Exam 17/18 B

Q1 (a) $\int \frac{e^{3x} - 3e^{-x-2}}{e^{x+1}}\,dx = \int \left[\frac{e^{3x}}{e^{x+1}} - 3\,\frac{e^{-x-2}}{e^{x+1}}\right] dx$

$= \int \left[e^{3x-(x+1)} - 3\,e^{-x-2-(x+1)}\right] dx$

$= \int e^{2x-1} - 3\,e^{-2x-3}\,]\,dx$

$= \frac{e^{2x-1}}{2} - 3\,\frac{e^{-2x-3}}{-2} + C$

(b) $\int x^3 \sec^2(x^4+2)\,dx$

$= \frac{1}{4}\int \sec^2(x^4+2)\;\underbrace{d(x^4+3)}_{4x^3\,dx}$

$= \frac{1}{4}\cancel{\int} \tan(x^4+2) + C$

$y = x^4+2 \Rightarrow \frac{dy}{dx} = 4x^3 \Rightarrow dx = \frac{dy}{4x^3}$

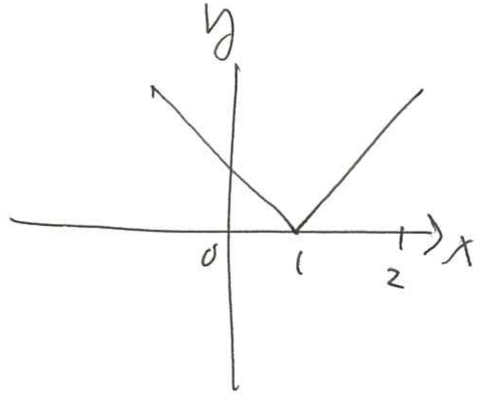
$\int x^3 \sec^2 y\,\frac{dy}{4x^3} = \frac{1}{4}\int \sec^2 y\,dy = \frac{1}{4}\tan y + C$

$= \frac{1}{4}\tan(x^4+2) + C$

(c) $\int_0^2 |x-1|\,dx = \int_0^1 \underbrace{|x-1|}_{-x+1}\,dx + \int_1^2 \underbrace{|x-1|}_{x-1}\,dx$

$= \left[-\frac{x^2}{2} + x\right]_0^1 + \left[\frac{x^2}{2} - 1\right]_1^2$

$= 1$ //



Q2 a) $\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

$y = 9 - x^2$ ordinary substitution doesn't work.

try ~~do~~ trigonometric substitution : $x = 3\sin\theta \Rightarrow \frac{dx}{d\theta} = 3\cos\theta$

$\cancel{\sqrt{9-x^2}} = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta$

(2)

$$= \int \frac{9\sin^2\theta}{\sqrt{9\cos^2\theta}}\, 3\cos\theta\, d\theta$$

$$= \int \frac{9\sin^2\theta}{\cancel{3\cos\theta}}\, \cancel{3\cos\theta}\, d\theta$$

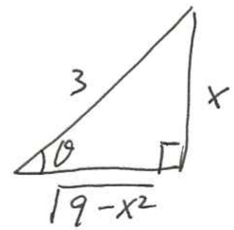
$$= 9\int \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \frac{9}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + C$$

$$= \frac{9}{2}\theta - \frac{9}{4}\underbrace{\sin 2\theta}_{2\sin\theta\cos\theta}$$

$\sin\theta = \frac{x}{3}$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$



$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\,\frac{x}{3}\,\frac{\sqrt{9-x^2}}{3} + C$$ //

(b) $\int (x+1)\tan^{-1}x\,dx = \int \underbrace{\tan^{-1}x}_{u}\ \underbrace{(x+1)\,dx}_{dv}$

$dv \Rightarrow v = \int (x+1)\,dx = \frac{x^2}{2} + x$

$$\overset{IBPs}{=} \left(\frac{x^2}{2}+x\right)\tan^{-1}x - \int \left(\frac{x^2}{2}+x\right) \underbrace{d(\tan^{-1}x)}_{\frac{1}{1+x^2}dx}$$

$$= \left(\frac{x^2}{2}+x\right)\tan^{-1}x - \underbrace{\int \frac{\frac{x^2}{2}+x}{x^2+1}\,dx}_{I_1}$$

