MA2001 Solutions **Assignment for Chapter 2** **Vector Integral Calculus**

1. Find the work done in moving a particle from $(0,0)$ to $(1,1)$ in the force field $\vec{F}=\left(xy+2y^2\right)\vec{i}+\left(3x^2+y\right)\vec{j}$ along the paths, (a) $y=x^2$; (b) $y=x$; (c) the *y*-axis and then $y=1$. What work is done if the particle moves from $(0,0)$ to $(1,1)$ along path (b) and returns to the origin along path (a)?

Solution:

(a)

$C_1: y=x^2$ from $(0,0)$ to $(1,1)$.

$$\int_{C_1}\vec{F}\cdot d\vec{r}=\int_{C_1}\left[\left(xy+2y^2\right)\vec{i}+\left(3x^2+y\right)\vec{j}\right]\cdot\left(dx\vec{i}+dy\vec{j}\right)=\int_{C_1}\left(xy+2y^2\right)dx+\left(3x^2+y\right)dy$$

$$=\int_0^1\left(x^3+2x^4\right)dx+\int_0^1 4y\,dy=\frac{53}{20}$$

(b)

$C_2: y=x$ from $(0,0)$ to $(1,1)$.

$$\int_{C_2}\vec{F}\cdot d\vec{r}=\int_{C_2}\left[\left(xy+2y^2\right)\vec{i}+\left(3x^2+y\right)\vec{j}\right]\cdot\left(dx\vec{i}+dy\vec{j}\right)=\int_{C_2}\left(xy+2y^2\right)dx+\left(3x^2+y\right)dy$$

$$=\int_0^1\left(x^2+2x^2\right)dx+\int_0^1\left(3y^2+y\right)dy=\frac{5}{2}$$

(c)

$C_3:$ the *y*-axis and then $y=1$.

$$\int_{C_3}\vec{F}\cdot d\vec{r}=\int_{C_3}\left[\left(xy+2y^2\right)\vec{i}+\left(3x^2+y\right)\vec{j}\right]\cdot\left(dx\vec{i}+dy\vec{j}\right)=\int_{C_3}\left(xy+2y^2\right)dx+\left(3x^2+y\right)dy$$

$$=\int_0^1 y\,dy+\int_0^1\left(x+2\right)dx=3$$

The work done when the particle moves from $(0,0)$ to $(1,1)$ along path (b) and returns to the origin along path (a):

$$\oint_{C_2-C_1}\vec{F}\cdot d\vec{r}=\oint_{C_2}\vec{F}\cdot d\vec{r}-\oint_{C_1}\vec{F}\cdot d\vec{r}=\frac{5}{2}-\frac{53}{20}=-\frac{3}{20}$$

2. Prove that the vector field $\vec{F}=\left(3x^2-y\right)\vec{i}+\left(2yz^2-x\right)\vec{j}+2y^2z\vec{k}$ is conservative, but not solenoidal. Hence find a scalar function $f(x,y,z)$ such that $F=\nabla f$ and evaluate $\int_C\vec{F}\cdot d\vec{r}$ along any curve C joining the point $(0,0,0)$ to the point $(1,2,3)$.

Solution:

$$curl\vec{F}=\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 3x^2-y & 2yz^2-x & 2y^2z\end{vmatrix}=\left(4yz-4yz\right)\vec{i}+\left[-1-(-1)\right]\vec{k}=\vec{0}$$

$$div\vec{F}=\nabla\cdot\vec{F}=\frac{\partial}{\partial x}\left(3x^2-y\right)+\frac{\partial}{\partial y}\left(2yz^2-x\right)+\frac{\partial}{\partial z}\left(2y^2z\right)=6x+2z^2+2y^2.$$

$\vec{F}$ is not solenoidal.

$$\nabla f=\vec{F}\Rightarrow\begin{cases}\dfrac{\partial f}{\partial x}=3x^2-y\\ \dfrac{\partial f}{\partial y}=2yz^2-x\Rightarrow f\left(x,y,z\right)=x^3-yx+g\left(y,z\right)\Rightarrow\dfrac{\partial f}{\partial y}=-x+\dfrac{\partial g}{\partial y}\\ \dfrac{\partial f}{\partial z}=2y^2z\end{cases}$$

