MA1201 Calculus and Basic Linear Algebra II **Solution of Problem Set 4** **Vector Algebra**

Problem 1

(a) The position vectors of A, B and C are given by
$\overrightarrow{OA} = \vec{\imath} + \vec{\jmath}$, $\overrightarrow{OB} = 2\vec{\jmath} + 3\vec{k}$ and $\overrightarrow{OC} = 2\vec{\imath} - \vec{\jmath}$

(b) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \left(2\vec{\jmath} + 3\vec{k}\right) - (\vec{\imath} + \vec{\jmath}) = -\vec{\imath} + \vec{\jmath} + 3\vec{k}$.
$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = (\vec{\imath} + \vec{\jmath}) - (2\vec{\imath} - \vec{\jmath}) = -\vec{\imath} + 2\vec{\jmath}$.

(c) Note that
$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (2\vec{\imath} - \vec{\jmath}) - \left(2\vec{\jmath} + 3\vec{k}\right) = 2\vec{\imath} - 3\vec{\jmath} - 3\vec{k}$.
Since the $\vec{\imath}$-component of $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are not the same, thus $\overrightarrow{AB} \neq \overrightarrow{BC}$.

(d)
$$\widehat{AB} = \frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} = \frac{-\vec{\imath} + \vec{\jmath} + 3\vec{k}}{\sqrt{(-1)^2 + 1^2 + 3^2}} = -\frac{1}{\sqrt{11}}\vec{\imath} + \frac{1}{\sqrt{11}}\vec{\jmath} + \frac{3}{\sqrt{11}}\vec{k}$$
$$\widehat{BC} = \frac{\overrightarrow{BC}}{\left|\overrightarrow{BC}\right|} = \frac{2\vec{\imath} - 3\vec{\jmath} - 3\vec{k}}{\sqrt{(2)^2 + (-3)^2 + (-3)^2}} = \frac{2}{\sqrt{22}}\vec{\imath} - \frac{3}{\sqrt{22}}\vec{\jmath} - \frac{3}{\sqrt{22}}\vec{k}$$

(e) (i) $\vec{a} = 3 \times \left(-\widehat{AB}\right) = \frac{3}{\sqrt{11}}\vec{\imath} - \frac{3}{\sqrt{11}}\vec{\jmath} - \frac{9}{\sqrt{11}}\vec{k}$

(ii) $\vec{b} = 5 \times \left(\widehat{BC}\right) = \frac{10}{\sqrt{22}}\vec{\imath} - \frac{15}{\sqrt{22}}\vec{\jmath} - \frac{15}{\sqrt{22}}\vec{k}$.

Problem 2

(a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (\vec{\imath} + 2\vec{\jmath}) - \left(\vec{\jmath} - \vec{k}\right) = \vec{\imath} + \vec{\jmath} + \vec{k}$.
$$\overrightarrow{AX} = \underbrace{\frac{2}{3}\left|\overrightarrow{AB}\right|}_{\text{magnitude}} \times \underbrace{\left(\widehat{AB}\right)}_{\text{direction}} = \frac{2}{3}\left|\overrightarrow{AB}\right| \times \frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} = \frac{2}{3}\left(\vec{\imath} + \vec{\jmath} + \vec{k}\right) = \frac{2}{3}\vec{\imath} + \frac{2}{3}\vec{\jmath} + \frac{2}{3}\vec{k}.$$

(b) Using the fact that $\overrightarrow{AX} = \overrightarrow{OX} - \overrightarrow{OA}$, we have
$$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX} = \left(\vec{\jmath} - \vec{k}\right) + \left(\frac{2}{3}\vec{\imath} + \frac{2}{3}\vec{\jmath} + \frac{2}{3}\vec{k}\right) = \frac{2}{3}\vec{\imath} + \frac{5}{3} - \frac{1}{3}\vec{k}.$$

Problem 3

(a) $|\vec{a}| = \sqrt{2^2 + (-3)^2 + 5^2} = \sqrt{38}$
$\left|\vec{a} - 2\vec{b}\right| = \left|-9\vec{\jmath} + 5\vec{k}\right| = \sqrt{(-9)^2 + 5^2} = \sqrt{106}$.

(b)
$$\hat{b} = \frac{\vec{b}}{\left|\vec{b}\right|} = \frac{\vec{\imath} + 3\vec{\jmath}}{\sqrt{1^2 + 3^2}} = \frac{1}{\sqrt{10}}\vec{\imath} + \frac{3}{\sqrt{10}}\vec{\jmath}.$$

(c) Note that
$$\left|2\vec{a} + \vec{b}\right| = \left|5\vec{\imath} - 3\vec{\jmath} + 10\vec{k}\right| = \sqrt{5^2 + (-3)^2 + 10^2} = \sqrt{134}.$$
Thus the vector $\vec{c}$ is given by
$$\vec{c} = \underbrace{\left|2\vec{a} + \vec{b}\right|}_{\text{magnitude}} \times \underbrace{\hat{b}}_{\text{direction}} = \sqrt{134}\left(\frac{1}{\sqrt{10}}\vec{\imath} + \frac{3}{\sqrt{10}}\vec{\jmath}\right) = \frac{\sqrt{134}}{\sqrt{10}}\vec{\imath} + \frac{3\sqrt{134}}{\sqrt{10}}\vec{\jmath}.$$

Problem 4

(a) $\vec{a}\cdot\vec{b} = 1(-2)+3(1)-2(3) = -5.$

(b) Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$. Then we have

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-5}{\sqrt{1^2+3^2+(-2)^2}\sqrt{(-2)^2+1^2+3^2}} = \frac{-5}{\sqrt{14}\sqrt{14}} \Rightarrow \theta \approx 110.92°.$$

(c) $\vec{c}$ and $\vec{b}$ are perpendicular, then $\vec{c}\cdot\vec{b} = |\vec{c}||\vec{b}|\cos 90° = 0$

$\Rightarrow \left(3\vec{\imath}+x\vec{\jmath}-2\vec{k}\right)\cdot\left(-2\vec{\imath}+\vec{\jmath}+3\vec{k}\right) = 0$

