Q1.

(b) $\int \left(\frac{1}{x^3} - \sqrt{x}\right) dx$

(d) $\int \frac{1}{1+16x^2} dx$

(f) $\int \frac{1}{(2x+1)^2} dx$

(b) $$\int \left(\frac{1}{x^3} - \sqrt{x}\right) dx = \int (x^{-3} - x^{\frac{1}{2}})\, dx = \frac{x^{-3+1}}{-3+1} - \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C.$$
$$= -\frac{1}{2x^2} - \frac{2}{3}x^{\frac{3}{2}} + C.$$

$a=4.$

(d) $$\int \frac{1}{1+\underset{(4x)^2}{16x^2}} dx = \int \frac{1}{1+(4x)^2} dx = \frac{1}{4}\tan^{-1}(4x) + C.$$

$$\begin{cases} (\tan^{-1}(x))' = \frac{1}{1+x^2} \\ \int f(ax+b)dx = \frac{1}{a}F(ax+b). \end{cases}$$

(f) $$\int \frac{1}{(2x+1)^2} dx = \int (2x+1)^{-2} dx = \frac{1}{2}\cdot\frac{(2x+1)^{-2+1}}{-2+1} = -\frac{1}{2}\frac{1}{2x+1}$$

Q2

(b) $\int \frac{2x^2}{x^2+1} dx$

(d) $\int \sin 3x \sin 2x\, dx$

(f) $\int \frac{1}{(x-1)(2x-3)} dx$

(h) $\int \frac{1}{2x^2-4x+9} dx$

(j) $\int \tan^2 x\, dx$

(b) $$\int \frac{2x^2}{x^2+1} dx = \int \frac{2x^2+2-2}{x^2+1} dx = \int 2 - \frac{2}{1+x^2} dx = \int 2\,dx - \int 2\cdot\frac{1}{1+x^2} dx$$

$(\tan^{-1}(x))' = \frac{1}{1+x^2}$

$$= 2x + C_1 - 2\tan^{-1}(x) + C_2.$$

$= 2x - 2\tan^{-1}(x) + C.$

(d) $$\int \sin 3x \sin 2x \, dx = \int -\frac{1}{2}[\cos(5x) - \cos x] \, dx$$

$$= -\frac{1}{2}\int \cos(5x)\,dx + \frac{1}{2}\int \cos x \, dx.$$

$$= -\frac{1}{2}\cdot\frac{1}{5}\sin(5x) + C_1 + \frac{1}{2}\cdot\sin x + C_2.$$

$\sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)].$ $\quad = -\frac{1}{10}\sin(5x) + \frac{1}{2}\sin x + C.$

(f) $$\int \frac{1}{(x-1)(2x-3)} dx = \int -\frac{1}{x-1}dx + 2\int \frac{1}{2x-3}dx.$$

$$= -\ln|x-1| + 2\cdot\frac{1}{2}\ln|2x-3| + C.$$

$$= -\ln|x-1| + \ln|2x-3| + C.$$

$$\frac{1}{(x-1)(2x-3)} = \frac{A}{x-1} + \frac{B}{2x-3}. = -\frac{1}{x-1} + \frac{2}{2x-3}$$

$1 = A(2x-3) + B(x-1).$ $\begin{cases} x = \frac{3}{2} \Rightarrow 1 = \frac{1}{2}B \Rightarrow B = 2 \\ x = 1 \Rightarrow 1 = -A \Rightarrow A = -1. \end{cases}$

$(\ln|x|)' = \frac{1}{x}.$

(h) $$\int \frac{1}{2x^2 - 4x + 9} dx = \int \frac{1}{2(x-1)^2 + 7} dx. = \int \frac{1}{7}\cdot\frac{1}{\frac{2}{7}(x-1)^2 + 1} dx.$$

$\int \frac{1}{1+(\ )^2} \Rightarrow \tan^{-1}(\ )$ $$= \frac{1}{7}\int \frac{1}{1 + (\underbrace{\sqrt{\tfrac{2}{7}}x - \sqrt{\tfrac{2}{7}}}_{ax+b.})^2} dx.$$

$$= \frac{1}{\sqrt{14}}\tan^{-1}\left(\sqrt{\tfrac{2}{7}}x - \sqrt{\tfrac{2}{7}}\right) + C.$$

$\tan^2 x = \sec^2 x - 1$ $\quad (\tan x)' = \sec^2 x$

(j) $$\int \tan^2 x \, dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C.$$

Q3

(b) $\displaystyle\int_{-1}^{1} \cos(3x+1)\,dx$

(d) $\displaystyle\int_{0}^{\frac{\pi}{2}} \sin^2 x\,dx$

(f) $\displaystyle\int_{-\pi}^{\pi} |\sin x|dx$

(h) $\displaystyle\int_{-1}^{1} x^4 \sin^9 x\,dx$

*(j) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x^2\cos x + \cos x + \sin^3 x}{x^2+1}dx$

odd

$\sin(-x) = -\sin x$

$\cos(ax+b)$

(b) $\displaystyle\int_{-1}^{1} \cos(3x+1)\,dx = \frac{1}{3}\sin(3x+1)\Big|_{-1}^{1} = \frac{1}{3}\sin 4 - \frac{1}{3}\sin(-2)$

