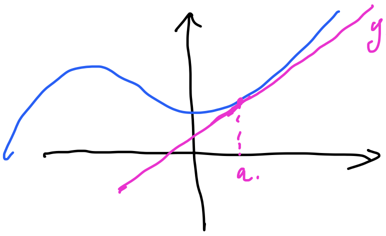
1 Taylor Series

Single variable Case:

Let f be a function, which has continuous derivatives at $x = a$ up to any order.

$y = f(x)$



$y = f(a) + f'(a)(x-a)$.

- Linear approximation:

(Linearization)

$$T_1(x) = f(a) + f'(a)(x-a).$$

eg. $y = e^x$, $x=0$

$f(0) = f'(0) = 1 = f''(0)$

$T_1(x) = 1 + 1(x-0)$
$= x+1$

$T_2(x) = 1 + x + \frac{1}{2}(x-0)^2$

- Quadratic approximation:

$$T_2(x) = \underbrace{f(a) + f'(a)(x-a)}_{T_1(x)} + \frac{f''(a)}{2!}(x-a)^2.$$

$2! = 1 \cdot 2$

$3! = 1 \cdot 2 \cdot 3$

$n! = 1 \cdot 2 \cdot 3 \cdots n$

- Cubic approximation:

$$T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3.$$

- Taylor Series of order n:

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$= f(a) + \sum_{k=1}^{n} \frac{1}{k!}\left[(x-a)\frac{d}{dx}\right]^k f(a)$

- When $a = 0$, Maclaurin Series of order n:

$$M_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

Example. Find linear, quadratic and cubic approximation of $f(x) = e^x$ and $g(x) = \sin x$ at $x = 0$.

$g(0) = 0$

$g'(0) = 1$

$g''(0) = 0$

$g^{(3)}(0) = -1$

$g^{(4)}(0) = 0$

$\vdots$

$T_1(x) = 0 + \frac{1}{1!}x$

$T_2(x) = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2$

$T_3(x) = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3$

$T_n(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7$

$T_n(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n$

$f(0) = f'(0) = f''(0) = e^0 = f^{(n)}(0) = 1$

$$\begin{cases} T_1(x) = 1 + x \\ T_2(x) = 1 + x + \frac{1}{2}x^2 \\ T_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 \end{cases}$$

$f\cdots f$

Single variable Case:

If a function $f(x)$ of one variable is differentiable through order $n+1$ at $x = a$, then for any real number x such that $a - l < x < a + l$, we have

$$f(x) = T_n(x) + R_n(x)$$

where $T_n(x)$ is the Taylor series of f at $x = a$ of order n and

$$R_n(x) = \frac{f^{(n+1)}[a + \theta(x-a)]}{(n+1)!}(x-a)^{n+1}$$

for some $0 < \theta < 1$.

$$f(x) - T_n(x) = \frac{f^{(n+1)}(a+\theta(x-a))}{(n+1)!}(x-a)^{n+1}.$$

$\Downarrow$

$$|f(x) - T_n(x)| \leq \frac{\max_{z\in D} f^{(n+1)}(z)}{(n+1)!}|x-a|^{n+1}.$$

for all $|x-a| \leq c$, $D = \{z : |z-a| \leq c\}$.

eg. $f(x) = e^x$. $T_2(x) = 1 + x + \frac{1}{2!}x^2$.

Let $|x - 0| \leq 0.1$.

$$|f(x) - T_2(x)| \leq \frac{\max_{|z|\leq 0.1} f^{(3)}(z)}{3!}|x|^3$$

$$\leq \frac{\max_{|z|\leq 0.1} e^z}{3!} 0.1^3 \leq \boxed{\frac{e^{0.1}}{6}\cdot 0.001}$$

Multi-Variable Case:

Given a two-variable function $z = f(x, y)$, its linearization or linear approximation of f at $P_0(a, b)$ is the function

$$L(x, y) = f(P_0) + \frac{\partial f}{\partial x}(P_0)(x - a) + \frac{\partial f}{\partial y}(P_0)(y - b),$$

which is the Tangent plane of the surface of f at $P_0(a, b)$ with equation

$$z = f(P_0) + \frac{\partial f}{\partial x}(P_0)(x - a) + \frac{\partial f}{\partial y}(P_0)(y - b).$$

Example $z = f(x, y) = 9 - x^2 - y^2$. Find the tangent plane of f at $P(1, 2, 4)$.

linearization of f.