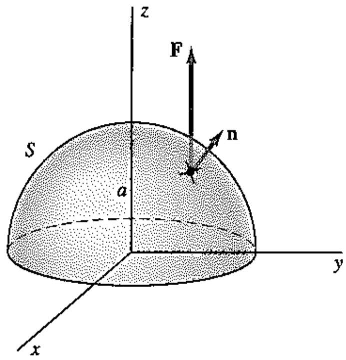
$$2yz^2-x=-x+\frac{\partial g}{\partial y}\Rightarrow\frac{\partial g}{\partial y}=2yz^2\Rightarrow g\left(y,z\right)=y^2z^2+h\left(z\right)$$

$$\Rightarrow f\left(x,y,z\right)=x^3-yx+y^2z^2+h\left(z\right)$$

$$\Rightarrow\frac{\partial f}{\partial z}=2y^2z+h'\left(z\right)\Rightarrow 2y^2z=2y^2z+h'\left(z\right)\Rightarrow h'\left(z\right)=0\Rightarrow h\left(z\right)=C$$

So $f\left(x,y,z\right)=x^3-yx+y^2z^2\ +c$. And $\int\limits_c\vec{F}\cdot d\vec{r}=f\left(1,2,3\right)-f\left(0,0,0\right)=35$.

3. Calculate the flux $\iint\limits_S\vec{f}\cdot\vec{n}dS$ where $\vec{f}=v_0\vec{k}$ and S is the hemispherical surface of radius a with equation $z=\sqrt{a^2-x^2-y^2}$ and with outer unit normal vector $\vec{n}$.



Solution:

$$z=\sqrt{a^2-x^2-y^2}\Rightarrow\varphi\left(x,y,z\right)=x^2+y^2+z^2=a^2\Rightarrow\nabla\varphi=2x\vec{i}+2y\vec{j}+2z\vec{k}\ ,\ \varphi_z=2z.$$

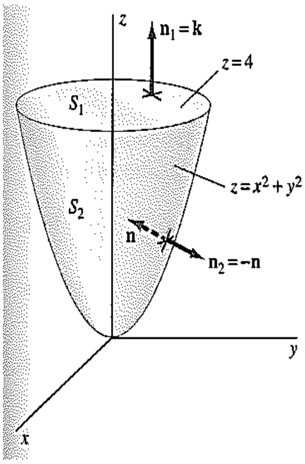
$$\iint\limits_S\vec{f}\cdot\vec{n}dS=\iint\limits_{\sigma_{xy}}\vec{f}\cdot\frac{\nabla\varphi}{\left|\varphi_z\right|}dxdy=\iint\limits_{\sigma_{xy}}v_0\vec{k}\cdot\frac{2x\vec{i}+2y\vec{j}+2z\vec{k}}{2z}dxdy=\iint\limits_{\sigma_{xy}}v_0dxdy.$$

σ_{xy} is a circular disk of radius a.

Introducing polar coordinates $dxdy=adrd\theta$ we have

$$\iint\limits_S\vec{f}\cdot\vec{n}dS=\iint\limits_{\sigma_{xy}}\vec{f}\cdot\frac{\nabla\varphi}{\left|\varphi_z\right|}dxdy=\iint\limits_{\sigma_{xy}}v_0dxdy=\int\limits_0^a\left(\int\limits_0^{2\pi}v_0rd\theta\right)dr=\int\limits_0^a2\pi v_0rdr=\pi v_0r^2\Big|_0^a=\pi a^2v_0.$$

4. Find the flux of the vector field $\vec{f} = x\vec{i} + y\vec{j} + 3\vec{k}$ out of *S*, where *S* is the closed surface of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$. (i.e. to find $\iint_S \vec{F} \cdot d\vec{S}$.)