$\Rightarrow 3(-2)+x(1)+(-2)(3) = 0 \Rightarrow x = 12.$

(d) $\vec{d} = y\vec{a}+3\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular, then $\vec{d}\cdot\left(\vec{a}-\vec{b}\right) = |\vec{d}||\vec{a}-\vec{b}|\cos 90° = 0$

$\Rightarrow \left(y\vec{a}+3\vec{b}\right)\cdot\left(\vec{a}-\vec{b}\right) = 0$

$\Rightarrow y(\vec{a}\cdot\vec{a}) + (3-y)\left(\vec{a}\cdot\vec{b}\right) - 3\left(\vec{b}\cdot\vec{b}\right) = 0$

$\Rightarrow y|\vec{a}|^2 + (3-y)\left(\vec{a}\cdot\vec{b}\right) - 3|\vec{b}|^2 = 0$

$\Rightarrow y\left(\sqrt{14}\right)^2 + (3-y)(-5) - 3\left(\sqrt{14}\right)^2 = 0$

$\Rightarrow y = 3.$

Problem 5

(a) Let $\theta = \angle ABC$, then θ satisfies

$$\cos\theta = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} \ldots\ldots (*)$$

Since $\overrightarrow{BA} = \overrightarrow{OA}-\overrightarrow{OB} = \vec{\imath}-2\vec{k}$, $\overrightarrow{BC} = \overrightarrow{OC}-\overrightarrow{OB} = 2\vec{\imath}-2\vec{k}$ and $\overrightarrow{BA}\cdot\overrightarrow{BC} = (1)(2)+(0)(0)+(-2)(-2) = 6$, then we can deduce from $(*)$ that

$$\cos\theta = \frac{6}{\sqrt{(1)^2+(-2)^2}\sqrt{(2)^2+(-2)^2}} = \frac{6}{\sqrt{5}\sqrt{8}} \Rightarrow \theta \approx 18.43°.$$

(b) Given that DE is perpendicular to EF, we then have $\overrightarrow{ED}\cdot\overrightarrow{EF} = |\overrightarrow{ED}||\overrightarrow{EF}|\cos 90° = 0.$

Since $\begin{cases}\overrightarrow{ED} = \overrightarrow{OD}-\overrightarrow{OE} = (x-1)\vec{\imath}-\vec{\jmath} \\ \overrightarrow{EF} = \overrightarrow{OF}-\overrightarrow{OE} = 3\vec{\imath}-6\vec{\jmath}-2\vec{k}\end{cases}$, then we have

$\overrightarrow{ED}\cdot\overrightarrow{EF} = 0 \Rightarrow (x-1)(3)+(-1)(-6)+(0)(-2) = 0 \Rightarrow x = -1.$

Problem 6

(a) Let $\theta = \angle ABC$, θ is found to be

$$\cos\theta = \frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} \ldots\ldots (*)$$

Note that $\begin{cases}\overrightarrow{BA} = \overrightarrow{OA}-\overrightarrow{OB} = 3\vec{\imath}+\vec{\jmath} \\ \overrightarrow{BC} = \overrightarrow{OC}-\overrightarrow{OB} = \vec{\imath}+2\vec{\jmath}\end{cases}$, then we obtain from $(*)$ that

$$\cos\theta = \frac{3(1)+(1)(2)}{\sqrt{3^2+(1)^2}\sqrt{1^2+2^2}} = \frac{5}{\sqrt{10}\sqrt{5}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45°.$$

(b) We need to check whether $\angle ABD = \frac{1}{2}\angle ABC.$

Let $\phi = \angle ABD$, then we have

$$\cos\phi = \frac{\overrightarrow{BA}\cdot\overrightarrow{BD}}{|\overrightarrow{BA}||\overrightarrow{BD}|} \ldots\ldots (**)$$

Since $\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = 2\vec{\imath} + 2\vec{\jmath}$, then we can obtain from $(**)$ that

$$\cos\phi = \frac{3(2)+(1)(2)}{\sqrt{10}\sqrt{8}} = \frac{8}{\sqrt{80}} = \frac{2}{\sqrt{5}} \Rightarrow \phi \approx 26.57°.$$

It is obvious that $\angle ABD \neq \frac{1}{2}\angle ABC$. Therefore BD is not an angle bisector of $\angle ABC$.

Problem 7

(Method 1)

Let $\theta = \angle BAC$, then

$$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{2}{(4)(4)} = \frac{1}{8}.$$

Using cosine law, we have

$$BC^2 = AB^2 + AC^2 - 2(AB)(AC)\cos\theta = 4^2 + 4^2 - 2(4)(4)\left(\frac{1}{8}\right) = 28 \Rightarrow BC = \sqrt{28}.$$

(Method 2)

Using the fact that $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = \overrightarrow{AC} - \overrightarrow{AB}$ and $|\vec{a}|^2 = \vec{a}\cdot\vec{a}$, we get

$$BC = |\overrightarrow{BC}| = \sqrt{\overrightarrow{BC}\cdot\overrightarrow{BC}} = \sqrt{\left(\overrightarrow{AC}-\overrightarrow{AB}\right)\cdot\left(\overrightarrow{AC}-\overrightarrow{AB}\right)}$$

$$= \sqrt{\left(\overrightarrow{AC}\cdot\overrightarrow{AC}\right) - 2\left(\overrightarrow{AC}\cdot\overrightarrow{AB}\right) + \left(\overrightarrow{AB}\cdot\overrightarrow{AB}\right)} = \sqrt{\left|\overrightarrow{AC}\right|^2 - 2\left(\overrightarrow{AB}\cdot\overrightarrow{AC}\right) + \left|\overrightarrow{AB}\right|^2} = \sqrt{4^2 - 2(2) + 4^2} = \sqrt{28}.$$

Problem 8

(a) Let θ be the angle between $\vec{a}$ and $\vec{b}$, then we have

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{1}{1(2)} = \frac{1}{2} \Rightarrow \theta = 60°.$$

