

CITY UNIVERSITY OF HONG KONG,

# **EE3210 Signals & Systems**

Lecture note

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Semester A, 2021/22

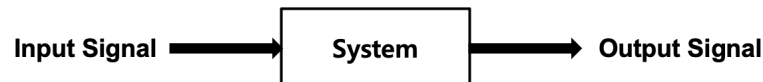
# 1 Signals and Systems

Major References:

- Chapter 1, *Signals and Systems* by Alan V. Oppenheim et. al., 2nd edition, Prentice Hall
- Chapter 1, *Schaum's Outline of Signals and Systems*, 2nd Edition, 2010, McGraw-Hill

## Introduction

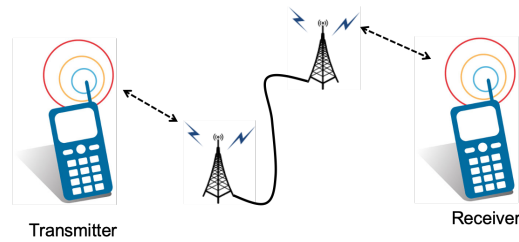
- **Objective:** Describe a **system** (physical, mathematical, or computational) by the way it transforms an **input signal** into an **output signal**.



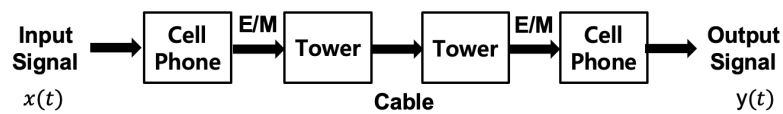
- **System level approach**
  - **Abstraction**
    - \* Identify the system input and output signals → characterize the signal types
    - \* Write input-output relation of the system → Operational transformations (e.g., Fourier analysis, Laplace Transformation).
    - \* Characterize the system types by the input-output relation → system types
  - **Modular design**
    - \* Break down the system into a number of interconnected subsystems (**module**)
    - \* Each module performs some specific task.
    - \* Focus on the **flow of signal**, abstract everything else away
  - **Composite system**
    - \* Determine the input-output relationship between each modules
    - \* Combine the components (**module**) to composite the overall system
    - \* **Component** and **composite systems** have the same form, and are analyzed with same methods.

## Example 1.1

Let's consider a typical mobile communication between the transmitter and receiver. Abstract the system input and output signals, then determine the input-output relationship.



**Sol)** We first describe each module as a cascade of component systems.

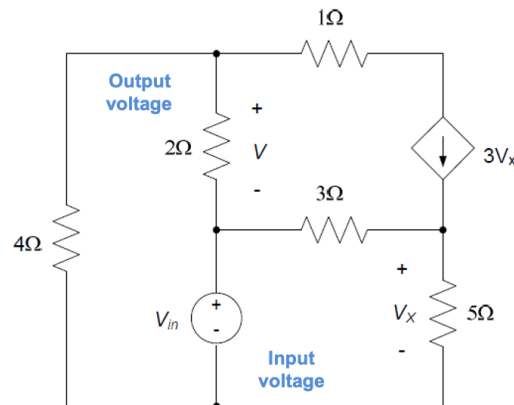


Then we combine the modules into a composite system.

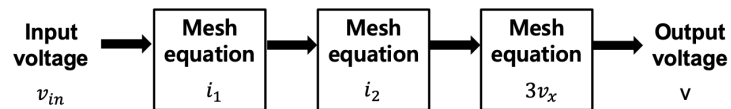
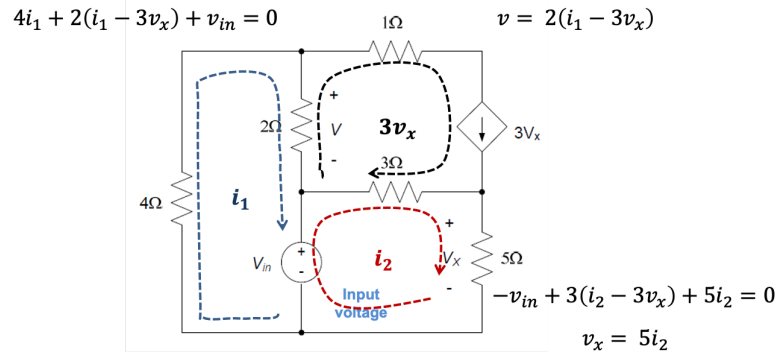


## Example 1.2

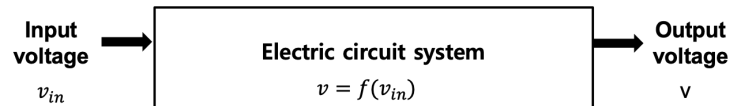
Determine the input-output relationship of the following electrical circuit.



**Sol)** We first characterize the input and output signals, then perform modular design by breaking the circuit into several modular circuits



Then we combine the modules into a composite system



## 1.1 Classification of Signal Types

- [Def] **Signal**  $x(t)$  is a function of an independent variable  $t$  representing a physical behavior of the phenomenon.

### 1. Continuous-Time (CT) and Discrete-Time (DT) Signals

A signal  $x(t)$  is a **continuous-time signal** if  $t$  is a continuous variable. If  $t$  is a discrete variable, i.e.,  $x(t)$  is defined at discrete times, then  $x(t)$  is a **discrete-time signal**.

