Properties of logarithms:

(1) For any real number x, $\boxed{\log_b b^x = x}$.

(2) For any real number $x > 0$, $\boxed{b^{\log_b x} = x}$.

(3) For any real numbers $x > 0$ and n, $\boxed{\log_b x^n = n \log_b x}$.

(4) For any real numbers $x > 0$ and $y > 0$, $\boxed{\log_b(xy) = \log_b x + \log_b y}$.

(5) For any real numbers $x > 0$ and $y > 0$, $\boxed{\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y}$.

useful in Ch.7 (Logarithmic differentiation)

(6) For any real numbers $x > 0$, $a > 1$ and $b > 1$, $\boxed{\log_b x = \frac{\log_a x}{\log_a b}}$.

$= \frac{\log x}{\log b} = \frac{\ln x}{\ln b}$

base 10 base e

In general,

1. $\log_b(x + y) \neq \log_b x + \log_b y$.

2. $(\log_b x)^2 \neq \log_b(x^2)$.

3. $\frac{\log_b x}{\log_b y} \neq \frac{x}{y}$, $\frac{\log_b x}{\log_b y} \neq \log_b \left(\frac{x}{y}\right)$

Example 4

Simplify each of the following:

(a) $\log_3\left(\frac{1}{81}\right)$

(b) $2\log_{10}5+\log_{10}4-5^{\log_5 3}+\log_2 16$

Solution

(a) $\log_3\left(\frac{1}{81}\right)=\log_3\left(\frac{1}{3^4}\right)=\log_3(3^{-4})=-4\underbrace{\log_3 3}_{=1}=-4$

(b) $2\log_{10}5+\log_{10}4-5^{\log_5 3}+\log_2 16=\log_{10}(5^2\times 4)-3+\log_2(2^4)$

$=\log_{10}(100)-3+4$

$=\log_{10}(10^2)+1$

$=2+1$

$=3$

Example 5

If $2^x = 3^y = 12^z$, show that $xy = z(x + 2y)$.

Solution

Let $2^x = 3^y = 12^z = k$.

Taking ln on both sides, we have

$$\begin{cases} \ln(2^x) = \ln k \\ \ln(3^y) = \ln k \\ \ln(12^z) = \ln k \end{cases} \Rightarrow \begin{cases} x\ln 2 = \ln k \\ y\ln 3 = \ln k \\ z\ln 12 = \ln k \end{cases} \Rightarrow \begin{cases} \ln 2 = \frac{\ln k}{x} \\ \ln 3 = \frac{\ln k}{y} \\ \ln 12 = \frac{\ln k}{z} \end{cases}$$

$$\because \ln 12 = \ln(3 \times 2^2) = \ln 3 + 2\ln 2$$

$$\therefore \frac{\ln k}{z} = \frac{\ln k}{y} + \frac{2\ln k}{x}$$

$$\Rightarrow \frac{1}{z} = \frac{1}{y} + \frac{2}{x}$$

$$\Rightarrow xy = xz + 2yz$$

$$= z(x+2y)$$

Natural logarithm

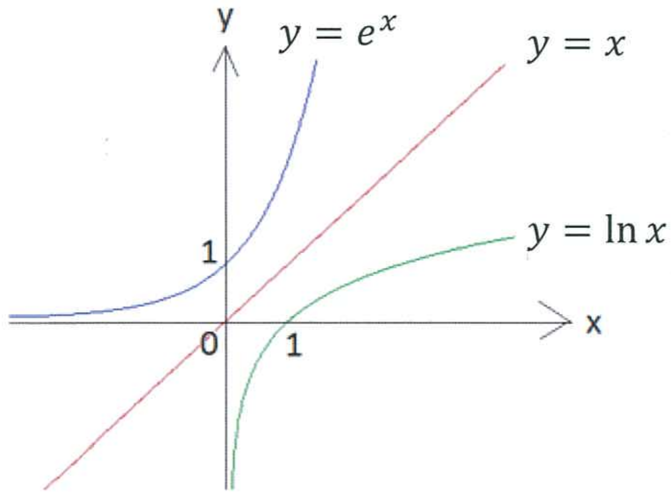
Logarithmic function with base e is called **natural logarithmic function**, denoted by $f(x) = \log_e x$ or $f(x) = \ln x$.

$$\boxed{y = \ln x \quad \Leftrightarrow \quad x = e^y}$$

Exponential function is the inverse function of logarithmic function, that is, the inverse function of $f(x) = \ln x$ is $f^{-1}(x) = e^x$.

