4. Vector Algebra

4.1 Vectors

Definition

A *vector* is a directed line segment. The directed line segment $\overrightarrow{AB}$ has *initial point* A and *terminal point* B; its *length* is denoted by $|\overrightarrow{AB}|$. Two vectors are *equal* if they have the same length and direction.

Component Form

- *Definition*. Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\vec{a}$ with representation $\overrightarrow{AB}$ is
$$\vec{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$
- *Length*. The length or magnitude of the vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is
$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Vector Algebra Operations

- *Definition*. If $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$ are vectors and k is a scalar, then
$$\begin{aligned} \text{Vector addition/subtraction}: &\qquad \vec{a} \pm \vec{b} = \langle a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3 \rangle, \\ \text{Scalar multiplication}: &\qquad k\vec{a} = \langle ka_1, ka_2, ka_3 \rangle. \end{aligned}$$
- *Properties of vector operations*. If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and k and l are scalars, then
  1. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.
  2. $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$.
  3. $\vec{a} + \vec{0} = \vec{a}$.
  4. $\vec{a} + (-\vec{a}) = \vec{0}$.
  5. $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$.
  6. $(k + l)\vec{a} = k\vec{a} + l\vec{a}$.
  7. $(kl)\vec{a} = k(l\vec{a})$.
  8. $0\vec{a} = \vec{0}$.
  9. $1\vec{a} = \vec{a}$.
  10. $|k\vec{a}| = |k||\vec{a}|$.

Unit Vectors

- *Definition*. A vector $\vec{a}$ of length 1 is called a *unit vector*.
- *Direction vector*. The (unit) *direction vector* of the vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{a} \neq \vec{0}$ is
$$\frac{\vec{a}}{|\vec{a}|} = \left\langle \frac{a_1}{|\vec{a}|}, \frac{a_2}{|\vec{a}|}, \frac{a_3}{|\vec{a}|} \right\rangle, \qquad \text{where} \qquad |\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$
  The equation $\vec{a} = |\vec{a}| \cdot \dfrac{\vec{a}}{|\vec{a}|}$ expresses $\vec{a}$ in terms of its length and direction.
- *Standard unit vectors*. The *standard unit vectors* in three-dimensional space are
$$\vec{i} = \langle 1, 0, 0 \rangle, \qquad \vec{j} = \langle 0, 1, 0 \rangle, \qquad \vec{k} = \langle 0, 0, 1 \rangle.$$

4.2 The Dot Product

Definition

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the *dot product* of $\vec{a}$ and $\vec{b}$ is the scalar $\vec{a} \cdot \vec{b}$ given by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

It is also called the *scalar product* or *inner product* of $\vec{a}$ and $\vec{b}$.

Properties of the Dot Product

If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and k is a scalar, then

1. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$.
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$.
4. $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (k\vec{b})$.
5. $\vec{0} \cdot \vec{a} = 0$.

Geometric and Physical Facts Related to the Dot Product

- *Angle between two vectors*. If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, then

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta \qquad \text{or} \qquad \cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}.$$

  $\vec{a}$ and $\vec{b}$ are orthogonal ($\theta = \frac{1}{2}\pi$) if and only if $\vec{a} \cdot \vec{b} = 0$.

- *Direction angles and direction cosines*. If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is a nonzero vector, then

$$\frac{\vec{a}}{|\vec{a}|} = \left\langle \frac{a_1}{|\vec{a}|}, \frac{a_2}{|\vec{a}|}, \frac{a_3}{|\vec{a}|} \right\rangle = \langle \cos \alpha, \cos \beta, \cos \gamma \rangle,$$

  where α, β, and γ are the *direction angles* that $\vec{a}$ makes with the positive x-, y-, and z-axes.

- *Projections*.

  Scalar projection of $\vec{a}$ onto $\vec{b}$: $\quad \text{comp}_{\vec{b}}\vec{a} = \vec{a} \cdot \dfrac{\vec{b}}{|\vec{b}|} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}.$

  Vector projection of $\vec{a}$ onto $\vec{b}$: $\quad \text{proj}_{\vec{b}}\vec{a} = \left(\vec{a} \cdot \dfrac{\vec{b}}{|\vec{b}|}\right) \dfrac{\vec{b}}{|\vec{b}|} = \left(\dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\right) \vec{b}.$

- *Work*. The work done by a constant force $\vec{F}$ acting through a displacement $\vec{D} = \overrightarrow{PQ}$ is

$$W = \vec{F} \cdot \vec{D} = |\vec{F}||\vec{D}| \cos \theta,$$

  where θ is the angle between $\vec{F}$ and $\vec{D}$.

Finding the Distance from a Point to a Line

To find the distance d from a point P to a line passing through the points A and B, take the following steps:

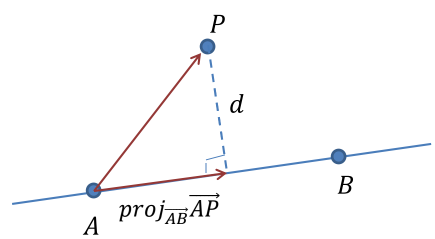
(a) compute the vectors $\overrightarrow{AB}$ and $\overrightarrow{AP}$;

(b) compute the *scalar* projection of $\overrightarrow{AP}$ onto $\overrightarrow{AB}$:

$$\mathrm{comp}_{\overrightarrow{AB}}\overrightarrow{AP} = \frac{\overrightarrow{AP}\cdot\overrightarrow{AB}}{|\overrightarrow{AB}|};$$

(c) compute the distance d using the Pythagorean theorem (note that $|\mathrm{comp}_{\overrightarrow{AB}}\overrightarrow{AP}| = |\mathrm{proj}_{\overrightarrow{AB}}\overrightarrow{AP}|$):

$$d = \sqrt{|\overrightarrow{AP}|^2 - |\mathrm{comp}_{\overrightarrow{AB}}\overrightarrow{AP}|^2}.$$



4.3 The Cross Product

Definition

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the *cross product* of $\vec{a}$ and $\vec{b}$ is the vector

$$\vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

It is also called the *vector product* of $\vec{a}$ and $\vec{b}$.

Properties of the Cross Product

If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and k is a scalar, then

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$.
2. $(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b}) = \vec{a} \times (k\vec{b})$.
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$.
5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$.
6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$.

Warning. $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$! e.g. $(\vec{i} \times \vec{i}) \times \vec{j} = \vec{0} \times \vec{j} = \vec{0}$, but $\vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}$.

Geometric and Physical Facts Related to the Cross Product

- *The direction of a cross product*.
  (a) The vector $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$, pointing in a direction determined by the *right-hand rule*.
  (b) If θ is the angle between $\vec{a}$ and $\vec{b}$, then
  $$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta \qquad \text{or} \qquad \sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}.$$
  $\vec{a}$ and $\vec{b}$ are parallel ($\theta = 0$ or π) if and only if $\vec{a} \times \vec{b} = 0$.
- *The length of a cross product*. The length of the cross product $\vec{a} \times \vec{b}$ is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.
- *The scalar triple product*. The volume of the parallelepiped determined by the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is equal to
  $$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|,$$
  where $\vec{a} \cdot (\vec{b} \times \vec{c})$ is the *scalar triple product* of $\vec{a}$, $\vec{b}$, $\vec{c}$ and can be written as:
  $$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$
- *Torque*. The torque (relative to a point P) produced by a force $\vec{F}$ acting on a rigid body at a point Q is
  $$\boldsymbol{\tau} = \overrightarrow{PQ} \times \vec{F}.$$

Finding the Distance from a Point to a Plane

To find the distance d from a point D to a plane containing the points A, B and C, take the following steps:

(a) compute the vectors $\overrightarrow{AD}$, $\overrightarrow{AB}$, and $\overrightarrow{AC}$;

