Chapter 5. Line and Surface Integrals

1 Mathematical Representation of Lines

Question 1: What is a line in 3-dimensional (2-dimensional) space?

Question 2: How to represent it in mathematics?

1.1 Parameterization of lines

Example Parameterize the following curves:

(1) The parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.

(2) The upper half of the unit circle $x^2 + y^2 = 1$.

(3) The line segment from $(-1, 5, 0)$ to $(1, 6, 4)$.

(4) The intersection of $x^2 + y^2 = 1$ and $y + z = 2$.

1.2 Tangent Vector of Curves

Definition Given a curve C with a parametric equation

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k},$$

its tangent vector at $P_0 = \vec{r}(t_0)$ is

$$\vec{r}\,'\left(t_0\right) = x'\left(t_0\right)\vec{i} + y'\left(t_0\right)\vec{j} + z'\left(t_0\right)\vec{k} = \lim_{t \to t_0} \frac{\vec{r}(t) - \vec{r}(t_0)}{t - t_0}.$$

Example Find the tangent vector of $\vec{r}(t) = \left(1 + t^2\right)\vec{i} + \left(te^{-t}\right)\vec{j} + \sin t\vec{k}$ at the point $t = 0$.

Physical Interpretation of Tangent Vector

Let $\vec{r}(t)$ represent the position of a particle A at time t. Then

(1) $\vec{r}\,'(t)$ represents the velocity vector of A at time t.

(2) $|\vec{r}\,'(t)|$ represents the speed of A at time t.

(3) $\vec{r}\,''(t)$ represents the acceleration vector of A at time t.

(4) $|\vec{r}\,''(t)|$ represents the acceleration magnitude of A at time t.

Example $\vec{r}(t) = (1 + t^2)\,\vec{i} + (te^{-t})\,\vec{j} + \sin t\ \vec{k}$, find the speed at $t = 0$.

2 Line Integral

Definition:

Physical Interpretation Let $f(x, y, z)$ be the point density of a thin-wire shaped curve C. Then

- $\int_C f(x, y, z)dC$ is the mass of the wire.
- $(\int_C xf(x, y, z)dC, \int_C yf(x, y, z)dC, \int_C zf(x, y, z)dC)$ is the center of mass of the wire.

Example Evaluate $\int_C y \sin z \, dC$, where C is the circular helix given by the equations

$$\vec{r}(t) = \cos t \, \vec{i} + \sin t \, \vec{j} + t \, \vec{k}, \, 0 \leq t \leq 2\pi.$$

Example $f(x, y) = 2 + x^2y$, C is the upper half of the unit circle $x^2 + y^2 = 1$. Evaluate $\int_C f \, dC$

Example Evaluate $\int_C 2x \ dC$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by the vertical line segment C_2 from $(1, 1)$ to $(1, 2)$.

3 Line integral of 2nd kind

Definition:

Example Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$ and C is the twisted cubic given by $x = t, y = t^2, z = t^3, 0 \leq t \leq 1$.