MA1200 CALCULUS AND BASIC LINEAR ALGEBRA I

LECTURE: CG1

Chapter 5
Exponential and Logarithmic Functions

Exponential Functions

The **exponential function with base b** is defined by

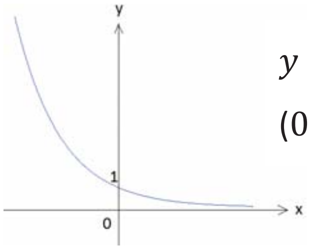
$$f(x) = b^x,$$

where the constant b (with $b > 0$ and $b \neq 1$) is called the **base**, and $x \in \mathbb{R}$ is called the **exponent**.

E.g. $f(x) = 10^x$, $g(x) = \left(\frac{1}{2}\right)^x$, $h(x) = 5^{3x+2}$ are examples of exponential functions.

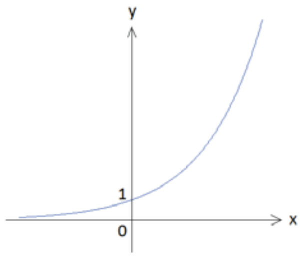
$k(x) = x^{10}$ is NOT an exponential function.

Graphs of exponential functions:



$y = b^x$

$(0 < b < 1)$



$y = b^x$

$(b > 1)$

Note that:

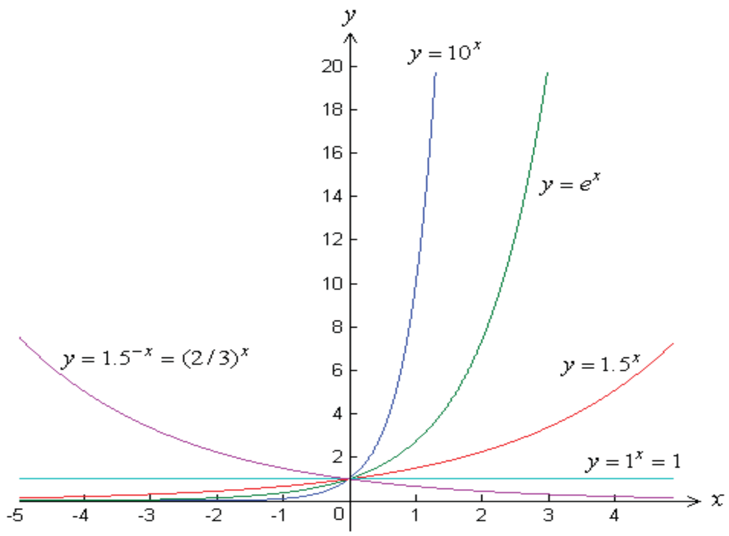
1. The largest possible domain of $f(x) = b^x$ is $\boxed{Dom(f) = \mathbb{R}}$.
2. The largest possible range of $f(x) = b^x$ is $\boxed{Ran(f) = (0, \infty)}$.
3. For $0 < b < 1$, $f(x) = b^x$ is a **strictly decreasing** function.

   $f(x) \to \infty$ as $x \to -\infty$ and $f(x) \to 0$ as $x \to \infty$
4. For $b > 1$, $f(x) = b^x$ is a **strictly increasing** function.

   $f(x) \to 0$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$
5. For any b (where $b > 0$ and $b \neq 1$), the graph of $f(x) = b^x$ always cuts the y-axis at $y = 1$, since $f(0) = b^0 = 1$ for all $b > 0$. However, it never touches the x-axis, since $f(x) = b^x$ is always positive.
6. Since the exponential function $f(x) = b^x$ is either strictly decreasing (for $0 < b < 1$) or strictly increasing (for $b > 1$), $f(x) = b^x$ is a one-to-one function and its inverse $f^{-1}(x)$ exists.

The graphs of exponential functions with different values of b are shown below.

Note that $y = 1^x$ is not an exponential function, since $y = 1^x = 1$ is a constant function.



Question 1: Compare the graphs of $y = \left(\frac{3}{2}\right)^x$ and $y = 10^x$. What do you observe?

