2. Dynamic Circuits: First-Order Transient

Alexander & Sadiku,
"Fundamentals of Electric Circuits"
5th Edition Chapters 6, 9

I. Components

- Capacitors (C) [Section 6.2, 6.3]
- Inductors (L) [Section 6.4, 6.5]
- C & L vs. R → Storage vs. Dissipation [Section 6.1]

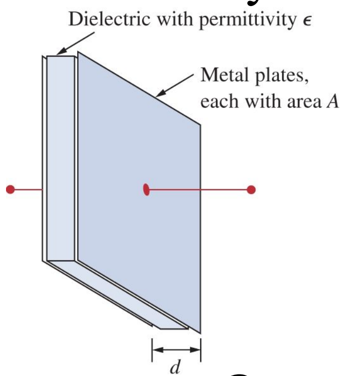
II. Simple RC and RL Circuits

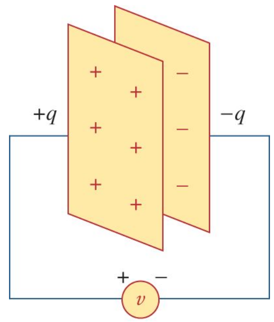
III. Transient Solutions

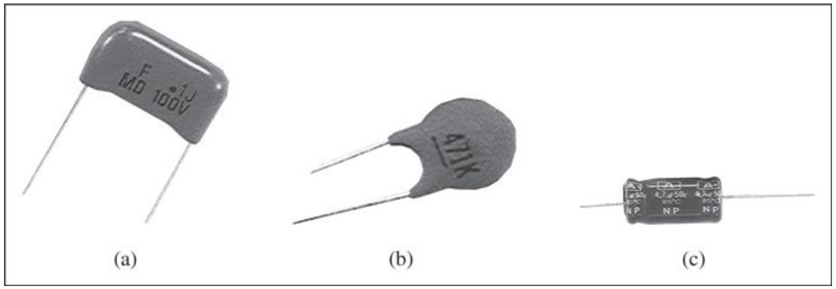
Capacitor and Capacitance

Capacitor

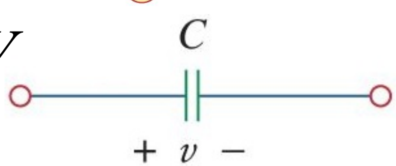
- Simplest way to form a capacitor is to sandwich an insulator (technical term is dielectric) between a pair of parallel conducting plates
- Hence the symbol for a capacitor is two parallel lines separated by a gap







$$Q = CV$$

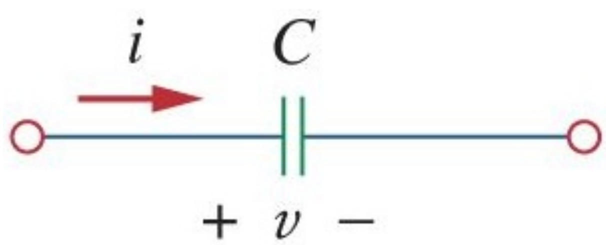


Capacitance

- Commonly symbolized by the letter C with unit of Farads (F)
- Relates the amount of charge stored for a given voltage applied
- No energy dissipated unlike resistors
- Energy is stored (in keeping the plates apart)

I-V relation in a capacitor

If the voltage across the capacitor is time-varying, then the charge stored on the capacitor must also be time-varying:



$$Q(t) = CV(t)$$

Differentiating with respect to time (t):

$$i(t) = C\frac{dV(t)}{dt}$$

Current depends on the **rate of change** of voltage

Case study to consider:

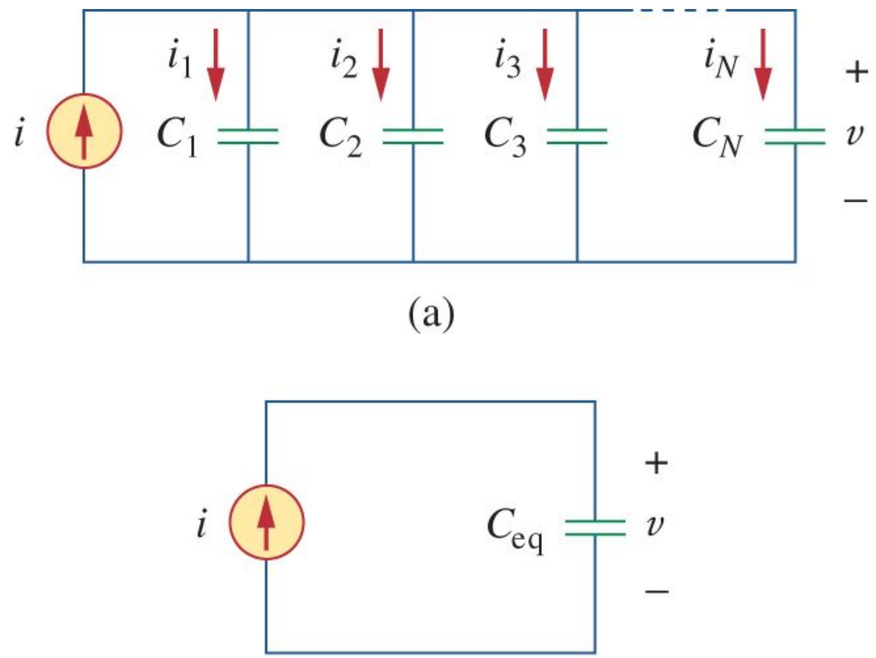
Given that C = 1 nF

If V = 1V → Q = 1 nC

If we reverse V, such that now V = -1V → Q = ________

If the above change was made gradually over 1 ms, what would be the resulting current?

