CITY UNIVERSITY OF HONG KONG

Course code and title	:	MA1200 Calculus and Basic Linear Algebra I Section CA1, CB1, CC1 and CD1 Test 1
Session	:	Semester A, 2015/2016
Time	:	09:00 - 10:00, 15 October 2015 (Thursday)
Time allowed	:	1 hour

This paper has **TWO** pages (including this cover page).

Instructions to candidates:

1. This paper has **SIX** questions.
2. Attempt **ALL** questions.

*This is a **closed-book** test.*

Candidates are allowed to use the following materials/aids:

Non-programmable calculators

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorised materials or aids are found on them.

NOT TO BE TAKEN AWAY

Question 1

(a) The straight line $y=\frac{3}{4}x+3$ meets the x-axis at P and the y-axis at Q.
Find the equation of the perpendicular bisector of PQ. (8 marks)

(b) Find the equation of the circle which passes through the points (0, 0), (0, 3), (-4, 0). (9 marks)

Question 2

Let $f(x)$ be a periodic function of x with period 2 and $f(x)=|x|-x$ for $-1<x\le 1$.
Sketch the graph of the curve $y=f(x)$ in the interval $[-3,3]$. (17 marks)

Question 3

Express $\frac{2x+30}{(x+3)(x^2-9)}$ in partial fractions. (17 marks)

Question 4

Show that the equation $20x^2+36y^2+40x-108y-79=0$ represents an ellipse, and find the coordinates of its centre and foci.
(Hint: You may use the method of completing the squares.) (17 marks)

Question 5

The following functions are one-to-one. Find the inverse function for each and state its largest possible domain.
(a) $F(x)=-2x-1$ for $x\in[0,\infty)$, (7 marks)
(b) $G(x)=(x-2)^2-1$ for $x\in[2,\infty)$. (10 marks)

Question 6

It is given that $\sin A=\frac{5}{13}$, where $0°<A<90°$, and that $\sin B=\frac{3}{5}$, where $90°<B<180°$.
Without using a calculator, find the values of
(a) $\sin(A+B)$,
(b) $\cos(A+B)$.
(Hint: You may use $\sin(A+B)=\sin A\cos B+\cos A\sin B$
$\cos(A+B)=\cos A\cos B-\sin A\sin B$) (17 marks)

MA1200, CA1, CB1, CC1 and CD1, Test 1.

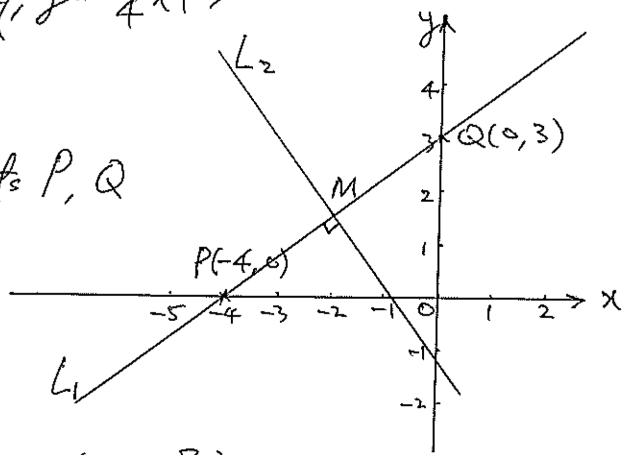
Sem. A, 2015/2016

Q.1.(a)

Equation of straight line L_1, $y=\frac{3}{4}x+3$

x	0	−4
y	3	0

The coordinates of points P, Q are $(-4,0)$ and $(0,3)$ respectively.



The coordinates of the mid-point of PQ are

$\left(\frac{-4+0}{2}, \frac{0+3}{2}\right)$, that is $\left(-2, \frac{3}{2}\right)$.

Slope of straight line L_1 is $\frac{3}{4}$.

Slope of straight line L_2 is $-\frac{4}{3}$.

