MA1201 Calculus and Basic Linear Algebra II

Chapter 4

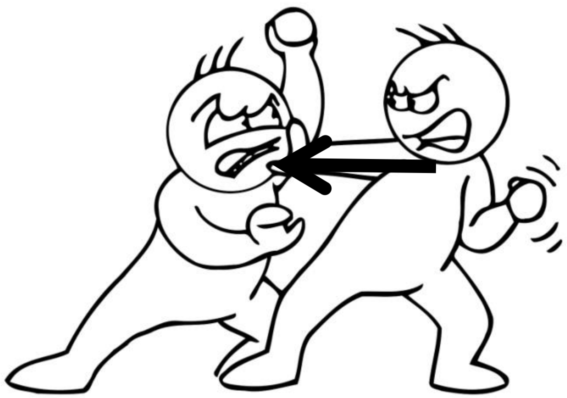
Vector Algebra

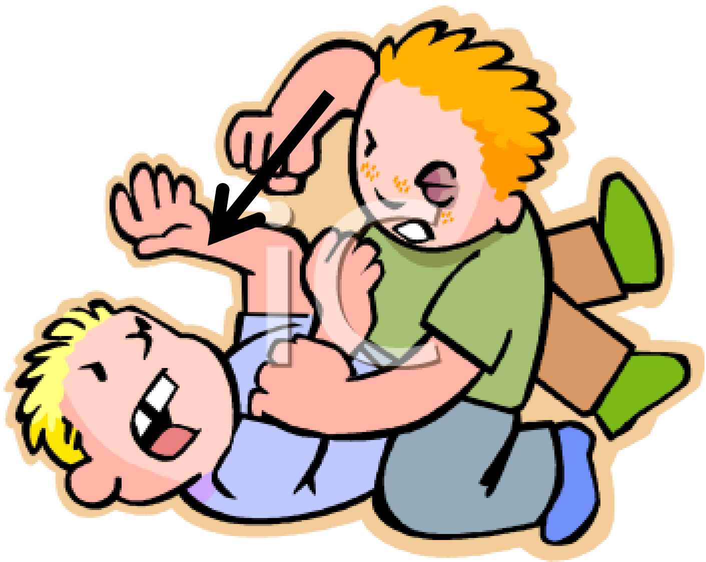
Introduction

We use numbers (real numbers) to describe a lot of things (e.g. temperature, price of products, your GPA, your achievement in Candy Crush Saga etc.) in our real life. Most of these quantities are just a single number (also called *scalar*).

Sometimes, a single number may not be enough to describe certain things. When people study force in Physics, they are also interested in the direction of the force besides its magnitude. Here are some examples:

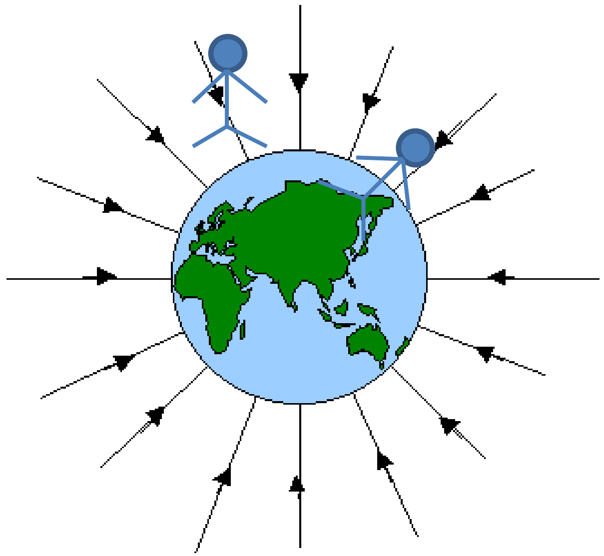
1. Force





2. Magnetic Field

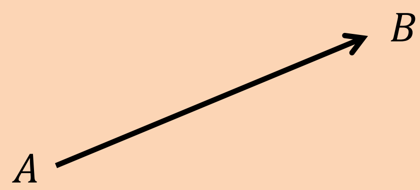
3. Gravitational force of earth $\vec{G}(x, y, z)$:



In order to describe the above phenomena precisely, one has to develop a notation which can describe the magnitude and the direction of the force at the same time. This notation is called *vectors*.

Definition (Vectors)

A vector is a directed line segment joining two points A and B. We denote a vector as $\overrightarrow{AB}$ or $\vec{a}$.



- Here, A is called an initial point and B is called a terminal point. These two points are used to indicate the direction of the vector.
- The magnitude of the vector, denoted by $\left|\overrightarrow{AB}\right|$ or $|\vec{a}|$, is defined as the length of the segment AB.
- One has to be careful that $\overrightarrow{AB} \neq \overrightarrow{BA}$. These two vectors are different in *direction* although they are the same in magnitude.

Some terminologies about vectors

1. Equality of vectors (comparison of two vectors)

We say two vectors $\vec{a}$ and $\vec{b}$ are equal if they both have the same magnitude and direction.

☺Note:

Since the vector only records the direction and magnitude. It is possible that two vectors at *different position* are equal. For example: the vectors $\overrightarrow{PQ}$ and $\overrightarrow{AB}$ shown below are equal since they both have magnitude 2 and same direction (pointing to the east).

$$A \xrightarrow{\quad 2 \quad} B$$

$$P \xrightarrow{\quad 2 \quad} Q$$

2. Zero vector (denoted by $\vec{0}$)

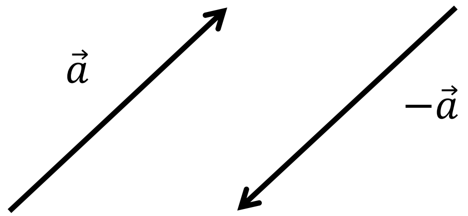
Zero vector is a vector which has magnitude 0.

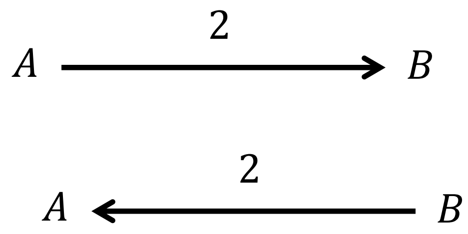
- Geometrically, it is simply a point in a plane.
- It does not have any direction.

3. "Negative" of a vector

The negative of a vector $\vec{a}$, denoted by $-\vec{a}$, is a vector having the <u>same magnitude</u> as $\vec{a}$ but <u>opposite direction</u> to $\vec{a}$.

Example: $\overrightarrow{BA} = -\overrightarrow{AB}$

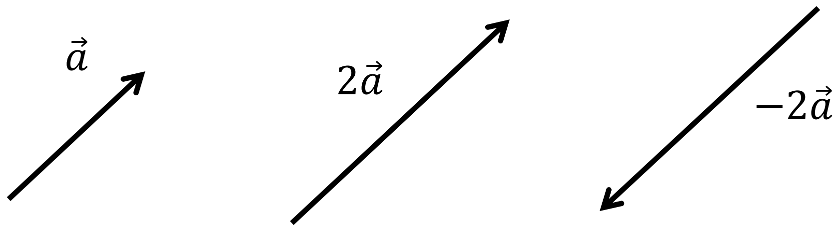




4. Scalar multiplication of a vector

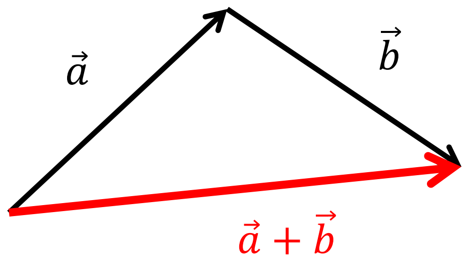
Given a vector $\vec{a}$ and a scalar number c,

- If $c > 0$, the vector $c\vec{a}$ has the magnitude $c|\vec{a}|$ and the direction is the same as that of $\vec{a}$.
- If $c < 0$, the vector $c\vec{a}$ has the magnitude $-c|\vec{a}|$ and the direction is opposite to that of $\vec{a}$.



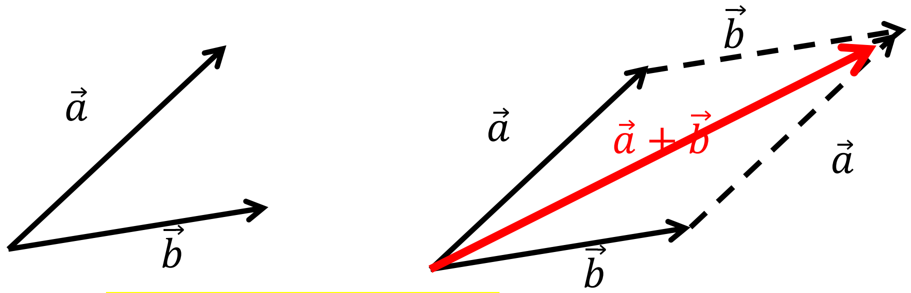
5. Addition of vectors

Let $\vec{a}$ and $\vec{b}$ be two vectors, we define the sum $\vec{a} + \vec{b}$ to be the *vector from the tail of* $\vec{a}$ *to the tip of* $\vec{b}$, i.e.,



Remark of 5

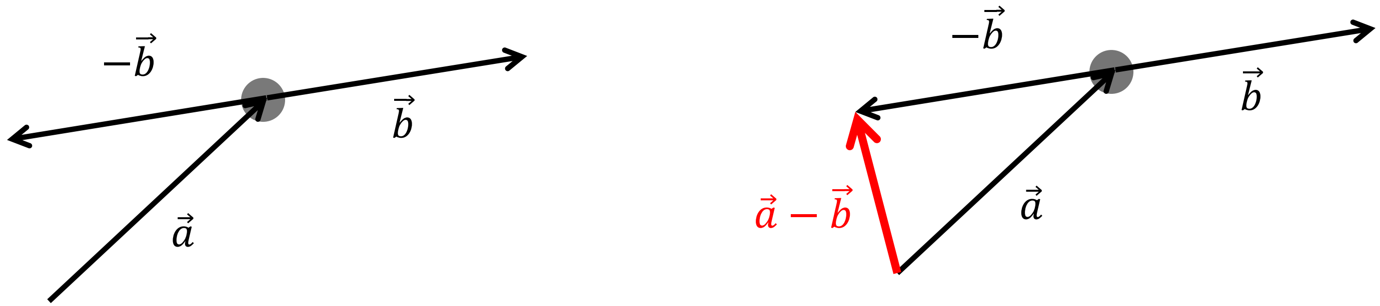
Alternatively, one can interpret the sum $\vec{a}+\vec{b}$ as follows



This is known as *parallelogram* law.

6. Subtraction of vectors

We define the difference $\vec{a}-\vec{b}$ to be $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$.



7. Unit Vector (denoted by $\hat{a}$)

It is the vector with magnitude 1. It is often used to describe the direction of a vector ONLY.

For any vector $\vec{a} \neq \vec{0}$, one can express $\vec{a}$ in the following form:

$$\vec{a} = \underbrace{|\vec{a}|}_{\text{magnitude}} \times \underbrace{\frac{\vec{a}}{|\vec{a}|}}_{\text{unit direction of } \vec{a}} .$$

In other word, any non-zero vector can be expressed as the product of a magnitude (number) and a direction (a unit vector).

Example 1

We let $\vec{a} \neq \vec{0}$ be a vector with magnitude 5. Using the definition of vector,

(a) find the magnitude of the vectors $5\vec{a}$ and $-2\vec{a}$.

(b) We let $\vec{b}$ be another vector having the magnitude 3 and its direction is the same to that of $\vec{a}$. Express $\vec{b}$ in terms of $\vec{a}$.

☺Solution of (a)

The magnitude of $5\vec{a}$ is $5 \times \underbrace{|\vec{a}|}_{=5} = 25$ and the magnitude of $-2\vec{a}$ is $2 \times |\vec{a}| = 10$.

☺Solution of (b)

Note that the unit vector of $\vec{a}$ (direction of $\vec{a}$) is $\frac{\vec{a}}{|\vec{a}|} = \frac{1}{5}\vec{a}$, then

$$\vec{b} = \underbrace{3}_{\text{magnitude}} \times \underbrace{\left(\frac{1}{5}\vec{a}\right)}_{\text{unit direction}} = \frac{3}{5}\vec{a}.$$

Mathematical representation of a vector

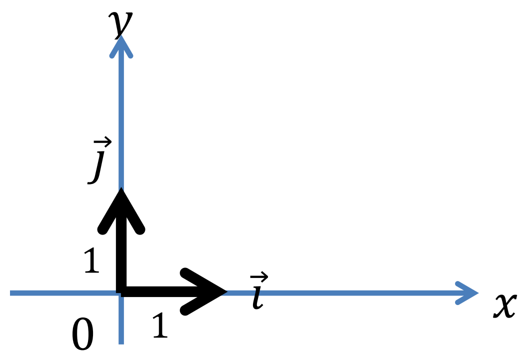
In this section, we will discuss how to present a vector mathematically in $\mathbb{R}^2$ and $\mathbb{R}^3$. In particular, given the coordinates of two points A and B, we would like to express the vector $\overrightarrow{AB}$ in terms of the coordinates of A and B.

The vector in 2D-space $\mathbb{R}^2$

The analysis is divided into three steps:

Step 1: 2 fundamental vectors

We define $\vec{\imath}$ and $\vec{\jmath}$ to be two *orthogonal* (perpendicular to each other) unit vectors in the directions of the positive x, y axes respectively.

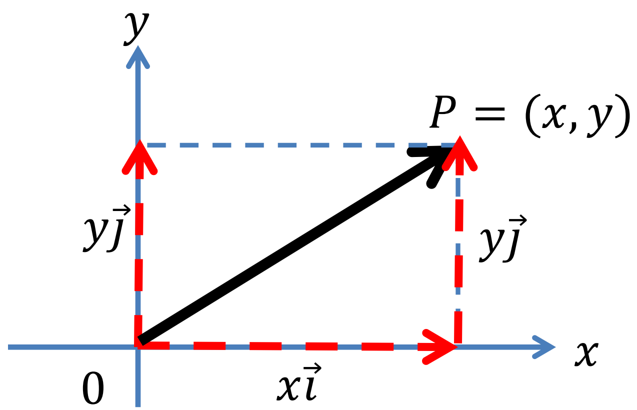


Step 2: Position Vectors

Let $P = (x,\ y)$ be a point in 2D-plane, the position vector of P, denoted by $\overrightarrow{OP}$, is defined as the vector from the origin $O = (0,0)$ to the point P.

