Chapter 0 Review

(1) Vectors

* $\bar{a}$ or $\underline{a} \in \mathbb{R}^n$

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \leftarrow \text{real \#}$$

* magnitude (length).

$$|\underline{a}| = \sqrt{\sum_{i=1}^{n} a_i^2} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

* Unit Vector

$|\underline{a}| = 1$, Standard Unit Vector

$$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \underline{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

* Zero Vector

$$\underline{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

* $\underline{a} \neq 0 \Rightarrow \dfrac{\underline{a}}{|\underline{a}|} \rightarrow \hat{a}$ is a unit Vector **

2/. Vector Operations

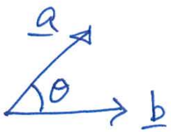
$\underline{c} = \underline{a} \pm \underline{b}$, $\quad m\underline{a}$, $\quad m(\underline{a}+\underline{b}) = m\underline{a} + m\underline{b}$

Scalar

$(m+n)\underline{a} = m\underline{a} + n\underline{a}$

3/. Scalar Product (dot Product) **

$$\underline{a}\cdot\underline{b} = \sum_{i=1}^{n} a_i b_i = a_1b_1 + a_2b_2 + \cdots + a_nb_n \Rightarrow \begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix}\cdot\begin{pmatrix}b_1\\b_2\\\vdots\\b_n\end{pmatrix}$$



$$= |a|\,|b|\cos\theta$$

$$\left[\ \cos\theta = \frac{\underline{a}\cdot\underline{b}}{|a|\,|b|}\ \right]$$

If $\underline{b} = \underline{a}$

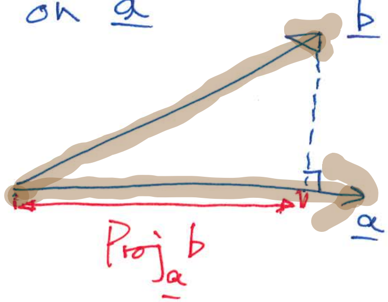
$|a|\,|a| = \underline{a}\cdot\underline{a}$

$\boxed{|a| = \sqrt{\underline{a}\cdot\underline{a}}}$ **

$$\begin{cases} |\underline{a}\cdot\underline{b}| \leq |\underline{a}||\underline{b}| & (\text{Schwarz inequality}). \\ |\underline{a}+\underline{b}| \leq |\underline{a}| + |\underline{b}| & (\Delta \text{ inequality}) \end{cases}$$

$\text{Proj}_{\underline{a}}\underline{b}$ projection vector of $\underline{b}$ on $\underline{a}$

$$= \left(\frac{\underline{a}\cdot\underline{b}}{\underline{a}\cdot\underline{a}}\right)\underline{a} = \left(\frac{\underline{a}\cdot\underline{b}}{|\underline{a}|^2}\right)\underline{a}$$



Exercise Compute $\text{Proj}_{\underline{a}}\underline{b}$ given $\underline{a} = \begin{pmatrix} 4 \\ 0 \\ 3 \\ 0 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 3 \\ 3 \\ 1 \\ 2 \end{pmatrix}$

Solution:

$$\text{Proj}_{\underline{a}}\underline{b} = \left(\frac{\underline{a}\cdot\underline{b}}{\underline{a}\cdot\underline{a}}\right)\underline{a} = \frac{15}{25}\begin{pmatrix} 4 \\ 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{12}{5} \\ 0 \\ 9/5 \\ 0 \end{pmatrix}$$

4/. Linear dependence / independence

check $\underset{\underline{c}}{\begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}} = \overset{3}{m_1} \underset{\underline{a}}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} + \overset{4}{m_2} \underset{\underline{b}}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$

$$\begin{cases} 3 = m_1 + m_2 \cdot 0 \\ 4 = m_1 \cdot 0 + m_2 \cdot 1 \\ 3 = m_1 \times 1 + m_2 \times 0 \end{cases}$$

scalar Value

Linear coefficients, m_1 m_2

$\underline{c}$ is dependent on $\underline{a}$ + $\underline{b}$

is Linearly combination of $\underline{a}$ and $\underline{b}$

check $\underline{a}$ + $\underline{b}$

$$\underset{\underline{a}}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} = \overset{?}{m} \underset{\underline{b}}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

m does not exist

$\underline{a}$ + $\underline{b}$ are independent of each other

$\underline{a} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}$, $\underline{c} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ linearly dependent (L. I: dependent) or linearly independent?

