MA1201 Calculus and Basic Linear Algebra II Solution of Problem Set 5

Complex Number

Problem 1

(a) $i^5 - i^7 + i^{10} = i - (-i) + (-1) = -2 + 2i.$

(b) $$\frac{1-i}{3+2i} = \frac{1-i}{3+2i}\left(\frac{3-2i}{3-2i}\right) = \frac{3-5i+2i^2}{9-4i^2} = \frac{3-5i-2}{9+4} = \frac{1}{13} - \frac{5}{13}i.$$

(c) $$(2+i)^2(3-i) = \left(4 + 4i + \underbrace{i^2}_{=-1}\right)(3-i) = (3+4i)(3-i) = 9 + 9i - 4i^2 = 13 + 9i.$$

(d) $$(1-3i)^{-1} = \frac{1}{1-3i} = \frac{1}{1-3i}\left(\frac{1+3i}{1+3i}\right) = \frac{1+3i}{1-9i^2} = \frac{1+3i}{1+9} = \frac{1}{10} + \frac{3}{10}i.$$

Problem 2

Note that

$$\frac{1}{z} = 1 + 3i \Rightarrow z = \frac{1}{1+3i} = \frac{1}{1+3i}\left(\frac{1-3i}{1-3i}\right) = \frac{1-3i}{1-9i^2} = \frac{1}{10} - \frac{3}{10}i.$$

Problem 3

(a) The modulus and argument of $z_1 = 3 - 3i$ are given by

$$r = \sqrt{3^2 + (-3)^2} = \sqrt{18}, \ \ \theta = -\tan^{-1}\frac{3}{3} = -\frac{\pi}{4}.$$

The Polar form and Euler form of z_1 are then given by

$$z_1 = \underbrace{\sqrt{18}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right)}_{\text{Polar Form}} = \underbrace{\sqrt{18}e^{-\frac{i\pi}{4}}}_{\text{Euler Form}}.$$

(b) The modulus and argument of $z_2 = \sqrt{6} + \sqrt{2}i$ are given by

$$r = \sqrt{\left(\sqrt{6}\right)^2 + \left(\sqrt{2}\right)^2} = \sqrt{8}, \ \ \theta = \tan^{-1}\frac{\sqrt{2}}{\sqrt{6}} = \frac{\pi}{6}.$$

The Polar form and Euler form of z_2 are then given by

$$z_2 = \underbrace{\sqrt{8}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)}_{\text{Polar Form}} = \underbrace{\sqrt{8}e^{\frac{i\pi}{6}}}_{\text{Euler Form}}.$$

(c) The modulus and argument of $z_3 = -2i$ is given by

$$r = \sqrt{0^2 + (-2)^2} = 2, \qquad \theta = -\frac{\pi}{2}$$

The Polar form and Euler form of z_3 are then given by

$$z_3 = \underbrace{2\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)}_{\text{Polar Form}} = \underbrace{2e^{-\frac{i\pi}{2}}}_{\text{Euler Form}}.$$

(d) The modulus and argument of $z_4 = -4 - \sqrt{48}i$ are given by

$$r = \sqrt{(-4)^2 + \left(-\sqrt{48}\right)^2} = 8, \ \ \theta = -\left(\pi - \tan^{-1}\frac{\sqrt{48}}{4}\right) = -\left(\pi - \frac{\pi}{3}\right) = -\frac{2\pi}{3}.$$

The Polar form and Euler form of z_4 are then given by

$$z_4 = \underbrace{8\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)}_{\text{Polar Form}} = \underbrace{8e^{-i\frac{2\pi}{3}}}_{\text{Euler Form}}.$$

(e) The modulus and argument of $z_5 = -1 + 5i$ are given by

$$r = \sqrt{(-1)^2 + 5^2} = \sqrt{26},\ \ \theta = \pi - \tan^{-1}\frac{5}{1} = \pi - \tan^{-1} 5.$$