$\therefore$ Equation of straight line L_2, the perpendicular bisector of PQ is given by

$$\frac{y-\frac{3}{2}}{x-(-2)} = -\frac{4}{3}, \text{ that is } 8x+6y+7=0.$$

(b) Method I.

Using the results in Q.1(a),

Centre of the circle is $M\left(-2,\frac{3}{2}\right)$,

radius of the circle is equal to $\frac{5}{2}$ units

$\therefore$ The equation of the circle is

$$(x-(-2))^2+\left(y-\frac{3}{2}\right)^2=\left(\frac{5}{2}\right)^2, \text{ that is } x^2+y^2+4x-3y=0$$

Method II.

Let $x^2+y^2+Dx+Ey+F=0$ be the equation of the circle through the points $(0,0)$, $(0,3)$ and $(-4,0)$, where D,E,F are unknown constants.

Then

$$\left.\begin{aligned} 0^2+0^2+0\cdot D+0\cdot E+F&=0\\ 0^2+3^2+0\cdot D+3E+F&=0\\ (-4)^2+0^2-4D+0\cdot E+F&=0 \end{aligned}\right\} \Rightarrow \begin{cases} F=0\\ E=-3\\ D=4 \end{cases}$$

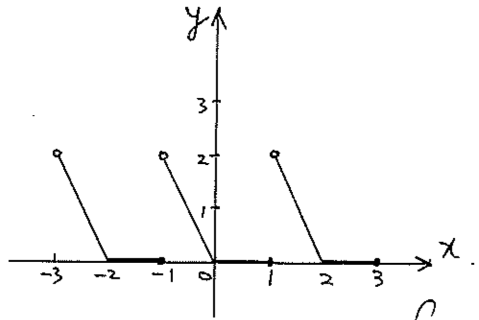
$\therefore$ The equation of the circle is $x^2+y^2+4x-3y=0$

Q.2.

$$f(x)=|x|-x \quad \text{for } -1<x\le 1$$

$$=\begin{cases} -x-x & \text{for } -1<x<0\\ x-x & \text{for } 0\le x\le 1 \end{cases}$$

$$=\begin{cases} -2x & \text{for } -1<x<0\\ 0 & \text{for } 0\le x\le 1 \end{cases}$$



Graph of $y=f(x)$ in $[-3,3]$.

Q.3.

Let $\dfrac{2x+30}{(x+3)(x^2-9)} = \dfrac{A}{x-3} + \dfrac{B}{x+3} + \dfrac{C}{(x+3)^2}$.

Then $2x+30 = A(x+3)^2 + B(x-3)(x+3) + C(x-3)$

Put $x=3$, then $6+30 = A(3+3)^2 + 0 + 0$, $\therefore A = 1$

So, $2x+30-(x+3)^2 = (x-3)[B(x+3)+C]$

$= -x^2-4x+21$

$= (x-3)(-x-7)$

$\Rightarrow -x-7 = B(x+3)+C$

Put $x=-3$, then $3-7 = 0+C$, $\therefore C = -4$

So, $-x-7+4 = B(x+3)$

$= -(x+3)$, $\therefore B = -1$

$\therefore \dfrac{2x+30}{(x+3)(x^2-9)} = \dfrac{1}{x-3} - \dfrac{1}{x+3} - \dfrac{4}{(x+3)^2}$. //

Q.4. $20x^2+36y^2+40x-108y-79=0$

$20(x^2+2x)+36(y^2-3y)-79=0$

$20[(x+1)^2-1]+36[(y-\frac{3}{2})^2-\frac{9}{4}]-79=0$

$20(x+1)^2+36(y-\frac{3}{2})^2 = 79+20+81$

$\dfrac{(x-(-1))^2}{3^2} + \dfrac{(y-\frac{3}{2})^2}{(\sqrt{5})^2} = 1$

Comparing with the standard form of the equation of an ellipse, $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, we have

$a^2=9$, $b^2=5$

$b^2=a^2(1-e^2)$, $1-e^2=\frac{5}{9}$, $e=\frac{2}{3}$, $ae=2$.

