MA1201 Calculus and Basic Linear Algebra II   Problem Set 4   Vector Algebra

Part A: Basic Concept

Problem 1

Let $A=(1,1,0)$, $B=(0,2,3)$ and $C=(2,-1,0)$ be three points in a plane.

(a) Write down the position vectors of A, B and C.

(b) Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{CA}$.

(c) Is $\overrightarrow{AB}=\overrightarrow{BC}$? Explain your answer.

(d) Find the unit vector of $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

(e) (i) Let $\vec{a}$ be a vector with magnitude 3 and the direction is opposite to that of $\overrightarrow{AB}$. Find the vector $\vec{a}$.

(ii) Let $\vec{b}$ be a vector with magnitude 5 and the direction is that of $\overrightarrow{BC}$. Find the vector $\vec{b}$.

Problem 2

Let $A=(0,1,-1)$ and $B=(1,2,0)$ be two points in a plane. Let X be a point between A and B such that $AX:XB=2:1$.

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AX}$.

(b) Hence, find the coordinate of X by finding its position vector $\overrightarrow{OX}$. (Hint: $\overrightarrow{AX}=\overrightarrow{OX}-\overrightarrow{OA}$).

Problem 3

Let $\vec{a}=2\vec{\imath}-3\vec{\jmath}+5\vec{k}$ and $\vec{b}=\vec{\imath}+3\vec{\jmath}$ be two vectors.

(a) Find $|\vec{a}|$ and $|\vec{a}-2\vec{b}|$.

(b) Find the unit vector of $\vec{b}$.

(c) Let $\vec{c}$ be another vector with magnitude $|2\vec{a}+\vec{b}|$ and its direction is same as that of $\vec{b}$. Find the vector $\vec{c}$.

Part B: Scalar Product and its application

Problem 4

Let $\vec{a}=\vec{\imath}+3\vec{\jmath}-2\vec{k}$ and $\vec{b}=-2\vec{\imath}+\vec{\jmath}+3\vec{k}$ be two vectors.

(a) Find $\vec{a}\cdot\vec{b}$.

(b) Find the angle between the vectors $\vec{a}$ and $\vec{b}$.

(c) Let $\vec{c}=3\vec{\imath}+x\vec{\jmath}-2\vec{k}$ be a vector which is perpendicular to $\vec{b}$, find the value of x.

(d) Let $\vec{d}=y\vec{a}+3\vec{b}$ be a vector which is perpendicular to $\vec{a}-\vec{b}$, find the value of y.

Problem 5

(a) Let $A=(1,1,0)$, $B=(0,1,2)$ and $C=(2,1,0)$ be three points in a plane, find $\angle ABC$.

(b) It is given that $D=(x,1,3)$, $E=(1,2,3)$ and $F=(4,-4,1)$ are three points in a plane. Suppose that DE is pperpendicular to EF, find the value of x.

Problem 6

Let $A=(4,2)$, $B=(1,1)$ and $C=(2,3)$ be three points in a 2D-plane. Let $D=(3,3)$ be another point in the same plane.

(a) Find $\angle ABC$.

(b) Is BD an angle bisector of $\angle ABC$? Explain your answer. (Hint: A clear figure may help)

Problem 7

Let A, B and C be three points in a plane such that $|\overrightarrow{AB}| = |\overrightarrow{AC}| = 4$ and $\overrightarrow{AB} \cdot \overrightarrow{AC} = 2$. Find the length of BC.

Problem 8

Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{a}| = 1$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = 1$.

(a) Find the angle between the vectors $\vec{a}$ and $\vec{b}$.

(b) Find the value of $(3\vec{a} - 2\vec{b}) \cdot (\vec{a} + 3\vec{b})$ and $|\vec{a} - 2\vec{b}|$.

(c) Find the angle between two vectors $\vec{a} - 2\vec{b}$ and $2\vec{a} + 3\vec{b}$.

Problem 9

Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{a}| = 2$ and $|\vec{b}| = 3$ and the angle between these two vectors is $\cos^{-1}\frac{3}{5}$.

