Example 26

Is $f(x) = \begin{cases} \frac{1-\cos x}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ continuous everywhere?

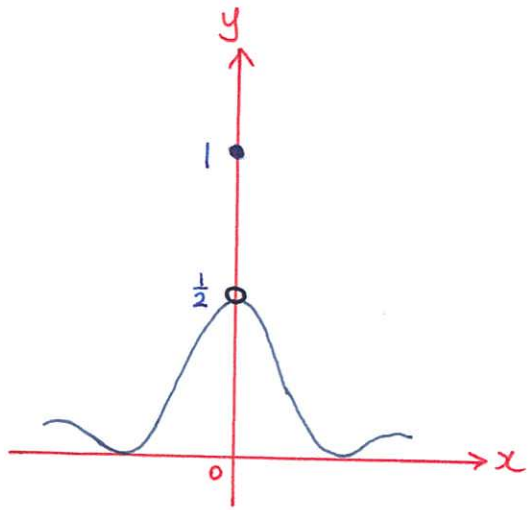
Solution

Both $1 - \cos x$ and x^2 are continuous at every $x \neq 0$, so $\frac{1-\cos x}{x^2}$ is continuous at every $x \neq 0$. Now we determine whether $f(x)$ is continuous at $x = 0$ or not.

$$\lim_{x\to 0} f(x) \underbrace{=}_{\because x\neq 0} \lim_{x\to 0} \frac{1-\cos x}{x^2} \overset{(\frac{0}{0}\text{ form})}{=} \lim_{x\to 0}\left[\frac{(1-\cos x)}{x^2}\cdot\frac{(1+\cos x)}{(1+\cos x)}\right]$$

$$= \lim_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)} = \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)}$$

$$= \lim_{x\to 0}\left[\frac{\sin^2 x}{x^2}\cdot\frac{1}{(1+\cos x)}\right] = 1^2\cdot\frac{1}{(1+\cos 0)} = 1^2\cdot\frac{1}{(1+1)} = \frac{1}{2}$$



Since $\lim_{x\to 0} f(x) = \frac{1}{2} \neq 1 = f(0)$, f is discontinuous at $x = 0$.

$\therefore$ f is continuous everywhere except at $x = 0$.

(Note: If we define $f(0) = \frac{1}{2}$, then f is continuous at $x = 0$.)

Example 27

Let $f(x) = \begin{cases} \cos\left(\frac{\pi x}{2}\right) & \text{if } \boxed{|x| \le 1} \\ |x-1| & \text{if } \boxed{|x| > 1} \end{cases}$. Determine the values of x at which f is continuous.

$\Rightarrow -1 \leqslant x \leqslant 1$

$\Rightarrow x > 1$ or $x < -1$

$|x-1| = \begin{cases} x-1 & \text{if } x \geqslant 1 \\ -(x-1) & \text{if } x < 1. \end{cases}$

Solution

Rewrite $f(x)$ as $f(x) = \begin{cases} -(x-1) & \text{if } x < -1 \\ \cos\left(\frac{\pi x}{2}\right) & \text{if } -1 \le x \le 1 \\ x-1 & \text{if } x > 1 \end{cases}$.

$f(x)$ is continuous at every $x \neq \pm 1$, since $\cos\left(\frac{\pi x}{2}\right)$ and $|x-1|$ are continuous at every $x \neq \pm 1$.

Is $f(x)$ continuous at $x = -1$? Check whether $\lim_{x \to -1} f(x) = f(-1)$.

$$\lim_{x \to -1^-} f(x) \underbrace{=}_{x<-1} \lim_{x \to -1^-} -(x-1) = -(-1-1) = 2$$

$$\lim_{x \to -1^+} f(x) \underbrace{=}_{x>-1} \lim_{x \to -1^+} \cos\left(\frac{\pi x}{2}\right) = \cos\left(\frac{-\pi}{2}\right) = 0$$

Since $\lim_{x\to-1^-} f(x) \neq \lim_{x\to-1^+} f(x)$, the limit $\lim_{x\to-1} f(x)$ does not exist. Hence $f(x)$ is not continuous at $x = -1$.

Is $f(x)$ continuous at $x = 1$? Check whether $\lim_{x\to 1} f(x) = f(1)$.

$$\lim_{x\to1^-} f(x) \underbrace{=}_{x<1} \lim_{x\to1^-} \cos\left(\frac{\pi x}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$\lim_{x\to1^+} f(x) \underbrace{=}_{x>1} \lim_{x\to1^+} (x-1) = 1 - 1 = 0$$

Since $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = 0$, the limit $\lim_{x\to1} f(x)$ exists and $\lim_{x\to1} f(x) = 0$.