Solution:
Let S_1 denote the circular top, which has outer unit normal vector $\vec{n_1} = \vec{k}$.
Let S_2 be the parabolic part of this surface, with outer unit normal vector $\vec{n_2}$.
The flux across S_1 is $\iint_{S_1} \left(x\vec{i} + y\vec{j} + 3\vec{k}\right) \cdot \vec{k} dS = \iint_{S_1} 3dS = 3 \times 2^2 \pi = 12\pi$, because S_1 is a circular disk of radius 2.

To compute $\iint_{S_2} \vec{f} \cdot \vec{n_2} dS$.
Let $\varphi(x, y, z) = z - x^2 - y^2$, then $\pm\nabla\varphi(x, y, z) = \pm\left(-2x\vec{i} - 2y\vec{j} + \vec{k}\right)$
We select $-\nabla\varphi(x, y, z) = -\left(-2x\vec{i} - 2y\vec{j} + \vec{k}\right) = 2x\vec{i} + 2y\vec{j} - \vec{k}$ and $|\varphi_z| = 1$.

$$\iint_{S_2} \vec{f} \cdot \vec{n_2} dS = \iint_{\sigma_{xy}} \vec{f} \cdot \frac{-\nabla\varphi}{|\varphi_z|} dxdy = \iint_{\sigma_{xy}} \left(x\vec{i} + y\vec{j} + 3\vec{k}\right) \cdot \left(2x\vec{i} + 2y\vec{j} - \vec{k}\right) dxdy = \iint_{\sigma_{xy}} \left(2x^2 + 2y^2 - 3\right) dxdy$$

σ_{xy} is a circular disk of radius 2.
Introducing polar coordinates $dxdy = adrd\theta$ we have

$$\iint_{\sigma_{xy}} \left(2x^2 + 2y^2 - 3\right) dxdy = \int_0^{2\pi} \left(\int_0^2 \left(2r^2 - 3\right) rdr \right) d\theta = 2\pi \left(\frac{r^4}{2} - \frac{3r^2}{2} \right) \Bigg|_0^2 = 4\pi.$$

Hence the total flux of $\vec{f}$ out of *T* is 16π.

Remark: You can also evaluate with the use of the Divergence Theorem for this question.

5. Consider the magnetic field $\vec{B} = (x+2)\vec{i} + (1-3y)\vec{j} + 2z\vec{k}$ and evaluate the total magnetic flux through each of the faces of the cube bounded by the planes $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1$. Check that your result is consistent with the divergence theorem.

Solution:

$$\iint_S \vec{B}\cdot d\vec{S} = \iint_{ABCD} \vec{B}\cdot d\vec{S} + \iint_{OEFG} \vec{B}\cdot d\vec{S} + \iint_{OADG} \vec{B}\cdot d\vec{S} + \iint_{EBCF} \vec{B}\cdot d\vec{S} + \iint_{OABE} \vec{B}\cdot d\vec{S} + \iint_{GDCF} \vec{B}\cdot d\vec{S}$$
$$= 3-2-1-2+0+2 = 0$$

$\nabla\cdot\vec{B} = \frac{\partial}{\partial x}(x+2) + \frac{\partial}{\partial y}(1-3y) + \frac{\partial}{\partial z}(2z) = 0$. $\iint_S \vec{B}\cdot d\vec{S} = \iiint_V \nabla\cdot\vec{B}dxdydz = \iiint_V 0dxdydz = 0$.

6. Use the divergence theorem to show that $\iint_S (x^2+y+z)\,dS = \frac{4}{3}\pi$ where W is the solid ball $x^2+y^2+z^2 \le 1$ and S is its boundary.

Solution:

Let $\phi(x,y,z) = x^2+y^2+z^2$.

Then $\vec{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2x\vec{i}+2y\vec{j}+2zk}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\vec{i}+y\vec{j}+z\vec{k}}{\sqrt{x^2+y^2+z^2}} = x\vec{i}+y\vec{j}+z\vec{k}$, note that $x^2+y^2+z^2=1$.

$\vec{F}\cdot\vec{n} = (F_1\vec{i}+F_2 j+F_3\vec{k})\cdot(x\vec{i}+y\vec{j}+z\vec{k}) = xF_1+yF_2+zF_3 = x^2+y+z$

Let $xF_1 = x^2, yF_2 = y, zF_3 = z$, then $F_1 = x, F_2 = 1, F_3 = 1$. So $\vec{F} = x\vec{i}+\vec{j}+\vec{k}$ and $\nabla\cdot\vec{F} = 1$.

