MA1200 CALCULUS AND BASIC LINEAR ALGEBRA I

LECTURE: CG1

Chapter 7
Techniques of Differentiation

Brief table of derivatives of some elementary functions with respect to x

Function $y = f(x)$	**Derivative of y with respect to x** $\frac{dy}{dx} = \frac{d}{dx}(f(x))$ **or** $\frac{dy}{dx} = f'(x)$
$y = c$, where c is a constant	$\frac{dy}{dx} = 0$
$y = x^a$, where a is a constant	$\frac{dy}{dx} = ax^{a-1}$
$y = a^x$	$\frac{dy}{dx} = a^x \ln a$
$y = e^x$	$\frac{dy}{dx} = e^x$
$y = \ln x$	$\frac{dy}{dx} = \frac{1}{x}$
$y = \log_a x$, where $a > 0$	$\frac{dy}{dx} = \frac{1}{x} \log_a e$
$y = \sin x$	$\frac{dy}{dx} = \cos x$
$y = \cos x$	$\frac{dy}{dx} = -\sin x$

etc.

Rules of Differentiation

Let $u = u(x)$ and $v = v(x)$ be differentiable functions of x.

Then $\frac{du}{dx}$ (also denoted as $u'(x)$) and $\frac{dv}{dx}$ (also denoted as $v'(x)$) are the derivatives of u and v, respectively, with respect to x.

Rule #1: The derivative of any **constant** c is 0.

$$\boxed{\frac{d}{dx}(c) = 0}.$$

Rule #2: For any constant c, the derivative of a **scalar multiple** of $u(x)$ is given by

$$\boxed{\frac{d}{dx}(c \cdot u) = c \cdot \frac{du}{dx}}.$$

It is also denoted as

$$\frac{d}{dx}[c \cdot u(x)] = c \cdot u'(x).$$

Rule #3: The derivative of a **sum** or a **difference** of two functions of x is given by

$$\boxed{\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}}.$$

It follows that $\boxed{\frac{d}{dx}(c_1 \cdot u \pm c_2 \cdot v) = c_1 \cdot \frac{du}{dx} \pm c_2 \cdot \frac{dv}{dx}}$ where c_1 and c_2 are constants.

It is also denoted as $\frac{d}{dx}[c_1 \cdot u(x) \pm c_2 \cdot v(x)] = c_1 \cdot u'(x) \pm c_2 \cdot v'(x)$.

Example 1

If $y = 2e^x + 3\cos x - 5x^3 + 4x - \sqrt{x} + \frac{2}{\sqrt[3]{x}} + 8$, then

$$\frac{dy}{dx} = 2\frac{d}{dx}(e^x) + 3\frac{d}{dx}(\cos x) - 5\frac{d}{dx}(x^3) + 4\frac{d}{dx}(x) - \frac{d}{dx}\left(x^{\frac{1}{2}}\right) + 2\frac{d}{dx}\left(x^{-\frac{1}{3}}\right) + \frac{d}{dx}(8)$$

$$= 2e^x + 3(-\sin x) - 5 \cdot 3x^2 + 4 \cdot 1 - \frac{1}{2}x^{-\frac{1}{2}} + 2\left(-\frac{1}{3}\right)x^{-\frac{1}{3}-1} + 0$$

$$= 2e^x - 3\sin x - 15x^2 + 4 - \frac{1}{2\sqrt{x}} - \frac{2}{3}x^{-\frac{4}{3}}$$

Rule #4: (**Product rule**) The derivative of a **product** of two functions of x is given by

$$\boxed{\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}}.$$

It is also denoted as

$$\frac{d}{dx}[u(x) \cdot v(x)] = u(x) \cdot v'(x) + v(x) \cdot u'(x).$$

Example 2

If $y = x^5 \cos x$, then

$$\frac{dy}{dx} = x^5 \cdot \frac{d}{dx}(\cos x) + \cos x \cdot \frac{d}{dx}(x^5) \quad \text{by using **Product rule**}$$

$$= x^5 \cdot (-\sin x) + \cos x \cdot 5x^4$$

$$= -x^5 \sin x + 5x^4 \cos x.$$

Rule #5: (**Quotient rule**) The derivative of a **quotient** of two functions of x is given by

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}}.$$

It is also denoted as

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}.$$

Example 3

If $y = \frac{\sin x}{x^2+1}$, then

$$\frac{dy}{dx} = \frac{(x^2+1)\cdot\frac{d}{dx}(\sin x) - \sin x \cdot \frac{d}{dx}(x^2+1)}{(x^2+1)^2} \quad \text{by using **Quotient rule**}$$

$$= \frac{(x^2+1) \cdot \cos x - \sin x \cdot (2x)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)\cos x - 2x \sin x}{(x^2+1)^2}$$

Rule #6: (**Chain rule**) Let f be a function of u, and u be a function of x. Then the derivative of the **composite** function $(f \circ u)(x) = f\big(u(x)\big)$ is given by

$$\boxed{\frac{d}{dx}[f(u)] = \frac{d[f(u)]}{du} \cdot \frac{du}{dx}}.$$

It is also denoted as

$$\frac{d}{dx}[f(u(x))] = f'\big(u(x)\big) \cdot u'(x).$$

Example 4

If $y = (3x^2 + 5x - 1)^{\frac{3}{2}}$, then

$$\frac{dy}{dx} = \frac{d\left[(3x^2+5x-1)^{\frac{3}{2}}\right]}{d(3x^2+5x-1)} \cdot \frac{d(3x^2+5x-1)}{dx}$$

$$= \frac{3}{2}(3x^2 + 5x - 1)^{\frac{3}{2}-1} \cdot [3(2x) + 5]$$

$$= \frac{3}{2}(3x^2 + 5x - 1)^{\frac{1}{2}} \cdot (6x + 5)$$

Example 5

Find the derivatives of the following functions:

(a) $\tan x$ (b) $\cot x$

Solution

(a) $$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot \frac{d}{dx}(\cos x)}{(\cos x)^2}, \quad \text{by the } \textbf{Quotient rule}$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

(b) $\dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}[(\tan x)^{-1}]$

$$= \frac{d[(\tan x)^{-1}]}{d(\tan x)} \cdot \frac{d(\tan x)}{dx}\ , \qquad \text{by the } \textbf{Chain rule}$$

$$= (-1)(\tan x)^{-2} \cdot \sec^2 x\ , \qquad \text{by (a)}$$

$$= \frac{-1}{\tan^2 x} \cdot \frac{1}{\cos^2 x}$$

$$= \frac{-1}{\frac{\sin^2 x}{\cos^2 x}} \cdot \frac{1}{\cos^2 x}$$

$$= \frac{-1}{\sin^2 x}$$

$$= -\csc^2 x$$

Note:

Similarly, we can show that $\frac{d}{dx}(\sec x) = \sec x \tan x$ and $\frac{d}{dx}(\csc x) = -\csc x \cot x$.

Example 6

Find the derivatives of the following hyperbolic functions:

(a) $\sinh x$ (b) $\cosh x$ (c) $\tanh x$

Solution

(a) $\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left[\frac{1}{2}(e^x - e^{-x})\right]$

$$= \frac{1}{2}\left[\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})\right]$$

$$= \frac{1}{2}\left[\frac{d}{dx}(e^x) - \frac{d(e^{-x})}{d(-x)} \cdot \frac{d(-x)}{dx}\right] \qquad \text{by } \textbf{Chain} \text{ rule}$$

$$= \frac{1}{2}[e^x - e^{-x} \cdot (-1)] = \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

(b) $\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left[\frac{1}{2}(e^x + e^{-x})\right] = \frac{1}{2}\left[\frac{d}{dx}(e^x) + \frac{d}{dx}(e^{-x})\right]$

$$= \frac{1}{2}\left[e^x + \frac{d(e^{-x})}{d(-x)} \cdot \frac{d(-x)}{dx}\right] \qquad \text{by } \textbf{Chain rule}$$

$$= \frac{1}{2}[e^x + e^{-x} \cdot (-1)] = \frac{1}{2}(e^x - e^{-x}) = \sinh x$$

(c) $\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)$

$= \frac{\cosh x \cdot \frac{d}{dx}(\sinh x) - \sinh x \cdot \frac{d}{dx}(\cosh x)}{(\cosh x)^2}$ by the **Quotient rule**

$= \frac{\cosh x \cdot \cosh x - \sinh x \cdot (\sinh x)}{\cosh^2 x}$ by (a) and (b)

$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$

$= \frac{1}{\cosh^2 x}$ (since $\boldsymbol{\cosh^2 x - \sinh^2 x = 1}$, from **Chapter 5**)

$= \operatorname{sech}^2 x$

Homework:

Given that $\coth x = \frac{1}{\tanh x}$, $\operatorname{sech} x = \frac{1}{\cosh x}$ and $\operatorname{csch} x = \frac{1}{\sinh x}$.

