Sandwich Theorem (or Squeeze Theorem)

Let I be an interval containing c. Suppose that for every $x \in I$ with $x \neq c$, we have

$$\boxed{f(x) \leq g(x) \leq h(x)}.$$

Furthermore, suppose that $\boxed{\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L}$.

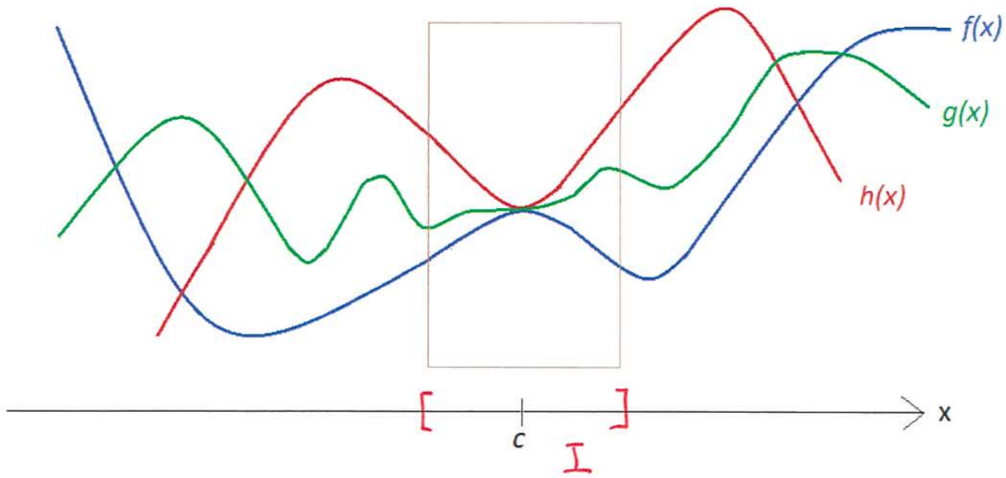
(That is, $f(x)$ and $h(x)$ approach the **same limit** L as x approaches c.)

Then $\boxed{\lim_{x \to c} g(x) = L}$.

$f(x) \leq g(x) \leq h(x)$

$\downarrow \quad \downarrow \quad \downarrow$

$L \quad L \quad L$



The functions $f(x)$ and $h(x)$ are called the **lower** and **upper bounds**, respectively, of $g(x)$.

Example 10

If $2 - x^2 \leq g(x) \leq 2\cos x$ for all $x \in \mathbb{R}$, find $\lim_{x \to 0} g(x)$.

Solution

Limit of lower bound: $\lim_{x \to 0}(2 - x^2) = 2 - 0^2 = 2$.

Limit of upper bound: $\lim_{x \to 0} 2\cos x = 2\underbrace{\cos 0}_{=1} = 2$.

$\because \lim_{x \to 0}(2 - x^2) = \lim_{x \to 0} 2\cos x = 2$

$\therefore$ By the Sandwich (or Squeeze) Theorem,

$$\lim_{x \to 0} g(x) = 2.$$

Example 11

Evaluate the limit $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$.

Solution

First note that the function $x^2 \sin\left(\frac{1}{x}\right)$ is not defined at $x = 0$.

For any $x \neq 0$, we know that $-1 \le \sin\left(\frac{1}{x}\right) \le 1$. lower bound

upper bound

Multiplying both sides by $\overset{>0}{x^2}$, we get $-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$.

Limit of lower bound: $\lim_{x\to 0}(-x^2) = -0^2 = 0$

Limit of upper bound: $\lim_{x\to 0} x^2 = 0^2 = 0$

Since $\lim_{x\to 0}(-x^2) = \lim_{x\to 0} x^2 = 0$, by the **Sandwich Theorem**, we have

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

Example:

Evaluate $\lim_{x \to 0} x \sin(\frac{1}{x})$.

Note: It is not correct to say that

$$-x \leq x \sin(\tfrac{1}{x}) \leq x$$

because x could be positive or negative.

Solution:

For any $x \neq 0$,

$$-1 \leq \sin(\tfrac{1}{x}) \leq 1.$$

Then $$-|x| \leq x \sin(\tfrac{1}{x}) \leq |x|.$$

This is always true no matter x is positive or negative.

$$\lim_{x \to 0} -|x| = -|0| = 0 = \lim_{x \to 0} |x|$$

$\therefore$ By the Sandwich Theorem,

$$\lim_{x \to 0} x \sin(\tfrac{1}{x}) = 0$$

Example 12

Show that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ (where θ is in radians) by using the Sandwich Theorem.