The Polar form and Euler form of z_5 are then given by

$$z_5 = \underbrace{\sqrt{26}(\cos(\pi - \tan^{-1} 5) + i\sin(\pi - \tan^{-1} 5))}_{\text{Polar Form}} = \underbrace{\sqrt{26}e^{i(\pi - \tan^{-1} 5)}}_{\text{Euler Form}}.$$

(f) The modulus and argument of $z_6 = 5 - 8i$ are given by

$$r = \sqrt{5^2 + (-8)^2} = \sqrt{89},\ \ \theta = -\tan^{-1}\frac{8}{5}.$$

The Polar form and Euler form of z_6 are then given by

$$z_6 = \underbrace{\sqrt{89}\left(\cos\left(-\tan^{-1}\frac{8}{5}\right) + i\sin\left(-\tan^{-1}\frac{8}{5}\right)\right)}_{\text{Polar Form}} = \underbrace{\sqrt{89}e^{i\left(-\tan^{-1}\frac{8}{5}\right)}}_{\text{Euler Form}}.$$

(g) $$ie^{\frac{i\pi}{4}} = e^{\frac{i\pi}{2}}e^{\frac{i\pi}{4}} = e^{i\left(\frac{\pi}{2}+\frac{\pi}{4}\right)} = \underbrace{e^{i\frac{3\pi}{4}}}_{\text{Euler Form}} = \underbrace{\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}}_{\text{Polar Form}}.$$

(h) $$-2e^{\frac{i\pi}{3}} = 2e^{i\pi}e^{\frac{i\pi}{3}} = 2e^{i\left(\pi+\frac{\pi}{3}\right)} = 2e^{i\frac{4\pi}{3}} = 2e^{i\left(\frac{4\pi}{3}-2\pi\right)} = \underbrace{2e^{-i\frac{2\pi}{3}}}_{\text{Euler Form}} = \underbrace{2\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)}_{\text{Polar Form}}.$$

(i) $$e^{\frac{i\pi}{5}} + e^{-i\pi} = e^{i\left(\frac{\frac{\pi}{5}+(-\pi)}{2}\right)}\left[e^{\frac{i\pi}{5}-i\left(\frac{\frac{\pi}{5}+(-\pi)}{2}\right)} + e^{-i\pi-i\left(\frac{\frac{\pi}{5}+(-\pi)}{2}\right)}\right] = e^{-i\frac{2\pi}{5}}\left(e^{i\frac{3\pi}{5}} + e^{-i\frac{3\pi}{5}}\right)$$

$$= \underbrace{2\cos\frac{3\pi}{5}}_{\text{negative!}}e^{-i\frac{2\pi}{5}} = \underbrace{\left(-2\cos\frac{3\pi}{5}\right)}_{\text{positive}}(-1)e^{-i\frac{2\pi}{5}} = \left(-2\cos\frac{3\pi}{5}\right)e^{i\pi}e^{-i\frac{2\pi}{5}}$$

$$= \underbrace{\left(-2\cos\frac{3\pi}{5}\right)e^{i\frac{3\pi}{5}}}_{\text{Euler Form}} = \underbrace{\left(-2\cos\frac{3\pi}{5}\right)\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)}_{\text{Polar Form}}.$$

(j) $$1 - e^{\frac{i\pi}{4}} = e^{i0} - e^{\frac{i\pi}{4}} = e^{i\left(\frac{0+\frac{\pi}{4}}{2}\right)}\left[e^{i0-i\left(\frac{0+\frac{\pi}{4}}{2}\right)} - e^{\frac{i\pi}{4}-i\left(\frac{0+\frac{\pi}{4}}{2}\right)}\right] = e^{\frac{i\pi}{8}}\left(e^{-\frac{i\pi}{8}} - e^{\frac{i\pi}{8}}\right) = e^{\frac{i\pi}{8}}\left(-2i\sin\frac{\pi}{8}\right)$$

$$= 2\sin\frac{\pi}{8}(-i)e^{\frac{i\pi}{8}} = 2\sin\frac{\pi}{8}e^{-\frac{i\pi}{2}}e^{\frac{i\pi}{8}}$$

$$= \underbrace{2\sin\frac{\pi}{8}e^{-i\frac{3\pi}{8}}}_{\text{Euler Form}} = \underbrace{2\sin\frac{\pi}{8}\left(\cos\left(-\frac{3\pi}{8}\right) + i\sin\left(-\frac{3\pi}{8}\right)\right)}_{\text{Polar Form}}.$$

Problem 4

Recall that $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$.