Show that $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$, $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

and $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$.

Table of derivatives of $y = f(u)$ with respect to x, where u is a function of x

Function $y = f(u)$	Derivative of y with respect to x
$y = c$, where c is a constant	$\frac{dy}{dx} = 0$
$y = cu$, where c is a constant	$\frac{dy}{dx} = c\,\frac{du}{dx}$
$y = u^p$, where p is a constant	$\frac{dy}{dx} = p\,u^{p-1}\,\frac{du}{dx}$
$y = u + v$	$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = f(u)$, where u is a function of x	$\frac{df(u)}{du} \cdot \frac{du}{dx}$, the chain rule
$y = \log_a u$, $a > 0$	$\frac{dy}{dx} = \frac{1}{u}\log_a e\,\frac{du}{dx}$
$y = a^u$, $a > 0$	$\frac{dy}{dx} = a^u \log_e a\,\frac{du}{dx}$
$y = e^u$	$\frac{dy}{dx} = e^u\,\frac{du}{dx}$
$y = u^v$	$\frac{dy}{dx} = v\,u^{v-1}\,\frac{du}{dx} + u^v\,\log_e u\,\frac{dv}{dx}$

Function $y=f(u)$	**Derivative of** y **with respect to** x
$y=\sin u$	$\frac{dy}{dx}=\cos u\ \frac{du}{dx}$
$y=\cos u$	$\frac{dy}{dx}=-\sin u\ \frac{du}{dx}$
$y=\tan u$	$\frac{dy}{dx}=\sec^2 u\ \frac{du}{dx}$
$y=\cot u$	$\frac{dy}{dx}=-\operatorname{cosec}^2 u\ \frac{du}{dx}$
$y=\sec u$	$\frac{dy}{dx}=\sec u\tan u\ \frac{du}{dx}$
$y=\operatorname{cosec} u$	$\frac{dy}{dx}=-\operatorname{cosec} u\cot u\ \frac{du}{dx}$
$y=\sin^{-1} u$	$\frac{dy}{dx}=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y=\cos^{-1} u$	$\frac{dy}{dx}=\frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y=\tan^{-1} u$	$\frac{dy}{dx}=\frac{1}{1+u^2}\frac{du}{dx}$
$y=\cot^{-1} u$	$\frac{dy}{dx}=-\frac{1}{1+u^2}\frac{du}{dx}$
$y=\sec^{-1} u$	$\frac{dy}{dx}=\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$
$y=\operatorname{cosec}^{-1} u$	$\frac{dy}{dx}=-\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$

Function $y=f(u)$	**Derivative of** y **with respect to** x
$y=\sinh u$	$\frac{dy}{dx}=\cosh u\ \frac{du}{dx}$
$y=\cosh u$	$\frac{dy}{dx}=\sinh u\ \frac{du}{dx}$
$y=\sinh^{-1} u$	$\frac{dy}{dx}=\frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$
$y=\cosh^{-1} u$	$\frac{dy}{dx}=\frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$
$y=\tanh^{-1} u$	$\frac{dy}{dx}=\frac{1}{1-u^2}\frac{du}{dx}$
$y=\coth^{-1} u$	$\frac{dy}{dx}=\frac{1}{1-u^2}\frac{du}{dx}$

Example 7

Find the derivatives of the following functions using the Table of Derivatives on p.12 – 14:

(a) $\sin^3 x$ (b) $\sin(x^3)$ (c) $\cos^3(x^2)$

(d) $\ln(\tan 2x)$ (e) $\sec\left(\frac{x^2-1}{x^3+3x}\right)$ (d) $e^{\sinh(x^4\cos 3x)}$

Solution

(a) $\frac{d}{dx}(\sin^3 x) = \frac{d}{dx}[(\sin x)^3] = 3(\sin x)^2 \cdot \frac{d}{dx}(\sin x)$ ← **using** $\frac{d}{dx}[u^p] = p\,u^{p-1}\frac{du}{dx}$

$= 3(\sin x)^2 \cdot \cos x \cdot \underbrace{\frac{dx}{dx}}_{=1}$ ← **using** $\frac{d}{dx}[\sin u] = \cos u\,\frac{du}{dx}$

$= 3\sin^2 x\cos x$

(b) $\frac{d}{dx}[\sin(x^3)] = \cos(x^3)\cdot\frac{d}{dx}(x^3)$ ← **using** $\frac{d}{dx}[\sin u] = \cos u\,\frac{du}{dx}$

$= \cos(x^3)\cdot 3x^2 \cdot \underbrace{\frac{dx}{dx}}_{=1}$ ← **using** $\frac{d}{dx}[u^p] = p\,u^{p-1}\frac{du}{dx}$

$= 3x^2\cos(x^3)$

(c) $\frac{d}{dx}[\cos^3(x^2)] = \frac{d}{dx}[(\cos(x^2))^3]$

$= 3(\cos(x^2))^2\cdot\frac{d}{dx}[\cos(x^2)]$ ← **using** $\frac{d}{dx}[u^p] = p\,u^{p-1}\frac{du}{dx}$

$= 3(\cos(x^2))^2\cdot\left[-\sin(x^2)\cdot\frac{d}{dx}(x^2)\right]$ ← **using** $\frac{d}{dx}[\cos u] = -\sin u\,\frac{du}{dx}$

$= -3\cos^2(x^2)\cdot\sin(x^2)\cdot 2x$ ← **using** $\frac{d}{dx}[u^p] = p\,u^{p-1}\frac{du}{dx}$

$= -6x\cos^2(x^2)\,\sin(x^2)$

(d) $\frac{d}{dx}[\ln(\tan 2x)] = \frac{1}{\tan 2x}\cdot\frac{d}{dx}(\tan 2x)$ ← **using** $\frac{d}{dx}[\log_a u] = \frac{1}{u}\log_a e\,\frac{du}{dx}$

$\therefore\ \frac{d}{dx}[\ln u] = \frac{d}{dx}[\log_e u] = \frac{1}{u}\underbrace{\log_e e}_{=\ln e=1}\frac{du}{dx} = \frac{1}{u}\cdot\frac{du}{dx}$

$= \frac{1}{\tan 2x}\cdot\sec^2(2x)\cdot\frac{d}{dx}(2x)$ ← **using** $\frac{d}{dx}[\tan u] = \sec^2 u\,\frac{du}{dx}$

$= \frac{1}{\tan 2x}\cdot\sec^2(2x)\cdot 2$ ← **using** $\frac{d}{dx}[cu] = c\,\frac{du}{dx}$

$= 2\cdot\frac{\cos 2x}{\sin 2x}\cdot\frac{1}{\cos^2 2x}$

$= \frac{2}{\sin 2x\,\cos 2x}$

(e) $\frac{d}{dx}\left[\sec\left(\frac{x^2-1}{x^3+3x}\right)\right]$

$$= \sec\left(\frac{x^2-1}{x^3+3x}\right)\tan\left(\frac{x^2-1}{x^3+3x}\right)\cdot\frac{d}{dx}\left(\frac{x^2-1}{x^3+3x}\right)$$ **← using $\frac{d}{dx}[\sec u] = \sec u \tan u \frac{du}{dx}$**

$$= \sec\left(\frac{x^2-1}{x^3+3x}\right)\tan\left(\frac{x^2-1}{x^3+3x}\right)\cdot\frac{(x^3+3x)\cdot\frac{d}{dx}(x^2-1)-(x^2-1)\cdot\frac{d}{dx}(x^3+3x)}{(x^3+3x)^2}$$ **← using Quotient rule**

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\cdot\frac{du}{dx} - u\cdot\frac{dv}{dx}}{v^2}$$

$$= \sec\left(\frac{x^2-1}{x^3+3x}\right)\tan\left(\frac{x^2-1}{x^3+3x}\right)\cdot\frac{(x^3+3x)(2x)-(x^2-1)(3x^2+3)}{(x^3+3x)^2}$$ **← using $\frac{d}{dx}[u^p] = p\,u^{p-1}\frac{du}{dx}$**

$$= \sec\left(\frac{x^2-1}{x^3+3x}\right)\tan\left(\frac{x^2-1}{x^3+3x}\right)\cdot\frac{(-x^4+6x^2+3)}{(x^3+3x)^2}$$