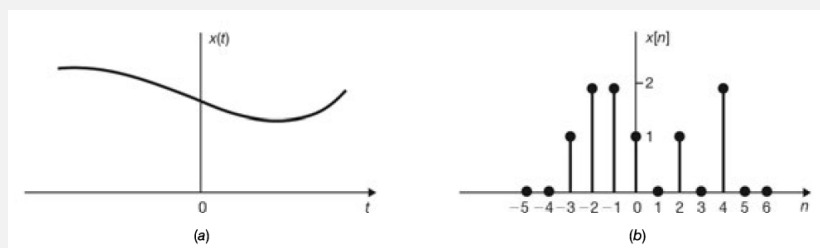


Fig. 1 (a) Continuous-time and (b) discrete-time signals

DT signal  $x[n]$  may be obtained by *sampling* a CT signal  $x(t)$  such as

$$x_n = x[n] = x(t_n)$$

Two representations of the DT signals

- Functional form. For example,

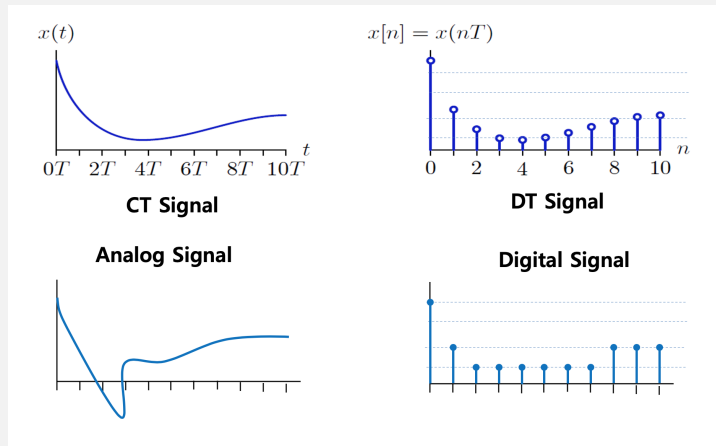
$$x_n = x[n] = \begin{cases} \left(\frac{1}{2}\right)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

- Sequence form. For instance,

$$\{x_n\} = \{\dots, 0, 0, 1, 2, 2, 1, 0, 1, 0, 2, 0, 0, \dots\}$$

## 2. Analog and Digital Signals

A CT signal  $x(t)$  is an **analog signal** if  $x(t)$  can take on any value in the continuous interval  $(a, b)$ . A DT signal  $x[n]$  is a **digital signal** if  $x[n]$  can take on only a finite number of distinct values.



## 3. Periodic and Aperiodic (or nonperiodic) Signals

A signal  $x(t)$  (or  $x[n]$ ) is **periodic** with period  $T$  (or  $N$ ) if there is a positive non-zero value of  $T$  (or  $N$ ) for which the following equality holds

$$\begin{aligned} x(t + T) &= x(t), & \text{for CT signal } x(t), \\ x[n + N] &= x[n], & \text{for DT signal } x[n] \end{aligned} \quad (1.1)$$

Any signal that is not periodic is called a **nonperiodic** (or **aperiodic**) signal.

- Property 1. For a given periodic signal,  $\{T, 2T, 3T, \dots, mT, \dots\}$  are all available period at all  $t$  and any integer  $m$ .
- Property 2. **Fundamental period**  $T_0$  is the smallest positive value of the period  $T$  for which (1.1) holds.

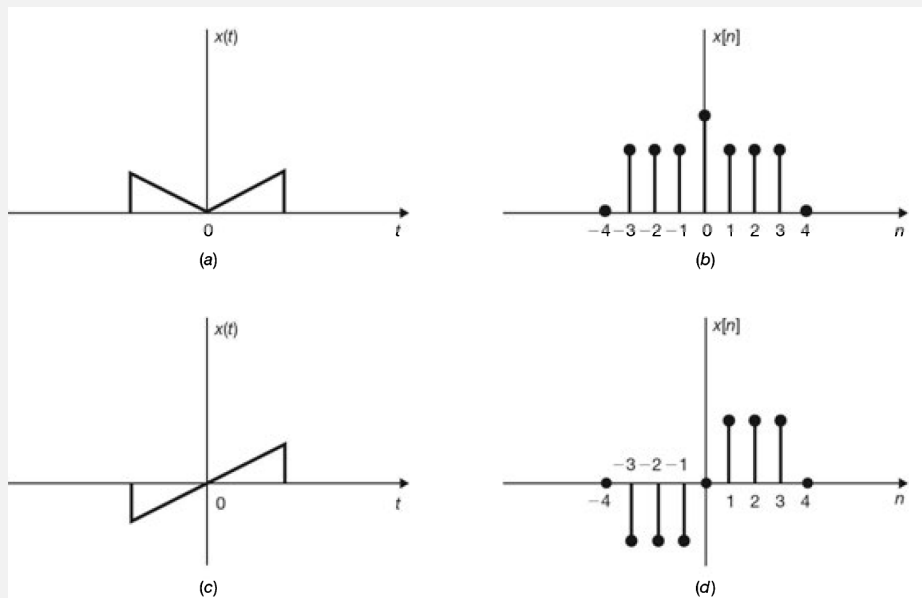
## 4. Even and Odd Signals

A signal is referred to as an **even signal** if

$$x(-t) = x(t) \quad \text{for CT signal } x(t), \quad x[-n] = x[n] \quad \text{for DT signal } x[n] \quad (1.2)$$

A signal is referred to as an **odd signal** if

$$x(-t) = -x(t) \quad \text{for CT signal } x(t), \quad x[-n] = -x[n] \quad \text{for DT signal } x[n] \quad (1.3)$$



- Property 1. Any signal  $x(t)$  or  $x[n]$  can be expressed as a sum of two signals

$$\begin{aligned} x(t) &= x_e(t) + x_o(t) \quad \text{for CT signal,} \\ x[n] &= x_e[n] + x_o[n] \quad \text{for DT signal,} \end{aligned} \quad (1.4)$$

where  $x_e(t)$  (or  $x_e[n]$ ) is the even part,  $x_o(t)$  (or  $x_o[n]$ ) is the odd part, and these components are related to the original signal  $x(t)$  (or  $x[n]$ ) as follows

$$\begin{aligned} x_e(t) &= \frac{1}{2} [x(t) + x(-t)], \\ x_o(t) &= \frac{1}{2} [x(t) - x(-t)]. \end{aligned} \quad (1.5)$$

