MA1201 Calculus and Basic Linear Algebra II

Chapter 3

Application of Integration

Application of Integration

1. Geometric Application
    - Area of region bounded by curves and / or axes (pp. 3-17)

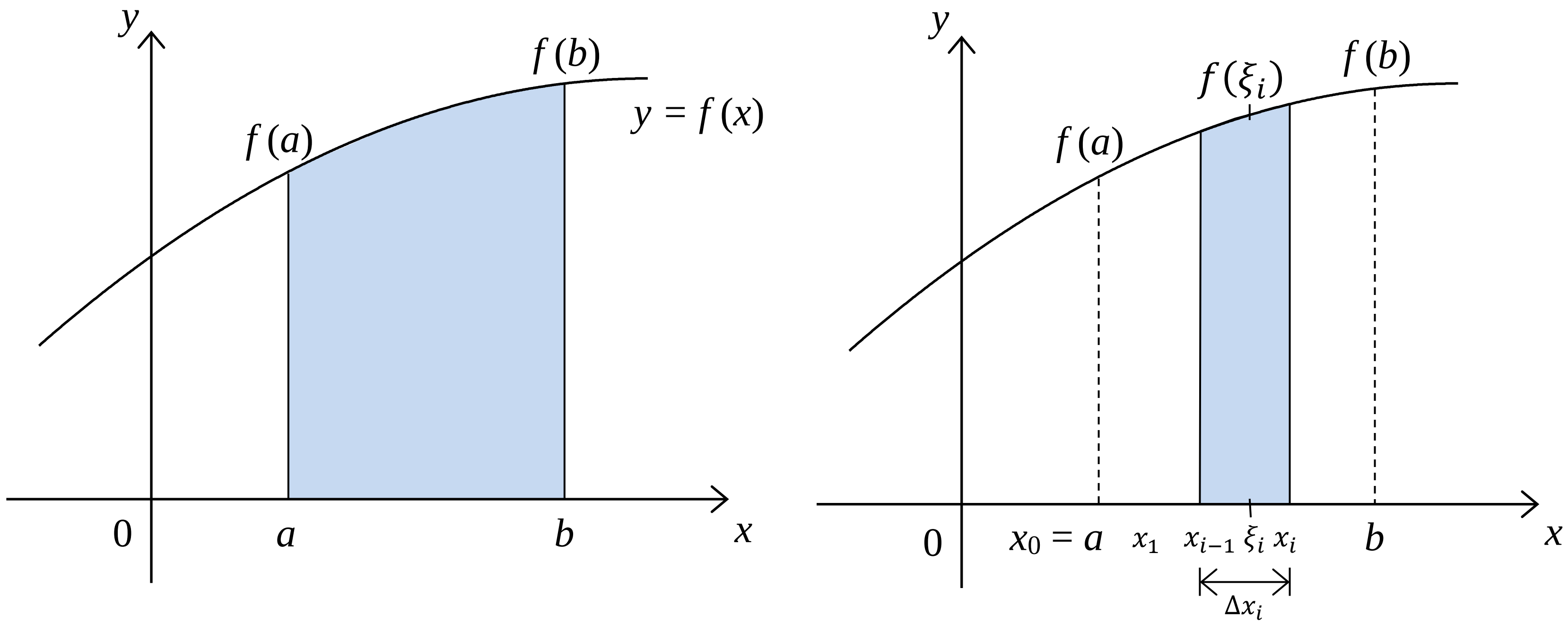
    - Volume of a solid generated by revolving a region bounded by curves about a horizontal line or vertical line (pp. 18-38)

    - Arc length of a curve (pp. 39-46 and pp. 60-67)

    - Surface area of a solid generated by revolving a region bounded by curves about a horizontal line or vertical line (pp. 47-57 and pp. 68-69)

Area of region bounded by curves, axes

Recall in Chapter 1, the definite integral is defined as the area under the curve $y = f(x)$.



$$\int_a^b f(x)dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x_i.$$

As mentioned in Chapter 1, this integral gives the signed area of the region under the graph:

- **If the curve $\boldsymbol{y = f(x)}$ <u>lies above the $\boldsymbol{x}$-axis</u>,**
  $\int_a^b f(x)dx$ returns the true area of the region bounded by the curve $y = f(x)$, the x-axis, the lines $x = a$ and $x = b$.

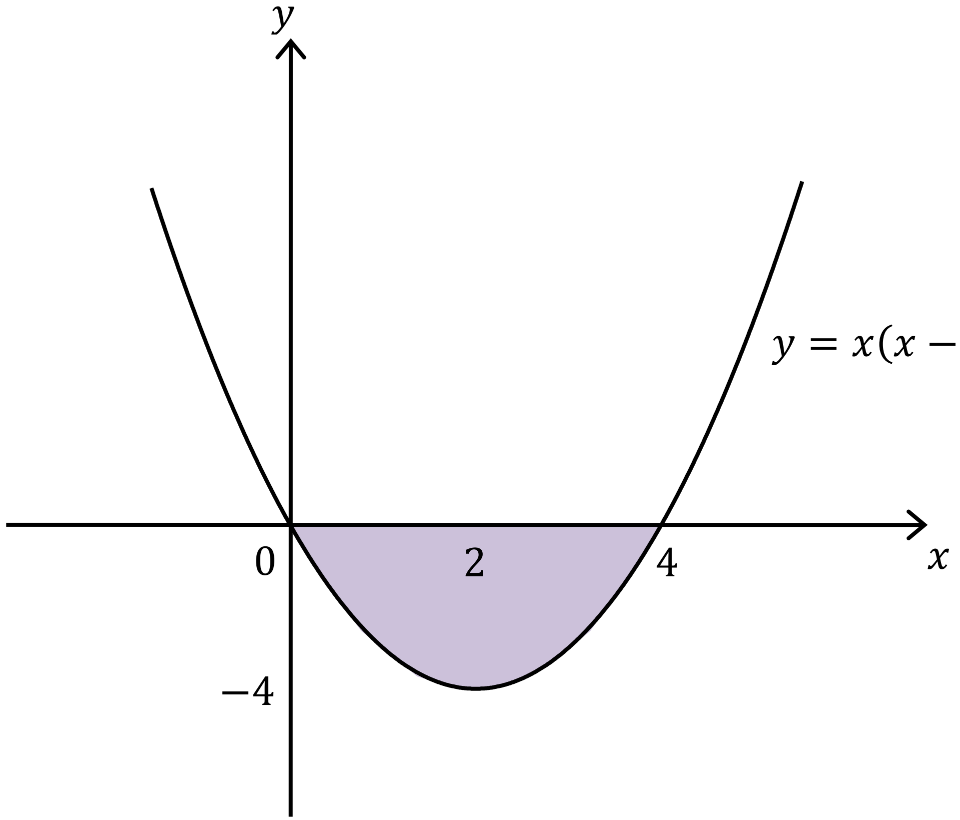
- **If the curve $\boldsymbol{y = f(x)}$ <u>lies below the $\boldsymbol{x}$-axis</u>**,
  $\int_a^b f(x)dx$ will become a negative number. However, $-\int_a^b f(x)dx$ (which is positive) represents the area of the region bounded by the curve $y = f(x)$, the x-axis, the lines $x = a$ and $x = b$.

Thus, we should use the following formula when finding the area under the curve:

$$\int_a^b |f(x)|dx.$$

Example 1

Calculate the area bounded by the x-axis and the curve $y = x(x-4)$.



☺Solution:

Note that $x(x-4) = 0 \Rightarrow x = 0 \quad \text{or} \quad x = 4$.

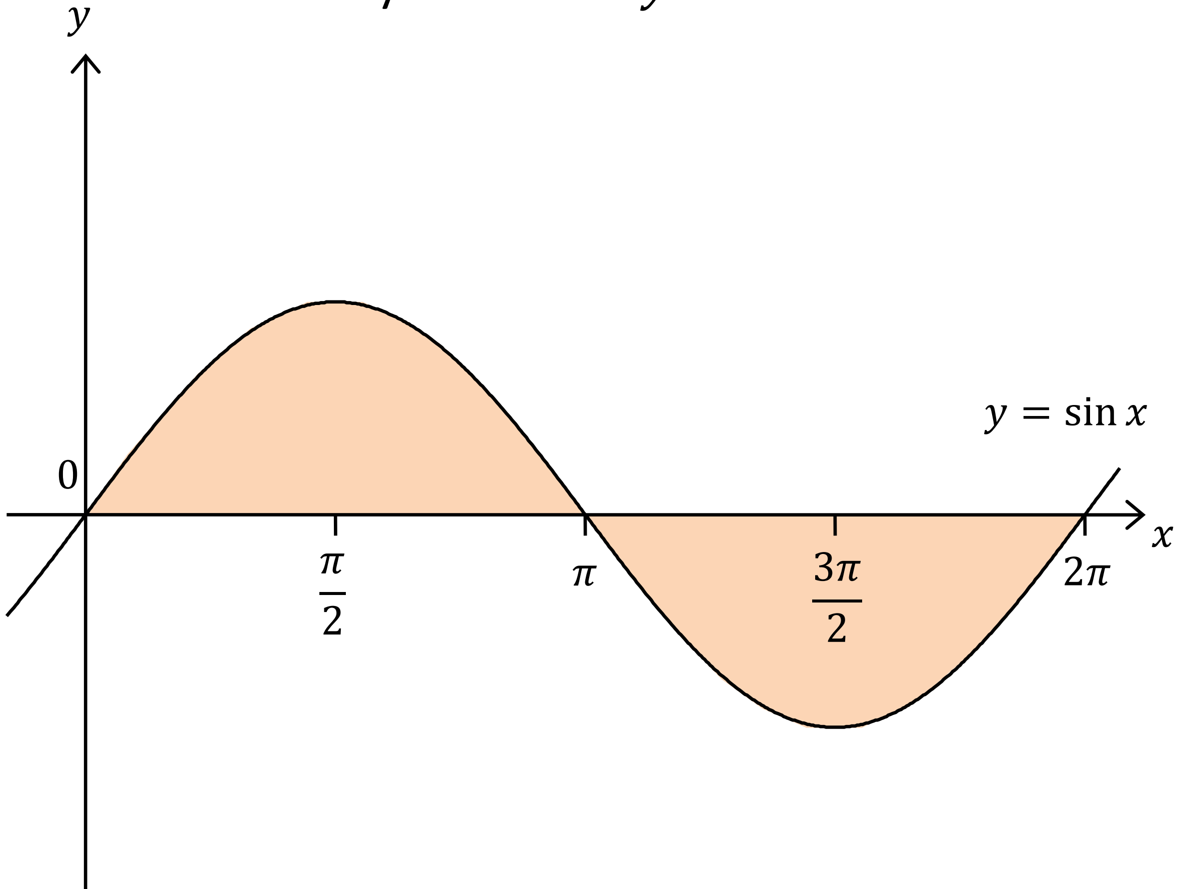
Since the entire shaded region is below the x-axis. The shaded area is given by

$$-\int_0^4 x(x-4)dx = \int_0^4 (4x - x^2)dx = 4\int_0^4 xdx - \int_0^4 x^2dx$$

$$= 4\left[\frac{x^2}{2}\right]_0^4 - \left[\frac{x^3}{3}\right]_0^4 = 4(8) - \frac{64}{3} = \frac{32}{3}\,\text{(square units)}.$$

Example 2

Find the whole area enclosed by the curve $y = \sin x$ and the x-axis between $x = 0$ and $x = 2\pi$.



☺Solution:

The curve $y = \sin x$ lies above the x-axis when $0 \leq x \leq \pi$ and lies below the x-axis when $\pi \leq x \leq 2\pi$.

The area of shaded region is given by

$$= \int_0^{2\pi} |\sin x| dx$$

$$= \underbrace{\int_0^{\pi} \sin x \, dx}_{\text{lies above the } x\text{-axis}} + \left(\underbrace{-\int_{\pi}^{2\pi} \sin x \, dx}_{\text{lies below the } x\text{-axis}} \right) = [-\cos x]_0^{\pi} - [-\cos x]_{\pi}^{2\pi} = 4.$$

Example 3

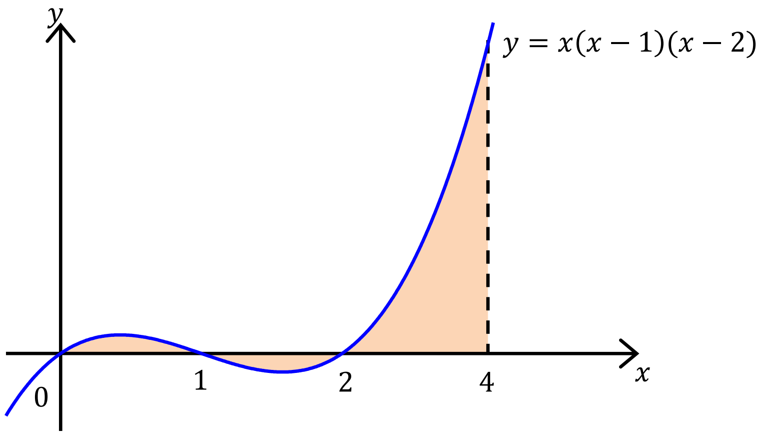
Find the area enclosed by the curve $y = x(x-1)(x-2)$ and the x-axis between $x = 0$ and $x = 4$.

☺Solution:

Note that $f(x) = x(x-1)(x-2) = 0 \Rightarrow x = 0, \; x = 1, \; x = 2.$

	$x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$1 < x < 2$	$x = 2$	$x > 2$
$f(x)$	$-$	0	$+$	0	$-$	0	$+$

The figure (and required shaded region) is shown below:



The area of shaded region is given by

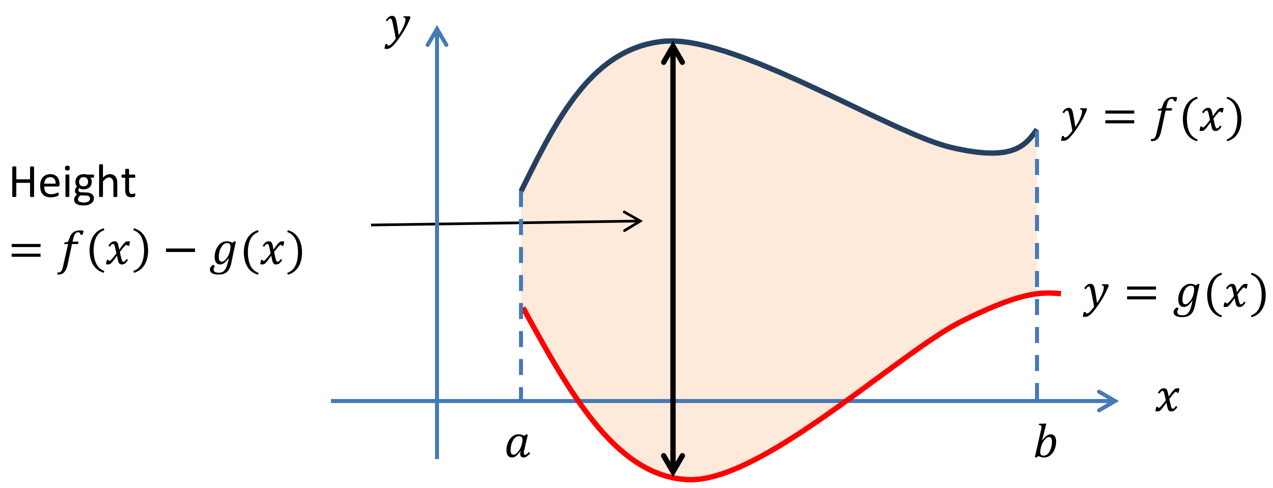
$$= \int_0^4 |x(x-1)(x-2)|dx$$

$$= \int_0^1 x(x-1)(x-2)dx + \left(-\int_1^2 x(x-1)(x-2)dx\right) + \int_2^4 x(x-1)(x-2)dx$$

$$= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 - \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2 + \left[\frac{x^4}{4} - x^3 + x^2\right]_2^4 = 16.5 \text{ (square units)}.$$

