CITY UNIVERSITY OF HONG KONG

Course code and title	:	MA2001 Multi-variable Calculus and Linear Algebra
Session	:	Semester A, 2020/2021
Time allowed	:	Two hours

This paper has **THREE** pages (including this cover page).

Instructions to candidates:

1. Attempt **ALL** questions in this paper.
2. The total mark of this paper is **110** marks.
3. The maximum obtainable mark is **100** marks.
4. Start each main question on a new page.
5. Show all steps.
6. *This is a **closed-book** examination.*
7. MA department hotline: 3442 8646

Candidates are allowed to use the following materials/aids:

Non-programmable portable battery operated calculator

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorised materials or aids are found on them.

<u>Question 1</u> (15 marks)

Let $\vec{F}(x, y, z) = (2xz + y^2)\mathbf{i} + 2xy\mathbf{j} + (x^2 + 3z^2)\mathbf{k}$ be a force field.

(a) Is $\vec{F}$ conservative? Explain. If so, determine a potential function φ on $\mathbb{R}^3$ such that $\nabla\varphi = \vec{F}$ and $\varphi(0, 0, 0) = 0$. (10 marks)

(b) Find the work done by the force field $\vec{F}$ on a particle that moves from $(0, 1, -1)$ to $(1, 2, 1)$ along the intersection of $x = (y - 1)^2$ and $2y - z = 3$. (5 marks)

<u>Question 2</u> (10 marks)

Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y, z) = x\mathbf{i} - z\mathbf{j} + y\mathbf{k}$ and C is given by $\vec{r}(t) = 2t\mathbf{i} + 3t\mathbf{j} - t^2\mathbf{k}$, $-1 \le t \le 1$.

<u>Question 3</u> (15 marks)

Calculate the flux of a field $\vec{F}$ across S , where

$$\vec{F}(x, y, z) = (2\cos z + y^2)\mathbf{i} + (xe^{-z} + x^2y)\mathbf{j} + (\sin y + x^2z)\mathbf{k} ,$$

and S is the surface of a solid bounded by the spheres $x^2 + y^2 + z^2 = 9$, $x^2 + y^2 + z^2 = 16$ and $y \ge 0$. [Hint: Use the Divergence Theorem]

<u>Question 4</u> (10 marks)

(a) Show that any vector field of the form

$$\vec{F}(x, y, z) = f(y, z)\mathbf{i} + g(x, z)\mathbf{j} + h(x, y)\mathbf{k}$$

is solenoidal (incompressible). (5 marks)

(b) Determine if there is a vector field $\vec{G}$ on $\mathbb{R}^3$ such that

$$\text{curl } \vec{G} = x \sin y\mathbf{i} + \cos y\mathbf{j} + (2z - xy)\mathbf{k} .$$

Explain. (5 marks)

Question 5 (12 marks)

Suppose $\begin{cases} x^2 + y^2 = \frac{1}{3} z^2 \\ x + y + z = 5 \end{cases}$

are uniquely solved for x, y as a function of z near $x = 1,\ y = 0$.

Find $\frac{dx}{dz}$, $\frac{dy}{dz}$, $\frac{d^2x}{dz^2}$, $\frac{d^2y}{dz^2}$ when $x = 1$, $y = 0$, $z = 3$.

Question 6 (23 marks)

(a) Change the order of integration and evaluate $\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y} \, dy \, dx$. (8 marks)

(b) Evaluate $\iiint_v xyz \, dx \, dy \, dz$ over the region v bounded by the planes $x = 0,\ y = 0,\ z = 0,\ z = 1$ and the cylinder $x^2 + y^2 = 1$. (15 marks)

Question 7 (25 marks)

Consider a quadratic form $Q = 3{x_1}^2 + 3{x_2}^2 + 3{x_3}^2 + 6x_1x_2 + 6x_2x_3 + 6x_1x_3$.

(a) Express $Q = \underline{x}^T A \underline{x}$, where A is a symmetric matrix and $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

(b) Find the eigenvalues of A and the corresponding eigenvectors.

(c) Determine the nature of the quadratic form Q .

(d) Find a real orthogonal matrix P such that $\underline{x} = P\underline{y}$, where $\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

(e) Transform Q to a sum of squares terms of y_1, y_2, y_3 .

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