EXERCISES 5.5

Evaluating Integrals

Evaluate the indefinite integrals in Exercises 1–12 by using the given substitutions to reduce the integrals to standard form.

1. $\displaystyle\int \sin 3x\,dx, \quad u = 3x$

2. $\displaystyle\int x \sin(2x^2)\,dx, \quad u = 2x^2$

3. $\displaystyle\int \sec 2t \tan 2t\,dt, \quad u = 2t$

4. $\displaystyle\int \left(1 - \cos\frac{t}{2}\right)^2 \sin\frac{t}{2}\,dt, \quad u = 1 - \cos\frac{t}{2}$

5. $\displaystyle\int 28(7x-2)^{-5}\,dx, \quad u = 7x - 2$

6. $\displaystyle\int x^3(x^4-1)^2\,dx, \quad u = x^4 - 1$

7. $\displaystyle\int \frac{9r^2\,dr}{\sqrt{1-r^3}}, \quad u = 1 - r^3$

8. $\displaystyle\int 12(y^4+4y^2+1)^2(y^3+2y)\,dy, \quad u = y^4 + 4y^2 + 1$

9. $\displaystyle\int \sqrt{x}\sin^2(x^{3/2}-1)\,dx, \quad u = x^{3/2} - 1$

10. $\displaystyle\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx, \quad u = -\frac{1}{x}$

11. $\displaystyle\int \csc^2 2\theta \cot 2\theta\,d\theta$

a. Using $u = \cot 2\theta$ **b.** Using $u = \csc 2\theta$

12. $\displaystyle\int \frac{dx}{\sqrt{5x+8}}$

a. Using $u = 5x + 8$ **b.** Using $u = \sqrt{5x+8}$

Evaluate the integrals in Exercises 13–48.

13. $\displaystyle\int \sqrt{3-2s}\,ds$

14. $\displaystyle\int (2x+1)^3\,dx$

15. $\displaystyle\int \frac{1}{\sqrt{5s+4}}\,ds$

16. $\displaystyle\int \frac{3\,dx}{(2-x)^2}$

17. $\displaystyle\int \theta\sqrt[4]{1-\theta^2}\,d\theta$

18. $\displaystyle\int 8\theta\sqrt[3]{\theta^2-1}\,d\theta$

19. $\displaystyle\int 3y\sqrt{7-3y^2}\,dy$

20. $\displaystyle\int \frac{4y\,dy}{\sqrt{2y^2+1}}$

21. $\displaystyle\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$

22. $\displaystyle\int \frac{(1+\sqrt{x})^3}{\sqrt{x}}\,dx$

23. $\displaystyle\int \cos(3z+4)\,dz$

24. $\displaystyle\int \sin(8z-5)\,dz$

25. $\displaystyle\int \sec^2(3x+2)\,dx$

26. $\displaystyle\int \tan^2 x \sec^2 x\,dx$

27. $\displaystyle\int \sin^5\frac{x}{3}\cos\frac{x}{3}\,dx$

28. $\displaystyle\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\,dx$

29. $\displaystyle\int r^2\left(\frac{r^3}{18}-1\right)^5 dr$

30. $\displaystyle\int r^4\left(7-\frac{r^5}{10}\right)^3 dr$

31. $\displaystyle\int x^{1/2}\sin(x^{3/2}+1)\,dx$

32. $\displaystyle\int x^{1/3}\sin(x^{4/3}-8)\,dx$

33. $\displaystyle\int \sec\left(v+\frac{\pi}{2}\right)\tan\left(v+\frac{\pi}{2}\right)dv$

34. $\displaystyle\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv$

35. $\displaystyle\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$

36. $\displaystyle\int \frac{6\cos t}{(2+\sin t)^3}\,dt$

37. $\displaystyle\int \sqrt{\cot y}\csc^2 y\,dy$

38. $\displaystyle\int \frac{\sec z\tan z}{\sqrt{\sec z}}\,dz$

39. $\displaystyle\int \frac{1}{t^2}\cos\left(\frac{1}{t}-1\right)dt$

40. $\displaystyle\int \frac{1}{\sqrt{t}}\cos(\sqrt{t}+3)\,dt$

41. $\displaystyle\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta$

42. $\displaystyle\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta$

43. $\displaystyle\int (s^3+2s^2-5s+5)(3s^2+4s-5)\,ds$

44. $\displaystyle\int (\theta^4-2\theta^2+8\theta-2)(\theta^3-\theta+2)\,d\theta$

45. $\displaystyle\int t^3(1+t^4)^3\,dt$

46. $\displaystyle\int \sqrt{\frac{x-1}{x^5}}\,dx$

47. $\displaystyle\int x^3\sqrt{x^2+1}\,dx$

48. $\displaystyle\int 3x^5\sqrt{x^3+1}\,dx$

Simplifying Integrals Step by Step

If you do not know what substitution to make, try reducing the integral step by step, using a trial substitution to simplify the integral a bit and then another to simplify it some more. You will see what we mean if you try the sequences of substitutions in Exercises 49 and 50.

49. $\displaystyle\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx$

a. $u = \tan x$, followed by $v = u^3$, then by $w = 2 + v$

b. $u = \tan^3 x$, followed by $v = 2 + u$

c. $u = 2 + \tan^3 x$

50. $\displaystyle\int \sqrt{1+\sin^2(x-1)}\sin(x-1)\cos(x-1)\,dx$

a. $u = x - 1$, followed by $v = \sin u$, then by $w = 1 + v^2$

b. $u = \sin(x-1)$, followed by $v = 1 + u^2$

c. $u = 1 + \sin^2(x-1)$

Evaluate the integrals in Exercises 51 and 52.

51. $\displaystyle\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\,dr$

52. $\displaystyle\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}\,d\theta$

Initial Value Problems

Solve the initial value problems in Exercises 53–58.

53. $\displaystyle\frac{ds}{dt} = 12t(3t^2-1)^3, \quad s(1) = 3$

54. $\displaystyle\frac{dy}{dx} = 4x(x^2+8)^{-1/3}, \quad y(0) = 0$

55. $\displaystyle\frac{ds}{dt} = 8\sin^2\left(t+\frac{\pi}{12}\right), \quad s(0) = 8$

56. $\displaystyle\frac{dr}{d\theta} = 3\cos^2\left(\frac{\pi}{4}-\theta\right), \quad r(0) = \frac{\pi}{8}$

57. $\dfrac{d^2s}{dt^2} = -4 \sin\left(2t - \dfrac{\pi}{2}\right), \quad s'(0) = 100, \quad s(0) = 0$

58. $\dfrac{d^2y}{dx^2} = 4 \sec^2 2x \tan 2x, \quad y'(0) = 4, \quad y(0) = -1$

59. The velocity of a particle moving back and forth on a line is $v = ds/dt = 6 \sin 2t$ m/sec for all t. If $s = 0$ when $t = 0$, find the value of s when $t = \pi/2$ sec.

60. The acceleration of a particle moving back and forth on a line is $a = d^2s/dt^2 = \pi^2 \cos \pi t$ m/sec^2 for all t. If $s = 0$ and $v = 8$ m/sec when $t = 0$, find s when $t = 1$ sec.

Theory and Examples

61. It looks as if we can integrate $2 \sin x \cos x$ with respect to x in three different ways:

a. $$\int 2 \sin x \cos x \, dx = \int 2u \, du \qquad u = \sin x,$$
$$= u^2 + C_1 = \sin^2 x + C_1$$

b. $$\int 2 \sin x \cos x \, dx = \int -2u \, du \qquad u = \cos x,$$
$$= -u^2 + C_2 = -\cos^2 x + C_2$$

c. $$\int 2 \sin x \cos x \, dx = \int \sin 2x \, dx \qquad 2 \sin x \cos x = \sin 2x$$
$$= -\frac{\cos 2x}{2} + C_3.$$

Can all three integrations be correct? Give reasons for your answer.

62. The substitution $u = \tan x$ gives

$$\int \sec^2 x \tan x \, dx = \int u \, du = \frac{u^2}{2} + C = \frac{\tan^2 x}{2} + C.$$

The substitution $u = \sec x$ gives

$$\int \sec^2 x \tan x \, dx = \int u \, du = \frac{u^2}{2} + C = \frac{\sec^2 x}{2} + C.$$

Can both integrations be correct? Give reasons for your answer.

63. (*Continuation of Example 9.*)

a. Show by evaluating the integral in the expression

$$\frac{1}{(1/60) - 0} \int_0^{1/60} V_{\max} \sin 120\,\pi t \, dt$$

that the average value of $V = V_{\max} \sin 120\,\pi t$ over a full cycle is zero.

b. The circuit that runs your electric stove is rated 240 volts rms. What is the peak value of the allowable voltage?

c. Show that

$$\int_0^{1/60} (V_{\max})^2 \sin^2 120\,\pi t \, dt = \frac{(V_{\max})^2}{120}.$$

EXERCISES 5.6

Evaluating Definite Integrals

Use the Substitution Formula in Theorem 6 to evaluate the integrals in Exercises 1–24.

