Example 12

Consider the function $f(x) = 2x^3 - 3x^2 - 12x + 1$, where $Dom(f) = [-2, 4]$. Find all local extrema and the absolute extrema of $f(x)$.

Solution

$f'(x) = 6x^2 - 6x - 12$

Set $f'(x) = 0 \quad \Rightarrow \quad 6x^2 - 6x - 12 = 0 \quad \Rightarrow \quad 6(x^2 - x - 2) = 0$

$\Rightarrow \quad 6(x-2)(x+1) = 0 \quad \Rightarrow \quad x = -1,\ 2 \ \in Dom(f)$

Use the **First Derivative Test:** $f'(x) = 6(x-2)(x+1)$

	$-2 \le x < -1$	$x = -1$	$-1 < x < 2$	$x = 2$	$2 < x \le 4$
Sign of $(x-2)$	$-$	$-$	$-$	0	$+$
Sign of $(x+1)$	$-$	0	$+$	$+$	$+$
Sign of $f'(x)$	$+$	0 $\frown$ (local max.)	$-$	0 $\smile$ (local min.)	$+$

$\therefore$ f has a **local maximum** at $(-1, 8)$ and a **local minimum** at $(2, -19)$.

OR use the **Second Derivative Test:**

$$f''(x) = 12x - 6$$

$f''(-1) = 12(-1) - 6 = -18 < 0 \quad \Rightarrow \quad f$ has a **local maximum** at $x = -1,\ y = 8$.

$f''(2) = 12(2) - 6 = 18 > 0 \quad \Rightarrow \quad f$ has a **local minimum** at $x = 2,\ y = -19$.

To find the absolute maximum and minimum of $f(x)$, we compare the local maximum and minimum values with the values of f at the end points of the domain.

$Dom(f) = [-2, 4]$.

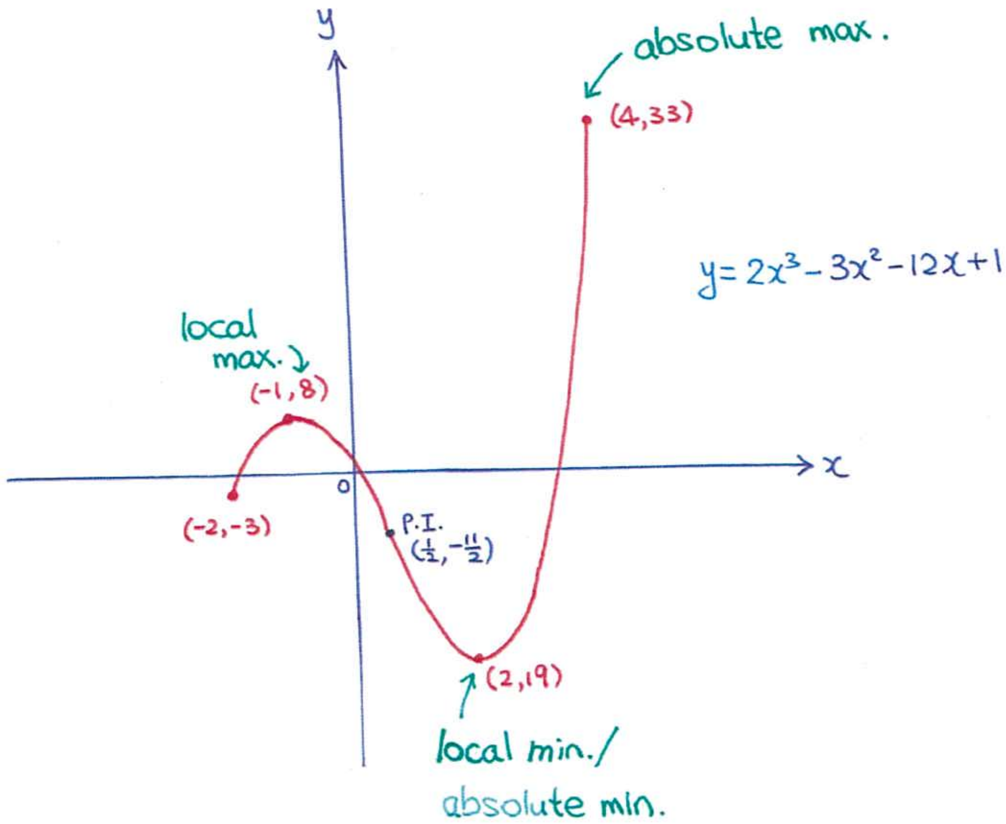
$f(-2) = 2(-2)^3 - 3(-2)^2 - 12(-2) + 1 = \mathbf{-3}$