Area between curves

Let $y = f(x)$ and $y = g(x)$ be two graphs of functions shown below:



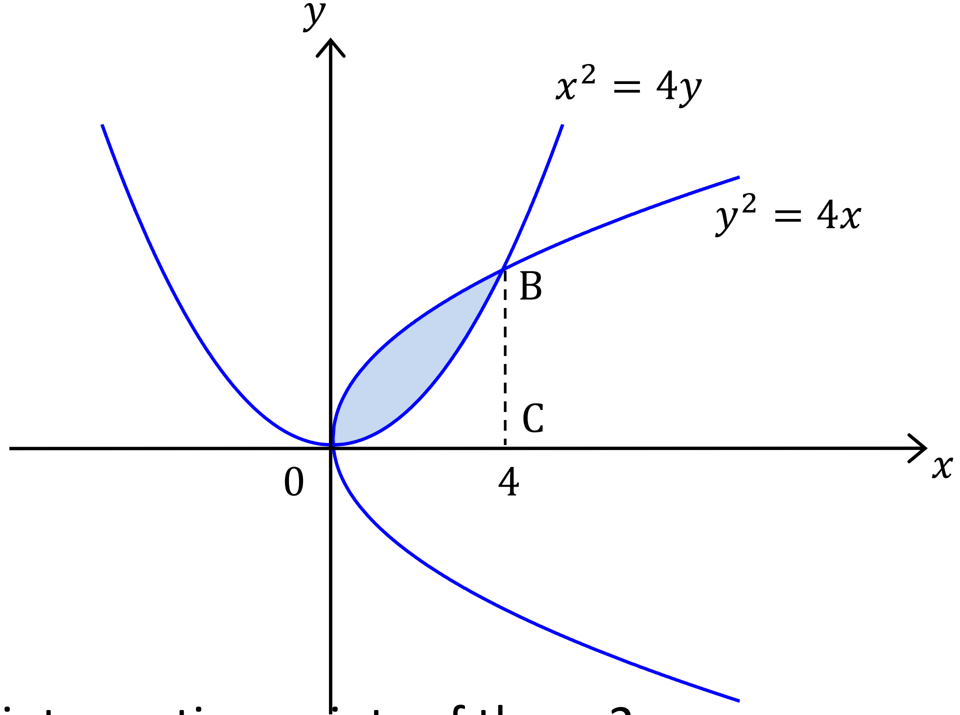
Then the area of the shaded region (area between curves) is given by

$$A = \int_a^b \big(f(x) - g(x)\big)dx.$$

- One has to make sure that the upper function is placed in the first term of the formula or otherwise "negative" area will be resulted.

Example 4

Find the area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$.



☺Solution:

We first find the intersection points of these 2 curves.

$$\begin{cases} y^2 = 4x \\ x^2 = 4y \end{cases} \Rightarrow \left(\frac{y^2}{4}\right)^2 = 4y \Rightarrow y^4 - 64y = 0 \Rightarrow y(y^3 - 64) = 0$$

$\Rightarrow \; y(y-4)(y^2+4y+16) = 0 \Rightarrow y = 0 \;\; (x = 0)$ or $y = 4 \;\; (x = 4)$.

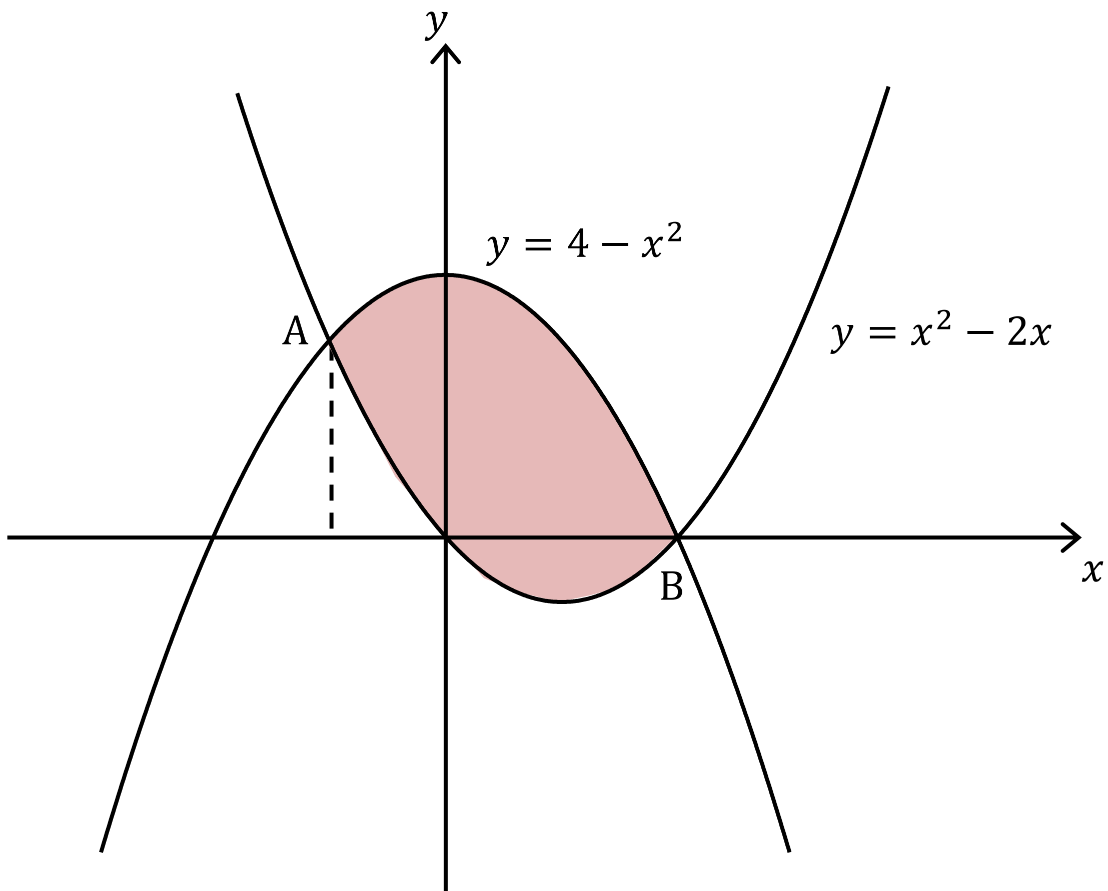
So $O = (0, 0)$ and $B = (4, 4)$.

The area of shaded region is then given by

$$\int_0^4 \left(\sqrt{4x} - \frac{x^2}{4}\right) dx = 2\int_0^4 \sqrt{x}\,dx - \frac{1}{4}\int_0^4 x^2\,dx = 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4 - \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4 = \frac{32}{3} - \frac{16}{3} = \frac{16}{3}.$$

Example 5

Find the area of the region bounded by $y = 4 - x^2$ and $y = x^2 - 2x$.



☺Solution

We first find the coordinates of the intersection between two curves.

$$\begin{cases} y = 4 - x^2 \\ y = x^2 - 2x \end{cases} \Rightarrow x^2 - 2x = 4 - x^2 \Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x-2)(x+1) = 0 \Rightarrow x = 2 \ \ (y = 0) \ \text{ or } \ x = -1 \ (y = 3).$$

Then the area of the region is given by

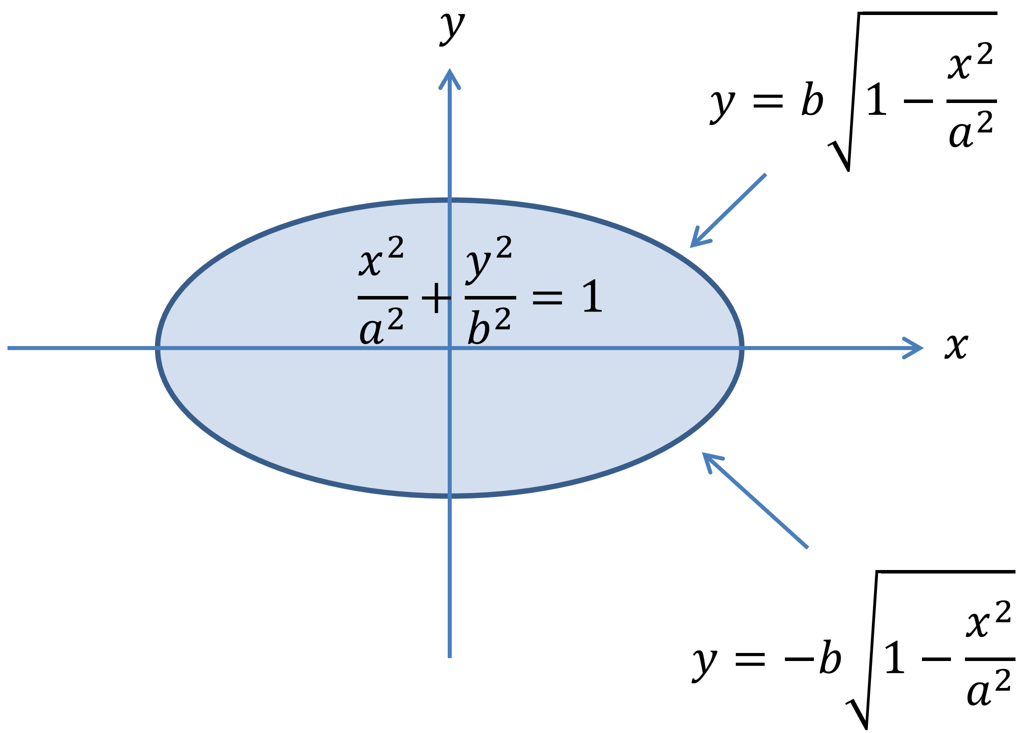
$$\int_{-1}^{2} [(4 - x^2) - (x^2 - 2x)]dx = \int_{-1}^{2} (-2x^2 + 2x + 4)dx$$

$$= -2\int_{-1}^{2} x^2 dx + 2\int_{-1}^{2} x dx + 4\int_{-1}^{2} dx$$

$$= -2\left[\frac{x^3}{3}\right]_{-1}^{2} + 2\left[\frac{x^2}{2}\right]_{-1}^{2} + 4[x]_{-1}^{2} = 9 \text{ (square units)}$$

Example 6

Find the area of the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a,\ b > 0$.



☺Solution:

By symmetry the area of the ellipse is then given by

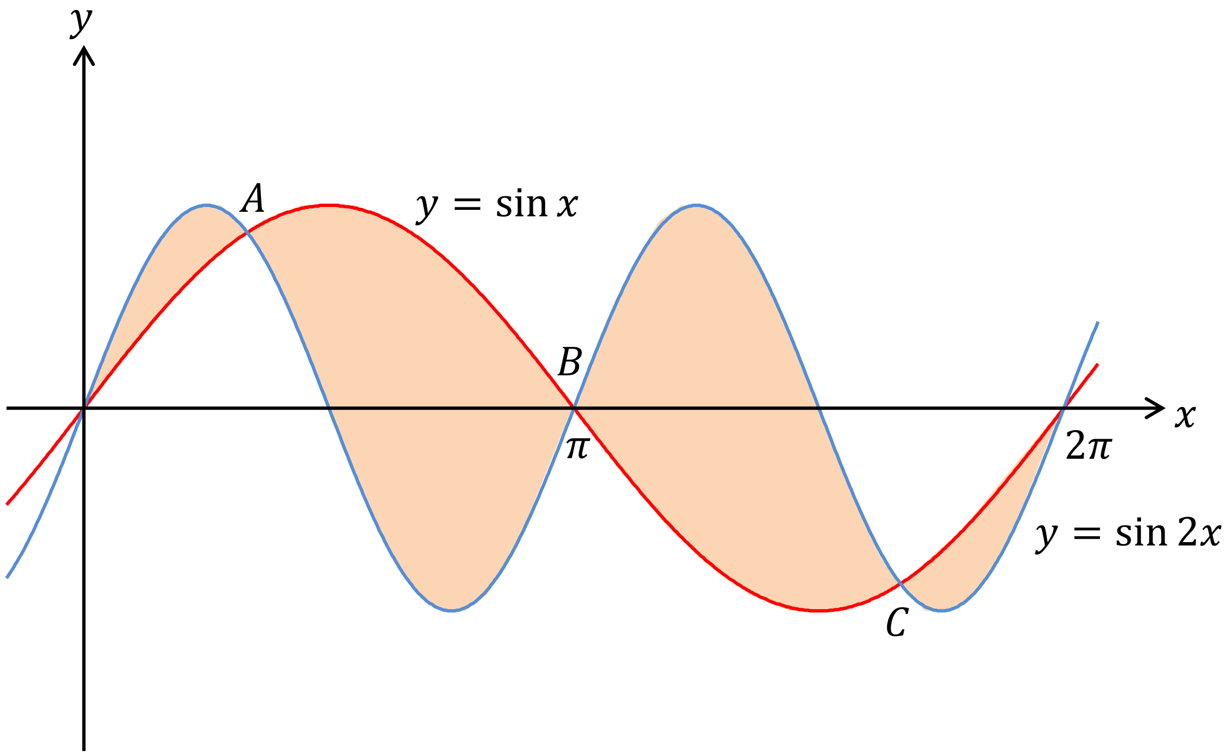
$$4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx = \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$\overset{x=a\sin\theta,\ \frac{dx}{d\theta}=a\cos\theta}{\cong} \quad = \frac{4b}{a}\int_0^{\frac{\pi}{2}} \sqrt{a^2-(a\sin\theta)^2}(a\cos\theta\, d\theta) = 4ab\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= 4ab\int_0^{\frac{\pi}{2}} \frac{(\cos 2\theta+1)}{2}\,d\theta = 2ab\left[\frac{\sin 2\theta}{2}+\theta\right]_0^{\frac{\pi}{2}} = \pi ab.$$

Example 7

Find the area of the region bounded by the curves $y = \sin 2x$ and $y = \sin x$ for $0 \le x \le 2\pi$.



☺Solution

We need to compute the area part by part. First of all, we need to find the intersection points A, B, C between these two curves. We solve the equation

$$\sin 2x = \sin x \Rightarrow \sin 2x - \sin x = 0$$

$$\overset{\substack{\text{compound angle}\\ \text{formula}}}{\overbrace{\Rightarrow}} \qquad 2\sin x\cos x - \sin x = 0$$

$$\Rightarrow \sin x\,(2\cos x - 1) = 0 \Rightarrow \sin x = 0 \ \text{ or } \ \cos x = \frac{1}{2}$$

$$\Rightarrow x = 0, \pi, 2\pi \ \text{ or } \ \frac{\pi}{3}, \frac{5\pi}{3}.$$

Hence, the coordinates of A, B and C are given by $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$, $(\pi, 0)$, $\left(\frac{5\pi}{3}, -\frac{\sqrt{3}}{2}\right)$, respectively.

