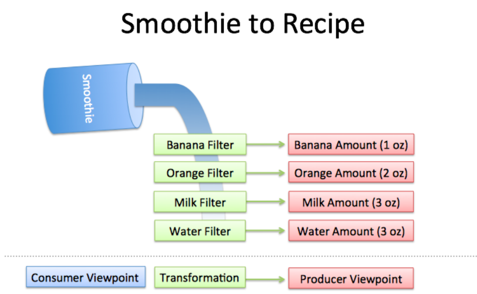
3 Fourier Series

Major References:

- *Chapter 3, Signals and Systems by Alan V. Oppenheim et. al., 2nd edition, Prentice Hall*
- *Chapter 5.2 & 6.2, Schaum's Outline of Signals and Systems, 2nd Edition, 2010, McGraw-Hill*

3.1 Introduction

Jean-Baptiste Joseph Fourier (1768-1830)



1st Metaphor of the Fourier Analysis (Source: https://betterexplained.com)

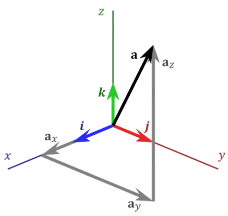
What is Fourier Analysis?

1. **What does the Fourier Transform do?** Given a smoothie, it finds the recipe.
2. **How?** Run the smoothie through filters to extract each ingredient.
3. **Why?** Recipes are easier to analyze, compare, and modify than the smoothie itself.
4. **How do we get the smoothie back?** Blend the ingredients.

Important Points to consider

1. **Filters must be independent**
2. **Filters must be complete**
3. **Ingredients must be combineable.** The ingredients must make the same result when separated and combined in any order.

2nd **Metaphor of the Fourier Analysis**



$$\mathbf{a} = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z$$

- Arbitrary vector can be expressed via unit vectors and the magnitude toward each unit vector.
- Can we break a function into it simple functions (referred to as base functions) just like vector case?
- Can we combine the base functions to represent arbitrary signals?

$$\Rightarrow x(t) = \sum_{n=-\infty}^{\infty} a_n\psi_n(t), \quad \text{where } \psi_n(t) \text{ is the base function.}$$

1. Fourier analysis is the study of the way general functions may be represented or approximated by sums of simpler trigonometric functions.
   - Analysis: breaking up a signal into simpler constituent parts
   - Synthesis: reassembling a signal from its constituent parts

   $\Rightarrow$ **Fourier analysis is all about breaking & reassembling a function**
2. Fourier Series: fourier analysis for periodic signals
3. Fourier Transform: fourier analysis for non-periodic signals

	Periodic signal	Aperiodic Signal
Continuous Time	Fourier Series (FS)	Fourier Transform (FT)
Discrete Time	Discrete time FS	Discrete time FT

3.2 Continuous Time Fourier Series

For a periodic signal $x(t)$ with fundamental period T_0, we adopt sinusoidal signals as the base function

$$\left\{ \begin{array}{l} \text{fundamental period:} T_0, \quad \text{fundamental frequency:} f_0 = \frac{1}{T_0}, \\ \text{fundamental angular frequency:} w_0 = 2\pi f_0 = \frac{2\pi}{T_0} \end{array} \right\}$$

Then the CT-Fourier series can be expressed into the following two representations. All of the proof for Chapter 3.2 are summarized at the end of the section.

1. Fourier Series (Complex Exponential Series Form)

The base function for this form is $\psi_k(t) = e^{jkw_0t} = e^{j2\pi kf_0t}$

1. Synthesis:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \psi_k(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkw_0t} \tag{3.1}$$

2. Analysis

$$c_k = \frac{1}{T_0} \int_{\mathbf{T}_0} x(t) e^{-jkw_0t} dt = \frac{1}{T_0} \int_{\mathbf{T}_0} x(t) \psi_k^*(t) dt, \tag{3.2}$$

where the integration interval $\mathbf{T}_0$ is any period with length T_0, *e.g.*, $[0, T_0]$ or $\left[-\frac{T_0}{2}, \frac{T_0}{2}\right]$

[Properties]

1. The set of base functions $\{\psi_k(t)\}$ is orthogonal on any interval over a period T_0, $(\alpha, \alpha + T_0)$

$$\int_{\alpha}^{\alpha+T_0} \psi_m(t) \psi_k^*(t) dt = \begin{cases} 0, & m \neq k \\ T_0, & m = k \end{cases} \tag{3.3}$$

2. If $x(t)$ is a real funcction, then $c_{-k} = c_k^*$

By using the Euler's Formula, $e^{jkw_0t} = \cos(kw_0t) + \sin(kw_0t)$, the Fourier series in the complex exponential series form can be converted to a trigonometric series form as follows.