Capacitors in parallel

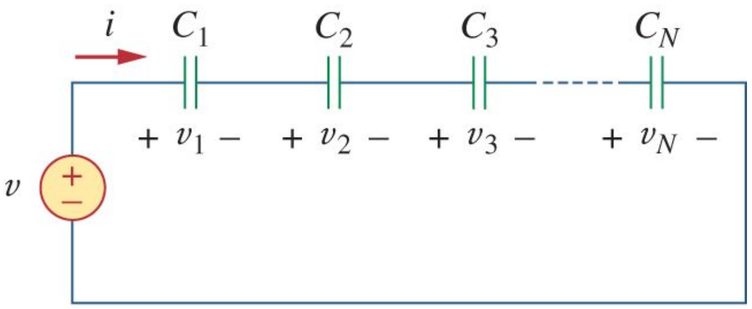


(a)
(b)

$$C_{eq} = C_1 + C_2 + C_3 + \cdots + C_N$$

What is the value of Q_{eq}?

Capacitors in series



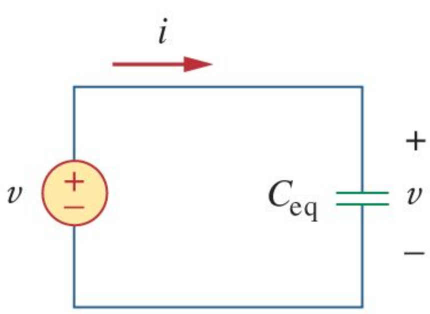
(a)

$$v = v_1 + v_2 + v_3 + v_4$$

$$\rightarrow \frac{Q_{eq}}{C_{eq}} = \frac{Q_1}{C_1} + \frac{Q_2}{C_2} + \frac{Q_3}{C_3} + \frac{Q_4}{C_4}$$

$$\rightarrow \frac{1}{C_{eq}}\frac{dQ_{eq}}{dt} = \frac{1}{C_1}\frac{dQ_1}{dt} + \frac{1}{C_2}\frac{dQ_2}{dt} + \frac{1}{C_3}\frac{dQ_3}{dt} + \frac{1}{C_4}\frac{dQ_4}{dt}$$

$$C_{eq} v = Q_{eq}$$

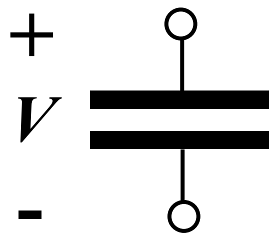


(b)

Since $i = \frac{dQ_{eq}}{dt} = \frac{dQ_1}{dt} = \frac{dQ_2}{dt} = \frac{dQ_3}{dt} = \frac{dQ_4}{dt} \rightarrow$

$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_N}}$$

Response in Capacitor



$$Q(t) = CV(t) \xrightarrow[\text{with respect to time}]{\text{Differentiate}} \boxed{i(t) = C\frac{dV(t)}{dt}}$$

If voltage is constant with time: **dV/dt = 0 → I = 0**

No voltage change → No current

If voltage changes with time: **dV/dt ≠ 0 → I ≠ 0**

Voltage change → Charge changes → Current

Summary: Capacitor response to DC

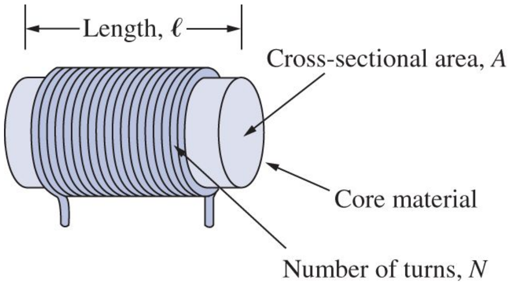
If applied voltage is DC

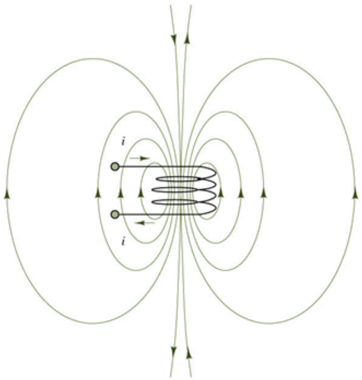
- Insulating dielectric blocks the current from flowing through
- Plates will charge up
- At DC, capacitor blocks current from flowing through

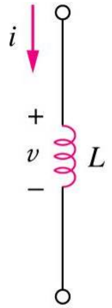
Inductors and Inductance

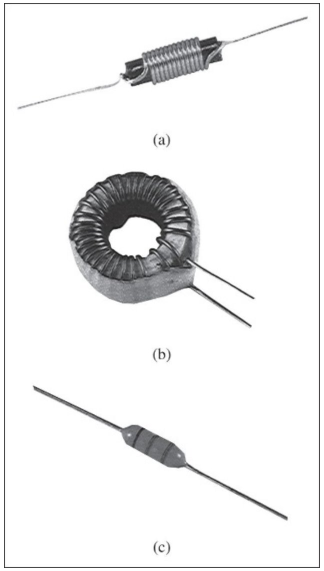
Inductor

- Simplest way to form an inductor is to winding a coil around a core that concentrates magnetic field lines (flux)
- Hence symbol of an inductor is coil between two terminals









