

EE3210 Signals & Systems

Due on Midnight, April 2, 2020

Homework #2

1. Total mark is 20 points ($= 4$ points per problem $\times 5$ problems)
2. Solution will be posted on April 3rd on Canvas website
3. Submission due by April 2, 2020, midnight. We will not accept late submission.
4. Online submission through Canvas
 - Scan or taking a photo of your answer sheet, then upload to Canvas
 - After initial submission to Canvas, you can resubmit through email to yjchun@cityu.edu.hk
 - For revision purpose or if the submitted file is corrupted

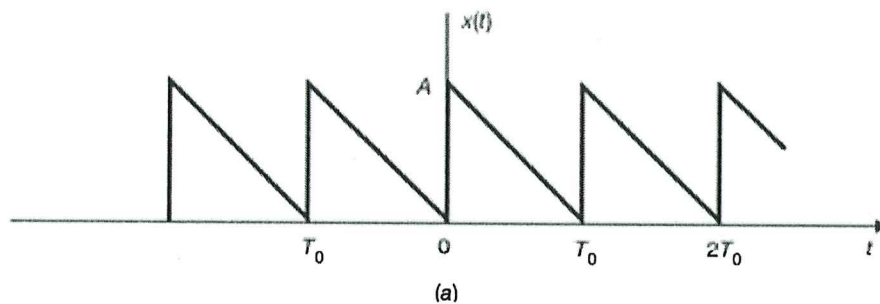
Problem 1 (4 pts)

Let's consider the triangular wave $x(t)$ as shown below.

$$x(t) = A \left(1 - \frac{t}{T_0} \right), \quad 0 \leq t < T_0, \quad \text{and } x(t + T_0) = x(t)$$

(2 pts) a) Find the complex exponential Fourier series of $x(t)$

(2 pts) b) Find the triangular Fourier series of $x(t)$



Solution)

$$a) \quad C_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt = \frac{A}{2}$$

$$C_k = \frac{A}{T_0} \int_0^{T_0} \left(1 - \frac{t}{T_0} \right) e^{-jk\omega_0 t} dt = \frac{A}{T_0^2} \int_0^{T_0} \tau e^{jk\omega_0 \tau} d\tau \cdot e^{-jk\omega_0 T_0}$$

change of
variable
 $\tau = T_0 - t$
 $d\tau = -dt$

$$\begin{cases} -\omega_0 T_0 = 2\pi \quad \text{and} \quad e^{-jk\omega_0 T_0} = (e^{-j2\pi})^k = 1 \\ \int_0^{T_0} \tau e^{jk\omega_0 \tau} d\tau = \frac{\tau e^{jk\omega_0 \tau}}{jk\omega_0} \Big|_0^{T_0} - \frac{1}{jk\omega_0} \int_0^{T_0} e^{jk\omega_0 \tau} d\tau = \frac{T_0}{jk\omega_0} \end{cases}$$

$$\text{Hence, } C_0 = \frac{A}{2}, \quad C_k = \frac{A}{jk(2\pi)}, \quad \text{for } k \neq 0$$

$$b) \quad a_0 = 2C_0 = A, \quad a_k = 2\operatorname{Re}[C_k] = 0 \quad \text{for } k \neq 0$$

$$b_k = -2\operatorname{Im}[C_k] = \frac{A}{k\pi}$$

Problem 2 (4 pts)Find the Fourier transform of the following signals ($\alpha > 0$)

