MA1201 Calculus and Basic Linear Algebra II

Chapter 5

Complex Number

What is a Complex Number?

- To start with, we consider the following quadratic equations

$$x^2 + 2x + 2 = 0.$$

- One can use quadratic formula to find

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2}.$$

- One cannot use any real number to represent $\sqrt{-4}$ since the square of any real number is always non-negative and there is no real number u such that

$$u^2 = -4.$$

- One would like to extend the current number system (real number) so that it can include the square root of a negative number.

- To start with, we define i to be the number satisfying the equation

$$i^2 = -1 \quad \text{or} \quad i = \sqrt{-1}.$$

Definition (Complex Number)

We say a number z is complex number if and only if it can be expressed in the following form:

$$z = a + bi \ \left(\text{or} \quad z = a + b\sqrt{-1}\right)$$

where a and b are both real numbers.

We also denote the set of these numbers to be $\mathbb{C}$.

Different from the classical number system you have learnt before, a complex number is the sum of a real number a (real part) and multiple of $i = \sqrt{-1}$ (imaginary part).

$$z = \underbrace{a}_{\substack{\text{Real Part} \\ a = \operatorname{Re} z}} + \underbrace{b}_{\substack{\text{Imaginary Part} \\ b = \operatorname{Im} z}} i$$

Equality of two complex numbers

We say two complex numbers are equal $a + bi = c + di$ if and only if

$$a = c \quad \text{and} \quad b = d.$$

In other word, two complex numbers are equal only when *both* real part and imaginary part are the same.

Some Examples of Complex Numbers

- $\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3i$ is a complex number.
- $4 - 3i$ is a complex number.
- $3i = 0 + 3i$ is also a complex number.
  (Remark: We call this complex number to be *purely imaginary* since its real part is zero)
- Any real number (e.g.: $8 = 8 + 0i$) is also a complex number.
  (Remark: We said this complex number to be *purely real*).

In order words, the complex number is an extension of real number $\mathbb{R}$ by introducing a new imaginary number $i = \sqrt{-1}$.

Basic Operation of Complex Number

For consistency, the operation of complex number must be same as that of real numbers since the complex numbers includes all real numbers.

Let $z_1 = a + bi$ and $z_2 = c + di$ be two complex numbers, we then have

Operation of Complex Number (Addition, subtraction and multiplication)

(1) $z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i$.

(2) $z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i$.

(3) $$\begin{aligned} z_1 z_2 &= (a + bi)(c + di) = a(c + di) + bi(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci + bd\underbrace{(-1)}_{i^2=-1} \\ &= (ac - bd) + (ad + bc)i. \end{aligned}$$

Question: How to do division $\frac{z_1}{z_2} = \frac{a+bi}{c+di}$?

Objective: Express the fraction into a complex number, i.e.

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} = \frac{a+b\sqrt{-1}}{c+d\sqrt{-1}} = \cdots = x + y\sqrt{-1} = \underbrace{x+yi}_{\text{Target!}}.$$

We can eliminate the $\sqrt{-1}$ in the denominator by rationalization: multiplying both numerator and denominator by a common factor $c - di$.

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} = \frac{a+bi}{c+di}\left(\frac{c-di}{c-di}\right) = \frac{ac+bci-adi-bdi^2}{c^2+cdi-cdi-d^2i^2}$$

$$= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \left(\frac{bc-ad}{c^2+d^2}\right)i.$$

Remark: There is no need for you to remember this formula. Just need to know how it is derived.

Example 1

Express the following in the form of $a + bi$, where a and b are real numbers.

$(a)\ i^{2013}$, $(b)\ (4+i)(2-3i)$, $(c)\ \dfrac{3+i}{1-i}$

☺Solution of (a)

$$i^{2013} = i^{2012}i = (i^2)^{1006}i = (-1)^{1006}i = i$$

☺Solution of (b)

$$(4+i)(2-3i) = 8 - 10i - 3i^2 = 8 - 10i - 3(-1) = 11 - 10i.$$

☺Solution of (c)

$$\frac{3+i}{1-i} = \frac{3+i}{1-i}\left(\frac{1+i}{1+i}\right) = \frac{3+4i+i^2}{1-i^2} = \frac{3+4i+(-1)}{1-(-1)} = \frac{2+4i}{2} = 1 + 2i.$$

Example 2 (Power of complex number)

Compute $(1+2i)^4$ and $(1-3i)^{-3}$

☺Solution

Using Binomial Theorem, we get

$$\begin{aligned}(1+2i)^4 &= C_0^4(1)^4 + C_1^4(1)^3(2i) + C_2^4(1)^2(2i)^2 + C_3^4(1)(2i)^3 + C_4^4(2i)^4 \\ &= 1 + 8i + 24i^2 + 32i^3 + 16i^4 = 1 + 8i + 24(-1) + 32(-i) + 16(1) \\ &= -7 - 24i.\end{aligned}$$

$$\begin{aligned}(1-3i)^{-3} &= \frac{1}{(1-3i)^3} = \frac{1}{C_0^3(1)^3 + C_1^3(1)^2(-3i) + C_2^3(1)(-3i)^2 + C_3^3(-3i)^3} \\ &= \frac{1}{1 - 9i + 27i^2 - 27i^3} = \frac{1}{1 - 9i + 27(-1) - 27(-i)} = \frac{1}{-26 + 18i} \\ &= \frac{1}{-26+18i}\left(\frac{-26-18i}{-26-18i}\right) = \frac{-26-18i}{676 - 324i^2} = \frac{-26-18i}{676+324} = \frac{-13-9i}{500}.\end{aligned}$$

Example 3 (Square Root of a complex number)

Similar to the case for real number, we define the *square root* $(\sqrt{z}\,)$ of a complex number $z = x + yi$ as a complex number $a + bi$ satisfying

$$(a + bi)^2 = z = x + yi.$$

Using this definition, find the value(s) of $\sqrt{3 + 4i}$.

☺Solution:

According to the definition, we have to find a complex number $a + bi$ satisfying

$$(a + bi)^2 = 3 + 4i.$$

To find the unknowns a and b, we expand the equation and get

$$a^2 + 2abi + b^2 \underbrace{i^2}_{-1} = 3 + 4i \Rightarrow (a^2 - b^2) + 2abi = 3 + 4i.$$

Comparing the real part and the imaginary part, we have

$$a^2 - b^2 = 3, \qquad 2ab = 4.$$

From the second equation, we have $b = \frac{2}{a}$.

Substitute this into the first equation, we have

$$a^2 - \left(\frac{2}{a}\right)^2 = 3 \Rightarrow a^4 - 3a^2 - 4 = 0$$

$$\Rightarrow (a^2 - 4)(a^2 + 1) = 0$$

$$\Rightarrow (a - 2)(a + 2)(a^2 + 1) = 0$$

$$\Rightarrow a - 2 = 0 \text{ or } a + 2 = 0 \text{ or } a^2 + 1 = 0 \text{ (rejected as } a \text{ is real)}$$

$$\Rightarrow a = 2 \ (b = 1) \text{ or } a = -2 \ (b = -1).$$

Therefore, we conclude that

$$\sqrt{3 + 4i} = a + bi = 2 + i \text{ or } -2 - i.$$

Similar to the case of real number, the square root of a complex number has more than one value.

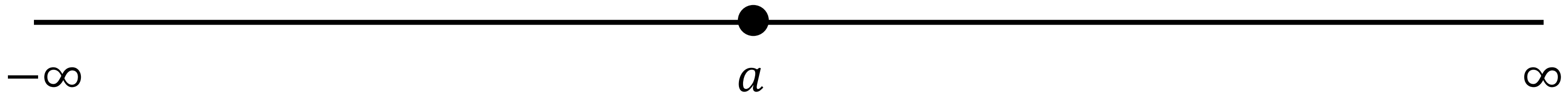
Question: How to compute $(a+bi)^n$ or $\sqrt[n]{a+bi}$ in general?

- Although we can use the previous method to do the computation, it is not efficient when n is large (say $n = 10,\ 100$ etc.). One has to seek other alternatives in order to reduce the computation cost.

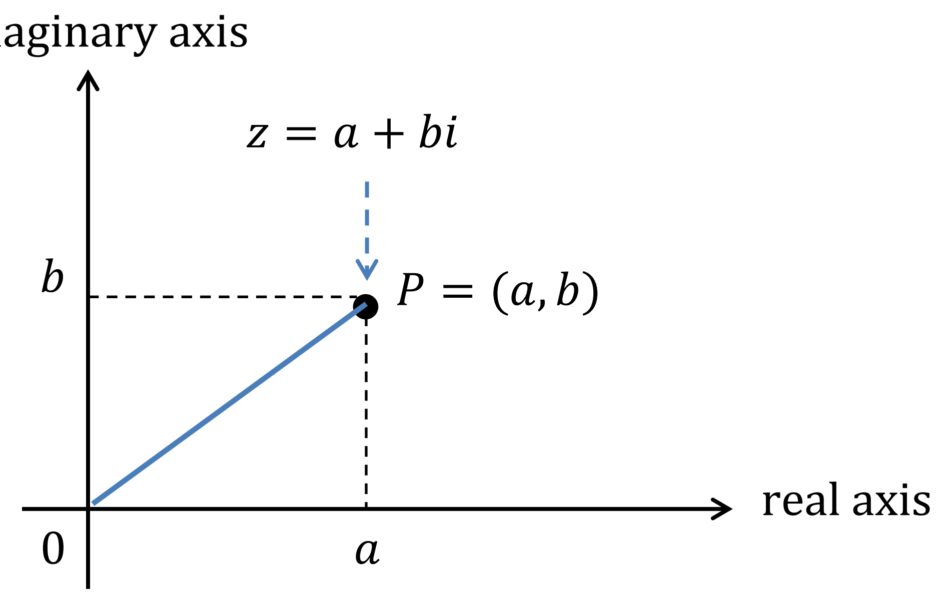
- In order to develop a better method, one has to represent a complex number $(a+bi)$ in other ways: polar form $z = r(\cos\theta + i\sin\theta)$ and Euler (or exponential) form.

