Problem 2

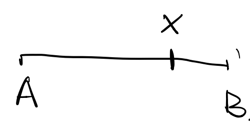
Let $A = (0,1,-1)$ and $B = (1,2,0)$ be two points in a plane. Let X be a point between A and B such that $AX{:}XB = 2{:}1$.

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AX}$.

(b) Hence, find the coordinate of X by finding its position vector $\overrightarrow{OX}$. (Hint: $\overrightarrow{AX} = \overrightarrow{OX} - \overrightarrow{OA}$).

(a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (\vec{i} + 2\vec{j}) - (\vec{j} - \vec{k})$

$= \vec{i} + \vec{j} + \vec{k}$.



$AX : XB = 2:1$.

$|\overrightarrow{AX}| = \frac{2}{3}|\overrightarrow{AB}|$

$\overrightarrow{AX} = \frac{2}{3}|\overrightarrow{AB}| \times |\hat{AB}| = \frac{2}{3}|\overrightarrow{AB}| \times \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{2}{3}(\vec{i} + \vec{j} + \vec{k})$

magnitude × direction

$= \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$.

(b). $\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$

$= (\vec{j} - \vec{k}) + (\frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k})$

$= \frac{2}{3}\vec{i} + \frac{5}{3}\vec{j} - \frac{1}{3}\vec{k}$.

$X(\frac{2}{3}, \frac{5}{3}, -\frac{1}{3})$.

Problem 3

Let $\vec{a} = 2\vec{i} - 3\vec{j} + 5\vec{k}$ and $\vec{b} = \vec{i} + 3\vec{j}$ be two vectors.

(a) Find $|\vec{a}|$ and $|\vec{a} - 2\vec{b}|$.

(b) Find the unit vector of $\vec{b}$.

(c) Let $\vec{c}$ be another vector with magnitude $|2\vec{a} + \vec{b}|$ and its direction is same as that of $\vec{b}$. Find the vector $\vec{c}$.

(a) $|\vec{a}| = \sqrt{2^2 + (-3)^2 + 5^2} = \sqrt{4+9+25} = \sqrt{38}$.

$|\vec{a} - 2\vec{b}| = |2\vec{i} - 3\vec{j} + 5\vec{k} - 2(\vec{i} + 3\vec{j})|$

$= |-9\vec{j} + 5\vec{k}|$

$= \sqrt{(-9)^2 + 5^2} = \sqrt{106}$

(b) $\hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{\vec{i} + 3\vec{j}}{\sqrt{1^2 + 3^2}} = \frac{1}{\sqrt{10}}\vec{i} + \frac{3}{\sqrt{10}}\vec{j}$.

(c). $\vec{c} = |2\vec{a}+\vec{b}| \times \hat{b} = \sqrt{134} \times \left(\frac{1}{\sqrt{10}}\vec{i} + \frac{3}{\sqrt{10}}\vec{j}\right) = \frac{\sqrt{134}}{\sqrt{10}}\vec{i} + \frac{3\sqrt{134}}{\sqrt{10}}\vec{j}$

magnitude × direction.

$\vec{a} = 2\vec{i} - 3\vec{j} + 5\vec{k}$ and $\vec{b} = \vec{i} + 3\vec{j}$

$|2\vec{a}+\vec{b}| = |4\vec{i} - 6\vec{j} + 10\vec{k} + \vec{i} + 3\vec{j}|$

$= |5\vec{i} - 3\vec{j} + 10\vec{k}| = \sqrt{5^2 + (-3)^2 + 10^2} = \sqrt{134}$

Problem 4

Let $\vec{a} = \vec{i} + 3\vec{j} - 2\vec{k}$ and $\vec{b} = -2\vec{i} + \vec{j} + 3\vec{k}$ be two vectors.

(a) Find $\vec{a} \cdot \vec{b}$.

(b) Find the angle between the vectors $\vec{a}$ and $\vec{b}$.

(c) Let $\vec{c} = 3\vec{i} + x\vec{j} - 2\vec{k}$ be a vector which is perpendicular to $\vec{b}$, find the value of x.

