MA1201 section E1F1 G1H1 Test 1

1. $\vec{u} = \vec{i} - 2\vec{j} + 3\vec{k}$, $\vec{v} = -2\vec{i} + \vec{j} - 4\vec{k}$, $\vec{w} = -\vec{i} - 2\vec{j} + 2\vec{k}$

(a) [15] $\vec{v} + \vec{w} = (-2\vec{i} + \vec{j} - 4\vec{k}) + (-\vec{i} - 2\vec{j} + 2\vec{k}) = -3\vec{i} - \vec{j} - 2\vec{k}$ ②

$\vec{u} \cdot (\vec{v} + \vec{w}) = (\vec{i} - 2\vec{j} + 3\vec{k}) \cdot (-3\vec{i} - \vec{j} - 2\vec{k})$

$= (1)(-3) + (-2)(-1) + (3)(-2) = -3 + 2 - 6 = -7$ ③

$|\vec{u}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$ ②

$|\vec{v} + \vec{w}| = \sqrt{(-3)^2 + (-1)^2 + (-2)^2} = \sqrt{9+1+4} = \sqrt{14}$ ②

$\cos\theta = \dfrac{\vec{u} \cdot (\vec{v} + \vec{w})}{|\vec{u}||\vec{v} + \vec{w}|}$ ② $= \dfrac{-7}{\sqrt{14}\sqrt{14}} = \dfrac{-7}{14} = \dfrac{-1}{2}$ ② $\Rightarrow \theta = 120^\circ$ ②

b) [15] Volume of parallelepiped $(\vec{u}, \vec{v}, \vec{w}) = |\vec{u} \cdot (\vec{v} \times \vec{w})|$ ②

$\vec{v} \times \vec{w} = (-2\vec{i} + \vec{j} - 4\vec{k}) \times (-\vec{i} - 2\vec{j} + 2\vec{k})$

$= 2\underbrace{(\vec{i} \times \vec{i})}_{\vec{0}} + 4\cancel{(\vec{i} \times \vec{j})}^{\vec{k}} - 4\cancel{(\vec{i} \times \vec{k})}^{-\vec{j}} - \cancel{(\vec{j} \times \vec{i})}^{-\vec{k}} - 2\cancel{(\vec{j} \times \vec{j})}^{0} + 2\cancel{(\vec{j} \times}$

$+ 4\cancel{(\vec{k} \times \vec{i})}^{\vec{j}} + 8\cancel{(\vec{k} \times \vec{j})}^{-\vec{i}} - 8\cancel{(\vec{k} \times \vec{k})}^{0}$

$= \vec{0} + 4\vec{k} + 4\vec{j} + \vec{k} - \vec{0} + 2\vec{i} + 4\vec{j} - 8\vec{i} - \vec{0}$

$= -6\vec{i} + 8\vec{j} + 5\vec{k}$ ⑧

$\therefore \vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{i} - 2\vec{j} + 3\vec{k}) \cdot (-6\vec{i} + 8\vec{j} + 5\vec{k})$

$= (1)(-6) + (-2)(8) + (3)(5) = -6 - 16 + 15 = -7$ ④

$\therefore V = |\vec{u} \cdot (\vec{v} \times \vec{w})| = |-7| = 7$ ①

2 (a) [10] $\displaystyle\int \frac{\sqrt{x} + 1}{x} = \int [x^{-\frac{1}{2}} + x^{-1}]\,dx$ ② $= \dfrac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$ ② $+ \ln|x|$ ② $+ C$ ① $= 2x^{\frac{1}{2}} + \ln|x|$

b) [10] $\displaystyle\int (2x-1)e^{-x}\,dx = \int (2x-1)\,d(-e^{-x})$ ② $= (2x-1)(-e^{-x}) - \int (-e^{-x})\,d(2x-1)$ ②

$= -(2x-1)e^{-x} + \int e^{-x}\,2dx$ ② $= -(2x-1)e^{-x} + 2\dfrac{e^{-x}}{-1} + C$

$= -(2x-1)e^{-x} - 2e^{-x} + C$ ④

c) [15] $\displaystyle\int \frac{dx}{\sqrt{x^2 - 2x + 5}} = \int \frac{dx}{\sqrt{(x-1)^2 + 4}}$

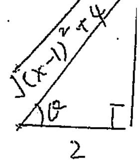
substitution $(x-1)^2 = 4\tan\theta$ or $x - 1 = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$ ②

