$y = f(x)$ $f'(x)$

- Multivariables Diff Cal.

	Book chapter	
Planes & Traces.	~~Chapt~~ "4.1 — 4.3"	Notes.
* Limits & Continuity	4.4	Page
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✓ Rates of Changes		
✓ geometric Interpretation		
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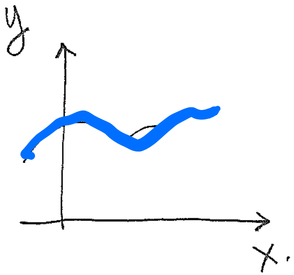
$y = f(x_1, x_2, x_3, \ldots x_n)$

Single Variable

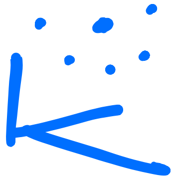
$$y = f(x)$$



Multivariable variables

$$y = f\Big(\overbrace{\underbrace{x_1, x_2, x_3, \dots, x_n}_{\text{independent variables.}}}^{\underline{x}}\Big).$$

dependent variable

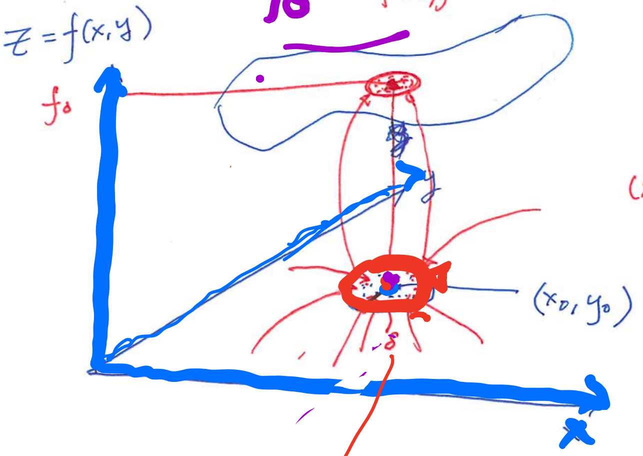


$$\text{Temp} = T(x, y, z, t)$$

at position x, y, z.

at time t.

$f_0 = f(x_0,y_0)$

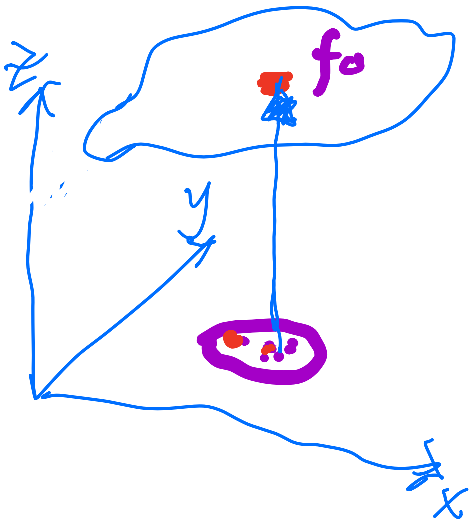


$\varepsilon \to 0, \ |f(x,y) - f_0| \le \varepsilon$

$\Downarrow$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f_0$$

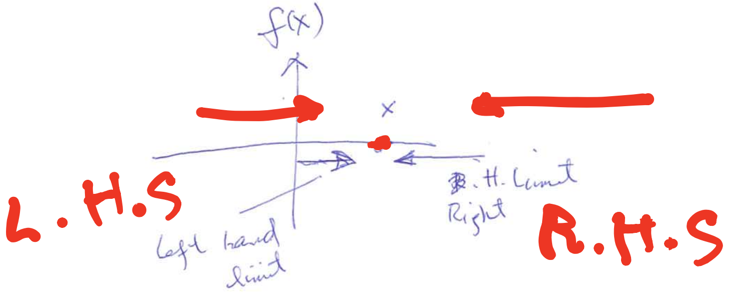
f(x,y)