(i) <u>Triple Scalar product</u>

$$\underline{a} \cdot \underline{b} \times \underline{c} = \begin{vmatrix} 3 & 5 & -2 \\ 0 & 4 & 2 \\ 1 & 1 & -1 \end{vmatrix} = 0 \Rightarrow \text{linearly dependent}$$

det

(ii) If $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3$, $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \in \mathbb{R}^3$, $\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \in \mathbb{R}^3$? linearly independent.

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\} ?$$

If you have $(n+1)$ vectors in $\mathbb{R}^n$, {all the} vectors in the group are dependent.

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

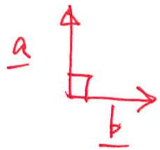
Orthogonality

$\underline{a} \perp \underline{b}$

$\underline{a}$ and $\underline{b}$ is said to be orthogonal to each other

iff

$(\underline{a} \perp \underline{b})$



$\underline{a} \cdot \underline{b} = 0$

$\underline{a} \cdot \underline{b} = |a|\,|b| \cos\theta = 0$ $\quad \theta = 90^\circ$, $\cos\theta = 0$

$\{\underline{a}_1, \underline{a}_2, \dots, \underline{a}_k\} \in \mathbb{R}^n$

is said to be orthogonal if

$\underline{a}_i \cdot \underline{a}_j = 0$ for all $i \neq j$

$\{\underline{a}_1, \underline{a}_2, \dots \underline{a}_k\} \in \mathbb{R}^n$ is said to be orthonormal if

$$\begin{cases} \underline{a}_i \cdot \underline{a}_j = 0 \\ \underline{a}_i \cdot \underline{a}_i = 1 \Rightarrow |\underline{a}_i| = 1, \text{ unit vector for all } i \end{cases}$$

Linear Algebra

Vectors

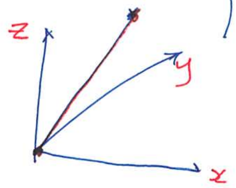
Matrices

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^{n}$$

$n \rightarrow$ n-dimensional

$\mathbb{R} \leftarrow$ real #

$$\bar{a} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$



$\mathbb{R}^3$

Magnitude

$$|\vec{a}| \text{ or } \|\vec{a}\|$$

$$= \sqrt{\sum_{i=1}^{n} a_i^2} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

(distance)

(norm)

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$|\vec{b} - \vec{a}|$$

$$= \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (a_3 - b_3)^2}$$

Unit Vector

$$|\vec{a}| = 1$$

eg: $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3$

$\vec{e}_{☺} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$ position

$\vec{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^n$

standard unit vector.

Zero Vector $\neq$ zero

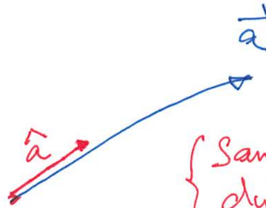
$\mathbb{R}$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3$$

$\vec{a} \neq 0 \quad \Rightarrow \quad \vec{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3$ $\quad \boxed{\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}}$ is a unit vector

$$|a| = \sqrt{1^2+0^2+1^2} = \sqrt{2} \qquad \hat{a} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

check. $|\hat{a}| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + 0^2 + \left(\frac{1}{\sqrt{2}}\right)} = 1$



$\vec{a}$ and $\hat{a}$ are in the same direction

* "$\boxed{\text{normalization}}$" *
*

2/. Vector Operations

✓ $\vec{a} \pm \vec{b}$

✓ $m \vec{a}$ $\qquad\qquad$ $3 * \vec{a} = 3\vec{a}$

scalar.