$f(P) = 4 = f(1,2)$

$f_x(x,y) = 0 - 2x - 0$ $\quad f_x(1,2) = -2\cdot 1 = -2$

$f_y(x,y) = -2y$ $\quad f_y(1,2) = -2\cdot 2 = -4$

$L(x,y) = f(P) + f_x(P)(x-1) + f_y(P)(y-2)$

$= 4 - 2(x-1) - 4(y-2)$

$=$

Example Find the linear approximation of $z = f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$ at the point $(3, 2)$.

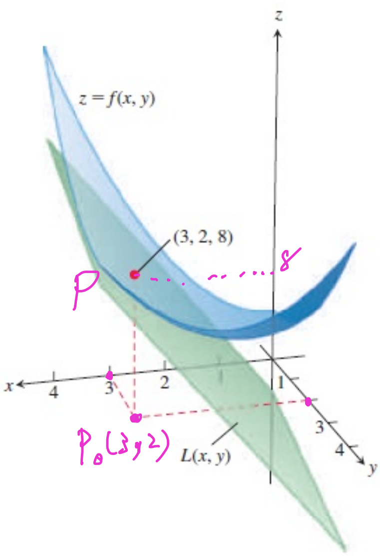
$f(P) = 8$

$f_x(P) = 4$

$f_y(P) = -1$

$L(x,y) = f(P) + f_x(P)(x-3) + f_y(P)(y-2)$

$= 8 + 4(x-3) + (-1)(y-2)$



Given a two-variable function $z = f(x, y)$, its **quadratic approximation** of f at $P\,(a, b)$ is the function

$$Q(x,y) = f(a,b) + \frac{\partial f}{\partial x}(P)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

$$+\frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}(P)(x-a)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(P)(x-a)(y-b) + \frac{\partial^2 f}{\partial y^2}(y-b)^2\right]$$

$$= f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^1 f$$

$$+ \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f \longrightarrow \frac{1}{2!}\left[(x-a)^2\frac{\partial^2}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2}{\partial x \partial y} + (y-b)^2\frac{\partial^2 f}{\partial y^2}\right]$$

Cubic Approximation of $z = f(x, y)$ at $P_0(a, b)$

$$C(x,y) = f(P) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^1 f(P)$$

$$+ \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(P)$$

$$+ \frac{1}{3!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^3 f(P)$$

$$= Q(x,y) + \frac{1}{3!}\left[(x-a)^3 f_{xxx}(P) + 3(x-a)^2(y-b) f_{xxy}(P) + 3(x-a)(y-b)^2 f_{xyy}(P) + (y-b)^3 f_{yyy}(P)\right]$$

Taylor Series of $f(x, y)$ up to n order at $P_0(a, b)$

$$T_n(x,y) = f(P) + \frac{1}{1!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(P)$$

$$+ \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(P)$$

$$+ \cdots + \frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^n f(P).$$

$$= f(P) + \sum_{k=1}^{n} \frac{1}{k!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^k f(P)$$

Particularly when $P(0,0)$, it's called Maclaurin Series.

Example Find the quadratic approximation of $f(x,y) = e^x \sin y$ at $(0,0)$.

