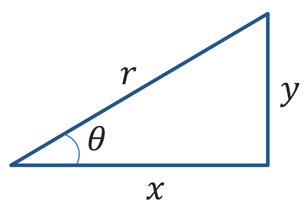
MA1200 CALCULUS AND BASIC LINEAR ALGEBRA I

LECTURE: CG1

Chapter 4
Trigonometric Functions
and Inverse Trigonometric Functions

Trigonometric Functions

In elementary trigonometry, the 3 basic trigonometric functions $(\sin\theta, \cos\theta, \tan\theta)$ are defined as the ratios of sides of a right-angled triangle, and the angles θ are restricted to acute angles, i.e. $0° \le \theta < 90°$



y = Opposite side

x = Adjacent side

r = Hypotenuse

By definition, $\sin\theta = \frac{opp.}{hyp.} = \frac{y}{r}$, $\cos\theta = \frac{adj.}{hyp.} = \frac{x}{r}$ and $\tan\theta = \frac{opp.}{adj.} = \frac{y}{x}$.

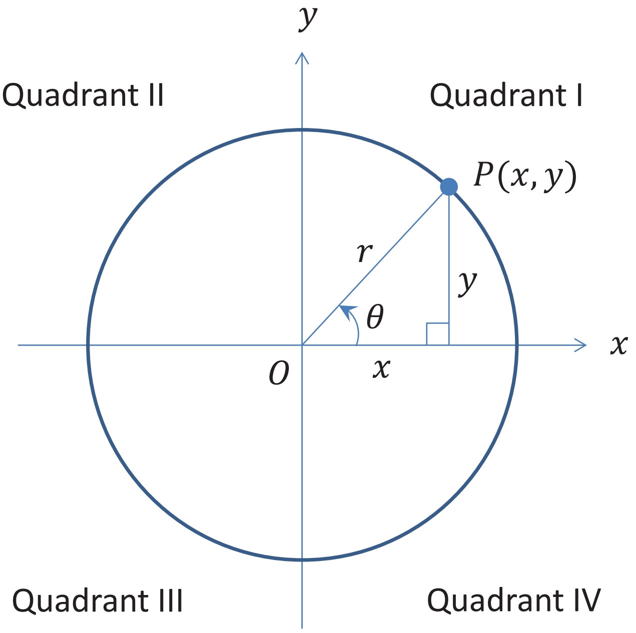
The special angles of sine, cosine and tangent are summarized below.

	$30°$	$45°$	$60°$
sin	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

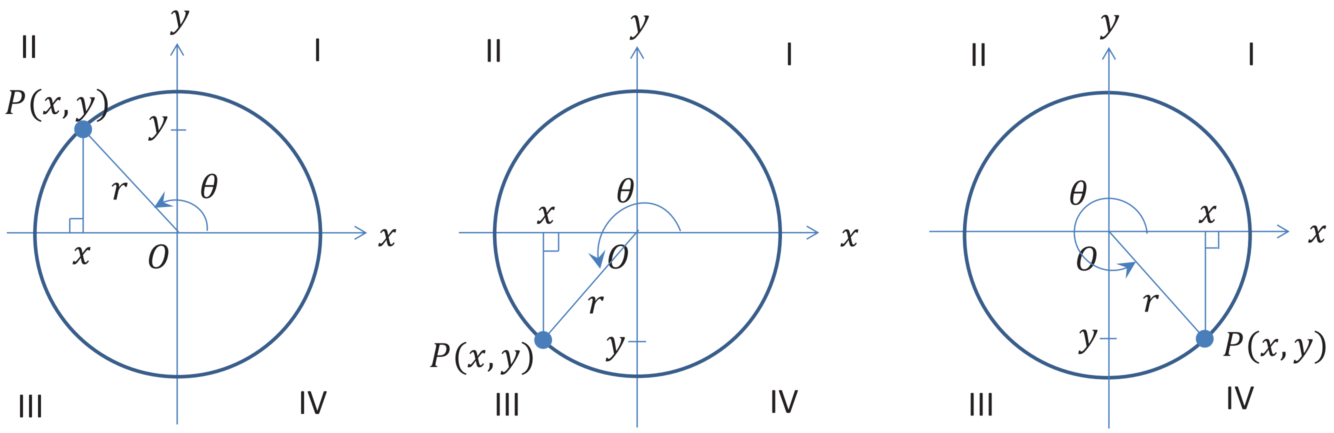
The six trigonometric functions are **sine**, **cosine**, **tangent**, **cosecant**, **secant** and **cotangent**, which are written as **sin**, **cos**, **tan**, **csc** (or **cosec**), **sec** and **cot**, respectively.

They are defined as follows:

$$\sin\theta = \frac{y}{r}$$
$$\cos\theta = \frac{x}{r}$$
$$\tan\theta = \frac{y}{x}, \quad \text{i.e. } \tan\theta = \frac{\sin\theta}{\cos\theta}$$
$$\csc\theta = \frac{r}{y}, \quad \text{i.e. } \csc\theta = \frac{1}{\sin\theta}$$
$$\sec\theta = \frac{r}{x}, \quad \text{i.e. } \sec\theta = \frac{1}{\cos\theta}$$
$$\cot\theta = \frac{x}{y}, \quad \text{i.e. } \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$



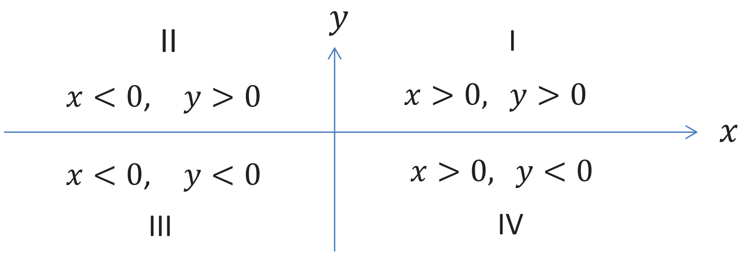
The above results are also true when the point P lies in other quadrants.



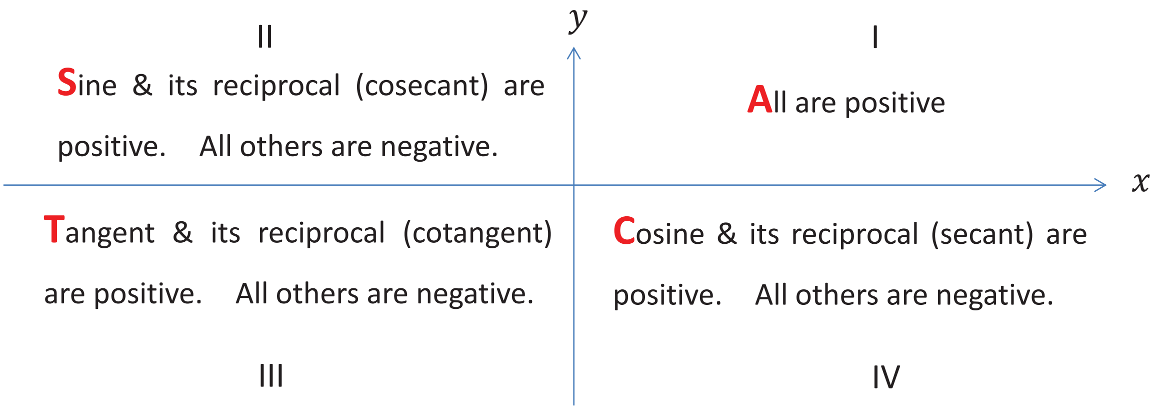
Here,

- P is a point in the xy-plane with Cartesian coordinates (x, y).
- θ is the angle measured from the positive x-axis to the line OP in anticlockwise direction.
  (θ is positive if the angle is measured in anticlockwise direction; and it is negative if the angle is measured in clockwise direction.)
- $r = \sqrt{x^2 + y^2}$ (> 0) is the distance from the origin O to the point P.

The signs of x and y depend on the quadrant in which the point P lies.



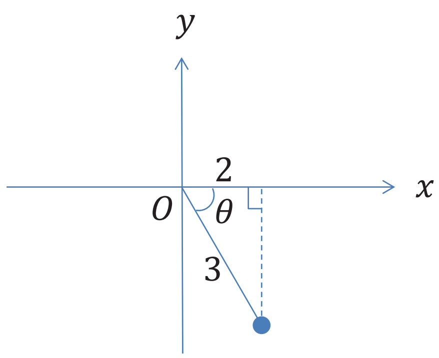
By using the definitions of the six trigonometric functions and also the fact that r is always positive, the signs of the trigonometric functions can be deduced and the results are summarized by the **CAST rule**:



Example 1

If $\cos\theta = \frac{2}{3}$ and θ is in Quadrant IV, find $\tan\theta$.

Solution



Since θ is in Quadrant IV, x must be positive and y must be negative.

$\cos\theta = \frac{x}{r}$ is the ratio of x to r.

Take $x = 2$ and $r = 3$. Then $y = -\sqrt{r^2 - x^2} = -\sqrt{3^2 - 2^2} = -\sqrt{5}$.

Thus, $\tan\theta = \frac{y}{x} = \frac{-\sqrt{5}}{2}$.

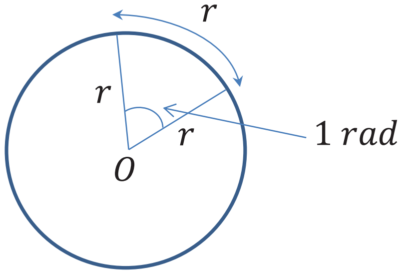
Example 2

If $\sin\theta = -\frac{3}{5}$ and $180^\circ < \theta < 270^\circ$, find $\sec\theta$ and $\cot\theta$.

