EXERCISES 5.1

Area

In Exercises 1–4 use finite approximations to estimate the area under the graph of the function using

a. a lower sum with two rectangles of equal width.

b. a lower sum with four rectangles of equal width.

c. an upper sum with two rectangles of equal width.

d. an upper sum with four rectangles of equal width.

1. $f(x) = x^2$ between $x = 0$ and $x = 1$.

2. $f(x) = x^3$ between $x = 0$ and $x = 1$.

3. $f(x) = 1/x$ between $x = 1$ and $x = 5$.

4. $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$.

Using rectangles whose height is given by the value of the function at the midpoint of the rectangle's base (*the midpoint rule*) estimate the area under the graphs of the following functions, using first two and then four rectangles.

5. $f(x) = x^2$ between $x = 0$ and $x = 1$.

6. $f(x) = x^3$ between $x = 0$ and $x = 1$.

7. $f(x) = 1/x$ between $x = 1$ and $x = 5$.

8. $f(x) = 4 - x^2$ between $x = -2$ and $x = 2$.

Distance

9. Distance traveled The accompanying table shows the velocity of a model train engine moving along a track for 10 sec. Estimate the distance traveled by the engine using 10 subintervals of length 1 with

a. left-endpoint values.

b. right-endpoint values.

Time (sec)	Velocity (in./sec)	Time (sec)	Velocity (in./sec)
0	0	6	11
1	12	7	6
2	22	8	2
3	10	9	6
4	5	10	0
5	13		

10. Distance traveled upstream You are sitting on the bank of a tidal river watching the incoming tide carry a bottle upstream. You record the velocity of the flow every 5 minutes for an hour, with the results shown in the accompanying table. About how far upstream did the bottle travel during that hour? Find an estimate using 12 subintervals of length 5 with

a. left-endpoint values.

b. right-endpoint values.

Time (min)	Velocity (m/sec)	Time (min)	Velocity (m/sec)
0	1	35	1.2
5	1.2	40	1.0
10	1.7	45	1.8
15	2.0	50	1.5
20	1.8	55	1.2
25	1.6	60	0
30	1.4		

11. Length of a road You and a companion are about to drive a twisty stretch of dirt road in a car whose speedometer works but whose odometer (mileage counter) is broken. To find out how long this particular stretch of road is, you record the car's velocity at 10-sec intervals, with the results shown in the accompanying table. Estimate the length of the road using

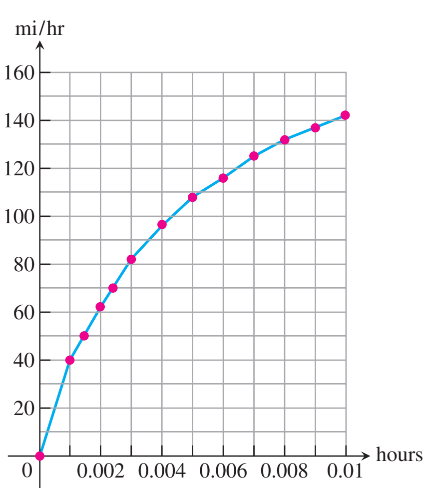
a. left-endpoint values.

b. right-endpoint values.

Time (sec)	Velocity (converted to ft/sec) (30 mi/h = 44 ft/sec)	Time (sec)	Velocity (converted to ft/sec) (30 mi/h = 44 ft/sec)
0	0	70	15
10	44	80	22
20	15	90	35
30	35	100	44
40	30	110	30
50	44	120	35
60	35		

12. Distance from velocity data The accompanying table gives data for the velocity of a vintage sports car accelerating from 0 to 142 mi/h in 36 sec (10 thousandths of an hour).

Time (h)	Velocity (mi/h)	Time (h)	Velocity (mi/h)
0.0	0	0.006	116
0.001	40	0.007	125
0.002	62	0.008	132
0.003	82	0.009	137
0.004	96	0.010	142
0.005	108		



a. Use rectangles to estimate how far the car traveled during the 36 sec it took to reach 142 mi/h.

b. Roughly how many seconds did it take the car to reach the halfway point? About how fast was the car going then?

Velocity and Distance

13. Free fall with air resistance An object is dropped straight down from a helicopter. The object falls faster and faster but its acceleration (rate of change of its velocity) decreases over time because of air resistance. The acceleration is measured in ft/sec^2 and recorded every second after the drop for 5 sec, as shown:

t	0	1	2	3	4	5
a	32.00	19.41	11.77	7.14	4.33	2.63

a. Find an upper estimate for the speed when $t = 5$.

b. Find a lower estimate for the speed when $t = 5$.

c. Find an upper estimate for the distance fallen when $t = 3$.

14. Distance traveled by a projectile An object is shot straight upward from sea level with an initial velocity of 400 ft/sec.

a. Assuming that gravity is the only force acting on the object, give an upper estimate for its velocity after 5 sec have elapsed. Use $g = 32$ ft/sec^2 for the gravitational acceleration.

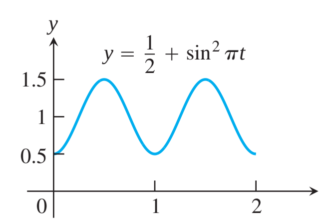
b. Find a lower estimate for the height attained after 5 sec.

Average Value of a Function

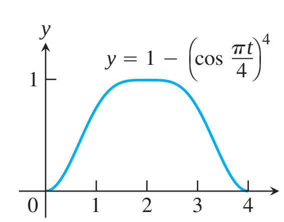
In Exercises 15–18, use a finite sum to estimate the average value of f on the given interval by partitioning the interval into four subintervals of equal length and evaluating f at the subinterval midpoints.

15. $f(x) = x^3$ on $[0, 2]$ **16.** $f(x) = 1/x$ on $[1, 9]$

17. $f(t) = (1/2) + \sin^2 \pi t$ on $[0, 2]$



18. $f(t) = 1 - \left(\cos \frac{\pi t}{4}\right)^4$ on $[0, 4]$



Pollution Control

19. Water pollution Oil is leaking out of a tanker damaged at sea. The damage to the tanker is worsening as evidenced by the increased leakage each hour, recorded in the following table.