(b) Using properties of scalar product, we get

$$\left(3\vec{a}-2\vec{b}\right)\cdot\left(\vec{a}+3\vec{b}\right) = 3(\vec{a}\cdot\vec{a}) + 7\left(\vec{a}\cdot\vec{b}\right) - 6\left(\vec{b}\cdot\vec{b}\right) = 3|\vec{a}|^2 + 7\left(\vec{a}\cdot\vec{b}\right) - 6\left|\vec{b}\right|^2$$
$$= 3(1)^2 + 7(1) - 6(2)^2 = -14.$$

$$\left|\vec{a}-2\vec{b}\right| = \sqrt{\left(\vec{a}-2\vec{b}\right)\cdot\left(\vec{a}-2\vec{b}\right)} = \sqrt{(\vec{a}\cdot\vec{a}) - 4\left(\vec{a}\cdot\vec{b}\right) + 4\left(\vec{b}\cdot\vec{b}\right)} = \sqrt{|\vec{a}|^2 - 4\left(\vec{a}\cdot\vec{b}\right) + 4\left|\vec{b}\right|^2}$$
$$= \sqrt{1^2 - 4(1) + 4(2)^2} = \sqrt{13}.$$

(c) Let ϕ be the angle between the vectors $\vec{a}-2\vec{b}$ and $2\vec{a}+3\vec{b}$. We have

$$\cos\phi = \frac{\left(\vec{a}-2\vec{b}\right)\cdot\left(2\vec{a}+3\vec{b}\right)}{\left|\vec{a}-2\vec{b}\right|\left|2\vec{a}+3\vec{b}\right|} \dots\dots (*)$$

Since

$$\left(\vec{a}-2\vec{b}\right)\cdot\left(2\vec{a}+3\vec{b}\right) = 2(\vec{a}\cdot\vec{a}) - \left(\vec{a}\cdot\vec{b}\right) - 6\left(\vec{b}\cdot\vec{b}\right) = 2|\vec{a}|^2 - \left(\vec{a}\cdot\vec{b}\right) - 6\left|\vec{b}\right|^2$$
$$= 2(1)^2 - (1) - 6(2)^2 = -23.$$

and

$$\left|2\vec{a}+3\vec{b}\right| = \sqrt{\left(2\vec{a}+3\vec{b}\right)\cdot\left(2\vec{a}+3\vec{b}\right)} = \sqrt{4(\vec{a}\cdot\vec{a}) + 12\left(\vec{a}\cdot\vec{b}\right) + 9\left(\vec{b}\cdot\vec{b}\right)}$$
$$= \sqrt{4|\vec{a}|^2 + 12\left(\vec{a}\cdot\vec{b}\right) + 9\left|\vec{b}\right|^2} = \sqrt{4(1)^2 + 12(1) + 9(2)^2} = \sqrt{52},$$

then we can conclude from $(*)$ that

$$\cos\phi = \frac{-23}{\sqrt{13}\sqrt{52}} = -\frac{23}{26} \Rightarrow \phi \approx 152.2°.$$

Problem 9

(a) Let θ be the angle between $\vec{a}-2\vec{b}$ and $-9\vec{a}+2\vec{b}$, then we have

$$\cos\theta=\frac{(\vec{a}-2\vec{b})\cdot(-9\vec{a}+2\vec{b})}{|\vec{a}-2\vec{b}||-9\vec{a}+2\vec{b}|}\ldots\ldots(*)$$

We first note that

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\left(\cos^{-1}\frac{3}{5}\right)=2(3)\left(\frac{3}{5}\right)=\frac{18}{5}.$$

Then we have

$$(\vec{a}-2\vec{b})\cdot(-9\vec{a}+2\vec{b})=-9(\vec{a}\cdot\vec{a})+20(\vec{a}\cdot\vec{b})-4(\vec{b}\cdot\vec{b})$$

$$=-9|\vec{a}|^2+20(\vec{a}\cdot\vec{b})-4|\vec{b}|^2=-9(2)^2+20\left(\frac{18}{5}\right)-4(3)^2=0.$$

Then we obtain from $(*)$ that

$$\cos\theta=\frac{0}{|\vec{a}-2\vec{b}||-9\vec{a}+2\vec{b}|}=0\Rightarrow\theta=90^\circ.$$

This shows that the two vectors are perpendicular to each other.

(b) Let ϕ be the angle between $\vec{a}$ and $\vec{a}+k\vec{b}$, then we have

$$\cos\underbrace{\phi}_{=60^\circ}=\frac{\vec{a}\cdot(\vec{a}+k\vec{b})}{|\vec{a}||\vec{a}+k\vec{b}|}\Rightarrow\frac{\vec{a}\cdot(\vec{a}+k\vec{b})}{|\vec{a}||\vec{a}+k\vec{b}|}=\frac{1}{2}\ldots\ldots(**).$$

Note that

$$\vec{a}\cdot(\vec{a}+k\vec{b})=|\vec{a}|^2+k(\vec{a}\cdot\vec{b})=4+\frac{18}{5}k,$$

$$|\vec{a}+k\vec{b}|=\sqrt{(\vec{a}+k\vec{b})\cdot(\vec{a}+k\vec{b})}=\sqrt{|\vec{a}|^2+2k(\vec{a}\cdot\vec{b})+k^2|\vec{b}|^2}=\sqrt{4+\frac{36}{5}k+9k^2},$$

then we obtain from $(**)$ that

$$\frac{4+\frac{18}{5}k}{2\sqrt{4+\frac{36}{5}k+9k^2}}=\frac{1}{2}\Rightarrow\cdots\Rightarrow 99k^2+540k+300=0$$

$$\Rightarrow k=-\frac{40\sqrt{3}}{33}-\frac{30}{11}\text{ or }k=\frac{40\sqrt{3}}{33}-\frac{30}{11}.$$

Problem 10

In the following, θ refers to the angle between the vectors $\vec{a}$ and $\vec{b}$

