MA2001 Assignment Chapter 1 Vector Differential calculus

1. If $\vec{r}(t) = r\cos(\omega t)\vec{i} + r\sin(\omega t)\vec{j}$ is the position vector of a point at time *t*, $\vec{v}(t)$ is the velocity vector of $\vec{r}(t)$ and $\vec{a}(t)$ is the acceleration vector of $\vec{r}(t)$, show that
   (a) $\vec{r}\cdot\vec{v} = 0$, (b) $\vec{r}\times\vec{v} =$ constant vector,
   (c) $\vec{a} = \omega^2\vec{r}$.

2. (a) Compute the divergence and curl of the vector functions:
   (i) $\vec{v} = e^x \cos y\vec{i} + xy^2\vec{j} + yz^3\vec{k}$
   (ii) $\vec{v} = yz\vec{i} + 3zx\vec{j} + z\vec{k}$
   (b) (i) Find $div(grad\ f)$, for $f(x,y,z) = 1 - x^2 - 4y^2 + 2z^2$
   (ii) Find $\nabla\times\nabla(\nabla\cdot\vec{v})$, for $\vec{v}(x,y,z) = e^x\vec{i} + e^y\vec{j} + e^z\vec{k}$
   (c) Verify the formula $div(f\vec{v}) = f\ div\vec{v} + \vec{v}\cdot grad\ f$ for $f = e^{xyz}$ and $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$.
   (d) Prove that for any vector fields $\vec{v}$ and $\vec{w}$ on $\mathbf{R}^3$,
   (i) $curl(\vec{v}+\vec{w}) = curl\,\vec{v} + curl\,\vec{w}$
   (ii) $div(curl\,\vec{v}) = 0$

3. It is given that $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{p} = a\vec{i} + b\vec{j} + c\vec{k}$ is a constant vector and $\vec{u} = (\vec{p}\cdot\vec{r})\vec{r}$.
   (a) Evaluate $\vec{u} = (\vec{p}\cdot\vec{r})\vec{r}$.
   (b) Show that
   (i) $\nabla\cdot\vec{u} = 4\vec{p}\cdot\vec{r}$, (ii) $\nabla\times\vec{u} = \vec{p}\times\vec{r}$, (iii) $\nabla\times(\vec{p}\times\vec{r}) = 2\vec{p}$.

4. Let $\vec{F}(x,y,z) = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ be a vector field on $\mathbf{R}^3$, where *a*, *b* and *c* are real constants.
   (a) Find the values of *a*, *b* and *c* such that $\vec{F}$ is irrotational.
   (b) With the values of *a*, *b* and *c* obtained in (a), determine a potential function φ on $\mathbf{R}^3$ for which $\nabla\varphi = \vec{F}$.

5. Let $\vec{G}(x,y,z) = 3yz\vec{i} + x^2\vec{j} + x\cos y\vec{k}$ be a vector field on $\mathbf{R}^3$.
   (a) Show that $\vec{G}$ is solenoidal.
   (b) Find a vector field $\vec{F}(x,y,z) = f_1(x,y,z)\vec{i} + f_2(x,y,z)\vec{j}$ on $\mathbf{R}^3$ such that $\nabla\times\vec{F} = \vec{G}$.

6. (a) A vector field $\vec{F}$ is said to be **solenoidal** if $\nabla\cdot\vec{F} = 0$. Let $\vec{F} = (y+z)\vec{i} + (x+z)\vec{j} + (x+y)\vec{k}$.
   Show that $\vec{F}$ is solenoidal.
   (b) As a consequence of $\vec{F}$ being solenoidal, there exists a vector field $\vec{H}$ such that $\vec{F} = \nabla\times\vec{H}$.
   Find a vector field $\vec{H} = h_1(x,y,z)\vec{i} + h_2(x,y,z)\vec{j} + h_3(x,y,z)\vec{k}$ with $h_2(x,y,z) \equiv 0$ such that $\vec{F} = \nabla\times\vec{H}$.
   (c) Observe that if φ is a scalar field and $\vec{H},\vec{F}$ are vector fields such that $\vec{F} = \nabla\times\vec{H}$, then we have
   $\nabla\times(\vec{H}+\nabla\varphi) = \nabla\times\vec{H} + \nabla\times\nabla\varphi = \nabla\times\vec{H} = \vec{F}\cdots\cdots$(I).
   Using (b) and observation (I), find a vector field $\vec{G} = g_1(x,y,z)\vec{i} + g_2(x,y,z)\vec{j} + g_3(x,y,z)\vec{k}$ such that $\vec{F} = \nabla\times\vec{G}$ and $g_2(x,y,z) = 2y$.

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