✓ $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$

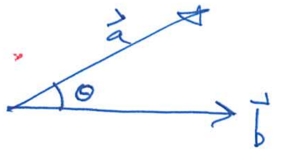
✓ $(m+n)(\vec{a}) = m\vec{a} + n\vec{a}$

3/. "***" Dot product of Vectors (Scalar product)

$$\vec{a} \cdot \vec{b} = \begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}4\\5\\6\end{pmatrix} \in \mathbb{R}$$

scalar Value

$$\hookrightarrow \sum_{i=1}^{n} a_i b_i = 1\times4 + 2\times5 + 3\times6 = 4 + 10 + 18 = 32$$

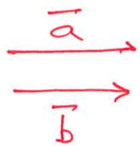


$$\boxed{\vec{a} \cdot \vec{b} = |a||b| \cos\theta}$$

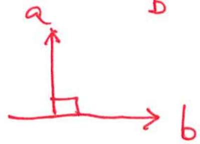
$$32 = \left(\sqrt{1^2+2^2+3^2}\right)\left(\sqrt{4^2+5^2+6^2}\right)\cos\theta$$

If

$\theta = 0$, $\cos\theta = 1$



$\theta = 90^\circ$, $\cos\theta = 0$



a "⊥" b

"$\vec{a}$" is orthogonal to "b"

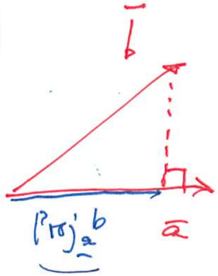
$$\vec{a} \cdot \vec{b} = \sum a_i b_i$$

$$\vec{a} \cdot \vec{a} = \sum a_i a_i$$

$$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} \quad \text{magnitude}$$

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}|\,|\vec{b}| \quad \text{(Schwartz inequality)}$$

$$|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}| \quad (\triangle \text{ inequality})$$



"$\mathrm{Proj}_{\underline{a}}\, \underline{b}$" projection vector of $\underline{b}$ onto $\underline{a}$

$$= \left(\frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}\right)\vec{a} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\right)\vec{a}$$

Ex: Compute $\text{Proj}_a b$ given $\vec{a} = \begin{pmatrix} 4 \\ 0 \\ 3 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 3 \\ 3 \\ 1 \\ 2 \end{pmatrix}$

$$\text{Proj}_{\underline{a}} \underline{b} = \frac{15}{25} \begin{pmatrix} 4 \\ 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 12/5 \\ 0 \\ 9/5 \\ 0 \end{pmatrix}$$

4/. Linear dependence / independence

Linear coefficients

$$\text{check} \quad \underset{\underline{c}}{\begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}} = 3 \underset{\underline{a}}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} + 4 \underset{\underline{b}}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

$\underline{c}$ is dependent on $\underline{a}$ & $\underline{b}$

"$\underline{c}$" is a linearly combination of $\underline{a}$ & $\underline{b}$

check $\underline{a}$ & $\underline{b}$?

?

$$\underset{\underline{a}}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} = m \underset{\underline{b}}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

m does not exist

$\underline{a}$ & $\underline{b}$ are independent of each other.

$\underline{a} = \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}$ $\underline{c} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ linearly dependent or linearly independent ?

(i) <u>Triple Scalar product</u>

$$\vec{a} \cdot \vec{b} \times \vec{c} = \begin{vmatrix} 3 & 5 & -2 \\ 0 & 4 & 2 \\ 1 & 1 & -1 \end{vmatrix} = 0 \Rightarrow$$ "linearly" dependent

(ii) If $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \in \mathbb{R}^3, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \in \mathbb{R}^3$ linearly independent ?

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$$

$\overset{3}{(n+1)}$ Vectors in $\mathbb{R}^n$ are always linearly dependent.