Thus, by Divergence Theorem,

$$\iint_S (x^2+y+z)\,dS = \iint_S (x\vec{i}+\vec{j}+\vec{k})\cdot(x\vec{i}+y\vec{j}+z\vec{k})\,dS$$
$$= \iiint_W \nabla\cdot(x\vec{i}+y\vec{j}+z\vec{k})\,dxdydz = \iiint_W 1dxdydz = \text{Volume}(W) = \frac{4}{3}\pi(\text{radius of the ball})^3$$
$$= \frac{4}{3}\pi\times 1^3 = \frac{4}{3}\pi$$

7. Verify the divergence theorem for the vector field $\vec{F} = (8+z)\vec{j} + z^2\vec{k}$ and the region bounded by the planes $z=0,\ z=6,\ x=2,\ y=0$ and the surface $y^2 = 8x$ in the first octant.

Solutions:

For the plane in $x=2$, the outward normal to it is $\vec{i}$ .So $\vec{F}\cdot\vec{i} = 0$ and $\iint_{x=2} \vec{F}\cdot\vec{i}dS = 0$.

For the plane in $z=0$, the outward normal to it is $-\vec{k}$. So $\vec{F}\cdot -\vec{k} = -z^2$, however, $z=0$ implies the surface integral on it $=0$.

For the plane in $y=0$, the outward normal to it is $-\vec{j}$. So $\vec{F}\cdot -\vec{j} = 8+z$ and

$$\iint_{y=0} -(8+z)dxdz = \int_0^2\left[\int_0^6 -(8+z)\,dz\right]dx = -132.$$

For the plane in $z=6$, the outward normal to it is $\vec{k}$. $\vec{F}\cdot\vec{k} = z^2$ and project the plane in $z=6$ onto the *x-y* plane. So $\int_0^2\left(\int_0^{\sqrt{8x}} 36dy\right)dx = 192$

For the curved surface in $y^2 = 8x$, let $\phi(x,y,z) = y^2-8x$ then $\nabla\varphi = -8\vec{i}+2y\vec{j}$. Project the curved surface in $y^2 = 8x$ onto the *x-z* plane, suppose we have, say, σ_{xz} .

$$\iint_{\sigma_{xz}} \vec{F}\cdot\frac{\nabla\varphi}{|\varphi_y|}dxdz=\int_0^2\left[\int_0^6 \frac{2y(8+z)}{2y}dz\right]dx=132$$

The total surface integral $=192$.

We have $\nabla\cdot\vec{F}=2z$ and project the region bounded by the planes $z=0, z=6, x=2, y=0$ and the surface $y^2=8x$ onto the *x*-*y* plane, suppose we have, say, σ_{xz}. Then

$$\iiint_V 2zdxdydz=\iint_{\sigma_{xy}}\left(\int_0^6 z^2dz\right)dxdy=\int_0^2\left(\int_0^{\sqrt{8x}} 36dy\right)dx=192$$

8. Verify Stokes's theorem for the vector field $\vec{F}=(x-y)\vec{i}+2z\vec{j}+x^2\vec{k}$ where S is the cone $z=\sqrt{x^2+y^2}$ for $x^2+y^2\le 4$.

Solution:

$$z=\sqrt{x^2+y^2}\Rightarrow\varphi(x,y,z)=z-\sqrt{x^2+y^2}\Rightarrow\nabla\varphi=\pm\left(-\frac{x}{z}\vec{i}-\frac{y}{z}\vec{j}+\vec{k}\right)$$

Now, we choose $\nabla\varphi=\left(-\frac{x}{z}\vec{i}-\frac{y}{z}\vec{j}+\vec{k}\right)$, the upper normal which makes an acute angle with the positive direction of *z*-axis.