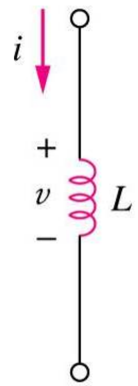
$$\varphi = Li$$

Inductance

- Inductance is commonly symbolized by letter L with unit of Henrys (H)
- Passing a current through an inductor produces a magnetic flux (φ) that is related to the inductance (L)
- No energy dissipated unlike resistors (note that wires are assumed to have no resistance by definition)

I-V relation in an inductor

If the current through an inductor is time-varying, then the generated voltage must also be time-varying:



Flux: $\varphi = Li$

Faraday's Law: $$v(t) = \frac{d\emptyset(t)}{dt}$$

Differentiating with respect to time (t): $$v(t) = L\frac{di(t)}{dt}$$

Voltage depends on the **rate of change** of current

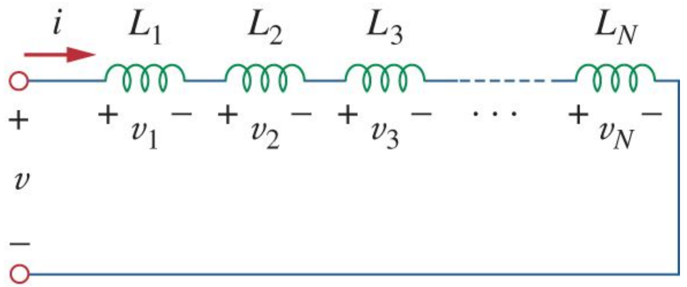
If current is constant with time → di/dt = 0 → V = 0 (No voltage)

No current change → No voltage difference

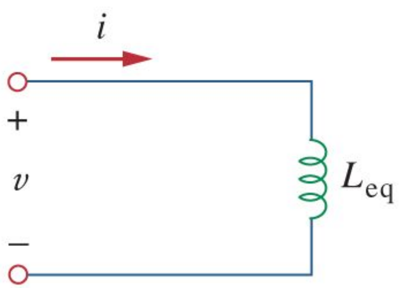
If current changes with time → di/dt ≠ 0 → V ≠ 0 (There is voltage)

Current change → Voltage difference

Inductors in series



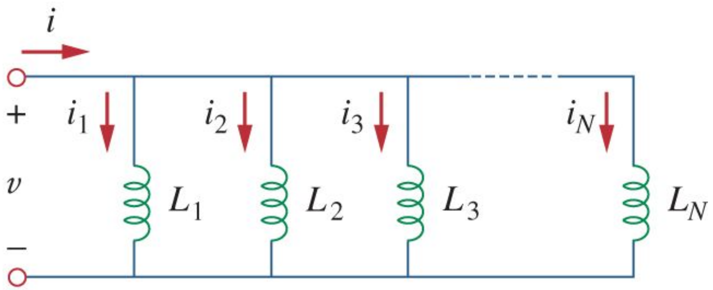
(a)



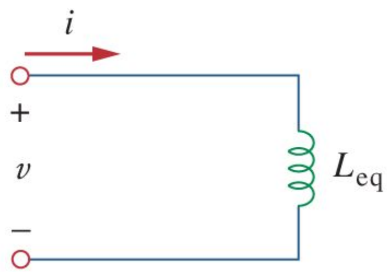
(b)

$$L_{eq} = L_1 + L_2 + L_3 + \cdots + L_N$$

Inductors in parallel



(a)



(b)

$$\frac{1}{\mathrm{L_{eq}}}\int v(t)dt = \frac{1}{\mathrm{L_1}}\int v(t)dt + \frac{1}{\mathrm{L_2}}\int v(t)dt + \frac{1}{\mathrm{L_3}}\int v(t)dt + \cdots + \frac{1}{\mathrm{L_N}}\int v(t)dt$$

$$\frac{1}{\mathrm{L_{eq}}} = \frac{1}{\mathrm{L_1}} + \frac{1}{\mathrm{L_2}} + \frac{1}{\mathrm{L_3}} + \cdots + \frac{1}{\mathrm{L_N}}$$

Important characteristics of the basic elements

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i	v = iR	$v = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)$	$v = L\frac{di}{dt}$
i-v	i = v/R	$i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau + i(t_0)$
Two in series	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel	$R_{eq} = \frac{R_1R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1L_2}{L_1 + L_2}$
At DC	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly	Not applicable	v	i

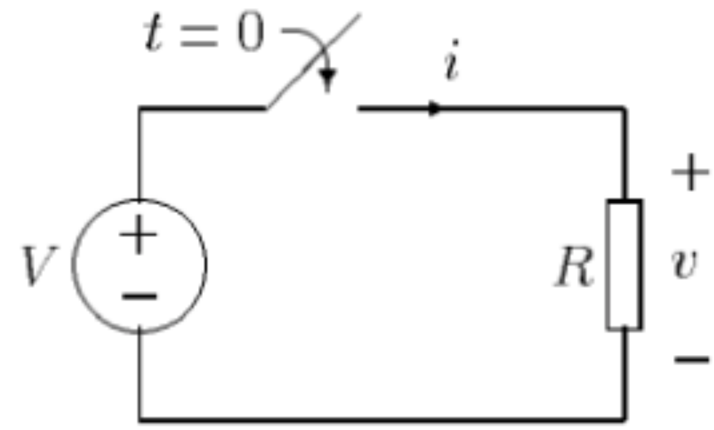
When a circuit has C and/or L, the circuit becomes dynamic.

