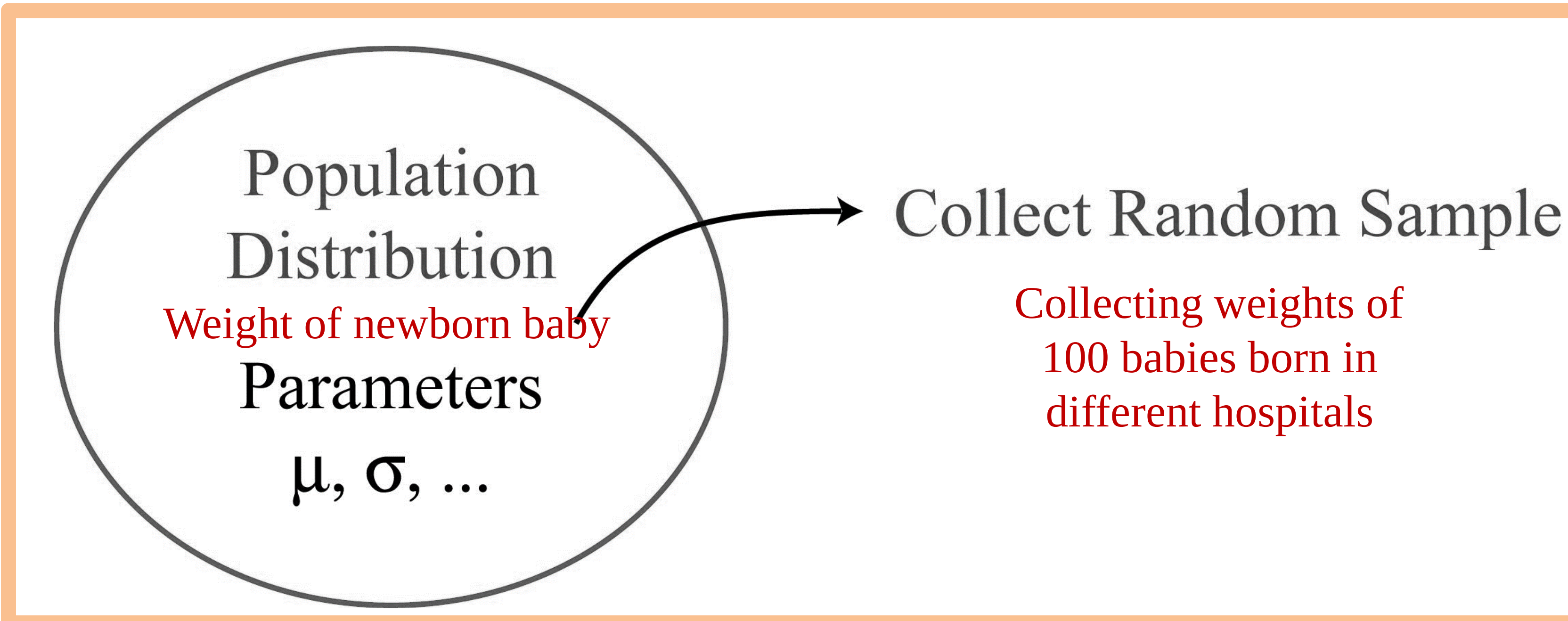
SDSC2102
Statistical Methods and Data Analysis