Hence, the given equation is the equation of an ellipse with centre at $C(-1, \frac{3}{2})$ and foci at $F_1(1, \frac{3}{2})$ and $F_2(-3, \frac{3}{2})$. //

Q.5. (a) Let $y = F(x) = -2x-1$.

Then $x = -\dfrac{y+1}{2}$

$\therefore F^{-1}(x) = -\dfrac{x+1}{2}$ //

$\text{Dom}(F^{-1}) = \text{Ran}(F) = (-\infty, -1]$. //

(b) Let $y = G(x) = (x-2)^2-1$.

Then $(x-2)^2 = y+1$

$x = 2 \pm \sqrt{y+1}$

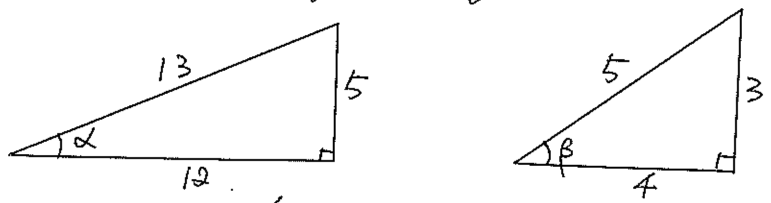
$G^{-1}(x) \neq 2-\sqrt{x+1}$ (rejected), since $\text{Ran}(G^{-1}) = \text{Dom}(G) = [2, \infty)$.

$\therefore G^{-1}(x) = 2+\sqrt{x+1}$ //

$\text{Dom}(G^{-1}) = \text{Ran}(G) = [-1, \infty)$. //

Q. 6.

With two right angled triangles.



$\because \sin A = \frac{5}{13}$, where $0° < A < 90°$

$\therefore \cos A = \frac{12}{13}$

$\because \sin B = \frac{3}{5}$, where $90° < B < 180°$

$\therefore \cos B = -\frac{4}{5}$

(a) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$= \left(\frac{5}{13}\right)\left(-\frac{4}{5}\right) + \left(\frac{12}{13}\right)\left(\frac{3}{5}\right)$$

$$= \underline{\underline{\frac{16}{65}}}$$

(b) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$= \left(\frac{12}{13}\right)\left(-\frac{4}{5}\right) - \left(\frac{5}{13}\right)\left(\frac{3}{5}\right)$$

$$= \underline{\underline{-\frac{63}{65}}}$$

CITY UNIVERSITY OF HONG KONG

Course code and title	:	MA1200 Calculus and Basic Linear Algebra I Section CE1, CF1, CG1 and CH1 Test 1
Session	:	Semester A, 2015/2016
Time	:	12:00 - 13:00, 15 October 2015 (Thursday)
Time allowed	:	1 hour

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NOT TO BE TAKEN AWAY

Question 1

The circle $x^2+y^2+2x-4y-4=0$ and the straight line $y=x$ intersect at P and Q.
Find the equation of the circle on PQ as diameter. (17 marks)

Question 2

Show that if $f(x)$ is a function of x defined for all real values of x, then
(a) $F(x)=f(x)+f(-x)$ is an even function of x, (8 marks)
(b) $G(x)=f(x)-f(-x)$ is an odd function of x. (9 marks)

Question 3

Express $\dfrac{4x^2-x+6}{x^3+3x^2}$ in partial fractions. (17 marks)

Question 4

Show that the equation $y^2+8x-2y-23=0$ represents a parabola whose vertex is at the point (3, 1).
Find the coordinates of its focus and the equation of its directrix.
(Hint: You may use the method of completing the squares.) (17 marks)

Question 5

Let $g(x)=\dfrac{3x+1}{x-2}$,

$h(x)=\sqrt{1-x^2}$.