Geometric Representation of a Complex Number

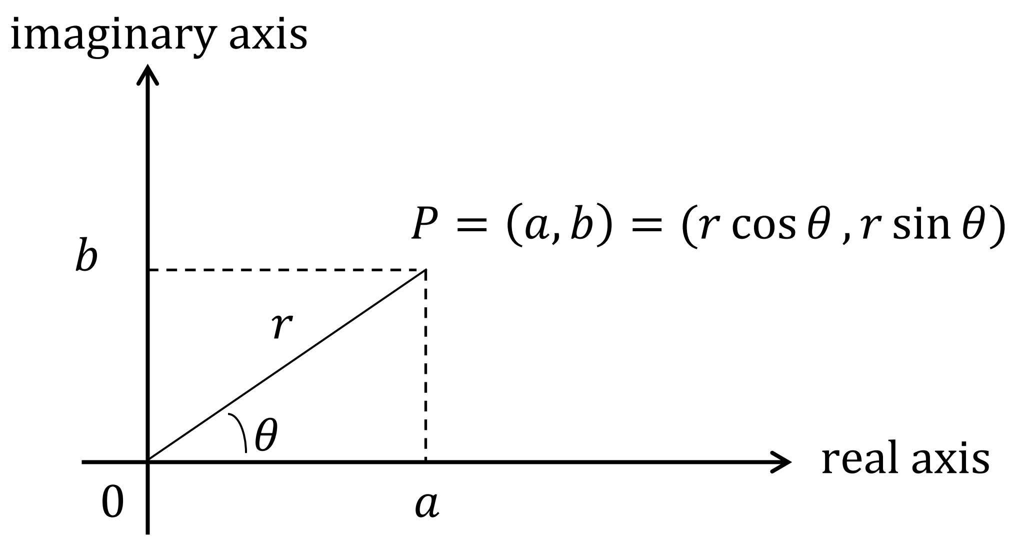
- Recall that every real number can be represented by a point of a straight line (real line from $-\infty$ to $+\infty$).



- Using similar logic, one can represent a complex number $a + bi$ by a point P (with coordinate (x, y) in a 2-D plane (also called an Argand diagram, a complex plane):



In MA1200, we knew that every point in 2-D can be expressed using **polar coordinates** $(\boldsymbol{r}, \boldsymbol{\theta})$.



- $r = \sqrt{a^2 + b^2} > 0$ represents the distance between O and the point (a, b). It is called ***modulus*** of $z = a + bi$ and is denoted by $|z|$
- θ (in radian) represents the angle between the line OP and the positive real axis. It is called ***argument*** of z and is denoted by $\arg z$. Usually, θ is taken to be within $-\pi < \theta \le \pi$. Such θ is called ***principal value*** of $\arg z$.

Polar form of a complex number

Recall that the parameters r, θ, a and b are related by

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta.$$

Then the complex number $z = a + bi$ can be rewritten as

$$z = r\cos\theta + (r\sin\theta)i = r(\cos\theta + i\sin\theta) \ldots\ldots (*)$$

This is called polar form of the complex number.

Remark:

One has to be careful that the complex number is in Polar form only when it is expressed in the form of $(*)$.

Example of polar form

1. $z_0 = 2\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)$ is in polar form with $|z_0| = 2$ and $\arg z_0 = \frac{\theta}{3}$.

Example of non-polar form

2. $z_1 = \cos\frac{\theta}{3} - i\sin\frac{\theta}{3}$ is NOT in polar form since there is a "$-$" between $\cos?$ And $\sin?$.
3. $z_2 = \cos 2\theta + i\sin\theta$ is NOT in polar form since the angles inside the cosine and sine function are not the same

4. $z_3 = -2\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right)$ is NOT in polar form since -2 is negative.

5. $z_4 = \sin\theta + i\cos\theta$ is NOT in polar form since the real part is not of the form $\cos??$.

6. $z_5 = i(\cos\theta + i\sin\theta)$ is NOT in polar form since $r = i$ (1st term) is not a real number.

How to express a complex number $a + bi$ *in polar form?*

Step 1: Given $z = a + bi$, one can find r (modulus of z) by

$$r = |z| = \sqrt{a^2 + b^2}.$$

Step 2: Next, we need to find the argument of z ($\arg z$). The whole process is divided into two main steps:

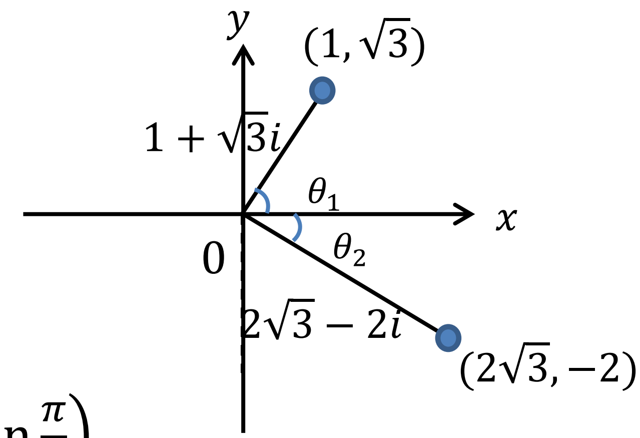
- Draw the diagram (xy-plane) and locate the position of $z = a + bi$. Identify the appropriate value of θ ($-\pi < \theta \leq \pi$).

- Use this diagram to obtain the value of θ using geometric method.

Example 4

Express the complex number $z_1 = 1 + \sqrt{3}i$ and $z_2 = 2\sqrt{3} - 2i$ into polar form.

☺Solution



Note that $r_1 = |z_1| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$

Then $\theta_1 = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$.

Thus, $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.

Note that $r_2 = |z_2| = \sqrt{\left(2\sqrt{3}\right)^2 + (-2)^2} = 4$

Then $\theta_2 = -\tan^{-1}\frac{2}{2\sqrt{3}} = -\frac{\pi}{6}$.

Thus, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2) = 4\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$.

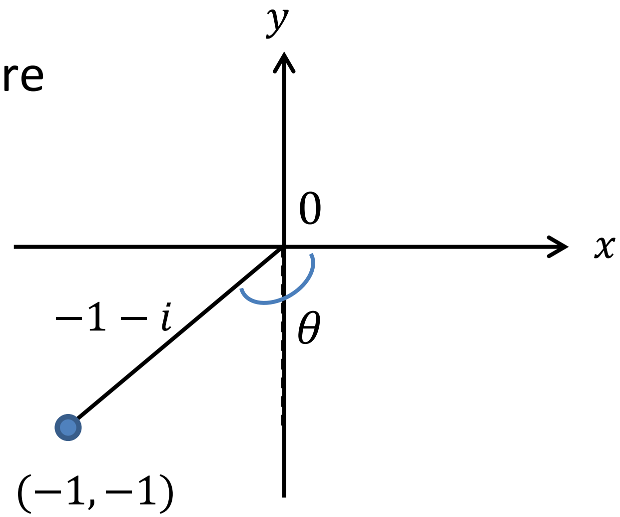
Example 5

Express the complex number $z = -1 - i$ into polar form.

☺Solution:

Note that $r = |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$.

To find the argument $\arg z$, we consider the figure on the right, it is easy to observe that



$$\theta = -\left(\pi - \tan^{-1}\left(\frac{1}{1}\right)\right)$$

$$= -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}$$

(Note: $\pi(\text{rad}) = 180°$)

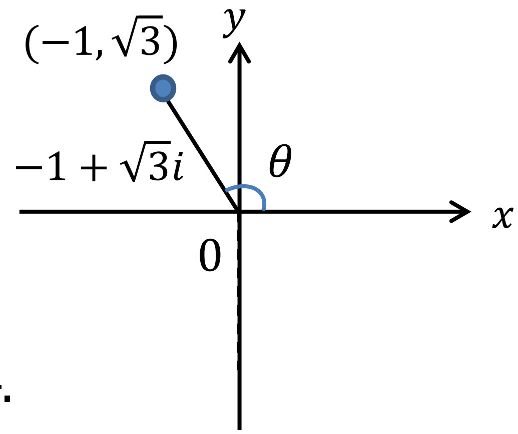
Hence, $z = \sqrt{2}\left[\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right]$.

Example 6

Express the complex number $z = -1 + \sqrt{3}i$ into polar form.