③

$$I_1 = \int \frac{\frac{x^2}{2} + x}{x^2+1}\, dx$$

improper rational function, need long division

$$\begin{array}{r l} & \frac{1}{2} \\ \cline{2-2} x^2+1 \,) & \frac{x^2}{2} + x \\ & \frac{x^2}{2} + \frac{1}{2} \\ \cline{2-2} & x - \frac{1}{2} \end{array}$$

$$= \int \left[\frac{1}{2} + \frac{(x - \frac{1}{2})}{x^2+1} \right] dx$$

$$= \int \frac{1}{2}\, dx + \underbrace{\int \frac{x}{x^2+1}\, dx}_{\frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} = \frac{1}{2}\ln|x^2+1|} - \frac{1}{2} \underbrace{\int \frac{1}{x^2+1}\, dx}_{\tan^{-1}x}$$

$$= \frac{1}{2}x + \frac{1}{2}\ln|x^2+1| - \frac{1}{2}\tan^{-1}x + C$$

$$\therefore \int (x+1)\tan^{-1}x\, dx = \left(\frac{x^2}{2} + x\right)\tan^{-1}x - \frac{1}{2}x - \frac{1}{2}\ln|x^2+1| + \frac{1}{2}\tan^{-1}x + C$$

2(c) $I = \int \frac{10x}{(x+3)(x^2+4x+13)}\,dx$

Resolve into partial fractions

$$\frac{10x}{(x+3)(x^2+4x+13)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+4x+13}$$

$\Rightarrow \quad 10x = A(x^2+4x+13) + (Bx+C)(x+3) \quad (*)$

$x=-3: \quad -30 = A(9-12+13) = 10A \Rightarrow A = -3$

compare the coefficient x^2: $\quad 0 = A + B \Rightarrow B = -A = 3$

compare the constant term: $\quad 0 = 13A + 3C \Rightarrow C = -\frac{13}{3}A = 13$

$$I = \underbrace{\int \frac{-3}{x+3}\,dx}_{-3\ln|x+3|} + \underbrace{\int \frac{3x+13}{x^2+4x+13}\,dx}_{I_1}$$

$y = x^2+4x+13 \Rightarrow \frac{dy}{dx} = 2x+4$

express $3x+13 = a(2x+4)+b$

$= 2ax + (4a+b)$

$\Rightarrow \begin{cases} 2a = 3 \Rightarrow a = \frac{3}{2} \\ 4a+b = 13 \Rightarrow b = 13-4a = 13-6 = 7 \end{cases}$

$$I_1 = \int \frac{3x+13}{x^2+4x+13}\,dx = \frac{3}{2}\underbrace{\int \frac{2x+4}{x^2+4x+13}\,dx}_{\int \frac{d(x^2+4x+13)}{x^2+4x+13} = \ln|x^2+4x+13|} + 7\underbrace{\int \frac{1}{x^2+4x+13}\,dx}$$

$\int \frac{1}{(x+2)^2+9}\,dx = \frac{1}{9}\int \frac{1}{\left(\frac{x+2}{3}\right)^2+1}\,dx$

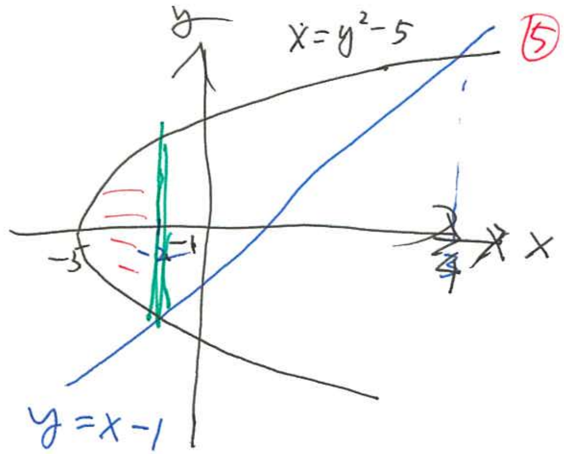
$= \frac{1}{9} \frac{\tan^{-1}\left(\frac{x+2}{3}\right)}{\frac{1}{3}}$

$= \frac{3}{2}\ln|x^2+4x+13| + \frac{7}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + C$

$I = -3\ln|x+3| + \frac{3}{2}\ln|x^2+4x+13| + \frac{7}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + C$ //

Q 3(a) (5)

$R: \begin{cases} x = y^2 - 5 & ① \\ y = x - 1 & ② \end{cases}$



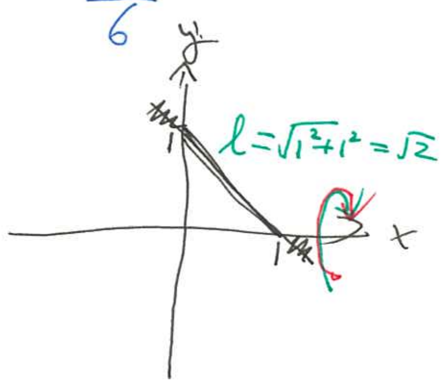
② $y = (y^2 - 5) - 1 \Rightarrow \underbrace{y^2 - y - 6}_{(y+2)(y-3)} = 0 \Rightarrow y = -2$ or 3