2. Fourier Series (Trigonometric Series Form)

1. Synthesis

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(kw_0t) + b_k \sin(kw_0t)\right) \tag{3.4}$$

2. Analysis

$$a_k = \frac{2}{T_0}\int_{\mathrm{T}_0} x(t)\cos(kw_0t)\,dt, \quad b_k = \frac{2}{T_0}\int_{\mathrm{T}_0} x(t)\sin(kw_0t)\,dt \tag{3.5}$$

[Properties]

1. The convertion between two representations

$$\begin{cases} \dfrac{a_0}{2} = c_0, & \\ a_k = c_k + c_{-k}, & b_k = j\left(c_k - c_{-k}\right) \end{cases} \Leftrightarrow \begin{cases} c_k = \dfrac{1}{2}\left(a_k - jb_k\right), \\ c_{-k} = \dfrac{1}{2}\left(a_k + jb_k\right), \end{cases} \tag{3.6}$$

2. If $x(t)$ is a real funcction, then $a_k = 2\,\mathrm{Re}\,[c_k]$, $b_k = -2\,\mathrm{Im}\,[c_k]$.

A periodic signal $x(t)$ has a Fourier series representation if it satisfies the Dirichlet conditions. In other words, Dirichlet conditions are the sufficient conditions (but not necessary condition) for the Fourier series to converge.

Peter Gustav Lejeune Dirichlet (1805-1859)

3. Dirichlet Condition (Sufficient conditions for FS to exist)

1. $x(t)$ is absolutely integrable over any period $\int_{\mathrm{T}_0} |x(t)|dt < \infty$
2. $x(t)$ has a finite number of maxima and minima within any finite interval of t.
3. $x(t)$ has a finite number of discontinuities within any finite interval of t, and each of these discontinuities is finite.

If $x(t)$ satisfies the Dirichlet condition, then the corresponding Fourier series is convergent and its sum is $x(t)$, except at any point t_0 at which $x(t)$ is discontinuous.

$$x(t_0) = \frac{1}{2}\left[x\left(t_0^+\right) + x\left(t_0^-\right)\right]$$

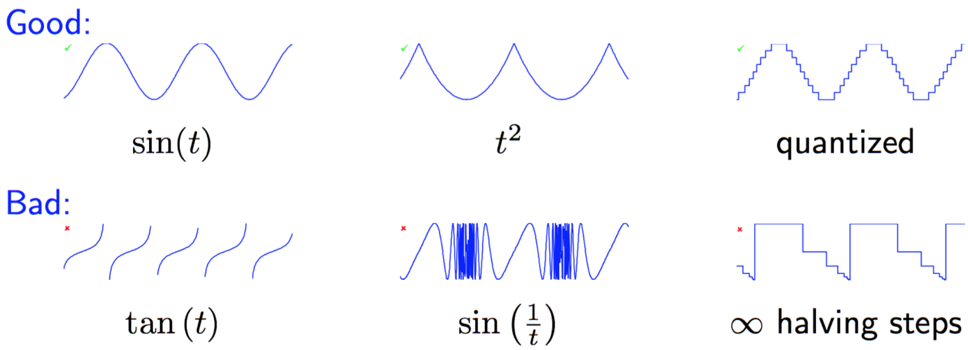


Figure 3.1: Examples of the Dirichlet Condition

4. Parseval's Theorem

Average signal power can be calculated by integral over time domain or infinite sum over frequency domain.

$$\mathbf{P} = \frac{1}{T_0}\int_{\mathrm{T}_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2 \tag{3.7}$$

[Proof of Chapter 3.2]

1. Prove that the set $\{\psi_k(t)\} = \{e^{jkw_0t}\}$ of base functions is orthogonal.

   Proof) Refer [Schaum's text, Problem 5.1]

   $$\int_{\alpha}^{\alpha+T_0} e^{jmw_0T}e^{-jkw_0T}dt = \int_{\alpha}^{\alpha+T_0} e^{j(m-k)w_0T}dt$$

   If $m = k$, then the complex exponential in the integral will be one ($e^0 = 1$) and the integration results will be T_0. If $m \neq k$, then denote $m - k = l$, which is a nonzero integer, and the integral can be further expressed as

   $$\int_{\alpha}^{\alpha+T_0} e^{jlw_0T}dt = \frac{1}{jlw_0}e^{jlw_0t}\Big|_{\alpha}^{\alpha+T_0} = \frac{e^{\alpha lw_0 l}}{jlw_0}\left[e^{j2\pi l} - 1\right] = 0, \tag{3.8}$$

   where the second equality follows by $w_0 = 2\pi/T_0$.

2. Derive $c_k = \frac{1}{T_0}\int_{\mathrm{T}_0} x(t)\psi_k^*(t)dt$ using the orthogonality condition of $\{\psi_k(t)\}$.

   Proof) Refer [Schaum's text, Problem 5.2]

   $$\begin{aligned}\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)\psi_k^*(t)dt &= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0}\left[\sum_{m=-\infty}^{\infty} c_m\psi_m(t)\right]\psi_k^*(t)dt \\ &= \frac{1}{T_0}\sum_{m=-\infty}^{\infty} c_m \underbrace{\int_{\alpha}^{\alpha+T_0}\psi_m(t)\psi_k^*(t)dt}_{=T_0\delta_{m,k}} = c_k,\end{aligned} \tag{3.9}$$

   where $\delta_{m,k} = 1$ if $m = k$ and $\delta_{m,k} = 0$ if $m \neq k$.