Topic 1. Basic Probability and Statistics Theory

Estimation and Hypothesis Testing

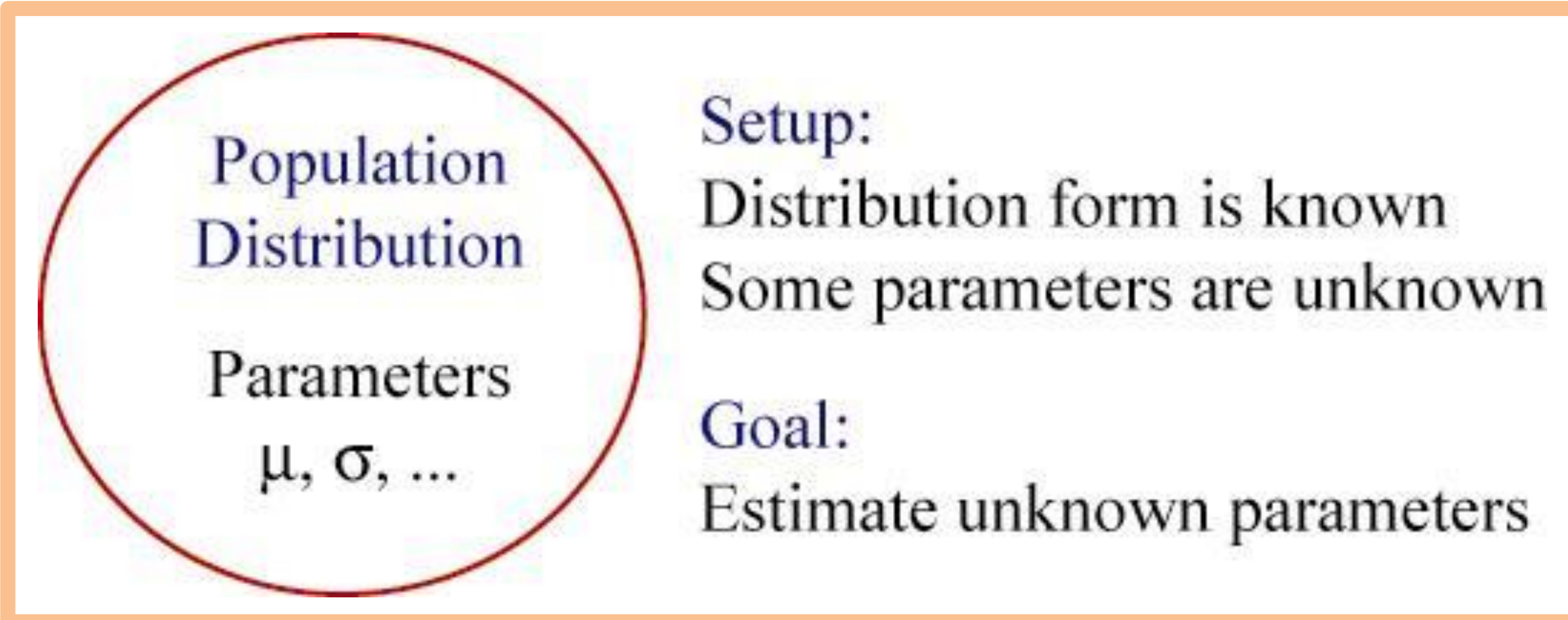
Random Sampling



➢ Random Sample

- Before data: $X_1, X_2, \ldots, X_n$ are independent and identically distributed (IID) r.v.'s
- After data: Denote observed values by $x_1, x_2, \ldots, x_n$

Two Basic Problems



- Estimation
- Hypothesis tests

Point Estimation

- A point estimator is designed to estimate an unknown parameter with a single value

- Point estimators of mean and variance
  - Sample mean $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is estimator of μ
  - Sample variance $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$ is estimator of σ^2

- An estimate is a calculated value (not random) of an estimator

Example

- If we take 4 samples of pie pumpkin, the estimators of mean and variance are

$$\bar{X} = \frac{X_1 + X_2 + X_3 + X_4}{4}$$

$$S^2 = \frac{\left(X_1 - \bar{X}\right)^2 + \left(X_2 - \bar{X}\right)^2 + \left(X_3 - \bar{X}\right)^2 + \left(X_4 - \bar{X}\right)^2}{3}$$

- Assume their weights are 10, 16, 16 and 22 pounds, the estimates of mean and variance are

$$\bar{x} = \frac{10 + 16 + 16 + 22}{4} = 16$$

$$s^2 = \frac{(10 - 16)^2 + (16 - 16)^2 + (16 - 16)^2 + (22 - 16)^2}{3} = 24$$

Unbiased Estimator

- **Theorem:** If $X_1, X_2, \ldots, X_n$ are IID random variables with

$$E[X_1] = E[X_2] = \cdots = E[X_n] = \mu \text{ and}$$

$$Var[X_1] = Var[X_2] = \cdots = Var[X_n] = \sigma^2$$

Then

$$E[\bar{X}] = \mu$$

$$E[S^2] = \sigma^2$$

*An estimator is called an unbiased estimator if the expected value of the estimator is equal to the unknown parameter to be estimated.

Confidence Interval Estimation

- Estimate an unknown parameter with an interval
  - A confidence interval (C.I.) is constructed such that $P(L \leq parameter \leq U)$ achieves a specified confidence level of $1 - \alpha$
  - Before data: Estimator is a r.v.