Time (h)	0	1	2	3	4	5	6	7	8
Leakage (gal/h)	50	70	97	136	190	265	369	516	720

a. Give an upper and a lower estimate of the total quantity of oil that has escaped after 5 hours.

b. Repeat part (a) for the quantity of oil that has escaped after 8 hours.

c. The tanker continues to leak 720 gal/h after the first 8 hours. If the tanker originally contained 25,000 gal of oil, approximately how many more hours will elapse in the worst case before all the oil has spilled? In the best case?

20. Air pollution A power plant generates electricity by burning oil. Pollutants produced as a result of the burning process are removed by scrubbers in the smokestacks. Over time, the scrubbers become less efficient and eventually they must be replaced when the amount of pollution released exceeds government standards. Measurements are taken at the end of each month determining the rate at which pollutants are released into the atmosphere, recorded as follows.

Month	**Jan**	**Feb**	**Mar**	**Apr**	**May**	**Jun**
Pollutant Release rate (tons/day)	0.20	0.25	0.27	0.34	0.45	0.52

Month	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
Pollutant Release rate (tons/day)	0.63	0.70	0.81	0.85	0.89	0.95

a. Assuming a 30-day month and that new scrubbers allow only 0.05 ton/day released, give an upper estimate of the total tonnage of pollutants released by the end of June. What is a lower estimate?

b. In the best case, approximately when will a total of 125 tons of pollutants have been released into the atmosphere?

Area of a Circle

21. Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of the polygon for the following values of n:

a. 4 (square) **b.** 8 (octagon) **c.** 16

d. Compare the areas in parts (a), (b), and (c) with the area of the circle.

22. (*Continuation of Exercise 21*)

a. Inscribe a regular n-sided polygon inside a circle of radius 1 and compute the area of one of the n congruent triangles formed by drawing radii to the vertices of the polygon.

b. Compute the limit of the area of the inscribed polygon as $n \to \infty$.

c. Repeat the computations in parts (a) and (b) for a circle of radius r.

COMPUTER EXPLORATIONS

In Exercises 23–26, use a CAS to perform the following steps.

a. Plot the functions over the given interval.

b. Subdivide the interval into $n = 100, 200,$ and 1000 subintervals of equal length and evaluate the function at the midpoint of each subinterval.

c. Compute the average value of the function values generated in part (b).

d. Solve the equation $f(x) = $ (average value) for x using the average value calculated in part (c) for the $n = 1000$ partitioning.

23. $f(x) = \sin x$ on $[0, \pi]$ **24.** $f(x) = \sin^2 x$ on $[0, \pi]$

25. $f(x) = x \sin \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$

26. $f(x) = x \sin^2 \frac{1}{x}$ on $\left[\frac{\pi}{4}, \pi\right]$

EXERCISES 5.2

Sigma Notation

Write the sums in Exercises 1–6 without sigma notation. Then evaluate them.

1. $\displaystyle\sum_{k=1}^{2} \frac{6k}{k+1}$

2. $\displaystyle\sum_{k=1}^{3} \frac{k-1}{k}$

3. $\displaystyle\sum_{k=1}^{4} \cos k\pi$

4. $\displaystyle\sum_{k=1}^{5} \sin k\pi$

5. $\displaystyle\sum_{k=1}^{3} (-1)^{k+1} \sin \frac{\pi}{k}$

6. $\displaystyle\sum_{k=1}^{4} (-1)^{k} \cos k\pi$

7. Which of the following express $1 + 2 + 4 + 8 + 16 + 32$ in sigma notation?

   a. $\displaystyle\sum_{k=1}^{6} 2^{k-1}$ **b.** $\displaystyle\sum_{k=0}^{5} 2^{k}$ **c.** $\displaystyle\sum_{k=-1}^{4} 2^{k+1}$

8. Which of the following express $1 - 2 + 4 - 8 + 16 - 32$ in sigma notation?

   a. $\displaystyle\sum_{k=1}^{6} (-2)^{k-1}$ **b.** $\displaystyle\sum_{k=0}^{5} (-1)^{k} 2^{k}$ **c.** $\displaystyle\sum_{k=-2}^{3} (-1)^{k+1} 2^{k+2}$

9. Which formula is not equivalent to the other two?

   a. $\displaystyle\sum_{k=2}^{4} \frac{(-1)^{k-1}}{k-1}$ **b.** $\displaystyle\sum_{k=0}^{2} \frac{(-1)^{k}}{k+1}$ **c.** $\displaystyle\sum_{k=-1}^{1} \frac{(-1)^{k}}{k+2}$

10. Which formula is not equivalent to the other two?

   a. $\displaystyle\sum_{k=1}^{4} (k-1)^{2}$ **b.** $\displaystyle\sum_{k=-1}^{3} (k+1)^{2}$ **c.** $\displaystyle\sum_{k=-3}^{-1} k^{2}$

Express the sums in Exercises 11–16 in sigma notation. The form of your answer will depend on your choice of the lower limit of summation.

11. $1 + 2 + 3 + 4 + 5 + 6$

12. $1 + 4 + 9 + 16$

13. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$

14. $2 + 4 + 6 + 8 + 10$

15. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$

16. $-\frac{1}{5} + \frac{2}{5} - \frac{3}{5} + \frac{4}{5} - \frac{5}{5}$

Values of Finite Sums

17. Suppose that $\sum_{k=1}^{n} a_k = -5$ and $\sum_{k=1}^{n} b_k = 6$. Find the values of

a. $\sum_{k=1}^{n} 3a_k$ **b.** $\sum_{k=1}^{n} \frac{b_k}{6}$ **c.** $\sum_{k=1}^{n} (a_k + b_k)$

d. $\sum_{k=1}^{n} (a_k - b_k)$ **e.** $\sum_{k=1}^{n} (b_k - 2a_k)$

18. Suppose that $\sum_{k=1}^{n} a_k = 0$ and $\sum_{k=1}^{n} b_k = 1$. Find the values of

a. $\sum_{k=1}^{n} 8a_k$ **b.** $\sum_{k=1}^{n} 250b_k$

c. $\sum_{k=1}^{n} (a_k + 1)$ **d.** $\sum_{k=1}^{n} (b_k - 1)$

Evaluate the sums in Exercises 19–28.

