MA 1201 Semester B 2020/21

Assignment 2 — Due at 11:59 pm, 18/3/2021 (Thursday) online on Canvas

Instructions:

- Please show your work. Unsupported answers will receive **NO** credits.
- Make sure you write down the correct lecture session (A/B/C/D/E/F/G/H) you have registered for, together with your full name and student ID on the front page of your answer script. Scan your solution into a single pdf file and upload it to Canvas.
- **NO** late homework will be accepted. Homework submitted to wrong tutorial sessions will **NOT** be graded and will receive **0 POINTS**.

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1. (50 points) Evaluate the following integrals.

(a) (10 points) $\int \frac{\tan^{-1}\sqrt{x}}{\sqrt{x}(1+x)}\,dx$.

Solution. Let $u=\sqrt{x}$. Then $dx = d(u^2) = 2udu = 2\sqrt{x}du$. So

$$\int \frac{\tan^{-1}\sqrt{x}}{\sqrt{x}(1+x)}\,dx = 2\int \frac{\tan^{-1}u}{1+u^2}du = 2\int \tan^{-1} u d(\tan^{-1}u) = 2(\tan^{-1}u)^2 - 2\int \frac{\tan^{-1}u}{1+u^2}du.$$

So

$$\int \frac{\tan^{-1}\sqrt{x}}{\sqrt{x}(1+x)}\,dx = 2\int \frac{\tan^{-1}u}{1+u^2}du = (\tan^{-1}u)^2 + C = (\tan^{-1}\sqrt{x})^2 + C.$$

(b) (10 points) $\int_{\ln 2}^{\ln 3} \frac{dx}{e^x - e^{-x}}$.

Solution. Let $u = e^x$. Then $du = d(e^x) = e^x dx = udx$. So

$$\int_{\ln 2}^{\ln 3} \frac{dx}{e^x - e^{-x}} = \int_2^3 \frac{du}{(u-u^{-1})u} = \int_2^3 \frac{du}{(u-1)(u+1)} = \int_2^3 \frac{du}{2(u-1)} - \int_2^3 \frac{du}{2(u+1)}.$$

So

$$\int_{\ln 2}^{\ln 3} \frac{dx}{e^x - e^{-x}} = \frac{1}{2}\ln|u-1|\Big|_2^3 - \frac{1}{2}\ln|u+1|\Big|_2^3 = \frac{1}{2}\ln(\frac{3}{2}).$$

(c). (15 points) (15 points) Find the induction formula of the integration $I_n = \int \frac{x^n}{\sqrt{1-x^2}}\,dx$.

Solution. Let $x = \sin\theta$. Then $dx = d(\sin\theta) = \cos\theta d\theta$. So

$$I_n = \int \frac{x^n}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^n\theta}{\cos\theta}\cos\theta d\theta = \int \sin^n\theta d\theta$$

By the integration by part method,

$$I_n = \int \sin^{n-1}\theta d(-\cos\theta) = -\cos\theta\sin^{n-1}\theta + \int \cos\theta d(\sin^{n-1}\theta) = -\cos\theta\sin^{n-1}\theta + (n-1)\int \cos^2\theta\sin^{n-2}\theta d\theta$$

So

$$I_n = -\cos\theta \sin^{n-1}\theta + (n-1)\int (1-\sin^2\theta)\sin^{n-2}\theta d\theta = -\cos\theta\sin^{n-1}\theta + (n-1)I_{n-2} - (n-1)I_n.$$

Note that $\sin\theta = x$ and $\cos\theta = \sqrt{1-x^2}$. So

$$I_n = \frac{n-1}{n} I_{n-2} - \frac{1}{n} x^{n-1}\sqrt{1-x^2}.$$

(d). (15 points) $\int \frac{3x^2+3x-1}{(x-1)(x^2+2x+2)}\,dx.$

Solution. Note that

$$\frac{3x^2+3x-1}{(x-1)(x^2+2x+2)} = \frac{2x+3}{x^2+2x+2} + \frac{1}{x-1}.$$

So

$$\int \frac{3x^2+3x-1}{(x-1)(x^2+2x+2)}dx = \int \frac{2x+3}{x^2+2x+2}dx + \int \frac{1}{x-1}dx = \int \frac{d(x^2+2x+2)}{x^2+2x+2}dx + \int \frac{dx}{(x+1)^2+1} + \int \frac{1}{x-1}dx.$$