(f) $\frac{d}{dx}\left[e^{\sinh(x^4\cos 3x)}\right]$

$$= e^{\sinh(x^4\cos 3x)}\cdot\frac{d}{dx}[\sinh(x^4\cos 3x)]$$ **← using $\frac{d}{dx}[e^u] = e^u\frac{du}{dx}$**

$$= e^{\sinh(x^4\cos 3x)}\cdot\cosh(x^4\cos 3x)\cdot\frac{d}{dx}(x^4\cos 3x)$$ **← using $\frac{d}{dx}[\sinh u] = \cosh u\frac{du}{dx}$**

$$= e^{\sinh(x^4\cos 3x)}\cdot\cosh(x^4\cos 3x)\cdot\left[x^4\cdot\frac{d}{dx}(\cos 3x) + \cos 3x\cdot\frac{d}{dx}(x^4)\right]$$

← using Product rule $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$$= e^{\sinh(x^4\cos 3x)}\cdot\cosh(x^4\cos 3x)\cdot\left[x^4\cdot(-\sin 3x)\cdot\underbrace{\frac{d}{dx}(3x)}_{=3} + \cos 3x\cdot(4x^3)\right]$$

← using $\frac{d}{dx}[\cos u] = -\sin u\frac{du}{dx}$ and $\frac{d}{dx}[u^p] = p\,u^{p-1}\frac{du}{dx}$

$$= e^{\sinh(x^4\cos 3x)}\cdot\cosh(x^4\cos 3x)\cdot[-3x^4\sin 3x + 4x^3\cos 3x]$$

Differentiation of functions in the form of u^p, a^u, e^u and u^v

(where u and v are functions of x, and $a\ (>0)$ and p are constants.)

(i) If $\boxed{y = u^p}$, then $\boxed{\dfrac{dy}{dx} = p\,u^{p-1}\,\dfrac{du}{dx}}$.

Example 8

$$\frac{d}{dx}\left[\frac{1}{\sqrt{x+\sqrt{x}}}\right] = \frac{d}{dx}\left[\left(x + x^{\frac{1}{2}}\right)^{-\frac{1}{2}}\right]$$

$$= -\frac{1}{2}\left(x + x^{\frac{1}{2}}\right)^{-\frac{1}{2}-1} \cdot \frac{d}{dx}\left(x + x^{\frac{1}{2}}\right)$$

$$= -\frac{1}{2}\left(x + x^{\frac{1}{2}}\right)^{-\frac{3}{2}} \cdot \left(1 + \frac{1}{2}x^{\frac{1}{2}-1}\right)$$

$$= -\frac{1}{2}\left(x + x^{\frac{1}{2}}\right)^{-\frac{3}{2}} \cdot \left(1 + \frac{1}{2}x^{-\frac{1}{2}}\right)$$

(ii) If $\boxed{y = a^u}$, then $\boxed{\dfrac{dy}{dx} = a^u \log_e a\ \dfrac{du}{dx}}$, where $\log_e a = \ln a$.

Example 9

$$\frac{d}{dx}\left[3^{\sin^{-1}(5x)}\right] = 3^{\sin^{-1}(5x)} \log_e 3 \cdot \frac{d}{dx}\left[\sin^{-1}(5x)\right]$$

$$= 3^{\sin^{-1}(5x)}\,(\ln 3) \cdot \frac{1}{\sqrt{1-(5x)^2}} \cdot \underbrace{\frac{d}{dx}(5x)}_{=5}$$

$$= 3^{\sin^{-1}(5x)}\,(\ln 3) \cdot \frac{5}{\sqrt{1-(5x)^2}}$$

Remark:

$\sin^{-1}x \neq (\sin x)^{-1}$.

($\sin^{-1}x$ is the inverse function of $\sin x$, and $(\sin x)^{-1}$ is the reciprocal of $\sin x$.)

$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ but $\frac{d}{dx}[(\sin x)^{-1}] = \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x\ \cot x$

(iii) If $\boxed{y = e^u}$, then $\boxed{\frac{dy}{dx} = e^u \ \frac{du}{dx}}$.

Example 10

$\frac{d}{dx}\left[e^{\tan^2(3x)}\right] =$

(iv) If $\boxed{y = u^v}$, then $\boxed{\frac{dy}{dx} = v\,u^{v-1}\ \frac{du}{dx} + u^v\ \log_e u\ \frac{dv}{dx}}$, where $\log_e u = \ln u$.

Example 11

$\frac{d}{dx}\left[(\log_e(2x))^{x^3}\right] =$

Implicit Differentiation

A function written in the form $\boxed{y = f(x)}$ is called an **explicit function**, where x is the independent variable, y is the dependent variable, and $f(x)$ is a function of x.

The first derivative of y with respect to x is $\frac{dy}{dx} = f'(x)$.

When the function is written as the form $\boxed{F(x,y) = 0}$ instead of $y = f(x)$, where $F(x,y)$ is a function of both x and y, then we say that $F(x,y) = 0$ is an **implicit function**.

Question: How to find $\frac{dy}{dx}$ when the function is expressed in **implicit** form?

Answer: By using **implicit differentiation**:

Step 1: Differentiate (using chain rule) both sides of $F(x,y) = 0$ w.r.t. x.

Step 2: Rearrange the expression to get

$\frac{dy}{dx} = \cdots$ (where the R.H.S. may contain both x and y.)

Example 12

Given $y^2 = x^2 + x\sin y$. Find $\frac{dy}{dx}$.

Solution

Note that $y^2 = x^2 + x\sin y$ is an **implicit** function.

Differentiate both sides of the equation with respect to (w.r.t.) x, we get

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(x\sin y)$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 2x + \underbrace{\left[x\frac{d}{dx}(\sin y) + \sin y \cdot \frac{d}{dx}(x)\right]}_{\text{by product rule}}$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 2x + \left[x \cdot \cos y\frac{dy}{dx} + \sin y \cdot 1\right]$$

$$\Rightarrow \quad (2y - x\cos y)\frac{dy}{dx} = 2x + \sin y$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x + \sin y}{2y - x\cos y}$$

Note that when differentiating an implicit function, the answer may contain both x and y.

Example 13

Find $\frac{dy}{dx}$ of the function $\ln(\sec x) + \tan^{-1} y = e^{xy}$.

Solution

Differentiate both sides w.r.t. x:

$$\frac{d}{dx}[\ln(\sec x)] + \frac{d}{dx}(\tan^{-1} y) = \frac{d}{dx}(e^{xy})$$

$$\Rightarrow \quad \frac{1}{\sec x} \cdot \frac{d(\sec x)}{dx} + \frac{1}{1+y^2} \cdot \frac{dy}{dx} = e^{xy} \cdot \frac{d(xy)}{dx}$$

$$\Rightarrow \quad \frac{1}{\sec x} \cdot \sec x \tan x + \frac{1}{1+y^2}\frac{dy}{dx} = e^{xy}\left(x\frac{dy}{dx} + y\right)$$

$$\Rightarrow \quad \tan x + \frac{1}{1+y^2}\frac{dy}{dx} = e^{xy}\left(x\frac{dy}{dx} + y\right)$$

$$\Rightarrow \quad \left(\frac{1}{1+y^2} - xe^{xy}\right)\frac{dy}{dx} = y\,e^{xy} - \tan x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y\,e^{xy} - \tan x}{\frac{1}{1+y^2} - xe^{xy}}$$

Example 14

Given $xy^2 + 4(x+y)^3 = \frac{y}{x}$. Find $\frac{dy}{dx}$.

Solution

Differentiate both sides w.r.t. x:

$$\frac{d}{dx}(xy^2) + 4\frac{d}{dx}[(x+y)^3] = \frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad x \cdot \frac{d(y^2)}{dx} + y^2 \cdot \frac{d(x)}{dx} + 4\left\{3(x+y)^2 \cdot \frac{d(x+y)}{dx}\right\} = \frac{x \cdot \frac{dy}{dx} - y \cdot \frac{d(x)}{dx}}{x^2}$$

$$\Rightarrow \quad x \cdot 2y\frac{dy}{dx} + y^2 + 4\left\{3(x+y)^2 \cdot \left(1 + \frac{dy}{dx}\right)\right\} = \frac{x\frac{dy}{dx} - y}{x^2}$$

$$\Rightarrow \quad 2xy\frac{dy}{dx} + y^2 + 12(x+y)^2 + 12(x+y)^2\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2}$$

$$\Rightarrow \quad \left[2xy + 12(x+y)^2 - \frac{1}{x}\right]\frac{dy}{dx} = -\frac{y}{x^2} - y^2 - 12(x+y)^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-\frac{y}{x^2} - y^2 - 12(x+y)^2}{2xy + 12(x+y)^2 - \frac{1}{x}}$$

Homework: Given that $x^3 + 2xy^2 - y^3 + x - 1 = 0$, find $\frac{dy}{dx}$. [*Ans.*: $\frac{dy}{dx} = \frac{3x^2+2y^2+1}{3y^2-4xy}$]

Differentiation of inverse functions

Let $f: I \to \mathbb{R}$ be a differentiable and strictly monotonic (i.e. strictly increasing or strictly decreasing) function of x. Thus, f is a one-to-one function.