Find the largest possible domain and range of each of the above functions. (17 marks)

Question 6

Starting from the formulae

$\sin(A+B)=\sin A\cos B+\cos A\sin B$ and

$\cos(A+B)=\cos A\cos B-\sin A\sin B$,

show that
(a) $\sin 2\theta=2\sin\theta\cos\theta$, (5 marks)
(b) $\cos 2\theta=2\cos^2\theta-1$, (6 marks)
(c) $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$. (6 marks)

- END -

MA1200, CE1, CF1, CG1 and CH1, 15/10/2015

Test 1,

Q.1. Circle : $x^2+y^2+2x-4y-4=0$ — ①

Straight line : $y=x$ — ②

Substituting eq^n ② into eq^n ①, we have

$x^2+x^2+2x-4x-4=0$

$(x-2)(x+1)=0$

$\therefore$ The solutions of system ①-② are

$\begin{cases} x=2 \\ y=2 \end{cases}$ or $\begin{cases} x=-1 \\ y=-1 \end{cases}$

The mid-point of PQ is $\left(\frac{2-1}{2}, \frac{2-1}{2}\right)$.

$PQ=\sqrt{(2-(-1))^2+(2-(-1))^2} = 3\sqrt{2}$ units.

$\therefore$ The equation of the circle on PQ as diameter is

$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\left(\frac{3\sqrt{2}}{2}\right)^2$,

that is $x^2+y^2-x-y-4=0$. //

Method II. The equation

$x^2+y^2+2x-4y-4+\lambda(x-y)=0$

represents a circle through P and Q.

The coordinates of its centre is $\left(-1-\frac{\lambda}{2}, 2+\frac{\lambda}{2}\right)$.

Since the centre lies on the line $x-y=0$.

Hence $-1-\frac{\lambda}{2}-2-\frac{\lambda}{2}=0$, $\therefore \lambda=-3$

$\therefore$ The equation of the circle on PQ as diameter is $x^2+y^2-x-y-4=0$. //

Q.2.

(a) $F(x)$ is defined as $F(x)=f(x)+f(-x)$.

$\because F(-x)=f(-x)+f(-(-x))=f(-x)+f(x)=F(x)$

$\therefore F(x)$ is an even function of x. //

(b) $G(x)$ is defined as $G(x)=f(x)-f(-x)$.

$\because G(-x)=f(-x)-f(-(-x))$

$=f(-x)-f(x)=-(f(x)-f(-x))$

$=-G(x)$

$\therefore G(x)$ is an odd function of x. //

Q.3.

Let $\frac{4x^2-x+6}{x^3+3x^2}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+3}$.

Then $4x^2-x+6=Ax(x+3)+B(x+3)+Cx^2$.

Put $x=0$, then $6=0+B(0+3)+0$

$\therefore B=2$

So, $4x^2-x+6-2(x+3)=x[A(x+3)+Cx]$

$=4x^2-3x$

$=x(4x-3)$

$\Rightarrow 4x-3=A(x+3)+Cx$

Put $x=0$, then $-3=A(0+3)+0$, $\therefore A=-1$

So, $4x-3+(x+3)=Cx$

$=5x$, $\therefore C=5$

$\therefore \dfrac{4x^2-x+6}{x^3+3x^2} = -\dfrac{1}{x} + \dfrac{2}{x^2} + \dfrac{5}{x+3}$. //

Q.4. $y^2+8x-2y-23=0$ —— (1)

$y^2-2y = -8x+23$

$y^2-2y+1 = -8(x-3)$

$(y-1)^2 = -4(2)(x-3)$ —— (*)

Comparing equation (*) with the standard form of the equation of a parabola

$(y-k)^2 = -4a(x-h)$,

we have $a=2$, $h=3$, $k=1$

$\therefore$ Equation (1) represents a parabola with vertex $V(3,1)$. //

The coordinates of its focus are $(1,1)$. //

The equation of its directrix is $x=5$. //

Q.5. $g(x) = \dfrac{3x+1}{x-2} = \dfrac{3(x-2)+7}{x-2} = 3 + \dfrac{7}{x-2}$.