Question 2: Compare the graphs of $y = \left(\frac{3}{2}\right)^x$ and $y = \left(\frac{2}{3}\right)^x$. What do you observe?

Laws of indices:

If $a > 0$, $b > 0$, x and y are real numbers, then

(1) $a^0 = 1$ (2) $a^{x+y} = a^x \cdot a^y$ (3) $a^{-x} = \frac{1}{a^x}$

(4) $a^{x-y} = \frac{a^x}{a^y}$ (5) $(a^x)^y = a^{xy}$ (6) $(ab)^x = a^x \cdot b^x$

(7) $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

Natural Base e

A special case, in which we consider $b = e$, where e is defined by the limit of the sequence

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = 2.7182818 \ldots.$$

That is, the value of $\left(1 + \frac{1}{n}\right)^n$ approaches the irrational number $e = 2.7182818\ldots$ as n gets larger and larger (i.e. as $n \to \infty$). The number e is called the **natural base**. The exponential function with base e, $f(x) = e^x$, is called the **natural exponential function**.

Example 1

For each of the following functions, find its largest possible domain and largest possible range, and then sketch its graph.

(a) $f(x) = e^{x+1} - 5$ (b) $g(x) = 3 + 2e^{-x}$ (c) $h(x) = 1 - 3\left(\frac{1}{2}\right)^x$

Solution

(a) Since e^{x+1} is well-defined for all real values of x, the function $f(x) = e^{x+1} - 5$ is also well-defined for all real values of x. $\therefore$ $Dom(f) = \mathbb{R}$

For any $x \in Dom(f) = \mathbb{R}$, e^{x+1} is always greater than 0, and thus $f(x) = e^{x+1} - 5$ is always greater than -5. $\therefore$ $Ran(f) = (-5, \infty)$.

(b) $g(x) = 3 + 2e^{-x}$ is well-defined for all real values of x, so $Dom(g) = \mathbb{R}$.

For any $x \in Dom(g) = \mathbb{R}$, we have $e^{-x} > 0$ and thus $g(x) = 3 + 2e^{-x} > 3$.

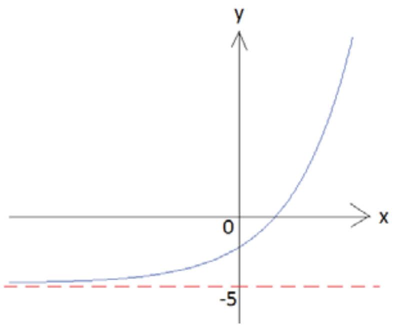
$\therefore$ $Ran(f) = (3, \infty)$.

(c) $h(x) = 1 - 3\left(\frac{1}{2}\right)^x$ is well-defined for all real values of x, so $Dom(h) = \mathbb{R}$.

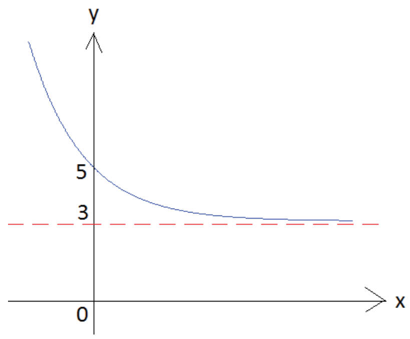
For any $x \in Dom(h) = \mathbb{R}$, we have $\left(\frac{1}{2}\right)^x > 0 \Rightarrow -3\left(\frac{1}{2}\right)^x < 0$

$\Rightarrow h(x) = 1 - 3\left(\frac{1}{2}\right)^x < 1$. Thus, $Ran(f) = (-\infty, 1)$.

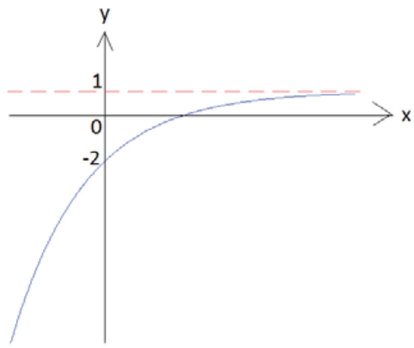
<u>*Graphs:*</u>



(a) $f(x) = e^{x+1} - 5$



(b) $g(x) = 3 + 2e^{-x}$



(c) $h(x) = 1 - 3\left(\frac{1}{2}\right)^x$

<u>Inverse function of b^x</u>

The exponential function $f(x) = b^x$ (with $b > 0$ and $b \neq 1$) is a one-to-one function and thus it has an inverse. Its inverse is

$$f^{-1}(x) = \log_b x,$$

which is called the logarithmic function with base b.