Thus, $Dom(f) = Ran(f^{-1}) = (0, \infty)$ and $Ran(f) = Dom(f^{-1}) = \mathbb{R}$.

The graphs of $y = \ln x$ and $y = e^x$ are shown below.



Note that:

(i) The graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ about the line $y = x$.

(ii) Both $y = \ln x$ and $y = e^x$ are strictly increasing functions.

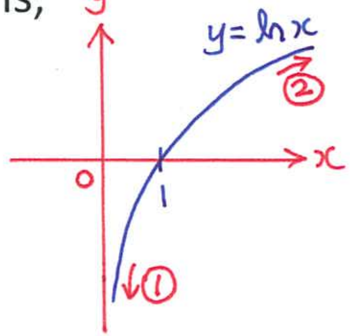
(iii) $\ln 1 = 0$ (i.e. the graph of $y = \ln x$ crosses the x-axis at $x = 1$.)

(iv) $\ln x < 0$ for $0 < x < 1$

(v) $\ln x > 0$ for $x > 1$

(vi) The value of $\ln x$ approaches to $-\infty$ as x tends to 0 from the right. That is,

$$\lim_{x \to 0^+} \ln x = -\infty \quad (1)$$



The value of $\ln x$ approaches to ∞ as x gets larger and larger. That is,

$$\lim_{x \to \infty} \ln x = \infty \quad (2)$$

(The limit of a function will be discussed in Chapter 6.)

$\rightarrow$ means "tends to"

Example 6

For each of the following functions, find its largest possible domain and largest possible range, and then sketch its graph.

(a) $f(x) = 2 + \ln\frac{1}{x}$

(b) $g(x) = 4 + \log\frac{x+1}{1000}$

Solution

(a) $f(x) = 2 + \ln\frac{1}{x}$ is well-defined when $\frac{1}{x} > 0$ and $x \neq 0$, i.e. when $x > 0$.

$\therefore$ $Dom(f) = (0, \infty)$.

The function can be written as

$$f(x) = 2 + \ln\frac{1}{x} = 2 + \ln(x^{-1}) = 2 + (-1)\ln x = 2 - \ln x.$$

For any $x \in Dom(f) = (0, \infty)$, $\ln x$ can be any real number and thus $2 - \ln x$ can be any real number.

$\therefore$ $Ran(f) = \mathbb{R}$.

$\ln x \xrightarrow[\text{about } x\text{-axis}]{\text{reflect}} -\ln x \xrightarrow[\text{units upward}]{\text{shifted 2}} 2-\ln x$

base 10

(b) $g(x) = 4 + \log\frac{x+1}{1000}$ is well-defined when $\frac{x+1}{1000} > 0$, i.e. $x > -1$.

$\therefore$ $Dom(g) = (-1, \infty)$.

The function can be written as

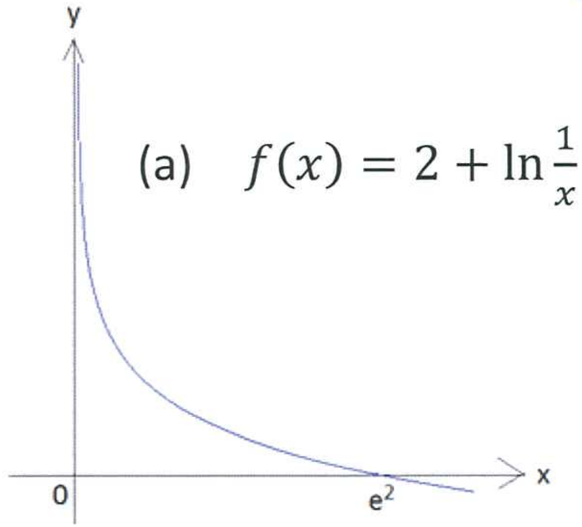
$$g(x) = 4 + \log\frac{x+1}{1000} = 4 + \log(x+1) - \log 1000 = 4 + \log(x+1) - \log 10^3$$

