MA1200 Calculus and Basic Linear Algebra I
Chapter 7 Techniques of Differentiation

1 Differentiability and Differentiation

We say that a function $y = f(x)$ is differentiable at $x = a$ if the limit

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

exists. If it exists, we denote it by $f'(a)$. Now, if we consider all those points x at which f is differentiable, then we can establish a function f' which gives the value of the limit (that is, $f'(x)$) at each x. This function is called the derivative of f with respect to x (or the first derivative of f with respect to x), and is denoted by $\frac{dy}{dx}$, y', $f'(x)$, $\frac{df(x)}{dx}$ or $Df(x)$. Note that $f'(a)$, $\left.\frac{dy}{dx}\right|_{x=a}$, $Df(a)$ are representing the same thing. We will establish techniques which help us to find the derivative of a function.

1.1 Differentiation from the first principle (p.283 – p.284)

A natural way to find the derivative of a function is to evaluate the limit directly. It is called d*ifferentiation from the first principle*. The following examples illustrate the procedure.

Example 1

Let $f(x) = x^n$ where n is a positive integer. Now,

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}.$$

By the Binomial Theorem for a positive index n,

$$(x+h)^n = x^n + {}_nC_1x^{n-1}h + {}_nC_2x^{n-2}h^2 + \cdots + {}_nC_rx^{n-r}h^r + \cdots + h^n$$

Hence $f'(x) = \lim_{h\to 0}\left({}_nC_1x^{n-1} + {}_nC_2x^{n-2}h + \cdots + h^{n-1}\right) = {}_nC_1x^{n-1} = nx^{n-1}$.

∎

Example 2

Let $f(x) = \sin x$.

$$f'(x) = \lim_{h\to 0}\left[\frac{\sin(x+h)-\sin x}{h}\right] = \lim_{h\to 0}\left(\frac{2}{h}\sin\frac{h}{2}\cos\left(x+\frac{h}{2}\right)\right), \text{ since } \sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}.$$

$$= \lim_{h\to 0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\cos\left(x+\frac{h}{2}\right)\right) = \lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\lim_{h\to 0}\cos\left(x+\frac{h}{2}\right) = (1)(\cos x) = \cos x.$$

∎

1.2 Simple differentiation rules (p.294 – p.303, p.318 – p.323, p.325 – p.330, p.344 – p.346)
In order to speed up the calculations of derivatives, the following rules are set up.

(1) Differentiate a constant function:
If $f(x)=k$ where k is a constant, then $f'(x)=0$.

(2) Differentiate a linear combination of functions:
If $f(x)=\lambda u(x)+\mu v(x)$ where λ and μ are real numbers and $u(x)$, $v(x)$ are differentiable functions of x, then $f'(x)=\lambda u'(x)+\mu v'(x)$.

(3) Differentiate a product of functions:
If $f(x)=u(x)v(x)$ where $u(x),v(x)$ are both differentiable functions of x, then $f'(x)=u'(x)v(x)+u(x)v'(x)$.

(4) Differentiable a quotient of functions:
If $f(x)=\dfrac{u(x)}{v(x)}$ where $u(x),v(x)$ are both differentiable functions of x, then $f'(x)=\dfrac{v(x)u'(x)-u(x)v'(x)}{v^2(x)}$, provided $v(x)\neq 0$.

(5) Differentiate a composite function $f\circ u(x)=f(u(x))$ (Chain Rule):
If $f\circ u(x)=f(u(x))$ where f is differentiable at u and u is differentiable at x, then $f'(x)=\dfrac{df}{du}\dfrac{du}{dx}$.

The following examples illustrate how to apply these rules.