(2 pts) a) $x(t) = e^{-\alpha t^2}$

(2 pts) b) $x(t) = e^{-\alpha|t|}$

Solution)

$$\begin{aligned}
 \text{a) } X(f) &= \int_{-\infty}^{\infty} e^{-\alpha t^2} e^{-j2\pi f t} dt = \int_{-\infty}^{\infty} e^{-\alpha \left[t^2 + \frac{j2\pi f}{\alpha} t + \left(\frac{j\pi f}{\alpha} \right)^2 - \left(\frac{j\pi f}{\alpha} \right)^2 \right]} dt \\
 &= \int_{-\infty}^{\infty} e^{-\alpha \left(t + \frac{j\pi f}{\alpha} \right)^2} dt \cdot e^{-\frac{(\pi f)^2}{\alpha}} \quad \leftarrow \begin{array}{l} \text{change of variable} \\ t + \frac{j\pi f}{\alpha} = \gamma \\ dt = d\gamma \end{array} \\
 &= \int_{-\infty}^{\infty} e^{-\alpha \gamma^2} d\gamma \cdot e^{-\frac{(\pi f)^2}{\alpha}} \quad \leftarrow \int_{-\infty}^{\infty} e^{-a x^2} dx = \sqrt{\frac{\pi}{a}} \\
 &= \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\pi^2}{\alpha} f^2}
 \end{aligned}$$

$$\text{b) } x(t) = e^{-\alpha|t|} = e^{\alpha t} u(-t) + e^{-\alpha t} u(t)$$

$$\begin{array}{cc}
 \uparrow \mathcal{F} & \uparrow \mathcal{F}
 \end{array}$$

$$X(f) = \mathcal{F}\{x(t)\} = \frac{1}{\alpha - j2\pi f} + \frac{1}{\alpha + j2\pi f} = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$$

Problem 3 (4 pts)

Consider a continuous time LTI system where the input and the output are related by the following differential equations

$$\frac{d^2 y(t)}{dt^2} + 6 \frac{dy(t)}{dt} + 8y(t) = 2x(t)$$

(2 pts) a) Find the impulse response of this system.

(2 pts) b) Find the output of this system if $x(t) = e^{-4t}u(t) - te^{-4t}u(t)$.

(Solution)

a) The FT of the differential equation is given by

$$[(j2\pi f)^2 + 6(j2\pi f) + 8] Y(f) = 2X(f)$$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{2}{(4+j2\pi f)(2+j2\pi f)} = \frac{1}{2+j2\pi f} - \frac{1}{4+j2\pi f}$$

By using the FT table

$$h(t) = e^{-2t} u(t) - e^{-4t} u(t)$$

$$b) X(f) = \mathcal{F}\{x(t)\} = \frac{1}{4+j2\pi f} - \frac{1}{(4+j2\pi f)^2} = \frac{3+j2\pi f}{(4+j2\pi f)^2}$$

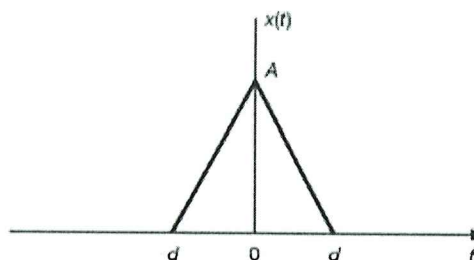
$$Y(f) = H(f) X(f) = \frac{2(3+j2\pi f)}{(4+j2\pi f)^3(2+j2\pi f)} = \frac{1}{(4+j2\pi f)^3} + \frac{-\frac{1}{2}}{(4+j2\pi f)^2} + \frac{-\frac{1}{4}}{(4+j2\pi f)} + \frac{\frac{1}{4}}{(2+j2\pi f)}$$

Based on the FT table

$$y(t) = \frac{1}{2} t^2 e^{-4t} u(t) - \frac{1}{2} t e^{-4t} u(t) - \frac{1}{4} e^{-4t} u(t) + \frac{1}{4} e^{-2t} u(t)$$

Problem 4 (4pts)

- (1pts) a) Find the Fourier transform of the triangular pulse signal shown below



