Tutorial 7 (with solution)

Vectors

Cauchy-Schwarz Inequality

$$\boxed{|a^T b| \leq \|a\|\|b\|}$$

- To apply this inequality, the key is to appropriately choose the two vectors a and b.

Question 1

- Show that for any angle θ,

$$|\cos^2\theta - \sin^2\theta| \leq 1.$$

Q.1 (solution)

- Let the two vectors be $(\cos\theta, \sin\theta)$ and $(\cos\theta, -\sin\theta)$.

Watch the following video (until 2:14)

- https://www.youtube.com/watch?v=NBGuDgJ5kjg

Question 2

- Let $a_1, a_2, \ldots, a_n$ be real numbers. Show that

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^2 \leq \frac{1}{n}\sum_{i=1}^{n} a_i^2$$

(Average)^2 $\leq$ Average of the Squares

or equivalently,

$|\mathbf{avg}(a)| \leq \mathbf{rms}(a)$

Q.2 (solution)

- Let the two vectors be $(a_1, a_2, \ldots, a_n)$ and $\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$.
- Continue watching the previous video until the end.

Question 3

- Let $a_1, a_2, \ldots, a_n$ be positive. Show that

$$a_1 + a_2 + \cdots + a_n \leq \frac{a_1^2}{a_2} + \frac{a_2^2}{a_3} + \cdots + \frac{a_n^2}{a_1}.$$

Q.3 (solution)

Let the two vectors be

$$x = \left(\frac{a_1}{\sqrt{a_2}}, \frac{a_2}{\sqrt{a_3}}, \dots, \frac{a_n}{\sqrt{a_1}} \right),$$

$$y = (\sqrt{a_2}, \sqrt{a_3}, \dots, \sqrt{a_n}, \sqrt{a_1}).$$

Then

$$(x^T y)^2 = (a_1 + a_2 + \cdots + a_n)^2$$

$$\|x\|^2 = \frac{a_1^2}{a_2} + \frac{a_2^2}{a_3} + \cdots + \frac{a_n^2}{a_1}$$

$$\|y\|^2 = a_1 + a_2 + \cdots + a_n$$

The given inequality then follows from

$$(x^T y)^2 \leq \|x\|^2 \|y\|^2.$$

Question 4

- The triangle inequality is given by

$$\boxed{\|a+b\| \leq \|a\| + \|b\|}$$

- When does it hold with equality?

Q.4 (solution)

- The triangle inequality is proved by using Cauchy-Schwarz inequality as follows:

$$\begin{aligned} \|a+b\|^2 &= \|a\|^2 + 2a^T b + \|b\|^2 && \text{(quadratic formula)} \\ &\leq \|a\|^2 + 2\|a\|\|b\| + \|b\|^2 && \text{(Cauchy-Schwarz)} \\ &= (\|a\| + \|b\|)^2 \end{aligned}$$

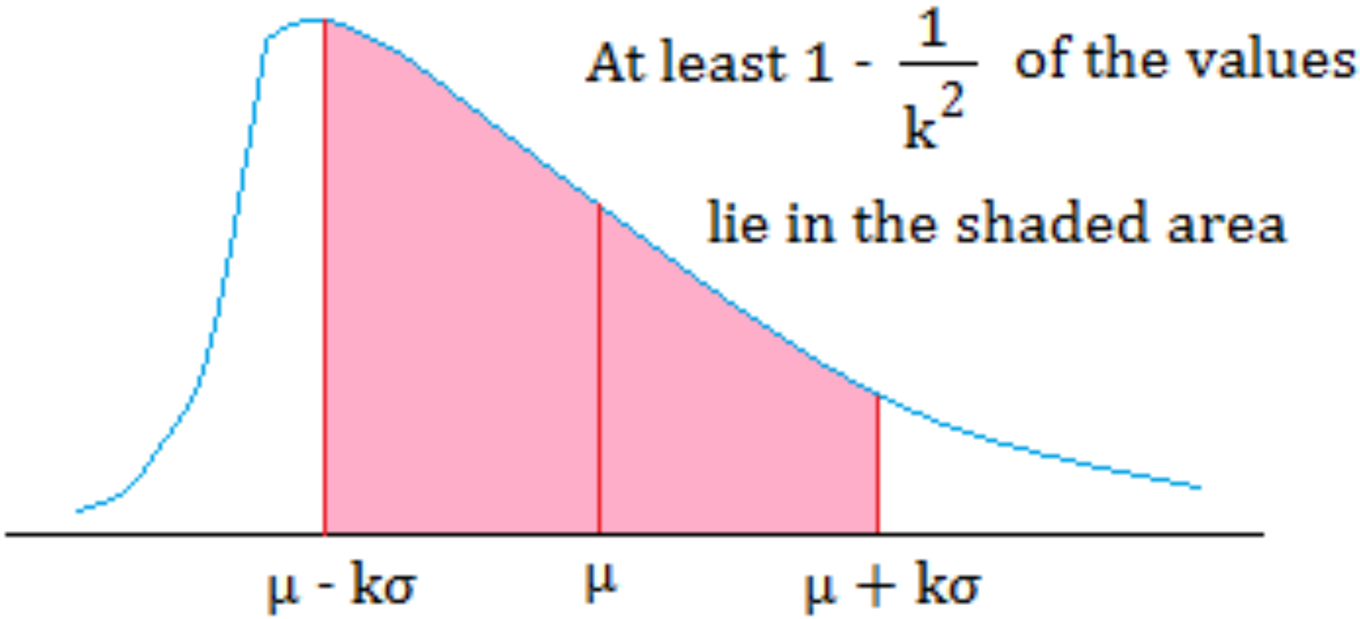
- Since $a^T b = \|a\|\|b\| \cos\theta$, the inequality holds when $\theta = 0$, i.e., when a and b point to the same direction.

Chebyshev's Inequality for Data Set

The proportion of entries of x that satisfy

$$|x_i - \mu| \geq m\sigma$$

is less than or equal to $\frac{1}{m^2}$.



The proportion of entries of x that satisfy

$$|x_i - \mu| < m\sigma$$

is greater than or equal to $1 - \frac{1}{m^2}$.

- Chebyshev’s Theorem Explained (22 min)
  - https://www.youtube.com/watch?v=OM0K22pmkuY
  - Watch the first 10’30’’ min during tutorial. Then 14’28’’
- Type 1 Application
  - Determine the percentage p for a given interval (a, b).
- Type 2 Application
  - Determine an interval given the percentage p.

Question 5

- Consider the marks of the students obtained in Test 1.
- The mean and the standard deviation are 53 and 21, respectively.
- At least what percentage of students obtain marks between 21.5 and 84.5?

Q.5 (solution)

- $m = \frac{84.5-\mu}{\sigma} = \frac{84.5-53}{21} = 1.5$
- It can be verified that $\frac{\mu-21.5}{\sigma} = 1.5$
- By Chebyshev's inequality, at least $1 - \frac{1}{m^2} =$ 56% of students obtain marks between the given range.