Then

$$\int \frac{3x^2+3x-1}{(x-1)(x^2+2x+2)}dx = \ln(x^2+2x+2) + \tan^{-1}(x+1) + \ln|x-1| + C.$$

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2. (15 points) Find the volume of the solid generated by revolving the region bounded by the curves $y = \sin x,\ y = 0,\ 0 \le x \le \pi$,

(a) (5 points) about the x-axis.

Solution. By the disk method, the volume $V = \int_0^\pi \pi \sin^2 x dx = \frac{\pi}{2}\int_0^\pi (1-\cos 2x)dx = \frac{\pi^2}{2}$.

(b) (5 points) about the y-axis.

Solution. By the shell method, the volume

$$V = \int_0^\pi 2\pi x \sin x dx = 2\pi \int_0^\pi x d(-\cos x) = -2\pi x\cos x\Big|_0^\pi + 2\pi\int_0^\pi \cos x dx = 2\pi^2.$$

(c) (5 points) about the line $x = 2\pi$.

Solution. By the shell method, the volume

$$V = \int_0^\pi 2\pi(2\pi - x)\sin x dx = 2\pi\int_0^\pi (2\pi - x)d(-\cos x) = -2\pi(2\pi - x)\cos x\Big|_0^\pi - 2\pi\int_0^\pi \cos x dx = 6\pi^2.$$

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3. (20 points) Let S_1 be the area of the region bounded by the curve $y = x^2$ and the lines $y = ax, (0 < a < 1)$, and let S_1 be the area of the region bounded by the curve $y = x^2$, the lines $y = ax, (0 < a < 1)$ and the line $x = 1$.

(a) (10 points) Find the value of a that minimizes the value of S_1+S_2. What is the minimal value?

Solution. For any $a \in (0,1)$, the intersecting point of $y=x^2$ and $y=ax$ is $y=x^2=ax$. That is, $x=0$ or $x=a$. So $S_1=\int_0^a (ax-x^2)dx=\frac{1}{6}a^3$. $S_2=\int_a^1 (x^2-ax)dx=\frac{1}{6}(a-1)^2(a+2)=\frac{1}{6}(a^3-3a+2)$. So

$$S_1+S_2=\frac{1}{6}(2a^3-3a+2).$$

Note that $(S_1+S_2)'(a)=\frac{1}{2}(2a^2-1)$. So S_1+S_2 takes the minimum when $a=\frac{\sqrt{2}}{2}$. And the minimal value is $\frac{1}{6}(2-\sqrt{2})$.

(b) (10 points) Find the volume of the solid generated by revolving the region, that takes the minimum value of S_1+S_2, about the x-axis.

Solution. By the disk method, the volume of the solid for any $a \in (0,1)$ is

$$V=\int_0^a (\pi(ax)^2-\pi(x^2)^2)dx+\int_a^1 (\pi(x^4)-\pi(ax)^2)dx=\frac{4\pi}{15}a^5-\frac{\pi}{3}a^2+\frac{\pi}{5}.$$

So when $a=\frac{\sqrt{2}}{2}$, $V=\frac{\sqrt{2}\pi}{30}+\frac{\pi}{30}$.

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4. (15 points) Find the length of the curve $y=\ln\cos x,\ 0\le x\le a<\frac{\pi}{2}$.

Solution. Note that $\frac{dy}{dx}=\frac{d\ln\cos x}{d(\cos x)}\frac{d\cos x}{dx}=-tanx$. Then the arclength is

$$s=\int_0^a \sqrt{1+\tan^2 x}dx=\int_0^a \sec x dx=\ln|\sec x+\tan x|\Big|_0^a=\ln(\sec a+\tan a).$$

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