Solution

Radian Measures

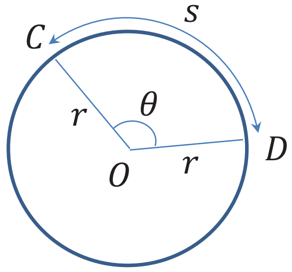
A **radian measure**, denoted by **rad**, is the measure of an angle subtended at the centre of a circle by an arc with length equal to its radius.



Let $r =$ radius of the circle;

$s =$ length of the arc CD; and

$\theta = \angle COD$, measured in radians.



Then θ is the ratio of the arc length to the radius of the circle,

i.e. $\theta = \frac{s}{r}$.

Recall that the circumference of a circle is $2\pi r$.

If $\theta = 360°$ (i.e. 360 degrees), then the arc length is $s = 2\pi r$.

Thus, if θ is measured in radians, we have $\theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi$ rad.

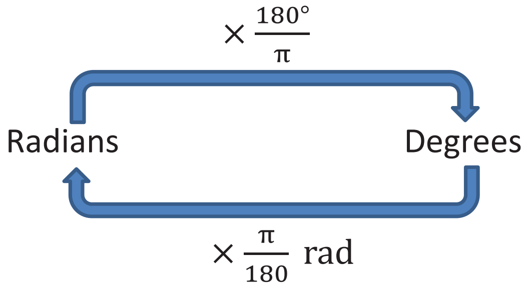
That is, $\theta = 360° = 2\pi \text{ rad} \quad \Rightarrow \quad \pi \text{ rad} = 180°$.

Therefore, we obtain

$$1° = \frac{\pi}{180} \text{ rad} \approx 0.017453 \text{ rad}$$

and $$1 \text{ rad} = \frac{180}{\pi} \text{ degrees} \approx 57.296°.$$

<u>Conversion between radians and degrees</u>:



<u>Example 3</u>

Convert the following angles from degrees to radians.

(a) $120°$ (b) $45°$ (c) $-720°$

<u>Solution</u>

(a) $120° = 120 \times \frac{\pi}{180} \text{ rad} = \frac{2\pi}{3} \text{ rad}$ (b) $45° = 45 \times \frac{\pi}{180} \text{ rad} = \frac{\pi}{4} \text{ rad}$

(c) $-720° = -720 \times \frac{\pi}{180} \text{ rad} = -4\pi \text{ rad}$

<u>Example 4</u>

Convert the following angles from radians to degrees.

(a) $\frac{\pi}{2}$ rad (b) $\frac{5\pi}{4}$ rad (c) $-\frac{5\pi}{6}$ rad

<u>Solution</u>

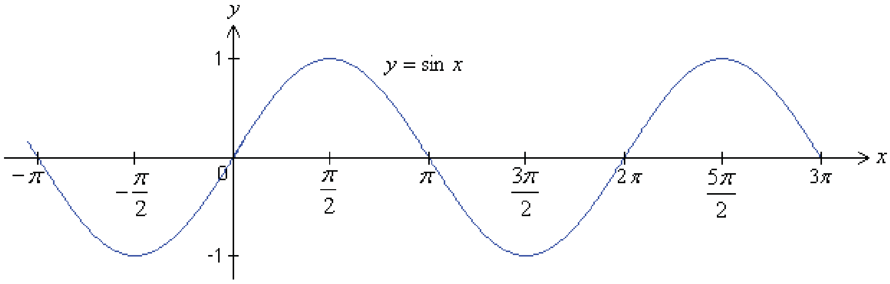
(a) $\frac{\pi}{2} \text{ rad} = \frac{\pi}{2} \times \frac{180°}{\pi} = 90°$ (b) $\frac{5\pi}{4} \text{ rad} = \frac{5\pi}{4} \times \frac{180°}{\pi} = 225°$

(c) $-\frac{5\pi}{6} \text{ rad} = -\frac{5\pi}{6} \times \frac{180°}{\pi} = -150°$

Graphs of Trigonometric Functions

The graphs of the six trigonometric functions are shown below. Note that x is measured in radians in the following graphs.

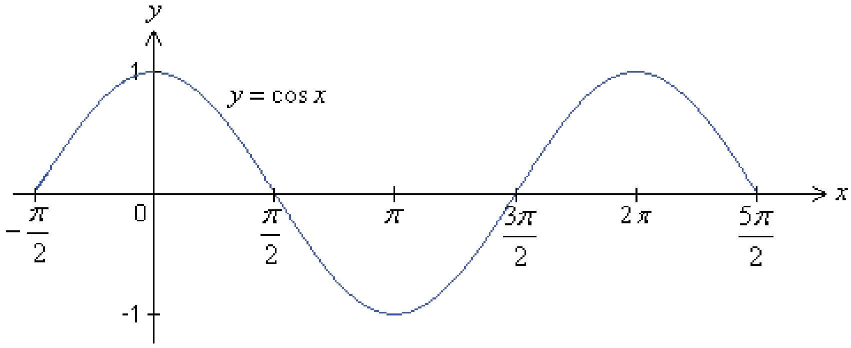
1. $f(x) = \sin x$



- $Dom(f) = \mathbb{R}$
- $Ran(f) = [-1, 1]$
- $f(x) = \sin x$ is an odd function, i.e. $f(-x) = -f(x)$, i.e. $\sin(-x) = -\sin x$.
- $f(x) = \sin x$ is a periodic function with period 2π, i.e. $f(x + 2\pi) = f(x)$,

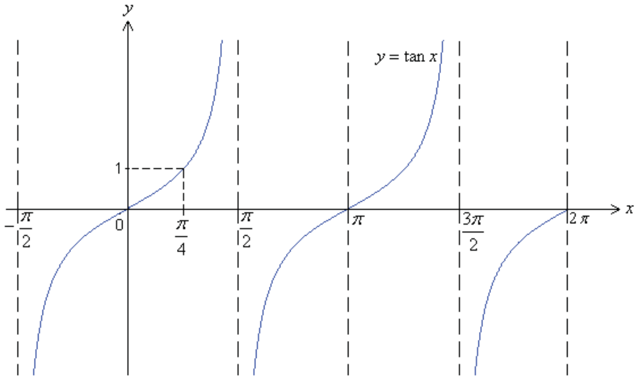
  i.e. $\sin(x + 2\pi) = \sin x$.

2. $f(x) = \cos x$



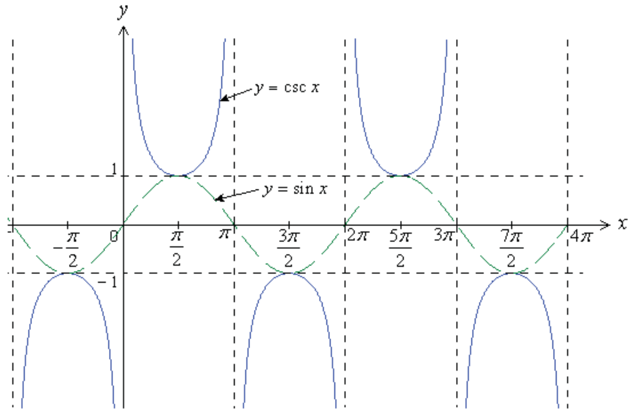
- $Dom(f) = \mathbb{R}$
- $Ran(f) = [-1, 1]$
- $f(x) = \cos x$ is an even function, i.e. $f(-x) = f(x)$, i.e. $\cos(-x) = \cos x$.
- $f(x) = \cos x$ is a periodic function with period 2π, i.e. $\cos(x + 2\pi) = \cos x$.

3. $f(x) = \tan x \left(= \frac{\sin x}{\cos x}\right)$



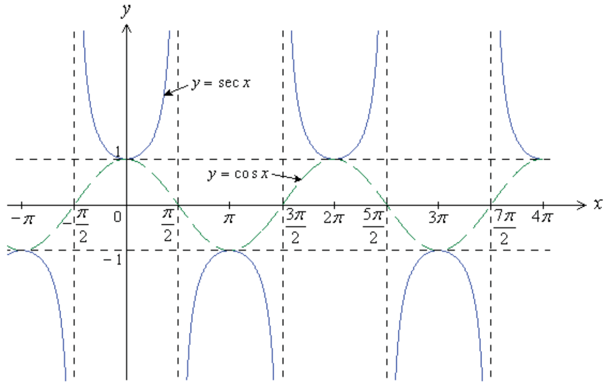
- $f(x) = \tan x = \frac{\sin x}{\cos x}$ is <u>not</u> defined when $\cos x = 0$,
  i.e. when $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots = \frac{(2n+1)\pi}{2}$ for $n \in \mathbb{Z}$.
  $\therefore\ Dom(f) = \mathbb{R}\backslash\left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\right\} = \mathbb{R}\backslash\left\{x \in \mathbb{R} \middle| x = \frac{(2n+1)\pi}{2} \text{ for } n \in \mathbb{Z}\right\}$
- $Ran(f) = \mathbb{R}$
- $f(x) = \tan x$ is an odd function, i.e. $\tan(-x) = -\tan x$.
- $f(x) = \tan x$ is a periodic function with period π, i.e. $\tan(x+\pi) = \tan x$.

4. $f(x) = \csc x \left(= \frac{1}{\sin x}\right)$



- $f(x) = \csc x = \frac{1}{\sin x}$ is <u>not</u> defined when $\sin x = 0$,
  i.e. when $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots = n\pi$ for $n \in \mathbb{Z}$.
  $\therefore\ Dom(f) = \mathbb{R}\backslash\{0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots\} = \mathbb{R}\backslash\{x \in \mathbb{R} | x = n\pi \text{ for } n \in \mathbb{Z}\}$
- $Ran(f) = (-\infty, -1] \cup [1, \infty)$
- $f(x) = \csc x$ is an odd function, i.e. $\csc(-x) = -\csc x$.
- $f(x) = \csc x$ is a periodic function with period 2π, i.e. $\csc(x + 2\pi) = \csc x$.