1. a. $\int_0^3 \sqrt{y+1}\, dy$ **b.** $\int_{-1}^0 \sqrt{y+1}\, dy$

2. a. $\int_0^1 r\sqrt{1-r^2}\, dr$ **b.** $\int_{-1}^1 r\sqrt{1-r^2}\, dr$

3. a. $\int_0^{\pi/4} \tan x \sec^2 x\, dx$ **b.** $\int_{-\pi/4}^0 \tan x \sec^2 x\, dx$

4. a. $\int_0^{\pi} 3\cos^2 x \sin x\, dx$ **b.** $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\, dx$

5. a. $\int_0^1 t^3(1+t^4)^3\, dt$ **b.** $\int_{-1}^1 t^3(1+t^4)^3\, dt$

6. a. $\int_0^{\sqrt{7}} t(t^2+1)^{1/3}\, dt$ **b.** $\int_{-\sqrt{7}}^0 t(t^2+1)^{1/3}\, dt$

7. a. $\int_{-1}^1 \frac{5r}{(4+r^2)^2}\, dr$ **b.** $\int_0^1 \frac{5r}{(4+r^2)^2}\, dr$

8. a. $\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\, dv$ **b.** $\int_1^4 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\, dv$

9. a. $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\, dx$ **b.** $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\, dx$

10. a. $\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\, dx$ **b.** $\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}}\, dx$

11. a. $\int_0^{\pi/6} (1-\cos 3t)\sin 3t\, dt$ **b.** $\int_{\pi/6}^{\pi/3} (1-\cos 3t)\sin 3t\, dt$

12. a. $\int_{-\pi/2}^0 \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\, dt$ **b.** $\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\, dt$

13. a. $\int_0^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\, dz$ **b.** $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\, dz$

14. a. $\int_{-\pi/2}^0 \frac{\sin w}{(3+2\cos w)^2}\, dw$ **b.** $\int_0^{\pi/2} \frac{\sin w}{(3+2\cos w)^2}\, dw$

15. $\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\, dt$

16. $\int_1^4 \frac{dy}{2\sqrt{y}\,(1+\sqrt{y})^2}$

17. $\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\, d\theta$

18. $\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right) d\theta$

19. $\int_0^{\pi} 5(5-4\cos t)^{1/4}\sin t\, dt$

20. $\int_0^{\pi/4} (1-\sin 2t)^{3/2}\cos 2t\, dt$

21. $\int_0^1 (4y-y^2+4y^3+1)^{-2/3}(12y^2-2y+4)\, dy$

22. $\int_0^1 (y^3+6y^2-12y+9)^{-1/2}(y^2+4y-4)\, dy$

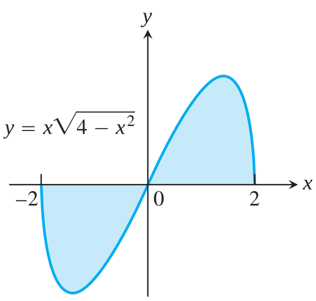
23. $\displaystyle\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\cos^2(\theta^{3/2})\,d\theta$

24. $\displaystyle\int_{-1}^{-1/2} t^{-2}\sin^2\left(1+\frac{1}{t}\right)dt$

Area

Find the total areas of the shaded regions in Exercises 25–40.

25.



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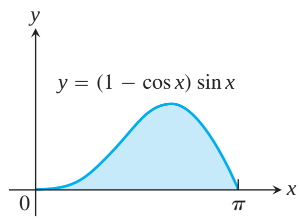

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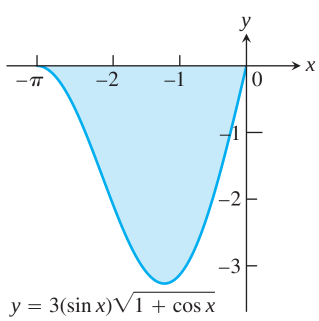

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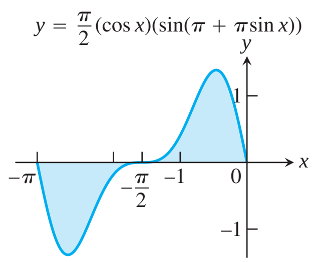

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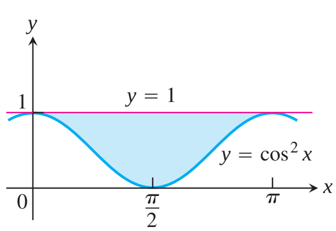

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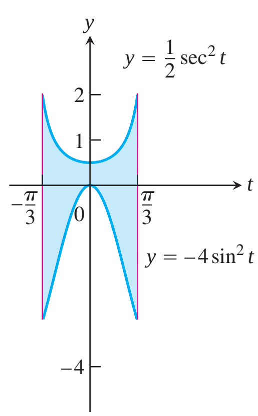

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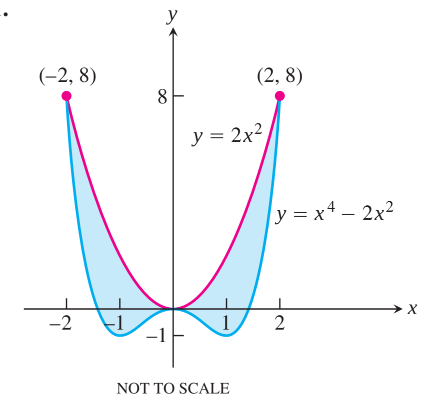

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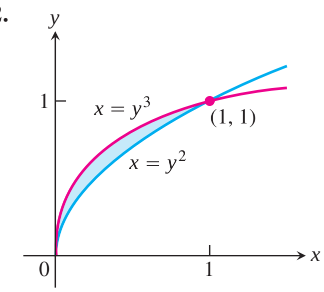

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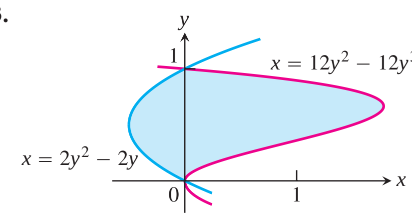

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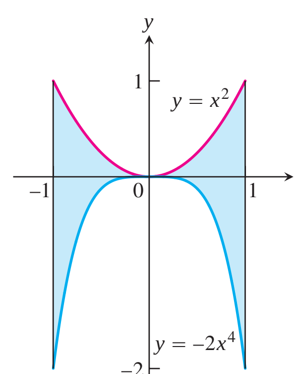

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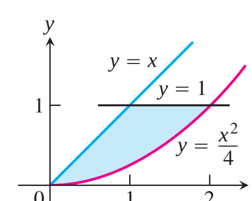

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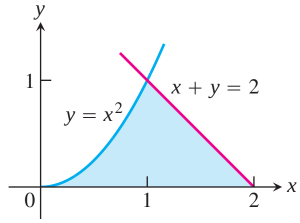

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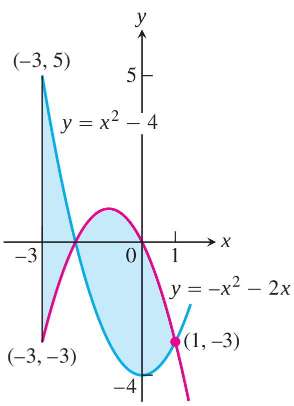

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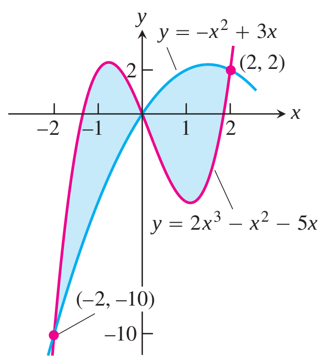

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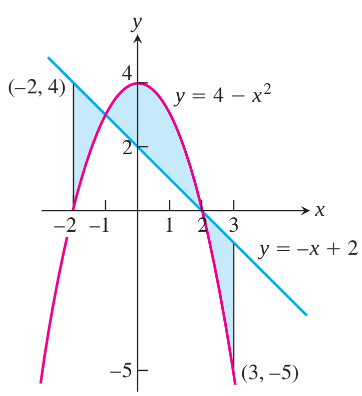

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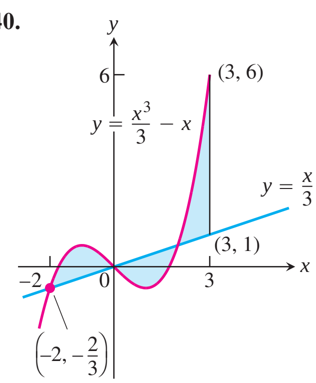


Find the areas of the regions enclosed by the lines and curves in Exercises 41–50.