$x = -2+1 = -1$ $\quad$ $\cancel{y} = 3 + 1 = 4$

$A = \int_{-\cancel{1}}^{\cancel{4}} \left(\sqrt{x+5} - (x-1) \right) dx$ ✗

$A = \int_{-2}^{3} [\underbrace{x_{outer}}_{y+1} - \underbrace{x_{inner}}_{y^2-5}] \, d\cancel{x}y = \int_{-2}^{3} (y + 1 - y^2 + 5) \, d\cancel{x}y = \frac{125}{6}$

(b) $\begin{cases} x = \cos^2 t \\ y = \sin^2 t \quad 0 \le t \le \frac{\pi}{2} \end{cases} \Rightarrow x + y = 1$



cylinder $V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi 1^2 1 = \frac{\pi}{3}$

$S = \pi r l = \pi \, 1 (\sqrt{2}) = \sqrt{2} \pi$

Alternatively

$\frac{dx}{dt} = 2 \cos t (-\sin t) = -2 \cos t \sin t$

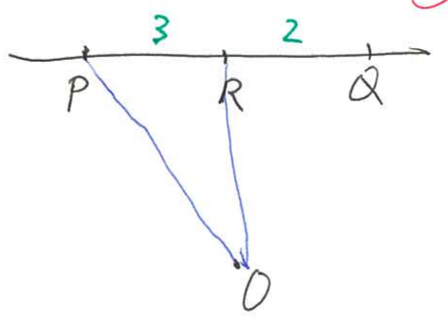
$\frac{dy}{dt} = 2 \sin t \cos t$

$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{4 \cos^2 t \sin^2 t + 4 \sin^2 \cos^2 t} = \sqrt{8} \, |\sin t \cos t|$

$S_x = \int_0^{\frac{\pi}{2}} 2\pi y \, ds = \int_0^{\frac{\pi}{2}} 2\pi \sin^2 t \, |\sin t \cos t| \, dt = = 2\pi \int_0^{\frac{\pi}{2}} \sin^3 t \cos t \, dt = \sqrt{2} \pi$ //

Q 4(a) $P(1,2,3)$, $Q(-3,1,-2)$, $2|\vec{PR}| = 3|\vec{QR}| \Rightarrow \frac{|\vec{PR}|}{|\vec{QR}|} = \frac{3}{2}$

R between P, Q $\Rightarrow 2\vec{PR} = 3\vec{RQ}$



method (I) $2\underbrace{\vec{PR}}_{(x-1)\vec{i}+(y-2)\vec{j}+(z-3)\vec{k}} = 3\vec{RQ}$ $\qquad \vec{OR} = x\vec{i}+y\vec{j}+z\vec{k}$

$$2[(x-1)\vec{i}+(y-2)\vec{j}+(z-3)\vec{k}] = 3[(-3-x)\vec{i}+(1-y)\vec{j}+(-2-z)\vec{k})$$

$\Rightarrow$ ① $2(x-1) = 3(-3-x) \Rightarrow 2x-2 = -9-3x = 5x = -7 \Rightarrow x = -\frac{7}{5}$

② $2(y-2) = 3(1-y) \Rightarrow 2y-4 = 3-3y \Rightarrow 5y = 7 \Rightarrow y = \frac{7}{5}$

③ $2(z-3) = 3(-2-z) \Rightarrow \cancel{-6-3z \Rightarrow}\ 2z-6 = -6-3z \Rightarrow 5z = 0 \Rightarrow z = 0$

$\therefore R = (-\frac{7}{5}, \frac{7}{5}, 0)$

Method (II) $\vec{OR} = \underbrace{\vec{OP}}_{\vec{i}+2\vec{j}+3\vec{k}} + \underbrace{\vec{PR}}_{\frac{3}{5}\vec{PQ}} = (\vec{i}+2\vec{j}+3\vec{k}) + \frac{3}{5}(-4\vec{i}-\vec{j}-5\vec{k})$

$= -\frac{7}{5}\vec{i} + \frac{7}{5}\vec{j}$

$\therefore R = (-\frac{7}{5}, \frac{7}{5}, 0)$ //

(b) $A = (-1,-2,-3)$, $B = (3,-1,2)$, $C = (1,3,0)$

$\vec{AB} = \vec{OB} - \vec{OA} = 4\vec{i}+\vec{j}+5\vec{k}$, $\vec{AC} = \vec{OC} - \vec{OA} = 2\vec{i}+5\vec{j}+3\vec{k}$

$\vec{n} = \vec{AB} \times \vec{AC} = -22\vec{i} - 2\vec{j} + 18\vec{k}$

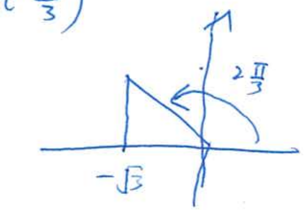
$P(x,y,z) \Rightarrow \cancel{\vec{OP} = x\vec{i}+y\vec{j}+z\vec{k}}$ $\quad \vec{AP} = \vec{OP} - \vec{OA} = (x+1)\vec{i} + (y+2)\vec{j} + (z+3)\vec{k})$