The inverse of f, denoted by f^{-1}, is defined as

$$f^{-1}(y) = x, \quad \text{where } y = f(x).$$

If $f'(x) \neq 0$ for all x in I and f^{-1} is differentiable at y, where $y = f(x)$, then

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

or written as

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.$$

This is known as the **inverse function theorem.** Hence,

$$\boxed{\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}}.$$

Example 15

Consider $f: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$, where $f(x) = \sin x$. Then $f(x)$ is one-to-one and its inverse function is $f^{-1}(x) = \sin^{-1} x$. If $y = \sin^{-1} x$, what is $\frac{dy}{dx}$?

Solution

Method 1: Use the inverse function theorem

$y = \sin^{-1} x \quad \Longrightarrow \quad x = \sin y$

Differentiate both sides with respect to y:

$$\frac{dx}{dy} = \cos y$$

By the **inverse function theorem**,

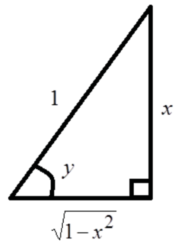
$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}},$$

which is only defined when $-1 < x < 1$.

Alternatively, we can deduce the relationship $\cos y = \sqrt{1-x^2}$ by considering the following right-angled triangle:

$$x = \sin y \;\Rightarrow\; \sin y = \frac{x}{1}$$

$$\therefore\; \cos y = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}.$$



Method 2: ***Implicit Differentiation***

$y = \sin^{-1} x \quad \Longrightarrow \quad x = \sin y$

Differentiate both sides with respect to x:

$$\underbrace{\frac{dx}{dx}}_{=1} = \cos y \,\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

Homework: Use the **inverse function theorem** to show that $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$.

(Hint: start with $y = \cos^{-1} x$.)

Example 16

Show that $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$.

Solution

Method 1: ***Inverse function theorem***

Let $y = \tan^{-1} x \quad \Longrightarrow \quad x = \tan y$

Differentiate both sides with respect to y:

$$\frac{dx}{dy} = \sec^2 y$$

By the **inverse function theorem**, $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$.

Method 2: ***Implicit Differentiation***

Let $y = \tan^{-1} x \quad \Longrightarrow \quad x = \tan y$ (which is an **implicit** function)

Differentiate both sides with respect to x:

$$1 = \sec^2 y \,\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

Example 17

Given that $\cosh^2 u - \sinh^2 u = 1$ for all $u \in \mathbb{R}$. Use **implicit differentiation** to show that

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}.$$

Solution

Note: $\sinh^{-1}x \neq (\sinh x)^{-1}$.

Example 18

Differentiate each of the following functions with respect to x:

(a) $x^2 \sin^{-1}(x^3+1)$ (b) $\tan^{-1}\left(\frac{1-x}{1+x}\right)$ (c) $\cos^{-1}\left(\frac{1}{1+x^2}\right)$

Solution

(a)
$$\begin{aligned}\frac{d}{dx}[x^2 \sin^{-1}(x^3+1)] &= x^2 \cdot \frac{d}{dx}[\sin^{-1}(x^3+1)] + \sin^{-1}(x^3+1) \cdot \frac{d}{dx}(x^2) \\ &= x^2 \cdot \frac{1}{\sqrt{1-(x^3+1)^2}} \cdot \frac{d}{dx}(x^3+1) + \sin^{-1}(x^3+1) \cdot 2x \\ &= x^2 \cdot \frac{1}{\sqrt{1-(x^3+1)^2}} \cdot 3x^2 + 2x \sin^{-1}(x^3+1) \\ &= \frac{3x^4}{\sqrt{1-(x^3+1)^2}} + 2x \sin^{-1}(x^3+1)\end{aligned}$$

(b) $$\frac{d}{dx}\left[\tan^{-1}\left(\frac{1-x}{1+x}\right)\right]=\frac{1}{1+\left(\frac{1-x}{1+x}\right)^2}\cdot\frac{d}{dx}\left(\frac{1-x}{1+x}\right)$$

$$=\frac{1}{1+\left(\frac{1-x}{1+x}\right)^2}\cdot\frac{(1+x)\frac{d}{dx}(1-x)-(1-x)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$=\frac{(1+x)^2}{(1+x)^2+(1-x)^2}\cdot\frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}$$

$$=\frac{-1-x-1+x}{(1+x)^2+(1-x)^2}=\frac{-2}{(1+2x+x^2)+(1-2x+x^2)}=\frac{-2}{2(1+x^2)}=\frac{-1}{1+x^2}$$

(c) $$\frac{d}{dx}\left[\cos^{-1}\left(\frac{1}{1+x^2}\right)\right]=\frac{-1}{\sqrt{1-\left(\frac{1}{1+x^2}\right)^2}}\cdot\frac{d}{dx}\left(\frac{1}{1+x^2}\right)=\frac{-1}{\sqrt{1-\left(\frac{1}{1+x^2}\right)^2}}\cdot\frac{d}{dx}[(1+x^2)^{-1}]$$

$$=\frac{-1}{\sqrt{1-\left(\frac{1}{1+x^2}\right)^2}}\cdot(-1)(1+x^2)^{-2}\cdot\frac{d}{dx}(1+x^2)$$

$$=\frac{-1}{\sqrt{1-\left(\frac{1}{1+x^2}\right)^2}}\cdot(-1)(1+x^2)^{-2}\cdot 2x$$

$$=\frac{2x}{(1+x^2)^2\sqrt{1-\left(\frac{1}{1+x^2}\right)^2}}=\frac{2x}{(1+x^2)\sqrt{(1+x^2)^2-1}}$$

Derivatives of Exponential and Logarithmic functions

Recall that exponential and logarithmic functions are the inverse functions of each other.

That is,

$$y=a^x \quad\Leftrightarrow\quad x=\log_a y \quad \text{for } x\in\mathbb{R} \text{ and } y\in(0,\infty),$$

where the base $a>0$ and $a\neq 1$. Now consider the exponential and logarithmic functions with base $a=e$, where $e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\approx 2.71828$ ….

We know that

$$\frac{d}{dx}(e^x)=e^x \quad\text{and}\quad \frac{d}{dx}(\ln x)=\frac{1}{x}.$$

Recall the basic properties of natural logarithm:

- $\ln(ab)=\ln a+\ln b$, where $a,b>0$.
- $\ln\left(\frac{a}{b}\right)=\ln a-\ln b$, where $a,b>0$.
- $\ln(a^b)=b\ln a$, where $a\ (>0)$ and b are constants.
- $\ln\left(e^{f(x)}\right)=f(x)\ln e=f(x)$, since $\ln e=1$.

Example 19

Find the derivatives of the following functions:

(a) $f(x) = a^x$, where $a > 0$ and $a \neq 1$

(b) $g(x) = \log_a x$, where $a > 0$ and $a \neq 1$

(c) $h(x) = x^a$, where $a \in \mathbb{R}$. (Note that x^a is not an exponential function.)

Solution

(a) $f(x) = a^x = e^{\ln(a^x)} = e^{x \ln a}$

$\therefore\ f'(x) = \frac{d}{dx}\left(e^{x \ln a}\right) = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) = e^{x \ln a} \cdot \ln a = a^x(\ln a)$

(b) $g(x) = \log_a x = \dfrac{\ln x}{\ln a}$

$\therefore\ g'(x) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a} \cdot \frac{d}{dx}(\ln x) = \frac{1}{x \ln a}$

(c) $h(x) = x^a = e^{\ln(x^a)} = e^{a \ln x}$

$\therefore\ h'(x) = \frac{d}{dx}\left(e^{a \ln x}\right) = e^{a \ln x} \cdot \frac{d}{dx}(a \ln x) = e^{a \ln x} \cdot \frac{a}{x} = x^a \cdot \frac{a}{x} = a x^{a-1}$

Example 20

If $f(x) = e^{-\frac{x}{n}} \cos\left(\frac{x}{a}\right)$, find the value of $f(0) + af'(0)$, where $a \neq 0$ and $n \neq 0$ are constants.

