$a=13$, $b=5$, $b^2=a^2(1-e^2)$

$$\Rightarrow 25=169(1-e^2)$$

$$\Rightarrow e^2=1-\frac{25}{169}=\frac{144}{169}$$

$e=\frac{12}{13}$, $ae=(13)\left(\frac{12}{13}\right)=12$

3

The coordinates of its centre are $(-1,2)$

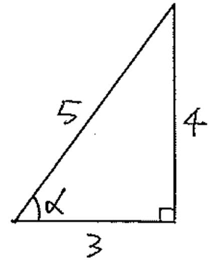
The coordinates of its vertices are $(-14,2)$, $(12,2)$, 2, 2

$(-1,-3)$, $(-1,7)$. 2, 2

The coordinates of its foci are $(-13,2)$, $(11,2)$. 2, 2

Question 2.

(a)



$0^\circ<\alpha<90^\circ$

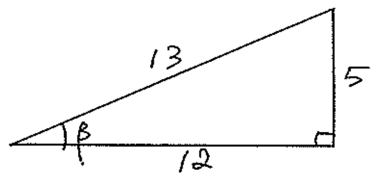
$\cos\alpha=\frac{3}{5}$

$\sin\alpha=\frac{4}{5}$

$270^\circ<A<360^\circ$

$\cos A=\frac{3}{5}$

$\sin A=-\frac{4}{5}$



$0^\circ<\beta<90^\circ$

$\sin\beta=\frac{5}{13}$

$\cos\beta=\frac{12}{13}$

$90^\circ<B<180^\circ$

$\sin B=\frac{5}{13}$

$\cos B=-\frac{12}{13}$

(i) $$\cos(A-B)=\cos A\cos B+\sin A\sin B$$
$$=\left(\frac{3}{5}\right)\left(-\frac{12}{13}\right)+\left(-\frac{4}{5}\right)\left(\frac{5}{13}\right)$$
$$=-\frac{36}{65}-\frac{20}{65}$$
$$=-\frac{56}{65}$$

5

(ii) $$\cos\frac{A}{2}=-\sqrt{\frac{1+\cos A}{2}}<0,\ \text{since } 135^\circ<\frac{A}{2}<180^\circ$$
$$=-\sqrt{\frac{1+\frac{3}{5}}{2}}$$
$$=-\sqrt{\frac{4}{5}}$$
$$=-0.894427\quad(\text{corr. to 6 d.p.})$$

5

(b) $2\sin x = -1 \Rightarrow \sin x = -\frac{1}{2}$

The general solution of the trigonometric equation is

$$x = n\pi + (-1)^n\left(-\frac{\pi}{6}\right), \quad n = 0, \pm 1, \pm 2, \cdots$$ (10)

Question 3

$$f(x) = \sqrt{(1+x)(3-x)}$$
$$= \sqrt{3+2x-x^2}$$
$$= \sqrt{4-(x^2-2x+1)} = \sqrt{4-(x-1)^2}$$

$(1+x)(3-x) \geq 0$

Solution of the inequality

$x \in [-1, 3]$.

	$\leftarrow x=-1$		$x=3 \rightarrow$
$1+x$	$-$	$+$	$+$
$3-x$	$+$	$+$	$-$
$(1+x)(3-x)$	$(-)$	$(+)$	$(-)$
	X	✓	X

(6)

$\text{Dom}(f) = [-1, 3]$. (7)

$\text{Ran}(f) = [0, 2]$. (7)

Question 4.

Let $\dfrac{6x^2+30x+52}{(x-3)(2x+1)^2} = \dfrac{A}{x-3} + \dfrac{B}{2x+1} + \dfrac{C}{(2x+1)^2}$. (5)

Then

$6x^2+30x+52 = A(2x+1)^2 + B(x-3)(2x+1) + C(x-3)$ (5)

Put $x=3$, then

$54+90+52 = 49A+0+0$, $\therefore A = \frac{196}{49} = 4$

So, $6x^2+30x+52-4(2x+1)^2 = (x-3)[B(2x+1)+C]$

$= -10x^2+14x+48$

$= (x-3)(-10x-16)$

$\Rightarrow -10x-16 = B(2x+1)+C$

Put $x = -\frac{1}{2}$, then $5-16 = 0+C$, $\therefore C = -11$

So, $-10x-16+11 = B(2x+1)$

$= -5(2x+1)$, $\therefore B = -5$

$\therefore \dfrac{6x^2+30x+52}{(x-3)(2x+1)^2} = \dfrac{4}{x-3} - \dfrac{5}{2x+1} - \dfrac{11}{(2x+1)^2}$. (10)

Question 5.