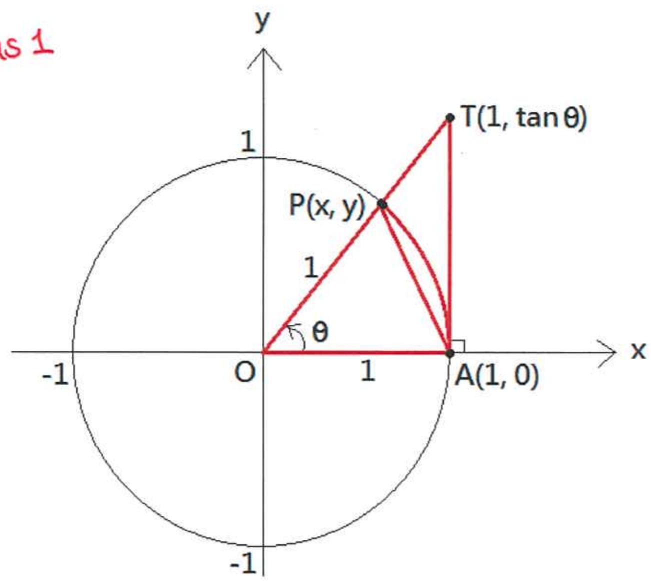
Solution

First note that the function $\frac{\sin\theta}{\theta}$ is not defined at $\theta = 0$. ($\frac{\sin\theta}{\theta}$ is of the $\frac{0}{0}$ **form** at $\theta = 0$.)

We want to show that $\lim_{\theta \to 0^-} \frac{\sin\theta}{\theta} = \lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$.

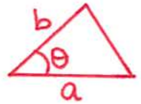
radius 1

To evaluate the right hand limit $\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta}$, we consider a unit circle centered at the origin. Let $P(x, y)$ be a point on the circle in the first quadrant, and θ be the angle (in **radians**) measured from the positive x-axis to the line segment OP. Since $P(x, y)$ lies in the first quadrant, we have $0 < \theta < \frac{\pi}{2}$.



From the diagram, we see that

Area of $\Delta = \frac{1}{2}ab\sin\theta$



$$\text{Area of } \Delta\text{OAP} < \text{Area of sector OAP} < \text{Area of } \Delta\text{OAT}.$$

$$\text{Area of } \Delta\text{OAP} = \frac{1}{2}\cdot(OP)\cdot(OA)\cdot\sin\theta = \frac{1}{2}\cdot 1\cdot 1\cdot\sin\theta = \frac{\sin\theta}{2}$$

Area of circle

$$\text{Area of sector OAP} = \boxed{\pi r^2}\cdot\boxed{\frac{\theta}{2\pi}} = \pi\cdot 1^2\cdot\frac{\theta}{2\pi} = \frac{\theta}{2}$$

proportion of the sector within the circle (θ is measured in radians)

$$\text{Area of } \Delta\text{OAT} = \frac{(OA)\cdot(AT)}{2} = \frac{(1)\cdot(\tan\theta)}{2} = \frac{\tan\theta}{2}$$

Therefore, $\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\tan\theta}{2}$, i.e. $\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\sin\theta}{2\cos\theta}$.

Dividing both sides by $\frac{\sin\theta}{2}$, we get

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Taking reciprocal on both sides, we get

lower bound upper bound

$$1 > \frac{\sin\theta}{\theta} > \frac{\cos\theta}{1}, \quad \text{i.e. } \cos\theta < \frac{\sin\theta}{\theta} < 1.$$

Since $\lim_{\theta\to 0^+}\cos\theta = \cos 0 = 1$ and $\lim_{\theta\to 0^+} 1 = 1$, by the **Sandwich Theorem**, we have

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1.$$

To evaluate the left hand limit $\lim\limits_{\theta\to 0^-}\dfrac{\sin\theta}{\theta}$ (where $\theta < 0$), we let $\theta = -\alpha$ where $\alpha > 0$. Then

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = \lim_{(-\alpha)\to 0^-}\frac{\sin(-\alpha)}{-\alpha} \overset{\text{sin is odd}}{=} \lim_{\alpha\to 0^+}\frac{-\sin\alpha}{-\alpha} = \lim_{\alpha\to 0^+}\frac{\sin\alpha}{\alpha} = 1 \text{ (from the above result).}$$

Since $\lim\limits_{\theta\to 0^+}\dfrac{\sin\theta}{\theta} = \lim\limits_{\theta\to 0^-}\dfrac{\sin\theta}{\theta} = 1$, we have $\boxed{\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1}$. □

$-\alpha \rightarrow 0 \leftarrow \alpha$

Useful result:

$$\boxed{\lim_{x\to 0}\frac{\sin x}{x}=1}.$$

It follows that

➢ $\boxed{\lim_{x\to 0}\frac{x}{\sin x}=1}$

It is because $\lim_{x\to 0}\frac{x}{\sin x}=\lim_{x\to 0}\frac{1}{\frac{\sin x}{x}}=\frac{1}{\lim_{x\to 0}\frac{\sin x}{x}}=\frac{1}{1}=1.$

➢ $\boxed{\lim_{x\to 0}\frac{\sin(cx)}{cx}=1}$, where c is a non-zero constant.