(a) Note that

$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{3(1)+(-4)(-18)}{5\sqrt{325}}=\frac{3}{\sqrt{13}}\Rightarrow\theta<90^\circ.$$

Then the required projection vector is given by

$$proj_{\vec{b}}\vec{a}=\underbrace{(|\vec{a}|\cos\theta)}_{\text{magnitude}}\times\underbrace{(\hat{b})}_{\text{direction}}\overset{\hat{b}=\frac{\vec{b}}{|\vec{b}|}}{\widehat{=}}5\left(\frac{3}{\sqrt{13}}\right)\times\frac{\vec{\imath}-18\vec{\jmath}}{\sqrt{325}}=\frac{3}{13}\vec{\imath}-\frac{54}{13}\vec{\jmath}.$$

(b) Note that

$$\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}=\frac{2(6)+(-3)(-2)+(-6)(11)}{\sqrt{49}\sqrt{161}}=\frac{-48}{7\sqrt{161}}\Rightarrow\theta>90^\circ.$$

Then the required projection vector is given by

$$proj_{\vec{b}}\vec{a} = \underbrace{(|\vec{a}|\cos(180^\circ - \theta))}_{\text{magnitude}} \times \underbrace{(-\hat{b})}_{\text{direction}} \overset{\hat{b}=\frac{\vec{b}}{|\vec{b}|}}{\cong} -|\vec{a}|\cos\theta \times \left(-\frac{6\vec{\imath} - 2\vec{\jmath} + 11\vec{k}}{\sqrt{161}}\right)$$

$$= -7\left(\frac{-48}{7\sqrt{161}}\right) \times \frac{-6\vec{\imath} + 2\vec{\jmath} - 11\vec{k}}{\sqrt{161}} = -\frac{288}{161}\vec{\imath} + \frac{96}{161}\vec{\jmath} - \frac{528}{161}\vec{k}.$$

(c) Note that

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{(-1)(1) + (2)(7) + (2)(7)}{3\sqrt{99}} = \frac{9}{\sqrt{99}} \Rightarrow \theta < 90^\circ.$$

Then the required projection vector is given by

$$proj_{\vec{b}}\vec{a} = \underbrace{(|\vec{a}|\cos\theta)}_{\text{magnitude}} \times \underbrace{(\hat{b})}_{\text{direction}} \overset{\hat{b}=\frac{\vec{b}}{|\vec{b}|}}{\cong} 3\left(\frac{9}{\sqrt{99}}\right) \times \frac{\vec{\imath} + 7\vec{\jmath} + 7\vec{k}}{\sqrt{99}} = \frac{3}{11}\vec{\imath} + \frac{21}{11}\vec{\jmath} + \frac{21}{11}\vec{k}.$$

Problem 11

(a) Using the same method as in Example 10, we need to find $|\overrightarrow{AC}|$ and $|proj_{\overrightarrow{AB}}\overrightarrow{AC}|$.
First, we note that

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -\vec{\imath}, \qquad \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\vec{\imath} + \vec{\jmath} + 3\vec{k}$$

It remains to find the magnitude of $proj_{\overrightarrow{AB}}\overrightarrow{AC}$.
Let θ be the angle between $\overrightarrow{AC}$ and $\overrightarrow{AB}$

$$\cos\theta = \frac{\overrightarrow{AC}\cdot\overrightarrow{AB}}{|\overrightarrow{AC}||\overrightarrow{AB}|} = \frac{(-1)(-2) + 0(1) + 0(3)}{(1)\sqrt{14}} = \frac{2}{\sqrt{14}} \Rightarrow \theta < 90^\circ.$$

Thus the magnitude of $proj_{\overrightarrow{AB}}\overrightarrow{AC}$ is given by

$$|\overrightarrow{AC}|\cos\theta = \sqrt{1}\left(\frac{2}{\sqrt{14}}\right) = \frac{2}{\sqrt{14}}.$$

Therefore the required distance d is given by

$$d = \sqrt{|\overrightarrow{AC}|^2 - |proj_{\overrightarrow{AB}}\overrightarrow{AC}|^2} = \sqrt{(1)^2 - \left(\frac{2}{\sqrt{14}}\right)^2} = \sqrt{\frac{10}{14}}.$$

(b) Using the same method as in Example 10, we need to find $|\overrightarrow{DF}|$ and $|proj_{\overrightarrow{DE}}\overrightarrow{DF}|$.
First, we note that

$$\overrightarrow{DF} = \overrightarrow{OF} - \overrightarrow{OD} = -\vec{\imath} + 4\vec{\jmath} - 2\vec{k}, \qquad \overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD} = -2\vec{\imath} + \vec{\jmath}$$

It remains to find the magnitude of $proj_{\overrightarrow{DE}}\overrightarrow{DF}$.
Let ϕ be the angle between $\overrightarrow{DF}$ and $\overrightarrow{DE}$

$$\cos\phi = \frac{\overrightarrow{DF}\cdot\overrightarrow{DE}}{|\overrightarrow{DF}||\overrightarrow{DE}|} = \frac{(-1)(-2) + 4(1) + (-2)(0)}{\sqrt{21}\sqrt{5}} = \frac{6}{\sqrt{105}} \Rightarrow \phi < 90^\circ.$$

Thus the magnitude of $proj_{\overrightarrow{DE}}\overrightarrow{DF}$ is given by

$$|\overrightarrow{DF}|\cos\phi = \sqrt{21}\left(\frac{6}{\sqrt{105}}\right) = \frac{6}{\sqrt{5}}.$$

Therefore the required distance d is given by

$$d = \sqrt{|\overrightarrow{DF}|^2 - |proj_{\overrightarrow{DE}}\overrightarrow{DF}|^2} = \sqrt{(\sqrt{21})^2 - \left(\frac{6}{\sqrt{5}}\right)^2} = \sqrt{\frac{69}{5}}.$$