$$= 4 + \log(x+1) - 3 = 1 + \log(x+1)$$

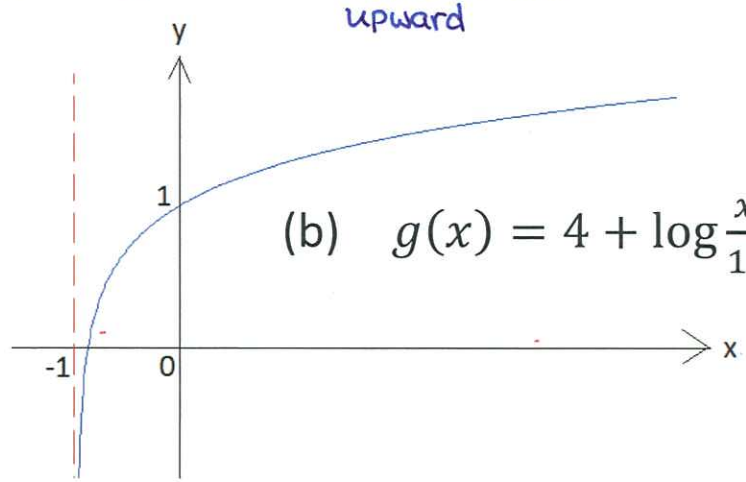
For any $x \in Dom(g) = (-1, \infty)$, $\log(x+1)$ can be any real number and thus $g(x) = 1 + \log(x+1)$ can be any real number. $\therefore$ $Ran(f) = \mathbb{R}$.

$$\log x \xrightarrow[\text{to the left}]{\text{shifted 1 unit}} \log(x+1) \xrightarrow[\text{upward}]{\text{shifted 1 unit}} 1 + \log(x+1)$$

<u>*Sketches*</u>



(a) $f(x) = 2 + \ln\frac{1}{x}$



(b) $g(x) = 4 + \log\frac{x+1}{1000}$

Example 7

For each of the following functions, find its inverse function.

(a) $f(x) = e^{x+1} - 5$ (b) $g(x) = 3 + 2e^{-x}$ (c) $h(x) = 1 - 3\left(\frac{1}{2}\right)^x$

(d) $f(x) = 2 + \ln\frac{1}{x}$ (e) $g(x) = 4 + \log\frac{x+1}{1000}$

Solution

(a) Let $y = e^{x+1} - 5$.

Then $e^{x+1} = y + 5 \Rightarrow \underbrace{\ln(e^{x+1})}_{=x+1} = \ln(y+5) \Rightarrow x = \ln(y+5) - 1$

$\therefore f^{-1}(x) = \ln(x+5) - 1$

(b) Let $y = 3 + 2e^{-x}$.

Then $e^{-x} = \frac{y-3}{2} \Rightarrow -x = \ln\left(\frac{y-3}{2}\right) \Rightarrow x = -\ln\left(\frac{y-3}{2}\right) = \ln\left(\frac{2}{y-3}\right)$

$\therefore g^{-1}(x) = \ln\left(\frac{2}{x-3}\right)$

(c) Let $y = 1 - 3\left(\frac{1}{2}\right)^x = 1 - 3(2)^{-x}$

$\Rightarrow \quad 2^{-x} = \frac{1-y}{3} \quad \Rightarrow \quad -x = \log_2\left(\frac{1-y}{3}\right) \quad \Rightarrow \quad x = -\log_2\left(\frac{1-y}{3}\right) = \log_2\left(\frac{3}{1-y}\right)$

$\therefore \quad h^{-1}(x) = \log_2\left(\frac{3}{1-x}\right)$

(d) Let $y = 2 + \ln\frac{1}{x}$.

