# 4. Conjugate Property

$$x^*_{(t)} \xrightarrow{\mathcal{F}} X^*_{(-f)}$$

$$X^*_{(f)} = \int_{-\infty}^{\infty} x^*_{(t)} e^{j2\pi f t} \, dt$$

$$X^*_{(-f)} = \int_{-\infty}^{\infty} x^*_{(t)} e^{-j2\pi f t} \, dt$$

$$\mathcal{F}\left(x^*_{(t)}\right) = X^*_{(-f)}$$

if $x_{(t)}$ is a real ft,

$$x_{(t)} = x^*_{(t)} \xleftrightarrow{\mathcal{F}} \mathcal{F}_{(x_{(t)})} = X_{(f)}$$

$$\mathcal{F}\left(x^*_{(t)}\right) = X^*_{(-f)} = X_{(f)}$$

$$X(f) = X^*(-f)$$

## 5. Convolution/Multiplication

$$\left( \underbrace{x(t) * y(t)}_{} \xrightarrow{\ \mathcal{F}\ } X(f)\,Y(f) \right.$$

$$\left. x(t)\,y(t) \xleftarrow{\ \mathcal{F}^{-1}\ } X(f) * Y(f) \right)$$

Proof) $\mathcal{F}(x(t) * y(t))$

$$= \int_{-\infty}^{\infty} \left( \underbrace{x(t) * y(t)} \right) e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(s)\, y(t-s)\, e^{-j2\pi f \boxed{t}}\, ds\, dt$$

$$t - s = \ell$$

Given $s$, $d\ell = dt$.

$$t = (t - s) + s$$

$$= \ell + s$$

$$\Rightarrow \left\{ \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x(s)\, y(t-s)\, e^{-j2\pi f(t-s+s)} \right] dt \, ds \right\}$$

1st, $s \to$ Fixed, $t \to$ change

$$\int_{-\infty}^{\infty} y(\ell)\, e^{-j2\pi f \ell}\, d\ell \qquad Y(f)$$

$$\int_{-\infty}^{\infty} x(s)\, e^{-j2\pi f s}\, ds \qquad X(f)$$

$$\Rightarrow X(f) Y(f)$$

## 6. Duality

$$\mathcal{F}( x(t)) = x(-f)$$

$$
\boxed{
\begin{aligned}
&\boxed{x(t)} \longleftrightarrow \boxed{X(f)} \\
&X(t) \longleftrightarrow x(-f)
\end{aligned}
}
$$

Proof)

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

$$( t \Rightarrow -f )$$

$$X(-f) = \int_{-\infty}^{\infty} X(t') \, e^{-j2\pi f' t'} \, dt'$$

$$e^{j2\pi f' t}$$

$$\mathcal{F}(X(t)) = X(-f)$$

7. Diff / Integ

$$\mathcal{F}\left(\frac{d^n X(t)}{dt^n}\right) = (j2\pi f)^n X(f)$$

$$\mathcal{F}\left((-j2\pi t)^n X(t)\right) = \frac{d^n X(f)}{df^n}$$

**Proof)**

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} \, df$$

$$\boxed{\frac{dx(t)}{dt}} = \int_{-\infty}^{\infty} X(f) \left( \frac{d e^{j2\pi ft}}{dt} \right) df$$

ft of t.

$$= (j2\pi f) e^{j2\pi ft}$$

$$= \int_{-\infty}^{\infty} (j2\pi f) X(f) e^{j2\pi ft} \, df$$

$$\int_{-\infty}^{\infty} \underline{\qquad\qquad} Y(f) \cdot df$$

$$\frac{d^n x_{(t)}}{dt^n} = \int_{-\infty}^{\infty} (j2\pi f)^n X_{(f)} e^{j2\pi f t} df$$

$$\mathcal{F}\left(\frac{d^n x_{(t)}}{dt^n}\right) = (j2\pi f)^n X_{(f)}$$

$$\mathcal{F}\left(\int_{-\infty}^{t} x_{(s)} ds\right) = \frac{1}{2} X_{(0)} \delta_{(f)}$$

$$+$$

$$\frac{1}{j2\pi f} X_{(f)}$$

$$\int_{-\infty}^{t} x_{(s)} ds = x_{(t)} * U_{(t)}$$

1

$$\mathcal{F}\left(\int_{-\infty}^{t} x(\zeta)\, d\zeta\right)$$

$$= \mathcal{F}\left(x(t) * u(t)\right)$$

$$= \mathcal{F}(x(t)) \cdot \underline{\mathcal{F}(u(t))}$$

$$= X(f) \cdot \left[\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right]$$

$$= \frac{1}{2}X(0)\delta(f) + \frac{1}{j2\pi f}X(f)$$

if $X(0) = 0$.