$f(4) = 2(4)^3 - 3(4)^2 - 12(4) + 1 = \mathbf{33}$ **(← absolute maximum)**

Local minimum at $x = 2,\ f(2) = \mathbf{-19}$ **(← absolute minimum)**

Local maximum at $x = -1,\ f(-1) = \mathbf{8}$

$\therefore\ f$ has an **absolute maximum** at $(4, 33)$ and an **absolute minimum** at $(2, -19)$.



<u>Remark:</u> (For your reference.)

To find point of inflection:

Set $f''(x) = 0 \Rightarrow 12x - 6 = 0$

$\Rightarrow x = \frac{1}{2}$

$f''(x) < 0$ when $x < \frac{1}{2}$

$f''(x) > 0$ when $x > \frac{1}{2}$

$\because$ Sign of f'' changes at $x = \frac{1}{2}$

$\therefore$ f has a point of inflection at $x = \frac{1}{2}$

$(f(\frac{1}{2}) = -\frac{11}{2}.)$

Remark of Ex. 12:

If the domain is not restricted, i.e. $Dom(f) = \mathbb{R}$, then we will find the absolute maximum and absolute minimum (if they exist) by comparing the values of f at the local max. / local min. with the limits of $f(x)$ as $x \to \pm\infty$.

Local maximum at $x=-1$, $f(-1) = 8$

Local minimum at $x=2$, $f(2) = -19$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (2x^3 - 3x^2 - 12x + 1) = -\infty$$

Dominant term: $2x^3 \to -\infty$ as $x \to -\infty$

$\leftarrow$ $f(x)$ decreases indefinitely as $x \to -\infty$

$\therefore$ $f(x)$ has no absolute minimum.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (2x^3 - 3x^2 - 12x + 1) = \infty$$

Dominant term: $2x^3 \to \infty$ as $x \to \infty$

$\leftarrow$ $f(x)$ increases indefinitely as $x \to \infty$

$\therefore$ $f(x)$ has no absolute maximum.

Example :

Find and classify the local extreme values of the function $f(x) = x - x^{\frac{2}{3}}$.

Solution:

$$f'(x) = 1 - \frac{2}{3}x^{-\frac{1}{3}} = \frac{x^{\frac{1}{3}} - \frac{2}{3}}{x^{\frac{1}{3}}}.$$

Note that $f'(x)$ does not exist at $x = 0$ (but $x = 0$ is in $Dom(f)$).

We call this a singular point.

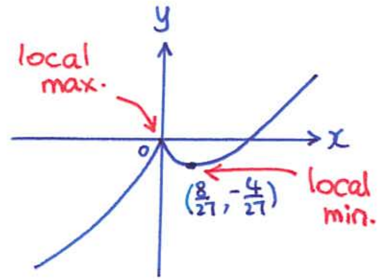
This point could be a local max./min.

Set $f'(x) = 0 \Rightarrow \frac{x^{\frac{1}{3}} - \frac{2}{3}}{x^{\frac{1}{3}}} = 0 \Rightarrow x^{\frac{1}{3}} - \frac{2}{3} = 0 \Rightarrow x = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

stationary point / critical point

Using the First Derivative Test,

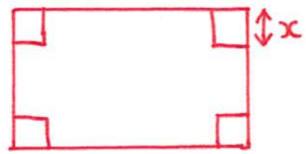
	$x<0$	$x=0$	$0<x<\frac{8}{27}$	$x=\frac{8}{27}$	$x>\frac{8}{27}$
sign of f'	$+$	X	$-$	0	$+$

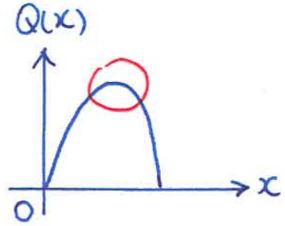


$\therefore$ $f(x)$ has a local maximum at $x = 0$, $f(0) = 0$

and a local minimum at $x = \frac{8}{27}$, $f(\frac{8}{27}) = -\frac{4}{27}$.