(d) Let $\vec{d} = y\vec{a} + 3\vec{b}$ be a vector which is perpendicular to $\vec{a} - \vec{b}$, find the value of y.

(a). $\vec{a} \cdot \vec{b} = 1 \cdot (-2) + 3 \cdot 1 + (-2) \cdot 3 = -5$.

(b). $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cdot \cos\theta$. $\rightarrow$ θ is the angle between $\vec{a}$ and $\vec{b}$.

$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{-5}{\sqrt{1^2+3^2+(-2)^2} \cdot \sqrt{(-2)^2+1^2+3^2}} = \frac{-5}{\sqrt{14} \cdot \sqrt{14}} = \frac{-5}{14}$.

$\cos\theta < 0$, $\theta > 90°$. $\theta \simeq 110.92°$.

(c) $\vec{c} \cdot \vec{b} = |\vec{c}||\vec{b}| \cdot \underbrace{\cos 90°}_{=0} = 0$.

$\vec{a} = \vec{i} + 3\vec{j} - 2\vec{k}$ and $\vec{b} = -2\vec{i} + \vec{j} + 3\vec{k}$

$\vec{c} \cdot \vec{b} = 3 \cdot (-2) + x \cdot (1) + (-2) \cdot 3 = 0$.

$x = 12$.

(d) $\vec{d} \cdot (\vec{a} - \vec{b}) = 0$.

$(y\vec{a} + 3\vec{b}) \cdot (\vec{a} - \vec{b}) = 0$

$y(\vec{a} \cdot \vec{a}) - y(\vec{a} \cdot \vec{b}) + 3(\vec{a} \cdot \vec{b}) - 3(\vec{b} \cdot \vec{b}) = 0$

$y|\vec{a}|^2 + (3-y)(\vec{a} \cdot \vec{b}) - 3|\vec{b}|^2 = 0$.

$14y + (3-y) \cdot (-5) - 3 \times 14 = 0$. $\Rightarrow$ $y = 3$.

Problem 8

Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{a}| = 1$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = 1$.

(a) Find the angle between the vectors $\vec{a}$ and $\vec{b}$.

(b) Find the value of $(3\vec{a} - 2\vec{b}) \cdot (\vec{a} + 3\vec{b})$ and $|\vec{a} - 2\vec{b}|$.

(c) Find the angle between two vectors $\vec{a} - 2\vec{b}$ and $2\vec{a} + 3\vec{b}$.

(a) $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \dfrac{1}{1\cdot 2} = \dfrac{1}{2} \Rightarrow \theta = 60^\circ$.

(b). $(3\vec{a} - 2\vec{b}) \cdot (\vec{a} + 3\vec{b})$

$= 3\cdot(\vec{a}\cdot\vec{a}) + 9(\vec{a}\cdot\vec{b}) - 2(\vec{a}\cdot\vec{b}) - 6(\vec{b}\cdot\vec{b})$

$= 3|\vec{a}|^2 + 7(\vec{a}\cdot\vec{b}) - 6|\vec{b}|^2$

$= 3\cdot 1 + 7\cdot 1 - 6\cdot 4 = -14.$

$\vec{a}\cdot\vec{a} = |\vec{a}|^2$

$|\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}}$.

$|\vec{a} - 2\vec{b}| = \sqrt{(\vec{a}-2\vec{b})\cdot(\vec{a}-2\vec{b})}$

$= \sqrt{\vec{a}\cdot\vec{a} - 4(\vec{a}\cdot\vec{b}) + 4(\vec{b}\cdot\vec{b})}$

$= \sqrt{|\vec{a}|^2 - 4(\vec{a}\cdot\vec{b}) + 4|\vec{b}|^2} = \sqrt{1 - 4 + 4\times 4} = \sqrt{13}$

(c) $\cos\theta = \dfrac{(\vec{a}-2\vec{b})\cdot(2\vec{a}+3\vec{b})}{|\vec{a}-2\vec{b}|\cdot|2\vec{a}+3\vec{b}|}$ (*)

$(\vec{a}-2\vec{b})\cdot(2\vec{a}+3\vec{b}) = 2(\vec{a}\cdot\vec{a}) + 3(\vec{a}\cdot\vec{b}) - 4(\vec{a}\cdot\vec{b}) - 6(\vec{b}\cdot\vec{b})$

