1. Let $w = f(x,y,z) = z\left(x^2+y^2\right)^{-1}$. At $(2,1,1)$, find the rate of change of w with respect to *y*. Suppose $x = u+v,\ y = u$ and $z = uv$, find the rates of change of w with respect to *u* and *v*, respectively. Observe that $\frac{\partial f}{\partial y} \neq \frac{\partial f}{\partial u}$ even though $y = u$. Why is this so?

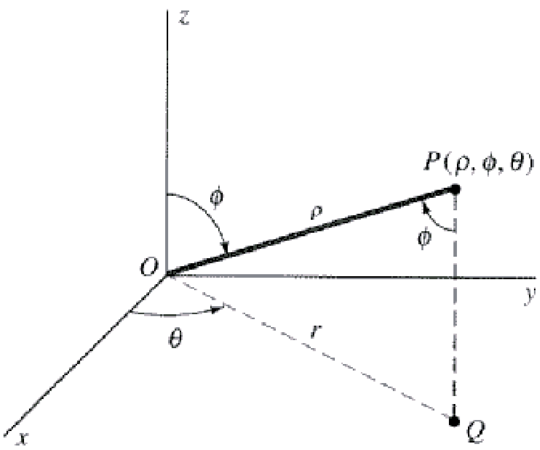
2. Suppose *w* is a function of *u*, *v*, that is, $w = w(u,v)$. Suppose $u = x+y,\ v = x-y,\ w = xy - z$, therefore, *z* is a function of *x*, *y*. Transform the following partial differential equation of *z* in *x*, *y*, $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ into a partial differential equation of *w* in *u* and *v*.

3. Let $z = f\left(x, \frac{x}{y}\right)$ and $s = x, t = \frac{x}{y}$. Assume *f* has continuous second order partial derivatives. Find $\frac{\partial z}{\partial x}$ and show that $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial s^2} + \frac{2}{y}\frac{\partial^2 f}{\partial s \partial t} + \frac{1}{y^2}\frac{\partial^2 f}{\partial t^2}$.

4. Suppose $u = u(x,y)$ satisfies the wave equation $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$. Let $y = y(x) = 2x$. Along the line, $y = 2x$, we have $u(x, y(x)) = x$ and $\left.\frac{\partial u}{\partial x}\right|_{\substack{x=x\\ y=2x}} = x^2$. Find along the line $y = 2x$, $\left.\frac{\partial u}{\partial y}\right|_{\substack{x=x\\ y=2x}}$, $\left.\frac{\partial^2 u}{\partial y \partial x}\right|_{\substack{x=x\\ y=2x}}$, $\left.\frac{\partial^2 u}{\partial x^2}\right|_{\substack{x=x\\ y=2x}}$, $\left.\frac{\partial^2 u}{\partial y^2}\right|_{\substack{x=x\\ y=2x}}$.

5. One of the most popular alternative coordinate systems to Cartesian coordinates is Spherical polar coordinates. Spherical polar coordinates represent a point $P(x,y,z)$ in space by ordered triples (ρ, θ, φ) in which
   a. ρ is the distance from *P* to the origin.
   b. ϕ is the angle $\overrightarrow{OP}$ makes with the positive *z*-axis.
   c. θ is the angle measured counterclockwise from the positive *x*-axis to $\overrightarrow{OQ}$ which is the projection of $\overrightarrow{OP}$ on *xy*-plane.

   The equations relating spherical polar coordinates to Cartesian coordinates are:
   $x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$, where $\rho \geq 0,\ 0 \leq \phi \leq \pi,\ 0 \leq \theta \leq 2\pi$.



(i) Show that

$$\frac{\partial V}{\partial x}=\frac{\partial V}{\partial \rho}\frac{\partial \rho}{\partial x}+\frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial x}+\frac{\partial V}{\partial \phi}\frac{\partial \phi}{\partial x}=\frac{\partial V}{\partial \rho}\sin\phi\cos\theta-\frac{\partial V}{\partial \theta}\frac{\sin\theta}{\rho\sin\phi}+\frac{\partial V}{\partial \phi}\frac{\cos\theta\cos\phi}{\rho},$$

and find also $\frac{\partial V}{\partial y}$, $\frac{\partial V}{\partial z}$.