$\sqrt{(x-1)^2 + 4} = \sqrt{4\tan^2\theta + 4} = 2\sqrt{\tan^2 + 1} = 2$

$= \displaystyle\int \frac{2\sec^2\theta\,d\theta}{2\sec\theta} = \int \sec\theta\,d\theta$ ② $= \ln|\sec\theta + \tan\theta| + C$ ③

$= \ln\left|\dfrac{\sqrt{(x-1)^2 + 4}}{2} + \dfrac{x-1}{2}\right| + C$ ③ $= \ln\left|\sqrt{x^2 - 2x + 5} + x - 1\right| + C'$ ①



2(d) [15]

$$\int_{-1}^{4} |x(x-2)|\,dx = \int_{-1}^{0} x(x-2)\,dx + \int_{0}^{2} -x(x-2)\,dx + \cancel{\int_{2}^{4} x(x-2)\,dx} \quad (5)$$

$$= \int_{-1}^{0} [x^2-2x]\,dx - \int_{0}^{2} [x^2-2x]\,dx + \cancel{\int_{2}^{4} [x^2-2x]\,dx} \quad (3)$$

$$= \left[\frac{x^3}{3} - x^2\right]_{-1}^{0} - \left[\frac{x^3}{3} - x^2\right]_{0}^{2} + \cancel{\left[\frac{x^3}{3} - x^2\right]_{2}^{4}}$$

$$= \left\{\frac{1}{3}[0-(-1)^3] - [0^2-(-1)^2]\right\} - \left\{\frac{1}{3}[2^3-0^3] - [2^2-0^2]\right\} \cancel{+ \left\{\frac{1}{3}[4^3-2^3] - [4^2 - 2^2]\right\}}$$

$$= \left(\frac{1}{3}+1\right) - \left(\frac{8}{3}-4\right) \cancel{+ \left\{\frac{1}{3}(64-8) - (16-4)\right\}}$$

$$= \frac{1}{3} + 1 - \frac{8}{3} + 4 \cancel{+ \frac{56}{3} - 12} \cancel{+ \left(\frac{1}{3} - \frac{8}{3} + \frac{56}{3}\right) + (1+4-12)} = 8/3$$

$$\cancel{= \frac{49}{3} - 7 = \frac{49-21}{3} = \frac{28}{3}} \quad (7)$$

2(e) [20]

$$\frac{13-x}{(2x-1)(x^2+2x+5)} = \frac{A}{2x-1} + \frac{Bx+C}{x^2+2x+5} \quad (2)$$

$$\Rightarrow 13-x = A(x^2+2x+5) + (Bx+C)(2x-1)$$

Set $x = \frac{1}{2}$: $13 - \frac{1}{2} = A\left(\frac{1}{4}+1+5\right) = \frac{25}{4}A = \frac{25}{2} \Rightarrow A = 2$ (2)

Compare coefficients of x^2 : $0 = A + 2B \Rightarrow 2B = -A = -2 \Rightarrow B = -1$ (2)

Compare constant term : $13 = 5A - C \Rightarrow C = 5A - 13 = 10 - 13 = -3$ (2)

$$\therefore \int \frac{13-x}{(2x-1)(x^2+2x+5)}\,dx = \int \frac{2}{2x-1}\,dx + \int \frac{-x-3}{x^2+2x+5}\,dx$$

$$= \int \frac{d(2x-1)}{2x-1} - \frac{1}{2}\int \frac{2x+2}{x^2+2x+5}\,dx - 2\int \frac{1}{x^2+2x+5}\,dx \quad (2)$$

$$= \ln|2x-1| \; (2) - \frac{1}{2}\int \frac{d(x^2+2x+5)}{x^2+2x+5} - 2\int \frac{dx}{(x+1)^2+4}$$

$$= \ln|2x-1| - \frac{1}{2}\ln|x^2+2x+5| \; (3) - \frac{2}{4}\int \frac{dx}{\left(\frac{x+1}{2}\right)^2+1}$$

$$= \ln|2x-1| - \frac{1}{2}\ln|x^2+2x+5| - \frac{1}{2}\cdot\frac{1}{\frac{1}{2}}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

$$= \ln|2x-1| - \frac{1}{2}\ln|x^2+2x+5| - \tan^{-1}\left(\frac{x+1}{2}\right) \; (4) + C \; (1)$$