5. $f(x) = \sec x \ \left(= \frac{1}{\cos x}\right)$

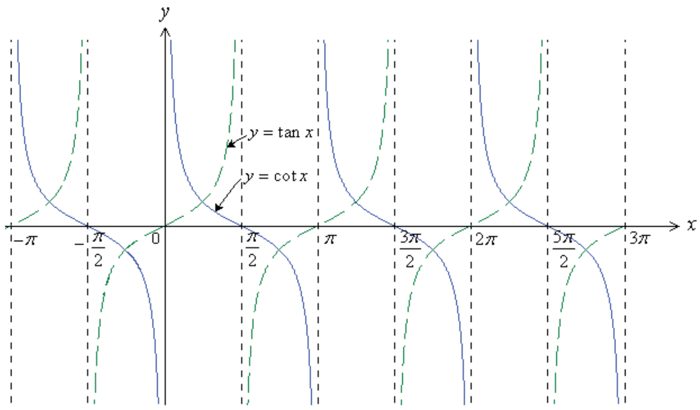


- $f(x) = \sec x = \frac{1}{\cos x}$ is <u>not</u> defined when $\cos x = 0$,

  i.e. when $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots = \frac{(2n+1)\pi}{2}$ for $n \in \mathbb{Z}$.

  $\therefore\ Dom(f) = \mathbb{R}\backslash\left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\right\} = \mathbb{R}\backslash\left\{x \in \mathbb{R} | x = \frac{(2n+1)\pi}{2} \text{ for } n \in \mathbb{Z}\right\}$
- $Ran(f) = (-\infty, -1] \cup [1, \infty)$
- $f(x) = \sec x$ is an even function, i.e. $\sec(-x) = \sec x$.
- $f(x) = \sec x$ is a periodic function with period 2π, i.e. $\sec(x + 2\pi) = \sec x$.

6. **$f(x) = \cot x \ \left(= \frac{\cos x}{\sin x} = \frac{1}{\tan x}\right)$**



- $f(x) = \cot x = \frac{\cos x}{\sin x}$ is <u>not</u> defined when $\sin x = 0$,

  i.e. when $x = 0, \pm\pi, \pm2\pi, \pm3\pi, \ldots = n\pi$ for $n \in \mathbb{Z}$.

  $\therefore\ Dom(f) = \mathbb{R}\backslash\{0, \pm\pi, \pm2\pi, \pm3\pi, \ldots\} = \mathbb{R}\backslash\{x \in \mathbb{R} | x = n\pi \text{ for } n \in \mathbb{Z}\}$
- $Ran(f) = \mathbb{R}$
- $f(x) = \cot x$ is an odd function, i.e. $\cot(-x) = -\cot x$.
- $f(x) = \cot x$ is a periodic function with period π, i.e. $\cot(x + \pi) = \cot x$.

Example 5

Consider the function $f(x) = -3\cos\left(2x - \frac{\pi}{2}\right) + 1$.

(a) Sketch the graph of $f(x)$ from $x = -\pi$ to $x = 2\pi$.

(b) State the largest possible domain and largest possible range of $f(x)$.

(c) State the period of $f(x)$.

Solution

(a) Consider the function $g(x) = \cos x$. We perform a sequence of transformations to obtain the graph of $f(x) = -3\cos\left(2x - \frac{\pi}{2}\right) + 1$.

Step 1: The graph of $g(x)$ is shifted $\frac{\pi}{2}$ units to the right.

Function obtained: $g_1(x) = g\left(x - \frac{\pi}{2}\right) = \cos\left(x - \frac{\pi}{2}\right)$

Step 2: The graph of $g_1(x)$ is compressed horizontally by a factor of 2.

Function obtained: $g_2(x) = g_1(2x) = \cos\left(2x - \frac{\pi}{2}\right)$

Step 3: The graph of $g_2(x)$ is reflected about the x-axis.

Function obtained: $g_3(x) = -g_2(x) = -\cos\left(2x - \frac{\pi}{2}\right)$

Step 4: The graph of $g_3(x)$ is stretched vertically by a factor of 3.

Function obtained: $g_4(x) = 3\,g_3(x) = -3\cos\left(2x - \frac{\pi}{2}\right)$

Step 5: The graph of $g_4(x)$ is shifted 1 unit upward.

Function obtained: $g_5(x) = g_4(x) + 1 = -3\cos\left(2x - \frac{\pi}{2}\right) + 1 = f(x)$

Sketch:

(b) $\mathrm{Dom}(f) =$, $\mathrm{Ran}(f) =$

(c) The period of $f(x)$ is .

Trigonometric Identities

➢ Basic identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}, \qquad \csc\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\cos(-\theta) = \cos\theta, \qquad \sin(-\theta) = -\sin\theta, \qquad \tan(-\theta) = -\tan\theta$$

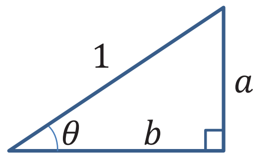
since $\cos\theta$ is an even function; whereas $\sin\theta$ and $\tan\theta$ are odd functions.

➢ Other useful identities

$$\sin^2\theta + \cos^2\theta = 1, \qquad 1 + \tan^2\theta = \sec^2\theta, \qquad 1 + \cot^2\theta = \csc^2\theta$$

Proof (for reference):

Consider the right-angled triangle as shown on the right:



Then $\sin\theta = \frac{a}{1} \quad \Rightarrow \quad a = \sin\theta$

and $\cos\theta = \frac{b}{1} \quad \Rightarrow \quad b = \cos\theta.$

By Pythagoras Theorem, $a^2 + b^2 = 1^2 \quad \Rightarrow \quad \boxed{\sin^2\theta + \cos^2\theta = 1}$,

where $\sin^2\theta = (\sin\theta)^2$ and $\cos^2\theta = (\cos\theta)^2$.

Dividing both sides of $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$, we have

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \quad \Rightarrow \quad \left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1 = \left(\frac{1}{\cos\theta}\right)^2 \quad \Rightarrow \quad \boxed{\tan^2\theta + 1 = \sec^2\theta}.$$

Dividing both sides of $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$, we have

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \quad \Rightarrow \quad 1 + \left(\frac{\cos\theta}{\sin\theta}\right)^2 = \left(\frac{1}{\sin\theta}\right)^2 \quad \Rightarrow \quad \boxed{1 + \cot^2\theta = \csc^2\theta}. \qquad \square$$

➢ Compound Angle Formulae

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

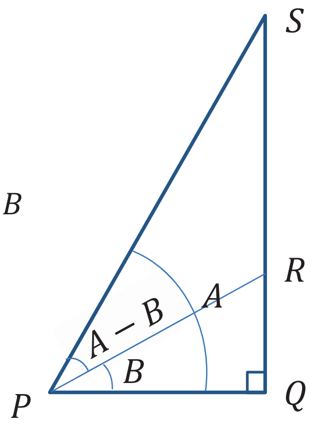
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Note that in general, $\sin(A+B) \neq \sin A + \sin B$, $\cos(A-B) \neq \cos A - \cos B$, etc.

Proof (for reference):

Consider the right-angled triangle as shown on the right:



$$\text{Area of } \Delta PRS = \text{Area of } \Delta PQS - \text{Area of } \Delta PQR$$

$$\Rightarrow \quad \tfrac{1}{2}(PR)(PS)\sin(A-B) = \tfrac{1}{2}(PQ)(PS)\sin A - \tfrac{1}{2}(PQ)(PR)\sin B$$

$$\Rightarrow \quad \sin(A-B) = \underbrace{\frac{PQ}{PR}}_{=\cos B}\sin A - \underbrace{\frac{PQ}{PS}}_{=\cos A}\sin B$$

$\therefore$ $\boxed{\sin(A-B) = \sin A \cos B - \cos A \sin B}$ ...... (1)

Replace B with $-B$ in (1):

$$\sin(A+B) = \sin A \underbrace{\cos(-B)}_{=\cos B} - \cos A \underbrace{\sin(-B)}_{=-\sin B} = \sin A \cos B + \cos A \sin B$$

$\therefore$ $\boxed{\sin(A+B) = \sin A \cos B + \cos A \sin B}$ ...... (2)

The proofs of the results for $\cos(A \pm B)$ *and* $\tan(A \pm B)$ *are omitted.* □

➢ **Trigonometric Functions of $(90°n \pm \theta)$ or $\left(\frac{n\pi}{2} \pm \theta\right)$**

For $n = 1$: Trigonometric Functions of $(90° \pm \theta)$ or $\left(\frac{\pi}{2} \pm \theta\right)$:

	$90° - \theta$ $\left(or\ \frac{\pi}{2} - \theta\right)$	$90° + \theta$ $\left(or\ \frac{\pi}{2} + \theta\right)$
sin	$\cos\theta$	$\cos\theta$
cos	$\sin\theta$	$-\sin\theta$
tan	$\cot\theta$	$-\cot\theta$

For $n = 2$: Trigonometric Functions of $(180° \pm \theta)$ or $(\pi \pm \theta)$:

	$180° - \theta$ $(or\ \pi - \theta)$	$180° + \theta$ $(or\ \pi + \theta)$
sin	$\sin\theta$	$-\sin\theta$
cos	$-\cos\theta$	$-\cos\theta$
tan	$-\tan\theta$	$\tan\theta$

For $n = 3$: Trigonometric Functions of $(270° \pm \theta)$ or $\left(\frac{3\pi}{2} \pm \theta\right)$:

	$270° - \theta$ $\left(or\ \frac{3\pi}{2} - \theta\right)$	$270° + \theta$ $\left(or\ \frac{3\pi}{2} + \theta\right)$
$\sin$	$-\cos\theta$	$-\cos\theta$
$\cos$	$-\sin\theta$	$\sin\theta$
$\tan$	$\cot\theta$	$-\cot\theta$

For $n = 4$: Trigonometric Functions of $(360° \pm \theta)$ or $(2\pi \pm \theta)$:

	$360° - \theta$ $(or\ 2\pi - \theta)$	$360° + \theta$ $(or\ 2\pi + \theta)$
$\sin$	$-\sin\theta$	$\sin\theta$
$\cos$	$\cos\theta$	$\cos\theta$
$\tan$	$-\tan\theta$	$\tan\theta$

Remarks:

1. The results for $450° \pm \theta = (360° + 90°) \pm \theta$ are the same as the corresponding results for $90° \pm \theta$.