$f(1) = \cos\left(\frac{\pi}{2}\right) = 0$

Since $\lim_{x\to1} f(x) = 0 = f(1)$, $f(x)$ is continuous at $x = 1$.

Hence, $f(x)$ is continuous at every $x \in \mathbb{R}\backslash\{-1\}$.

Continuity on an interval

- A function f is **continuous on the open interval** (a, b) if it is continuous at every point inside the interval.

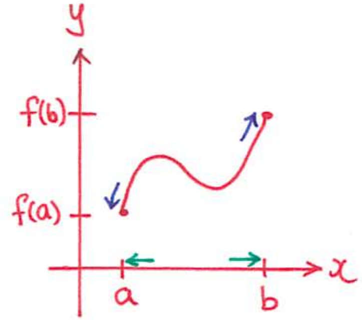
- A function f is **continuous on the closed interval** $[a, b]$ if it is

(1) **continuous** on the open interval (a, b);

(2) **right continuous** at the left endpoint a (i.e. $\lim_{x \to a^+} f(x) = f(a)$); and

(3) **left continuous** at the right endpoint b (i.e. $\lim_{x \to b^-} f(x) = f(b)$).

If any one of the above conditions fails, f is not continuous on $[a, b]$.



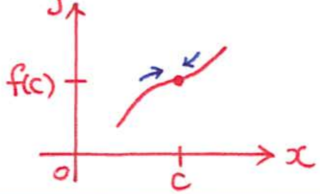
Remark:

A function f is continuous at c if and only if it is both left continuous and right continuous at c, i.e.

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$$

and therefore

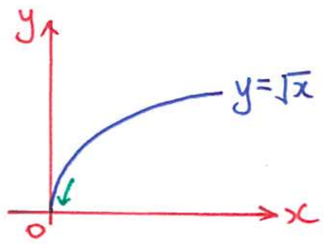
$$\lim_{x \to c} f(x) = f(c).$$



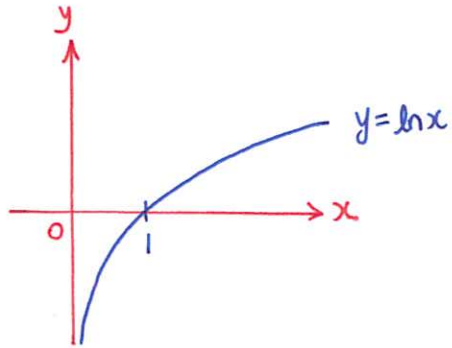
Examples

- The function $f(x) = \sqrt{x}$ is continuous on the interval $[0, \infty)$, since f is continuous at every x in the open interval $(0, \infty)$, and also it is right continuous at the left endpoint $x = 0$, i.e.

$$\lim_{x \to 0^+} f(x) = 0 = f(0).$$



- The function $g(x) = \ln x$ is not continuous on the interval $[0, 1]$, since g is not defined at $x = 0$ (i.e. 0 is not in the domain of g).



Intermediate Value Theorem (IVT)

Suppose f is continuous on $[a, b]$, and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

at least one c

Example 28

Show that there is a root of the equation $4x^3 - 6x^2 = -3x + 2$ between 1 and 2.

Solution

$$4x^3 - 6x^2 = -3x + 2 \Rightarrow 4x^3 - 6x^2 + 3x - 2 = \mathbf{0}$$

Let $f(x) = 4x^3 - 6x^2 + 3x - 2$.

Then $f(\mathbf{1}) = -1 < \mathbf{0}$ and $f(\mathbf{2}) = 12 > \mathbf{0}$.

0 is between -1 and 2

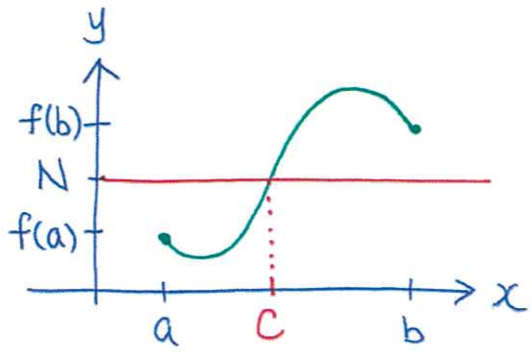
N f(a) f(b)