- Voltage and/or current is a function of time.
- Voltage and/or current is described by differential equation.
- The circuit has
  - **transient response** (circuit response immediately after certain initial condition) AND
  - **steady state response** (as $t \to \infty$)

Section 4.3, Linear Circuit Analysis, C. K. Tse, Addison-Wesley, 1998

Note that pure resistive circuits have no transient!

When we turn on the switch at t = 0, potential difference across the resistor R becomes V immediately.

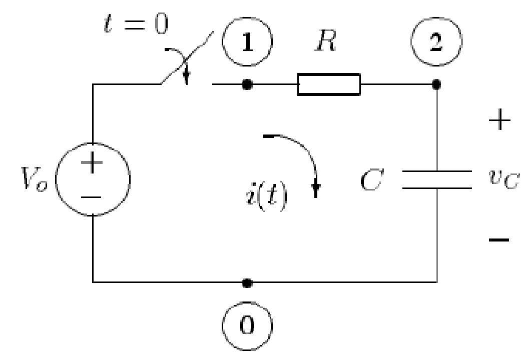


For all $t \geq 0$,

- v = V = iR
- i = V/R

Chapter 4, Linear Circuit Analysis, C. K. Tse, Addison-Wesley, 1998

Simple first-order RC circuit

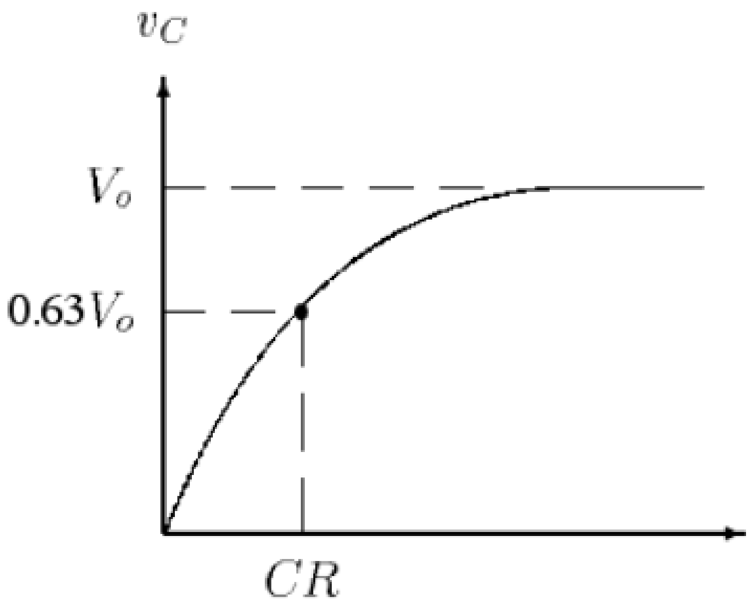


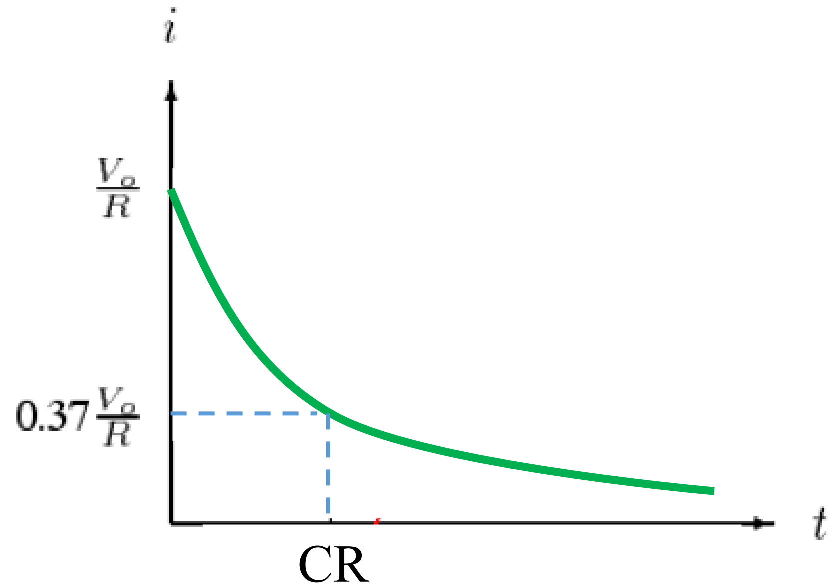
- At t = 0, $v_c(0) = 0$
- After t = 0, the circuit is closed:
  - $i(t) = \frac{v_R}{R} = C\frac{dv_c}{dt}$, and
  - $v_R(t) = V_0 - v_c(t)$
- Hence, $\frac{V_0 - v_c(t)}{CR} = \frac{dv_c}{dt} \quad \Rightarrow \quad \frac{dv_c}{dt} + \frac{v_c}{CR} = \frac{V_0}{CR}$
- The general solution is $v_c(t) = Ae^{-\frac{t}{CR}} + V_0$ for $t \geq 0$ (A is a constant)
- As the **initial condition is $v_c(0^+) = 0$**, $A + V_0 = 0 \Rightarrow A = -V_0$
- Hence,