Then, the area of the entire region is given by

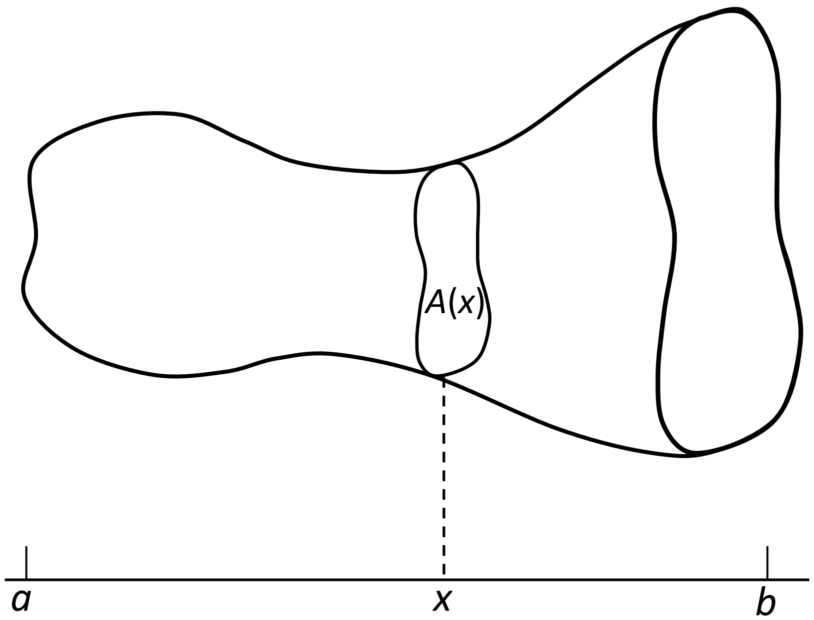
$$A = \int_0^{\frac{\pi}{3}} (\sin 2x - \sin x)dx + \int_{\frac{\pi}{3}}^{\pi} (\sin x - \sin 2x)dx + \int_{\pi}^{\frac{5\pi}{3}} (\sin 2x - \sin x)dx$$
$$+ \int_{\frac{5\pi}{3}}^{2\pi} (\sin x - \sin 2x)dx$$

$$= \left[-\frac{1}{2}\cos 2x + \cos x\right]_0^{\frac{\pi}{3}} + \left[-\cos x + \frac{1}{2}\cos 2x\right]_{\frac{\pi}{3}}^{\pi} + \left[-\frac{1}{2}\cos 2x + \cos x\right]_{\pi}^{\frac{5\pi}{3}}$$
$$+ \left[-\cos x + \frac{1}{2}\cos 2x\right]_{\frac{5\pi}{3}}^{2\pi}$$

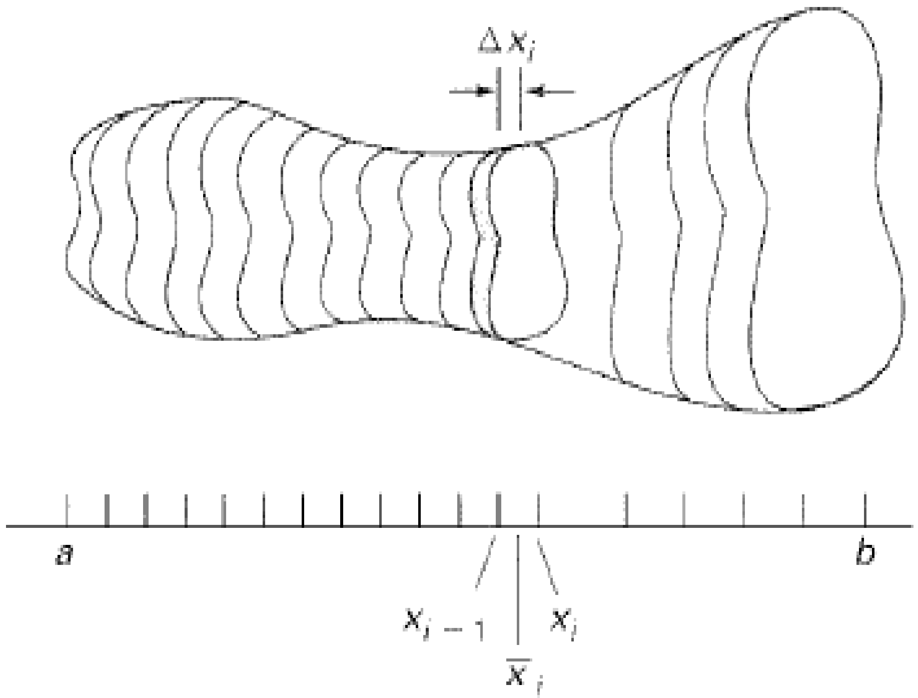
$$= \frac{1}{4} + \frac{9}{4} + \frac{9}{4} + \frac{1}{4} = 5.$$

Calculation of volumes of some objects

- Consider a solid which the cross sections are perpendicular to a given line (say the x-axis).
- As an example, suppose we would like to find the volume of the solid shown below



Suppose the cross section area $A(x)$ is known, one can find the volume of the solid using integration. To do this, we first try to cut the solid into n small pieces.



Each small piece “looks like” a cylinder.

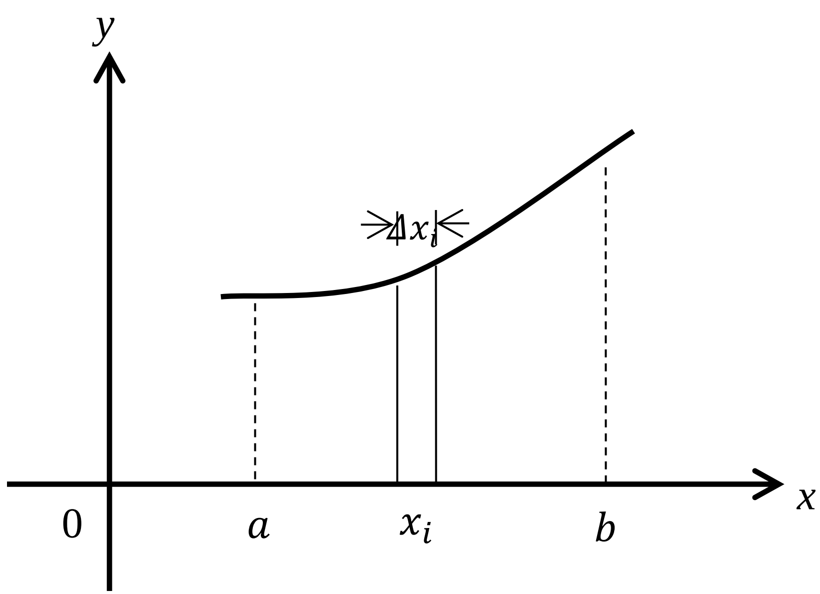
The volume of each small piece is approximated by $\Delta V_i \approx A(\bar{x}_i)\Delta x_i$ where $\bar{x}_i$ is some point between x_{i-1} and x_i .

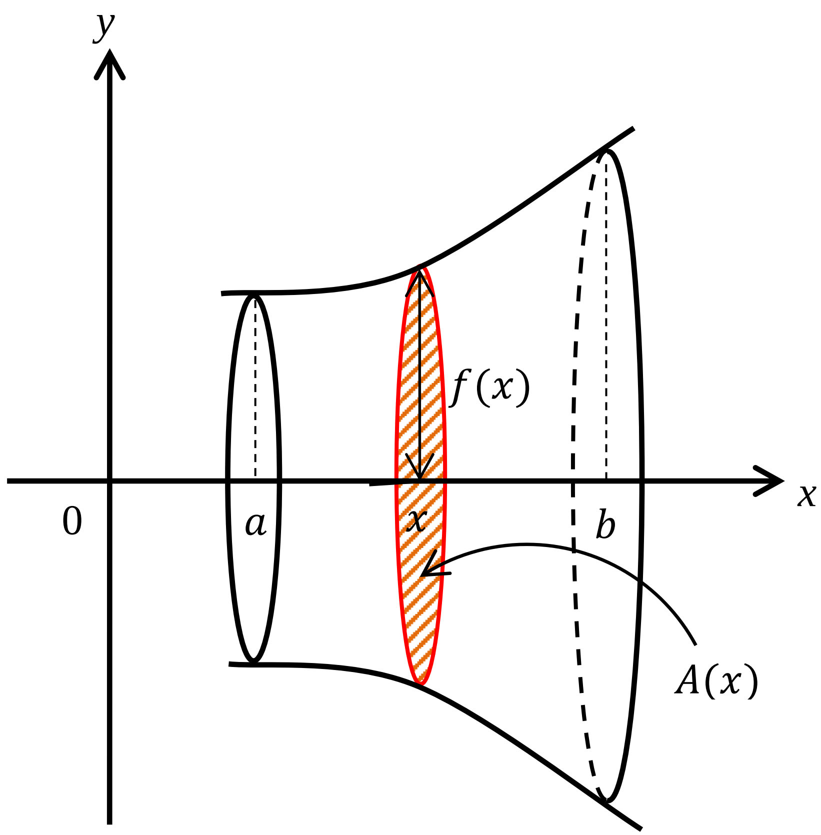
Therefore, the volume of the solid is approximated by $V \approx \sum_{i=1}^{n} A(\bar{x}_i)\Delta x_i$.

If we cut the solid into more pieces ($n \to \infty$ and $\Delta x_i \to 0$), then

$$V = \lim_{n\to\infty}\sum_{i=1}^{n} A(\bar{x}_i)\Delta x_i = \int_a^b A(x)dx.$$

Suppose we would like to find the volume of the solid formed by rotating a continuous curve $y = f(x)$ about the x-axis.





- Each cross-section is a circle. Therefore

$$A(x) = \pi[f(x)]^2.$$

- Therefore, the volume of the solid is given by

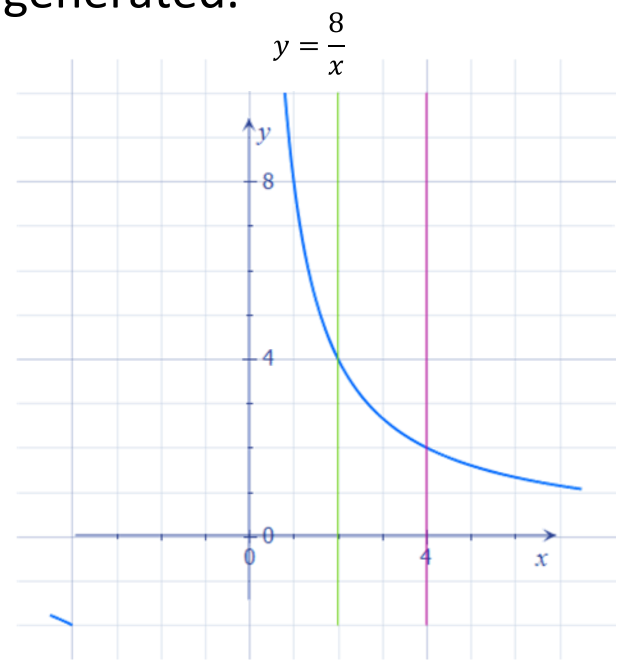
$$V_x = \int_a^b A(x)dx = \int_a^b \pi[f(x)]^2 dx.$$

Remark:

- Different from finding the area bounded by curves, we need not to care whether the curve is above the x-axis or below the x-axis.

Example 8

The portion of the curve $xy = 8$ from $x = 2$ to $x = 4$ is rotated about the x-axis. Find the volume of the solid generated.



☺Solution:

The volume

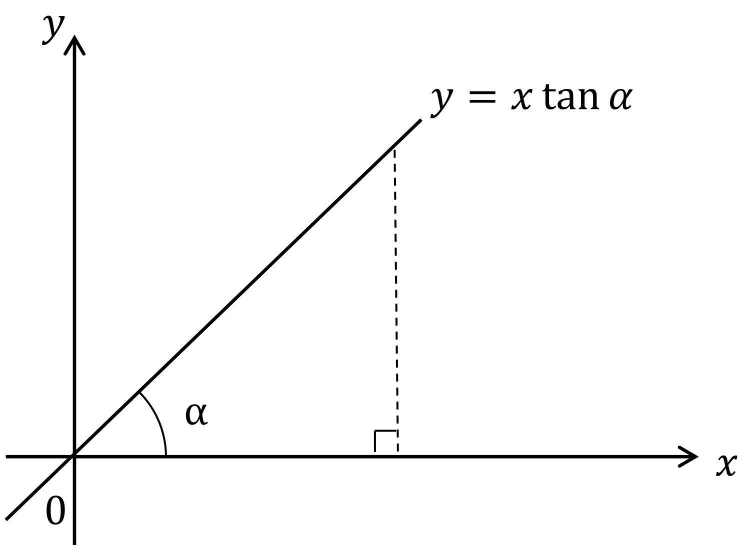
$$V_x = \int_2^4 \pi y^2 dx = \int_2^4 \pi \left(\frac{8}{x}\right)^2 dx = 64\pi \int_2^4 \frac{1}{x^2} dx = 64\pi \left[-\frac{1}{x}\right]_2^4 = 16\pi.$$

Example 9

Find the volume of a right circular cone of height h and semi-vertical angle α.

☺Solution:

The required circular cone can be generated by rotating the straight line $y = x\tan\alpha$ from $x = 0$ to $x = h$ about the x-axis.



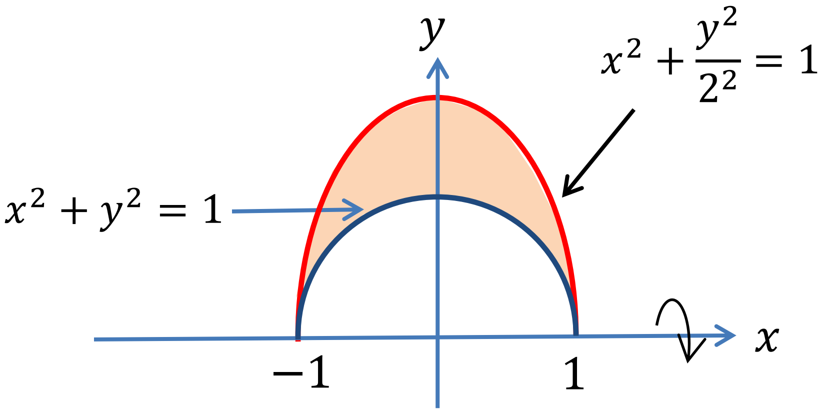
The volume is then given by

$$V_x = \int_0^h \pi y^2 dx = \int_0^h \pi (x \tan \alpha)^2 dx = \pi \tan^2 \alpha \int_0^h x^2 dx$$

$$= \pi \tan^2 \alpha \left[\frac{x^3}{3}\right]_0^h = \frac{1}{3} \pi \tan^2 \alpha \, h^3.$$

Example 10

Find the volume of the solid formed by rotating the region bounded by the upper half of the ellipse $x^2 + \frac{y^2}{2^2} = 1$ and the upper half of the circle $x^2 + y^2 = 1$ about the x-axis.



☺Solution:

Using the graph above, the volume of the solid is then given by

$$V_x = \underbrace{\int_{-1}^{1} \pi \left(2\sqrt{1-x^2}\right)^2 dx}_{\text{volume of ellipoid}} - \underbrace{\int_{-1}^{1} \pi \left(\sqrt{1-x^2}\right)^2 dx}_{\text{volume of sphere}}$$

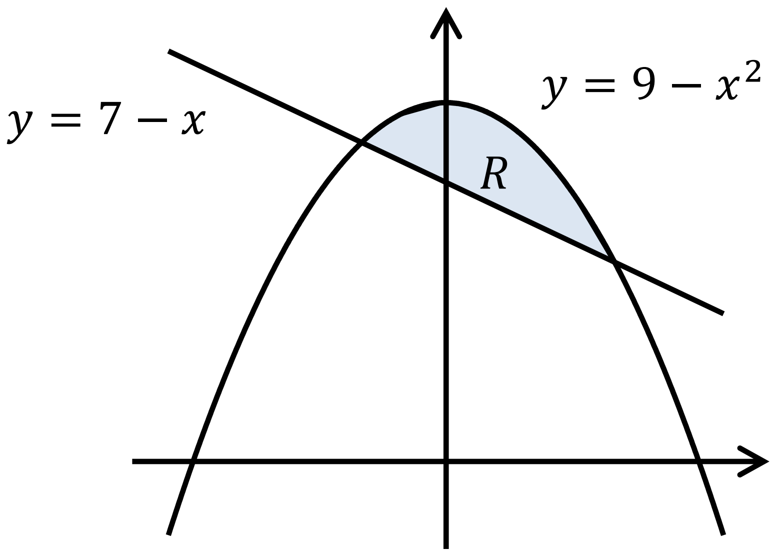
$$= 3\pi \int_{-1}^{1} (1-x^2)dx = 6\pi \int_{0}^{1} (1-x^2)dx \quad \text{as the integrand is an even function}$$

$$= 6\pi \left[x - \frac{x^3}{3}\right]_0^1$$

$$= 4\pi.$$

Example 11

Let R be the region bounded by $y = 9 - x^2$ and $y = 7 - x$. Find the volume of the solid generated by rotating the region R about the x-axis.