Using the fundamental vectors and the graph below, one can express the position vector $\overrightarrow{OP}$ as

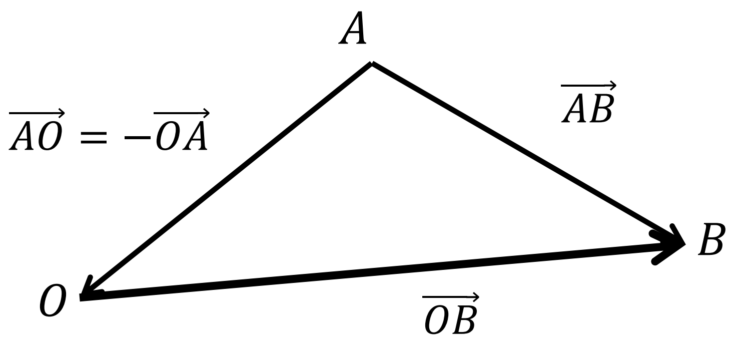
$$\overrightarrow{OP} = x\vec{i} + y\vec{j}$$



Step 3: Define vector in 2D plane

We let $A = (x_1,\ y_1)$ and $B = (x_2,\ y_2)$ be two points in a plane. Using vector addition, the vector from A to B $(\overrightarrow{AB})$ is found to be

$$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} = -(x_1\vec{\imath} + y_1\vec{\jmath}) + (x_2\vec{\imath} + y_2\vec{\jmath})$$
$$= (x_2 - x_1)\vec{\imath} + (y_2 - y_1)\vec{\jmath}.$$

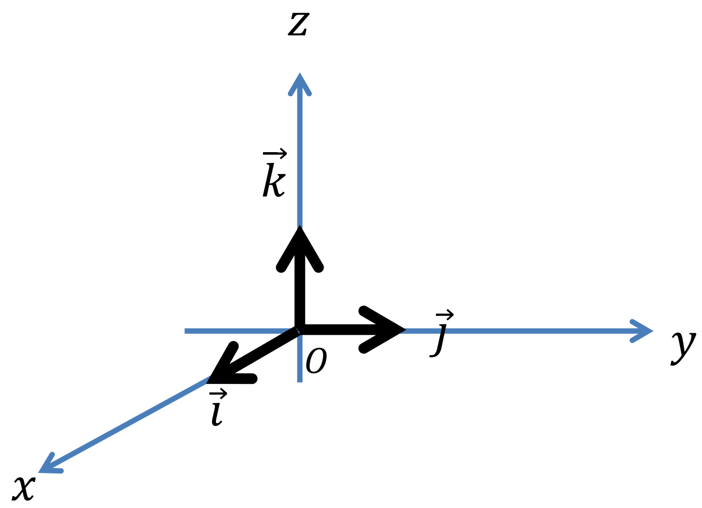


The vectors in 3D-space $\mathbb{R}^3$

Given the coordinates of two points $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ in 3D-space, one can use similar method to express the vector $\overrightarrow{AB}$.

Step 1: 3 fundamental vectors

We let $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$ to be three *orthogonal* (perpendicular to each other) unit vectors in the directions of the positive x, y and z axes respectively.



Step 2: Position

Let $P = (x,\ y,\ z)$. The position vector of P is defined as the vector from the origin $O = (0, 0, 0)$ to P and is given by

$$\overrightarrow{OP} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}.$$

Step 3: Define vector in 3D plane

Given the coordinates of two points $A = (x_1,\ y_1,\ z_1)$ and $B = (x_2,\ y_2,\ z_2)$, then the vector $\overrightarrow{AB}$ can be expressed as

$$\begin{aligned}\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\overrightarrow{OA} + \overrightarrow{OB} &= -\left(x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k}\right) + \left(x_2\vec{\imath} + y_2\vec{\jmath} + z_2\vec{k}\right) \\ &= (x_2 - x_1)\vec{\imath} + (y_2 - y_1)\vec{\jmath} + (z_2 - z_1)\vec{k}.\end{aligned}$$

Previously, we define the equality of vector, sum of vectors, scalar multiplication of vectors, unit vectors etc., in an intuitive way. We shall restate those statements in a mathematical way. We summarize the result as follows:

Properties of vectors

Let $\vec{a} = x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k}$ and $\vec{b} = x_2\vec{\imath} + y_2\vec{\jmath} + z_2\vec{k}$ be two vectors, then we have

1. *Equality of vectors*

$$\vec{a} = \vec{b} \;\; \Rightarrow \;\; x_1 = x_2, \qquad y_1 = y_2, \qquad z_1 = z_2.$$

2. *Zero Vector*

$$\vec{0} = 0\vec{\imath} + 0\vec{\jmath} + 0\vec{k}$$

3. *Addition and Subtraction of vectors*

$$\vec{a} \pm \vec{b} = (x_1 \pm x_2)\vec{\imath} + (y_1 \pm y_2)\vec{\jmath} + (z_1 \pm z_2)\vec{k}.$$

4. Scalar multiplication

$$c\vec{a} = cx_1\vec{\imath} + cy_1\vec{\jmath} + cz_1\vec{k}.$$

5. Magnitude of $\vec{a}$

$$|\vec{a}| = \sqrt{x_1^2 + y_1^2 + z_1^2}.$$

6. Unit Vector of $\vec{a}$

$$\frac{\vec{a}}{|\vec{a}|} = \underbrace{\frac{x_1}{\sqrt{x_1^2 + y_1^2 + z_1^2}}}_{\cos\alpha}\vec{\imath} + \underbrace{\frac{y_1}{\sqrt{x_1^2 + y_1^2 + z_1^2}}}_{\cos\beta}\vec{\jmath} + \underbrace{\frac{z_1}{\sqrt{x_1^2 + y_1^2 + z_1^2}}}_{\cos\gamma}\vec{k}.$$

where α, β, γ *are the angles between* $\vec{a}$ *and the positive* x*-axis,* y*-axis,* z*-axis, respectively and* $\cos\alpha$, $\cos\beta$, $\cos\gamma$ *are called direction cosines.*

Remark

All these properties can be proved using the definition of vectors in pp. 4-15. Although these properties are intuitive, it requires certain level of mathematical argument.

Example 2

Let $P = (1, 2, -4)$ and $Q = (3, -2, 1)$ be two points in $\mathbb{R}^3$.

(a) Find $\overrightarrow{PQ}$.

(b) Find a vector $\vec{a}$ which has magnitude 3 and direction is *opposite* to that of $\overrightarrow{PQ}$.

☺Solution

(a) $\overrightarrow{PQ} = \underbrace{\left(3\vec{\imath} - 2\vec{\jmath} + \vec{k}\right)}_{\overrightarrow{OQ}} - \underbrace{\left(\vec{\imath} + 2\vec{\jmath} - 4\vec{k}\right)}_{\overrightarrow{OP}} = 2\vec{\imath} - 4\vec{\jmath} + 5\vec{k}$.

(b) The unit vector of $\overrightarrow{PQ}$ is given by $\widehat{PQ} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{2\vec{\imath} - 4\vec{\jmath} + 5\vec{k}}{\sqrt{2^2 + (-4)^2 + 5^2}} = \frac{2}{\sqrt{45}}\vec{\imath} - \frac{4}{\sqrt{45}}\vec{\jmath} + \frac{5}{\sqrt{45}}\vec{k}$.

Thus the vector $\vec{a}$ is then given by

$$\vec{a} = \underbrace{3}_{\text{magnitude}} \underbrace{\left(-\widehat{PQ}\right)}_{\substack{\text{direction} \\ \left(\text{opposite to } \overrightarrow{PQ}\right)}} = 3\left(-\frac{2}{\sqrt{45}}\vec{\imath} + \frac{4}{\sqrt{45}}\vec{\jmath} - \frac{5}{\sqrt{45}}\vec{k}\right)$$

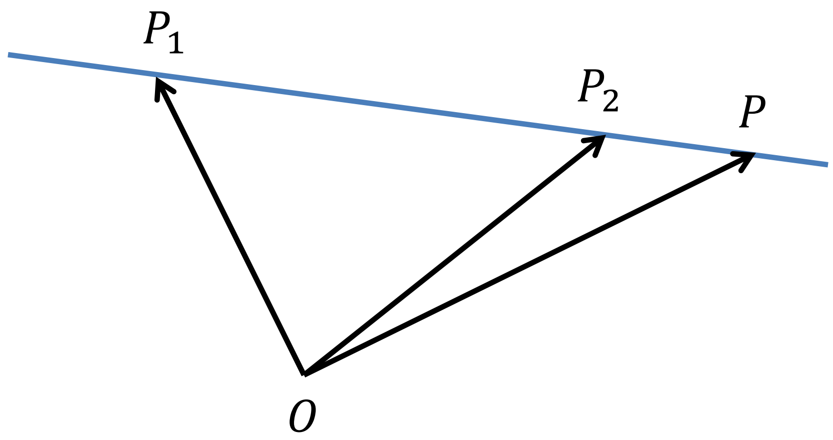
$$= -\frac{6}{\sqrt{45}}\vec{\imath} + \frac{12}{\sqrt{45}}\vec{\jmath} - \frac{15}{\sqrt{45}}\vec{k}$$

Example 3

Let L be a straight line passing through $P_1 = (3, 2, 3)$ and $P_2 = (0, 2, 7)$. Let P be a point on L that is NOT on the line segment P_1P_2 and is 2 units from P_2.

(a) Find the vector $\overrightarrow{P_1P}$.

(b) Hence, find the coordinates of P.



☺IDEA of (a)

To find the vector $\overrightarrow{P_1P}$, we need to know its magnitude (which is the length of $\left|\overrightarrow{P_1P_2}\right| + 2$) as well as its direction (same as that of $\overrightarrow{P_1P_2}$)

☺Solution of (a):

Note that $\overrightarrow{P_1P_2} = \underbrace{\left(2\vec{j} + 7\vec{k}\right)}_{\overrightarrow{OP_2}} - \underbrace{\left(3\vec{\imath} + 2\vec{j} + 3\vec{k}\right)}_{\overrightarrow{OP_1}} = -3\vec{\imath} + 4\vec{k}$.

Then $\left|\overrightarrow{P_1P_2}\right| = \sqrt{(-3)^2 + (0)^2 + 4^2} = 5$ and $\widehat{P_1P_2} = \frac{\overrightarrow{P_1P_2}}{\left|\overrightarrow{P_1P_2}\right|} = -\frac{3}{5}\vec{\imath} + \frac{4}{5}\vec{k}$. Hence, the vector $\overrightarrow{P_1P}$ is found to be

$$\overrightarrow{P_1P} = \underbrace{\left(\left|\overrightarrow{P_1P_2}\right| + 2\right)}_{\text{magnitude}} \underbrace{\widehat{P_1P_2}}_{\text{direction}} = 7\left(-\frac{3}{5}\vec{\imath} + \frac{4}{5}\vec{k}\right) = -\frac{21}{5}\vec{\imath} + \frac{28}{5}\vec{k}.$$

☺Solution of (b)

We need to find the vector $\overrightarrow{OP}$, we note from the above figure that

$$\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P} = \left(3\vec{\imath} + 2\vec{j} + 3\vec{k}\right) + \left(-\frac{21}{5}\vec{\imath} + \frac{28}{5}\vec{k}\right) = -\frac{6}{5}\vec{\imath} + 2\vec{j} + \frac{43}{5}\vec{k}.$$

The coordinates of P is $\left(-\frac{6}{5},\ 2,\ \frac{43}{5}\right)$.

"Multiplication" of vectors: Scalar Product and Vector Product

There are two products developed in vectors: Scalar product $\vec{a}\cdot\vec{b}$ (also called dot product) and vector product $\vec{a}\times\vec{b}$ (also called cross product). The difference between these two products is their outputs: The outcome of a scalar product is a number and the outcome of vector product is a vector.

Scalar Product

Definition (Scalar Product)

Let $\vec{a} = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$ and $\vec{b} = b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k}$ be two vectors, the scalar product of these two vectors, denoted by $\vec{a}\cdot\vec{b}$ is defined as

$$\vec{a}\cdot\vec{b} = \underbrace{a_1b_1}_{\substack{\text{product}\\\text{of the 1st}\\\text{component}}} + \underbrace{a_2b_2}_{\substack{\text{product}\\\text{of the 2nd}\\\text{component}}} + \underbrace{a_3b_3}_{\substack{\text{product}\\\text{of the 3rd}\\\text{component}}}.$$

Properties of scalar product

Using the definition of scalar product, one can derive the following properties

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors, we have

1. $\vec{a} \cdot \vec{0} = \vec{0} \cdot \vec{a} = 0$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\left(k\vec{a} + l\vec{b}\right) \cdot \vec{c} = k(\vec{a} \cdot \vec{c}) + l\left(\vec{b} \cdot \vec{c}\right)$
   where k, l are two constants.
4. $\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2 = |\vec{a}|^2$

Remark:

- We observe that the scalar product has similar properties as that of product of real numbers.
- The 4th property is useful in finding the vector's magnitude in theoretic context (doing proof).