It is because $\lim_{x\to 0}\frac{\sin(cx)}{cx}=\lim_{(cx)\to 0}\frac{\sin(cx)}{cx}=1.$

➢ $\boxed{\lim_{x\to 0}\frac{\sin^n x}{x^n}=1}$, where n is an integer.

It is because $\lim_{x\to 0}\frac{\sin^n x}{x^n}=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^n=1^n=1.$

Note:

* $\lim_{x \to 0} \frac{\sin x}{x} = 1$

* $\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{\cos x} \right) = 1 \cdot \frac{1}{\underbrace{\boxed{\cos 0}}_{=1}} = 1$

* $\lim_{x \to 0} \frac{\cos x}{x} = \frac{1}{0}$ which is undefined.

$\therefore$ The limit does not exist.

Example 13

Evaluate the following limits, if they exist.

(a) $\lim\limits_{x\to 0}\dfrac{\sin 2x}{\sin 5x}$ ($\frac{0}{0}$ form) (b) $\lim\limits_{x\to 0}\dfrac{\tan^2(3x)}{5x^2}$ ($\frac{0}{0}$ form) (c) $\lim\limits_{x\to \frac{\pi}{2}}\dfrac{\sin\left(x-\frac{\pi}{2}\right)}{2x-\pi}$ ($\frac{0}{0}$ form)

Solution

(a) $$\lim_{x\to 0}\frac{\sin 2x}{\sin 5x}=\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\frac{5x}{\sin 5x}\cdot\frac{2\cancel{x}}{5\cancel{x}}=\frac{2}{5}\underbrace{\left(\lim_{x\to 0}\frac{\sin 2x}{2x}\right)}_{=1}\cdot\underbrace{\left(\lim_{x\to 0}\frac{5x}{\sin 5x}\right)}_{=1}=\frac{2}{5}\cdot 1\cdot 1=\frac{2}{5}$$

(b) $$\lim_{x\to 0}\frac{\tan^2(3x)}{5x^2}=\lim_{x\to 0}\frac{\sin^2(3x)}{\cos^2(3x)}\cdot\frac{1}{5x^2}=\lim_{x\to 0}\frac{\sin^2(3x)}{(3x)^2}\cdot\frac{1}{\cos^2(3x)}\cdot\frac{1}{5\cancel{x^2}}\cdot(3\cancel{x})^2$$

$$=\frac{9}{5}\cdot\underbrace{\left(\lim_{x\to 0}\frac{\sin 3x}{3x}\right)^2}_{=1^2=1}\cdot\underbrace{\left(\lim_{x\to 0}\frac{1}{\cos^2(3x)}\right)}_{=\frac{1}{\cos^2 0}=\frac{1}{1^2}=1}=\frac{9}{5}\cdot 1^2\cdot 1=\frac{9}{5}$$

(c) $$\lim_{x\to\frac{\pi}{2}}\frac{\sin\left(x-\frac{\pi}{2}\right)}{2x-\pi}=\lim_{x-\frac{\pi}{2}\to 0}\frac{\sin\left(x-\frac{\pi}{2}\right)}{2\left(x-\frac{\pi}{2}\right)}\underbrace{=}_{\text{put }\theta=x-\frac{\pi}{2}}\lim_{\theta\to 0}\frac{\sin\theta}{2\theta}=\frac{1}{2}\left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)=\frac{1}{2}\cdot 1=\frac{1}{2}$$

Example 14

Do the limits (a) $\lim_{x\to 0}\frac{\sin x}{|x|}$ and (b) $\lim_{x\to 0}\frac{|\sin x|}{x}$ exist?

Solution

(a) First note that $\frac{\sin x}{|x|}$ is not defined when $x = 0$. Recall that $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.

Then $\frac{\sin x}{|x|} = \begin{cases} \frac{\sin x}{x} & \text{if } x > 0 \\ \frac{\sin x}{-x} & \text{if } x < 0 \end{cases}$.

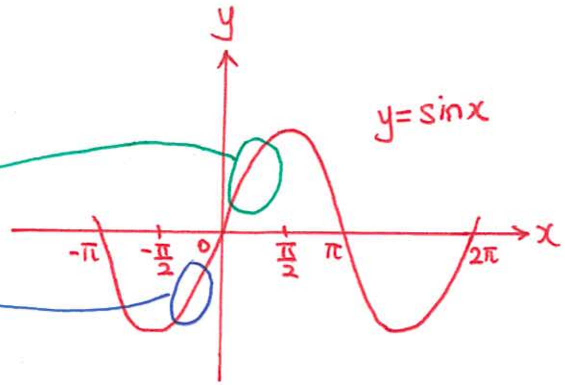
Since $\frac{\sin x}{|x|}$ has different formulas when x is to the left of 0 and to the right of 0, we have to consider the left-hand and right-hand limits separately.