<u>Logarithmic Functions</u>

The **logarithmic function with base b** is defined as

$$f(x) = \log_b x$$

for $x > 0$. For $y = \log_b x$, the constant b (with $b > 0$ and $b \neq 1$) is called the **base**, and y is called the **exponent**.

$$\boxed{y = \log_b x \quad \Leftrightarrow \quad x = b^y}$$

Here, $y = \log_b x$ is the <u>**logarithmic form**</u> and $b^y = x$ is the <u>**exponential form**</u>.

Note that exponential function is the inverse function of logarithmic function.

Example 2

Write down each equation in its equivalent exponential form.

(a) $2 = \log_5 x$ (b) $3 = \log_b 64$ (c) $\log_3 7 = y$

Solution

(a) $2 = \log_5 x$ means $5^2 = x$.

(b) $3 = \log_b 64$ means $b^3 = 64$.

(c) $\log_3 7 = y$ means $3^y = 7$.

Example 3

Write down each equation in its equivalent logarithmic form.

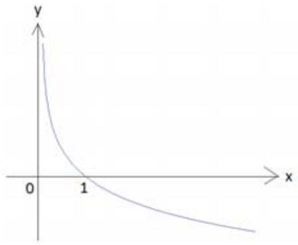
(a) $12^2 = r$ (b) $b^3 = 8$ (c) $e^a = 9$

Solution

(a) $12^2 = r$ means $2 = \log_{12} r$. (b) $b^3 = 8$ means $3 = \log_b 8$.

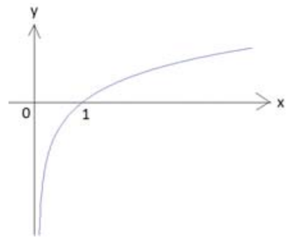
(c) $e^a = 9$ means $a = \log_e 9$.

Graphs of logarithmic functions:



$y = \log_b x$

$(0 < b < 1)$



$y = \log_b x$

$(b > 1)$

Note that:

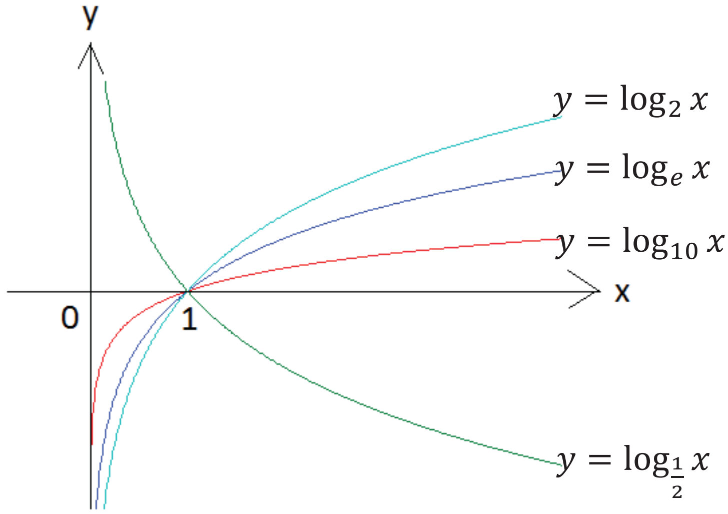
1. The logarithmic function $f(x) = \log_b x$ is only defined for positive values of x.

   $\therefore$ The largest possible domain of $f(x) = \log_b x$ is $\boxed{Dom(f) = (0, \infty)}$.
2. The largest possible range of $f(x) = \log_b x$ is $\boxed{Ran(f) = \mathbb{R}}$.
3. For $0 < b < 1$, $f(x) = \log_b x$ is a **strictly decreasing** function.
4. For $b > 1$, $f(x) = \log_b x$ is a **strictly increasing** function.
5. For any b (where $b > 0$ and $b \neq 1$), the graph of $f(x) = \log_b x$ always cuts the x-axis at $x = 1$, i.e. $f(1) = \log_b 1 = 0$ for all $b > 0$ and $b \neq 1$. However, it never cuts the y-axis, since $f(x) = \log_b x$ is not defined at zero or negative values of x.

