MA 1201 Semester B 2019/20

Midterm Exam (E/F/G/H, 100 mins)

Instructions:

- Please show your work. Unsupported answers will receive <u>**NO**</u> credits.
- Make sure you write down the correct lecture session (E/F/G/H) you have registered for, together with your full name and student ID on the front page of your answer script.
- Exams submitted to wrong lecture sessions will <u>**NOT**</u> be graded and will receive <u>**0 POINTS**</u>.

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1. (25 points) Let $A(-1,2,1)$, $B(2,3,-1)$, and $C(0,-1,3)$ be three points in $\mathbb{R}^3$. Using vector method:

(a) (8 points) Find the angle $\angle ABC$.

Solution. The cosine of the angle is given by

$$\cos\angle ABC = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}|\cdot|\overrightarrow{BC}|},$$ 1

where

$$\overrightarrow{BA} = A - B = \langle -1-2, 2-3, 1-(-1)\rangle = \langle -3,-1,2\rangle,$$ 1

$$\overrightarrow{BC} = C - B = \langle 0-2, -1-3, 3-(-1)\rangle = \langle -2,-4,4\rangle.$$ 1

Thus

$$\begin{aligned}\cos\angle ABC &= \frac{\langle -3,-1,2\rangle\cdot\langle -2,-4,4\rangle}{|\langle -3,-1,2\rangle|\cdot|\langle -2,-4,4\rangle|}\\ &= \frac{(-3)\cdot(-2)+(-1)\cdot(-4)+2\cdot 4}{\sqrt{(-3)^2+(-1)^2+2^2}\sqrt{(-2)^2+(-4)^2+4^2}} = \frac{18}{\sqrt{14}\sqrt{36}} = \frac{3}{\sqrt{14}},\end{aligned}$$ 3

which shows that

$$\angle ABC = \cos^{-1}\frac{3}{\sqrt{14}} \approx 0.6405.$$ 2

(b) (9 points) Find the equation of the plane that contains A, B, and C.

Solution. The equation of the plane is given by

$$n_1(x-x_0)+n_2(y-y_0)+n_3(z-z_0)=0,$$ 1

where $\vec{n}=\langle n_1,n_2,n_3\rangle$ is a normal vector of the plane and $P(x_0,y_0,z_0)$ is a point on the plane. Since (see page 13 of the supplementary notes for the determinant formula for computing vector products)

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$$\vec{n} = \overrightarrow{BA}\times\overrightarrow{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k}\\ -3 & -1 & 2\\ -2 & -4 & 4\end{vmatrix} = \vec{i}\begin{vmatrix}-1 & 2\\ -4 & 4\end{vmatrix} - \vec{j}\begin{vmatrix}-3 & 2\\ -2 & 4\end{vmatrix} + \vec{k}\begin{vmatrix}-3 & -1\\ -2 & -4\end{vmatrix}$$

$$= [(-1)\cdot 4-(-4)\cdot 2]\vec{i} - [(-3)\cdot 4-(-2)\cdot 2]\vec{j} + [(-3)\cdot(-4)-(-2)\cdot(-1)]\vec{k} = \langle 4,8,10\rangle = 2\langle 2,4,5\rangle,$$ 3

and $P(x_0,y_0,z_0) = A(-1,2,1)$ is a point on the plane, the equation takes the form 1

$$2[x-(-1)]+4(y-2)+5(z-1)=0, \qquad \text{or} \qquad 2x+4y+5z=11.$$ 3

(c) (8 points) Find the distance from $D(1,0,-2)$ to the plane containing A, B, and C.

