$\sin(2e^x+1)$. "compositional function" $\sin y$: $y = 2e^x+1$

Problem 3

Compute the following integrals using suitable method. You may need to use method of substitution or integration by parts or both.

(a) $\int e^{2x}\sin(2e^x+1)\,dx$

(b) $\int_0^1 \sin(2\sqrt{x})\,dx$

(c) $\int_0^1 \ln(1+\sqrt[3]{x})\,dx$

(d) $\int \cos(\ln x)\,dx$

(e) $\int \sin 2x \ln(\sin x)\,dx$

(f) $\int (x+1)\ln(x+3)\,dx$

(g) $\int_1^2 \frac{e^{2x}}{e^x-1}\,dx$

(h) $\int x^3\cos(3x^2)\sin(x^2)\,dx$

(a) Let $y = 2e^x+1 \Rightarrow \frac{dy}{dx} = 2e^x \Rightarrow dx = \frac{1}{2e^x}dy$.

method of substitution

$$\int e^{2x}\sin(2e^x+1)\frac{1}{2e^x}dy = \frac{1}{2}\int \underbrace{e^x}_{=\frac{y-1}{2}}\sin(2e^x+1)\,dy$$

$$= \frac{1}{2}\int \frac{y-1}{2}\sin y\,dy = \frac{1}{4}\int y\sin y\,dy - \frac{1}{4}\int \sin y\,dy.$$

let $u = y$

$dv = \sin y\,dy \Rightarrow v = -\cos y$

integration by parts.

$=$ $\boxed{x}$

$\int x^2\sin x\,dx$

$\int x^2\cos x\,dx$

$\int x e^x\,dx$

$u = x$, $dv = \sin x\,dx$

$= \cos x\,dx$

$= e^x dx$.

$\int x\ln x\,dx$, $u = \ln x$, $dv = x\,dx$.

(c) $\int_0^1 \ln(1+\sqrt[3]{x})\,dx$

Let $y = 1+\sqrt[3]{x} \Rightarrow \frac{dy}{dx} = \frac{1}{3x^{\frac{2}{3}}} \Rightarrow dx = 3x^{\frac{2}{3}}dy$.

when $x=0$, $y=1$; $x=1$, $y=2$.

$$\int_1^2 \ln(1+\sqrt[3]{x})(3\underbrace{x^{\frac{2}{3}}}_{x^{\frac{1}{3}}=y-1}dy) = 3\int_1^2 \ln y\cdot(y-1)^2\,dy.$$

let $u = \ln y$

$dv = (y-1)^2 dy \Rightarrow v = \frac{(y-1)^3}{3}$

$$= \frac{(y-1)^3}{3}\ln y\Big|_1^2 - \int_1^2 \frac{(y-1)^3}{3}\cdot\frac{1}{y}\,dy$$

$$= \ln 2 - \int_1^2 \frac{y^3-3y^2+3y-1}{y}\,dy$$

$$= \ln 2 - \int_1^2 \left(y^2 - 3y + 3 - \frac{1}{y}\right)dy$$

$$= \dots$$

(d) $\int \cos(\ln x)\,dx$

$y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x} \Rightarrow dx = x\,dy.$

$$\int \cos(\ln x)\, \overset{e^y}{x}\, dy = \boxed{\int \cos y \cdot e^y\, dy}$$

let $u = \cos y$

$dv = e^y dy \Rightarrow v = e^y$

$$e^y \cos y + \int e^y \sin y\, dy.$$

let $u = \sin y$

$dv = e^y dy \Rightarrow v = e^y$

$$e^y \cos y + e^y \sin y - \boxed{\int e^y \cos y\, dy}$$

$$\Rightarrow 2\int e^y \cos y\, dy = e^y\cos y + e^y \sin y + C.$$

$$\Rightarrow \int e^y \cos y\, dy = \frac{1}{2}\left[e^y \cos y + e^y \sin y\right] + C.$$

$$= \frac{1}{2}\left[\boxed{e^{\ln x}}^{=x}\cos(\ln x) + \boxed{e^{\ln x}}^{=x}\sin(\ln x)\right] + C$$

$$= \frac{1}{2}\left[x\cos(\ln x) + x\sin(\ln x)\right] + C.$$

(g) $\int_1^2 \frac{e^{2x}}{e^x - 1} dx$

let $y = e^x - 1 \Rightarrow \frac{dy}{dx} = e^x \Rightarrow dx = \frac{1}{e^x} dy$.