Problem 12

(a)
$$\vec{a}\times\vec{b} = (\vec{\imath} + 3\vec{\jmath}) \times (-2\vec{\jmath} + 5\vec{k}) = -2(\vec{\imath}\times\vec{\jmath}) + 5(\vec{\imath}\times\vec{k}) - 6(\vec{\jmath}\times\vec{\jmath}) + 15(\vec{\jmath}\times\vec{k})$$
$$= -2\vec{k} + 5(-\vec{\jmath}) - 6(\vec{0}) + 15\vec{\imath} = 15\vec{\imath} - 5\vec{\jmath} - 2\vec{k}.$$

(b) $\vec{a}\times\vec{b} = (\vec{\imath}+\vec{\jmath}-2\vec{k})\times(-3\vec{\imath}+2\vec{\jmath}+5\vec{k})$

$= -3(\vec{\imath}\times\vec{\imath}) + 2(\vec{\imath}\times\vec{\jmath}) + 5(\vec{\imath}\times\vec{k}) - 3(\vec{\jmath}\times\vec{\imath}) + 2(\vec{\jmath}\times\vec{\jmath}) + 5(\vec{\jmath}\times\vec{k}) + 6(\vec{k}\times\vec{\imath}) - 4(\vec{k}\times\vec{\jmath}) - 10(\vec{k}\times\vec{k})$

$= -3(\vec{0}) + 2\vec{k} + 5(-\vec{\jmath}) - 3(-\vec{k}) + 2(\vec{0}) + 5\vec{\imath} + 6\vec{\jmath} - 4(-\vec{\imath}) - 10(\vec{0}) = 9\vec{\imath}+\vec{\jmath}+5\vec{k}.$

(c) $\vec{a}\times\vec{b} = (-3\vec{\imath}+\vec{\jmath}+3\vec{k})\times(6\vec{\jmath}+\vec{k})$

$= -18(\vec{\imath}\times\vec{\jmath}) - 3(\vec{\imath}\times\vec{k}) + 6(\vec{\jmath}\times\vec{\jmath}) + (\vec{\jmath}\times\vec{k}) + 18(\vec{k}\times\vec{\jmath}) + 3(\vec{k}\times\vec{k})$

$= -18\vec{k} - 3(-\vec{\jmath}) + 6(\vec{0}) + \vec{\imath} + 18(-\vec{\imath}) + 3(\vec{0}) = -17\vec{\imath}+3\vec{\jmath}-18\vec{k}.$

(d) $\vec{a}\times\vec{b} = (\vec{\jmath}+\vec{k})\times(3\vec{\imath}-\vec{\jmath}+2\vec{k})$

$= 3(\vec{\jmath}\times\vec{\imath}) - (\vec{\jmath}\times\vec{\jmath}) + 2(\vec{\jmath}\times\vec{k}) + 3(\vec{k}\times\vec{\imath}) - (\vec{k}\times\vec{\jmath}) + 2(\vec{k}\times\vec{k})$

$= 3(-\vec{k}) - (\vec{0}) + 2\vec{\imath} + 3\vec{\jmath} - (-\vec{\imath}) + 2(\vec{0}) = 3\vec{\imath}+3\vec{\jmath}+3\vec{k}.$

Problem 13

According to the definition of vector product, the vector $\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. Then we conclude that

$$\vec{a}\cdot(\vec{a}\times\vec{b}) = |\vec{a}||\vec{a}\times\vec{b}|\cos 90^\circ = 0.$$

Problem 14

(a) We first note that $\begin{cases}\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA} = 2\vec{\imath}-3\vec{\jmath}-2\vec{k}\\ \overrightarrow{AC} = \overrightarrow{OC}-\overrightarrow{OA} = -3\vec{\imath}-2\vec{\jmath}+\vec{k}\end{cases}$. According to the definition of vector product, the vector $\overrightarrow{AB}\times\overrightarrow{AC}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Thus the required vector is found to be

$\overrightarrow{AB}\times\overrightarrow{AC} = (2\vec{\imath}-3\vec{\jmath}-2\vec{k})\times(-3\vec{\imath}-2\vec{\jmath}+\vec{k})$

$= -6(\vec{\imath}\times\vec{\imath}) - 4(\vec{\imath}\times\vec{\jmath}) + 2(\vec{\imath}\times\vec{k}) + 9(\vec{\jmath}\times\vec{\imath}) + 6(\vec{\jmath}\times\vec{\jmath}) - 3(\vec{\jmath}\times\vec{k}) + 6(\vec{k}\times\vec{\imath}) + 4(\vec{k}\times\vec{\jmath}) - 2(\vec{k}\times\vec{k})$

$= -6(\vec{0}) - 4(\vec{k}) + 2(-\vec{\jmath}) + 9(-\vec{k}) + 6(\vec{0}) - 3(\vec{\imath}) + 6(\vec{\jmath}) + 4(-\vec{\imath}) - 2(\vec{0})$

$= -7\vec{\imath}+4\vec{\jmath}-13\vec{k}.$

(b) Note that $\overrightarrow{BC} = \overrightarrow{OC}-\overrightarrow{OB} = -5\vec{\imath}+\vec{\jmath}+3\vec{k}$, the required vector is given by

$$\vec{a} = |\overrightarrow{BC}|\times(\widehat{AB\times AC}) = \sqrt{35}\times\frac{\overrightarrow{AB}\times\overrightarrow{AC}}{|\overrightarrow{AB}\times\overrightarrow{AC}|} = \sqrt{35}\times\frac{-7\vec{\imath}+4\vec{\jmath}-13\vec{k}}{\sqrt{234}}$$

$$= -\frac{7\sqrt{35}}{\sqrt{234}}\vec{\imath} + \frac{4\sqrt{35}}{\sqrt{234}}\vec{\jmath} - \frac{13\sqrt{35}}{\sqrt{234}}\vec{k}.$$

(c) For any point $P=(x,y,z)$ in the plane, the vector $\overrightarrow{AP}$ lies on the same plane and its perpendicular to the vector $\overrightarrow{AB}\times\overrightarrow{AC}$. Note that $\overrightarrow{AP} = \overrightarrow{OP}-\overrightarrow{OA} = (x-1)\vec{\imath}+(y-2)\vec{\jmath}+z\vec{k}$, then we have

$$\overrightarrow{AP}\cdot(\overrightarrow{AB}\times\overrightarrow{AC}) = 0$$
$$\Rightarrow -7(x-1)+4(y-2)-13(z) = 0$$
$$\Rightarrow 7x-4y+13z = -1.$$

Thus the equation of plane is $7x-4y+13z=-1$.

