MA1200 Notes 9 (Part 1) Binomial Theorem

<u>Binomial Coefficient</u>

If *n* and *r* are integers with $0 \le r \le n$, then the **binomial coefficient** ${}_nC_r$, is defined by

$${}_nC_r = \frac{n!}{r!(n-r)!},$$

where $r! = r(r-1)(r-2)\ldots 3\cdot 2\cdot 1$ for $r > 0$ (called the **factorial** of *r*)
and $0! = 1$.

For example, ${}_5C_2 = \frac{5!}{2!(5-2)!} = \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{(2\cdot 1)(3\cdot 2\cdot 1)} = 10$, ${}_5C_0 = \frac{5!}{0!(5-0)!} = \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{1(5\cdot 4\cdot 3\cdot 2\cdot 1)} = 1$

Question: Evaluate each of the following.
(a) ${}_7C_3$ (b) ${}_7C_4$ (c) ${}_6C_2 + {}_6C_3$

Meaning: The binomial coefficient ${}_nC_r$ represents the number of different ways of choosing *r* distinct objects from *n* distinct objects $(n \ge r \ge 0)$ in an unordered manner. For example, there are 10 different ways of choosing 2 distinct letters from the five letters A, B, C, D and E in an unordered manner.

Remarks:

1. The binomial coefficient ${}_nC_r$ can also be written as C_r^n and $\binom{n}{r}$.
2. It should be noted that the binomial coefficient ${}_nC_r$ $(n \ge r \ge 0)$ is a positive integer for all *n* and *r* with $n \ge r \ge 0$.

We will use the notation ${}_nC_r$ in the following parts.

Example 1 Evaluate each of the following.
(a) ${}_nC_0$ (b) ${}_nC_1$ (c) ${}_nC_2$ (d) ${}_nC_3$ (e) ${}_nC_{n-2}$

Solutions

(a) ${}_nC_0 = \frac{n!}{0!(n-0)!} = \frac{n!}{n!} = 1$

(b) ${}_nC_1 = \frac{n!}{1!(n-1)!} = \frac{n\cdot(n-1)!}{1\cdot(n-1)!} = n$

(c) ${}_nC_2 = \frac{n!}{2!(n-2)!} = \frac{n\cdot(n-1)\cdot(n-2)!}{2\cdot 1\cdot(n-2)!} = \frac{n(n-1)}{2}$

(d) ${}_nC_3 = \frac{n!}{3!(n-3)!} = \frac{n\cdot(n-1)\cdot(n-2)\cdot(n-3)!}{3\cdot 2\cdot 1\cdot(n-3)!} = \frac{n(n-1)(n-2)}{6}$

(e) ${}_nC_{n-2} = \frac{n!}{(n-2)![n-(n-2)]!} = \frac{n!}{(n-2)!2!} = \frac{n\cdot(n-1)\cdot(n-2)!}{(n-2)!\cdot 2\cdot 1} = \frac{n(n-1)}{2}$

The binomial coefficient has the following properties:

(i) If r and n are two non-negative integers with $n \geq r \geq 0$, then

$$_{n}C_{r} = {}_{n}C_{n-r} .$$

For example, $_{5}C_{2} = {}_{5}C_{3} = 10$, $_{15}C_{2} = {}_{15}C_{13} = \frac{15 \cdot 14}{2} = 105$ and $_{n}C_{3} = {}_{n}C_{n-3} = \frac{n \cdot (n-1)(n-2)}{6}$.

(ii) If r and n are two non-negative integers with $n \geq r \geq 1$, then

$$_{n}C_{r} = {}_{n-1}C_{r-1} + {}_{n-1}C_{r}$$

For example, $_{7}C_{3} = {}_{6}C_{2} + {}_{6}C_{3} = 35$.

Pascal Triangle

The binomial coefficients can be obtained from the list called the Pascal triangle, showing below:

$$\begin{array}{lccccccccccc}
n=0 & & & & & & 1 & & & & & \\
n=1 & & & & & 1 & & 1 & & & & \\
n=2 & & & & 1 & & 2 & & 1 & & & \\
n=3 & & & 1 & & 3 & & 3 & & 1 & & \\
n=4 & & 1 & & 4 & & 6 & & 4 & & 1 & \\
n=5 & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

The summation notation

The summation notation is a convenient way of expressing the sum of n numbers which are formulated in a common way.

For example, $\sum_{r=1}^{5} r^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$

$$\sum_{r=1}^{18} \frac{1}{r} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{16} + \frac{1}{17} + \frac{1}{18}$$

Question: Write out each of the sums below.