41. $y = x^2 - 2$ and $y = 2$

42. $y = 2x - x^2$ and $y = -3$

43. $y = x^4$ and $y = 8x$

44. $y = x^2 - 2x$ and $y = x$

45. $y = x^2$ and $y = -x^2 + 4x$

46. $y = 7 - 2x^2$ and $y = x^2 + 4$

47. $y = x^4 - 4x^2 + 4$ and $y = x^2$

48. $y = x\sqrt{a^2 - x^2}$, $a > 0$, and $y = 0$

49. $y = \sqrt{|x|}$ and $5y = x + 6$ (How many intersection points are there?)

50. $y = |x^2 - 4|$ and $y = (x^2/2) + 4$

Find the areas of the regions enclosed by the lines and curves in Exercises 51–58.

51. $x = 2y^2$, $x = 0$, and $y = 3$

52. $x = y^2$ and $x = y + 2$

53. $y^2 - 4x = 4$ and $4x - y = 16$

54. $x - y^2 = 0$ and $x + 2y^2 = 3$

55. $x + y^2 = 0$ and $x + 3y^2 = 2$

56. $x - y^{2/3} = 0$ and $x + y^4 = 2$

57. $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$

58. $x = y^3 - y^2$ and $x = 2y$

Find the areas of the regions enclosed by the curves in Exercises 59–62.

59. $4x^2 + y = 4$ and $x^4 - y = 1$

60. $x^3 - y = 0$ and $3x^2 - y = 4$

61. $x + 4y^2 = 4$ and $x + y^4 = 1$, for $x \geq 0$

62. $x + y^2 = 3$ and $4x + y^2 = 0$

Find the areas of the regions enclosed by the lines and curves in Exercises 63–70.

63. $y = 2\sin x$ and $y = \sin 2x$, $0 \leq x \leq \pi$

64. $y = 8\cos x$ and $y = \sec^2 x$, $-\pi/3 \leq x \leq \pi/3$

65. $y = \cos(\pi x/2)$ and $y = 1 - x^2$

66. $y = \sin(\pi x/2)$ and $y = x$

67. $y = \sec^2 x$, $y = \tan^2 x$, $x = -\pi/4$, and $x = \pi/4$

68. $x = \tan^2 y$ and $x = -\tan^2 y$, $-\pi/4 \leq y \leq \pi/4$

69. $x = 3\sin y\sqrt{\cos y}$ and $x = 0$, $0 \leq y \leq \pi/2$

70. $y = \sec^2(\pi x/3)$ and $y = x^{1/3}$, $-1 \leq x \leq 1$

71. Find the area of the propeller-shaped region enclosed by the curve $x - y^3 = 0$ and the line $x - y = 0$.

72. Find the area of the propeller-shaped region enclosed by the curves $x - y^{1/3} = 0$ and $x - y^{1/5} = 0$.

73. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x-axis.

74. Find the area of the "triangular" region in the first quadrant bounded on the left by the y-axis and on the right by the curves $y = \sin x$ and $y = \cos x$.

75. The region bounded below by the parabola $y = x^2$ and above by the line $y = 4$ is to be partitioned into two subsections of equal area by cutting across it with the horizontal line $y = c$.

a. Sketch the region and draw a line $y = c$ across it that looks about right. In terms of c, what are the coordinates of the points where the line and parabola intersect? Add them to your figure.

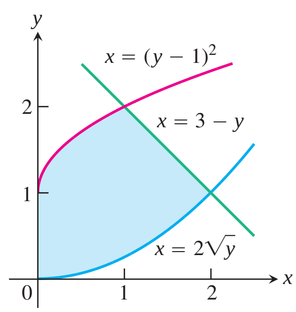
b. Find c by integrating with respect to y. (This puts c in the limits of integration.)

c. Find c by integrating with respect to x. (This puts c into the integrand as well.)

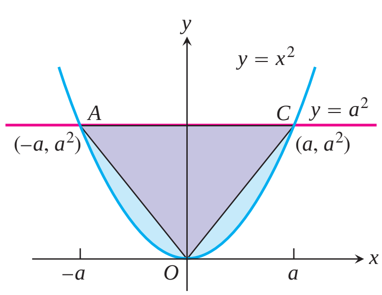
76. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to **a.** x, **b.** y.

77. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$.

78. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y - 1)^2$, and above right by the line $x = 3 - y$.



79. The figure here shows triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero.

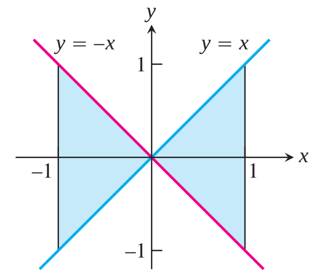


80. Suppose the area of the region between the graph of a positive continuous function f and the x-axis from $x = a$ to $x = b$ is 4 square units. Find the area between the curves $y = f(x)$ and $y = 2f(x)$ from $x = a$ to $x = b$.

81. Which of the following integrals, if either, calculates the area of the shaded region shown here? Give reasons for your answer.

a. $\displaystyle\int_{-1}^{1} (x - (-x))\, dx = \int_{-1}^{1} 2x\, dx$

b. $\displaystyle\int_{-1}^{1} (-x - (x))\, dx = \int_{-1}^{1} -2x\, dx$



82. True, sometimes true, or never true? The area of the region between the graphs of the continuous functions $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$ $(a < b)$ is

$$\int_a^b [f(x) - g(x)]\, dx.$$

Give reasons for your answer.

Theory and Examples

83. Suppose that $F(x)$ is an antiderivative of $f(x) = (\sin x)/x$, $x > 0$. Express

$$\int_1^3 \frac{\sin 2x}{x}\, dx$$

in terms of F.

84. Show that if f is continuous, then

$$\int_0^1 f(x)\, dx = \int_0^1 f(1 - x)\, dx.$$

85. Suppose that

$$\int_0^1 f(x)\, dx = 3.$$

Find

$$\int_{-1}^0 f(x)\, dx$$

if **a.** f is odd, **b.** f is even.

86. **a.** Show that if f is odd on $[-a, a]$, then

$$\int_{-a}^{a} f(x)\, dx = 0.$$

b. Test the result in part (a) with $f(x) = \sin x$ and $a = \pi/2$.

87. If f is a continuous function, find the value of the integral

$$I = \int_0^a \frac{f(x)\, dx}{f(x) + f(a - x)}$$

by making the substitution $u = a - x$ and adding the resulting integral to I.

88. By using a substitution, prove that for all positive numbers x and y,

$$\int_x^{xy} \frac{1}{t}\,dt = \int_1^y \frac{1}{t}\,dt.$$

The Shift Property for Definite Integrals

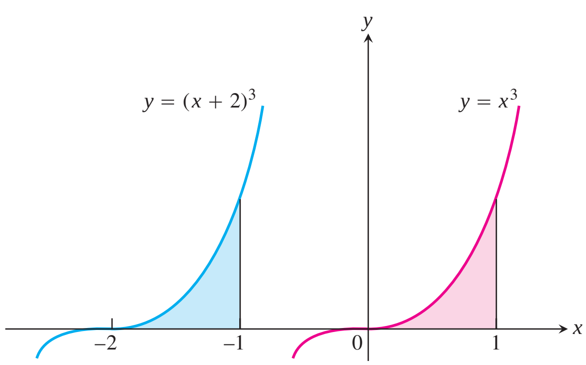
A basic property of definite integrals is their invariance under translation, as expressed by the equation.

$$\int_a^b f(x)\,dx = \int_{a-c}^{b-c} f(x+c)\,dx. \tag{1}$$

The equation holds whenever f is integrable and defined for the necessary values of x. For example in the accompanying figure, show that

$$\int_{-2}^{-1} (x+2)^3\,dx = \int_0^1 x^3\,dx$$

because the areas of the shaded regions are congruent.



89. Use a substitution to verify Equation (1).

90. For each of the following functions, graph $f(x)$ over $[a, b]$ and $f(x + c)$ over $[a - c, b - c]$ to convince yourself that Equation (1) is reasonable.

a. $f(x) = x^2, \quad a = 0, \quad b = 1, \quad c = 1$

b. $f(x) = \sin x, \quad a = 0, \quad b = \pi, \quad c = \pi/2$

c. $f(x) = \sqrt{x - 4}, \quad a = 4, \quad b = 8, \quad c = 5$

COMPUTER EXPLORATIONS

In Exercises 91–94, you will find the area between curves in the plane when you cannot find their points of intersection using simple algebra. Use a CAS to perform the following steps:

a. Plot the curves together to see what they look like and how many points of intersection they have.

b. Use the numerical equation solver in your CAS to find all the points of intersection.

c. Integrate $|f(x) - g(x)|$ over consecutive pairs of intersection values.

d. Sum together the integrals found in part (c).