<u>Example 3</u>
Let $f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ where $a_0,a_1,\cdots,a_n$ are real numbers. Then,

$$\begin{aligned} f'(x)&=\left(a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0\right)' \\ &=\left(a_nx^n\right)'+\left(a_{n-1}x^{n-1}\right)'+\cdots+\left(a_1x\right)'+a_0' \quad \text{, from 6.2(2).} \\ &=na_nx^{n-1}+(n-1)a_{n-1}x^{n-2}+\cdots+a_1 . \end{aligned}$$

∎

<u>Example 4</u>
If $f(x)=\dfrac{\sin x}{x^2+1}$, find $f'(x)$.
<u>Solution</u>:

$$f'(x)=\frac{\left(x^2+1\right)(\sin x)'-(\sin x)\left(x^2+1\right)'}{\left(x^2+1\right)^2}=\frac{\left(x^2+1\right)\cos x-2x\sin x}{\left(x^2+1\right)^2}.$$

∎

Example 5

Let $f(x)=\sin^5(x^5)$. Find $f'(x)$.

Solution:

$f'(x)=5\sin^4(x^5)\left[\sin(x^5)\right]'=5\sin^4(x^5)\cos(x^5)(x^5)'=5\sin^4(x^5)\cos(x^5)5x^4=25x^4\sin^4(x^5)\cos(x^5)$.

□

Example 6

If $y=\sqrt{ax^2+2bx+c}$, where a,b,c are constants, prove that $\dfrac{dy}{dx}=\dfrac{ax+b}{y}$.

Proof:

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sqrt{ax^2+2bx+c}\right)=\frac{d}{dx}\left[\left(ax^2+2bx+c\right)^{\frac{1}{2}}\right]=\frac{1}{2}\left(ax^2+2bx+c\right)^{-\frac{1}{2}}\left(2ax+2b\right)$$

$$=\frac{ax+b}{\left(ax^2+2bx+c\right)^{\frac{1}{2}}}=\frac{ax+b}{y}.$$

□

1.3 Implicit functions and their derivatives (p.340 – p.344)

Most functions that we have encountered are expressed explicitly as

$$y=f(x).$$

However, the dependence between two variables x and y can also be described by writing

$$F(x,y)=0.$$

A typical example of such a relationship is

$$x^2+y^2-1=0.$$

Observe that there are two continuous functions (both with domain $(-1,1)$)

$$y=\sqrt{1-x^2}$$

$$y=-\sqrt{1-x^2}$$

defined by this equation implicitly.

The derivative of an implicit function can be calculated in a completely straightforward fashion.

Example 7

Curve: $x^2+y^2-1=0-(*)$

Differentiating both sides of (*) with respect to x

$$\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}\left(y^2\right)-\frac{d}{dx}(1)=0$$

$$2x+2y\frac{dy}{dx}=0$$

It follows that $\dfrac{dy}{dx}=-\dfrac{x}{y}$.

□

Example 8

Find $\frac{dy}{dx}$ from the implicit relationship

$$xy^2+3x^3=\frac{y}{x}.$$

Solution:

$$\frac{d}{dx}\left(xy^2+3x^3\right)=\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$2xy\frac{dy}{dx}+y^2+9x^2=\frac{x\frac{dy}{dx}-y}{x^2}$$

$$2xy\frac{dy}{dx}+y^2+9x^2=\frac{1}{x}\frac{dy}{dx}-\frac{y}{x^2}$$

$$2xy\frac{dy}{dx}-\frac{1}{x}\frac{dy}{dx}=-\frac{y}{x^2}-y^2-9x^2$$

$$\therefore\ \frac{dy}{dx}=\frac{-\frac{y}{x^2}-y^2-9x^2}{2xy-\frac{1}{x}}.$$

□

1.4 Inverse functions and their derivatives (p.618 – p.624, p.669 – p.680)

Review

A function *f* takes a number *x* from its domain *D* and assigns to it a single value *y* from its range *R*. For some functions, say, for example, $f(x)=2x-3$, we can reverse f. That is, for any given *y* in *R*, we can unambiguously go back and find the x from which it came. This new function (takes *y* and assigns *x* to it) is denoted by f^{-1} and is called the *inverse* of *f*.

Recall from Chapter 2 that, the criterion that a function f possesses an inverse function f^{-1} is that it must be one-to-one.