19. a. $\sum_{k=1}^{10} k$ **b.** $\sum_{k=1}^{10} k^2$ **c.** $\sum_{k=1}^{10} k^3$

20. a. $\sum_{k=1}^{13} k$ **b.** $\sum_{k=1}^{13} k^2$ **c.** $\sum_{k=1}^{13} k^3$

21. $\sum_{k=1}^{7} (-2k)$

22. $\sum_{k=1}^{5} \frac{\pi k}{15}$

23. $\sum_{k=1}^{6} (3 - k^2)$

24. $\sum_{k=1}^{6} (k^2 - 5)$

25. $\sum_{k=1}^{5} k(3k + 5)$

26. $\sum_{k=1}^{7} k(2k + 1)$

27. $\sum_{k=1}^{5} \frac{k^3}{225} + \left(\sum_{k=1}^{5} k\right)^3$

28. $\left(\sum_{k=1}^{7} k\right)^2 - \sum_{k=1}^{7} \frac{k^3}{4}$

Rectangles for Riemann Sums

In Exercises 29–32, graph each function $f(x)$ over the given interval. Partition the interval into four subintervals of equal length. Then add to your sketch the rectangles associated with the Riemann sum $\Sigma_{k=1}^{4} f(c_k)\,\Delta x_k$, given that c_k is the (a) left-hand endpoint, (b) right-hand endpoint, (c) midpoint of the kth subinterval. (Make a separate sketch for each set of rectangles.)

29. $f(x) = x^2 - 1, \quad [0, 2]$

30. $f(x) = -x^2, \quad [0, 1]$

31. $f(x) = \sin x, \quad [-\pi, \pi]$

32. $f(x) = \sin x + 1, \quad [-\pi, \pi]$

33. Find the norm of the partition $P = \{0, 1.2, 1.5, 2.3, 2.6, 3\}$.

34. Find the norm of the partition $P = \{-2, -1.6, -0.5, 0, 0.8, 1\}$.

Limits of Upper Sums

For the functions in Exercises 35–40 find a formula for the upper sum obtained by dividing the interval $[a, b]$ into n equal subintervals. Then take a limit of these sums as $n \to \infty$ to calculate the area under the curve over $[a, b]$.

35. $f(x) = 1 - x^2$ over the interval $[0, 1]$.

36. $f(x) = 2x$ over the interval $[0, 3]$.

37. $f(x) = x^2 + 1$ over the interval $[0, 3]$.

38. $f(x) = 3x^2$ over the interval $[0, 1]$.

39. $f(x) = x + x^2$ over the interval $[0, 1]$.

40. $f(x) = 3x + 2x^2$ over the interval $[0, 1]$.

EXERCISES 5.3

Expressing Limits as Integrals

Express the limits in Exercises 1–8 as definite integrals.

1. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} c_k^2 \, \Delta x_k$, where P is a partition of $[0, 2]$

2. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} 2c_k^3 \, \Delta x_k$, where P is a partition of $[-1, 0]$

3. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (c_k^2 - 3c_k) \, \Delta x_k$, where P is a partition of $[-7, 5]$

4. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \left(\frac{1}{c_k}\right) \Delta x_k$, where P is a partition of $[1, 4]$

5. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \frac{1}{1 - c_k} \, \Delta x_k$, where P is a partition of $[2, 3]$

6. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{4 - c_k^2} \, \Delta x_k$, where P is a partition of $[0, 1]$

7. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (\sec c_k) \, \Delta x_k$, where P is a partition of $[-\pi/4, 0]$

8. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (\tan c_k) \, \Delta x_k$, where P is a partition of $[0, \pi/4]$

Using Properties and Known Values to Find Other Integrals

9. Suppose that f and g are integrable and that

$$\int_1^2 f(x)\,dx = -4,\quad \int_1^5 f(x)\,dx = 6,\quad \int_1^5 g(x)\,dx = 8.$$

Use the rules in Table 5.3 to find

a. $\int_2^2 g(x)\,dx$ **b.** $\int_5^1 g(x)\,dx$

c. $\int_1^2 3f(x)\,dx$ **d.** $\int_2^5 f(x)\,dx$

e. $\int_1^5 [f(x) - g(x)]\,dx$ **f.** $\int_1^5 [4f(x) - g(x)]\,dx$

10. Suppose that f and h are integrable and that

$$\int_1^9 f(x)\,dx = -1,\quad \int_7^9 f(x)\,dx = 5,\quad \int_7^9 h(x)\,dx = 4.$$

Use the rules in Table 5.3 to find

a. $\int_1^9 -2f(x)\,dx$ **b.** $\int_7^9 [f(x) + h(x)]\,dx$

c. $\int_7^9 [2f(x) - 3h(x)]\,dx$ **d.** $\int_9^1 f(x)\,dx$

e. $\int_1^7 f(x)\,dx$ **f.** $\int_9^7 [h(x) - f(x)]\,dx$

11. Suppose that $\int_1^2 f(x)\,dx = 5$. Find

a. $\int_1^2 f(u)\,du$ **b.** $\int_1^2 \sqrt{3}f(z)\,dz$

c. $\int_2^1 f(t)\,dt$ **d.** $\int_1^2 [-f(x)]\,dx$

12. Suppose that $\int_{-3}^0 g(t)\,dt = \sqrt{2}$. Find

a. $\int_0^{-3} g(t)\,dt$ **b.** $\int_{-3}^0 g(u)\,du$

c. $\int_{-3}^0 [-g(x)]\,dx$ **d.** $\int_{-3}^0 \frac{g(r)}{\sqrt{2}}\,dr$

13. Suppose that f is integrable and that $\int_0^3 f(z)\,dz = 3$ and $\int_0^4 f(z)\,dz = 7$. Find

a. $\int_3^4 f(z)\,dz$ **b.** $\int_4^3 f(t)\,dt$

14. Suppose that h is integrable and that $\int_{-1}^1 h(r)\,dr = 0$ and $\int_{-1}^3 h(r)\,dr = 6$. Find

a. $\int_1^3 h(r)\,dr$ **b.** $-\int_3^1 h(u)\,du$

Using Area to Evaluate Definite Integrals

In Exercises 15–22, graph the integrands and use areas to evaluate the integrals.

15. $\int_{-2}^4 \left(\frac{x}{2} + 3\right) dx$ **16.** $\int_{1/2}^{3/2} (-2x + 4)\,dx$

17. $\int_{-3}^3 \sqrt{9 - x^2}\,dx$ **18.** $\int_{-4}^0 \sqrt{16 - x^2}\,dx$

19. $\int_{-2}^1 |x|\,dx$ **20.** $\int_{-1}^1 (1 - |x|)\,dx$

21. $\int_{-1}^1 (2 - |x|)\,dx$ **22.** $\int_{-1}^1 \left(1 + \sqrt{1 - x^2}\right) dx$

Use areas to evaluate the integrals in Exercises 23–26.