- Property 2. Product of signals
  - Even signal  $\times$  Even signal = Even signal, Odd  $\times$  odd = Even signal
  - Even signal  $\times$  odd signal = odd signal

## 5. Energy and Power Signals

Energy  $E$  of a signal  $x(t)$  (or  $x[n]$ ) is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad \text{for CT signal,} \quad E = \sum_{n=-\infty}^{\infty} |x[n]|^2 \quad \text{for DT signal,} \quad (1.6)$$

whereas the Power  $P$  of a signal is defined as follows

$$P = \begin{cases} \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt & \text{for CT signal,} \\ \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x[n]|^2 & \text{for DT signal} \end{cases} \quad (1.7)$$

- **Energy signal** has finite energy and zero power

$$0 < E < \infty, \quad P = 0.$$

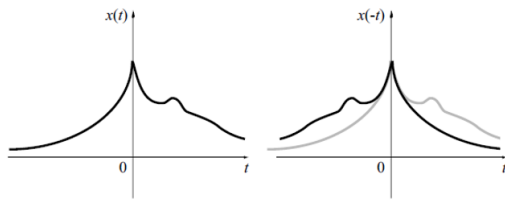
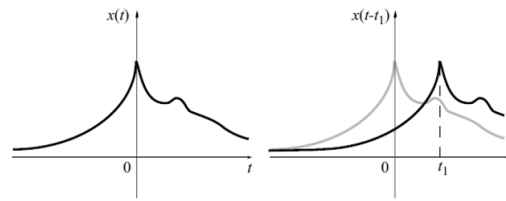
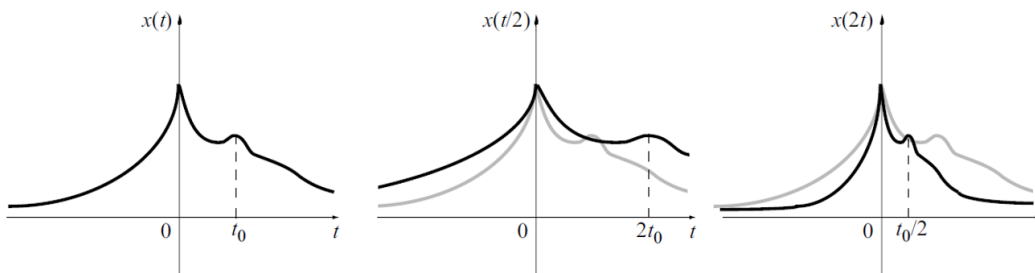
- **Power signal** has finite power and infinite energy

$$0 < P < \infty, \quad E = \infty.$$

- Signals that satisfy neither property are neither energy signals nor power signals.

## 1.2 Basic Signal Operations

- **Time Reversal:** Flip the signal around the vertical axis  $x(t) \rightarrow x(-t)$
- **Time Shifts:** Shift the signal to left or right  $x(t) \rightarrow x(t - t_0)$ 
  - **Right-shift** if  $t_0 > 0$ , **Left-shift** if  $t_0 < 0$ .
- **Time Scaling:** Linearly stretch or compress the signal  $x(t) \rightarrow x(ct)$ 
  - **Compression** if  $|c| > 1$ , **Expansion** if  $|c| < 1$ .
- **Affine Transformation:**  $x(t) \rightarrow x(\alpha t + \beta) = x(\alpha(t + \beta/\alpha))$  for any real  $\alpha, \beta$ 
  - Step 1. **Scale** by  $\alpha$ . If  $\alpha < 0$ , reflection across  $y$ -axis
  - Step 2. **Shift** by  $-\beta/\alpha$ .
    - \* If  $\alpha$  and  $\beta$  have different signs, right-shift.
    - \* If  $\alpha$  and  $\beta$  have same signs, left shift.

**Time Reflection:**  $x(t) \longrightarrow x(-t)$ **Time shifts:**  $x(t) \longrightarrow x(t - t_1)$ **Time scaling:**  $x(t) \longrightarrow x(ct)$ 

### 1.3 Example of Important Signals

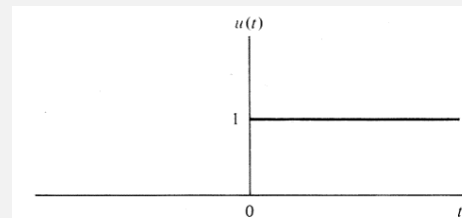
#### 1. Unit Step Function (also referred as Heaviside unit function)

- **Definition**

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0, \end{cases} \quad (1.8)$$

- **Properties**

- Aperiodic signal
- Power signal  $P = 1/2$
- Infinite Energy  $E = \infty$



**Functions related to the step function  $u(t)$**

#### a) Signum Function

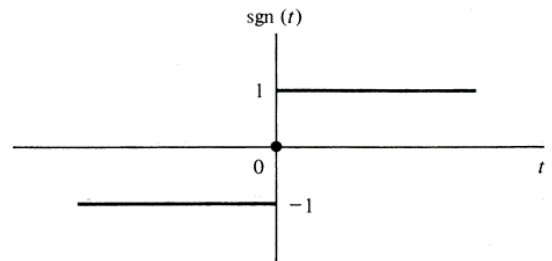
- **Definition**

$$\text{sgn}(t) = 2u(t) - 1 = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$