(a) $$-\cos\theta - i\sin\theta = -\frac{e^{i\theta}+e^{-i\theta}}{2} - i\frac{e^{i\theta}-e^{-i\theta}}{2i} = -e^{i\theta} = e^{i\pi}e^{i\theta} = \underbrace{e^{i(\theta+\pi)}}_{\text{Euler Form}}$$

$$= \underbrace{\cos(\theta+\pi) + i\sin(\theta+\pi)}_{\text{Polar Form}}.$$

(b) $$-\sin\theta + i\cos\theta = -\frac{e^{i\theta}-e^{-i\theta}}{2i} + i\frac{e^{i\theta}+e^{-i\theta}}{2} = -\frac{i\left(e^{i\theta}-e^{-i\theta}\right)}{-2} + i\frac{e^{i\theta}+e^{-i\theta}}{2} = ie^{i\theta} = e^{\frac{i\pi}{2}}e^{i\theta}$$

$$= \underbrace{e^{i\left(\theta+\frac{\pi}{2}\right)}}_{\text{Euler Form}} = \underbrace{\cos\left(\theta+\frac{\pi}{2}\right) + i\sin\left(\theta+\frac{\pi}{2}\right)}_{\text{Polar Form}}.$$

(c)
$$1-\sin\theta+i\cos\theta \overset{\text{from (b)}}{\cong} 1+e^{i\left(\theta+\frac{\pi}{2}\right)}=e^{i0}+e^{i\left(\theta+\frac{\pi}{2}\right)}$$
$$=e^{i\left(\frac{0+\theta+\frac{\pi}{2}}{2}\right)}\left[e^{i0-i\left(\frac{0+\theta+\frac{\pi}{2}}{2}\right)}+e^{i\left(\theta+\frac{\pi}{2}\right)-i\left(\frac{0+\theta+\frac{\pi}{2}}{2}\right)}\right]=e^{i\left(\frac{\theta}{2}+\frac{\pi}{4}\right)}\left[e^{-i\left(\frac{\theta}{2}+\frac{\pi}{4}\right)}+e^{i\left(\frac{\theta}{2}+\frac{\pi}{4}\right)}\right]$$
$$=\underbrace{2\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)e^{i\left(\frac{\theta}{2}+\frac{\pi}{4}\right)}}_{\text{Euler Form}}=\underbrace{2\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\left[\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)+i\sin\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right]}_{\text{Polar Form}}.$$

(d)
$$1+\cos\theta-i\sin\theta=1+\frac{e^{i\theta}+e^{-i\theta}}{2}-i\,\frac{e^{i\theta}-e^{-i\theta}}{2i}=1+e^{-i\theta}=e^{i0}+e^{-i\theta}$$
$$=e^{i\left(\frac{0-\theta}{2}\right)}\left[e^{i0-i\left(\frac{0-\theta}{2}\right)}+e^{-i\theta-i\left(\frac{0-\theta}{2}\right)}\right]=e^{-\frac{i\theta}{2}}\left(e^{\frac{i\theta}{2}}+e^{-\frac{i\theta}{2}}\right)=\underbrace{2\cos\frac{\theta}{2}e^{-\frac{i\theta}{2}}}_{\text{Euler Form}}$$
$$=\underbrace{2\cos\frac{\theta}{2}\left[\cos\left(-\frac{\theta}{2}\right)+i\sin\left(-\frac{\theta}{2}\right)\right]}_{\text{Polar Form}}.$$

Problem 5

(a)
$$\frac{1+\sqrt{3}i}{2-2i}=\frac{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}{\sqrt{8}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)}$$
$$=\frac{1}{\sqrt{2}}\left(\cos\left(\frac{\pi}{3}-\left(-\frac{\pi}{4}\right)\right)+i\sin\left(\frac{\pi}{3}-\left(-\frac{\pi}{4}\right)\right)\right)=\frac{1}{\sqrt{2}}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right).$$