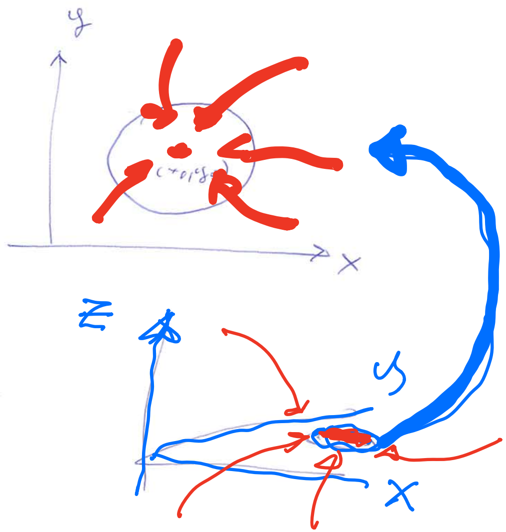


neighbourhood

$$\delta = \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

$$\boxed{\lim_{\substack{x\to x_0\\ y\to y_0}} f(x,y) = f_0}$$





Show that $z = f(x,y) = \dfrac{x^2}{x^2+y^2}$ has no limit at the origin

$(x,y) \to (0,0)$ along the line $y=0$, limiting value is 1
$f(x,0)$

But $(x,y) \to (0,0)$ along the line $x=0$ and the limiting value is 0
$f(0,y)$ $\qquad \dfrac{0}{0+y^2}$

Two directions approach the origin have different limiting values $\Rightarrow$ No limit at the origin

$(x,y) \to (0,0)$

$$f(x,y) = \frac{x^3}{x^2+y^2}$$

(i) along the line $y=0$ (approach the $(0,0)$ along x-axis), $f(\underset{\underset{0}{\Downarrow}}{x},0) = \frac{x^2}{x^2} = 1$ ✓

(ii) along the line $x=0$ (along y-axis), $f(0,\underset{\underset{0}{\Downarrow}}{y}) = \frac{0}{0+y^2} = \frac{0}{y^2} = 0$ ✓

$x \to 0$, $y \to 0$ $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$ does not exist

(i)

$$\lim_{(x,y)\to(1,2)} (x^2+3y^2+xy) = 1^2+3\times 2^2+1\times 2 = 15$$

$\Rightarrow$ any direction $\Rightarrow$ continuous everywhere

$x \to 1$
$y \to 2$

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{(x,y)\to(0,0)} f(x,y)$$

$$= \lim_{x\to 0}\lim_{y\to 0}$$

$$= \lim_{y\to 0}\lim_{x\to 0}$$

(ii) Continuous everywhere except at the origin.

($f(x,y)$ is not defined)

(iii) Show that

$$\left[\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0\right]$$

$$f(x,y)=\begin{cases}\dfrac{x^2}{(x^2+y^2)^{\frac{1}{2}}} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

Then $x = r\cos\theta$, $y = r\sin\theta$, so

$$\frac{r^2\cos^2\theta}{\sqrt{r^2(\cos^2\theta+\sin^2\theta)}} = r\cos^2\theta \quad \text{for } r>0$$

Since $r = \sqrt{x^2+y^2}$, it is clear that $r \to 0$ as both x and y approach zero

$$\lim \frac{x^2}{\sqrt{x^2+y^2}} = \lim_{r\to 0} r\cos^2\theta = 0$$

Sketch the graph

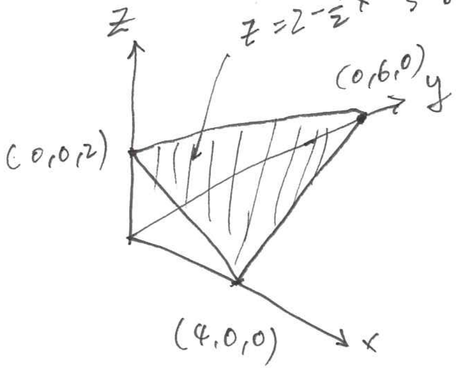
$\cancel{f(x)}\ f(x,y) = 2 - \frac{1}{2}x - \frac{1}{3}y$

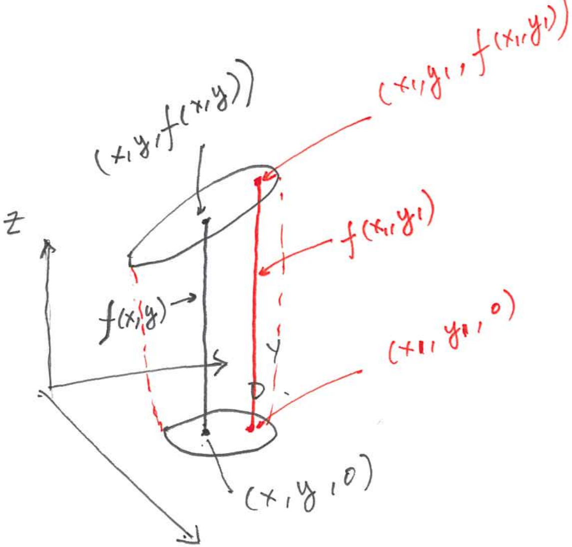
Let $z = 2 - \frac{1}{2}x - \frac{1}{3}y$