In practice, the inverse function f^{-1} is calculated from f by the following procedure:

1) check whether the function $y=f(x)$ is one-to-one,
2) solve x in terms of y (if possible),
3) rewrite the independent variable as x and the dependent variable as y.

Example 9

(a) Let $f(x)=|2x-4|$ for $x\geq 2$. Note that $f(x)=2x-4$ for $x\geq 2$.

The function $f(x)$ is one-to-one.

$$y=2x-4$$

$$\Rightarrow\qquad x=\frac{1}{2}(y+4)$$

The inverse function is $f^{-1}(x)=\frac{1}{2}(x+4)$ for $x\geq 0$.

(b) Let $f(x)=|2x-4|$ for all $x\in\mathbf{R}$.

$f(x)$ takes the same value twice for $x\neq 2$. (e.g. $f(6)=8=f(-2)$)

Therefore $f(x)$ has no inverse.

Derivative of an Inverse Function

The derivative of a function and the derivative of its inverse have the following relationship:

Theorem (Inverse function theorem)
Let *f* be differentiable and strictly monotonic on an interval **I**. If $f'(x)\neq 0$ at a certain *x* in **I**, then f^{-1} is differentiable at the corresponding point $y=f(x)$ in the range of *f* and

$$\left(f^{-1}\right)'(y)=\frac{1}{f'(x)}.$$

It is customarily written as

$$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}}.$$

This theorem will be applied frequently in the following sections.

Example 10 (Inverse trigonometric functions)
We know that the function $\sin x$ (with domain **R**) has no inverse. However, by restricting the domain, the function $\sin x$ can have well-defined inverses. For example, the following function has inverse:

$$f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to\mathbf{R},\ f(x)=\sin x$$

To be definite, we define the inverse sine function by

$$\sin^{-1}:[-1,1]\to\left[-\frac{\pi}{2},\frac{\pi}{2}\right],$$

where $\sin^{-1}y=x$ for $y=\sin x$.

The interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ is called the principal range of $\sin^{-1}y$.

Similarly, the inverse cosine function and inverse tangent function can be defined by choosing $[0,\pi]$ and $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ as their corresponding principal ranges.

□

Now, we can apply the inverse function theorem to find the derivatives for the functions $\sin^{-1}y$, $\cos^{-1}y$ and $\tan^{-1}y$. Since $y=\sin x\Rightarrow\frac{dy}{dx}=\cos x$. Hence,

$$y=\sin^{-1}x\Rightarrow x=\sin y\Rightarrow\frac{dx}{dy}=\cos y\Rightarrow\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-\sin^2 y}}=\frac{1}{\sqrt{1-x^2}}\quad,-1<x<1.$$

Similarly, we can show

$$\left(\cos^{-1} x\right)' = \frac{-1}{\sqrt{1-x^2}}, \quad -1 < x < 1 \quad \text{and} \quad \left(\tan^{-1} x\right)' = \frac{1}{1+x^2}.$$

<u>Example 11</u>
Differentiate $x\sin^{-1}(2x)$ with respect to *x*.
<u>Solution</u>:

$$\frac{d}{dx}\left(x\sin^{-1}(2x)\right) = x\frac{d}{dx}\left(\sin^{-1}(2x)\right) + \sin^{-1}(2x)\frac{d}{dx}(x) = \frac{2x}{\sqrt{1-4x^2}} + \sin^{-1}(2x).$$

□

<u>Example 12</u>
Differentiate $\tan^{-1}(x+1)$ with respect to x.
<u>Solution</u>:

$$\frac{d}{dx}\left(\tan^{-1}(x+1)\right) = \frac{1}{1+(x+1)^2}(x+1)' = \frac{1}{x^2+2x+2}.$$

□

<u>Example 13</u>
Differentiate $\cos^{-1}\frac{2x}{1+x^2}$ with respect to x.
<u>Solution</u>:
Let $y = \cos^{-1}\frac{2x}{1+x^2}$.