Two commonly used logarithms:

- If the base $b = 10$, $\log_{10} x$ is called the **common logarithm**, usually denoted by $\log x$.
- If the base $b = e$ (the natural number), $\log_e x$ is called the **natural logarithm**, usually denoted by $\ln x$.

The graphs of logarithmic functions with different values of b are shown below.



Properties of logarithms:

(1) For any real number x, $\boxed{\log_b b^x = x}$.

(2) For any real number $x > 0$, $\boxed{b^{\log_b x} = x}$.

(3) For any real numbers $x > 0$ and n, $\boxed{\log_b x^n = n \log_b x}$.

(4) For any real numbers $x > 0$ and $y > 0$, $\boxed{\log_b(xy) = \log_b x + \log_b y}$.

(5) For any real numbers $x > 0$ and $y > 0$, $\boxed{\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y}$.

(6) For any real numbers $x > 0$, $a > 1$ and $b > 1$, $\boxed{\log_b x = \frac{\log_a x}{\log_a b}}$.

In general,

1. $\log_b(x + y) \neq \log_b x + \log_b y$.
2. $(\log_b x)^2 \neq \log_b(x^2)$.
3. $\frac{\log_b x}{\log_b y} \neq \frac{x}{y}$, $\frac{\log_b x}{\log_b y} \neq \log_b\left(\frac{x}{y}\right)$

Example 4

Simplify each of the following:

(a) $\log_3\left(\frac{1}{81}\right)$

(b) $2\log_{10}5+\log_{10}4-5^{\log_5 3}+\log_2 16$

Solution

(a) $\log_3\left(\frac{1}{81}\right)=\log_3\left(\frac{1}{3^4}\right)=\log_3(3^{-4})=-4\underbrace{\log_3 3}_{=1}=-4$

(b) $2\log_{10}5+\log_{10}4-5^{\log_5 3}+\log_2 16=$

Example 5

If $2^x=3^y=12^z$, show that $xy=z(x+2y)$.

Solution

Natural logarithm

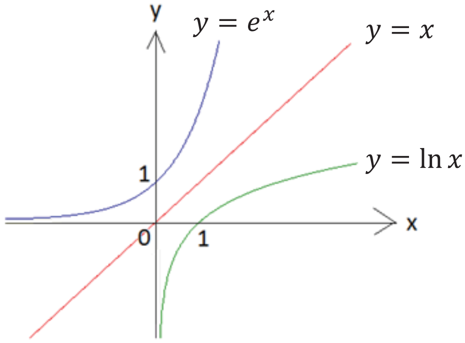
Logarithmic function with base e is called **natural logarithmic function**, denoted by $f(x) = \log_e x$ or $f(x) = \ln x$.

$$\boxed{y = \ln x \quad \Leftrightarrow \quad x = e^y}$$

Exponential function is the inverse function of logarithmic function, that is, the inverse function of $f(x) = \ln x$ is $f^{-1}(x) = e^x$.

Thus, $Dom(f) = Ran(f^{-1}) = (0, \infty)$ and $Ran(f) = Dom(f^{-1}) = \mathbb{R}$.

The graphs of $y = \ln x$ and $y = e^x$ are shown below.



Note that:

(i) The graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ about the line $y = x$.

(ii) Both $y = \ln x$ and $y = e^x$ are strictly increasing functions.

(iii) $\ln 1 = 0$ (i.e. the graph of $y = \ln x$ crosses the x-axis at $x = 1$.)

(iv) $\ln x < 0$ for $0 < x < 1$

(v) $\ln x > 0$ for $x > 1$

(vi) The value of $\ln x$ approaches to $-\infty$ as x tends to 0 from the right. That is,

$$\lim_{x \to 0^+} \ln x = -\infty$$

The value of $\ln x$ approaches to ∞ as x gets larger and larger. That is,

$$\lim_{x \to \infty} \ln x = \infty$$

(The limit of a function will be discussed in Chapter 6.)