$$\boxed{\longrightarrow \frac{1}{j2\pi f}X(f).}$$

## 8. Parseval's thm.

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$= \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

Ex (4-2).

$$e^{-\alpha t} u(t) \longleftrightarrow \frac{1}{\alpha + j2\pi f}$$

$$e^{\alpha t} u(-t) \longleftrightarrow \frac{1}{\alpha - j2\pi f}$$

$$e^{-\alpha t} u(t) - e^{\alpha t} u(-t) \longleftrightarrow \frac{1}{\alpha + j2\pi f} \ominus \frac{1}{\alpha - j2\pi f}$$
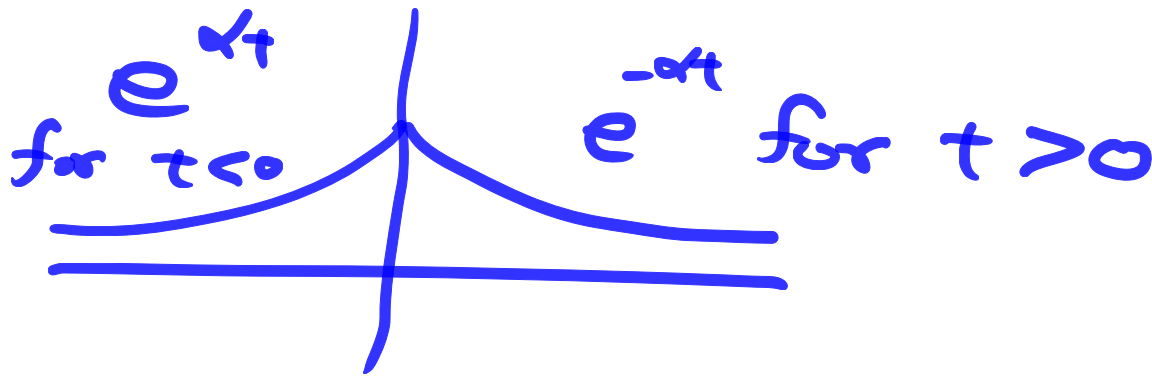
$$= \frac{-j4\pi}{\alpha^2 + 4(\pi f)^2}$$

$$\alpha \to 0 \qquad \text{sgn}(t) = \frac{1}{j\pi f}$$

$$U(t) = \frac{1}{2} + \frac{1}{2} \text{sgn}(t).$$

$$\mathcal{F}\left(e^{-\alpha t} U(t) + e^{\alpha t} U(-t)\right) = \frac{2\alpha}{\alpha^2 + 4(\pi f)^2}$$

$e^{\alpha t}$ for $t < 0$      $e^{-\alpha t}$ for $t > 0$

$$e^{-\alpha |t|} = e^{-\alpha t} U(t) + e^{\alpha t} U(-t)$$

$$\mathcal{F}\left(e^{-\alpha |t|}\right) = \boxed{\frac{2\alpha}{\alpha^2 + 4(\pi f)^2}}$$

$t \to -f$

$$\mathcal{F}\left\{\frac{2\alpha}{\alpha^2 + 4(\pi t)^2}\right\} = e^{-\alpha |-f|}$$

$$= e^{-\alpha |f|}$$

$$\mathcal{F}\left\{\frac{1}{\alpha^2 + 4(\pi t)^2}\right\} = \frac{1}{2\alpha}e^{-\alpha|f|}$$