(b) compute a vector $\vec{n}$ normal to the plane:

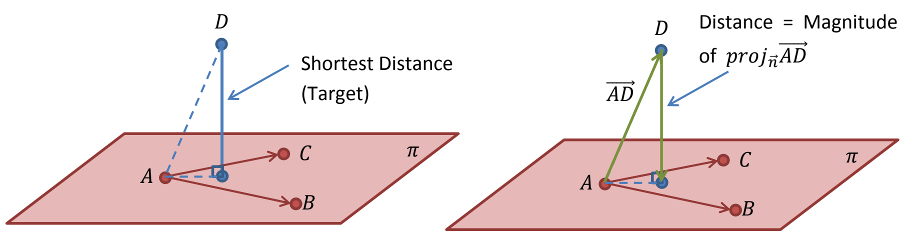
$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC};$$

(c) compute the *scalar* projection of $\overrightarrow{AD}$ onto $\vec{n}$:

$$\text{comp}_{\vec{n}}\overrightarrow{AD} = \frac{\overrightarrow{AD} \cdot \vec{n}}{|\vec{n}|};$$

(d) compute the distance d as the absolute value of $\text{comp}_{\vec{n}}\overrightarrow{AD}$:

$$d = |\text{proj}_{\vec{n}}\overrightarrow{AD}| = |\text{comp}_{\vec{n}}\overrightarrow{AD}|.$$



Finding the Distance between Two Lines

To find the distance d between two lines L_1 and L_2 passing through the points A, B and C, D, respectively, take the following steps:

(a) compute the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ lying on L_1 and L_2;

(b) compute a vector $\vec{n}$ perpendicular to both L_1 and L_2:

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{CD};$$

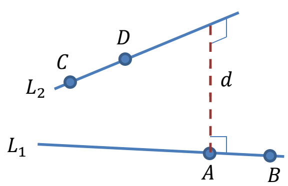
(c) compute a vector connecting two arbitrary points on L_1 and L_2, say $\overrightarrow{AD}$; any other vector $\overrightarrow{AC}$, $\overrightarrow{BC}$, or $\overrightarrow{BD}$ will do;

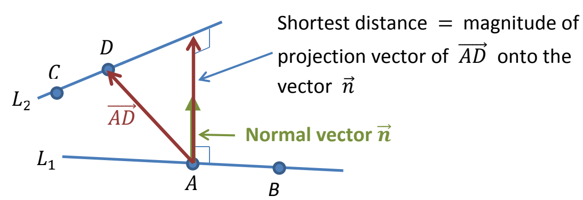
(d) compute the *scalar* projection of $\overrightarrow{AD}$ onto $\vec{n}$:

$$\mathrm{comp}_{\vec{n}}\overrightarrow{AD} = \frac{\overrightarrow{AD} \cdot \vec{n}}{|\vec{n}|};$$

(e) compute the distance d as the absolute value of $\mathrm{comp}_{\vec{n}}\overrightarrow{AD}$:

$$d = |\mathrm{proj}_{\vec{n}}\overrightarrow{AD}| = |\mathrm{comp}_{\vec{n}}\overrightarrow{AD}|.$$





1. Integrals

1.1 Antiderivatives and Indefinite Integrals

Definition

Let $f(x)$ be a given function. A function $F(x)$ is called an *antiderivative* of $f(x)$ on an interval $I = (a,b)$ if

$$\frac{d}{dx}F(x) = F'(x) = f(x)$$

for all x in I. The set of all antiderivatives of $f(x)$ is called the *indefinite integral* of $f(x)$ (with respect to x), and is denoted by

$$\int f(x)\,dx.$$

If $F(x)$ is an antiderivative of $f(x)$, then

$$\int f(x)\,dx = F(x) + C,$$

where C is an arbitrary constant.

Preliminary Table of Antiderivatives

Function	Particular antiderivative	Function	Particular antiderivative
x^n $(n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\dfrac{1}{x}$	$\log\lvert x\rvert$
e^x	e^x		
$\cos x$	$\sin x$	$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$	$\sec x \tan x$	$\sec x$
$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$

Properties of Indefinite Integrals

- *Linearity*.
  1. $\displaystyle\int \left[f(x) \pm g(x)\right] dx = \int f(x)\,dx \pm \int g(x)\,dx.$
  2. $\displaystyle\int c f(x)\,dx = c\int f(x)\,dx.$

- *Special integrals.*

  Constant and powers

1. $\int k\,dx = kx + C.$

2. $\int x^n\,dx = \begin{cases} \dfrac{x^{n+1}}{n+1} + C, & n \neq -1 \\ \log|x| + C, & n = -1 \end{cases}.$

Exponentials

3. $\int \mathrm{e}^x\,dx = \mathrm{e}^x + C.$

4. $\int a^x\,dx = \dfrac{a^x}{\log a} + C,\ a \neq 1,\ a > 0.$

Trigonometric functions

5. $\int \sin x\,dx = -\cos x + C.$

6. $\int \cos x\,dx = \sin x + C.$

7. $\int \sec^2 x\,dx = \tan x + C.$

8. $\int \csc^2 x\,dx = -\cot x + C.$

9. $\int \sec x \tan x\,dx = \sec x + C.$

10. $\int \csc x \cot x\,dx = -\csc x + C.$

Algebraic functions

11. $\int \dfrac{1}{x^2+1}\,dx = \tan^{-1} x + C.$

12. $\int \dfrac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C.$

1.2 Areas and Distances

- *The area problem.* The *area* A of the region R that lies under the graph of a continuous function $f(x)$ and between the vertical lines $x = a$ and $x = b$ is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty}\left[f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x\right] = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x, \qquad c_i \in [x_{i-1}, x_i],\ \Delta x = \frac{b-a}{n}.$$

  Specifically,

$$\begin{aligned}
A &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x, && \text{if } c_i = x_i \text{ (right Riemann sum)},\\
A &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_{i-1})\Delta x, && \text{if } c_i = x_{i-1} \text{ (left Riemann sum)},\\
A &= \lim_{n\to\infty}\sum_{i=1}^{n} f(\bar{x}_i)\Delta x, && \text{if } c_i = \bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) \text{ (midpoint Riemann sum)}.
\end{aligned}$$

- *The distance problem.* The *distance* traveled by an object moving with velocity $v = f(t)$ during the time interval $[a, b]$ is:

$$d = \lim_{n\to\infty}\sum_{i=1}^{n} f(t_{i-1})\Delta t = \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)\Delta t, \qquad \Delta t = \frac{b-a}{n}.$$

1.3 The Definite Integral

Definition

Let $f(x)$ be a function defined on a closed interval $[a,b]$. We say that $f(x)$ is *(Riemann) integrable* over $[a,b]$ if the Riemann sums $\sum_{i=1}^{n} f(c_i)\Delta x_i$ converge to a number I as the maximum subinterval width $\|P\|$ in the partition P tends to 0. The number I is called the *definite integral* of $f(x)$ over $[a,b]$, and it is denoted by

$$I = \int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i.$$

Remark. Geometrically, $\int_a^b f(x)\,dx$ gives the *signed area* of the region trapped between the curve $y = f(x)$ and the x-axis on the interval $[a,b]$, meaning that a *positive* sign is attached to areas of parts *above* the x-axis and a *negative* sign is attached to areas of parts *below* the x-axis.

Properties of Finite Sums

- *Linearity*.

  1. $\displaystyle\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i.$
  2. $\displaystyle\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i.$

- *Special sums*.

  1. $\displaystyle\sum_{i=1}^{n} c = nc.$
  2. $\displaystyle\sum_{i=1}^{n} i = \frac{1}{2}n(n+1).$
  3. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1).$
  4. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2.$

Properties of Definite Integrals

- *Linearity*.