91. $f(x) = \dfrac{x^3}{3} - \dfrac{x^2}{2} - 2x + \dfrac{1}{3}, \quad g(x) = x - 1$

92. $f(x) = \dfrac{x^4}{2} - 3x^3 + 10, \quad g(x) = 8 - 12x$

93. $f(x) = x + \sin(2x), \quad g(x) = x^3$

94. $f(x) = x^2 \cos x, \quad g(x) = x^3 - x$

EXERCISES 8.1

Basic Substitutions

Evaluate each integral in Exercises 1–36 by using a substitution to reduce it to standard form.

1. $\displaystyle\int \frac{16x\,dx}{\sqrt{8x^2+1}}$

2. $\displaystyle\int \frac{3\cos x\,dx}{\sqrt{1+3\sin x}}$

3. $\displaystyle\int 3\sqrt{\sin v}\cos v\,dv$

4. $\displaystyle\int \cot^3 y\csc^2 y\,dy$

5. $\displaystyle\int_0^1 \frac{16x\,dx}{8x^2+2}$

6. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sec^2 z}{\tan z}\,dz$

7. $\displaystyle\int \frac{dx}{\sqrt{x}\,(\sqrt{x}+1)}$

8. $\displaystyle\int \frac{dx}{x-\sqrt{x}}$

9. $\displaystyle\int \cot(3-7x)\,dx$

10. $\displaystyle\int \csc(\pi x-1)\,dx$

11. $\displaystyle\int e^{\theta}\csc(e^{\theta}+1)\,d\theta$

12. $\displaystyle\int \frac{\cot(3+\ln x)}{x}\,dx$

13. $\displaystyle\int \sec\frac{t}{3}\,dt$

14. $\displaystyle\int x\sec(x^2-5)\,dx$

15. $\displaystyle\int \csc(s-\pi)\,ds$

16. $\displaystyle\int \frac{1}{\theta^2}\csc\frac{1}{\theta}\,d\theta$

17. $\displaystyle\int_0^{\sqrt{\ln 2}} 2x\,e^{x^2}\,dx$

18. $\displaystyle\int_{\pi/2}^{\pi} (\sin y)e^{\cos y}\,dy$

19. $\displaystyle\int e^{\tan v}\sec^2 v\,dv$

20. $\displaystyle\int \frac{e^{\sqrt{t}}\,dt}{\sqrt{t}}$

21. $\displaystyle\int 3^{x+1}\,dx$

22. $\displaystyle\int \frac{2^{\ln x}}{x}\,dx$

23. $\displaystyle\int \frac{2^{\sqrt{w}}\,dw}{2\sqrt{w}}$

24. $\displaystyle\int 10^{2\theta}\,d\theta$

25. $\displaystyle\int \frac{9\,du}{1+9u^2}$

26. $\displaystyle\int \frac{4\,dx}{1+(2x+1)^2}$

27. $\displaystyle\int_0^{1/6} \frac{dx}{\sqrt{1-9x^2}}$

28. $\displaystyle\int_0^{1} \frac{dt}{\sqrt{4-t^2}}$

29. $\displaystyle\int \frac{2s\,ds}{\sqrt{1-s^4}}$

30. $\displaystyle\int \frac{2\,dx}{x\sqrt{1-4\ln^2 x}}$

31. $\displaystyle\int \frac{6\,dx}{x\sqrt{25x^2-1}}$

32. $\displaystyle\int \frac{dr}{r\sqrt{r^2-9}}$

33. $\displaystyle\int \frac{dx}{e^x+e^{-x}}$

34. $\displaystyle\int \frac{dy}{\sqrt{e^{2y}-1}}$

35. $\displaystyle\int_1^{e^{\pi/3}} \frac{dx}{x\cos(\ln x)}$

36. $\displaystyle\int \frac{\ln x\,dx}{x+4x\ln^2 x}$

Completing the Square

Evaluate each integral in Exercises 37–42 by completing the square and using a substitution to reduce it to standard form.

37. $\displaystyle\int_1^2 \frac{8\,dx}{x^2-2x+2}$

38. $\displaystyle\int_2^4 \frac{2\,dx}{x^2-6x+10}$

39. $\displaystyle\int \frac{dt}{\sqrt{-t^2+4t-3}}$

40. $\displaystyle\int \frac{d\theta}{\sqrt{2\theta-\theta^2}}$

41. $\displaystyle\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

42. $\displaystyle\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

Trigonometric Identities

Evaluate each integral in Exercises 43–46 by using trigonometric identities and substitutions to reduce it to standard form.

43. $\displaystyle\int (\sec x+\cot x)^2\,dx$

44. $\displaystyle\int (\csc x-\tan x)^2\,dx$

45. $\displaystyle\int \csc x\sin 3x\,dx$

46. $\displaystyle\int (\sin 3x\cos 2x-\cos 3x\sin 2x)\,dx$

Improper Fractions

Evaluate each integral in Exercises 47–52 by reducing the improper fraction and using a substitution (if necessary) to reduce it to standard form.

47. $\displaystyle\int \frac{x}{x+1}\,dx$

48. $\displaystyle\int \frac{x^2}{x^2+1}\,dx$

49. $\displaystyle\int_{\sqrt{2}}^{3} \frac{2x^3}{x^2-1}\,dx$

50. $\displaystyle\int_{-1}^{3} \frac{4x^2-7}{2x+3}\,dx$

51. $\displaystyle\int \frac{4t^3-t^2+16t}{t^2+4}\,dt$

52. $\displaystyle\int \frac{2\theta^3-7\theta^2+7\theta}{2\theta-5}\,d\theta$

Separating Fractions

Evaluate each integral in Exercises 53–56 by separating the fraction and using a substitution (if necessary) to reduce it to standard form.

53. $\displaystyle\int \frac{1-x}{\sqrt{1-x^2}}\,dx$

54. $\displaystyle\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx$

55. $\displaystyle\int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\,dx$

56. $\displaystyle\int_0^{1/2} \frac{2-8x}{1+4x^2}\,dx$

Multiplying by a Form of 1

Evaluate each integral in Exercises 57–62 by multiplying by a form of 1 and using a substitution (if necessary) to reduce it to standard form.

57. $\displaystyle\int \frac{1}{1+\sin x}\,dx$

58. $\displaystyle\int \frac{1}{1+\cos x}\,dx$

59. $\displaystyle\int \frac{1}{\sec\theta+\tan\theta}\,d\theta$

60. $\displaystyle\int \frac{1}{\csc\theta+\cot\theta}\,d\theta$

61. $\displaystyle\int \frac{1}{1-\sec x}\,dx$

62. $\displaystyle\int \frac{1}{1-\csc x}\,dx$

Eliminating Square Roots

Evaluate each integral in Exercises 63–70 by eliminating the square root.

63. $\displaystyle\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\,dx$

64. $\displaystyle\int_0^{\pi} \sqrt{1-\cos 2x}\,dx$

65. $\displaystyle\int_{\pi/2}^{\pi} \sqrt{1 + \cos 2t}\, dt$ **66.** $\displaystyle\int_{-\pi}^{0} \sqrt{1 + \cos t}\, dt$

67. $\displaystyle\int_{-\pi}^{0} \sqrt{1 - \cos^2 \theta}\, d\theta$ **68.** $\displaystyle\int_{\pi/2}^{\pi} \sqrt{1 - \sin^2 \theta}\, d\theta$

69. $\displaystyle\int_{-\pi/4}^{\pi/4} \sqrt{1 + \tan^2 y}\, dy$ **70.** $\displaystyle\int_{-\pi/4}^{0} \sqrt{\sec^2 y - 1}\, dy$

Assorted Integrations

Evaluate each integral in Exercises 71–82 by using any technique you think is appropriate.