4. Optimization problem





Aim: To **maximize** or **minimize** the quantity Q.

E.g. To maximize the volume of a box

E.g. To minimize the surface area of a container, etc.

Procedure for solving optimization problem

Suppose the question asks you to maximize or minimize the quantity Q.

Step 1: Read the question carefully. Draw a diagram if appropriate.

Step 2: Define any variables you wish to use that are not already specified in the question.

Step 3: Use the diagram or the given information from the question to write down one or more constraints which link the variables.

Step 4: Express the quantity Q to be maximized or minimized as a function of one or more variables.

Step 5: If Q depends on more than one variable, use the constraints in Step 3 to express Q as a function of only one variable (say x). Determine the interval in which this variable must lie for the problem to make sense.

Step 6: Differentiate the function Q with respect to the variable x (in Step 5), i.e. find $Q'(x)$.

Step 7: Set $Q'(x) = 0$ and solve for the value(s) of x.

Step 8: Eliminate any values of x obtained in Step 7 that do not make sense.

Step 9: Use either the First Derivative Test or the Second Derivative Test to determine which of the remaining critical point(s) is the one that you are looking for. If there is only one remaining critical point after Step 8, you are still required to check that the extreme value is really a maximum or minimum point.

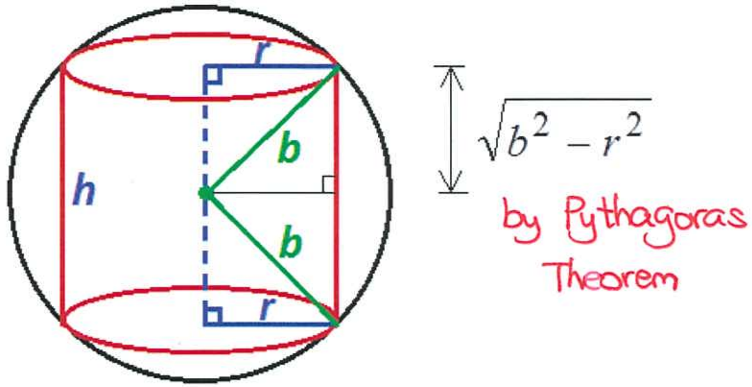
Step 10: Write down the final answer and the optimized value of Q (if necessary).

Example 13

Find the greatest volume of a cylinder that can be inscribed in a sphere of radius b, where b is a given constant.

Solution

Let h and r be the height and radius of the cylinder, respectively.



The height of the cylinder is $h = 2\sqrt{b^2 - r^2}$,

where $0 \le h \le 2b$.

Thus, $r^2 = b^2 - \left(\frac{h}{2}\right)^2$.

Volume of the cylinder is $V = \pi r^2 h = \pi\left[b^2 - \left(\frac{h}{2}\right)^2\right]h = \pi\left(b^2 h - \frac{h^3}{4}\right)$. ← Express V in terms of one variable, i.e. h

Differentiate both sides w.r.t. h:

$$V'(h) = \pi\left(b^2 - \frac{3h^2}{4}\right).$$

Set $V'(h) = 0 \quad \Rightarrow \quad \pi\left(b^2 - \frac{3h^2}{4}\right) = 0 \quad \Rightarrow \quad b^2 - \frac{3h^2}{4} = 0 \quad \Rightarrow \quad h^2 = \frac{4b^2}{3}$

$\Rightarrow \quad h = \pm\sqrt{\frac{4b^2}{3}} = \frac{2b}{\sqrt{3}}$ or $-\frac{2b}{\sqrt{3}}$ (rejected since h must be ≥ 0.)

First Derivative Test: $V'(h) = \pi\left(b^2 - \frac{3h^2}{4}\right) = \frac{3}{4}\pi\left(\frac{4b^2}{3} - h^2\right) = \frac{3}{4}\pi\left(\frac{2b}{\sqrt{3}} + h\right)\left(\frac{2b}{\sqrt{3}} - h\right)$

	$0 \leq h < \frac{2b}{\sqrt{3}}$	$h = \frac{2b}{\sqrt{3}}$	$\frac{2b}{\sqrt{3}} < h \leq 2b$
Sign of $V'(h)$	$+$	0	$-$

(local max.)

$\therefore$ The volume of the cylinder is maximized at $h = \frac{2b}{\sqrt{3}}$.