$= \frac{1}{3}(\sin 4 + \sin 2)$

(d) $\displaystyle\int_{0}^{\frac{\pi}{2}} \sin^2 x\,dx = \int_0^{\frac{\pi}{2}} -\frac{1}{2}[\cos(2x) - \cos 0]\,dx$

$\displaystyle= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2x\,dx + \int_0^{\frac{\pi}{2}} \frac{1}{2}\,dx$

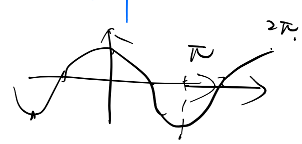
$\displaystyle= -\frac{1}{2}\cdot\frac{1}{2}\sin 2x\Big|_0^{\frac{\pi}{2}} + \frac{1}{2}x\Big|_0^{\frac{\pi}{2}}$

$= -\frac{1}{4}(\sin\pi - \sin 0) + \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\cdot 0 = \frac{1}{4}\pi.$

$\sin A\sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$

$|\sin x| = \begin{cases} \sin x & \text{if } \sin x \geq 0 \quad 0 \leq x \leq \pi \\ -\sin x & \text{if } \sin \leq 0 \quad -\pi \leq x \leq 0 \end{cases}$

$\displaystyle= \int_{-\pi}^{0} |\sin x|\,dx + \int_0^{\pi} |\sin x|\,dx$



(f) $\displaystyle\int_{-\pi}^{\pi} |\sin x|dx = \int_{-\pi}^{0} -\sin x\,dx + \int_0^{\pi} \sin x\,dx$

$= \cos x\Big|_{-\pi}^{0} - \cos x\Big|_0^{\pi} = \cos 0 - \cos(-\pi) - \cos\pi + \cos 0$

$= 1+1+1+1 = 4.$

(h) $\int_{-1}^{1} x^4 \sin^9 x \, dx = 0.$

$f(-x) = -f(x) \quad \int_{-a}^{a} f(x) = 0.$

$f(-x) = (-x)^4 \sin^9(-x) = x^4 \cdot (-\sin x)^9 = -x^4 \sin^9 x = -f(x)$

$\sin(-x) = -\sin x.$ $\quad$ $x^4 \sin^9 x$ is an odd function.

*(j) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x^2 \cos x + \cos x + \sin^3 x}{x^2+1} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(x^2+1)\cos x + \sin^3 x}{x^2+1} dx.$

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x \, dx + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^3 x}{x^2+1} dx, = \sin x \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + 0.$ $\Rightarrow$ odd.

$f(-x) = \frac{\sin^3(-x)}{(-x)^2+1} = \frac{(-\sin x)^3}{x^2+1} = \frac{-\sin^3 x}{x^2+1} = -f(x)$

$= \sin\left(\frac{\pi}{4}\right) - \sin\left(-\frac{\pi}{4}\right) = 2 \cdot \underset{\downarrow \, \frac{1}{\sqrt{2}}}{\sin\left(\frac{\pi}{4}\right)} = 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}.$

Q4

(b) $\frac{d}{dx} \int_{2x}^{x^2} \cos(y^2) \, dy$ $\qquad \frac{dG(y)}{dy} = \cos(y^2)$

$G(y) = \int \cos(y^2) dy \Rightarrow \int_{2x}^{x^2} \cos(y^2) dy = G(y) \Big|_{2x}^{x^2} = G(x^2) - G(2x).$

$\frac{d}{dx} \int_{2x}^{x^2} \cos(y^2) dy = \frac{d}{dx}\left(G(x^2) - G(2x)\right) = \frac{dG(x^2)}{dx^2} \cdot \frac{dx^2}{dx} - \frac{dG(2x)}{2x} \cdot \frac{d2x}{x}.$

$\Rightarrow y = x^2 \qquad y = 2x$

$= \cos(x^4) \cdot 2x - \cos(4x^2) \cdot 2$

$= 2x\cos(x^4) - 2\cos(4x^2)$

Problem 5

(a) Using fundamental theorem of calculus, show that

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

(b) It is given that $g(x)$ is a periodic function with period 1 (i.e. $g(x+1) = g(x)$ for any x). Using fundamental theorem of calculus, show that

(i) $\int_0^4 g(x)dx = 4\int_0^1 g(x)dx$ (ii) $\int_0^1 g(3x)dx = \frac{1}{3}\int_0^3 g(x)dx$

$\int_0^a f(x)\,dx = F(x)\big|_0^a = F(a) - F(0)$

$\int f(x)\,dx = F(x)$

$\int_0^a f(a-x)\,dx = \frac{1}{-1}F(-x+a)\big|_0^a = -F(-x+a)\big|_0^a = -F(0) + F(a) = F(a) - F(0)$

$f(-x+a)$

$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)$ $g(x) = g(x-1) = g(x-2) = g(x-3)$

(b)(i) $\int_0^4 g(x)dx = \int_0^1 g(x)\,dx + \int_1^2 g(x)\,dx + \int_2^3 g(x)\,dx + \int_3^4 g(x)\,dx.$

$= \int_0^1 g(x)\,dx + \int_1^2 g(x-1)\,dx + \int_2^3 g(x-2)\,dx + \int_3^4 g(x-3)\,dx.$

let $\int g(x)dx = G(x)$ $= G(x)\big|_0^1 + G(x-1)\big|_1^2 + G(x-2)\big|_2^3 + G(x-3)\big|_3^4$

$= 4(G(1) - G(0))$

$= 4\int_0^1 g(x)\,dx.$