Problem 15

(a) According to the definition, the vector $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. The required vector is found to be

$$\vec{c} = \vec{a} \times \vec{b} = \left(2\vec{\imath} - \vec{\jmath} + 2\vec{k}\right) \times \left(4\vec{\imath} - 4\vec{\jmath} + 3\vec{k}\right)$$
$$= 8(\vec{\imath} \times \vec{\imath}) - 8(\vec{\imath} \times \vec{\jmath}) + 6\left(\vec{\imath} \times \vec{k}\right) - 4(\vec{\jmath} \times \vec{\imath}) + 4(\vec{\jmath} \times \vec{\jmath}) - 3\left(\vec{\jmath} \times \vec{k}\right) + 8\left(\vec{k} \times \vec{\imath}\right) - 8\left(\vec{k} \times \vec{\jmath}\right) + 6\left(\vec{k} \times \vec{k}\right)$$
$$= 8\left(\vec{0}\right) - 8\left(\vec{k}\right) + 6(-\vec{\jmath}) - 4\left(-\vec{k}\right) + 4\left(\vec{0}\right) - 3(\vec{\imath}) + 8(\vec{\jmath}) - 8(-\vec{\imath}) + 6\left(\vec{0}\right) = 5\vec{\imath} + 2\vec{\jmath} - 4\vec{k}.$$

(b) The area of the triangle is

$$= \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\sqrt{5^2 + 2^2 + (-4)^2} = \frac{\sqrt{45}}{2}.$$

(c) For any point $P = (x, y, z)$ in the plane, the vector $\overrightarrow{AP}$ lies on the same plane and its perpendicular to the vector $\vec{c} = \vec{a} \times \vec{b}$. Note that $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = (x - 1)\vec{\imath} + (y - 1)\vec{\jmath} + (z - 1)\vec{k}$, then we have

$$\overrightarrow{AP} \cdot \vec{c} = 0$$
$$\Rightarrow 5(x - 1) + 2(y - 1) - 4(z - 1) = 0$$
$$\Rightarrow 5x + 2y - 4z = 3.$$

Thus the equation of the plane is $5x + 2y - 4z = 3$.

(d) The volume of parallelepiped is given by

$$V = \left|\vec{d} \cdot \left(\vec{a} \times \vec{b}\right)\right| = \left|\left(\vec{\imath} + 2\vec{k}\right) \cdot \left(5\vec{\imath} + 2\vec{\jmath} - 4\vec{k}\right)\right| = |(1)(5) + (0)(2) + 2(-4)| = |-3| = 3.$$

Since the volume is non-zero, thus the vectors $\vec{a}$, $\vec{b}$ and $\vec{d}$ are not coplanar.

Problem 16

Note that $\begin{cases} \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -3\vec{\imath} + 8\vec{\jmath} - 5\vec{k} \\ \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -12\vec{\imath} + 4\vec{\jmath} - 6\vec{k} \end{cases}$ and

$$\overrightarrow{AB} \times \overrightarrow{AC} = \left(-3\vec{\imath} + 8\vec{\jmath} - 5\vec{k}\right) \times \left(-12\vec{\imath} + 4\vec{\jmath} - 6\vec{k}\right)$$
$$= 36(\vec{\imath} \times \vec{\imath}) - 12(\vec{\imath} \times \vec{\jmath}) + 18\left(\vec{\imath} \times \vec{k}\right) - 96(\vec{\jmath} \times \vec{\imath}) + 32(\vec{\jmath} \times \vec{\jmath}) - 48\left(\vec{\jmath} \times \vec{k}\right) + 60\left(\vec{k} \times \vec{\imath}\right) - 20\left(\vec{k} \times \vec{\jmath}\right) + 30\left(\vec{k} \times \vec{k}\right)$$
$$= 36\left(\vec{0}\right) - 12\left(\vec{k}\right) + 18(-\vec{\jmath}) - 96\left(-\vec{k}\right) + 32\left(\vec{0}\right) - 48(\vec{\imath}) + 60(\vec{\jmath}) - 20(-\vec{\imath}) - 30\left(\vec{0}\right)$$
$$= -28\vec{\imath} + 42\vec{\jmath} + 84\vec{k}.$$

Then

- the area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{(-28)^2 + 42^2 + 84^2} = 49.$
- the area of parallelogram $= \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = 98.$

Problem 17

Recall that the points A, B and C are colinear if and only if $\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = 0$

(a) We note that $\begin{cases} \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 3\vec{\imath} + 4\vec{\jmath} \\ \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 2\vec{\imath} + \vec{\jmath} - \vec{k} \end{cases}$ and $\overrightarrow{AB} \times \overrightarrow{AC} = \cdots = -4\vec{\imath} + 3\vec{\jmath} - 5\vec{k}$. Since $\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-4)^2 + 3^2 + (-5)^2} = \sqrt{50} \neq 0$, so A, B and C are not colinear.

(b) We note that $\begin{cases} \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\vec{\imath} - \vec{\jmath} + 3\vec{k} \\ \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 2\vec{\imath} + \vec{\jmath} - 3\vec{k} \end{cases}$ and $\overrightarrow{AB} \times \overrightarrow{AC} = \cdots = 0\vec{\imath} + 0\vec{\jmath} + 0\vec{k}$. Since $\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{0^2 + 0^2 + 0^2} = 0$, so A, B and C are colinear.