OR use Second Derivative Test:

$V''(h) = -\frac{3\pi h}{2}$

$V''\left(\frac{2b}{\sqrt{3}}\right) = -\frac{3\pi}{2}\left(\frac{2b}{\sqrt{3}}\right) < 0$

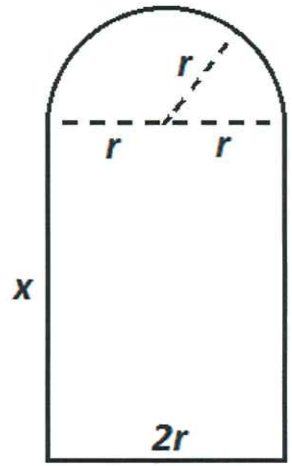
$\therefore$ $V(h)$ is maximized when $h = \frac{2b}{\sqrt{3}}$.

$\therefore$ The greatest volume of the cylinder is

$$V\left(\frac{2b}{\sqrt{3}}\right) = \pi\left[b^2\left(\frac{2b}{\sqrt{3}}\right) - \frac{\left(\frac{2b}{\sqrt{3}}\right)^3}{4}\right] = \pi\left(\frac{2}{\sqrt{3}}b^3 - \frac{2}{3\sqrt{3}}b^3\right) = \frac{4\pi}{3\sqrt{3}}b^3 \quad (\text{unit}^3)$$

Example 14

A window is in the shape as shown in the figure on the right. Suppose the perimeter of the window is fixed to be 240 cm. Find the dimensions of the window so that the area of the window is maximized.



Solution

Perimeter of the window is $2x + 2r + \pi r = 240$

$\therefore\ x = \frac{240-2r-\pi r}{2} = 120 - r - \frac{\pi}{2}r$

Area of the window: $A(r) = \frac{\pi r^2}{2} + 2rx = \frac{\pi r^2}{2} + 2r\left(120 - r - \frac{\pi}{2}r\right)$ ← Express A in terms of one variable, i.e. r

$$= 240r - 2r^2 - \frac{\pi r^2}{2}$$

Differentiate both sides w.r.t. r:

$$A'(r) = 240 - 4r - \pi r$$

Set $A'(r) = 0 \quad \Rightarrow \quad 240 - 4r - \pi r = 0$

$\Rightarrow \quad r = \frac{240}{\pi+4}$

Using the Second Derivative Test:

$$A''(r) = -4 - \pi$$

$\because\ A''\left(\frac{240}{\pi+4}\right) = -4 - \pi < 0,$

$\therefore\ A(r)$ is maximized when

$$r = \frac{240}{\pi + 4} \approx \boxed{33.6\text{ cm}}$$

and

$$x = 120 - r - \frac{\pi}{2}r = 120 - \frac{240}{\pi + 4} - \frac{\pi}{2}\left(\frac{240}{\pi + 4}\right) = \frac{240}{\pi + 4} \approx \boxed{33.6\text{ cm}}.$$

<u>OR</u> Use the First Derivative Test:

$A'(r) > 0$ when $r < \frac{240}{\pi+4}$

$A'(r) < 0$ when $r > \frac{240}{\pi+4}$

$\therefore$ $A(r)$ is maximized when $r = \frac{240}{\pi+4}$

& $x = \ldots = \frac{240}{\pi+4}$.

Example 15

Find the coordinates of a point on the parabola $2y = x^2$ that is closest to the point $(-6, 0)$.

Hence find that shortest distance.

Solution

Let (x, y) be a point on the parabola $2y = x^2$.

Then the distance between (x, y) and $(-6, 0)$ is

$$l = \sqrt{[x-(-6)]^2 + (y-0)^2} = \sqrt{x^2 + 12x + 36 + \left(\frac{1}{2}x^2\right)^2} = \left(\frac{x^4}{4} + x^2 + 12x + 36\right)^{\frac{1}{2}}$$

Differentiating both sides w.r.t. x:

$$\frac{dl}{dx} = \frac{1}{2}\left(\frac{x^4}{4} + x^2 + 12x + 36\right)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left(\frac{x^4}{4} + x^2 + 12x + 36\right)$$

$$= \frac{1}{2} \cdot \frac{x^3 + 2x + 12}{\sqrt{\frac{x^4}{4} + x^2 + 12x + 36}}$$

