MA1200 CALCULUS AND BASIC LINEAR ALGEBRA I

LECTURE: CG1

Chapter 2
Sets and Functions

Set Notation

A ***set*** is a collection of distinct objects. Each object in a set is called an ***element*** or a ***member*** of that set. A set may contain a finite number of elements, infinitely many elements, or even no elements.

For example,

- $V = \{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{u}\}$ is the set of all vowels of the English alphabets.
- $\mathrm{A} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is the set of all integers from 1 to 10.
- $\mathrm{B} = \{2, 4, 6, 8, 10, \dots\}$ is the set of all positive even numbers.

They are all sets, and their elements are listed inside the brackets "{ }".

A set is a collection, but not a list. The order in which the elements are written is not important. For example, $\mathrm{S} = \{a, b, c\} = \{b, a, c\} = \{c, b, a\}$.

In general, we use the notation

$$\{x|x \text{ processes certain properties}\}$$

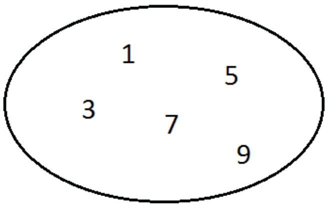
to denote a set of objects that share some common properties. The vertical line "|" means "such that".

Example 1

- $C = \{x|x \text{ is an odd number and } 0 < x < 10\} = \{1, 3, 5, 7, 9\}$.
- $D = \{x|x \text{ is negative and } x \text{ is a multiple of } 5\} = \{-5, -10, -15, -20, -25, \ldots\}$.

A set can be represented using ***Venn diagram***.

For example, a Venn diagram for the set $C = \{1, 3, 5, 7, 9\}$ is shown on the right.



Some notations:

- "∈" means "belongs to" or "is an element of".

  If "a belongs to S" or "a is an element of S", we write $a \in S$.

- "∉" means "does not belong to" or "is not an element of".

  If "b does not belong to S" or "b is not an element of S", we write $b \notin S$.

- "⊂" means "is a **subset** of".

  If every element in set A also belongs to set B, we say that "A is a **subset** of B" and we write $A \subset B$.

- "⊄" means "is not a subset of".

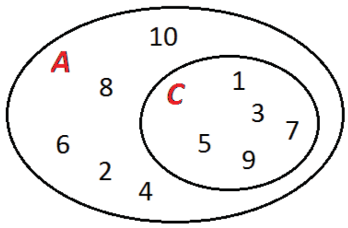
  If there is at least one element which belongs to set A but does not belong to set B, we say that "A is not a subset of B" and we write $A \not\subset B$.

Remarks:

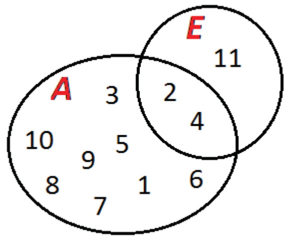
1. Some authors use "⊆" to denote "is a subset of", and "⊈" to denote "is not a subset of".
2. By the definition of subsets, any set is a subset of itself.

Example 2

Given the sets $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $C = \{1, 3, 5, 7, 9\}$ and $E = \{2, 4, 11\}$. Then C is a subset of A, denoted by $C \subset A$, since every element in C also belongs to A. The set E is not a subset of A, because $11 \in E$ but $11 \notin A$. We write $E \not\subset A$.



$C \subset A$



$E \not\subset A$

Equality of sets

Two sets *A* and *B* are equal (written as $A = B$) if they contain the same elements. For example, if $A = \{1, 2, 3\}$, $B = \{3, 2, 1\}$ and $C = \{1, 2\}$, then we have $A = B$ but $A \neq C$.

Some commonly used sets in Mathematics include:

$\emptyset = \{\ \}$ is called an "empty set", which contains no elements.

$\mathbb{N} = \{1, 2, 3, 4, \dots\}$ is the set of all natural numbers (positive integers).

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$ is the set of all integers.

$\mathbb{Q} = \left\{\frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0\right\}$ = the set of all rational numbers.

