Take Home Assignment MA2001 #1

For each of the following questions, write down your solution with details of steps. Marks will not be given if only final answers are provided.

1. Use diagonalization to compute A^8, where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$.

2. Let $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$, $\vec{v_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\vec{v_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Suppose you are told that $\vec{v_1}$ and $\vec{v_2}$ are eigenvectors of A. Use this information to diagonalize A.

3. Let
$$A = \begin{bmatrix} 4 & 2 & 3 \\ -1 & 1 & -3 \\ 2 & 4 & 9 \end{bmatrix}.$$
It is known $\lambda = 3$ is one eigenvalue of A. Find linearly independent eigenvectors corresponding to λ.

4. Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, $\vec{v_1} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\vec{v_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Verify that $\vec{v_1}$ and $\vec{v_2}$ are eigenvectors of A. Then orthogonally diagonalize A.

5. A quadratic form Q in the components $x_1, \ldots, x_n$ of a vector $\vec{x} = [x_1, \ldots, x_n]^\top$ with symmetric coefficient matrix $A = (a_{ij})_{1 \le i,j \le n}$ is defined to be
$$Q(\vec{x}) := \vec{x}^\top A \vec{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j.$$
Use **THREE** methods to determine whether the following quadratic forms in three variables is positive or negative definite or semidefinite, or indefinite.
$$Q(\vec{x}) = 2(x_1^2 + x_2^2 + x_3^2 + x_1 x_3).$$

6. Matrices are $n \times n$ and vectors are in $\mathbb{R}^n$. Mark each statement True or False. Justify each answer.

   (a) If A is diagonalizable, then A is invertible.

(b) If $A\vec{x} = \lambda\vec{x}$ for some vector $\vec{x}$, then λ is an eigenvalue of A.

(c) A matrix A is not invertible if and only if 0 is an eigenvalue of A.

(d) A number c is an eigenvalue of A if and only if the equation $(A - cI)\vec{x} = \vec{0}$ has a nontrivial solution.

(e) A positive definite quadratic form Q satisfies $Q(\vec{x}) > 0$ for all $\vec{x}$ in $\mathbb{R}^n$.

(f) If the eigenvalues of a symmetric matrix A are all positive, then the quadratic form $\vec{x}^T A\vec{x}$ is positive definite.

7. (a) Construct a matrix A which is diagonalizable, but not invertible.

   (b) Construct another matrix B which is invertible, but not diagonalizable.

   (c) Construct two vectors which are linearly independent, but not orthogonal.

8. **Discovery Question**. Read "https://en.wikipedia.org/wiki/Gram-Schmidt process" and use the Gram-Schmidt process to find an orthogonal basis spanning the same space of $\mathbb{R}^n$ as the given of vectors:

   (a) $< 1, 4, 0 >, < 2, -5, 0 >$ in $\mathbb{R}^3$.

   (b) $< 0, 2, 1, -1 >, < 0, -1, 1, 6 >, < 0, 2, 2, 3 >$ in $\mathbb{R}^4$.