☺Solution:

The intersection points between two given curves are found to be

$$\begin{cases} y = 9 - x^2 \\ y = 7 - x \end{cases} \Rightarrow 9 - x^2 = 7 - x \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$$

$$\Rightarrow x = 2 \ \ (y = 5), \qquad x = -1 \ \ (y = 8)$$

Using similar technique as in Example 10, the required volume is given by

$$V_x = \pi \int_{-1}^{2} (9 - x^2)^2 dx - \pi \int_{-1}^{2} (7 - x)^2 dx$$

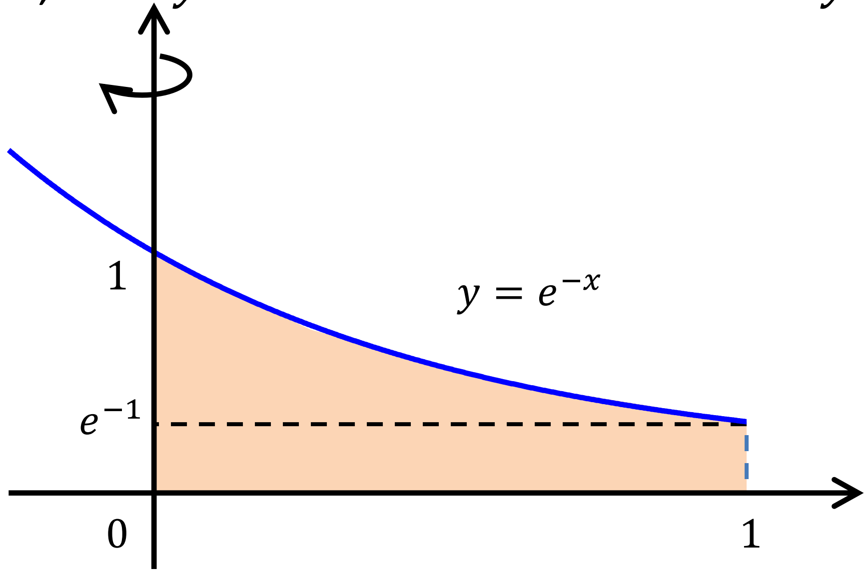
$$= \pi \int_{-1}^{2} (x^4 - 19x^2 + 14x + 32) dx$$

$$= \pi \left[\frac{x^5}{5} - \frac{19x^3}{3} + 7x^2 + 32x \right]_{-1}^{2} = \frac{333\pi}{5}.$$

The rotation axis is the y-axis: Volume $V_y = \int_c^d \pi[g(y)]^2 dy$ for revolving the region bounded by the curve $x = g(y)$ over $[c, d]$ about the y-axis

Example 12

Find the volume of the solid generated by revolving the area bounded by the curve $y = e^{-x}$, the x-axis, the y-axis and $x = 1$ about the y-axis.



☺Solution:

Although the rotation axis is not the x-axis, one can use the formula to obtain the volume by *interchanging the role of x and y.*

The equation of the graph $y = e^{-x}$ can be rewritten as $x = -\ln y$. The volume is then given by

$$V_y = \int_0^{e^{-1}} \pi(1)^2 dy + \int_{e^{-1}}^{1} \pi(-\ln y)^2 dy = \pi \int_0^{e^{-1}} 1 dy + \pi \int_{e^{-1}}^{1} \underbrace{(\ln y)^2}_{u} \underbrace{dy}_{dv}$$

$$= \pi e^{-1} + \pi \left[y(\ln y)^2 |_{e^{-1}}^{1} - \int_{e^{-1}}^{1} y \, d(\ln y)^2 \right] \text{ by integration by parts}$$

$$= \pi e^{-1} - \pi e^{-1} - \pi \int_{e^{-1}}^{1} y \left(\frac{2 \ln y}{y} \right) dy = -2\pi \int_{e^{-1}}^{1} \underbrace{\ln y}_{u} \underbrace{dy}_{dv}$$

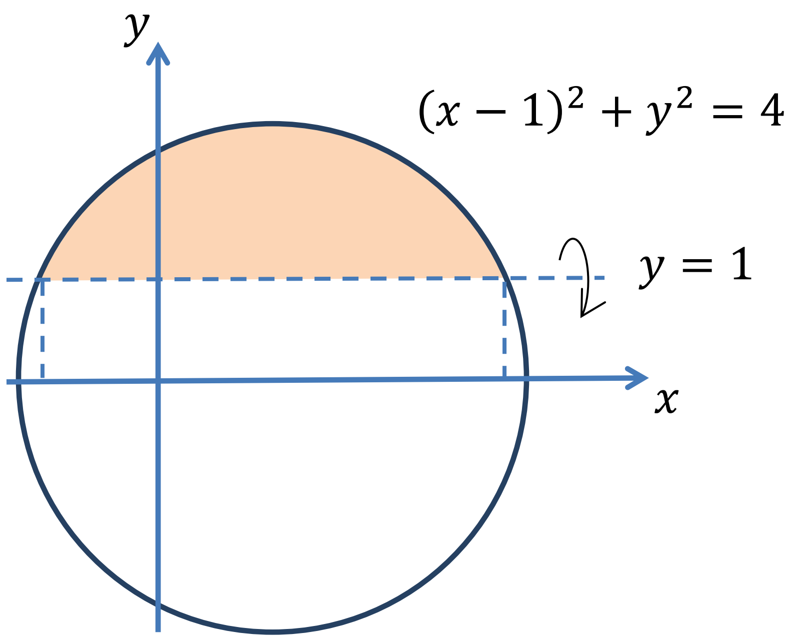
$$= -2\pi \left[y \ln y \, |_{e^{-1}}^{1} - \int_{e^{-1}}^{1} y \, d(\ln y) \right] \text{ by integration by parts}$$

$$= -2\pi e^{-1} + 2\pi \int_{e^{-1}}^{1} 1 dy$$

$$= -2\pi e^{-1} + 2\pi(1 - e^{-1}) = 2\pi(1 - 2e^{-1}).$$

Example 13

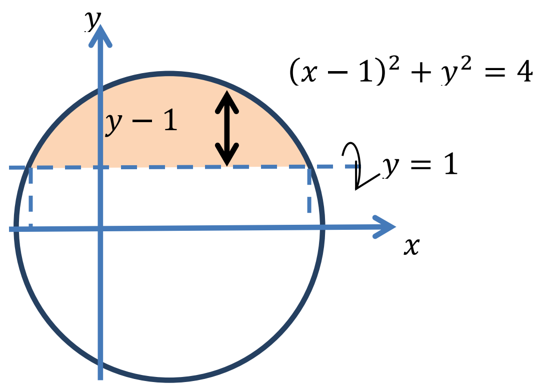
Let R be the region bounded by upper half of a circle $(x-1)^2+y^2=4$ and the line $y=1$. Find the volume of the solid generated by revolving the region R about the line $y=1$ (not the x-axis or the y-axis).



☺Solution: Lower and upper limits:

$$(x-1)^2+1^2=4 \quad \Rightarrow \quad x=1\pm\sqrt{3}.$$

Using the graph on the R.H.S, the volume is given by



$$V=\int_{1-\sqrt{3}}^{1+\sqrt{3}} \pi\left(\sqrt{4-(x-1)^2}-1\right)^2 dx$$

$$=\pi\int_{1-\sqrt{3}}^{1+\sqrt{3}} \left(5-(x-1)^2-2\sqrt{4-(x-1)^2}\right) dx$$

$$\overset{(x-1)=2\sin\theta}{\triangleq} \quad \pi\int_{1-\sqrt{3}}^{1+\sqrt{3}} (4+2x-x^2)dx-2\pi\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sqrt{4-4\sin^2\theta}\,(2\cos\theta\, d\theta)$$

$$=\pi\left[4x+x^2-\frac{x^3}{3}\right]_{1-\sqrt{3}}^{1+\sqrt{3}}-8\pi\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos^2\theta\, d\theta$$

$$=8\sqrt{3}\pi-8\pi\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\cos 2\theta+1}{2} d\theta=\cdots=6\sqrt{3}\pi-\frac{8\pi^2}{3}.$$

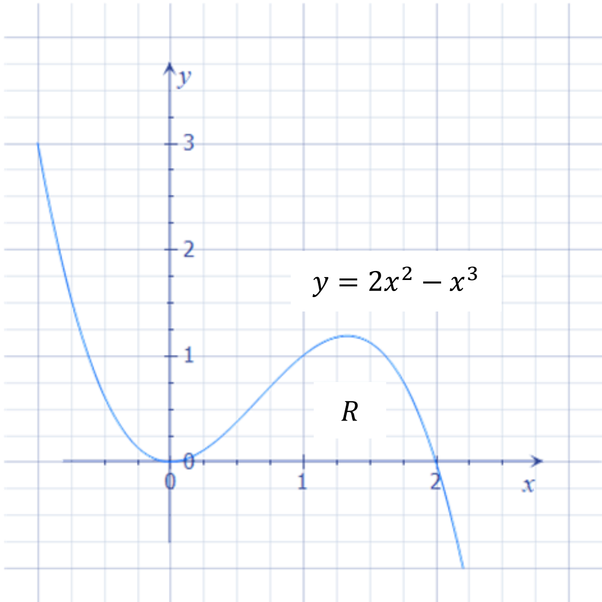
Volume by cylindrical shells: Shell method

The volume of the solid generated by revolving the region bounded by the curve $y = f(x)$ over $[a, b]$ one complete revolution about the y-axis:

$$V_y = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{2\pi x_i f(x_i)}_{\substack{\text{area of the shell} \\ \text{with radius } r_i = x_i \\ \text{and height } h_i = f(x_i)}} \underbrace{\Delta x_i}_{\substack{\text{shell} \\ \text{width}}} = \int_a^b 2\pi x f(x)\,dx$$

Example 14

Find the volume of the solid generated by revolving the region R bounded by the curve $y = 2x^2 - x^3$ and the x-axis about the y-axis.



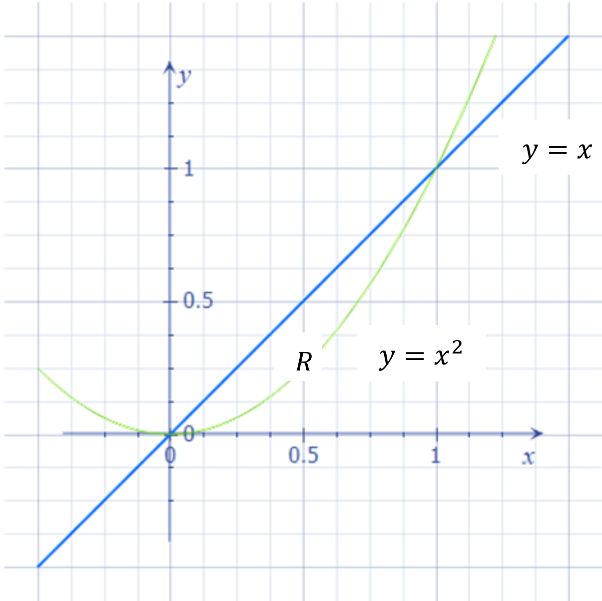
☺Solution:

Lower and upper limits: $0 = 2x^2 - x^3 = x^2(2 - x) \Rightarrow x = 0 \text{ or } 2.$

$$V_y = 2\pi \int_0^2 xy dx = 2\pi \int_0^2 x(2x^2 - x^3) dx = 2\pi \int_0^2 (2x^3 - x^4) dx = \frac{16\pi}{5}.$$

Example 15

Find the volume of the solid generated by revolving the region R bounded by the curve $y = x$ and $y = x^2$ about the y-axis.



☺Solution:

Lower and upper limits: $x = x^2 \Rightarrow 0 = x^2 - x = x(x-1) \Rightarrow x = 0,\ 1.$

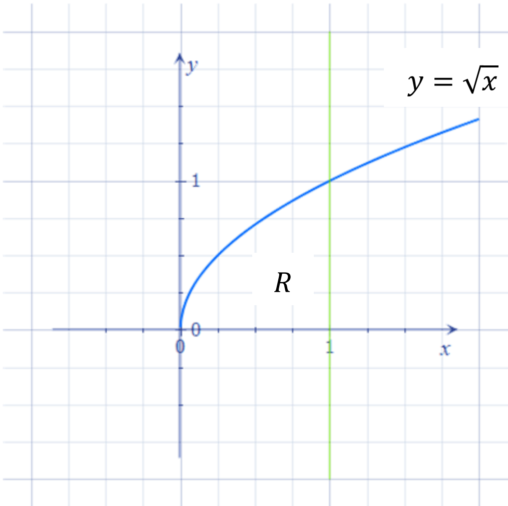
Height: $h = x - x^2$

$$V_y = 2\pi \int_0^1 xh\,dx = 2\pi \int_0^1 x(x - x^2)dx = 2\pi \int_0^1 (x^2 - x^3)dx$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi \left[\frac{1}{3} - \frac{1}{4}\right] = \frac{2\pi}{12} = \frac{\pi}{6}.$$

Example 16

Find the volume of the solid generated by revolving the region R bounded by the curve $y = \sqrt{x}$, the x-axis and $x = 1$ about the x-axis.



☺Solution:

Lower and upper limits of y: $x = 0 \Rightarrow y = \sqrt{0} = 0$; $x = 1 \Rightarrow y = \sqrt{1} = 1$.

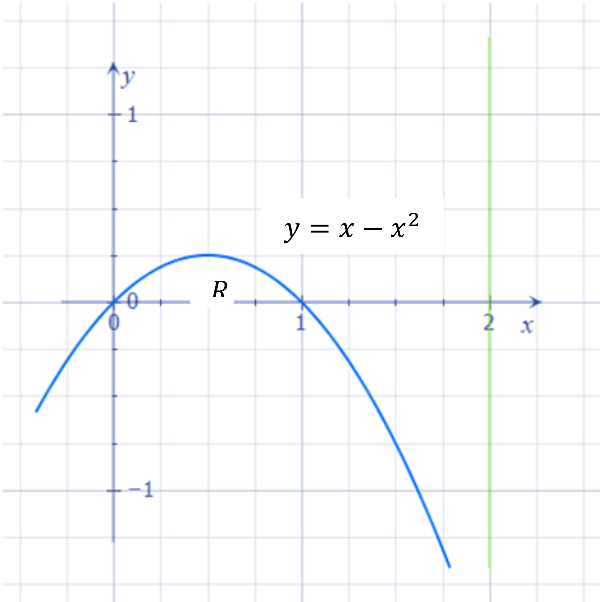
Horizontal height: $h = 1 - y^2$

$$V_x = 2\pi \int_0^1 yh dy = 2\pi \int_0^1 y(1 - y^2) dy = 2\pi \int_0^1 (y - y^3) dy$$

$$= 2\pi \left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = 2\pi \left[\frac{1}{2} - \frac{1}{4}\right] = \frac{2\pi}{4} = \frac{\pi}{2}.$$

Example 17

Find the volume of the solid generated by revolving the region R bounded by the curve $y = x - x^2$ and the x-axis about the line $x = 2$.