$\mathbb{R}$ = the set of all real numbers.

$\mathbb{C}$ = the set of all complex numbers *(will be discussed in MA1201)*

Using notations of subsets, we have

$$\emptyset \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

Caution!! Be careful when using the notations "$\in$" and "$\subset$". For example, $1 \in \mathbb{Z}$ (which means "1 is an element of $\mathbb{Z}$") and $\{1\} \subset \mathbb{Z}$ (which means "the set containing the number 1 is a subset of $\mathbb{Z}$"), but **never write** $1 \subset \mathbb{Z}$. (This doesn't make sense!!)

Example 3

Use set notations to represent each of the following sets.

(a) The set of integers which are smaller than -6 and greater than -13.

(b) The set of integers which are greater than 2 but less than or equal to 15.

Solution

(a) $\{-12, -11, -10, -9, -8, -7\}$ or $\{x | x \in \mathbb{Z} \text{ and } -13 < x < -6\}$

or $\{x \in \mathbb{Z} | -13 < x < -6\}$

(b) $\{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ or $\{x | x \in \mathbb{Z} \text{ and } 2 < x \leq 15\}$

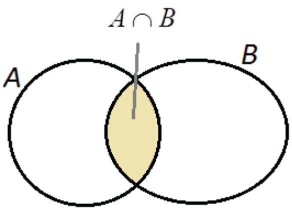
or $\{x \in \mathbb{Z} | \, 2 < x \leq 15\}$

Operations of sets

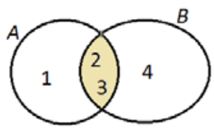
Given two sets A and B. We can combine the two sets to form new sets by using set operations:

- **Intersection**

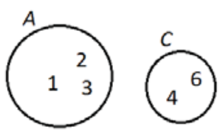
  The **intersection of sets *A* and *B***, written as $A \cap B$, is a set whose elements belong to **both A and B**. That is, $A \cap B = \{x | x \in A \text{ and } x \in B\}$.



  E.g. If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, then $A \cap B = \{2, 3\}$.

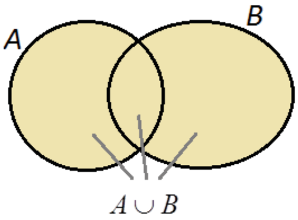


  E.g. If $A = \{1, 2, 3\}$ and $C = \{4, 6\}$, then $A \cap C = \emptyset$. That is, A and C are disjoint sets.

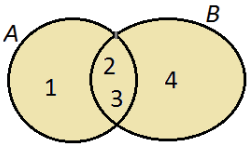


- **Union**

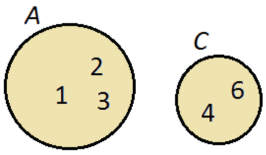
The **union of sets A and B**, written as $A \cup B$, is a set whose elements belong to **either A or B or both of them**. That is, $A \cup B = \{x | x \in A \text{ or } x \in B\}$.



E.g. If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, then $A \cup B = \{1, 2, 3, 4\}$.

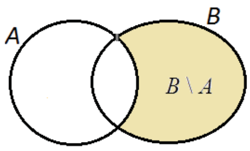


E.g. If $A = \{1, 2, 3\}$ and $C = \{4, 6\}$, then $A \cup C = \{1, 2, 3, 4, 6\}$.

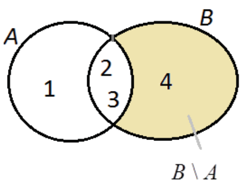


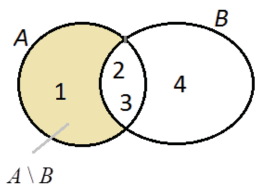
- **Complement**

The **complement of A with respect to B**, written as $B \backslash A$, is a set whose elements **belong to B but not belong to A**. That is, $B \backslash A = \{x | x \in B \text{ but } x \notin A\}$. The line "\" means "exclude".