Example 6

For each of the following functions, find its largest possible domain and largest possible range, and then sketch its graph.

(a) $f(x) = 2 + \ln\frac{1}{x}$

(b) $g(x) = 4 + \log\frac{x+1}{1000}$

Solution

(a) $f(x) = 2 + \ln\frac{1}{x}$ is well-defined when $\frac{1}{x} > 0$ and $x \neq 0$, i.e. when $x > 0$.

$\therefore$ $Dom(f) = (0, \infty)$.

The function can be written as

$$f(x) = 2 + \ln\frac{1}{x} = 2 + \ln(x^{-1}) = 2 + (-1)\ln x = 2 - \ln x.$$

For any $x \in Dom(f) = (0, \infty)$, $\ln x$ can be any real number and thus $2 - \ln x$ can be any real number.

$\therefore$ $Ran(f) = \mathbb{R}$.

(b) $g(x) = 4 + \log\frac{x+1}{1000}$ is well-defined when $\frac{x+1}{1000} > 0$, i.e. $x > -1$.

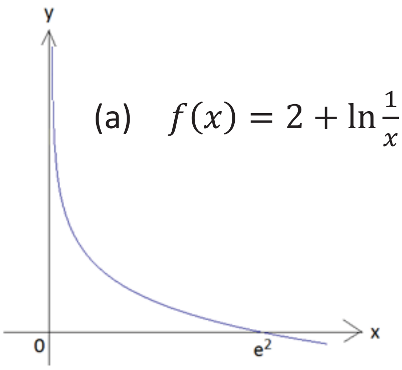
$\therefore$ $Dom(g) = (-1, \infty)$.

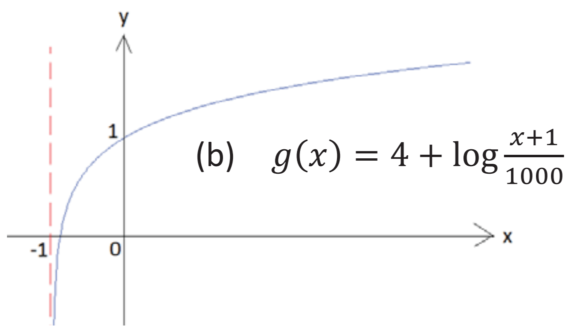
The function can be written as

$$g(x) = 4 + \log\frac{x+1}{1000} = 4 + \log(x+1) - \log 1000 = 4 + \log(x+1) - \log 10^3$$
$$= 4 + \log(x+1) - 3 = 1 + \log(x+1)$$

For any $x \in Dom(g) = (-1, \infty)$, $\log(x+1)$ can be any real number and thus $g(x) = 1 + \log(x+1)$ can be any real number. $\therefore$ $Ran(f) = \mathbb{R}$.

Sketches





Example 7

For each of the following functions, find its inverse function.

(a) $f(x) = e^{x+1} - 5$ (b) $g(x) = 3 + 2e^{-x}$ (c) $h(x) = 1 - 3\left(\frac{1}{2}\right)^x$

(d) $f(x) = 2 + \ln\frac{1}{x}$ (e) $g(x) = 4 + \log\frac{x+1}{1000}$

Solution

(a) Let $y = e^{x+1} - 5$.

Then $e^{x+1} = y + 5 \Rightarrow \underbrace{\ln(e^{x+1})}_{=x+1} = \ln(y+5) \Rightarrow x = \ln(y+5) - 1$

$\therefore f^{-1}(x) = \ln(x+5) - 1$

(b) Let $y = 3 + 2e^{-x}$.

Then $e^{-x} = \frac{y-3}{2} \Rightarrow -x = \ln\left(\frac{y-3}{2}\right) \Rightarrow x = -\ln\left(\frac{y-3}{2}\right) = \ln\left(\frac{2}{y-3}\right)$

$\therefore g^{-1}(x) = \ln\left(\frac{2}{x-3}\right)$

(c) Let $y = 1 - 3\left(\frac{1}{2}\right)^x$.