23. $\int_0^b \frac{x}{2}\,dx,\quad b > 0$ **24.** $\int_0^b 4x\,dx,\quad b > 0$

25. $\int_a^b 2s\,ds,\quad 0 < a < b$ **26.** $\int_a^b 3t\,dt,\quad 0 < a < b$

Evaluations

Use the results of Equations (1) and (3) to evaluate the integrals in Exercises 27–38.

27. $\int_1^{\sqrt{2}} x\,dx$ **28.** $\int_{0.5}^{2.5} x\,dx$ **29.** $\int_\pi^{2\pi} \theta\,d\theta$

30. $\int_{\sqrt{2}}^{5\sqrt{2}} r\,dr$ **31.** $\int_0^{\sqrt[3]{7}} x^2\,dx$ **32.** $\int_0^{0.3} s^2\,ds$

33. $\int_0^{1/2} t^2\,dt$ **34.** $\int_0^{\pi/2} \theta^2\,d\theta$ **35.** $\int_a^{2a} x\,dx$

36. $\int_a^{\sqrt{3}a} x\,dx$ **37.** $\int_0^{\sqrt[3]{b}} x^2\,dx$ **38.** $\int_0^{3b} x^2\,dx$

Use the rules in Table 5.3 and Equations (1)–(3) to evaluate the integrals in Exercises 39–50.

39. $\int_3^1 7\,dx$ **40.** $\int_0^{-2} \sqrt{2}\,dx$

41. $\int_0^2 5x\,dx$ **42.** $\int_3^5 \frac{x}{8}\,dx$

43. $\int_0^2 (2t - 3)\,dt$ **44.** $\int_0^{\sqrt{2}} \left(t - \sqrt{2}\right) dt$

45. $\int_2^1 \left(1 + \frac{z}{2}\right) dz$ **46.** $\int_3^0 (2z - 3)\,dz$

47. $\int_1^2 3u^2\,du$ **48.** $\int_{1/2}^1 24u^2\,du$

49. $\int_0^2 (3x^2 + x - 5)\,dx$ **50.** $\int_1^0 (3x^2 + x - 5)\,dx$

Finding Area

In Exercises 51–54 use a definite integral to find the area of the region between the given curve and the x-axis on the interval $[0, b]$.

51. $y = 3x^2$ **52.** $y = \pi x^2$

53. $y = 2x$ **54.** $y = \frac{x}{2} + 1$

Average Value

In Exercises 55–62, graph the function and find its average value over the given interval.

Exercises

55. $f(x) = x^2 - 1$ on $[0, \sqrt{3}]$

56. $f(x) = -\dfrac{x^2}{2}$ on $[0, 3]$ **57.** $f(x) = -3x^2 - 1$ on $[0, 1]$

58. $f(x) = 3x^2 - 3$ on $[0, 1]$

59. $f(t) = (t - 1)^2$ on $[0, 3]$

60. $f(t) = t^2 - t$ on $[-2, 1]$

61. $g(x) = |x| - 1$ on **a.** $[-1, 1]$, **b.** $[1, 3]$, and **c.** $[-1, 3]$

62. $h(x) = -|x|$ on **a.** $[-1, 0]$, **b.** $[0, 1]$, and **c.** $[-1, 1]$

Theory and Examples

63. What values of a and b maximize the value of

$$\int_a^b (x - x^2)\,dx?$$

(*Hint:* Where is the integrand positive?)

64. What values of a and b minimize the value of

$$\int_a^b (x^4 - 2x^2)\,dx?$$

65. Use the Max-Min Inequality to find upper and lower bounds for the value of

$$\int_0^1 \frac{1}{1 + x^2}\,dx.$$

66. (*Continuation of Exercise 65*) Use the Max-Min Inequality to find upper and lower bounds for

$$\int_0^{0.5} \frac{1}{1 + x^2}\,dx \quad \text{and} \quad \int_{0.5}^1 \frac{1}{1 + x^2}\,dx.$$

Add these to arrive at an improved estimate of

$$\int_0^1 \frac{1}{1 + x^2}\,dx.$$

67. Show that the value of $\int_0^1 \sin(x^2)\,dx$ cannot possibly be 2.

68. Show that the value of $\int_1^0 \sqrt{x + 8}\,dx$ lies between $2\sqrt{2} \approx 2.8$ and 3.

69. Integrals of nonnegative functions Use the Max-Min Inequality to show that if f is integrable then

$$f(x) \ge 0 \quad \text{on} \quad [a, b] \quad \Rightarrow \quad \int_a^b f(x)\,dx \ge 0.$$

70. Integrals of nonpositive functions Show that if f is integrable then

$$f(x) \le 0 \quad \text{on} \quad [a, b] \quad \Rightarrow \quad \int_a^b f(x)\,dx \le 0.$$

71. Use the inequality $\sin x \le x$, which holds for $x \ge 0$, to find an upper bound for the value of $\int_0^1 \sin x\,dx$.

72. The inequality $\sec x \ge 1 + (x^2/2)$ holds on $(-\pi/2, \pi/2)$. Use it to find a lower bound for the value of $\int_0^1 \sec x\,dx$.

73. If $\text{av}(f)$ really is a typical value of the integrable function $f(x)$ on $[a, b]$, then the number $\text{av}(f)$ should have the same integral over $[a, b]$ that f does. Does it? That is, does

$$\int_a^b \text{av}(f)\,dx = \int_a^b f(x)\,dx?$$

Give reasons for your answer.

74. It would be nice if average values of integrable functions obeyed the following rules on an interval $[a, b]$.

a. $\text{av}(f + g) = \text{av}(f) + \text{av}(g)$

b. $\text{av}(kf) = k\,\text{av}(f)$ (any number k)

c. $\text{av}(f) \le \text{av}(g)$ if $f(x) \le g(x)$ on $[a, b]$.

Do these rules ever hold? Give reasons for your answers.