Ex 4-3

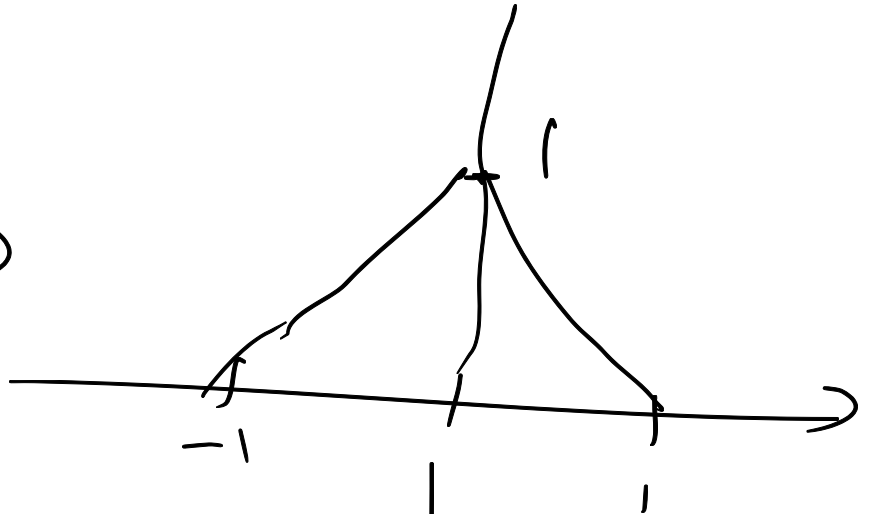$$\text{rect}\left(\frac{t}{J}\right) \xrightarrow{\mathcal{F}} J\,\text{Sinc}(f \cdot J)$$

$$J\,\text{Sinc}(t \cdot J) \longleftrightarrow \text{rect}\left(-\frac{f}{J}\right)$$

$$= \text{rect}\left(\frac{f}{J}\right)$$
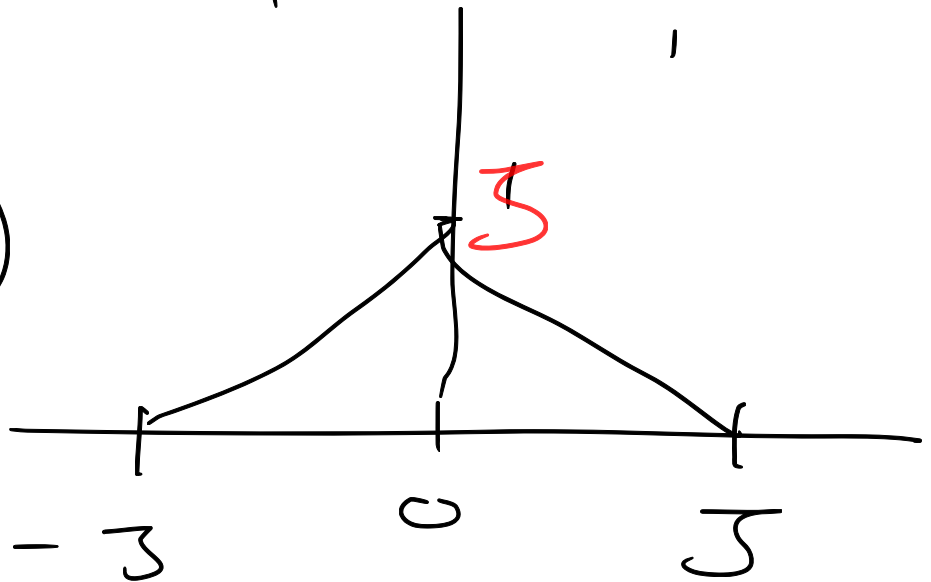
1) $J = 2B$, 2) $\frac{1}{2B}$

$$\text{Sinc}(2Bt) \longrightarrow \frac{1}{2B}\text{rect}\left(\frac{f}{2B}\right)$$

$$\mathcal{F}\left(5 \cdot tri\left(\frac{t}{5}\right)\right) \quad \left(\begin{smallmatrix} Ex\ 2-1 \\ p_{23} \end{smallmatrix}\right)$$

$$tri(t) \Rightarrow$$



$$5\ tri\left(\frac{t}{5}\right)$$



c)

$$\mathcal{F}\left(5\ tri\left(\frac{t}{5}\right)\right) = \mathcal{F}\left(rect\left(\frac{t}{5}\right) * rect\left(\frac{t}{5}\right)\right)$$

$$= \mathcal{F}\left(rect\left(\frac{t}{5}\right)\right) \cdot \mathcal{F}\left(rect\left(\frac{t}{5}\right)\right)$$

$$= 5^2 sinc^2(f5)$$

$$\mathcal{F}\left(\underbrace{rect\left(\frac{t}{5}\right) * rect\left(\frac{t}{5}\right) * \cdots * rect\left(\frac{t}{5}\right)}_{n\text{-th order convolution}}\right)$$