If $x = y = 0 \Rightarrow z = 2$

$x = z = 0 \Rightarrow y = 6$

$y = z = 0 \Rightarrow x = 4$



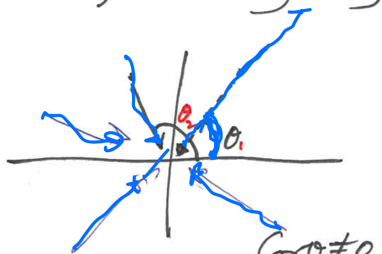


Show that if $f(x,y)=\dfrac{2xy^2}{x^2+y^4}$, does the limit exist at $(0,0)$?

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$$

(i) The function $f(x,y)$ passing through the origin along any straight line

$$\begin{cases} x = t\cos\theta \\ y = t\sin\theta \end{cases} \qquad \boxed{t\to 0}$$



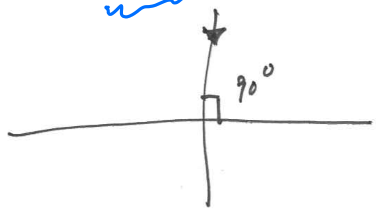
$\cos\theta \neq 0$

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4} = \lim_{t\to 0} \frac{2\cancel{t^3}^{\,t}\cos\theta\sin^2\theta}{\cancel{t^2}\cos^2\theta + \cancel{t^4}_{\,t^2}\sin^4\theta} = \frac{0}{\underline{\cos^2\theta}+0} = 0 = f(0,0)$$

$\cos^2\theta \neq 0$

For $\cos\theta = 0 \Rightarrow \theta = 90°$, $\Rightarrow$ approach $(0,0)$ along the y axis

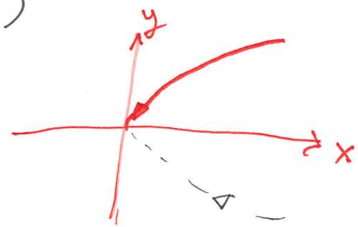
i.e $(0, y)$



$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} f(0,y) = 0 = f(0,0)$$

However, if (x,y) is approching $(0,0)$ along $\begin{cases} x = t^2 \\ y = t \end{cases}$ $t \in \mathbb{R}$

(This is not a straight line!)



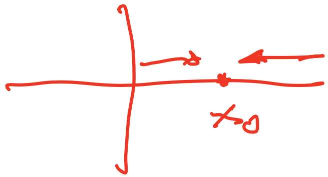
$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0} f(t^2, t)$$

$$= \lim_{t\to 0} \frac{2t^4}{t^4 + t^4} = \frac{2\cancel{t^4}}{\cancel{t^4}(2)} = 1, \neq f(0,0) = 0$$

$\therefore$ limit does not exist

$f(x)$

$$\boxed{\frac{df(x)}{dx} = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}}$$



$$z = f(x,y) = \begin{cases} \dfrac{\partial f}{\partial x}(x_0,y_0) = \displaystyle\lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h} \\[2ex] \dfrac{\partial f}{\partial y}(x_0,y_0) = \displaystyle\lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h} \end{cases}$$

Keep y as a constant

Keep x as a constant

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x(x,y) = D_x f = D_1[f(x,y)]$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y(x,y) = D_y f = D_2[f(x,y)]$$

Geometric Interpretation of Partial Derivatives

(i) $\left(\dfrac{\partial z}{\partial x}\right)_y = \left(\dfrac{\partial f}{\partial x}\right)_y$.

(ii) $\left(\dfrac{\partial z}{\partial y}\right)_x = \left(\dfrac{\partial f}{\partial y}\right)_x$

MA2170 Illustration on Partial Derivatives

$$\frac{\partial f}{\partial x}\bigg|_{y=b} = \frac{\Delta z}{\Delta x}$$

$$\left[\frac{\partial f}{\partial x}\right]_{y=b}$$

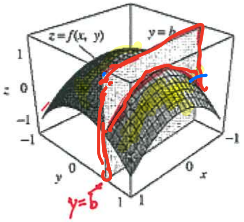


FIGURE 4.5.1 A vertical plane parallel to the xz-plane intersects the surface $z = f(x, y)$ in an x-curve.

