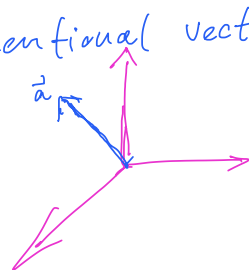


# Chapter 0 Review of Linear Algebra

## 1. Vectors

\*  $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n$

eg.  $\vec{a} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \rightarrow$  three dimensional vector  
 $|\vec{a}| = \sqrt{1^2 + 0^2 + 2^2} = \sqrt{5}$



\* magnitude (length)  $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$

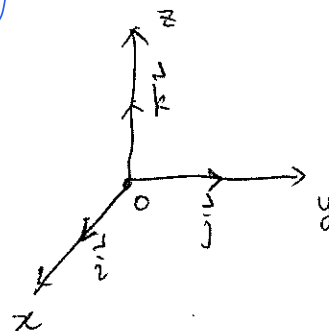
\* Unit vector  $|\vec{a}| = 1$ , standard unit vector

$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \vec{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$

Zero vector  $\vec{0} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

\*  $\vec{a} \neq \vec{0}$ , unit vector  $\frac{\vec{a}}{|\vec{a}|}$

$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$  normalize  $\frac{\vec{v}}{|\vec{v}|} = \begin{pmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{pmatrix}$



\*  $\mathbb{R}^3: \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$

$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

## 2. Vector Operations

linear operation  $\left\{ \begin{array}{l} * \vec{a} \pm \vec{b} \\ * m \vec{a} \end{array} \right.$   
 $\uparrow$   
 scalar

$3 \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 \\ 3 \cdot 0 \\ 3 \cdot 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 6 \end{pmatrix}$

\*  $m(\vec{a} \pm \vec{b}) = m\vec{a} \pm m\vec{b}$

\*  $(m \pm n)\vec{a} = m\vec{a} \pm n\vec{a}$

### 3. Scalar product, Cross product, and Triple scalar product

\* Scalar product,

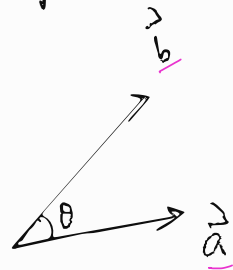
$$\vec{a} \cdot \vec{b} = \sum_{i=1}^n a_i b_i = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

$$\equiv |\vec{a}| |\vec{b}| \cos \theta$$

eg.  $\vec{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$   $\vec{b} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}$

$$\vec{a} \cdot \vec{b} = 1 \cdot (-1) + 0 \cdot 3 + 1 \cdot 5 = 4$$

If  $\vec{a} \cdot \vec{b} = 0$ , then  $\theta = \frac{\pi}{2}$ ,  $\vec{a} \perp \vec{b}$ .



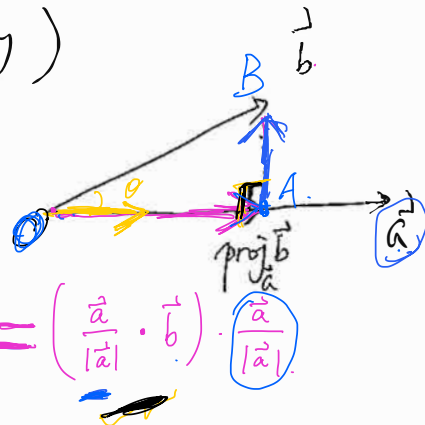
$$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}$$

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}| |\vec{b}| \quad (\text{Schwarz inequality})$$

$$|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}| \quad (\text{Triangle inequality})$$

\*  $\text{proj}_{\vec{a}} \vec{b}$ : projection vector of  $\vec{b}$  on  $\vec{a}$

$$\text{proj}_{\vec{a}} \vec{b} = \underbrace{\left( \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \right)}_{\text{Scalar}} \underbrace{\vec{a}}_{\text{vector}} = \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a} = \left( \frac{\vec{a}}{|\vec{a}|} \cdot \vec{b} \right) \left( \frac{\vec{a}}{|\vec{a}|} \right)$$

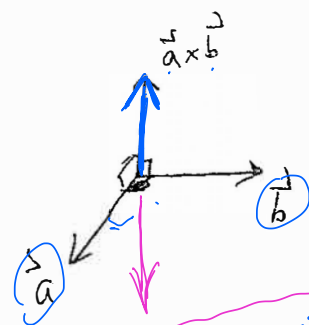


Example: Compute  $\text{proj}_{\vec{a}} \vec{b}$  given  $\vec{a} = \begin{pmatrix} 4 \\ 0 \\ 3 \\ 0 \end{pmatrix}$  and  $\vec{b} = \begin{pmatrix} 3 \\ 3 \\ 2 \\ 0 \end{pmatrix}$

$$\text{Solution: } \text{proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a} = \frac{15}{25} \begin{pmatrix} 4 \\ 0 \\ 3 \\ 0 \end{pmatrix} = \frac{3}{5} \begin{pmatrix} 4 \\ 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 9 \\ 0 \end{pmatrix}$$

\* Cross product,  $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ ,  $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \neq \vec{b} \times \vec{a}$$