(a) $F(x)$ and $G(x)$ are one-to-one functions of x

Let $y = F(x) = 2x-3$. Then $x = \frac{1}{2}(y+3)$.

$\therefore F^{-1}(x) = \frac{1}{2}(x+3)$ for $x \in [-5, \infty)$. (6)

$\text{Dom}(F^{-1}) = [-5, \infty)$.

Let $z = G(x) = x^3$. Then $x = z^{\frac{1}{3}}$.

$\therefore G^{-1}(x) = x^{\frac{1}{3}}$ for $x \in R$. (6)

$\text{Dom}(G^{-1}) = R$.

(b) $\left(\frac{G}{F}\right)(x) = \dfrac{G(x)}{F(x)} = \dfrac{x^3}{2x-3}$. (4)

$\text{Dom}\left(\frac{G}{F}\right) = \text{Dom}(G) \cap \text{Dom}(F)$ except that values of x for which $F(x) = 0$. (4)

$= [-1, \frac{3}{2}) \cup (\frac{3}{2}, \infty)$

CITY UNIVERSITY OF HONG KONG

Course code and title : MA1200 Calculus and Basic Linear Algebra I
Section C61
Test 1

Session : Semester A, 2017/2018

Time : 18:00 - 19:00, 16 October 2017 (Monday)

Time allowed : 1 hour

This paper has **TWO** pages (including this cover page).

Instructions to candidates:

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NOT TO BE TAKEN AWAY

Question 1

Find the equation of the circle which passes through the points (-2, -2), (1, 3), (3, 3). (20 marks)

Question 2

If $A+B+C=180^\circ$, show that $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

(Hint: You may use

$$\sin x+\sin y=2\sin\frac{x+y}{2}\cos\frac{x-y}{2} ,$$

$$\cos x+\cos y=2\cos\frac{x+y}{2}\cos\frac{x-y}{2} ,$$

$$\sin(2x)=2\sin x\cos x ,$$

$$\sin(90^\circ-x)=\cos x \text{ .)}$$

(20 marks)

Question 3

Express $\dfrac{4x^2+11x+12}{(x+2)^3}$ in partial fractions. (20 marks)

Question 4

(a) Let $f(x)=x^2+2x-1$ for $x\in[-1,\infty)$. Sketch its graph.
Find $f^{-1}(x)$ and state its largest possible domain. (12 marks)

(b) Let $g(x)=\sqrt{\dfrac{x-2}{x+3}}$.
Find the largest possible domain of $g(x)$. (8 marks)

Question 5

Let $F(x)$ and $G(x)$ be two functions defined by
$F(x)=[x]$, where $[x]$ denotes the greatest integer not greater than x ,

$$G(x)=|x|=\begin{cases} x & \text{, if } x\ge 0 \\ -x & \text{, if } x<0 \end{cases} .$$

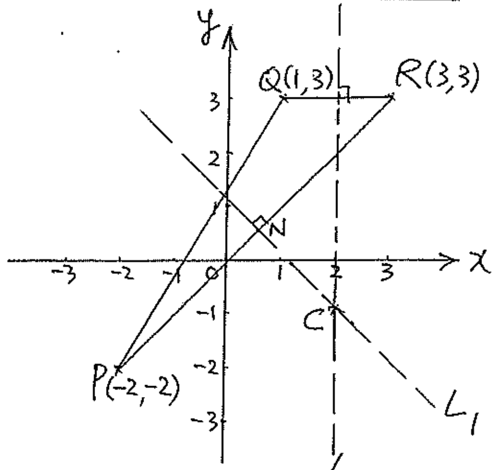
Find the largest possible ranges of $F(x)$ and $G(x)$, and sketch the graphs of these functions for $-2\le x\le 2$. (20 marks)

MA1200, C61, Test 1, 2017/2018, Sem. A.

Suggested solutions.

Q.1. Method I.