In addition, $\nabla\times\vec{F}=-2\vec{i}-2x\vec{j}+\vec{k}$.

$$\iint_S \nabla\times\vec{F}\cdot d\vec{S}=\iint_{x^2+y^2\le 4}\left(-2\vec{i}-2x\vec{j}+\vec{k}\right)\cdot\left(-\frac{x}{z}\vec{i}-\frac{y}{z}\vec{j}+\vec{k}\right)dxdy=\iint_{x^2+y^2\le 4}\left(\frac{2x}{\sqrt{x^2+y^2}}+\frac{2xy}{\sqrt{x^2+y^2}}+1\right)dxdy$$

$$=\int_0^{2\pi}\left[\int_0^2\left(\frac{2r\cos\theta}{r}+\frac{2r^2\cos\theta\sin\theta}{r}+1\right)rdr\right]d\theta=4\pi$$

According to right-handed rule , use the representation $x=2\cos\theta, y=2\sin\theta, z=2, 0\le\theta\le 2\pi$ (anti-clockwise)

$$\oint_C \vec{F}\cdot d\vec{r}=\int_0^{2\pi}\left((x-y)\vec{i}+2z\vec{j}+x^2\vec{k}\right)\cdot\left(dx\vec{i}+dy\vec{j}+dz\vec{k}\right)=\int_0^{2\pi}(x-y)dx+2zdy+x^2dz$$

$$=\int_0^{2\pi}(2\cos\theta-2\sin\theta)(-2\sin\theta)d\theta+8\cos\theta d\theta=4\pi$$

9. Verify Stokes's theorem by evaluating both sides of $\iint_S \nabla\times\vec{F}\cdot\vec{n}dS=\oint_C \vec{F}\cdot d\vec{r}$ for the vector field $\vec{F}=(2x-y)\vec{i}-yz^2\vec{j}-y^2z\vec{k}$ where *S* is the curved surface of the hemisphere $x^2+y^2+z^2=16,\ z\le 0$ and C is its boundary.

Solution:

$$x^2+y^2+z^2=16,\ z\le 0\Rightarrow\varphi(x,y,z)=16-x^2-y^2-z^2\Rightarrow\pm\nabla\varphi=\pm\left(-2x\vec{i}-2y\vec{j}-2z\vec{k}\right).$$

Now, we choose $\nabla\varphi=-2x\vec{i}-2y\vec{j}-2z\vec{k}$, the upper normal which makes an acute angle with the positive direction of *z*-axis (notice $z\le 0$).

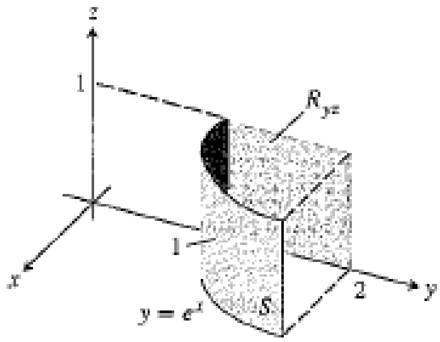
$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2x-y & -yz^2 & -y^2z\end{vmatrix}=\vec{k}$$

$$\iint_S \nabla\times\vec{F}\cdot d\vec{S}=\iint_{x^2+y^2\le 16}\vec{k}\cdot\left(\frac{-2x\vec{i}-2y\vec{j}-2z\vec{k}}{|-2z|}\right)dxdy=\iint_{x^2+y^2\le 16}\vec{k}\cdot\left(\frac{-2x\vec{i}-2y\vec{j}-2z\vec{k}}{-2z}\right)dxdy$$
$$=\iint_{x^2+y^2\le 16}1dxdy=16\pi$$

According to right-handed rule, use the representation $x=4\cos\theta, y=4\sin\theta, z=0$
$0\le\theta\le 2\pi$ (anti-clockwisely)

$$\oint_C \vec{F}\cdot d\vec{r}=\int_0^{2\pi}\left((2x-y)\vec{i}-yz^2\vec{j}-y^2z\vec{k}\right)\cdot\left(dx\vec{i}+dy\vec{j}+dz\vec{k}\right)=\int_0^{2\pi}(2x-y)dx-yz^2dy-y^2zdz$$
$$=\int_0^{2\pi}(8\cos\theta-4\sin\theta)(-4\sin\theta)d\theta=\int_0^{2\pi}16\sin^2\theta d\theta=\int_0^{2\pi}16\left(\frac{1-\cos 2\theta}{2}\right)d\theta=16\pi$$