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}}\frac{d}{dx}\left(\frac{2x}{1+x^2}\right) = -\frac{\frac{(1+x^2)2-2x(2x)}{(1+x^2)^2}}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}} = \frac{2x^2-2}{(1+x^2)^2\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}}.$$

□

1.5 Logarithmic functions, exponential functions and their derivatives (p.628 – p.652)
Exponential functions arise in many real life problems. For example, in an ideal environment, the mass *m* of a cell will grow according to the following equation:

$$\frac{dm}{dt} = km, \quad \text{where } k \text{ is a constant.}$$

In words, it means that the cell's growth rate is proportional to its mass at each instance of time. The solution of the above equation is given by:

$$m = m_0 e^{kt},$$

where m_0 is the initial mass of the cell.

In fact, we can define the exponential function as follow:
The function $y = f(x) = e^x$ is the solution of the differential equation

$$\frac{dy}{dx} = y \quad \text{with} \quad y(0) = 1$$

Note that $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.71828$.

The general properties of the function $y = e^x$ are:

- By definition, we know that
  $\frac{d}{dx}\left(e^x\right) = e^x$ and $e^0 = 1$.
- Since e is a positive number, all index laws can be applied. (For example $e^{x_1}\cdot e^{x_2} = e^{x_1+x_2}$).
- It is continuous (because it is differentiable) and is an increasing function of x.
- The domain of e^x is **R** and its range is the set of all positive numbers.

Since e^x is strictly increasing on its domain, it has an inverse. This inverse is called the natural logarithm and is denoted by

$$\log_e x \text{ or } \ln x.$$

The natural logarithm function has the following properties:

- (i) $\ln xy = \ln x + \ln y$ where $x, y > 0$.

  (ii) $\ln\frac{x}{y} = \ln x - \ln y$.

  (iii) $\ln x^a = a\ln x$.
- Since it is the inverse of e^x, by inverse function theorem, we have
  $$\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}$$
- The domain and range of this function are:
  Domain: The set of all positive real numbers. Range: The set of all real numbers.
- It is differentiable and increasing everywhere on its domain.

Example 14

Consider the following three functions:

$$f\left(x\right) = a^x = e^{x\ln a}$$

$$g\left(x\right) = \log_a x = \frac{\ln x}{\ln a}$$

$$h\left(x\right) = x^a = e^{a\ln x}$$

where a is a real constant. We see that

(i) $y = a^x \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(a^x\right) = \frac{d}{dx}\left(e^{x\ln a}\right) = e^{x\ln a}\frac{d}{dx}\left(x\ln a\right) = e^{x\ln a}\ln a = a^x\ln a$.

(ii) $y = \log_a x \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}\left(\ln x\right) = \frac{1}{x\ln a}$.

(iii) $y = x^a \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(e^{a\ln x}\right) = e^{a\ln x}\frac{d}{dx}\left(a\ln x\right) \Rightarrow \frac{dy}{dx} = \frac{a}{x}e^{a\ln x} = \frac{a}{x}a^x = ax^{a-1}$.

□

Example 15

We can apply the natural logarithm function to find the derivatives of some functions. Find $f'\left(x\right)$ where $f\left(x\right) = \frac{x}{\left(x-1\right)\left(x-2\right)\left(x-3\right)}$.

Solution:

We put $y=\dfrac{x}{(x-1)(x-2)(x-3)}$

$$\Rightarrow \ln y=\ln x-\ln(x-1)-\ln(x-2)-\ln(x-3)\Rightarrow \frac{dy}{dx}=\frac{x}{(x-1)(x-2)(x-3)}\left(\frac{1}{x}-\frac{1}{x-1}-\frac{1}{x-2}-\frac{1}{x-3}\right).$$

This method simplifies the calculations.

Example 16

If $f(x)=e^{-\frac{x}{n}}\cos\dfrac{x}{a}$, find the value of $f(0)+af'(0)$.