$$\boxed{v_c(t) = V_0\left(1 - e^{-\frac{t}{CR}}\right)}$$

Section 4.4, Linear Circuit Analysis, C. K. Tse, Addison-Wesley, 1998

Transient response of a simple RC circuit





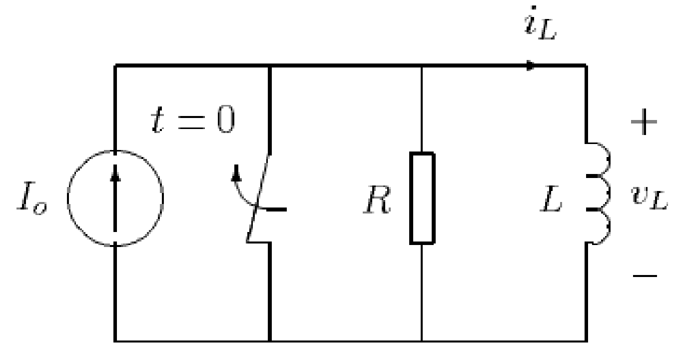
$$v_c(t) = V_0\left(1 - e^{-\frac{t}{CR}}\right) \Rightarrow i(t) = C\frac{dv_c}{dt} = \frac{V_0}{R}e^{-\frac{t}{CR}}$$

When t = CR,

$$v_c(CR) = V_0(1 - e^{-1}) = 0.63V_0 \text{ and } i(CR) = \frac{V_0}{R}e^{-1} = 0.37V_0$$

Section 4.4.1, Linear Circuit Analysis, C. K. Tse, Addison-Wesley, 1998

Simple first-order RL circuit

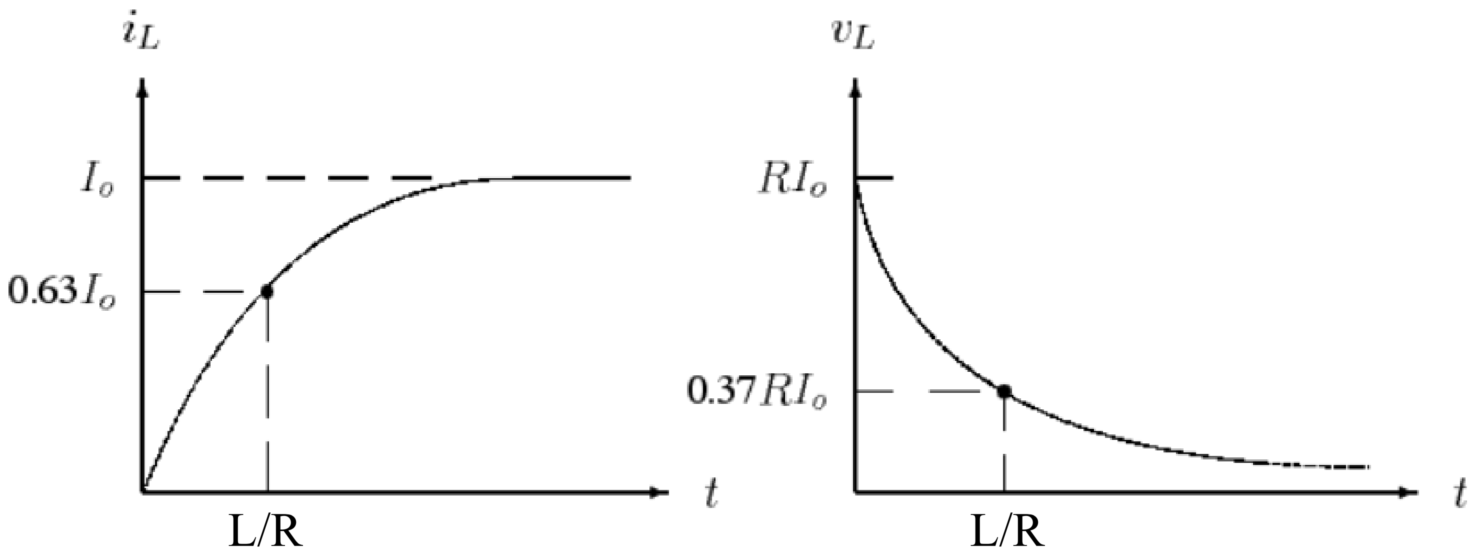


- When t < 0, short-circuit at switch, $i_L = 0$
- For $t \geq 0$,
  - $v_L(t) = L\left(\frac{di_L}{dt}\right)$, and
  - $v_R(t) = \left(I_0 - i_L(t)\right)R$
- Hence, $\left(I_0 - i_L(t)\right)R = L\left(\frac{di_L}{dt}\right) \Rightarrow \frac{di_L}{dt} + \frac{R}{L} i_L(t) = \frac{RI_0}{L}$
- The general solution is $i_L(t) = Ae^{-\frac{Rt}{L}} + I_0$ where A is a constant
- $i_L(0^+) = 0 = A + I_0 \Rightarrow A = -I_0$
- Hence, for $t \geq 0$, we obtain