☺Solution: Lower and upper limits: $0 = x - x^2 = x(1-x) \Rightarrow x = 0,\ 1.$

Radius: $r = 2 - x$; Height: $y = x - x^2$

$$V = 2\pi \int_0^1 rh dx = 2\pi \int_0^1 (2-x)(x-x^2) dx = 2\pi \int_0^1 (2x - 3x^2 + x^3) dx$$

$$= 2\pi \left[x^2 - x^3 + \frac{x^4}{4} \right]_0^1 = 2\pi \left[1 - 1 + \frac{1}{4} \right] = \frac{2\pi}{4} = \frac{\pi}{2}.$$

Remark: If we apply the shell method for Example 12,

$$V_y = 2\pi \int_0^1 xy dx = 2\pi \int_0^1 x e^{-x} dx = 2\pi \int_0^1 x d(-e^{-x})$$

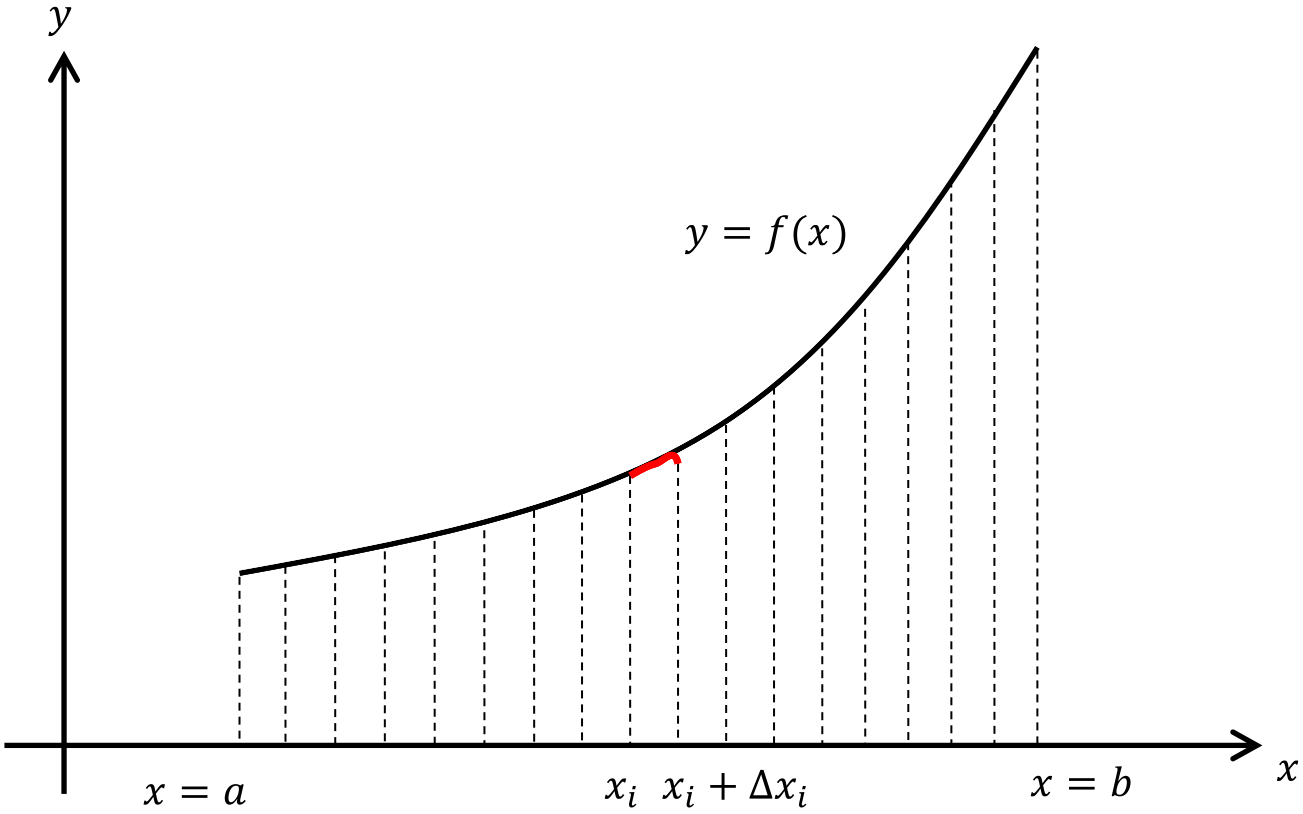
$$= 2\pi\{[-xe^{-x}]_0^1 + \int_0^1 e^{-x} dx\} \quad \text{(integration by parts)}$$

$$= 2\pi\{[-e^{-1} + 0] + [-e^{-x}]_0^1\} = 2\pi\{-e^{-1} + [-e^{-1} + 1]]$$

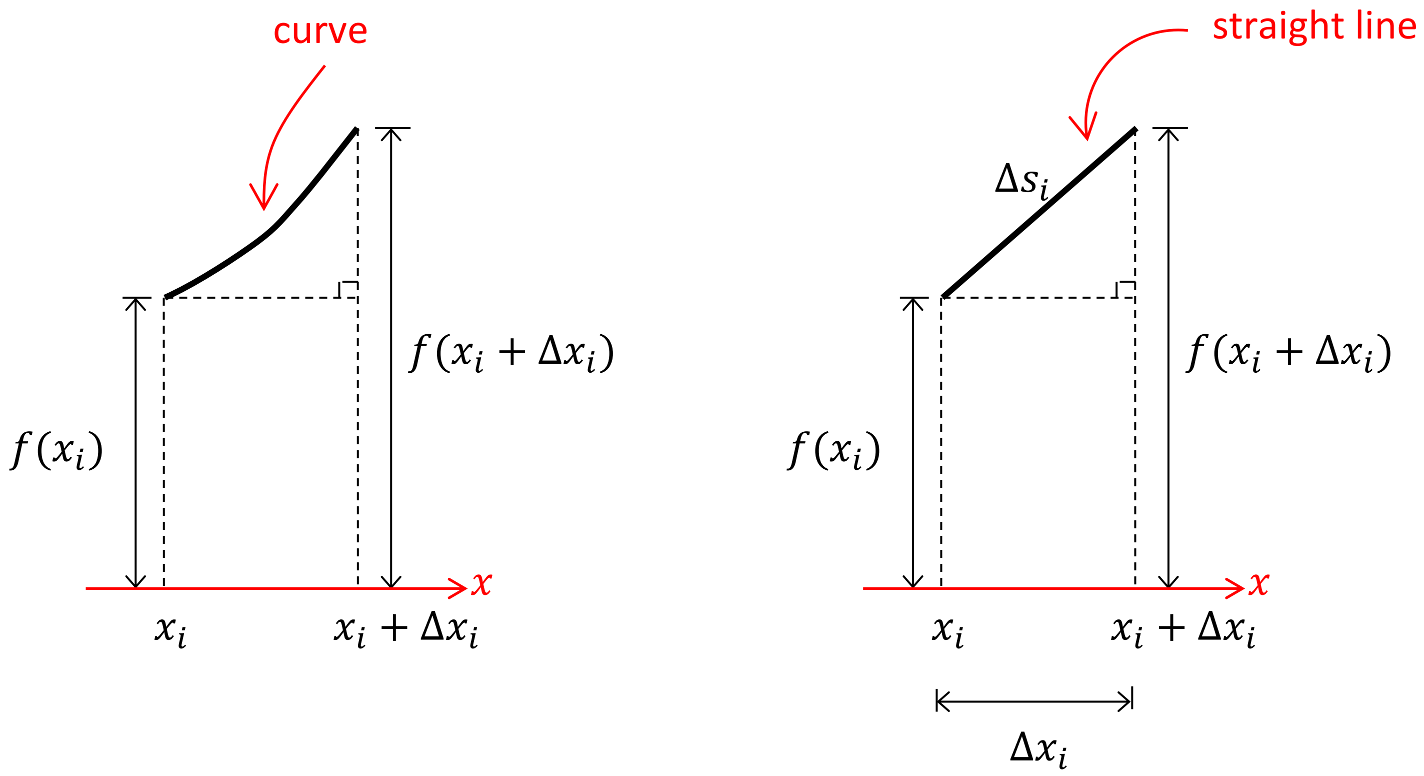
$$= 2\pi(1 - 2e^{-1}).$$

Arc length of a curve

- We adopt the similar procedure as we did when finding the volume of the solid. We first divide the curve into n parts:



- Each part of curve is approximated by a straight line joining the endpoints of the curve.



- The length of straight line Δs_i is found to be

$$\Delta s_i = \sqrt{(\Delta x_i)^2 + [f(x_i + \Delta x_i) - f(x_i)]^2}$$

$$= \Delta x_i \sqrt{1 + \left[\frac{f(x_i + \Delta x_i) - f(x_i)}{\Delta x_i}\right]^2} \approx \Delta x_i \sqrt{1 + \left(f'(x_i)\right)^2}.$$

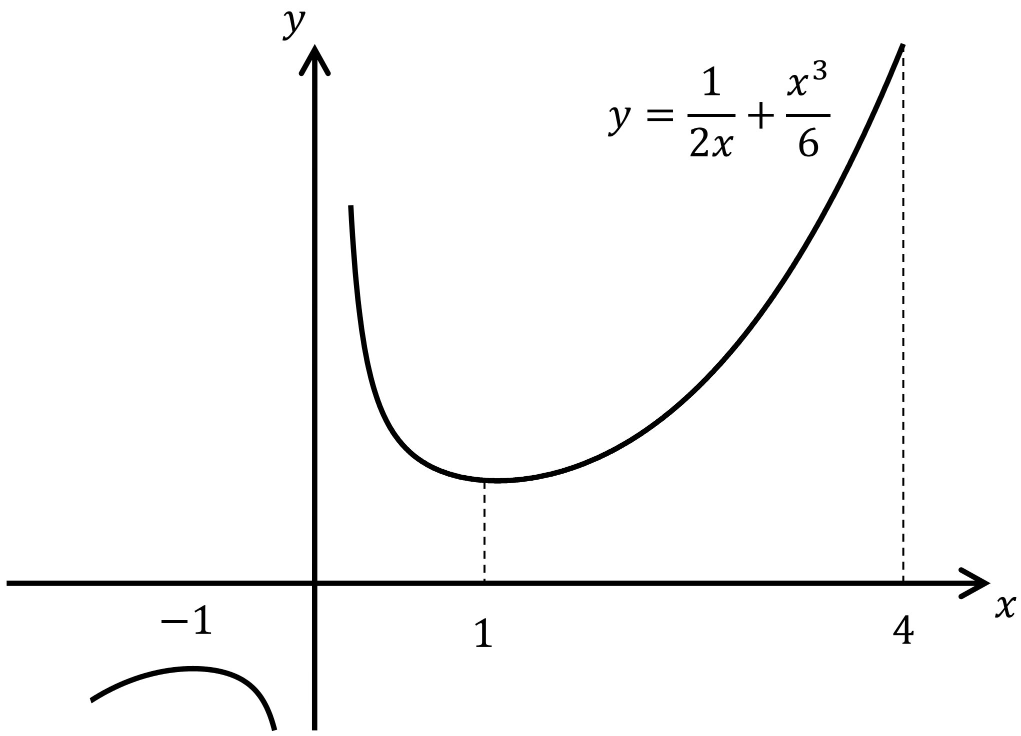
Therefore, the total length of the curve (arc length)

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left(f'(x_i)\right)^2}\,\Delta x_i$$

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

Example 18

Find the arc length of the curve $6xy = 3 + x^4$ (or $y = \frac{3+x^4}{6x}$) between the points whose abscissa (x-coordinate) are 1 and 4.



☺Solution:

Take $f(x) = \frac{3+x^4}{6x}$, note that

$$f'(x) = \frac{1}{6}\frac{x(4x^3) - (3 + x^4)}{x^2} = \frac{1}{6}\ \frac{3x^4 - 3}{x^2} = \frac{x^4 - 1}{2x^2}.$$

So the arc length is seen to be

$$s = \int_1^4 \sqrt{1 + [f'(x)]^2}dx = \int_1^4 \sqrt{1 + \left(\frac{x^4 - 1}{2x^2}\right)^2}\ dx$$

$$= \int_1^4 \sqrt{\frac{(2x^2)^2 + (x^4 - 1)^2}{(2x^2)^2}}dx$$

$$= \int_1^4 \sqrt{\frac{4x^4 + x^8 - 2x^4 + 1}{4x^4}}dx$$

$$= \int_1^4 \sqrt{\frac{x^8 + 2x^4 + 1}{4x^4}}\, dx$$

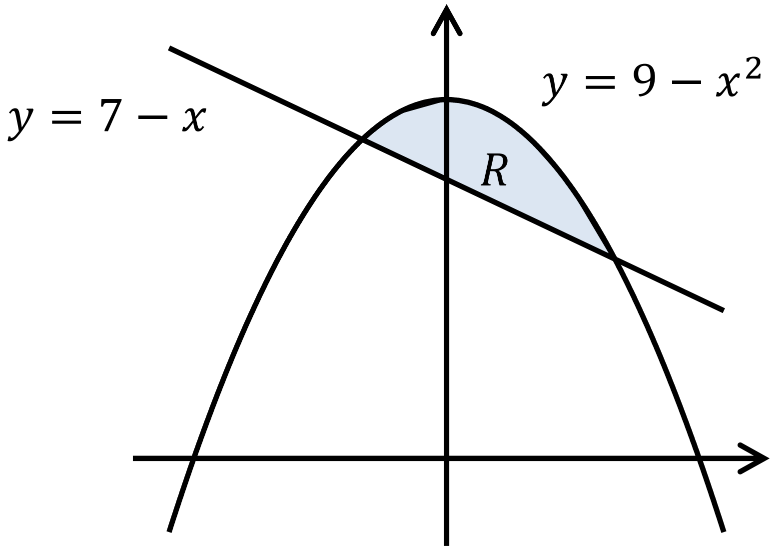
$$= \int_1^4 \sqrt{\frac{(x^4 + 1)^2}{4x^4}}\, dx$$

$$= \int_1^4 \frac{x^4 + 1}{2x^2}\, dx = \frac{1}{2}\int_1^4 \left(\frac{x^4}{x^2} + \frac{1}{x^2}\right) dx$$

$$= \frac{1}{2}\int_1^4 \left(x^2 + \frac{1}{x^2}\right) dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{1}{x}\right]_1^4 = \frac{87}{8}\,(\text{units}).$$

Example 19

Let R be the region bounded by $y = 9 - x^2$ and $y = 7 - x$. Find the arc length of the boundary curve of the region R.