(d) Let $y = 2 + \ln\frac{1}{x}$.

Then $\ln\frac{1}{x} = y - 2 \Rightarrow \frac{1}{x} = e^{y-2} \Rightarrow x = \frac{1}{e^{y-2}} = e^{2-y}$

$\therefore f^{-1}(x) = e^{2-x}$

(e) Let $y = 4 + \log\frac{x+1}{1000}$.

Example 8

Determine the largest possible domain and largest possible range of the function $f(x) = \ln\left(\frac{x+2}{x-1}\right)$.

Solution

The function $f(x) = \ln\left(\frac{x+2}{x-1}\right)$ is well-defined when $\frac{x+2}{x-1} > 0$ and $x - 1 \neq 0$.

	$x < -2$	$x = -2$	$-2 < x < 1$	$x = 1$	$x > 1$
Sign of $x + 2$	$-$	0	$+$		$+$
Sign of $x - 1$	$-$	$-$	$-$		$+$
Sign of $\frac{x+2}{x-1}$	$+$	0	$-$		$+$

$\therefore$ The largest possible domain of $f(x)$ is $Dom(f) = (-\infty, -2) \cup (1, \infty)$.

Let $y = \ln\left(\frac{x+2}{x-1}\right)$. Then $e^y = \frac{x+2}{x-1} \Rightarrow e^y(x-1) = x + 2 \Rightarrow x(e^y - 1) = e^y + 2$

$$\Rightarrow x = \frac{e^y+2}{e^y-1}.$$

In the last expression, y can be any real number <u>except</u> when $e^y = 1$, i.e. $y = \ln 1 = 0$.

$\therefore$ The largest possible range of $f(x)$ is $Ran(f) = \mathbb{R}\backslash\{0\}$.

Example 9

Solve each of the following equations for x.

(a) $2^x = 16$ (b) $3^{x-1} = 81$ (c) $3^x = 17$ (d) $3 \cdot 5^{2x-1} + 2 = 17$

Solution

(a) $2^x = 16 = 2^4 \Rightarrow x = 4$

(b) $3^{x-1} = 81 = 3^4 \Rightarrow x - 1 = 4 \Rightarrow x = 5$

(c) $3^x = 17$

Take natural logarithm on both sides:

$$\ln 3^x = \ln 17 \Rightarrow x \ln 3 = \ln 17 \Rightarrow x = \frac{\ln 17}{\ln 3} \approx 2.5789$$

(d) $3 \cdot 5^{2x-1} + 2 = 17 \Rightarrow 3 \cdot 5^{2x-1} = 15 \Rightarrow 5^{2x-1} = 5 \Rightarrow 2x - 1 = 1 \Rightarrow x = 1$

Example 10

Solve each of the following equations for x.

(a) $\ln(x^3) = 2\ln 5$ (b) $5^{2x-1} = 12 \cdot 3^x$

(c) $\log_2(x+5) + \log_2(x-2) = 3$ (d) $e^x - 8e^{-x} = 7$

Solution

(a) $\ln(x^3) = 2\ln 5$

Taking natural exponential on both sides, we get

$$e^{\ln(x^3)} = e^{2\ln 5} \quad \Rightarrow \quad x^3 = e^{\ln(5^2)} \quad \Rightarrow \quad x^3 = 5^2 \quad \Rightarrow \quad \boxed{x = 5^{\frac{2}{3}}}.$$

(b) $5^{2x-1} = 12 \cdot 3^x$

Taking natural logarithm on both sides, we get

$$\ln(5^{2x-1}) = \ln(12 \cdot 3^x) \quad \Rightarrow \quad (2x-1)\ln 5 = \ln 12 + \underbrace{\ln 3^x}_{=x\ln 3}$$

$$\Rightarrow \quad x(2\ln 5 - \ln 3) = \ln 12 + \ln 5$$

$$\Rightarrow \quad \boxed{x = \frac{\ln 12 + \ln 5}{2\ln 5 - \ln 3}}$$

(c) $\log_2(x+5) + \log_2(x-2) = 3 \quad \Rightarrow \quad \log_2[(x+5)(x-2)] = 3$