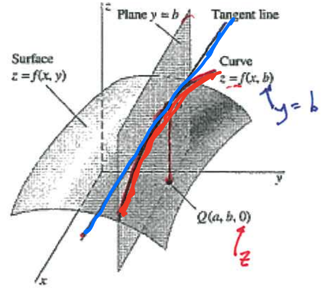


FIGURE 4.5.2 An x-curve and its tangent line at P.

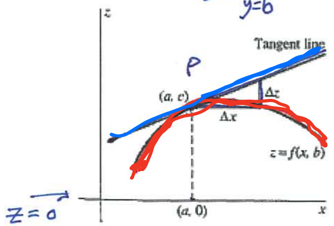


FIGURE 4.5.3 Projection into the xz-plane of the x-curve through $P(a, b, c)$ and its tangent line.

Geometric Interpretation of $\dfrac{\partial f}{\partial x}$

The value $f_x(a, b)$ is the slope of the line tangent at $P(a, b, c)$ to the x-curve through P on the surface $z = f(x, y)$.

x=a

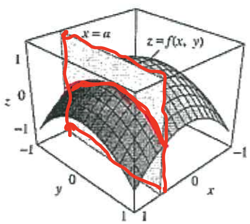


FIGURE 4.5.4 A vertical plane parallel to the yz-plane intersects the surface $z = f(x, y)$ in a y-curve.

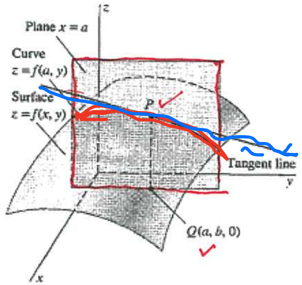


FIGURE 4.5.5 A y-curve and its tangent line at P.

$$\frac{\partial f}{\partial y}\bigg|_{x=a} = \frac{\Delta z}{\Delta y}$$

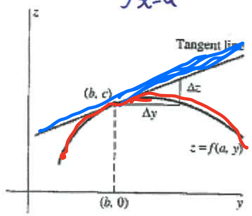


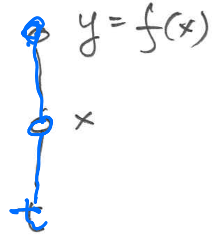
FIGURE 4.5.6 Projection into the yz-plane of the y-curve through $P(a, b, c)$ and its tangent line.

$$\frac{\partial f}{\partial y}\bigg|_{x=a}$$

Geometric Interpretation of $\dfrac{\partial f}{\partial y}$

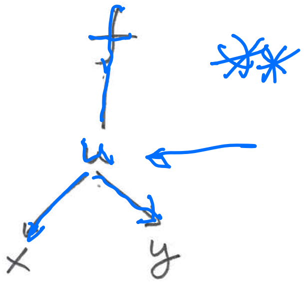
The value $f_y(a, b)$ is the slope of the line tangent at $P(a, b, c)$ to the y-curve through P on the surface $z = f(x, y)$.

Chain Rule



$$y = f(x) \qquad x = X(t)$$

$$\boxed{\frac{dy}{dt} = \frac{df}{dx}\,\frac{dx}{dt}}$$



$$f = f(u) \qquad u(x, y)$$

$$\left\{ \begin{aligned} \left(\frac{\partial f}{\partial x}\right)_y &= \frac{df}{du}\left(\frac{\partial u}{\partial x}\right)_y \\ \left(\frac{\partial f}{\partial y}\right)_x &= \frac{df}{du}\left(\frac{\partial u}{\partial y}\right)_x \end{aligned} \right.$$

$$\left(\frac{\partial f}{\partial x}\right)_{y,z} = \frac{df}{du}\left(\frac{\partial u}{\partial x}\right)_{y,z}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,z} = \frac{df}{du}\left(\frac{\partial u}{\partial y}\right)_{x,z}$$

$$\left(\frac{\partial f}{\partial z}\right)_{x,y} = \frac{df}{du}\left(\frac{\partial u}{\partial z}\right)_{x,y}$$

$$\vdots$$

$$\begin{tikzcd} & f \arrow[d, dash] & \\ & u \arrow[dl] \arrow[d] \arrow[dr] & \\ x & y & z \end{tikzcd}$$

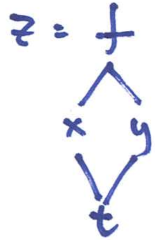
$$z = f(x, y) \quad , \quad x = x(t) \, , \quad y = y(t)$$

$$\frac{dz}{dt} = \underbrace{\left(\frac{\partial f}{\partial x}\right)_y \frac{dx}{dt}} + \underbrace{\left(\frac{\partial f}{\partial y}\right)_x \frac{dy}{dt}} \quad ** **$$

