MA1201 Calculus and Basic Linear Algebra II Problem Set 2 Techniques of Integration

Problem 1 (Method of substitution)

Compute the following integral using method of substitution:

(a) $\displaystyle\int \frac{e^{1+\frac{1}{x^2}}}{x^3}\,dx$

(b) $\displaystyle\int x^2 \sec(1-2x^3)\,dx$

(c) $\displaystyle\int x^{11}\sqrt{1+x^4}\,dx$

(d) $\displaystyle\int x\cos^2(x^2)\,dx$

(e) $\displaystyle\int \sin 2x\sqrt{\cos x}\,dx$

(f) $\displaystyle\int \frac{e^{2x}}{(1+e^x)^3}\,dx$

(g) $\displaystyle\int_1^2 xe^{x^2-1}\,dx$

(h) $\displaystyle\int_1^5 \frac{\sin^2(\ln x)}{x}\,dx$

(i) $\displaystyle\int \frac{3x+2}{x^2+4}\,dx$

(j) $\displaystyle\int \frac{2x+1}{x^2-2x+5}\,dx$

(k) $\displaystyle\int \frac{4x}{3x^2+6x+19}\,dx$

(l) $\displaystyle\int \frac{1}{x^2\sqrt{1-x^2}}\,dx$

(m) $\displaystyle\int \frac{1}{(1+3x^2)^{\frac{3}{2}}}\,dx$

(n) $\displaystyle\int \frac{3x}{\sqrt{4x^2+1}}\,dx$

(o) $\displaystyle\int \sqrt{9-16x^2}\,dx$

(p) $\displaystyle\int \frac{1}{(x^2+6x+10)^{\frac{3}{2}}}\,dx$

(q) $\displaystyle\int \sin^7 x\,dx$

(r) $\displaystyle\int \sin^3 x\cos^5 x\,dx$

Problem 2 (Integration by parts)

Compute the following integrals using integration by parts:

(a) $\displaystyle\int xe^{-3x}\,dx$

(b) $\displaystyle\int_1^e \sqrt{x}\ln x\,dx$

(c) $\displaystyle\int x^2\sin x\,dx$

(d) $\displaystyle\int x\sin^2 x\,dx$

(e) $\displaystyle\int x^2\cos^{-1} x\,dx$

(f) $\displaystyle\int \tan^{-1} x\,dx$

(g) $\displaystyle\int \csc^3 x\,dx$

(h) $\displaystyle\int \cos^3 x\,dx$

(i) $\displaystyle\int_1^e \left(\frac{\ln x}{x}\right)^2 dx$

(j) $\displaystyle\int e^x\sin 3x\,dx$

Problem 3

Compute the following integrals using suitable method. You may need to use method of substitution or integration by parts or both.

(a) $\displaystyle\int e^{2x}\sin(2e^x+1)\,dx$

(b) $\displaystyle\int_0^1 \sin(2\sqrt{x})\,dx$

(c) $\displaystyle\int_0^1 \ln\left(1+\sqrt[3]{x}\right)dx$

(d) $\displaystyle\int \cos(\ln x)\,dx$

(e) $\displaystyle\int \sin 2x\ln(\sin x)\,dx$

(f) $\displaystyle\int (x+1)\ln(x+3)\,dx$

(g) $\displaystyle\int_1^2 \frac{e^{2x}}{e^x-1}\,dx$

(h) $\displaystyle\int x^3\cos(3x^2)\sin(x^2)\,dx$

(i) $\displaystyle\int x^2\sqrt{4-x^2}\,dx$

(j) $\displaystyle\int x^3\sin(4+x^2)\,dx$

Problem 4

(a) Compute the integrals

$$\int e^{2x} \sin 3x \, dx \quad \text{and} \quad \int e^{2x} \cos 3x \, dx.$$

(b) Hence, compute the integrals

$$\int x e^{2x} \cos 3x \, dx.$$

(Hint: You need to eliminate x in the integrand so that you can compute the integral using the result of (a). Which technique should you use: Method of substitution and/or integration by parts?)

Problem 5

Let $f(x)$ be a differentiable function on $[a, b]$ such that $\int_a^b f(x)dx = 0$ and $f(a) = f(b) = 1$. Find the value of $\int_a^b x f'(x)dx$.

Problem 6

Let $f(x)$ be a continuous function, show that for any $a > 0$, we must have

$$\int_0^a x^3 f(x^2)dx - \frac{1}{2}\int_0^{a^2} x f(x)dx = 0.$$

Problem 7

Let $f(x)$ be a twice differentiable function on $[0,1]$ such that $f(0) = f(1) = 1$ and $\int_0^1 f(x)dx = 1$. Using integration by parts, find the value of

$$\int_0^1 x(1-x)f''(x)dx.$$

(Hint: The technique in Problem 5 may be useful.)

Problem 8

For non-negative integer n, we define the integral

$$I_n = \int_0^1 x^n e^{-3x} dx.$$

(a) Deduce the following reduction formula for I_n:

$$I_n = -\frac{1}{3}e^{-3} + \frac{n}{3} I_{n-1}, \quad n \geq 1.$$

(b) Using the reduction formula in (a), find the value of

$$\int_0^1 x^3 e^{-3x} dx.$$

Problem 9

For any real number $a \neq 1$ and non-negative integer n, we define the integral as

$$I_n = \int_1^e x^a (\ln x)^n dx.$$

(a) Deduce the following reduction formula for I_n:

$$I_n = \frac{e^{a+1}}{a+1} - \frac{n}{a+1} I_{n-1}, \quad n \geq 1.$$

(b) Using the reduction formula in (a), find the value of

$$\int_1^e x^2 (\ln x)^3 dx.$$

(c) What is the value of I_n when $a = -1$?