(b)
$$(1+i)^{-5}=\left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^{-5}=2^{-\frac{5}{2}}\left(\cos\left(-\frac{5\pi}{4}\right)+i\sin\left(-\frac{5\pi}{4}\right)\right).$$

(c) We first note that
$$\frac{(1-i)(\sqrt{3}+i)}{2i}=\frac{\left[\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)\right]\left[2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]}{2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)}$$
$$=\sqrt{2}\left(\cos\left(-\frac{\pi}{4}+\frac{\pi}{6}-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{4}+\frac{\pi}{6}-\frac{\pi}{2}\right)\right)$$
$$=\sqrt{2}\left[\cos\left(-\frac{7\pi}{12}\right)+i\sin\left(-\frac{7\pi}{12}\right)\right].$$
Therefore, we have
$$\left(\frac{(1-i)(\sqrt{3}+i)}{2i}\right)^{12}=\left[\sqrt{2}\left[\cos\left(-\frac{7\pi}{12}\right)+i\sin\left(-\frac{7\pi}{12}\right)\right]\right]^{12}=64(\cos(-7\pi)+i\sin(-7\pi))$$
$$=-64.$$

(d)
$$\sqrt[6]{-\sqrt{48}+4i}=\sqrt[6]{8\left(\cos\frac{5\pi}{6}+\sin\frac{5\pi}{6}\right)}=8^{\frac{1}{6}}\left(\cos\frac{2k\pi+\frac{5\pi}{6}}{6}+i\sin\frac{2k\pi+\frac{5\pi}{6}}{6}\right)$$
$$=8^{\frac{1}{6}}\left(\cos\left(\frac{k\pi}{3}+\frac{5\pi}{36}\right)+i\sin\left(\frac{k\pi}{3}+\frac{5\pi}{36}\right)\right),\qquad k=0,1,2,\dots,5.$$

(e) We first note that
$$\frac{4+4i}{\left(-2+\sqrt{12}i\right)^3}=\frac{\sqrt{32}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}{\left[4\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)\right]^3}=\frac{\sqrt{32}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}{64\underbrace{(\cos 2\pi+i\sin 2\pi)}_{=1}}=\frac{1}{2\sqrt{32}}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right).$$
Thus, we have

$$\sqrt{\frac{4+4i}{(-2+\sqrt{12}i)^3}}=\sqrt{\frac{1}{2\sqrt{32}}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)}=2^{-\frac{7}{4}}\left(\cos\frac{2k\pi+\frac{\pi}{4}}{2}+i\sin\frac{2k\pi+\frac{\pi}{4}}{2}\right)$$
$$=2^{-\frac{7}{4}}\left(\cos\left(k\pi+\frac{\pi}{8}\right)+i\sin\left(k\pi+\frac{\pi}{8}\right)\right),\qquad k=0,1.$$

(f)
$$(-2-2i)^{\frac{3}{4}}=\sqrt[4]{\left[\sqrt{8}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)\right]^3}=\sqrt[4]{8^{\frac{3}{2}}\left(\cos\frac{15\pi}{4}+i\sin\frac{15\pi}{4}\right)}$$
$$=8^{\frac{3}{8}}\left(\cos\frac{2k\pi+\frac{15\pi}{4}}{4}+i\sin\frac{2k\pi+\frac{15\pi}{4}}{4}\right),\quad \text{for } k=0,1,2,3.$$

(g)
$$\sin\theta-i\cos\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}-i\frac{e^{i\theta}+e^{-i\theta}}{2}=\frac{i\left(e^{i\theta}-e^{-i\theta}\right)}{-2}-i\frac{e^{i\theta}+e^{-i\theta}}{2}=-ie^{i\theta}=e^{-\frac{i\pi}{2}}e^{i\theta}$$
$$=e^{i\left(\theta-\frac{\pi}{2}\right)}.$$