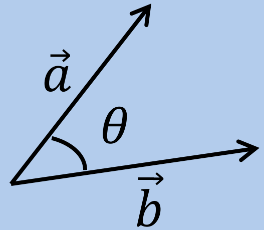
Alternative representation of scalar product

Theorem

Let $\vec{a} = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$ and $\vec{b} = b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k}$ be two vectors, then

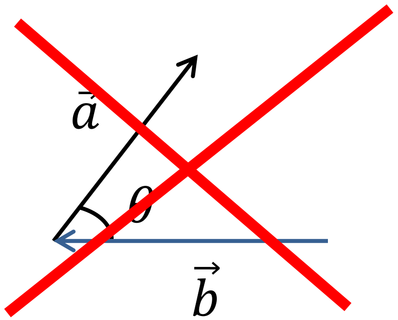
$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

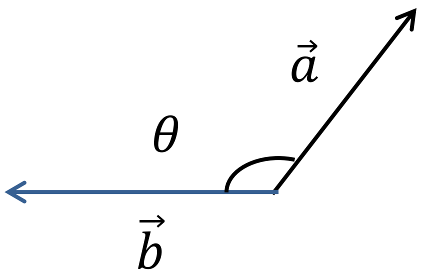
where $0 \le \theta \le \pi$ is the angle between two vectors.



Important Remark: The position of θ

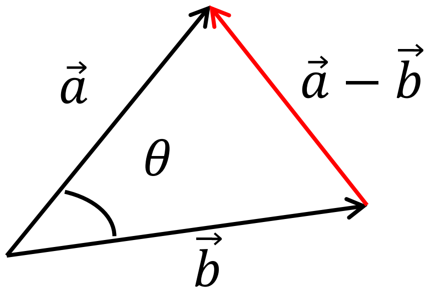
In the theorem, θ in the formula represents the (smaller) angle between two vectors and both vectors should point OUTWARDS.





☺Proof of theorem:

We consider the figure on the right.



Applying the cosine law for this triangle, we have

$$\left|\vec{a}-\vec{b}\right|^2 = |\vec{a}|^2 + \left|\vec{b}\right|^2 - 2|\vec{a}|\left|\vec{b}\right|\cos\theta$$

Using the fact that $|\vec{y}|^2 = \vec{y}\cdot\vec{y}$, one can rewrite the equation as

$$\left(\vec{a}-\vec{b}\right)\cdot\left(\vec{a}-\vec{b}\right) = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - 2|\vec{a}|\left|\vec{b}\right|\cos\theta$$

$$\Rightarrow \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - 2\left(\vec{a}\cdot\vec{b}\right) = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - 2|\vec{a}|\left|\vec{b}\right|\cos\theta$$

$$\Rightarrow \vec{a}\cdot\vec{b} = |\vec{a}|\left|\vec{b}\right|\cos\theta.$$

Example 4

Let $\vec{\boldsymbol{a}} = \vec{\boldsymbol{\imath}} - \vec{\boldsymbol{k}}$ and $\vec{\boldsymbol{b}} = 2\vec{\boldsymbol{\imath}} - \vec{\boldsymbol{\jmath}} + \vec{\boldsymbol{k}}$ be two vectors, find

(a) $\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}}$

(b) The angle between $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$

(c) Let $\vec{\boldsymbol{c}} = \boldsymbol{x}\vec{\boldsymbol{\imath}} + 2\vec{\boldsymbol{k}}$ be a vector such that $\vec{\boldsymbol{b}}$ and $\vec{\boldsymbol{c}}$ are orthogonal (or perpendicular). Find the value of $\boldsymbol{x}$.

☺Solution:

(a) $\vec{a} \cdot \vec{b} = (1)(2) + (0)(-1) + (-1)(1) = 1.$

(b) Let θ $(0 \leq \theta \leq \pi)$ be the angle between $\vec{a}$ and $\vec{b}$, then it must satisfy

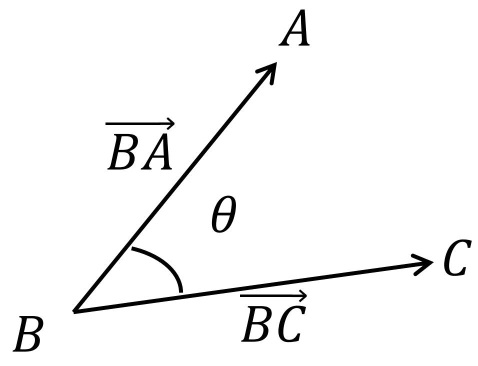
$$\vec{a} \cdot \vec{b} = |\vec{a}|\left|\vec{b}\right| \cos\theta \;\Rightarrow\; 1 = \left(\sqrt{2}\right)\left(\sqrt{6}\right)\cos\theta \;\Rightarrow\; \cos\theta = \frac{1}{\sqrt{12}} \;\Rightarrow\; \theta = 73.22°.$$

(c) If $\vec{b}$ and $\vec{c}$ are orthogonal, then $\cos\theta = 0$ and

$$\vec{b} \cdot \vec{c} = 0 \;\Rightarrow\; 2x + 2 = 0 \;\Rightarrow\; x = -1.$$

Example 5

Let $A = (2, 1, 0)$, $B = (1, 3, -1)$ and $C = (-9, 0, 3)$ be three points in the 3D space. Using vector method, find $\angle ABC$.



☺Solution

We let $\theta = \angle ABC$, then we have

$$\cos\theta = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|} \dots\dots (*)$$

Note that $\overrightarrow{BA} = (2\vec{\imath} + \vec{\jmath}) - \left(\vec{\imath} + 3\vec{\jmath} - \vec{k}\right) = \vec{\imath} - 2\vec{\jmath} + \vec{k}$ and $\overrightarrow{BC} = \left(-9\vec{\imath} + 3\vec{k}\right) - \left(\vec{\imath} + 3\vec{\jmath} - \vec{k}\right) = -10\vec{\imath} - 3\vec{\jmath} + 4\vec{k}$, we then get

$$\overrightarrow{BA}\cdot\overrightarrow{BC} = 1(-10) + (-2)(-3) + (1)(4) = 0.$$

Substitute them into the equation $(*)$, we finally get

$$\cos\theta = \frac{0}{\left(\sqrt{6}\right)\left(\sqrt{125}\right)} = \frac{0}{\sqrt{750}} \Rightarrow \theta = 90°.$$

Example 6

Let $\vec{c}$ and $\vec{d}$ be two <u>unit vectors</u> with $\vec{c} \cdot \vec{d} = \frac{1}{2}$.

(a) Compute $(\vec{c} + 2\vec{d}) \cdot (3\vec{c} - \vec{d})$

(b) Find $|\vec{c} + \vec{d}|$

☺Solution of (a)

Using the property of scalar product, we have

$$(\vec{c} + 2\vec{d}) \cdot (3\vec{c} - \vec{d}) = \vec{c} \cdot (3\vec{c} - \vec{d}) + 2\vec{d} \cdot (3\vec{c} - \vec{d})$$

$$= 3(\vec{c} \cdot \vec{c}) - (\vec{c} \cdot \vec{d}) + 6(\vec{d} \cdot \vec{c}) - 2(\vec{d} \cdot \vec{d})$$

$$\overset{\vec{c}\cdot\vec{d}=\vec{d}\cdot\vec{c}=\frac{1}{2}}{\cong} 3|\vec{c}|^2 - \frac{1}{2} + 6\left(\frac{1}{2}\right) - 2|\vec{d}|^2$$

$$= 3(1)^2 + \frac{5}{2} - 2(1)^2 = \frac{7}{2}.$$

☺Solution of (b)

☺IDEA:

Since $\vec{c}$ and $\vec{d}$ are unknown vectors, it is hard to find $\left|\vec{c}+\vec{d}\right|$ directly using $|\vec{a}| = \sqrt{x^2+y^2+z^2}$. Here, we may use scalar product to do our job.

Using the fact that $\vec{a}\cdot\vec{a} = |\vec{a}|^2$, we have

$$\left|\vec{c}+\vec{d}\right| = \sqrt{(\vec{c}+\vec{d})\cdot(\vec{c}+\vec{d})} = \sqrt{\vec{c}\cdot(\vec{c}+\vec{d})+\vec{d}\cdot(\vec{c}+\vec{d})}$$

$$= \sqrt{(\vec{c}\cdot\vec{c})+(\vec{c}\cdot\vec{d})+(\vec{d}\cdot\vec{c})+(\vec{d}\cdot\vec{d})}$$

$$\overset{\vec{c}\cdot\vec{d}=\vec{d}\cdot\vec{c}=\frac{1}{2}}{\cong} \sqrt{|\vec{c}|^2+\frac{1}{2}+\frac{1}{2}+\left|\vec{d}\right|^2}$$

$$= \sqrt{(1)^2+1+(1)^2} = \sqrt{3}.$$

Example 7 (Harder)

Let $\vec{a}$ and $\vec{b}$ be two vectors, show that

(a) $\left|\vec{a}+\vec{b}\right|^2+\left|\vec{a}-\vec{b}\right|^2=2\left(|\vec{a}|^2+\left|\vec{b}\right|^2\right)$

(b) $\left|\vec{a}+\vec{b}\right|\leq|\vec{a}|+\left|\vec{b}\right|$. Triangle inequality (Hint: Square it!)

☺Solution of (a)

Using the property that $|\vec{a}|^2=\vec{a}\cdot\vec{a}$, we get

$$\left|\vec{a}+\vec{b}\right|^2+\left|\vec{a}-\vec{b}\right|^2=\left(\vec{a}+\vec{b}\right)\cdot\left(\vec{a}+\vec{b}\right)+\left(\vec{a}-\vec{b}\right)\cdot\left(\vec{a}-\vec{b}\right)$$

$$=\left(\vec{a}\cdot\vec{a}+2\left(\vec{a}\cdot\vec{b}\right)+\vec{b}\cdot\vec{b}\right)+\left(\vec{a}\cdot\vec{a}-2\left(\vec{a}\cdot\vec{b}\right)+\vec{b}\cdot\vec{b}\right)$$

$$=2\left(\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}\right)$$

$$=2\left(|\vec{a}|^2+\left|\vec{b}\right|^2\right).$$

☺Solution of (b)

☺IDEA

We take the square on both sides of equation, we have

$$|\vec{a}+\vec{b}|^2 \le \left(|\vec{a}|+|\vec{b}|\right)^2 = |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2.$$

We consider

$$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = \vec{a}\cdot\vec{a} + 2(\vec{a}\cdot\vec{b}) + \vec{b}\cdot\vec{b}$$

$$\overset{\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta}{=} \quad |\vec{a}|^2 + 2|\vec{a}||\vec{b}|\cos\theta + |\vec{b}|^2$$

$$\overset{-1\le\cos\theta\le 1}{\le} \quad |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2 = \left(|\vec{a}|+|\vec{b}|\right)^2.$$

Taking square root on both sides, we finally get

$$|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|.$$

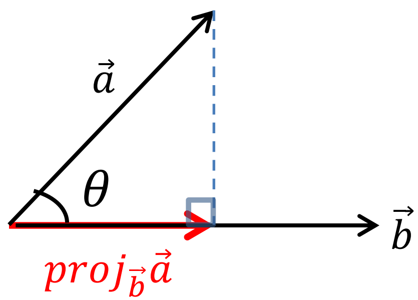
Projection Vector

Definition (Projection Vector)

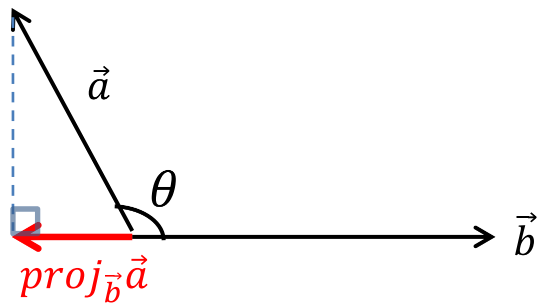
Let $\vec{b}$ be a fixed vector in 3D space $\mathbb{R}^3$ (or 2D space). For any vector $\vec{a}$, the projection vector of $\vec{a}$ onto $\vec{b}$, denoted by $proj_{\vec{b}}\vec{a}$ is defined as the *perpendicular projection* of $\vec{a}$ on $\vec{b}$.

Remark

Depending on the value of θ, the direction of the projection vector is different.



If $\theta < 90°$, $proj_{\vec{b}}\vec{a}$ has the *same* direction as that of $\vec{b}$



If $\theta > 90°$, $proj_{\vec{b}}\vec{a}$ has the *opposite* direction as that of $\vec{b}$

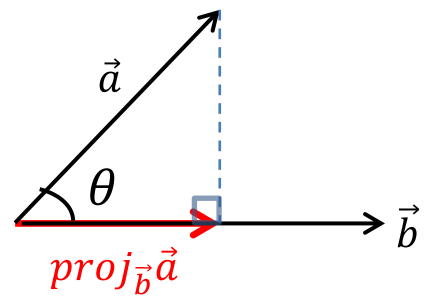
General Procedure of finding the projection vector

Step 1: Finding the angle between $\vec{a}$ and $\vec{b}$ to determine the direction of the projector vector. This can be done by using scalar product:

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \Rightarrow \begin{cases} \theta < 90^\circ & \text{if } \cos\theta > 0 \\ \theta > 90^\circ & \text{if } \cos\theta < 0 \end{cases}.$$

Step 2: Determine the projection vector using the formula

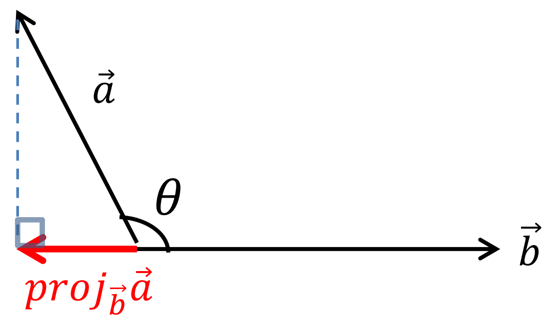
$$proj_{\vec{b}}\vec{a} = (\text{magnitude}) \times (\text{direction}) = |\vec{a}|\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\frac{\vec{b}}{|\vec{b}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}.$$



If $\theta < 90^\circ$

Magnitude $= |\vec{a}|\cos\theta$

Direction $= \hat{b}$



If $\theta > 90^\circ$

Magnitude $= |\vec{a}|\cos(180^\circ - \theta) = -|\vec{a}|\cos\theta$

Direction $= -\hat{b}$

Example 8

Let $\vec{b} = 2\vec{j} + 4\vec{k}$ be a vector. Find the projection vector of $\vec{a} = 3\vec{i} + \vec{k}$ on $\vec{b}$.