☺Solution: As shown in Example 11,

Low and Upper limits: $9 - x^2 = 7 - x \;\Rightarrow\; x^2 - x - 2 = 0 \;\Rightarrow\; x = 2$ or $x = -1$.
Note that the boundary curves consists of two different parts: (1) the curve $y = 9 - x^2$ from $x = -1$ to $x = 2$ and (2) the straight line $y = 7 - x$ from $x = -1$ to $x = 2$.
Hence the total arc length is given by

$$\underbrace{\int_{-1}^{2}\sqrt{1+\left(\frac{d}{dx}(9-x^2)\right)^2}\,dx}_{\text{arc length of upper curve}}+\underbrace{\int_{-1}^{2}\sqrt{1+\left(\frac{d}{dx}(7-x)\right)^2}\,dx}_{\text{arc length of lower curve}}=\int_{-1}^{2}\sqrt{1+4x^2}dx+\int_{-1}^{2}\sqrt{2}dx$$

$$\overset{\substack{x=\frac{1}{2}\tan\theta\\ \Rightarrow\frac{dx}{d\theta}=\frac{1}{2}\sec^2\theta}}{\equiv}\int_{-\tan^{-1}2}^{\tan^{-1}4}\sqrt{1+4\left(\frac{1}{4}\tan^2\theta\right)}\left(\frac{1}{2}\sec^2\theta\,d\theta\right)+\left[\sqrt{2}x\right]_{-1}^{2}$$

$$=\frac{1}{2}\int_{-\tan^{-1}2}^{\tan^{-1}4}\sec^3\theta\,d\theta+3\sqrt{2}$$

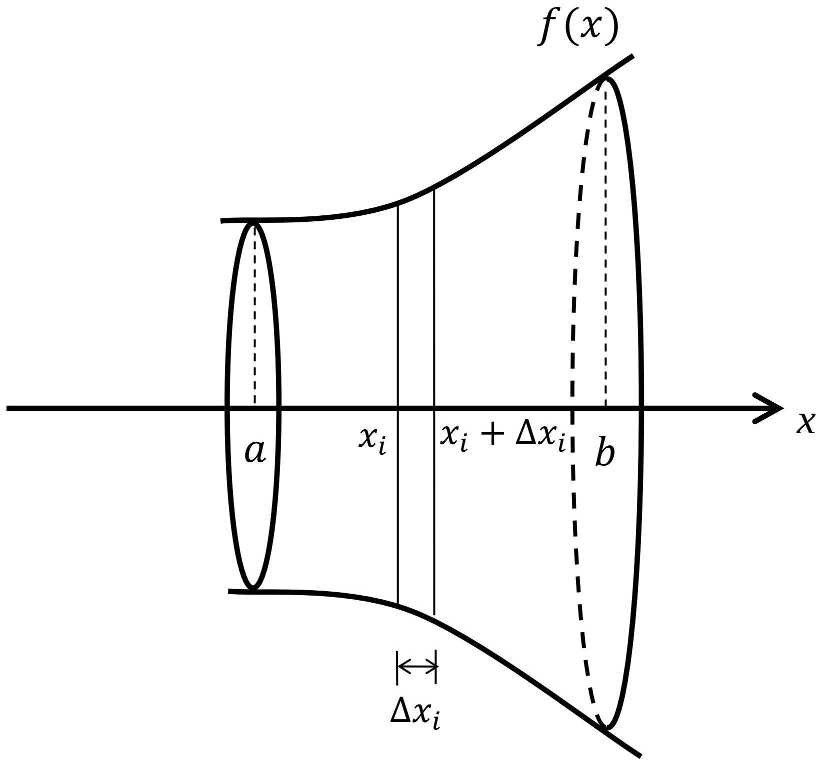
$$=\frac{1}{2}\left[\frac{1}{2}(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|)\right]_{-\tan^{-1}2}^{\tan^{-1}4}+3\sqrt{2}=\cdots$$

$$=\sqrt{17}+\frac{\sqrt{5}}{2}+\frac{1}{4}\ln\left|\sqrt{17}+4\right|-\frac{1}{4}\ln\left|\sqrt{5}-2\right|+3\sqrt{2}$$

$$=\sqrt{17}+\frac{\sqrt{5}}{2}+3\sqrt{2}+\frac{1}{4}\ln\left|\sqrt{17}+4\right|+\frac{1}{4}\ln\left|\sqrt{5}+2\right|.$$

Area of surface of revolution

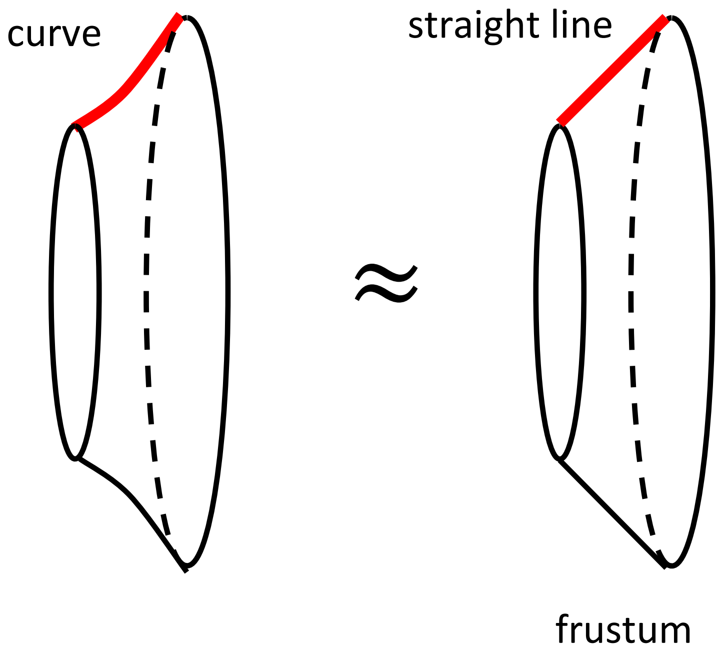
Suppose we would like to find the area of surface generated by rotating a continuous curve $y = f(x)$ about the x-axis.



We cut the object into n parts.

- Each smaller solid is approximated by the frustum.

- The surface area of each smaller solid approximately equals the surface area of the frustum.

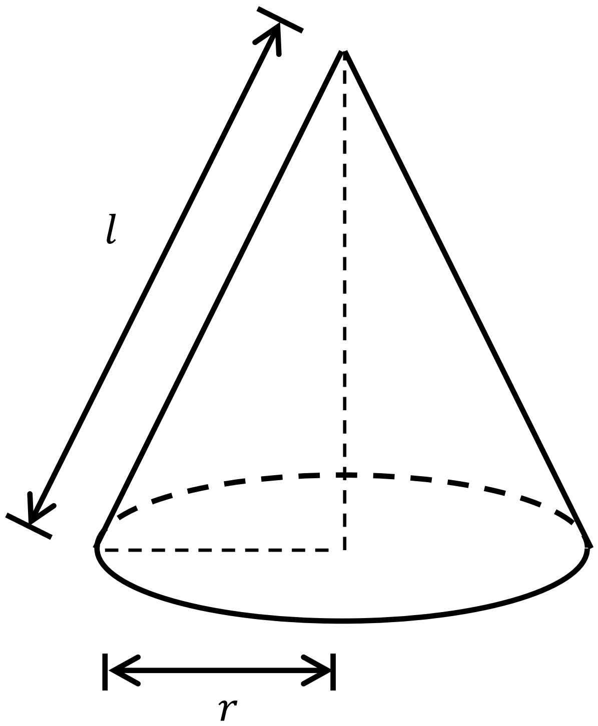


Question: How to find the surface area of the frustum?

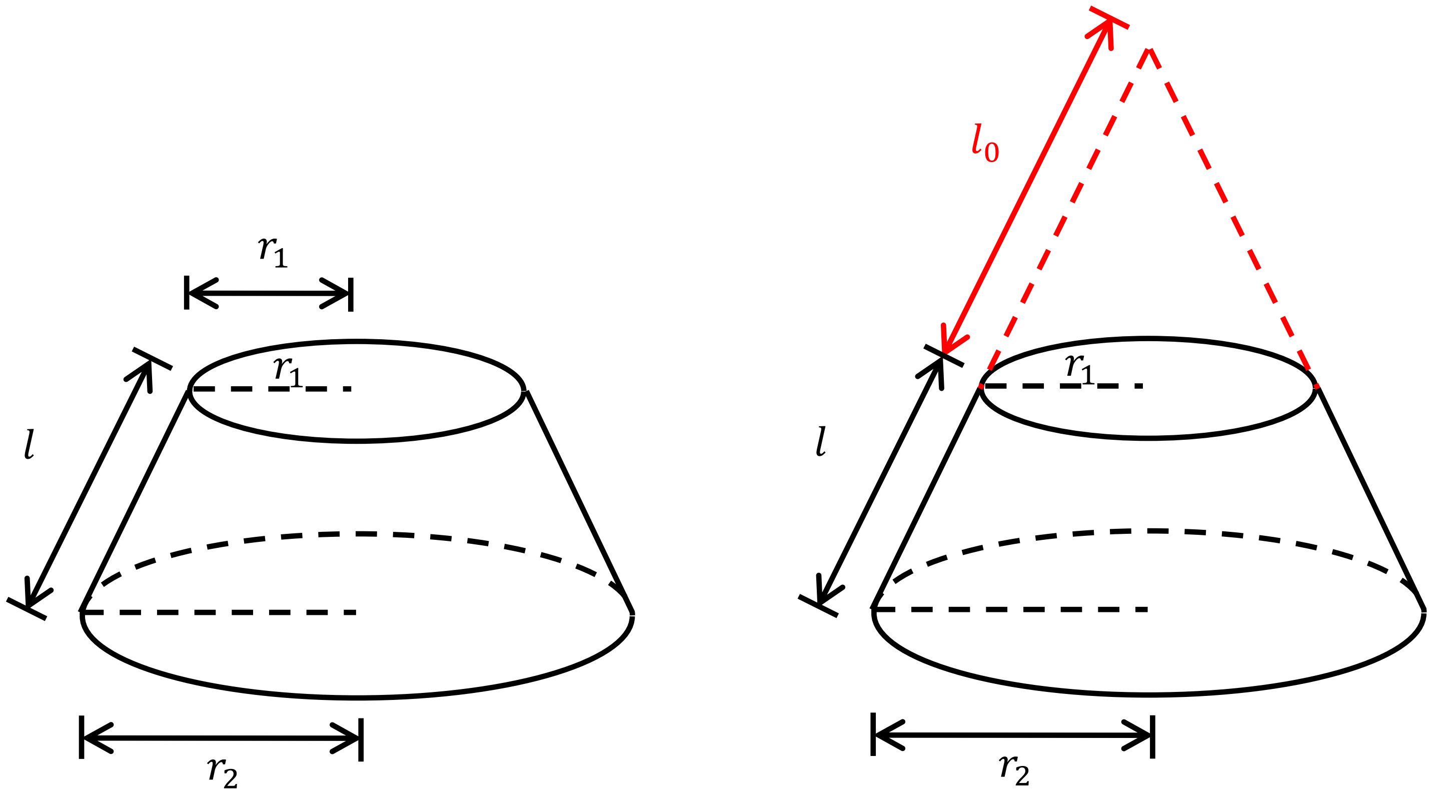
Recall that the surface area of a circular cone (shown below) is given by

$$S = \pi r l,$$

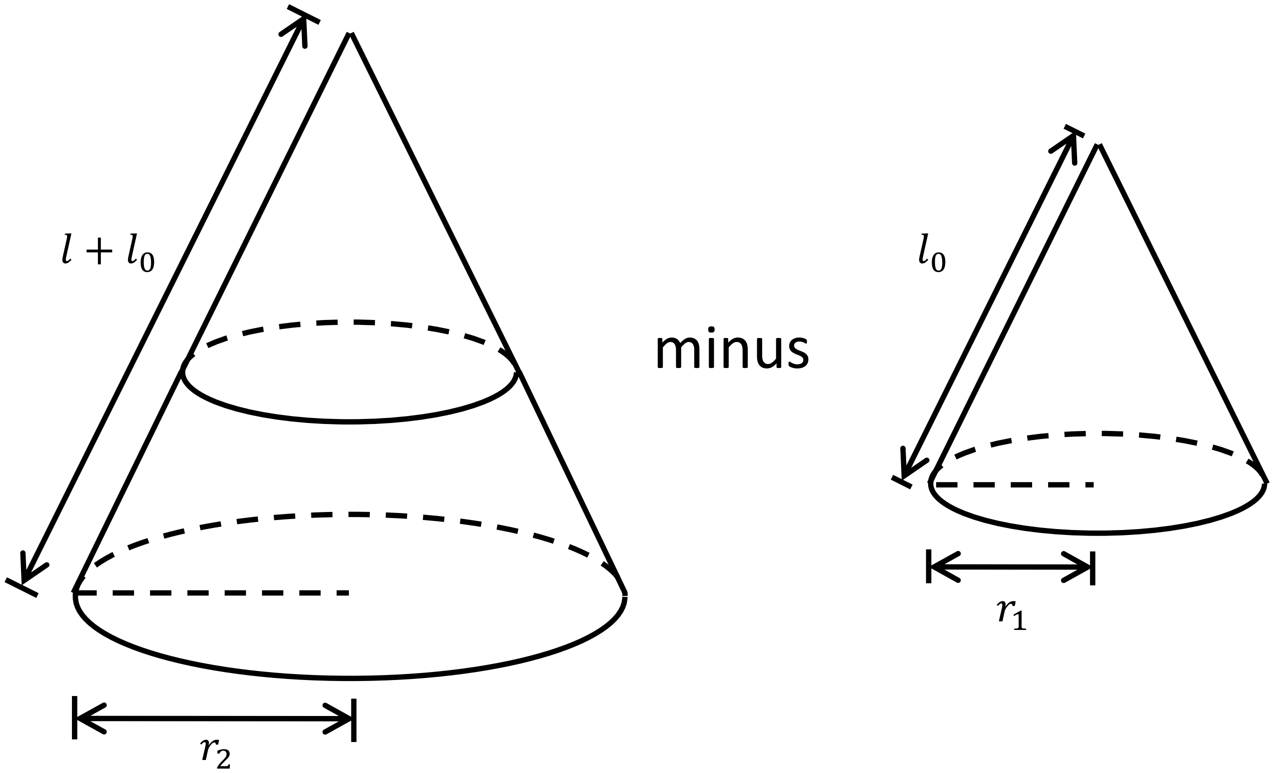
where r is the base radius and l is the slant height.



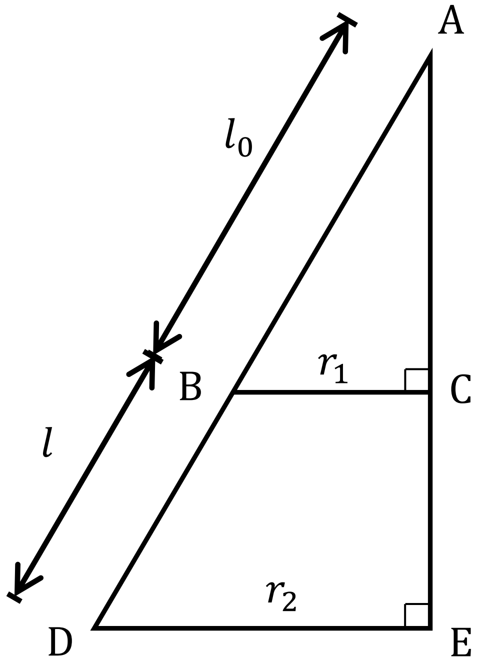
If we add a circular cone (red part) onto the frustum, the solid becomes a bigger cone.



Therefore, the surface area of the frustum is given by the difference of surface areas between two cones, i.e.



