Classification of conic sections

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The four types of conic sections that we have discussed are: (or)

- **Parabola**: $(x-h)^2 = 4a(y-k)$ or $(y-k)^2 = 4a(x-h)$
- **Circle**: $(x-h)^2 + (y-k)^2 = r^2$
- **Ellipse**: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$ Thin $(a > b > 0)$

  Fat
- **Hyperbola**: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ North-South openings

  East-West openings

Each of the above equations could be expressed into the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0 \quad \text{............}(*)$$

where A, C, D, E, F are constants.

(Note that there is no xy term in equation (*).)

If the equation of a conic section is expressed in the form (*), one can identify the type of conic section by first using the technique "***completing the square***" to write the equation into the standard form of the corresponding type [see Example 7(b), 13 and 15].

Example 16

Classify the type of conic section described by each of the following equations by completing the square.

(a) $4x^2 - 16x + 25y^2 - 84 = 0$

(b) $4x^2 + 4y^2 + 8x - 24y + 15 = 0$

Solution

(a) $4x^2 - 16x + 25y^2 - 84 = 0 \Rightarrow 4(x^2 - 4x) + 25y^2 - 84 = 0$

$$\Rightarrow 4[(x-2)^2 - 2^2] + 25y^2 - 84 = 0$$

$$\Rightarrow 4(x-2)^2 + 25y^2 = 100$$

$$\Rightarrow \frac{4(x-2)^2}{100} + \frac{25y^2}{100} = \frac{100}{100}$$

$$\Rightarrow \frac{(x-2)^2}{25} + \frac{y^2}{4} = 1$$

"fat"

$\Rightarrow \frac{(x-2)^2}{5^2} + \frac{y^2}{2^2} = 1$ $\quad \therefore$ The above equation represents an **ellipse**.

Note that this is a "fat" ellipse.

- It is centred at $(2,0)$.
- Its vertices are at $(-5+2,0)$, $(5+2,0)$, $(0+2,-2)$ and $(0+2,2)$,

  i.e. $(-3,0)$, $(7,0)$, $(2,-2)$ and $(2,2)$.

  $c=\sqrt{5^2-2^2}=\sqrt{21}$
- Its foci are at $(-\sqrt{21}+2,0)$ and $(\sqrt{21}+2,0)$.

(b) $4x^2+4y^2+8x-24y+15=0$

$\Rightarrow\ 4x^2+8x+4y^2-24y+15=0$

$\Rightarrow\ 4(x^2+2x)+4(y^2-6y)+15=0$

$\Rightarrow\ 4[(x+1)^2-1^2]+4[(y-3)^2-3^2]+15=0$

$\Rightarrow\ 4(x+1)^2+4(y-3)^2=25$

$\Rightarrow\ (x+1)^2+(y-3)^2=\frac{25}{4} \quad \Rightarrow\ (x-(-1))^2+(y-3)^2=\left(\frac{5}{2}\right)^2$

$\therefore$ The above equation represents a **circle**, centred at $(-1,3)$ and its radius is $\frac{5}{2}$.

Example 17

The following equations represent typical degenerate conic sections. Identify the graph of each equation by completing the square.

(a) $2x^2+y^2-4y+16=0$

(b) $2x^2+4x+y^2-4y+6=0$

Solution

(a) $2x^2+y^2-4y+16=0 \Rightarrow 2x^2+(y-2)^2-2^2+16=0 \Rightarrow 2x^2+(y-2)^2=-12$

Since the LHS $=2x^2+(y-2)^2\geq 0$ for all real values of x and y while the RHS $=-12<0$, the graph contains **no point**.

(b) $2x^2+4x+y^2-4y+6=0 \Rightarrow 2(x^2+2x)+y^2-4y+6=0$

$\Rightarrow 2[(x+1)^2-1^2]+(y-2)^2-2^2+6=0$

$\Rightarrow 2(x+1)^2+(y-2)^2=0$

$\Rightarrow x=-1,\ y=2$ is the only solution.

$\therefore$ The graph contains only **one point** $(-1,2)$.

Classification of conic section by considering the coefficients of x^2 and y^2:

NOT VERY RELIABLE !

Given an equation of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$, in which at least one of A and C is non-zero. If it is given that the equation belongs one of the four types of conic sections (circle, ellipse, parabola and hyperbola), then one can identify its type by considering the coefficients of x^2 and y^2, that is, A and C:

- It is a circle if $A = C$.
- It is a parabola if $AC = 0$ (i.e. either $A = 0$ or $C = 0$)
- It is an ellipse if $A \neq C$ and $AC > 0$ (i.e. A and C are unequal but have the same sign)
- It is a hyperbola if $AC < 0$ (i.e. A and C have different signs).

From slide p.43:

Example 12

Identify the graph of the following functions

(a) $3x^2 - 2y^2 + 5x - y - 5 = 0$
(b) $2x^2 + 2y^2 - x + y - 7 = 0$
(c) $y^2 - 4x + 2y - 1 = 0$

☺Solution:

(a) Note that $AC = 3 \times (-2) = -6 < 0$, the graph is a hyperbola.

(b) Note that $AC = 2 \times 2 = 4 > 0$, the graph is either circle or ellipse. Since $A = C = 2$, we conclude that the graph is a circle.

(c) Note that $AC = 0 \times 1 = 0$ and coefficient of x $(= -4)$ is non-zero, the graph is a parabola.