Solution:

$$f(x)=e^{-\frac{x}{n}}\cos\frac{x}{a}\Rightarrow f(0)=1.$$

$$\frac{df(x)}{dx}=e^{-\frac{x}{n}}\frac{d\left(\cos\frac{x}{a}\right)}{dx}+\cos\frac{x}{a}\frac{d\left(e^{-\frac{x}{n}}\right)}{dx}=-e^{-\frac{x}{n}}\frac{1}{a}\sin\frac{x}{a}-\frac{1}{n}e^{-\frac{x}{n}}\cos\frac{x}{a}\Rightarrow f'(0)=-\frac{1}{n}.$$

So $f(0)+af'(0)=1-\dfrac{a}{n}$.

Example 17

If $y=\left(\dfrac{a}{x}\right)^{ax}$, find $\dfrac{dy}{dx}$.

Solution:

Method 1:

$$\ln y=\ln\left(\frac{a}{x}\right)^{ax}\Rightarrow \ln y=ax\ln\left(\frac{a}{x}\right).$$

Differentiate both sides of $\ln y=ax\ln\left(\dfrac{a}{x}\right)$ with respect to x, we have

$$\frac{1}{y}\cdot\frac{dy}{dx}=ax\frac{-\frac{a}{x^2}}{\frac{a}{x}}+a\ln\left(\frac{a}{x}\right)\Rightarrow\frac{dy}{dx}=y\left[-a+a\ln\left(\frac{a}{x}\right)\right]=-a\left(\frac{a}{x}\right)^{ax}+a\left(\frac{a}{x}\right)^{ax}\ln\left(\frac{a}{x}\right).$$

In general for function $y=u(x)^{v(x)}$, where $u(x)>0$, we may use the above method to obtain $\dfrac{dy}{dx}$.

Method 2:

$$y=\left(\frac{a}{x}\right)^{ax}\Rightarrow\frac{dy}{dx}=ax\left(\frac{a}{x}\right)^{ax-1}\frac{d}{dx}\left(\frac{a}{x}\right)+\left(\ln\left(\frac{a}{x}\right)\right)\left(\frac{a}{x}\right)^{ax}\frac{d}{dx}(ax)=-ax\frac{\left(\frac{a}{x}\right)^{ax}}{\frac{a}{x}}\frac{a}{x^2}+a\left(\frac{a}{x}\right)^{ax}\ln\left(\frac{a}{x}\right)$$

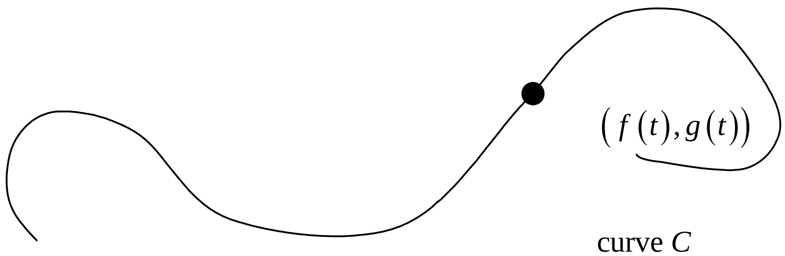
$$=-a\left(\frac{a}{x}\right)^{ax}+a\left(\frac{a}{x}\right)^{ax}\ln\left(\frac{a}{x}\right)$$

2 Differentiation of parametric equations (p.330 – p.336)

A plane curve C is usually described by a pair of equations

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

where $t \in I$ with *f* and *g* continuous on the interval I . t is called the parameter and the above equations are called the parametric equation for the curve C .



Now, to determine the slope of the tangent at a given point $(x(t), y(t))$, we have to calculate the value

$$\left.\frac{dy}{dx}\right|_{(x(t),y(t))}$$

This value can be calculated by the following rule:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Example 18

The parametric equation of a parabola is given by $\begin{cases} x = at^2 \\ y = 2at \end{cases}$.

As $\frac{dx}{dt} = 2at$ and $\frac{dy}{dt} = 2a$, we have

$$\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$$

That is, the slope of the tangent at a given point $(x(t), y(t))$ is equal to 1/*t*.

