4. Conjugate Property

$$\boxed{x^*(t) \xrightarrow{\mathcal{F}} X^*(-f)}$$

$$X^*(f) = \int_{-\infty}^{\infty} x^*(t)\, e^{j2\pi \boxed{f}t}\, dt.$$

$$\boxed{X^*(-f) = \int_{-\infty}^{\infty} \boxed{x^*(t)}\, e^{-j2\pi ft}\, dt.}$$

$$\mathcal{F}(x^*(t)) = X^*(-f)$$

If $x(t)$ is a real ft,

$$\boxed{x(t) = x^*(t)} \xleftrightarrow{\mathcal{F}} \mathcal{F}(x(t)) = \boxed{X(f)}$$

$$\mathcal{F}(x^*(t)) = \boxed{X^*(-f)} = \boxed{X(f)}$$

$$X(f) = X^*(-f)$$

5. Convolution / Multiplication

$$\left(\begin{array}{l} \underbrace{x(t) * y(t)}_{} \xrightarrow{\mathcal{F}} X(f)\,Y(f) \\ x(t)\,y(t) \xleftarrow{\mathcal{F}^{-1}} X(f) * Y(f) \end{array} \right.$$

Proof) $\mathcal{F}(x(t) * y(t))$

$$= \int_{-\infty}^{\infty} \left(\underbrace{x(t) * y(t)}_{} \right) e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\, y(\underline{t-\tau})\, e^{-j2\pi f \boxed{t}}\, \underline{d\tau\, dt}$$

$t - \tau = l$

Given ζ, $dl = dt$.

$$t = (t-\zeta) + \zeta$$

$$= l + \zeta$$

$$\Rightarrow \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\zeta)\, y(t-\zeta)\, e^{-j2\pi f(t-\zeta+\zeta)}\, dt \right] d\zeta$$

1st, $\zeta \rightarrow$ Fixed, $t \rightarrow$ change

$$\underbrace{\int_{-\infty}^{\infty} y(l)\, e^{-j2\pi f l}\, dl}_{Y(f)}$$

$$\underbrace{\int_{-\infty}^{\infty} x(\zeta)\, e^{-j2\pi f \zeta}\, d\zeta}_{X(f)}$$

$\Rightarrow X(f)\,Y(f)$

6. Duality

$$\mathcal{F}(X(t)) = x(-f)$$

$$\boxed{\begin{array}{c} x(t) \longleftrightarrow \boxed{X(f)} \\ X(t) \longleftrightarrow x(-f) \end{array}}$$

Proof)

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t}\, df$$

$(t \rightarrow -f'$

$f \to t'$

$$x(-f) = \int_{-\infty}^{\infty} \boxed{X(t')} \, e^{-j2\pi f' t'} \, \boxed{dt'}$$

$e^{-j2\pi f' t'}$

$$\mathcal{F}(X(t)) = x(-f)$$

7. Diff / Integ

$$\mathcal{F}\left(\frac{d^n x(t)}{dt^n}\right) = (j2\pi \boxed{f})^n X(f)$$

$$\mathcal{F}\left((-j2\pi \boxed{t})^n x(t)\right) = \frac{d^n \boxed{X(f)}}{df^n}$$

Proof)

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

$$\boxed{\frac{dx(t)}{dt}} = \int_{-\infty}^{\infty} X(f) \left(\frac{d e^{j2\pi ft}}{dt} \right) df$$

ft of t

$$= (j2\pi f) e^{j2\pi ft}$$

$$= \int_{-\infty}^{\infty} (j2\pi f) X(f) e^{j2\pi ft} df$$

$$\int_{-\infty}^{\infty} \underline{\qquad\qquad} Y(f) . df$$

$$\frac{d^n x(t)}{dt^n} = \int_{-\infty}^{\infty} (j2\pi f)^n X(f)\, e^{j2\pi f t}\, df$$

$$\mathcal{F}\left(\frac{d^n x(t)}{dt^n}\right) = (j2\pi f)^n X(f)$$

$$\mathcal{F}\left(\int_{-\infty}^{t} x(\tau)\, d\tau\right) = \frac{1}{2} X(0)\,\delta(f) + \frac{1}{j2\pi f} X(f)$$