(a) Are the vector $\vec{a} - 2\vec{b}$ and $-9\vec{a} + 2\vec{b}$ perpendicular to each other? Explain your answer.

(b) If the angle between the vectors $\vec{a}$ and $\vec{a} + k\vec{b}$ is $60°$, find the value of k.

Problem 10

Find the projection vector of $\vec{a}$ onto $\vec{b}$ ($proj_{\vec{b}}\vec{a}$) for each of the following set of vectors $\vec{a}$ and $\vec{b}$.

(a) $\vec{a} = 3\vec{\imath} - 4\vec{\jmath}$ and $\vec{b} = \vec{\imath} - 18\vec{\jmath}$.

(b) $\vec{a} = 2\vec{\imath} - 3\vec{\jmath} - 6\vec{k}$ and $\vec{b} = 6\vec{\imath} - 2\vec{\jmath} + 11\vec{k}$.

(c) $\vec{a} = -\vec{\imath} + 2\vec{\jmath} + 2\vec{k}$ and $\vec{b} = \vec{\imath} + 7\vec{\jmath} + 7\vec{k}$.

Problem 11

(a) Let L_1 be a line passing through the points $A = (1,1,0)$ and $B = (-1,2,3)$, find the shortest distance between a point $C = (0,1,0)$ and the line L_1.

(b) Let L_2 be a line passing through the points $D = (2,-1,1)$ and $E = (0,0,1)$, find the shortest distance between a point $F = (1,3,-1)$ and the line L_2.

Part C: Vector Product and Scalar Triple Product

Problem 12

Find the value of $\vec{a} \times \vec{b}$ for each of following set of the vectors $\vec{a}$ and $\vec{b}$.

(a) $\vec{a} = \vec{\imath} + 3\vec{\jmath}$ and $\vec{b} = -2\vec{\jmath} + 5\vec{k}$.

(b) $\vec{a} = \vec{\imath} + \vec{\jmath} - 2\vec{k}$ and $\vec{b} = -3\vec{\imath} + 2\vec{\jmath} + 5\vec{k}$

(c) $\vec{a} = -3\vec{\imath} + \vec{\jmath} + 3\vec{k}$ and $\vec{b} = 6\vec{\jmath} + \vec{k}$

(d) $\vec{a} = \vec{\jmath} + \vec{k}$ and $\vec{b} = 3\vec{\imath} - \vec{\jmath} + 2\vec{k}$.

Problem 13

Let $\vec{a}$ and $\vec{b}$ be two vectors in a plane, what is the value of $\vec{a} \cdot (\vec{a} \times \vec{b})$? (Hint: Think about the relationship between the vector $\vec{a}$ and $\vec{a} \times \vec{b}$.)

Problem 14

Let $A = (1,2,0)$, $B = (3,-1,-2)$ and $C = (-2,0,1)$ be three points in the plane.

(a) Find a vector which is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(b) Let $\vec{a}$ be a vector with the same magnitude as that of $\overrightarrow{BC}$ and it is perpendicular to both vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Find the vector $\vec{a}$.

(c) (A bit harder) Find the equation of the plane containing the points A, B and C. (Hint: See the remark of Example 12 of Chapter 4.)

Problem 15

Let $\vec{a} = 2\vec{i} - \vec{j} + 2\vec{k}$ and $\vec{b} = 4\vec{i} - 4\vec{j} + 3\vec{k}$ be two vectors.

(a) Find a vector $\vec{c}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$.

(b) Find the area of the triangle with $\vec{a}$ and $\vec{b}$ as its adjacent sides.

(c) Find the equation of the plane passing through a point $(1,1,1)$ and containing the vectors $\vec{a}$ and $\vec{b}$. (Hint: See the remark of Example 12 of Chapter 4.)

(d) Let $\vec{d} = \vec{i} + 2\vec{k}$ be a vector. Determine whether the vectors $\vec{a}$, $\vec{b}$ and $\vec{d}$ are coplanar by finding the volume of parallelepiped with $\vec{a}$, $\vec{b}$ and $\vec{d}$ as adjacent sides.