   Similarly, the results for $540° \pm \theta = (360° + 180°) \pm \theta$ are the same as the corresponding results for $180° \pm \theta$, etc.

2. The results for $\csc(90°n \pm \theta)$, $\sec(90°n \pm \theta)$ and $\cot(90°n \pm \theta)$ can be deduced from the fact that $\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$, together with the results from the tables on the previous 2 pages.

   For example, $\csc(90° - \theta) = \frac{1}{\sin(90°-\theta)} = \frac{1}{\cos\theta} = \sec\theta$,

   $$\sec(270° + \theta) = \frac{1}{\cos(270°+\theta)} = \frac{1}{\sin\theta} = \csc\theta,$$

   $$\cot(180° - \theta) = \frac{1}{\tan(180°-\theta)} = \frac{1}{-\tan\theta} = -\cot\theta, \quad \text{etc.}$$

Proof (for reference)

E.g. By using the compound angle formulae

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

and $$\cos(A+B) = \cos A\cos B - \sin A\sin B,$$

we obtain $\sin(90°+\theta) = \underbrace{\sin 90°}_{=1}\cos\theta + \underbrace{\cos 90°}_{=0}\sin\theta = \cos\theta$

and $\cos(90°+\theta) = \underbrace{\cos 90°}_{=0}\cos\theta - \underbrace{\sin 90°}_{=1}\sin\theta = -\sin\theta.$

Thus, $\tan(90°+\theta) = \frac{\sin(90°+\theta)}{\cos(90°+\theta)} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta.$

Similar method can be used to find the other results.

To proof the other results, we will use the following results of sine and cosine as well:

$$\sin 90° = 1, \quad \sin 180° = 0, \quad \sin 270° = -1, \quad \sin 360° = 0$$

$$\cos 90° = 0, \quad \cos 180° = -1, \quad \cos 270° = 0, \quad \cos 360° = 1$$

The proofs of the other results are omitted. □

➢ Double Angle Formulae

$$\boxed{\begin{aligned} \sin 2A &= 2\sin A\cos A \\ \cos 2A &= \cos^2 A - \sin^2 A \end{aligned}}$$

Proof (for reference)

By putting $B = A$ into the compound angle formula

$$\sin(A+B) = \sin A\cos B + \cos A\sin B,$$

we get $\sin(A+A) = \sin A\cos A + \cos A\sin A = 2\ \sin A\cos A,$

i.e. $\boxed{\sin(2A) = 2\ \sin A\cos A}$.

Similarly, by putting $B = A$ into the compound angle formula

$$\cos(A+B) = \cos A\cos B - \sin A\sin B,$$

we get $\cos(A+A) = \cos A\cos A - \sin A\sin A,$

i.e. $\boxed{\cos 2A = \cos^2 A - \sin^2 A}$. □

➢ Half Angle Formulae

$$\boxed{\begin{aligned} \cos^2 A &= \frac{1}{2}(1+\cos 2A) \\ \sin^2 A &= \frac{1}{2}(1-\cos 2A) \end{aligned}}$$

By replacing A with $\frac{A}{2}$, we obtain the following results:

$$\boxed{\begin{aligned} \cos^2 \frac{A}{2} &= \frac{1}{2}(1+\cos A) \\ \sin^2 \frac{A}{2} &= \frac{1}{2}(1-\cos A) \end{aligned}}$$

Proof (for reference)

Consider the double angle formula

$$\cos 2A = \cos^2 A - \underbrace{\sin^2 A}_{=1-\cos^2 A} = 2\cos^2 A - 1 \quad \Rightarrow \quad \boxed{\cos^2 A = \tfrac{1}{2}(1+\cos 2A)}$$

Also, $\cos 2A = \underbrace{\cos^2 A}_{=1-\sin^2 A} - \sin^2 A = 1 - 2\sin^2 A \quad \Rightarrow \quad \boxed{\sin^2 A = \tfrac{1}{2}(1-\cos 2A)}$ □

➢ Product-to-Sum Formulae

$$\boxed{\begin{aligned} \sin A \cos B &= \frac{1}{2}[\sin(A+B)+\sin(A-B)] \\ \cos A \sin B &= \frac{1}{2}[\sin(A+B)-\sin(A-B)] \\ \cos A \cos B &= \frac{1}{2}[\cos(A+B)+\cos(A-B)] \\ \sin A \sin B &= -\frac{1}{2}[\cos(A+B)-\cos(A-B)] \end{aligned}}$$

Proof (for reference)

Consider the compound angle formulae $\sin(A+B) = \sin A\cos B + \cos A \sin B$ ... (1)

and $\sin(A-B) = \sin A\cos B - \cos A \sin B$ ... (2).

(1) + (2) gives $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$

$\Rightarrow \boxed{\sin A\cos B = \tfrac{1}{2}[\sin(A+B)+\sin(A-B)]}$

Similar method can be used to prove the other results. □

➢ **Sum-to-Product Formulae**

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Proof (for reference)

By putting $A = \frac{x+y}{2}$ and $B = \frac{x-y}{2}$ into the product-to-sum formula

$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$, we obtain

$$\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) = \frac{1}{2}\left\{\sin\left[\left(\frac{x+y}{2}\right)+\left(\frac{x-y}{2}\right)\right] + \sin\left[\left(\frac{x+y}{2}\right)-\left(\frac{x-y}{2}\right)\right]\right\} = \frac{1}{2}(\sin x + \sin y)$$

$$\Rightarrow \boxed{\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}.$$

Similar method can be used to prove the other results. □

Example 6

Evaluate each of the following without using calculators. Leave your answers in surd form.

(a) $\sin\frac{5\pi}{6}$ (b) $\tan\left(-\frac{3\pi}{4}\right)$ (c) $\sec(-75°)$

Solution

(a) $\sin\frac{5\pi}{6} = \sin\left(\pi - \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. (Note that $\frac{\pi}{6}\ rad. = 30°$.)

(b) $\tan\left(-\frac{3\pi}{4}\right) = -\tan\left(\frac{3\pi}{4}\right) = -\tan\left(\pi - \frac{\pi}{4}\right) = -\left[-\tan\left(\frac{\pi}{4}\right)\right] = 1$

(Note that $\frac{\pi}{4}\ rad. = 45°$.)

(c) $\sec(-75°) = \frac{1}{\cos(-75°)} = \frac{1}{\cos(75°)} = \frac{1}{\cos(45°+30°)}$

$= \frac{1}{\cos(45°)\cos(30°) - \sin(45°)\sin(30°)}$, by using the **compound angle formula**

$= \frac{1}{\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)} = \frac{1}{\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}} = \frac{4}{\sqrt{6}-\sqrt{2}} = \frac{4(\sqrt{6}+\sqrt{2})}{(\sqrt{6}-\sqrt{2})(\sqrt{6}+\sqrt{2})} = \frac{4(\sqrt{6}+\sqrt{2})}{6-2}$

$= \sqrt{6} + \sqrt{2}$

Example 7

Simplify each of the following expressions.

(a) $1+\tan^2(270°-\theta)$

(b) $\dfrac{\cos(360°-A)\sin(90°-A)\tan(A-180°)}{\sin(-A)\sin(180°+A)\cot(-A)}$

(c) $\dfrac{\cos\left(A-\frac{3\pi}{2}\right)\cot(A-\pi)}{\tan\left(A+\frac{\pi}{2}\right)}$

(d) $\dfrac{\cos\left(\frac{3\pi}{2}-2x\right)\csc\left(\frac{\pi}{2}+x\right)}{\tan(x-\pi)}$

Solution

(a) $1+\tan^2(270°-\theta)=1+[\tan(270°-\theta)]^2=1+(\cot\theta)^2=1+\cot^2\theta=\csc^2\theta$

(b)
$$\frac{\cos(360°-A)\sin(90°-A)\tan(A-180°)}{\sin(-A)\sin(180°+A)\cot(-A)}=\frac{\cos A\cos A\tan[-(180°-A)]}{(-\sin A)(-\sin A)(-\cot A)}$$
$$=\frac{\cos A\cos A\,[-\tan(180°-A)]}{-\sin A\sin A\cot A}$$
$$=\frac{\cos^2 A\,[-(-\tan A)]}{-\sin^2 A\cot A}=-\frac{\cos^2 A\tan^2 A}{\sin^2 A}=-1$$

(c)
$$\frac{\cos\left(A-\frac{3\pi}{2}\right)\cot(A-\pi)}{\tan\left(A+\frac{\pi}{2}\right)}=\frac{\cos\left[-\left(\frac{3\pi}{2}-A\right)\right]\cot[-(\pi-A)]}{-\cot A}$$
$$=\frac{\cos\left(\frac{3\pi}{2}-A\right)[-\cot(\pi-A)]}{-\cot A}$$
$$=\frac{(-\sin A)[-(-\cot A)]}{-\cot A}$$
$$=\sin A$$

(d)
$$\frac{\cos\left(\frac{3\pi}{2}-2x\right)\csc\left(\frac{\pi}{2}+x\right)}{\tan(x-\pi)}=\frac{(-\sin 2x)\cdot\dfrac{1}{\sin\left(\frac{\pi}{2}+x\right)}}{\tan[-(\pi-x)]}$$
$$=\frac{(-2\sin x\cos x)\cdot\dfrac{1}{\cos x}}{-\tan(\pi-x)}$$
$$=\frac{-2\sin x}{-(-\tan x)}=\frac{-2\sin x}{\dfrac{\sin x}{\cos x}}=-2\cos x$$

Example 8

Prove that $\sin(30° + x) + \cos(60° + x) - \cos x = 0$.

