(b) $\int \frac{2x^2}{x^2+1}dx$

(d) $\int \sin 3x \sin 2x \, dx$

(f) $\int \frac{1}{(x-1)(2x-3)}dx$

(h) $\int \frac{1}{2x^2-4x+9}dx$

(j) $\int \tan^2 x \, dx$

Q2.

constant + $\frac{\text{constant}}{x^2+1}$

(b) $\int \frac{2x^2}{x^2+1}dx = \int \frac{2x^2+2-2}{x^2+1}dx = \int \frac{2(x^2+1)}{x^2+1} - \frac{2}{x^2+1}\, dx$

$= \int 2 - 2\cdot\frac{1}{x^2+1}\, dx = 2x - 2\tan^{-1}x + C.$

$(\tan^{-1}x)' = \frac{1}{x^2+1}$

$(\sin x)' = \cos x,$

$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)+C.$

(d) $\int \sin 3x \sin 2x \, dx = \int -\frac{1}{2}[\cos(5x) - \cos x]\, dx = -\frac{1}{2}\left(\frac{1}{5}\sin(5x) - \sin x\right) + C$

$\sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)].$

$= -\frac{1}{10}\sin(5x) + \frac{1}{2}\sin x + C$

$(\ln|x|)' = \frac{1}{x}$ & $\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)+C.$

(f) $\int \frac{1}{(x-1)(2x-3)}dx = \int \left(-\frac{1}{x-1} + 2\cdot\frac{1}{2x-3}\right)dx$

$= -\ln|x-1| + \ln|2x-3| + C$

$\frac{1}{(x-1)(2x-3)} = \frac{A}{x-1} + \frac{B}{2x-3} = -\frac{1}{x-1} + \frac{2}{2x-3}$

$\Rightarrow 1 = A(2x-3) + B(x-1)$

$x=1 \Rightarrow 1 = -A \Rightarrow A = -1.$

$x = \frac{3}{2} \Rightarrow 1 = B\cdot\frac{1}{2} \Rightarrow B = 2$

$(\tan^{-1}x)' = \frac{1}{x^2+1}$

(h) $\int \frac{1}{2x^2-4x+9}dx = \int \frac{1}{2(x-1)^2+7}dx = \int \frac{1}{7}\cdot\frac{1}{\frac{2}{7}(x-1)^2+1}dx.$

$= \int \frac{1}{7}\cdot\frac{1}{\left(\sqrt{\frac{2}{7}}(x-1)\right)^2+1}dx$

$$= \frac{1}{7}\cdot\sqrt{\frac{7}{2}}\tan^{-1}\left(\sqrt{\frac{2}{7}}x-\sqrt{\frac{2}{7}}\right)+C.$$

$$= \frac{1}{\sqrt{14}}\tan^{-1}\left(\sqrt{\frac{2}{7}}x-\sqrt{\frac{2}{7}}\right)+C.$$

$\tan^2 x = \sec^2 x - 1$ $\qquad \tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos^2 x}{\cos^2 x} = \sec^2 x - 1.$

(j) $\displaystyle\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C.$

$(\tan x)' = \sec^2 x$

Q3

(b) $\displaystyle\int_{-1}^{1} \cos(3x+1)\,dx$

(d) $\displaystyle\int_{0}^{\frac{\pi}{2}} \sin^2 x\,dx$

(f) $\displaystyle\int_{-\pi}^{\pi} |\sin x|dx$

(h) $\displaystyle\int_{-1}^{1} x^4 \sin^9 x\,dx$

*(j) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x^2\cos x + \cos x + \sin^3 x}{x^2+1}dx$

(b) $\displaystyle\int_{-1}^{1} \cos(3x+1)\,dx = \frac{1}{3}\sin(3x+1)\Big|_{-1}^{1}$ $\qquad \sin(-x) = -\sin x$

$$= \frac{1}{3}\sin 4 - \frac{1}{3}\sin(-2) = \frac{1}{3}(\sin 4 + \sin 2)$$

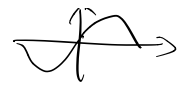
(d) $\displaystyle\int_{0}^{\frac{\pi}{2}} \sin^2 x\,dx = \int_0^{\frac{\pi}{2}} -\frac{1}{2}\left[\cos(2x) - \cos 0\right]dx = \int_0^{\frac{\pi}{2}}\left[\frac{1}{2} - \frac{1}{2}\cos(2x)\right]dx$

$$= \left(\frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{2}\sin 2x\right)\Big|_0^{\frac{\pi}{2}}$$

$$= \left(\frac{1}{2}\frac{\pi}{2} - \frac{1}{4}\sin\pi\right) - \left(0 - \frac{1}{4}\sin 0\right) = \frac{\pi}{4}$$

