Exercises on quadratic forms with solutions

1. Determine whether each of the following quadratic forms in two variables is positive or negative definite or semidefinite, or indefinite.
   a. $x^2 + 2xy$.
   b. $-x^2 + 4xy - 4y^2$
   c. $-x^2 + 2xy - 3y^2$.
   d. $4x^2 + 8xy + 5y^2$.
   e. $-x^2 + xy - 3y^2$.
   f. $x^2 - 6xy + 9y^2$.
   g. $4x^2 - y^2$.
   h. $(1/2)x^2 - xy + (1/4)y^2$.
   i. $6xy - 9y^2 - x^2$.

   Solution

   a. The matrix is
   $$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$
   The determinant is $-1 < 0$, so the quadratic form is indefinite.
   b. The matrix is
   $$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}.$$
   The first-order principal minors are -1 and -4; the determinant is 0. Thus the quadratic form is negative semidefinite (but not negative definite, because of the zero determinant).
   c. The matrix is
   $$\begin{pmatrix} -1 & 1 \\ 1 & -3 \end{pmatrix};$$
   the leading principal minors are -1 and 2, so the quadratic form is negative definite.
   d. The associated matrix is
   $$\begin{pmatrix} 4 & 4 \\ 4 & 5 \end{pmatrix}.$$
   The leading principal minors are $4 > 0$ and $(4)(5) - (4)(4) = 4 > 0$. Thus the matrix is positive definite.
   e. The associated matrix is
   $$\begin{pmatrix} -1 & 1/2 \\ 1/2 & -3 \end{pmatrix}.$$
   The leading principal minors are $-1 < 0$ and $(-1)(-3) - (1/2)(1/2) > 0$. Thus the matrix is negative definite.
   f. The associated matrix is
   $$\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}.$$
   The principal minors are $1 > 0$, $9 > 0$, and $(1)(9) - (-3)(-3) = 0$. Thus the matrix is positive semidefinite.
   g. The associated matrix is
   $$\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}.$$

The determinant is $-4 < 0$. Thus the matrix is indefinite.

h. The associated matrix is

$$\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/4 \end{pmatrix}.$$

The determinant is $(1/2)(1/4) - (-1/2)(-1/2) < 0$. Thus the matrix is indefinite.

i. The associated matrix is

$$\begin{pmatrix} -1 & 3 \\ 3 & -9 \end{pmatrix}.$$

The principal minors are $-1 < 0$, $-9 < 0$, and $(-1)(-9) - (3)(3) = 0$. Thus the matrix is negative semidefinite.

2. Determine whether each of the following quadratic forms in three variables is positive or negative definite or semidefinite, or indefinite.
   a. $-x^2 - y^2 - 2z^2 + 2xy$
   b. $x^2 - 2xy + xz + 2yz + 2z^2 + 3zx$
   c. $-4x^2 - y^2 + 4xz - 2z^2 + 2yz$
   d. $-x^2 - y^2 + 2xz + 4yz + 2z^2$
   e. $-x^2 + 2xy - 2y^2 + 2xz - 5z^2 + 2yz$
   f. $y^2 + xy + 2xz$
   g. $-3x^2 + 2xy - y^2 + 4yz - 8z^2$
   h. $2x^2 + 2xy + 2y^2 + 4z^2$

Solution

a. The matrix is

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The first-order minors are -1, -1 and -2, the second-order minors are 0, 2, and 2, and the determinant is 0. Thus the matrix is negative semidefinite.

b. The matrix is

$$\begin{pmatrix} 1 & -1 & 2 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix}.$$

The first-order principal minors are 1, 0, and 2, so the only possibility is that the quadratic form is positive semidefinite. However, the first second-order principal minor is -1. So the matrix is indefinite.

