MA1201 (A/B/C/D) Test B 17/18

1(a) [7] $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 3x \cos 3x\,dx = \frac{1}{3}\int_0^{\frac{\pi}{2}} \sin^2 3x \, d(\sin 3x)$ (2) $= \frac{1}{9}\sin^3 3x \Big|_0^{\frac{\pi}{2}}$ (2)

$\displaystyle = \frac{1}{9}\Big[\underbrace{\sin^3\frac{3\pi}{2}}_{(-1)^3=-1} - \underbrace{\sin^3 0}_{0}\Big] = \frac{-1}{9}$ (3)

(b) [7] $\displaystyle\int \frac{4x+5}{\sqrt{2x+1}}\,dx = \int \frac{2(2x+1)+3}{\sqrt{2x+1}}\,dx$ (2) $= \int\Big[2(2x+1)^{\frac{1}{2}} + 3(2x+1)^{-\frac{1}{2}}\Big]dx$ (2)

$\displaystyle = \frac{2}{2}\frac{(2x+1)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{3}{2}\frac{(2x+1)^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{3}(2x+1)^{3/2} + 3(2x+1)^{1/2} + C$ (3)

(c) [6] $\displaystyle\int_0^2 e^{|-x+1|}\,dx = \int_0^1 e^{-x+1}\,dx + \int_1^2 e^{-(-x+1)}\,dx$ (2)

$\displaystyle = -e^{-x+1}\Big|_0^1 + e^{x-1}\Big|_1^2$ (2) $= -[e^0 - e^1] + [e^1 - e^0] = 2(e-1)$ (2)

2.(a) [12] Let $x = 2\sec\theta$ (2), $dx = 2\sec\theta\tan\theta\,d\theta$ (1)

$\displaystyle\int\sqrt{x^2-4}\,dx = \int\underbrace{\sqrt{4\sec^2\theta-4}}_{2\tan\theta\ (1)}\,2\sec\theta\tan\theta\,d\theta = 4\int\sec\theta\tan^2\theta\,d\theta = 4\int\sec\theta(\sec^2\theta-1)\,d\theta$ (1)

$\displaystyle = 4\int(\sec^3\theta - \sec\theta)\,d\theta$ (1) $= 4\int\sec^3\theta\,d\theta - 4\int\sec\theta\,d\theta$

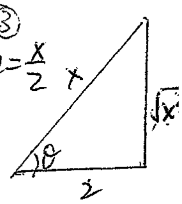
$= 2\sec\theta\tan\theta + 2\ln|\sec\theta+\tan\theta| - 4\ln|\sec\theta+\tan\theta| + C$ (3)

$= 2\sec\theta\tan\theta - 2\ln|\sec\theta+\tan\theta| + C$

$\displaystyle = 2\,\frac{x}{2}\,\frac{\sqrt{x^2-4}}{2} - 2\ln\Big|\frac{x}{2}+\frac{\sqrt{x^2-4}}{2}\Big| + C$

$\displaystyle = \frac{x}{2}\sqrt{x^2-4} - 2\ln|x+\sqrt{x^2-4}| + \underbrace{2\ln 2 + C}_{C'}$ (3)

$\sec\theta = \frac{x}{2}$



(b) [8] $\displaystyle\int\sqrt{x}\ln x\,dx = \int\ln x\, d\Big(\frac{x^{3/2}}{3/2}\Big)$ (2) $\overset{IP}{=} \frac{2}{3}x^{3/2}\ln x - \int\frac{2}{3}x^{3/2}\underbrace{d\ln x}_{\frac{1}{x}dx}$ (2)

$\displaystyle = \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{\frac{1}{2}}\,dx$ (1) $= \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C$ (3)

(c) [18] $\displaystyle\frac{5x^2}{(x-2)(x^2-6x+13)} = \frac{A}{x-2} + \frac{Bx+C}{x^2-6x+13}$ (2)

$\Rightarrow\quad 5x^2 = A(x^2-6x+13) + (Bx+C)(x-2)$

$x=2:\ 20 = A(4-12+13) = 5A \Rightarrow A = 4$ (2)

Compare the coefficient of x^2: $5 = A+B \Rightarrow B = 5-A = 5-4 = 1$ (2)

Compare the constant term: $0 = 13A - 2C \Rightarrow 2C = 13A = 52 \therefore C = 26$ (2)

$\displaystyle\int\frac{5x^2}{(x-2)(x^2-6x+13)}\,dx = \int\frac{4}{x-2}\,dx + \int\frac{x+26}{x^2-6x+13}\,dx$

$\displaystyle = 4\ln|x-2|$ (2) $\displaystyle + \frac{1}{2}\int\frac{2x-6}{x^2-6x+13}\,dx + \int\frac{29}{x^2-6x+13}\,dx$ (2)

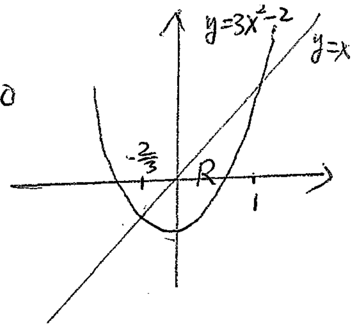
$\displaystyle = 4\ln|x-2| + \frac{1}{2}\int\frac{d(x^2-6x+13)}{x^2-6x+13} + 29\int\frac{1}{(x-3)^2+4}\,dx$