  1. $\displaystyle\int_a^b \big[f(x) \pm g(x)\big]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$
  2. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$

- *Interval properties*.

  1. $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$
  2. $\displaystyle\int_a^a f(x)\,dx = 0.$
  3. $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$

- *Comparison properties*.

  1. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,dx \geq 0.$

2. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
3. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

- *Special integrals*.

1. $\int_a^b c\,dx = c(b-a)$.

Hint. You may remember the interval properties by noting that $[b,a] = -[a,b]$, that the length of $[a,a] = 0$, and that $[a,c]+[c,b] = [a,b]$.

1.4 The Fundamental Theorem of Calculus

Let $f(x)$ be a function defined on a closed interval $[a,b]$.

(a) If $G(x) = \int_a^x f(t)\,dt$, then $G'(x) = f(x)$.

(b) $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F(x)$ is *any* antiderivative of $f(x)$.

We may also rewrite the theorem as

(a) $\dfrac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

(b) $\int_a^b \dfrac{d}{dx}F(x)\,dx = F(b) - F(a)$.

*The Generalized First Fundamental Theorem of Calculus

Let $f(x)$ be continuous and $a(x),\ b(x)$ be differentiable. Then

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(t)\,dt\right) = f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x).$$

Remark. In particular, we have

$$\frac{d}{dx}\left(\int_{a}^{b(x)} f(t)\,dt\right) = f(b(x))\cdot b'(x), \qquad \frac{d}{dx}\left(\int_{a(x)}^{b} f(t)\,dt\right) = -f(a(x))\cdot a'(x).$$

For a proof of this theorem, see appendix.

Examples

[Hw 1.4] Compute the following derivatives:

$$\frac{d}{dx}\int_3^x e^{2y^2+1}\,dy, \qquad \frac{d}{dx}\int_{2x}^{x^2} \cos(y^2)\,dy.$$

Appendix: The Proof of the Generalized Fundamental Theorem

The main idea of the proof is to use the chain rule and the first fundamental theorem of calculus. First, by the interval additive property,

$$\int_{a(x)}^{b(x)} f(t)\,dt = \int_{a(x)}^{c} f(t)\,dt + \int_{c}^{b(x)} f(t)\,dt$$

for any fixed real number c. To apply the first fundamental theorem of calculus, we also rewrite the first integral on the right side so that

$$\int_{a(x)}^{b(x)} f(t)\,dt = -\int_{c}^{a(x)} f(t)\,dt + \int_{c}^{b(x)} f(t)\,dt = \int_{c}^{b(x)} f(t)\,dt - \int_{c}^{a(x)} f(t)\,dt.$$

Now

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(t)\,dt\right) = \frac{d}{dx}\left(\int_{c}^{b(x)} f(t)\,dt\right) - \frac{d}{dx}\left(\int_{c}^{a(x)} f(t)\,dt\right).$$

To find the derivative

$$\frac{d}{dx}\left(\int_{c}^{b(x)} f(t)\,dt\right),$$

we define

$$F(x) = \int_{c}^{b(x)} f(t)\,dt, \qquad G(u) = \int_{c}^{u} f(t)\,dt$$

and introduce the intermediate variable

$$u = b(x).$$

Then clearly $F(x) = G(b(x)) = G(u)$, so by the chain rule

$$\frac{d}{dx}\left(\int_{c}^{b(x)} f(t)\,dt\right) = \frac{d}{dx}\big[F(x)\big] = \frac{d}{dx}\big[G(b(x))\big] = \frac{dG(u)}{du}\cdot\frac{du}{dx}.$$

By the first fundamental theorem of calculus,

$$\frac{dG(u)}{du} = \frac{d}{du}\left(\int_{c}^{u} f(t)\,dt\right) = f(u)$$

(remember that c is a fixed real number), so

$$\frac{d}{dx}\left(\int_{c}^{b(x)} f(t)\,dt\right) = \frac{dG(u)}{du}\cdot\frac{du}{dx} = f(u)\cdot\frac{du}{dx}.$$

After substituting $u = b(x)$ into above equation we obtain

$$\frac{d}{dx}\left(\int_{c}^{b(x)} f(t)\,dt\right) = f(b(x))\cdot b'(x).$$

Similarly we can show that

$$\frac{d}{dx}\left(\int_{c}^{a(x)} f(t)\,dt\right) = f(a(x))\cdot a'(x),$$

so

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(t)\,dt\right) = f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x).$$

2. Techniques of Integration

Review of Basic Integrals

Constant and powers

1. $\int k\,dx = kx + C.$

2. $\int x^n\,dx = \begin{cases} \dfrac{x^{n+1}}{n+1} + C, & n \neq -1 \\ \log|x| + C, & n = -1 \end{cases}.$

Exponentials

3. $\int \mathrm{e}^x\,dx = \mathrm{e}^x + C.$

4. $\int a^x\,dx = \dfrac{a^x}{\log a} + C,\ a \neq 1,\ a > 0.$

Trigonometric functions

5. $\int \sin x\,dx = -\cos x + C.$

6. $\int \cos x\,dx = \sin x + C.$

7. $\int \sec^2 x\,dx = \tan x + C.$

8. $\int \csc^2 x\,dx = -\cot x + C.$

9. $\int \sec x \tan x\,dx = \sec x + C.$

10. $\int \csc x \cot x\,dx = -\csc x + C.$

11. $\int \tan x\,dx = \log|\sec x| + C.$

12. $\int \cot x\,dx = \log|\sin x| + C.$

13. $\int \sec x\,dx = \log|\sec x + \tan x| + C.$

14. $\int \csc x\,dx = \log|\csc x - \cot x| + C.$

Algebraic functions

15. $\int \dfrac{1}{x^2 + a^2}\,dx = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C.$

16. $\int \dfrac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\dfrac{x}{a}\right) + C.$

17. $\int \dfrac{1}{x^2 - a^2}\,dx = \dfrac{1}{2a}\log\left|\dfrac{x-a}{x+a}\right| + C.$

18. $\int \dfrac{1}{\sqrt{x^2 \pm a^2}}\,dx = \log\left|x + \sqrt{x^2 \pm a^2}\right| + C.$

Hyperbolic functions

19. $\int \sinh x\,dx = \cosh x + C.$

20. $\int \cosh x\,dx = \sinh x + C.$

Review of Differentials

- *Definition*. The *differential* dx in a definite integral $\int_a^b f(x)\,dx$ is an infinitesimal analog of the subinterval width Δx introduced in a Riemann sum $\sum_{i=1}^n f(c_i)\Delta x_i$; it represents infinitesimal changes in the variable x. Likewise, if $y = f(x)$ is a differentiable function of x, then the differential dy represents the infinitesimal change in the variable y as an infinitesimal change of dx occurs in the variable x:

$$dy = f(x+dx) - f(x).$$

- *The law of differential*. If $y = f(x)$ is a differentiable function of x, then

$$dy = f'(x)\,dx = \frac{dy}{dx}\,dx.$$

- *Properties of differentials*. If $u(x),\ v(x)$ are differentiable functions and c is a constant, then

  1. $d(cu) = c\,du$.
  2. $d(u \pm v) = du \pm dv$.
  3. $d(uv) = v\,du + u\,dv$.
  4. $d\left(\dfrac{u}{v}\right) = \dfrac{v\,du - u\,dv}{v^2}$.

2.1 The Substitution Rule

(a) *The substitution rule for indefinite integrals.* Let $u = g(x)$ be a differentiable function whose range is an interval I and $f(u)$ be a continuous function on I. Then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

(b) *The substitution rule for definite integrals.* Let $g'(x)$ be continuous on the interval $[a,b]$ and $f(u)$ be continuous on the range of $g(x)$. Then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Integrals of Symmetric Functions

Let $f(x)$ be a continuous function on $[-a,a]$.

