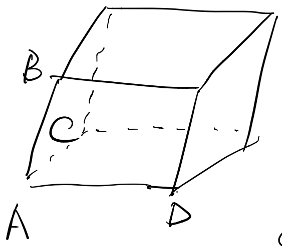
Problem 18

In each of the following, find the volume of parallelepiped with the given four points as the adjacent vertices. Hence determine if the given four points are coplanar.

(a) $A = (2,1,-1)$, $B = (0,1,1)$, $C = (-2,-1,5)$ and $D = (2,3,-3)$.



A B C D coplanar $\iff$ ▱ ABCD $= 0$.

$$V = |\vec{AB}\cdot(\vec{AC}\times\vec{AD})| = 0$$

$$\vec{AB}\cdot(\vec{AC}\times\vec{AD}) = 0.$$

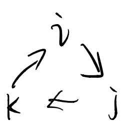
$$\begin{cases} \vec{AB} = \vec{OB}-\vec{OA} = -2\vec{i}+2\vec{k} \\ \vec{AC} = \vec{OC}-\vec{OA} = -4\vec{i}-2\vec{j}+6\vec{k} \\ \vec{AD} = \vec{OD}-\vec{OA} = 2\vec{j}-2\vec{k} \end{cases}$$

$$\vec{AC}\times\vec{AD} = (-4\vec{i}-2\vec{j}+6\vec{k})\times(2\vec{j}-2\vec{k})$$

$$= -8\overset{\vec{k}}{(\vec{i}\times\vec{j})} + 8\overset{-\vec{j}}{(\vec{i}\times\vec{k})} - 4\overset{0}{(\vec{j}\times\vec{j})} + 4\overset{\vec{i}}{(\vec{j}\times\vec{k})} + 12\overset{-\vec{i}}{(\vec{k}\times\vec{j})} - 12\overset{0}{(\vec{k}\times\vec{k})}$$

$+4\vec{i} - 12\vec{i}$

$$= -8\vec{k}-8\vec{j}-8\vec{i} = -8\vec{i}-8\vec{j}-8\vec{k}.$$

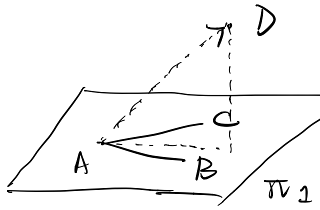


$$\vec{AB}\cdot(\vec{AC}\times\vec{AD}) = (-2)\cdot(-8) + 0\cdot(-8) + 2\cdot(-8) = 0.$$

Volume of paranelepiped $= 0$. $\Rightarrow$ 4 points A. B. C D are coplanar.

Problem 19

(a) Let π_1 be a plane containing the points $A=(3,-2,0)$, $B=(2,0,3)$ and $C=(1,-1,1)$, find the shortest distance between the point $D=(1,0,-1)$ and the plane π_1.



$$d=\left|\text{proj}_{\vec{w}}\vec{AD}\right|$$

$$\vec{w}=\vec{AB}\times\vec{AC}.$$

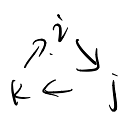
$$\vec{AD}=\vec{OD}-\vec{OA}=-2\vec{i}+2\vec{j}-\vec{k}$$

$$\vec{AB}=\vec{OB}-\vec{OA}=-\vec{i}+2\vec{j}+3\vec{k}$$

$$\vec{AC}=\vec{OC}-\vec{OA}=-2\vec{i}+\vec{j}+\vec{k}.$$

$$\vec{n}=\vec{AB}\times\vec{AC}=(-\vec{i}+2\vec{j}+3\vec{k})\times(-2\vec{i}+\vec{j}+\vec{k})$$

$$=-(\vec{i}\times\vec{j})-(\vec{i}\times\vec{k})-4(\vec{j}\times\vec{i})+2(\vec{j}\times\vec{k})$$