71. $\displaystyle\int_{\pi/4}^{3\pi/4} (\csc x - \cot x)^2\, dx$ **72.** $\displaystyle\int_{0}^{\pi/4} (\sec x + 4\cos x)^2\, dx$

73. $\displaystyle\int \cos\theta \csc(\sin\theta)\, d\theta$ **74.** $\displaystyle\int \left(1 + \frac{1}{x}\right) \cot(x + \ln x)\, dx$

75. $\displaystyle\int (\csc x - \sec x)(\sin x + \cos x)\, dx$

76. $\displaystyle\int 3\sinh\left(\frac{x}{2} + \ln 5\right) dx$

77. $\displaystyle\int \frac{6\, dy}{\sqrt{y}(1 + y)}$ **78.** $\displaystyle\int \frac{dx}{x\sqrt{4x^2 - 1}}$

79. $\displaystyle\int \frac{7\, dx}{(x - 1)\sqrt{x^2 - 2x - 48}}$ **80.** $\displaystyle\int \frac{dx}{(2x + 1)\sqrt{4x^2 + 4x}}$

81. $\displaystyle\int \sec^2 t \tan(\tan t)\, dt$ **82.** $\displaystyle\int \frac{dx}{x\sqrt{3 + x^2}}$

Trigonometric Powers

83. **a.** Evaluate $\int \cos^3 \theta\, d\theta$. (*Hint:* $\cos^2\theta = 1 - \sin^2\theta$.)

b. Evaluate $\int \cos^5 \theta\, d\theta$.

c. Without actually evaluating the integral, explain how you would evaluate $\int \cos^9 \theta\, d\theta$.

84. **a.** Evaluate $\int \sin^3 \theta\, d\theta$. (*Hint:* $\sin^2\theta = 1 - \cos^2\theta$.)

b. Evaluate $\int \sin^5 \theta\, d\theta$.

c. Evaluate $\int \sin^7 \theta\, d\theta$.

d. Without actually evaluating the integral, explain how you would evaluate $\int \sin^{13} \theta\, d\theta$.

85. **a.** Express $\int \tan^3 \theta\, d\theta$ in terms of $\int \tan\theta\, d\theta$. Then evaluate $\int \tan^3\theta\, d\theta$. (*Hint:* $\tan^2\theta = \sec^2\theta - 1$.)

b. Express $\int \tan^5 \theta\, d\theta$ in terms of $\int \tan^3 \theta\, d\theta$.

c. Express $\int \tan^7 \theta\, d\theta$ in terms of $\int \tan^5 \theta\, d\theta$.

d. Express $\int \tan^{2k+1} \theta\, d\theta$, where k is a positive integer, in terms of $\int \tan^{2k-1} \theta\, d\theta$.

86. **a.** Express $\int \cot^3 \theta\, d\theta$ in terms of $\int \cot\theta\, d\theta$. Then evaluate $\int \cot^3\theta\, d\theta$. (*Hint:* $\cot^2\theta = \csc^2\theta - 1$.)

b. Express $\int \cot^5 \theta\, d\theta$ in terms of $\int \cot^3 \theta\, d\theta$.

c. Express $\int \cot^7 \theta\, d\theta$ in terms of $\int \cot^5 \theta\, d\theta$.

d. Express $\int \cot^{2k+1} \theta\, d\theta$, where k is a positive integer, in terms of $\int \cot^{2k-1} \theta\, d\theta$.

Theory and Examples

87. Area Find the area of the region bounded above by $y = 2\cos x$ and below by $y = \sec x$, $-\pi/4 \le x \le \pi/4$.

88. Area Find the area of the "triangular" region that is bounded from above and below by the curves $y = \csc x$ and $y = \sin x$, $\pi/6 \le x \le \pi/2$, and on the left by the line $x = \pi/6$.

89. Volume Find the volume of the solid generated by revolving the region in Exercise 87 about the x-axis.

90. Volume Find the volume of the solid generated by revolving the region in Exercise 88 about the x-axis.

91. Arc length Find the length of the curve $y = \ln(\cos x)$, $0 \le x \le \pi/3$.

92. Arc length Find the length of the curve $y = \ln(\sec x)$, $0 \le x \le \pi/4$.

93. Centroid Find the centroid of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

94. Centroid Find the centroid of the region that is bounded by the x-axis, the curve $y = \csc x$, and the lines $x = \pi/6$, $x = 5\pi/6$.

95. The integral of csc x Repeat the derivation in Example 7, using cofunctions, to show that

$$\int \csc x\, dx = -\ln|\csc x + \cot x| + C.$$

96. Using different substitutions Show that the integral

$$\int ((x^2 - 1)(x + 1))^{-2/3}\, dx$$

can be evaluated with any of the following substitutions.

a. $u = 1/(x + 1)$

b. $u = ((x - 1)/(x + 1))^k$ for $k = 1, 1/2, 1/3, -1/3, -2/3$, and -1

c. $u = \tan^{-1} x$

d. $u = \tan^{-1}\sqrt{x}$ **e.** $u = \tan^{-1}((x - 1)/2)$

f. $u = \cos^{-1} x$ **g.** $u = \cosh^{-1} x$

What is the value of the integral? (*Source:* "Problems and Solutions," *College Mathematics Journal*, Vol. 21, No. 5 (Nov. 1990), pp. 425–426.)

EXERCISES 8.2

Integration by Parts

Evaluate the integrals in Exercises 1–24.

1. $\int x \sin \frac{x}{2}\, dx$
2. $\int \theta \cos \pi\theta\, d\theta$
3. $\int t^2 \cos t\, dt$
4. $\int x^2 \sin x\, dx$
5. $\int_1^2 x \ln x\, dx$
6. $\int_1^e x^3 \ln x\, dx$
7. $\int \tan^{-1} y\, dy$
8. $\int \sin^{-1} y\, dy$
9. $\int x \sec^2 x\, dx$
10. $\int 4x \sec^2 2x\, dx$
11. $\int x^3 e^x\, dx$
12. $\int p^4 e^{-p}\, dp$
13. $\int (x^2 - 5x)e^x\, dx$
14. $\int (r^2 + r + 1)e^r\, dr$
15. $\int x^5 e^x\, dx$
16. $\int t^2 e^{4t}\, dt$
17. $\int_0^{\pi/2} \theta^2 \sin 2\theta\, d\theta$
18. $\int_0^{\pi/2} x^3 \cos 2x\, dx$
19. $\int_{2/\sqrt{3}}^{2} t \sec^{-1} t\, dt$
20. $\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2)\, dx$
21. $\int e^\theta \sin \theta\, d\theta$
22. $\int e^{-y} \cos y\, dy$
23. $\int e^{2x} \cos 3x\, dx$
24. $\int e^{-2x} \sin 2x\, dx$

Substitution and Integration by Parts

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}}\,ds$

26. $\int_0^1 x\sqrt{1-x}\,dx$

27. $\int_0^{\pi/3} x\tan^2 x\,dx$

28. $\int \ln(x+x^2)\,dx$

29. $\int \sin(\ln x)\,dx$

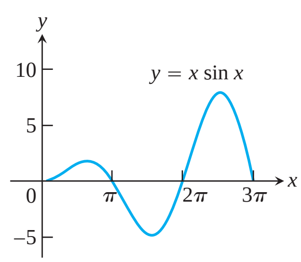
30. $\int z(\ln z)^2\,dz$

Theory and Examples

31. Finding area Find the area of the region enclosed by the curve $y = x\sin x$ and the x-axis (see the accompanying figure) for

a. $0 \le x \le \pi$ **b.** $\pi \le x \le 2\pi$ **c.** $2\pi \le x \le 3\pi$.

d. What pattern do you see here? What is the area between the curve and the x-axis for $n\pi \le x \le (n+1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.



32. Finding area Find the area of the region enclosed by the curve $y = x\cos x$ and the x-axis (see the accompanying figure) for

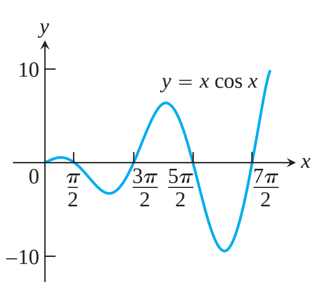
a. $\pi/2 \le x \le 3\pi/2$ **b.** $3\pi/2 \le x \le 5\pi/2$

c. $5\pi/2 \le x \le 7\pi/2$.

d. What pattern do you see? What is the area between the curve and the x-axis for

$$\left(\frac{2n-1}{2}\right)\pi \le x \le \left(\frac{2n+1}{2}\right)\pi,$$

n an arbitrary positive integer? Give reasons for your answer.



33. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

34. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$

a. about the y-axis. **b.** about the line $x = 1$.

35. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \le x \le \pi/2$, about

a. the y-axis. **b.** the line $x = \pi/2$.

36. Finding volume Find the volume of the solid generated by revolving the region bounded by the x-axis and the curve $y = x\sin x$, $0 \le x \le \pi$, about

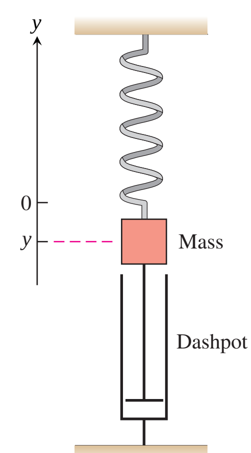
a. the y-axis. **b.** the line $x = \pi$.