Left hand limit: $\lim_{x\to 0^-}\frac{\sin x}{|x|} \overset{x<0}{=} \lim_{x\to 0^-}\frac{\sin x}{-x} = -\lim_{x\to 0^-}\frac{\sin x}{x} = -1$

Right hand limit: $\lim_{x\to 0^+}\frac{\sin x}{|x|} \overset{x>0}{=} \lim_{x\to 0^+}\frac{\sin x}{x} = 1$

Since $\lim_{x\to 0^-}\frac{\sin x}{|x|} \neq \lim_{x\to 0^+}\frac{\sin x}{|x|}$, the limit $\lim_{x\to 0}\frac{\sin x}{|x|}$ does not exist.

(b) First note that $\dfrac{|\sin x|}{x}$ is not defined when $x = 0$.



Recall that $|\sin x| = \begin{cases} \sin x & \text{if } \sin x \geq 0 \\ -\sin x & \text{if } \sin x < 0 \end{cases}$.

When $0 < x < \frac{\pi}{2}$, $\sin x > 0$.

When $-\frac{\pi}{2} < x < 0$, $\sin x < 0$.

Then $\dfrac{|\sin x|}{x} = \begin{cases} \dfrac{\sin x}{x} & \text{if } 0 < x < \dfrac{\pi}{2} \\ \dfrac{-\sin x}{x} & \text{if } -\dfrac{\pi}{2} < x < 0 \end{cases}$.

Since $\dfrac{|\sin x|}{x}$ has different formulas when x is to the left of 0 and to the right of 0, we have to consider the left-hand and right-hand limits separately.

Left hand limit: $\displaystyle \lim_{x\to 0^-} \frac{|\sin x|}{x} \overset{x<0}{=} \lim_{x\to 0^-} \frac{-\sin x}{x} = -\lim_{x\to 0^-} \frac{\sin x}{x} = -1$

Right hand limit: $\displaystyle \lim_{x\to 0^+} \frac{|\sin x|}{x} \overset{x>0}{=} \lim_{x\to 0^+} \frac{\sin x}{x} = 1$

Since $\displaystyle \lim_{x\to 0^-} \frac{|\sin x|}{x} \neq \lim_{x\to 0^+} \frac{|\sin x|}{x}$, the limit $\displaystyle \lim_{x\to 0} \frac{|\sin x|}{x}$ does not exist.

Example 15

Evaluate the limit $\lim_{x\to 0}\frac{2\sin x\cos x}{x}$. $\left(\frac{0}{0}\text{ form}\right)$

Solution

Method 1:

$$\lim_{x\to 0}\frac{2\sin x\cos x}{x}=\lim_{x\to 0}\frac{\sin 2x}{x}$$ (by using **double angle formula** $\boxed{\sin 2x = 2\sin x\cos x}$)

$$=\lim_{x\to 0}\frac{\sin 2x}{\mathbf{2}x}\cdot\mathbf{2}=2\cdot\underbrace{\left(\lim_{x\to 0}\frac{\sin 2x}{2x}\right)}_{=1}=2\cdot 1=2$$

Method 2:

$$\lim_{x\to 0}\frac{2\sin x\cos x}{x}=2\underbrace{\left(\lim_{x\to 0}\frac{\sin x}{x}\right)}_{=1}\cdot\underbrace{\left(\lim_{x\to 0}\cos x\right)}_{=\cos 0=1}=2\cdot 1\cdot 1=2$$

Example 16

Evaluate the limit $\lim_{x\to 0}\frac{1-\cos x}{x\sin x}$.

Solution

Note that the function $\frac{1-\cos x}{x\sin x}$ is of the $\frac{0}{0}$ **form** when $x=0$.

Method 1:

$$\lim_{x\to 0}\frac{1-\cos x}{x\sin x}=\lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)}{x\sin x}$$ by the **Half-angle formula** $\boxed{\sin^2\left(\frac{x}{2}\right)=\frac{1}{2}(1-\cos x)}$.

$$=2\lim_{x\to 0}\left[\frac{\sin\left(\frac{x}{2}\right)\cdot\sin\left(\frac{x}{2}\right)}{\frac{x}{2}\cdot\frac{x}{2}}\cdot\frac{\frac{\cancel{x}}{2}\cdot\frac{x}{2}}{\cancel{x}\sin x}\right]=\frac{2}{4}\lim_{x\to 0}\left[\underbrace{\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}}_{\to 1}\cdot\underbrace{\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}}}_{\to 1}\cdot\underbrace{\frac{x}{\sin x}}_{\to 1}\right]=\frac{2}{4}\cdot 1\cdot 1\cdot 1=\frac{1}{2}$$