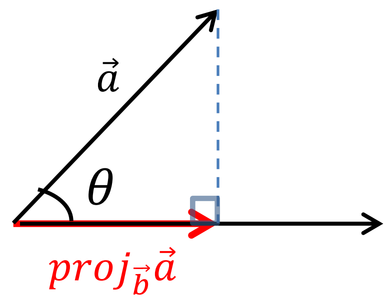
☺Solution:

Note that

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{3(0)+(0)(2)+4(1)}{\sqrt{10}\sqrt{20}} = \frac{4}{\sqrt{10}\sqrt{20}} > 0 \Rightarrow \theta < 90°.$$

Then the magnitude of the projection vector is $|\vec{a}|\cos\theta = \sqrt{10}\left(\frac{4}{\sqrt{10}\sqrt{20}}\right) = \frac{4}{\sqrt{20}}$ and its direction is $\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{2}{\sqrt{20}}\vec{j} + \frac{4}{\sqrt{20}}\vec{k}$. The projection vector is found to be

$$proj_{\vec{b}}\vec{a} = \underbrace{\frac{4}{\sqrt{20}}}_{\text{magnitude}} \underbrace{\left(\frac{2}{\sqrt{20}}\vec{j} + \frac{4}{\sqrt{20}}\vec{k}\right)}_{\text{direction}} = \frac{2}{5}\vec{j} + \frac{4}{5}\vec{k}.$$



Example 9

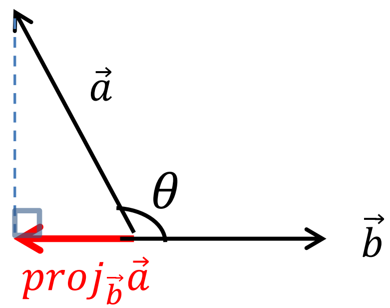
Let $\vec{a} = \vec{\imath} + \vec{\jmath} + \vec{k}$ and $\vec{b} = \vec{\jmath} - 4\vec{k}$ be two vectors, find the projection vector of $\vec{a}$ onto $\vec{b}$ ($proj_{\vec{b}}\vec{a}$).

☺Solution:

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-3}{\sqrt{3}\sqrt{17}} < 0 \Rightarrow \theta > 90°.$$

Then the magnitude of the projection vector is $|\vec{a}|\cos(180° - \theta) = -|\vec{a}|\cos\theta = -\sqrt{3}\left(\frac{-3}{\sqrt{3}\sqrt{17}}\right) = \frac{3}{\sqrt{17}}$ and its direction is $-\hat{b} = -\frac{\vec{b}}{|\vec{b}|} = -\frac{1}{\sqrt{17}}\vec{\jmath} + \frac{4}{\sqrt{17}}\vec{k}$. The projection vector is found to be

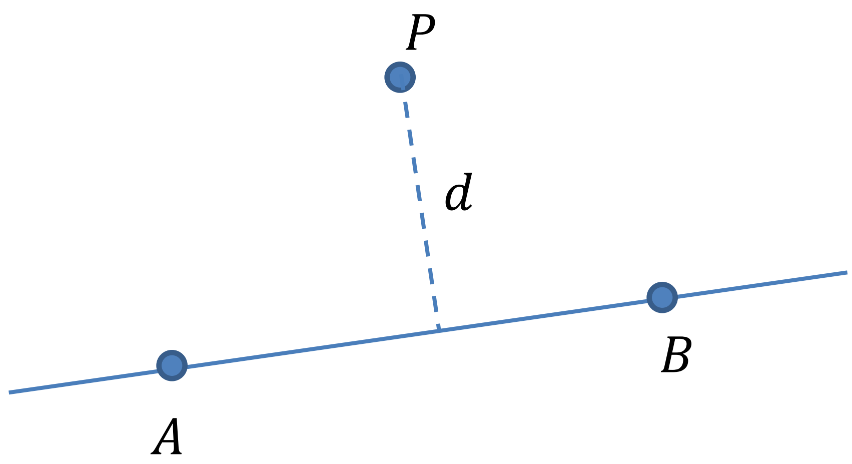
$$proj_{\vec{b}}\vec{a} = \underbrace{\frac{3}{\sqrt{17}}}_{\text{magnitude}}\underbrace{\left(-\frac{1}{\sqrt{17}}\vec{\jmath} + \frac{4}{\sqrt{17}}\vec{k}\right)}_{\text{direction}} = -\frac{3}{17}\vec{\jmath} + \frac{12}{17}\vec{k}.$$



Application of projection vector

Finding the distance between a point and a line

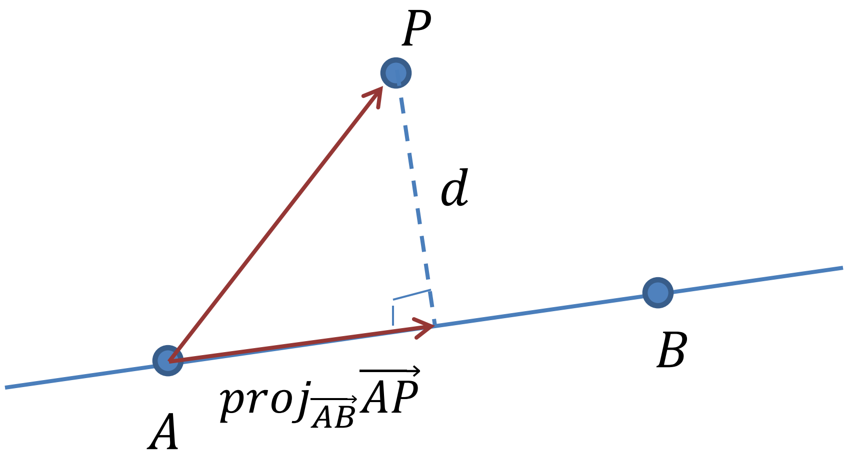
The projection vector is useful in finding the shortest distance between a point and a line. To see this, we consider the following figure



We would like to find the *shortest distance* d between the point P and the line passing through A and B.

- One can find the distance by considering the *projection vector*.

Suppose the coordinates of P, A and B are known, one can find the distance as follows



Since the triangle is right-angled, one can obtain the value of d by

$$d = \sqrt{\left|\overrightarrow{AP}\right|^2 - \left|proj_{\overrightarrow{AB}}\overrightarrow{AP}\right|^2}$$

provided that $proj_{\overrightarrow{AB}}\overrightarrow{AP}$ and $\overrightarrow{AP}$ are known.

Example 10

Let $A = (1, 3, -1)$, $B = (3, 6, 0)$ and $C = (-2, 4, -3)$ be three points. Find the shortest distance between C and the line passing through A and B.

☺Solution:

Let d be the shortest distance, one can sketch the graph and observe that

$$d = \sqrt{\left|\overrightarrow{AC}\right|^2 - \left|proj_{\overrightarrow{AB}}\overrightarrow{AC}\right|^2}.$$

Note that

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -3\vec{\imath} + \vec{\jmath} - 2\vec{k}, \qquad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 2\vec{\imath} + 3\vec{\jmath} + \vec{k}$$

It remains to find the magnitude of $proj_{\overrightarrow{AB}}\overrightarrow{AC}$. Note that

$$\cos\theta = \frac{\overrightarrow{AC} \cdot \overrightarrow{AB}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = -\frac{5}{14} < 0 \Rightarrow \ \theta > 90°.$$

Thus the magnitude of $proj_{\overrightarrow{AB}}\overrightarrow{AC}$ is given by

$$\left|\overrightarrow{AC}\right|\cos(180° - \theta) = -\left|\overrightarrow{AC}\right|\cos\theta = -\sqrt{14}\left(-\frac{5}{14}\right) = \frac{5}{\sqrt{14}}.$$

Therefore the required distance d is given by

$$d = \sqrt{\left|\overrightarrow{AC}\right|^2 - \left|proj_{\overrightarrow{AB}}\overrightarrow{AC}\right|^2} = \sqrt{\left(\sqrt{14}\right)^2 - \left(\frac{5}{\sqrt{14}}\right)^2} = \sqrt{\frac{171}{14}}.$$

Remark. Let Q be the foot of the perpendicular from C to A and B. Then

$$\overrightarrow{AQ} = proj_{\overrightarrow{AB}}\overrightarrow{AC} = \left(\overrightarrow{AC} \cdot \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}\right)\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = -\frac{5}{\sqrt{14}}\frac{2\vec{\imath} + 3\vec{\jmath} + \vec{k}}{\sqrt{14}} = -\frac{5}{7}\vec{\imath} - \frac{15}{14}\vec{\jmath} - \frac{5}{14}\vec{k},$$

$$\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ} = \left(\vec{\imath} + 3\vec{\jmath} - \vec{k}\right) + \left(-\frac{5}{7}\vec{\imath} - \frac{15}{14}\vec{\jmath} - \frac{5}{14}\vec{k}\right) = \frac{2}{7}\vec{\imath} + \frac{27}{14}\vec{\jmath} - \frac{19}{14}\vec{k}\,.$$

The coordinates of $Q = \left(\frac{2}{7}, \frac{27}{14}, -\frac{19}{14}\right)$.

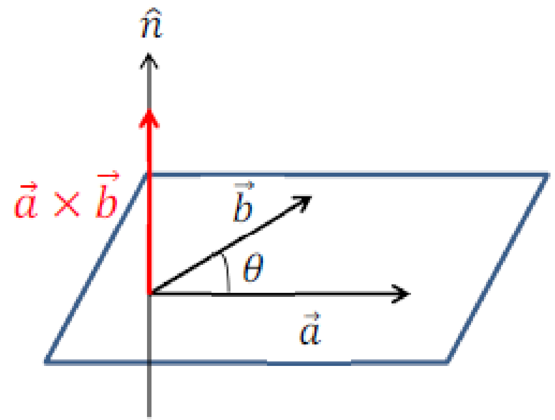
Vector Product

Definition (Vector Product)

Let $\vec{a} = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$ and $\vec{b} = b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k}$ be two vectors, the vector product of these two vectors, denoted by $\vec{a} \times \vec{b}$, is defined as

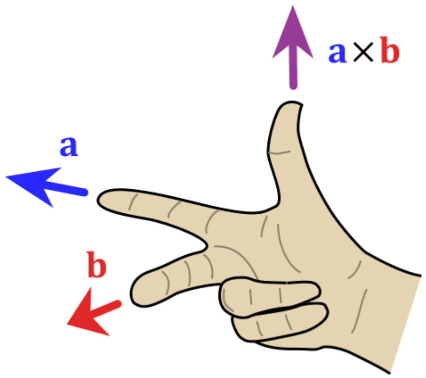
$$\vec{a} \times \vec{b} = \underbrace{|\vec{a}||\vec{b}|\sin\theta}_{\text{magnitude}} \quad \underbrace{\hat{n}}_{\text{direction}} \quad .$$

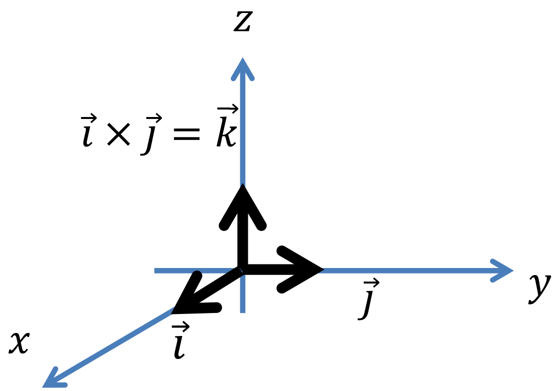
where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{n}$ is the unit vector *perpendicular to both* $\vec{a}$ and $\vec{b}$.



How to determine the direction of $\hat{n}$*?*

Given the vectors $\vec{a}$ and $\vec{b}$, there are two vectors (pointing upwards or pointing downwards) which are perpendicular to both $\vec{a}$ and $\vec{b}$. The final direction of $\hat{n}$ is often determined by using the right hand rule.





As an example, we would like to compute $\vec{\imath} \times \vec{\jmath}$. Note that the magnitude of $\vec{\imath} \times \vec{\jmath}$ is given by $|\vec{\imath}||\vec{\jmath}| \sin 90° = 1 \times 1 \times 1 = 1$. Using the right hand, we find that the direction of $\vec{\imath} \times \vec{\jmath}$ is $\hat{n} = \vec{k}$.

Thus $\vec{\imath} \times \vec{\jmath} = \underbrace{1}_{\text{magnitude}} \times \underbrace{\vec{k}}_{\text{direction}} = \vec{k}$.

Similarly, one can obtain the following elementary result:

$\vec{j} \times \vec{i} = -\vec{k}$,

$\vec{j} \times \vec{k} = \vec{i}$, $\qquad \vec{k} \times \vec{j} = -\vec{i}$.

$\vec{i} \times \vec{k} = -\vec{j}$, $\qquad \vec{k} \times \vec{i} = \vec{j}$.

$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = \vec{0}$ (Reason: Note that $\sin\theta = 0$ in this case)

Properties of Vector Product

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors in $\mathbb{R}^3$. Then

1. $\vec{b} \times \vec{a} = -\left(\vec{a} \times \vec{b}\right)$
2. $\vec{a} \times \left(\vec{b} \pm \vec{c}\right) = \left(\vec{a} \times \vec{b}\right) \pm (\vec{a} \times \vec{c})$
3. $(m\vec{a}) \times \vec{b} = \vec{a} \times \left(m\vec{b}\right) = m\left(\vec{a} \times \vec{b}\right)$
4. If $\vec{a}$ and $\vec{b}$ are parallel, then $\vec{a} \times \vec{b} = \vec{0}$.

Computation of Vector Products for General Case

Example 11

Let $\vec{a} = 2\vec{\imath} + \vec{\jmath}$, $\vec{b} = -\vec{\jmath} + 2\vec{k}$ and $\vec{c} = \vec{\imath} - \vec{\jmath} + \vec{k}$, find the value of $\vec{a} \times \vec{b}$, $\vec{b} \times \vec{c}$.

