GE2262 Business Statistics

Topic 2
Basic Probability

Reference

Levine, D.M., Krehbiel, T.C. and Berenson, M.L., *Business Statistics: A First Course*, Pearson Education Ltd, Chapter 4

Outline

- Basic Probability Concepts
- Conditional Probability
- Counting Rules

Basic Probability Concepts – Outcomes and Events

- An outcome is the most basic possible results of observations or experiments
- Each possible outcome or combination of outcomes of a variable is an event
- Consider a toss of 2 coins and count the number of heads:

  - There are 4 different ways the coins could fall, i.e. there are 4 possible outcomes for the 2-coin toss
  - Suppose we are interested only in the number of heads, i.e. there are 3 possible events: 0 heads, 1 head and 2 heads

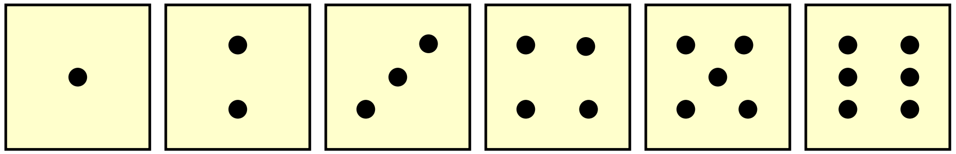
Basic Probability Concepts – Outcomes and Events

Cont'd

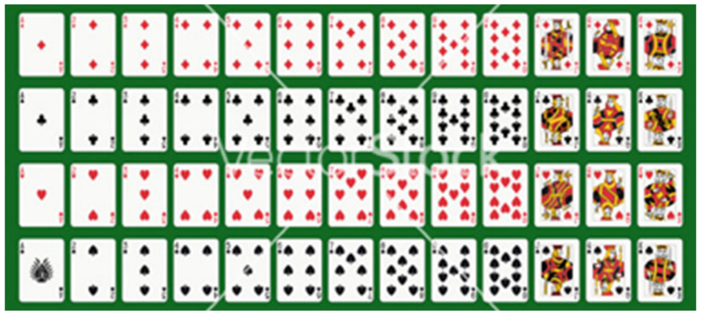
- Simple event – an event described by a single characteristic
  - E.g. A day in January from all days in 2015
- Joint event – an event described by two or more characteristics
  - E.g. A day in January that is also a Wednesday from all days in 2015
- Complement of an event A (denoted A') – all events that are not part of event A
  - E.g. All days in January that is not a Wednesday in 2015

Basic Probability Concepts – Sample Space

- Sample space is the collection of all possible outcomes
- E.g.
  - All 6 faces of a die



  - Full deck of playing cards



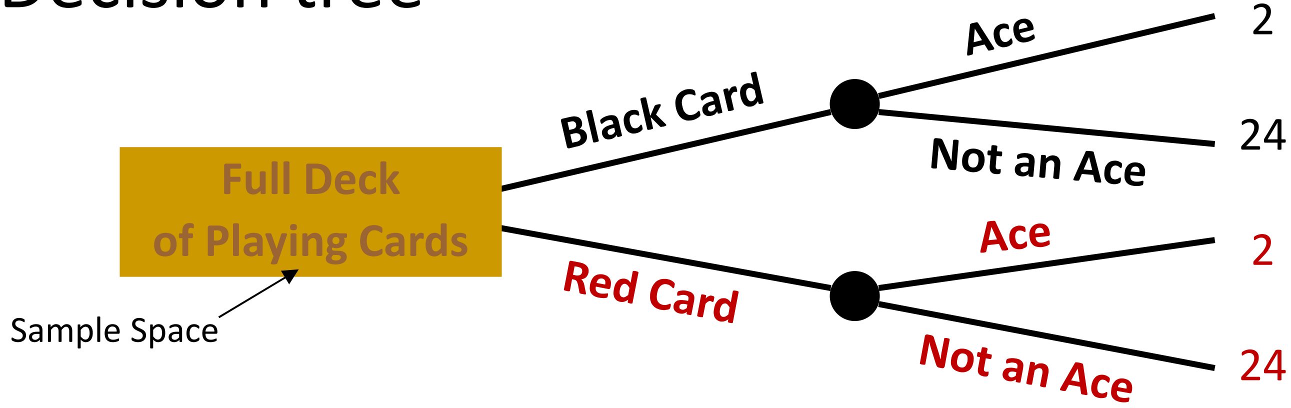
Basic Probability Concepts – Visualizing Events

- Contingency table

	Ace	Not Ace	Total
Black	2	24	**26**
Red	2	24	**26**
Total	4	48	52

Sample Space

- Decision tree



Basic Probability Concepts – Exercise

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- A sample of 1,000 households were asked about their intentions to purchase a large-screen HDTV sometime during the next 12 months. The same 1,000 households were surveyed 12 months later to see whether they purchased a television. The results are:

Planned to Purchase	Actually Purchased Yes	No	Total
Yes	200	50	250
No	100	650	750
Total	300	700	1000

- What is the sample space?
- Give examples of simple event and joint events.