$f(P) = 0$

$f_x = e^x \sin y \qquad f_x(P) = 0$

$f_y = e^x \cos y \qquad f_y(P) = 1$

$f_{xx} = e^x \sin y \qquad f_{xx}(P) = 0$

$f_{xy} = e^x \cos y \qquad f_{xy}(P) = 1$

$f_{yy} = -e^x \sin y \qquad f_{yy}(P) = 0$

$$Q(x,y) = f(P) + f_x(P)\,x + f_y(P)\,y + \frac{1}{2!}\left[x^2 f_{xx}(P) + 2 f_{xy}(P)\,xy + f_{yy}(P)\,y^2\right]$$

$$= 0 + 0x + y + \frac{1}{2}(0 + 2xy + 0)$$

$$= y + xy$$

Example $z^3 - 2xz + y = 0$ defines an implicit function $z = f(x,y)$. Find tangent plane approximation of z at $(0,1)$.

linear

$f(P) = -1 \leftarrow z^3 - 2\cdot 0 + 1 = 0 \Rightarrow z = -1$

$f_x(P) = \frac{-2}{3} \leftarrow$ Take $\frac{\partial}{\partial x}$ for both sides $\quad 3z^2 \cdot \frac{\partial z}{\partial x} - 2\left(z + x\frac{\partial z}{\partial x}\right) + 0 = 0 \xRightarrow[y=1,\ z=-1]{x=0} 3\cdot\frac{\partial z}{\partial x} - 2\cdot(-1) = 0 \Downarrow \frac{\partial z}{\partial x} = -\frac{2}{3}$

$f_y(P) = \frac{-1}{3} \leftarrow$ Take $\frac{\partial}{\partial y}$ for $\cdots$ $\quad 3z^2\cdot\frac{\partial z}{\partial y} - 2x\frac{\partial z}{\partial y} + 1 = 0 \xRightarrow[y=1,\ z=-1]{x=0} 3\frac{\partial z}{\partial y} + 1 = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{1}{3}$

$$L(x,y) = -1 + \frac{-2}{3}(x-0) - \frac{1}{3}(y-1)$$

Example Find the linear and quadratic approximation of $f(x, y) = \sin x \sin y$ at the origin $(0, 0)$.

$f(P) = 0$

$f_x = \cos x \sin y \quad f_x(P) = 0$

$f_y = \sin x \cos y \quad f_y(P) = 0$

$f_{xx} = -\sin x \sin y \quad f_{xx}(P) = 0$

$f_{xy} = \cos x \cos y \quad f_{xy}(P) = 1$

$f_{yy} = -\sin x \sin y \quad f_{yy}(P) = 0$

$L(x, y) = 0$

$Q(x, y) = 0 + \frac{1}{2!}[0 + 2 \cdot 1 (x-0)(y-0) + 0]$

$= xy$

Use the linear L and quadratic Q approximation of $f(x, y) = \sin x \sin y$ at the origin to estimate $f(0.1, 0.1) = \sin 0.1 \sin 0.1$.

$f(0.1, 0.1) \approx L(0.1, 0.1) = 0$

$f(0.1, 0.1) \approx Q(0.1, 0.1) = 0.1 \cdot 0.1 = 0.01 \rightarrow (\sin 0.1)^2$

Estimate the error between between f and Q in the region $|x| \leq 0.1$ and $|y| \leq 0.1$.

$|f(x, y) - T_2(x, y)| \leq \frac{M}{3!}(|x-0| + |y-0|)^3$

where $M = \max\{|f_{xxx}|, |f_{xxy}|, |f_{yyx}|, |f_{yyy}|\}$

$= \max\{|-\cos x \sin y|, |-\sin x \cos y|, \ldots\}$

≤ 1

f_{xxx}

$|f - Q| \leq \frac{1}{3!}(|x| + |y|)^3$

$= \frac{1}{6}(0.1 + 0.1)^3 = \boxed{\frac{0.008}{6}}$ when $|x| \leq 0.1$, $|y| \leq 0.1$

Remark

- When (x, y) is around (a, b), $L(x, y)$ provides a good approximation for $f(x, y)$. i.e. $f(x, y) \approx L(x, y)$.

  More precisely,

  $$|f(x,y) - L(x,y)| \leq \frac{M}{2!}(|x-a| + |y-b|)^2$$

  where $M = \max\{\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}\}$.