- Properties

- Aperiodic & odd signal
- Power signal  $P = 1$
- Infinite Energy  $E = \infty$



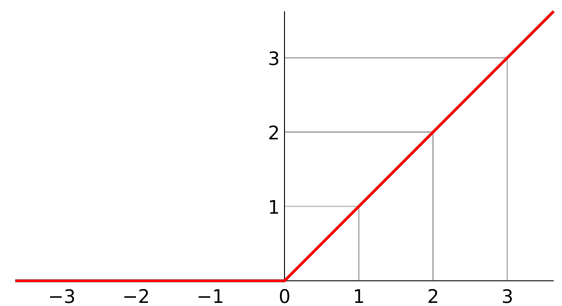
b) **Ramp Function**

- Definition

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}, \quad \int_{-\infty}^t u(\tau) d\tau = r(t)$$

- Properties

- Aperiodic
- Infinite Power  $P = \infty$
- Infinite Energy  $E = \infty$



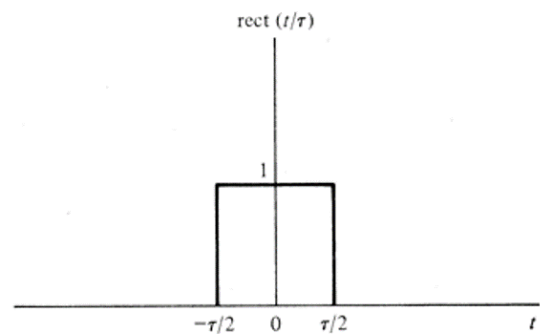
c) **Rectangular Pulse**

- Definition

$$\text{rect}(t/\tau) = \begin{cases} 1, & |t| < \frac{\tau}{2} \\ 0, & |t| > \frac{\tau}{2} \end{cases}, \quad \text{rect}(t/\tau) = u\left(t + \frac{\tau}{2}\right) - u\left(t - \frac{\tau}{2}\right) \quad (1.9)$$

- Properties

- Aperiodic & Even signal
- Zero Power  $P = 0$
- Energy Signal  $E = \tau$



2. Unit Impulse Function (also referred as *Dirac delta function*)

## • Definition

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}, \quad \int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (1.10)$$

## • Properties

## – Sampling Property

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

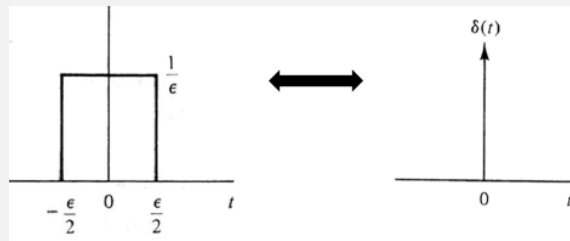
## – Sifting Property

$$\int_a^b x(t)\delta(t - t_0) dt = \begin{cases} x(t_0), & \text{if } a < t_0 < b \\ 0, & \text{otherwise} \end{cases}$$

- Impulse function is the *building block of any signal*, i.e., arbitrary signal can be represented as an infinite sum of impulse function and signal amplitude.

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau) d\tau \quad (1.11)$$

## Relationship between Rectangular Pulse and Impulse Function



- |  |  |
|--|--|
| • $\delta_\epsilon(t) = \frac{1}{\epsilon} \text{rect}\left(\frac{t}{\epsilon}\right)$ | • $\delta(t) = \lim_{\epsilon \rightarrow 0} \delta_\epsilon(t)$ |
| • $\delta_\epsilon(0) = \frac{1}{\epsilon}$  | • $\delta(0) \rightarrow \infty$                                 |
| • $\delta_\epsilon(t) = 0,  t  > \frac{\epsilon}{2}$                                   | • $\delta(t) = 0, t \neq 0$                                      |
| • $\int_{-\infty}^{\infty} \delta_\epsilon(t) dt = 1$                                  | • $\int_{-\infty}^{\infty} \delta(t) dt = 1$                     |

**Additional Properties of Unit impulse function**

- Scaling Property:

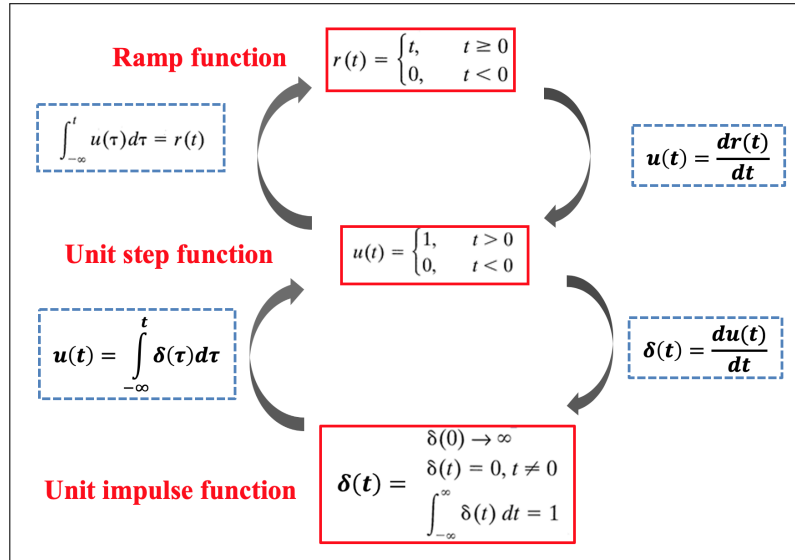
$$\delta(at) = \frac{1}{|a|} \delta(t)$$

- Even Function:

$$\delta(-t) = \delta(t)$$

- Derivative and Integral:

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau, \quad \delta(t) = \frac{du(t)}{dt}$$