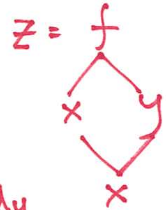
$\frac{dz}{dt}$ → Rate of change of z. w.r.t. t

$\left(\frac{\partial f}{\partial x}\right)_y \frac{dx}{dt}$ → along the x-direction (keep y as a constant)

$\left(\frac{\partial f}{\partial y}\right)_x \frac{dy}{dt}$ → along the y-direction keep x as a constant



Special case replace "t" by "x"



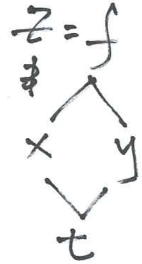
$$\left.\begin{aligned} \frac{dz}{dx} &= \left(\frac{\partial f}{\partial x}\right)_y \overset{1}{\frac{dx}{dx}} + \left(\frac{\partial f}{\partial y}\right)_x \frac{dy}{dx} \\ &= \left(\frac{\partial f}{\partial x}\right)_y + \left(\frac{\partial f}{\partial y}\right)_x \frac{dy}{dx} \end{aligned}\right\}$$

When $\frac{dz}{dt} = 0$

$$0 = \left(\frac{\partial f}{\partial x}\right)_y + \left(\frac{\partial f}{\partial y}\right)_x \frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = - \left(\frac{\partial f}{\partial x}\right)_y \Big/ \left(\frac{\partial f}{\partial y}\right)_x}$$

$$z = f(x, y), \quad x = x(t), \quad y = y(t)$$



$$\ast\ast \qquad \frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\left[\frac{d\,\cdot}{dt} = \frac{\partial\,\cdot}{\partial x}\frac{dx}{dt} + \frac{\partial\,\cdot}{\partial y}\frac{dy}{dt}\right]$$

Find $\dfrac{d^2 z}{dt^2}$?

$$\frac{d^2 z}{dt^2} = \frac{d}{dt}\left(\frac{dz}{dt}\right) = \frac{d}{dt}\left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right)$$

$$= \left(\underbrace{\frac{d}{dt}\left(\frac{\partial f}{\partial x}\right)\frac{dx}{dt}}_{①} + \underbrace{\frac{\partial f}{\partial x}\frac{d^2 x}{dt^2}}_{②}\right) + \left(\underbrace{\frac{d}{dt}\left(\frac{\partial f}{\partial y}\right)\frac{dy}{dt}}_{③} + \underbrace{\frac{\partial f}{\partial y}\frac{d^2 y}{dt^2}}_{④}\right)$$

$$\boxed{\frac{d\bullet}{dt} = \left(\frac{\partial \bullet}{\partial x}\right)_y \frac{dx}{dt} + \left(\frac{\partial \bullet}{\partial y}\right)_x \frac{dy}{dt}}$$

$\bullet$ is replaced by f ✓

$\bullet$ replaced by $\left(\frac{\partial f}{\partial x}\right)$

$$\frac{d\boxed{\left(\frac{\partial f}{\partial x}\right)}}{dt} = \left(\frac{\partial \boxed{\left(\frac{\partial f}{\partial x}\right)}}{\partial x}\right)_y \frac{dx}{dt} + \left(\frac{\partial \boxed{\left(\frac{\partial f}{\partial x}\right)}}{\partial y}\right)_x \frac{dy}{dt}$$

$$\boxed{\frac{d\left(\frac{\partial f}{\partial x}\right)}{dt} = \left(\frac{\partial^2 f}{\partial x^2}\right)_y \frac{dx}{dt} + \left(\frac{\partial^2 f}{\partial y\, \partial x}\right)_x \frac{dy}{dt}.}$$

Replace $\bullet$ by $\frac{\partial f}{\partial y}$

$$\frac{d\left(\frac{\partial f}{\partial y}\right)}{dt} = \frac{\partial \left(\frac{\partial f}{\partial y}\right)_x}{\partial x} \frac{dx}{dt} + \frac{\partial \left(\frac{\partial f}{\partial y}\right)_x}{\partial y} \frac{dy}{dt}.$$

$$= \frac{\partial^2 f}{\partial x\, \partial y} \frac{dx}{dt} + \left(\frac{\partial^2 f}{\partial y^2}\right)_x \frac{dy}{dt}$$

from (**/**)

$$= \left\{ \left[\frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial x}\frac{dx}{dt} + \frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial y}\frac{dy}{dt} \right]\frac{dx}{dt} \right\} + \frac{\partial f}{\partial x}\frac{d^2x}{dt^2}$$