Solution

$$f(x) = e^{-\frac{x}{n}} \cos\left(\frac{x}{a}\right) \ \Rightarrow\ f(0) = e^0 \cdot \cos 0 = 1$$

$$\begin{aligned} f'(x) &= e^{-\frac{x}{n}} \cdot \frac{d}{dx}\left[\cos\left(\frac{x}{a}\right)\right] + \cos\left(\frac{x}{a}\right) \cdot \frac{d}{dx}\left(e^{-\frac{x}{n}}\right) \\ &= e^{-\frac{x}{n}} \cdot \left[-\sin\left(\frac{x}{a}\right)\right] \cdot \frac{1}{a} + \cos\left(\frac{x}{a}\right) \cdot e^{-\frac{x}{n}} \cdot \left(-\frac{1}{n}\right) \\ &= -\frac{1}{a} e^{-\frac{x}{n}} \sin\left(\frac{x}{a}\right) - \frac{1}{n} e^{-\frac{x}{n}} \cos\left(\frac{x}{a}\right) \end{aligned}$$

$$\Rightarrow\ f'(0) = -\frac{1}{a} e^0 \cdot \sin 0 - \frac{1}{n} e^0 \cdot \cos 0 = -\frac{1}{n}$$

$$\therefore\ f(0) + af'(0) = 1 + a \cdot \left(-\frac{1}{n}\right) = 1 - \frac{a}{n}$$

Logarithmic Differentiation

This is used to differentiate functions of the form

(i) $\boxed{y = [u(x)]^{v(x)}}$, where $u(x)$ and $v(x)$ are both functions of x.

(Here, $u(x)$ could be a non-zero constant or a function of x.)

(ii) $\boxed{y = \dfrac{[u_1(x)]^{a_1} \cdot [u_2(x)]^{a_2} \cdot \cdots \cdot [u_n(x)]^{a_n}}{[v_1(x)]^{b_1} \cdot [v_2(x)]^{b_2} \cdot \cdots \cdot [v_m(x)]^{b_m}}}$, where $u_1(x), \dots, u_n(x), v_1(x), \dots, v_m(x)$ are functions of x; and $a_1, \dots, a_n, b_1, \dots, b_m$ could be non-zero constants or functions of x.

Example 21

Given that $y = (x^2+1)^{\cot x}$. Find $\frac{dy}{dx}$.

Solution

Method 1: Use logarithmic differentiation

$$y = (x^2+1)^{\cot x}$$

Take natural logarithm on both sides:

$$\begin{aligned} \ln y &= \ln[(x^2+1)^{\cot x}] \\ &= (\cot x)\ln(x^2+1) \quad \leftarrow \text{This is an } \textbf{implicit function}. \end{aligned}$$

Differentiate both sides w.r.t. x:

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}[(\cot x)\ln(x^2+1)]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \underbrace{(\cot x)\cdot\frac{d}{dx}[\ln(x^2+1)] + \ln(x^2+1)\cdot\frac{d}{dx}(\cot x)}_{\text{by product rule}}$$

$$= (\cot x)\cdot\frac{1}{x^2+1}\cdot\frac{d}{dx}(x^2+1) + \ln(x^2+1)\cdot[-\operatorname{cosec}^2 x]$$

$$= (\cot x)\cdot\frac{1}{x^2+1}\cdot 2x - \ln(x^2+1)\cdot\operatorname{cosec}^2 x$$

Multiply both sides by y and then replace y with $(x^2+1)^{\cot x}$:

$$\begin{aligned}\frac{dy}{dx} &= y\left[(\cot x)\cdot\frac{1}{x^2+1}\cdot 2x - \ln(x^2+1)\cdot\operatorname{cosec}^2 x\right]\\ &= (x^2+1)^{\cot x}\left[(\cot x)\cdot\frac{1}{x^2+1}\cdot 2x - \ln(x^2+1)\cdot\operatorname{cosec}^2 x\right]\end{aligned}$$

Note: For logarithmic differentiation, the right hand side should not contain any y.

Method 2: Use the Table of Derivatives

If $\boxed{y = u^v}$, then $\boxed{\frac{dy}{dx} = v\,u^{v-1}\,\frac{du}{dx} + u^v\,\log_e u\,\frac{dv}{dx}}$, where $\log_e u = \ln u$.

$y = (x^2+1)^{\cot x}$ is of the form $y = u^v$, where $u = x^2+1$ and $v = \cot x$.

Thus, $\frac{dy}{dx} = (\cot x)(x^2+1)^{\cot x - 1}\,\frac{d}{dx}(x^2+1) + (x^2+1)^{\cot x}\,\log_e(x^2+1)\,\frac{d}{dx}(\cot x)$

$$= (\cot x)(x^2+1)^{\cot x - 1}\cdot(2x) + (x^2+1)^{\cot x}\,\ln(x^2+1)\cdot(-\operatorname{cosec}^2 x)$$

Example 22

If $y = \left(\frac{a}{x}\right)^{ax}$, find $\frac{dy}{dx}$.

Solution

$$y = \left(\frac{a}{x}\right)^{ax}$$

Take natural logarithm on both sides:

$$\ln y = \ln\left[\left(\frac{a}{x}\right)^{ax}\right] = ax\ln\left(\frac{a}{x}\right) = ax\,(\ln a - \ln x)$$

Differentiate both sides w.r.t. x:

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}[ax(\ln a - \ln x)]$$

$$\Rightarrow\quad \frac{1}{y}\,\frac{dy}{dx} = \underbrace{ax\cdot\frac{d}{dx}(\ln a - \ln x) + (\ln a - \ln x)\cdot\frac{d}{dx}(ax)}_{\text{by product rule}}$$

$$= ax\cdot\left(0 - \frac{1}{x}\right) + (\ln a - \ln x)\cdot a$$

$$\Rightarrow\quad \frac{dy}{dx} = \left(\frac{a}{x}\right)^{ax}\left[-a + a\ln\left(\frac{a}{x}\right)\right]$$

Example 23

Given that $y = \frac{x}{(x-1)(x-2)(x-3)}$. Find $\frac{dy}{dx}$.

Solution

Method 1: Use Product rule and Quotient rule ← long calculation, tedious!

Not recommended.

Method 2: Use logarithmic differentiation ← more convenient!

Take natural logarithm on both sides:

$$\ln y = \ln\left[\frac{x}{(x-1)(x-2)(x-3)}\right] = \ln x - \ln(x-1) - \ln(x-2) - \ln(x-3)$$

Differentiate both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x-1} - \frac{1}{x-2} - \frac{1}{x-3}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{x} - \frac{1}{x-1} - \frac{1}{x-2} - \frac{1}{x-3}\right)$$

$$= \frac{x}{(x-1)(x-2)(x-3)}\left(\frac{1}{x} - \frac{1}{x-1} - \frac{1}{x-2} - \frac{1}{x-3}\right)$$

Example 24

Given that $y = \frac{(x+2)^3\,(x^2+4)^{\frac{1}{2}}\,e^{\sin 2x}}{(3x+5)^2\,(4x^3+1)}$. Find $\frac{dy}{dx}$.

Solution

We use **logarithmic differentiation**.

$$y = \frac{(x+2)^3\,(x^2+4)^{\frac{1}{2}}\,e^{\sin 2x}}{(3x+5)^2\,(4x^3+1)}$$

Take natural logarithm on both sides:

$$\ln y = \ln\left[\frac{(x+2)^3\,(x^2+4)^{\frac{1}{2}}\,e^{\sin 2x}}{(3x+5)^2\,(4x^3+1)}\right]$$

$$= \ln[(x+2)^3] + \ln\left[(x^2+4)^{\frac{1}{2}}\right] + \ln\left(e^{\sin 2x}\right) - \ln[(3x+5)^2] - \ln(4x^3+1)$$

$$= 3\ln(x+2) + \frac{1}{2}\ln(x^2+4) + \sin 2x - 2\ln(3x+5) - \ln(4x^3+1)$$

Differentiate both sides with respect to x:

$$\frac{1}{y}\frac{dy}{dx} = 3\cdot\frac{1}{x+2}\cdot\frac{d(x+2)}{dx} + \frac{1}{2}\cdot\frac{1}{x^2+4}\cdot\frac{d(x^2+4)}{dx} + \cos 2x\cdot\frac{d(2x)}{dx}$$
$$-2\cdot\frac{1}{3x+5}\cdot\frac{d(3x+5)}{dx} - \frac{1}{4x^3+1}\cdot\frac{d(4x^3+1)}{dx}$$
$$= \frac{3}{x+2}\cdot(1) + \frac{1}{2}\cdot\frac{1}{x^2+4}\cdot(2x) + \cos 2x\cdot(2) - \frac{2}{3x+5}\cdot(3) - \frac{1}{4x^3+1}\cdot(4\cdot 3x^2)$$
$$\Longrightarrow \frac{dy}{dx} = y\left[\frac{3}{x+2} + \frac{x}{x^2+4} + 2\cos 2x - \frac{6}{3x+5} - \frac{12x^2}{4x^3+1}\right]$$
$$= \left[\frac{(x+2)^3\,(x^2+4)^{\frac{1}{2}}\,e^{\sin 2x}}{(3x+5)^2\,(4x^3+1)}\right]\left[\frac{3}{x+2} + \frac{x}{x^2+4} + 2\cos 2x - \frac{6}{3x+5} - \frac{12x^2}{4x^3+1}\right]$$

Homework: If $y = \left(x + \frac{1}{x}\right)^{x^2}$, find $\frac{dy}{dx}$ by using logarithmic differentiation. Check your answer by using the table of derivatives. [*Ans.*: $\frac{dy}{dx} = \left(x+\frac{1}{x}\right)^{x^2}\left[2x\ln\left(x+\frac{1}{x}\right) + \frac{x(x^2-1)}{x^2+1}\right]$]

Example 25

Given that $y = (\cos x)^x + 3^x$. Find $\frac{dy}{dx}$.