(a) If $f(x)$ is even, i.e. $f(-x) = f(x)$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If $f(x)$ is odd, i.e. $f(-x) = -f(x)$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

Rules of Thumb

1. If there is a power (fraction) in the integrand, let the base (denominator) be the new variable.
[Hw 2.1(c)(e)(f)] $\displaystyle\int x^{11}\sqrt{1+x^4}\,dx$, $\displaystyle\int \sin 2x\sqrt{\cos x}\,dx$, $\displaystyle\int \frac{e^{2x}}{(1+e^x)^3}\,dx$.

2. If there is a trigonometric function in the integrand, let its argument be the new variable.
[Hw 2.1(b)(d)(h)] $\displaystyle\int x^2\sec(1-2x^3)\,dx$, $\displaystyle\int x\cos^2(x^2)\,dx$, $\displaystyle\int_1^5 \frac{\sin^2(\log x)}{x}\,dx$.

3. If there is an exponential function in the integrand, let its exponent be the new variable.
[Hw 2.1(a)(g)] $\displaystyle\int \frac{e^{1+(1/x^2)}}{x^3}\,dx$, $\displaystyle\int_1^2 xe^{x^2-1}\,dx$.

4. The key to success: after identifying the new variable u as part of the integrand, the rest of the integrand must be expressible *as a product* $g(u)u'$ for some function g.
[Hw 2.1(i)(j)(k)] $\displaystyle\int \frac{3x+2}{x^2+4}\,dx$, $\displaystyle\int \frac{2x+1}{x^2-2x+5}\,dx$, $\displaystyle\int \frac{4x}{3x^2+6x+19}\,dx$.

2.2 Integration by Parts

The Formula

(a) Indefinite integrals: $\displaystyle\int u\,dv = uv - \int v\,du.$

(b) Definite integrals: $\displaystyle\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$

Rules of Thumb

1. The ILATE rule: a rule of thumb for the choice of u and dv in the integration by parts formula is to let u be whichever function that comes first in the *ILATE list*:

I : inverse trigonometric functions: $\tan^{-1} x,\ \sin^{-1} x$, etc.;

L : logarithmic functions: $\log x$, etc.;

A : algebraic functions: $x^2,\ 3x^5$, etc.;

T : trigonometric functions: $\sin x,\ \tan x$, etc.;

E : exponential functions: e^x, etc.;

and then let dv be the rest of the integrand (including dx). Note that functions longer down the list have easier antiderivatives than the functions above them.

[Hw 2.2(a)(b)(f)] $\displaystyle\int x\mathrm{e}^{-3x}\,dx,\ \int_1^{\mathrm{e}} \sqrt{x}\log x\,dx,\ \int \tan^{-1} x\,dx.$

2. In some cases repeated integration by parts is needed.

[Hw 2.2(c)(e)(j)] $\displaystyle\int x^2 \sin x\,dx,\ \int x^2\cos^{-1} x\,dx,\ \int \mathrm{e}^x \sin 3x\,dx.$

3. In some cases a substitution is needed or preferred before integration by parts.

[Hw 2.3(a)(c)(e)] $\displaystyle\int \mathrm{e}^{2x}\sin(2\mathrm{e}^x+1)\,dx,\ \int_0^1 \log(1+\sqrt[3]{x})\,dx,\ \int \sin 2x\cdot\log(\sin x)\,dx.$

The Reduction Method

1. The reduction method is an integration technique which aims to compute integrals involving powers of elementary functions such as $\cos^n x$, $(ax^2+bx+c)^n$, and $(\log x)^n$.

2. The derivation of reduction formulas requires the use of integration by parts, where u is normally chosen to be the power involved in the integral.

Examples. For $I_n = \int x^n e^x \, dx$, let $u = x^n$ and $dv = e^x \, dx = d(e^x)$.

For $I_n = \int_1^e (\log x)^n \, dx$, let $u = (\log x)^n$ and $dv = dx$.

3. In some cases, integration by parts leads to an equation relating I_n to I_{n+1} instead of an equation relating I_n to I_{n-1}. In this case one simply solves for I_{n+1} in terms of I_n.

Example. The equation for $I_n = \int_0^1 \frac{1}{(1+x^2)^n} \, dx$ is $I_n = \frac{1}{2^n} + 2nI_n - 2nI_{n+1}$, which leads to the reduction formula $I_{n+1} = \frac{1}{2^{n+1}n} + \frac{2n-1}{2n} I_n$.

4. In some cases, it may be preferable to use a substitution before integration by parts.

Example. For $I_n = \int x^n e^{x^2+1} \, dx$, substitute $w = x^2+1$ to obtain $I_n = \frac{1}{2}\int (w-1)^{(n-1)/2} e^w \, dw$.

5. For integrals involving powers of trigonometric functions, it is usually necessary to write the power as a product of two factors, one being u, the other being v'.

Examples. For $S_n = \int_0^{\pi/2} \sin^n x \, dx$, let $u = \sin^{n-1} x$ and $dv = \sin x \, dx = d(-\cos x)$.

For $C_n = \int_0^{\pi/2} \cos^n x \, dx$, let $u = \cos^{n-1} x$ and $dv = \cos x \, dx = d(\sin x)$.

For $I_n = \int \sec^n x \, dx$, let $u = \sec^{n-2} x$ and $dv = \sec^2 x \, dx = d(\tan x)$.

2.3 Trigonometric Integrals

Useful Identities

Pythagorean identities

1. $\sin^2 x + \cos^2 x = 1$.
2. $1 + \tan^2 x = \sec^2 x$.
3. $1 + \cot^2 x = \csc^2 x$.

Half-angle identities

4. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.
5. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

Product-to-sum identities

6. $\sin mx \cos nx = \frac{1}{2}\left[\sin(m+n)x + \sin(m-n)x\right]$.
7. $\sin mx \sin nx = -\frac{1}{2}\left[\cos(m+n)x - \cos(m-n)x\right]$.
8. $\cos mx \cos nx = \frac{1}{2}\left[\cos(m+n)x + \cos(m-n)x\right]$.

Type 1 ($\int \sin^m x \cos^n x\,dx$)

(a) If the exponent of cosine is odd ($n = 2k+1$), save one factor of cosine and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining power of cosine as a function of sine; then substitute $u = \sin x$:

$$\begin{aligned}\int \sin^m x \cos^{2k+1} x\,dx &= \int (\sin x)^m (\cos^2 x)^k \cos x\,dx \\ &= \int (\underbrace{\sin x}_{u})^m (1 - \underbrace{\sin^2 x}_{u^2})^k \underbrace{\cos x\,dx}_{u'\,dx} = \int u^m (1-u^2)^k\,du.\end{aligned}$$

(b) If the exponent of sine is odd ($m = 2k+1$), save one factor of sine and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining power of sine as a function of cosine; then substitute $u = \cos x$:

$$\begin{aligned}\int \sin^{2k+1} x \cos^n x\,dx &= \int (\sin^2 x)^k (\cos x)^n \sin x\,dx \\ &= \int (1 - \underbrace{\cos^2 x}_{u^2})^k (\underbrace{\cos x}_{u})^n \underbrace{\sin x\,dx}_{-u'\,dx} = -\int (1-u^2)^k u^n\,du.\end{aligned}$$

(c) If the exponents of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x),$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Examples

[Hw 2.1(q)(r)] $\int \sin^7 x\,dx$, $\int \sin^3 x \cos^5 x\,dx$.