Thus

$$(\sin\theta-i\cos\theta)^5=\left[e^{i\left(\theta-\frac{\pi}{2}\right)}\right]^5=e^{i\left(5\theta-\frac{5\pi}{2}\right)}.$$

(h)
$$1+e^{\frac{i\pi}{4}}=e^{i0}+e^{\frac{i\pi}{4}}=e^{\frac{i\pi}{8}}\left(e^{-\frac{i\pi}{8}}+e^{\frac{i\pi}{8}}\right)=2\cos\frac{\pi}{8}e^{\frac{i\pi}{8}}=2\cos\frac{\pi}{8}\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right).$$

Thus, we conclude that

$$\sqrt[4]{1+e^{\frac{i\pi}{4}}}=\sqrt[4]{2\cos\frac{\pi}{8}\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right)}=\left(2\cos\frac{\pi}{8}\right)^{\frac{1}{4}}\left(\cos\frac{2k\pi+\frac{\pi}{8}}{4}+i\sin\frac{2k\pi+\frac{\pi}{8}}{4}\right),$$
$$k=0,1,2,3.$$

Problem 6

$$\left(\frac{1+i\tan\theta}{1-i\tan\theta}\right)^{\mathbf{5}}=\left(\frac{1+i\frac{\sin\theta}{\cos\theta}}{1-i\frac{\sin\theta}{\cos\theta}}\right)^{\mathbf{5}}=\left(\frac{\cos\theta+i\sin\theta}{\cos\theta-i\sin\theta}\right)^5=\frac{(\cos\theta+i\sin\theta)^5}{(\cos(-\theta)+i\sin(-\theta))^5}$$
$$=\frac{\cos5\theta+i\sin5\theta}{\cos(-5\theta)+i\sin(-5\theta)}=\frac{\cos5\theta+i\sin5\theta}{\cos5\theta-i\sin5\theta}=\frac{1+i\frac{\sin5\theta}{\cos5\theta}}{1-i\frac{\sin5\theta}{\cos5\theta}}=\frac{1+i\tan5\theta}{1-i\tan5\theta}.$$

Problem 7

(a) Since $|z|=r=1$, then the polar form of z is seen to be $z=\cos\theta+i\sin\theta$. Then

$$z_0=\frac{1+z}{1-z}=\frac{(1+\cos\theta)+i\sin\theta}{(1-\cos\theta)-i\sin\theta}=\frac{(1+\cos\theta)+i\sin\theta}{(1-\cos\theta)-i\sin\theta}\left(\frac{(1-\cos\theta)+i\sin\theta}{(1-\cos\theta)+i\sin\theta}\right)$$
$$=\frac{(1-\cos^2\theta)+[\sin\theta\,(1-\cos\theta)+\sin\theta\,(1+\cos\theta)]i+i^2\sin^2\theta}{(1-\cos\theta)^2+\sin^2\theta}$$
$$=\frac{\sin^2\theta-\sin^2\theta+2\sin\theta\, i}{2-2\cos\theta}=\frac{\sin\theta}{1-\cos\theta}i\overset{\text{Take } b=\frac{\sin\theta}{1-\cos\theta}}{\triangleq}bi.$$

Thus z_0 is purely imaginary.

(b) Since $z=\cos\theta+i\sin\theta$, then $\bar{z}=\cos\theta-i\sin\theta$. The remaining argument is similar to that of (a). We omit the detail here.

Problem 8

(a) $$\frac{3z}{1+\bar{z}}=\frac{3(1+3i)}{1+(1-3i)}=\frac{3+9i}{2-3i}=\frac{3+9i}{2-3i}\left(\frac{2+3i}{2+3i}\right)=\frac{6+24i+27i^2}{4-9i^2}=\frac{-21+24i}{13}.$$

(b) $$(z+3\bar{z})^2=[(1+3i)+3(1-3i)]^2=(4-6i)^2=16-48i+36i^2=-20-48i.$$

(c) $$\sqrt[4]{z+\bar{z}}=\sqrt[4]{(1+3i)+(1-3i)}=\sqrt[4]{2}=\sqrt[4]{2(\cos 0+i\sin 0)}=2^{\frac{1}{4}}\left(\cos\frac{2k\pi}{4}+i\sin\frac{2k\pi}{4}\right),$$
for $k=0,1,2,3$.