□

3 Higher Derivatives

The operation of differentiation takes a function $f(x)$ and produces a new function $f'(x)$. If $f'(x)$ is again differentiable, we produce still another function (by differentiation $f'(x)$). It is denoted by $f''(x)$ and called the second derivative of $f(x)$. In general, we can define the n^{th} derivative of $f(x)$ (written as $f^{(n)}(x)$) by $f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'$ for $n \geq 2$, provided that $f^{(n-1)}(x)$ is differentiable.

Note that the n^{th} derivative $f^{(n)}(x)$ can also be written as $\dfrac{d^n y}{dx^n}$ or $D^n y$.

Example 19

Find $D^n\left(x^m\right)$ where *m* is a positive integer.

Solution:

Consider separately the following 3 cases:

(a) $n < m$, $D^n\left(x^m\right) = m(m-1)\cdots(m-n+1)x^{m-n}$

(b) $n = m$, $D^n\left(x^m\right) = m!$

(c) $n > m$, $D^n\left(x^m\right) = 0$

□

Example 20

Find $D^n\left[\sin(ax+b)\right]$ where *a*, *b* are constants and n is a positive integer.

Solution:

$$D\left[\sin(ax+b)\right] = a\cos(ax+b) = a\sin\left(ax+b+\frac{\pi}{2}\right)$$

$$D^2\left[\sin(ax+b)\right] = D\left[a\sin\left(ax+b+\frac{\pi}{2}\right)\right] = a\left[a\sin\left(ax+b+\frac{\pi}{2}+\frac{\pi}{2}\right)\right] = a^2\sin\left(ax+b+2\times\frac{\pi}{2}\right)$$

$$D^3\left[\sin(ax+b)\right] = D\left[a^2\sin\left(ax+b+2\times\frac{\pi}{2}\right)\right] = a^2\left[a\sin\left(ax+b+2\times\frac{\pi}{2}+\frac{\pi}{2}\right)\right] = a^3\sin\left(ax+b+3\times\frac{\pi}{2}\right)$$

and in general

$$D^n\left[\sin(ax+b)\right] = a^n\sin\left(ax+b+\frac{n\pi}{2}\right).$$

□

Example 21

If any $ay^2 + by + c = x$, show that $\frac{d^2y}{dx^2} + 2a\left(\frac{dy}{dx}\right)^3 = 0$.

Solution:

Now, $ay^2 + by + c = x$. Differentiating with respect to x, we get $2ay\frac{dy}{dx} + b\frac{dy}{dx} = 1$. Therefore, $\frac{dy}{dx} = \frac{1}{2ay+b}$. Differentiating again with respect to x, we get $\frac{d^2y}{dx^2} = \frac{-2a}{(2ay+b)^2}\frac{dy}{dx} = \frac{-2a}{(2ay+b)^3}$.

So $\frac{d^2y}{dx^2} + 2a\left(\frac{dy}{dx}\right)^3 = \frac{-2a}{(2ay+b)^3} + 2a\left(\frac{1}{2ay+b}\right)^3 = 0$.

∎

3.1 Leibnitz' rule for higher derivatives of product of two differentiable functions

Leibnitz' rule for higher derivatives of product of differentiable functions says that $\frac{d^n}{dx^n}(fg) = \frac{d^n f}{dx^n}g + n\frac{d^{n-1}f}{dx^{n-1}}\frac{dg}{dx} + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!}\frac{d^{n-k}f}{dx^{n-k}}\frac{d^k g}{dx^k} + \cdots + n\frac{df}{dx}\frac{d^{n-1}g}{dx^{n-1}} + f\frac{d^n g}{dx^n}$, where $k! = k\times(k-1)\times\cdots\times 2\times 1$.

Example 22

Evaluate $\frac{d^{10}}{dx^{10}}\left[(x^2+3)\sin x\right]$.