Solution. The distance between D and the plane is given by $|\mathrm{proj}_{\vec{n}}\overrightarrow{AD}|$, the magnitude of the (orthogonal) projection of the vector

$$\overrightarrow{AD} = D - A = \langle 1-(-1), 0-2, -2-1\rangle = \langle 2,-2,-3\rangle$$

onto the normal vector $\vec{n} = \langle 2,4,5\rangle$:

$$d = |\mathrm{proj}_{\vec{n}}\overrightarrow{AD}| = \left|\frac{\overrightarrow{AD}\cdot\vec{n}}{|\vec{n}|}\right| = \left|\frac{\langle 2,-2,-3\rangle\cdot\langle 2,4,5\rangle}{|\langle 2,4,5\rangle|}\right| = \left|\frac{2\cdot 2+(-2)\cdot 4+(-3)\cdot 5}{\sqrt{2^2+4^2+5^2}}\right| = \left|\frac{-19}{\sqrt{45}}\right| = \frac{19}{\sqrt{45}}.$$

Note that projecting any one of the vectors $\overrightarrow{AD}$, $\overrightarrow{BD}$, or $\overrightarrow{CD}$ onto $\vec{n}$ yields the same result.

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2. (50 points) Evaluate the following integrals.

(a) (7 points) $\displaystyle\int \tan(3x+1)\,dx$.

Solution. Let

$$u = 3x+1,$$
$$du = (3x+1)'\,dx = 3\,dx, \qquad \text{or} \qquad dx = \frac{1}{3}\,du.$$

Then

$$\int \tan(3x+1)\,dx = \int \tan u\cdot\left(\frac{1}{3}\,du\right) = \frac{1}{3}\int \tan u\,du = \frac{1}{3}\ln|\sec u| + C = \frac{1}{3}\ln|\sec(3x+1)| + C.$$

(b) (8 points) $\displaystyle\int_0^2 \mathrm{e}^{1+|x-1|}\,dx$.

Motivation. Since, by definition,

$$|x-1| = \begin{cases} x-1, & \text{if } x-1\geq 0 \\ -(x-1), & \text{if } x-1<0 \end{cases} = \begin{cases} x-1, & \text{if } x\geq 1 \\ 1-x, & \text{if } x<1 \end{cases},$$

it's necessary to partition the interval of integration $[0,2]$ at $x=1$ into two parts, $[0,1]$ and $[1,2]$.

Solution.

$$\int_0^2 \mathrm{e}^{1+|x-1|}\,dx = \int_0^1 \mathrm{e}^{1+|x-1|}\,dx + \int_1^2 \mathrm{e}^{1+|x-1|}\,dx = \int_0^1 \mathrm{e}^{1+(1-x)}\,dx + \int_1^2 \mathrm{e}^{1+(x-1)}\,dx$$
$$= \int_0^1 \mathrm{e}^{2-x}\,dx + \int_1^2 \mathrm{e}^{x}\,dx = -\mathrm{e}^{2-x}\big|_0^1 + \mathrm{e}^{x}\big|_1^2 = -(\mathrm{e}-\mathrm{e}^2) + (\mathrm{e}^2-\mathrm{e}) = 2(\mathrm{e}^2-\mathrm{e}).$$

(c) (10 points) $\displaystyle\int \mathrm{e}^{2x}\sin(2\mathrm{e}^{x}+1)\,dx$.

Motivation. Observe that

$$\mathrm{e}^{2x}\,dx = \frac{1}{2}\mathrm{e}^{x}\cdot 2\mathrm{e}^{x}\,dx = \frac{1}{2}\mathrm{e}^{x}\,d(2\mathrm{e}^{x}) = \frac{1}{2}\mathrm{e}^{x}\,d(2\mathrm{e}^{x}+1).$$

This suggests that the integral can be simplified by the substitution $w = 2e^x + 1$.

Solution. Let

$$w = 2e^x + 1, \qquad \text{or} \qquad e^x = \frac{1}{2}(w-1),$$
$$dw = (2e^x+1)'\,dx = 2e^x\,dx, \qquad \text{or} \qquad e^x\,dx = \frac{1}{2}dw.$$

Then

$$\int e^{2x}\sin(2e^x+1)\,dx = \int \sin(2e^x+1)\cdot e^x\cdot e^x\,dx$$
$$= \int \sin w\cdot\left(\frac{1}{2}(w-1)\right)\cdot\left(\frac{1}{2}dw\right) = \frac{1}{4}\int \sin w\cdot(w-1)\,dw.$$