Problem 9

Note that

$$\operatorname{Re}\left(\frac{z-1}{z+1}\right)=\frac{1}{2}\left(\frac{z-1}{z+1}+\overline{\frac{z-1}{z+1}}\right)\overset{\bar{1}=1}{\cong}\frac{1}{2}\left(\frac{z-1}{z+1}+\frac{\bar{z}-1}{\bar{z}+1}\right)=\frac{1}{2}\frac{(z-1)(\bar{z}+1)+(\bar{z}-1)(z+1)}{(z+1)(\bar{z}+1)}$$
$$=\frac{1}{2}\frac{2z\bar{z}-2}{(z+1)(\bar{z}+1)}\overset{z\bar{z}=|z|^2}{\cong}\frac{|z|^2-1}{(z+1)(\bar{z}+1)}\overset{|z|=1}{\cong}\frac{1-1}{(z+1)(\bar{z}+1)}=0.$$
$$\operatorname{Im}\left(\frac{z-1}{z+1}\right)=\frac{1}{2i}\left(\frac{z-1}{z+1}-\overline{\frac{z-1}{z+1}}\right)\overset{\bar{1}=1}{\cong}\frac{1}{2i}\left(\frac{z-1}{z+1}-\frac{\bar{z}-1}{\bar{z}+1}\right)=\frac{1}{2i}\frac{(z-1)(\bar{z}+1)-(\bar{z}-1)(z+1)}{(z+1)(\bar{z}+1)}$$
$$=\frac{1}{2i}\frac{2z-2\bar{z}}{(z+1)(\bar{z}+1)}=\frac{1}{i}\frac{z-\bar{z}}{z\bar{z}+z+\bar{z}+1}=\frac{1}{i}\frac{2i\operatorname{Im}(z)}{|z|^2+2\operatorname{Re}(z)+1}=\frac{\operatorname{Im}(z)}{1+\operatorname{Re}(z)}.$$

Problem 10

$$|z_1+z_2|^2-|z_1-z_2|^2=(z_1+z_2)\overline{(z_1+z_2)}-(z_1-z_2)\overline{(z_1-z_2)}$$
$$=(z_1+z_2)(\bar{z}_1+\bar{z}_2)-(z_1-z_2)(\bar{z}_1-\bar{z}_2)$$
$$=z_1\bar{z}_1+z_1\bar{z}_2+z_2\bar{z}_1+z_2\bar{z}_2-z_1\bar{z}_1+z_1\bar{z}_2+z_2\bar{z}_1-z_2\bar{z}_2=2(z_1\bar{z}_2+z_2\bar{z}_1)$$
$$=2(z_1\bar{z}_2+\overline{z_1\bar{z}_2})=4\operatorname{Re}(z_1\bar{z}_2).$$

Remark: The last equality follows from the fact that $z+\bar{z}=2\operatorname{Re}(z)$.

Problem 11

(a)
$$z^6=-3+\sqrt{3}i\Rightarrow z=\sqrt[6]{-3+\sqrt{3}i}=\sqrt[6]{\sqrt{12}\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)}$$
$$=12^{\frac{1}{12}}\left(\cos\frac{2k\pi+\frac{5\pi}{6}}{6}+i\sin\frac{2k\pi+\frac{5\pi}{6}}{6}\right)$$
$$=12^{\frac{1}{12}}\left(\cos\left(\frac{k\pi}{3}+\frac{5\pi}{36}\right)+i\sin\left(\frac{k\pi}{3}+\frac{5\pi}{36}\right)\right),\quad\text{for } k=0,1,2,\dots,5.$$

(b)
$$(1-z)^7+(1+z)^7=0\Rightarrow\left(\frac{1-z}{1+z}\right)^7=-1\Rightarrow\frac{1-z}{1+z}=(\cos\pi+i\sin\pi)^{\frac{1}{7}}$$
$$\Rightarrow\frac{1-z}{1+z}=\underbrace{\cos\frac{2k\pi+\pi}{7}+i\sin\frac{2k\pi+\pi}{7}}_{\omega_k},\quad k=0,1,2,\dots,6.$$
$$\Rightarrow z=\frac{1-\omega_k}{1+\omega_k}$$