Problem 18

Recall that the points A, B, C and D are coplanar if and only if the volume of parallelepiped with A, B, C and D as adjacent vertices is 0, i.e.

$$\left|\overrightarrow{AB}\cdot\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)\right| = 0 \Leftrightarrow \overrightarrow{AB}\cdot\left(\overrightarrow{AC}\times\overrightarrow{AD}\right) = 0.$$

(a) Note that $\begin{cases} \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\vec{\imath} + 2\vec{k} \\ \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -4\vec{\imath} - 2\vec{\jmath} + 6\vec{k}, \\ \overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = 2\vec{\jmath} - 2\vec{k} \end{cases}$ then

$$\begin{aligned}\overrightarrow{AB}\cdot\left(\overrightarrow{AC}\times\overrightarrow{AD}\right) &= \cdots = \overrightarrow{AB}\cdot\left(-8\vec{\imath} - 8\vec{\jmath} - 8\vec{k}\right) = \left(-2\vec{\imath} + 2\vec{k}\right)\cdot\left(-8\vec{\imath} - 8\vec{\jmath} - 8\vec{k}\right) \\ &= (-2)(-8) + 0(-8) + 2(-8) = 0.\end{aligned}$$

Therefore volume of parallelepiped = 0 and the four points are coplanar.

(b) Note that $\begin{cases} \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\vec{\jmath} + 2\vec{k} \\ \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -2\vec{\imath} - \vec{\jmath} + \vec{k}, \\ \overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \vec{\imath} - 2\vec{\jmath} + \vec{k} \end{cases}$ then

$$\begin{aligned}\overrightarrow{AB}\cdot\left(\overrightarrow{AC}\times\overrightarrow{AD}\right) &= \cdots = \overrightarrow{AB}\cdot\left(\vec{\imath} + 3\vec{\jmath} + 5\vec{k}\right) = \left(-2\vec{\jmath} + 2\vec{k}\right)\cdot\left(\vec{\imath} + 3\vec{\jmath} + 5\vec{k}\right) \\ &= (0)(1) + (-2)(3) + 2(5) = 4.\end{aligned}$$

Therefore, volume of parallelepiped = 4 and the four points are not coplanar.

Problem 19

(a) Following the procedure as in Example 15, we need to obtain the vectors $\overrightarrow{AD}$ and $\vec{n} = \overrightarrow{AB}\times\overrightarrow{AC}$, one can find that

- $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = -2\vec{\imath} + 2\vec{\jmath} - \vec{k}$,
- $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -\vec{\imath} + 2\vec{\jmath} + 3\vec{k}$ and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -2\vec{\imath} + \vec{\jmath} + \vec{k}$
  $\Rightarrow \vec{n} = \overrightarrow{AB}\times\overrightarrow{AC} = \cdots = -\vec{\imath} - 5\vec{\jmath} + 3\vec{k}$.

Let θ be the angle between $\overrightarrow{AD}$ and $\vec{n}$, then we have

$$\cos\theta = \frac{\overrightarrow{AD}\cdot\vec{n}}{\left|\overrightarrow{AD}\right||\vec{n}|} = \frac{(-2)(-1) + (2)(-5) + (-1)(3)}{3\sqrt{35}} = \frac{-11}{3\sqrt{35}}.$$

Then the required distance is simply the magnitude of the projection vector $proj_{\vec{n}}\overrightarrow{AD}$ and is given by

$$\left|\left|\overrightarrow{AD}\right|\cos\theta\right| = \left|3\left(\frac{-11}{3\sqrt{35}}\right)\right| = \frac{11}{\sqrt{35}}.$$

(b) We first note that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -\vec{\imath} - 2\vec{\jmath} + 7\vec{k}$.

Let θ be the angle between $\overrightarrow{BA}$ and $\vec{n}$, then we have

$$\cos\theta = \frac{\overrightarrow{AB}\cdot\vec{n}}{\left|\overrightarrow{AB}\right||\vec{n}|} = \frac{(-1)(-1) + (-2)(-1) + (7)(-1)}{\sqrt{54}\sqrt{3}} = \frac{-4}{\sqrt{162}}.$$

The required distance is the magnitude of the projection vector $proj_{\vec{n}}\overrightarrow{AB}$ and is given by

$$\left|\left|\overrightarrow{AB}\right|\cos\theta\right| = \left|\sqrt{54}\left(\frac{-4}{\sqrt{162}}\right)\right| = \frac{4}{\sqrt{3}}.$$

Problem 20

(a) We let $A = (5,0,-1)$ and $B = (6,2,-2)$ be two points on the line L_1. We also let $C = (2,4,0)$ and $D = (3,3,1)$ be two points on the line L_2.

Following the procedure as in Example 16, we need to obtain the vectors $\overrightarrow{AD}$ and $\vec{n} = \overrightarrow{AB}\times\overrightarrow{CD}$.

- $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = -2\vec{\imath} + 3\vec{\jmath} + 2\vec{k}$.
- $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{\imath} + 2\vec{\jmath} - \vec{k}$ and $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \vec{\imath} - \vec{\jmath} + \vec{k}$.

The normal vector $\vec{n}$ is found to be
$\vec{n} = \overrightarrow{AB} \times \overrightarrow{CD} = \left(\vec{\imath} + 2\vec{\jmath} - \vec{k}\right) \times \left(\vec{\imath} - \vec{\jmath} + \vec{k}\right) = \vec{\imath} - 2\vec{\jmath} - 3\vec{k}.$

Let θ be the angle between $\overrightarrow{AD}$ and $\vec{n}$, then we have

$$\cos\theta = \frac{\overrightarrow{AD} \cdot \vec{n}}{\left|\overrightarrow{AD}\right||\vec{n}|} = \frac{(-2)(1) + (3)(-2) + (2)(-3)}{\sqrt{17}\sqrt{14}} = \frac{-14}{\sqrt{17}\sqrt{14}}.$$

Then the required distance is simply the magnitude of the projection vector $proj_{\vec{n}}\overrightarrow{AD}$ and is given by

$$|\left|\overrightarrow{AD}\right|\cos\theta\,| = |\sqrt{17}\left(\frac{-14}{\sqrt{17}\sqrt{14}}\right)| = \sqrt{14}.$$

(b) We let $A = (1,1,1)$ and $B = (2,1,2)$ be two points on the line L_1. We also let $C = (2,1,0)$ and $D = (3,2,0)$ be two points on the line L_2.

