MA1201 Calculus and Basic Linear Algebra II Problem Set 6

Matrix

Problem 1

Let $A = \begin{pmatrix} 2 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 4 & -1 & 2 \\ 2 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$ be two 3×3 matrices.

(a) Compute AB and A^2.

(b) Compute $A(B + I_3)$ where I_3 is identity matrix.

Problem 2

Let $A = (1 \quad 2 \quad 3)$, $B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ -2 & 5 \end{pmatrix}$ and $D = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ be four matrices.

(a) Find AD and DA.

(b) Find BC and BD.

Problem 3

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ be two matrices.
Compute A^3 and B^4.

Problem 4

Fill in the following table.

	Upper-Triangular	Lower-Triangular	Diagonal	Symmetric	Skew-symmetric
$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	Yes	Yes	Yes	Yes	No
$A = \begin{pmatrix} -2 & 1 \\ 0 & 3 \end{pmatrix}$					
$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$					
$C = \begin{pmatrix} 2 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$					
$D = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}^2$					

Determinant

Problem 5

Let $A = \begin{pmatrix} 2 & 0 & -3 \\ 1 & 5 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ be a 3×3 matrix. Compute the determinant of A by

(1) Expanding along the 1st row.

(2) Expanding along the 2nd column.

(3) Expanding along the 3rd row.

(Hint: If your calculations are correct, the answers of the three parts must be the same.)

Problem 6

Compute each of the following determinants using suitable method

(a) $\det\begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & 0 \\ -1 & 1 & 5 \end{pmatrix}$

(b) $\det\begin{pmatrix} 3 & -1 & 2 \\ 2 & 0 & 4 \\ 1 & 1 & 0 \end{pmatrix}$

(c) $\det\begin{pmatrix} 1 & a & 2 \\ a & -1 & 0 \\ 2 & 3 & 1 \end{pmatrix}$

(d) $\det\begin{pmatrix} 1 & 0 & -4 \\ 2 & 1 & 1 \\ 1 & 0 & 5 \end{pmatrix}^5$

(e) $\det\begin{pmatrix} 1 & 2 & 1 & 3 \\ 5 & 0 & 0 & -1 \\ 2 & -1 & -1 & 0 \\ 1 & 0 & 4 & 2 \end{pmatrix}$

(f) $\det\begin{pmatrix} 2 & 1 & 1 & 3 \\ 1 & 0 & 1 & -1 \\ -3 & 2 & -1 & 0 \\ 5 & 0 & 4 & 2 \end{pmatrix}$

Problem 7

Let D be a 3×3 diagonal matrix, show that the determinant of D is simply the product of all diagonal entries. Hence, find the determinant of identity matrix I_3.

(Hint: Let $D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, then you need to show $\det D = abc$)

Problem 8

Let A and B be two 4×4 matrices with $\det A = 3$ and $\det B = 1$.
Using the properties of determinant, find the value of

(a) $\det(A^3)$,
(b) $\det(A^{-1})$
(c) $\det A^{-1}B$
(d) $\det B^T A$
(e) $\det(2A)$ and $\det(3A^T B)$.
(f) $\det(2C^2)$ where C is 2×2 matrix with $\det C = 3$.

Inverse of matrix

Problem 9

Find the inverse (if exist) of the following matrices

(a) $A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

(b) $B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(c) $C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

(d) $D = \begin{pmatrix} 2 & -3 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

Problem 10

Let $C = \begin{pmatrix} 1 & a & 1 \\ a & 2 & 0 \\ 2a & 0 & a \end{pmatrix}$ be a 3×3 matrix.

(a) Find all possible a such that the matrix is *singular*.
(b) Pick $a = 2$, find the inverse of C.

Problem 11

Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -3 & 1 \end{pmatrix}$ be two matrices.

(a) Find AB^T and BA^T.

(b) For each of the matrices obtained in (a), determine whether it is invertible and find its inverse if the inverse exists.

Problem 12

Let $E = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 5 \\ 3 & 2 & 6 \end{pmatrix}$ be a matrix and suppose $EX = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 3 & 5 \\ 6 & 0 & 1 \end{pmatrix}$, find X.