☺Solution:

$\vec{a} \times \vec{b} = (2\vec{\imath} + \vec{\jmath}) \times \left(-\vec{\jmath} + 2\vec{k}\right) = 2\vec{\imath} \times \left(-\vec{\jmath} + 2\vec{k}\right) + \vec{\jmath} \times \left(-\vec{\jmath} + 2\vec{k}\right)$

$= -2(\vec{\imath} \times \vec{\jmath}) + 4\left(\vec{\imath} \times \vec{k}\right) - (\vec{\jmath} \times \vec{\jmath}) + 2\left(\vec{\jmath} \times \vec{k}\right)$

$= -2\vec{k} - 4\vec{\jmath} - \vec{0} + 2\vec{\imath} = 2\vec{\imath} - 4\vec{\jmath} - 2\vec{k}$.

$\vec{b} \times \vec{c} = \left(-\vec{\jmath} + 2\vec{k}\ \right) \times \left(\vec{\imath} - \vec{\jmath} + \vec{k}\right) = -\vec{\jmath} \times \left(\vec{\imath} - \vec{\jmath} + \vec{k}\right) + 2\vec{k} \times \left(\vec{\imath} - \vec{\jmath} + \vec{k}\right)$

$= -(\vec{\jmath} \times \vec{\imath}) + (\vec{\jmath} \times \vec{\jmath}) - \left(\vec{\jmath} \times \vec{k}\right) + 2\left(\vec{k} \times \vec{\imath}\right) - 2\left(\vec{k} \times \vec{\jmath}\right) + 2\left(\vec{k} \times \vec{k}\right)$

$= -\left(-\vec{k}\right) + \vec{0} - \vec{\imath} + 2\vec{\jmath} - 2(-\vec{\imath}) + 2\left(\vec{0}\right) = \vec{\imath} + 2\vec{\jmath} + \vec{k}$.

Example 12

Find a vector perpendicular to the plane containing the points $A = (1, 2, 3)$, $B = (-1, 4, 8)$ and $C = (5, 1, -2)$.

☺Solution

According to the definition of vector product, the vector $\overrightarrow{AB} \times \overrightarrow{AC}$ is the vector perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$. So this vector is also perpendicular to the plane containing A, B, C. Note that,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\vec{\imath} + 2\vec{\jmath} + 5\vec{k}, \qquad \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 4\vec{\imath} - \vec{\jmath} - 5\vec{k}.$$

Then the required vector is given by

$$\overrightarrow{AB} \times \overrightarrow{AC} = \left(-2\vec{\imath} + 2\vec{\jmath} + 5\vec{k}\right) \times \left(4\vec{\imath} - \vec{\jmath} - 5\vec{k}\right)$$

$$= -8(\vec{\imath} \times \vec{\imath}) + 2(\vec{\imath} \times \vec{\jmath}) + 10\left(\vec{\imath} \times \vec{k}\right) + 8(\vec{\jmath} \times \vec{\imath}) - 2(\vec{\jmath} \times \vec{\jmath}) - 10\left(\vec{\jmath} \times \vec{k}\right) + 20\left(\vec{k} \times \vec{\imath}\right)$$
$$- 5\left(\vec{k} \times \vec{\jmath}\right) - 25\left(\vec{k} \times \vec{k}\right)$$

$$= 2\vec{k} - 10\vec{\jmath} - 8\vec{k} - 10\vec{\imath} + 20\vec{\jmath} + 5\vec{\imath} = -5\vec{\imath} + 10\vec{\jmath} - 6\vec{k}.$$

Remark of Example 12 (Finding an equation of a plane)

The vector $\overrightarrow{AB} \times \overrightarrow{AC}$ obtained in Example 12 is called a *normal* to the plane. It is useful in determining the equation of the plane in coordinate geometry.

To see this, we consider a point $P = (x, y, z)$ in the same plane. Then the vector $\overrightarrow{AP} = (x-1)\vec{\imath} + (y-2)\vec{\jmath} + (z-3)\vec{k}$ lies on the plane and is perpendicular to the normal $\overrightarrow{AB} \times \overrightarrow{AC}$.

Then using scalar product, the point $P(x, y, z)$ should satisfy

$$\overrightarrow{AP} \cdot \left(\overrightarrow{AB} \times \overrightarrow{AC}\right) = 0$$

$$\Rightarrow \left[(x-1)\vec{\imath} + (y-2)\vec{\jmath} + (z-3)\vec{k}\,\right] \cdot \left(-5\vec{\imath} + 10\vec{\jmath} - 6\vec{k}\right) = 0$$

$$\Rightarrow -5(x-1) + 10(y-2) - 6(z-3) = 0$$

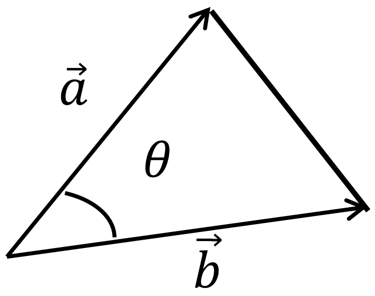
$$\Rightarrow 5x - 10y + 6z = 3.$$

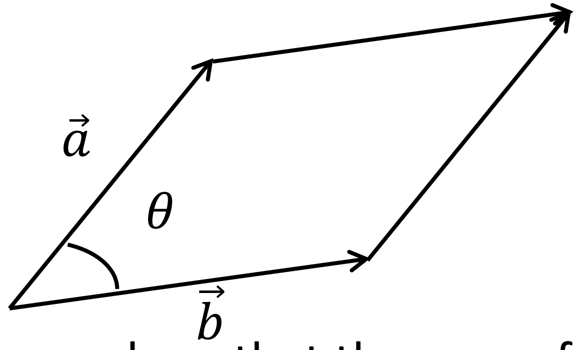
This is the equation of the plane containing A, B and C.

Application of vector product

1. Finding the area of triangle / parallelogram

We would like to find the area of a triangle and a parallelogram as shown below:





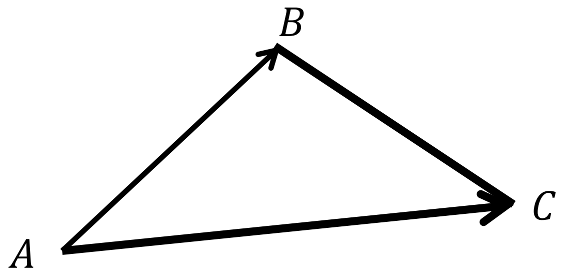
Using standard geometric argument, one can show that the area of the triangle and the area of the parallelogram are given by $\frac{1}{2}|\vec{a}||\vec{b}|\sin\theta$ and $|\vec{a}||\vec{b}|\sin\theta$, respectively.

Using the definition of vector product $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, we have

- the area of triangle is $\frac{1}{2}|\vec{a}\times\vec{b}|$.
- the area of parallelogram is $|\vec{a}\times\vec{b}|$.

Example 13

Let $A = (0, 1, -1)$, $B = (2, 1, 0)$ and $C = (1, 2, 1)$ be three points in the plane. Find the area of the triangle ΔABC.



☺Solution

Note that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 2\vec{\imath} + \vec{k}$ and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \vec{\imath} + \vec{\jmath} + 2\vec{k}$.

Then $\overrightarrow{AB} \times \overrightarrow{AC} = \left(2\vec{\imath} + \vec{k}\right) \times \left(\vec{\imath} + \vec{\jmath} + 2\vec{k}\right) = \cdots = -\vec{\imath} - 3\vec{\jmath} + 2\vec{k}$.

Therefore, the area of ΔABC is

$$\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\left(\sqrt{(-1)^2 + (-3)^2 + 2^2}\right) = \frac{\sqrt{14}}{2} \text{ (square units).}$$

Example 14

Show that the points $A = (2,4,3)$, $B = (4,8,6)$ and $C = (6,12,9)$ are collinear (i.e. the three points lie on the same straight line).

☺IDEA

A, B and C are collinear, then the area of ΔABC is simply zero.

☺Solution:

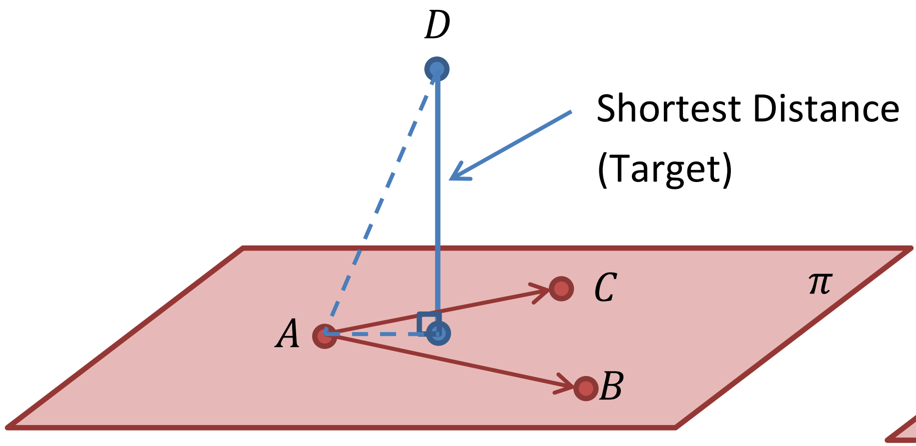
Note that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 2\vec{\imath} + 4\vec{\jmath} + 3\vec{k}$ and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 4\vec{\imath} + 8\vec{\jmath} + 6\vec{k}$.
Then

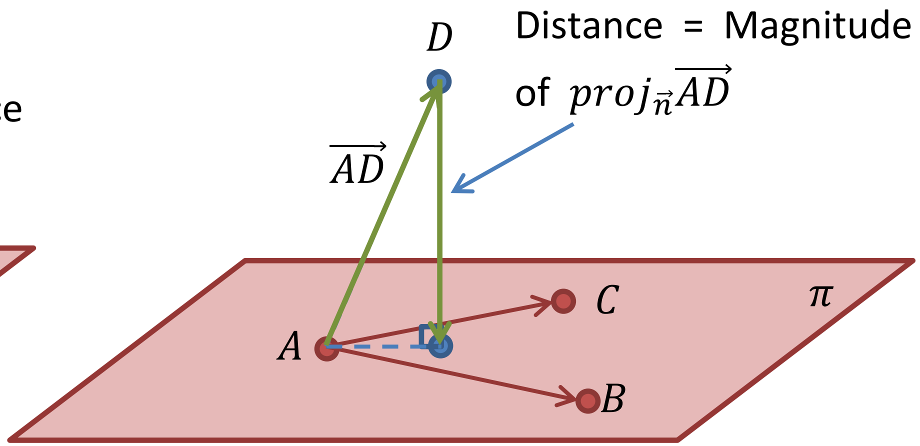
$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= \left(2\vec{\imath} + 4\vec{\jmath} + 3\vec{k}\right) \times \left(4\vec{\imath} + 8\vec{\jmath} + 6\vec{k}\right) = 2\left(2\vec{\imath} + 4\vec{\jmath} + 3\vec{k}\right) \times \left(2\vec{\imath} + 4\vec{\jmath} + 3\vec{k}\right) \\ &= \vec{0}.\end{aligned}$$

So the area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = 0$. Thus A, B and C are collinear.

2. Finding the distance between a point and a plane

Given a point D and a plane π containing points A, B, C as shown below, we would like to find the *shortest distance* (*or perpendicular distance*) between point D and the plane:





- We observe from the above figure that the shortest distance is simply the *magnitude* of the *projection vector of* $\overrightarrow{AD}$ onto the normal vector $\vec{n}$ (the vector perpendicular to the plane).

In order to obtain the magnitude of the projection vector, one needs to obtain the vectors $\overrightarrow{AD}$ and $\vec{n}$.

- Suppose the coordinates of the points A and D are given, one can obtain the vector $\overrightarrow{AD}$ using $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$.

- The normal vector $\vec{n}$ is perpendicular to the plane π and hence is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$ which lies on the plane π. Using the definition of vector product, such normal vector can be found as

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}.$$

Once the vectors $\overrightarrow{AD}$ and $\vec{n}$ are obtained, the projection vector $proj_{\vec{n}}\overrightarrow{AD}$ can be found using the method described in p. 32.

Example 15

Let $A=(3,-2,1)$, $B=(1,-3,2)$ and $C=(2,-1,-3)$ be three points on a plane π. Find the shortest distance from $D=(-4,-1,2)$ to the plane π.

☺Solution:

Following the procedure described in previous page, we need to obtain the vectors $\overrightarrow{AD}$ and $\vec{n}=\overrightarrow{AB}\times\overrightarrow{AC}$ first.

- $\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=\left(-4\vec{\imath}-\vec{\jmath}+2\vec{k}\right)-\left(3\vec{\imath}-2\vec{\jmath}+\vec{k}\right)=-7\vec{\imath}+\vec{\jmath}+\vec{k}$.
- $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=-2\vec{\imath}-\vec{\jmath}+\vec{k}$ and $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=-\vec{\imath}+\vec{\jmath}-4\vec{k}$.

  The normal vector $\vec{n}$ is found to be

$$\begin{aligned}\vec{n}=\overrightarrow{AB}\times\overrightarrow{AC}&=\left(-2\vec{\imath}-\vec{\jmath}+\vec{k}\right)\times\left(-\vec{\imath}+\vec{\jmath}-4\vec{k}\right)\\&=2(\vec{\imath}\times\vec{\imath})-2(\vec{\imath}\times\vec{\jmath})+8\left(\vec{\imath}\times\vec{k}\right)+(\vec{\jmath}\times\vec{\imath})-(\vec{\jmath}\times\vec{\jmath})+4\left(\vec{\jmath}\times\vec{k}\right)\\&-\left(\vec{k}\times\vec{\imath}\right)+\left(\vec{k}\times\vec{\jmath}\right)-4\left(\vec{k}\times\vec{k}\right)=3\vec{\imath}-9\vec{\jmath}-3\vec{k}.\end{aligned}$$

It remains to find the magnitude of the projection vector $proj_{\vec{n}}\overrightarrow{AD}$. Let θ be the angle between $\overrightarrow{AD}$ and $\vec{n}$, then θ is found to be

$$\cos\theta = \frac{\overrightarrow{AD}\cdot\vec{n}}{\left|\overrightarrow{AD}\right||\vec{n}|} = \frac{(-7)(3)+(1)(-9)+(1)(-3)}{\sqrt{51}\sqrt{99}} = \frac{-33}{\sqrt{51}\sqrt{99}} \Rightarrow \theta > 90°.$$

Then the magnitude of the projection vector $proj_{\vec{n}}\overrightarrow{AD}$ is given by

$$\text{Magnitude of } proj_{\vec{n}}\overrightarrow{AD} = \left|\overrightarrow{AD}\right|\cos(180° - \theta) = \sqrt{51}\left(\frac{33}{\sqrt{51}\sqrt{99}}\right) = \frac{33}{\sqrt{99}} = \sqrt{11}.$$

Therefore the required shortest distance is $\sqrt{11}$.