Solution

Note that $\ln y = \ln[(\cos x)^x + 3^x] \neq \ln[(\cos x)^x] + \ln(3^x)$.

Let $y_1 = (\cos x)^x$ and $y_2 = 3^x$.

Then $\ln y_1 = \ln[(\cos x)^x] = x\ln(\cos x)$ ⋯⋯ (1)

and $\ln y_2 = \ln(3^x) = x\ln 3$ ⋯⋯ (2).

Differentiate both sides of (1) w.r.t. x:

$$\frac{1}{y_1}\cdot\frac{dy_1}{dx} = x\cdot\frac{d}{dx}[\ln(\cos x)] + \ln(\cos x)\cdot\frac{d(x)}{dx}$$
$$= x\cdot\frac{1}{\cos x}\cdot\underbrace{\frac{d}{dx}(\cos x)}_{=-\sin x} + \ln(\cos x)\cdot 1$$
$$= -x\tan x + \ln(\cos x)$$
$$\Rightarrow \frac{dy_1}{dx} = (\cos x)^x\,[-x\tan x + \ln(\cos x)]$$

Differentiate both sides of (2) w.r.t. x:

$$\frac{1}{y_2}\cdot\frac{dy_2}{dx} = \ln 3$$

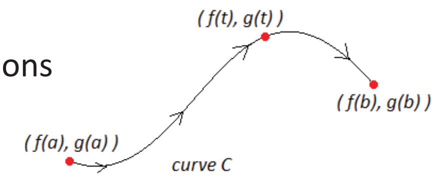
$$\Rightarrow \frac{dy_2}{dx} = 3^x \ln 3$$

$\because\ y = y_1 + y_2$

$\therefore \dfrac{dy}{dx} = \dfrac{dy_1}{dx} + \dfrac{dy_2}{dx} = (\cos x)^x\,[-x\tan x + \ln(\cos x)] + 3^x \ln 3$

Differentiation of Parametric Equations

Suppose that a curve C is described by the parametric equations



$$\begin{cases} x = f(t) \\ y = g(t) \end{cases},\quad t \in [a, b]$$

where $f(t)$ and $g(t)$ are differentiable functions of t, and t is a parameter.

Then the **first derivative** of y w.r.t. x is given by

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}},$$

and the **second derivative** of y w.r.t. x is given by

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left[\frac{g'(t)}{f'(t)}\right]}{f'(t)}}.$$

Remarks:

1. The second derivative of y w.r.t. x is the derivative of $\frac{dy}{dx}$ w.r.t. x.
2. For differentiation of parametric equations, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are usually expressed in terms of the parameter t.

Example 26

Given that $\begin{cases} x = 2t \\ y = t^2 \end{cases}$ where $-\infty < t < \infty$, describes a parabola. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Solution

$x = 2t \quad \Rightarrow \quad \frac{dx}{dt} = 2$

$y = t^2 \quad \Rightarrow \quad \frac{dy}{dt} = 2t$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{2} = t \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(t)}{2} = \frac{1}{2}$$

Remark: The parametric equations $\begin{cases} x = 2t \\ y = t^2 \end{cases}$ represent the parabola $y = \frac{x^2}{4}$.

Then $\frac{dy}{dx} = \frac{2x}{4} = \frac{x}{2}$ $(= t)$ & $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{x}{2}\right) = \frac{1}{2}$

Example 27

Given that $\begin{cases} x = 2\cos t \\ y = 3\sin t \end{cases}$ where $0 \le t < 2\pi$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Solution

$x = 2\cos t \quad \Rightarrow \quad \frac{dx}{dt} = -2\sin t$

$y = 3\sin t \quad \Rightarrow \quad \frac{dy}{dt} = 3\cos t$

Then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\cos t}{-2\sin t} = -\frac{3}{2}\cot t$

and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(-\frac{3}{2}\cot t\right)}{-2\sin t} = \frac{-\frac{3}{2}\left(-\operatorname{cosec}^2 t\right)}{-2\sin t} = -\frac{3}{4}\operatorname{cosec}^3 t \quad (t \ne 0, \pi)$

Higher derivatives

The operation of differentiation takes a differentiable function $f(x)$ and produces a new function $f'(x)$. If $f'(x)$ is also differentiable, we can differentiate $f'(x)$ and produce another function called the second derivative of $f(x)$ and it is denoted by $f''(x)$. We may repeat the process and suppose that $y=f(x)$ is a differentiable function such that $f'(x)$, $f''(x)$, …, up to its $(n-1)^{\text{th}}$ derivative are differentiable. Then the n^{th} derivative of f exists and we denote it by $f^{(n)}(x)$. These are summarized in the following table:

The function $f(x)$	$y=f(x)$
First derivative of $f(x)$ w.r.t. x (i.e. differentiate $f(x)$ once)	$y'=\frac{dy}{dx}=f'(x)$
Second derivative of $f(x)$ w.r.t. x (i.e. differentiate $f(x)$ twice)	$y''=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=f''(x)$

Third derivative of $f(x)$ w.r.t. x (i.e. differentiate $f(x)$ three times)	$y'''=\frac{d^3y}{dx^3}=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=f'''(x)$ (Also denoted as $y^{(3)}$ or $f^{(3)}(x)$.)
$\vdots$	$\vdots$
The n-th derivative of $f(x)$ w.r.t. x (i.e. differentiate $f(x)$ n times, where n is a positive integer)	$y^{(n)}=\frac{d^ny}{dx^n}=\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)=f^{(n)}(x)$ ($\frac{d^ny}{dx^n}$ is also denoted by D^ny.)

Note: $\boxed{f^{(n)}(x)\neq f^n(x)}$

$f^{(n)}(x)$ is the n-th derivative of $f(x)$ w.r.t. x, while $f^n(x)=[f(x)]^n$ is $f(x)$ to the power n.

E.g. $\frac{d^2y}{dx^2}\neq\left(\frac{dy}{dx}\right)^2$, $f^{(3)}(x)\neq f^3(x)=[f(x)]^3$, etc.

Example 28

If $ay^2 + by + c = x$ for any constants $a(\neq 0)$, b and c, show that

$$\frac{d^2y}{dx^2} + 2a\left(\frac{dy}{dx}\right)^3 = 0.$$

Solution

$$ay^2 + by + c = x$$

Differentiate both sides w.r.t. x:

$$a \cdot 2y\frac{dy}{dx} + b\frac{dy}{dx} + 0 = 1 \quad \Rightarrow \quad (2ay + b)\frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2ay + b} \quad \dots (*)$$

Differentiate both sides of $(2ay + b)\frac{dy}{dx} = 1$ w.r.t. x:

$$(2ay + b) \cdot \underbrace{\frac{d}{dx}\left(\frac{dy}{dx}\right)}_{=\frac{d^2y}{dx^2}} + 2a\underbrace{\frac{dy}{dx} \cdot \frac{dy}{dx}}_{=\left(\frac{dy}{dx}\right)^2} = 0$$

$$\Rightarrow \quad (2ay + b)\frac{d^2y}{dx^2} + 2a\left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + 2a\underbrace{\left(\frac{1}{2ay + b}\right)}_{=\frac{dy}{dx} \text{ by } (*)}\left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + 2a\left(\frac{dy}{dx}\right)^3 = 0$$

□

For some simple functions like those in the following examples, we may differentiate the function $y = f(x)$ a few times to obtain $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ etc., and then conjecture the general formula for $\frac{d^ny}{dx^n}$, where $n \in \mathbb{N}$.