3. Derive the trigonometric series representation of FS from the complex exponential form.

   Proof) Apply the Euler Formula $e^{jwt} = \cos(wt) + j\sin(wt)$ into the complex exponential FS formula as follows. *Refer [Schaum's text, Problem 5.3].*

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t} = c_0 + \sum_{k=1}^{\infty}\left[c_k e^{jkw_0 t} + c_{-k}e^{-jkw_0 t}\right] \\ &= c_0 + \sum_{k=1}^{\infty}\left[(c_k + c_{-k})\cos(kw_0 t) + j(c_k - c_{-k})\sin(kw_0 t)\right] \\ &= \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos(kw_0 t) + b_k\sin(kw_0 t)\right), \end{aligned} \tag{3.10}$$

   where $\frac{a_0}{2} = c_0$, $a_k = c_k + c_{-k}$, and $b_k = j(c_k - c_{-k})$.

4. Derive the Parseval's theorem in (3.7).

   Proof) Let's evaluate the power of a periodic signal $x(t)$. *Refer [Schaum's text, Problem 5.14].*

$$\begin{aligned} \mathbf{P} &= \frac{1}{T_0}\int_{\mathrm{T}_0}|x(t)|^2 dt = \frac{1}{T_0}\int_{\mathrm{T}_0} x(t)x^*(t)dt \\ &= \frac{1}{T_0}\int_{\mathrm{T}_0}\left\{\sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t} \times \sum_{m=-\infty}^{\infty} c_m^* e^{-jmw_0 t}\right\}dt \quad \left(\text{Applied } x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t}\right) \\ &= \frac{1}{T_0}\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_k c_m^* \underbrace{\int_{\mathrm{T}_0} e^{jkw_0 t}e^{-jmw_0 t}dt}_{=T_0\delta_{k,m}} = \sum_{k=-\infty}^{\infty}|c_k|^2 \end{aligned} \tag{3.11}$$

[Example 3-1] Derive the complex exponential FS representation of the following signals
Refer [Schaum's text, Problem 5.4]

a) $x(t) = \cos(w_0 t)$

b) $x(t) = \sin(w_0 t)$

c) $x(t) = \cos(4t) + \sin(6t)$

d) $x(t) = \sin^2(t)$

Solution) For sinusoidal functions, we can get the FS by simply expanding it in terms of the complex exponentials.

$$\begin{aligned} \cos(w_0 t) &= \frac{1}{2}\left(e^{jw_0 t} + e^{-jw_0 t}\right), \quad \sin(w_0 t) = \frac{1}{2j}\left(e^{jw_0 t} - e^{-jw_0 t}\right), \\ \cos(4t) + \sin(6t) &= \frac{1}{2}\left(e^{j4t} + e^{-j4t}\right) + \frac{1}{2j}\left(e^{j6t} - e^{-j6t}\right), \\ \sin^2(t) &= \frac{1}{2}(1 - \cos 2t) = \frac{1}{2} - \frac{1}{4}e^{j2t} - \frac{1}{4}e^{-j2t} \end{aligned} \tag{3.12}$$

Hence, the complex Fourier coefficients for each signals are derived as follows.

(a) $c_1 = \frac{1}{2}, c_{-1} = \frac{1}{2}$, and $c_k = 0$ for other k index.

(b) $c_1 = \frac{1}{2j}, c_{-1} = -\frac{1}{2j}$, and $c_k = 0$ for other k index.

(c) $c_{-3} = -\frac{1}{2j}, c_{-2} = \frac{1}{2}, c_2 = \frac{1}{2}, c_3 = \frac{1}{2j}$, and $c_k = 0$ for other k index.

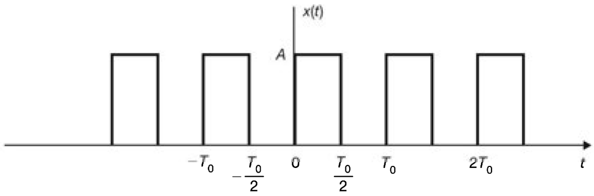
(d) $c_1 = -\frac{1}{4}, c_{-1} = -\frac{1}{4}, c_0 = \frac{1}{2}$, and $c_k = 0$ for other k index.

[Example 3-2] Determine the complex and trigonometric FS representation of the following signals. *Refer [Schaum's text, Problem 5.5, 5.6, 5.7]*

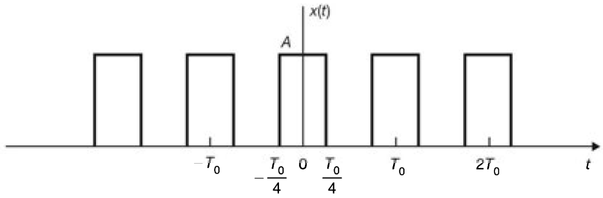
a) Case (a)

b) Case (b)

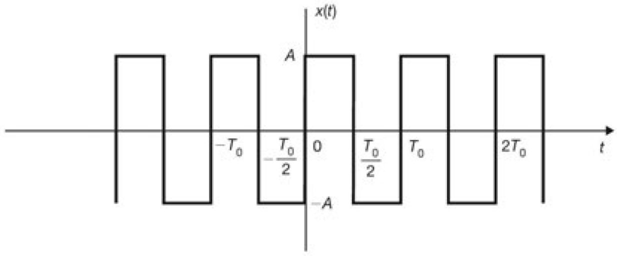
c) Case (c)