$\vec{a} \times \vec{b}$   $\begin{cases}$  magnitude  
direction:

$$|\vec{a}| |\vec{b}| \sin \theta$$

$$\vec{a} \times \vec{b} \perp \vec{a}$$

right-hand rule.

$$\vec{a} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}$$

$$\vec{a} \times \vec{b} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ -1 & 1 & 3 \end{pmatrix} = -2 \cdot \vec{i} + (-1) \cdot \vec{j} + 1 \cdot \vec{k} = -2\vec{i} - \vec{j} + \vec{k}$$

$$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$$

Vector

notation

$\vec{a} \in \mathbb{R}^n$ ,  $n$ -dimensional vector.  $= \begin{pmatrix} -2 \\ -5 \\ 1 \end{pmatrix}$ .

eg,  $\vec{a} \in \mathbb{R}^3$   $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ .

magnitude (length) :  $|\vec{a}| = \sqrt{\sum_{j=1}^n a_j^2}$ .

unit vector :  $|\vec{a}| = 1$  then  $\vec{a}$  is called unit

normalize a vector : If  $\vec{a} \neq \vec{0}$ , then  $\frac{\vec{a}}{|\vec{a}|}$  is a unit vector.

standard ~~the~~ unit vector :  $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ,  $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$  ...  $\vec{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$

zero vector :  $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ .

operations(I)

linear operation

$$\left\{ \begin{array}{l} \vec{a} \pm \vec{b} \\ m\vec{a} \\ m(\vec{a} \pm \vec{b}) = m\vec{a} + m\vec{b} \\ (m+n)\vec{a} = m\vec{a} + n\vec{a} \end{array} \right.$$

operations (II) { Scalar product (Inner product)  
 $\vec{a} \cdot \vec{b} = \sum_{j=1}^n a_j b_j$       $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$       $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$   
cross product.

$\vec{a} \times \vec{b}$  is still a vector whose  
magnitude is  $|\vec{a}| |\vec{b}| \sin \theta$ .  
direction is ~~orthog~~ perpendicular  
with both  $\vec{a}$ ,  $\vec{b}$ .

properties of a  
set of vectors. { linear dependent / linear independent.  
orthogonality.

$$\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$\vec{u} \qquad \qquad \vec{a} \qquad \qquad \vec{b}$

$$\vec{u} = \vec{a} + 2\vec{b}$$

$\{\vec{u}, \vec{a}, \vec{b}\}$  is linear dependent.

$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$  is linear dependent  $\Leftrightarrow a_1\vec{v}_1 + \dots + a_m\vec{v}_m = \vec{0}$   
 and at least one of coefficients is non-zero

if  $\vec{v}_j = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_{j-1}\vec{v}_{j-1} + a_{j+1}\vec{v}_{j+1} + \dots + a_m\vec{v}_m$

$$\vec{a} + 2\vec{b} - \vec{u} = \vec{0}$$

linear dependent.

$$\vec{v}_j = a_1\vec{v}_1 + \dots + a_{j-1}\vec{v}_{j-1} + \dots + a_m\vec{v}_m$$



$a_1\vec{v}_1 + \dots + a_m\vec{v}_m = \vec{0}$   
 non-trivial linear combination.

#### 4 Linear dependence and Linear Independence

\*  $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_k$  are linearly dependent if there exists  $1 \leq i \leq k$  such that  $\vec{a}_i = \sum_{\substack{j=1 \\ j \neq i}}^k m_j \vec{a}_j$ . If no vector can be represented by other vectors, then  $\vec{a}_1, \dots, \vec{a}_k$  are said to be linearly independent.

\*  $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_k$  are linearly independent iff.

$$\sum_{j=1}^k m_j \vec{a}_j = \vec{0} \Rightarrow m_j = 0, j=1, \dots, k.$$

\* Ex: Are  $\vec{a} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}$ ,  $\vec{b} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}$ ,  $\vec{c} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$  linearly indep?

$$\vec{a} \cdot \vec{b} \times \vec{c} = \begin{vmatrix} 3 & 5 & -2 \\ 0 & 4 & 2 \\ 1 & 1 & -1 \end{vmatrix} = 0 \Rightarrow \vec{a}, \vec{b}, \vec{c} \text{ are linearly dependent.}$$

Q: Are 4 vectors in  $\mathbb{R}^3$  linearly independent or not?

Thm:  $(n+1)$  vectors in  $\mathbb{R}^n$  are always linearly dependent!

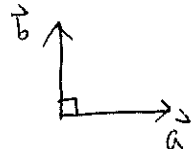
#### 5. Orthogonality

\*  $\vec{a}$  and  $\vec{b}$  are said to be orthogonal if  $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow (\vec{a} \perp \vec{b})$

\*  $\{a_1, \dots, a_k\}$  is said to be orthogonal if

$$a_i \perp a_j \text{ for } 1 \leq i, j \leq k \text{ and } i \neq j.$$

\*  $\{a_1, \dots, a_k\}$  is said to be orthonormal if it is orthogonal and  $|\vec{a}_j| = 1, j=1, \dots, k$ .