☺Solution:

Using similar method, we have



$$r = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2$$

and

$$\theta = \pi - \tan^{-1}\frac{\sqrt{3}}{1} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

Therefore the polar form of this complex is given by

$$z = -1 + \sqrt{3}i = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right).$$

Example 7

Express the complex number $z = \sin\theta + i\cos\theta$ into polar form where $0 < \theta < \pi$. What are $|z|$ and $\arg z$?

(Be careful: It is not in polar form yet!!!)

☺Solution:

Note that $\sin\theta > 0$ for $0 < \theta < \pi$, the point lies in either 1st/ 4th quadrant.

$$r = \sqrt{\sin^2\theta + \cos^2\theta} = \sqrt{1} = 1 \quad \text{and}$$

$$\phi = \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{1}{\tan\theta}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \theta\right)\right) = \frac{\pi}{2} - \theta.$$

Therefore the polar form of z is

$$\sin\theta + i\cos\theta = 1\left(\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right).$$

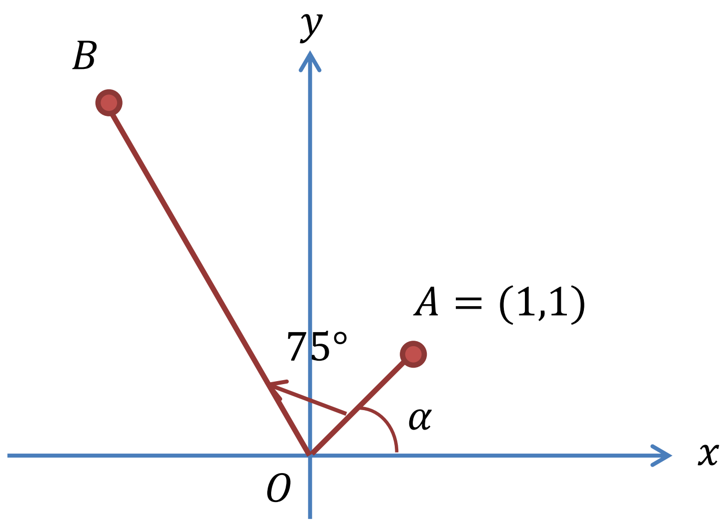
Here $|z| = r = 1$ and $\arg z = \frac{\pi}{2} - \theta$.

Example 8 (A bit harder)

Let $z_A = 1 + i$ be a complex number and A be the point in argand diagram representing the complex number z_A. Suppose the line OA is being rotated by $75°$ along anti-clockwise direction and is then stretched in length by 3 times (fixing the endpoint origin O), we let OB (where O, B are the endpoints) be the resulting line.

(a) Find the corresponding complex number z_B representing the point B.

(b) Find the coordinates of B.



☺Solution

(a) To find the complex number z_B, we need to find the *modulus* and *argument* of z_B.

Using the above figure, we get

- Modulus of $z_B =$ Length of OB

$$= 3 \times \text{Length of } OA = 3 \times \sqrt{1^2 + 1^2} = 3\sqrt{2}.$$

- Argument of $z_B = \alpha + 75° = \tan^{-1}\frac{1}{1} + 75°$

$$= 45° + 75° = 120° = \frac{2\pi}{3}.$$

Thus the complex number z_B is given by

$$z_B = 3\sqrt{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{6}}{2}i.$$

(b) Since B is a point representing the complex number z_B, hence the coordinates of B is $\left(-\frac{3\sqrt{2}}{2}, \frac{3\sqrt{6}}{2}\right)$.

Multiplication and division of complex numbers in polar form

In fact, the polar form can greatly reduce the computational cost in doing multiplication and division of complex numbers.

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers expressed in polar form. The product of z_1 and z_2 is given by

$$z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2(\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i^2\sin\theta_1\sin\theta_2)$$

$$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

$$= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

The last inequality follows from compound angle formula:

$$\begin{cases} \cos(A+B) = \cos A\cos B - \sin A\sin B \\ \sin(A+B) = \sin A\cos B + \cos A\sin B \end{cases}.$$

On the other hand, we consider the quotient $\frac{z_1}{z_2}$ and get

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}\left(\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2}\right)$$

$$= \frac{r_1}{r_2}\left(\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2}\right)\left(\frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}\right)$$

$$= \frac{r_1}{r_2}\left(\frac{\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 - i^2\sin\theta_1\sin\theta_2}{\underbrace{\cos^2\theta_2 + \sin^2\theta_2}_{=1}}\right)$$

$$= \frac{r_1}{r_2}[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]$$

$$= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)).$$

Again, the last inequality is due to the compound angle formula.

We summarize the result in the following theorem:

Theorem: (Multiplication and Division of complex numbers in polar forms)

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)),$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)).$$

Corollary: (Properties of modulus and argument of complex numbers)

$$|z_1 z_2| = r_1 r_2 = |z_1||z_2|, \qquad \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|},$$

$$\arg z_1 z_2 = \theta_1 + \theta_2 = \arg z_1 + \arg z_2,$$

$$\arg\frac{z_1}{z_2} = \theta_1 - \theta_2 = \arg z_1 - \arg z_2.$$

Example 9

Compute $(1+i)^3$ and $(1+i)^{-2}$

☺Solution:

It is not good for you to do computation by expanding. It may be better for you to transform the complex number $1+i$ in polar form first.

Note that $r=\sqrt{1^2+1^2}=\sqrt{2}$ and $\theta=\tan^{-1}\frac{1}{1}=\tan^{-1}1=\frac{\pi}{4}$, so

$$1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right).$$

Then

$$(1+i)^3=\left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^3$$

$$=\left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]\left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]\left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]$$

$$=2\sqrt{2}\left(\cos\frac{2\pi}{4}+i\sin\frac{2\pi}{4}\right)\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=2\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$$

$$= 2\sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\left(\frac{\sqrt{2}}{2}\right)\right) = -2 + 2i.$$

$$(1+i)^{-2} = \frac{1}{(1+i)^2} = \frac{1}{\left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]\left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]}$$

$$= \frac{1}{2\left(\cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}\right)} = \frac{\cos 0 + i\sin 0}{2\left(\cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4}\right)}$$

$$= \frac{1}{2}\left[\cos\left(0 - \frac{\pi}{2}\right) + i\sin\left(0 - \frac{\pi}{2}\right)\right] = \frac{1}{2}\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) = \frac{1}{2}i(-1)$$

$$= -\frac{1}{2}i.$$

Example 10

Compute

$$\frac{1+\sqrt{3}i}{\left(3-\sqrt{3}i\right)^2}.$$

☺Solution

Note that (left as exercise)

$$1+\sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right), \qquad 3-\sqrt{3}i = 2\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right).$$

Then

$$\left(3-\sqrt{3}i\right)^2 = \left[2\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)\right]\left[2\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)\right]$$

$$= 12\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right].$$

$$\frac{1+\sqrt{3}i}{\left(3-\sqrt{3}i\right)^2}$$

$$=\frac{2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)}{12\left[\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right]}$$

$$=\frac{1}{6}\left[\cos\left(\frac{\pi}{3}-\left(-\frac{\pi}{3}\right)\right)+i\sin\left(\frac{\pi}{3}-\left(-\frac{\pi}{3}\right)\right)\right]$$

$$=\frac{1}{6}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$$

$$=\frac{1}{6}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=-\frac{1}{12}+\frac{\sqrt{3}}{12}i.$$

Euler Form (Exponential Form)

Euler (1707-1783) established an important formula which provides a deep relationship between the trigonometric functions and the complex exponential function. The formula states that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Given the polar form of a complex number $z = r(\cos\theta + i\sin\theta)$, one can use Euler formula to express the complex number into following form

$$z = re^{i\theta}.$$

This is called Euler form (or Exponential form) of a complex number.

Rough Proof of Euler relation ($e^{i\theta} = \cos\theta + i\sin\theta$)

Recall that the Taylor's series of e^x about $x = 0$ is given by

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

Substituting $x = i\theta$, we have

$$e^{i\theta} = 1 + i\theta + i^2\frac{\theta^2}{2!} + i^3\frac{\theta^3}{3!} + i^4\frac{\theta^4}{4!} + i^5\frac{\theta^5}{5!} + i^6\frac{\theta^6}{6!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta.$$

Using the theorem in p.26, one can obtain the following theorem

Theorem: (Multiplication and Division of complex numbers in Euler form)

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}, \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$

Take Example 10 as an example, we have

$$1+\sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2e^{i\frac{\pi}{3}},$$

$$3-\sqrt{3}i = 2\sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = 2\sqrt{3}e^{-i\frac{\pi}{6}}.$$

$$\Rightarrow \frac{1+\sqrt{3}i}{\left(3-\sqrt{3}i\right)^2} = \frac{2e^{i\frac{\pi}{3}}}{\left(2\sqrt{3}e^{-i\frac{\pi}{6}}\right)^2} = \frac{1}{6}\frac{e^{i\frac{\pi}{3}}}{e^{-i\frac{\pi}{3}}} = \frac{1}{6}e^{i\frac{\pi}{3}-\left(-i\frac{\pi}{3}\right)} = \frac{1}{6}e^{i\frac{2\pi}{3}}$$

$$= \frac{1}{6}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right).$$

Example of Euler form of some complex numbers

1. $-1 = -1 + 0i = \cos\pi + i\sin\pi = e^{i\pi}$.
2. $i = 0 + i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$
3. $-i = 0 - i = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = e^{-i\frac{\pi}{2}}$.
4. (From Example 4)

$$1 + \sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2e^{i\frac{\pi}{3}}.$$

Example of non-Euler form

1. $z = ie^{i\theta}$ is not in Euler form! (There is a complex number i appeared in front of $e^{i\theta}$.