(c) $$z^{10}-5z^5-6=0\Rightarrow(z^5-6)(z^5+1)=0$$
$$\Rightarrow z^5=6\ \text{ or }\ z^5=-1$$

$$\Rightarrow z = \sqrt[5]{6(\cos 0 + i\sin 0)} = 6^{\frac{1}{5}}\left(\cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}\right) \quad \text{or}$$
$$z = \sqrt[5]{(\cos\pi + i\sin\pi)} = \cos\frac{2k\pi+\pi}{5} + i\sin\frac{2k\pi+\pi}{5}$$
where $k = 0, 1, 2, 3, 4$.

(d)
$$z^8 - 2\sqrt{3}z^4 + 4 = 0 \overset{y=z^4}{\Rightarrow} y^2 - 2\sqrt{3}y + 4 = 0$$
$$\Rightarrow y = \frac{2\sqrt{3} \pm \sqrt{\left(-2\sqrt{3}\right)^2 - 4(1)(4)}}{2} = \sqrt{3} \pm i$$
$$\Rightarrow z^4 = \sqrt{3} + i \ \text{ or } \ z^4 = \sqrt{3} - i$$
$$\Rightarrow z = \sqrt[4]{2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)} \quad \text{or} \quad z = \sqrt[4]{2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)}$$
$$\Rightarrow z = 2^{\frac{1}{4}}\left(\cos\frac{2k\pi + \frac{\pi}{6}}{4} + i\sin\frac{2k\pi + \frac{\pi}{6}}{4}\right) \quad \text{or} \quad z = 2^{\frac{1}{4}}\left(\cos\frac{2k\pi - \frac{\pi}{6}}{4} + i\sin\frac{2k\pi - \frac{\pi}{6}}{4}\right).$$
where $k = 0, 1, 2, 3$.

(e)
$$\frac{z^5}{1+z^5} = \sqrt{3}i \Rightarrow z^5 = \sqrt{3}i + \sqrt{3}iz^5$$
$$\Rightarrow z^5 = \frac{\sqrt{3}i}{1-\sqrt{3}i} = \frac{\sqrt{3}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}{2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)} = \frac{\sqrt{3}}{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$
$$\Rightarrow z = \sqrt[5]{\frac{\sqrt{3}}{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)} = \frac{3^{\frac{1}{10}}}{2^{\frac{1}{5}}}\left(\cos\frac{2k\pi + \frac{5\pi}{6}}{5} + i\sin\frac{2k\pi + \frac{5\pi}{6}}{5}\right)$$
where $k = 0, 1, 2, 3, 4$.

Problem 12

Since $3+i$ is one of the roots, it follows that $\overline{3+i} = 3-i$ is also a root of the same equation. By factor theorem, $[z-(3+i)]$ and $[z-(3-i)]$ are the factors of the polynomial on L.H.S. This implies that the product $[z-(3+i)][z-(3-i)] = z^2 - 6z + 10$ is also a factor of the same polynomial.

Using long division, the equation can be factorized as

$$z^4 - 8z^3 + 27z^2 - 50z + 50 = 0$$
$$\Rightarrow (z^2 - 6z + 10)(z^2 - 2z + 5) = 0$$
$$\Rightarrow z = 3 \pm i \ \text{ or } \ z = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(5)}}{2} = 1 \pm 2i.$$

Problem 13

(a) We first note that

$$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta \ \ldots (1)$$

On the other hand, one can use the Binomial theorem to obtain

$$(\cos\theta + i\sin\theta)^5$$
$$= \cos^5\theta + 5\cos^4\theta\sin\theta\, i + 10\cos^3\theta\sin^2\theta\, i^2 + 10\cos^2\theta\sin^3\theta\, i^3 + 5\cos\theta\sin^4\theta\, i^4 + \sin^5\theta\, i^5$$
$$= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta) \ \ldots (2).$$