Remark. Let P be the foot of the perpendicular from D to the plane π. Then

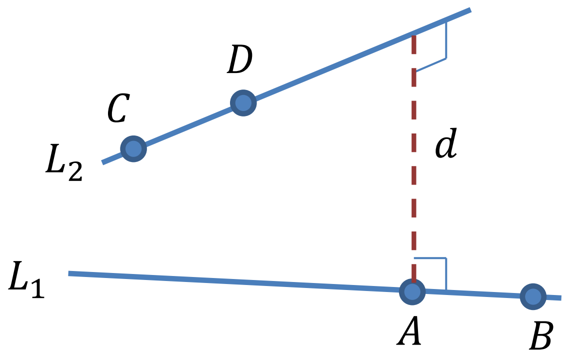
$$\overrightarrow{PD} = proj_{\vec{n}}\overrightarrow{AD} = \left(\overrightarrow{AD}\cdot\frac{\vec{n}}{|\vec{n}|}\right)\frac{\vec{n}}{|\vec{n}|} = -\frac{33}{\sqrt{99}}\frac{3\vec{\imath}-9\vec{\jmath}-3\vec{k}}{\sqrt{99}} = -\vec{\imath}+3\vec{\jmath}+\vec{k},$$

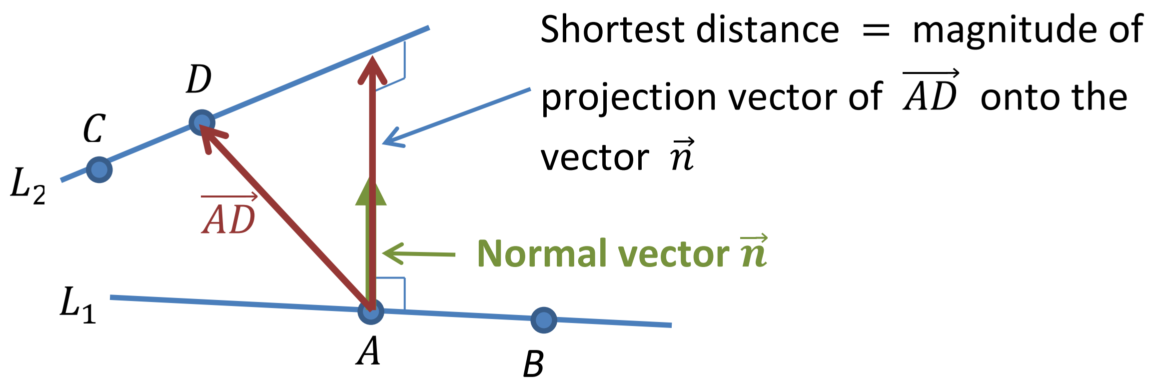
$$\overrightarrow{OP} = \overrightarrow{OD} + \overrightarrow{DP} = \left(-4\vec{\imath}-\vec{\jmath}+2\vec{k}\right) - \left(-\vec{\imath}+3\vec{\jmath}+\vec{k}\right) = -3\vec{\imath}-4\vec{\jmath}+\vec{k}\,.$$

The coordinates of $P = (-3,-4,1)$.

3. Finding the distance between two non-intersecting straight lines.

The techniques used in p. 48 can also be applied to find the *shortest distance* between two non-intersecting straight lines (L_1 and L_2) as shown:





Let A, B, C, D be four given points (the coordinates are known) on these two straight lines. As shown in the second figure, finding shortest distance is equivalent to find the magnitude of the projection vector of $\overrightarrow{AD}$ onto the vector $\vec{n}$.

In order to obtain the magnitude of the projection vector, one needs to obtain the vectors $\overrightarrow{AD}$ and $\vec{n}$.

- Suppose the coordinates of the points A and D are given, one can obtain the vector $\overrightarrow{AD}$ using $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$.

- The normal vector $\vec{n}$ is perpendicular to both lines L_1 and L_2 and hence is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{CD}$ which lie on L_1 and L_2 respectively. Using the definition of vector product, such normal vector can be found as
$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{CD}.$$

Once the vectors $\overrightarrow{AD}$ and $\vec{n}$ are obtained, the projection vector $proj_{\vec{n}}\overrightarrow{AD}$ can be found using the method described in p. 32.

Example 16

Find the shortest distance between the line L_1 passing through the points $A = (0, 1, 4)$ and $B = (3, 4, 1)$, and the line L_2 passing through the points $C = (-1, 4, 11)$ and $D = (1, 5, 9)$.

☺Solution:

Using the procedure described in previous page, we first obtain the vectors $\overrightarrow{AD}$ and $\vec{n} = \overrightarrow{AB} \times \overrightarrow{CD}$.

- $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \left(\vec{\imath} + 5\vec{\jmath} + 9\vec{k}\ \right) - \left(\vec{\jmath} + 4\vec{k}\ \right) = \vec{\imath} + 4\vec{\jmath} + 5\vec{k}.$
- $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 3\vec{\imath} + 3\vec{\jmath} - 3\vec{k}$ and $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = 2\vec{\imath} + \vec{\jmath} - 2\vec{k}.$

  The normal vector $\vec{n}$ is found to be

  $$\begin{aligned}\vec{n} = \overrightarrow{AB} \times \overrightarrow{CD} &= \left(3\vec{\imath} + 3\vec{\jmath} - 3\vec{k}\right) \times \left(2\vec{\imath} + \vec{\jmath} - 2\vec{k}\right) \\ &= 6(\vec{\imath} \times \vec{\imath}) + 3(\vec{\imath} \times \vec{\jmath}) - 6\left(\vec{\imath} \times \vec{k}\right) + 6(\vec{\jmath} \times \vec{\imath}) + 3(\vec{\jmath} \times \vec{\jmath}) - 6\left(\vec{\jmath} \times \vec{k}\right) \\ &\quad - 6\left(\vec{k} \times \vec{\imath}\right) - 3\left(\vec{k} \times \vec{\jmath}\right) + 6\left(\vec{k} \times \vec{k}\right) = -3\vec{\imath} - 3\vec{k}.\end{aligned}$$

It remains to find the magnitude of the projection vector $proj_{\vec{n}}\overrightarrow{AD}$. Let θ be the angle between $\overrightarrow{AD}$ and $\vec{n}$, then θ is found to be

$$\cos\theta = \frac{\overrightarrow{AD}\cdot\vec{n}}{\left|\overrightarrow{AD}\right||\vec{n}|} = \frac{(1)(-3)+(4)(0)+(5)(-3)}{\sqrt{42}\sqrt{18}} = \frac{-18}{\sqrt{42}\sqrt{18}} \Rightarrow \theta > 90°.$$

Then the magnitude of the projection vector $proj_{\vec{n}}\overrightarrow{AD}$ is given by

$$\text{Magnitude of } proj_{\vec{n}}\overrightarrow{AD} = \left|\overrightarrow{AD}\right|\cos(180° - \theta) = \sqrt{42}\left(\frac{18}{\sqrt{42}\sqrt{18}}\right) = \sqrt{18} = 3\sqrt{2}.$$

Therefore the required shortest distance is $3\sqrt{2}$.

Triple Scalar Product

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors in $\mathbb{R}^3$, the *triple scalar product* of these three vectors is defined as $\vec{a} \cdot (\vec{b} \times \vec{c})$. The output is simply a number.

Important Remark:

One has to compute the vector product first when computing the scalar triple product!

$$\vec{a} \cdot \vec{b} \times \vec{c} = (\vec{a} \cdot \vec{b}) \times \vec{c} = (\text{number}) \underbrace{\times}_{\substack{\text{vector}\\ \text{product}}} \underbrace{\vec{c}}_{\text{vector}} = ?????$$

$$\vec{a} \cdot \vec{b} \times \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c}) = \underbrace{\vec{a}}_{\text{vector}} \underbrace{\cdot}_{\substack{\text{scalar}\\ \text{product}}} (\text{vector}) = (\text{number}).$$

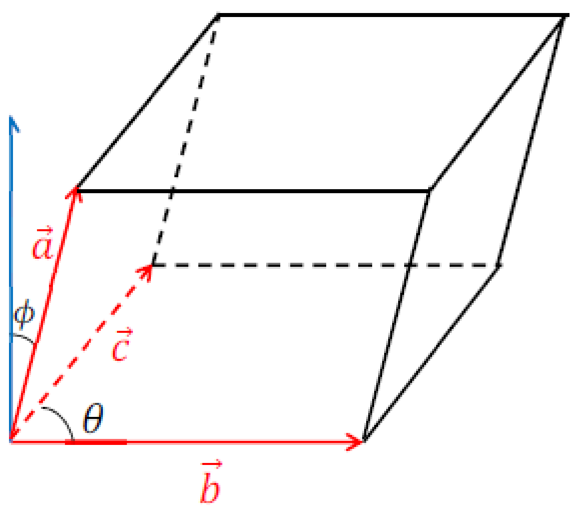
Geometric Interpretation of triple scalar product

Geometrically, one can use triple scalar product to obtain the volume of a parallelepiped.

Theorem

The volume V of the parallelepiped with $\vec{a}$, $\vec{b}$ and $\vec{c}$ as its adjacent sides is given by $V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$.

(Here, |? ? | is the absolute value sign.)



☺Proof of the theorem

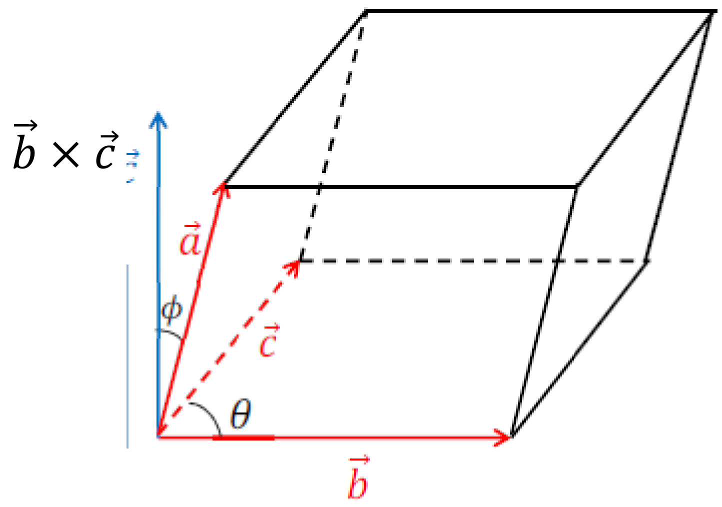
Note that the volume of the parallelepiped is given by

$$V = (\text{base area}) \times (\text{height})$$

$$= \underbrace{|\vec{b}||\vec{c}|\sin\theta}_{\text{base area}} \times \underbrace{|\vec{a}|\cos\phi}_{\text{height}}$$

$$= |\vec{b} \times \vec{c}||\vec{a}|\cos\phi$$

$$= \vec{a} \cdot (\vec{b} \times \vec{c})$$



Remark:

It may happen that $\vec{a} \cdot (\vec{b} \times \vec{c})$ becomes negative if $90° \le \phi \le 180°$. Therefore, we need to put the absolute value to $\vec{a} \cdot (\vec{b} \times \vec{c})$ so that the volume of parallelepiped is given by $|\vec{a} \cdot (\vec{b} \times \vec{c})|$.

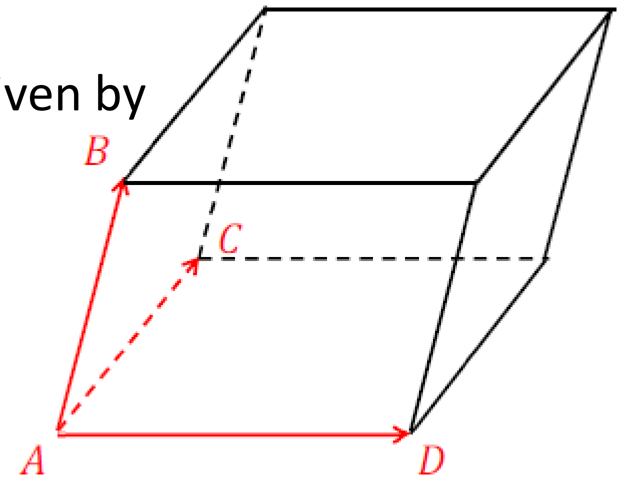
Note: The volume of tetrahedron with $\vec{a}$, $\vec{b}$ and $\vec{c}$ as its adjacent sides will be

$$\frac{1}{3}\ \underbrace{\frac{1}{2}|\vec{b}||\vec{c}|\sin\theta}_{\text{triangle base area}}\ \underbrace{|\vec{a}|\cos\phi}_{\text{height}} = \frac{1}{6}|\vec{a} \cdot (\vec{b} \times \vec{c})|$$

Example 17

Find the volumes of the parallelepiped and the tetrahedron with $A = (1, 3, -1)$, $B = (2, 1, 4)$, $C = (1, 3, 7)$ and $D = (5, 0, 2)$ as its adjacent vertices.