Solution:

$$\begin{aligned}
\frac{d^{10}}{dx^{10}}\left[(\sin x)(x^2+3)\right] &= (\sin x)^{(10)}(x^2+3) + 10(\sin x)^{(9)}(x^2+3)^{(1)} + \frac{10\times 9}{2!}(\sin x)^{(8)}(x^2+3)^{(2)} \\
&= \sin\left(x+\frac{10\pi}{2}\right)(x^2+3) + 10\sin\left(x+\frac{9\pi}{2}\right)(2x) + 45\sin\left(x+\frac{8\pi}{2}\right)(2) \\
&= -(x^2+3)\sin x + 20x\cos x + 90\sin x \\
&= (87-x^2)\sin x + 20x\cos x.
\end{aligned}$$

∎

Short Table of Derivatives of $y = f(u)$ with respect to x, where u is a function of x

Functions, $y = f(u)$	**Derivative of y with respect to x**
$y = c$, where c is a constant.	$\frac{dy}{dx} = 0$
$y = cu$, where c is a constant.	$\frac{dy}{dx} = c\frac{du}{dx}$
$y = u^p$, where p is a constant.	$\frac{dy}{dx} = pu^{p-1}\frac{du}{dx}$
$y = u + v$	$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = f(u)$, where u is a function of x.	$\frac{dy}{dx} = \frac{df(u)}{du} \cdot \frac{du}{dx}$, the chain rule
$y = \log_a u,\ a > 0$.	$\frac{dy}{dx} = \frac{1}{u}\log_a e\frac{du}{dx}$
$y = a^u,\ a > 0$.	$\frac{dy}{dx} = a^u \log_e a\frac{du}{dx}$
$y = e^u$	$\frac{dy}{dx} = e^u \frac{du}{dx}$
$y = u^v$	$\frac{dy}{dx} = vu^{v-1}\frac{du}{dx} + u^v \log_e u\frac{dv}{dx}$
$y = \sin u$	$\frac{dy}{dx} = \cos u \ \frac{du}{dx}$
$y = \cos u$	$\frac{dy}{dx} = -\sin u \ \frac{du}{dx}$
$y = \tan u$	$\frac{dy}{dx} = \sec^2 u \ \frac{du}{dx}$
$y = \cot u$	$\frac{dy}{dx} = -\mathrm{cosec}^2 u \ \frac{du}{dx}$
$y = \sec u$	$\frac{dy}{dx} = \sec u \tan u \ \frac{du}{dx}$
$y = \mathrm{cosec}\ u$	$\frac{dy}{dx} = -\mathrm{cosec}\ u \ \cot u \ \frac{du}{dx}$
$y = \sin^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \ \frac{du}{dx}$
$y = \cos^{-1} u$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}} \ \frac{du}{dx}$
$y = \tan^{-1} u$	$\frac{dy}{dx} = \frac{1}{1+u^2} \ \frac{du}{dx}$

$y=\cot^{-1}u$	$\frac{dy}{dx}=\frac{-1}{1+u^2}\ \frac{du}{dx}$
$y=\sec^{-1}u$	$\frac{dy}{dx}=\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\ \frac{du}{dx}$
$y=\mathrm{cosec}^{-1}u$	$\frac{dy}{dx}=\frac{-1}{\lvert u\rvert\sqrt{u^2-1}}\ \frac{du}{dx}$
$y=\sinh u$	$\frac{dy}{dx}=\cosh u\ \frac{du}{dx}$
$y=\cosh u$	$\frac{dy}{dx}=\sinh u\ \frac{du}{dx}$
$y=\sinh^{-1}u$	$\frac{dy}{dx}=\frac{1}{\sqrt{1+u^2}}\ \frac{du}{dx}$
$y=\cosh^{-1}u$	$\frac{dy}{dx}=\frac{1}{\sqrt{u^2-1}}\ \frac{du}{dx}$
$y=\tanh^{-1}u$	$\frac{dy}{dx}=\frac{1}{1-u^2}\ \frac{du}{dx}$
$y=\coth^{-1}u$	$\frac{dy}{dx}=\frac{-1}{u^2-1}\ \frac{du}{dx}$

<u>N.B.</u> $\frac{d}{dx}(\sinh u)=\frac{d}{dx}\left(\frac{1}{2}\left(e^u-e^{-u}\right)\right)=\frac{1}{2}\left(e^u+e^{-u}\right)\frac{du}{dx}=\cosh u\ \frac{du}{dx}.$