If  $\{\vec{v}_1, \dots, \vec{v}_n\}$  is orthogonal, then it must be linear independent.  
(the converse is not necessary).

eg.  $\{\vec{i}, \vec{j}, \vec{k}\}$  is orthonormal

$$x_1 \vec{i} + x_2 \vec{j} + x_3 \vec{k} = \vec{0} \Rightarrow x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} x_1 + 0 + 0 \\ 0 + x_2 + 0 \\ 0 + 0 + x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow x_1 = x_2 = x_3 = 0$$

eg.  $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$  is linear independent, but not orthogonal.

$\vec{v}_1 \cdot \vec{v}_2 = 1 \neq 0$  so  $\vec{v}_1, \vec{v}_2$  not orthogonal.

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 = \vec{0} \Rightarrow \begin{pmatrix} x_1 \\ 2x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{x_1 = x_2 = 0}, \text{ so } \vec{v}_1, \vec{v}_2 \text{ linear indep.}$$

$$0\vec{v}_1 + 0\vec{v}_2 = \vec{0}$$

nontrivial linear combination of

$$\underline{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}}$$

$$2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2 \quad -1 \quad +1$$

6. Matrix (Square matrix)  $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_{3 \times 1}$

\*  $m \times n$  matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_m & \dots & a_{mn} \end{pmatrix} = (\vec{a}_1, \dots, \vec{a}_n) = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$$

↑  
entry

↑  
column vector

↑  
row vector

$m \times n$

\* Diagonal matrix

eg  $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}_{3 \times 3}$

$$D = \begin{pmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & a_{nn} \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

Identity matrix

Zero matrix  $O = \begin{pmatrix} 0 & \dots & 0 \\ & & \\ 0 & \dots & 0 \end{pmatrix}$

7. Matrix Operations

\*  $A \pm B$ ,  $\in A$

eg.  $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}_{3 \times 3}$

$$A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

\* Transpose  $A^T$   $\underline{A^T = A}$  Symmetric matrix

eg.  $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 5 \\ 0 & 5 & 3 \end{pmatrix}$

\* Product

$$C = AB = (c_{ij})$$

↑     ↑     ↑  
 $m \times l$     $m \times n$     $n \times l$

$AB \neq BA$   
in general.

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kl}$$

eg  $\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{pmatrix}_{2 \times 3} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}_{3 \times 1} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 3 + 3 \cdot (-1) \\ -1 \cdot 1 + 0 \cdot 3 + 4 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}_{2 \times 1}$



\* Determinant (square matrix)

$A: n \times n$

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\det(A) = |A|$$

eg.  $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2 \times 2}$

$$|A| = 1 \cdot 4 - 2 \cdot 3 = -2$$

$$B = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$$\det(B) = |B| = 1 \cdot \begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix}$$

$$A = (a_{ij})_{1 \leq i, j \leq n} + (-1) \cdot \begin{vmatrix} 3 & 4 \\ 0 & 2 \end{vmatrix} = \dots$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_{3 \times 3}$$

$$= (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\det A = a_{i1} A_{i1} + a_{i2} A_{i2} + \dots + a_{in} A_{in}$$

Cofactor

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Cofactor of  $a_{ij}$

$$M_{ij} = \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}$$

Minor of  $a_{ij}$

\*  $\det(A) = \det(A^T)$ ,  $\det(AB) = \det A \cdot \det B$

\* Rank: rank  $A$  is the maximal number of linearly independent row vectors

$$\text{rank}(A) = \text{rank}(A^T)$$

eg.

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 3 & -1 \end{pmatrix}_{2 \times 3}$$

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}_{3 \times 3}$$

$$(AB)_{2 \times 3} = \begin{pmatrix} 1 \cdot 1 + 1 \cdot 2 + 3 \cdot (-1) & 1 \cdot 0 + 1 \cdot 1 + 3 \cdot 1 & (1, 1, 3) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ (2 \ 3 \ -1) \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} & (2 \ 3 \ -1) \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} & (2 \ 3 \ -1) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}$$

$BA$  is not defined  
 $3 \times 3$   $2 \times 3$

$$= \begin{pmatrix} 0 & 4 & 1 \\ 9 & 2 & 2 \end{pmatrix}_{2 \times 3}$$

Note : a vector  $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$  can be regarded as a matrix with only one column.

eg.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 0 \\ 2 & -1 & 2 \end{pmatrix}_{3 \times 3}$$

$$\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}_{3 \times 1}$$



$$A\vec{x} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 0 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 0 + 3 \cdot 2 \\ 4 \cdot 1 + 1 \cdot 0 + 0 \cdot 2 \\ 2 \cdot 1 + (-1) \cdot 0 + 2 \cdot 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 6 \end{pmatrix}$$

Note: usually,  $AB \neq BA$ .

• Determinant

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\det(A) = |A| = a_{11} a_{22} - a_{12} a_{21}$$

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}_{3 \times 3}$$

$$\det(A) = 1 \cdot \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} - 0 \cdot \begin{vmatrix} -1 & 1 \\ 0 & 4 \end{vmatrix} + 2 \cdot \begin{vmatrix} -1 & 3 \\ 0 & 2 \end{vmatrix}$$

= ...