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \underline{1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

Orthogonality

$\underline{a} \perp \underline{b}$

$\underline{a}$ and $\underline{b}$ is said to be orthogonal to each other iff

$(\vec{a} \perp \vec{b})$

$\vec{a} \cdot \vec{b} = 0$

$\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_k\} \in \mathbb{R}^n$

is said to be orthogonal if

$\boxed{\underline{a}_i \cdot \underline{a}_j = 0}$ for all $i \neq j \quad 1 \leq i, j \leq k$

$\{\underline{a}_1, \underline{a}_2, \ldots \underline{a}_k\} \in \mathbb{R}^n$

is said to be orthonormal if

$$\text{and} \begin{cases} \underline{a}_i \cdot \underline{a}_j = 0 & \Rightarrow \quad \underline{a}_i \perp \underline{a}_j \\ \underline{a}_i \cdot \underline{a}_i = 1 & \Rightarrow \quad |\bar{a}_i| = 1 \quad \text{unit vector} \end{cases}$$

Orthogonality

$\underline{a} \perp \underline{b}$ $\quad$ $\underline{a}$ and $\underline{b}$ is said to be orthogonal to each other

iff

$(\underline{a} \perp \underline{b})$

$\underline{a} \cdot \underline{b} = |a||b|\cos\theta = 0$

$\theta = 90^\circ$

$\cos\theta = 0$.

$\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\} \in \mathbb{R}^n$ is said to be orthogonal if $\underline{a}_i \cdot \underline{a}_j = 0$ for all $i \neq j$

$$\underbrace{\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}}_{} \in \mathbb{R}^3 \qquad \left\{ \underline{\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix}} \right\} \in \mathbb{R}^3$$

$\hookrightarrow$ ortho/normal set

$\{\underline{a}_1, \underline{a}_2, \ldots \underline{a}_n\} \in \mathbb{R}^n$ is said to be orthonormal if

unit Vector for all i $\left\{ \underline{a}_i \cdot \underline{a}_j = 0 \right.$ and $\underline{a}_i \cdot \underline{a}_i = 1$, $\underline{a}_j \cdot \underline{a}_j = 1$

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Let $\underline{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ and $\underline{c} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

(i) Show that $\underline{a}$, $\underline{b}$ and $\underline{c}$ are L.I independent

(ii) Are $\{\underline{a}, \underline{b}, \underline{c}\}$ ~~are~~ orthogonal ?

(iii) Normalize $\{\underline{a}, \underline{b}, \underline{c}\}$ to obtain orthonormal set

Let $\underline{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ and $\underline{c} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

(i) Show that $\underline{a}$, $\underline{b}$ and $\underline{c}$ are linearly independent.

(ii) Show that $\{\underline{a}, \underline{b}, \underline{c}\}$ are orthogonal

(iii) Normalize $\underline{a}, \underline{b}, \underline{c}$ to obtain ~~orthonormal~~ set.
orthonormal

(i)
$$\underline{a} \cdot \underline{b} \times \underline{c} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}$$
$$= -2 \quad -2 \quad -2 = -6 \neq 0$$

$\Rightarrow$ linearly independent

(ii)
$$\begin{cases} \underline{a} \cdot \underline{b} = 1.1 + 1.1 + 1.(-2) = 0 \\ \underline{b} \cdot \underline{c} = 1.1 + 1.-1 + -2.(0) = 0 \\ \underline{a} \cdot \underline{c} = 1.1 + 1.-1 + 1(0) = 0 \end{cases}$$

(iii)
$$\hat{a} = \frac{\underline{a}}{|\underline{a}|} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \hat{b} = \frac{\underline{b}}{|\underline{b}|} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \qquad \hat{c} = \frac{\underline{c}}{|\underline{c}|} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$\{\hat{a}, \hat{b}, \hat{c}\}$ is orthonormal.