Example 29

Let $y = x^3$. Find $\frac{d^n y}{dx^n}$, where n is a positive integer.

Solution

$$y = x^3$$

For $n = 1$: $\frac{dy}{dx} = 3x^2$

For $n = 2$: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3x^2) = 6x$

For $n = 3$: $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}(6x) = 6$

For all integers $n \geq 4$: $\frac{d^n y}{dx^n} = 0.$

Hence, $$\frac{d^n y}{dx^n} = \begin{cases} 3x^2 & , \text{ for } n = 1 \\ 6x & , \text{ for } n = 2 \\ 6 & , \text{ for } n = 3 \\ 0 & , \text{ for } n \geq 4 \end{cases}$$

Example 30

Let $y = x^m$, where m is a positive integer. Find $\frac{d^n y}{dx^n}$, where n is a positive integer.

Solution

$$\frac{dy}{dx} = mx^{m-1}$$

$$\frac{d^2y}{dx^2} = m(m-1)x^{m-2}$$

$$\frac{d^3y}{dx^3} = m(m-1)(m-2)x^{m-3}$$

$$\vdots$$

$$\frac{d^m y}{dx^m} = m(m-1)(m-2)\cdots\underbrace{[m-(m-1)]}_{=1}\underbrace{x^{m-m}}_{=x^0=1} = m!$$

$$\frac{d^n y}{dx^n} = 0 \quad \text{for } n > m.$$

Hence, $$\frac{d^n y}{dx^n} = \begin{cases} m(m-1)(m-2)\cdots[m-(n-1)]\,x^{m-n} & , \text{ if } n \leq m \\ 0 & , \text{ if } n > m \end{cases}$$

Example 31

Find $\frac{d^n}{dx^n}(e^{ax})$, where a is a non-zero constant and n is a positive integer.

Solution

$$\frac{d}{dx}(e^{ax}) = e^{ax} \cdot \frac{d}{dx}(ax) = ae^{ax}$$

$$\frac{d^2}{dx^2}(e^{ax}) = \frac{d}{dx}(ae^{ax}) = ae^{ax} \cdot a = a^2e^{ax}$$

$$\frac{d^3}{dx^3}(e^{ax}) = \frac{d}{dx}(a^2e^{ax}) = a^2e^{ax} \cdot a = a^3e^{ax}$$

$\therefore$ By conjecture,

$$\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}, \qquad n \in \mathbb{N}.$$

Example 32

Find $\frac{d^n}{dx^n}\left(\frac{1}{ax+b}\right)$, where $a \neq 0$ and b are constants and n is a positive integer.

Solution

$$\frac{d}{dx}\left(\frac{1}{ax+b}\right) = \frac{d}{dx}[(ax+b)^{-1}] = (-1)\cdot(ax+b)^{-2}\cdot\underbrace{\frac{d}{dx}(ax+b)}_{=a} = (-1)\cdot a\cdot(ax+b)^{-2}$$

$$\frac{d^2}{dx^2}\left(\frac{1}{ax+b}\right) = (-1)\cdot a\cdot\frac{d}{dx}[(ax+b)^{-2}] = \underbrace{(-1)}_{(-1)\cdot 1}\cdot a\cdot\underbrace{(-2)}_{=(-1)\cdot 2}\cdot(ax+b)^{-3}\cdot\underbrace{\frac{d}{dx}(ax+b)}_{=a}$$

$$= (-1)^2\cdot\underbrace{2!}_{=2\cdot 1}\cdot a^2\cdot(ax+b)^{-3}$$

$$\frac{d^3}{dx^3}\left(\frac{1}{ax+b}\right) = (-1)^2\cdot 2!\cdot a^2\cdot\frac{d}{dx}[(ax+b)^{-3}] = (-1)^2\cdot 2!\cdot a^2\cdot\underbrace{(-3)}_{=(-1)\cdot 3}\cdot(ax+b)^{-4}\cdot a$$

$$= (-1)^3\cdot\underbrace{3!}_{=3\cdot 2\cdot 1}\cdot a^3\cdot(ax+b)^{-4}$$

$\therefore$ By conjecture, $\frac{d^n}{dx^n}\left(\frac{1}{ax+b}\right) = (-1)^n\cdot n!\cdot a^n\cdot(ax+b)^{-(n+1)} = \frac{(-1)^n\, n!\, a^n}{(ax+b)^{n+1}}$

Example 33

Find $\frac{d^n}{dx^n}[\cos(ax+b)]$, where $a \neq 0$ and b are constants and n is a positive integer.

Solution

$$\frac{d}{dx}[\cos(ax+b)] = -\sin(ax+b)\cdot\underbrace{\frac{d}{dx}(ax+b)}_{=a}$$

$$= -a\sin(ax+b)$$

$$= a\cos\left(ax+b+\frac{\pi}{2}\right) \qquad \because \boxed{\cos\left(\theta+\frac{\pi}{2}\right) = -\sin\theta}$$

$$\frac{d^2}{dx^2}[\cos(ax+b)] = \frac{d}{dx}\left[a\cos\left(ax+b+\frac{\pi}{2}\right)\right]$$

$$= -a\sin\left(ax+b+\frac{\pi}{2}\right)\cdot\underbrace{\frac{d}{dx}\left(ax+b+\frac{\pi}{2}\right)}_{=a}$$

$$= a^2\cos\left(ax+b+\frac{\pi}{2}+\frac{\pi}{2}\right) \qquad \because \boxed{\cos\left(\theta+\frac{\pi}{2}\right) = -\sin\theta}$$

$$= a^2\cos\left(ax+b+\frac{2\pi}{2}\right)$$

$$\frac{d^3}{dx^3}[\cos(ax+b)] = \frac{d}{dx}\left[a^2\cos\left(ax+b+\frac{2\pi}{2}\right)\right]$$

$$= -a^2\sin\left(ax+b+\frac{2\pi}{2}\right)\cdot\underbrace{\frac{d}{dx}\left(ax+b+\frac{2\pi}{2}\right)}_{=a}$$

$$= a^3\cos\left(ax+b+\frac{2\pi}{2}+\frac{\pi}{2}\right) \qquad \because \boxed{\cos\left(\theta+\frac{\pi}{2}\right) = -\sin\theta}$$

$$= a^3\cos\left(ax+b+\frac{3\pi}{2}\right)$$

etc.

$$\therefore \quad \frac{d^n}{dx^n}[\cos(ax+b)] = a^n\cos\left(ax+b+\frac{n\pi}{2}\right), \qquad \text{where } n \in \mathbb{N}.$$

Homework: Show that $\frac{d^n}{dx^n}[\sin(ax+b)] = a^n\sin\left(ax+b+\frac{n\pi}{2}\right)$.

(Hint: $\boxed{\sin\left(\theta+\frac{\pi}{2}\right) = \cos\theta}$.)

Example 34

Let $y = \ln(2x+3)$. Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ and $\frac{d^4y}{dx^4}$, and then conjecture the formula for $\frac{d^ny}{dx^n}$, where $n \in \mathbb{N}$.

Solution

More on Higher Derivatives

If f and g are n-times differentiable functions of x, then

$$\boxed{\frac{d^n}{dx^n}[\alpha \cdot f(x) \pm \beta \cdot g(x)] = \alpha \cdot \frac{d^n}{dx^n}[f(x)] \pm \beta \cdot \frac{d^n}{dx^n}[g(x)]} \quad \ldots\ldots (*)$$

for all constants α and β.

Note that $\frac{d^n}{dx^n}[f(x) \cdot g(x)] \neq \left\{\frac{d^n}{dx^n}[f(x)]\right\} \cdot \left\{\frac{d^n}{dx^n}[g(x)]\right\}$.

Question: How to find $\frac{d^n}{dx^n}[f(x) \cdot g(x)]$?

Method 1: For simple functions, decompose $f(x) \cdot g(x)$ into sum or difference of functions in x, then use result (*) and other known results.

Method 2: Use Leibnitz' rule.

Example 35

(a) Resolve $\frac{1}{(x+1)(2x+1)}$ into partial fractions.

(b) Find $\frac{d^n}{dx^n}\left[\frac{1}{(x+1)(2x+1)}\right]$, where $n \in \mathbb{N}$.

Solution

(a)

(b)

Example 36

Find $\frac{d^n}{dx^n}(\cos^2 x)$, where $n \in \mathbb{N}$.

Solution

Leibnitz' Rule

This is used to determine the **n-th derivative of a product of two functions of x**.