    $\Rightarrow$ C.I. (L, U) is a random interval
  - After data: Estimate is a calculated value

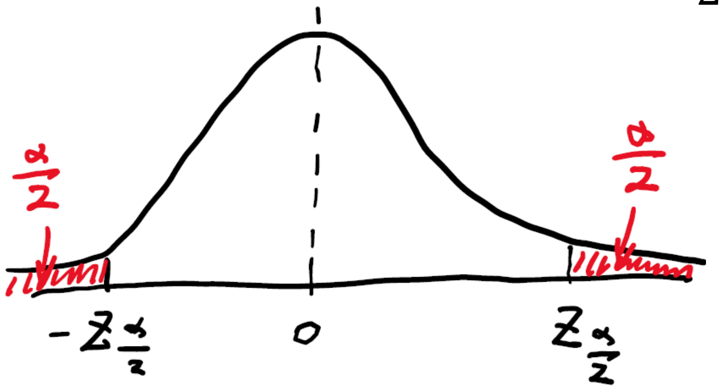
    $\Rightarrow$ C.I. (L, U) is a fixed interval

Confidence Interval for μ (σ^2 Known)

- Assume

$$\Phi(Z_{\alpha/2}) = 1 - \frac{\alpha}{2}$$

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$



$$P\left(-Z_{\alpha/2} \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq Z_{\alpha/2}\right) = 1 - \alpha$$

$$\Rightarrow P\left(\bar{X} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

Z-Intervals

➢ With known σ^2, the $100(1-\alpha)\%$ confidence interval for μ is

$$\left(\bar{x} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \quad \text{or} \quad \bar{x} \pm Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

Example: Suppose the population variance is 9, and a sample of size 16 results in a sample mean of 20. Given $\alpha = 0.05, \alpha/2 = 0.025$, from the standard normal table,

$$Z_{0.025} = 1.96$$

So the 95% confidence interval for μ is

$$\bar{x} \pm 1.96\frac{\sigma}{\sqrt{n}} = 20 \pm (1.96)\frac{3}{4} = (18.53, 21.47)$$

Interpretation of Confidence Interval

- Before data, the C.I. is random and satisfies the desired probability statement
- After data, the C.I. is fixed, e.g. (18.53, 21.47)
  - $P(18.53 \leq \mu \leq 21.47)$ is irrelevant because nothing is random.
  - We are 95% confident that the true population mean μ lies between 18.53 and 21.47.
- If many, many random samples are collected, then 95% of them should contain μ
  - Cannot be verified in reality since μ is unknown

Important Note

- In the derivation, we assumed that $\bar{X}$ is normally distributed.

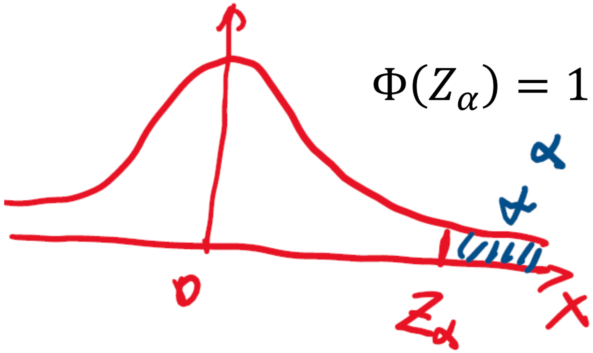
- When is $\bar{X}$ normally distributed?
  - When sample size is large, i.e., $n \longrightarrow \infty$, by CLT
  - When sample size is small,

    Theorem: If $X_1, \ldots, X_n$ are IID and normally distributed, then $\bar{X} \sim N(\mu, \sigma^2/n)$

- We assume that our data are sampled from normal distribution.

One-Sided C. I.

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$



$$\Phi(Z_\alpha) = 1-\alpha$$

$$P\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < Z_\alpha\right) = 1-\alpha$$

$$\Rightarrow P\left(\mu > \bar{X} - Z_\alpha \frac{\sigma}{\sqrt{n}}\right) = 1-\alpha$$

Lower bound

One-Sided Bounds for μ (σ^2 Known)

- $100(1-\alpha)\%$ one-sided C.I.'s
  - The upper bound and lower bound C.I.'s for μ :

$$\left(-\infty, \bar{x} + Z_{\alpha} \frac{\sigma}{\sqrt{n}}\right) \quad \left(\bar{x} - Z_{\alpha} \frac{\sigma}{\sqrt{n}}, \infty\right)$$

- Note: One-sided C.I.'s use Z_{α} instead of $Z_{\alpha/2}$
- Commonly used Z_{α} values:
  - $Z_{.10} = 1.282$, $Z_{.05} = 1.645$, $Z_{.025} = 1.96$
  - $Z_{.01} = 2.326$, $Z_{.005} = 2.576$

Example

- Suppose the population variance is 9, and a sample of size 16 results in a sample mean of 20. Find the 95% upper one-sided confidence bound for the mean.

Given $\alpha = 0.05, Z_{0.05} = 1.645$

$n = 16, \sigma^2 = 9, \bar{x} = 20$

$$\mu < \bar{x} + Z_\alpha \frac{\sigma}{\sqrt{n}} = 20 + 1.645 \times \frac{3}{4} = 21.234$$

So the 95% upper bound for μ is $(-\infty, 21.234)$.