(ii) (optional) Show that in spherical coordinates (ρ,θ,φ) Laplace's equation

$$\nabla^2 V=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

takes the form

$$\nabla^2 V=\frac{\partial^2 V}{\partial \rho^2}+\frac{2}{\rho}\frac{\partial V}{\partial \rho}+\frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 V}{\partial \theta^2}+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\cot\phi}{\rho^2}\frac{\partial V}{\partial \phi}=0.$$

6. Suppose the equations $\begin{cases} x^2-y^2-u^3+v^2+4=0 \\ 2xy+y^2-2u^2+3v^4+8=0 \end{cases}$ determine functions $u(x,y)$ and $v(x,y)$ near $x=2$ and $y=-1$ such that $u(2,-1)=2$ and $v(2,-1)=1$. Compute $\frac{\partial u}{\partial x}(2,-1)$.

7. It is given that $\begin{cases} x^2+y^2=\frac{1}{2}z^2 \\ x+y+z=2 \end{cases}$. Find $\frac{dx}{dz},\frac{dy}{dz},\frac{d^2x}{dz^2},\frac{d^2y}{dz^2}$ when $x=1,\ y=-1,\ z=2$.

8. Use Taylor's theorem to expand $f(x,y)=\sin xy$ about the point $\left(1,\frac{\pi}{3}\right)$, neglecting cubic and higher terms. Hence estimate $\sin 0.3\pi$.

9. In surveying a triangular plot of land, two of its sides were measured as 160m and 210m with maximum possible errors of 0.1m and the included angle was $\pi/3$ exactly. Estimate the maximum error in calculating the length of the third side from the cosine rule $c=\left(a^2+b^2-2ab\ \cos C\right)^{\frac{1}{2}}$.

10. Find and classify the stationary points of
   (a) $f(x,y)=x^3+3x^2-3y^2+6xy$,
   (b) $f(x,y)=x^3+3xy-3x^2-3y^2+4$,

(c) $z = f(x,y) = x^4 + y^4 - x^2 - 2xy - y^2$.

11. Find the minimum distance between the origin and the surface $z^2 = x^2 y + 4$.

12. The equation $x^3 - 3x + y^2 - 2y + z^3 + z + 1 = 0$ implicitly determines an implicit function $z = z(x, y)$ of x and y defined in R^2. Find the stationary point(s) of $z = z(x, y)$ and determine the local extreme value(s) of $z = z(x, y)$ that $z = z(x, y)$ will attain there (if any) and state what kind of local extreme value(s) it is or they are.

13. If $\varphi(x, y, z) = x^2 y^2 z^2$, find
    (a) the maximum rate of change of φ at the point $(1,1,1)$ and the direction in which this occurs;
    (b) the rate of change of φ at the point $(2,1,1)$ in the direction of $3\vec{i} + 4\vec{k}$.

14. A bomber is carrying a heat seeking missile which has the property that at any point (x, y, z) in space it moves in the direction of maximum temperature increase and the temperature at (x, y, z) is $T = T(x, y, z) = 2x^2 - xyz$.

    Suppose the bomber has just launched the missile at the point $P(1, 2, 3)$ and the missile can move at a speed of 50 km/minute in the direction specified.
    (a) In what direction will the missile move?
    (b) How fast is the temperature experienced by the missile changing in degree Celsius per kilometer at that instant when the missile has just left the bomber?
    (c) How fast is the temperature experienced by the missile changing in degree Celsius per minute at that instant when the missile has just left the bomber?
    (d) Due to the failure of an electronic device, the missile is no longer heat seeking but still can move at a speed of 50 km/minute in the direction specified.
    How fast is the temperature experienced by the missile changing in degree Celsius per minute at that instant when the missile has just left the bomber and moved in the direction specified by $\vec{v} = \vec{i} + \vec{j}$?

15. Let $u(x, y) = 3x^2 + y^2$.
    (a) Find the directional derivative of $u(x, y) = 3x^2 + y^2$ at the point (x, y) in the direction $(1,1)$.
    (b) Determine the points (x, y) and directions for which the directional derivative of $u(x, y) = 3x^2 + y^2$ has its largest value if (x, y) is restricted to lie on the circle $x^2 + y^2 = 1$.

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