Type 2 ($\int \tan^m x \sec^n x\,dx$)

(a) If the exponent of secant is even ($n = 2k,\ k \geq 1$), save one factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining power of secant as a function of tangent; then substitute $u = \tan x$:

$$\begin{aligned}\int \tan^m x \sec^{2k} x\,dx &= \int (\tan x)^m (\sec^2 x)^{k-1} \sec^2 x\,dx \\ &= \int (\underbrace{\tan x}_{u})^m (1 + \underbrace{\tan^2 x}_{u^2})^{k-1} \underbrace{\sec^2 x\,dx}_{u'\,dx} = \int u^m (1+u^2)^{k-1}\,du.\end{aligned}$$

(b) If the exponent of tangent is odd ($m = 2k+1$), save one factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining power of tangent as a function of secant; then substitute $u = \sec x$:

$$\begin{aligned}\int \tan^{2k+1} x \sec^n x\,dx &= \int (\tan^2 x)^k (\sec x)^{n-1} \sec x \tan x\,dx \\ &= \int (\underbrace{\sec^2 x}_{u^2} - 1)^k (\underbrace{\sec x}_{u})^{n-1} \underbrace{\sec x \tan x\,dx}_{u'\,dx} = \int (u^2-1)^k u^{n-1}\,du.\end{aligned}$$

(c) If the exponent of secant is zero or odd and the exponent of tangent is even, use the identities

$$\tan^2 x = \sec^2 - 1, \qquad \sec^2 x = \tan^2 x + 1,$$

and integrate by parts when necessary to reduce the higher powers to lower powers.

Examples

[Hw 2.2(g)] $\displaystyle\int \csc^3 x\,dx.$

Type 3 ($\int \sin mx \cos nx\,dx,\ \int \sin mx \sin nx\,dx,\ \int \cos mx \cos nx\,dx$)

Use product-to-sum identities.

Examples

[Hw 1.2(d)] $\displaystyle\int \sin 3x \sin 2x\,dx.$

2.4 Trigonometric Substitution

The Rules

Function	Substitution	Restriction on θ	Result
$\sqrt{a^2-x^2}$	$x=a\sin\theta$	$-\pi/2\le\theta\le\pi/2$	$a\cos\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$	$-\pi/2<\theta<\pi/2$	$a\sec\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$	$0\le\theta<\pi/2$	$a\tan\theta$
$\dfrac{1}{(a^2-x^2)^n}$	$x=a\sin\theta$	$-\pi/2\le\theta\le\pi/2$	$\dfrac{1}{a^{2n}\cos^{2n}\theta}$
$\dfrac{1}{(a^2+x^2)^n}$	$x=a\tan\theta$	$-\pi/2<\theta<\pi/2$	$\dfrac{1}{a^{2n}\sec^{2n}\theta}$
$\dfrac{1}{(x^2-a^2)^n}$	$x=a\sec\theta$	$0\le\theta<\pi/2$	$\dfrac{1}{a^{2n}\tan^{2n}\theta}$

Examples

[Hw 2.1(l)(m)(o)] $\displaystyle\int\frac{1}{x^2\sqrt{1-x^2}}\,dx,\ \int\frac{1}{(1+3x^2)^{3/2}}\,dx,\ \int\sqrt{9-16x^2}\,dx.$

2.5 Matching Integrals to Basic Formulas

The Techniques

1. Completing the square.

   [Hw 2.1(j)(k)(p)] $\displaystyle\int \frac{2x+1}{x^2-2x+5}\,dx,\ \int \frac{4x}{3x^2+6x+19}\,dx,\ \int \frac{1}{(x^2+6x+10)^{3/2}}\,dx.$

2. Eliminating a square root.

   [Hw 3.5(f)] $\displaystyle\int_0^{2\pi} \sqrt{2-2\cos t}\,dt.$

3. Reducing an "improper" fraction.

   [Hw 1.2(b)(i)] $\displaystyle\int \frac{2x^2}{x^2+1}\,dx,\ \int \frac{x+6}{(2x-1)^3}\,dx.$

4. Separating a fraction.

   [Hw 2.1(i)(j)(k)] $\displaystyle\int \frac{3x+2}{x^2+4}\,dx,\ \int \frac{2x+1}{x^2-2x+5}\,dx,\ \int \frac{4x}{3x^2+6x+19}\,dx.$

5. Multiplying by a form of 1.

2.6 Integration of Rational Functions by Partial Fractions

Finding the Partial Fractions of a Rational Function $f(x) = p(x)/q(x)$

Step 1. If $f(x)$ is improper, that is, if $p(x)$ is of degree at least that of $q(x)$, divide $p(x)$ by $q(x)$ to obtain

$$f(x) = \text{quotient (a polynomial)} + \frac{N(x)}{D(x)}.$$

Step 2. Factor $D(x)$ into a product of linear and irreducible quadratic factors with *real* coefficients.

Step 3. For each factor of the form $(ax+b)^k$, expect the decomposition to have the terms

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}.$$

Step 4. For each factor of the form $(ax^2+bx+c)^m$, expect the decomposition to have the terms

$$\frac{B_1x+C_1}{(ax^2+bx+c)} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_mx+C_m}{(ax^2+bx+c)^m}.$$

Step 5. Set $N(x)/D(x)$ equal to the sum of all the terms found in Step 3 and 4.

Step 6. Multiply both sides of the equation found in Step 5 by $D(x)$ and solve for the unknown constants; this can be done by either:

(a) equating coefficients of like powers (most general);

(b) assigning convenient values to the variables x (most effective for simple linear factors $(ax+b)$); or

(c) using differentiation (most effective for repeated linear factors).

Examples

[Hw 2.16(e)(h)(k)] $\displaystyle\int \frac{-7x+19}{(x^2-4x+9)(2x+1)}\,dx$, $\displaystyle\int \frac{x^5+2x^4-x+2}{x^3+2x^2-x-2}\,dx$, $\displaystyle\int \frac{6x^3-27x^2+5x-1}{(x-2)^2(4x^2+1)}\,dx$.

3. Applications of Integration

3.1 Areas between Curves

If $f(x)$ and $g(x)$ are continuous with $f(x) \geq g(x)$ throughout $[a,b]$, then the *area* of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b is the integral of $\left[f(x) - g(x)\right]$ from a to b:

$$A = \int_a^b \left[f(x) - g(x)\right] dx.$$

If the region's bounding curves are described by functions of y, say, $f(y)$ and $g(y)$ with $f(y) \geq g(y)$, then its area is given by the same formula with y replaced by x:

$$A = \int_c^d \left[f(y) - g(y)\right] dy.$$

Rules of Thumb

1. To apply the above formulas to calculate the area of a given region, take the following steps:

   (a) sketch the region and a typical rectangle;

   (b) find a formula for h, the height of a typical rectangle;

   (c) find the limits of integration;

   (d) integrate h using the Fundamental Theorem.

Note that h is

   (a) a function of x if the rectangle is vertical;

   (b) a function of y if the rectangle is horizontal.

2. If the upper or lower boundary of the region consists of two different curves, it might be a good idea to slice the region *horizontally*.

3.2 Volumes by Slicing: the Method of Slab

The *volume* of a solid of known integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of $A(x)$ from a to b:

$$V = \int_a^b A(x)\,dx.$$

If the solid has known integrable cross-sectional area $A(y)$ (or $A(z)$), then its volume is given by the same formula with x replaced by y (or z).

Rules of Thumb

To apply the above formulas to calculate the volume of a given solid, take the following steps:

(a) sketch the solid and a typical cross-section;

(b) find a formula for A, the area of a typical cross-section;

(c) find the limits of integration;

(d) integrate A using the Fundamental Theorem.

Note that A is

(a) a function of x if the cross-section is perpendicular to the x-axis;

(b) a function of y if the cross-section is perpendicular to the y-axis;

(c) a function of z if the cross-section is perpendicular to the z-axis.