**3. Complex Exponential Function**

- **Definition**

$$e^{j\omega_0 t} = \cos(\omega_0 t) + j \sin(\omega_0 t)$$

- **Properties**

- Periodic with  $T = \frac{2\pi n}{\omega_0}$  where  $n$  is an integer

- Fundamental period  $T_0 = \frac{2\pi}{|\omega_0|}$

- Infinite Energy  $E = \infty$

- Finite power  $P = \lim_{T_0 \rightarrow \infty} \frac{1}{T_0} \int_0^{T_0} |e^{j\omega_0 t}|^2 dt = \lim_{T_0 \rightarrow \infty} \frac{1}{T_0} \int_0^{T_0} 1 \cdot dt = 1$

**4. Sinusoidal Function**

$$A \cos(\omega_0 t + \theta) \quad \text{or} \quad A \sin(\omega_0 t + \theta),$$

where  $A$  is the *amplitude*,  $\theta$  is the *phase angle*,  $\omega_0$  is the *radian frequency* with

$$\text{Fundamental period } T_0 = \frac{2\pi}{\omega_0} \text{ (sec),} \quad \text{Fundamental frequency } f_0 = \frac{1}{T_0} \text{ hertz (Hz)}$$

## 1.4 Classification of System Types

- **[Def]** A *system* is a mathematical model of a physical process that relates the *input signal* to the *output signal* in the form  $y = Tx$ .

### 1. Invertible and Noninvertible System

A system is said to be **invertible** if distinct inputs lead to distinct outputs. Otherwise, the system is said to be **noninvertible**.

**[Examples]**

#### Invertible System

- $y(t) = 2x(t) \leftrightarrow w(t) = \frac{1}{2}y(t)$
- $y[n] = \sum_{k=-\infty}^n x[k] \leftrightarrow w[n] = y[n] - y[n-1]$

#### Noninvertible System

- $y[n] = 0$
- $y(t) = x^2(t)$

### 2. Memory and Memoryless System

A system is said to be **memoryless** if the output at any time depends only on the input at that same time. Otherwise, the system is said to have **memory**.

**[Examples]**

#### Memoryless System

- $y(t) = Rx(t)$
- $y[n] = (2x[n] - x^2[n])^2$
- 

#### System with Memory

- $y[n] = \sum_{k=-\infty}^n x[k]$
- $y[n] = x[n-1]$
- $y(t) = \frac{1}{c} \int_{-\infty}^t x(\tau) d\tau$

### 3. Causal and Noncausal System

A system is said to be **causal** if its output at the present time depends on only the present and/or past values of the input. If its output at the present time depends on future values of the input, the system is known as **noncausal**.

**[Examples]**

#### Causal System

- $y[n] = \sum_{k=-\infty}^n x[k]$
- $y(t) = x^2(t)$

#### Noncausal System

- $y[n] = x[n] + x[n+2]$
- $y[n] = x[-n]$  or  $y(t) = x(t+1)$

\* **Note)** All memoryless systems are causal, but not vice versa.

## 4. Linear and Nonlinear System

A system is said to be **linear** if the following superposition property (1.12) holds for a given operator  $T$ . If the system does not satisfy (1.12), it is a **nonlinear system**.

$$T\{\alpha_1 x_1 + \alpha_2 x_2\} = \alpha_1 T\{x_1\} + \alpha_2 T\{x_2\} \quad (1.12)$$

**[Examples]****Linear System**

- $y[n] = \sum_{k=-\infty}^n x[k]$
- $y(t) = tx(t)$

**Nonlinear System**

- $y(t) = x^2(t)$
- $y[n] = 2x[n] + 3$

\* **Note)** For a linear system, zero input always yields a zero output.

## 5. Time-invariant and Time-Varying System

A system is **time-invariant** if a time-shift of the input causes a corresponding shift in the output. In other words, the system response is independent of time.

$$\text{If } y(t) = T\{x(t)\}, \text{ then } y(t - t_0) = T\{x(t - t_0)\} \quad (1.13)$$

**[Examples]****Time invariant System**

- $y[n] = \sum_{k=-\infty}^n x[k]$
- $y[n] = x[n - n_0]$

**Time varying System**

- $y(t) = x(2t)$
- $y[n] = nx[n]$

## LTI System

**Linear time-invariant (LTI) system:** A system that is linear and also time-invariant.

## 6. Stable and Unstable System

A system is **stable** if every bounded input produces a bounded output for all time.

$$\text{If } |x(t)| < A, \text{ then } |y(t)| < B \text{ where } |A| < \infty, |B| < \infty \quad (1.14)$$

**[Examples]****Stable System**

- $y(t) = x^2(t)$
- $y[n] = x[n] + x[n + 2]$

**Unstable System**

- $y[n] = \frac{1}{x[n]}$
- $y[n] = nx[n]$

## 1.5 Examples

**[Example 1-1]** Determine whether or not each of the following signals is periodic. If a signal is periodic, determine its fundamental period.

- |   |  |
|---|--|
| a) $x(t) = \cos\left(t + \frac{\pi}{4}\right)$                                  | b) $x(t) = \sin\left(\frac{2\pi t}{3}\right)$    |
| c) $x(t) = \cos\left(\frac{\pi t}{3}\right) + \sin\left(\frac{\pi t}{4}\right)$ | d) $x(t) = \cos(t) + \sin(\sqrt{2}t)$            |
| e) $x(t) = \sin^2(t)$   | f) $x(t) = e^{j\left[\frac{\pi}{2}t - 1\right]}$ |
| g) $x(t) = \cos\left(2t + \frac{\pi}{4}\right)$                                 | h) $x(t) = \cos^2(t)$                            |
| i) $x(t) = (\cos(2\pi t))u(t)$  | j) $x(t) = e^{j\pi t}$                           |