Case (a)



Case (b)



Case (c)

Solution) For (a), the Complex Fourier coefficients can be derived as follows

$$
\begin{aligned}
c_0 &= \frac{A}{T_0}\int_0^{\frac{T_0}{2}} dt = \frac{A}{2} \\
c_k &= \frac{A}{T_0}\int_0^{\frac{T_0}{2}} e^{-jkw_0 t}dt = \frac{A}{jkw_0T_0}e^{-jkw_0 t}\Big|_{\frac{T_0}{2}}^{0} = \frac{A}{jkw_0T_0}\left[1-e^{-jk\pi}\right] = \frac{A}{j2\pi k}\left[1-(-1)^k\right] \\
&= \begin{cases} 0 & \text{for even } k = 2m, \\ \dfrac{A}{j\pi\,(2m+1)} & \text{for odd } k = 2m+1 \end{cases}
\end{aligned}
\tag{3.13}
$$

The trigonometric Fourier coefficients can be derived as follows.

$$
\begin{aligned}
a_0 &= 2c_0 = A, \quad a_k = 2\,\mathrm{Re}\,[c_k] = 0, \\
b_k &= -2\,\mathrm{Im}\,[c_k] = \begin{cases} 0 & \text{for even } k = 2m, \\ \dfrac{2A}{(2m+1)\,\pi} & \text{for odd } k = 2m+1 \end{cases}
\end{aligned}
\tag{3.14}
$$

The signals $x_b(t)$ for case (b) is a time-shifted version of the case (a), *i.e.*, $x_b(t) = x_a\left(t+\frac{T_0}{4}\right)$. Then, the Fourier coefficient of the time-shifted signal is derived as follows

$$
\frac{1}{T_0}\int_{T_0} x\left(t+\frac{T_0}{4}\right)e^{-jkw_0 t}dt = \frac{1}{T_0}e^{jkw_0\frac{T_0}{4}}\int_{T_0} x(\tau)\,e^{-jkw_0\tau}dt = c_k e^{jk\frac{\pi}{2}} = j^k c_k,
\tag{3.15}
$$

where we used a change of variable, *i.e.*, $\tau = t + \frac{T_0}{4}$. Hence, the Fourier coefficients of case (b) are given by

$$\begin{aligned}
c_0 &= \frac{A}{2} \times j^0 = \frac{A}{2} \\
c_k &= \begin{cases} 0 & \text{for even } k = 2m, \\ \dfrac{A \times j^{2m+1}}{j\pi\,(2m+1)} = \dfrac{A\,(-1)^m}{\pi\,(2m+1)} & \text{for odd } k = 2m+1 \end{cases} \\
a_0 &= 2c_0 = A, \quad b_k = -2\,\mathrm{Im}\,[c_k] = 0, \\
a_k &= 2\,\mathrm{Re}\,[c_k] = \begin{cases} 0 & \text{for even } k = 2m, \\ \dfrac{2A\,(-1)^m}{\pi\,(2m+1)} & \text{for odd } k = 2m+1 \end{cases}
\end{aligned} \tag{3.16}$$

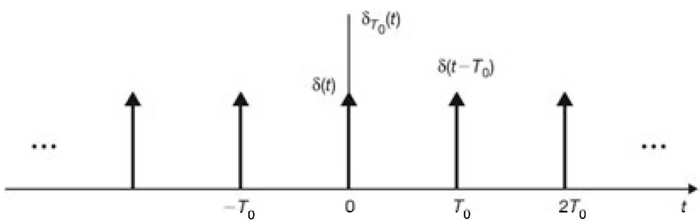
The signals $x_c(t)$ for case (c) is a scaled version of the case (a), *i.e.*, $x_c(t) = 2x_a(t) - A$. Then, the Fourier coefficient of the case (c) is derived as follows

$$\begin{aligned}
c_0 &= \frac{1}{T_0}\int_{\mathrm{T}_0} (2x_a(t) - A)\,dt = 2\,[c_0 \text{ for case (a)}] - A \\
c_k &= \frac{1}{T_0}\int_{\mathrm{T}_0} (2x_a(t) - A)\,e^{-jk w_0 t}dt = 2\,[c_k \text{ for case (a)}]\,,
\end{aligned} \tag{3.17}$$

where $\int_{\mathrm{T}_0} e^{-jk w_0 t}dt = 0$. Hence, the Fourier coefficients of case (c) are given by

$$\begin{aligned}
&c_0 = 2 \times \frac{A}{2} - A = 0, \quad c_k = \begin{cases} 0 & \text{for even } k = 2m, \\ \dfrac{2A}{j\pi\,(2m+1)} & \text{for odd } k = 2m+1 \end{cases}, \\
&a_0 = 2c_0 = 0, \quad a_k = 2\,\mathrm{Re}\,[c_k] = 0, \\
&b_k = -2\,\mathrm{Im}\,[c_k] = \begin{cases} 0 & \text{for even } k = 2m, \\ \dfrac{4A}{(2m+1)\,\pi} & \text{for odd } k = 2m+1 \end{cases}
\end{aligned} \tag{3.18}$$