P21, Cha 2
Prop. 3-c)

$$\int_{-\infty}^{t} x(\tau)\, d\tau = x(t) * u(t)$$

$$\mathcal{F}\left(\int_{-\infty}^{t} x(\tau)\,d\tau\right)$$

$$= \mathcal{F}\left(x(t) * u(t)\right)$$

$$= \mathcal{F}(x(t)) \cdot \mathcal{F}(u(t))$$

$$= X(f) \cdot \left[\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right]$$

$$= \frac{1}{2}X(0)\delta(f) + \frac{1}{j2\pi f}X(f)$$

if $X(0) = 0$.

$$\rightarrow \boxed{\frac{1}{j2\pi f}X(f)}$$

8. Parseval's thm.

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Ex 4-2).

$$e^{-\alpha t} u(t) \longleftrightarrow \frac{1}{\alpha + j2\pi f}$$

$$e^{\alpha t} u(-t) \longleftrightarrow \frac{1}{\alpha - j2\pi f}$$

$$e^{-\alpha t} u(t) - e^{\alpha t} u(-t) \longleftrightarrow \frac{1}{\alpha + j2\pi f} - \frac{1}{\alpha - j2\pi f}$$

$$= \frac{-j4\pi}{\alpha^2 + 4(\pi f)^2}$$

$\alpha \to 0$ . .

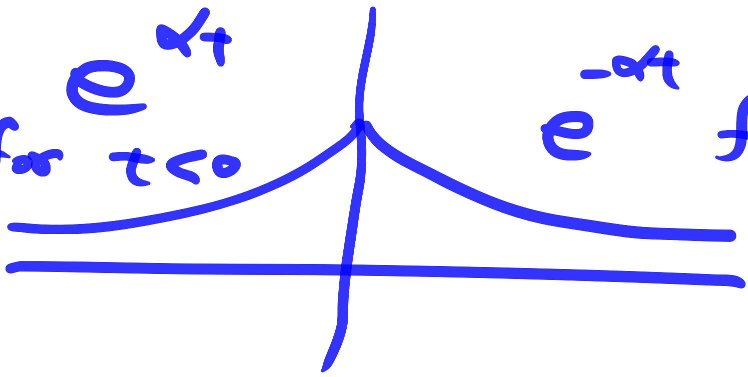
$$sgn(t) = \frac{1}{j\pi f}$$

$$u(t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{sgn}(t).$$

$$\mathcal{F}\{e^{-\alpha t}u(t) + e^{\alpha t}u(-t)\} = \frac{2\alpha}{\alpha^2+4(\pi f)^2}$$

$e^{\alpha t}$ for $t<0$

$e^{-\alpha t}$ for $t>0$



$$e^{-\alpha|t|} = e^{-\alpha t}u(t) + e^{\alpha t}u(-t)$$

$$\mathcal{F}(e^{-\alpha|t|}) = \boxed{\frac{2\alpha}{\alpha^2+4(\pi f)^2}}$$

$t \to -f$

$$\mathcal{F}\left\{\frac{2\alpha}{\alpha^2+4(\pi t)^2}\right\} = e^{-\alpha|-f|} = e^{-\alpha|f|}$$

$$\boxed{\mathcal{F}\left\{\frac{1}{\alpha^2+4(\pi t)^2}\right\}=\frac{1}{2\alpha}e^{-\alpha|f|}}$$

Ex 4-3

$$\operatorname{rect}\left(\frac{t}{\tau}\right) \xleftrightarrow{\mathcal{F}} \tau \operatorname{Sinc}(f\tau).$$

$$\tau \operatorname{Sinc}(t\cdot\tau) \longleftrightarrow \operatorname{rect}\left(-\frac{f}{\tau}\right)$$