$$\boxed{i_L(t) = I_0\left(1 - e^{-\frac{Rt}{L}}\right)}$$

Section 4.5, Linear Circuit Analysis, C. K. Tse, Addison-Wesley, 1998

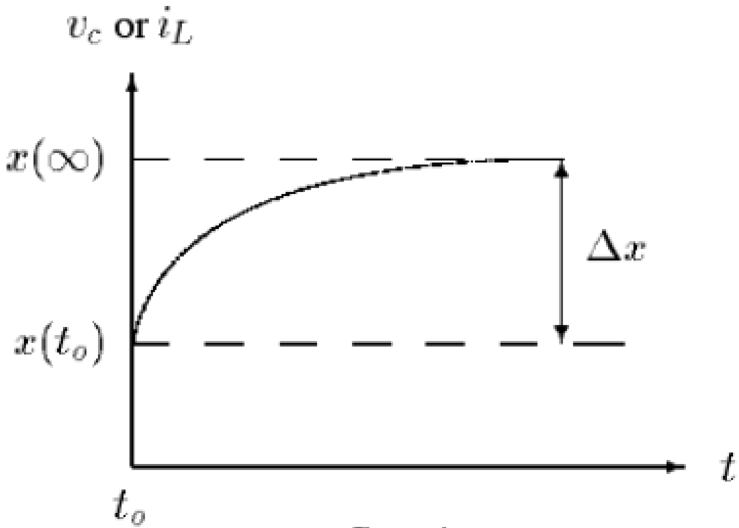
Transient response of a simple RL circuit



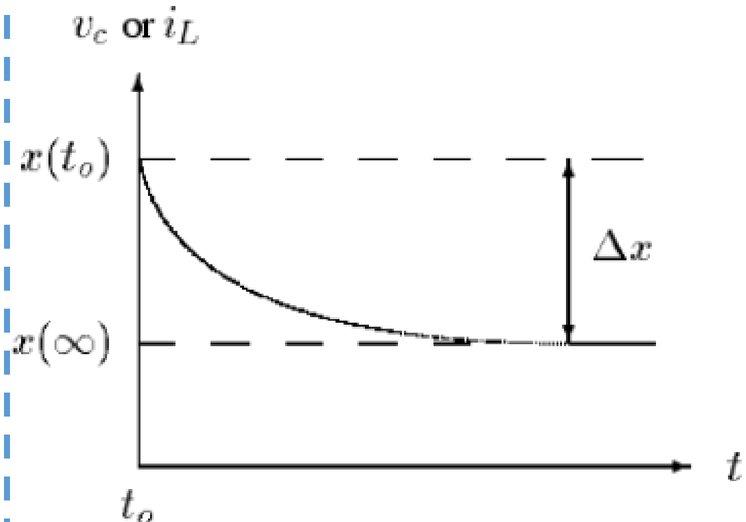
$$i_L(t) = I_0\left(1 - e^{-\frac{Rt}{L}}\right) \Rightarrow v_L(t) = L\frac{di_L(t)}{dt} = RI_0e^{-\frac{Rt}{L}}$$

Section 4.5, Linear Circuit Analysis, C. K. Tse, Addison-Wesley, 1998

General first-order solution



Case A



Case B

$$x(t) = x(t_0) + \Delta x(1 - e^{(-t-t_0)/\tau})$$

$$x(t) = x(\infty) + \Delta x e^{(-t-t_0)/\tau}$$

1. Find the time constant $\boldsymbol{\tau}$
2. Find initial value $\boldsymbol{x(t_0)}$ and final value $\boldsymbol{x(\infty)}$
3. Determine whether Case A or Case B expression shall be used

Section 4.6, Linear Circuit Analysis, C. K. Tse, Addison-Wesley, 1998

Time constant τ

- Simple 1st order RC circuit

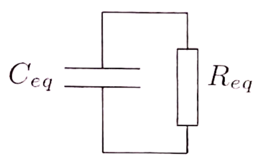
$$\tau = CR$$

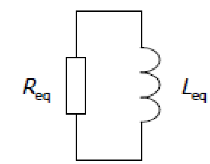
- Simple 1st order RL circuit

$$\tau = \frac{L}{R}$$

To find τ from an equivalent simple RC or RL circuit

1. Short-circuit all voltage sources and open-circuit all current sources
2. Place switches in their final positions
3. Reduce resistances to one equivalent resistance R_{eq}, if possible





Reduce capacitances to one equivalent capacitance $\boldsymbol{C_{eq}}$ (if possible) ➔

Time constant of any RC circuit

$$\boldsymbol{\tau = C_{eq} R_{eq}}$$

Reduce inductances to one equivalent inductance $\boldsymbol{L_{eq}}$ (if possible) ➔