By comparing the *real part* between the equations (1) and (2), we get

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
$$= \cos^5\theta - 10\cos^3\theta\,(1-\cos^2\theta) + 5\cos\theta\,(1-\cos^2\theta)^2$$
$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$

(b) We shall consider the expression $(\cos\theta + i\sin\theta)^3$. We first note that
$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta \ldots (1)$$
On the other hand, one can use the Binomial theorem to obtain
$$(\cos\theta + i\sin\theta)^3$$
$$= \cos^3\theta + 3\cos^2\theta\sin\theta\, i + 3\cos\theta\sin^2\theta\, i^2 + \sin^3\theta\, i^3$$
$$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta) \ldots (2).$$
By comparing the *imaginary part* between the equation (1) and (2), we obtain
$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = 3(1-\sin^2\theta)\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta.$$

Problem 14

(a) Using the fact that $z - \frac{1}{z} = 2i\sin\theta$, we obtain
$$\left(z - \frac{1}{z}\right)^5 = (2i\sin\theta)^5 = 32\sin^5\theta\, i^5 = 32\sin^5\theta\, i \ldots (1).$$
On the other hand, one can use the Binomial theorem to obtain
$$\left(z - \frac{1}{z}\right)^5 = z^5 - 5z^3 + 10z - 10\left(\frac{1}{z}\right) + 5\left(\frac{1}{z^3}\right) - \frac{1}{z^5} = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$
$$= 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta \ldots (2).$$
By comparing (1) and (2), we obtain
$$32\sin^5\theta\, i = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$$
$$\Rightarrow \sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta).$$

(b) Using the fact that $z - \frac{1}{z} = 2i\sin\theta$ and $z + \frac{1}{z} = 2\cos\theta$, we get
$$\left(z - \frac{1}{z}\right)^3\left(z + \frac{1}{z}\right)^4 = (2i\sin\theta)^3(2\cos^4\theta) = -128\sin^3\theta\cos^4\theta\, i \ldots (1)$$
One can use the Binomial theorem to obtain
$$\left(z - \frac{1}{z}\right)^3\left(z + \frac{1}{z}\right)^4 = \left[\left(z - \frac{1}{z}\right)\left(z + \frac{1}{z}\right)\right]^3\left(z + \frac{1}{z}\right) = \left(z^2 - \frac{1}{z^2}\right)^3\left(z + \frac{1}{z}\right)$$
$$= \left(z^6 - 3z^2 + \frac{3}{z^2} - \frac{1}{z^6}\right)\left(z + \frac{1}{z}\right) = z^7 - 3z^3 + \frac{3}{z} - \frac{1}{z^5} + z^5 - 3z + \frac{3}{z^3} - \frac{1}{z^7}$$
$$= \left(z^7 - \frac{1}{z^7}\right) + \left(z^5 - \frac{1}{z^5}\right) - 3\left(z^3 - \frac{1}{z^3}\right) - 3\left(z - \frac{1}{z}\right)$$
$$= (2\sin 7\theta + 2\sin 5\theta - 6\sin 3\theta - 6\sin\theta)i \ldots (2)$$
By comparing (1) and (2), we obtain
$$-128\sin^3\theta\cos^4\theta\, i = (2\sin 7\theta + 2\sin 5\theta - 6\sin 3\theta - 6\sin\theta)i$$
$$\Rightarrow \sin^3\theta\cos^4\theta = -\frac{1}{64}(\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta).$$

(c) Using the result obtained in (b), the integral can be computed as
$$\int_0^{\frac{\pi}{2}} \sin^3\theta\cos^4\theta\, d\theta = -\frac{1}{64}\int_0^{\frac{\pi}{2}}(\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta)d\theta$$
$$= -\frac{1}{64}\left[-\frac{1}{7}\cos 7\theta - \frac{1}{5}\cos 5\theta + \cos 3\theta + 3\cos\theta\right]_0^{\frac{\pi}{2}} = -\frac{2}{35}.$$