1 2

$$+ \left[\frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial x}\frac{dx}{dt} + \frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial y}\frac{dy}{dt} \right]\frac{dy}{dt} + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2}$$

3 4

$$= \left[\frac{\partial^2 f}{\partial x^2}\frac{dx}{dt} + \boxed{\frac{\partial^2 f}{\partial y \partial x}\frac{dy}{dt}} \right]\frac{dx}{dt} + \frac{\partial f}{\partial x}\frac{d^2x}{dt^2}$$

1 2

$$+ \left[\boxed{\frac{\partial^2 f}{\partial x \partial y}\frac{dx}{dt}} + \frac{\partial^2 f}{\partial y^2}\frac{dy}{dt} \right]\frac{dy}{dt} + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2}$$

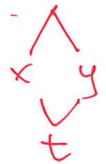
3 4

$$= \frac{\partial^2 f}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + 2\left(\frac{\partial^2 f}{\partial x \partial y}\frac{dx}{dt}\frac{dy}{dt}\right) + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dt}\right)^2 + \frac{\partial f}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial f}{\partial y}\frac{d^2y}{dt^2}$$

//

$z = f(x,y)$, $x = t^2$, $y = t$

$z = f(x,y)$



$$\frac{\partial f}{\partial x}(\underset{x}{1},\underset{y}{1}) = e = \frac{\partial f}{\partial y}(1,1)$$

$$\frac{\partial^2 f}{\partial x^2}(1,1) = e = \frac{\partial^2 f}{\partial y^2}(1,1) = e$$

$$\frac{\partial^2 f}{\partial x \partial y}(1,1) = \frac{\partial^2 f}{\partial y \partial x}(1,1) = 2e$$

Calculate $\frac{d^2 z}{dt^2} = ?$ for $t=1$.

Ans : $x = t^2 \rightarrow \frac{dx}{dt} = 2t$, $\frac{d^2x}{dt^2} = 2$

$y = t \Rightarrow \frac{dy}{dt} = 1$, $\frac{d^2y}{dt^2} = 0$ for $t=1$ then $\begin{cases} x=1 \\ y=1 \end{cases}$

$$\frac{d^2z}{dt^2} = \frac{\partial^2 f}{\partial x^2}(1,1)\left(\frac{dx}{dt}\right)^2_{t=1} + 2\frac{\partial^2 f}{\partial y \partial x}(1,1)\left(\frac{dy}{dt}\right)_{t=1}\left(\frac{dx}{dt}\right) + \frac{\partial^2 f}{\partial y^2}(1,1)\left(\frac{dy}{dt}\right)^2_{t=1}$$

$$+ \frac{\partial f}{\partial x}(1,1)\left.\frac{d^2x}{dt^2}\right|_{t=1} + \frac{\partial f}{\partial y}(1,1)\left.\frac{d^2y}{dt^2}\right|_{t=1}$$

$$= e(2)^2 + 2(2e)(1)2 + e(1)^2 + e \cdot 2 + e(0)$$

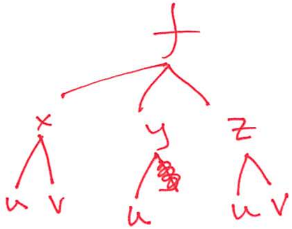
$$= 4e + 8e + e + 2e = 15e$$

$$w = f(x,y,z) = z\,(x^2+y^2)^{-1} = \frac{z}{x^2+y^2}$$

Find the rate of change of w with respect to y. at $(2,1,1)$

$$\left.\frac{\partial w}{\partial y}\right|_{xy} = \frac{-z}{(x^2+y^2)^2}\cdot 2y = -\frac{2yz}{(x^2+y^2)^2} =$$

$$\left.\frac{\partial w}{\partial y}\right|_{xy}(2,1,1) = -\frac{2}{25}$$



$$\frac{\partial w}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{dy}{du} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

$$= \frac{-2xz}{(x^2+y^2)^2} - \frac{2yz}{(x^2+y^2)^2} + \frac{v}{x^2+y^2}$$

$$\begin{cases} x = u+v \\ y = u \\ z = uv \end{cases}$$

$$\frac{\partial w}{\partial v} =$$

$$= \cdots = \frac{-2xz}{(x^2+y^2)^2} + \frac{u}{x^2+y^2}$$