Time constant of any RC circuit

$$\boldsymbol{\tau = \frac{L_{eq}}{R_{eq}}}$$

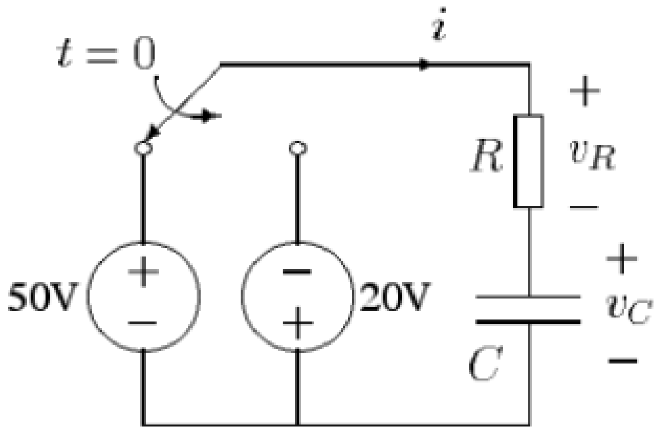
Section 4.7, Linear Circuit Analysis, C. K. Tse, Addison-Wesley, 1998

Example 1: Initial and final values given

- At t = 0, the switch is thrown to the right
- Initial and final values of capacitor voltage are:

  $v_c(0^+) = 50V$ and $v_c(\infty) = -20V$
- Decrease in v_c ➔ Case B ($\tau = CR$)

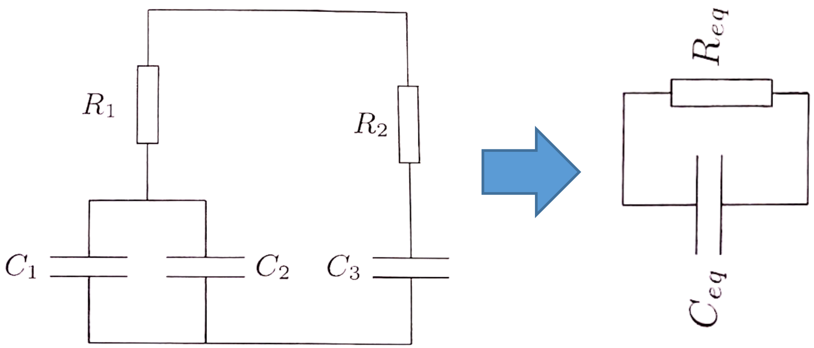
  $$v_c(t) = -20 + 70e^{-\frac{t}{CR}}$$
- $v_R(t) = -20 - v_c(t) = -70e^{-\frac{t}{CR}}$
- $i(t) = \frac{v_R(t)}{R} = -\frac{70}{R}e^{-\frac{t}{CR}}$

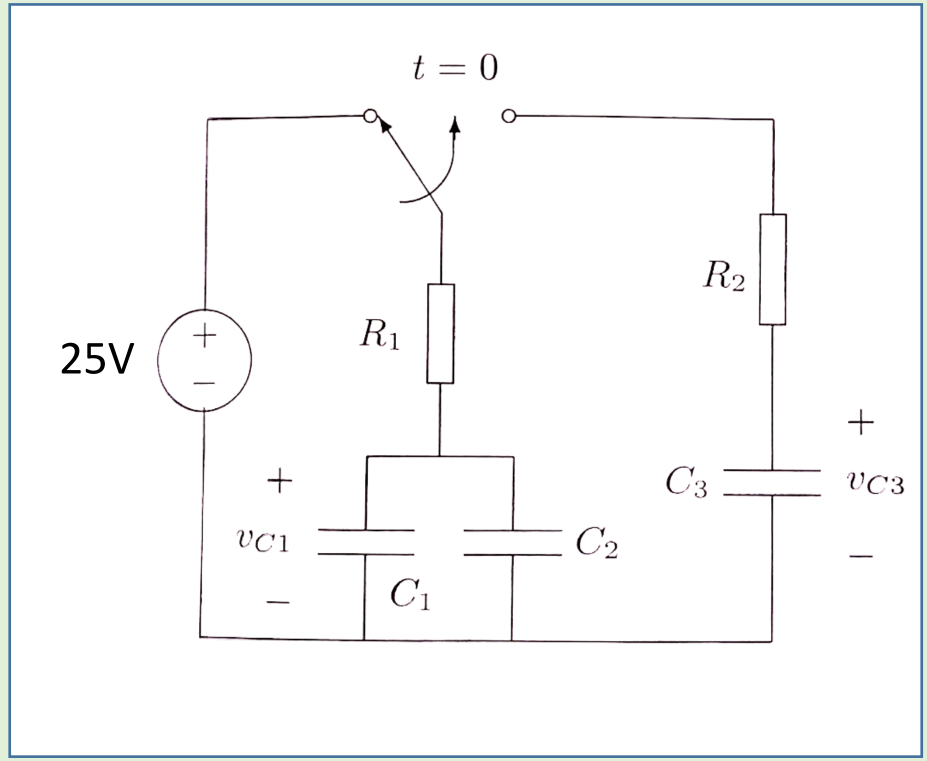


Example 4.4, Linear Circuit Analysis, C. K. Tse, Addison-Wesley, 1998

Example 2 (non-trivial boundary conditions)

- At t = 0, the switch is thrown to the right





➔ Find equivalent resistance:

$$R_{eq} = R_1 + R_2$$

➔ Find equivalent capacitance

$$\frac{1}{C_{eq}} = \frac{1}{C_1+C_2} + \frac{1}{C_3} \rightarrow$$

$$C_{eq} = \frac{C_3(C_1 + C_2)}{(C_1 + C_2 + C_3)}$$

Example 4.5, Linear Circuit Analysis, C. K. Tse, Addison-Wesley, 1998

Example 2 (non-trivial boundary conditions)