c. The matrix is

$$\begin{pmatrix} -4 & 0 & 2 \\ 0 & -1 & 1 \\ 2 & 1 & -2 \end{pmatrix}.$$

The first-order principal minors are -4, -1, and -2; the second-order principal minors are 4, 4, and 1, and the third-order principal minor is 0. Thus the matrix is negative semidefinite.

d. The matrix is

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$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

The leading principal minors are −1, 1, and 7, so the quadratic form is indefinite.

e. The matrix is

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -5 \end{pmatrix}.$$

The leading principal minors are −1, 1, and 0, so the matrix is not positive or negative definite, but may be negative semidefinite. The first order principal minors are −1, −2, and −5; the second-order principal minors are 1, 4, and 9; the third-order principal minor is 0. Thus the matrix is negative semidefinite.

f. The matrix is

$$\begin{pmatrix} 0 & 1/2 & 1 \\ 1/2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Thus the form is indefinite: one of the first-order principal minors is positive, but the second-order one that is obtained by deleting the third row and column of the matrix is negative.

g. The matrix is

$$\begin{pmatrix} -3 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 2 & -8 \end{pmatrix},$$

with leading principal minors −3, 2, and −4. So the form is negative definite.

h. The matrix is

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

The leading principal minors are 2, 3, and (2)(8) − (1)(4) = 12 > 0. Thus the matrix is positive definite.

3. Consider the quadratic form $2x^2 + 2xz + 2ayz + 2z^2$, where a is a constant. Determine the definiteness of this quadratic form for each possible value of a.

Solution

The matrix is

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & a \\ 1 & a & 2 \end{pmatrix}.$$

The first-order minors are 2, 0, and 2, the second-order minors are 0, 3, and $-a^2$, and determinant $-2a^2$. Thus for $a = 0$ the matrix is positive semidefinite, and for other values of a the matrix is indefinite.

4. Determine the values of a for which the quadratic form $x^2 + 2axy + 2xz + z^2$ is positive definite, negative definite, positive semidefinite, negative semidefinite, and indefinite.

Solution

The matrix is

$$\begin{pmatrix} 1 & a & 1 \\ a & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

The leading principal minors are 1, $-a^2$, and $-a^2$.

Thus if $a \neq 0$ the matrix is indefinite.

If $a = 0$, we need to examine all the principal minors to determine whether the matrix is positive semidefinite. In this case, the first-order principal minors are 1, 0, and 1; the second-order principal minors are 0, 0, and 0; and the third-order principal minor is 0. Thus the quadratic form is positive semidefinite.

Conclusion: If $a \neq 0$ the matrix is indefinite; if $a = 0$ it is positive semidefinite.

5. Consider the matrix

$$\begin{pmatrix} a & 1 & b \\ 1 & -1 & 0 \\ b & 0 & -2 \end{pmatrix}.$$

Find conditions on a and b under which this matrix is negative definite, negative semidefinite, positive definite, positive semidefinite, and indefinite. (There may be no values of a and b for which the matrix satisfies some of these conditions.)

Solution

The matrix is not positive definite or positive semidefinite for any values of a and b, because two of the first-order principal minors are negative. Necessary and sufficient conditions for it to be negative definite are

- $a < 0$
- $-a - 1 > 0$, or $a < -1$ (looking at first second-order principal minor)
- $2a + 2 + b^2 < 0$ (looking at determinant).

Thus it is negative definite if and only if $a < -1$ and $2a + 2 + b^2 < 0$.

It is negative semidefinite if and only if $a \leq -1$, $-2a - b^2 \geq 0$, and $2a + 2 + b^2 \leq 0$. The second condition implies the first, so the matrix is negative semidefinite if and only if $a \leq -1$ and $2a + 2 + b^2 \leq 0$.

Otherwise the matrix is indefinite.

6. Show that the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

is not positive definite.

Solution

The second order principal minor obtained by deleting the second and fourth rows and columns is 0, so the matrix is not positive definite. (Alternatively, the third-order leading principal minor is 0, and the principal minor obtained by deleting the second and third rows and columns is 0.)