3.3 Solids of Revolution: the Method of Disk and Washer

The volume of the solid generated by rotating a planar region about a horizontal axis is

$$V = \int_a^b \pi\left[R^2(x) - r^2(x)\right] dx,$$

where $r(x)$ and $R(x)$ are distances from the region's boundaries to the axis of revolution. If the axis of revolution is vertical, the volume of the solid is given by the same formula with x replaced by y:

$$V = \int_c^d \pi\left[R^2(y) - r^2(y)\right] dy,$$

where again $r(y)$ and $R(y)$ are distances from the region's boundaries to the axis of revolution.

Rules of Thumb

To apply the disk and washer method to calculate the volume of a solid of revolution, take the following steps:

(a) draw the planar region and sketch a typical line segment across it *perpendicular* to the axis of revolution; if the line segment touches the axis of revolution, then a disk results after revolution; otherwise a washer results after revolution;

(b) find the radius R of the disk or the inner and outer radii r and R of the washer that would be swept out by the line segment if it were revolved about the axis along with the region;

(c) find the limits of integration;

(d) integrate $\pi(R^2 - r^2)$, with $r = 0$ for disks, using the Fundamental Theorem.

Note that r and R are

(a) functions of x if the disk or washer is perpendicular to the x-axis;

(b) functions of y if the disk or washer is perpendicular to the y-axis.

3.4 Solids of Revolution: the Method of Shell

The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \geq 0,\ a \leq x \leq b$, about a vertical line $x = L \leq a$ is

$$V = \int_a^b 2\pi(\text{shell radius})(\text{shell height})\,dx = \int_a^b 2\pi(x-L)f(x)\,dx.$$

If the axis of revolution is a horizontal line $y = L$, and $L \leq c$ where $[c,d]$ is the interval over which the function $x = f(y)$ is defined, then the volume of the solid is given by the same formula with x replaced by y:

$$V = \int_c^d 2\pi(\text{shell radius})(\text{shell height})\,dy = \int_c^d 2\pi(y-L)f(y)\,dy.$$

Rules of Thumb

To apply the shell method to calculate the volume of a solid of revolution, take the following steps:

(a) draw the planar region and sketch a typical line segment across it *parallel* to the axis of revolution;

(b) find the radius r and height h of the shell that would be swept out by the line segment if it were revolved about the axis along with the region;

(c) find the limits of integration;

(d) integrate $2\pi rh$ using the Fundamental Theorem.

Note that r and h are

(a) functions of x if the axis of the shell is parallel to the y-axis;

(b) functions of y if the axis of the shell is parallel to the x-axis.

3.5 Lengths of Plane Curves

(a) If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where $f'(t)$ and $g'(t)$ are continuous and not simultaneously zero on $[a,b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then the *length* of C is the definite integral

$$L = \int_a^b \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

(b) If the curve is described by the graph of a continuously differentiable function $y = f(x)$, $a \le x \le b$, then its length is given by

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

(c) If the curve is described by the graph of a continuously differentiable function $x = g(y)$, $c \le y \le d$, then its length is given by

$$L = \int_c^d \sqrt{1 + \left[g'(y)\right]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$$

The Short Differential Formula

The curve length formulas can be conveniently remembered by

(a) introducing the *arc length differential*

$$ds = \sqrt{dx^2 + dy^2};$$

(b) recognizing from the law of differential that

$$ds = \sqrt{\left(\frac{dx}{dt}dt\right)^2 + \left(\frac{dy}{dt}dt\right)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt,$$

if $x = x(t)$ and $y = y(t)$ are functions of t, that

$$ds = \sqrt{dx^2 + \left(\frac{dy}{dx}dx\right)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx,$$

if $y = y(x)$ is a function of x, and that

$$ds = \sqrt{\left(\frac{dx}{dy}dy\right)^2 + dy^2} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy,$$

if $x = x(y)$ is a function of y;

(c) writing the curve length formulas collectively as

$$L = \int ds = \int \sqrt{dx^2 + dy^2},$$

and substituting in appropriate representations for ds depending on the specific problem at hand.

3.6 Areas of Surfaces of Revolution

(a) If a curve is described by the graph of a continuously differentiable function $y = f(x) \geq 0,\ a \leq x \leq b$, the *area* of the surface generated by revolving the curve about the x-axis is

$$S = \int_a^b 2\pi \underbrace{f(x)}_{\text{radius } y} \underbrace{\sqrt{1 + [f'(x)]^2}\,dx}_{\text{band width } ds}.$$

(b) If the curve is described by the graph of a continuously differentiable function $x = g(y) \geq 0,\ c \leq y \leq d$, the area of the surface generated by revolving the curve about the y-axis is

$$S = \int_c^d 2\pi \underbrace{g(y)}_{\text{radius } x} \underbrace{\sqrt{1 + [g'(y)]^2}\,dy}_{\text{band width } ds}.$$

(c) If the curve is defined parametrically by $x = f(t)$ and $y = g(t),\ a \leq t \leq b$, where $f'(t)$ and $g'(t)$ are continuous and not simultaneously zero on $[a, b]$, and it is traversed exactly once as t increases from $t = a$ to $t = b$, then the area of the surface generated by revolving the curve is

$$S = \int_a^b 2\pi \underbrace{g(t)}_{\text{radius } y} \underbrace{\sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt}_{\text{band width } ds},$$

if the revolution is about the x-axis (assuming $y = g(t) \geq 0$), and

$$S = \int_a^b 2\pi \underbrace{f(t)}_{\text{radius } x} \underbrace{\sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt}_{\text{band width } ds},$$

if the revolution is about the y-axis (assuming $x = f(t) \geq 0$).

The Short Differential Formula

All these formulas for calculating areas of surfaces of revolution can be written in a unified form as

$$S = \int 2\pi(\text{radius})(\text{band width}) = \int 2\pi\rho\,ds,$$

where ρ is the radius from the axis of revolution to an element of arc length ds. For any given problem, the task is then to express the radius function ρ and the arc length differential ds in terms of a common variable, and to supply limits of integration for that variable.

Rules of Thumb

To calculate the area of a surface of revolution, take the following steps:

(a) find the radius function ρ and the arc length differential ds; as a general rule, $\rho = x$ if the axis of revolution is the y-axis, and $\rho = y$ if the axis of revolution is the x-axis;

(b) choose the variable of integration; this can be $x,\ y$, or a parameter t, if the curve that is used to generate the surface is described by a function of $x,\ y$, or a pair of parametric equations in t;

(c) express the radius function ρ and the arc length differential ds as functions of the variable of integration;

(d) find the limits of integration;

(e) integrate $2\pi\rho\, ds$ using the Fundamental Theorem.

5. Complex Numbers

5.1 Complex Numbers

Definition

A *complex number* is a number of the form

$$z = x + \mathrm{i}y,$$

where $\mathrm{i} = \sqrt{-1}$ is the *imaginary unit* and $x,\ y$ are real numbers. The number x is called the *real part* of z, and the number y is called the *imaginary part* of z.

Geometric Representation, Polar Form and Euler's Form

- *Argand diagram.* A complex number $z = x + \mathrm{i}y$ has two geometric representations:

  (a) as a point $P(x, y)$ in the xy-plane;

  (b) as a vector $\overrightarrow{OP}$ from the origin to P.

  Both representations are *Argand diagrams* for $x + \mathrm{i}y$.

- *Polar form.* In terms of the polar coordinates of $P(x, y)$, the complex number $z = x + \mathrm{i}y$ can be written in *polar form* as

  $$z = x + \mathrm{i}y = r\cos\theta + \mathrm{i}r\sin\theta = r(\cos\theta + \mathrm{i}\sin\theta).$$

  The length $r = \sqrt{x^2 + y^2}$ is called the *modulus* of z, and the angle θ is called the *argument* of z.

