MA1200 TAKE HOME PROBLEM SET 3

The following is the third take-home assignment of MA1200, which counts 3 points of total 100 of your final score of the course.

Please submit it via canvas in a pdf file (you can handwrite the answers and take photos by your phone, then make it into a pdf file, see for example, https://www.wikihow.com/Convert-JPG-to-PDF) for how to combine jpg files to a pdf; you can also do it by note-taking apps on an iPad or an Surface)

Q1. Differentiate with respect to x

$$\begin{aligned}
&(0.3point)(a) \qquad \tan^{-1}(\sinh x) \\
&(0.3point)(b) \qquad 3^{\sqrt{x}} \\
&(0.4point)(c) \qquad \frac{\sin^2(x)e^{3x}}{(2x)^{1/2}\tan^{1/2}(3x)}
\end{aligned}$$

Q2. (0.5 point) Show from first principles that $\dfrac{d}{dx}(\cos x) = -\sin x$.

Q4. (0.5 point) $x = \cos t + \ln(\tan(t/2))$ and $y = \sin t$, compute $\dfrac{d^2y}{dx^2}$

Q5. Let $f(x) = \sin(2\ln(1+x))$

(a) (0.3 point) Show that $(1+x)^2 f''(x) + (1+x)f'(x) + 4f(x) = 0$.

(b) (0.4 point) Let n be a positive integer, show that

$$(1+x)^2 f^{(n+2)}(x) + (2n+1)(1+x)f^{(n+1)}(x) + (n^2+4)f^{(n)}(x) = 0.$$

Hint, Leibnitz' rule: $(uv)^{(n)} = \sum_{r=0}^{n} C_r^n u^{(r)} v^{(n-r)}$, $C_r^n = \frac{n!}{(n-r)!r!}$

(c) (0.3 point) Find $f^{(n)}(0)$ for $n = 0, 1, 2, 3, 4, 5, 6$.

The assignment is due on 23:59 of Nov 22, Sunday.

You will lose 1 point for each day of late submission. All submissions after the midnight of Nov 25 will be marked as 0.

Date: November 12, 2020.

HW3

1. ① $\left(\tan^{-1}(\sinh x)\right)' = \dfrac{\cosh x}{1+\sinh^2 x}$ 0.3

② $y=3^{\sqrt{x}}$, $\ln y = \sqrt{x}\ln 3$, $\dfrac{y'}{y} = \dfrac{\ln 3}{2\sqrt{x}}$

$\Rightarrow y' = \dfrac{3^{\sqrt{x}}}{2\sqrt{x}} \ln 3.$ 0.3

③ $y = \dfrac{\sin^2 x \, e^{3x}}{\sqrt{2x}\sqrt{\tan(3x)}}$, $\ln y = 2\ln\sin x + 3x - \frac{1}{2}\ln(2x) - \frac{1}{2}\ln(\tan 3x)$

$y' = \left(\dfrac{\sin^2 x \, e^{3x}}{\sqrt{2x}\,\tan(3x)}\right)\left(\dfrac{2\cos x}{\sin x} + 3 - \dfrac{1}{4x} - \dfrac{3\sec^2(3x)}{2\tan 3x}\right)$

0.4

Q2. $\dfrac{d}{dx}\cos x = \lim_{h\to 0}\dfrac{\cos(x+h) - \cos x}{h}$

0.5

$= \lim_{h\to 0}\dfrac{-2\sin\left(\frac{h}{2}+x\right)\sin\left(\frac{h}{2}\right)}{h} = \lim_{h\to 0}\dfrac{\sin\left(\frac{h}{2}\right)}{h/2}\left(-\sin(x+h/2)\right)$

$= -\sin(x)$

Q3. $x = \cos t + \ln(\tan(\frac{t}{2}))$ 0.5

$y = \sin t.$

$\dfrac{dy}{dx} = \dfrac{\cos t}{-\sin t + \dfrac{\sec^2(\frac{t}{2})}{\tan(\frac{t}{2})}\cdot\frac{1}{2}}$

$= \dfrac{\cos t}{-\sin t + \dfrac{\cos(\frac{t}{2})}{2\cos^2(\frac{t}{2})\sin(\frac{t}{2})}} = \dfrac{\cos t}{-\sin t + \frac{1}{\sin t}}$

$= \dfrac{\sin t\cos t}{-\sin^2 t + 1} = \dfrac{\sin t\cos t}{\cos^2 t} = \tan t.$

$\dfrac{dy^2}{dx^2} = \dfrac{\sec^2 t}{\frac{1}{\sin t} - \sin t} = \dfrac{\sin t}{\cos^4 t}$

Q4. $f = \sin(2\ln(1+x))$

$$f'(x) = \cos(2\ln(1+x))\frac{2}{1+x}$$

$$f''(x) = \frac{-2\sin(2\ln(1+x))\frac{2}{1+x}\cdot(1+x) - 2\cos(2\ln(1+x))}{(1+x)^2}$$

$$= \frac{\cancel{(1+x)} - 4f(x) - f'(x)(1+x)}{(1+x)^2}$$

$$\Rightarrow (1+x)^2 f''(x) + (1+x)f'(x) + 4f(x) = 0$$ 0.3

take n-th derivative $\Rightarrow$

$$(1+x)^2 f^{(n+2)}(x) + n(1+x)\cdot 2\cdot f^{(n+1)}(x) + \frac{2n(n-1)}{2} f^{(n)}(x)$$

$$+ (1+x)f^{(n+1)}(x) + n f^{(n)}(x) + 4f^{(n)}(x)$$ 0.4

$$= (1+x)^2 f^{(n+2)}(x) + (2n+1)(1+x)f^{(n+1)}(x) + (n^2+4)f^{(n)}(x) = 0$$

c). $x=0 \Rightarrow f^{(n+2)}(0) + (2n+1)f^{(n+1)}(0) + (n^2+4)f^{(n)}(0) = 0$

$$f(0) = \sin(2\cdot 0) = 0$$

$$f'(0) = 2, \quad f''(0) = -f'(0) - 4f(0) = -2$$ 0.3

$$f^{(3)}(0) + 3f^{(2)}(0) + 5f^{(1)}(0) = 0$$

$$\Rightarrow f^{(3)}(0) = -3f''(0) - 5f'(0) = 6 - 10 = -4$$

$$f^{(4)}(0) = -5f^{(3)}(0) - 8f^{(2)}(0) = -5\cdot(-4) - 8\cdot(-2)$$

$$= 36.$$

$$f^{(5)}(0) = -7f^{(4)}(0) - 13f^{(3)}(0) = -7\cdot(36) - 13(-4) = -200$$

$$f^{(6)}(0) = -9f^{(5)}(0) - 20f^{(4)}(0) = 1080$$