- Time constant $\tau = \frac{C_3(C_1+C_2)(R_1+R_2)}{C_1+C_2+C_3}$
- Initial values are

$$v_{c_1}(0^+) = 25, \quad v_{c_2}(0^+) = 25, v_{c_3}(0^+) = 0$$

- To find the final values after the switch throw to right

$$(C_1 + C_2)\frac{dv_{C_1}}{dt} + C_2\frac{dv_{c_3}}{dt} = 0$$

➔ $(C_1 + C_2)v_{c_1}(t) + C_3 v_{C_3}(t) = constant$ for all $t \geq 0$

- $v_{c_1}(0^+) = 25$ and $v_{c_3}(0^+) = 0 \Rightarrow 25(C_1 + C_2) = constant$

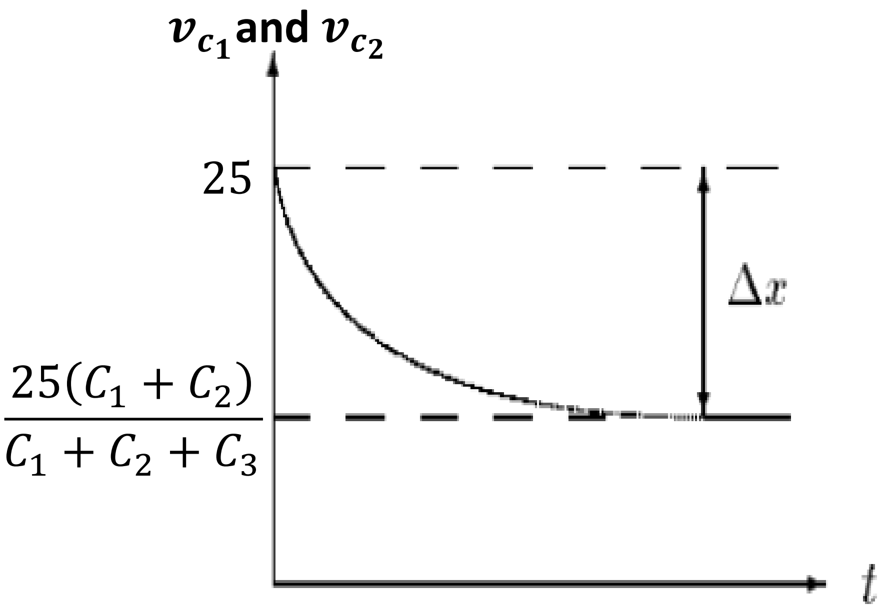
➔ $(C_1 + C_2)v_{c_1}(t) + C_3 v_{C_3}(t) = 25(C_1 + C_2)$

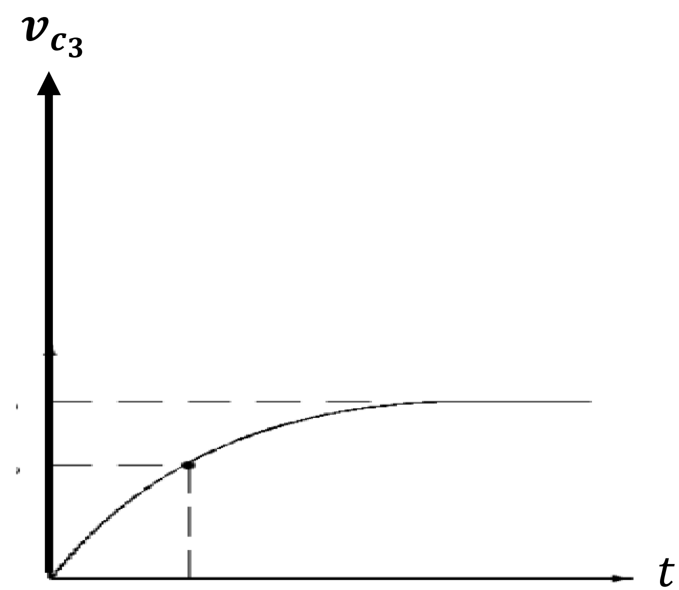
- The final values must satisfy $v_{c_1}(\infty) = v_{c_3}(\infty)$ ➔

$$v_{c_1}(\infty) = v_{c_2}(\infty) = v_{c_3}(\infty) = \frac{25(C_1 + C_2)}{C_1 + C_2 + C_3}$$

Example 4.5, Linear Circuit Analysis, C. K. Tse, Addison-Wesley, 1998

Example 2 (non-trivial boundary conditions)





- v_{c_1} **and** v_{c_2} **: Case B**

$v_{c_1}(t) = v_{c_2}(t)$

$$= \frac{25(C_1+C_2)}{C_1+C_2+C_3} + \frac{25C_3}{C_1+C_2+C_3} e^{-\frac{t}{C_{eq}R_{eq}}}$$

- v_{c_3} **: Case A**

$$v_{c_3}(t) = \frac{25(C_1+C_2)}{C_1+C_2+C_3}\left(1 - e^{-\frac{t}{C_{eq}R_{eq}}}\right)$$