$h(x) = \sqrt{1-x^2}$

$Dom(g) = \mathbb{R} \setminus \{2\}$. //

$Ran(g) = \mathbb{R} \setminus \{3\}$. //

Solving the inequality $1-x^2 \geq 0$.

$(1+x)(1-x) \geq 0$

$-1 \leq x \leq 1$

	$\leftarrow x=-1$		$x=1 \rightarrow$
$1+x$	$-$	$+$	$+$
$1-x$	$+$	$+$	$-$
$(1+x)(1-x)$	$(-)$	$(+)$	$(-)$
	×	✓	×

$\therefore Dom(h) = [-1, 1]$ //

$Ran(h) = [0, 1]$ //

Q.6. $\sin(A+B) = \sin A \cos B + \cos A \sin B$ —— (1)

$\cos(A+B) = \cos A \cos B - \sin A \sin B$ —— (2)

(a) Put $A=B=\theta$ in equation (1), then

$\sin(\theta+\theta) = \sin\theta\cos\theta + \cos\theta\sin\theta$

$\therefore \sin(2\theta) = 2\sin\theta\cos\theta$. //

(b) Put $A = B = \theta$ in equation (2), then

$$\cos(\theta+\theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$$

$$\therefore \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= \cos^2\theta - (1-\cos^2\theta),$$ since $\sin^2\theta + \cos^2\theta = 1$ for all θ.

$$= 2\cos^2\theta - 1 \quad //$$

(c)

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta}$$

$$= \frac{2\sin\theta\cos\theta}{2\cos^2\theta - 1},$$ from (a) and (b).

$$= \frac{\dfrac{2\sin\theta\cos\theta}{\cos^2\theta}}{\dfrac{2\cos^2\theta - 1}{\cos^2\theta}}$$

$$= \frac{2\left(\dfrac{\sin\theta}{\cos\theta}\right)}{2 - \sec^2\theta},$$ since $\sec\theta = \dfrac{1}{\cos\theta}$, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

$$= \frac{2\tan\theta}{2-(1+\tan^2\theta)}$$

$$= \frac{2\tan\theta}{1-\tan^2\theta} \quad //$$

CITY UNIVERSITY OF HONG KONG

Course code and title	:	MA1200 Calculus and Basic Linear Algebra I Section C61 Test 1
Session	:	Semester A, 2015/2016
Time	:	18:00 - 19:00, 12 October 2015 (Monday)
Time allowed	:	1 hour

This paper has **TWO** pages (including this cover page).

Instructions to candidates:

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2. Attempt **ALL** questions.

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NOT TO BE TAKEN AWAY

Question 1

(a) Find the equation of the straight line joining the points (0, -2), (3, 0) . (7 marks)

(b) Find the centre and radius of the circle $x^2+y^2+2x-4y-4=0$. (10 marks)

Question 2

If the equation of a parabola is $(y-3)^2=-8(x+1)$, find the coordinates of its vertex and focus. Sketch its graph. (17 marks)

Question 3

Express $\dfrac{9x^2-5x+16}{(x-2)(x^2+x+1)}$ in partial fractions. (17 marks)

Question 4

Let $f(x)=x-[x]$ for $-3\le x\le 3$, where $[x]$ denotes the greatest integer not greater than x. Find the range of $f(x)$, and sketch its graph. (17 marks)

Question 5

Let $F(x)$ and $G(x)$ be two functions defined by

$$F(x)=\frac{1}{1+x},$$

$$G(x)=1+\frac{1}{x}.$$

(a) Find their largest possible domains and ranges. (8 marks)
(b) Find $(F\circ G)(x)$ and state its largest possible domain. (9 marks)

