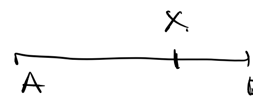
2020/01/19 ch4 practice problem.

Problem 2

Let $A = (0,1,-1)$ and $B = (1,2,0)$ be two points in a plane. Let X be a point between A and B such that $AX{:}XB = 2{:}1$.

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AX}$.

(b) Hence, find the coordinate of X by finding its position vector $\overrightarrow{OX}$. (Hint: $\overrightarrow{AX} = \overrightarrow{OX} - \overrightarrow{OA}$).



$$\text{(a)}\ \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (\vec{i}+2\vec{j}) - (\vec{j}-\vec{k}) = \vec{i}+\vec{j}+\vec{k}.$$

$AX : XB = 2 : 1.$

$|\overrightarrow{AX}| = \frac{2}{3}|\overrightarrow{AB}|.$

$$\overrightarrow{AX} = \frac{2}{3}|\overrightarrow{AB}| \times |\hat{AB}| = \frac{2}{3}\cancel{|\overrightarrow{AB}|} \times \frac{\overrightarrow{AB}}{\cancel{|\overrightarrow{AB}|}} = \frac{2}{3}(\vec{i}+\vec{j}+\vec{k})$$

magnitud × direction.

$$= \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}.$$

$$\text{(b)}.\ \overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$$

$$= (\vec{j}-\vec{k}) + \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k} = \frac{2}{3}\vec{i} + \frac{5}{3}\vec{j} - \frac{1}{3}\vec{k}$$

Problem 3

Let $\vec{a} = 2\vec{\imath} - 3\vec{\jmath} + 5\vec{k}$ and $\vec{b} = \vec{\imath} + 3\vec{\jmath}$ be two vectors.

(a) Find $|\vec{a}|$ and $|\vec{a} - 2\vec{b}|$.

(b) Find the unit vector of $\vec{b}$.

(c) Let $\vec{c}$ be another vector with magnitude $|2\vec{a} + \vec{b}|$ and its direction is same as that of $\vec{b}$. Find the vector $\vec{c}$.

$$\text{(a)}\ |\vec{a}| = \sqrt{2^2+(-3)^2+5^2} = \sqrt{4+9+25} = \sqrt{38}$$

$$|\vec{a}-2\vec{b}| = |(2\vec{i}-3\vec{j}+5\vec{k}) - 2(\vec{i}+3\vec{j})|$$

$$= |-9\vec{j}+5\vec{k}| = \sqrt{(-9)^2+5^2} = \sqrt{81+25} = \sqrt{106}$$

$$\text{(b)}.\ \hat{b} = \frac{\vec{b}}{|\vec{b}|} = \frac{\vec{i}+3\vec{j}}{\sqrt{1^2+3^2}} = \frac{1}{\sqrt{10}}\vec{i} + \frac{3}{\sqrt{10}}\vec{j}$$

$$\text{(c)}.\ \vec{c} = |2\vec{a}+\vec{b}| \times \hat{b} = \sqrt{134}\cdot\left(\frac{1}{\sqrt{10}}\vec{i} + \frac{3}{\sqrt{10}}\vec{j}\right) = \frac{\sqrt{134}}{\sqrt{10}}\vec{i} + \frac{3\sqrt{134}}{\sqrt{10}}\vec{j}$$

magnitude × direction.

$$|2\vec{a}+\vec{b}| = |4\vec{i}-6\vec{j}+10\vec{k}+\vec{i}+3\vec{j}| = |5\vec{i}-3\vec{j}+10\vec{k}|$$
$$= \sqrt{5^2+(-3)^2+10^2} = \sqrt{134}$$

Problem 4

Let $\vec{a} = \vec{\imath} + 3\vec{\jmath} - 2\vec{k}$ and $\vec{b} = -2\vec{\imath} + \vec{\jmath} + 3\vec{k}$ be two vectors.

(a) Find $\vec{a} \cdot \vec{b}$.

(b) Find the angle between the vectors $\vec{a}$ and $\vec{b}$.

(c) Let $\vec{c} = 3\vec{\imath} + x\vec{\jmath} - 2\vec{k}$ be a vector which is perpendicular to $\vec{b}$, find the value of x.