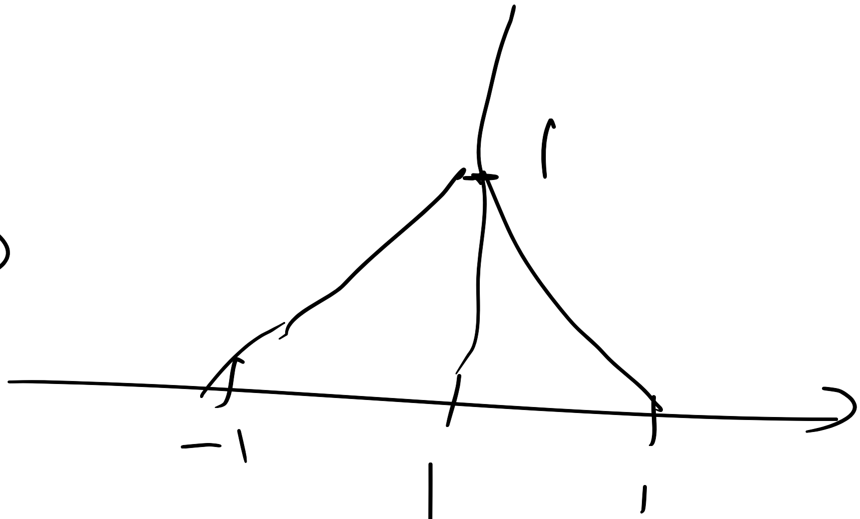
$$= \operatorname{rect}\left(\frac{f}{\tau}\right)$$

1) $\tau = 2B$, 2) $\frac{1}{2B}$

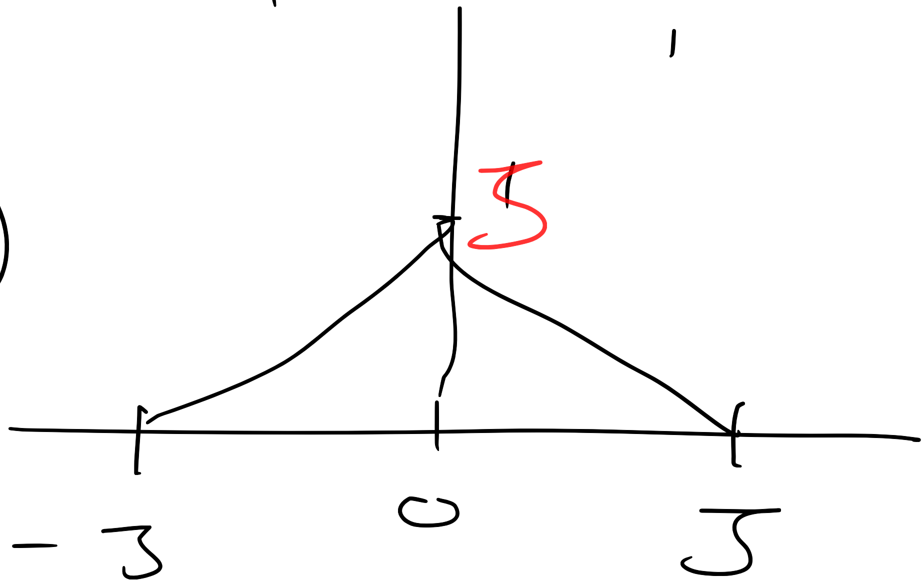
$$\operatorname{Sinc}(2Bt) \longleftrightarrow \frac{1}{2B}\operatorname{rect}\left(\frac{f}{2B}\right)$$

$$\mathcal{F}\left(5\,\mathrm{tri}\left(\frac{t}{5}\right)\right)$$ (Ex 2-1 p23)

$\mathrm{tri}(t) \Rightarrow$



$$5\,\mathrm{tri}\left(\frac{t}{5}\right)$$



c).

$$\mathcal{F}\left(5\,\mathrm{tri}\left(\frac{t}{5}\right)\right) = \mathcal{F}\left[\mathrm{rect}\left(\frac{t}{5}\right) * \mathrm{rect}\left(\frac{t}{5}\right)\right]$$

$$= \mathcal{F}\left(\mathrm{rect}\left(\frac{t}{5}\right)\right) \cdot \mathcal{F}\left(\mathrm{rect}\left(\frac{t}{5}\right)\right)$$

$$= 5^2\,\mathrm{sinc}^2(f5).$$

$$\mathcal{F}\left(\underbrace{\mathrm{rect}\left(\frac{t}{3}\right) * \mathrm{rect}\left(\frac{t}{3}\right) * \cdots * \mathrm{rect}\left(\frac{t}{3}\right)}_{n\text{-th order convolution}} \right)$$