Set $\dfrac{dl}{dx} = 0 \;\Rightarrow\; \dfrac{1}{2} \cdot \dfrac{x^3 + 2x + 12}{\sqrt{\dfrac{x^4}{4} + x^2 + 12x + 36}} = 0$

$$\Rightarrow\; x^3 + 2x + 12 = 0$$

$$\Rightarrow\; (x+2)(x^2 - 2x + 6) = 0$$

Discriminant $b^2 - 4ac = (-2)^2 - 4(1)(6) = -20 < 0$

$$\Rightarrow\; x + 2 = 0 \quad \text{or} \quad x^2 - 2x + 6 = 0 \;\; \text{(which has no real solution)}$$

$$\Rightarrow\; x = -2$$

Using the First Derivative Test:

$$\frac{dl}{dx} < 0 \;\text{ when }\; x < -2$$
$$\frac{dl}{dx} > 0 \;\text{ when }\; x > -2$$

$\therefore$ The distance between (x, y) and $(-6, 0)$ is minimized when $x = -2,\; y = 2$.

$\therefore$ The shortest distance is

$$l = \left[\frac{(-2)^4}{4} + (-2)^2 + 12(-2) + 36\right]^{\frac{1}{2}} = \sqrt{20} \;\text{(unit)}$$

5. L'Hôpitals rule

This is used to find **limits of indeterminate forms** such as

$$\boxed{\frac{0}{0}\ ,\ \frac{\infty}{\infty}}\ ,\ \boxed{0\times\infty\ ,\ \infty-\infty}\ \text{ or }\ \boxed{1^{\infty}\ ,\ \infty^{0}\ ,\ 0^{0}}.$$

Type $\frac{0}{0}$: If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then

$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}}.$$

Note: a could be any real number, a^-, a^+, $-\infty$ or ∞.

Remark: Always check that you have $\frac{0}{0}$ form **BEFORE** applying the L'Hôpital's rule. DO NOT use the L'Hôpital's rule if any one of the numerator and denominator is non-zero when taking limit.

Example 16

Evaluate the limit $\lim\limits_{x\to 2} \dfrac{x^2-4}{x^2+x-6}$.

Solution

Method 1:

$$\lim_{x\to 2} \frac{x^2-4}{x^2+x-6} \quad \left(\frac{\mathbf{0}}{\mathbf{0}}\ \textbf{form}\right)$$

$$= \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x+3)} = \lim_{x\to 2} \frac{x+2}{x+3} = \frac{2+2}{2+3} = \frac{4}{5}$$

Method 2:

$$\lim_{x\to 2} \frac{x^2-4}{x^2+x-6} \quad \left(\frac{\mathbf{0}}{\mathbf{0}}\ \textbf{form}\right)$$

$$= \lim_{x\to 2} \frac{2x}{2x+1} \quad \text{by } \mathbf{L'Hôpital's\ rule}$$

$$= \frac{2(2)}{2(2)+1} = \frac{4}{5}$$

Example 17

Evaluate $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2+x}$.

Solution

Method 1:

$$\lim_{x\to 0} \frac{1-\cos x}{x^2+x} \left(\frac{0}{0} \text{ form}\right) = \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)}{(x^2+x)(1+\cos x)} = \lim_{x\to 0} \frac{\boxed{1-\cos^2 x}^{\,=\sin^2 x}}{x(x+1)(1+\cos x)}$$

$$= \lim_{x\to 0}\left(\boxed{\frac{\sin x}{x}}^{\to 1} \cdot \frac{\sin x}{(x+1)(1+\cos x)}\right)$$

$$= 1 \cdot \frac{\sin 0}{(0+1)(1+\underbrace{\cos 0}_{=1})} = \frac{0}{2} = 0$$

Recall from Ch.6:

$\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

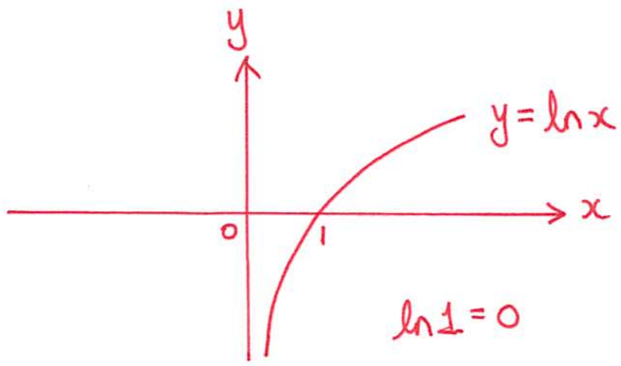
Method 2:

$$\lim_{x\to 0} \frac{1-\cos x}{x^2+x} \left(\frac{0}{0} \text{ form}\right) = \lim_{x\to 0} \frac{\sin x}{2x+1} \quad \text{by } \textbf{L'Hôpital's rule}$$

$$= \frac{\sin 0}{2(0)+1} = \frac{0}{1} = 0$$

Example 18

Find $\lim_{x \to 1} \frac{x-1}{\ln x}$.



Solution

$$\lim_{x \to 1} \frac{x-1}{\ln x} \quad \left(\frac{\mathbf{0}}{\mathbf{0}} \textbf{ form}\right)$$

$$= \lim_{x \to 1} \frac{1}{\frac{1}{x}} \quad \text{by } \textbf{L'Hôpital's rule}$$

$$= \lim_{x \to 1} x$$

$$= 1$$

Example 19

Evaluate $\lim\limits_{x\to 0} \frac{\cos(2x)-\cos x}{x^2}$.

Solution

$$\lim_{x\to 0} \frac{\cos(2x)-\cos x}{x^2} \quad \left(\frac{0}{0} \textbf{ form}\right)$$

$$= \lim_{x\to 0} \frac{-2\sin(2x)+\sin x}{2x} \quad \text{by } \textbf{L'Hôpital's rule} \quad \left(\frac{0}{0} \textbf{ form}\right)$$

$$= \lim_{x\to 0} \frac{-4\cos(2x)+\cos x}{2} \quad \text{by } \textbf{L'Hôpital's rule}$$

$$= \frac{-4\cos 0+\cos 0}{2}$$

$$= \frac{-4+1}{2}$$

$$= -\frac{3}{2}$$

Example 20

Evaluate $\lim_{x\to 0} \frac{\tan x - x}{x - \sin x}$.

Solution

$$\lim_{x\to 0} \frac{\tan x - x}{x - \sin x} \quad \left(\frac{\mathbf{0}}{\mathbf{0}}\ \textbf{form}\right)$$

$$= \lim_{x\to 0} \frac{\sec^2 x - 1}{1 - \cos x} \quad \text{by } \textbf{L'Hôpital's rule} \quad \left(\frac{\mathbf{0}}{\mathbf{0}}\ \textbf{form}\right)$$

$$= \lim_{x\to 0} \frac{2\sec x \cdot \sec x \tan x}{\sin x} \quad \text{by } \textbf{L'Hôpital's rule}$$

$$= \lim_{x\to 0} \left(\frac{2}{\cos^2 x \cdot \sin x} \cdot \frac{\sin x}{\cos x}\right)$$

$$= \lim_{x\to 0} \frac{2}{\cos^3 x}$$

$$= \frac{2}{\cos^3 0}$$

$$= \frac{2}{1^3} = 2$$

Example 21

Evaluate $\lim_{x\to 0} \frac{e^{x^3}-1-x^3}{x^6}$.

Solution

$$\lim_{x\to 0} \frac{e^{x^3}-1-x^3}{x^6} \quad \left(\frac{0}{0}\ \textbf{form}\right)$$

$$= \lim_{x\to 0} \frac{3x^2e^{x^3}-3x^2}{6x^5} \qquad \text{by } \textbf{L'Hôpital's rule}$$

$$= \lim_{x\to 0} \frac{e^{x^3}-1}{2x^3} \quad \left(\frac{0}{0}\ \textbf{form}\right)$$

$$= \lim_{x\to 0} \frac{3x^2e^{x^3}}{6x^2} \qquad \text{by } \textbf{L'Hôpital's rule}$$

$$= \lim_{x\to 0} \frac{e^{x^3}}{2}$$

$$= \frac{e^0}{2}$$

$$= \frac{1}{2}$$

Example 22

Evaluate $\lim_{x\to 0}\frac{\ln(\cos 5x)}{\ln(\cos 2x)}$.