- When (x, y) is around (a, b), $T_n(x, y)$ provides a good approximation for $f(x, y)$. i.e. $f(x, y) \approx T_n(x, y)$.

  More precisely,

  $$|f(x,y) - T_n(x,y)| \leq \frac{M}{(n+1)!}(|x-a| + |y-b|)^{n+1}$$

  where $M = \max_{s+t=n+1}\{\frac{\partial^{n+1} f}{\partial x^s \partial y^t}\}$.

- When (x, y) is far away from (a, b), $T_n(x, y)$ might be very different from $f(x, y)$.

Example

Find the tangent plane and quadratic surface approximations given by Taylor's Theorem at the point $(1, 0)$ for the cone $f(x, y) = (x^2 + y^2)^{\frac{1}{2}}$.
Hence estimate $f(0.9, 0.1)$.

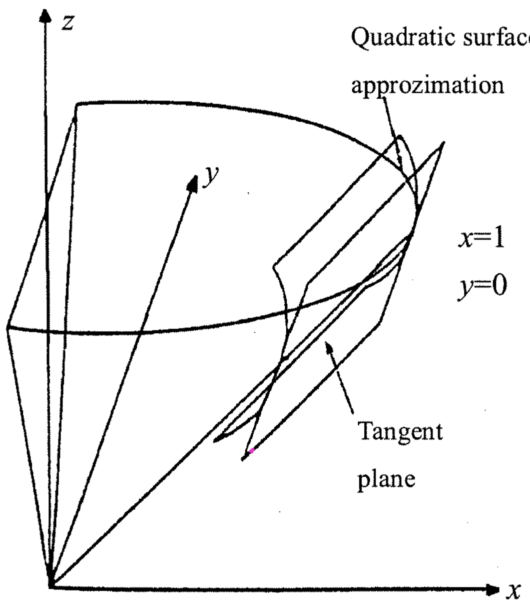
Example

Let $f(x)$ be a smooth function, that is, $f(x)$ has continuous derivatives up to any order.

Then the Taylor series of $f(x)$ about the point a up to order 2 is

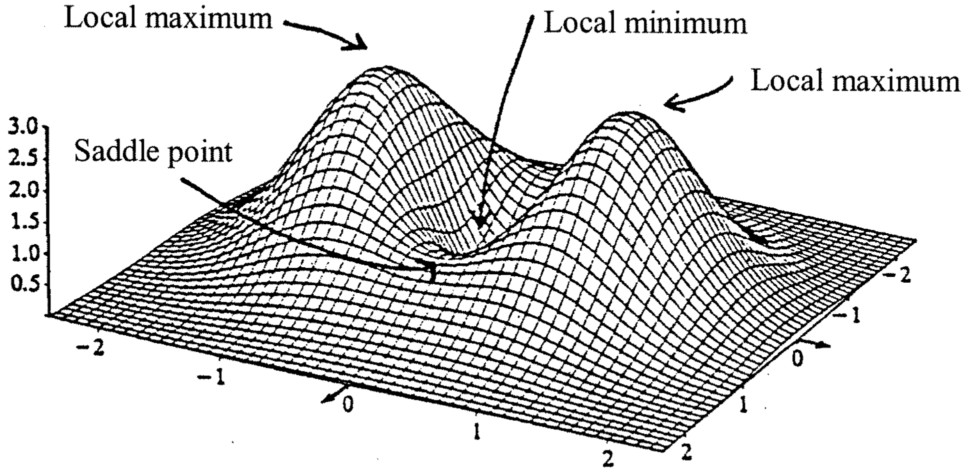
$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Hence or otherwise, find the Taylor series of $g(x, y, z) = \cos(x+y+z) - \cos x \cos y \cos z$ about the point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$ up to order 2 .



Linear and quadratic approximations to the cone

$$z = \sqrt{x^2 + y^2}, \quad (x, y) = (0, 0)$$



$$z = f(x, y) = (x^2 + 3y^2)\, e^{1-x^2-y^2}$$