$$= \mathcal{F}\left(\mathrm{rect}\left(\frac{t}{3}\right)\right)^n$$

$$= 3^n \,\mathrm{sinc}^n(f3)$$

$$\underbrace{3\,\mathrm{tri}\left(\frac{t}{3}\right)} \longleftrightarrow \underbrace{3^2 \mathrm{sinc}^2(f3)}$$

d) $\mathrm{sinc}^2(t\boxed{3}) \longleftrightarrow \frac{1}{3}\mathrm{tri}\left(\frac{f}{3}\right)$

$$\mathrm{sinc}^2(2Bt) \longleftrightarrow \frac{1}{2B}\mathrm{tri}\left(\frac{f}{2B}\right)$$

e) $\mathcal{F}\left(\mathrm{rect}\left(\frac{t-t_0}{\tau}\right)\right)$

$= \tau \,\mathrm{Sinc}\,(f\tau)$

$\times$

$e^{-j2\pi f t_0}.$

Ex 4-4) $e^{j2\pi f_0 t} \longleftrightarrow \delta(f-f_0)$

a) $\mathcal{F}(\cos(2\pi f_0 t))$

$= \mathcal{F}\left\{\frac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)\right\}$

$= \frac{1}{2}\left\{\delta(f-f_0) + \delta(f+f_0)\right\}$

b) $\mathcal{F}(\sin(2\pi f_0 t))$

$= \mathcal{F}\left\{\frac{1}{2j}\left(e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right)\right\}$

$= \frac{1}{2j}\left(\delta(f-f_0) - \delta(f+f_0)\right)$

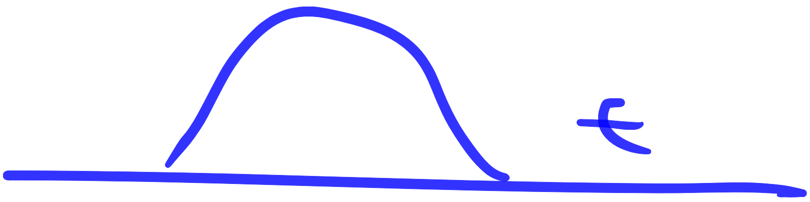
c) $\mathcal{F}(x(t)\cos(2\pi f_0 t))$

$\Rightarrow \mathcal{F}(x(t)) * \mathcal{F}(\cos(2\pi f_0 t))$

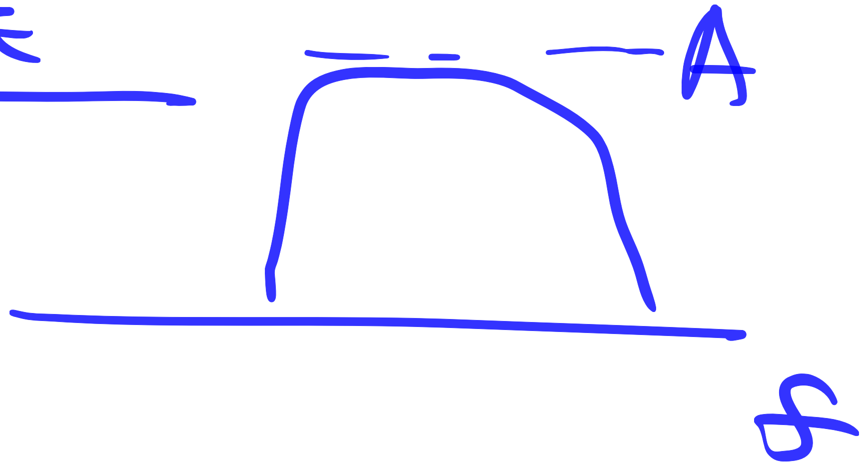
$X(f) * \left[\frac{1}{2}\left(\delta(f-f_0) + \delta(f+f_0)\right)\right]$