When σ^2 Is Unknown

- When σ^2 is unknown, we use S^2 to estimate σ^2.

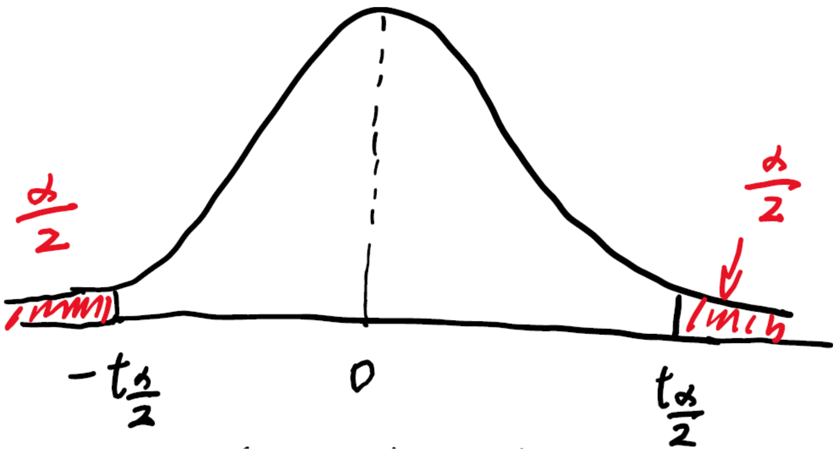
- If $X_1, \ldots, X_n$ are IID normally distributed random variables with mean μ and variance σ^2, then

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

has a (Student's) t-distribution with $v = n - 1$ degrees of freedom.

t-Intervals

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t(n-1)$$



$$P\left(-t_{\alpha/2,n-1} \le \frac{\bar{X} - \mu}{S/\sqrt{n}} \le t_{\alpha/2,n-1}\right) = 1 - \alpha$$

$$\Rightarrow P\left(\bar{X} - t_{\alpha/2,n-1}\frac{S}{\sqrt{n}} \le \mu \le \bar{X} + t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}\right) = 1 - \alpha$$

t-Intervals

- With unknown σ^2, the $100(1-\alpha)\%$ two-sided confidence interval for μ is

$$\left(\bar{x} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}, \bar{x} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}\right)$$

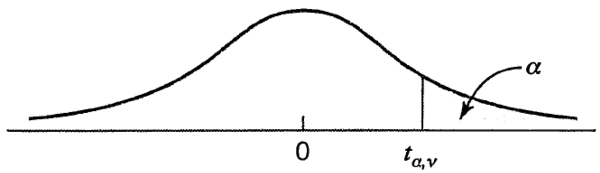
- The $100(1-\alpha)\%$ one-sided C.I. for μ
  - The upper bound and lower bound C.I.'s for μ :

$$\left(-\infty, \bar{x} + t_{\alpha,n-1}\frac{s}{\sqrt{n}}\right) \qquad \left(\bar{x} - t_{\alpha,n-1}\frac{s}{\sqrt{n}}, \infty\right)$$

t Table

■ APPENDIX IV

Percentage Points of the t Distribution[a]



	α									
v	**0.40**	**0.25**	**0.10**	**0.05**	**0.025**	**0.01**	**0.005**	**0.0025**	**0.001**	**0.0005**
1	0.325	1.000	3.078	6.314	12.706	31.821	63.657	127.32	318.31	636.62
2	0.289	0.816	1.886	2.920	4.303	6.965	9.925	14.089	23.326	31.598
3	0.277	0.765	1.638	2.353	3.182	4.541	5.841	7.453	10.213	12.924
4	0.271	0.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	0.267	0.727	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869
6	0.265	0.727	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	0.263	0.711	1.415	1.895	2.365	2.998	3.49	4.019	4.785	5.408
8	0.262	0.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	0.261	0.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	0.260	0.700	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	0.260	0.697	1.363	1.796	2.20	2.718	3.106	3.497	4.025	4.437
12	0.259	0.695	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	0.259	0.694	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221

Class Problem 1

1. A machine produces cylindrical pieces. A sample of pieces is taken and the diameters are 1.01, 0.97, 1.03, 1.04, 0.99, 0.98, 0.99, 1.01 and 1.03 centimeters. Find a 99% confidence interval for the mean diameter of pieces from this machine, assuming an approximate normal distribution (check the assumption using a normal probability plot).