$m \times n$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{m,n} \end{pmatrix} = \left(a_{ij} \right)$$

← row vector

col vector

scalar

$$A \pm B \quad , \quad cA$$

Transpose $A^T = \{ a_{ji} \}$

$$\underset{n \times p}{C} = \underset{n \times m}{A}\,\underset{m \times p}{B}$$

$$\boxed{c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}}$$

If $A = A^T$, A is symmetric

$$\det A = |A|^{n\times n}$$

$$= a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots a_{in}A_{in}$$

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Minor

$$\det(A * B) = \det A * \det B$$

$$\det(A) = \det(A^T)$$

Inverse of Matrix

$$A^{n\times n}$$

* A^{-1} : $A^{-1}A = AA^{-1} = I_n$
* $(AB)^{-1}$: $B^{-1}A^{-1} = (AB)^{-1}$
* $\boxed{(A^T)^{-1} = (A^{-1})^T}$

* $\det(A^{-1}) = (\det A)^{-1} = \dfrac{1}{\det(A)}$

  $\parallel$

* $(A^{-1})^{-1} = A$

* A^{-1} exists iff $\det(A) \neq 0$

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* A^{-1} exists iff rank$(A) = n$ $\quad \overset{n\times n}{A}$

rank(A) is the ~~#~~ maximal # of linearly independent row / col vectors.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad 3 \text{ L.I cols/rows} \underset{\text{rank}(A) = 3}{\parallel}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 2 \end{bmatrix} \quad 3 \text{ L.I cols/rows} \underset{\text{rank}(A) = 3}{\parallel}$$

3x3

n=3 Full rank

Orthogonal matrix A ****

$$A^TA = AA^T = I \qquad \boxed{A^{-1} = A^T}$$

$$\begin{bmatrix} A = a_{ij} \end{bmatrix} \begin{bmatrix} A^T = a_{ji} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \\ & & 1 \end{bmatrix}$$

Find the inverse of $A = A^{-1}$ by elementary row operations

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X is a matrix

$$\underset{3\times3}{A}\,\underset{3\times3}{X} = \underset{3\times3}{I} = \underset{3\times3}{[\underline{e}_1, \underline{e}_2, \underline{e}_3]}$$

$$A[\underline{x}_1, \underline{x}_2, \underline{x}_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{3\times3}$$

$$\left\{ \begin{array}{l} A\underline{x}_1 = \underline{e}_1 \\ A\underline{x}_2 = \underline{e}_2 \\ A\underline{x}_3 = \underline{e}_3 \end{array} \right. \Longleftrightarrow \left(\underset{3\times3}{A} \,\vdots\, \underset{3\times3}{I} \right) \longrightarrow \left(I \,\vdots\, \overset{A^{-1}}{X} \right)$$

Consider a linear system

unknown vector

$$\begin{cases} 3x_1 - 5x_2 = 6 \\ -2x_1 + 3x_2 = -1 \end{cases} \Longrightarrow \begin{bmatrix} 3 & -5 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$A \quad \underline{x} = \underline{b}$$

Method 1

$$\underline{x} = A^{-1}\underline{b} = \left(\frac{\operatorname{adj} A}{\det A}\right)\underline{b}$$

$$= \begin{bmatrix} 3 & -5 \\ -2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ -1 \end{bmatrix} = \frac{\begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}}{(-1)} \begin{bmatrix} 6 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 & -5 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} -13 \\ -9 \end{bmatrix}$$

Method 2

$$[A \,\vdots\, I]$$

$$\left[\begin{array}{cc:cc} 3 & -5 & 1 & 0 \\ -2 & 3 & 0 & 1 \end{array}\right] \xrightarrow[(r_1/3 \times 2 + r_2) \to r_2]{r_1/3 \to r_1} \left[\begin{array}{cc:cc} 1 & -5/3 & 1/3 & 0 \\ 0 & -\frac{1}{3} & 2/3 & 1 \end{array}\right] \xrightarrow[\substack{r_1 - 5r_2 \to r_1 \\ -3r_2 \to r_2}]{} \left[\begin{array}{cc:cc} 1 & 0 & -3 & -5 \\ 0 & 1 & -2 & -3 \end{array}\right]$$