- *Euler's form.* Using Euler's formula

  $$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta,$$

  the complex number $z = x + \mathrm{i}y$ can be written in *Euler's form* as

  $$z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{e}^{\mathrm{i}\theta}.$$

Algebraic Operations in Cartesian Form

- *Equality.* $a + \mathrm{i}b = c + \mathrm{i}d$ if and only if $a = c$ and $b = d$.
- *Addition.* $(a + \mathrm{i}b) + (c + \mathrm{i}d) = (a + c) + \mathrm{i}(b + d)$.
- *Subtraction.* $(a + \mathrm{i}b) - (c + \mathrm{i}d) = (a - c) + \mathrm{i}(b - d)$.
- *Multiplication.* $(a + \mathrm{i}b)(c + \mathrm{i}d) = (ac - bd) + \mathrm{i}(ad + bc)$.
- *Division.* $\dfrac{a + \mathrm{i}b}{c + \mathrm{i}d} = \dfrac{ac + bd}{c^2 + d^2} + \mathrm{i}\dfrac{bc - ad}{c^2 + d^2}$.

Algebraic Operations in Polar and Euler's Form

- *Multiplication*. If $z_1 = r_1 e^{i\theta_1} = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2 e^{i\theta_2} = r_2(\cos\theta_2 + i\sin\theta_2)$, then
$$z_1 z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)} = r_1 r_2 \big[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big].$$

- *Division*. If $z_1 = r_1 e^{i\theta_1} = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2 e^{i\theta_2} = r_2(\cos\theta_2 + i\sin\theta_2)$, then
$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} = \frac{r_1}{r_2}\big[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\big].$$

- *Powers*. If $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then
$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta).$$
  This formula is also known as *De Moivre's Theorem*.

- *Roots*. If $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then
$$z^{1/n} = \big[re^{i(\theta+2k\pi)}\big]^{1/n} = r^{1/n} e^{i(\theta+2k\pi)/n} = r^{1/n}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right),$$
  where $k = 0, 1, 2, \dots, n-1$.

Complex Conjugate

- *Definition*. The *complex conjugate* of a complex number $z = x + iy$, denoted by $\bar{z}$, is given by
$$\bar{z} = x - iy.$$

- *Euler's form*. In Euler's form, the complex conjugate of a complex number $z = re^{i\theta}$ is given by
$$\bar{z} = \overline{re^{i\theta}} = \overline{r\cos\theta + ir\sin\theta} = r\big[\cos(-\theta) + i\sin(-\theta)\big] = re^{-i\theta}.$$

- *Properties of complex conjugate*. If $z = x + iy$ is a complex number, then
  1. if z is real, then $\bar{z} = z$.
  2. $\bar{\bar{z}} = z$.
  3. $z\bar{z} = x^2 + y^2 = |z|^2$.
  4. $|\bar{z}| = |z|$.
  5. $\arg\bar{z} = -\theta = -\arg z$.
  6. $z + \bar{z} = 2x = 2\operatorname{Re}(z)$.
  7. $z - \bar{z} = 2iy = 2i\operatorname{Im}(z)$.
  8. $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$.
  9. $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$.
  10. $\overline{z_1/z_2} = \bar{z}_1/\bar{z}_2$.

Roots of Polynomials with Real Coefficients

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with real coefficients and degree $n \geq 2$. If $z = a + ib$ is a root of $p(x) = 0$, then $\bar{z} = a - ib$ is also a root of $p(x) = 0$.

6. Matrices, Determinants, and Systems of Linear Equations

6.1 Linear Independence

Definition

A set of vectors $\mathscr{S} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is said to be a *linearly independent set* if the only solution for the scalars α_i in the homogeneous equation

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n = 0$$

is the trivial solution $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. If a nontrivial solution exists with at least one $\alpha_i \neq 0$, then $\mathscr{S}$ is said to be a *linearly dependent set*. By convention, the empty set is always linearly independent.

Alternative Characterization of Linear Independence

Suppose $\mathscr{S} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a finite set of vectors having at least two elements. Then $\mathscr{S}$ is linearly dependent if and only if some vector $\vec{v}_k$ in $\mathscr{S}$ can be expressed as a linear combination of the other vectors in $\mathscr{S}$:

$$\vec{v}_k = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_{k-1}\vec{v}_{k-1} + c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n.$$

Testing for Linear Independence

To determine whether a finite nonempty set of vectors $\mathscr{S} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ in $\mathbb{R}^n$ is linearly independent:

(a) create the $n \times k$ matrix $A = [\vec{v}_1|\vec{v}_2|\cdots|\vec{v}_k]$ whose columns are the vectors in $\mathscr{S}$;

(b) use Gaussian elimination to find a row echelon form U of A;

(c) if there is a pivot in every column of U, then the homogeneous system $Ax = 0$ has only the trivial solution (see Section 6.4), and $\mathscr{S}$ is linearly independent; otherwise $\mathscr{S}$ is linearly dependent.

6.2 Matrices

Definition

An $m \times n$ *matrix* A is a rectangular array of real or complex numbers (called scalars) with m rows and n columns:

$$A = A_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Whenever $m = n$, A is called a *square matrix*, otherwise A is said to be *rectangular*. Matrices consisting of a single row or a single column are called *row* and *column vectors*, respectively.

Matrix Addition and Subtraction

- *Definition of matrix addition*. If A and B are $m \times n$ matrices, the *sum* of A and B is defined to be the $m \times n$ matrix $A + B$ obtained by adding the corresponding entries of A and B, that is,

$$[A+B]_{ij} = [A]_{ij} + [B]_{ij}, \qquad i = 1,2,\ldots,m, \; j = 1,2,\ldots,n.$$

- *Definition of matrix subtraction*. The matrix $(-A)$, called the *additive inverse* of A, is defined to be the matrix obtained by negating each entry of A, that is,

$$[-A]_{ij} = -[A]_{ij}, \qquad i = 1,2,\ldots,m, \; j = 1,2,\ldots,n.$$

  If A and B are $m \times n$ matrices, the *difference* of A and B is defined to be the $m \times n$ matrix $A - B = A + (-B)$ so that

$$[A-B]_{ij} = [A]_{ij} - [B]_{ij}, \qquad i = 1,2,\ldots,m, \; j = 1,2,\ldots,n.$$

- *Properties of matrix addition*. If $A,\ B$, and C are $m \times n$ matrices and $0_{m\times n}$ is the $m \times n$ matrix consisting of all zeros, then
  1. $A + B$ is again an $m \times n$ matrix.
  2. $(A+B)+C = A+(B+C)$.
  3. $A+B = B+A$.
  4. $A + 0_{m\times n} = 0_{m\times n} + A = A$.
  5. $A + (-A) = (-A) + A = 0_{m\times n}$.

Scalar Multiplication

- *Definition*. The *scalar multiplication* of a scalar α and an $m \times n$ matrix A, denoted by αA, is defined to be the matrix obtained by multiplying each entry of A by α, that is,

$$[\alpha A]_{ij} = \alpha [A]_{ij}, \qquad i = 1,2,\ldots,m, \; j = 1,2,\ldots,n.$$

- *Properties of scalar multiplication*. If $A,\ B$ are $m \times n$ matrices and $\alpha,\ \beta$ are scalars, then
  1. αA is again an $m \times n$ matrix.
  2. $(\alpha\beta)A = \alpha(\beta A)$.
  3. $\alpha(A+B) = \alpha A + \alpha B$.
  4. $(\alpha+\beta)A = \alpha A + \beta A$.
  5. $1A = A$.

Matrix Transposition

- *Definition*. If A is an $m \times n$ matrix, the *transpose* of A is defined to be the $n \times m$ matrix A^T obtained by interchanging rows and columns in A, that is,

$$[A^T]_{ij} = [A]_{ji}, \qquad i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m.$$

- *Properties of matrix transposition*. If A, B are $m \times n$ matrices and α is a scalar, then
  1. $(A^T)^T = A$.
  2. $(A+B)^T = A^T + B^T$.
  3. $(\alpha A)^T = \alpha A^T$.