Basic Probability Concepts – Exercise

Cont'd

- Sample space:
- Simple events are:

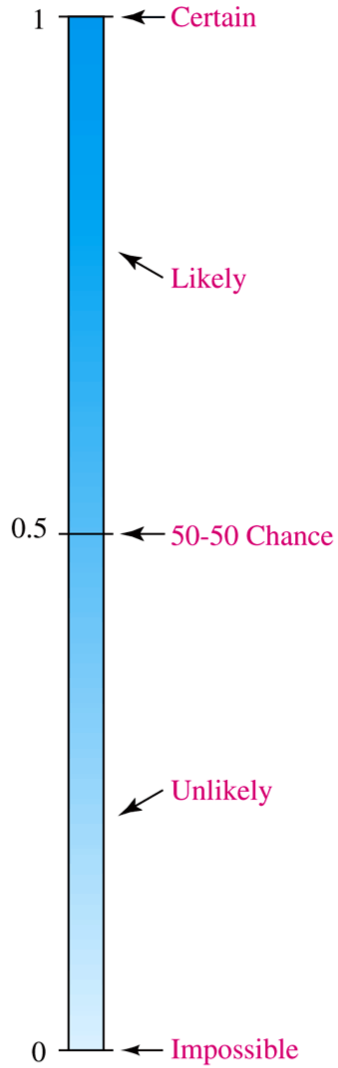
- Joint events are:

Basic Probability Concepts

- Probability is the numerical value representing the chance, likelihood, or possibility that a particular event will occur
- Its value is between 0 and 1, inclusive

  $0 \leq P(A) \leq 1$ for any event A

- Impossible event – an event that has no chance of occurring (probability = 0)
- Certain event – an event that is sure to occur (probability = 1)



Assessing Probability

- There are 3 possible approaches to assessing the probability of an event A
  - A priori – based on prior knowledge of the process

    $$P(A) = \frac{X}{T} = \frac{number\ of\ ways\ the\ event\ can\ occur}{total\ number\ of\ possible\ outcomes}$$

  - Empirical probability

    $$P(A) = \frac{number\ of\ times\ the\ event\ occurred}{total\ number\ of\ observations}$$

    Assuming all outcomes are equally likely

  - Subjective probability

    based on a combination of an individual's past experience, personal opinion, and analysis of a particular situation

Assessing Probability

Cont'd

- Example of prior probability: What is the probability of getting 1 head in a toss of two coins?

  - P(Getting 1 head) = 2 / 4 = 0.5

Assessing Probability

Cont'd

- Example of empirical probability: Refer to the 1,000 households interviewed for the intention of buying a HDTV. What is the probability of selecting a household that planned to purchase a large-screen HDTV in the next 12 months?

Planned to Purchase	Actually Purchased Yes	No	Total
Yes	200	50	250
No	100	650	750
Total	300	700	1000

- P(Planned to purchase) = 250 / 1000 = 0.25

Assessing Probability

Cont'd

- The assignment of a subjective probability is based on a person's experiences, opinions, and analysis of a particular situation
  - It may differ from person to person
  - It is useful in situations when an empirical or a priori probability cannot be computed
    - Example: A media development team assigns a 60% probability of success to its new ad campaign, but the chief media officer of the company is less optimistic and assigns a 40% of success to the same campaign

Joint Probability

- Joint probability refers to the probability of an occurrence involving two or more events, denoted as P(events A and B and C and D)
- Example: Refers to the 1,000 households interview. What is the probability of selecting a household that planned to purchase a HDTV and actually purchased?
  - P(Planned to purchase and actually purchased)

$$= \frac{\text{Planned to purchase and actually purchased}}{\text{Total number of households}} = \frac{200}{1000} = 0.2$$

Joint Probability

Cont'd

- Example: What is the probability you get heads on the first toss of a coin and heads in the second toss of a coin?