Problem 16

Let $A = (3,-1,3)$, $B = (0,7,-2)$ and $C = (-9,3,-3)$ be three points in a plane. Find the area of the triangle ABC. Also find the area of the parallelogram with AB and AC as the adjacent sides.

Problem 17

In each of the following, determine whether the given three points are collinear.

(a) $A = (-1,0,1)$, $B = (2,4,1)$ and $C = (1,1,0)$

(b) $A = (1,2,-1)$, $B = (-1,1,2)$ and $C = (3,3,-4)$

Problem 18

In each of the following, find the volume of parallelepiped with the given four points as the adjacent vertices. Hence determine if the given four points are coplanar.

(a) $A = (2,1,-1)$, $B = (0,1,1)$, $C = (-2,-1,5)$ and $D = (2,3,-3)$.

(b) $A = (1,1,1)$, $B = (1,-1,3)$, $C = (-1,0,2)$ and $D = (2,-1,2)$.

Problem 19

(a) Let π_1 be a plane containing the points $A = (3,-2,0)$, $B = (2,0,3)$ and $C = (1,-1,1)$, find the shortest distance between the point $D = (1,0,-1)$ and the plane π_1.

(b) Let π_2 be a plane passing through a point $A = (2,1,-6)$. It is also given that the vector $\vec{n} = -\vec{i} - \vec{j} - \vec{k}$ is perpendicular to the plane π_2. Find the shortest distance between $B = (1,-1,1)$ and the plane π_2.

Problem 20

(a) Let L_1 be a line passing through the points $(5,0,-1)$ and $(6,2,-2)$. We let L_2 be another line passing through the points $(2,4,0)$ and $(3,3,1)$. Find the shortest distance between the line L_1 and L_2.

(b) Let L_1 be a line passing through the points $(1,1,1)$ and $(2,1,2)$. We let L_2 be another line passing through the points $(2,1,0)$ and $(3,2,0)$. Find the shortest distance between the line L_1 and L_2.

Part D: Linear Independence of vectors

Problem 21

Determine if each of the following set of vectors are linearly independent.

(a) $\vec{a} = \vec{\imath} - 2\vec{\jmath}$ and $\vec{b} = 2\vec{\imath} + \vec{\jmath}$.

(b) $\vec{a} = \vec{\imath} - 2\vec{\jmath} + 3\vec{k}$, $\vec{b} = 2\vec{\imath} + 5\vec{\jmath} + \vec{k}$ and $\vec{c} = 3\vec{\imath} + 2\vec{\jmath} - 3\vec{k}$.

(c) $\vec{a} = \vec{\imath} + 2\vec{\jmath} - 5\vec{k}$, $\vec{b} = -\vec{\imath} + 2\vec{\jmath} - 3\vec{k}$ and $\vec{c} = 3\vec{\imath} - 2\vec{\jmath} + \vec{k}$.

Problem 22

Find the value of m such that the following sets of vectors are *linearly dependent*.

$$\vec{a} = (1-m)\vec{\imath} + 6\vec{\jmath} + 5\vec{k}, \qquad \vec{b} = 2\vec{\imath} - m\vec{\jmath}, \qquad \vec{c} = -5m\vec{\jmath} + 5\vec{k}.$$

Part E: A bit harder problems

Problem 23

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-zero vectors. Show that

(a) $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$.

(b) If $\vec{a}$ and $\vec{b}$ are perpendicular, then $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$.

(c) If $\vec{a}$ and $\vec{b}$ are parallel, then the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are also parallel. (Hint: If two vectors are parallel, what is the angle between them? What can you say about the vector product of these two vectors?)

(d) $\tan\theta = \frac{|\vec{a}\times\vec{b}|}{\vec{a}\cdot\vec{b}}$ where θ is the angle between $\vec{a}$ and $\vec{b}$.

(e) $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$.