E.g. If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, then $B \backslash A = \{4\}$ and $A \backslash B = \{1\}$.





E.g. If $A = \{1, 2, 3\}$ and $C = \{4, 6\}$, then $C \backslash A = \{4, 6\} = C$ and $A \backslash C = \{1, 2, 3\} = A$.

E.g. $\mathbb{R} \backslash \{1, 3\}$ is the set of all real numbers except 1 and 3.

E.g. $\mathbb{R} \backslash \mathbb{Q}$ is the set of all irrational numbers. For example, $\pi = 3.14159 \ldots$, $e = 2.71828 \ldots$ and $\sqrt{2} = 1.4142 \ldots$ are irrational numbers.

Example 4

Let $A = \{2, 4, 6, 8\}$ and $B = \{-3, 6, 8, 12.4\}$.

Write the set described by each of the following. List all the elements in the set.

(a) $A \cup B$ (b) $A \cap B$ (c) $B \cap \mathbb{Z}$ (d) $B \cap \mathbb{R}$

Solution

(a) $A \cup B = \{-3, 2, 4, 6, 8, 12.4\}$

(b) $A \cap B = \{6, 8\}$

(c) $B \cap \mathbb{Z} = \{-3, 6, 8\}$

(d) $B \cap \mathbb{R} = \{-3, 6, 8, 12.4\}$

Intervals

Recall that $\mathbb{R}$ is the set of all real numbers. Let a and b be two distinct real numbers where $a < b$. We use the following notations to describe some **subsets of real numbers** (known as **intervals**):

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$
$$[a, b) = \{x \in \mathbb{R} \mid a \le x < b\}$$
$$(a, b] = \{x \in \mathbb{R} \mid a < x \le b\}$$
$$[a, b] = \{x \in \mathbb{R} \mid a \le x \le b\}$$
$$[a, \infty) = \{x \in \mathbb{R} \mid x \ge a\}$$
$$(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$$
$$(-\infty, a] = \{x \in \mathbb{R} \mid x \le a\}$$
$$(-\infty, a) = \{x \in \mathbb{R} \mid x < a\}$$
$$\mathbb{R} = (-\infty, \infty)$$

Note: Never write $[a, \infty]$, $(a, \infty]$, $[-\infty, a]$ and $[-\infty, a)$.

Example 5

Express each of the following sets as interval notations:

(a) "The set of all real numbers which are smaller than or equal to 6" $=$

(b) $\{x \in \mathbb{R} | \ x > 2\} =$

(c) $\{x \in \mathbb{R} | \ x < 3 \text{ and } x \geq 1\} =$

(d) $\{x \in \mathbb{R} | \ x < 3 \text{ or } x \geq 1\} =$

Example 6

Simply each of the following:

(a) $(1,4) \cap [2,6] =$

(b) $[-2,0) \cap [0,3] =$

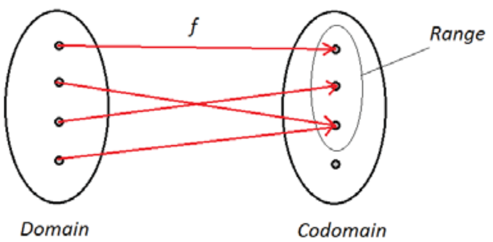
(c) $[-2,0) \cup [0,3] =$

(d) $[-2,3] \cup (3,\infty) =$

(e) $(-\infty,6] \cap (3,\infty) =$

Functions

- A **function** is a rule that assigns a unique value $f(x)$ to any x from a set called the domain.
- The **domain** of a function is the set of all possible input values (i.e. all possible values of x) for which the function is defined.
- The **codomain** of a function is the set which contains all possible output values.
- The **range** is the set of all output values (i.e. all values of y or $f(x)$), which actually result from using the function formula.
- In general, the range of a function is a subset of its codomain but not necessarily the same set.



- Clearly, the range of a function depends on what you put into the function (domain) and the function itself.