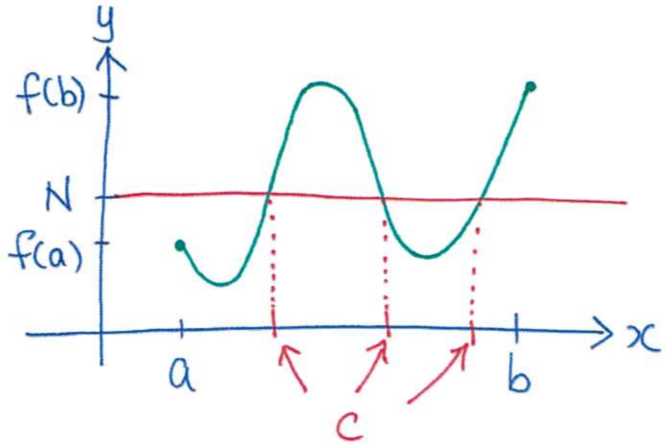
Note that $f(x)$ is continuous everywhere.

By the IVT, there is a number $\boldsymbol{c} \in (\mathbf{1}, \mathbf{2})$ such that $f(\boldsymbol{c}) = \mathbf{0}$,

i.e. there is a root of the equation $4x^3 - 6x^2 = -3x + 2$ between $\mathbf{1}$ and $\mathbf{2}$.

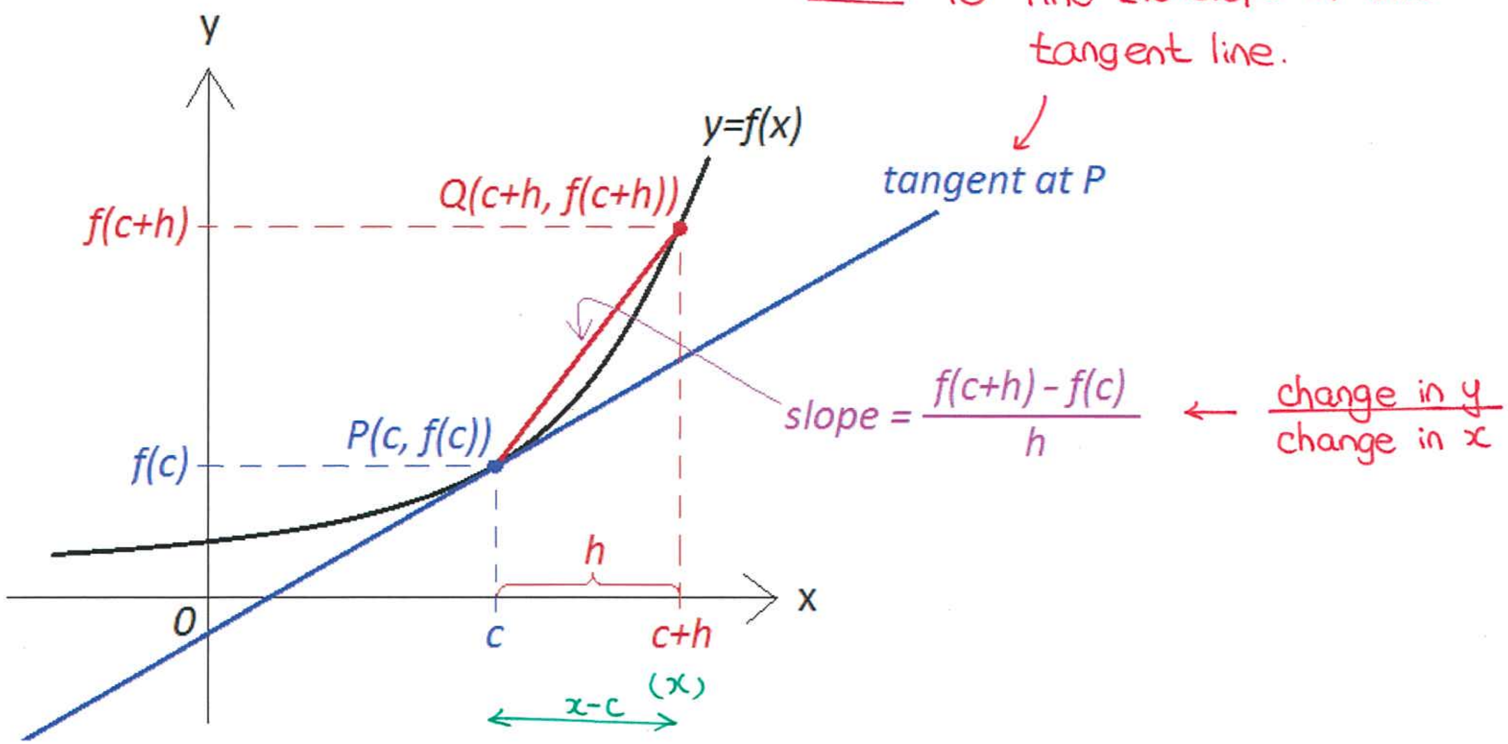
IVT





Differentiability of functions

Consider the graph of the function $y = f(x)$.