[Example 3-3] Determine the complex and trigonometric FS representation of the periodic impulse trains $\delta_{T_0}(t)$ signals, which is defined by $\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$. *Refer [Schaum's text, Problem 5.8]*



Solution) The Complex Fourier coefficients can be derived as follows

$$c_0 = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \delta(t)dt = \frac{1}{T_0}, \quad c_k = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sum_{k=-\infty}^{\infty} \delta(t - kT_0)\,e^{-jk w_0 t}dt = \frac{1}{T_0} \tag{3.19}$$

The trigonometric Fourier coefficients can be derived as follows.

$$a_0 = 2c_0 = \frac{2}{T_0}, \quad a_k = 2\,\mathrm{Re}\,[c_k] = \frac{2}{T_0}, \quad b_k = -2\,\mathrm{Im}\,[c_k] = 0 \tag{3.20}$$

Hence, we get

$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta\,(t - kT_0) = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} e^{jkw_0 t} = \frac{1}{T_0} + \frac{2}{T_0}\sum_{k=1}^{\infty}\cos\,(kw_0 t)$$

[Example 3-4] Determine the complex and trigonometric FS representation of the periodic signal $x(t)$ defined by

$$x(t) = t^2, \quad -\pi < t < \pi \quad \text{and} \quad x\,(t + 2\pi) = x(t).$$

Refer [Schaum's text, Problem 5.62]

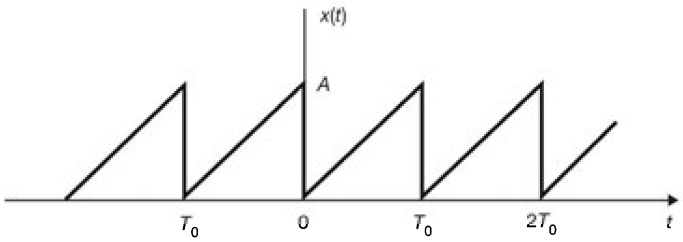
Solution) For the given signal, $T_0 = 2\pi$ and $w_0 = \frac{2\pi}{T_0} = 1$. Then, the Fourier coefficients can be derived as

$$\begin{aligned}
c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 dt = \frac{\pi^2}{3}, \\
c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 e^{-jkt} dt = \frac{1}{2\pi}\left[\frac{t^2 e^{-jkt}}{jk}\bigg|_{\pi}^{-\pi} + \frac{2}{jk}\int_{-\pi}^{\pi} t e^{-jkt} dt\right] \\
&= \frac{\pi}{k}\sin\,(\pi kt) + \frac{1}{jk\pi}\left[\frac{te^{-jkt}}{jk}\bigg|_{\pi}^{-\pi} + \frac{1}{(jk\pi)^2} e^{-jkt}\bigg|_{\pi}^{-\pi}\right] \\
&= \frac{2\cos\,(k\pi t)}{k^2} = \frac{2\,(-1)^k}{k^2}, \\
a_0 &= 2c_0 = \frac{2\pi^2}{3}, \quad b_k = -2\,\mathrm{Im}\,[c_k] = 0, \\
a_k &= 2\,\mathrm{Re}\,[c_k] = \frac{4\,(-1)^k}{k^2}
\end{aligned} \tag{3.21}$$

[Example 3-5] Determine the complex and trigonometric FS representation of the periodic signal $x(t)$ defined by

$$x(t) = \frac{A}{T_0}t, \quad 0 < t < T_0 \quad \text{and} \quad x\,(t + T_0) = x(t).$$

Refer [Schaum's text, Problem 5.63]



Solution) The Complex Fourier coefficients can be derived as follows

$$
\begin{aligned}
c_0 &= \frac{A}{T_0^2}\int_0^{T_0} t dt = \frac{A}{2}, \\
c_k &= \frac{A}{T_0^2}\int_0^{T_0} te^{-jkw_0 t} dt = \frac{A}{T_0^2}\left[\left.\frac{te^{-jkw_0 t}}{jkw_0}\right|_{T_0}^{0}\right] + \frac{A}{jkw_0 T_0^2}\int_0^{T_0} e^{-jkw_0 t} dt \\
&= \frac{Aj}{2\pi k} + \frac{A}{\cancel{(jkw_0 T_0)^2}}\cancelto{0}{\left[1 - e^{-j2k\pi}\right]} = \frac{Aj}{2\pi k}
\end{aligned}
\tag{3.22}
$$

The trigonometric Fourier coefficients can be derived as follows.

$$
a_0 = 2c_0 = A, \quad a_k = 2\,\mathrm{Re}\,[c_k] = 0, \quad b_k = -2\,\mathrm{Im}\,[c_k] = -\frac{A}{\pi k} \tag{3.23}
$$

※ **Properties of Fourier Series**. (*Refer [Oppenheim], Chapter 3.5 for detailed proof*)

1. **Linear Property**
   If the FS coefficients of $x_1(t)$ and $x_2(t)$ are $c_{1,k}$ and $c_{2,k}$, then the FS coefficients of $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ are

   $$\alpha_1 x_1(t) + \alpha_2 x_2(t) \quad \leftrightarrow \quad \alpha_1 c_{1,k} + \alpha_2 c_{2,k}$$