Solution

$$\begin{aligned}\text{L.H.S.} &= \sin(30° + x) + \cos(60° + x) - \cos x \\ &= (\sin 30° \cos x + \cos 30° \sin x) + (\cos 60° \cos x - \sin 60° \sin x) - \cos x\end{aligned}$$

by using **Compound angle formulae**

$$\begin{aligned}&= \left(\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x\right) + \left(\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x\right) - \cos x \\ &= 0 \\ &= \text{R.H.S.}\end{aligned}$$

$\therefore \quad \sin(30° + x) + \cos(60° + x) - \cos x = 0$

Example 9

Find the value of the following in surd form.

(a) $\sin 15°$ (b) $\cos\frac{5\pi}{12}$

Solution

(a) $\sin 15° = \sin(45° - 30°)$

$= \sin 45° \cos 30° - \cos 45° \sin 30°$ by using **Compound angle formula**

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

(b) $\cos\frac{5\pi}{12} = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}$ by using **Compound angle formula**

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$$

Example 10

Prove the following identities.

(a) $\sin x\,(\tan x + \cot x) = \sec x$

(b) $\dfrac{\tan x}{1+\tan^2 x} = \sin x \cos x$

(c) $2\sin^2\left(\dfrac{x}{2}\right)\tan x = \tan x - \sin x$

(d) $\sin^2\theta\cos^2\theta = \dfrac{1}{8} - \dfrac{1}{8}\cos 4\theta$

(e) $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$

(f) $\dfrac{\sin(A+B) - \sin 4B}{\cos(A+B) + \cos 4B} = -\tan\left(\dfrac{3B-A}{2}\right)$

(g) $\cos^4 x - \sin^4 x = \cos 2x$

(h) $\dfrac{1-\cos x}{\sin x} = \tan\left(\dfrac{x}{2}\right)$

Solution

(a) $\text{L.H.S.} = \sin x\,(\tan x + \cot x) = \sin x\left(\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}\right) = \sin x\left(\dfrac{\sin^2 x + \cos^2 x}{\sin x \cos x}\right) = \dfrac{1}{\cos x}$

$= \sec x = \text{R.H.S.}$

$\therefore \quad \sin x\,(\tan x + \cot x) = \sec x$

(b) $\text{L.H.S.} = \dfrac{\tan x}{1+\tan^2 x} = \dfrac{\frac{\sin x}{\cos x}}{\sec^2 x} = \dfrac{\sin x}{\cos x}\cdot\cos^2 x = \sin x\cos x = \text{R.H.S.}$

$\therefore \quad \frac{\tan x}{1+\tan^2 x} = \sin x\cos x$

(c) $\text{L.H.S.} = 2\sin^2\left(\dfrac{x}{2}\right)\tan x = 2\cdot\dfrac{1}{2}(1-\cos x)\tan x$ by **Half-angle formula**

$= \tan x - \cos x\cdot\dfrac{\sin x}{\cos x} = \tan x - \sin x = \text{R.H.S.}$

$\therefore \quad 2\sin^2\left(\frac{x}{2}\right)\tan x = \tan x - \sin x$

(d) $\text{L.H.S.} = \sin^2\theta\cos^2\theta = (\sin\theta\cos\theta)^2 = \left(\dfrac{1}{2}\sin 2\theta\right)^2$ by **Double-angle formula**

$= \dfrac{1}{4}\sin^2 2\theta = \dfrac{1}{4}\cdot\dfrac{1}{2}(1-\cos 4\theta)$ by **Half-angle formula**

$= \dfrac{1}{8} - \dfrac{1}{8}\cos 4\theta = \text{R.H.S.}$

$\therefore \quad \sin^2\theta\cos^2\theta = \frac{1}{8} - \frac{1}{8}\cos 4\theta$

(e) L.H.S. $= \sin(x+y)\sin(x-y)$

$= -\frac{1}{2}\{\cos[(x+y)+(x-y)] - \cos[(x+y)-(x-y)]\}$, by **Product-to-sum formula**

$= -\frac{1}{2}(\cos 2x - \cos 2y)$

$= -\frac{1}{2}[(\cos^2 x - \sin^2 x) - (\cos^2 y - \sin^2 y)]$, by **Double angle formula**

$= -\frac{1}{2}[(1 - \sin^2 x - \sin^2 x) - (1 - \sin^2 y - \sin^2 y)]$

$= -\frac{1}{2}[-2\sin^2 x - (-2\sin^2 y)]$

$= \sin^2 x - \sin^2 y =$ R.H.S.

$\therefore \quad \sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y$

(f) $$\text{L.H.S.} = \frac{\sin(A+B) - \sin 4B}{\cos(A+B) + \cos 4B}$$

$$= \frac{2\cos\left[\frac{(A+B)+4B}{2}\right]\sin\left[\frac{(A+B)-4B}{2}\right]}{2\cos\left[\frac{(A+B)+4B}{2}\right]\cos\left[\frac{(A+B)-4B}{2}\right]}$$ by **Sum-to-product formulae**

$$= \frac{\sin\left(\frac{A-3B}{2}\right)}{\cos\left(\frac{A-3B}{2}\right)} = \tan\left(\frac{A-3B}{2}\right) = \tan\left[-\left(\frac{3B-A}{2}\right)\right] = -\tan\left(\frac{3B-A}{2}\right) = \text{R.H.S.}$$

$$\therefore \quad \frac{\sin(A+B) - \sin 4B}{\cos(A+B) + \cos 4B} = -\tan\left(\frac{3B-A}{2}\right)$$

(g) $$\text{L.H.S.} = \cos^4 x - \sin^4 x = \left(\underbrace{\cos^2 x + \sin^2 x}_{=1}\right)\left(\underbrace{\cos^2 x - \sin^2 x}_{=\cos 2x}\right) = \cos 2x = \text{R.H.S.}$$

$\therefore \ \cos^4 x - \sin^4 x = \cos 2x$

(h) $$\text{L.H.S.} = \frac{1-\cos x}{\sin x} = \frac{2\sin^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$$ $\because \ \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ (**Half angle formula**)

& $\sin 2\theta = 2\sin\theta\cos\theta$ (**Double angle formula**)

$$= \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \tan\left(\frac{x}{2}\right) = \text{R.H.S.}$$

$$\therefore \ \frac{1-\cos x}{\sin x} = \tan\left(\frac{x}{2}\right)$$

Example 11

Simplify each of the following expressions.

(a) $\dfrac{\cos 3\theta - \cos\theta}{\sin\theta + \sin 3\theta}$

(b) $\dfrac{\cot\theta}{1+\cot^2\theta}$

(c) $\cos(x+y)\cos y + \sin(x+y)\sin y$

(d) $\dfrac{\cos^2 A - \sin^2 A}{\cos 3A\cos A + \sin 3A\sin A}$

Solution

(a)
$$\frac{\cos 3\theta - \cos\theta}{\sin\theta + \sin 3\theta} = \frac{-2\sin\left(\frac{3\theta+\theta}{2}\right)\sin\left(\frac{3\theta-\theta}{2}\right)}{2\sin\left(\frac{\theta+3\theta}{2}\right)\cos\left(\frac{\theta-3\theta}{2}\right)}$$ by **Sum-to-product formula**

$$= \frac{-\sin 2\theta\sin\theta}{\sin 2\theta\cos(-\theta)}$$

$$= -\frac{\sin\theta}{\cos\theta}$$

$$= -\tan\theta$$

(b)
$$\frac{\cot\theta}{1+\cot^2\theta} = \frac{\frac{\cos\theta}{\sin\theta}}{\csc^2\theta} = \frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin^2\theta}} = \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta,$$ by **Double angle formula**

(c)
$$\cos(x+y)\cos y + \sin(x+y)\sin y = \cos[(x+y)-y]$$ by **Compound angle formula**

$$= \cos x$$

(d)
$$\frac{\cos^2 A - \sin^2 A}{\cos 3A\cos A + \sin 3A\sin A} = \frac{\cos 2A}{\cos(3A-A)}$$ by **Double angle formula**

and Compound angle formula

$$= \frac{\cos 2A}{\cos 2A}$$

$$= 1$$

Example 12

Find the **largest possible domain** and **largest possible range** of $f(x) = 1 - 3\sin x \cos x$.

Solution

The function $f(x) = 1 - 3\sin x \cos x$ is defined for all real values of x.

$\therefore\ Dom(f) = \mathbb{R}$.

$f(x) = 1 - 3\sin x \cos x = 1 - \frac{3}{2}\sin 2x$ by using the **double angle formula**.

For any $x \in Dom(f) = \mathbb{R}$, we have $-1 \le \sin 2x \le 1$

$$\Rightarrow \quad -\frac{3}{2} \le \frac{3}{2}\sin 2x \le \frac{3}{2}$$

$$\Rightarrow \quad -\left(-\frac{3}{2}\right) \ge -\frac{3}{2}\sin 2x \ge -\frac{3}{2}$$

$$\Rightarrow \quad -\frac{3}{2} \le -\frac{3}{2}\sin 2x \le \frac{3}{2}$$

$$\Rightarrow \quad \underbrace{1 - \frac{3}{2}}_{=-\frac{1}{2}} \le \underbrace{1 - \frac{3}{2}\sin 2x}_{=f(x)} \le \underbrace{1 + \frac{3}{2}}_{=\frac{5}{2}}$$

$\therefore\ Ran(f) = \left[-\frac{1}{2}, \frac{5}{2}\right]$.