(a) $\sum_{r=1}^{5} 3r$ (b) $\sum_{r=1}^{7} \frac{1}{10-r}$ (c) $\sum_{r=1}^{n} \frac{1}{r}$

(d) $\sum_{r=1}^{n} \frac{n-r}{r}$ (e) $\sum_{r=1}^{10} 5$

Question: Express each of the following sums using summation notation.

(a) $x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 6x^6$

(b) $3(4) + 4(5) + 5(6) + \ldots + 15(16) + 16(17)$

(c) $a_3 + a_4 + a_5 + \ldots$

Binomial Theorem

Let a and b be two numbers and n be a positive integer. Then

$$\begin{aligned}(a+b)^n &= a^n + {}_nC_1\, a^{n-1} b + {}_nC_2\, a^{n-2} b^2 + \ldots + {}_nC_r\, a^{n-r} b^r + \ldots + {}_nC_{n-1}\, a b^{n-1} + b^n \\ &= \sum_{r=0}^{n} {}_nC_r\, a^{n-r} b^r\end{aligned}$$

Note: (i) The number of terms in the expansion is $n + 1$.

(ii) The sum of the powers of a and b in each term is n.

(iii) The $(r + 1)^{\text{th}}$ term is ${}_nC_r\, a^{n-r} b^r$.

Example 2 Expand the following with the *binomial theorem*.

(a) $(3-i)^4$ (b) $\left(z+\dfrac{1}{z}\right)^6$

(c) $(\cos\theta + i\sin\theta)^3$ (d) $\left(\sqrt{2}\cos\theta + \sqrt{2}i\sin\theta\right)^4$

Solutions

(a) $$\begin{aligned}(3-i)^4 &= (3)^4 + {}_4C_1(3)^3(-i)^1 + {}_4C_2(3)^2(-i)^2 + {}_4C_3(3)^1(-i)^3 + (-i)^4 \\ &= 81 - 108i + 54i^2 - 12i^3 + i^4\end{aligned}$$

(Note: the positive and negative signs occur alternatively with the first term positive.)

(b) $$\begin{aligned}\left(z+\frac{1}{z}\right)^6 &= z^6 + {}_6C_1\, z^5\left(\frac{1}{z}\right)^1 + {}_6C_2\, z^4\left(\frac{1}{z}\right)^2 + {}_6C_3\, z^3\left(\frac{1}{z}\right)^3 + {}_6C_4\, z^2\left(\frac{1}{z}\right)^4 + {}_6C_5\, z^1\left(\frac{1}{z}\right)^5 + \left(\frac{1}{z}\right)^6 \\ &= z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}\end{aligned}$$

(c) $$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= (\cos\theta)^3 + {}_3C_1(\cos\theta)^2(i\sin\theta)^1 + {}_3C_2(\cos\theta)^1(i\sin\theta)^2 + (i\sin\theta)^3 \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta\end{aligned}$$

(d) $$\begin{aligned}\left(\sqrt{2}\cos\theta + \sqrt{2}\,i\sin\theta\right)^4 &= \left(\sqrt{2}\right)^4(\cos\theta + i\sin\theta)^4 \\ &= 4\left[(\cos\theta)^4 + {}_4C_1(\cos\theta)^3(i\sin\theta)^1 + {}_4C_2(\cos\theta)^2(i\sin\theta)^2 + {}_4C_3(\cos\theta)^1(i\sin\theta)^3 + (i\sin\theta)^4\right] \\ &= 4\left[\cos^4\theta + 4i\cos^3\theta\sin\theta + 6i^2\cos^2\theta\sin^2\theta + 4i^3\cos\theta\sin^3\theta + i^4\sin^4\theta\right]\end{aligned}$$

Remark: This question demonstrates a typical calculation involving complex numbers (where $i = \sqrt{-1}$). More kinds of calculations will be discussed in the chapter ‘Complex Number’.

Example 3 Determine the coefficients of the terms specified in the expansions of the following.

(a) $(1+2x)^8$, the term in x^5

(b) $(3y-1)^5$, the fourth term in descending powers of y

(c) $\left(2z-\dfrac{1}{z}\right)^6$, the constant term

Solutions

(a) The term in x^5 is ${}_8C_5\,(1)^{8-5}(2x)^5$. Thus, the coefficient of x^5 is

${}_8C_5\,(1)^3(2)^5 = 1792$.

(b) The fourth term in descending powers of y is ${}_5C_3\,(3y)^{5-3}\,(-1)^3$. Thus, the coefficient of this term is $-{}_5C_3\,(3)^2 = -90$.

(c) The general term in the expansion is ${}_6C_r\,(2z)^{6-r}\left(-\frac{1}{z}\right)^r$.

It is the constant term when $6-r=r$, i.e. $r=3$. Thus the constant term is ${}_6C_3\,(2)^3\,(-1)^3 = -160$.