Find C.I. for μ, σ^2 unknown $\Rightarrow$ t-interval

Given $n = 9, \alpha = 0.01 \Rightarrow t_{\alpha/2,n-1} = t_{0.005,8} = 3.355$

$\bar{x} = 1.006, s = 0.02455$

$$\bar{x} \pm t_{\alpha/2,n-1} \frac{s}{\sqrt{n}} = 1.006 \pm 3.355 \times \frac{0.02455}{\sqrt{9}}$$

$$= (0.9785, 1.0335)$$

Two Independent Samples

- $X_{11}, \ldots, X_{1\,n_1}$ are IID random variables from a population distribution $N(\mu_1, \sigma_1^2)$, and $X_{21}, \ldots, X_{2\,n_2}$ are IID random variables from a population distribution $N(\mu_2, \sigma_2^2)$. The two populations are independent.

Parameter of interest: $\mu_1 - \mu_2$

Point estimator: $\bar{X}_1 - \bar{X}_2$

Variance of estimator: $\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}$

Three Cases

- **Case 1:** σ_1^2 and σ_2^2 are known

- **Case 2:** σ_1^2 and σ_2^2 are unknown but equal

- **Case 3:** σ_1^2 and σ_2^2 are unknown and unequal (Not required)

Case 1 – Z-Intervals

- When σ_1^2 and σ_2^2 are known
  - The $100(1-\alpha)\%$ C.I. for $\mu_1 - \mu_2$ is

$$\bar{x}_1 - \bar{x}_2 \pm Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

$$\bar{x}_1 - \bar{x}_2 - Z_{\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < \mu_1 - \mu_2 < \bar{x}_1 - \bar{x}_2 + Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

  - The $100(1-\alpha)\%$ one-sided C.I. for $\mu_1 - \mu_2$ is

$$\left(-\infty, \bar{x}_1 - \bar{x}_2 + Z_{\alpha}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right) \quad \left(\bar{x}_1 - \bar{x}_2 - Z_{\alpha}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}, \infty\right)$$

Example

- **Example:** A new drug is tested to determine whether it can lower the blood glucose level of diabetic rats. A sample of 6 rats are given the drug and 5 others are given a "placebo". The measured blood glucose levels of the two groups are given below. Construct a 99% C.I. for the difference in the means.
  - $n_1 = 6,\ \bar{x}_1 = 19,\ \sigma_1^2 = 0.05$
  - $n_2 = 5,\ \bar{x}_2 = 20,\ \sigma_2^2 = 0.03$

$$(19 - 20) \pm Z_{0.005}\sqrt{\frac{0.05}{6} + \frac{0.03}{5}}$$

$$= -1 \pm 2.576 \times 0.12 = (-1.31, -0.69)$$

Case 2 – *t*-Intervals

- Assume σ_1^2 and σ_2^2 are unknown, but equal:
  - $\sigma_1^2 = \sigma_2^2 = \sigma^2$ common variance
  - $V\left(\bar{X}_1 - \bar{X}_2\right) = \dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2} = \sigma^2\left[\dfrac{1}{n_1} + \dfrac{1}{n_2}\right]$
- Estimate σ^2 with a pooled sample variance

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

Case 2 – *t*-Intervals

- When σ_1^2 and σ_2^2 are unknown but equal
  - The $100(1-\alpha)\%$ C.I. for $\mu_1 - \mu_2$ is

$$\bar{x}_1 - \bar{x}_2 \pm t_{\alpha/2, n_1+n_2-2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

$$\bar{x}_1 - \bar{x}_2 - t_{\alpha/2, n_1+n_2-2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} < \mu_1 - \mu_2 < \bar{x}_1 - \bar{x}_2 + t_{\alpha/2, n_1+n_2-2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

  - The $100(1-\alpha)\%$ one-sided C.I. for $\mu_1 - \mu_2$ is

$$\left(-\infty, \bar{x}_1 - \bar{x}_2 + t_{\alpha, n_1+n_2-2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right) \quad \left(\bar{x}_1 - \bar{x}_2 - t_{\alpha, n_1+n_2-2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}, \infty\right)$$

Example

- **Example:** Two independent samples are taken to see how many hours students spend in course work. 12 students take the survey in Course 1, with $\bar{x}_1 = 3.11$ and a standard deviation $s_1 = 0.771$. 10 students take the survey in Course 2 with $\bar{x}_2 = 2.04$ and a standard deviation $s_2 = 0.448$. Find a 90% confidence interval for $\mu_1 - \mu_2$, assuming that the populations are approximately normally distributed with equal variance.

