Tutorial 5 (with solution)

Modular Arithmetic

Question 1: Divisibility by 9

Let x be an n-digit number. Prove that

$$x \equiv a_{n-1} + a_{n-2} + \cdots + a_1 + a_0 \pmod 9,$$

where a_i is the $(i+1)$-th digit of x.

- Example 1:
  - Suppose $x = 6213$. $x \bmod 9 = 6 + 2 + 1 + 3 \bmod 9 = 3$.

- Example 2:
  - Suppose $x = 7218$. Since the digit sum $\pmod 9 = 7 + 2 + 1 + 8 \pmod 9 = 0$, x must be divisible by 9.

Q.1 (solution)

An n-digit number x can be represented by

$$x = a_{n-1}10^{n-1} + a_{n-2}10^{n-2} + \cdots + a_1 10 + a_0.$$

Since $10 \equiv 1 \bmod 9$,

$$10^i \equiv 1 \bmod 9, \text{ for any integer } i.$$

Then ,

$$x \equiv a_{n-1}\times 1 + a_{n-2}\times 1 + \cdots + a_1\times 1 + a_0\times 1 \pmod 9.$$
$$\equiv a_{n-1} + a_{n-2} + \cdots + a_1 + a_0 \pmod 9.$$

Q.E.D

Question 2: Diophantie Equation

- Solve the equation

$$98x + 35y = 14,$$

where x and y are integers.

Q.2 (solution)

98	35		
1	0	98	a
0	1	35	b
1	-2	28	c = a – 2b
-1	3	7	d = b – c

- First, find gcd(98,35).
- By extended Euclidean algorithm,
$$98(-1) + 35(3) = 7.$$
- Multiplying both sides by 2, we obtain
$$98(-2) + 35(6) = 14.$$
- Therefore, $x_0 = -2, y_0 = 6$ is a particular solution.
- Dividing both sides by 7, we obtain
$$14(-2) + 5(6) = 2.$$
- It can be seen that
$$14(-2 + 5t) + 5(6 - 14t) = 2.$$
- The general solution is
$$x = -2 + 5t, y = 6 - 14t,$$
where t is an integer.

Question 3: Application

- The admission fee at a small fair is \$1.5 for children and \$4.00 for adult.
- On a certain day, the fair collected \$5,050.
- It was known that there attended more children than adults, and that in total there were not more than 3,000 people.
- How many possible combinations of number of children and number of adults could be used to satisfy the given conditions?

Q.3 (solution)

- Let x be the number of adults.
- Let y be the number of children.
- Set up the equations and inequalities:

a) $4x + 1.5y = 5050$, or $40x + 15y = 50500$

b) $0 \leq x < y$

c) $x + y \leq 3000$

$40\,(-1) + 15\,(3) = 5$

Multiply both sides by 10100,

$$40(-10100) + 15(30300) = 50500.$$

40	15		
1	0	40	a
0	1	15	b
1	-2	10	$c = a - 2\mathrm{b}$
-1	3	5	$d = b - c$

A particular solution:

$$x_0 = -10100 \text{ and } y_0 = 30300$$

All possible solutions

$$x = x_0 + tb/d = -10100 + 3t.$$

$$y = y_0 - ta/d = 30300 - 8t.$$

$d = \mathrm{gcd}(40{,}15) = 5$

$$x = -10100 + 3t.$$
$$y = 30300 - 8t.$$

Since $x \geq 0$, $t \geq 10100/3$ or $t \geq 3367$ (since t is an integer)

Since $x < y$,

$$-10100 + 3t < 30300 - 8t.$$

$$t < 3672.73 \text{ or } t \leq 3672$$ (since t is an integer)

Since $x + y \leq 3000$,

$$20200 - 5t \leq 3000$$
$$t \geq \frac{17200}{5} = 3440$$

Combining them, $3440 \leq t \leq 3672$.

No. of possible combinations = 233.

Question 4: Repeat-and-Multiply

a) Use the Repeat-and-Multiply method to compute $3^{94} \bmod 17$.

b) User Fermat's Little Theorem to compute $40^{110} \bmod 37$.

Q.4(a) (solution)

- $3^2 \equiv 9 \pmod{17}$
- $3^4 \equiv (9)^2 \equiv 81 \equiv 13 \equiv -4 \pmod{17}$
- $3^8 \equiv (-4)^2 \equiv 16 \equiv -1 \pmod{17}$
- $3^{16} \equiv (-1)^2 \equiv 1 \pmod{17}$
- $3^{32} \equiv 1 \pmod{17}$
- $3^{64} \equiv 1 \pmod{17}$
- Therefore, $3^{94} \equiv 3^{64}3^{16}3^{8}3^{4}3^{2}$

$$\equiv (1)(1)(-1)(-4)(9)$$

$$\equiv 36 \equiv 2 \pmod{17}$$

Q.4(b) (solution)

- First, note that

$$40^{110} \equiv 3^{110} \bmod 37.$$

- Since 37 is a prime, we can use Fermat's Little Theorem, which implies $3^{36} \equiv 1 \bmod 37$.
- Hence,

$$40^{110} \equiv 3^{110} \equiv 3^{36\times 3} 3^2 \equiv 9 \pmod{37}.$$

Question 5: Fermat's Little Theorem

- Solve $x^{103} \equiv 4 \bmod 11$.

Q.5 (solution)

- By Fermat's Little Theorem, $x^{10} \equiv 1 \bmod 11$.
- Therefore, $x^{103} \equiv x^3 \bmod 11$.
- We only need to solve $x^3 \equiv 4 \bmod 11$.
- If we try all values from $x = 1$ through $x = 10$, we find that
$$x \equiv 5 \bmod 11.$$