**Solution)** To solve this type of problem, try to find the minimum  $T$  that satisfy  $x(t+T) = x(t)$ . For instance, in (a), if the following equality holds with a nonzero constant  $T$ , then it is periodic

$$\cos\left(t + \frac{\pi}{4}\right) = \cos\left(t + T + \frac{\pi}{4}\right) \rightarrow \cos(t') = \cos(t' + T), \quad (1.15)$$

where we used a *change of variable*  $t' = t + \frac{\pi}{4}$  in the second equality. Since the minimum  $T$  that satisfy (1.15) is  $2\pi$ , (a) is a periodic signal with period  $T = 2\pi$ . Similarly, for (b),

$$\sin\left(\frac{2\pi t}{3}\right) = \sin\left(\frac{2\pi t}{3} + \frac{2\pi T}{3}\right) \rightarrow \frac{2\pi T}{3} = 2\pi, \quad (1.16)$$

and by denoting  $t' = \frac{2\pi t}{3}$ , the minimum  $T$  that satisfy (1.16) is 3.

For (c) and (d), we can use (1.18); The period  $T_1$  for  $\cos\left(\frac{\pi t}{3}\right)$  in (c) is  $T_1 = 6$  and  $T_2$  for  $\sin\left(\frac{\pi t}{4}\right)$  is  $T_2 = 8$ . Since  $T_1/T_2 = 3/4$ , (c) is a periodic signal with period  $T = 24$ . In (d), the period  $T_1$  for  $\cos(t)$  is  $T_1 = 2\pi$  and  $T_2$  for  $\sin(\sqrt{2}t)$  is  $T_2 = \sqrt{2}\pi$ . Since  $T_1/T_2 = \sqrt{2}$ , (d) is aperiodic signal.

For (e) and (h), convert  $x(t)$  as follows, then apply similar approach as (a).

$$\cos^2(t) = \frac{1}{2}(1 + \cos(2t)), \quad \sin^2(t) = \frac{1}{2}(1 - \cos(2t)), \quad (1.17)$$

and the remaining can be solved using similar method. The solutions are summarized below.

- |                             |                            |                           |
|-----------------------------|----------------------------|---------------------------|
| a) Periodic with $T = 2\pi$ | b) Periodic with $T = 3$   | c) Periodic with $T = 24$ |
| d) Aperiodic                | e) Periodic with $T = \pi$ | f) Periodic with $T = 4$  |
| g) Periodic with $T = \pi$  | h) Periodic with $T = \pi$ | i) Aperiodic              |
| j) Periodic with $T = 2$    |                            |                           |

## Sum of Periodic Signals

- Let  $x_1(t)$  and  $x_2(t)$  be periodic signals with fundamental periods  $T_1$  and  $T_2$ , respectively. The sum  $x(t) = x_1(t) + x_2(t)$  is periodic if and only if the following condition holds

$$\frac{T_1}{T_2} = \frac{k}{m} = \text{rational number} \quad (1.18)$$

where the fundamental period  $T$  is the least common multiple of  $T_1$  and  $T_2$ .

- Let  $x_1[n]$  and  $x_2[n]$  be periodic sequence with fundamental periods  $N_1$  and  $N_2$ , respectively. The sum  $x[n] = x_1[n] + x_2[n]$  is periodic given the following condition

$$mN_1 = kN_2 = N \quad (1.19)$$

where the fundamental period  $N$  is the least common multiple of  $N_1$  and  $N_2$ .

Refer [Schaum's text, Problem 1.14 & 1.15]



**[Example 1-2]** Determine whether the following signals are energy signals, power signals, or neither.

a)  $x(t) = e^{-at}u(t)$ ,  $a > 0$

b)  $x(t) = A \cos(\omega_0 t + \theta)$

**Solution)** To solve this type of problem, **(Step 1.)** you need to calculate the energy  $E$  first. If  $E$  is finite, the signal is a Energy signal. Otherwise, **(Step 2.)** if  $E$  is infinite, you need to calculate the power  $P$  as well. If  $P$  is finite, the signal is a Power signal. Otherwise, if  $P$  is infinite, then it is neither a energy nor a power signal. For example, in (a),

$$E = \int_{-\infty}^{\infty} e^{-2at}u(t)dt = \int_0^{\infty} e^{-2at}dt = \frac{1}{2a}, \quad (1.20)$$

where we used the definition of the step function in the second equality. Since  $\frac{1}{2a}$  is finite,  $x(t)$  in (a) is a energy signal. For a periodic signal, the integration interval  $T$  in (1.7) is equal to the period. In (b), the period is  $T = \frac{2\pi}{\omega_0}$  and the signal power can be calculated as follows

$$\begin{aligned} P &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T A^2 \cos^2(\omega_0 t + \theta) dt = \lim_{T \rightarrow \infty} \frac{A^2}{2\pi} \int_{\theta}^{2\pi+\theta} \cos^2(l) dl \\ &= \lim_{T \rightarrow \infty} \frac{A^2}{4\pi} \int_{\theta}^{2\pi+\theta} [1 + \cos(2l)] dl = \frac{A^2}{2}, \end{aligned} \quad (1.21)$$

where we used  $T = \frac{2\pi}{\omega_0}$  and a change of variable,  $l = \omega_0 t + \theta$  or  $\omega_0 dt = dl$ , in the second equality, then applied the Cosine rule  $\cos^2(t) = \frac{1}{2}(1 + \cos(2t))$  in the third equality. Since  $\frac{A^2}{2}$  is finite,  $x(t)$  in (b) is a power signal. In summary, the solutions are

a) Energy signal

b) Power signal

## Definition of Energy and Power Signals

- **Energy signal** has finite energy and zero power, i.e.,  $0 < E < \infty$ ,  $P = 0$
- **Power signal** has finite power and infinite energy, i.e.,  $0 < P < \infty$ ,  $E = \infty$ , where