➢ A function f is **differentiable at $x = c$** if the limit

$$\boxed{\lim_{h \to 0} \frac{f(c+h)-f(c)}{h}} \quad \text{(or equivalently, } \boxed{\lim_{x \to c} \frac{f(x)-f(c)}{x-c}}\text{)}$$

put $x = c+h \Rightarrow h = x-c$

exists.

- The **derivative of $f(x)$ at $x = c$** (i.e. the slope of the tangent to the curve of $y = f(x)$ at $P(c, f(c))$) is given by

at a particular point

$$\star \quad \boxed{f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}},$$

or equivalently,

$$\star \quad \boxed{f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}},$$

if the limit exists.

evaluated at $x=c$

(Note: $f'(c)$ is also denoted by $\left.\frac{dy}{dx}\right|_{x=c}$ if $y = f(x)$.)

Now, if we consider all those points x at which f is differentiable, then we can establish a function f' which gives the value of the limit at each x. This function is called the **derivative of f with respect to x** or the **first derivative of f with respect to x** and is denoted by $f'(x)$ or $\frac{df(x)}{dx}$. or $Df(x)$

$D \equiv \frac{d}{dx}$

➢ From the **First Principle**, the **derivative of $f(x)$** is given by

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}.$$

This gives the slope of the tangent to the curve of $y = f(x)$ at the point $P(x, f(x))$ for every x.

(Note: $f'(x)$ is also denoted by $\frac{dy}{dx}$ or y' if $y = f(x)$.)

$\frac{dy}{dx}$ is the rate of change of y with one unit increase in x.

Example 29

Is $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ differentiable at $x = 0$?

check differentiability of f(x) at a point.
∴ use the formula on p.56

Solution

$f(x)$ is differentiable at $x = 0$ if $\lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ exists.

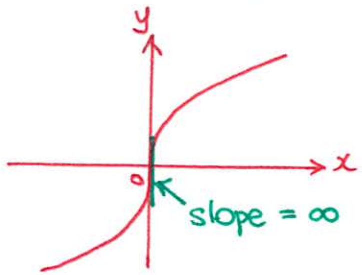
$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^{\frac{1}{3}} - 0^{\frac{1}{3}}}{x - 0} = \lim_{x \to 0} x^{\frac{1}{3} - 1} = \lim_{x \to 0} x^{-\frac{2}{3}} = \lim_{x \to 0} \frac{1}{x^{\frac{2}{3}}} = \infty$$

not a real number

$$\left[\text{Or using} \quad \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^{\frac{1}{3}} - 0^{\frac{1}{3}}}{h} = \lim_{h \to 0} h^{-\frac{2}{3}} = \lim_{h \to 0} \frac{1}{h^{\frac{2}{3}}} = \infty \right]$$

∴ The limit does not exist.

Hence, $f(x)$ is not differentiable at $x = 0$.



Tangent line is vertical at x=0.
∴ Infinite slope at x=0.
∴ Not differentiable at x=0.

Remark: We say that f is differentiable in an open interval I if it is differentiable at every point of I. For example, $f(x) = \sqrt[3]{x}$ is differentiable at every real number x except at $x = 0$, i.e. it is differentiable in the open intervals $(\infty, 0)$ and $(0, \infty)$.

Example 30

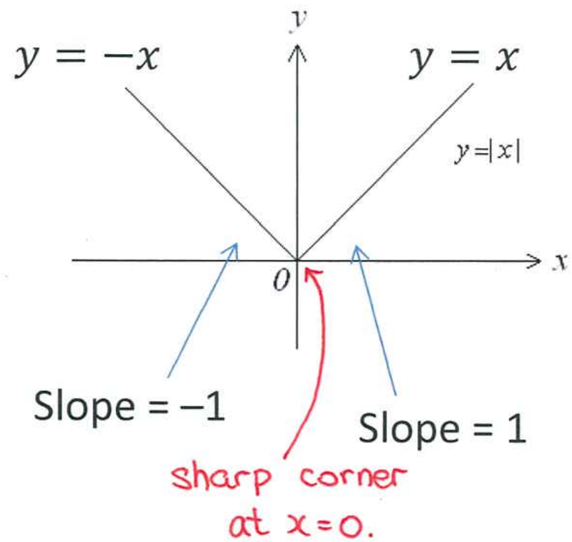
Is $f(x) = |x|$ differentiable at $x = 0$?