  - P(Heads on the 1st toss and heads of the 2nd toss) = $\frac{1}{4}$ = 0.25

Marginal Probability

- Marginal probability is the probability of the occurrence of the single event (simple probability)
- It is often derived from summing up all possible joint events that involves the concerned event
- Formal definition
  - $P(A) = P(A \text{ and } B_1)+P(A \text{ and } B_2)+……+P(A \text{ and } B_k)$

    where $B_1, B_2, ……, B_k$ are mutually exclusive and collectively exhaustive events

Marginal Probability

Cont'd

- The events are mutually exclusive if they cannot occur simultaneously
  - If events A and B are mutually exclusive, P(A and B) = 0
  - Example: A = queen of diamonds B = queen of clubs

    Events A and B are mutually exclusive

Marginal Probability

Cont'd

- A set of events is collectively exhaustive if one of the events must occur, and the set of events cover the whole sample space
  - If event A and B are mutually exclusive and collective exhaustive, P(A)+P(B) = 1
  - Example: A = aces B = black C = diamonds D = hearts

    Events A, B, C and D are collectively exhaustive but not mutually exclusive as an ace may also be a heart

    Events B, C and D are collectively exhaustive and also mutually exclusive

Marginal Probability

Cont'd

- Example: Refer to the 1,000 households interviewed for the intention of buying a HDTV

Planned to Purchase	Actually Purchased Yes	No	Total
Yes	200	50	250
No	100	650	750
Total	300	700	1000

- P(Planned to purchase)

  = P(Planned to purchase and actually purchased) + P(Planned to purchase and did not actually purchased)

  = 0.2 + 0.05 = 0.25

General Addition Rule

- The probability of either event A or event B occurs is denoted as P(A or B) and is defined as

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

- If A and B are mutually exclusive events, the rule is simplified as

$$P(A \text{ or } B) = P(A) + P(B)$$

General Addition Rule

Cont'd

- Example: Refer to the 1,000 households interviewed for the intention of buying a HDTV

Planned to Purchase	Actually Purchased Yes	No	Total
Yes	200	50	250
No	100	650	750
Total	300	700	1000

- P(Planned to purchase or actually purchased)

  = P(Planned to purchase) + P(Actually purchased) – P(Planned to purchase and actually purchased)

  = 0.25 + 0.30 – 0.20 = 0.35

Conditional Probability

- A conditional probability is the probability of one event, given that another event has occurred
- Symbolically, conditional probability of event A given event B is denoted as P(A|B)
  - P(A|B) may or may not equal to P(A), it depends on the relationship between the two events

Conditional Probability

Cont'd

- Example: Consider throwing a dice. What is the probability that you'll get an even number if you are told that the number is less than 4?

  - P(Even number | The number is less than 4) = $\frac{1}{3}$

$$= \frac{1/6}{3/6} = \frac{\text{P(Even number and The number is less than 4)}}{\text{P(The number is less than 4)}}$$

Conditional Probability

Cont'd

- The conditional probability of A given B is defined as

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} \qquad \text{with } P(B) > 0$$

- Similarly, the conditional probability of B given A is defined as

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)} \qquad \text{with } P(A) > 0$$

- Noted that P(A and B) = P(A|B)P(B) = P(B|A)P(A)
  - This is called the general multiplication rule

Conditional Probability

Cont'd

- Example: Refer to the households interviewed for the intention of buying a HDTV. What is the probability that a household planned to purchase a HDTV actually purchased the television?

Planned to Purchase	Actually Purchased Yes	No	Total
Yes	200	50	250
No	100	650	750
Total	300	700	1000

- P(Actually purchased |Planned to purchase)

$$= \frac{\text{P(Actually purchased and Planned to purchase)}}{\text{P(Planned to purchase)}}$$

$$= \frac{200/1000}{250/1000} = 0.80$$

Conditional Probability

Cont'd

- Example:

$$P(Red\ Card\ given\ that\ it\ is\ an\ Ace)$$
$$= P(Red|Ace) = \frac{2\ Red\ Aces}{4\ Aces} = \frac{1}{2}$$
$$= \frac{P(Red\ and\ Ace)}{P(Ace)} = \frac{2/52}{4/52} = \frac{1}{2}$$

	Ace	Not Ace	Total
Black	2	24	**26**
Red	2	24	**26**
Total	4	48	52

Red Ace Ace Card

- Example:

$$P(Ace\ card\ given\ that\ it\ is\ Red)$$
$$= P(Ace|Red) = \frac{2\ Red\ Aces}{26\ Red} = \frac{1}{13}$$
$$= \frac{P(Red\ and\ Ace)}{P(Red)} = \frac{2/52}{26/52} = \frac{1}{13}$$

	Ace	Not Ace	Total
Black	2	24	**26**
Red	2	24	**26**
Total	4	48	52

Red Ace Red Card

Conditional Probability

Cont'd

- Conditional probability
  - Probability of event A given that event B has occurred