2. **Time Shifting**
   If $x(t)$ is a CT periodic signal with period T_0 and FS coefficients c_k, the FS coefficients of $x(t - t_0)$ are

   $$x(t - t_0) \quad \leftrightarrow \quad e^{-jkw_0 t_0} c_k$$

   Proof) The FS coefficients of the delayed signal are given by

   $$\frac{1}{T_0}\int_{\mathrm{T}_0} x(t - t_0)\, e^{-jkw_0 t} dt = \left\{\frac{1}{T_0}\int_{\mathrm{T}_0} x(\tau)\, e^{-jkw_0 \tau} d\tau\right\} \times e^{-jkw_0 t_0} = e^{-jkw_0 t_0}\, c_k,$$

   where we used a change of variable $(t - t_0 = \tau)$ in the first equality and (3.2) in the second equality.

3. **Conjugate Property**
   If $x(t)$ is a CT periodic signal with period T_0 and FS coefficients c_k, the FS coefficients of $x^*(t)$ are given by

   $$x^*(t) \quad \leftrightarrow \quad c_{-k}^*$$

   If $x(t)$ is a real-valued signal, then $c_k = c_{-k}^*$ follows by the conjugate property.
   Proof) By taking the conjugate of (3.2), we get

   $$c_k^* = \frac{1}{T_0}\int_{\mathrm{T}_0} x(t)\, e^{jkw_0 t} dt, \quad \Rightarrow \quad c_{-k}^* = \frac{1}{T_0}\int_{\mathrm{T}_0} x(t)\, e^{-jkw_0 t} dt,$$

   where the second expression follows by substituting k to $-k$. Then, by comparing (3.2) to the last expression, it follows that $\left\{c_{-k}^*\right\}$ are the FS coefficients of the conjuget signal $x^*(t)$.

4. **Frequency Shifting**
   If $x(t)$ is a CT periodic signal with period T_0 and FS coefficients c_k, the FS coefficients of $e^{jk_0 w_0 t} x(t)$ are

   $$e^{jk_0 w_0 t} x(t) \quad \leftrightarrow \quad c_{k-k_0}$$

Proof) Based on (3.2), the Fourier series coefficients of $e^{jk_0 w_0 t}x(t)$ are given by

$$\frac{1}{T_0}\int_{\mathrm{T}_0} e^{jk_0 w_0 t} x\left(t\right) e^{-jkw_0 t} dt = \frac{1}{T_0}\int_{\mathrm{T}_0} x\left(t\right) e^{-j(k-k_0)w_0 t} dt = c_{k-k_0}$$

5. **Time Reversal Property**
   If $x(t)$ is a CT periodic signal with period T_0 and FS coefficients c_k, the FS coefficients of $x\left(-t\right)$ are given by

$$x\left(-t\right) \quad \leftrightarrow \quad c_{-k}$$

6. **Time Scaling Property**
   If $x(t)$ is a CT periodic signal with period T_0 and FS coefficients c_k, the FS coefficients of $x\left(\alpha t\right)$, $\alpha > 0$, are

$$x\left(\alpha t\right) \quad \leftrightarrow \quad c_k \quad \text{with period } \frac{T_0}{\alpha}$$

   Proof) Since the period of $x\left(\alpha t\right)$ reduces to $\widehat{T}_0 = \frac{T_0}{\alpha}$, the Fourier series coefficients of $x\left(\alpha t\right)$ are

$$\frac{1}{\widehat{T}_0}\int_{\widehat{T}_0} x\left(\alpha t\right) e^{-jk\frac{2\pi}{\widehat{T}_0}t} dt = \frac{\alpha}{T_0}\int_{\frac{T_0}{\alpha}} x\left(\alpha t\right) e^{-jk\frac{2\pi\alpha}{T_0}t} dt = \frac{1}{T_0}\int_{T_0} x\left(\tau\right) e^{-jk\frac{2\pi}{T_0}\tau} d\tau = c_k,$$

   where we used a change of variable ($t = \alpha\tau$) in the second equality.

7. **Differentiation**
   If $x(t)$ is a CT periodic signal with period T_0 and FS coefficients c_k, the FS coefficients of $\frac{dx(t)}{dt}$ are

$$\frac{dx(t)}{dt} \quad \leftrightarrow \quad jkw_0 c_k$$

   Proof) By differentiating (3.1), we get

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t} \quad \rightarrow \quad \frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} jkw_0 c_k e^{jkw_0 t}$$

8. **Integration**
   If $x(t)$ is a CT periodic signal with period T_0 and FS coefficients c_k, the FS coefficients of $\int x(t)dt$ are

$$\int x(t)dt \quad \leftrightarrow \quad \frac{c_k}{jkw_0}$$

   Proof) By integrating (3.1), we get

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkw_0 t} \quad \rightarrow \quad \int x(t)dt = \sum_{k=-\infty}^{\infty} \frac{c_k}{jkw_0} e^{jkw_0 t}$$