Question 6

(a) Let θ be an angle lies between $180°$ and $270°$, and $\tan\theta=\frac{3}{4}$. Without using a calculator, find the values of
 (i) $10\sin\theta-5\cos\theta$,
 (ii) $\cos\frac{\theta}{2}$.
 (Hint: You may use the identity, $\cos 2x=2\cos^2 x-1$) (8 marks)

(b) Express $\cos x+\sqrt{3}\sin x$ in the form $R\sin(x+\alpha)$, where $R>0$ and $0<\alpha<\frac{\pi}{2}$.
 (Hint: You may use $\sin(A+B)=\sin A\cos B+\cos A\sin B$) (9 marks)

- END -

MA1200, C61, Test 1. Sem. A, 2015/2016

Q.1 (a) Equation of the straight line joining the points $(0,-2)$, $(3,0)$ is given by

$$\frac{y-(-2)}{x-0}=\frac{0-(-2)}{3-0}, \text{ that is } 2x-3y-6=0.$$

(b) Equation of the circle

$$x^2+2x+y^2-4y-4=0$$
$$(x+1)^2-1+(y-2)^2-4-4=0$$
$$(x-(-1))^2+(y-2)^2=3^2.$$

centre of the circle is $(-1,2)$.
radius of the circle $=3$ units.

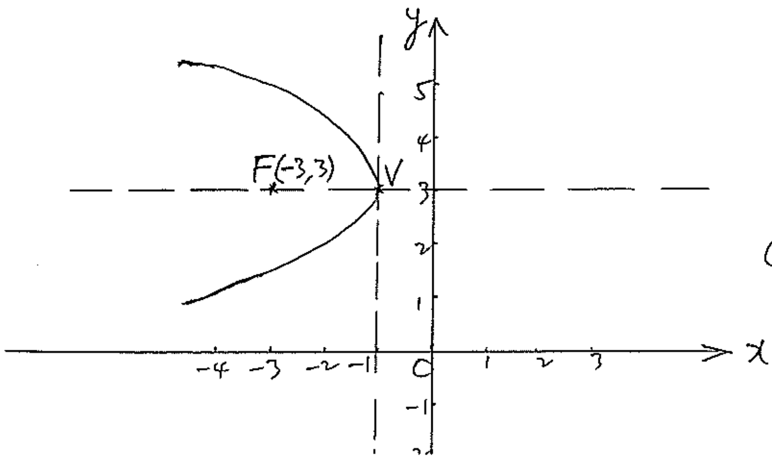
Q.2. Equation of the parabola

$$(y-3)^2=-4(2)(x-(-1)). \quad \text{— (1)}$$

Comparing eqn (1) with the standard form,

$$(y-k)^2=-4a(x-h).$$

The coordinates of its vertex is $V(-1,3)$.
The coordinates of its focus is $F(-3,3)$.



Graph.

Q.3.

Let $\dfrac{9x^2-5x+16}{(x-2)(x^2+x+1)}=\dfrac{A}{x-2}+\dfrac{Bx+C}{x^2+x+1}$.

Then $9x^2-5x+16=A(x^2+x+1)+(Bx+C)(x-2)$

Put $x=2$, then $36-10+16=A(4+2+1)+0$.