Then $\ln\frac{1}{x} = y - 2 \Rightarrow \frac{1}{x} = e^{y-2} \Rightarrow x = \frac{1}{e^{y-2}} \overset{= e^{-(y-2)}}{=} e^{2-y}$

$\therefore f^{-1}(x) = e^{2-x}$

(e) Let $y = 4 + \underset{\text{base } 10}{\log}\frac{x+1}{1000}$. $= 1 + \log(x+1)$ (from Ex. 6(b))

$\Rightarrow \quad \log(x+1) = y - 1$

$\Rightarrow \quad x + 1 = 10^{y-1}$

$\Rightarrow \quad x = 10^{y-1} - 1$

$\therefore \quad g^{-1}(x) = 10^{x-1} - 1$

Example 8

Determine the largest possible domain and largest possible range of the function $f(x) = \ln\left(\frac{x+2}{x-1}\right)$.

Solution

Not $\geq$

The function $f(x) = \ln\left(\frac{x+2}{x-1}\right)$ is well-defined when $\frac{x+2}{x-1} > 0$ and $x - 1 \neq 0$.

	$x < -2$	$x = -2$	$-2 < x < 1$	~~$x = 1$~~	$x > 1$
Sign of $x+2$	$-$	0	$+$		$+$
Sign of $x-1$	$-$	$-$	$-$		$+$
Sign of $\frac{x+2}{x-1}$	$+$	0	$-$		$+$

$\therefore$ The largest possible domain of $f(x)$ is $Dom(f) = (-\infty, -2) \cup (1, \infty)$.

Let $y = \ln\left(\frac{x+2}{x-1}\right)$. Then $e^y = \frac{x+2}{x-1} \Rightarrow e^y(x-1) = x+2 \Rightarrow x(e^y - 1) = e^y + 2$

$$\Rightarrow x = \frac{e^y+2}{e^y-1}.$$

In the last expression, y can be any real number except when $e^y = 1$, i.e. $y = \ln 1 = 0$.

$\therefore$ The largest possible range of $f(x)$ is $Ran(f) = \mathbb{R}\backslash\{0\}$.

Note:

$f^{-1}(x) = \frac{e^x+2}{e^x-1}$

$Dom(f^{-1}) = Ran(f) = \mathbb{R}\backslash\{0\}$

$Ran(f^{-1}) = Dom(f) = (-\infty,-2)\cup(1,\infty)$

Example 9

Solve each of the following equations for x.

(a) $2^x = 16$ (b) $3^{x-1} = 81$ (c) $3^x = 17$ (d) $3 \cdot 5^{2x-1} + 2 = 17$

Solution

(a) $2^x = 16 = 2^4 \Rightarrow x = 4$

same

(b) $3^{x-1} = 81 = 3^4 \Rightarrow x - 1 = 4 \Rightarrow x = 5$

(c) $3^x = 17$

Take natural logarithm on both sides:

$$\ln 3^x = \ln 17 \Rightarrow x \ln 3 = \ln 17 \Rightarrow x = \frac{\ln 17}{\ln 3} \approx 2.5789$$

(d) $3 \cdot 5^{2x-1} + 2 = 17 \Rightarrow 3 \cdot 5^{2x-1} = 15 \Rightarrow 5^{2x-1} = 5 \Rightarrow 2x - 1 = 1 \Rightarrow x = 1$

Example 10

Solve each of the following equations for x.

(a) $\ln(x^3) = 2\ln 5$ (b) $5^{2x-1} = 12 \cdot 3^x$

(c) $\log_2(x+5) + \log_2(x-2) = 3$ (d) $e^x - 8e^{-x} = 7$

Solution

(a) $\ln(x^3) = 2\ln 5$

Taking natural exponential on both sides, we get

$$e^{\ln(x^3)} = e^{2\ln 5} \quad \Rightarrow \quad x^3 = e^{\ln(5^2)} \quad \Rightarrow \quad x^3 = 5^2 \quad \Rightarrow \quad \boxed{x = 5^{\frac{2}{3}}}.$$