Solution

$f(x)$ is differentiable at $x = 0$ if the limit $\lim\limits_{h\to 0}\frac{f(0+h)-f(0)}{h}$, or equivalently $\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}$, exists.

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{|0+h|-|0|}{h} = \lim_{h\to 0}\frac{|h|}{h}$$

The above limit involves the absolute value function, so we consider the left hand limit and right hand limit separately.



Right hand limit: $\lim\limits_{h\to 0^+}\frac{|h|}{h} \overset{h>0}{=} \lim\limits_{h\to 0^+}\frac{h}{h} = \lim\limits_{h\to 0^+} 1 = 1$

Left hand limit: $\lim\limits_{h\to 0^-}\frac{|h|}{h} \overset{h<0}{=} \lim\limits_{h\to 0^-}\frac{-h}{h} = \lim\limits_{h\to 0^-}(-1) = -1$

Since $\lim\limits_{h\to 0^+}\frac{|h|}{h} \neq \lim\limits_{h\to 0^-}\frac{|h|}{h}$, the limit $\lim\limits_{h\to 0}\frac{|h|}{h}$ does not exist.

Hence, $f(x) = |x|$ is not differentiable at $x = 0$.

(Note: A function is not differentiable at points with sharp corners.)

Summary:

- $f(x)$ is continuous at $x = c$ iff $\lim_{x \to c} f(x) = f(c)$.
- $f(x)$ is differentiable at $x = c$ iff $\lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$ (or $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$) exists.

Theorem

If f is differentiable at $x = c$, then f is continuous at $x = c$.

Proof: (For your reference)

a real number

If f is differentiable at $x = c$, then $\boxed{f'(c)} = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ exists.

Consider $f(x) - f(c) = \frac{f(x)-f(c)}{x-c} \cdot (x - c)$ for $x \neq c$. Take limits on both sides:

$$\lim_{x \to c}\left(f(x) - f(c)\right) = \lim_{x \to c}\left[\frac{f(x)-f(c)}{x-c} \cdot (x-c)\right] = \left(\lim_{x \to c}\frac{f(x)-f(c)}{x-c}\right)\left(\lim_{x \to c}(x-c)\right) = f'(c) \cdot 0 = 0$$

$$\Rightarrow \lim_{x \to c} f(x) = f(c)$$

$\therefore$ f is continuous at $x = c$. □

The above theorem says that if a function f is differentiable at $x = c$, then f is continuous at $x = c$. However, the converse is not true. That is, if a function f is continuous at $x = c$, then f is not necessarily differentiable at $x = c$.

For example, $f(x) = |x|$ is continuous at $x = 0$ but it is not differentiable at $x = 0$ (see Example 30).

$$\boxed{\text{Differentiability of } f(x) \text{ at } x = c \;\underset{\nLeftarrow}{\Rightarrow}\; \text{Continuity of } f(x) \text{ at } x = c}$$

Example 31

Is $f(x) = |x|^3$ differentiable at $x = 0$?

Solution

$f(x)$ is differentiable at $x = \mathbf{0}$ if the limit $\lim_{h\to 0}\frac{f(\mathbf{0}+h)-f(\mathbf{0})}{h}$, or equivalently $\lim_{x\to \mathbf{0}}\frac{f(x)-f(\mathbf{0})}{x-\mathbf{0}}$, exists.

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{|0+h|^3-|0|^3}{h} = \lim_{h\to 0}\frac{|h|^3}{h} = \lim_{h\to 0}\frac{|h|^2\cdot|h|}{h}$$

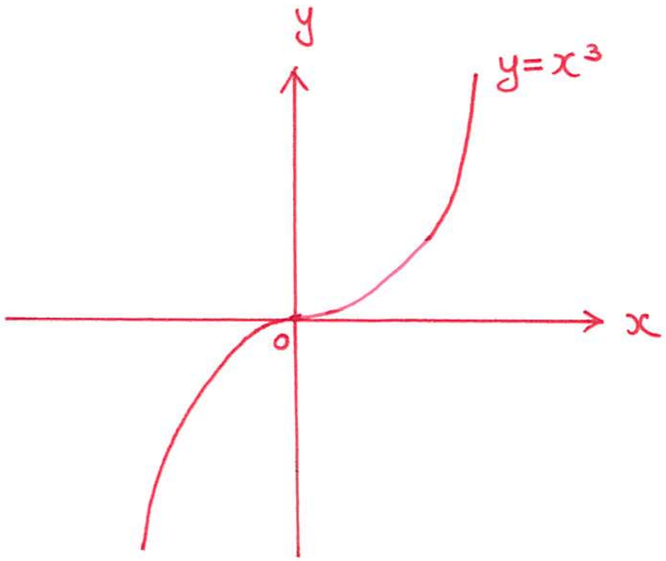
$$= \lim_{h\to 0}\frac{h^2\cdot|h|}{h} \quad (\text{since } |h|^2 = h^2)$$