Example 13

Express $\sin 3x$ in terms of $\sin x$ and powers of $\sin x$.

Solution

$\sin 3x = \sin(2x + x)$

$= \sin 2x \cos x + \cos 2x \sin x$, by using **Compound angle formula**

$= (2\sin x \cos x)\cos x + (\cos^2 x - \sin^2 x)\sin x$, by using **Double angle formulae**

$= 2\sin x \cos^2 x + \sin x \cos^2 x - \sin^3 x$

$= 3\sin x\,(1 - \sin^2 x) - \sin^3 x$, since $\boldsymbol{\cos^2 x = 1 - \sin^2 x}$

$= 3\sin x - 4\sin^3 x$

Example 14

Express $\sin^4 x$ as the sum of a constant and various $\cos kx$ terms, for some $k \in \mathbb{N}$.

Solution

$$\sin^4 x = (\sin^2 x)^2$$

$$= \left[\frac{1}{2}(1-\cos 2x)\right]^2 \quad \text{by } \textbf{Half angle formula}$$

$$= \frac{1}{4}[1 - 2\cos 2x + \cos^2 2x]$$

$$= \frac{1}{4}\left[1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right] \quad \text{by } \textbf{Half angle formula}$$

$$= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

Additional Exercise

It is given that $\sin A = -\frac{12}{13}$ where $-90^\circ < A < 0^\circ$, and that $\cos B = -\frac{4}{5}$ where $180^\circ < B < 270^\circ$. Without using calculator,

(i) find the value of $\sin(A+B)$,

(ii) find the value of $\cos(A+B)$,

(iii) deduce that $90^\circ < A+B < 180^\circ$,

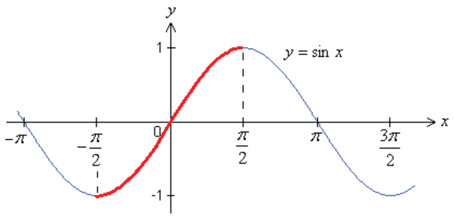
(iv) find the value of $\cos\left(\frac{B}{2}\right)$.

Inverse Trigonometric Functions

In this section, we will study the inverse functions of $\sin x$, $\cos x$ and $\tan x$.

➢ Inverse function of $\sin x$:

Consider the graph of $y = \sin x$.



The function $g(x) = \sin x$, where $x \in \mathbb{R}$, is not one-to-one, so $g(x)$ has no inverse.

The **principal part** of sine function is defined as $f(x) = \sin x$, where $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then $f(x)$ is one-to-one and therefore its inverse $f^{-1}(x)$ exists.

$$f^{-1}(x) = \sin^{-1} x \ , \quad \text{for} \quad x \in [-1, 1].$$

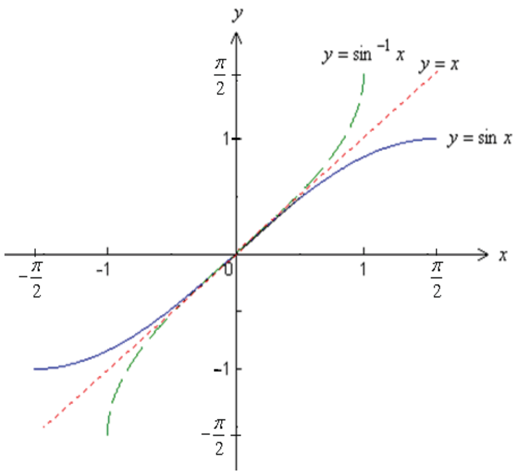
This is called the **inverse sine** (or **arcsine**) function, denoted by $\sin^{-1}$ (or arcsin).

$$\boxed{y = \sin^{-1} x \quad \Leftrightarrow \quad x = \sin y \quad \text{for } -1 \le x \le 1 \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}.}$$

Thus, (i) $\sin(\sin^{-1} x) = x$ for $-1 \le x \le 1$.

(ii) $\sin^{-1}(\sin y) = y$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Graphs of $y = \sin x$ and its inverse $y = \sin^{-1} x$:



$$\boxed{\begin{array}{c} f(x) = \sin x \\ f^{-1}(x) = \sin^{-1} x \end{array}}$$

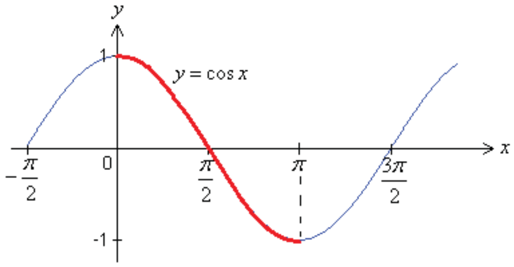
Note: $Dom(f) = Ran(f^{-1}) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ← principal range

$Ran(f) = Dom(f^{-1}) = [-1, 1]$

Is $f^{-1}(x) = \sin^{-1} x$ an odd function, even function, or neither of them?

➢ Inverse function of $\cos x$:

Consider the graph of $y = \cos x$.



The function $g(x) = \cos x$, where $x \in \mathbb{R}$, is not one-to-one, so $g(x)$ has no inverse.

The **principal part** of cosine function is defined as $f(x) = \cos x$, where $x \in [0, \pi]$. Then $f(x)$ is one-to-one and therefore its inverse $f^{-1}(x)$ exists.

$$f^{-1}(x) = \cos^{-1} x \ , \quad \text{for} \quad x \in [-1, 1].$$

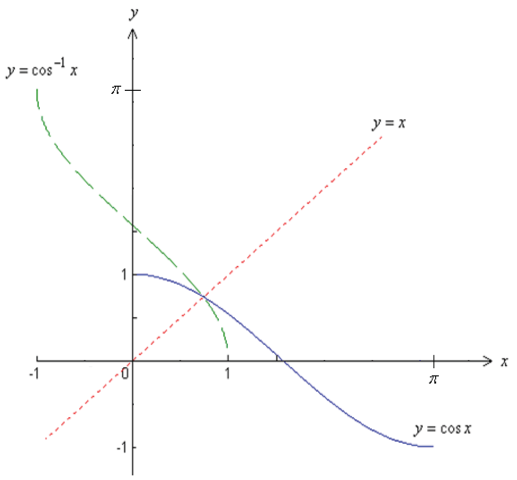
This is called the **inverse cosine** (or **arccosine**) function, denoted by $\cos^{-1}$ (or arccos).

$$\boxed{y = \cos^{-1} x \quad \Leftrightarrow \quad x = \cos y \quad \text{for } -1 \le x \le 1 \text{ and } 0 \le y \le \pi.}$$

Thus, (i) $\cos(\cos^{-1} x) = x$ for $-1 \le x \le 1$.

(ii) $\cos^{-1}(\cos y) = y$ for $0 \le y \le \pi$.

Graphs of $y = \cos x$ and its inverse $y = \cos^{-1} x$:



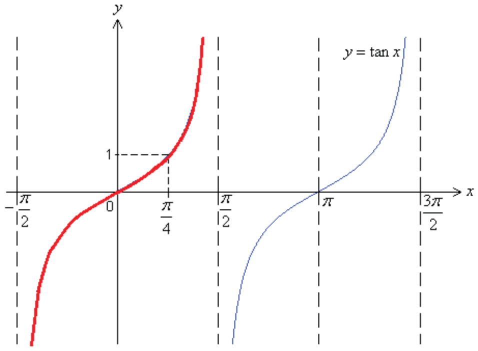
$$\boxed{\begin{array}{c} f(x) = \cos x \\ f^{-1}(x) = \cos^{-1} x \end{array}}$$

Note: $Dom(f) = Ran(f^{-1}) = [0, \pi]$ ← principal range

$Ran(f) = Dom(f^{-1}) = [-1, 1]$

Is $f^{-1}(x) = \cos^{-1} x$ an odd function, even function, or neither of them?

➢ **Inverse function of $\tan x$:**



The function $g(x) = \tan x$, where $x \in \mathbb{R}\backslash\left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\right\}$, is not one-to-one, so $g(x)$ has no inverse.

The **principal part** of tangent function is defined as $f(x) = \tan x$, where $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $f(x)$ is one-to-one and therefore its inverse $f^{-1}(x)$ exists.

$$f^{-1}(x) = \tan^{-1} x \ , \quad \text{for} \quad x \in \mathbb{R}.$$

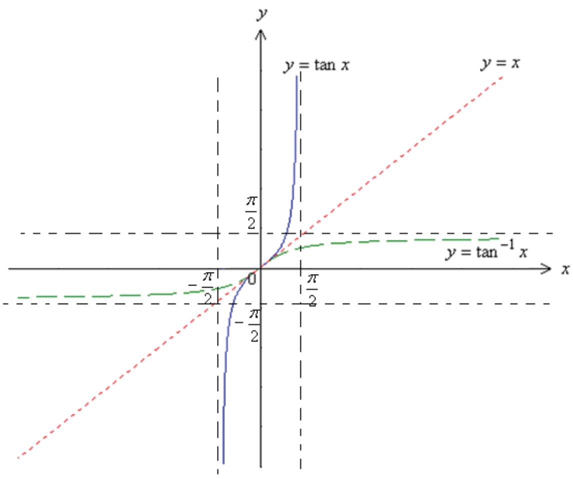
This is called the **inverse tangent** (or **arctangent**) function, denoted by $\tan^{-1}$ (or arctan).

$$\boxed{y = \tan^{-1} x \iff x = \tan y \quad \text{for every real number } x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}.}$$

Thus, (i) $\tan(\tan^{-1} x) = x$ for $x \in \mathbb{R}$.