Symmetries

Let A be an $n \times n$ square matrix.

(a) A is said to be *symmetric* if $A^T = A$, i.e. if $[A]_{ij} = [A]_{ji}$.

(b) A is said to be *skew-symmetric* if $A^T = -A$, i.e. if $[A]_{ij} = -[A]_{ji}$.

Matrix Multiplication

- *Definition*. Matrices A and B are said to be *conformable* for multiplication in the order AB if A has exactly as many columns as B has rows, i.e. A is $m \times p$ and B is $p \times n$. For conformable matrices $A_{m \times p} = [a_{ij}]$ and $B_{p \times n} = [b_{ij}]$, the *matrix product* AB is defined to be the $m \times n$ matrix whose (i, j)-entry is the inner product of the i-th row of A with the j-th column in B, that is,

$$[AB]_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}.$$

  In case A and B fail to be conformable, no product AB is defined.

- *Properties of matrix multiplication*. If A, B, and C are conformable matrices and α is a scalar, then
  1. $A(B+C) = AB + AC$.
  2. $(A+B)C = AC + BC$.
  3. $A(BC) = (AB)C$.
  4. $\alpha(AB) = (\alpha A)B = A(\alpha B)$.
  5. $(AB)^T = B^T A^T$.

Matrix Inversion

- *Definition*. For a given square matrix $A_{n \times n}$, the matrix $B_{n \times n}$ that satisfies the conditions

$$AB = I_n, \qquad BA = I_n,$$

  is called the *(multiplicative) inverse* of A and is denoted by $B = A^{-1}$. An invertible matrix is said to be *nonsingular*, and a square matrix with no inverse (such as the zero matrix) is called a *singular matrix*.

- *Properties of matrix inversion*. If A and B are nonsingular $n \times n$ matrices, then
  1. $(A^{-1})^{-1} = A$.
  2. the product AB is also nonsingular.
  3. $(AB)^{-1} = B^{-1}A^{-1}$.
  4. $(A^{-1})^T = (A^T)^{-1}$.

- *Finding inverse using adjoint*. If A is a *nonsingular* $n \times n$ matrix, then $A^{-1} = \mathscr{A} / \det(A)$, where

$$\mathscr{A} = \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}$$

  is the *classical adjoint* of A. The (i,j)-entry A_{ji} in $\mathscr{A}$ is the (j,i)-*cofactor* of A (see Section 6.3).

- *Finding inverse using Gaussian elimination*. To find the inverse of an *arbitrary* $n \times n$ matrix A:

  (a) augment A to an $n \times 2n$ matrix $[A|I_n]$;

  (b) use Gaussian elimination to transform $[A|I_n]$ into reduced row echelon form;

  (c) if the first n columns of $[A|I_n]$ cannot be transformed into I_n (a row of zeros emerges), then A is singular; stop;

  (d) otherwise, A is nonsingular, and the last n columns of the augmented matrix in reduced row echelon form constitute A^{-1}; that is, $[A|I_n]$ row reduces to $[I_n|A^{-1}]$.

6.3 Determinants

Definition

- *Minor*. The (i, j)-*minor* M_{ij} of an $n \times n$ matrix A $(n \geq 2)$ is the determinant of the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the i-th row and j-th column of A.
- *Cofactor*. The (i, j)-*cofactor* A_{ij} of an $n \times n$ matrix A $(n \geq 2)$ is defined to be $A_{ij} = (-1)^{i+j} M_{ij}$.
- *Definition of determinant*. The *determinant* of an $n \times n$ matrix A, denoted by $\det(A)$ or $|A|$, is defined recursively as:
  (a) $\det(A) = a_{11}$ if $n = 1$;
  (b) $\det(A) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$ if $n > 1$.

Effect of Row Operations on Determinant

Let B be the matrix obtained from $A_{n\times n}$ by one of the three elementary row operations:

(a) type I: interchange rows i and j;

(b) type II: multiply row i by $\alpha \neq 0$;

(c) type III: add α times row i to row j.

Then $\det(B)$ is given by:

(a) $\det(B) = -\det(A)$ for type I operations;

(b) $\det(B) = \alpha \det(A)$ for type II operations;

(c) $\det(B) = \det(A)$ for type III operations.

Similar conclusions hold for column operations.

Further Properties of Determinant

If A and B are $n \times n$ matrices, then

1. $\det(A) = a_{11}a_{22}\cdots a_{nn}$ if A is triangular.
2. $\det(A^T) = \det(A)$.
3. A is nonsingular if and only if $\det(A) \neq 0$.
4. $\det(AB) = \det(A)\det(B)$.
5. $\det(A^{-1}) = 1/\det(A)$ if A is nonsingular.
6. $\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$ for each $i = 1, 2, \ldots, n$.
7. $\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$ for each $j = 1, 2, \ldots, n$.

6.4 Systems of Linear Equations

Definition

A *system of m (simultaneous) linear equations* in n unknowns $x_1, x_2, \ldots, x_n$ is a collection of m equations, each containing a linear combination of the same n unknowns summing to a scalar b_i:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2, \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m. \end{array}$$

In matrix terms, the system can be written as $Ax = b$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Gaussian Elimination

Gaussian elimination systematically transforms a linear system into another simpler, but equivalent, system[1] by successively eliminating unknowns using one of the three elementary operations:

(a) type I: interchanging the i-th and j-th equations;

(b) type II: replacing the i-th equation by a nonzero multiple of itself;

(c) type III: replacing the j-th equation by a combination of itself plus a multiple of the i-th equation.

In matrix terms, Gaussian elimination is executed on the *augmented matrix* $[A|b]$ by performing the three elementary operations to the rows of $[A|b]$. The end product of Gaussian elimination is a matrix $[U|c]$, where U is in *row echelon form*:

$$U = \begin{pmatrix} \boxed{*} & * & * & * & * & * & * & * \\ 0 & 0 & \boxed{*} & * & * & * & * & * \\ 0 & 0 & 0 & \boxed{*} & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{*} & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

[1] Two systems are said to be *equivalent* if they possess the same solution sets.

The Gauss-Jordan Method

The *Gauss-Jordan method* is a variation of Gaussian elimination and is distinct in the following two aspects:

(a) at each step, the pivot element is forced to be 1;

(b) at each step, all entries *above* and below the pivot are eliminated.

The end product of Gauss-Jordan applied to the augmented matrix $[A|b]$ is a matrix $[E_A|c]$, where E_A is in *reduced row echelon form*:

$$E_A = \begin{pmatrix} \boxed{1} & * & 0 & 0 & * & * & 0 & * \\ 0 & 0 & \boxed{1} & 0 & * & * & 0 & * \\ 0 & 0 & 0 & \boxed{1} & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Trichotomy

Let $Ax = b$ be a system of linear equations. Let $[U|c]$ be the row echelon form augmented matrix obtained by row reducing $[A|b]$.

(a) If there is a row of $[U|c]$ having the form $(0\,0\,\cdots\,0|\alpha)$, $\alpha \neq 0$, then $Ax = b$ has no solution.

(b) If not, but if one of the columns of U has no nonzero pivot entry, then $Ax = b$ has infinite many solutions. The non-pivot columns correspond to *free variables* that can take on any value, and the values of the remaining variables are determined from those.

(c) Otherwise, $Ax = b$ has a unique solution.

Cramer's Rule

Let $Ax = b$ be a system of n equations in n unknowns with $\det(A) \neq 0$. For $1 \leq i \leq n$, let A_i be the $n \times n$ matrix obtained by replacing the i-th column of A with b. Then the entries of the unique solution x are

$$x_1 = \frac{\det(A_1)}{\det(A)}, \qquad x_2 = \frac{\det(A_2)}{\det(A)}, \qquad \ldots, \qquad x_n = \frac{\det(A_n)}{\det(A)}.$$