To compute l_0, one can consider the following figure:



Note that $\Delta ABC \sim \Delta ADE$, then we have

$$\frac{AB}{AD} = \frac{BC}{DE} \Rightarrow \frac{l_0}{l + l_0} = \frac{r_1}{r_2} \Rightarrow l_0 = \frac{r_1 l}{r_2 - r_1}.$$

Surface area of frustum

= (surface area of bigger cone) – (surface area of smaller cone)

$$= \pi r_2(l + l_0) - \pi r_1(l_0)$$

$$= \pi r_2\left(l + \frac{r_1 l}{r_2 - r_1}\right) - \pi r_1\left(\frac{r_1 l}{r_2 - r_1}\right)$$

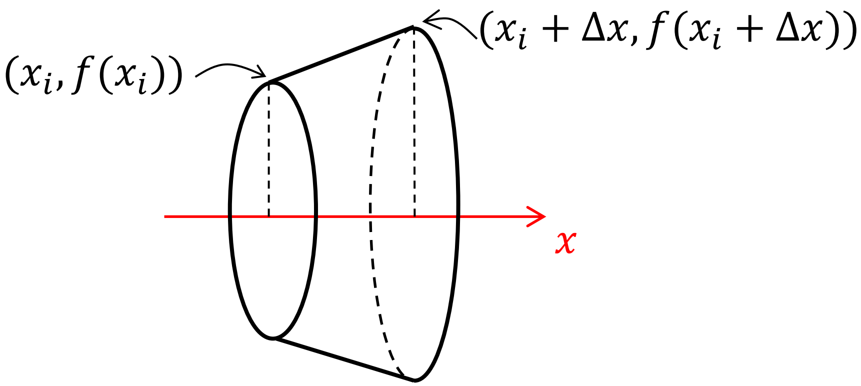
$$= \frac{\pi l(r_2^2 - r_1^2)}{r_2 - r_1} = \frac{\pi l(r_2 - r_1)(r_2 + r_1)}{(r_2 - r_1)} = \pi l(r_1 + r_2)$$

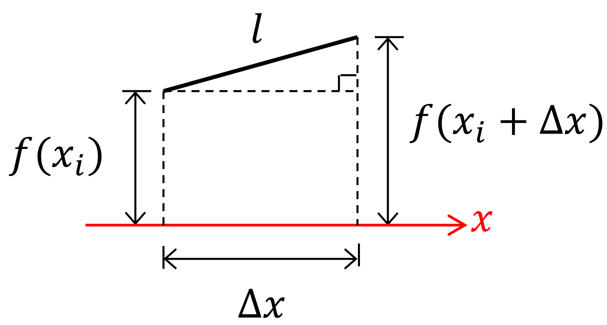
Back to our case, we know

- $r_1 = f(x_i)$ and $r_2 = f(x_i + \Delta x)$
- l is given by

$$l = \sqrt{(\Delta \mathrm{x})^2 + [f(x_i + \Delta x) - f(x_i)]^2}$$

$$= \Delta x \sqrt{1 + \left[\frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}\right]^2} \approx \Delta x \sqrt{1 + (f'(x_i))^2}.$$





Then the surface area of small frustum is given by

$$A(x_i) \approx \pi[f(x_i + \Delta x) + f(x_i)]\sqrt{1 + (f'(x_i))^2}\Delta x$$

$$\approx \pi[f(x_i) + f(x_i)]\sqrt{1 + (f'(x_i))^2}\Delta x \approx 2\pi f(x_i)\sqrt{1 + (f'(x_i))^2}\Delta x$$

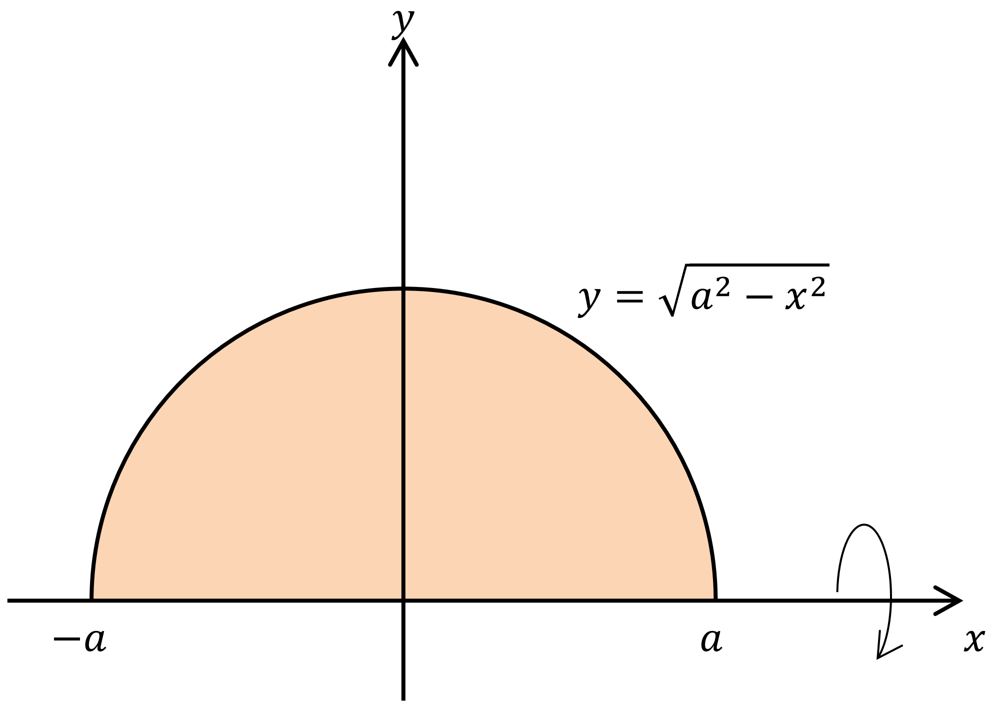
Therefore surface area of the solid generated by revolving the curve $y = f(x)$ over $[a, b]$ about the x-axis:

$$S_x = \lim_{n\to\infty}\sum_{i=1}^{n} A(x_i) = \lim_{n\to\infty}\sum_{i=1}^{n} \underbrace{2\pi f(x_i)}_{\substack{\text{perimeter}\\ \text{with raduis}\\ r_i = f(x_i)}} \underbrace{\sqrt{1 + (f'(x_i))^2}\Delta x_i}_{\text{width } \Delta s_i}$$

$$S_x = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\,dx.$$

Example 20

The sphere with radius a $(a > 0)$ can be formed by rotating the upper half of the curve $y = \sqrt{a^2 - x^2}$ (or $x^2 + y^2 = a^2$) from $x = -a$ to $x = a$ about the x-axis. Find the surface area of the sphere.



☺Solution:

Let $f(x) = \sqrt{a^2 - x^2}$.

Step 1: Compute $\boldsymbol{f'(x)}$

$$f'(x) = \frac{d}{dx}\sqrt{a^2 - x^2} = \frac{d}{dx}(a^2 - x^2)^{\frac{1}{2}}$$

$$= \frac{d\left((a^2 - x^2)^{\frac{1}{2}}\right)}{d(a^2 - x^2)} \frac{d(a^2 - x^2)}{dx} \qquad \text{(Chain Rule)}$$

$$= -\frac{x}{\sqrt{a^2 - x^2}}.$$

Step 2: Calculate the surface area

$$A = 4\pi \int_0^a f(x)\sqrt{1 + [f'(x)]^2}dx \quad \text{by symmetry}$$

$$= 4\pi \int_0^a \sqrt{a^2 - x^2} \sqrt{1 + \left(-\frac{x}{\sqrt{a^2 - x^2}}\right)^2} dx$$

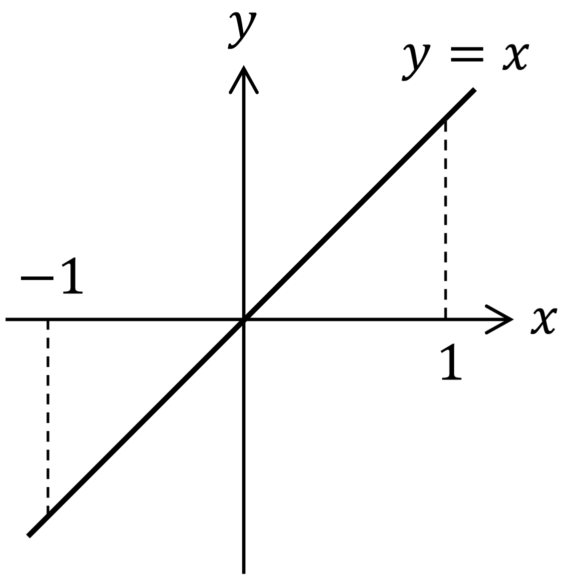
$$= 4\pi \int_0^a \sqrt{a^2 - x^2} \sqrt{1 + \frac{x^2}{a^2 - x^2}} dx$$

$$= 4\pi \int_0^a \sqrt{a^2}dx = 4\pi \int_0^a adx = 4\pi a \int_0^a dx$$

$$= 4\pi a[x]_0^a = 4\pi a(a) = 4\pi a^2.$$

Example 21

Find the area A of the surface generated by rotating the line $y = x$ from $x = -1$ to $x = 1$ about the x-axis.



☹Wrong Solution

Take $f(x) = x$, we have $f'(x) = 1$.

The required surface area

$$A = 2\pi \int_{-1}^{1} f(x)\sqrt{1 + [f'(x)]^2}dx$$

$$= 2\pi \int_{-1}^{1} x\sqrt{1+1}dx$$

$$= 2\sqrt{2}\pi \int_{-1}^{1} xdx$$

$$= 2\sqrt{2}\pi \left[\frac{x^2}{2}\right]_{-1}^{1} = 0 \ < - - - - - - - \text{(Oops)}$$

Remark about the surface area formula

- The formula works well when the entire curve $y = f(x)$ lies above the x-axis (i.e., $f(x)$ is positive).

- When the curve $y = f(x)$ lies below the x-axis, the integral
$$2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}dx$$
becomes a negative number.

- If such case happens, split the surface into two main parts (lies above the x-axis / below the x-axis) and compute the surface area separately.

☺Correct Solution of Example 21

Take $f(x) = x$, we have $f'(x) = 1$.

The required surface area

$$A = 2\pi\left[\int_0^1 f(x)\sqrt{1+[f'(x)]^2}dx + \left(-\int_{-1}^0 f(x)\sqrt{1+[f'(x)]^2}dx\right)\right]$$

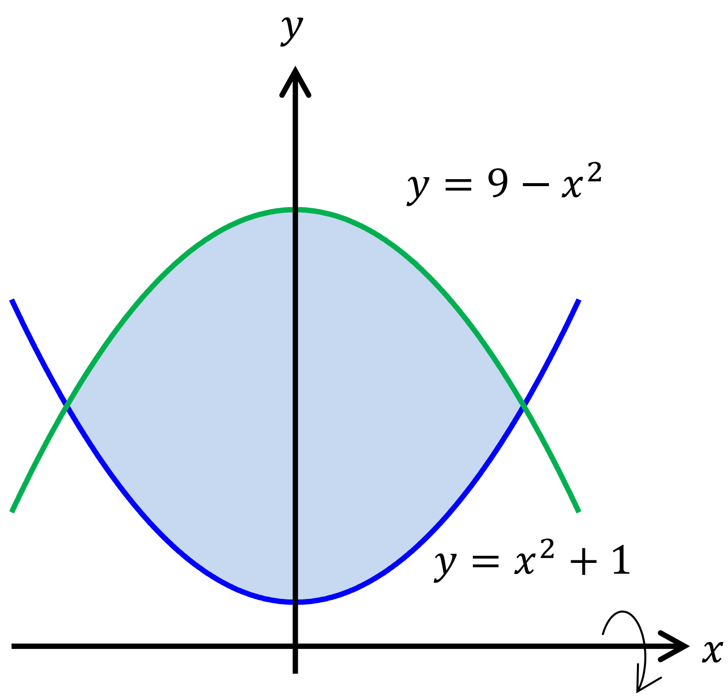
$$= 2\pi\left[\int_0^1 x\sqrt{1+1}dx - \int_{-1}^0 x\sqrt{1+1}dx\right]$$

$$= 2\pi\left\{\sqrt{2}\left[\frac{x^2}{2}\right]_0^1 - \sqrt{2}\left[\frac{x^2}{2}\right]_{-1}^0\right\}$$

$$= 2\sqrt{2}\pi.$$

Example 22

Let R be the region bounded by the curves $y = x^2 + 1$ and $y = 9 - x^2$. Find the area A of the surface generated by rotating the region R about the x-axis.



☺Solution

The surface area of the entire object

$$= \text{surface area of outer part of the solid} + \text{surface area of inner part of the solid}$$

$$= 2\pi\int_{-2}^{2}(9-x^2)\sqrt{1+[-2x]^2}dx + 2\pi\int_{-2}^{2}(1+x^2)\sqrt{1+[2x]^2}dx$$

$$= 20\pi\int_{-2}^{2}\sqrt{1+4x^2}dx = 40\pi\int_{0}^{2}\sqrt{1+4x^2}dx \quad \text{by symmetry}$$

$$\overset{x=\frac{\tan\theta}{2}}{\cong} 40\pi\int_{0}^{\tan^{-1}4}\sqrt{1+\tan^2\theta}\left(\frac{\sec^2\theta}{2}d\theta\right) = 20\pi\int_{0}^{\tan^{-1}4}\sec^3\theta\, d\theta$$

$$= 10\pi[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|]_0^{\tan^{-1}4}$$

$$= 40\sqrt{17}\pi + 10\pi\ln\left|\sqrt{17}+4\right|.$$

Parametric equations of curve

- Alternatively, the curve in 2-D plane can be expressed in the following form:
$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}, \quad a \leq t \leq b$$
where t is another variable called a parameter.

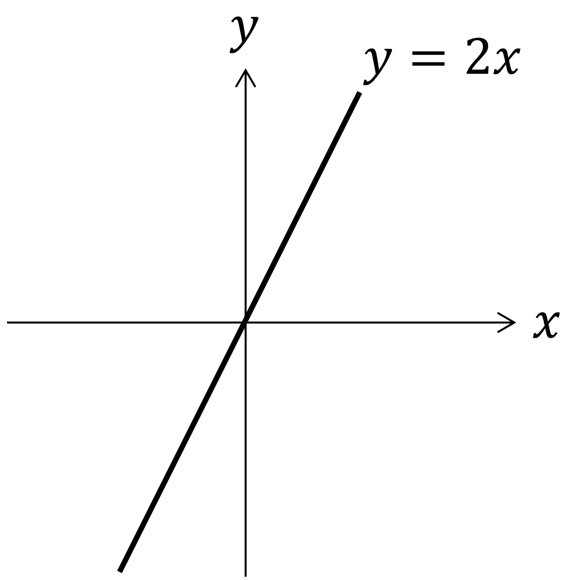
- This notion is commonly used in Physics. If we interpret t as time, then the above pair of equations describes the position of a moving particle at different time t.

Example 23a

Suppose (x, y) is governed by the following parametric equations:

$$\begin{cases} x = \;\; t \\ y = 2t \end{cases} \qquad \text{where } \; t \geq 0.$$

- Then the particle is moving along the straight line $y = 2x$ (starting from $(0, 0)$).

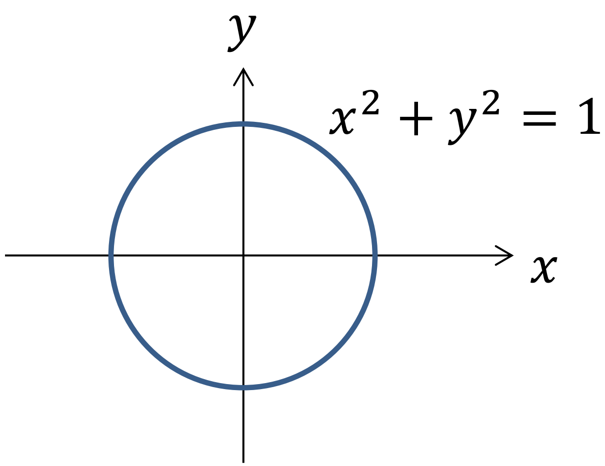


Example 23b

Suppose (x, y) is governed by the following parametric equations:

$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad \text{where } 0 \le t \le 2\pi.$$

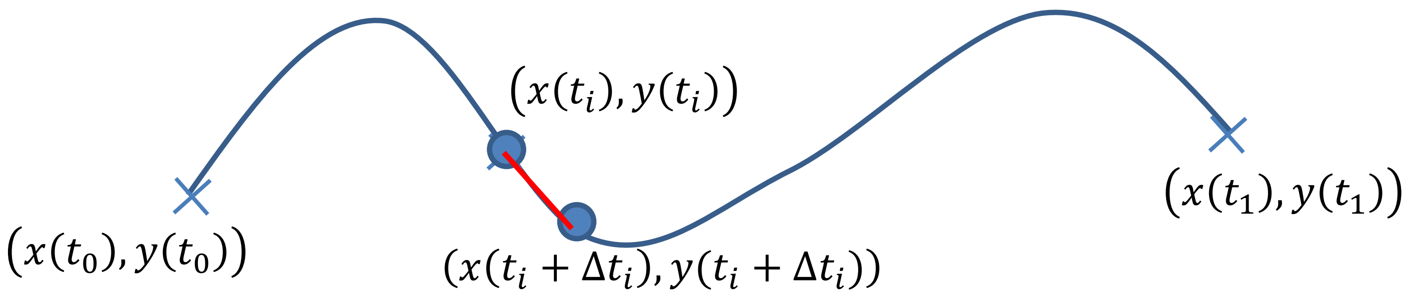
- Then the particle is moving along the circle with equation $x^2 + y^2 = 1$ along anti-clockwise direction once.