$$\therefore A=\frac{42}{7}=6$$

So, $9x^2-5x+16-6(x^2+x+1)=(Bx+C)(x-2)$

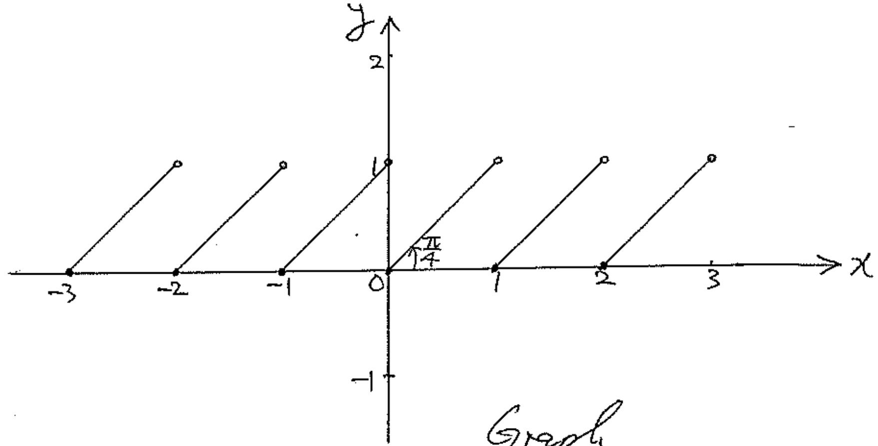
$=3x^2-11x+10$

$=(3x-5)(x-2)$, $\therefore Bx+C=3x-5$

$$\therefore \frac{9x^2-5x+16}{(x-2)(x^2+x+1)}=\frac{6}{x-2}+\frac{3x-5}{x^2+x+1}.$$

Q.4. $f(x)=x-[x]$ for $-3\le x\le 3$.

$\operatorname{Ran}(f)=[0,1)$.



Graph

Q.5. $F(x)=\dfrac{1}{1+x}$, $G(x)=1+\dfrac{1}{x}$.

(a) $\text{Dom}(F)=\mathbb{R}\setminus\{-1\}$,

$\text{Ran}(F)=\mathbb{R}\setminus\{0\}$.

$\text{Dom}(G)=\mathbb{R}\setminus\{0\}$.

$\text{Ran}(G)=\mathbb{R}\setminus\{1\}$.

(b) $(F\circ G)(x)=F(G(x))=F\left(1+\frac{1}{x}\right)=\dfrac{1}{1+1+\frac{1}{x}}$

$$=\frac{1}{2+\frac{1}{x}}.$$

$\begin{cases} 2+\frac{1}{x}\neq 0 \\ x\neq 0 \end{cases}$ $\begin{cases} x\neq -\frac{1}{2} \\ x\neq 0 \end{cases}$

$\therefore$ $\text{Dom}(F\circ G)=\mathbb{R}\setminus\left\{-\frac{1}{2},0\right\}$.

Q.6. (a) $\tan\theta=\frac{3}{4}$, where $180^\circ<\theta<270^\circ$.

(i) $\sin\theta=-\frac{3}{5}$, $\cos\theta=-\frac{4}{5}$

$\therefore$ $10\sin\theta-5\cos\theta=10\left(-\frac{3}{5}\right)-5\left(-\frac{4}{5}\right)=\underline{\underline{-2}}$

(ii) $\cos\frac{\theta}{2}=\pm\sqrt{\dfrac{1+\cos\theta}{2}}=\pm\sqrt{\dfrac{1-\frac{4}{5}}{2}}=\pm\sqrt{\dfrac{1}{10}}$

$\because$ $\cos\frac{\theta}{2}\neq+\sqrt{\frac{1}{10}}>0$, for $90^\circ<\frac{\theta}{2}<135^\circ$

$\therefore$ $\cos\frac{\theta}{2}=-\sqrt{\frac{1}{10}}$. //

(b) $\cos x+\sqrt{3}\sin x=R\sin(x+\alpha)$

$=R\sin\alpha\cos x+R\cos\alpha\sin x$

Comparing the corresponding terms, we have

$\begin{cases} R\sin\alpha=1 \\ R\cos\alpha=\sqrt{3} \end{cases}\Rightarrow\begin{cases} R^2=1^2+(\sqrt{3})^2 \\ \dfrac{\sin\alpha}{\cos\alpha}=\dfrac{1}{\sqrt{3}} \\ =\tan\alpha \end{cases}\Rightarrow\begin{cases} R=2; \\ \alpha=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ =\frac{\pi}{6} \end{cases}$, since $R>0$.

$\therefore$ $\cos x+\sqrt{3}\sin x=2\sin\left(x+\frac{\pi}{6}\right)$. //