$$= \mathcal{F}^{n}\left(rect\left(\frac{t}{5}\right)\right)$$

$$= 5^{n} \, sinc^{n}(f5)$$

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$$5 \, tri\left(\frac{t}{5}\right) \longleftrightarrow 5^{2} sinc^{2}(f5)$$

d) $sinc^{2}(t\boxed{5}) \longleftrightarrow \frac{1}{5} tri\left(\frac{f}{5}\right)$

$$sinc^{2}(2Bt) \longleftrightarrow \frac{1}{2B} tri\left(\frac{f}{2B}\right)$$

e) $\mathcal{F}\left(\text{rect}\left(\frac{t-t_0}{J}\right)\right)$

$$= J \, \text{sinc}\,(fJ)$$
$$\times$$
$$e^{-j2\pi f t_0}$$

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Ex 4-4)

$e^{j2\pi f_0 t} \longleftrightarrow \delta(f-f_0)$

a) $\mathcal{F}\left(\cos\left(2\pi f_0 t\right)\right)$

$$= \mathcal{F}\left\{\frac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)\right\}$$

$$= \frac{1}{2}\left\{\delta(f-f_0) + \delta(f+f_0)\right\}$$

b) $\mathcal{F}\left(\sin\left(2\pi f_0 t\right)\right)$

$= \mathcal{F}\left\{\frac{1}{2j}\left(e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right)\right\}$
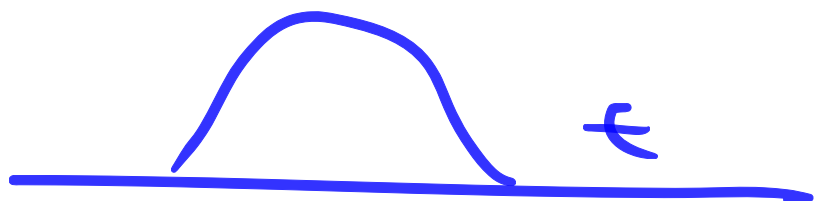
$= \frac{1}{2j}\left(\delta(f-f_0) - \delta(f+f_0)\right\}$

c) $\underline{\mathcal{F}\left(x(t)\cos\left(2\pi f_0 t\right)\right.}$

$\Rightarrow \mathcal{F}(x(t)) * \mathcal{F}\left(\cos\left(2\pi f_0 t\right)\right)$