$$-6(\vec{k}\times\vec{i})+3(\vec{k}\times\vec{j})$$

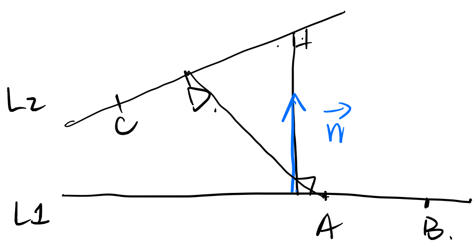
$$=\cancel{-\vec{k}}\cancel{+\vec{j}}+\cancel{4\vec{k}}+\cancel{2\vec{i}}\cancel{-6\vec{j}}\cancel{-3\vec{i}}=-\vec{i}-5\vec{j}+3\vec{k}$$

$$\cos\theta=\frac{\vec{AD}\cdot\vec{n}}{|\vec{AD}||\vec{w}|}=\frac{2-10-3}{\sqrt{(-2)^2+2^2+(-1)^2}\cdot\sqrt{(-1)^2+(-5)^2+3^2}}=\frac{-11}{3\sqrt{35}}$$

$$\left|\text{proj}_{\vec{n}}\vec{AD}\right|=\left||\vec{AD}|\cdot\cos\theta\right|=\left|3\cdot\frac{-11}{3\sqrt{35}}\right|=\frac{11}{\sqrt{35}}$$

Problem 20

(a) Let L_1 be a line passing through the points $\overset{A}{(5,0,-1)}$ and $\overset{B}{(6,2,-2)}$. We let L_2 be another line passing through the points $\underset{C}{(2,4,0)}$ and $\underset{D}{(3,3,1)}$. Find the shortest distance between the line L_1 and L_2.



$d = |\operatorname{proj}_{\vec{n}} \vec{AD}|$

$\vec{n} = \vec{AB} \times \vec{CD}$.

$\vec{AD} = \vec{OD} - \vec{OA} = -2\vec{i} + 3\vec{j} + 2\vec{k}$

$\vec{AB} = \vec{OB} - \vec{OA} = \vec{i} + 2\vec{j} - \vec{k}$

$\vec{CD} = \vec{OD} - \vec{OC} = \vec{i} - \vec{j} + \vec{k}$.

$\vec{n} = \vec{AB} \times \vec{CD} = (\vec{i} + 2\vec{j} - \vec{k}) \times (\vec{i} - \vec{j} + \vec{k})$

$= -\vec{i} \times \vec{j} + \vec{i} \times \vec{k} + 2(\vec{j} \times \vec{i}) + 2(\vec{j} \times \vec{k}) - \vec{k} \times \vec{i} + \vec{k} \times \vec{j}$

i
$k \leftarrow j$

$= \cancel{-\vec{k}} \cancel{-\vec{j}} \cancel{-2\vec{k}} + \cancel{2\vec{i}} \cancel{-\vec{j}} \cancel{-\vec{i}} = \vec{i} - 2\vec{j} - 3\vec{k}$

$$\cos\theta = \frac{\vec{AD} \cdot \vec{n}}{|\vec{AD}| \cdot |\vec{n}|} = \frac{-2 \cdot 1 + 3 \cdot (-2) + 2 \cdot (-3)}{\sqrt{(-2)^2 + 3^2 + 2^2} \cdot \sqrt{1^2 + (-2)^2 + (-3)^2}} = \frac{-14}{\sqrt{17} \cdot \sqrt{14}}$$

$$|\operatorname{proj}_{\vec{n}} \vec{AD}| = ||\vec{AD}| \cdot \cos\theta| = \left| \cancel{\sqrt{17}} \cdot \frac{-14}{\cancel{\sqrt{17}} \cdot \sqrt{14}} \right| = \sqrt{14}.$$

Problem 21

Determine if each of the following set of vectors are linearly independent.

(b) $\vec{a} = \vec{\imath} - 2\vec{\jmath} + 3\vec{k}$, $\vec{b} = 2\vec{\imath} + 5\vec{\jmath} + \vec{k}$ and $\vec{c} = 3\vec{\imath} + 2\vec{\jmath} - 3\vec{k}$.