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt, \quad P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

## Properties of Periodic Signals

The following equalities hold for a periodic signal  $x(t+T) = x(t)$

$$\int_{\alpha}^{\beta} x(t) dt = \int_{\alpha+T}^{\beta+T} x(t) dt, \quad \int_0^T x(t) dt = \int_a^{a+T} x(t) dt,$$

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \frac{1}{T_0} \int_0^{T_0} |x(t)|^2 dt,$$

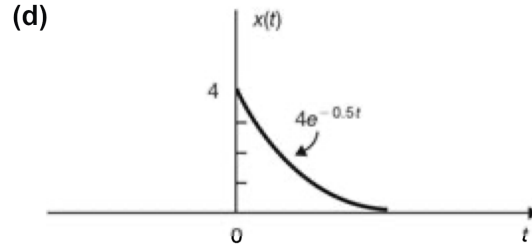
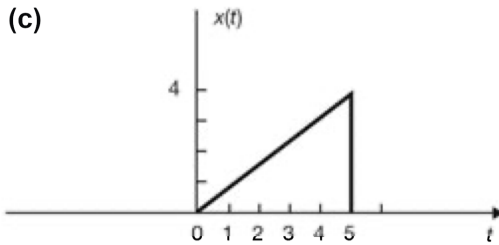
where  $T_0$  is the fundamental period and  $\alpha, \beta, a$  are arbitrary real valued constants.  
Refer [Schaum's text, Problem 1.17 & 1.18]



**[Example 1-3]** Determine the even and odd component of the following signals

a)  $x(t) = u(t)$

b)  $x(t) = \sin(\omega_0 t + \frac{\pi}{4})$



**Solution)** To solve this type of problem, you need to apply (1.22). In (a),  $x(-t) = u(-t) = 1$  for  $t < 0$  and  $u(-t) = 0$  for  $t > 0$ . Then, the following results can be derived

$$x_e(t) = \frac{1}{2} [u(t) + u(-t)] = \frac{1}{2},$$

$$x_o(t) = \frac{1}{2} [u(t) - u(-t)] = \frac{1}{2} \text{sgn}(t) = \begin{cases} 0.5, & t > 0, \\ -0.5, & t < 0 \end{cases}$$

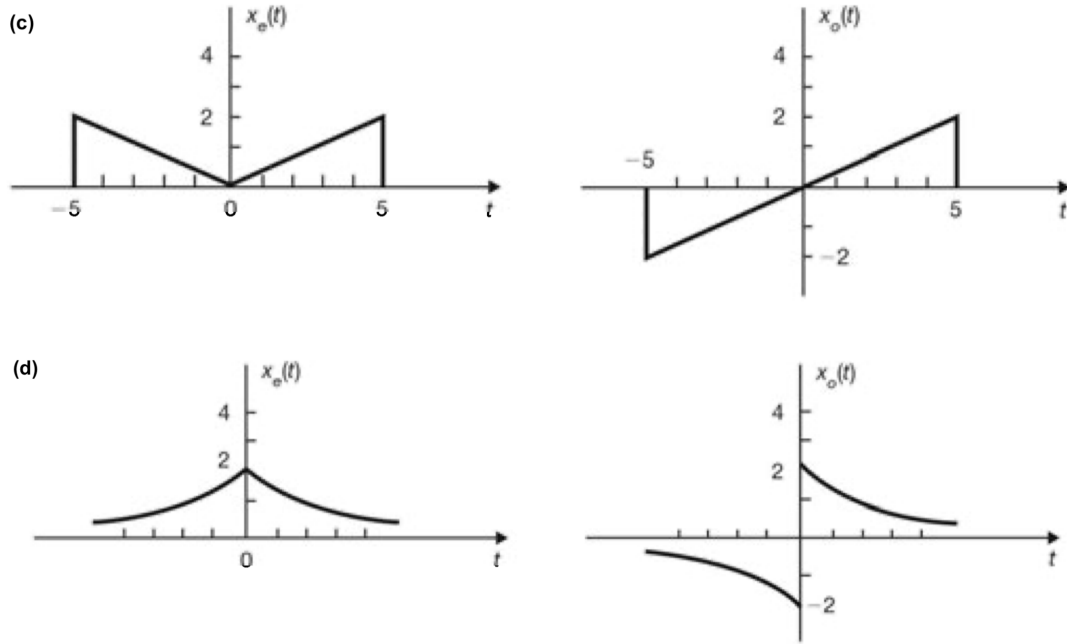


In (b), we first use Sine rule to expand the Sine function, then the following results can be derived.

$$\sin\left(\omega_0 t + \frac{\pi}{4}\right) = \sin(\omega_0 t) \cos\left(\frac{\pi}{4}\right) + \cos(\omega_0 t) \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} (\sin(\omega_0 t) + \cos(\omega_0 t)).$$

$$x_e(t) = \frac{1}{\sqrt{2}} \cos(\omega_0 t), \quad x_o(t) = \frac{1}{\sqrt{2}} \sin(\omega_0 t).$$

Similarly, the even and odd component of (c) and (d) can be found as follows



#### Even and Odd Component

Any signal  $x(t)$  can be expressed as a sum of two signals

$$x(t) = x_e(t) + x_o(t),$$

where  $x_e(t)$  and  $x_o(t)$  are related to the original signal  $x(t)$  as follows

$$x_e(t) = \frac{1}{2} [x(t) + x(-t)], \quad x_o(t) = \frac{1}{2} [x(t) - x(-t)]. \quad (1.22)$$