A^{-1}

$$[I \,\vdots\, A^{-1}]$$

$$\underline{x} = \begin{bmatrix} -3 & -5 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} -13 \\ -9 \end{bmatrix}$$

Method 3

$$\left[\begin{array}{c:c} A & \underline{b} \end{array}\right] \longrightarrow \left[\begin{array}{c:c} I & \underline{x} \end{array}\right]$$

$$\left[\begin{array}{cc:c} 3 & -5 & 6 \\ -2 & 3 & -1 \end{array}\right] \xrightarrow[\frac{2}{3}r_1 + r_2]{r_1/3 \to r_1} \left[\begin{array}{cc:c} 1 & -\frac{5}{3} & 2 \\ 0 & -1/3 & 3 \end{array}\right] \xrightarrow[-3r_2 \to r_2]{} \left[\begin{array}{cc:c} 1 & -5/3 & 2 \\ 0 & 1 & -9 \end{array}\right]$$

$$\xrightarrow[\frac{5}{3}r_2 + r_1 \to r_1]{} \left[\begin{array}{cc:c} 1 & 0 & -13 \\ 0 & 1 & -9 \end{array}\right]$$

$\underline{x}$

$A\underline{x} = \underline{b}$

$$\begin{cases} x_1 + 2x_2 = 0 \\ x_2 = 0 \end{cases}$$ "homogeneous Eqns"

$$\Downarrow$$

unknowns

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$B \quad \underline{x} = \underline{0}$$

$$B^{-1} = \frac{\operatorname{adj} B}{\det(B)}$$

(i) If B^{-1} exists, $(\det(B) \neq 0)$

$$B^{-1} B \underline{x} = B^{-1} \underline{0}$$

$$\underline{x} = \underline{0} \quad \text{(trivial)}$$

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eg: $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ B^{-1} exists $(\det(B) = 1)$ $\neq 0$

$$x_1 = x_2 = 0$$

(ii) If $\underline{x} \neq \underline{0}$

"**"

$$B\,\underline{x} = 0$$

We cannot find B^{-1} ~~such that~~ given $\underline{x} \neq 0$

B^{-1} does not exist $\iff \det(B) = 0$

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eg $\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ $\rightarrow$ $\boxed{x_1 + 3x_2 = 0}$ ✓

1 Eqn 2 unknowns

$\det(B) = 0$

x_1, x_2 have infinite sols

$x_2 = k$

$x_1 = -3k$ $\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

(x_1, x_2), $(-3, 1)$, $(-6, 2)$ ....

(iii) If $\underline{x} = 0$

$\Rightarrow B^{-1}$ may / may not exist.

$\underline{x} \neq 0 \quad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad n = 3$

$$B\,\underline{x} = \underline{0}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 1 & 3 & 7 & 0 \end{array}\right) \xrightarrow{r_3 - r_1 - r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

2 eqns but 3 unknowns

$\text{rank}(B) = \boxed{2}$

# of parameters = 1

= 3 unknowns − 2 eqn

= 1

$x_3 = s$

$x_2 = -4x_3 = -4s$

$x_1 = 8s - 3s = 5s$

in a vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5s \\ -4s \\ s \end{bmatrix} = s \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \neq 0$$

$$B\underline{x} = 0$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 1 & 3 & 7 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underline{x} \neq 0 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad n = 3$$

$$\underset{\underset{\text{2 Eqns}}{||}}{2 \text{ pivots}} < \underset{\underset{n}{||}}{3}$$

$$\text{rank}.(B) = 2.$$

$$\boxed{x_3 = s} \longrightarrow n - \#\text{ of Eqn} \qquad 3 - 2$$

$$3\,(\text{unknowns}) - 2\,(\text{pivots}) = 1\,(\text{chosen parameters})$$

$$\begin{cases} x_3 = s \\ x_2 = -4x_3 = -4s \\ x_1 = 8s - 3s = 5s \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s\begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix} \neq 0 \qquad \text{"}\underline{s \neq 0}\text{"}$$