In the figure,



Slope of $PR = \dfrac{-2-3}{-2-3} = 1$,

Mid-point of PR is

$N\left(\frac{-2+3}{2}, \frac{-2+3}{2}\right)$,

slope of $L_1 = -1$

Equation of L_1 is given by

$$\frac{y-\frac{1}{2}}{x-\frac{1}{2}} = -1, \text{ that is } y = -x+1 \quad \text{—— ①} \qquad (3)$$

Equation of L_2 (the perpendicular bisector of QR) is

$$x = 2 \quad \text{—— ②} \qquad (2)$$

Solving the equations ① and ②, we obtain $x=2$, $y=-1$.

The coordinates of the circum-centre, C of $\triangle PQR$ is

$(2, -1)$ (5)

The radius of the circum-circle of $\triangle PQR$ is given by

$$r = \sqrt{(3-2)^2+(3-(-1))^2} \quad (= RC) \qquad (3)$$

$$= \sqrt{17}$$

$\therefore$ The equation of the required circle is

$(x-2)^2+(y+1)^2 = 17$, that is

$$x^2+y^2-4x+2y-12=0. \; // \qquad (7)$$

Method II.

Let $x^2+y^2+Dx+Ey+F=0$ be the equation of the circle which passes through the points $(-2,-2)$, $(1,3)$, $(3,3)$, where D, E, F are unknown constants.

Then

$(-2)^2+(-2)^2-2D-2E+F=0$ —— ①

$1^2+3^2+D+3E+F=0$ —— ②

$3^2+3^2+3D+3E+F=0$ —— ③ (10)

③ − ②: $8+2D=0$, $\therefore D=-4$

③ + $\frac{3}{2}$①: $18+\frac{3}{2}\times 8+\frac{5}{2}F=0$, $\therefore F=-12$

Substituting $D=-4$, $F=-12$ into ①, we get

$8+8-2E-12=0$, $\therefore E=2$

$\therefore$ The equation of the required circle is

$$x^2+y^2-4x+2y-12=0. \; // \qquad (10)$$

Q.2. Proof:

(5)

$$LHS = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$= 2\sin\left(90^\circ-\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2}$$

$$= 2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)+\sin\left(90^\circ-\frac{A+B}{2}\right)\right]$$

$$= 2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{A+B}{2}\right)\right] \qquad (5)$$

$$= 2\cos\frac{C}{2}\left[2\cos\left(\frac{\frac{A+B}{2}+\frac{A-B}{2}}{2}\right)\cos\left(\frac{\frac{A+B}{2}-\frac{A-B}{2}}{2}\right)\right]$$

$$= 2\cos\frac{C}{2}\left[2\cos\frac{A}{2}\cos\frac{B}{2}\right] = RHS. \text{ Hence the result } // \qquad (10)$$

Question 3. Method I

Let $\dfrac{4x^2+11x+12}{(x+2)^3} = \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2} + \dfrac{C}{(x+2)^3}$. 5

Then $4x^2+11x+12 = A(x+2)^2 + B(x+2) + C$ 5

Put $x=-2$, we have

$4(-2)^2+11(-2)+12 = 0+0+C$, $\therefore C = 6$

So, $4x^2+11x+12-6 = (x+2)[A(x+2)+B]$

$= (x+2)(4x+3)$

$\Rightarrow 4x+3 = A(x+2)+B$

Put $x=-2$, we have $4(-2)+3 = 0+B$

$\therefore B = -5$

So, $4x+3+5 = A(x+2)$

$= 4(x+2)$, $\therefore A = 4$

Therefore, $\dfrac{4x^2+11x+12}{(x+2)^3} = \dfrac{4}{x+2} - \dfrac{5}{(x+2)^2} + \dfrac{6}{(x+2)^3}$ // 10

Method II. By synthetic division, -2

4	11	12
	−8	−6
4	3	$\triangle$ 6
	−8	
$\triangle$ 4	$\triangle$ −5	

10

$\therefore 4x^2+11x+12 = 4(x+2)^2 - 5(x+2) + 6$

$\therefore \dfrac{4x^2+11x+12}{(x+2)^3} = \dfrac{4}{x+2} - \dfrac{5}{(x+2)^2} + \dfrac{6}{(x+2)^3}$ // 10

Method III.