(See Exercise 31 for a graph.)

37. Average value A retarding force, symbolized by the dashpot in the figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t}\cos t, \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.



38. Average value In a mass-spring-dashpot system like the one in Exercise 37, the mass's position at time t is

$$y = 4e^{-t}(\sin t - \cos t), \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.

Reduction Formulas

In Exercises 39–42, use integration by parts to establish the *reduction formula*.

39. $\int x^n \cos x\, dx = x^n \sin x - n \int x^{n-1} \sin x\, dx$

40. $\int x^n \sin x\, dx = -x^n \cos x + n \int x^{n-1} \cos x\, dx$

41. $\int x^n e^{ax}\, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax}\, dx, \quad a \neq 0$

42. $\int (\ln x)^n\, dx = x(\ln x)^n - n \int (\ln x)^{n-1}\, dx$

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\int f^{-1}(x)\, dx = \int y f'(y)\, dy \qquad \begin{aligned} &y = f^{-1}(x), \quad x = f(y) \\ &dx = f'(y)\, dy \end{aligned}$$

$$= y f(y) - \int f(y)\, dy \qquad \begin{aligned} &\text{Integration by parts with} \\ &u = y,\, dv = f'(y)\, dy \end{aligned}$$

$$= x f^{-1}(x) - \int f(y)\, dy$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\int \ln x\, dx = \int y e^y\, dy \qquad \begin{aligned} &y = \ln x, \quad x = e^y \\ &dx = e^y\, dy \end{aligned}$$

$$= y e^y - e^y + C$$

$$= x \ln x - x + C.$$

For the integral of $\cos^{-1} x$ we get

$$\int \cos^{-1} x\, dx = x \cos^{-1} x - \int \cos y\, dy \qquad y = \cos^{-1} x$$

$$= x \cos^{-1} x - \sin y + C$$

$$= x \cos^{-1} x - \sin(\cos^{-1} x) + C.$$

Use the formula

$$\int f^{-1}(x)\, dx = x f^{-1}(x) - \int f(y)\, dy \qquad y = f^{-1}(x) \qquad (4)$$

to evaluate the integrals in Exercises 43–46. Express your answers in terms of x.

43. $\int \sin^{-1} x\, dx$

44. $\int \tan^{-1} x\, dx$

45. $\int \sec^{-1} x\, dx$

46. $\int \log_2 x\, dx$

Another way to integrate $f^{-1}(x)$ (when f^{-1} is integrable, of course) is to use integration by parts with $u = f^{-1}(x)$ and $dv = dx$ to rewrite the integral of f^{-1} as

$$\int f^{-1}(x)\, dx = x f^{-1}(x) - \int x \left(\frac{d}{dx} f^{-1}(x) \right) dx. \qquad (5)$$

Exercises 47 and 48 compare the results of using Equations (4) and (5).

47. Equations (4) and (5) give different formulas for the integral of $\cos^{-1} x$:

a. $\int \cos^{-1} x\, dx = x \cos^{-1} x - \sin(\cos^{-1} x) + C$ — Eq. (4)

b. $\int \cos^{-1} x\, dx = x \cos^{-1} x - \sqrt{1 - x^2} + C$ — Eq. (5)

Can both integrations be correct? Explain.

48. Equations (4) and (5) lead to different formulas for the integral of $\tan^{-1} x$:

a. $\int \tan^{-1} x\, dx = x \tan^{-1} x - \ln \sec(\tan^{-1} x) + C$ — Eq. (4)

b. $\int \tan^{-1} x\, dx = x \tan^{-1} x - \ln \sqrt{1 + x^2} + C$ — Eq. (5)

Can both integrations be correct? Explain.

Evaluate the integrals in Exercises 49 and 50 with **(a)** Eq. (4) and **(b)** Eq. (5). In each case, check your work by differentiating your answer with respect to x.

49. $\int \sinh^{-1} x\, dx$

50. $\int \tanh^{-1} x\, dx$

EXERCISES 8.3

Expanding Quotients into Partial Fractions

Expand the quotients in Exercises 1–8 by partial fractions.

1. $\dfrac{5x - 13}{(x - 3)(x - 2)}$

2. $\dfrac{5x - 7}{x^2 - 3x + 2}$

3. $\dfrac{x + 4}{(x + 1)^2}$

4. $\dfrac{2x + 2}{x^2 - 2x + 1}$

5. $\dfrac{z + 1}{z^2(z - 1)}$

6. $\dfrac{z}{z^3 - z^2 - 6z}$

7. $\dfrac{t^2 + 8}{t^2 - 5t + 6}$

8. $\dfrac{t^4 + 9}{t^4 + 9t^2}$

Nonrepeated Linear Factors

In Exercises 9–16, express the integrands as a sum of partial fractions and evaluate the integrals.

9. $\displaystyle\int \frac{dx}{1 - x^2}$

10. $\displaystyle\int \frac{dx}{x^2 + 2x}$

11. $\displaystyle\int \frac{x + 4}{x^2 + 5x - 6}\,dx$

12. $\displaystyle\int \frac{2x + 1}{x^2 - 7x + 12}\,dx$

13. $\displaystyle\int_4^8 \frac{y\,dy}{y^2 - 2y - 3}$

14. $\displaystyle\int_{1/2}^1 \frac{y + 4}{y^2 + y}\,dy$

15. $\displaystyle\int \frac{dt}{t^3 + t^2 - 2t}$

16. $\displaystyle\int \frac{x + 3}{2x^3 - 8x}\,dx$

Repeated Linear Factors

In Exercises 17–20, express the integrands as a sum of partial fractions and evaluate the integrals.

17. $\displaystyle\int_0^1 \frac{x^3\,dx}{x^2 + 2x + 1}$

18. $\displaystyle\int_{-1}^0 \frac{x^3\,dx}{x^2 - 2x + 1}$

19. $\displaystyle\int \frac{dx}{(x^2 - 1)^2}$

20. $\displaystyle\int \frac{x^2\,dx}{(x - 1)(x^2 + 2x + 1)}$

Irreducible Quadratic Factors

In Exercises 21–28, express the integrands as a sum of partial fractions and evaluate the integrals.

21. $\displaystyle\int_0^1 \frac{dx}{(x + 1)(x^2 + 1)}$

22. $\displaystyle\int_1^{\sqrt{3}} \frac{3t^2 + t + 4}{t^3 + t}\,dt$

23. $\displaystyle\int \frac{y^2 + 2y + 1}{(y^2 + 1)^2}\,dy$

24. $\displaystyle\int \frac{8x^2 + 8x + 2}{(4x^2 + 1)^2}\,dx$

25. $\displaystyle\int \frac{2s + 2}{(s^2 + 1)(s - 1)^3}\,ds$

26. $\displaystyle\int \frac{s^4 + 81}{s(s^2 + 9)^2}\,ds$

27. $\displaystyle\int \frac{2\theta^3 + 5\theta^2 + 8\theta + 4}{(\theta^2 + 2\theta + 2)^2}\,d\theta$

28. $\displaystyle\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta^2 + 1)^3}\,d\theta$

Improper Fractions

In Exercises 29–34, perform long division on the integrand, write the proper fraction as a sum of partial fractions, and then evaluate the integral.

29. $\displaystyle\int \frac{2x^3 - 2x^2 + 1}{x^2 - x}\,dx$

30. $\displaystyle\int \frac{x^4}{x^2 - 1}\,dx$

31. $\displaystyle\int \frac{9x^3 - 3x + 1}{x^3 - x^2}\,dx$

32. $\displaystyle\int \frac{16x^3}{4x^2 - 4x + 1}\,dx$

33. $\displaystyle\int \frac{y^4 + y^2 - 1}{y^3 + y}\,dy$

34. $\displaystyle\int \frac{2y^4}{y^3 - y^2 + y - 1}\,dy$

Evaluating Integrals

Evaluate the integrals in Exercises 35–40.

35. $\displaystyle\int \frac{e^t\,dt}{e^{2t} + 3e^t + 2}$

36. $\displaystyle\int \frac{e^{4t} + 2e^{2t} - e^t}{e^{2t} + 1}\,dt$

37. $\displaystyle\int \frac{\cos y\,dy}{\sin^2 y + \sin y - 6}$

38. $\displaystyle\int \frac{\sin\theta\,d\theta}{\cos^2\theta + \cos\theta - 2}$

39. $\displaystyle\int \frac{(x-2)^2\tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2}\,dx$

40. $\displaystyle\int \frac{(x+1)^2\tan^{-1}(3x) + 9x^3 + x}{(9x^2+1)(x+1)^2}\,dx$

Initial Value Problems

Solve the initial value problems in Exercises 41–44 for x as a function of t.