9. **Periodic Convolution**
   If $x_1(t)$ and $x_2(t)$ are periodic signals with common period T_0 and FS coefficients $c_{1,k}$ and $c_{2,k}$, the FS coefficients of the periodic convolution defined in (2.2) are given by

$$x_1(t) \circledast x_2(t) = \int_{\mathrm{T}_0} x_1\left(\tau\right) x_2\left(t-\tau\right) d\tau \quad \leftrightarrow \quad T_0 c_{1,k} c_{2,k}$$

Proof) The FS coefficients of $x_1(t) \circledast x_2(t)$ are given by

$$\begin{aligned}\frac{1}{T_0}\int_{T_0} x_1(t) \circledast x_2(t) e^{-jkw_0} dt &= \frac{1}{T_0}\int_{T_0}\left[\int_{T_0} x_1(\tau)\, x_2(t-\tau)\, d\tau\right] e^{-jkw_0 t} dt \\ &= \frac{1}{T_0}\int_{T_0}\int_{T_0} x_1(\tau)\, x_2(t_1)\, d\tau e^{-jkw_0\tau} e^{-jkw_0 t_1} d\tau dt_1 \\ &= \frac{1}{T_0}\int_{T_0} x_1(\tau)\, e^{-jkw_0\tau} d\tau \cdot \int_{T_0} x_2(t_1)\, e^{-jkw_0 t_1} d\tau dt_1 = T_0 c_{1,k} c_{2,k}\end{aligned}$$

where we used the definition of periodic convoltion in the first equality, applied a change of variable ($t-\tau = t_1$) in the second equality, and employed (3.2) for $c_{1,k}$ and $c_{2,k}$ in the last equality

10. **Multiplication Property**
If $x_1(t)$ and $x_2(t)$ are periodic signals with common period T_0 and FS coefficients $c_{1,k}$ and $c_{2,k}$, the FS coefficients of the product of two signals $x_1(t)x_2(t)$ are given by

$$x_1(t)x_2(t) \quad \leftrightarrow \quad \sum_{l=-\infty}^{\infty} c_{1,l} c_{2,k-l}$$

Proof) The FS coefficients of $x_1(t)x_2(t)$ are given by

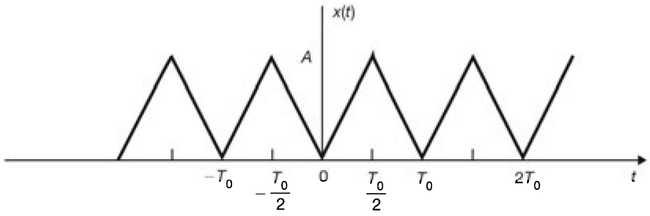
$$\begin{aligned}\frac{1}{T_0}\int_{T_0} x_1(t)x_2(t)e^{-jkw_0 t} dt &= \frac{1}{T_0}\int_{T_0}\sum_{l=-\infty}^{\infty} c_{1,l} e^{jlw_0 t} x_2(t) e^{-jkw_0 t} dt \\ &= \sum_{l=-\infty}^{\infty} c_{1,l}\left[\frac{1}{T_0}\int_{T_0} x_2(t) e^{-j(k-l)w_0 t} dt\right] = \sum_{l=-\infty}^{\infty} c_{1,l} c_{2,k-l}\end{aligned}$$

where we applied (3.1) for $x_1(t)$ in the first equality, then used (3.2) in the last equality.

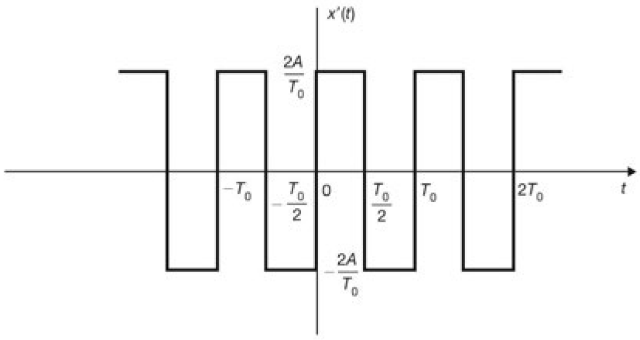
Properties of Fourier Series

1. **Linear Property**: $\alpha_1 x_1(t) + \alpha_2 x_2(t) \quad \leftrightarrow \quad \alpha_1 c_{1,k} + \alpha_2 c_{2,k}$
2. **Time Shifting**: $x(t-t_0) \quad \leftrightarrow \quad e^{-jkw_0 t_0} c_k$
3. **Conjugate Property**: $x^*(t) \quad \leftrightarrow \quad c^*_{-k}$
4. **Frequency Shifting**: $e^{jk_0 w_0 t} x(t) \quad \leftrightarrow \quad c_{k-k_0}$
5. **Time Reversal Property**: $x(-t) \quad \leftrightarrow \quad c_{-k}$
6. **Time Scaling Property**: $x(\alpha t) \quad \leftrightarrow \quad c_k$ with period $\frac{T_0}{\alpha}$
7. **Differentiation**: $\frac{dx(t)}{dt} \quad \leftrightarrow \quad jkw_0 c_k$
8. **Integration**: $\int x(t)dt \quad \leftrightarrow \quad \frac{c_k}{jkw_0}$
9. **Periodic Convolution**: $x_1(t) \circledast x_2(t) = \int_{T_0} x_1(\tau)\, x_2(t-\tau)\, d\tau \quad \leftrightarrow \quad T_0 c_{1,k} c_{2,k}$
10. **Multiplication Property**: $x_1(t)x_2(t) \quad \leftrightarrow \quad \sum_{l=-\infty}^{\infty} c_{1,l} c_{2,k-l}$