$X(f) * \left[\frac{1}{2}\left(\delta(f-f_0) + \delta(f+f_0)\right]\right.$

$= \frac{1}{2}\left\{X(f-f_0) + X(f+f_0)\right\}$

$x(t)$

$X(F)$



$$\mathcal{F}\left(x(t)\,\cos(2\pi f_0 t)\right)$$

$$\frac{1}{2}A$$
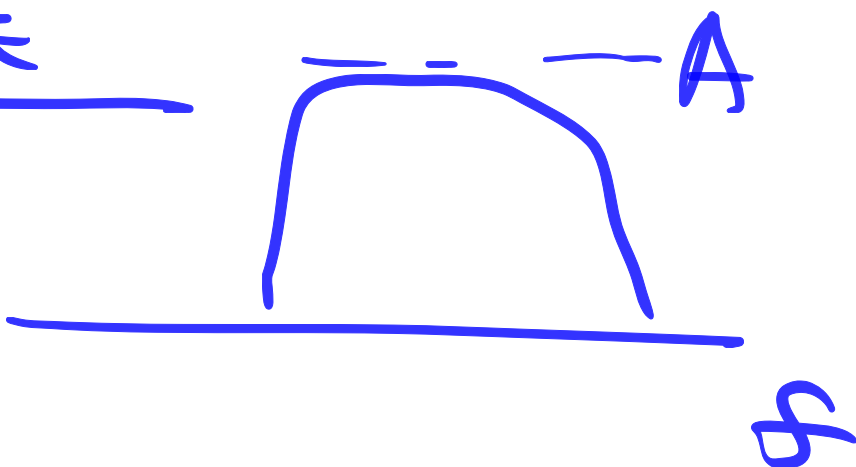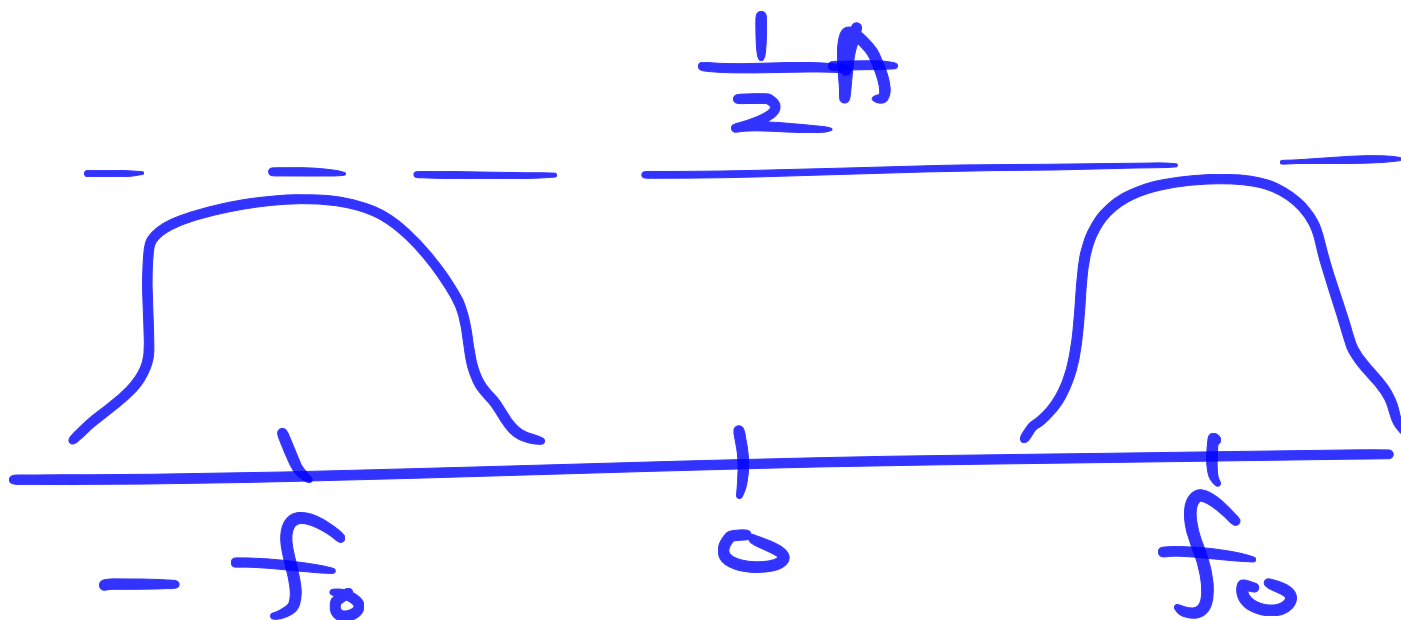
c)

$$\mathcal{F}\left(\cos(2\pi f_0 t) U(t)\right)$$

$$= \mathcal{F}\left(\cos(2\pi f_0 t)\right) \times \mathcal{F}\left(U(t)\right)$$

$$\frac{1}{2}\left(\delta(f-f_0) + \delta(f+f_0)\right) * \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$

$$\frac{1}{4}\left(\delta(f-f_0) + \delta(f+f_0)\right) \quad \frac{2f}{f^2 - f_0^2}$$

$$+ \frac{1}{j4\pi}\left\{\frac{1}{f-f_0} + \frac{1}{f+f_0}\right\}$$

$$\overline{\quad j\pi \quad}$$

$$= \left| \frac{1}{4}\left(\delta(f-f_0) + \delta(f+f_0)\right) \right.$$

$$\left. + \frac{j2\pi f}{(2\pi f_0)^2 - (2\pi f)^2} \right|$$

g)

$$\mathcal{F}\left( e^{-\alpha t} u_{(t)} \cos(2\pi f_0 t) \right)$$

$$= \mathcal{F}\left( e^{-\alpha t} u_{(t)} \right) * \mathcal{F}\left( \cos(2\pi f_0 t) \right)$$

$$= \left( \frac{1}{\alpha + j2\pi f} \right) * \frac{1}{2}\left( \delta(f - f_0) + \delta(f + f_0) \right)$$

$$= \frac{1}{2}\left[ \frac{1}{\alpha + j2\pi(f - f_0)} + \frac{1}{\alpha + j2\pi(f + f_0)} \right]$$

$$= \frac{\alpha + j2\pi f}{(\alpha + j2\pi f)^2 + 4(\pi f_0)^2}$$