Following the procedure as in Example 16, we need to obtain the vectors $\overrightarrow{AD}$ and $\vec{n} = \overrightarrow{AB} \times \overrightarrow{CD}$.

- $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = 2\vec{\imath} + \vec{\jmath} - \vec{k}$.
- $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{\imath} + \vec{k}$ and $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \vec{\imath} + \vec{\jmath}$.
  The normal vector $\vec{n}$ is found to be
  $\vec{n} = \overrightarrow{AB} \times \overrightarrow{CD} = \left(\vec{\imath} + \vec{k}\right) \times (\vec{\imath} + \vec{\jmath}) = -\vec{\imath} + \vec{\jmath} + \vec{k}.$

Let θ be the angle between $\overrightarrow{AD}$ and $\vec{n}$, then we have

$$\cos\theta = \frac{\overrightarrow{AD} \cdot \vec{n}}{\left|\overrightarrow{AD}\right||\vec{n}|} = \frac{(2)(-1) + (1)(1) + (-1)(1)}{\sqrt{6}\sqrt{3}} = \frac{-2}{\sqrt{18}}.$$

Then the required distance is simply the magnitude of the projection vector $proj_{\vec{n}}\overrightarrow{AD}$ and is given by

$$|\left|\overrightarrow{AD}\right|\cos\theta| = |\sqrt{6}\left(\frac{-2}{\sqrt{18}}\right)| = \frac{2}{\sqrt{3}}.$$

Problem 21

(a) Since

$$\left|\vec{a} \times \vec{b}\right| = |(\vec{\imath} - 2\vec{\jmath}) \times (2\vec{\imath} + \vec{\jmath})| = \left|5\vec{k}\right| = 5 \neq 0,$$

thus $\vec{a}$ and $\vec{b}$ are not collinear and these two vectors are linearly independent.

(b) Note that

$$\begin{aligned}\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) &= \vec{a} \cdot \left(-17\vec{\imath} + 9\vec{\jmath} - 11\vec{k}\right) = \left(\vec{\imath} - 2\vec{\jmath} + 3\vec{k}\right) \cdot \left(-17\vec{\imath} + 9\vec{\jmath} - 11\vec{k}\right) \\ &= (1)(-17) - 2(9) + (3)(-11) = -68 \neq 0.\end{aligned}$$

The vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are not coplanar, these three vectors are linearly independent.

(c) Note that

$$\begin{aligned}\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) &= \vec{a} \cdot \left(-4\vec{\imath} - 8\vec{\jmath} - 4\vec{k}\right) = \left(\vec{\imath} + 2\vec{\jmath} - 5\vec{k}\right) \cdot \left(-4\vec{\imath} - 8\vec{\jmath} - 4\vec{k}\right) \\ &= (1)(-4) + 2(-8) + (-5)(-4) = 0.\end{aligned}$$

The vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar, these three vectors are linearly dependent.

Problem 22

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are linearly dependent, then we may have

$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = 0 \Rightarrow \vec{a} \cdot \left(-5m\vec{\imath} - 10\vec{\jmath} - 10m\vec{k}\right) = 0$

$\Rightarrow \left((1-m)\vec{\imath} + 6\vec{\jmath} + 5\vec{k}\right) \cdot \left(-5m\vec{\imath} - 10\vec{\jmath} - 10m\vec{k}\right) = 0$

$\Rightarrow -5m(1-m) + 6(-10) + 5(-10m) = 0$

$\Rightarrow m^2 - 11m - 12 = 0 \Rightarrow (m-12)(m+1) = 0$

$\Rightarrow m = 12 \text{ or } m = -1.$

Problem 23

(a) $$(\vec{a}-\vec{b})\times(\vec{a}+\vec{b}) = \underbrace{(\vec{a}\times\vec{a})}_{\vec{0}} + (\vec{a}\times\vec{b}) - (\vec{b}\times\vec{a}) - \underbrace{(\vec{b}\times\vec{b})}_{=\vec{0}} = (\vec{a}\times\vec{b}) - \big(-(\vec{a}\times\vec{b})\big)$$
$$= 2(\vec{a}\times\vec{b}).$$

(b) Given that $\vec{a}$ and $\vec{b}$ are perpendicular, we have $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos 90° = 0.$
$$|\vec{a}+\vec{b}| = \sqrt{(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})} = \sqrt{(\vec{a}\cdot\vec{a}) + 2(\vec{a}\cdot\vec{b}) + (\vec{b}\cdot\vec{b})}$$
$$\overset{\vec{a}\cdot\vec{b}=0}{\cong} \sqrt{(\vec{a}\cdot\vec{a}) - 2(\vec{a}\cdot\vec{b}) + (\vec{b}\cdot\vec{b})} = \sqrt{(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})} = |\vec{a}-\vec{b}|.$$

(c) If $\vec{a}$ and $\vec{b}$ are parallel, the angle between these two vectors is either $0°$ or $180°$. For either case, we always have $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\underbrace{\sin\theta}_{=0}\,\hat{n} = \vec{0}.$

We consider
$$(\vec{a}-\vec{b})\times(\vec{a}+\vec{b}) \overset{\text{from (a)}}{\cong} 2(\vec{a}\times\vec{b}) = \vec{0}.$$
Since the cross product is 0, the vector $\vec{a}-\vec{b}$ and $\vec{a}+\vec{b}$ may be collinear and hence are parallel.

(d) $$\frac{|\vec{a}\times\vec{b}|}{\vec{a}\cdot\vec{b}} = \frac{|\vec{a}||\vec{b}|\sin\theta}{|\vec{a}||\vec{b}|\cos\theta} = \tan\theta.$$

(e) $$|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2\sin^2\theta = |\vec{a}|^2|\vec{b}|^2(1-\cos^2\theta) = |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta$$
$$= |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2.$$