$\sin x \sin x = -\frac{1}{2}\left[\cos(2x) - \cos 0\right]$

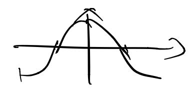
(f) $\displaystyle\int_{-\pi}^{\pi} |\sin x| dx = \int_0^{\pi} \sin x\, dx + \int_{-\pi}^{0} -\sin x\, dx$



$$|\sin x| = \begin{cases} \sin x & x \in [0,\pi] \\ -\sin x & x \in [-\pi, 0] \end{cases}$$

$$= -\cos x \Big|_0^{\pi} + \cos x \Big|_{-\pi}^{0}$$

cos



$$= -\overset{-1}{\cos\pi} + \overset{1}{\cos 0} + \overset{1}{\cos 0} - \overset{-1}{\cos(-\pi)}$$

$$= 4$$

(h) $\displaystyle\int_{-1}^{1} \overset{f(x)}{\overset{\uparrow}{x^4 \sin^9 x}}\, dx = 0$

$f(-x) = -f(x) \Rightarrow \int_{-a}^{a} f(x) = 0$

$f(-x) = (-x)^4 [\sin(-x)]^9 = x^4(-\sin x)^9 = -x^4 \sin^9 x = -f(x). \Rightarrow f(x)$ odd

*(j) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x^2 \cos x + \cos x + \sin^3 x}{x^2+1} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos x (x^2+1)}{x^2+1} + \frac{\sin^3 x}{x^2+1}\, dx$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x + \overset{=f(x)}{\frac{\sin^3 x}{x^2+1}}\, dx. = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x\, dx + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^3 x}{x^2+1}\, dx$$

$f(-x) = \dfrac{(\sin(-x))^3}{(-x)^2+1} = \dfrac{(-\sin x)^3}{x^2+1} = -\dfrac{\sin^3 x}{x^2+1} = -f(x) \Rightarrow f(x)$ is odd.

$$= \sin x \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + 0 = \sin\left(\tfrac{\pi}{4}\right) - \sin\left(-\tfrac{\pi}{4}\right)$$

$$= 2 \cdot \sin\left(\tfrac{\pi}{4}\right) = 2 \cdot \tfrac{1}{\sqrt{2}} = \sqrt{2}$$

Q4

(b) $\displaystyle\frac{d}{dx}\int_{2x}^{x^2} \cos(y^2)\, dy$

let $G(y) = \int \cos(y^2)\, dy \Rightarrow \int_{2x}^{x^2} \cos(y^2)\, dx = G(y)\Big|_{2x}^{x^2} = G(x^2) - G(2x)$

$$\frac{d}{dx}\int_{2x}^{x^2} \cos(y^2)\, dy = \frac{d}{dx}\left(G(x^2) - G(2x)\right) = \frac{d}{dx}G(x^2) - \frac{d}{dx}G(2x)$$

$$= \frac{dG(\overset{=y}{x^2})}{dx^2} \cdot \frac{dx^2}{dx} - \frac{dG(2x)}{d(2x)} \cdot \frac{d(2x)}{dx}.$$

$$\frac{dG(x^2)}{dx^2} = \frac{dG(y)}{dy} = \frac{d}{dy}\int \cos(y^2)dy = \cos(y^2) = \cos(x^4)$$

$$= \cos(x^4)\cdot 2x - \cos(4x^2)\cdot 2 = 2x\cos(x^4) - 2\cos(4x^2)$$

Problem 5

(a) Using fundamental theorem of calculus, show that

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

let $F(x) = \int f(x)\,dx$.

$\int_0^a f(x)dx = F(x)\Big|_0^a = F(a) - F(0)$

$\int_0^a f(-x+a)dx = -F(a-x)\Big|_0^a = -F(0) + F(a) = F(a) - F(0)$

$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)$, $a = -1$

(b) It is given that $g(x)$ is a periodic function with period 1 (i.e. $g(x+1) = g(x)$ for any x). Using fundamental theorem of calculus, show that

(i) $\displaystyle\int_0^4 g(x)dx = 4\int_0^1 g(x)dx$ (ii) $\displaystyle\int_0^1 g(3x)dx = \frac{1}{3}\int_0^3 g(x)dx$

(i) let $G(x) = \int g(x)\,dx$.

$$\int_0^4 g(x)dx = \int_0^1 g(x)dx + \int_1^2 \underbrace{g(x)}_{=g(x-1)}dx + \int_2^3 \underbrace{g(x)}_{=g(x-2)}dx + \int_3^4 \underbrace{g(x)}_{=g(x-3)}dx.$$

$$= \int_0^1 g(x)dx + \int_1^2 g(x-1)dx + \int_2^3 g(x-2)dx + \int_3^4 g(x-3)dx.$$

$$= G(x)\Big|_0^1 + G(x-1)\Big|_1^2 + G(x-2)\Big|_2^3 + G(x-3)\Big|_3^4$$

$$= 4(G(1) - G(0)) = 4\int_0^1 g(x)dx.$$

(ii) $\displaystyle\int_0^1 g(3x)\,dx = \frac{1}{3}G(3x)\Big|_0^1 = \frac{1}{3}(G(3) - G(0)) = \frac{1}{2}\int_0^3 g(x)dx.$