☺Solution



Using the graph on the right, the volume V is given by

$V = \left|\overrightarrow{AB} \cdot \left(\overrightarrow{AD} \times \overrightarrow{AC}\right)\right| \dots \dots (*)$

Note that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{\imath} - 2\vec{\jmath} + 5\vec{k}$,

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = 4\vec{\imath} - 3\vec{\jmath} + 3\vec{k}$ and

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 8\vec{k}$,

then we have

$\overrightarrow{AB} \cdot \left(\overrightarrow{AD} \times \overrightarrow{AC}\right) = \overrightarrow{AB} \cdot \left(\left(4\vec{\imath} - 3\vec{\jmath} + 3\vec{k}\right) \times \left(8\vec{k}\right)\right)$

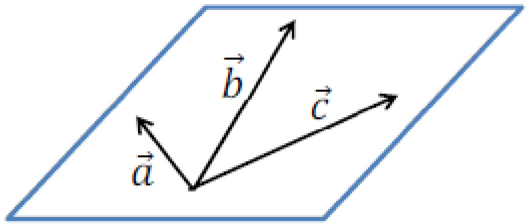
$= \overrightarrow{AB} \cdot \left[32\left(\vec{\imath} \times \vec{k}\right) - 24\left(\vec{\jmath} \times \vec{k}\right) + 24\left(\vec{k} \times \vec{k}\right)\right]$

$= (\vec{\imath} - 2\vec{\jmath} + 5\vec{k}) \cdot (-24\vec{\imath} - 32\vec{\jmath}) = -24 + 64 = 40.$

From $(*)$, the volume of the parallelepiped is $V = |40| = 40$ and so the volume of tetrahedron is $\frac{40}{6} = \frac{20}{3}$.

Example 18

Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ in $\mathbb{R}^3$ are coplanar if they lies on a common plane. Suppose that the vectors $\vec{a} = \vec{\imath} - 2\vec{\jmath} + 3\vec{k}$, $\vec{b} = -\vec{\imath} + \vec{\jmath} + z\vec{k}$ and $\vec{c} = z\vec{\jmath} + \vec{k}$ are coplanar, find the value of z.



☺IDEA:

$\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar $\Leftrightarrow$ The volume of parallelepiped with $\vec{a}$, $\vec{b}$ and $\vec{c}$ as adjacent edges $= 0$.

☺Solution

Since $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, then we have

$$\left|\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right| = 0 \Rightarrow \vec{a}\cdot\left(\vec{b}\times\vec{c}\right) = 0$$

$$\Rightarrow \vec{a}\cdot\left[\left(-\vec{\imath}+\vec{\jmath}+z\vec{k}\right)\times\left(z\vec{\jmath}+\vec{k}\right)\right] = 0$$

$$\Rightarrow \left(\vec{\imath}-2\vec{\jmath}+3\vec{k}\right)\cdot\left((1-z^2)\vec{\imath}+\vec{\jmath}-z\vec{k}\right) = 0$$

$$\Rightarrow (1-z^2)-2-3z = 0$$

$$\Rightarrow z^2+3z+1 = 0$$

$$\Rightarrow z = \frac{-3\pm\sqrt{3^2-4(1)(1)}}{2(1)} = \frac{-3\pm\sqrt{5}}{2}.$$

Triple Vector Product

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three vectors in $\mathbb{R}^3$, the triple vector product of these three vectors is defined as $\vec{a} \times (\vec{b} \times \vec{c})$.

Remark

- The outcome of triple vector product is a vector.
- One has to be careful that $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$ in general. As an example, we take $\vec{a} = \vec{\imath}$ and $\vec{b} = \vec{c} = \vec{\jmath}$. Then we have
$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{\imath} \times (\vec{\jmath} \times \vec{\jmath}) = \vec{\imath} \times \vec{0} = \vec{0},$$
$$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{\imath} \times \vec{\jmath}) \times \vec{\jmath} = \vec{k} \times \vec{\jmath} = -\vec{\imath}.$$
Thus one cannot change the order of multiplication when doing triple vector product.

Linear Independence and Dependence

Linear Combination of vectors

Let $\overrightarrow{a_1}$, $\overrightarrow{a_2}, \cdots$, $\overrightarrow{a_n}$ be n vectors, the vector

$$\vec{m} = c_1\overrightarrow{a_1} + c_2\overrightarrow{a_2} + \cdots + c_n\overrightarrow{a_n}, \qquad c_1, c_2, \dots, c_n \text{ are constants}$$

is called *linear combination* *of* $\overrightarrow{a_1}$, $\overrightarrow{a_2}, \cdots$, $\overrightarrow{a_n}$.

Example 19

The vector $\vec{a} = 3\vec{\imath} + 2\vec{\jmath} - 5\vec{k}$ can be seen as the linear combination of the vectors $\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$.

The vector $\vec{b} = 3\vec{k}$ is not linear combination of the vectors $\vec{\imath}$, $\vec{\jmath}$ since it is impossible that

$$\vec{b} = \underbrace{3\vec{k}}_{\text{not on the } xy-\text{plane}} = \underbrace{\overbrace{c_1\vec{\imath} + c_2\vec{\jmath}}^{\text{linear combination}}}_{\text{lies on the } xy-\text{plane}}$$

Definition (Linear Dependence and Linear Independence)

We say the non-zero vectors $\overrightarrow{a_1}, \overrightarrow{a_2}, \cdots, \overrightarrow{a_n}$ are linearly dependent if there is a vector $\overrightarrow{a_k}$ which can be expressed as a linear combination of other vectors $\overrightarrow{a_j}$, i.e.,

$$\overrightarrow{a_k} = \underbrace{c_1\overrightarrow{a_1} + c_2\overrightarrow{a_2} + \cdots + c_{k-1}\overrightarrow{a_{k-1}} + c_{k+1}\overrightarrow{a_{k+1}} + \cdots + c_n\overrightarrow{a_n}}_{\text{linear combination of other vectors } \overrightarrow{a_j} \text{ (without } \overrightarrow{a_k})}.$$

If the above situation does not happen, we say these vectors $\overrightarrow{a_1}, \overrightarrow{a_2}, \cdots, \overrightarrow{a_n}$ are linearly independent.

Example 20

The vectors $\vec{a} = 4\vec{\imath}$, $\vec{b} = \vec{\jmath}$ and $\vec{c} = 2\vec{\imath} + 5\vec{\jmath}$ are linearly dependent since the vector $\vec{c}$ can be expressed as the linear combination of $\vec{a}$ and $\vec{b}$, i.e.

$$\vec{c} = 2\vec{\imath} + 5\vec{\jmath} = \frac{1}{2}(4\vec{\imath}) + 5\vec{\jmath} = \frac{1}{2}\vec{a} + 5\vec{b}.$$

Example 21

The vectors $\vec{a} = \vec{\imath}$, $\vec{b} = \vec{\jmath}$ and $\vec{c} = 2\vec{\imath} + \vec{k}$ are linearly independent since the vector $\vec{c}$ cannot be expressed as the linear combination of another two vectors,

$$\vec{a} = \vec{\imath} \neq 2x_2\vec{\imath} + x_1\vec{\jmath} + x_2\vec{k} = x_1\vec{b} + x_2\vec{c},$$

$$\vec{b} = \vec{\jmath} \neq (x_1 + 2x_2)\vec{\imath} + x_2\vec{k} = x_1\vec{a} + x_2\vec{c},$$

$$\vec{c} = 2\vec{\imath} + \vec{k} \neq x_1\vec{\imath} + x_2\vec{\jmath} = x_1\vec{a} + x_2\vec{b}.$$

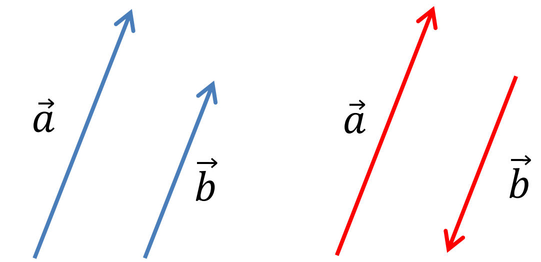
How do the linearly independent/ linearly dependent vectors look like?

Two vectors case: $\vec{a}$ and $\vec{b}$

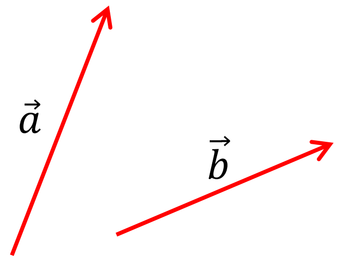
If two vectors are linearly dependent, then one of them can be written as a linear combination of another vector, i.e.,

$$\vec{a} = k\vec{b}$$

In order words, $\vec{a}$ and $\vec{b}$ are *parallel* to each other.



$\vec{a}$, $\vec{b}$ are linearly dependent



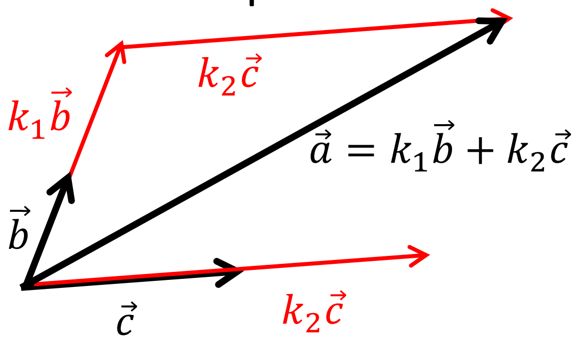
$\vec{a}$, $\vec{b}$ are linearly independent

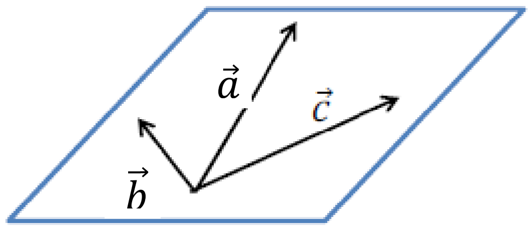
Three vectors case $\vec{a}$, $\vec{b}$ and $\vec{c}$

If three vectors are linearly dependent, then one of them (say $\vec{a}$) can be written as the linear combination of other two vectors:

$$\vec{a} = k_1\vec{b} + k_2\vec{c}.$$

It implies that the vector $\vec{a}$ lies on the plane containing $\vec{b}$ and $\vec{c}$. In particular, three vectors are coplanar.





How do we check whether a set of vectors are linearly independent in $\mathbb{R}^3$*?*

Using the geometric interpretation above, one can establish the following useful test for checking the linear dependency of vectors in 3D space $\mathbb{R}^3$.

<u>Linear independency of two vectors $\vec{a}$ and $\vec{b}$ in $\mathbb{R}^3$</u>

$\vec{a}$ and $\vec{b}$ are linearly dependent $\Leftrightarrow$ The angle between $\vec{a}$ and $\vec{b}$ is $0°$ or $180°$

$\Leftrightarrow \sin\theta = 0 \Leftrightarrow \vec{a} \times \vec{b} = \vec{0}$.

<u>Linear independency of three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ in $\mathbb{R}^3$</u>

$\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly dependent $\Leftrightarrow$ $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar

$\Leftrightarrow \left|\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)\right| = 0.$

<u>Linear independency of four or more vectors $\overrightarrow{a_1}$, $\overrightarrow{a_2}, \cdots, \overrightarrow{a_n}$ in $\mathbb{R}^3$</u>

$\overrightarrow{a_1}$, $\overrightarrow{a_2}$, $\cdots$, $\overrightarrow{a_n}$ are always <u>linearly dependent.</u>

Remark

- This test is useful for vectors in 3-D ONLY. For vectors with higher dimensions (4-D, 5-D, 6-D, etc.), we will have a more general method to verify the linear independent. We will discuss the detail later in Chapter 6.

- To understand why the set of four or more vectors in 3D are always linearly dependent. One can consider the following particular examples: We consider the vectors

$$\vec{a} = \vec{\imath}, \qquad \vec{b} = \vec{\jmath}, \qquad \vec{c} = \vec{k}, \qquad \vec{d} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}.$$

  Since $\vec{d}$ is the vector in 3D-plane, it can be expressed as the linear combination of another three (linearly independent) vectors $\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$.
  This argument works even we replace $\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$ by other three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ which are linearly independent.

Example 22

(a) Determine whether the vectors $\vec{a} = \vec{\imath} + 2\vec{\jmath} - 3\vec{k}$, $\vec{b} = \vec{\imath} + 4\vec{\jmath} - 9\vec{k}$ are linearly independent.

(b) Let $\vec{c} = \vec{\jmath} - 3\vec{k}$ be another vector, determine whether the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly independent.

☺Solution of (a)

Note that $\vec{a} \times \vec{b} = \left(\vec{\imath} + 2\vec{\jmath} - 3\vec{k}\right) \times \left(\vec{\imath} + 4\vec{\jmath} - 9\vec{k}\right)$

$$= (\vec{\imath} \times \vec{\imath}) + 4(\vec{\imath} \times \vec{\jmath}) - 9\left(\vec{\imath} \times \vec{k}\right) + 2(\vec{\jmath} \times \vec{\imath}) + 8(\vec{\jmath} \times \vec{\jmath}) - 18\left(\vec{\jmath} \times \vec{k}\right) - 3\left(\vec{k} \times \vec{\imath}\right) - 12\left(\vec{k} \times \vec{\jmath}\right) + 27\left(\vec{k} \times \vec{k}\right)$$

$$= \vec{0} + 4\vec{k} + 9\vec{\jmath} - 2\vec{k} + 8\left(\vec{0}\right) - 18\vec{\imath} - 3\vec{\jmath} + 12\vec{\imath} + 27\left(\vec{0}\right) = -6\vec{\imath} + 6\vec{\jmath} + 2\vec{k} \neq \vec{0}.$$

So they are linearly independent.

☺Solution of (b)

Note that $\left|\vec{c} \cdot \left(\vec{a} \times \vec{b}\right)\right| = \left|\left(\vec{\jmath} - 3\vec{k}\right) \cdot \left(-6\vec{\imath} + 6\vec{\jmath} + 2\vec{k}\right)\right| = |0 + 6 - 6| = 0.$

Thus they are linearly dependent.

Example 23

Let $\vec{a} = 2\vec{\imath} - 4\vec{\jmath} + \vec{k}$, $\vec{b} = 3\vec{\imath} - \vec{\jmath}$ and $\vec{c} = x\vec{\imath} + \vec{k}$ be three vectors. Suppose the vectors are linearly independent. Find the values of x.