   However, one can transform this number into Euler form as follows:

$$ie^{i\theta} = e^{i\left(\frac{\pi}{2}\right)}e^{i\theta} = e^{i\left(\theta+\frac{\pi}{2}\right)}.$$

2. $z = -e^{i\theta}$ is not in Euler form! (The leading coefficient is a negative number -1). Using similar technique as in 1, we can transform the number into Euler form:

$$-e^{i\theta} = (-1)e^{i\theta} \underbrace{=}_{-1=e^{i\pi}} e^{i\pi}e^{i\theta} = e^{i(\theta+\pi)}.$$

Example 11

It is clear that the complex number $z = e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{2}}$ is not in Euler's form since there are two exponential terms. Express z in Euler's form.

IDEA: To express the number in Euler form, one has to combine those two exponential terms.

☺Solution:

$$z = e^{i\frac{\pi}{3}} + e^{-i\frac{\pi}{2}} = e^{i\left(\frac{\frac{\pi}{3}+\left(-\frac{\pi}{2}\right)}{2}\right)}\left[e^{i\frac{\pi}{3}-i\left(\frac{\frac{\pi}{3}+\left(-\frac{\pi}{2}\right)}{2}\right)} + e^{-i\frac{\pi}{2}-i\left(\frac{\frac{\pi}{3}+\left(-\frac{\pi}{2}\right)}{2}\right)}\right]$$

$$= e^{-i\frac{\pi}{12}}\left[e^{i\frac{5\pi}{12}} + e^{-i\frac{5\pi}{12}}\right] = e^{-i\frac{\pi}{12}}\left[2\cos\frac{5\pi}{12}\right] = \underbrace{2\cos\frac{5\pi}{12}}_{r>0}\underbrace{e^{i\left(\frac{-\pi}{12}\right)}}_{e^{i\theta}}.$$

The second last equality follows from $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$

Euler Form v.s. Polar Form

- Euler form is useful in handling the multiplication and division of complex numbers which involves the trigonometric functions, e.g.
$$z_1 = \sin\theta \pm i\cos\theta\,, \qquad z_2 = 1 + \sin\theta + i\cos\theta.$$

- It is hard to change the above complex numbers into polar form, since their arguments $\theta = \arg z_i$ is hard to obtain.

- It is often easier to transform these numbers into Euler form instead of polar form.

How to change these numbers into Euler form?

- Let's recall the Euler relation:

$$e^{i\theta} = \cos\theta + i\sin\theta \,\ldots\ldots (1)$$

- On the other hand, we replace θ in (1) by $-\theta$, we obtain another equation:

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta \,\ldots\ldots (2)$$

- (1) + (2) implies

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \Rightarrow \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

(1) – (2) implies

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta \Rightarrow \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Example 12

Transform the following numbers into Euler form (where $0 < \theta < \pi$). Find the modulus and the principle value of the argument.

(a) $-\cos\theta + i\sin\theta$

(b) $1 - \cos\theta + i\sin\theta$

☺Solution

(a) $-\cos\theta + i\sin\theta = -\frac{e^{i\theta}+e^{-i\theta}}{2} + i\,\frac{e^{i\theta}-e^{-i\theta}}{2i}$

$$= -\frac{e^{i\theta} + e^{-i\theta}}{2} + \frac{e^{i\theta} - e^{-i\theta}}{2}$$

$$= -\frac{2e^{-i\theta}}{2} = -e^{-i\theta}$$

$$= e^{i\pi}e^{-i\theta} = e^{i(\pi-\theta)}.$$

Modulus $= 1$ and Argument $= \pi - \theta$.

(b) $1 - \cos\theta + i\sin\theta = 1 - \frac{e^{i\theta}+e^{-i\theta}}{2} + i\,\frac{e^{i\theta}-e^{-i\theta}}{2i}$

$$= 1 - \frac{e^{i\theta} + e^{-i\theta}}{2} + \frac{e^{i\theta} - e^{-i\theta}}{2}$$

$$= 1 - e^{-i\theta}$$

$$= e^{i0} - e^{-i\theta}$$

$$= e^{-i\frac{\theta}{2}}\left(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}\right)$$

$$= e^{-i\frac{\theta}{2}}\left(2i\sin\frac{\theta}{2}\right)$$

$$= 2\sin\frac{\theta}{2}\,ie^{-i\frac{\theta}{2}}$$

$$= 2\sin\frac{\theta}{2}\,e^{i\frac{\pi}{2}}e^{-i\frac{\theta}{2}} = 2\sin\frac{\theta}{2}\,e^{i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}.$$

Modulus $= 2\sin\frac{\theta}{2}$; Argument $= \frac{\pi}{2} - \frac{\theta}{2}$.

Example 13

Compute $\frac{1-\cos\theta+i\sin\theta}{-\cos\theta+i\sin\theta}$.

☺Solution:

Using the result in Example 12, we have

$$\frac{1-\cos\theta+i\sin\theta}{-\cos\theta+i\sin\theta}=\frac{2\sin\frac{\theta}{2}e^{i\left(\frac{\pi}{2}-\frac{\theta}{2}\right)}}{e^{i(\pi-\theta)}}$$

$$=2\sin\frac{\theta}{2}e^{i\left[\left(\frac{\pi}{2}-\frac{\theta}{2}\right)-(\pi-\theta)\right]}$$

$$=2\sin\frac{\theta}{2}e^{i\left(\frac{\theta}{2}-\frac{\pi}{2}\right)}.$$

Summary: (Relations between Cartesian form, Polar form and Euler form)

$$r = \sqrt{a^2 + b^2} \geq 0$$

Polar Form

$$z = r(\cos\theta + i\sin\theta)$$

Cartesian Form

$$z = a + bi$$

$$a = r\cos\theta$$
$$b = r\sin\theta$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$a = r\cos\theta$$
$$b = r\sin\theta$$

Euler Form

$$z = re^{i\theta}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\pm i = e^{i\left(\pm\frac{\pi}{2}\right)}, \quad -1 = e^{i\pi}$$

Power and n^{th} root of complex numbers

Question: How to compute $(a+bi)^n$ and $\sqrt[n]{a+bi}$?

- It is not efficient to do the computation if the complex number is expressed in the form of $a+bi$.
- The computation will be easier if we express the complex number into polar form: $r(\cos\theta + i\sin\theta)$.

Power of complex number

Let $z = r(\cos\theta + i\sin\theta)$, then for any positive integer n ($n = 1, 2, 3$ ...), we have

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos\theta + i\sin\theta)^n$$

$$= r^n[\cos(\theta + \theta + \cdots + \theta) + i\sin(\theta + \theta + \cdots + \theta)]$$

$$= r^n(\cos n\theta + i\sin n\theta)$$

This theorem is known as DeMoivre’s theorem.

DeMoivre's theorem (Non-negative integer index power)

If n is nonnegative integer, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

or in more general,

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

Remark:

This theorem only works when the complex number is expressed in POLAR FORM!
We cannot adopt the DeMoivre's theorem directly to the following:

$$(\sin\theta + i\cos\theta)^n \neq (\sin n\theta + i\cos n\theta),$$

$$(-\cos\theta + i\sin\theta)^n \neq -\cos n\theta + i\sin n\theta.$$

Example 14

Compute $\left(\dfrac{1-\sqrt{3}i}{1+i}\right)^{12}$

☺Solution:

Step 1: Transform the complex numbers into polar form.

One can find that

$$1-\sqrt{3}i = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right], \qquad 1+i = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\frac{\pi}{4}\right).$$

$$\Rightarrow \frac{1-\sqrt{3}i}{1+i} = \frac{2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]}{\sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]}$$

$$= \sqrt{2}\left[\cos\left(-\frac{\pi}{3}-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{3}-\frac{\pi}{4}\right)\right] = \sqrt{2}\left[\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right].$$

Step 2: Use DeMoivre’s theorem

$$\left(\frac{1-\sqrt{3}i}{1+i}\right)^{12} = \left(\sqrt{2}\left[\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right]\right)^{12}$$

$$= \left(\sqrt{2}\right)^{12}\left[\cos\left(12\left(-\frac{7\pi}{12}\right)\right) + i\sin\left(12\left(-\frac{7\pi}{12}\right)\right)\right]$$

$$= 64[\cos(-7\pi) + i\sin(-7\pi)]$$

$$= 64(-1+0i)$$

$$= -64.$$

DeMoivre's Theorem for negative integer index m

We write $m = -n$ (where n is positive integer), then

$$(\cos\theta + i\sin\theta)^m = (\cos\theta + i\sin\theta)^{-n} = \left(\frac{1}{\cos\theta + i\sin\theta}\right)^n$$

$$= \left(\frac{\cos 0 + i\sin 0}{\cos\theta + i\sin\theta}\right)^n$$

$$= (\cos(-\theta) + i\sin(-\theta))^n$$

$$= \cos(-n\theta) + i\sin(-n\theta)$$

$$= \cos m\theta + i\sin m\theta.$$

So we have the similar result

$$\boxed{(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta.}$$

where m is a negative integer. Hence, the DeMoivre's theorem can be applied for any *integer power* n!

