MA1200 Applications of Derivatives

Rate of change

1. An airplane, flying horizontally at an altitude of 1 km, passes directly over an observer. If the constant speed of the plane is 240 km/hr, how fast is its distance from the observer increasing 30 seconds later?

2. A particle P is moving along the graph $y=\sqrt{x^2-4}, x\geq 2$ so that the x-coordinate of P is increasing at the rate of 5 units/s. How fast is the y-coordinate of P increasing when $x=3$?

3. Water is pumped at a uniform rate of 2 liters per minute into a tank shaped like a frustum of a right circular cone. The tank has altitude 80 centimeters and lower end and upper radii of 20 and 40 centimeters, respectively. How fast is the water level rising when the depth of the water is 30 centimeters? (The volume, V of a frustum of a right circular cone of altitude h and lower and upper radii d and b is $V=\frac{1}{3}\pi h\left(d^2+db+b^2\right)$.

4. Ship A is 15 meters east of O and moving west at 20 meters per hour; ship B is 60 meters south of O and moving north at 15 m/hr.

(a) Are they approaching or separating after one hour and at what rate?

(b) When are they nearest to one another?

Local Extrema

5. Let $y=f\left(x\right)=\dfrac{\sqrt[3]{x^2}}{x^2+1}$

   (a) Evaluate $f'\left(x\right)$ for $x\neq 0$ and prove that $f'\left(0\right)$ does not exist.

   (b) Determine those values of x for which $f'\left(x\right)>0$ and those values of x for which $f'\left(x\right)<0$.

   (c) Find the relative extreme points of $y=f\left(x\right)$.

   (d) Evaluate $f''(x)$ for $x\neq 0$. Hence determine the points of inflexion of $y=f\left(x\right)$.

Optimization

6. At which point on the parabola $y=1-x^2$ does the tangent have the property that it cuts from the first quadrant a triangle of minimum area?

7. During the course of an epidemic, the proportion of the population infected after time t is equal to $P(t)=\dfrac{t^2}{5\left(1+t^2\right)^2}$, where t is measured in months and the epidemic starts at $t=0$. Find the maximum proportion of population that becomes infected.

8. A rectangular box with a square base with an open top is to have a volume of 60m^3. Find the dimensions of the box that minimize the amount of materials used.

9. The yield y (tons per acre) of a certain crop of wheat is given by $y=a\left(1-e^{-kx}\right)+b$, where a, b, and k are constants, and x is the number of pounds per acre of fertilizer. The profit from the sale of the wheat is given by $P=\rho y-c_0-cx$, where ρ is the profit per ton, c is the cost per kg of fertilizer, and c_0 is an overhead cost. Determine how much fertilizer must be used in order to maximum the profit P.

Taylor/Maclaurin Series

10.

(a) Find the Taylor series for $(1-x)^{-2}$ in $-1<x<1$ at $x=0.5$.

(b) Find the Maclaurin series for $\sqrt{1+x}$ and use it to approximate $\sqrt{1.1}$ to five decimal places.

(c) Find the Maclaurin series for $\ln(1+x)$ and use it to find an expression for ln2.

(d) Find the Taylor series for $\sin^2 x$ at $x=\pi/4$.

*11. If $y=(1+x^2)^{-\frac{1}{2}}$,

(a) show that $(1+x^2)\dfrac{dy}{dx}+xy=0\ldots$(*).

(b) Use (*) to deduce that

$$(1+x^2)\frac{d^{n+1}y}{dx^{n+1}}+(2n+1)x\frac{d^n y}{dx^n}+n^2\frac{d^{n-1}y}{dx^{n-1}}=0 \text{ for } n=1,2,3,\ldots$$

(c) Hence or otherwise, find the Taylor series expansion for $(1+x^2)^{-\frac{1}{2}}$ at $x=1$ for the first four nonzero terms.

L'Hopital's rule

12. Evaluate each of the following limits.

(a) $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2+3x}$

(b) $\lim\limits_{x\to 0^+} x^x$

(c) $\lim\limits_{x\to\infty}\dfrac{x^a}{\ln x}$, where $a>0$

(d) $\lim\limits_{x\to 0^+}(x+1)^{\cot x}$

(e) $\lim\limits_{x\to\frac{\pi}{2}^+}\tan x\ln(\sin x)$

(f) $\lim\limits_{x\to 0}\dfrac{\sqrt{a+x}-\sqrt{a-x}}{x}$

(g) $\lim\limits_{x\to 0}\dfrac{\arcsin 2x}{x}$

(h) $\lim\limits_{x\to 0^+}\left(\cot x-\csc x\right)$

(i) $\lim\limits_{x\to\infty}(\sqrt{x}-1)^{1/\sqrt{x}}$

(j) $\lim\limits_{x\to\infty}(x^3+1)^{1/\ln x}$

(k) $\lim\limits_{x\to\frac{1}{2}}\dfrac{\cos^2\pi x}{e^{2x}-2ex}$

(l) $\lim\limits_{x\to a^+}\dfrac{\ln\sin(x-a)}{\ln\tan(x-a)}$

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