$|\vec{a}\cdot(\vec{b}\times\vec{c})| = 0 \Rightarrow \vec{a}, \vec{b}, \vec{c}$ are coplanar $\Rightarrow \vec{a}, \vec{b}, \vec{c}$ are linear depedent.

$$\vec{b}\times\vec{c} = (2\vec{i}+5\vec{j}+\vec{k})\times(3\vec{i}+2\vec{j}-3\vec{k})$$
$$= \cancel{4\vec{k}} + \cancel{6\vec{j}} + 15(-\vec{k}) - \cancel{15\vec{i}} + \cancel{3\vec{j}} + \cancel{2(-\vec{i})}$$
$$= -17\vec{i} + 9\vec{j} - 11\vec{k}$$

$\vec{i} \to \vec{j}$, $\vec{k} \leftarrow$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = 1\cdot(-17) + (-2)\cdot 9 + 3\cdot(-11)$$
$$= -17 - 8 - 33 \neq 0.$$

$\vec{a}, \vec{b}, \vec{c}$ are not coplanar, $\Rightarrow \vec{a}, \vec{b}, \vec{c}$ are linear independent.

Part E: A bit harder problems

Problem 23

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-zero vectors. Show that

(a) $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$.

(b) If $\vec{a}$ and $\vec{b}$ are perpendicular, then $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$.

(c) If $\vec{a}$ and $\vec{b}$ are parallel, then the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are also parallel. (Hint: If two vectors are parallel, what is the angle between them? What can you say about the vector product of these two vectors?) $\theta = 0 / 180°$. $|\vec{a}||\vec{b}|\sin\theta$.

(d) $\tan\theta = \frac{|\vec{a}\times\vec{b}|}{\vec{a}\cdot\vec{b}}$ where θ is the angle between $\vec{a}$ and $\vec{b}$.

(e) $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$.

(a) $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b}) = \vec{a}\times\vec{a} + \vec{a}\times\vec{b} - \vec{b}\times\vec{a} - \vec{b}\times\vec{b}$

$\vec{a}\times\vec{a} = |\vec{a}||\vec{a}|\sin\theta\cdot\vec{n}$. $\quad = \vec{a}\times\vec{b} - \vec{b}\times\vec{a}$

$\theta = 0 \quad = \vec{a}\times\vec{b} + \vec{a}\times\vec{b} = 2(\vec{a}\times\vec{b})$

$\sin\theta = 0$.

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cdot\cos\theta$

(b) $\theta = 90°$. $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$.

$$|\vec{a}+\vec{b}| = \sqrt{(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})} = \sqrt{\vec{a}\cdot\vec{a} + 2(\vec{a}\cdot\vec{b}) + \vec{b}\cdot\vec{b}}$$
$$= \sqrt{\vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b}} = \sqrt{\vec{a}\cdot\vec{a} - 2(\vec{a}\cdot\vec{b}) + \vec{b}\cdot\vec{b}}$$

$$= \sqrt{(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})} = |\vec{a}-\vec{b}|.$$

(c) If $\vec{a}$ and $\vec{b}$ are parallel, then the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are also parallel. (Hint: If two vectors are parallel, what is the angle between them? What can you say about the vector product of these two vectors?) $\theta = 0 / 180^\circ$. $|\vec{a}||\vec{b}|\sin\theta$.

(c) $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b}) = 2(\vec{a}\times\vec{b}) = 2(|\vec{a}||\vec{b}|\underset{0}{\sin\theta})\cdot\vec{n} = 0.$

(d) $\tan\theta = \frac{|\vec{a}\times\vec{b}|}{\vec{a}\cdot\vec{b}}$ where θ is the angle between $\vec{a}$ and $\vec{b}$.

(e) $|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$.

(d). $\frac{|\vec{a}\times\vec{b}|}{\vec{a}\cdot\vec{b}} = \frac{\cancel{|\vec{a}||\vec{b}|}\sin\theta}{\cancel{|\vec{a}||\vec{b}|}\cos\theta} = \tan\theta.$

(e). $|\vec{a}\times\vec{b}|^2 = (|\vec{a}||\vec{b}|\sin\theta)^2 = |\vec{a}|^2|\vec{b}|^2\sin^2\theta$

$= |\vec{a}|^2|\vec{b}|^2(1-\cos^2\theta)$

$= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta.$

$= |a|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2.$