41. $(t^2 - 3t + 2)\dfrac{dx}{dt} = 1 \quad (t > 2), \quad x(3) = 0$

42. $(3t^4 + 4t^2 + 1)\dfrac{dx}{dt} = 2\sqrt{3}, \quad x(1) = -\pi\sqrt{3}/4$

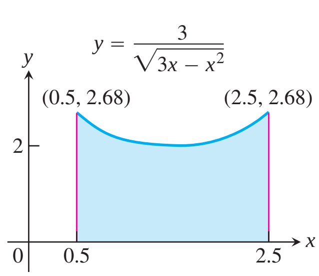
43. $(t^2 + 2t)\dfrac{dx}{dt} = 2x + 2 \quad (t, x > 0), \quad x(1) = 1$

44. $(t + 1)\dfrac{dx}{dt} = x^2 + 1 \quad (t > -1), \quad x(0) = \pi/4$

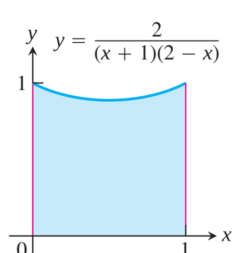
Applications and Examples

In Exercises 45 and 46, find the volume of the solid generated by revolving the shaded region about the indicated axis.

45. The x-axis

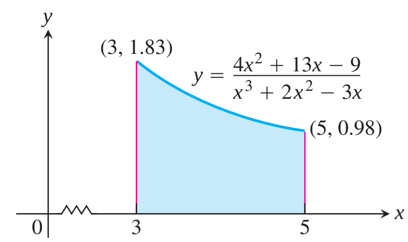


46. The y-axis



47. Find, to two decimal places, the x-coordinate of the centroid of the region in the first quadrant bounded by the x-axis, the curve $y = \tan^{-1} x$, and the line $x = \sqrt{3}$.

48. Find the x-coordinate of the centroid of this region to two decimal places.



49. Social diffusion Sociologists sometimes use the phrase "social diffusion" to describe the way information spreads through a population. The information might be a rumor, a cultural fad, or news about a technical innovation. In a sufficiently large population, the number of people x who have the information is treated as a differentiable function of time t, and the rate of diffusion, dx/dt, is assumed to be proportional to the number of people who have the information times the number of people who do not. This leads to the equation

$$\frac{dx}{dt} = kx(N - x),$$

where N is the number of people in the population.

Suppose t is in days, $k = 1/250$, and two people start a rumor at time $t = 0$ in a population of $N = 1000$ people.

a. Find x as a function of t.

b. When will half the population have heard the rumor? (This is when the rumor will be spreading the fastest.)

50. Second-order chemical reactions Many chemical reactions are the result of the interaction of two molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If a is the amount of substance A and b is the amount of substance B at time $t = 0$, and if x is the amount of product at time t, then the rate of formation of x may be given by the differential equation

$$\frac{dx}{dt} = k(a - x)(b - x),$$

or

$$\frac{1}{(a - x)(b - x)}\frac{dx}{dt} = k,$$

where k is a constant for the reaction. Integrate both sides of this equation to obtain a relation between x and t **(a)** if $a = b$, and **(b)** if $a \neq b$. Assume in each case that $x = 0$ when $t = 0$.

51. An integral connecting π to the approximation 22/7

a. Evaluate $\displaystyle\int_0^1 \frac{x^4(x-1)^4}{x^2+1}\,dx$.

b. How good is the approximation $\pi \approx 22/7$? Find out by expressing $\left(\dfrac{22}{7} - \pi\right)$ as a percentage of π.

c. Graph the function $y = \dfrac{x^4(x-1)^4}{x^2+1}$ for $0 \le x \le 1$. Experiment with the range on the y-axis set between 0 and 1, then between 0 and 0.5, and then decreasing the range until the graph can be seen. What do you conclude about the area under the curve?

52. Find the second-degree polynomial $P(x)$ such that $P(0) = 1$, $P'(0) = 0$, and

$$\int \frac{P(x)}{x^3(x-1)^2}\,dx$$

is a rational function.

EXERCISES 8.4

Products of Powers of Sines and Cosines

Evaluate the integrals in Exercises 1–14.

Exercises

1. $\displaystyle\int_0^{\pi/2} \sin^5 x\, dx$

2. $\displaystyle\int_0^{\pi} \sin^5 \frac{x}{2}\, dx$

3. $\displaystyle\int_{-\pi/2}^{\pi/2} \cos^3 x\, dx$

4. $\displaystyle\int_0^{\pi/6} 3\cos^5 3x\, dx$

5. $\displaystyle\int_0^{\pi/2} \sin^7 y\, dy$

6. $\displaystyle\int_0^{\pi/2} 7\cos^7 t\, dt$

7. $\displaystyle\int_0^{\pi} 8\sin^4 x\,dx$

8. $\displaystyle\int_0^{1} 8\cos^4 2\pi x\,dx$

9. $\displaystyle\int_{-\pi/4}^{\pi/4} 16\sin^2 x\cos^2 x\,dx$

10. $\displaystyle\int_0^{\pi} 8\sin^4 y\cos^2 y\,dy$

11. $\displaystyle\int_0^{\pi/2} 35\sin^4 x\cos^3 x\,dx$

12. $\displaystyle\int_0^{\pi} \sin 2x\cos^2 2x\,dx$

13. $\displaystyle\int_0^{\pi/4} 8\cos^3 2\theta\sin 2\theta\,d\theta$

14. $\displaystyle\int_0^{\pi/2} \sin^2 2\theta\cos^3 2\theta\,d\theta$

Integrals with Square Roots

Evaluate the integrals in Exercises 15–22.

15. $\displaystyle\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\,dx$

16. $\displaystyle\int_0^{\pi} \sqrt{1-\cos 2x}\,dx$

17. $\displaystyle\int_0^{\pi} \sqrt{1-\sin^2 t}\,dt$

18. $\displaystyle\int_0^{\pi} \sqrt{1-\cos^2\theta}\,d\theta$

19. $\displaystyle\int_{-\pi/4}^{\pi/4} \sqrt{1+\tan^2 x}\,dx$

20. $\displaystyle\int_{-\pi/4}^{\pi/4} \sqrt{\sec^2 x-1}\,dx$

21. $\displaystyle\int_0^{\pi/2} \theta\sqrt{1-\cos 2\theta}\,d\theta$

22. $\displaystyle\int_{-\pi}^{\pi} (1-\cos^2 t)^{3/2}\,dt$

Powers of Tan x and Sec x

Evaluate the integrals in Exercises 23–32.

23. $\displaystyle\int_{-\pi/3}^{0} 2\sec^3 x\,dx$

24. $\displaystyle\int e^x\sec^3 e^x\,dx$

25. $\displaystyle\int_0^{\pi/4} \sec^4\theta\,d\theta$

26. $\displaystyle\int_0^{\pi/12} 3\sec^4 3x\,dx$

27. $\displaystyle\int_{\pi/4}^{\pi/2} \csc^4\theta\,d\theta$

28. $\displaystyle\int_{\pi/2}^{\pi} 3\csc^4\frac{\theta}{2}\,d\theta$

29. $\displaystyle\int_0^{\pi/4} 4\tan^3 x\,dx$

30. $\displaystyle\int_{-\pi/4}^{\pi/4} 6\tan^4 x\,dx$

31. $\displaystyle\int_{\pi/6}^{\pi/3} \cot^3 x\,dx$

32. $\displaystyle\int_{\pi/4}^{\pi/2} 8\cot^4 t\,dt$

Products of Sines and Cosines

Evaluate the integrals in Exercises 33–38.

33. $\displaystyle\int_{-\pi}^{0} \sin 3x\cos 2x\,dx$

34. $\displaystyle\int_0^{\pi/2} \sin 2x\cos 3x\,dx$

35. $\displaystyle\int_{-\pi}^{\pi} \sin 3x\sin 3x\,dx$

36. $\displaystyle\int_0^{\pi/2} \sin x\cos x\,dx$

37. $\displaystyle\int_0^{\pi} \cos 3x\cos 4x\,dx$

38. $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x\cos 7x\,dx$

Theory and Examples

39. **Surface area** Find the area of the surface generated by revolving the arc

$$x = t^{2/3},\quad y = t^2/2,\quad 0 \le t \le 2,$$

about the x-axis.

40. **Arc length** Find the length of the curve

$$y = \ln(\cos x),\quad 0 \le x \le \pi/3.$$

41. **Arc length** Find the length of the curve

$$y = \ln(\sec x),\quad 0 \le x \le \pi/4.$$

42. **Center of gravity** Find the center of gravity of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

43. **Volume** Find the volume generated by revolving one arch of the curve $y = \sin x$ about the x-axis.