Solution

$$\lim_{x\to 0}\frac{\ln(\cos 5x)}{\ln(\cos 2x)} \quad \left(\frac{\mathbf{0}}{\mathbf{0}}\ \textbf{form}\right)$$

$$= \lim_{x\to 0}\frac{\frac{-5\sin 5x}{\cos 5x}}{\frac{-2\sin 2x}{\cos 2x}} \quad \text{by } \textbf{L'Hôpital's rule}$$

$$= \frac{5}{2}\lim_{x\to 0}\left(\frac{\sin 5x}{5x}\cdot\frac{2x}{\sin 2x}\cdot\frac{\cos 2x}{\cos 5x}\cdot\frac{5}{2}\right)$$

$$= \frac{5}{2}\cdot 1\cdot 1\cdot\frac{1}{1}\cdot\frac{5}{2}$$

$$= \frac{25}{4}$$

$$\frac{d}{dx}[\ln(\cos ax)] = \frac{1}{\cos ax}\cdot\frac{d}{dx}(\cos ax)$$

$$= \frac{1}{\cos ax}\cdot(-\sin ax)\cdot\frac{d}{dx}(ax)$$

$$= \frac{-a\sin(ax)}{\cos(ax)}$$

Type $\frac{\infty}{\infty}$: If $\lim_{x\to a}|f(x)| = \lim_{x\to a}|g(x)| = \infty$, then

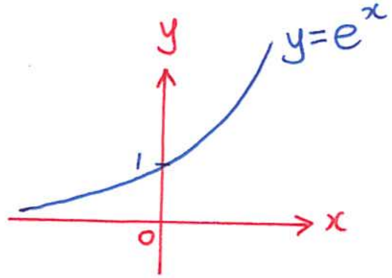
$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}}.$$

Note: a could be any real number, a^-, a^+, $-\infty$ or ∞.

Remark: Always check that you have $\frac{\infty}{\infty}$ form **BEFORE** applying the L'Hôpital's rule.

Example 23

Find $\lim_{x\to\infty}\frac{x}{e^x}$.



Solution

$$\lim_{x\to\infty}\frac{x}{e^x}\ \left(\frac{\infty}{\infty}\ \textbf{form}\right) = \lim_{x\to\infty}\frac{1}{e^x} \text{ by } \textbf{L'Hôpital's rule}$$

$$= 0$$

$\frac{1}{\infty} \to 0$

Example 24

Find $\lim_{x \to \infty} \frac{e^x}{x}$, if it exists.

Solution

$$\lim_{x \to \infty} \frac{e^x}{x} \left(\frac{\infty}{\infty} \textbf{ form}\right) = \lim_{x \to \infty} \frac{e^x}{1} \quad \text{by } \textbf{L'Hôpital's rule}$$

$$= \infty$$

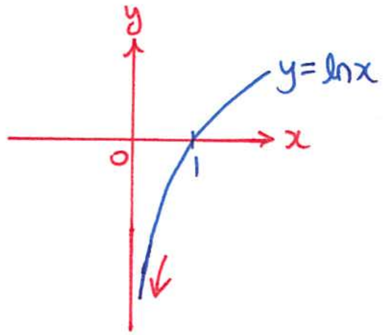
$\therefore$ The limit does not exist.

Example 25

Evaluate $\lim_{x \to 0^+} \frac{\ln x}{\cot x}$.

Solution

$$\lim_{x \to 0^+} \frac{\ln x}{\cot x} \quad \left(\frac{-\infty}{\infty} \textbf{ form}\right)$$



$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\operatorname{cosec}^2 x} \quad \text{by } \textbf{L'Hôpital's rule}$$

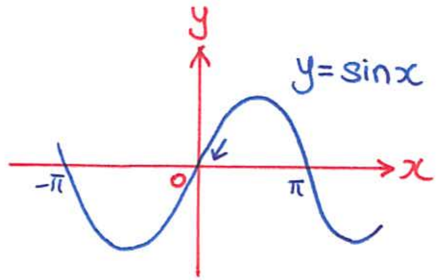
$\cot x = \frac{\cos x}{\sin x}$

$$= -\lim_{x \to 0^+} \frac{\sin^2 x}{x}$$

$$= -\lim_{x \to 0^+} \left(\underbrace{\frac{\sin x}{x}}_{\to 1} \cdot \underbrace{\sin x}_{\to 0}\right)$$

$$= -1 \cdot 0$$

$$= 0$$



As $x \to 0^+$,
$\sin x \to 0^+$
& $\cos x \to 1$
$\therefore \cot x \to +\infty$