$$= \lim_{h\to 0} h\cdot|h|$$

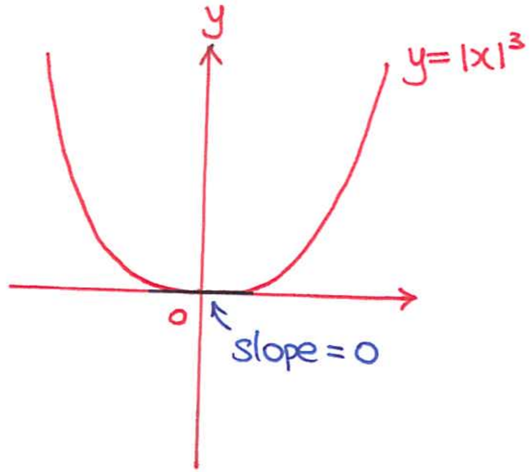
$$= 0\cdot|0|$$

$$= \mathbf{0}$$

Since the limit $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ exists, $f(x) = |x|^3$ is differentiable at $x = \mathbf{0}$ and $f'(\mathbf{0}) = \mathbf{0}$. ← i.e. the slope of the tangent line to the graph $y=|x|^3$ at $x=0$ is 0.



y
$y=x^3$
x
0



y
$y=|x|^3$
0
slope = 0

Example 32

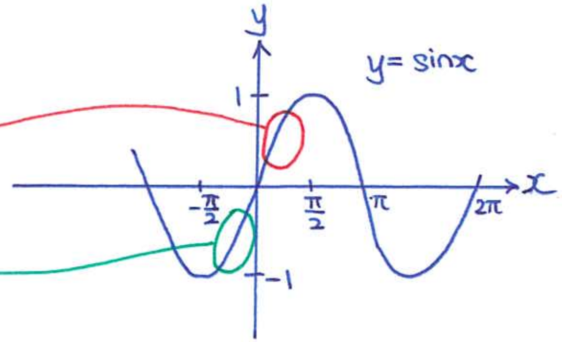
Is $f(x) = |\sin x|$ differentiable at $x = 0$?

Solution

$f(x) = |\sin x|$ is differentiable at $x = 0$ if the limit $\lim\limits_{x\to 0} \dfrac{f(x)-f(0)}{x-0}$ exists.

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{|\sin x|-|\sin 0|}{x-0} = \lim_{x\to 0} \frac{|\sin x|-0}{x-0} = \lim_{x\to 0} \frac{|\sin x|}{x}$$

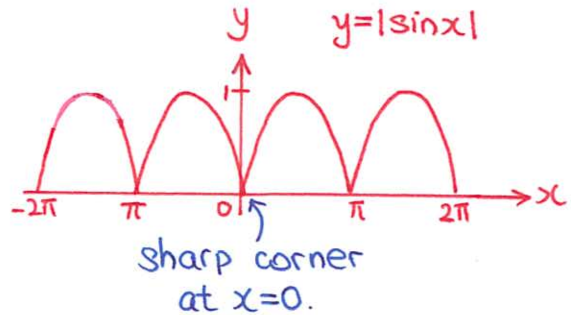
Recall that $|\sin x| = \begin{cases} \sin x & \text{if } 0 \le x \le \frac{\pi}{2} \\ -\sin x & \text{if } -\frac{\pi}{2} \le x < 0 \end{cases}$.



Left-hand limit: $\lim\limits_{x\to 0^-} \dfrac{|\sin x|}{x} \overset{x<0}{=} \lim\limits_{x\to 0^-} \dfrac{-\sin x}{x} = -\lim\limits_{x\to 0^-} \dfrac{\sin x}{x} = -1.$

Right-hand limit: $\lim\limits_{x\to 0^+} \dfrac{|\sin x|}{x} \overset{x>0}{=} \lim\limits_{x\to 0^+} \dfrac{\sin x}{x} = 1.$

Since $\lim\limits_{x\to 0^-} \dfrac{|\sin x|}{x} \ne \lim\limits_{x\to 0^+} \dfrac{|\sin x|}{x}$, the limit $\lim\limits_{x\to 0} \dfrac{|\sin x|}{x}$ does not exist.