Problem 10

For non-negative integer n, we define the integral

$$I_n = \int (x^2 + a^2)^n dx.$$

(a) Show that

$$I_n = \frac{1}{2n+1} x(x^2 + a^2)^n + \frac{2n}{2n+1} a^2 I_{n-1}, \quad n \geq 1.$$

(b) Compute the integral

$$\int (x^2 + a^2)^4 dx.$$

Problem 11

For any non-negative integer n, we define the integral I_n as

$$I_n = \int x^n \cos 3x \, dx.$$

(a) Deduce the following reduction formula for I_n:

$$I_n = \frac{1}{3} x^n \sin 3x + \frac{n}{9} x^{n-1} \cos 3x - \frac{n(n-1)}{9} I_{n-2}, \ \ n \geq 2.$$

(Hint: You need to use integration by parts twice)

(b) Using the result of (a), compute the integral

$$\int x^4 \cos 3x \, dx.$$

Problem 12

Consider the integral

$$I_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n x \, dx.$$

(a) Show that

$$I_n = \frac{n-1}{n} I_{n-2}, \quad n \geq 2.$$

(b) Using the reduction formula obtained in (a), find the value of

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5 x \, dx, \qquad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^4 x \, dx.$$

(c) Find the value of

$$\int_{-1}^{1} (1 - x^2)^{\frac{5}{2}} dx.$$

Problem 13

Consider the integral

$$I_n = \int_0^{\frac{\pi}{4}} \sec^n x \, dx.$$

(a) Deduce the following reduction formula for I_n:

$$I_n = \frac{\left(\sqrt{2}\right)^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}, \quad n \geq 2.$$

(b) Using the reduction formula in (a), find the value of

$$\int_0^1 (1 + x^2)^{\frac{3}{2}} dx \qquad \text{and} \qquad \int_2^{2\sqrt{2}} x^2 \sqrt{x^2 - 4} \, dx.$$

Problem 14 (A bit harder)

For any non-negative integer n, we define the integral

$$I_n = \int_0^a x^n \sqrt{a^2 - x^2}\,dx\,,\quad a > 0.$$

(a) Deduce the reduction formula for I_n:

$$I_n = \left(\frac{n-1}{n+2}\right) a^2 I_{n-2},\quad n \geq 2.$$

(b) Compute the integral

$$\int_0^a x^2 (a^2 - x^2)^{\frac{3}{2}} dx.$$

(Hint: $(a^2 - x^2)^{\frac{3}{2}} = (a^2 - x^2)\sqrt{a^2 - x^2}$.)

Problem 15 (Harder)

For any non-negative integer n, we define the integral

$$I_n = \int (\sin^{-1} x)^n dx.$$

(a) Show that

$$I_n = x(\sin^{-1} x)^n + n(\sin^{-1} x)^{n-1}\sqrt{1 - x^2} - n(n-1)I_{n-2},\qquad n \geq 2.$$

(b) Hence, compute the integrals

$$\int (\sin^{-1} x)^3 dx,\quad \int \frac{x(\sin^{-1} x)^5}{\sqrt{1 - x^2}} dx.$$

(Hint: You need to apply integration by parts on the second integral.)

Problem 16 (Method of Partial Fractions)

Compute the following integrals using method of partial fraction.

(☺Hint: In some cases (say, (a), (c), (f), (h), you may need to factorize the denominator)

(a) $\displaystyle\int \frac{1}{3x - x^2} dx$

(b) $\displaystyle\int \frac{2x^2 - 5x + 5}{(x-1)^2(x-2)} dx$

(c) $\displaystyle\int \frac{3x^4 - 5x^3 + x^2 + 2x + 1}{3x^3 - 2x^2 - x} dx$

(d) $\displaystyle\int \frac{3x^2 + 10}{(x+3)(x^2 - 6x + 10)} dx$

(e) $\displaystyle\int \frac{-7x + 19}{(x^2 - 4x + 9)(2x + 1)} dx$

(f) $\displaystyle\int \frac{8x^2 - 3x - 2}{4x^3 - 3x + 1} dx$

(g) $\displaystyle\int \frac{x^2 - 5x - 5}{(x-2)(x^2 + 2x + 3)} dx$

(h) $\displaystyle\int \frac{x^5 + 2x^4 - x + 2}{x^3 + 2x^2 - x - 2} dx$

(i) $\displaystyle\int \frac{2x^2 - x + 1}{x^3(x-1)} dx$

(j) $\displaystyle\int \frac{3x^3 - 2x - 20}{(x^2 + 3)(2x^2 - 6x + 5)} dx$

*(k) $\displaystyle\int \frac{6x^3 - 27x^2 + 5x - 1}{(x-2)^2(4x^2 + 1)} dx$

*(l) $\displaystyle\int \frac{x^2 + 2x + 4}{x(x^2 + 2)^2} dx$