Example 15

If n is an integer and $z = \cos\theta + i\sin\theta$, show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i\sin n\theta,$$

☺Solution:

Note that

$$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

$$\frac{1}{z^n} = z^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta.$$

So by simple computation, we get

$$z^n + \frac{1}{z^n} = 2\cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i\sin n\theta.$$

$\boldsymbol{n^{th}}$ root of a complex number $\boldsymbol{\sqrt[n]{z} = z^{\frac{1}{n}}}$

With the help of polar form, one can find a simple way to compute the n^{th} root of the complex number $z^{\frac{1}{n}}$.

As an example, we would like to find the value of $\sqrt[3]{2-2i}$. According to the definition of cubic root, it is equivalent to find a complex number z satisfying

$$z^3 = 2 - 2i.$$

To solve for z, first express the complex numbers z and $2-2i$ into polar form.

Let $z = \underbrace{\rho}_{\text{modulus}}\left(\cos \underbrace{\phi}_{\text{argument}} + i\sin\phi\right)$ and $2-2i = \sqrt{8}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$. Using DeMoivre's theorem, we have

$$[\rho(\cos\phi + i\sin\phi)]^3 = \sqrt{8}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

$$\Rightarrow \underbrace{\rho^3}_{\substack{\text{modulus}\\ \text{of } z^3}} \left(\cos \underbrace{3\phi}_{\substack{\text{argument}\\ \text{of } z^3}} + i\sin 3\phi\right) = \underbrace{\sqrt{8}}_{\substack{\text{modulus}\\ \text{of } 2-2i}} \left[\cos \underbrace{\left(-\frac{\pi}{4}\right)}_{\substack{\text{argument}\\ \text{of } 2-2i}} + i\sin\left(-\frac{\pi}{4}\right)\right]$$

Note that if the two complex numbers z^3 and $2-2i$ are equal, then we have

$$\underbrace{\rho^3 = \sqrt{8}}_{\text{modulus}}, \qquad \underbrace{3\phi = 2k\pi + \left(-\frac{\pi}{4}\right)}_{\text{argument}}, \qquad k = 0, 1, 2, \ldots$$

$$\Rightarrow \rho = \left(\sqrt{8}\right)^{\frac{1}{3}} = \sqrt{2}, \qquad \phi = \frac{2k\pi}{3} - \frac{\pi}{12}, \qquad k = 0, 1, 2, \ldots.$$

Thus $z = \sqrt[3]{2-2i}$ is then given by

$$z = \sqrt{2}\left[\cos\left(\frac{2k\pi}{3} - \frac{\pi}{12}\right) + i\sin\left(\frac{2k\pi}{3} - \frac{\pi}{12}\right)\right], \qquad k = 0, 1, 2, \ldots.$$

Putting $k = 0, 1, 2, \ldots,$ we have

$$z_0 = \sqrt{2}\left[\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right] = \cdots$$

$$z_1 = \sqrt{2}\left[\cos\left(\frac{2\pi}{3} - \frac{\pi}{12}\right) + i\sin\left(\frac{2\pi}{3} - \frac{\pi}{12}\right)\right] = \cdots$$

$$z_2 = \sqrt{2}\left[\cos\left(\frac{4\pi}{3} - \frac{\pi}{12}\right) + i\sin\left(\frac{4\pi}{3} - \frac{\pi}{12}\right)\right] = \cdots$$

$$z_3 = \sqrt{2}\left[\cos\left(\frac{6\pi}{3} - \frac{\pi}{12}\right) + i\sin\left(\frac{6\pi}{3} - \frac{\pi}{12}\right)\right] = \sqrt{2}\left[\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right]$$

$$z_4 = \sqrt{2}\left[\cos\left(\frac{8\pi}{3} - \frac{\pi}{12}\right) + i\sin\left(\frac{8\pi}{3} - \frac{\pi}{12}\right)\right] = \sqrt{2}\left[\cos\left(\frac{2\pi}{3} - \frac{\pi}{12}\right) + i\sin\left(\frac{2\pi}{3} - \frac{\pi}{12}\right)\right]$$

… (repeated)

Thus, we conclude that

$$z = \sqrt{2}\left[\cos\left(\frac{2k\pi}{3} - \frac{\pi}{12}\right) + i\sin\left(\frac{2k\pi}{3} - \frac{\pi}{12}\right)\right], \qquad k = 0, 1, 2.$$

General Case

Suppose we would like to find the value of $\sqrt[n]{(a+bi)}$ (where n is a positive integer), one can let $z=\sqrt[n]{(a+bi)}$ and so

$$z^n = a + bi \ldots\ldots (*)$$

Following the idea of the above numerical example, we first express the complex numbers z and $a+bi$ into polar forms: Let $z=\rho(\cos\phi + i\sin\phi)$ and $a+bi = r(\cos\theta + i\sin\theta)$.

From equation $(*)$, we then have

$$[\rho(\cos\phi + i\sin\phi)]^n = r(\cos\theta + i\sin\theta),$$

$$\Rightarrow \underbrace{\rho^n}_{\substack{\text{modulus}\\ \text{of } z^n}} \left(\cos \underbrace{n\phi}_{\substack{\text{argument}\\ \text{of } z^n}} + i\sin n\phi\right) = \underbrace{r}_{\substack{\text{modulus}\\ \text{of } a+bi}} \left(\cos \underbrace{\theta}_{\substack{\text{argument}\\ \text{of } a+bi}} + i\sin\theta\right)$$

Since two complex numbers are equal, then we must have

$$\underbrace{\rho^n = r}_{\text{modulus}} \quad \text{and} \quad \underbrace{n\phi = 2k\pi + \theta}_{\text{argument}}, \qquad k = 0, 1, 2, 3, \ldots$$

$$\Rightarrow \rho = r^{\frac{1}{n}}, \qquad \phi = \frac{2k\pi + \theta}{n}, k = 0,1,2, \ldots$$

Since the value of ϕ will be repeated after $k = n,\ n+1,..,$ thus we just need to concentrate on the values of ϕ for $k = 0, 1, 2, \ldots, n-1$. Thus we have the following result:

Theorem: (n^{th} root of a complex number)

If $z = r(\cos\theta + i\sin\theta)$ is the complex number expressed in polar form, then

$$\sqrt[n]{r(\cos\theta + i\sin\theta)} = \underbrace{r^{\frac{1}{n}}}_{\rho}\left(\cos\underbrace{\frac{2k\pi + \theta}{n}}_{\phi} + i\sin\frac{2k\pi + \theta}{n}\right),$$

We see that the n^{th} root of a complex number **<u>has n possible values</u>**.

Example 16

Compute $\sqrt[4]{1-i}$. Express your answer in Cartesian form.

☺Solution:

Step 1: Write $1-i$ in polar form

One can find that

$$1-i=\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right).$$

Step 2: Do computation

$$\sqrt[4]{1-i}=(1-i)^{\frac{1}{4}}=\left[\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)\right]^{\frac{1}{4}}$$

$$=2^{\frac{1}{8}}\left[\cos\frac{2k\pi-\frac{\pi}{4}}{4}+i\sin\frac{2k\pi-\frac{\pi}{4}}{4}\right],\qquad k=0,1,2,3.$$

Take $k=0$, then

$$z_0 = 2^{\frac{1}{8}}\left[\cos\left(-\frac{\pi}{16}\right) + i\sin\left(-\frac{\pi}{16}\right)\right] = 1.0696 - 0.2128i.$$

Take $k = 1,$ then

$$z_1 = 2^{\frac{1}{8}}\left[\cos\left(\frac{7\pi}{16}\right) + i\sin\left(\frac{7\pi}{16}\right)\right] = 0.2128 + 1.0696i.$$

Take $k = 2,$ then

$$z_2 = 2^{\frac{1}{8}}\left[\cos\left(\frac{15\pi}{16}\right) + i\sin\left(\frac{15\pi}{16}\right)\right] = -1.0696 + 0.2128i.$$

Take $k = 3,$ then

$$z_3 = 2^{\frac{1}{8}}\left[\cos\left(\frac{23\pi}{16}\right) + i\sin\left(\frac{23\pi}{16}\right)\right] = 2^{\frac{1}{8}}\left[\cos\left(\frac{-9\pi}{16}\right) + i\sin\left(\frac{-9\pi}{16}\right)\right]$$
$$= -0.2128 - 1.0696i.$$

Example 17

In real number world, we know that $\sqrt[5]{1} = (1)^{\frac{1}{5}} = 1$. What are the values of $\sqrt[5]{1}$ in complex number world?