(ii) $\tan^{-1}(\tan y) = y$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Graphs of $y = \tan x$ and its inverse $y = \tan^{-1} x$:



$$\boxed{\begin{array}{c} f(x) = \tan x \\ f^{-1}(x) = \tan^{-1} x \end{array}}$$

Note: $Dom(f) = Ran(f^{-1}) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ← principal range

$Ran(f) = Dom(f^{-1}) = \mathbb{R}$

Is $f^{-1}(x) = \tan^{-1} x$ an odd function, even function, or neither of them?

Remarks:

1. $\sin^2 x = (\sin x)^2$, $\sin^3 x = (\sin x)^3$, etc.

   However, $\sin^{-1} x \neq (\sin x)^{-1} = \frac{1}{\sin x}$. (Similarly for $\cos^{-1} x$ and $\tan^{-1} x$.)
2. The ranges of the inverse trigonometric functions are known as the **principal ranges**.

Inverse functions of cosecant, secant and cotangent

Similarly, we use the notations $\csc^{-1}$, $\sec^{-1}$ and $\cot^{-1}$ to denote the inverse functions of cosecant, secant and cotangent, respectively.

Example 15

Find the value of each of the following.

(a) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$ (b) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (c) $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

(d) $\cos\left(\cos^{-1}\left(\frac{1}{4}\right)\right)$ (e) $\sin^{-1}(\sin 10°)$ (f) $\sin^{-1}(\sin 380°)$

(g) $\sin^{-1}\left(\sin\left(-\frac{5\pi}{6}\right)\right)$ (h) $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$ (i) $\cos^{-1}(\cos 300°)$

(j) $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$ (k) $\sin^{-1}(\cos 390°)$ (l) $\cos^{-1}\left(\sin\frac{5\pi}{4}\right)$

Solution

(a) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$ (in radians) or $45°$ (in degrees)

(b) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$ (in radians) or $150°$ (in degrees)

(c) $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6}$ (in radians) or $-30°$ (in degrees)

(d) $\cos\left(\cos^{-1}\left(\frac{1}{4}\right)\right) = \frac{1}{4}$

(e) $\sin^{-1}(\sin 10°) = 10°$ (since $10°$ lies in the principal range $[-90°, 90°]$.)

(f) $\sin^{-1}(\sin 380°) = \sin^{-1}(\sin(360° + 20°)) = \sin^{-1}(\sin(20°)) = 20°$

(which lies in the principal range $[-90°, 90°]$.)

(g) $$\sin^{-1}\left(\sin\left(-\frac{5\pi}{6}\right)\right) = \sin^{-1}\left(\sin\left(\frac{\pi}{6} - \pi\right)\right) = \sin^{-1}\left(-\sin\left(\pi - \frac{\pi}{6}\right)\right)$$
$$= \sin^{-1}\left(-\sin\frac{\pi}{6}\right) = \sin^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right) = -\frac{\pi}{6}$$

(which lies in the principal range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.)

(h) $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right) = \cos^{-1}\left(\cos\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$ (which lies in the principal range $[0, \pi]$.)

(i) $\cos^{-1}(\cos 300°) = \cos^{-1}(\cos(360° - 60°)) = \cos^{-1}(\cos 60°) = 60°$

(which lies in the principal range $[0°, 180°]$.)

(j) $\tan^{-1}\left(\tan\frac{7\pi}{6}\right) = \tan^{-1}\left(\tan\left(\pi + \frac{\pi}{6}\right)\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$

(which lies in the principal range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.)

(k) $$\sin^{-1}(\cos 390°) = \sin^{-1}(\cos(360° + 30°)) = \sin^{-1}(\cos(30°))$$
$$= \sin^{-1}(\cos(90° - 60°)) = \sin^{-1}(\sin(60°)) = 60°$$

(which lies in the principal range $[-90°, 90°]$.)

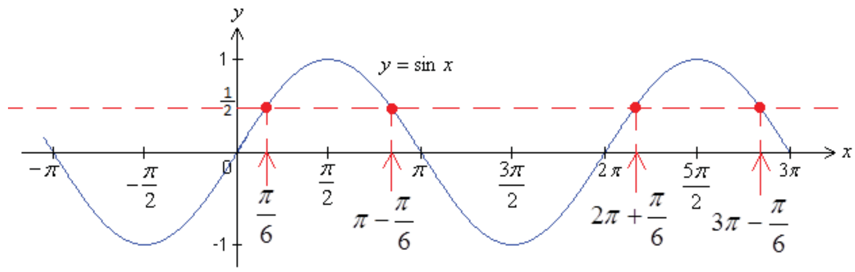
(l) $\cos^{-1}\left(\sin\frac{5\pi}{4}\right) = \cos^{-1}\left(\sin\left(\frac{\pi}{2} + \frac{3\pi}{4}\right)\right) = \cos^{-1}\left(\cos\left(\frac{3\pi}{4}\right)\right) = \frac{3\pi}{4}$

(which lies in the principal range $[0, \pi]$.)

General Solutions of Trigonometric Equations

Sine function:

Find the general solution of $\sin x = \frac{1}{2}$.



$\alpha = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ ($\because$ the principal range of $\alpha = \sin^{-1}(x)$ is $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$.)

The solutions of $\sin x = \frac{1}{2}$ in $[0, 2\pi)$ are

$$x = \alpha = \frac{\pi}{6} \quad \text{and} \quad x = \pi - \alpha = \pi - \frac{\pi}{6} = \frac{5\pi}{6}.$$

Since $y = \sin x$ is periodic with period 2π,

$$x = \frac{\pi}{6} + (2\pi)m \quad \text{and} \quad x = \pi - \frac{\pi}{6} + (2\pi)m,$$

where $m \in \mathbb{Z}$, are also solutions of $\sin x = \frac{1}{2}$.

That is, $x = \underbrace{(2m)}_{even\ no.}\pi + \frac{\pi}{6}$ and $x = \underbrace{(2m+1)}_{odd\ no.}\pi - \frac{\pi}{6}$, where $m \in \mathbb{Z}$.

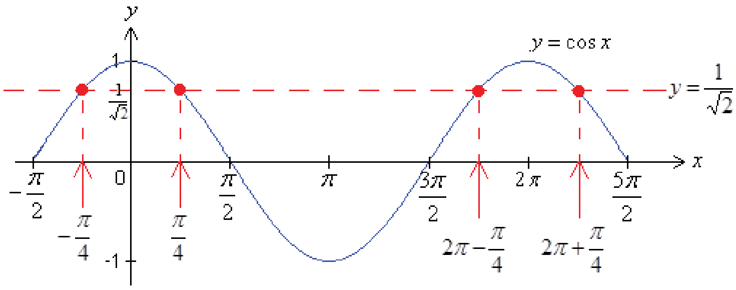
$\therefore$ The **general solution** of the equation $\sin x = \frac{1}{2}$ is

$$\boxed{x = n\pi + (-1)^n \cdot \alpha}\text{, where } n \in \mathbb{Z} \text{ and } \boxed{\alpha = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}}.$$

That is, $x = n\pi + (-1)^n \cdot \frac{\pi}{6}$, where $n \in \mathbb{Z}$.

<u>*Cosine function*</u>:

Find the general solution of $\cos x = \frac{1}{\sqrt{2}}$.



$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ ($\because$ the principal range of $\alpha = \cos^{-1}(x)$ is $0 \le \alpha \le \pi$.)

The solutions of $\cos x = \frac{1}{\sqrt{2}}$ in $(-\pi, \pi]$ are

$$x = \alpha = \frac{\pi}{4} \quad \text{and} \quad x = -\alpha = -\frac{\pi}{4} \quad (\because \cos(-x) = \cos x).$$

Since $y = \cos x$ is periodic with period 2π,

$$x = \frac{\pi}{4} + (2\pi)n = 2n\pi + \frac{\pi}{4} \quad \text{and} \quad x = -\frac{\pi}{4} + (2\pi)n = 2n\pi - \frac{\pi}{4},$$

where $n \in \mathbb{Z}$, are also solutions of $\cos x = \frac{1}{\sqrt{2}}$.

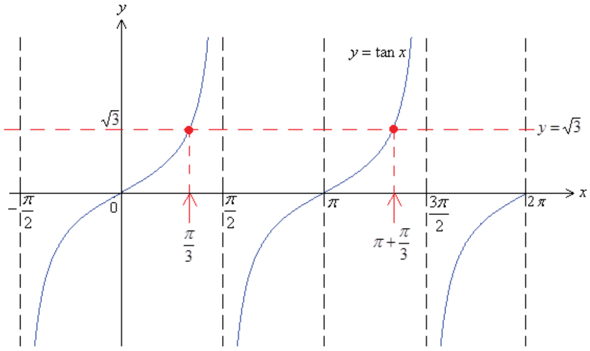
$\therefore$ The **general solution** of the equation $\cos x = \frac{1}{\sqrt{2}}$ is

$$\boxed{x = 2n\pi \pm \alpha}\ \text{, where } n \in \mathbb{Z} \text{ and } \boxed{\alpha = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}}\ (\text{since } 0 \le \alpha \le \pi).$$

That is, $x = 2n\pi \pm \frac{\pi}{4}$, where $n \in \mathbb{Z}$.

Tangent function:

Find the general solution of $\tan x = \sqrt{3}$.



$\alpha = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ ($\because$ the principal range of $\alpha = \tan^{-1}(x)$ is $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.)

The solution of $\tan x = \sqrt{3}$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is

$$x = \alpha = \frac{\pi}{3}.$$

Since $y = \tan x$ is periodic with period π,

$$x = n\pi + \frac{\pi}{3},$$

where $n \in \mathbb{Z}$, are also solutions of $\tan x = \sqrt{3}$.

$\therefore$ The **general solution** of the equation $\tan x = \sqrt{3}$ is

$$\boxed{x = n\pi + \alpha}\text{ , where } n \in \mathbb{Z} \text{ and } \boxed{\alpha = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}} \text{ (since } -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\text{).}$$

(since $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$).