75. Use limits of Riemann sums as in Example 4a to establish Equation (2).

76. Use limits of Riemann sums as in Example 4a to establish Equation (3).

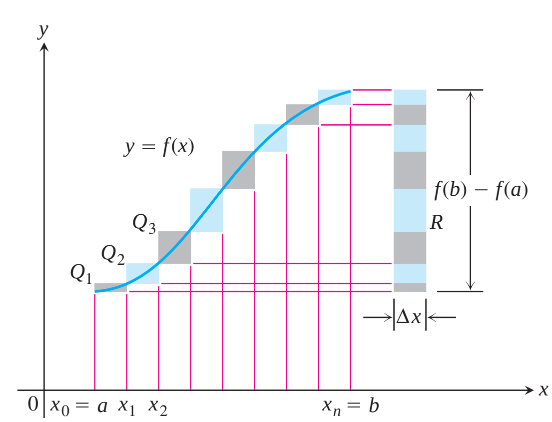
77. Upper and lower sums for increasing functions

a. Suppose the graph of a continuous function $f(x)$ rises steadily as x moves from left to right across an interval $[a, b]$. Let P be a partition of $[a, b]$ into n subintervals of length $\Delta x = (b - a)/n$. Show by referring to the accompanying figure that the difference between the upper and lower sums for f on this partition can be represented graphically as the area of a rectangle R whose dimensions are $[f(b) - f(a)]$ by Δx. (*Hint*: The difference $U - L$ is the sum of areas of rectangles whose diagonals $Q_0Q_1, Q_1Q_2, \ldots, Q_{n-1}Q_n$ lie along the curve. There is no overlapping when these rectangles are shifted horizontally onto R.)

b. Suppose that instead of being equal, the lengths Δx_k of the subintervals of the partition of $[a, b]$ vary in size. Show that

$$U - L \le |f(b) - f(a)|\,\Delta x_{\max},$$

where $\Delta x_{\max}$ is the norm of P, and hence that $\lim_{\|P\| \to 0} (U - L) = 0$.



78. Upper and lower sums for decreasing functions (*Continuation of Exercise 77*)

a. Draw a figure like the one in Exercise 77 for a continuous function $f(x)$ whose values decrease steadily as x moves from left to right across the interval $[a, b]$. Let P be a partition of $[a, b]$ into subintervals of equal length. Find an expression for $U - L$ that is analogous to the one you found for $U - L$ in Exercise 77a.

b. Suppose that instead of being equal, the lengths Δx_k of the subintervals of P vary in size. Show that the inequality

$$U - L \leq |f(b) - f(a)|\ \Delta x_{\max}$$

of Exercise 77b still holds and hence that $\lim_{\|P\|\to 0} (U - L) = 0$.

79. Use the formula

$$\sin h + \sin 2h + \sin 3h + \cdots + \sin mh = \frac{\cos(h/2) - \cos((m + (1/2))h)}{2\sin(h/2)}$$

to find the area under the curve $y = \sin x$ from $x = 0$ to $x = \pi/2$ in two steps:

a. Partition the interval $[0, \pi/2]$ into n subintervals of equal length and calculate the corresponding upper sum U; then

b. Find the limit of U as $n \to \infty$ and $\Delta x = (b - a)/n \to 0$.

80. Suppose that f is continuous and nonnegative over $[a, b]$, as in the figure at the right. By inserting points

$$x_1, x_2, \ldots, x_{k-1}, x_k, \ldots, x_{n-1}$$

as shown, divide $[a, b]$ into n subintervals of lengths $\Delta x_1 = x_1 - a$, $\Delta x_2 = x_2 - x_1, \ldots, \Delta x_n = b - x_{n-1}$, which need not be equal.

a. If $m_k = \min\{f(x) \text{ for } x \text{ in the } k\text{th subinterval}\}$, explain the connection between the *lower sum*

$$L = m_1\,\Delta x_1 + m_2\,\Delta x_2 + \cdots + m_n\,\Delta x_n$$

and the shaded region in the first part of the figure.

b. If $M_k = \max\{f(x) \text{ for } x \text{ in the } k\text{th subinterval}\}$, explain the connection between the *upper sum*

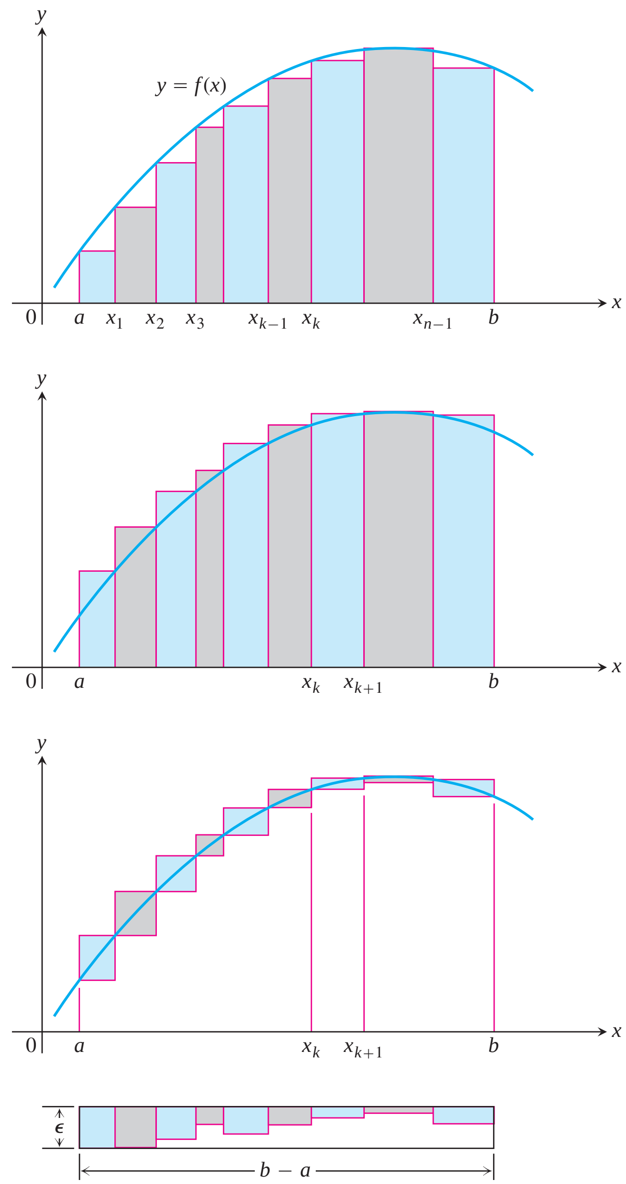
$$U = M_1\,\Delta x_1 + M_2\,\Delta x_2 + \cdots + M_n\,\Delta x_n$$

and the shaded region in the second part of the figure.

c. Explain the connection between $U - L$ and the shaded regions along the curve in the third part of the figure.