$n_1 = 12, \bar{x}_1 = 3.11, s_1 = 0.771$

$n_2 = 10, \bar{x}_2 = 2.04, s_2 = 0.448$

$\alpha = 0.1$

Example

$$t_{\frac{\alpha}{2},n_1+n_2-2} = t_{0.05,12+10-2} = t_{0.05,20} = 1.725$$

$$S_p^2 = \frac{11 \cdot 0.771^2 + 9 \cdot 0.448^2}{12 + 10 - 2} = 0.4173 \Rightarrow S_p = \sqrt{0.4173} = 0.646$$

The 90% C. I. for $\mu_1 - \mu_2$ is:

$$\bar{x}_1 - \bar{x}_2 \pm t_{\frac{\alpha}{2},n_1+n_2-2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = 3.11 - 2.04 \pm 1.725 \cdot 0.646 \cdot \sqrt{\frac{1}{12} + \frac{1}{10}}$$

$$= (0.593, 1.547)$$

$$0.593 < \mu_1 - \mu_2 < 1.547$$

Hypothesis Testing

- A hypothesis test considers two hypotheses
  - H_0 is the null hypothesis
  - H_a (or H_1) is the alternative hypothesis
- The result of the test is to
  - **Reject $\boldsymbol{H_0}$**: the data indicate that H_0 is false
  - **Fail to reject $\boldsymbol{H_0}$**: the data do not strongly contradict H_0

Test Forms

- There are 3 test forms:
  - Two-sided test
  $$H_0 : \mu = \mu_0 \;\; \text{vs.} \;\; H_1 : \mu \neq \mu_0$$
  - One-sided lowertail test
  $$H_0 : \mu = \mu_0 \;\; \text{vs.} \;\; H_1 : \mu < \mu_0$$
  $$(H_0 : \mu \geq \mu_0 \;\; \text{vs.} \;\; H_1 : \mu < \mu_0)$$
  - One-sided uppertail test
  $$H_0 : \mu = \mu_0 \;\; \text{vs.} \;\; H_1 : \mu > \mu_0$$
  $$(H_0 : \mu \leq \mu_0 \;\; \text{vs.} \;\; H_1 : \mu > \mu_0)$$
- Note: Equality (=) is always in H_0

Errors in Hypothesis Testing

- Define the error probabilities
  - $\alpha = P$[Type I error] $= P$[Reject $H_0 \mid H_0$ is true]
  - $\beta = P$[Type II error] $= P$[Fail to reject $H_0 \mid H_0$ is false]
  - α is called significance level of the test
  - The same α as we use for C.I.'s

	H_0 is true	**H_0 is false**
Fail to reject H_0	Correct	Type II error
Reject H_0	Type I error	Correct

- We control α, but usually do not control β
  - Common choices for α: 0.01, 0.05, 0.10

Test for Single Mean μ (σ^2 Known)

- Two-sided Test

$$H_0 : \mu = \mu_0 \text{ vs. } H_1 : \mu \neq \mu_0$$

- When H_0 is true

$$\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$P\left(-Z_{\alpha/2} \leq \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \leq Z_{\alpha/2}\right) = 1 - \alpha$$

$100(1-\alpha)\%$ confidence interval for μ is

$$\bar{x} - Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \leq \mu_0 \leq \bar{x} + Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \quad \Leftrightarrow \quad \left|\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}\right| \leq Z_{\alpha/2}$$

Z-Tests (Two-Sided)

- With known σ^2, the decision rule for the two-sided test is

Reject H_0 when

μ_0 is NOT in the C.I. $\left(\bar{x} - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$

or

$$|Z_0| > Z_{\alpha/2}$$

$$Z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

Z-Tests (One-Sided)

➢ One-sided Tests

$$H_0\!: \mu = \mu_0 \qquad \text{vs.} \qquad H_1\!: \mu > \mu_0$$

Reject $\boldsymbol{H_0}$ when

μ_0 is not in the one-sided lowerbound C.I. $\left(\bar{x} - Z_\alpha \frac{\sigma}{\sqrt{n}}, \infty\right)$

or

$$Z_0 > Z_\alpha$$

$$H_0\!: \mu = \mu_0 \qquad \text{vs.} \qquad H_1\!: \mu < \mu_0$$

Reject $\boldsymbol{H_0}$ when

μ_0 is not in the one-sided upperbound C.I. $\left(-\infty, \bar{x} + Z_\alpha \frac{\sigma}{\sqrt{n}}\right)$

or

$$Z_0 < -Z_\alpha$$

Recall: When σ^2 Is Unknown

- When σ^2 is unknown, we use S^2 to estimate σ^2.