10. Let S be the portion of the cylinder $y=e^x$ in the first octant that projects parallel to the x-axis onto the rectangle $R_{yz}:1\le y\le 2,\ 0\le z\le 1$ in the yz-plane. Let $\vec{n}$ be the unit vector normal to S that points away from the yz-plane. Find the flux of the field $\vec{F}(x,y,z)=-2\vec{i}+2y\vec{j}+z\vec{k}$ across S in the direction of $\vec{n}$.



Solutions:
Let $\varphi(x,y,z)=y-e^x$. Then $y=e^x\Leftrightarrow\varphi(x,y,z)=0$. $\varphi(x,y,z)=y-e^x\Rightarrow\pm\nabla\varphi=\pm\left(-e^x\vec{i}+\vec{j}\right)$.
Observe that the normal to S makes an acute angle with $\vec{i}$, as a result, we choose
$-\nabla\varphi=-\left(-e^x\vec{i}+\vec{j}\right)=e^x\vec{i}-\vec{j}$ with $|\varphi_x|=e^x$.

$$\iint_S\vec{F}\cdot d\vec{S}=\iint_S\vec{F}\cdot\vec{n}dS=\iint_S\vec{F}\cdot(-\nabla\varphi)\frac{1}{|\varphi_x|}dydz=\iint_S\left(-2\vec{i}+2y\vec{j}+z\vec{k}\right)\cdot\left(e^x\vec{i}-\vec{j}\right)\frac{1}{e^x}dydz$$
$$=\iint_S\left(\frac{-2e^x-2y}{e^x}\right)dydz\underset{y=e^x}{=}\iint_{R_{yz}}\left(\frac{-2y-2y}{y}\right)dydz=\iint_{R_{yz}}-4dydz=\int_0^1\left(\int_1^2-4dy\right)dz=-4$$

11. Let $\vec{F} = \left(y^2 - z^2\right)\vec{i} + \left(z^2 - x^2\right)\vec{j} + \left(x^2 - y^2\right)\vec{k}$. Use Stoke's Theorem to calculate $\oint_C \vec{F} \cdot d\vec{r}$ where C is the path which is the intersection of the plane $x + y + z = 2$ and the faces of a parallelepiped bounded by the planes $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 2$. The direction of the path C is anticlockwise when looking from the positive direction of x-axis.

Solution:

According to Stokes's Theorem, $\oint_C \vec{F} \cdot d\vec{r} = \iint_S curl\vec{F} \cdot d\vec{S}$.

$$\nabla \times \vec{F} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{pmatrix} = \left(-2y - 2z\right)\vec{i} - \left(2x + 2z\right)\vec{j} + \left(-2x - 2y\right)\vec{k}.$$

$\phi(x, y, z) = x + y + z \Rightarrow \nabla\phi = \vec{i} + \vec{j} + \vec{k}$ and $\phi_z = 1$.

The normal to $x + y + z = 2$ makes an acute angle with the positive direction of z-axis, and the projection of the plane enclosed by C onto xy-plane is $\left\{\left(x, y\right) : 0 \le x \le 1, 0 \le y \le 1\right\}$

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_{\substack{0\le x\le 1 \\ 0\le y\le 1}} \left[\left(-2y - 2z\right)\vec{i} - \left(2x + 2z\right)\vec{j} + \left(-2x - 2y\right)\vec{k}\right] \cdot \frac{\vec{i} + \vec{j} + \vec{k}}{1} dxdy$$

$$= \iint_{\substack{0\le x\le 1 \\ 0\le y\le 1}} \left(-4x - 4y - 4z\right) dxdy = \int_0^1 \left(\int_0^1 \left[-4x - 4y - 4\left(2 - x - y\right)\right] dy \right) dx = \int_0^1 \left(\int_0^1 -8dy \right) dx = -8$$