☺Solution

Step 1: Write 1 in polar form

One can find that

$$1 = \cos 0 + i \sin 0.$$

Step 2: Do computation

$\sqrt[5]{1} = (\cos 0 + i \sin 0)^{\frac{1}{5}}$

$= \left[\cos \dfrac{2k\pi + 0}{5} + i \sin \dfrac{2k\pi + 0}{5}\right], \qquad k = 0, 1, 2, 3, 4.$

The values of $\sqrt[5]{1}$ are given by

$$z_0 = \cos 0 + i \sin 0 = 1,$$

$$z_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = 0.3090 + 0.9511i$$

$$z_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} = -0.8090 + 0.5878i$$

$$z_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} = \cos\frac{-4\pi}{5} + i\sin\frac{-4\pi}{5} = -0.8090 - 0.5878i$$

$$z_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = \cos\frac{-2\pi}{5} + i\sin\frac{-2\pi}{5} = 0.3090 - 0.9511i$$

Remark:

The number $z = \sqrt[n]{1}$ (or $z^n = 1$) is also called the n^{th} root of unity and is given by

$$\sqrt[n]{1} = \left[\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right],\quad k = 0, 1, 2, \cdots, n-1.$$

Example 18

Let $\omega\ (\neq 1)$ be a complex cube root of unity. Find

(a) $(1+3\omega+7\omega^2)(1+7\omega+3\omega^2)$.
(b) $(\mathbf{1}+\omega)(1+\omega^2)(1-\omega^4)(1-\omega^5)$.

☺Solution:

Notice that $\omega\ (\neq 1)$ is a root of unity if $\omega^3=1$ implies $(\omega-1)(\omega^2+\omega+1)=0$.

As $\omega\neq 1$, we have $\omega^2+\omega+1=0$ or $\omega^2=-1-\omega$.

(a) $(1+3\omega+7\omega^2)(1+7\omega+3\omega^2)$

$$=[1+3\omega+7(-1-\omega)][1+7\omega+3(-1-\omega)]=(-6-4\omega)(-2+4\omega)$$
$$=12+8\omega-24\omega-16\omega^2=12-16\omega-16(-1-\omega)=28$$

(b) $(\mathbf{1}+\omega)(1+\omega^2)(1-\omega^4)(1-\omega^5)=(\mathbf{1}+\omega)(1+\omega^2)(1-\omega)(1-\omega^2)$

$$=(1+\omega)(1-\omega)(1+\omega^2)(1-\omega^2)=(1-\omega^2)(1-\omega^4)=(1-\omega^2)(1-\omega)$$
$$=1-\omega-\omega^2+\omega^3=1-\omega-(-1-\omega)+1=3$$

Example 19 (Fractional power of a complex number)

Find all possible values of $\left(-\sqrt{3}+i\right)^{\frac{2}{5}}$.

☺Solution:

The polar form of $-\sqrt{3}+i$ is given by $-\sqrt{3}+i = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$.

Then

$$\left(-\sqrt{3}+i\right)^{\frac{2}{5}} = \left\{\left[2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\right]^2\right\}^{\frac{1}{5}}$$

$$= \left[2^2\left(\cos\frac{10\pi}{6} + i\sin\frac{10\pi}{6}\right)\right]^{\frac{1}{5}} = 2^{\frac{2}{5}}\left(\cos\frac{2k\pi+\frac{5\pi}{3}}{5} + i\sin\frac{2k\pi+\frac{5\pi}{3}}{5}\right)$$

$$= 2^{\frac{2}{5}}\left(\cos\left(\frac{2k\pi}{5}+\frac{\pi}{3}\right) + i\sin\left(\frac{2k\pi}{5}+\frac{\pi}{3}\right)\right), \qquad k = 0, 1, 2, 3, 4.$$

Example 20 (DeMoivre's Theorem for fractional index)

Compute $(\cos\theta + i\sin\theta)^{\frac{m}{n}}$ where m is an integer and n is a positive integer.

☺Solution:

$$(\cos\theta + i\sin\theta)^{\frac{m}{n}} = [(\cos\theta + i\sin\theta)^m]^{\frac{1}{n}}$$

$$= [\cos m\theta + i\sin m\theta]^{\frac{1}{n}}$$

$$= \cos\frac{2k\pi + m\theta}{n} + i\sin\frac{2k\pi + m\theta}{n},$$

where $k = 0, 1, 2, \dots, (n-1)$.

Remark: m and n in the index should be relative prime, i.e., they have no common factors other than 1. For example,

$1^{\frac{1}{3}}$ has three distinct complex roots: $1,\ e^{i\frac{2\pi}{3}},\ e^{i\frac{4\pi}{3}} = e^{-i\frac{2\pi}{3}}$;

$1^{\frac{2}{6}} = (1^2)^{\frac{1}{6}} = 1^{\frac{1}{6}}$ has six distinct complex roots: $1,\ e^{i\frac{2\pi}{6}} = e^{i\frac{\pi}{3}},\ e^{i\frac{4\pi}{6}} = e^{i\frac{2\pi}{3}},$

$e^{i\frac{6\pi}{6}} = e^{i\pi} = -1,\ e^{i\frac{8\pi}{6}} = e^{i\frac{4\pi}{3}} = e^{-i\frac{2\pi}{3}},\ e^{i\frac{10\pi}{6}} = e^{i\frac{5\pi}{3}} = e^{-i\frac{\pi}{3}}$;

However, $1^{\frac{2}{6}} = \left(1^{\frac{1}{6}}\right)^2$ has complex roots: $1^2 = 1,\ \left(e^{i\frac{\pi}{3}}\right)^2 = e^{i\frac{2\pi}{3}},$

$\left(e^{i\frac{2\pi}{3}}\right)^2 = e^{i\frac{4\pi}{3}} = e^{-i\frac{2\pi}{3}},\ (-1)^2 = 1,\ \left(e^{-i\frac{2\pi}{3}}\right)^2 = e^{-i\frac{4\pi}{3}} = e^{i\frac{2\pi}{3}},\ \left(e^{-i\frac{\pi}{3}}\right)^2 = e^{-i\frac{2\pi}{3}}.$

We still have three distinct complex roots: $1,\ e^{i\frac{2\pi}{3}},\ e^{-i\frac{2\pi}{3}}$ and the remaining three are repeated the previous roots.

Solving algebraic equations using DeMoivre's Theorem

Example 21

(a) Solve $(1+z)^8 = (1+i)$
(b) Solve $(1+z)^8 = (2-z)^8$

Express your answer in form of $a+bi$.

☺Solution of (a)

$$(1+z)^8 = (1+i) \Rightarrow (1+z) = (1+i)^{\frac{1}{8}} \Rightarrow z = (1+i)^{\frac{1}{8}} - 1$$

$$\Rightarrow z = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^{\frac{1}{8}} - 1$$

$$\Rightarrow z = 2^{\frac{1}{16}}\left(\cos\frac{2k\pi + \frac{\pi}{4}}{8} + i\sin\frac{2k\pi + \frac{\pi}{4}}{8}\right) - 1\,, \qquad k = 0, 1, 2, \dots, 7.$$

$$\Rightarrow z = \left(2^{\frac{1}{16}} \cos \frac{2k\pi + \frac{\pi}{4}}{8} - 1 \right) + 2^{\frac{1}{16}} \sin \frac{2k\pi + \frac{\pi}{4}}{8} i, \qquad k = 0, 1, 2, \dots, 7.$$

☺Solution of (b)

$$(1+z)^8 = (2-z)^8 \Rightarrow \left(\frac{1+z}{2-z}\right)^8 = 1$$

$$\Rightarrow \frac{1+z}{2-z} = 1^{\frac{1}{8}} = (\cos 0 + i \sin 0)^{\frac{1}{8}}$$

$$\Rightarrow \frac{1+z}{2-z} = \cos \frac{2k\pi}{8} + i \sin \frac{2k\pi}{8} \quad \text{for } k = 0, 1, 2, \dots., 7.$$

For simplicity, we let $\omega_k = \cos \frac{k\pi}{4} + i \sin \frac{k\pi}{4}$, then

$$\frac{1+z_k}{2-z_k} = \omega_k \Rightarrow 1 + z_k = 2\omega_k - \omega_k z_k \Rightarrow (1+\omega_k) z_k = 2\omega_k - 1 \Rightarrow z_k = \frac{2\omega_k - 1}{1+\omega_k}.$$

$$z_k = \frac{2\cos\frac{k\pi}{4} - 1 + 2i\sin\frac{k\pi}{4}}{1 + \cos\frac{k\pi}{4} + i\sin\frac{k\pi}{4}}$$

$$= \frac{2\cos\frac{k\pi}{4} - 1 + 2i\sin\frac{k\pi}{4}}{1 + \cos\frac{k\pi}{4} + i\sin\frac{k\pi}{4}}\left(\frac{\left(1 + \cos\frac{k\pi}{4}\right) - i\sin\frac{k\pi}{4}}{1 + \cos\frac{k\pi}{4} - i\sin\frac{k\pi}{4}}\right)$$

$$= \frac{\left[\left(2\cos\frac{k\pi}{4} - 1\right)\left(1 + \cos\frac{k\pi}{4}\right) + 2\sin^2\frac{k\pi}{4}\right] + i\left[\left(1 + \cos\frac{k\pi}{4}\right)2\sin\frac{k\pi}{4} - \left(2\cos\frac{k\pi}{4} - 1\right)\sin\frac{k\pi}{4}\right]}{\left(1 + \cos\frac{k\pi}{4}\right)^2 + \left(\sin\frac{k\pi}{4}\right)^2}$$

$$= \frac{1 + \cos\frac{k\pi}{4} + 3\sin\frac{k\pi}{4}i}{2 + 2\cos\frac{k\pi}{4}} = \left(\frac{1 + \cos\frac{k\pi}{4}}{2 + 2\cos\frac{k\pi}{4}}\right) + \left(\frac{3\sin\frac{k\pi}{4}}{2 + 2\cos\frac{k\pi}{4}}\right)i,$$

$$= \frac{1}{2} + \left(\frac{3\sin\frac{k\pi}{4}}{2 + 2\cos\frac{k\pi}{4}}\right)i, \qquad k = 0, 1, \dots, 7.$$

Example 22

Solve $z^6 - z^3 - 2 = 0$.