To proceed, apply integration by parts with

$$u = w-1, \qquad dv = \sin w\,dw.$$

Direct integration yields

$$v = \int dv = \int \sin w\,dw = -\cos w,$$

so

$$\int \sin w\cdot(w-1)\,dw = \int (w-1)\cdot\sin w\,dw = \int (w-1)\,d(-\cos w)$$
$$= (w-1)\cdot(-\cos w) - \int(-\cos w)\,d(w-1) = -(w-1)\cos w + \int \cos w\,dw$$
$$= -(w-1)\cos w + \sin w + C = -2e^x\cos(2e^x+1) + \sin(2e^x+1) + C,$$

and thus

$$\int e^{2x}\sin(2e^x+1)\,dx = \frac{1}{4}\left[-2e^x\cos(2e^x+1)+\sin(2e^x+1)\right]+C.$$

(d) (10 points) $\displaystyle\int \frac{1}{(x^2-4)^{3/2}}\,dx.$

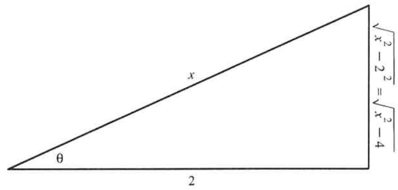
Motivation. Since the integrand contains a square root function $(x^2-4)^{3/2}$, but no extra factor of x ($=$ a constant multiple of $(x^2-4)'$), the simple substitution $u = x^2-4$ does not work. Thus it is a good idea to try a trig substitution instead.

Solution. The Pythagorean identity $1+\tan^2\theta = \sec^2\theta$, or equivalently $4\sec^2\theta - 4 = 4\tan^2\theta$, motivates the (trig) substitution

$$x^2 = 4\sec^2\theta, \quad \text{or} \quad x = 2\sec\theta, \quad \text{so that} \quad (x^2-4)^{3/2} = (4\tan^2\theta)^{3/2} = 8\tan^3\theta,$$
$$dx = (2\sec\theta)'\,d\theta = 2\sec\theta\tan\theta\,d\theta.$$

It follows that

$$\int \frac{1}{(x^2-4)^{3/2}}\,dx = \int \frac{1}{8\tan^3\theta}\cdot(2\sec\theta\tan\theta\,d\theta) = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$
$$= \frac{1}{4}\int \sec\theta\cot^2\theta\,d\theta = \frac{1}{4}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int\frac{\cos\theta}{\sin^2\theta}\,d\theta$$
$$= \frac{1}{4}\int \frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}\,d\theta = \frac{1}{4}\int \csc\theta\cot\theta\,d\theta = -\frac{1}{4}\csc\theta + C = -\frac{1}{4}\cdot\frac{x}{\sqrt{x^2-4}}+C,$$



where in the last step the following right triangle has been used to find $\csc\theta$. This triangle is constructed based on the observation that

$$x = 2\sec\theta, \qquad \text{or} \qquad \sec\theta = \frac{x}{2} = \frac{\text{hyp}}{\text{adj}},$$

and its opposite is determined using the Pythagorean theorem.

(e) (15 points) $\displaystyle\int \frac{11x+29}{(x-1)(x^2+6x+13)}\,dx.$

Solution. The idea is to decompose the integrand, a rational function, into partial fractions. To begin with, note that

$$\deg(\text{numerator}) = 1 < \deg(\text{denominator}) = 1+2 = 3.$$

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This shows that the rational function is proper, and thus a long division is not needed. Next, note that the denominator is already in a factored form, and the quadratic polynomial $(x^2+6x+13)$ is irreducible ($\Delta = 6^2 - 4\cdot 1\cdot 13 < 0$). Thus $(x-1)$ and $(x^2+6x+13)$ are the only factors of the denominator, and the partial fraction decomposition of the rational function is given by

$$\frac{11x+29}{(x-1)(x^2+6x+13)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+6x+13}.$$