[Example 3-6] Derive the complex exponential FS coefficients of the following signal using the differentiation / or integration property. *Refer [Schaum's text, Problem 5.9]*



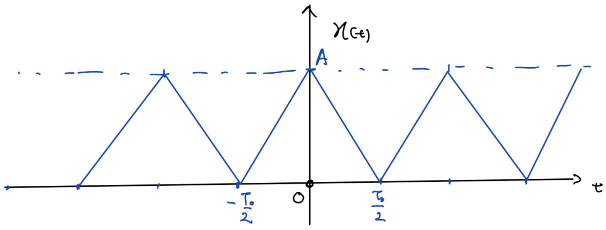
Solution) The differentiation of the given signal is plotted below, which is similar to the signal in Example 3-2. Case (c). The amplitude of Ex 3-2. Case (c) was A, whereas the that of the illustrated signal is $\frac{2A}{T_0}$.



By using the integration property, the complex exponential FS coefficients of the given signal is $c_k = \frac{c_{c,k}}{jkw_0}$ for $k \neq 0$, where $c_{c,k}$ represents the FS coefficients of Ex 3-2. Case (c). The constant c_0 for $k = 0$ can not determined by the integration property, so it should be derived through the following equation

$$c_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{T_0}\cdot\frac{AT_0}{2} = \frac{A}{2},$$
$$c_k = \begin{cases} 0 & \text{for even } k = 2m, \\ \dfrac{4A}{(jk)^2 w_0 T_0} = -\dfrac{2A}{\pi^2(2m+1)^2} & \text{for odd } k = 2m+1 \end{cases} \tag{3.24}$$

[Example 3-7] Derive the complex exponential FS coefficients of the following signal using the property of FS.



$$x(t) = \begin{cases} A\left(1-\frac{2}{T_0}t\right) & \text{for } 0 \le t < \frac{T_0}{2} \\ A\left(1+\frac{2}{T_0}t\right) & \text{for } -\frac{T_0}{2} \le t < 0. \end{cases}$$

$$x(t+T_0) = x(t)$$

Solution) The signal is a time-shifted version of Example 3-6, *i.e.*, $x(t) = x\left(t+\frac{T_0}{2}\right)_{ex3-6}$. Due to the time shifting property, the FS coefficients are determined by $e^{jkw_0T_0/2}c_k = (-1)^k c_k$.

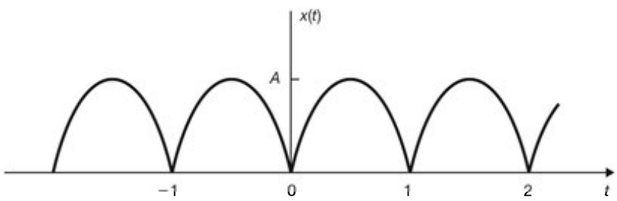
$$\begin{aligned} c_0 &= \frac{A}{2}, \\ c_k &= \begin{cases} 0 & \text{for even } k = 2m, \\ \dfrac{(-1)^{k+1}\,2A}{(k\pi)^2} = \dfrac{2A}{\pi^2\,(2m+1)^2} & \text{for odd } k = 2m+1 \end{cases} \end{aligned} \tag{3.25}$$

[Example 3-8] Consider a signal defined by $x(t) = |A\sin(\pi t)|$. *Refer [Schaum's text, Problem 5.61]*

a) Sketch $x(t)$ and find its fundamental period T_0 and angular frequency w_0

b) Find the complex exponential FS series and the trigonometric FS series of $x(t)$

Solution)



The fundamental period and fundamental angular frequency of the given signal are

$$T_0 = 1, \quad w_0 = \frac{2\pi}{T_0} = 2\pi \tag{3.26}$$

The complex exponential FS coefficients can be derived as follows

$$\begin{aligned} c_0 &= A\int_0^1 \sin(\pi t)\,dt = \frac{A}{\pi}\cos(\pi t)\Big|_1^0 = \frac{2A}{\pi}, \\ c_k &= A\int_0^1 \sin(\pi t)\,e^{jk2\pi t}dt = \frac{2A}{\pi}\left(\frac{1}{1-4k^2}\right), \end{aligned} \tag{3.27}$$

where the integral $\int_0^1 \sin(\pi t)\,e^{jk2\pi t}dt$ can be derived by using integration by parts two times

$$\int_0^1 \sin(\pi t)\,e^{jk2\pi t}dt = \frac{2}{\pi}\left(\frac{1}{1-4k^2}\right). \tag{3.28}$$

The trigonometric Fourier coefficients can be derived as follows.

$$a_0 = 2c_0 = \frac{4A}{\pi}, \quad a_k = 2\,\mathrm{Re}\,[c_k] = \frac{4A}{\pi}\left(\frac{1}{1-4k^2}\right), \quad b_k = -2\,\mathrm{Im}\,[c_k] = 0 \tag{3.29}$$