☺Solution:

To make your life easier, we first let $y = z^3$, then the equation becomes

$$y^2 - y - 2 = 0$$

$$\Rightarrow (y-2)(y+1) = 0$$

$$\Rightarrow y = 2 \quad \text{or} \quad y = -1$$

For $y = 2$, then $z^3 = 2$. So

$$z = \sqrt[3]{2} = [2(\cos 0 + i\sin 0)]^{\frac{1}{3}} = 2^{\frac{1}{3}}\left(\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}\right), \quad k = 0, 1, 2.$$

For $y = -1$, then $z^3 = -1$. So

$$z = \sqrt[3]{-1} = (\cos\pi + i\sin\pi)^{\frac{1}{3}} = \left(\cos\frac{2k\pi + \pi}{3} + i\sin\frac{2k\pi + \pi}{3}\right)$$

$$= \cos\frac{(2k+1)\pi}{3} + i\sin\frac{(2k+1)\pi}{3}, \quad k = 0, 1, 2.$$

Example 23 (A bit harder)

Let z be a complex number such that $z+3i$ is a 5^{th} root of unity, find and <u>list all possible values of z</u>. Express your answer in the Cartesian form $a+bi$.

☺Solution:

Given that $z+3i$ is the 5^{th} root of unity, we have

$(z+3i)^5 = 1$

$\Rightarrow z+3i = \sqrt[5]{1} = \sqrt[5]{\cos 0 + i \sin 0}$

$$\Rightarrow z+3i = \cos\frac{2k\pi+0}{5} + i\sin\frac{2k\pi+0}{5}, \qquad k = 0, 1, 2, 3, 4.$$

$$\Rightarrow z = \cos\frac{2k\pi}{5} + i\left(\sin\frac{2k\pi}{5} - 3\right), \quad k = 0, 1, 2, 3, 4.$$

Substitute $k = 0, 1, 2, 3, 4,$ we then get

$z_0 = \cos 0 + i(\sin 0 - 3) = 1 - 3i;$

$$z_1 = \cos\frac{2\pi}{5} + i\left(\sin\frac{2\pi}{5} - 3\right) = 0.3090 - 2.0489i;$$

$$z_2 = \cos\frac{4\pi}{5} + i\left(\sin\frac{4\pi}{5} - 3\right) = -0.8090 - 2.4122i;$$

$$z_3 = \cos\frac{6\pi}{5} + i\left(\sin\frac{6\pi}{5} - 3\right) = -0.8090 - 3.5878i;$$

$$z_4 = \cos\frac{8\pi}{5} + i\left(\sin\frac{8\pi}{5} - 3\right) = 0.3090 - 3.9511i;$$

Application of Complex Numbers

1. Deriving some useful identities of trigonometric functions

Example 24

By considering $(\cos\theta + i\sin\theta)^4$, show that

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1.$$

☺IDEA:
One will compute $(\cos\theta + i\sin\theta)^4$ in two (or more) different ways and compare the result obtained.

☺Solution:

Way 1: Use DeMoivre's theorem, we get

$$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta \; \ldots . (1)$$

Way 2: Use Binomial theorem to expand $(\cos\theta + i\sin\theta)^4$ by brute force

Recall: Binomial Theorem

For any numbers a, b and positive integer n, we have

$$(a+b)^n = \sum_{r=0}^{n} C_r^n a^{n-r} b^r = a^n + C_1^n a^{n-1} b + C_2^n a^{n-2} b^2 + \cdots + b^n$$

where $C_r^n = \frac{n!}{r!(n-r)!}$ and $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1.$

$$(\cos\theta + i\sin\theta)^4 = \sum_{r=0}^{4} C_r^4 (\cos\theta)^{4-r} (i\sin\theta)^r$$

$$= \cos^4\theta + 4\cos^3\theta\sin\theta\, i + 6\cos^2\theta\sin^2\theta\, i^2 + 4\cos\theta\sin^3\theta\, i^3 + \sin^4\theta\, i^4$$

$$= \cos^4\theta + 4\cos^3\theta\sin\theta\, i - 6\cos^2\theta\sin^2\theta - 4\cos\theta\sin^3\theta\, i + \sin^4\theta$$

$$= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta) \ldots (2)$$

$$z = (\cos\theta + i\sin\theta)^4$$

Way 1:
DeMoivre's Theorem

Way 2:
Binomial Theorem

$$z = \cos 4\theta + i\sin 4\theta$$

Equal

$$z = (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$$

We compare the real part of the expression (1) and (2), we get

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$= \cos^4\theta - 6\cos^2\theta\,(1 - \cos^2\theta) + (1 - \cos^2\theta)^2$$

$$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$

$$= 8\cos^4\theta - 8\cos^2\theta + 1.$$

Example 25

By considering $\left(z+\frac{1}{z}\right)^6$ where $z = \cos\theta + i\sin\theta$ and using the fact that $z^n + \frac{1}{z^n} = 2\cos n\theta$ for all n. Show that

$$\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10).$$

☺Solution:

Similar to Example 23, we compute $\left(z+\frac{1}{z}\right)^6$ in two different ways and compare the results again.

Way 1: Using the given fact (put $n = 1$), we get

$$\left(z+\frac{1}{z}\right)^6 = (2\cos\theta)^6 = 64\cos^6\theta \dots . (1)$$

Way 2: Expand the expression by brute force using binomial theorem

$$\left(z+\frac{1}{z}\right)^6 = \sum_{r=0}^{6} C_r^6 z^{6-r}\left(\frac{1}{z}\right)^r = \sum_{r=0}^{6} C_r^6 z^{6-2r}$$

$$= z^6 + 6z^4 + 15z^2 + 20z^0 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$$

$$= \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$$

Using the given fact with $n = 2, 4, 6$ we get

$$= 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20 \ldots .(2)$$

We compare the result in (1) and (2), we get

$$64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$$

$$\Rightarrow \cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10).$$

2. Computation of integral

Sometimes, complex number theory allows to compute some integrals in a easier way. It can also allow us to compute some integrals which cannot be computed under real number framework (say $\int_0^\infty e^{-x^2} dx$)

Example 26

Using complex number and compute the integral

$$\int e^{3x} \cos 2x \, dx.$$

☺IDEA:

The Euler formula $(e^{i\theta} = \cos\theta + i\sin\theta)$ in complex number theory allows us to "transform" the trigonometric functions into exponential function so that they can "combine" with the exponential function in the integral and simplify the calculation.

☺Solution:

Using the fact that $\cos 2x = \frac{1}{2}\left(e^{i(2x)} + e^{i(-2x)}\right)$, the integral can be computed as

$$\int e^{3x}\cos 2x\,dx = \int e^{3x}\left[\frac{1}{2}\left(e^{i(2x)} + e^{i(-2x)}\right)\right]dx$$

$$= \frac{1}{2}\int e^{(3+2i)x}dx + \frac{1}{2}\int e^{(3-2i)x}dx$$

$$= \frac{1}{2(3+2i)}e^{(3+2i)x} + \frac{1}{2(3-2i)}e^{(3-2i)x} + C$$

$$= \frac{1}{2(3+2i)}\left(\frac{3-2i}{3-2i}\right)e^{(3+2i)x} + \frac{1}{2(3-2i)}\left(\frac{3+2i}{3+2i}\right)e^{(3-2i)x} + C$$

$$= \frac{3-2i}{26}e^{(3+2i)x} + \frac{3+2i}{26}e^{(3-2i)x} + C$$

$$= \frac{3-2i}{26}e^{3x}e^{2ix} + \frac{3+2i}{26}e^{3x}e^{-2ix} + C$$

$$= \frac{3-2i}{26} e^{3x}(\cos 2x + i \sin 2x) + \frac{3+2i}{26} e^{3x}\left(\underbrace{\cos(-2x)}_{\cos 2x} + i\, \underbrace{\sin(-2x)}_{-\sin 2x}\right) + C$$

$$= \frac{e^{3x}}{26}\left[\left(3\cos 2x + 3\sin 2x\, i - 2\cos 2x\, i - 2\sin 2x \underbrace{i^2}_{=-1}\right)\right.$$
$$\left. + \left(3\cos 2x - 3\sin 2x\, i + 2\cos 2x\, i - 2\sin 2x \underbrace{i^2}_{-1}\right)\right] + C$$

$$= \frac{e^{3x}}{26}[(3\cos 2x + 2\sin 2x + 3\cos 2x + 2\sin 2x)$$
$$+ (3\sin 2x - 2\cos 2x - 3\sin 2x + 2\cos 2x)i] + C$$

$$= \frac{e^{3x}}{26}(6\cos 2x + 4\sin 2x) = \frac{3}{13} e^{3x}\cos 2x + \frac{2}{13} e^{3x}\sin 2x + C.$$

Complex Conjugate

- In doing the division of complex number $\frac{a+bi}{c+di}$, we try to multiply both numerator and denominator by a common factor $(c-di)$. By doing so, the "i" in the denominator can be eliminated.