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

  - Probability of event B given that event A has occurred

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

- General multiplication rule for joint event, A and B

$$P(A \text{ and } B) = P(A|B)P(B)$$

E.g. $P(Red\ Card\ and\ Ace)$
$= P(Red|Ace)P(Ace)$
$= \frac{1}{2} \times \frac{4}{52}$
$= \frac{2}{52} = \frac{1}{26}$

$$P(A \text{ and } B) = P(B|A)P(A)$$

E.g. $P(Red\ Card\ and\ Ace)$
$= P(Ace|Red)P(Red)$
$= \frac{1}{13} \times \frac{26}{52}$
$= \frac{2}{52} = \frac{1}{26}$

Conditional Probability

Cont'd

- Statistical independence
  - Two events, A and B, are independent if the occurrence of event A does not affect the probability of occurrence of event B, or vice versa
  - Two events are independent if and only if

    $P(A|B) = P(A)$ or $P(B|A) = P(B)$

  - This implies P(A and B) = P(A)P(B) if A and B are independent

Conditional Probability

Cont'd

- Example: Throwing a fair dice twice. If you get a six in the first throw, what is the probability that you get a six in the second throw? Are the two outcomes independent?
  - P(6 in the 2^{nd} throw | 6 in the 1^{st} throw)

    $$= \frac{\text{P(6 in the 2}^{nd}\text{ throw and 6 in the 1}^{st}\text{ throw)}}{\text{P(6 in the 1}^{st}\text{ throw)}} = \frac{1/36}{1/6} = \frac{1}{6}$$
  - Since P(6 in the 2^{nd} throw)

    $$= \text{P(6 in the 2}^{nd}\text{ throw | 6 in the 1}^{st}\text{ throw)} = \frac{1}{6}$$
  - Hence, the two outcomes are independent

Conditional Probability

Cont'd

- Example: Refer to the households interview data. Suppose for the 300 households that actually purchased HDTVs with either standard or faster refresh rate, the households were asked if they were satisfied with their purchases. Their responses to the question as follows:

TV Refresh Rate	Satisfied with Purchase? Yes	No	Total
Faster	64	16	80
Standard	176	44	220
Total	240	60	300

Conditional Probability

Cont'd

- Determine whether being satisfied with the purchase and the refresh rate of the television purchased are independent
  - P(Satisfied | Faster refresh rate) = $\frac{64/300}{80/300}$ = 0.80
  - P(Satisfied) = $\frac{240}{300}$ = 0.80
  - The two probabilities are equal. Thus being satisfied with the purchase and the refresh rate of the TV purchased are independent

Conditional Probability – Exercise

Cont'd

- A company is considering changing its starting business hour from 8am to 7:30am. The company has 1200 workers, including 450 office and 750 production workers. A census shows that 370 production workers favor the change, and a total of 715 office and production workers favor the change. Is the relationship between worker type and opinion independent?

Counting Rules

- For a sample space with a large number of possible outcomes, counting rules can be used to compute probabilities
- Counting rule 1:
  - If any one of k different mutually exclusive and collectively exhaustive events can occur on each of n trials, the number of possible outcomes is equal to

    k^n
  - Example: If you roll a fair die 3 times then there are $6^3 = 216$ possible outcomes

Counting Rules

Cont'd

- Counting rule 2:
  - If there are k_1 events on the first trial, k_2 events on the second trial, … and k_n events on the n^{th} trial, the number of possible outcomes is

    $(k_1)(k_2)\cdots\cdots(k_n)$

  - Example: You want to go to a park, eat at a restaurant, and see a movie. There are 3 parks, 4 restaurants, and 6 movie choices. How many different possible combinations are there?
    - Answer: (3)(4)(6) = 72 different possibilities

Counting Rules

Cont'd

- Counting rule 3:
  - The number of ways that n items can be arranged in order is

    $$n! = (n)(n-1)\cdots(1)$$

  - Example: You have five books to put on a bookshelf. How many different ways can these books be placed on the shelf?
    - Answer: $5! = (5)(4)(3)(2)(1) = 120$ different possibilities

Counting Rules

Cont'd

- Counting rule 4:
  - Permutations: The number of ways of arranging X objects selected from n objects in order is

$$ {}_{n}P_{X} = \frac{n!}{(n-X)!} $$

  - Example: You have five books and are going to put three on a bookshelf. How many different ways can the books be ordered on the bookshelf?
    - Answer: ${}_5P_3 = \frac{5!}{(5-3)!} = \frac{120}{2} = 60$ different possibilities

Counting Rules

Cont'd

- Counting rule 5:
  - Combinations: The number of ways of selecting X objects from n objects, irrespective of order, is

$$ {}_{n}C_{X} = \frac{n!}{X!(n-X)!} $$

  - Example: You have five books and are going to select three to read. How many different combinations are there, ignoring the order in which they are selected?
    - Answer: ${}_{5}C_{3} = \frac{5!}{3!(5-3)!} = \frac{120}{(6)(2)} = 10$ different possibilities