- If $X_1, \dots, X_n$ are IID normally distributed random variables with mean μ and variance σ^2, then

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

has a (Student's) t-distribution with $v = n - 1$ degrees of freedom.

t-Tests (Two-Sided)

- Two-sided Test

$$H_0 : \mu = \mu_0 \ \text{ vs. } \ H_1 : \mu \neq \mu_0$$

Reject H_0 when

μ_0 is not in the C.I. $\left(\bar{x} - t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}, \bar{x} + t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}\right)$

or

$$|t_0| > t_{\alpha/2,n-1}$$

$$t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

t-Tests (One-Sided)

➢ One-sided Tests

$$H_0\!: \mu = \mu_0 \qquad \text{vs.} \qquad H_1\!: \mu > \mu_0$$

Reject $\boldsymbol{H_0}$ when

μ_0 is not in the one-sided lowerbound C.I. $\left(\bar{x} - t_{\alpha,n-1}\frac{S}{\sqrt{n}}, \infty\right)$

or

$$t_0 > t_{\alpha,n-1}$$

$$H_0\!: \mu = \mu_0 \qquad \text{vs.} \qquad H_1\!: \mu < \mu_0$$

Reject $\boldsymbol{H_0}$ when

μ_0 is not in the one-sided upperbound C.I. $\left(-\infty, \bar{x} + t_{\alpha,n-1}\frac{S}{\sqrt{n}}\right)$

or

$$t_0 < -t_{\alpha,n-1}$$

Class Problem 2

2. Test the hypothesis that the average content of containers of a particular lubricant is 10 liters if the contents of a random sample of 10 containers are 10.2, 9.7, 10.1, 10.3, 10.1, 9.8, 9.9, 10.4, 10.3 and 9.8 liters. Use a 0.01 level of significance and assume that the distribution of contents is normal (also check if that assumption is reasonable). Find a 95% confidence interval for the mean content level.

$$H_0\colon \mu = 10 \qquad \text{vs.} \qquad H_1\colon \mu \neq 10$$

You can also put the hypotheses as

$H_0\colon \mu = 10$

$H_1\colon \mu \neq 10$

Test for μ, σ^2 unknown $\Rightarrow$ t-test (two-sided)

$$\bar{x} = 10.06, s = 0.2459$$

$$n = 10, \alpha = 0.01 \Longrightarrow t_{\alpha/2,n-1} = t_{0.005,9} = 3.25$$

Class Problem 2

Method 1: Find the 99% confidence interval

$$\bar{x} \pm t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} = 10.06 \pm 3.25 \times \frac{0.2459}{\sqrt{10}} = (9.81, 10.31)$$

$$\mu_0 = 10 \in (9.81, 10.31) \Longrightarrow \text{Fail to reject } H_0$$

Method 2: Find t_0

$$t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{10.06 - 10}{0.2459/\sqrt{10}} = 0.7716 < t_{0.005,9}$$

$$\Longrightarrow \text{Fail to reject } H_0$$

Conclusion: The average content of containers is 10.

Example of One-Sided Tests

- **Example:** An online gaming platform claimed that after the outbreak of COVID-19, the average hours that each user spends on that platform per week is 46 hours. If we randomly sampled 12 people and found that the sample has a mean 42 hours and standard deviation 11.9 hours. Does this suggest at 0.05 significance level that the average hours people spend in gaming each week is less than 46 hours?

$H_0\colon\ \mu = 46$

$H_1\colon\ \mu < 46$

Test for μ, σ^2 unknown $\Rightarrow$ t-test (one-sided)

Example of One-Sided Tests

$\bar{x} = 42, s = 11.9$

$n = 12, \alpha = 0.05 \Longrightarrow -t_{\alpha,n-1} = -t_{0.05,11} = -1.796$

$$t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{42 - 46}{11.9/\sqrt{12}} = -1.1644 > -t_{0.05,11}$$

$\Longrightarrow$ Fail to reject H_0

The data do not suggest that the average hours people spend in gaming each week is less than 46 hours.

Recall: Two Independent Samples

- $X_{11}, \ldots, X_{1\,n_1}$ are IID random variables from a population distribution $N(\mu_1, \sigma_1^2)$, and $X_{21}, \ldots, X_{2\,n_2}$ are IID random variables from a population distribution $N(\mu_2, \sigma_2^2)$. The two populations are independent.

Parameter of interest: $\mu_1 - \mu_2$

Point estimator: $\bar{X}_1 - \bar{X}_2$

Variance of estimator: $\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$

Two-Sample Tests for Means

$$H_0: \mu_1 = \mu_2 \leftrightarrow H_0: \mu_1 - \mu_2 = 0$$

$$H_1: \mu_1 \neq \mu_2 \leftrightarrow H_1: \mu_1 - \mu_2 \neq 0$$

$$H_1: \mu_1 > \mu_2 \leftrightarrow H_1: \mu_1 - \mu_2 > 0$$

$$H_1: \mu_1 < \mu_2 \leftrightarrow H_1: \mu_1 - \mu_2 < 0$$

Use tests for $\mu_1 - \mu_2$ with $\mu_0 = 0$.