81. We say f is **uniformly continuous** on $[a, b]$ if given any $\epsilon > 0$ there is a $\delta > 0$ such that if x_1, x_2 are in $[a, b]$ and $|x_1 - x_2| < \delta$ then $|f(x_1) - f(x_2)| < \epsilon$. It can be shown that a continuous function on $[a, b]$ is uniformly continuous. Use this and the figure at the right to show that if f is continuous and $\epsilon > 0$ is given, it is possible to make $U - L \leq \epsilon \cdot (b - a)$ by making the largest of the Δx_k's sufficiently small.

82. If you average 30 mi/h on a 150-mi trip and then return over the same 150 mi at the rate of 50 mi/h, what is your average speed for the trip? Give reasons for your answer. (Source: David H.



Pleacher, *The Mathematics Teacher*, Vol. 85, No. 6, pp. 445–446, September 1992.)

COMPUTER EXPLORATIONS

Finding Riemann Sums

If your CAS can draw rectangles associated with Riemann sums, use it to draw rectangles associated with Riemann sums that converge to the integrals in Exercises 83–88. Use $n = 4$, 10, 20, and 50 subintervals of equal length in each case.

83. $\displaystyle\int_0^1 (1 - x)\,dx = \frac{1}{2}$

84. $\displaystyle\int_0^1 (x^2 + 1)\,dx = \frac{4}{3}$

85. $\int_{-\pi}^{\pi} \cos x \, dx = 0$

86. $\int_{0}^{\pi/4} \sec^2 x \, dx = 1$

87. $\int_{-1}^{1} |x| \, dx = 1$

88. $\int_{1}^{2} \frac{1}{x} \, dx$ (The integral's value is about 0.693.)

Average Value

In Exercises 89–92, use a CAS to perform the following steps:

a. Plot the functions over the given interval.

b. Partition the interval into $n = 100, 200$, and 1000 subintervals of equal length, and evaluate the function at the midpoint of each subinterval.

c. Compute the average value of the function values generated in part (b).

d. Solve the equation $f(x) = (\text{average value})$ for x using the average value calculated in part (c) for the $n = 1000$ partitioning.

89. $f(x) = \sin x \quad \text{on} \quad [0, \pi]$

90. $f(x) = \sin^2 x \quad \text{on} \quad [0, \pi]$

91. $f(x) = x \sin \frac{1}{x} \quad \text{on} \quad \left[\frac{\pi}{4}, \pi\right]$

92. $f(x) = x \sin^2 \frac{1}{x} \quad \text{on} \quad \left[\frac{\pi}{4}, \pi\right]$

EXERCISES 5.4

Evaluating Integrals

Evaluate the integrals in Exercises 1–26.

1. $\displaystyle\int_{-2}^{0}(2x+5)\,dx$

2. $\displaystyle\int_{-3}^{4}\left(5-\frac{x}{2}\right)dx$

3. $\displaystyle\int_{0}^{4}\left(3x-\frac{x^3}{4}\right)dx$

4. $\displaystyle\int_{-2}^{2}(x^3-2x+3)\,dx$

5. $\displaystyle\int_{0}^{1}\left(x^2+\sqrt{x}\right)dx$

6. $\displaystyle\int_{0}^{5}x^{3/2}\,dx$

7. $\displaystyle\int_{1}^{32}x^{-6/5}\,dx$

8. $\displaystyle\int_{-2}^{-1}\frac{2}{x^2}\,dx$

9. $\displaystyle\int_{0}^{\pi}\sin x\,dx$

10. $\displaystyle\int_{0}^{\pi}(1+\cos x)\,dx$

11. $\displaystyle\int_{0}^{\pi/3}2\sec^2 x\,dx$

12. $\displaystyle\int_{\pi/6}^{5\pi/6}\csc^2 x\,dx$

13. $\displaystyle\int_{\pi/4}^{3\pi/4}\csc\theta\cot\theta\,d\theta$

14. $\displaystyle\int_{0}^{\pi/3}4\sec u\tan u\,du$

15. $\displaystyle\int_{\pi/2}^{0}\frac{1+\cos 2t}{2}\,dt$

16. $\displaystyle\int_{-\pi/3}^{\pi/3}\frac{1-\cos 2t}{2}\,dt$

17. $\displaystyle\int_{-\pi/2}^{\pi/2}(8y^2+\sin y)\,dy$

18. $\displaystyle\int_{-\pi/3}^{-\pi/4}\left(4\sec^2 t+\frac{\pi}{t^2}\right)dt$

19. $\displaystyle\int_{1}^{-1}(r+1)^2\,dr$

20. $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}}(t+1)(t^2+4)\,dt$

21. $\displaystyle\int_{\sqrt{2}}^{1}\left(\frac{u^7}{2}-\frac{1}{u^5}\right)du$

22. $\displaystyle\int_{1/2}^{1}\left(\frac{1}{v^3}-\frac{1}{v^4}\right)dv$

23. $\displaystyle\int_{1}^{\sqrt{2}}\frac{s^2+\sqrt{s}}{s^2}\,ds$

24. $\displaystyle\int_{9}^{4}\frac{1-\sqrt{u}}{\sqrt{u}}\,du$

25. $\displaystyle\int_{-4}^{4}|x|\,dx$

26. $\displaystyle\int_{0}^{\pi}\frac{1}{2}(\cos x+|\cos x|)\,dx$

Derivatives of Integrals

Find the derivatives in Exercises 27–30

a. by evaluating the integral and differentiating the result.

b. by differentiating the integral directly.

27. $\displaystyle\frac{d}{dx}\int_{0}^{\sqrt{x}}\cos t\,dt$

28. $\displaystyle\frac{d}{dx}\int_{1}^{\sin x}3t^2\,dt$

29. $\displaystyle\frac{d}{dt}\int_{0}^{t^4}\sqrt{u}\,du$

30. $\displaystyle\frac{d}{d\theta}\int_{0}^{\tan\theta}\sec^2 y\,dy$

Find dy/dx in Exercises 31–36.