Example 18

For the equations in Example 16,

(a) $4x^2 - 16x + 25y^2 - 84 = 0$, (b) $4x^2 + 4y^2 + 8x - 24y + 15 = 0$,

identify the types of conic sections described by the equations by considering the coefficients of x^2 and y^2.

Solution

(a) For the equation $4x^2 - 16x + 25y^2 - 84 = 0$, we have $A = 4$ and $C = 25$.
Since $A \neq C$ and $AC > 0$, the above equation represents an ellipse.

(b) For the equation $4x^2 + 4y^2 + 8x - 24y + 15 = 0$, we have $A = 4$ and $C = 4$.
Since $A = C$, the above equation represents a circle.

However, the above mentioned method, which only requires comparison between A and C, is not very reliable.

Consider the following examples:

E.g. 1: For the equation $x^2 + y^2 + 1 = 0$, we have $A = 1$ and $C = 1$. Since $A = C$, one may say that the equation represents a circle. However, $x^2 + y^2 + 1 = 0 \Rightarrow x^2 + y^2 = -1$ which represents no points at all ($x^2 + y^2 \geq 0$ for all real numbers x and y).

E.g. 2: For the equation $x^2 - 4y^2 + 2x + 1 = 0$, we have $A = 1$ and $C = -4$. Since $AC < 0$, one may say that it is a hyperbola. However,

$$x^2 - 4y^2 + 2x + 1 = 0 \Rightarrow (x+1)^2 - 4y^2 = 0 \Rightarrow y = \pm\frac{1}{2}(x+1)$$

which represents a pair of intersecting straight lines.

E.g. 3: For the equation $x^2 + 4y^2 = 0$, we have $A = 1$ and $C = 4$. Since $A \neq C$ and $AC > 0$, one may say that it is an ellipse. However,

$$x^2 + 4y^2 = 0 \Rightarrow x = 0,\ y = 0$$ (which is a single point (0,0).)

Note: The above 3 examples are ***degenerate*** cases.

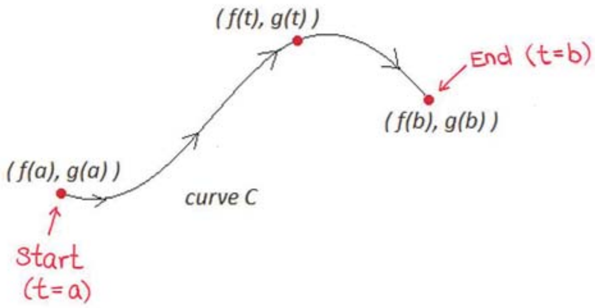
Parametric Equations

Suppose that an object moves around in the xy-plane so that the coordinates of its position at any time t are functions of the variable t:

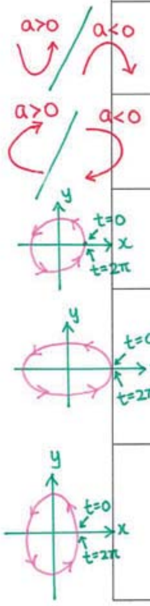
$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad a \le t \le b.$$

The point (x, y) moves and traces a curve C as t varies.

These equations are called the ***parametric equations*** for the curve C. The independent variable t is called a **parameter**.

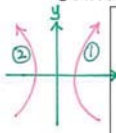


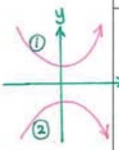
The equations for the four conics (circle, ellipse, parabola, and hyperbola) can be expressed in parametric form.

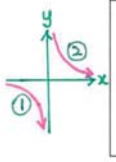


Type of Conics	Equation in Rectangular Coordinate Form	Equation in Parametric Form
Parabola Type 1	$x^2 = 4ay$	$\begin{cases} x = 2at \\ y = at^2 \end{cases}, -\infty < t < \infty$
Parabola Type 2	$y^2 = 4ax$	$\begin{cases} x = at^2 \\ y = 2at \end{cases}, -\infty < t < \infty$
Circle	$x^2 + y^2 = r^2$	$\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}, 0 \le t \le 2\pi$
Ellipse "Fat"	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (a > b)$	$\begin{cases} x = a\cos t \\ y = b\sin t \end{cases} (a > b)$, $0 \le t \le 2\pi$
Ellipse "Thin"	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \ (a > b)$	$\begin{cases} x = b\cos t \\ y = a\sin t \end{cases} (a > b)$, $0 \le t \le 2\pi$

$\sin^2 t + \cos^2 t = 1$







Hyperbola (East-West Openings)	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\begin{cases} x = a \sec t \\ y = b \tan t \end{cases}$, $-\frac{\pi}{2} < \overset{1}{t} < \frac{\pi}{2}$ and $\frac{\pi}{2} < \overset{2}{t} < \frac{3\pi}{2}$
Hyperbola (North-South Openings)	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$	$\begin{cases} x = b \tan t \\ y = a \sec t \end{cases}$, $-\frac{\pi}{2} < \overset{1}{t} < \frac{\pi}{2}$ and $\frac{\pi}{2} < \overset{2}{t} < \frac{3\pi}{2}$
Rectangular Hyperbola	$xy = c^2$; c is a constant	$\begin{cases} x = t \\ y = \frac{c^2}{t} \end{cases}$, $-\infty < \overset{1}{t} < 0$ and $0 < \overset{2}{t} < \infty$

$\sec^2 t - \tan^2 t = 1$ (Ch4)

<u>Remark</u>: In the parametric equations for circles, ellipses and hyperbolae, the parameter t is measured in radians. (π radians = 180°). Topic on radian measure will be discussed in Chapter 4.