(d) Let $\vec{d} = y\vec{a} + 3\vec{b}$ be a vector which is perpendicular to $\vec{a} - \vec{b}$, find the value of y.

(a). $\vec{a}\cdot\vec{b} = 1\times(-2)+3\times 1+(-2)\times 3. = -2+3-6 = -5.$

(b). $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cdot\cos\theta. \longrightarrow$ θ is the angle between $\vec{a}$ and $\vec{b}$.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{-5}{\sqrt{1^2+3^2+(-2)^2}\cdot\sqrt{(-2)^2+1^2+3^2}} = \frac{-5}{\sqrt{14}\cdot\sqrt{14}} = -\frac{5}{14}.$$

$\theta > 90^\circ$, $\theta \approx 110.92^\circ$

(c) $\vec{b}\cdot\vec{c} = |\vec{b}|\cdot|\vec{c}|\cdot\cos 90^\circ = 0.$

$\vec{b}\cdot\vec{c} = -2\cdot 3 + 1\cdot x + 3\cdot(-2) = 0.$

$x = 6+6 = 12.$

(d) $\vec{d}\cdot(\vec{a}-\vec{b}) = 0.$

$\vec{a} = \vec{\imath} + 3\vec{\jmath} - 2\vec{k}$ and $\vec{b} = -2\vec{\imath} + \vec{\jmath} + 3\vec{k}$

$(y\vec{a}+3\vec{b})\cdot(\vec{a}-\vec{b}) = 0.$

$y(\vec{a}\cdot\vec{a}) - y(\vec{a}\cdot\vec{b}) + 3(\vec{a}\cdot\vec{b}) - 3(\vec{b}\cdot\vec{b}) = 0.$

$y|\vec{a}|^2 + (3-y)(\vec{a}\cdot\vec{b}) - 3|\vec{b}|^2 = 0.$

$y\cdot 14 + (3-y)\cdot(-5) - 3\cdot 14 = 0.$

$19y = 42+15 = 57$

$y = 3.$

Problem 8

Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{a}| = 1$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = 1$.

(a) Find the angle between the vectors $\vec{a}$ and $\vec{b}$.

(b) Find the value of $(3\vec{a} - 2\vec{b}) \cdot (\vec{a} + 3\vec{b})$ and $|\vec{a} - 2\vec{b}|$.

(c) Find the angle between two vectors $\vec{a} - 2\vec{b}$ and $2\vec{a} + 3\vec{b}$.

(a). θ is the angle between $\vec{a}$ and $\vec{b}$.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{1}{1\cdot 2} = \frac{1}{2} \Rightarrow \theta = 60^\circ.$$

(b). $(3\vec{a}-2\vec{b})\cdot(\vec{a}+3\vec{b})$

$$= 3(\vec{a}\cdot\vec{a}) + 9(\vec{a}\cdot\vec{b}) - 2(\vec{a}\cdot\vec{b}) - 6(\vec{b}\cdot\vec{b})$$

$$= 3|\vec{a}|^2 + 7(\vec{a}\cdot\vec{b}) - 6|\vec{b}|^2$$

$$= 3\cdot 1 + 7\cdot 1 - 6\cdot 4 = -14.$$

$$|\vec{a}-2\vec{b}| = \sqrt{(\vec{a}-2\vec{b})(\vec{a}-2\vec{b})}$$

$$= \sqrt{(\vec{a}\cdot\vec{a}) - 4(\vec{a}\cdot\vec{b}) + 4(\vec{b}\cdot\vec{b})}$$

$$= \sqrt{|\vec{a}|^2 - 4(\vec{a}\cdot\vec{b}) + 4|\vec{b}|^2}$$

$$= \sqrt{1-4+4\times 4} = \sqrt{13}.$$

(c) $$\cos\theta = \frac{(\vec{a}-2\vec{b})\cdot(2\vec{a}+3\vec{b})}{|\vec{a}-2\vec{b}|\cdot|2\vec{a}+3\vec{b}|} \quad (*).$$

$\vec{a} - 2\vec{b}$ and $2\vec{a} + 3\vec{b}$.