Thus, $f(x) = |\sin x|$ is <u>not</u> differentiable at $x = 0$.

Example 33

Let $f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 1 \\ x^2 & \text{if } x < 1 \end{cases}$. $f(1) = \sqrt{1} = 1$

(a) Is f continuous at $x = 1$?

(b) Is f differentiable at $x = 1$?

Solution

(a) f is continuous at $x = 1$ iff $\lim_{x\to 1} f(x) = f(1)$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1^2 = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \sqrt{x} = \sqrt{1} = 1$$

$$f(1) = \sqrt{1} = 1$$

$$\because \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = 1$$

$\therefore$ f is continuous at $x = 1$.

(b) f is differentiable at $x = 1$ iff $\lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$ exists.

$$\lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^-} \frac{x^2-\sqrt{1}}{x-1} = \lim_{x\to 1^-} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1^-}(x+1) = 1+1 = 2$$

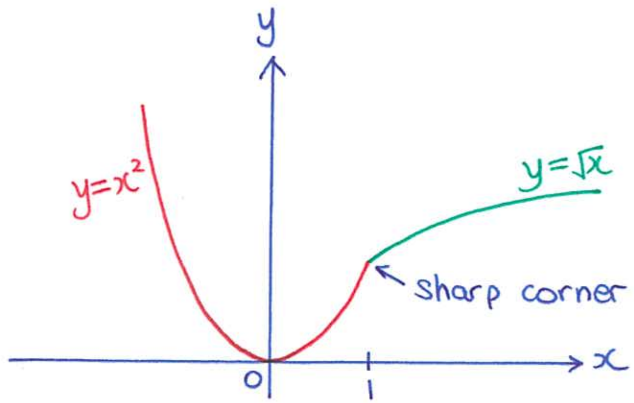
$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{\sqrt{x}-\sqrt{1}}{x-1} = \lim_{x\to 1^+} \frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}{(x-1)\left(\sqrt{x}+1\right)} = \lim_{x\to 1^+} \frac{1}{\sqrt{x}+1}$$

$$= \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

$\because \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1} \neq \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$

$\therefore \lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$ does not exist.

$\therefore$ f is not differentiable at $x = 1$.

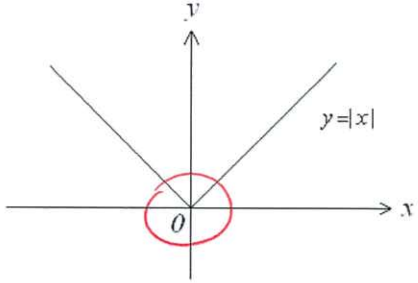


Function f is **not differentiable** at $x = c$ if one of the following situations is true:

(i) **f has a sharp corner at c**

E.g. $f(x) = |x|$ has a sharp corner at $x = 0$.

$\therefore$ f is not differentiable at $x = 0$.

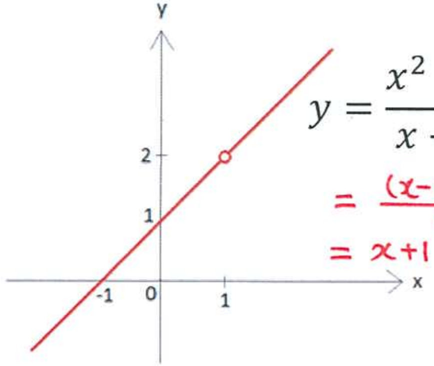


(ii) **f is discontinuous at c**

(i.e. f is not defined at c, or $\lim_{x \to c} f(x)$ does not exist, or $\lim_{x \to c} f(x) \neq f(c)$).

E.g. $f(x) = \frac{x^2-1}{x-1}$ is not defined at $x = 1$, so it is discontinuous at $x = 1$.

$\therefore$ f is not differentiable at $x = 1$.

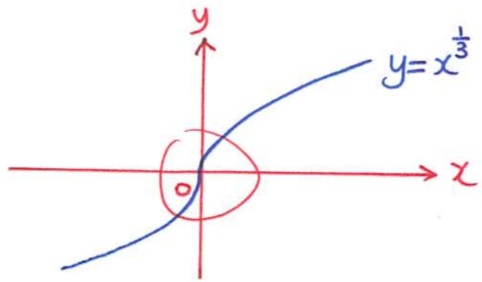


$$y = \frac{x^2 - 1}{x - 1}$$
$$= \frac{(x-1)(x+1)}{x-1}$$
$$= x+1 \text{ for } x \neq 1$$

(iii) **f has a vertical tangent line at c** (i.e. $\lim_{x \to c} |f'(x)| = \infty$).