- When we solve the quadratic equation such as $x^2+x+1=0$, we find that the roots are given by $x=\frac{-1\pm\sqrt{1^2-4(1)}}{2}=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$. We observe that if $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ is a root of the equation, then $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ is also a root.

- Given a complex number $z=a+bi$, the complex number $a-bi$ plays an important role. We call this number to be complex conjugate of z.

Definition (Complex Conjugate)

The *complex conjugate* of a complex number $z = a + bi$, denoted by $\bar{z}$ is defined as

$$\bar{z} = a - bi$$

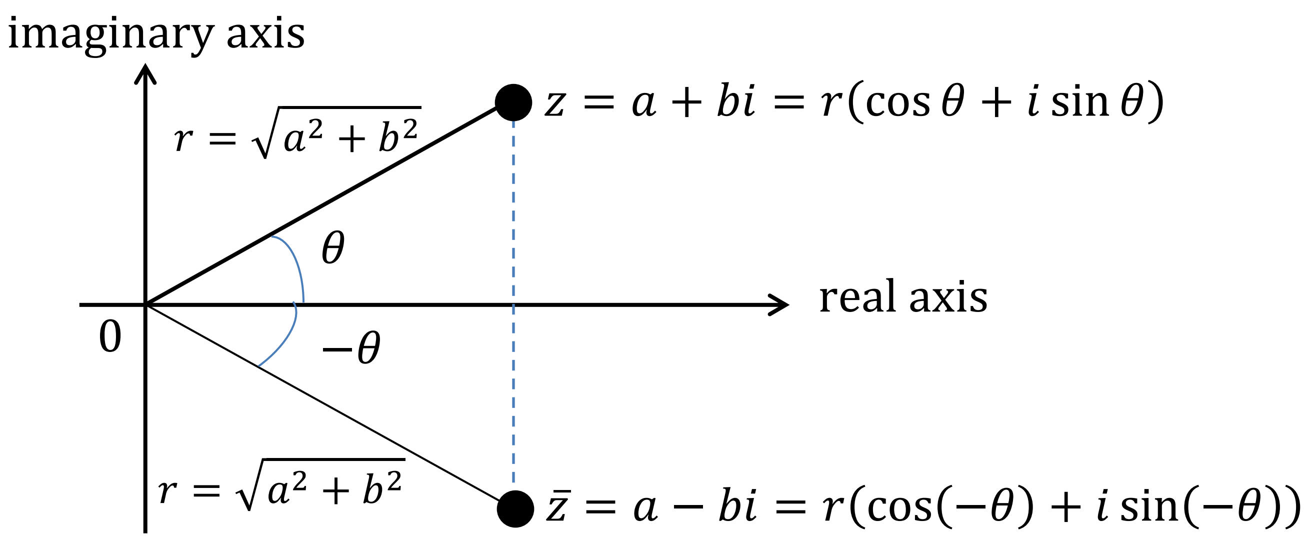
Example 27

- The complex conjugate of $4 + 3i$ is $4 - 3i$.
- The complex conjugate of $4 - 3i$ is $4 - (-3)i = 4 + 3i$.
- The complex conjugate of a real number a is also a.
  (Reason: Note that $a = a + 0i$, then the complex conjugate of a is $\bar{a} = a - 0i = a$)

Geometric Representation of Complex Conjugate

- The real parts of z and its complex conjugate $\bar{z}$ are equal and the imaginary parts are equal in magnitude but opposite in sign.

- In an Argand diagram, z and $\bar{z}$ are symmetrical about the real axis (the x-axis).



Properties of Complex Conjugate

Let $z = a + bi$ be a complex number. Then

(1) If z is real, then $\bar{z} = z$.

(2) $\bar{\bar{z}} = z$

(3) $z\bar{z} = a^2 + b^2 = |z|^2$

(4) $|\bar{z}| = |z|$

(5) $\arg \bar{z} = -\theta = -\arg z$

(6) $z + \bar{z} = 2a = 2\mathrm{Re}(z)$. (Here, $a = \mathrm{Re}(z)$ denote real part of z)

(7) $z - \bar{z} = 2bi = 2i\,\mathrm{Im}(z)$. (Here, $b = \mathrm{Im}(z)$ denote imaginary part of z)

(Operation)

(8) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

(9) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(10) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

The complex conjugate allows us to prove various identities about complex numbers and derive other facts.

Example 28

Prove that, for any complex numbers z_1 and z_2, we have

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

☺Solution:

Note that

$$|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$$

$$= z_1\bar{z}_1 + z_2\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_2$$

$$|z_1 - z_2|^2 = (z_1 - z_2)\overline{(z_1 - z_2)} = (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$$

$$= z_1\bar{z}_1 - z_2\bar{z}_1 - z_1\bar{z}_2 + z_2\bar{z}_2$$

Therefore

$|z_1 + z_2|^2 + |z_1 - z_2|^2$

$= (z_1\bar{z}_1 + z_2\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_2) + (z_1\bar{z}_1 - z_2\bar{z}_1 - z_1\bar{z}_2 + z_2\bar{z}_2)$

$= 2z_1\bar{z}_1 + 2z_2\bar{z}_2$

$= 2|z_1|^2 + 2|z_2|^2$

Example 29

If $|z - 3| = |z + 3|$, show that $z = bi$ for some real number b.

IDEA: Let play around with the condition given $|z - 3| = |z + 3|$ and see how the thing evolve.

☺Solution:

$$|z-3| = |z+3| \Rightarrow |z-3|^2 = |z+3|^2$$

$$\Rightarrow (z-3)\overline{(z-3)} = (z+3)\overline{(z+3)}$$

$$\Rightarrow (z-3)(\bar{z}-\bar{3}) = (z+3)(\bar{z}+\bar{3})$$

$$\Rightarrow (z-3)(\bar{z}-3) = (z+3)(\bar{z}+3)$$

$$\Rightarrow z\bar{z} - 3z - 3\bar{z} + 9 = z\bar{z} + 3z + 3\bar{z} + 9$$

$$\Rightarrow 6z + 6\bar{z} = 0$$

$$\Rightarrow z + \bar{z} = 0 \Rightarrow 2\mathrm{Re}(z) = 0$$

$$\Rightarrow \mathrm{Re}(z) = 0.$$

This implies the real part of z must be 0. So $z = bi$ for some real number b.

Roots of Polynomials with real coefficient

Recall that if $a + bi$ is a solution of a quadratic equation $a_2x^2 + a_1x + a_0 = 0$, then $a - bi$ is also a solution of the same equation.

Question: Does the statement hold for the general polynomial equation?

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = 0.$$

Theorem

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial with real coefficients and degree n $(n \geq 2)$.

If $z = a + bi$ is a root of $f(x) = 0$, then $\bar{z} = a - bi$ is also a root of $f(x) = 0$.

Proof of the theorem

It is sufficient to show $f(\bar{z}) = 0$.

Note that

$$f(\bar{z}) = a_n\bar{z}^n + a_{n-1}\bar{z}^{n-1} + \cdots + a_2\bar{z}^2 + a_1\bar{z} + a_0$$

$$= a_n\overline{z^n} + a_{n-1}\overline{z^{n-1}} + \cdots + a_2\overline{z^2} + a_1\bar{z} + a_0$$

$$= \overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \cdots + \overline{a_2}\,\overline{z^2} + \overline{a_1}\bar{z} + \overline{a_0}$$

$$= \overline{(a_nz^n + a_{n-1}z^{n-1} + \cdots + a_2z^2 + a_1z + a_0)}$$

$$= \overline{f(z)}$$

$$= \bar{0}$$

$$= 0.$$

Example 30

Solve the equation $2x^3 - 7x^2 + 6x + 5 = 0$ given that $2 + i$ is one of the roots.

☺Solution:

Note that $2 - i$ is also the solution of the same equation.

By factor theorem, $\big(x - (2 + i)\big)$ and $(x - (2 - i))$ are "factor" of $2x^3 - 7x^2 + 6x + 5$. This implies that $\big(x - (2 + i)\big)\big(x - (2 - i)\big) = x^2 - 4x + 5$ is also a factor of the same expression.

Using long division, the expression on L.H.S. can be factorized as:

$2x^3 - 7x^2 + 6x + 5 = 0$

$\Rightarrow (x^2 - 4x + 5)(2x + 1) = 0$

$\Rightarrow x^2 - 4x + 5 = 0 \quad \text{or} \quad 2x + 1 = 0$

$\Rightarrow x = 2 + i \quad \text{or} \quad x = 2 - i \quad \text{or} \quad x = -\dfrac{1}{2}.$