- (1pts) b) Find the inverse Fourier transform of

$$X(f) = \frac{1}{2 - f^2 + j3f}$$

- (2pts) c) Find the 80 percent energy containment bandwidth for the signal

$$x(t) = \frac{1}{t^2 + a^2}, \quad a > 0$$

$$a) \quad x(t) = A \left(1 - \frac{|t|}{d} \right) = A \cdot \text{tri} \left(\frac{t}{d} \right) = \frac{A}{d} \cdot \text{rect} \left(\frac{t}{d} \right) * \text{rect} \left(\frac{t}{d} \right)$$

$$\text{Since } \mathcal{F} \left(\text{rect} \left(\frac{t}{d} \right) \right) = d \text{Sinc}(f \cdot d),$$

$$X(f) = \frac{A}{d} \cdot d^2 \text{Sinc}^2(f \cdot d) = (A \cdot d) \text{Sinc}^2(f \cdot d)$$

$$b) \quad X(f) = \frac{1}{(2+jf)(1+jf)} = 2\pi \left[\frac{1}{2\pi + j2\pi f} - \frac{1}{4\pi + j2\pi f} \right] \rightarrow x(t) = 2\pi \left[e^{-2\pi t} - e^{-4\pi t} \right] u(t)$$

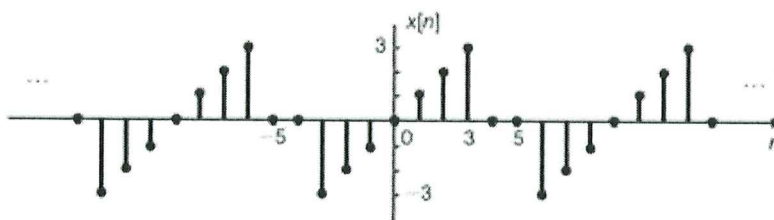
$$c) \quad x(t) = \frac{(2\pi)^2}{(2\pi t)^2 + (2\pi a)^2} \rightarrow X(f) = \frac{\pi}{a} e^{-2\pi a |f|}$$

$$\int_{-f_{80\%}}^{f_{80\%}} |X(f)|^2 df = 0.8 \int_{-\infty}^{\infty} |X(f)|^2 df \Rightarrow 1 - e^{-4\pi a f_{80\%}} = 0.8$$

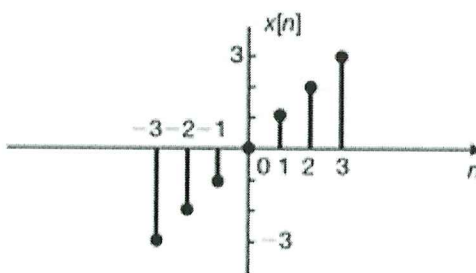
$$\Rightarrow f_{80\%} = -\frac{\ln(0.2)}{4\pi a}$$

Problem 5 (4 pts)

- (2 pts) a) Find the discrete-time Fourier series of the sequence
- $x[n]$
- as plotted below



- (2 pts) b) Find the discrete-time Fourier transform of the sequence
- $x[n]$
- as shown below



- a) The period is
- $N=9$
- ,
- $\pi_0 = \frac{2\pi}{N} = \frac{2\pi}{9}$

$$C_k = \frac{1}{9} \sum_{n=0}^{8} x[n] e^{-jk\pi_0 n} = \frac{1}{9} \left[-3 \cdot e^{j3k\pi_0} + 3 e^{-j3k\pi_0} - 2 e^{j2k\pi_0} + 2 e^{-j2k\pi_0} - e^{jk\pi_0} + e^{-jk\pi_0} \right] = \frac{-2j}{9} \left[3 \cdot \sin\left(\frac{6\pi}{9}k\right) + 2 \sin\left(\frac{4\pi}{9}k\right) + \sin\left(\frac{2\pi}{9}k\right) \right]$$

$$\begin{aligned} b) X(f) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi f n} = \sum_{n=-3}^3 x[n] \cdot e^{-j2\pi f n} \\ &= \left[-3 e^{j6\pi f} + 3 e^{-j6\pi f} - 2 e^{j4\pi f} + 2 e^{-j4\pi f} - e^{j2\pi f} + e^{-j2\pi f} \right] \\ &= -2j \left[3 \sin(6\pi f) + 2 \sin(4\pi f) + \sin(2\pi f) \right] \end{aligned}$$