Method 2:

$$\lim_{x\to 0}\frac{1-\cos x}{x\sin x}=\lim_{x\to 0}\left(\frac{1-\cos x}{x\sin x}\cdot\frac{1+\cos x}{1+\cos x}\right)=\lim_{x\to 0}\frac{1-\cos^2 x}{x\sin x(1+\cos x)}$$

$$=\lim_{x\to 0}\frac{\sin^{\cancel{2}} x}{x\cancel{\sin x}(1+\cos x)}=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{1}{1+\cos x}\right)=1\cdot\frac{1}{1+1}=\frac{1}{2}$$

$\nwarrow \cos 0=1$

Example 17

Evaluate the limit $\lim_{x\to 0}\frac{(\sin 3x)^2}{x^2\cos x}$.

Solution

$$\lim_{x\to 0}\frac{(\sin 3x)^2}{x^2\cos x} \quad \left(\frac{\mathbf{0}}{\mathbf{0}}\ \textbf{form}\right)$$

$$= \lim_{x\to 0}\left[\frac{(\sin 3x)^2}{(\mathbf{3}x)^{\mathbf{2}}}\cdot\frac{1}{\cos x}\cdot\mathbf{3^2}\right]$$

$$= 1^2\cdot\frac{1}{\cos 0}\cdot 3^2$$

$$= 9$$

Example 18

Evaluate the limit $\lim_{x\to 0} \frac{\cos(3x)-\cos(7x)}{x^2}$.

Solution

$$\lim_{x\to 0} \frac{\cos(3x)-\cos(7x)}{x^2} \quad \left(\frac{0}{0} \textbf{ form}\right)$$

$$= \lim_{x\to 0} \frac{-2\sin\left(\frac{3x+7x}{2}\right)\sin\left(\frac{3x-7x}{2}\right)}{x^2}$$

by using the **sum-to-product formula**

$$\boxed{\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}$$

$$= \lim_{x\to 0} \frac{-2\sin(5x)\sin(-2x)}{x^2}$$

$$= \lim_{x\to 0} \frac{-2\sin(5x)[-\sin(2x)]}{x^2}$$

since $\sin(2x)$ is an odd function

$$= 2\lim_{x\to 0} \underbrace{\frac{\sin(5x)}{5x}}_{\to 1} \cdot \underbrace{\frac{\sin(2x)}{2x}}_{\to 1} \cdot 5 \cdot 2$$

$$= 2 \cdot 1 \cdot 1 \cdot 5 \cdot 2 = 20$$

Example 19

Evaluate the limit $\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}$.

Solution

$$\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} \quad \left(\frac{0}{0} \textbf{ form}\right)$$

$$= \lim_{x\to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3}$$

$$= \lim_{x\to 0} \frac{\sin x\left(\frac{1}{\cos x} - 1\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{\sin x\left(\frac{1-\cos x}{\cos x}\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} \cdot \frac{1-\cos x}{x^2}$$

$$= 2 \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} \cdot \frac{\sin^2\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)^2} \cdot \frac{\left(\frac{\cancel{x}}{2}\right)^2}{\cancel{x^2}}$$

by the half angle formula $\sin^2\left(\frac{x}{2}\right) = \frac{1}{2}(1 - \cos x)$.

$$= 2 \cdot 1 \cdot 1 \cdot 1^2 \cdot \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{2}$$

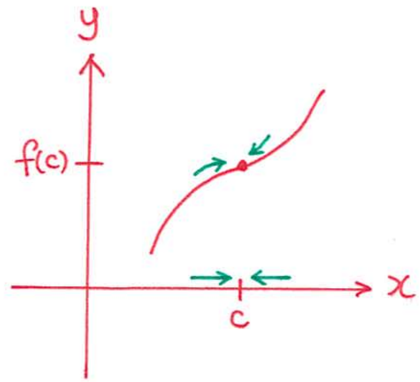
Continuity of functions

Definition (Continuity at a point)

Let f be defined on an open interval containing c.

Then f is **continuous** at $x = c$ if and only if $\Leftrightarrow$

$$\boxed{\lim_{x \to c} f(x) = f(c).}$$



By this definition, there are 3 conditions for continuity of f at $x = c$:

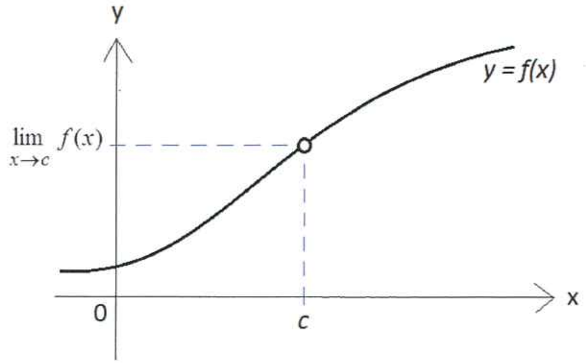
(i) $f(c)$ exists (i.e. c is in the domain of f)

(ii) $\lim_{x \to c} f(x)$ exists

(iii) $\lim_{x \to c} f(x) = f(c)$

If any one of these three conditions fails, then f is **discontinuous** at $x = c$ (i.e. there is a break on the graph of $y = f(x)$ at $x = c$).