Problem 13

Using matrix method, solve the following system of linear equations

$$\begin{cases} x - 2y + z = 0 \\ 2x + y - 3z = -5. \\ -x + 4z = 11 \end{cases}$$

(Hint: Write the system in the matrix form)

System of Linear Equations

Problem 14

Using Gaussian Elimination, solve the following systems

(a) $\begin{cases} x - y + 3z = 15 \\ -3x + 2y + z = 4 \\ 2x - 3y + 2z = 9 \end{cases}$

(b) $\begin{cases} 2x + y - 3z = 12 \\ 4x + z = 5 \\ 3x - y + 2z = 1 \end{cases}$

(c) $\begin{cases} x - 2y + 3z = 3 \\ 3x - 5y + z = 4 \end{cases}$

(d) $\begin{cases} x + y + 2z = 0 \\ -3x + 4y + z = 0 \\ -2x + 5y + 3z = 0 \end{cases}$

(e) $\begin{cases} x + y + z = 9 \\ 2x + 5y + 7z = 52 \\ 2x + y - z = 0 \end{cases}$

(f) $\begin{cases} x + 3y - 2z - w = 1 \\ 2x + 5y - z + 3w = 2 \\ -x - y - 3z + 2w = -3 \end{cases}$

(g) $\begin{cases} x + 2y + 3z + 4w = -2 \\ 2x + 4y + 5z + 9w = 1 \\ -3x - 6y + w = 4 \end{cases}$

(h) $\begin{cases} x - 3y + 4z + 7w = 1 \\ 2x - 6y - 3z + 5w = 2 \\ 4x - 12y - 17z + w = 4 \end{cases}$

Problem 15

Consider the following system

$$\begin{cases} x - 2y + z = 1 \\ 2x - 4y - 5z = 3. \\ 3x - 6y + 24z = a \end{cases}$$

(a) If $a = 3$, does the system have any solutions?

(b) Find all possible values of a such that the system is consistent.

Problem 16

Find all possible values of c such that the system

$$\begin{cases} 3x - \ \ y - \ \ z = 1 \\ 2x - 4y + 5z = 1 \\ 4x + 2y - 7z = c \end{cases}$$

is consistent.

Problem 17

Consider the system

$$\begin{cases} x + 2y - z = c \\ -x + 4y + z = c^2 \\ x + 8y - z = c^3 \end{cases}.$$

Find all possible values of c such that the system

(a) has a unique solution.
(b) has infinitely many solutions.
(c) has no solution.

Problem 18

Consider the system

$$\begin{cases} x - 2y + \ \ z = 1 \\ x - \ \ y + 2z = 2 \\ \qquad y + c^2 z = c \end{cases}.$$

Find all possible values of c such that the system

(a) has a unique solution.
(b) has infinitely many solutions.
(c) has no solution.

Problem 19

Consider the system

$$\begin{cases} 2x + \ \ y - bz = 3 \\ \qquad ay - \ \ z = 2 \\ -2x + 5y \qquad = 1 \end{cases}$$

Find all possible values of a and b such that the system

(a) has a unique solution.
(b) has infinitely many solutions.
(c) has no solution.

Problem 20

Consider the following system of linear equations

$$\begin{cases} x + 3y - \qquad 2z - \qquad\qquad w = 1 \\ 2x + 5y - \qquad z + \qquad\qquad 3w = 2 \\ -x - \ \ y + (a-3)z + (a^2 - 10)w = b \end{cases}$$

Find all possible values of a and b such that the system

(a) has a unique solution.
(b) has infinitely many solutions.
(c) has no solution.

Problem 21

Solve the following system of linear equations using Cramer's rule.

$$\begin{cases} x - 2y + 3z = 1 \\ 4x + y - 2z = 5 \\ 2x - y + 3z = 6 \end{cases}$$

Problem 22

Find the inverse (if exists) of each of the following matrix using Gauss-Jordan method

(a) $A = \begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$

(b) $B = \begin{pmatrix} 3 & 1 \\ -6 & 4 \end{pmatrix}$

(c) $C = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 2 & 3 \\ 4 & 2 & 1 \end{pmatrix}$

(d) $D = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{pmatrix}$

(e) $E = \begin{pmatrix} 0 & -1 & 2 \\ 2 & 1 & 4 \\ 1 & -1 & 5 \end{pmatrix}$

(f) $F = \begin{pmatrix} -2 & 3 & 4 \\ 1 & 0 & -1 \\ -1 & 5 & 8 \end{pmatrix}$

(g) $G = \begin{pmatrix} 1 & 1 & 2 & -3 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & -1 & 1 & 1 \end{pmatrix}$

(h) $H = \begin{pmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 1 & 2 & -2 \\ 2 & 0 & -1 & 3 \end{pmatrix}$

Problem 23

Determine whether the following set of vectors are linearly independent. (For simplicity, each vector will be expressed in column vector form. For example: $\vec{\imath} + 2\vec{\jmath} + 3\vec{k} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.)

(a) $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ -3 \end{pmatrix}$

(d) $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Problem 24

Find the rank of each of the following matrices

(a) $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \end{pmatrix}$