When $x=1$, $y = e-1$; $x=2$, $y = e^2-1$.

$$\int_{e-1}^{e^2-1} \frac{e^{2x}}{e^x-1} \frac{1}{e^x} dy = \int_{e-1}^{e^2-1} \frac{e^x}{e^x-1} dy$$

$$= \int_{e-1}^{e^2-1} \frac{y+1}{y} dy$$

$$= \int_{e-1}^{e^2-1} \left(1+\frac{1}{y}\right) dy.$$

$\vdots$

Problem 4

(a) Compute the integrals

$$\int e^{2x}\sin 3x\,dx \quad \text{and} \quad \int e^{2x}\cos 3x\,dx.$$

(b) Hence, compute the integrals

$$\int x e^{2x}\cos 3x\,dx.$$

(Hint: You need to eliminate x in the integrand so that you can compute the integral using the result of (a). Which technique should you use: Method of substitution and/or integration by parts?)

$$\boxed{\int u\,dv} = uv - \boxed{\int v\,du}$$

$u = x$

$$\boxed{\int e^{2x}\sin 3x\,dx}$$

let $u = \sin 3x$

$dv = e^{2x}dx \Rightarrow v = \frac{1}{2}e^{2x}$

$$\frac{1}{2}e^{2x}\cdot\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\,dx.$$

let $u = \cos 3x$

$dv = e^{2x}dx \Rightarrow v = \frac{1}{2}e^{2x}$

$$\frac{1}{2}e^{2x}\cdot\sin 3x - \frac{3}{2}\left[\frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\,dx.\right]$$

$$= \frac{1}{2}e^{2x}\cdot\sin 3x - \frac{3}{4}e^{2x}\cos 3x - \frac{9}{4}\boxed{\int e^{2x}\sin 3x\,dx}.$$

$$\Rightarrow \frac{13}{4}\int e^{2x}\sin 3x\,dx = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x + C.$$

$$\Rightarrow \int e^{2x}\sin 3x\,dx = \frac{2}{13}e^{2x}\sin 3x - \frac{3}{13}e^{2x}\cos 3x + C.$$

(b). let $u = x$. $dv = e^{2x}\cos 3x\,dx \Rightarrow v = \int e^{2x}\cos 3x\,dx = \frac{3}{13}e^{2x}\sin 3x + \frac{2}{13}e^{2x}\cos 3x.$

$$\int x e^{2x}\cos 3x\,dx = x\left[\frac{3}{13}e^{2x}\sin 3x + \frac{2}{13}e^{2x}\cos 3x\right] - \int \frac{3}{13}e^{2x}\sin 3x + \frac{2}{13}e^{2x}\cos 3x\,.\,dx$$

$$= x\ \ldots\ - \frac{3}{13}\int e^{2x}\sin 3x\,dx - \frac{2}{13}\int e^{2x}\cos 3x\,dx.$$

Problem 5

Let $f(x)$ be a differentiable function on $[a,b]$ such that $\int_a^b f(x)dx = 0$ and $f(a) = f(b) = 1$. Find the value of $\int_a^b xf'(x)dx$.

$$\int_a^b x \cdot f'(x)\,dx$$

Let $u = x$, $dv = f'(x)dx \Rightarrow v = f(x)$.

$$= x f(x)\Big|_a^b - \underbrace{\int_a^b f(x)\,dx}_{=0}$$

$$= b\underbrace{f(b)}_{=1} - a\underbrace{f(a)}_{=1}$$

$$= b - a.$$

Problem 7

Let $f(x)$ be a twice differentiable function on $[0,1]$ such that $f(0) = f(1) = 1$ and $\int_0^1 f(x)dx = 1$. Using integration by parts, find the value of

$$\int_0^1 x(1-x)f''(x)dx.$$

(Hint: The technique in Problem 5 may be useful.)

let $u = x(1-x)$. $dv = f''(x)\,dx \Rightarrow v = f'(x)$

$$\int_0^1 x(1-x)f''(x)dx = \underbrace{x(1-x)f'(x)\Big|_0^1}_{} - \int_0^1 f'(x)\,d[x(1-x)]$$

$1-x+(-1)x$
$= 1-2x$

$$= 0 - \int_0^1 f'(x)(1-2x)dx$$

$$= -\int_0^1 f'(x)dx + 2\int_0^1 x f'(x)dx$$

$$= \underbrace{-f(x)\Big|_0^1}_{=0} + 2\int_0^1 x f'(x)dx$$

let $u = x$
$dv = f'(x)dx \Rightarrow v = f(x)$

$$= 2xf(x)\Big|_0^1 - 2\underbrace{\int_0^1 f(x)dx}_{=1}$$

$$= 2\underbrace{f(1)}_{=1} - 0 - 2$$

$$= 2-2$$

$$= 0.$$