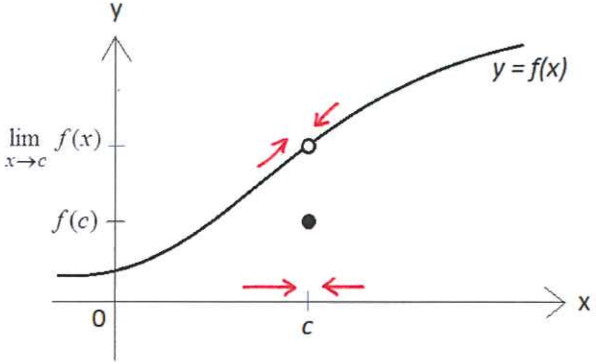
Examples



$c \notin Dom(f)$ condition (i) fails

$f(x)$ is not defined at $x = c$, i.e. $f(c)$ does not exist.

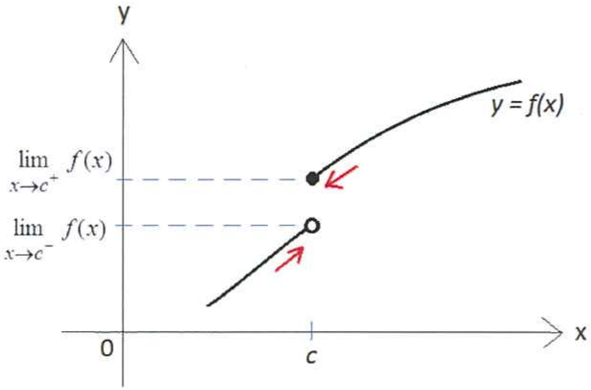
$\therefore$ f is **discontinuous** at $x = c$.



Both $f(c)$ and $\lim\limits_{x\to c} f(x)$ exist.

However, $\lim\limits_{x\to c} f(x) \neq f(c)$ ← condition (iii) fails

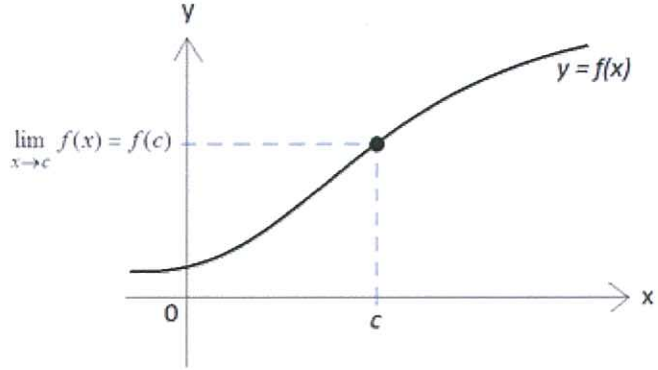
$\therefore$ f is **discontinuous** at $x = c$.



Since $\lim\limits_{x\to c^-} f(x) \neq \lim\limits_{x\to c^+} f(x)$,

the limit $\lim\limits_{x\to c} f(x)$ does not exist. ← condition (ii) fails

$\therefore$ f is **discontinuous** at $x = c$.



Both $f(c)$ and $\lim_{x\to c} f(x)$ exist.

Moreover, $\lim_{x\to c} f(x) = f(c)$ ← All 3 conditions hold

$\therefore$ f is **continuous** at $x = c$.

Example 20

Is $f(x) = \frac{x^2-1}{x-1}$ continuous at $x = 1$?

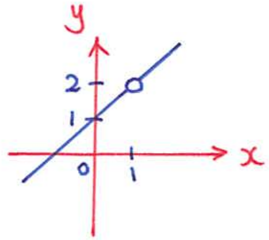
Solution

$1 \notin \mathrm{Dom}(f)$

$f(x) = \frac{x^2-1}{x-1}$ is not defined at $x = 1$, i.e. $f(1)$ does not exist. ← condition (i) fails

$\therefore$ $f(x) = \frac{x^2-1}{x-1}$ is not continuous at $x = 1$.

(i.e. f is discontinuous at $x = 1$.)



$$y = \frac{x^2-1}{x-1}$$
$$= \frac{(x-1)(x+1)}{x-1}$$
$$= x+1 \text{ for } x \neq 1.$$

Example 21

Let $g(x)=\begin{cases}\dfrac{x}{|x|} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$. Is g continuous at $x=0$?