(b) $5^{2x-1} = 12 \cdot 3^x$

Taking natural logarithm on both sides, we get

$$\ln(5^{2x-1}) = \ln(12 \cdot 3^x) \quad \Rightarrow \quad (2x-1)\ln 5 = \ln 12 + \underbrace{\ln 3^x}_{= x\ln 3}$$

$$\Rightarrow \quad x(2\ln 5 - \ln 3) = \ln 12 + \ln 5$$

$$\Rightarrow \quad \boxed{x = \frac{\ln 12 + \ln 5}{2\ln 5 - \ln 3}}$$

(c) $\log_2(x+5)+\log_2(x-2)=3 \quad \Rightarrow \quad \log_2[(x+5)(x-2)]=3$

Taking exponential with base 2 on both sides, we get

$$2^{\log_2[(x+5)(x-2)]}=2^3$$

$$\Rightarrow \quad (x+5)(x-2)=8$$

$$\Rightarrow \quad x^2+3x-10=8$$

$$\Rightarrow \quad x^2+3x-18=0$$

$$\Rightarrow \quad (x-3)(x+6)=0$$

$$\Rightarrow \quad x-3=0 \quad \text{or} \quad x+6=0$$

$\Rightarrow \quad x=3 \quad$ or $\quad x=-6$ (rejected since $\log_2(\overset{x>-5}{x+5})$ and $\log_2(\overset{x>2}{x-2})$ are not defined when $x=-6$)

$$\therefore \quad \boxed{x=3}$$

(d) $e^x - 8e^{-x} = 7$ ← Note: We cannot take ln on both sides, because $\ln(e^x - 8e^{-x})$ cannot be simplified.

$\ln(a-b) \neq \ln a - \ln b$

Multiply both sides by e^x:

$$e^{2x} - 8 = 7e^x \Rightarrow \underbrace{e^{2x}}_{=(e^x)^2} - 7e^x - 8 = 0$$

← quadratic equation in e^x

$$\Rightarrow (e^x - 8)(e^x + 1) = 0$$

$\Rightarrow e^x = 8$ or $e^x = -1$ (rejected since $e^x > 0$ for all $x \in \mathbb{R}$)

$$\Rightarrow x = \ln 8$$

Hyperbolic functions

For any real value x, the **hyperbolic sine** function ($\sinh x$) and the **hyperbolic cosine** function ($\cosh x$) are defined as

$$\boxed{\sinh x = \frac{1}{2}(e^x - e^{-x})} \quad \text{and} \quad \boxed{\cosh x = \frac{1}{2}(e^x + e^{-x})}\ , \quad \text{respectively.}$$

Note that $\sinh x \neq \sin(hx)$, $\cosh x \neq \cos(hx)$.

Remark:

$\cos^2 t + \sin^2 t = 1$

Recall that cosine and sine are called **circular functions** because, for any $t \in \mathbb{R}$, the point $(\overset{x}{\cos} t, \overset{y}{\sin} t)$ lies on the circle with equation $x^2 + y^2 = 1$. Similarly, hyperbolic cosine and hyperbolic sine are called **hyperbolic functions** because, for any $t \in \mathbb{R}$, the point $(\overset{x}{\cosh} t, \overset{y}{\sinh} t)$ lies on the hyperbola with equation $x^2 - y^2 = 1$ (see Example 11(a)).

$\cosh^2 t - \sinh^2 t = 1$

Example 11

Prove the following:

(a) $\cosh^2 x - \sinh^2 x = 1$

(b) $\cosh^2 x + \sinh^2 x = \cosh(2x)$

Solution

(a) $$\cosh^2 x - \sinh^2 x = \left[\frac{1}{2}(e^x + e^{-x})\right]^2 - \left[\frac{1}{2}(e^x - e^{-x})\right]^2$$

$$= \left[\frac{1}{4}\left(e^{2x} + 2\underbrace{e^x \cdot e^{-x}}_{=1} + e^{-2x}\right)\right] - \left[\frac{1}{4}\left(e^{2x} - 2\underbrace{e^x \cdot e^{-x}}_{=1} + e^{-2x}\right)\right]$$

$$= \frac{4}{4}$$

$$= 1$$

$\therefore\ \cosh^2 x - \sinh^2 x = 1.$

(b) $$\cosh^2 x + \sinh^2 x = \left[\frac{1}{2}(e^x + e^{-x})\right]^2 + \left[\frac{1}{2}(e^x - e^{-x})\right]^2$$

$$= \left[\frac{1}{4}\left(e^{2x} + 2\underbrace{e^x \cdot e^{-x}}_{=1} + e^{-2x}\right)\right] + \left[\frac{1}{4}\left(e^{2x} - 2\underbrace{e^x \cdot e^{-x}}_{=1} + e^{-2x}\right)\right]$$

$$= \frac{1}{2}(e^{2x} + e^{-2x})$$

$$= \cosh(2x)$$

$\therefore\ \cosh^2 x + \sinh^2 x = \cosh(2x)$

Other identities

- $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$
- $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
- $\cosh(2x) = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x = 2\cosh^2 x - 1$
- $\sinh(2x) = 2\sinh x \cosh x$

Exercise: Prove each of the above identities by using the definitions of $\sinh x$ and $\cosh x$.