Let $y = (fg)(x) = f(x) \cdot g(x)$, where f and g are n-times differentiable functions. Then the n-th derivative of fg is:

$$\boxed{\begin{aligned}(fg)^{(n)}(x) &= \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) \cdot g^{(n-k)}(x) \\ &= \binom{n}{0} f^{(0)}(x)\ g^{(n)}(x) + \binom{n}{1} f^{(1)}(x)\ g^{(n-1)}(x) \\ &\quad + \binom{n}{2} f^{(2)}(x)\ g^{(n-2)}(x) + \cdots + \binom{n}{n} f^{(n)}(x)\ g^{(0)}(x),\end{aligned}}$$

where $f^{(k)}(x) = \frac{d^k}{dx^k}[f(x)]$, $f^{(0)}(x) = f(x)$,

$g^{(n-k)}(x) = \frac{d^{n-k}}{dx^{n-k}}[g(x)]$, $g^{(0)}(x) = g(x)$,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)\cdot(n-k)!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!},$$

$$k! = k(k-1)(k-2)\cdot\dots\cdot 3\cdot 2\cdot 1 \quad \text{for } k \in \mathbb{N},$$

and $0! = 1$ (by definition).

Compare the Leibnitz' rule with the Binomial Theorem.

Recall the **Binomial Theorem**:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k\, b^{n-k}$$

$$= \binom{n}{0} a^0\, b^n + \binom{n}{1} a^1\, b^{n-1} + \binom{n}{2} a^2\, b^{n-2} + \cdots + \binom{n}{n} a^n\, b^0$$

Note that $f^{(k)}(x) \neq f^k(x)$.

E.g. $f^{(0)}(x) = f(x)$ but $f^0(x) = [f(x)]^0 = 1$, where $f(x)$ is not identically equal to 0.

Example 37

If $y = e^{2x}(x^3 + 5x - 1)$, find $\frac{d^{10}y}{dx^{10}}$.

Solution

By the **Leibnitz' rule**,

$$\frac{d^{10}y}{dx^{10}} = \sum_{k=0}^{10} \binom{10}{k} (x^3+5x-1)^{(k)}\ (e^{2x})^{(10-k)}$$

$$= \binom{10}{0} (x^3+5x-1)^{(0)}\ (e^{2x})^{(10)} + \binom{10}{1} (x^3+5x-1)^{(1)}\ (e^{2x})^{(9)}$$

$$+ \binom{10}{2} (x^3+5x-1)^{(2)}\ (e^{2x})^{(8)} + \binom{10}{3} (x^3+5x-1)^{(3)}\ (e^{2x})^{(7)}$$

$$+ \binom{10}{4} (x^3+5x-1)^{(4)}\ (e^{2x})^{(6)} + \cdots + \binom{10}{10} (x^3+5x-1)^{(10)}\ (e^{2x})^{(0)}$$

$$= 1\cdot(x^3+5x-1)\cdot 2^{10}\, e^{2x} + 10\cdot(3x^2+5)\cdot 2^9\, e^{2x} + 45\cdot(6x)\cdot 2^8\, e^{2x}$$

$$+ 120\cdot(6)\cdot 2^7\, e^{2x}$$ **using** $\boxed{\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}}$ **(Example 31)**

$+\mathbf{210}\cdot(\mathbf{0})\cdot\mathbf{2^6}\,e^{2x}+\cdots$ **← all the remaining terms are 0,**

since $\frac{d^n}{dx^n}\left(x^3+5x-1\right)=0$ **for** $n\geq 4$

$$=2^7\,e^{2x}[2^3\cdot(x^3+5x-1)+10\cdot(3x^2+5)\cdot 2^2+45\cdot(6x)\cdot 2+120\cdot(6)]$$
$$=2^7\,e^{2x}\,(8x^3+120x^2+580x+912)$$
$$=2^9\,e^{2x}\,(2x^3+30x^2+145x+228)$$

Remark:

We take $f(x)=x^3+5x-1$ and $g(x)=e^{2x}$ so that the first $1+3=4$ terms are non-zero (i.e. $f^{(k)}(x)\neq 0$ when $k=0,1,2,3$) and all the remaining terms are zeros.

If we take $f(x)=e^{2x}$ and $g(x)=x^3+5x-1$, then the last 4 terms are non-zero and all the remaining terms are zeros.

Example 38

Given that $y=(2x^2+3x-7)\cos(3x+2)$. Find $\frac{d^ny}{dx^n}$, where $n\in\mathbb{N}$.

Solution

By using the **Leibnitz' rule**,

$$\frac{d^ny}{dx^n}=\sum_{k=0}^{n}\binom{n}{k}\,(2x^2+3x-7)^{(k)}\;[\cos(3x+2)]^{(n-k)}$$

Example 39 (This is hard!)

Given that $y = e^{\sin^{-1} x}$.

(a) Show that $(1-x^2)\,\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} - y = 0 \quad \ldots\ldots \ (*)$.

(b) Using part (a) and the Leibnitz' rule, show that

$$(1-x^2)\,y^{(n+2)} - (2n+1)x\,y^{(n+1)} - (n^2+1)\,y^{(n)} = 0\,,$$

where $y^{(k)} = \frac{d^k y}{dx^k}$.

Solution

(a) $y = e^{\sin^{-1} x}$

$$\frac{dy}{dx} = e^{\sin^{-1} x} \cdot \frac{d}{dx}(\sin^{-1} x) = e^{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \sqrt{1-x^2}\,\frac{dy}{dx} = e^{\sin^{-1} x}$$

$$\Rightarrow \quad (1-x^2)^{\frac{1}{2}}\,\frac{dy}{dx} = y \quad \ldots\ldots (**)$$

Differentiate both sides of $(**)$ w.r.t x:

$$\underbrace{(1-x^2)^{\frac{1}{2}} \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx} \cdot \frac{d}{dx}\left[(1-x^2)^{\frac{1}{2}}\right]}_{by\ product\ rule} = \frac{d}{dx}(y)$$

$$\Rightarrow\ (1-x^2)^{\frac{1}{2}} \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \underbrace{\frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot \frac{d}{dx}(1-x^2)}_{by\ chain\ rule} = \frac{dy}{dx}$$

$$\Rightarrow\ (1-x^2)^{\frac{1}{2}} \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (-2x) = \frac{dy}{dx}$$

Multiply both sides by $(1-x^2)^{\frac{1}{2}}$:

$$(1-x^2) \cdot \frac{d^2y}{dx^2} - x\ \frac{dy}{dx} = \underbrace{(1-x^2)^{\frac{1}{2}}\ \frac{dy}{dx}}_{=y,\ \ from\ (**)}$$

$$\therefore\ \ (1-x^2)\ \frac{d^2y}{dx^2} - x\ \frac{dy}{dx} - y = 0 \quad \dots\dots (*)$$

(b) Use the Leibnitz' rule to differentiate both sides of $(*)$ n times w.r.t. x:

$$[(1-x^2)\,y'']^{(n)} - (x\,y')^{(n)} - y^{(n)} = (0)^{(n)}$$

$$\Rightarrow\ \left[\sum_{k=0}^{n}\binom{n}{k}\ (1-x^2)^{(k)}\ \ (y'')^{(n-k)}\right] - \left[\sum_{k=0}^{n}\binom{n}{k}\ (x)^{(k)}\ \ (y')^{(n-k)}\right] - y^{(n)} = 0$$

$$\Rightarrow\ \left[\tbinom{n}{0}(1-x^2)^{(0)}\ (y'')^{(n)} + \tbinom{n}{1}(1-x^2)^{(1)}\ (y'')^{(n-1)} + \tbinom{n}{2}(1-x^2)^{(2)}\ (y'')^{(n-2)} + 0\right]$$

$$-\left[\tbinom{n}{0}\,(x)^{(0)}\ \ (y')^{(n)} + \tbinom{n}{1}\,(x)^{(1)}\ \ (y')^{(n-1)} + 0\right] - y^{(n)} = 0$$

$$\Rightarrow\ \left[1 \cdot (1-x^2) \cdot y^{(n+2)} + n \cdot (-2x) \cdot y^{(n+1)} + \tfrac{n(n-1)}{2} \cdot (-2) \cdot y^{(n)}\right]$$

$$-\left[1 \cdot x \cdot y^{(n+1)} + n \cdot (1) \cdot y^{(n)}\right] - y^{(n)} = 0$$

$$\Rightarrow\ (1-x^2)\ \ y^{(n+2)} + (-2nx - x)\ \ y^{(n+1)} + [-n(n-1) - n - 1]\ \ y^{(n)} = 0$$

Hence,

$$(1-x^2)\,y^{(n+2)} - (2n+1)x\,y^{(n+1)} - (n^2+1)\,y^{(n)} = 0.$$