31. $\displaystyle y=\int_{0}^{x}\sqrt{1+t^2}\,dt$

32. $\displaystyle y=\int_{1}^{x}\frac{1}{t}\,dt,\quad x>0$

33. $\displaystyle y=\int_{\sqrt{x}}^{0}\sin(t^2)\,dt$

34. $\displaystyle y=\int_{0}^{x^2}\cos\sqrt{t}\,dt$

35. $\displaystyle y=\int_{0}^{\sin x}\frac{dt}{\sqrt{1-t^2}},\quad |x|<\frac{\pi}{2}$

36. $\displaystyle y=\int_{\tan x}^{0}\frac{dt}{1+t^2}$

Area

In Exercises 37–42, find the total area between the region and the x-axis.

37. $y=-x^2-2x,\quad -3\le x\le 2$

38. $y=3x^2-3,\quad -2\le x\le 2$

39. $y=x^3-3x^2+2x,\quad 0\le x\le 2$

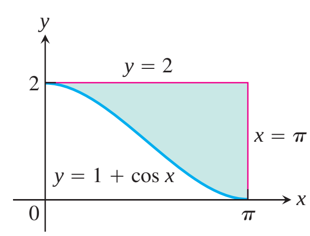
40. $y=x^3-4x,\quad -2\le x\le 2$

41. $y=x^{1/3},\quad -1\le x\le 8$

42. $y=x^{1/3}-x,\quad -1\le x\le 8$

Find the areas of the shaded regions in Exercises 43–46.

43.



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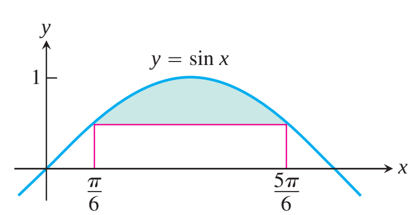

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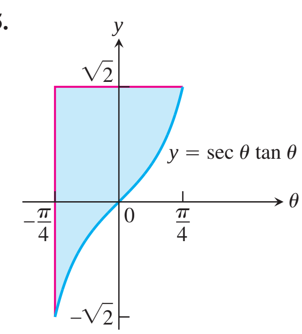
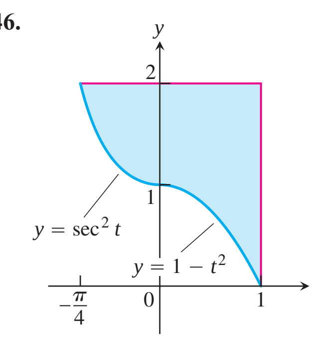


46.



Initial Value Problems

Each of the following functions solves one of the initial value problems in Exercises 47–50. Which function solves which problem? Give brief reasons for your answers.

a. $y = \int_1^x \frac{1}{t}\,dt - 3$ **b.** $y = \int_0^x \sec t\,dt + 4$

c. $y = \int_{-1}^x \sec t\,dt + 4$ **d.** $y = \int_\pi^x \frac{1}{t}\,dt - 3$

47. $\frac{dy}{dx} = \frac{1}{x}, \quad y(\pi) = -3$ **48.** $y' = \sec x, \quad y(-1) = 4$

49. $y' = \sec x, \quad y(0) = 4$ **50.** $y' = \frac{1}{x}, \quad y(1) = -3$

Express the solutions of the initial value problems in Exercises 51–54 in terms of integrals.

51. $\frac{dy}{dx} = \sec x, \quad y(2) = 3$

52. $\frac{dy}{dx} = \sqrt{1 + x^2}, \quad y(1) = -2$

53. $\frac{ds}{dt} = f(t), \quad s(t_0) = s_0$

54. $\frac{dv}{dt} = g(t), \quad v(t_0) = v_0$

Applications

55. Archimedes' area formula for parabolas Archimedes (287–212 B.C.), inventor, military engineer, physicist, and the greatest mathematician of classical times in the Western world, discovered that the area under a parabolic arch is two-thirds the base times the height. Sketch the parabolic arch $y = h - (4h/b^2)x^2$, $-b/2 \le x \le b/2$, assuming that h and b are positive. Then use calculus to find the area of the region enclosed between the arch and the x-axis.

56. Revenue from marginal revenue Suppose that a company's marginal revenue from the manufacture and sale of egg beaters is

$$\frac{dr}{dx} = 2 - 2/(x + 1)^2,$$

where r is measured in thousands of dollars and x in thousands of units. How much money should the company expect from a production run of $x = 3$ thousand egg beaters? To find out, integrate the marginal revenue from $x = 0$ to $x = 3$.

57. Cost from marginal cost The marginal cost of printing a poster when x posters have been printed is

$$\frac{dc}{dx} = \frac{1}{2\sqrt{x}}$$

dollars. Find $c(100) - c(1)$, the cost of printing posters 2–100.

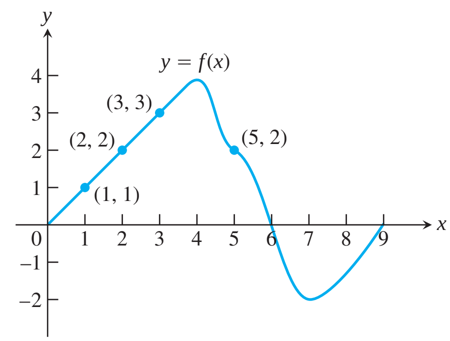
58. (*Continuation of Exercise 57.*) Find $c(400) - c(100)$, the cost of printing posters 101–400.

Drawing Conclusions About Motion from Graphs

59. Suppose that f is the differentiable function shown in the accompanying graph and that the position at time t (sec) of a particle moving along a coordinate axis is

$$s = \int_0^t f(x)\,dx$$

meters. Use the graph to answer the following questions. Give reasons for your answers.



a. What is the particle's velocity at time $t = 5$?

b. Is the acceleration of the particle at time $t = 5$ positive, or negative?

c. What is the particle's position at time $t = 3$?

d. At what time during the first 9 sec does s have its largest value?

e. Approximately when is the acceleration zero?

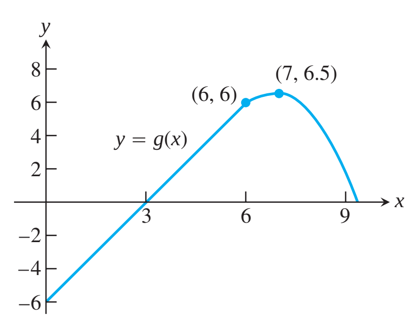
f. When is the particle moving toward the origin? away from the origin?

g. On which side of the origin does the particle lie at time $t = 9$?