Example 12

For each of the hyperbolic functions $\sinh x$ and $\cosh x$, determine whether it is an even function , odd function, or neither of them.

Solution

Let $f_1(x) = \sinh x$, then

$$f_1(-x) = \sinh(-x) = \frac{1}{2}\left(e^{-x} - e^{-(-x)}\right) = -\frac{1}{2}(e^{x} - e^{-x}) = -\sinh x = -f_1(x).$$

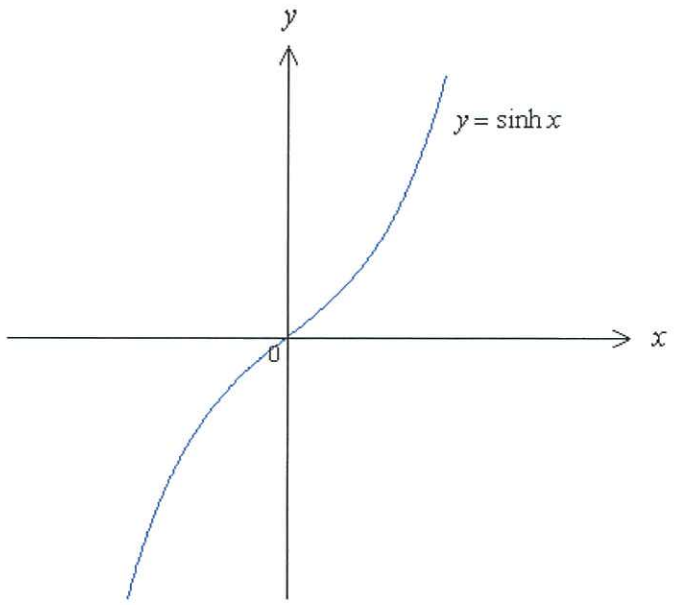
$\therefore$ $f_1(x) = \sinh x$ is an **odd** function.

Let $f_2(x) = \cosh x$, then

$$f_2(-x) = \cosh(-x) = \frac{1}{2}\left(e^{-x} + e^{-(-x)}\right) = \cosh x = f_2(x).$$

$\therefore$ $f_2(x) = \cosh x$ is an **even** function.

Graphs of hyperbolic sine and hyperbolic cosine functions



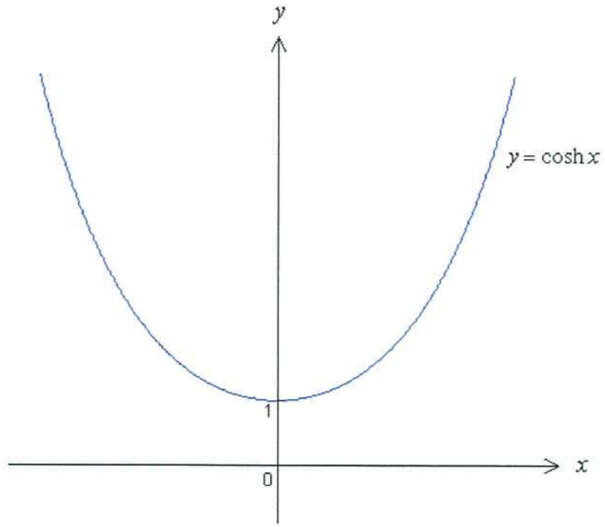
$$y = \sinh x = \frac{1}{2}(e^x - e^{-x})$$

Domain = $\mathbb{R}$

Range = $\mathbb{R}$

Odd function

$\sinh(-x) = -\sinh x$



$$y = \cosh x = \frac{1}{2}(e^x + e^{-x})$$

Domain = $\mathbb{R}$

Range = $[1, \infty)$

Even function

$\cosh(-x) = \cosh x$

Example 13

Solve each of the following equations for x.