44. **Area** Find the area between the x-axis and the curve $y = \sqrt{1+\cos 4x}$, $0 \le x \le \pi$.

45. **Orthogonal functions** Two functions f and g are said to be **orthogonal** on an interval $a \le x \le b$ if $\int_a^b f(x)g(x)\,dx = 0$.

   a. Prove that $\sin mx$ and $\sin nx$ are orthogonal on any interval of length 2π provided m and n are integers such that $m^2 \ne n^2$.

   b. Prove the same for $\cos mx$ and $\cos nx$.

   c. Prove the same for $\sin mx$ and $\cos nx$ even if $m = n$.

46. **Fourier series** A finite Fourier series is given by the sum

$$\begin{aligned} f(x) &= \sum_{n=1}^{N} a_n \sin nx \\ &= a_1\sin x + a_2\sin 2x + \cdots + a_N\sin Nx \end{aligned}$$

Show that the mth coefficient a_m is given by the formula

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx.$$

EXERCISES 8.5

Basic Trigonometric Substitutions

Evaluate the integrals in Exercises 1–28.

1. $\displaystyle\int \frac{dy}{\sqrt{9+y^2}}$

2. $\displaystyle\int \frac{3\,dy}{\sqrt{1+9y^2}}$

3. $\displaystyle\int_{-2}^{2} \frac{dx}{4+x^2}$

4. $\displaystyle\int_{0}^{2} \frac{dx}{8+2x^2}$

5. $\displaystyle\int_{0}^{3/2} \frac{dx}{\sqrt{9-x^2}}$

6. $\displaystyle\int_{0}^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$

7. $\displaystyle\int \sqrt{25-t^2}\,dt$

8. $\displaystyle\int \sqrt{1-9t^2}\,dt$

9. $\displaystyle\int \frac{dx}{\sqrt{4x^2-49}}, \quad x > \frac{7}{2}$

10. $\displaystyle\int \frac{5\,dx}{\sqrt{25x^2-9}}, \quad x > \frac{3}{5}$

11. $\displaystyle\int \frac{\sqrt{y^2-49}}{y}\,dy, \quad y > 7$

12. $\displaystyle\int \frac{\sqrt{y^2-25}}{y^3}\,dy, \quad y > 5$

13. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-1}}, \quad x > 1$

14. $\displaystyle\int \frac{2\,dx}{x^3\sqrt{x^2-1}}, \quad x > 1$

15. $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+4}}$

16. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}}$

17. $\displaystyle\int \frac{8\,dw}{w^2\sqrt{4-w^2}}$

18. $\displaystyle\int \frac{\sqrt{9-w^2}}{w^2}\,dw$

19. $\displaystyle\int_{0}^{\sqrt{3}/2} \frac{4x^2\,dx}{(1-x^2)^{3/2}}$

20. $\displaystyle\int_{0}^{1} \frac{dx}{(4-x^2)^{3/2}}$

21. $\displaystyle\int \frac{dx}{(x^2-1)^{3/2}}, \quad x > 1$

22. $\displaystyle\int \frac{x^2\,dx}{(x^2-1)^{5/2}}, \quad x > 1$

23. $\displaystyle\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$

24. $\displaystyle\int \frac{(1-x^2)^{1/2}}{x^4}\,dx$

25. $\displaystyle\int \frac{8\,dx}{(4x^2+1)^2}$

26. $\displaystyle\int \frac{6\,dt}{(9t^2+1)^2}$

27. $\displaystyle\int \frac{v^2\,dv}{(1-v^2)^{5/2}}$

28. $\displaystyle\int \frac{(1-r^2)^{5/2}}{r^8}\,dr$

In Exercises 29–36, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

29. $\displaystyle\int_{0}^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}}$

30. $\displaystyle\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1+e^{2t})^{3/2}}$

31. $\displaystyle\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}+4t\sqrt{t}}$

32. $\displaystyle\int_{1}^{e} \frac{dy}{y\sqrt{1+(\ln y)^2}}$

33. $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$

34. $\displaystyle\int \frac{dx}{1+x^2}$

35. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-1}}$

36. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$

Initial Value Problems

Solve the initial value problems in Exercises 37–40 for y as a function of x.

37. $\displaystyle x\frac{dy}{dx} = \sqrt{x^2-4}, \quad x \geq 2, \quad y(2) = 0$

38. $\displaystyle \sqrt{x^2-9}\,\frac{dy}{dx} = 1, \quad x > 3, \quad y(5) = \ln 3$

39. $\displaystyle (x^2+4)\frac{dy}{dx} = 3, \quad y(2) = 0$

40. $\displaystyle (x^2+1)^2\frac{dy}{dx} = \sqrt{x^2+1}, \quad y(0) = 1$

Applications

41. Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve $y = \sqrt{9 - x^2/3}$.

42. Find the volume of the solid generated by revolving about the x-axis the region in the first quadrant enclosed by the coordinate axes, the curve $y = 2/(1 + x^2)$, and the line $x = 1$.

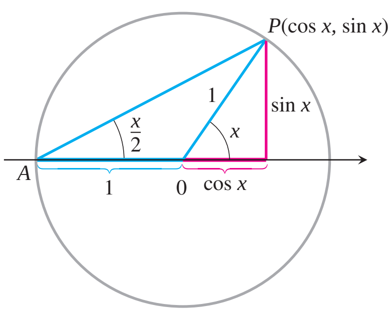
The Substitution $z = \tan(x/2)$

The substitution

$$z = \tan\frac{x}{2} \tag{1}$$

reduces the problem of integrating a rational expression in $\sin x$ and $\cos x$ to a problem of integrating a rational function of z. This in turn can be integrated by partial fractions.

From the accompanying figure



we can read the relation

$$\tan\frac{x}{2} = \frac{\sin x}{1 + \cos x}.$$

To see the effect of the substitution, we calculate

$$\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1 = \frac{2}{\sec^2(x/2)} - 1$$

$$= \frac{2}{1 + \tan^2(x/2)} - 1 = \frac{2}{1 + z^2} - 1$$

$$\cos x = \frac{1 - z^2}{1 + z^2}, \tag{2}$$

and

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{\sin(x/2)}{\cos(x/2)}\cdot\cos^2\left(\frac{x}{2}\right)$$

$$= 2\tan\frac{x}{2}\cdot\frac{1}{\sec^2(x/2)} = \frac{2\tan(x/2)}{1 + \tan^2(x/2)}$$

$$\sin x = \frac{2z}{1 + z^2}. \tag{3}$$

Finally, $x = 2\tan^{-1} z$, so

$$dx = \frac{2\,dz}{1 + z^2}. \tag{4}$$

Examples

a.
$$\int \frac{1}{1 + \cos x}\,dx = \int \frac{1 + z^2}{2}\,\frac{2\,dz}{1 + z^2}$$

$$= \int dz = z + C$$

$$= \tan\left(\frac{x}{2}\right) + C$$

b.
$$\int \frac{1}{2 + \sin x}\,dx = \int \frac{1 + z^2}{2 + 2z + 2z^2}\,\frac{2\,dz}{1 + z^2}$$

$$= \int \frac{dz}{z^2 + z + 1} = \int \frac{dz}{(z + (1/2))^2 + 3/4}$$

$$= \int \frac{du}{u^2 + a^2}$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2z + 1}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{1 + 2\tan(x/2)}{\sqrt{3}} + C$$

Use the substitutions in Equations (1)–(4) to evaluate the integrals in Exercises 43–50. Integrals like these arise in calculating the average angular velocity of the output shaft of a universal joint when the input and output shafts are not aligned.

43. $\displaystyle\int \frac{dx}{1 - \sin x}$

44. $\displaystyle\int \frac{dx}{1 + \sin x + \cos x}$

45. $\displaystyle\int_0^{\pi/2} \frac{dx}{1 + \sin x}$

46. $\displaystyle\int_{\pi/3}^{\pi/2} \frac{dx}{1 - \cos x}$

47. $\displaystyle\int_0^{\pi/2} \frac{d\theta}{2 + \cos\theta}$

48. $\displaystyle\int_{\pi/2}^{2\pi/3} \frac{\cos\theta\,d\theta}{\sin\theta\cos\theta + \sin\theta}$

49. $\displaystyle\int \frac{dt}{\sin t - \cos t}$

50. $\displaystyle\int \frac{\cos t\,dt}{1 - \cos t}$

Use the substitution $z = \tan(\theta/2)$ to evaluate the integrals in Exercises 51 and 52.

51. $\displaystyle\int \sec\theta\,d\theta$

52. $\displaystyle\int \csc\theta\,d\theta$