Recall: Three Cases

- **Case 1:** σ_1^2 and σ_2^2 are known

- **Case 2:** σ_1^2 and σ_2^2 are unknown but equal

- **Case 3:** σ_1^2 and σ_2^2 are unknown and unequal (Not required)

Case 1 – Z-Tests

- Two-sided test

$$H_0: \mu_1 = \mu_2 \quad \text{vs.} \quad H_1: \mu_1 \neq \mu_2$$

Reject H_0 when

μ_0 is NOT in the C.I.

$$\left(\bar{x}_1 - \bar{x}_2 - Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}, \bar{x}_1 - \bar{x}_2 + Z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right)$$

or

$$|Z_0| > Z_{\alpha/2}$$

$$Z_0 = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Case 1 – Z-Tests

➢One-sided test

$$H_0: \mu_1 = \mu_2 \quad \text{vs.} \quad H_1: \mu_1 > \mu_2$$

Reject H_0 when

μ_0 is not in the one-sided lowerbound C.I. $\left(\bar{x}_1 - \bar{x}_2 - Z_\alpha\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}, \infty\right)$

or

$$Z_0 > Z_\alpha$$

$$H_0: \mu_1 = \mu_2 \quad \text{vs.} \quad H_1: \mu_1 < \mu_2$$

Reject H_0 when

μ_0 is not in the one-sided upperbound C.I. $\left(-\infty, \bar{x}_1 - \bar{x}_2 + Z_\alpha\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right)$

or

$$Z_0 < -Z_\alpha$$

Case 2 – *t*-Tests

- Two-sided test

$$H_0: \mu_1 = \mu_2 \quad \text{vs.} \quad H_1: \mu_1 \neq \mu_2$$

Reject H_0 when

μ_0 is NOT in the C.I.

$$\left(\bar{x}_1 - \bar{x}_2 - t_{\alpha/2, n_1+n_2-2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}, \bar{x}_1 - \bar{x}_2 + t_{\alpha/2, n_1+n_2-2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right)$$

or

$$|t_0| > t_{\alpha/2, n_1+n_2-2}$$

$$t_0 = \frac{\bar{x}_1 - \bar{x}_2}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

Case 1 – t-Tests

➢One-sided test

$$H_0: \mu_1 = \mu_2 \quad \text{vs.} \quad H_1: \mu_1 > \mu_2$$

Reject H_0 when

μ_0 is not in the one-sided lowerbound C.I. $\left(\bar{x}_1 - \bar{x}_2 - t_{\alpha,n_1+n_2-2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}, \infty\right)$

or

$$t_0 > t_{\alpha,n_1+n_2-2}$$

$$H_0: \mu_1 = \mu_2 \quad \text{vs.} \quad H_1: \mu_1 < \mu_2$$

Reject H_0 when

μ_0 is not in the one-sided upperbound C.I. $\left(-\infty, \bar{x}_1 - \bar{x}_2 + t_{\alpha,n_1+n_2-2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right)$

or

$$t_0 < -t_{\alpha,n_1+n_2-2}$$

Class Problem 6

6. To find out whether a new serum will arrest leukemia, 9 mice, all with an advanced stage of the disease are selected. Five mice receive the treatment and 4 do not. Survival times, in years, from the time the experiment commences are as follows: for the 5 mice that received treatment they are 2.1, 5.3, 1.4, 4.6 and 0.9; for the 4 mice that did not receive treatment they are 1.9, 0.5, 2.8 and 3.1. At the 0.05 level of significance can the serum be said to be effective? Assume the two distributions to be normal with equal variances.

μ_1: mean survival time of Group 1 (treated)

μ_2: mean survival time of Group 2 (not treated)

$H_0: \mu_1 = \mu_2$
$H_1: \mu_1 > \mu_2$

Unknown and equal variance ⇒ *t*-test (one-sided)

$\bar{x}_1 = 2.86, s_1 = 1.97$

$\bar{x}_2 = 2.075, s_2 = 1.167$

$$S_p^2 = \frac{(n_1 - 1) \cdot s_1^2 + (n_2 - 1) \cdot s_2^2}{n_1 + n_2 - 2} = 2.801 \Rightarrow S_p = \sqrt{2.801} = 1.674$$

Class Problem 6

$$t_{\alpha, n_1+n_2-2} = t_{0.05,7} = 1.895$$

$$t_0 = \frac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{2.86 - 2.075}{1.674\sqrt{\frac{1}{5} + \frac{1}{4}}} = 0.699 < 1.895$$

$\Longrightarrow$ Fail to reject H_0

There is no evidence that the serum is effective in treating the disease.