$0 = \vec{n}\cdot\vec{OP} = -22(x+1) - 2(y+2) + 18(z+3) \Rightarrow 11x + y - 9z = 14$ //

Q5(a) $\left(\frac{1+i}{1-i}\right)^{2018} = \left(\frac{1+i}{1-i} \cdot \frac{1+i}{1+i}\right)^{2018} = \left(\frac{1+2i+(-1)}{1^2+1^2}\right)^{2018} \Rightarrow i^{2018} = (-1)^{1009} = -$

$= 1(\cos\pi + i\sin\pi)$ polar form

(b) $(\underbrace{iz)^3}_{-iz^3} = 3+\sqrt{3}i \Rightarrow z^3 = \frac{3+\sqrt{3}i}{-i} \cdot \frac{i}{i} = -\sqrt{3} + 3i = \sqrt{12}\, e^{i\left(\frac{2\pi}{3}\right)}$



$z_k = (\sqrt{12})^{\frac{1}{3}} e^{i\left(\frac{2\pi}{3}+2k\pi\right)/3}, \quad k=0,1,2)$

$z_0 = 12^{\frac{1}{6}} e^{i\frac{2\pi}{9}}$

$z_1 = 12^{\frac{1}{6}} e^{i\left(\frac{2\pi}{3}+2\pi\right)/3} = 12^{\frac{1}{6}} e^{i\frac{8\pi}{9}}$

$z_2 = 12^{\frac{1}{6}} e^{i\left(\frac{2\pi}{3}+4\pi\right)/3} = 12^{\frac{1}{6}} e^{i\left(\frac{14\pi}{9}\right)} = 12^{\frac{1}{6}} e^{i\left(\frac{14\pi}{9}-2\pi\right)} = 12^{\frac{1}{6}} e^{i\left(-\frac{4\pi}{9}\right)}$

not principal

Q 6(a) $|A| = \begin{vmatrix} 3 & 1 & -2 \\ -2 & 2 & 2 \\ 0 & -1 & -1 \end{vmatrix} \overset{R_3}{=} 0 \begin{vmatrix} 1 & -2 \\ 2 & 2 \end{vmatrix} - (-1) \begin{vmatrix} 3 & -2 \\ -2 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix} = -6$ (8)

$$|A^T A^{-2}| = \underbrace{|A^T|}_{|A|} \underbrace{|A^{-1}|^2}_{\left(\frac{1}{|A|}\right)^2} = \frac{1}{|A|} = \frac{1}{-6} = \frac{-1}{6}$$

(b) $A = \begin{pmatrix} 3 & 1 & -2 \\ -2 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} -2 & 2 & 2 \\ 3 & 1 & -2 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{-\frac{1}{2}R_1} \begin{pmatrix} 1 & -1 & -1 \\ 3 & 1 & -2 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{R_2 - 3R_1} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 4 & 1 \\ 0 & -1 & -1 \end{pmatrix}$

$\xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 4 & 1 \end{pmatrix} \xrightarrow{R_3 + 4R_2} \begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -3 \end{pmatrix}$ row echelon form

(c) Gauss - Jordan elimination

$(A|I) = \left(\begin{array}{ccc|ccc} 3 & 1 & -2 & 1 & 0 & 0 \\ -2 & 2 & 2 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} -2 & 2 & 2 & 0 & 1 & 0 \\ 3 & 1 & -2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right)$

$\xrightarrow{-\frac{1}{2}R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 0 & -\frac{1}{2} & 0 \\ 3 & 1 & -2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - 3R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 0 & -\frac{1}{2} & 0 \\ 0 & 4 & 1 & 1 & +\frac{3}{2} & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \end{array}\right)$

$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 0 & -\frac{1}{2} & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 4 & 1 & 1 & \frac{3}{2} & 0 \end{array}\right) \xrightarrow{R_2 + 4R_2} \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 0 & -\frac{1}{2} & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & -3 & 1 & \frac{3}{2} & 4 \end{array}\right)$ row-echelon form

$$\xrightarrow[-\frac{1}{3}R_3]{-R_2} \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{2} & -\frac{4}{3} \end{array}\right) \xrightarrow[R_2-R_3]{R_1+R_3} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & -\frac{1}{3} & -1 & -\frac{4}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{2} & -\frac{4}{3} \end{array}\right)$$

$$\xrightarrow{R_1+R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\frac{1}{2} & -1 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{2} & -\frac{4}{3} \end{array}\right)$$

A^{-1}