Solution

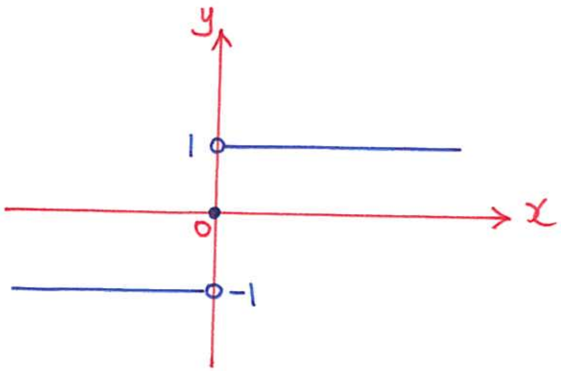
The function g is defined at $x=0$, so $g(0)$ exists. ← condition (i) holds

Left hand limit: $\lim\limits_{x\to 0^-} g(x)=\lim\limits_{x\to 0^-}\dfrac{x}{|x|}\underbrace{=}_{\because\, x<0}\lim\limits_{x\to 0^-}\dfrac{\cancel{x}}{-\cancel{x}}=\lim\limits_{x\to 0^-}(-1)=-1$

Right hand limit: $\lim\limits_{x\to 0^+} g(x)=\lim\limits_{x\to 0^+}\dfrac{x}{|x|}\underbrace{=}_{\because\, x>0}\lim\limits_{x\to 0^+}\dfrac{\cancel{x}}{\cancel{x}}=\lim\limits_{x\to 0^+}(1)=1$

Since $\lim\limits_{x\to 0^-} g(x)\neq\lim\limits_{x\to 0^+} g(x)$, the limit $\lim\limits_{x\to 0} g(x)$ does not exist. ← condition (ii) fails

$\therefore$ $g(x)$ is discontinuous at $x=0$.



Example 22

Let $h(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$. Is h continuous at $x = 0$?

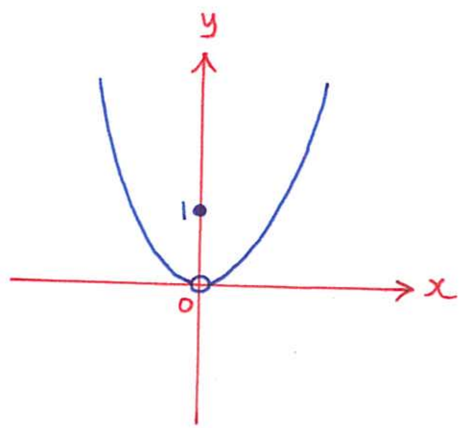
Solution

$h(x)$ is defined at $x = 0$, so $h(0)$ exists.

$$\lim_{x\to 0} h(x) \underbrace{=}_{\because x\neq 0} \lim_{x\to 0} x^2 = 0^2 = 0.$$

condition (iii) fails

Since $\lim_{x\to 0} h(x) = 0 \neq 1 = h(0)$, h is discontinuous at $x = 0$.



Example 23

The function $f(x) = \frac{x^2-3x-10}{x-5}$ is undefined at $x = 5$, then $f(5)$ doesn't exist and so the function f is not continuous at $x = 5$. If we define $g(x) = \begin{cases} \frac{x^2-3x-10}{x-5} & \text{if } x \neq 5 \\ c & \text{if } x = 5 \end{cases}$, where c is a constant, find the value of c such that g is continuous at $x = 5$.

Solution

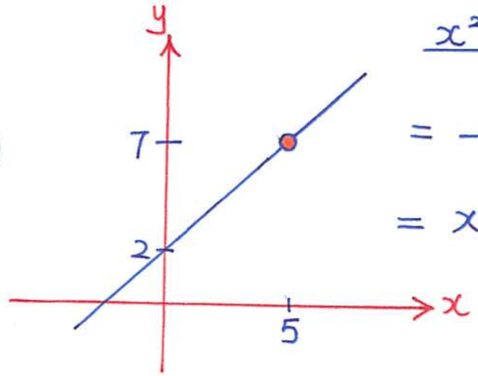
The function g is defined at $x = 5$, so $g(5)$ exists.

$$\lim_{x\to5} g(x) = \lim_{x\to5} \frac{x^2-3x-10}{x-5} \overset{(\frac{0}{0}\text{ form})}{=} \lim_{x\to5} \frac{\cancel{(x-5)}(x+2)}{\cancel{x-5}} = \lim_{x\to5}(x+2) = 5+2 = 7$$

If we put $g(5) = c = 7$, then

$$\lim_{x\to5} g(x) = 7 = g(5)$$

and therefore the function g is continuous at $x = 5$.



$\frac{x^2-3x-10}{x-5}$

$= \frac{(x-5)(x+2)}{x-5}$

$= x+2$ for $x \neq 5$

Example 24

Let $f(x) = \begin{cases} k\left|\frac{2+x}{x^2-3}\right| & \text{if } -1 \le x < 0 \\ c & \text{if } x = 0 \\ \frac{x}{\sqrt{2+3x}-\sqrt{2}} & \text{if } 0 < x \le 1 \end{cases}$, where k and c are constants.