Let $u = x+2$. Then $x = u-2$ 10

$\dfrac{4x^2+11x+12}{(x+2)^3} = \dfrac{4(u-2)^2+11(u-2)+12}{u^3}$

$= \dfrac{4(u^2-4u+4)+11u-22+12}{u^3}$

$= \dfrac{4u^2-5u+6}{u^3}$

$= \dfrac{4}{x+2} - \dfrac{5}{(x+2)^2} + \dfrac{6}{(x+2)^3}$. // 10

Question 4.

(a) $f(x) = x^2+2x-1$

$= (x+1)^2-2$ for $x \in [-1, \infty)$

$f(x)$ is one to one.

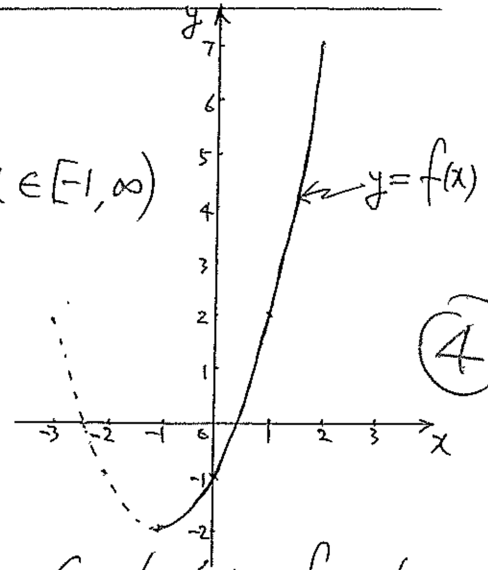
Let $y = f(x)$

$= (x+1)^2 - 2$.

Then $x = -1 \pm \sqrt{y+2}$.

$f^{-1}(x) \neq -1-\sqrt{x+2} < 0$, for all x. (rejected)

$\therefore f^{-1}(x) = -1+\sqrt{x+2}$ for $x \in [-2, \infty)$. 5 //



Graph of $y = f(x)$, for $x \geq -1$

4

Its largest possible domain is

$Dom(f^{-1}) = Ran(f) = [-2, \infty)$. // 3

(b) $g(x) = \sqrt{\dfrac{x-2}{x+3}}$

$$\left.\begin{aligned} &\frac{x-2}{x+3} \geq 0, \\ &x+3 \neq 0 \end{aligned}\right\}$$

	$\leftarrow$	$x=-3$	$x=2$ $\rightarrow$
$x-2$	$-$	$-$	$+$
$x+3$	$-$	$+$	$+$
$\frac{x-2}{x+3}$	$(+)$	$(-)$	$(+)$
	✓	X	✓

$\Rightarrow x \in (-\infty, -3) \cup [2, \infty)$

$\therefore$ The largest possible domain of $g(x)$ is

$\text{Dom}(g) = (-\infty, -3) \cup [2, \infty)$. //

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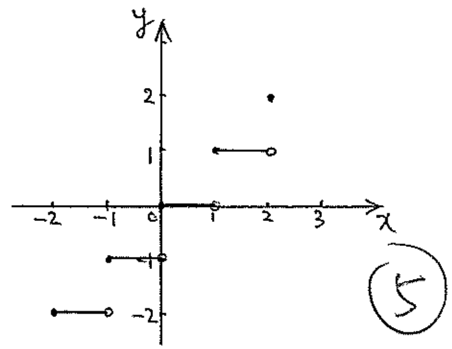
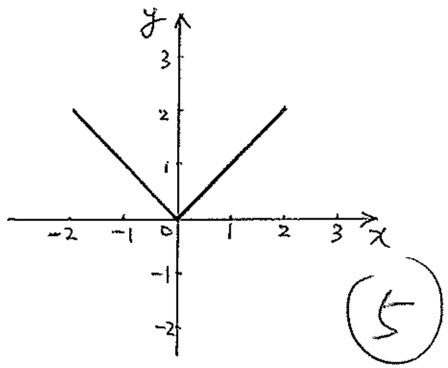
Question 5





Graph of $y = [x]$ for $x \in [-2, 2]$ Graph of $y = |x|$ for $-2 \leq x \leq 2$

$\text{Ran}(F) = \mathbb{Z}$ = the set of all integers. 5

$\text{Ran}(G) = [0, \infty)$. 5