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To find the constants A, B, and C, multiply both sides of the equation by the denominator $(x-1)(x^2+6x+13)$ and rearrange. This yields

$$11x+29 = A(x^2+6x+13) + (Bx+C)(x-1).$$

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Setting $x=1$, the zero of the linear factor $(x-1)$, on both sides of the equation yields

$$11\cdot 1 + 29 = A\left[1^2 + 6\cdot 1 + 13\right],$$

which implies

$$40 = 20A, \qquad \text{or} \qquad A = 2.$$

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To find B and C, one could substitute other convenient values of x (e.g. $x=0$ and $x=-1$) into the equation, but it's better here to expand the right side of the equation, combine like powers of x, and then compare coefficients:

$$\begin{aligned}
11x+29 &= A(x^2+6x+13) + (Bx+C)(x-1)\\
&= (Ax^2+6Ax+13A) + (Bx^2 - Bx + Cx - C)\\
&= (A+B)x^2 + (6A-B+C)x + (13A-C).
\end{aligned}$$

This yields the linear system

$$\begin{aligned} A+B \quad &= 0 \\ 6A-B+C &= 11\,, \\ 13A \quad -C &= 29 \end{aligned}$$

where the first and third equation immediately implies (recall $A=2$)

$$B=-A=-2, \qquad C=13A-29=-3. \quad 2$$

It follows that

$$\frac{11x+29}{(x-1)(x^2+6x+13)} = \frac{2}{x-1} + \frac{-2x-3}{x^2+6x+13},$$

and thus

$$\int \frac{11x+29}{(x-1)(x^2+6x+13)}\,dx = \int \frac{2}{x-1}\,dx + \int \frac{-2x-3}{x^2+6x+13}\,dx = 2\ln|x-1| + \int \frac{-2x-3}{x^2+6x+13}\,dx. \quad 2$$

To evaluate the last integral on the right side, observe that the substitution

$$\begin{aligned} u &= x^2+6x+13, \\ du &= (x^2+6x+13)'\,dx = (2x+6)\,dx = 2(x+3)\,dx, \end{aligned}$$

indicates that an extra factor of $(x+3)$ (= a constant multiple of u') is needed to simplify the integral. This suggests the decomposition of the numerator:

$$-2x-3 = -2(x+3-3)-3 = -2(x+3)+3,$$

and hence

$$\begin{aligned} \int \frac{-2x-3}{x^2+6x+13}\,dx &= \int \left(\frac{-2(x+3)}{x^2+6x+13} + \frac{3}{x^2+6x+13}\right) dx \\ &= -2\int \frac{x+3}{x^2+6x+13}\,dx + \int \frac{3}{x^2+6x+13}\,dx. \quad 2 \end{aligned}$$

The first integral on the right side can be directly solved using the substitution

$$\begin{aligned} u &= x^2+6x+13, \\ du &= 2(x+3)\,dx, \qquad \text{or} \qquad (x+3)\,dx = \frac{1}{2}\,du, \end{aligned}$$

which gives

$$\begin{aligned} -2\int \frac{x+3}{x^2+6x+13}\,dx &= -2\int \frac{1}{x^2+6x+13}\cdot(x+3)\,dx \\ &= -2\int \frac{1}{u}\cdot\left(\frac{1}{2}\,du\right) = -\int \frac{1}{u}\,du = -\ln|u|+C = -\ln|x^2+6x+13|+C. \quad 2 \end{aligned}$$

As for the second integral, the fact that the quadratic polynomial $(x^2+6x+13)$ is irreducible implies that

$$\begin{aligned} \int \frac{3}{x^2+6x+13}\,dx &= \int \frac{3}{(x^2+2\cdot x\cdot 3)+13}\,dx = \int \frac{3}{(x^2+2\cdot x\cdot 3+3^2-3^2)+13}\,dx \\ &= \int \frac{3}{(x+3)^2+4}\,dx = \frac{3}{4}\int \frac{1}{(1/4)(x+3)^2+1}\,dx = \frac{3}{4}\cdot\frac{\tan^{-1}[(1/2)(x+3)]}{(1/2)}+C. \quad 2 \end{aligned}$$

In conclusion,

$$\int \frac{11x+29}{(x-1)(x^2+6x+13)}\,dx = 2\ln|x-1| - \ln|x^2+6x+13| + \frac{3}{2}\tan^{-1}\left[(1/2)(x+3)\right] + C.$$