E.g. $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ has a vertical tangent line at $x = 0$.

$\therefore$ f is not differentiable at $x = 0$.



Differentiation from the First Principle

From the **First Principle**, the **derivative of $f(x)$** is given by

$$\boxed{f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}.$$

Example 34

Let $f(x)=\frac{1}{x}$. Find $f'(x)$ from the **First Principle**.

Solution

From the First Principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\lim_{h\to 0}\frac{x-(x+h)}{x(x+h)h}$$

$$=\lim_{h\to 0}\frac{-\cancel{h}}{x(x+h)\cancel{h}}=\lim_{h\to 0}\frac{-1}{x(x+h)}=\frac{-1}{x(x+0)}=\frac{-1}{x^2}$$

Note: $f(x)=\frac{1}{x}$ is differentiable at every real number x except at $x=0$.

Example 35

Let $f(x) = x^n$, where n is a positive integer. Find $f'(x)$ from the **First Principle**.

Solution

From the **First Principle**,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{\left[\cancel{x^n} + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n\right] - \cancel{x^n}}{h}$$

$$= \lim_{h\to 0}\left[\binom{n}{1}x^{n-1} + \underbrace{\binom{n}{2}x^{n-2}h}_{\substack{\to 0 \\ \text{as } h\to 0}} + \cdots + \underbrace{\binom{n}{n-1}xh^{n-2}}_{\substack{\to 0 \\ \text{as } h\to 0}} + \underbrace{h^{n-1}}_{\substack{\to 0 \\ \text{as } h\to 0}}\right]$$

$$= \binom{n}{1}x^{n-1}$$

$$\binom{n}{1} = \frac{n!}{1!\,(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n$$

$$= nx^{n-1}$$

Note: In the above calculation, we have used the **Binomial Theorem** to expand $(x+h)^n$.

Binomial Theorem: For all positive integers n,

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a\, b^{n-1} + b^n$$

$$= \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \quad \leftarrow n+1 \text{ terms}$$

where $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ (called the **binomial coefficient**),

also written as nCr or C^n_r, is the number of ways of choosing r objects from n objects in an unordered manner.

$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$ and

$0! = 1$ (by definition).

$! \leftarrow$ factorial

E.g. $3! = 3 \times 2 \times 1 = 6$

$\binom{n}{0} = 1$

$\binom{n}{1} = n$

$\binom{n}{2} = \dfrac{n(n-1)}{2}$

$\binom{n}{3} = \dfrac{n(n-1)(n-2)}{3!}$

$\vdots$

$\binom{n}{n} = 1$

Example 36

Let $g(x) = \sin x$. Find $g'(x)$ from the **First Principle**.

Solution

From the **First Principle**,

Do not use L'Hôpital's rule (Ch.8) in First Principle.

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{2\cos\left[\frac{(x+h)+x}{2}\right]\sin\left[\frac{(x+h)-x}{2}\right]}{h}$$

(using the **sum-to-product formula:** $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$)

$$= \lim_{h\to 0}\frac{\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} = \lim_{h\to 0}\left[\underbrace{\cos\left(x+\frac{h}{2}\right)}_{\substack{\to\cos(x)\\ \text{as } h\to 0}}\cdot\underbrace{\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}_{\to 1 \text{ as } h\to 0}\right]$$

$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

$$= \cos x$$

Similarly, it can be shown from the First Principle that if $f(x) = \cos x$, then $f'(x) = -\sin x$.

Hint: $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

Example 37

Let $f(x) = \sqrt{x^2+1}$. Find $f'(x)$ from the **First Principle**.

Solution

From the First Principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h}$$

$a^2-b^2=(a-b)(a+b)$

$$= \lim_{h\to 0}\left(\frac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h}\cdot\frac{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}\right)$$

$$= \lim_{h\to 0}\frac{[(x+h)^2+1]-(x^2+1)}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)} = \lim_{h\to 0}\frac{(\cancel{x^2}+2xh+h^2+\cancel{1})-(\cancel{x^2}+\cancel{1})}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}$$

$$= \lim_{h\to 0}\frac{2x\cancel{h}+h^{\cancel{2}}}{\cancel{h}\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)} = \lim_{h\to 0}\frac{2x+h}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}$$

$$= \frac{2x+0}{\sqrt{(x+0)^2+1}+\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$$