$= \frac{1}{2}\left\{X(f-f_0) + X(f+f_0)\right\}$

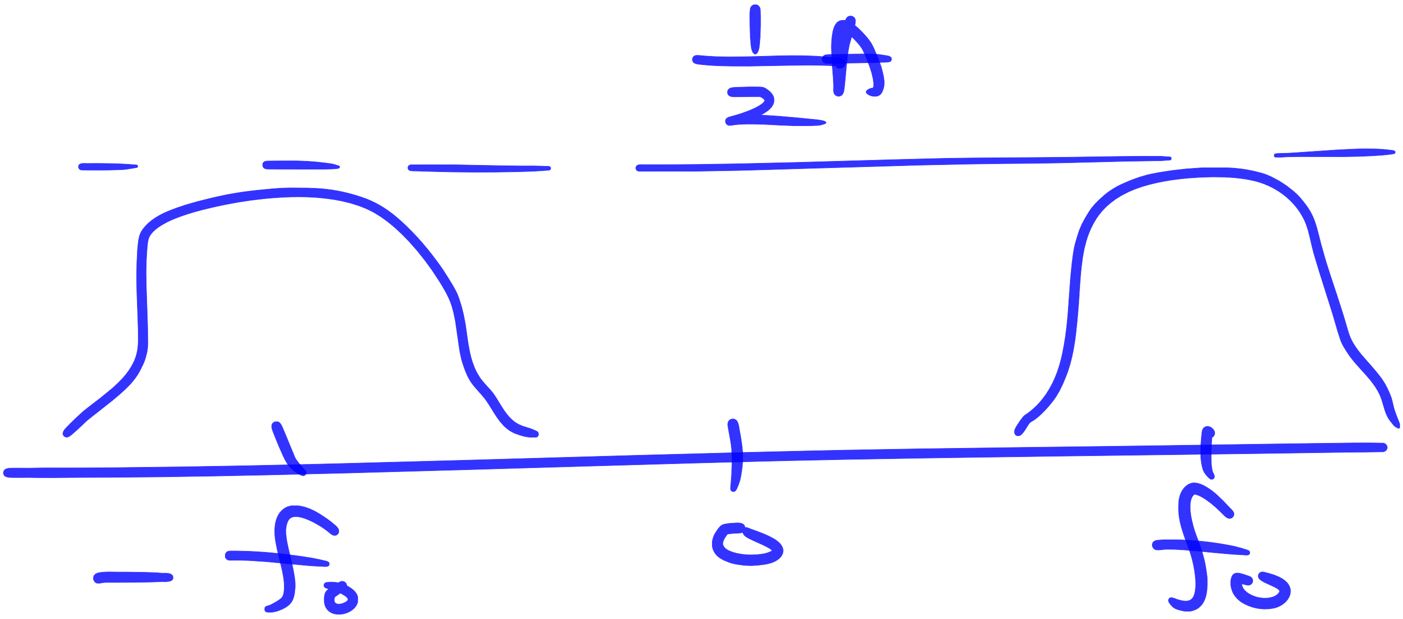
$x(t)$



$X(f)$



$\mathcal{F}\left(x(t)\cos(2\pi f_0 t)\right)$



c)

$$\mathcal{F}(\cos(2\pi f_0 t)u(t))$$

$$= \mathcal{F}(\cos(2\pi f_0 t)) * \mathcal{F}(u(t))$$

$$\boxed{\frac{1}{2}\left(\delta(f-f_0)+\delta(f+f_0)\right)} * \left\{\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right\}$$

$$\frac{1}{4}\left(\delta(f-f_0)+\delta(f+f_0)\right) \quad \frac{2f}{f^2-f_0^2}$$

$$+\frac{1}{j4\pi}\left\{\frac{1}{f-f_0}+\frac{1}{f+f_0}\right\}$$

$$j\pi$$

$$= \frac{1}{4}\left(\delta(f-f_0)+\delta(f+f_0)\right) + \frac{j2\pi f}{(2\pi f_0)^2-(2\pi f)^2}$$

g)

$$\mathcal{F}\left(e^{-\alpha t}u(t)\cos(2\pi f_0 t)\right)$$

$$= \mathcal{F}\left(e^{-\alpha t}u(t)\right) * \mathcal{F}\left(\cos(2\pi f_0 t)\right)$$

$$\left(\frac{1}{\alpha + j2\pi f}\right) * \frac{1}{2}\left(\delta(f-f_0) + \delta(f+f_0)\right)$$

$$= \frac{1}{2}\left[\frac{1}{\alpha + j2\pi(f-f_0)} + \frac{1}{\alpha + j2\pi(f+f_0)}\right]$$

$$= \left(\frac{\alpha + j2\pi f}{(\alpha + j2\pi f)^2 + 4(\pi f_0)^2}\right)$$