60. Suppose that g is the differentiable function graphed here and that the position at time t (sec) of a particle moving along a coordinate axis is

$$s = \int_0^t g(x)\,dx$$

meters. Use the graph to answer the following questions. Give reasons for your answers.



a. What is the particle's velocity at $t = 3$?

b. Is the acceleration at time $t = 3$ positive, or negative?

c. What is the particle's position at time $t = 3$?

d. When does the particle pass through the origin?

e. When is the acceleration zero?

f. When is the particle moving away from the origin? toward the origin?

g. On which side of the origin does the particle lie at $t = 9$?

Theory and Examples

61. Show that if k is a positive constant, then the area between the x-axis and one arch of the curve $y = \sin kx$ is $2/k$.

62. Find

$$\lim_{x \to 0} \frac{1}{x^3} \int_0^x \frac{t^2}{t^4 + 1}\, dt.$$

63. Suppose $\int_1^x f(t)\, dt = x^2 - 2x + 1$. Find $f(x)$.

64. Find $f(4)$ if $\int_0^x f(t)\, dt = x \cos \pi x$.

65. Find the linearization of

$$f(x) = 2 - \int_2^{x+1} \frac{9}{1 + t}\, dt$$

at $x = 1$.

66. Find the linearization of

$$g(x) = 3 + \int_1^{x^2} \sec(t - 1)\, dt$$

at $x = -1$.

67. Suppose that f has a positive derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of the function

$$g(x) = \int_0^x f(t)\, dt?$$

Give reasons for your answers.

a. g is a differentiable function of x.

b. g is a continuous function of x.

c. The graph of g has a horizontal tangent at $x = 1$.

d. g has a local maximum at $x = 1$.

e. g has a local minimum at $x = 1$.

f. The graph of g has an inflection point at $x = 1$.

g. The graph of dg/dx crosses the x-axis at $x = 1$.

68. Suppose that f has a negative derivative for all values of x and that $f(1) = 0$. Which of the following statements must be true of the function

$$h(x) = \int_0^x f(t)\, dt?$$

Give reasons for your answers.

a. h is a twice-differentiable function of x.

b. h and dh/dx are both continuous.

c. The graph of h has a horizontal tangent at $x = 1$.

d. h has a local maximum at $x = 1$.

e. h has a local minimum at $x = 1$.

f. The graph of h has an inflection point at $x = 1$.

g. The graph of dh/dx crosses the x-axis at $x = 1$.

T **69. The Fundamental Theorem** If f is continuous, we expect

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)\, dt$$

to equal $f(x)$, as in the proof of Part 1 of the Fundamental Theorem. For instance, if $f(t) = \cos t$, then

$$\frac{1}{h} \int_x^{x+h} \cos t\, dt = \frac{\sin(x + h) - \sin x}{h}. \tag{7}$$

The right-hand side of Equation (7) is the difference quotient for the derivative of the sine, and we expect its limit as $h \to 0$ to be $\cos x$.

Graph $\cos x$ for $-\pi \le x \le 2\pi$. Then, in a different color if possible, graph the right-hand side of Equation (7) as a function of x for $h = 2, 1, 0.5$, and 0.1. Watch how the latter curves converge to the graph of the cosine as $h \to 0$.

T **70.** Repeat Exercise 69 for $f(t) = 3t^2$. What is

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} 3t^2\, dt = \lim_{h \to 0} \frac{(x + h)^3 - x^3}{h}?$$

Graph $f(x) = 3x^2$ for $-1 \le x \le 1$. Then graph the quotient $((x + h)^3 - x^3)/h$ as a function of x for $h = 1, 0.5, 0.2$, and 0.1. Watch how the latter curves converge to the graph of $3x^2$ as $h \to 0$.

COMPUTER EXPLORATIONS

In Exercises 71–74, let $F(x) = \int_a^x f(t)\, dt$ for the specified function f and interval $[a, b]$. Use a CAS to perform the following steps and answer the questions posed.

a. Plot the functions f and F together over $[a, b]$.

b. Solve the equation $F'(x) = 0$. What can you see to be true about the graphs of f and F at points where $F'(x) = 0$? Is your observation borne out by Part 1 of the Fundamental Theorem coupled with information provided by the first derivative? Explain your answer.

c. Over what intervals (approximately) is the function F increasing and decreasing? What is true about f over those intervals?

d. Calculate the derivative f' and plot it together with F. What can you see to be true about the graph of F at points where $f'(x) = 0$? Is your observation borne out by Part 1 of the Fundamental Theorem? Explain your answer.

71. $f(x) = x^3 - 4x^2 + 3x, \quad [0, 4]$

72. $f(x) = 2x^4 - 17x^3 + 46x^2 - 43x + 12, \left[0, \frac{9}{2}\right]$

73. $f(x) = \sin 2x \cos \frac{x}{3}, \quad [0, 2\pi]$

74. $f(x) = x \cos \pi x, \quad [0, 2\pi]$

In Exercises 75–78, let $F(x) = \int_a^{u(x)} f(t)\, dt$ for the specified a, u, and f. Use a CAS to perform the following steps and answer the questions posed.

a. Find the domain of F.

b. Calculate $F'(x)$ and determine its zeros. For what points in its domain is F increasing? decreasing?

c. Calculate $F''(x)$ and determine its zero. Identify the local extrema and the points of inflection of F.

d. Using the information from parts (a)–(c), draw a rough hand-sketch of $y = F(x)$ over its domain. Then graph $F(x)$ on your CAS to support your sketch.

75. $a = 1, \quad u(x) = x^2, \quad f(x) = \sqrt{1 - x^2}$

76. $a = 0, \quad u(x) = x^2, \quad f(x) = \sqrt{1 - x^2}$

77. $a = 0, \quad u(x) = 1 - x, \quad f(x) = x^2 - 2x - 3$

78. $a = 0, \quad u(x) = 1 - x^2, \quad f(x) = x^2 - 2x - 3$

In Exercises 79 and 80, assume that f is continuous and $u(x)$ is twice-differentiable.

79. Calculate $\frac{d}{dx}\int_a^{u(x)} f(t)\, dt$ and check your answer using a CAS.

80. Calculate $\frac{d^2}{dx^2}\int_a^{u(x)} f(t)\, dt$ and check your answer using a CAS.