$= 2|\vec{a}|^2 - (\vec{a}\cdot\vec{b}) - 6|\vec{b}|^2 = 2 - 1 - 6\times 4 = -23.$

$|2\vec{a}+3\vec{b}| = \sqrt{(2\vec{a}+3\vec{b})\cdot(2\vec{a}+3\vec{b})}$

$= \sqrt{4(\vec{a}\cdot\vec{a}) + 12(\vec{a}\cdot\vec{b}) + 9(\vec{b}\cdot\vec{b})}$

$= \sqrt{4\underbrace{|\vec{a}|^2}_{1} + 12\underbrace{(\vec{a}\cdot\vec{b})}_{1} + 9\underbrace{|\vec{b}|^2}_{4}} = \sqrt{4+12+36} = \sqrt{52}$

$\cos\theta = \dfrac{(\vec{a}-2\vec{b})\cdot(2\vec{a}+3\vec{b})}{|\vec{a}-2\vec{b}|\cdot|2\vec{a}+3\vec{b}|} = \dfrac{-23}{\sqrt{13}\cdot\underbrace{\sqrt{52}}_{13\times 4}} = \dfrac{-23}{13\times 2} = \dfrac{-23}{26}$

$\theta \approx 152.2^\circ$.

Problem 9

Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{a}| = 2$ and $|\vec{b}| = 3$ and the angle between these two vectors is $\cos^{-1}\frac{3}{5}$. $= \theta$.

(a) Are the vector $\vec{a} - 2\vec{b}$ and $-9\vec{a} + 2\vec{b}$ perpendicular to each other? Explain your answer.

(b) If the angle between the vectors $\vec{a}$ and $\vec{a} + k\vec{b}$ is $60°$, find the value of k.

(a) if $(\vec{a}-2\vec{b})$ and $-9\vec{a}+2\vec{b}$ are perpendicular,

$(\vec{a}-2\vec{b})\cdot(-9\vec{a}+2\vec{b}) = 0$.

$(\vec{a}-2\vec{b})\cdot(-9\vec{a}+2\vec{b})$

$= -9|\vec{a}|^2 + 2(\vec{a}\cdot\vec{b}) + 18(\vec{a}\cdot\vec{b}) - 4|\vec{b}|^2$

$= -9|\vec{a}|^2 + 20(\vec{a}\cdot\vec{b}) - 4|\vec{b}|^2$.

$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\theta = 2\cdot3\cdot\left(\frac{3}{5}\right) = \frac{18}{5}$.

$= -9\times4 + 20\times\frac{18}{5} - 4\times9 = 0$.

36 $\quad$ 4×18. $\quad$ −36

$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$|\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}}$.

(b) $\cos 60^\circ = \frac{1}{2}$.

$\cos 60^\circ = \frac{\vec{a}\cdot(\vec{a}+k\vec{b})}{|\vec{a}|\cdot|\vec{a}+k\vec{b}|} = \frac{1}{2}$. $\quad (*)$

$\vec{a}\cdot(\vec{a}+k\vec{b}) = \vec{a}\cdot\vec{a} + k\,\vec{a}\cdot\vec{b} = 4 + \frac{18}{5}k$

$|\vec{a}| = 2$

$|\vec{a}+k\vec{b}| = \sqrt{(\vec{a}+k\vec{b})\cdot(\vec{a}+k\vec{b})}$

$= \sqrt{|\vec{a}|^2 + 2k(\vec{a}\cdot\vec{b}) + k^2|\vec{b}|^2}$

$= \sqrt{4 + \frac{36}{5}k + 9k^2}$

$$\frac{4+\frac{18}{5}k}{2\sqrt{4+\frac{36}{5}k+9k^2}} = \frac{1}{2} \Rightarrow 99k^2+540k+300=0.$$

$$k = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$k = \frac{-540 \pm \sqrt{540^2-4\times 300\times 99}}{2\times 99}.$$

$$\Rightarrow k = \pm\frac{40\sqrt{3}}{33} - \frac{30}{11}.$$