$\displaystyle = 4\ln|x-2| + \frac{1}{2}\ln|x^2-6x+13|$ (1) $\displaystyle + \frac{29}{4}\int\frac{1}{(\frac{x-3}{2})^2+1}\,dx$ (1)

$= 4\ln|x-2| + \frac{1}{2}\ln|x^2-6x+13| + 29 \ldots$

3(a) [10] $R\begin{cases} y = 3x^2 - 2 \\ y = x \end{cases} \Rightarrow x = 3x^2 - 2$ or $3x^2 - x - 2 = 0 \Rightarrow (3x+2)(x-1) = 0$

$\Rightarrow x = -\frac{2}{3}$ or 1 (2)



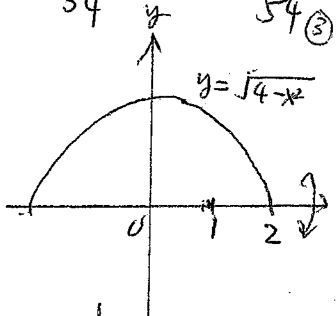
$$A = \int_{-\frac{2}{3}}^{1} [y_{upper} - y_{lower}]\, dx$$

$$= \int_{-\frac{2}{3}}^{1} [x - (3x^2 - 2)]\, dx \ (3) = \left[\frac{x^2}{2} - x^3 + 2x\right]_{-\frac{2}{3}}^{1} \ (2)$$

$$= \frac{1}{2}\left[1^2 - \left(-\frac{2}{3}\right)^2\right] - \left[1^3 - \left(-\frac{2}{3}\right)^3\right] + 2\left[1 - \left(-\frac{2}{3}\right)\right]$$

$$= \frac{1}{2}\left(1 - \frac{4}{9}\right) - \left(1 + \frac{8}{27}\right) + 2\left(1 + \frac{2}{3}\right) = \frac{5}{18} - \frac{35}{27} + \frac{10}{3} = \frac{15 - 70 + 180}{54} = \frac{125}{54} \ (3)$$

b) [10] $y = \sqrt{4 - x^2}, \quad 0 \le x \le 1$



$$\frac{dy}{dx} = \frac{1}{2}\frac{-2x}{\sqrt{4-x^2}} = \frac{-x}{\sqrt{4-x^2}} \ (2)$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{-x}{\sqrt{4-x^2}}\right)^2} = \sqrt{\frac{(4-x^2) + x^2}{4 - x^2}} = \frac{2}{\sqrt{4-x^2}} \ (2)$$

$$S_x = \int_0^1 2\pi y\, ds \ (2) = 2\pi \int_0^1 \sqrt{4-x^2}\, \frac{2}{\sqrt{4-x^2}}\, dx = 4\pi \int_0^1 dx = 4\pi\, [x]_0^1 = 4\pi \ (4)$$

4 a) [8] $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}, \quad b = -3\vec{i} + \vec{j} - 2\vec{k}$

$$\vec{a} \cdot \vec{b} = 1(-3) + 2(1) + 3(-2) = -3 + 2 - 6 = -7 \ (2)$$

$$|\vec{b}| = \sqrt{(-3)^2 + 1^2 + (-2)^2} = \sqrt{9 + 1 + 4} = \sqrt{14} \ (2)$$

$$\mathrm{proj}_{\vec{b}}\vec{a} = \left(\vec{a} \cdot \frac{\vec{b}}{|\vec{b}|}\right)\frac{\vec{b}}{|\vec{b}|} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b} \ (2) = \frac{-7}{(\sqrt{14})^2}\vec{b} = \frac{-7}{14}\vec{b} = -\frac{1}{2}(-3\vec{i} + \vec{j} - 2\vec{k})$$

$$= \frac{3}{2}\vec{i} - \frac{1}{2}\vec{j} + \vec{k} \ (2)$$

b) [14] $A = (-1, 0, 2),\ B = (2, 1, -3),\ C = (0, 1, 3)$

$\vec{AB} = \vec{OB} - \vec{OA} = 3\vec{i} + \vec{j} - 5\vec{k}$ (1) $\quad \vec{AC} = \vec{OC} - \vec{OA} = \vec{i} + \vec{j} + \vec{k}$ (1)

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -5 \\ 1 & 1 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & -5 \\ 1 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & -5 \\ 1 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= \vec{i}(1 + 5) - \vec{j}(3 + 5) + \vec{k}(3 - 1)$$

$$= 6\vec{i} - 8\vec{j} + 2\vec{k} \ (4)$$

Let $P = (x, y, z)$ (1) be on the plane containing A, B, C.

$$\vec{AP} = \vec{OP} - \vec{OA} = (x+1)\vec{i} + y\vec{j} + (z-2)\vec{k} \ (2)$$

$$0 = \vec{AP} \cdot \vec{n} \ (1) = 6(x+1) - 8(y) + 2(z-2) \ (2)$$

$$\Rightarrow 0 = 3(x+1) - 4y + (z-2) = 3x + 3 - 4y + z - 2 = 3x - 4y + z + 1$$

$\therefore$ plane equation $3x - 4y + z + 1 = 0$ (2)