How to compute the arc length for this case?

Given a curve with parametrization $x = x(t)$ and $y = y(t)$, we can find the arc length using the following approach:

We first shall divide the whole curve into n small segments.



Since each small segment is close to a straight line, the arc length is given by

$$\Delta s_i = \sqrt{[x(t_i + \Delta t_i) - x(t_i)]^2 + [y(t_i + \Delta t_i) - y(t_i)]^2}$$
$$= \sqrt{\left(\frac{x(t_i + \Delta t_i) - x(t_i)}{\Delta t_i}\right)^2 + \left(\frac{y(t_i + \Delta t_i) - y(t_i)}{\Delta t_i}\right)^2}\,\Delta t_i$$
$$\approx \sqrt{[x'(t_i)]^2 + [y'(t_i)]^2}\Delta t_i$$

Then the entire arc length is given by

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta s_i = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[x'(t_i)]^2 + [y'(t_i)]^2}\Delta t_i = \int_{t_0}^{t_1} \sqrt{[x'(t)]^2 + [y'(t)]^2}dt$$

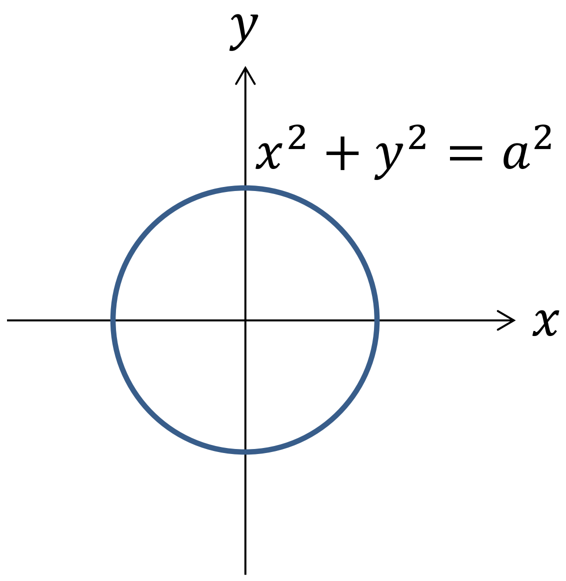
Example 24

Find the arc length of the curve with parametric equations (circle)

$$x(t) = a\cos t\,, \qquad y(t) = a\sin t$$

for $0 \le t \le 2\pi$.

(Remark: It traces out a circle with equation $x^2 + y^2 = a^2$ once)



☺Solution:

Take $x(t) = a\cos t$ and $y(t) = a\sin t$, then

$$x'(t) = -a\sin t\,, \qquad y'(t) = a\cos t.$$

So the arc length is then given by

$$s = \int_0^{2\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2}dt = \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (a\cos t)^2}dt$$

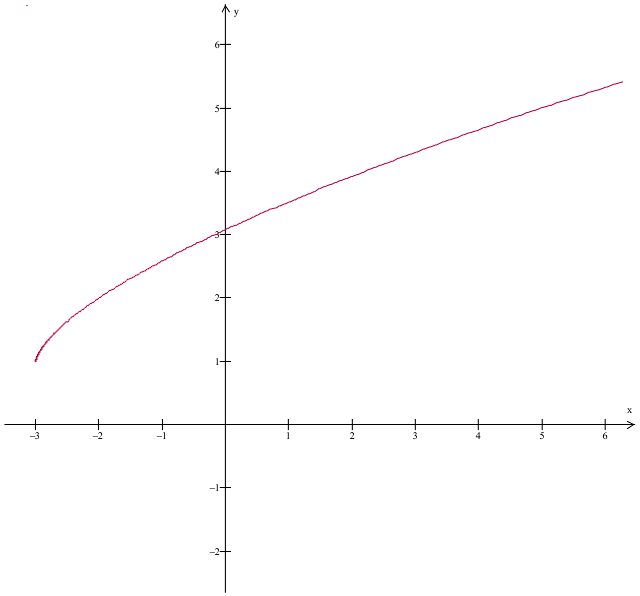
$$= \int_0^{2\pi} \sqrt{a^2(\sin^2 t + \cos^2 t)}dt$$

$$= \int_0^{2\pi} a dt = a\int_0^{2\pi} dt = a[t]_0^{2\pi} = 2\pi a.$$

Example 25

Find the arc length of the curve with parametric equations

$$x = t^3 - 3,\ y = t^2 + 1 \quad \text{for } 1 \le t \le 2.$$



☺Solution:

Let $x(t) = t^3 - 3$ and $y(t) = t^2 + 1$, then

$$x'(t) = 3t^2 \text{ and } y'(t) = 2t.$$

Then the arc length is given by

$$s = \int_1^2 \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt = \int_1^2 \sqrt{9t^4 + 4t^2}\,dt = \int_1^2 t\sqrt{9t^2 + 4}\,dt$$

Let $u = 9t^2 + 4$, then $\frac{du}{dt} = 18t \Rightarrow dt = \frac{1}{18t}du.$

When $t = 2$, $u = 40$; when $t = 1$, $u = 13$.

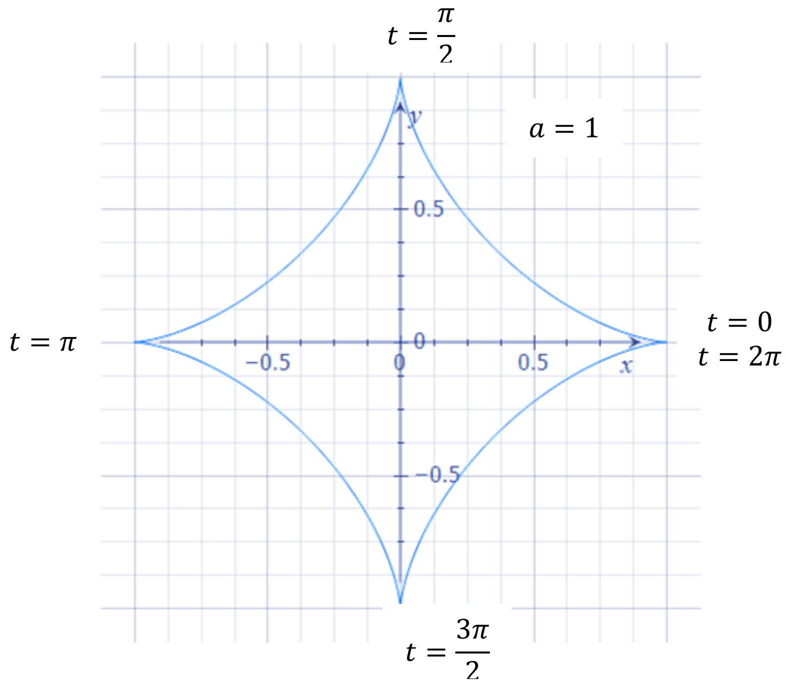
$$\int_1^2 t\sqrt{9t^2 + 4}\,dt = \int_{13}^{40} t\sqrt{u}\left(\frac{1}{18t}du\right) = \frac{1}{18}\int_{13}^{40} \sqrt{u}\,du = \frac{1}{18}\int_{13}^{40} u^{\frac{1}{2}}du = \frac{1}{18}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_{13}^{40}$$

$$= \frac{1}{27}\left(40^{\frac{3}{2}} - 13^{\frac{3}{2}}\right).$$

Example 26

Find the arc length of the curve with parametric equations (Astroid: $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$):

$$\begin{cases} x(t) = a\cos^3 t \\ y(t) = a\sin^3 t \end{cases}, \quad 0 \le t \le 2\pi, \quad a > 0.$$



☺Solution:

$$s = 4\int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{d}{dt}a\cos^3 t\right)^2 + \left(\frac{d}{dt}a\sin^3 t\right)^2}\,dt \quad \text{by symmetry}$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{(-3a\cos^2 t\sin t)^2 + (3a\sin^2 t\cos t)^2}\,dt$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t}\,dt$$

$$= 4\int_0^{\frac{\pi}{2}} \sqrt{9a^2\cos^2 t\sin^2 t\,\underbrace{(\cos^2 t + \sin^2 t)}_{=1}}\,dt$$

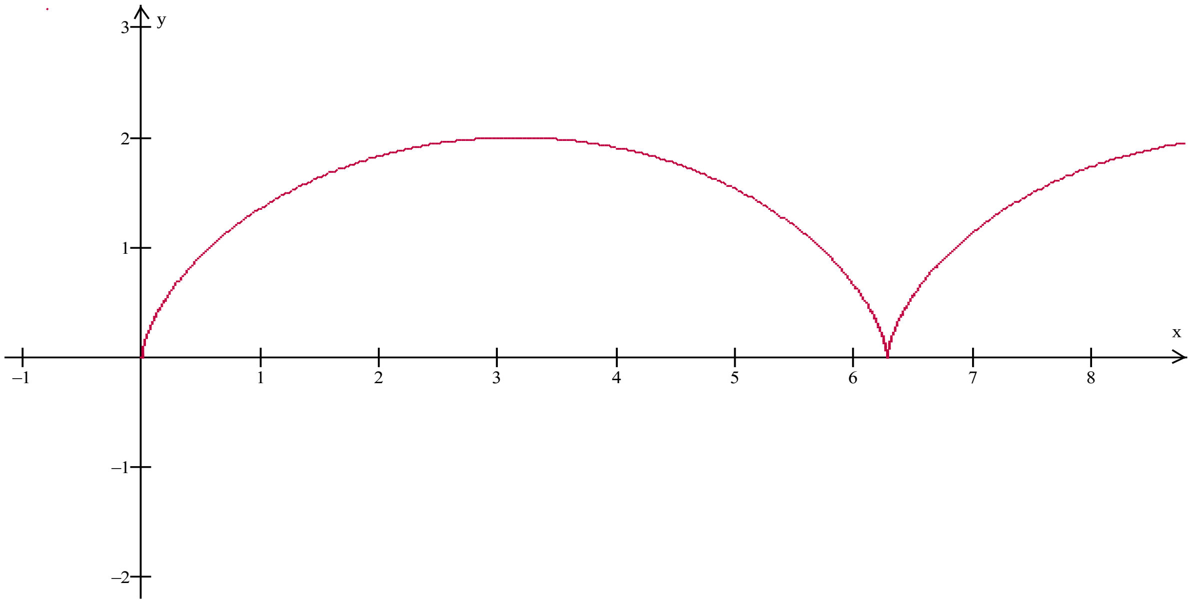
$$= 4\int_0^{\frac{\pi}{2}} |3a\cos t\sin t|\,dt = 12a\int_0^{\frac{\pi}{2}} \sin t\,d\sin t$$

$$= 6a[\sin^2 t]_0^{\frac{\pi}{2}} = 6a[1^2 - 0^2] = 6a.$$

Example 27

Find the surface area of the solid generated by revolving the Cycloid about the x-axis:

$$x = a(t - \sin t),\ \ y = a(1 - \cos t),\ \ t: 0 \to 2\pi,\ \ a > 0$$



$a = 1$

☺Solution

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t}$$
$$= a\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2}a\sqrt{1-\cos t}$$

$$S_x = \int_0^{2\pi a} 2\pi y ds = 2\pi\int_0^{2\pi} a(1-\cos t)\sqrt{2}a\sqrt{1-\cos t}\, dt$$
$$= 2\sqrt{2}\pi a^2 \int_0^{2\pi}\Bigg(\underbrace{1-\cos t}_{2\sin^2\frac{t}{2}}\Bigg)^{\frac{3}{2}} dt = 8\pi a^2\int_0^{2\pi}\sin^3\frac{t}{2}\, dt$$
$$= 16\pi a^2\int_0^{2\pi}(1-\cos^2\frac{t}{2})\, d(-\cos\frac{t}{2}) = -16\pi a^2\left[\cos\frac{t}{2} - \frac{\cos^3\frac{t}{2}}{3}\right]_0^{2\pi}$$
$$= -16\pi a^2\left[\left(\underbrace{\cos\pi}_{-1} - \underbrace{\cos 0}_{1}\right) - \frac{1}{3}\left(\underbrace{\cos^3\pi}_{-1} - \underbrace{\cos^3 0}_{1}\right)\right] = \frac{64}{3}\pi a^2.$$

Summary of formula

	Formula	$x(t_0)=a$, $x(t_1)=b$ or $y(t_0)=c$, $y(t_1)=d$
Area bounded by the region R_x: $y_{\text{top}}=f(x)$, $y_{\text{bottom}}=g(x)$, $x=a$ and $x=b$	$\int_a^b [f(x)-g(x)]dx$	$\int_{t_0}^{t_1} [f(x(t))-g(x(t))]x'(t)dt$
Area bounded by the region R_y: $x_{\text{right}}=\phi(y)$, $x_{\text{left}}=\psi(y)$, $y=c$ and $y=d$	$\int_c^d [\phi(y)-\psi(y)]dy$	$\int_{t_0}^{t_1} [\phi(y(t))-\psi(y(t))]y'(t)dt$
Volume of solid generated by rotating R_x about the line $y=y_0$ (not cut R_x)	$\pi\int_a^b \lvert [f(x)-y_0]^2 - [g(x)-y_0]^2\rvert dx$	$\pi\int_{t_0}^{t_1} \lvert [f(x(t))-y_0]^2 - [g(x(t))-y_0]^2\rvert x'(t)dt$
Volume of solid generated by rotating R_x about the line $x=x_0$ (not cut R_x)	$2\pi\int_a^b \lvert x-x_0\rvert [f(x)-g(x)]dx$	$2\pi\int_{t_0}^{t_1} \lvert x(t)-x_0\rvert [f(x(t)) - g(x(t))]x'(t)dt$

Volume of solid generated by rotating R_y about the line $x = x_0$ (not cut R_y)	$\pi \int_c^d \|[\phi(y) - x_0]^2 - [\psi(y) - x_0]^2\| dy$	$\pi \int_{t_0}^{t_1} \|[\phi(y(t)) - x_0]^2 - [\psi(y(t)) - x_0]^2\| y'(t) dt$
Volume of solid generated by rotating R_y about the line $y = y_0$ (not cut R_y)	$2\pi \int_c^d \|y - y_0\| [\phi(y) - \psi(y)] dy$	$2\pi \int_{t_0}^{t_1} \|y(t) - y_0\| [\phi(y(t) - \psi(y(t)] y'(t) dt$
Arc length of $y = f(x)$ over $[a, b]$	$\int_a^b \sqrt{1 + [f'(x)]^2} dx$	$\int_{t_0}^{t_1} \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$
Area of surface generated by rotating $y = f(x)$ over $[a, b]$ about the line $y = y_0$	$2\pi \int_a^b \|f(x) - y_0\| \sqrt{1 + [f'(x)]^2} dx$	$2\pi \int_{t_0}^{t_1} \|y(t) - y_0\| \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$
Area of surface generated by rotating $x = \phi(y)$ over $[c, d]$ about the line $x = x_0$	$2\pi \int_c^d \|\phi(y) - x_0\| \sqrt{1 + [\phi'(y)]^2} dy$	$2\pi \int_{t_0}^{t_1} \|x(t) - x_0\| \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$

Note: the x-axis: $y = 0$; the y-axis: $x = 0$