**[Example 1-4] [Part 1]** Sketch the following signals.

a)  $x_1(t) = u(t) + 5u(t-1) - 2u(t-2)$

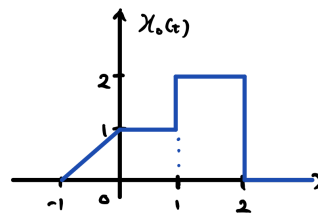
b)  $x_2(t) = r(t) - r(t-1) - u(t-2)$

c)  $x_3(t) = u(t)u(a-t), a > 0$

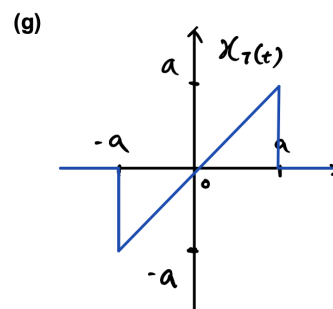
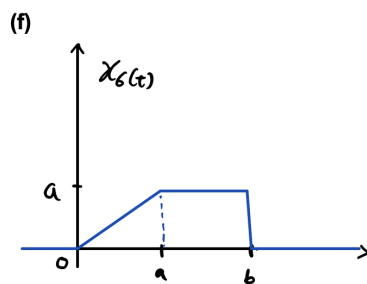
d)  $x_4(t) = x_0(t)u(1-t)$

e)  $x_5(t) = x_0(t) [u(t) - u(t - 1)]$

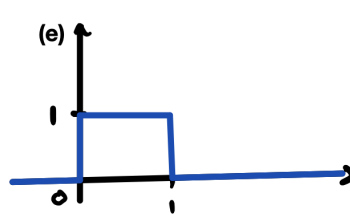
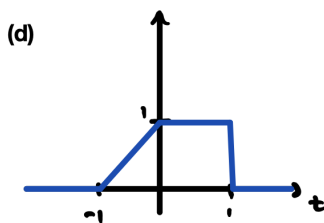
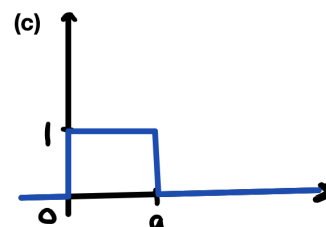
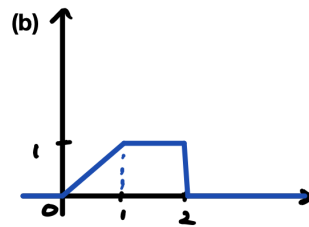
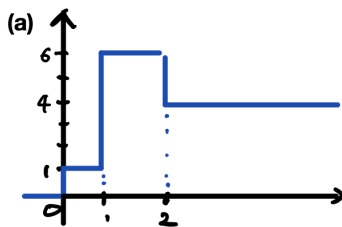
where the signal  $x_0(t)$  is plotted below.



[Part 2] For each of the signals plotted below, write an expression in terms of unit step and unit ramp functions.



**Solution)**[Part 1]



[Part 2]

(f)  $x_6(t) = r(t) - r(t - a) - au(t - b)$ ,    (g)  $x_7(t) = (r(t) - r(-t))(u(t + a) - u(t - a))$

- **Time Reversal:** Flip the signal around the vertical axis  $x(t) \rightarrow x(-t)$
- **Time Shifts:** Shift the signal to left or right  $x(t) \rightarrow x(t - t_0)$ 
  - **Right-shift** if  $t_0 > 0$ ,      **Left-shift** if  $t_0 < 0$ .
- **Time Scaling:** Linearly stretch or compress the signal  $x(t) \rightarrow x(ct)$ 
  - **Compression** if  $|c| > 1$ ,      **Expansion** if  $|c| < 1$ .



**[Example 1-5]** Evaluate the following integrals.

- |   |   |
|---|---|
| a) $\int_{-\infty}^t \cos(\tau) u(\tau) d\tau$                                      | b) $\int_{-\infty}^t \cos(\tau) \delta(\tau) d\tau$   |
| c) $\int_{-\infty}^{\infty} \cos(t) u(t-1) \delta(t) dt$                            | d) $\int_0^{2\pi} t \sin\left(\frac{t}{2}\right) \delta(t-\pi) dt$  |
| e) $\int_{-\infty}^{\infty} \left(\frac{2}{3}t - \frac{3}{2}\right) \delta(t-1) dt$ | f) $\int_{-3}^2 \left[\exp(1-t) + \sin\left(\frac{2\pi t}{3}\right)\right] \delta\left(t - \frac{3}{2}\right) dt$ |

**Solution)**

(a)

$$\int_{-\infty}^t \cos(\tau) u(\tau) d\tau = \begin{cases} \text{If } t > 0, \int_0^t \cos(\tau) d\tau = \sin(t) \\ \text{If } t < 0, 0 \end{cases} = u(t) \sin(t)$$

(b)

$$\int_{-\infty}^t \cos(\tau) \delta(\tau) d\tau = \begin{cases} \text{If } t > 0, \cos 0 = 1 \\ \text{If } t < 0, 0 \end{cases} = u(t)$$

(c)

$$\int_{-\infty}^{\infty} \cos(t) u(t-1) \delta(t) dt = \cos(0) u(-1) = 0$$

(d)

$$\int_0^{2\pi} t \sin\left(\frac{t}{2}\right) \delta(t-\pi) dt = \pi \sin\left(\frac{\pi}{2}\right) = \pi$$

(e)

$$\int_{-\infty}^{\infty} \left(\frac{2}{3}t - \frac{3}{2}\right) \delta(t-1) dt = \frac{2}{3} - \frac{3}{2} = -\frac{5}{6}$$

(f)

$$\int_{-3}^2 \left[\exp(1-t) + \sin\left(\frac{2\pi t}{3}\right)\right] \delta\left(t - \frac{3}{2}\right) dt = \exp\left(-\frac{1}{2}\right) + \sin(\pi) = \exp(-0.5)$$