☺Solution:

Since the vectors are linearly independent, then we must have

$$\left|a \cdot \left(\vec{b} \times \vec{c}\right)\right| \neq 0 \quad \text{or} \quad a \cdot \left(\vec{b} \times \vec{c}\right) \neq 0.$$

Note that $\vec{b} \times \vec{c} = 3x\,\underbrace{(\vec{\imath} \times \vec{\imath})}_{=0} + 3\,\underbrace{\left(\vec{\imath} \times \vec{k}\right)}_{=-\vec{\jmath}} - x\,\underbrace{(\vec{\jmath} \times \vec{\imath})}_{=-\vec{k}} - \underbrace{\left(\vec{\jmath} \times \vec{k}\right)}_{=\vec{\imath}} = -\vec{\imath} - 3\vec{\jmath} + x\vec{k}$.

Then $a \cdot \left(\vec{b} \times \vec{c}\right) = \left(2\vec{\imath} - 4\vec{\jmath} + \vec{k}\right) \cdot \left(-\vec{\imath} - 3\vec{\jmath} + x\vec{k}\right) = -2 + 12 + x = x + 10.$

Hence, $a \cdot \left(\vec{b} \times \vec{c}\right) \neq 0 \;\Rightarrow\; x + 10 \neq 0 \;\Rightarrow\; x \neq -10.$

The general procedure of checking the linear dependency of the vectors

The procedure above is only useful for 3D vectors. This approach is no longer to be useful if the vector is in higher dimensions: $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$ where $\vec{v}_1$, $\vec{v}_2, \cdots$, $\vec{v}_n$ are n fundamental vectors in $\mathbb{R}^n$ (just like $\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$ in 3D space). The following theorem provides a more general procedure in checking the linear dependency and it requires some knowledge of the system of linear equations

Theorem

The vectors $\overrightarrow{a_1}$, $\overrightarrow{a_2}, \cdots$, $\overrightarrow{a_n}$ are linearly independent if and only if the equation $x_1\overrightarrow{a_1} + x_2\,\overrightarrow{a_2} + \cdots + x_n\overrightarrow{a_n} = \vec{0}$ has only *trivial solution* (i.e., $x_1 = x_2 = \cdots = x_n = 0$).

On the other hand, if the equation $x_1\overrightarrow{a_1} + x_2\,\overrightarrow{a_2} + \cdots + x_n\overrightarrow{a_n} = \vec{0}$ has other solutions $x_1, x_2, \dots, x_n$ (other than the trivial solution). Then the vectors $\overrightarrow{a_1}$, $\overrightarrow{a_2}, \cdots$, $\overrightarrow{a_n}$ are linearly dependent.

Example 24

Determine whether the vectors $\vec{a} = 3\vec{\imath} + 2\vec{\jmath}$ and $\vec{b} = \vec{\imath} - \vec{\jmath}$ are linearly independent.

☺Solution:

To check this, we solve the following equation for $x_1,\ x_2$

$$x_1\vec{a} + x_2\,\vec{b} = \vec{0} \ \Rightarrow\ (3x_1 + x_2)\vec{\imath} + (2x_1 - x_2)\vec{\jmath} = 0\vec{\imath} + 0\vec{\jmath} \ \Rightarrow\ \begin{cases} 3x_1 + x_2 = 0 \\ 2x_1 - x_2 = 0 \end{cases}.$$

Adding the second equation into the first one, we get $5x_1 = 0 \ \Rightarrow\ x_1 = 0.$

Substitute $x_1 = 0$ into the first equation, we then get $x_2 = 0.$

Since $(x_1,\ x_2) = (0, 0)$ is the only solution, thus $\vec{a}$ and $\vec{b}$ are linearly independent.

Review Example 1

Let $\vec{a} = 3\vec{\imath} + 2\vec{\jmath} - \vec{k}$ and $\vec{b} = \vec{\imath} - 2\vec{\jmath} + 3\vec{k}$

(a) Find $\vec{a} \cdot \vec{b}$.

(b) Using scalar product, find $|\vec{a}|$.

☺Solution:

(a) Using the definition of scalar product

$$\vec{a} \cdot \vec{b} = \left(3\vec{\imath} + 2\vec{\jmath} - \vec{k}\right) \cdot \left(\vec{\imath} - 2\vec{\jmath} + 3\vec{k}\right) = (3)(1) + (2)(-2) + (-1)(3) = -4.$$

(b) Using the fact that $\vec{a} \cdot \vec{a} = |\vec{a}|^2$, we have

$$|\vec{a}| = \sqrt{(\vec{a} \cdot \vec{a})} = \sqrt{\left(3\vec{\imath} + 2\vec{\jmath} - \vec{k}\right) \cdot \left(3\vec{\imath} + 2\vec{\jmath} - \vec{k}\right)}$$

$$= \sqrt{(3)(3) + (2)(2) + (-1) \times (-1)} = \sqrt{14}.$$

Review Example 2

Let $\vec{a} = 3\vec{\imath} + \vec{\jmath} - \vec{k}$, $\vec{b} = \vec{\imath} - 2\vec{\jmath} + 3\vec{k}$ and $\vec{c} = \vec{\imath} + \vec{\jmath} + 4\vec{k}$ be three vectors.

(a) Are $\vec{a}$ and $\vec{b}$ parallel to each other?

(b) Are $\vec{a}$ and $\vec{c}$ orthogonal?

☺Solution:

(a) The angle between $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{3-2-3}{\sqrt{11}\sqrt{14}} = -\frac{2}{\sqrt{154}} \quad\Rightarrow\quad \theta = 99.27°$$

Since $\theta \neq 0°$ or $180°$, these two vectors are not parallel.

(b) The angle between $\vec{a}$ and $\vec{c}$ is given by

$$\cos\phi = \frac{\vec{a}\cdot\vec{c}}{|\vec{a}||\vec{c}|} = \frac{3+1-4}{\sqrt{11}\sqrt{18}} = 0 \quad\Rightarrow\quad \phi = 90°.$$

These two vectors are orthogonal.

Review Example 3

(a) If two vectors $\vec{a}$ and $\vec{b}$ are parallel, what can you say about $\vec{a} \times \vec{b}$?

(b) Determine whether the vectors $\vec{a} = 3\vec{\imath} - \vec{\jmath} + \vec{k}$ and $\vec{b} = \vec{\imath} - 2\vec{\jmath}$ are parallel.

(c) It is given that the vectors $\vec{c}$ and $\vec{d}$ are not parallel. Show that the vectors $\vec{c} + \vec{d}$ and $\vec{c} - \vec{d}$ cannot be parallel.

☺Solution of (a)

Note that if $\vec{a}$ and $\vec{b}$ are parallel, the angle between these two vectors must be either $0°$ (same direction) or $180°$ (opposite direction). In either case, we have

$$\vec{a} \times \vec{b} = |\vec{a}|\left|\vec{b}\right| \sin\theta\, \hat{n} = \vec{0}.$$

☺Solution of (b)

Using the result of (a), we consider

$$\vec{a} \times \vec{b} = \left(3\vec{\imath} - \vec{\jmath} + \vec{k}\right) \times (\vec{\imath} - 2\vec{\jmath})$$

$$= 3(\vec{\imath} \times \vec{\imath}) - 6(\vec{\imath} \times \vec{\jmath}) - (\vec{\jmath} \times \vec{\imath}) + 2(\vec{\jmath} \times \vec{\jmath}) + \left(\vec{k} \times \vec{\imath}\right) - 2\left(\vec{k} \times \vec{\jmath}\right)$$

$$= -6\vec{k} + \vec{k} + \vec{\jmath} + 2\vec{\imath} = 2\vec{\imath} + \vec{\jmath} - 5\vec{k}.$$

Since $\vec{a} \times \vec{b} \neq \vec{0}$, thus these two vectors cannot be parallel.

☺Solution of (c)

Again, we consider the vector product

$$\left(\vec{c} + \vec{d}\right) \times \left(\vec{c} - \vec{d}\right) = (\vec{c} \times \vec{c}) + \left(\vec{d} \times \vec{c}\right) - \left(\vec{c} \times \vec{d}\right) - \left(\vec{d} \times \vec{d}\right) = \left(\vec{d} \times \vec{c}\right) - \left(\vec{c} \times \vec{d}\right)$$
$$= -\left(\vec{c} \times \vec{d}\right) - \left(\vec{c} \times \vec{d}\right) = -2\left(\vec{c} \times \vec{d}\right).$$

Since $\vec{c} \times \vec{d} \neq \vec{0}$, then $\left(\vec{c} + \vec{d}\right) \times \left(\vec{c} - \vec{d}\right) \neq \vec{0}$ and they are not parallel.

Review Example 4

Using vector product and the techniques in Example 12, find the equation of the plane containing the points $A = (-2, 1, -3)$, $B = (-3, -2, 4)$ and $C = (1, -3, -2)$.

☺Solution:

We need to find the normal vector which is perpendicular to the plane. More precisely, we need to find a vector which is perpendicular to both $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

Note that

$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \vec{\imath} + 3\vec{\jmath} - 7\vec{k} \text{ and } \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = 4\vec{\imath} - \vec{\jmath} - 6\vec{k}$$

Then the required normal vector is given by

$$\vec{n} = \overrightarrow{BA} \times \overrightarrow{BC} = \left(\vec{\imath} + 3\vec{\jmath} - 7\vec{k}\right) \times \left(4\vec{\imath} - \vec{\jmath} - 6\vec{k}\right) = \cdots = -25\vec{\imath} - 22\vec{\jmath} - 13\vec{k}.$$

To find the equation of the plane, we pick a point $P = (x,\ y,\ z)$ in the plane, then the vector $\overrightarrow{BP}$ is the vector on the plane and thus is perpendicular to the normal vector. Thus the scalar product of $\overrightarrow{BP}$ and $\vec{n}$ must 0 i.e., $\overrightarrow{BP} \cdot \vec{n} = \left|\overrightarrow{BP}\right||\vec{n}| \cos 90° = 0$.

Note that

$$\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{OB} = (x+3)\vec{\imath} + (y+2)\vec{\jmath} + (z-4)\vec{k}\ .$$

Thus the equation of the plane is given by

$$\overrightarrow{BP} \cdot \vec{n} = 0$$

$$\Rightarrow -25(x+3) - 22(y+2) - 13(z-4) = 0$$

$$\Rightarrow 25x + 22y + 13z + 67 = 0.$$

Review Example 5

Let $A = (1, 2, 0)$, $B = (3, -1, -2)$ and $C = (-2, 0, 1)$ be three points in the plane

(a) Determine if the line AB is perpendicular to the line AC.

(b) Find the area of ΔABC.

(c) Let $\vec{a}$ be a vector with same magnitude as that of $\overrightarrow{BC}$ and it is perpendicular to both vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Find the vector $\vec{a}$.

☺Solution

Note that $\begin{cases} \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 2\vec{\imath} - 3\vec{\jmath} - 2\vec{k}, \\ \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -3\vec{\imath} - 2\vec{\jmath} + \vec{k}. \end{cases}$

(a) Let θ be the angle between AB and AC (i.e. $\angle BAC$), then θ satisfies

$$\cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{(2)(-3) + (-3)(-2) + (-2)(1)}{\sqrt{2^2 + (-3)^2 + (-2)^2}\sqrt{(-3)^2 + (-2)^2 + 1^2}} = \frac{-2}{\sqrt{17}\sqrt{14}}$$

$\Rightarrow \theta \neq 90°$

Thus AB and AC are not perpendicular.

(b) The area of ΔABC is

$$\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\left|\left(2\vec{\imath} - 3\vec{\jmath} - 2\vec{k}\right) \times \left(-3\vec{\imath} - 2\vec{\jmath} + \vec{k}\right)\right|$$

$$= \frac{1}{2}\left| -6\underbrace{(\vec{\imath}\times\vec{\imath})}_{\vec{0}} - 4\underbrace{(\vec{\imath}\times\vec{\jmath})}_{\vec{k}} + 2\underbrace{(\vec{\imath}\times\vec{k})}_{-\vec{\jmath}} + 9\underbrace{(\vec{\jmath}\times\vec{\imath})}_{-\vec{k}} + 6\underbrace{(\vec{\jmath}\times\vec{\jmath})}_{\vec{0}} - 3\underbrace{(\vec{\jmath}\times\vec{k})}_{\vec{\imath}} + 6\underbrace{(\vec{k}\times\vec{\imath})}_{\vec{\jmath}} + 4\underbrace{(\vec{k}\times\vec{\jmath})}_{-\vec{\imath}} - 2\underbrace{(\vec{k}\times\vec{k})}_{\vec{0}} \right|$$

$$= \frac{1}{2}\left|-7\vec{\imath} + 4\vec{\jmath} - 13\vec{k}\right| = \frac{1}{2}\sqrt{(-7)^2 + (4)^2 + (-13)^2} = \frac{\sqrt{234}}{2}.$$

(c) Note that $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = -5\vec{\imath} + \vec{\jmath} + 3\vec{k}$ and recall that the vector $\overrightarrow{AB}\times\overrightarrow{AC}$ is perpendicular to both $\overrightarrow{AB}$, $\overrightarrow{AC}$. Then

- Magnitude of $\vec{a} = \left|\overrightarrow{BC}\right| = \sqrt{(-5)^2 + 1^2 + 3^2} = \sqrt{35}.$
- Direction of $\vec{a} = \underbrace{\frac{\overrightarrow{AB}\times\overrightarrow{AC}}{\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|}}_{\text{unit vector!!}} = \frac{-7\vec{\imath}+4\vec{\jmath}-13\vec{k}}{\sqrt{234}}.$

Thus the vector $\vec{a}$ is found to be

$$\vec{a} = \underbrace{(\sqrt{35})}_{\text{magnitude}}\ \underbrace{\frac{-7\vec{\imath} + 4\vec{\jmath} - 13\vec{k}}{\sqrt{234}}}_{\text{direction}} = -\frac{7\sqrt{35}}{\sqrt{234}}\vec{\imath} + \frac{4\sqrt{35}}{\sqrt{234}}\vec{\jmath} - \frac{13\sqrt{35}}{\sqrt{234}}\vec{k}.$$