Taking exponential with base 2 on both sides, we get

$$2^{\log_2[(x+5)(x-2)]} = 2^3$$

$$\Rightarrow \quad (x+5)(x-2) = 8$$

$$\Rightarrow \quad x^2 + 3x - 10 = 8$$

$$\Rightarrow \quad x^2 + 3x - 18 = 0$$

$$\Rightarrow \quad (x-3)(x+6) = 0$$

$\Rightarrow \quad x - 3 = 0$ or $x + 6 = 0$

$\Rightarrow \quad x = 3$ or $x = -6$ (rejected since $\log_2(x+5)$ and $\log_2(x-2)$ are not defined when $x = -6$)

$$\therefore \quad \boxed{x = 3}$$

(d) $e^x - 8e^{-x} = 7$

Hyperbolic functions

For any real value x, the **hyperbolic sine** function ($\sinh x$) and the **hyperbolic cosine** function ($\cosh x$) are defined as

$\boxed{\sinh x = \frac{1}{2}(e^x - e^{-x})}$ and $\boxed{\cosh x = \frac{1}{2}(e^x + e^{-x})}$, respectively.

Note that $\sinh x \neq \sin(hx)$, $\cosh x \neq \cos(hx)$.

Remark:

Recall that cosine and sine are called **circular functions** because, for any $t \in \mathbb{R}$, the point $(\cos t, \sin t)$ lies on the circle with equation $x^2 + y^2 = 1$. Similarly, hyperbolic cosine and hyperbolic sine are called **hyperbolic functions** because, for any $t \in \mathbb{R}$, the point $(\cosh t, \sinh t)$ lies on the hyperbola with equation $x^2 - y^2 = 1$ (see Example 11(a)).

Example 11

Prove the following:

(a) $\cosh^2 x - \sinh^2 x = 1$

(b) $\cosh^2 x + \sinh^2 x = \cosh(2x)$

Solution

(a) $$\cosh^2 x - \sinh^2 x = \left[\frac{1}{2}(e^x + e^{-x})\right]^2 - \left[\frac{1}{2}(e^x - e^{-x})\right]^2$$

$$= \left[\frac{1}{4}\left(e^{2x} + 2\underbrace{e^x \cdot e^{-x}}_{=1} + e^{-2x}\right)\right] - \left[\frac{1}{4}\left(e^{2x} - 2\underbrace{e^x \cdot e^{-x}}_{=1} + e^{-2x}\right)\right]$$

$$= \frac{4}{4}$$

$$= 1$$

$\therefore\ \cosh^2 x - \sinh^2 x = 1.$

(b) $$\cosh^2 x + \sinh^2 x = \left[\frac{1}{2}(e^x + e^{-x})\right]^2 + \left[\frac{1}{2}(e^x - e^{-x})\right]^2$$

$$= \left[\frac{1}{4}\left(e^{2x} + 2\underbrace{e^x \cdot e^{-x}}_{=1} + e^{-2x}\right)\right] + \left[\frac{1}{4}\left(e^{2x} - 2\underbrace{e^x \cdot e^{-x}}_{=1} + e^{-2x}\right)\right]$$

$$= \frac{1}{2}(e^{2x} + e^{-2x})$$

$$= \cosh(2x)$$

$\therefore\ \cosh^2 x + \sinh^2 x = \cosh(2x)$

Other identities

- $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$
- $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
- $\cosh(2x) = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x = 2\cosh^2 x - 1$
- $\sinh(2x) = 2\sinh x \cosh x$

Exercise: Prove each of the above identities by using the definitions of $\sinh x$ and $\cosh x$.

Example 12

For each of the hyperbolic functions $\sinh x$ and $\cosh x$, determine whether it is an even function , odd function, or neither of them.

Solution

Let $f_1(x) = \sinh x$, then

$f_1(-x) = \sinh(-x) = \frac{1}{2}\left(e^{-x} - e^{-(-x)}\right) = -\frac{1}{2}(e^x - e^{-x}) = -\sinh x = -f_1(x)$.