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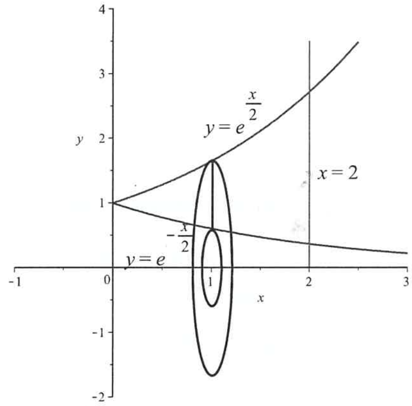
3. (25 points)

(a) (12 points) Find the volume of the solid generated by revolving the region in the first quadrant bounded from above by $y = \mathrm{e}^{x/2}$, from below by $y = \mathrm{e}^{-x/2}$, and on the right by $x = 2$ about the x-axis.

Solution. As shown in the following figure, the slice is chosen to be vertical (and hence the integra-



tion has to be carried out in x), since this avoids partitioning the interval of integration into multiple subintervals. Since the slice is perpendicular to the axis of rotation, the revolution of the slice generates a washer, which then suggests that the washer method has to be used. Now the inner and outer radii $r(x)$ and $R(x)$ of the washer, *expressed as functions of* x, are given by

$$r(x) = \mathrm{e}^{-x/2} - 0 = \mathrm{e}^{-x/2}, \qquad R(x) = \mathrm{e}^{x/2} - 0 = \mathrm{e}^{x/2}.$$

The bounds of integration, on the other hand, are obviously given by $x = 0$ (which is the intersection of $y = \mathrm{e}^{x/2}$ with $y = \mathrm{e}^{-x/2}$) and $x = 2$. The volume of the solid is then given by

$$V = \int_0^2 \pi\left[R^2(x) - r^2(x)\right]dx = \int_0^2 \pi\left[(\mathrm{e}^{x/2})^2 - (\mathrm{e}^{-x/2})^2\right]dx = \pi\int_0^2 (\mathrm{e}^x - \mathrm{e}^{-x})\,dx$$
$$= \pi(\mathrm{e}^x + \mathrm{e}^{-x})|_0^2 = \pi\left[(\mathrm{e}^2 - 1) + (\mathrm{e}^{-2} - 1)\right] = \pi(\mathrm{e}^2 - 2 + \mathrm{e}^{-2}).$$

(b) (13 points) Find the length of the curve $x(t) = at^2$, $y(t) = 2at$, $0 \le t \le a$ where $a > 0$ is a constant.

Solution. Since the curve is described by functions of t, the curve length should be expressed as an integral of t:

$$L = \int ds = \int_0^a \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2}\,dt,$$

where

$$\left[x'(t)\right]^2 + \left[y'(t)\right]^2 = (2at)^2 + (2a)^2 = 4a^2t^2 + 4a^2 = 4a^2(t^2+1).$$

This shows that

$$L = \int_0^a \sqrt{4a^2(t^2+1)}\,dt = \int_0^a 2a\sqrt{t^2+1}\,dt = 2a\int_0^a \sqrt{t^2+1}\,dt.$$

To proceed, observe that the Pythagorean identity $1+\tan^2\theta = \sec^2\theta$ motivates the (trig) substitution

$$t^2 = \tan^2\theta, \quad \text{or} \quad t = \tan\theta, \qquad \text{so that} \qquad \sqrt{t^2+1} = \sqrt{\sec^2\theta} = \sec\theta,$$
$$dt = (\tan\theta)'\,d\theta = \sec^2\theta\,d\theta.$$

As a result,

$$\begin{aligned}\int \sqrt{t^2+1}\,dt &= \int \sec\theta\cdot(\sec^2\theta\,d\theta) = \int \sec^3\theta\,d\theta \\ &= \frac{1}{2}\left[\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right] = \frac{1}{2}\left[t\sqrt{t^2+1} + \ln\left|t+\sqrt{t^2+1}\right|\right],\end{aligned}$$

which then implies

$$L = 2a\int_0^a \sqrt{t^2+1}\,dt = 2a\cdot\frac{1}{2}\left[t\sqrt{t^2+1} + \ln\left|t+\sqrt{t^2+1}\right|\right]\Big|_0^a = a\left[a\sqrt{a^2+1} + \ln\left|a+\sqrt{a^2+1}\right|\right].$$

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