That is, $x = n\pi + \frac{\pi}{3}$, where $n \in \mathbb{Z}$.

The results are summarized on the next page.

Summary

- The **general solution** of $\sin x = k$ (where $-1 \le k \le 1$) is

$$x = n\pi + (-1)^n\,\alpha,$$

for $n \in \mathbb{Z}$, where $\alpha = \sin^{-1} k$ and $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$.

- The **general solution** of $\cos x = k$ (where $-1 \le k \le 1$) is

$$x = 2n\pi \pm \alpha,$$

for $n \in \mathbb{Z}$, where $\alpha = \cos^{-1} k$ and $0 \le \alpha \le \pi$.

- The **general solution** of $\tan x = k$ (where $k \in \mathbb{R}$) is

$$x = n\pi + \alpha,$$

for $n \in \mathbb{Z}$, where $\alpha = \tan^{-1} k$ and $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.

Example 16

Find, in radians, the general solution of the equation $\sin\theta + \cos\theta = 0$, and give all the values of θ which lie between 0 and 2π.

Solution

$$\sin\theta + \cos\theta = 0 \quad\Rightarrow\quad \sin\theta = -\cos\theta \quad\Rightarrow\quad \frac{\sin\theta}{\cos\theta} = -1 \quad\Rightarrow\quad \tan\theta = -1$$

$\therefore$ The general solution of the equation is

$$\theta = n\pi + \alpha,$$

where $\alpha = \tan^{-1}(-1) = -\frac{\pi}{4}$ and $n \in \mathbb{Z}$,

i.e. $\boxed{\theta = n\pi - \frac{\pi}{4}}$ for $n \in \mathbb{Z}$.

When $n = 1$, $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$.

When $n = 2$, $\theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$.

$\therefore$ The solutions of the equation which lie between 0 and 2π are

$$\boxed{\theta = \frac{3\pi}{4}} \text{ and } \boxed{\theta = \frac{7\pi}{4}}.$$

Example 17

Find, in radians, the general solution of the equation $2\sin 5x = -1$.

Solution

$2\sin 5x = -1 \quad \Rightarrow \quad \sin 5x = -\frac{1}{2}$

$\therefore$ The general solution of the equation is

$$5x = n\pi + (-1)^n\,\alpha,$$

where $\alpha = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ and $n \in \mathbb{Z}$.

That is, $\boxed{x = \frac{n\pi}{5} + \frac{(-1)^n\left(-\frac{\pi}{6}\right)}{5} = \frac{n\pi}{5} + (-1)^n\left(-\frac{\pi}{30}\right)}$ for $n \in \mathbb{Z}$

Example 18

Find the general solution of the equation $\sin x = \cos 2x$.

Solution By using the **Double angle formula**, we have

$$\cos 2x = \cos^2 x - \sin^2 x = (1 - \sin^2 x) - \sin^2 x = 1 - 2\sin^2 x.$$

Then $\sin x = \cos 2x \quad \Rightarrow \quad \sin x = 1 - 2\sin^2 x$

$\Rightarrow \quad 2\sin^2 x + \sin x - 1 = 0$

$\Rightarrow \quad (2\sin x - 1)(\sin x + 1) = 0$

$\Rightarrow \quad 2\sin x - 1 = 0 \quad$ or $\quad \sin x + 1 = 0$

$\Rightarrow \quad \sin x = \frac{1}{2} \quad$ or $\quad \sin x = -1$

$\therefore$ The general solution of the equation is

$x = n\pi + (-1)^n\,\alpha_1$, where $\alpha_1 = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$, for $n \in \mathbb{Z}$,

and $x = n\pi + (-1)^n\,\alpha_2$, where $\alpha_2 = \sin^{-1}(-1) = -\frac{\pi}{2}$, for $n \in \mathbb{Z}$.

That is, $\boxed{x = n\pi + (-1)^n\left(\frac{\pi}{6}\right)}$ or $\boxed{x = n\pi + (-1)^n\left(-\frac{\pi}{2}\right)}$, for $n \in \mathbb{Z}$.

Example 19

Find the general solution of the equation $2\sin^2 4x + 3\cos 4x = 3$.

Solution

$2\sin^2 4x + 3\cos 4x = 3$

$\Rightarrow \quad 2(1-\cos^2 4x) + 3\cos 4x = 3$

$\Rightarrow \quad 2\cos^2 4x - 3\cos 4x + 1 = 0$

$\Rightarrow \quad (2\cos 4x - 1)(\cos 4x - 1) = 0$

$\Rightarrow \quad 2\cos 4x - 1 = 0$ or $\cos 4x - 1 = 0$

$\Rightarrow \quad \cos 4x = \frac{1}{2}$ or $\cos 4x = 1$

$\therefore$ The general solution of the equation is

$4x = 2n\pi \pm \alpha_1$, where $\alpha_1 = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$, for $n \in \mathbb{Z}$,

and $4x = 2n\pi \pm \alpha_2$, where $\alpha_2 = \cos^{-1}(1) = 0$, for $n \in \mathbb{Z}$.

That is, $\boxed{x = \frac{2n\pi \pm \frac{\pi}{3}}{4} = \frac{n\pi}{2} \pm \frac{\pi}{12}}$ or $\boxed{x = \frac{2n\pi \pm 0}{4} = \frac{n\pi}{2}}$, for $n \in \mathbb{Z}$.

Example 20

Find, in radians, the general solution of the equation

$$2\sin^2(2x) - 2\sin x\cos x - 1 = 0,$$

and give all the values of x which lie between 0 and 2π.

Solution

$2\sin^2(2x) - \underbrace{2\sin x\cos x}_{=\sin 2x} - 1 = 0$

$\Rightarrow \quad 2\sin^2(2x) - \sin(2x) - 1 = 0$ (by using the **Double angle formula**)

$\Rightarrow \quad [2\sin(2x)+1][\sin(2x)-1] = 0$

$\Rightarrow \quad 2\sin(2x)+1 = 0$ or $\sin(2x) - 1 = 0$

$\Rightarrow \quad \sin(2x) = -\frac{1}{2}$ or $\sin(2x) = 1$

$\therefore$ The general solution of the equation is

$2x = n\pi + (-1)^n\alpha_1$, where $\alpha_1 = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$, for $n \in \mathbb{Z}$,

and $2x = n\pi + (-1)^n\alpha_2$, where $\alpha_2 = \sin^{-1}(1) = \frac{\pi}{2}$, for $n \in \mathbb{Z}$.

That is, $\boxed{x = \frac{n\pi}{2} + (-1)^n \cdot \left(-\frac{\pi}{12}\right)}$ or $\boxed{x = \frac{n\pi}{2} + (-1)^n \cdot \frac{\pi}{4}}$, where $n \in \mathbb{Z}$.

For $x = \frac{n\pi}{2} + (-1)^n \cdot \left(-\frac{\pi}{12}\right)$:

When $n = 1$, $x = \frac{\pi}{2} + \frac{\pi}{12} = \frac{7\pi}{12}$

When $n = 2$, $x = \frac{2\pi}{2} - \frac{\pi}{12} = \frac{11\pi}{12}$

When $n = 3$, $x = \frac{3\pi}{2} + \frac{\pi}{12} = \frac{19\pi}{12}$

When $n = 4$, $x = \frac{4\pi}{2} - \frac{\pi}{12} = \frac{23\pi}{12}$

For $x = \frac{n\pi}{2} + (-1)^n \cdot \frac{\pi}{4}$:

When $n = 1$, $x = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

When $n = 2$, $x = \frac{2\pi}{2} + \frac{\pi}{4} = \frac{5\pi}{4}$

(When $n = 3$, $x = \frac{3\pi}{2} - \frac{\pi}{4} = \frac{5\pi}{4}$)

Hence, the solutions which lie between 0 and 2π are

$$\boxed{\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{4}}.$$

Example 21

Find the general solution of the equation $\sin x + \cos x = 1$.

Solution

$$\begin{aligned}
\sin x + \cos x = 1 \quad &\Rightarrow \quad \sin x = 1 - \cos x \\
&\Rightarrow \quad 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = 2\sin^2\left(\frac{x}{2}\right) \\
&\Rightarrow \quad 2\sin\left(\frac{x}{2}\right)\left[\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right] = 0 \\
&\Rightarrow \quad 2\sin\left(\frac{x}{2}\right) = 0 \quad \text{or} \quad \cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right) = 0 \\
&\Rightarrow \quad \sin\left(\frac{x}{2}\right) = 0 \quad \text{or} \quad \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = 1 \\
&\Rightarrow \quad \sin\left(\frac{x}{2}\right) = 0 \quad \text{or} \quad \tan\left(\frac{x}{2}\right) = 1
\end{aligned}$$

$\therefore$ The general solution of the equation is

$$\frac{x}{2} = n\pi + (-1)^n \alpha_1, \quad \text{where } \alpha_1 = \sin^{-1}(0) = 0, \quad \text{for } n \in \mathbb{Z},$$

and $$\frac{x}{2} = n\pi + \alpha_2, \quad \text{where } \alpha_2 = \tan^{-1}(1) = \frac{\pi}{4}, \quad \text{for } n \in \mathbb{Z}.$$

That is, $\boxed{x = 2n\pi}$ or $\boxed{x = 2n\pi + \frac{\pi}{2}}$, for $n \in \mathbb{Z}$.

Additional Exercise

(a) Express $3\cos 4x + \sqrt{3}\sin 4x$ in the form $R\cos(4x - \phi)$, where $R > 0$ and $0 < \phi < \frac{\pi}{2}$.

(b) Find the general solution of $\cos 4x + \frac{1}{\sqrt{3}}\sin 4x = 1$.