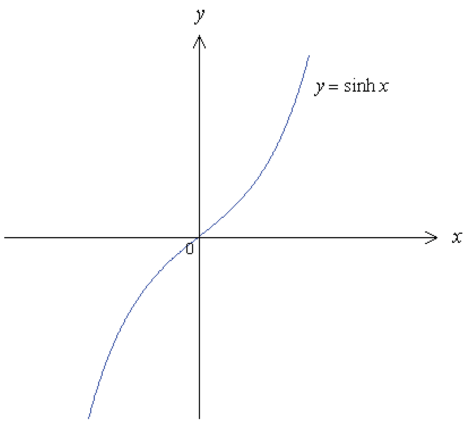
$\therefore$ $f_1(x) = \sinh x$ is an **odd** function.

Let $f_2(x) = \cosh x$, then

$f_2(-x) = \cosh(-x) = \frac{1}{2}\left(e^{-x} + e^{-(-x)}\right) = \cosh x = f_2(x)$.

$\therefore$ $f_2(x) = \cosh x$ is an **even** function.

Graphs of hyperbolic sine and hyperbolic cosine functions



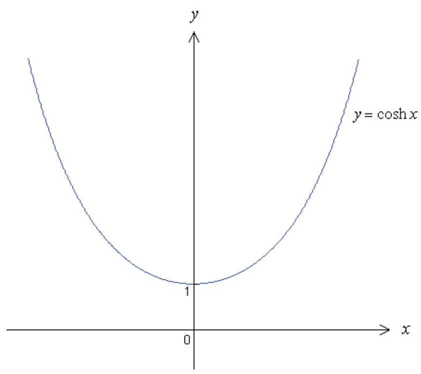
$y = \sinh x = \frac{1}{2}(e^x - e^{-x})$

Domain = $\mathbb{R}$

Range = $\mathbb{R}$

Odd function

$\sinh(-x) = -\sinh x$



$y = \cosh x = \frac{1}{2}(e^x + e^{-x})$

Domain = $\mathbb{R}$

Range = $[1, \infty)$

Even function

$\cosh(-x) = \cosh x$

Example 13

Solve each of the following equations for x.

(a) $\cosh 3x = 2$ (b) $4\sinh x = 3\cosh x$ (c) $\cosh 2x = 3\sinh x$

Solution

(a) $\cosh 3x = 2 \Rightarrow \frac{1}{2}(e^{3x} + e^{-3x}) = 2$

$\Rightarrow e^{3x} + e^{-3x} = 4$

$\Rightarrow e^{6x} + 1 = 4e^{3x}$

$\Rightarrow e^{6x} - 4e^{3x} + 1 = 0$

Let $y = e^{3x}$. Then we have $y^2 - 4y + 1 = 0$. By the quadratic equation formula,

$$y = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

$\therefore\ e^{3x} = 2 + \sqrt{3}$ or $e^{3x} = 2 - \sqrt{3}$

$\Rightarrow 3x = \ln(2 + \sqrt{3})$ or $3x = \ln(2 - \sqrt{3})$

$\Rightarrow \boxed{x = \frac{1}{3}\ln(2 + \sqrt{3})}$ or $\boxed{x = \frac{1}{3}\ln(2 - \sqrt{3})}$

(b) $4\sinh x = 3\cosh x \Rightarrow \frac{\sinh x}{\cosh x} = \frac{3}{4}$

$\Rightarrow \frac{\frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x})} = \frac{3}{4}$

$\Rightarrow 4(e^x - e^{-x}) = 3(e^x + e^{-x})$

$\Rightarrow e^x - 7e^{-x} = 0$

$\Rightarrow e^{2x} - 7 = 0$

$\Rightarrow e^{2x} = 7$

$\Rightarrow 2x = \ln 7$

$\Rightarrow \boxed{x = \frac{1}{2}\ln 7}$

(c) $\cosh 2x = 3\sinh x$

Other hyperbolic functions (*for your reference*)

The **hyperbolic tangent** ($\tanh x$), **hyperbolic secant** ($\operatorname{sech} x$), **hyperbolic cosecant** ($\operatorname{csch} x$), and **hyperbolic cotangent** ($\coth x$) functions are defined as follows:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$