Find the values of c and k so that $f(x)$ is continuous at $x = 0$.

Solution

Left-hand limit:

$$\lim_{x\to 0^-} f(x) \overset{x<0}{=} \lim_{x\to 0^-} k\left|\frac{2+x}{x^2-3}\right| = k\left|\frac{2+0}{0^2-3}\right| = k\left|-\frac{2}{3}\right| = \frac{2}{3}k$$

Right-hand limit:

$$\lim_{x\to 0^+} f(x) \overset{x>0}{=} \lim_{x\to 0^+} \frac{x}{\sqrt{2+3x}-\sqrt{2}} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0^+} \frac{x(\sqrt{2+3x}+\sqrt{2})}{(\sqrt{2+3x}-\sqrt{2})(\sqrt{2+3x}+\sqrt{2})}$$

$$= \lim_{x\to 0^+} \frac{x\left(\sqrt{2+3x}+\sqrt{2}\right)}{(2+3x)-2}$$

$$= \lim_{x\to 0^+} \frac{\cancel{x}\left(\sqrt{2+3x}+\sqrt{2}\right)}{3\cancel{x}}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{2+3x}+\sqrt{2}}{3}$$

$$= \frac{\sqrt{2+0}+\sqrt{2}}{3}$$

$$= \frac{2\sqrt{2}}{3}$$

$\lim\limits_{x\to 0} f(x)$ exists iff $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x)$, i.e.

$$\frac{2}{3}k = \frac{2\sqrt{2}}{3} \Rightarrow k = \sqrt{2}.$$

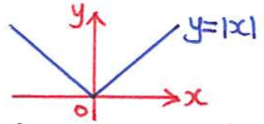
$f(x)$ is continuous at $x = 0$ iff $\boxed{\lim\limits_{x\to 0} f(x)} = \boxed{f(0)}$, i.e.

c

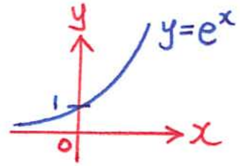
$\frac{2\sqrt{2}}{3}$

$$c = \frac{2\sqrt{2}}{3}.$$

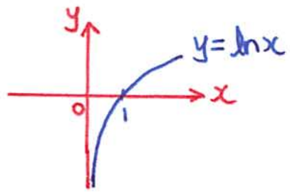
Examples of continuous functions



- All polynomials, $\sin x$, $\cos x$ and $|x|$ are continuous at every $x = c$ where $c \in \mathbb{R}$.
- A rational function $\dfrac{f(x)}{g(x)}$ is continuous at every $x = c$ where $c \in \mathbb{R}$, provided $g(c) \neq 0$, and $f(x)$ and $g(x)$ are both continuous at $x = c$.



- e^x is continuous at every $x = c$ where $c \in \mathbb{R}$.
- $\ln x$ is continuous at every $x = c$ where $c > 0$.



Theorems on continuity

1. If f and g are continuous at c, then so are
   - kf (where k is any real number),
   - $f + g$, $f - g$, fg, $\dfrac{f}{g}$ (where $g(c) \neq 0$), and
   - f^n (where n is a positive integer).

2. If $\lim_{x\to c} g(x) = l$ and if f is continuous at l, then

$$\boxed{\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f(l)}.$$

outer function is continuous at l

E.g. $\lim_{x\to c}\left[e^{f(x)}\right] = e^{\lim_{x\to c} f(x)}$.

exponential function is continuous everywhere

3. If g is continuous at c and f is continuous at $g(c)$, then the composite function $f\circ g$ is continuous at c.

$l = g(c)$ $\because$ g is continuous at c.

$$\boxed{\lim_{x\to c}(f\circ g)(x) = \lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right) = f\big(g(c)\big)}$$

E.g. $\cos x$ is continuous at every $x\in\mathbb{R}$, and e^x is continuous at every $x\in[-1,1]$.

$\therefore$ $e^{\cos x}$ is continuous at every $x\in\mathbb{R}$.

Example 25

Find $\lim_{x\to 1} \sin \frac{(x^2-1)\pi}{x-1}$.

Solution

$$\begin{aligned}
\lim_{x\to 1} \sin \frac{(x^2-1)\pi}{x-1} &= \sin\left[\lim_{x\to 1} \frac{(x^2-1)\pi}{x-1}\right], \quad \text{since sin function is continuous everywhere} \\
&= \sin\left[\lim_{x\to 1} \frac{(x-1)(x+1)\pi}{x-1}\right] \\
&= \sin\left[\lim_{x\to 1} (x+1)\pi\right] \\
&= \sin(2\pi) \\
&= 0
\end{aligned}$$