$$(\vec{a}-2\vec{b})\cdot(2\vec{a}+3\vec{b}) = 2(\vec{a}\cdot\vec{a}) + 3(\vec{a}\cdot\vec{b}) - 4(\vec{a}\cdot\vec{b}) - 6(\vec{b}\cdot\vec{b})$$

$$= 2|\vec{a}|^2 - (\vec{a}\cdot\vec{b}) - 6|\vec{b}|^2.$$

$$= 2\cdot 1 - 1 - 6\cdot 4 = -23.$$

$$|2\vec{a}+3\vec{b}| = \sqrt{(2\vec{a}+3\vec{b})\cdot(2\vec{a}+3\vec{b})}$$

$$= \sqrt{4|\vec{a}|^2 + 12(\vec{a}\cdot\vec{b}) + 9|\vec{b}|^2}$$

$$= \sqrt{4+12+9\times 4} = \sqrt{52}$$

$$\cos\theta = \frac{(\vec{a}-2\vec{b})\cdot(2\vec{a}+3\vec{b})}{|\vec{a}-2\vec{b}|\cdot|2\vec{a}+3\vec{b}|} \quad (*)$$

$$= \frac{-23}{\sqrt{13}\cdot\sqrt{52}} = \frac{-23}{13\cdot 2} = -\frac{23}{26} < 0.$$

$\sqrt{52}$: 13×4.

$\theta \approx 152.2^\circ.$

Problem 9

Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{a}| = 2$ and $|\vec{b}| = 3$ and the angle between these two vectors is $\cos^{-1}\frac{3}{5}$.

(a) Are the vector $\vec{a} - 2\vec{b}$ and $-9\vec{a} + 2\vec{b}$ perpendicular to each other? Explain your answer.

(b) If the angle between the vectors $\vec{a}$ and $\vec{a} + k\vec{b}$ is $60°$, find the value of k.

$$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\theta = 2\cdot 3\cdot\frac{3}{5} = \frac{18}{5}$$

($\theta = \cos^{-1}\frac{3}{5}$.)

(a).

$$(\vec{a}-2\vec{b})\cdot(-9\vec{a}+2\vec{b}) \overset{?}{=} 0$$

$$= -9(\vec{a}\cdot\vec{a}) + 2(\vec{a}\cdot\vec{b}) + 18(\vec{a}\cdot\vec{b}) - 4(\vec{b}\cdot\vec{b})$$

$$= -9|\vec{a}|^2 + 20(\vec{a}\cdot\vec{b}) - 4|\vec{b}|^2. = -9\cdot 4 + 20\cdot\frac{18}{5} - 4\cdot 9$$

$$= -36 + 72 - 36 = 0.$$

$\vec{a}-2\vec{b}$ and $-9\vec{a}+2\vec{b}$ are perpendicular.

$$\vec{a}\cdot(\vec{a}+k\vec{b}) = |\vec{a}|\cdot|\vec{a}+k\vec{b}|\cdot\cos\theta. \quad \Rightarrow \theta = 60^\circ.\ \cos\theta = \frac{1}{2}.$$

$$\vec{a}\cdot(\vec{a}+k\vec{b}) = (\vec{a}\cdot\vec{a}) + k(\vec{a}\cdot\vec{b}) = |\vec{a}|^2 + k(\vec{a}\cdot\vec{b})$$

$$|\vec{a}+k\vec{b}| = \sqrt{(\vec{a}+k\vec{b})\cdot(\vec{a}+k\vec{b})} = \sqrt{|\vec{a}|^2 + 2k(\vec{a}\cdot\vec{b}) + k^2|\vec{b}|^2}$$

$$= \sqrt{4 + 2k\cdot\frac{18}{5} + 9k^2}$$

$$|\vec{a}|\cdot|\vec{a}+k\vec{b}|\cdot\cos\theta = \cancel{2}\sqrt{4+\frac{36}{5}k+9k^2}\cdot\cancel{\frac{1}{2}} = \sqrt{4+\frac{36}{5}k+9k^2}.$$

$$|\vec{a}|^2 + k(\vec{a}\cdot\vec{b}) = 4 + k\cdot\frac{18}{5}.$$

$$4 + \frac{18}{5}k = \sqrt{4 + \frac{36}{5}k + 9k^2}.$$

$$\left(4 + \frac{18}{5}k\right)^2. = 4 + \frac{36}{5}k + 9k^2.$$

$$k = -\frac{40\sqrt{3}}{33} - \frac{30}{11} \quad \text{or} \quad k = \frac{40\sqrt{3}}{33} - \frac{30}{11}.$$