(a) $\cosh 3x = 2$ (b) $4\sinh x = 3\cosh x$ (c) $\cosh 2x = 3\sinh x$

Solution

(a) $\cosh 3x = 2 \Rightarrow \frac{1}{2}(e^{3x} + e^{-3x}) = 2$

$$\Rightarrow e^{3x} + e^{-3x} = 4$$

$$\Rightarrow e^{6x} + 1 = 4e^{3x}$$

$$\Rightarrow e^{6x} - 4e^{3x} + 1 = 0$$

Let $y = e^{3x}$. Then we have $y^2 - 4y + 1 = 0$. By the quadratic equation formula,

$$y = \frac{-(-4)\pm\sqrt{(-4)^2-4(1)(1)}}{2(1)} = \frac{4\pm\sqrt{12}}{2} = \frac{4\pm2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

$\therefore\ e^{3x} = 2 + \sqrt{3}$ or $e^{3x} = 2 - \sqrt{3}$

$\Rightarrow\ 3x = \ln\left(2 + \sqrt{3}\right)$ or $3x = \ln\left(2 - \sqrt{3}\right)$

$\Rightarrow\ \boxed{x = \frac{1}{3}\ln\left(\underbrace{2 + \sqrt{3}}_{>0}\right)}$ or $\boxed{x = \frac{1}{3}\ln\left(\underbrace{2 - \sqrt{3}}_{>0}\right)}$

(b) $4\sinh x = 3\cosh x \Rightarrow \dfrac{\sinh x}{\cosh x} = \dfrac{3}{4}$

$$\Rightarrow \frac{\frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x})} = \frac{3}{4}$$

$$\Rightarrow 4(e^x - e^{-x}) = 3(e^x + e^{-x})$$

$$\Rightarrow e^x - 7e^{-x} = 0$$

$$\Rightarrow e^{2x} - 7 = 0$$

$$\Rightarrow e^{2x} = 7$$

$$\Rightarrow 2x = \ln 7$$

$$\Rightarrow \boxed{x = \frac{1}{2}\ln 7}$$

(c) $\cosh 2x = 3\sinh x$

$\Rightarrow \cosh^2 x + \sinh^2 x = 3\sinh x$ $\quad \because \cosh^2 x + \sinh^2 x = \cosh(2x)$ (Ex.11 (b))

$\Rightarrow (1+\sinh^2 x) + \sinh^2 x = 3\sinh x$ $\quad \because \cosh^2 x - \sinh^2 x = 1$ (Ex. 11 (a))

$\Rightarrow 2\sinh^2 x - 3\sinh x + 1 = 0$

$\Rightarrow (2\sinh x - 1)(\sinh x - 1) = 0$

$\Rightarrow \sinh x = \frac{1}{2}$ or $\sinh x = 1$

$\Rightarrow \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}$ or $\frac{1}{2}(e^x - e^{-x}) = 1$

$\Rightarrow e^x - e^{-x} = 1$ or $e^x - e^{-x} = 2$

$\Rightarrow (e^x)^2 - e^x - 1 = 0$ or $(e^x)^2 - 2e^x - 1 = 0$

$\Rightarrow e^x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$ or $e^x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$

$= \frac{1 \pm \sqrt{5}}{2}$ $\qquad = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$

$e^x \neq \frac{1-\sqrt{5}}{2} < 0$ (rejected) & $e^x \neq 1 - \sqrt{2} < 0$ (rejected), since $e^x > 0$ for all $x \in \mathbb{R}$.

$\therefore e^x = \frac{1+\sqrt{5}}{2}$ or $1+\sqrt{2}$

$$\boxed{\therefore x = \ln\left(\frac{1+\sqrt{5}}{2}\right) \text{ or } x = \ln(1+\sqrt{2})}$$

Other hyperbolic functions (*for your reference*)

The **hyperbolic tangent** ($\tanh x$), **hyperbolic secant** ($\operatorname{sech} x$), **hyperbolic cosecant** ($\operatorname{csch} x$), and **hyperbolic cotangent** ($\coth x$) functions are defined as follows:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^{x} - e^{-x}}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^{x} + e^{-x}}$$

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} = \frac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$$