- If set A is the domain of f and set B is the codomain of f, we write
$$f: A \to B.$$
For example, we may write the following to define a function:
Let $f: \mathbb{R} \to \mathbb{R}$ be defined by
$$f(x) = x^2 + x + 1.$$

- If $x \in A$ and $y = f(x) \in B$ (for example, $y = x^2 + x + 1$), then x is called the **independent** variable and y is called the **dependent** variable.

- We use the term "**largest possible domain**" to denote the largest possible **set** of the input values x, not just the largest possible number that x can take.

- We use the notations $Dom(f)$ and $Ran(f)$ to denote the **largest possible domain** and the **largest possible range** of the function f, respectively. Then $x \in Dom(f)$ and $f(x) \in Ran(f)$.

In this course, we will mainly study those functions whose domains and codomains are subsets of $\mathbb{R}$, i.e. they are real-valued functions.

Summary of the domain, codomain and range of a function:

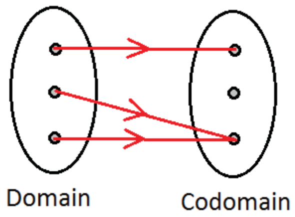
Domain: What can be put into the function?

Codomain: What may possibly come out of a function?

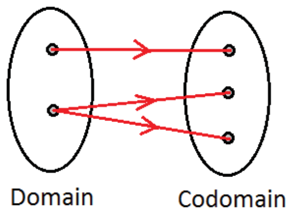
Range: What actually comes out of a function?

Note that every element of the domain A (input) must have exactly one output (in the codomain B).

Consider the following figures:



This is a well-defined function. (Why?)



This is not a well-defined function. (Why?)

Here are some examples of equations which define y as a function of x (where $x \in \mathbb{R}$):

- $y = 3x^2 + 5x + 1, \quad y = 3x - 1$ (These are examples of **polynomials** (Ch.3))
- $y = \sin x, \quad y = \cos x$ (These are examples of **trigonometric functions** (Ch.4))
- $y = e^x, \quad y = 10^x$ (These are examples of **exponential functions** (Ch.5))
- $y = \ln x, \quad y = \log x$ (for $\boldsymbol{x > 0}$) (These are examples of **logarithmic functions** (Ch.5))

Examples of equations which do not define y as a function of x (where $x \in \mathbb{R}$):

- $x^2 + y^2 = 4$ **(Why?)**
- $x = y^2 + 1$ **(Why?)**

Example 7

For each of the following functions, determine the largest possible domain and the largest possible range of f.

(a) $f(x) = x^2 + 1$ (b) $f(x) = 25 - x$

(c) $f(x) = \sqrt{x+4}$ (d) $f(x) = 3 + \frac{1}{x-5}$

(e) $f(x) = 5 + \sin x$

Solution

(a) The function $f(x) = x^2 + 1$ is well-defined for every real number x.

$\therefore$ The largest possible domain of f is $Dom(f) = \mathbb{R}$ (the set of all real numbers)

Since $x^2 \geq 0$ for any $x \in Dom(f) = \mathbb{R}$, we have $x^2 + 1 \geq 1$ for any $x \in \mathbb{R}$.

$\therefore$ The largest possible range of f is $Ran(f) = [1, \infty)$ (the set of all real numbers greater than or equal to 1)

(b) The function $f(x) = 25 - x$ is well-defined for every real number x.

$\therefore$ The largest possible domain of f is $Dom(f) = \mathbb{R}$.

For any $x \in Dom(f) = \mathbb{R}$, $25 - x$ can be any real number.

$\therefore$ The largest possible range of f is $Ran(f) = \mathbb{R}$.

(c) The function $f(x) = \sqrt{x+4}$ is well-defined when $x + 4 \geq 0$, i.e. $x \geq -4$.

$\therefore$ The largest possible domain of f is $Dom(f) = [-4, \infty)$.

For any $x \in Dom(f) = [-4, \infty)$, we have $x + 4 \geq 0$ and therefore $\sqrt{x+4} \geq 0$.

$\therefore$ The largest possible range of f is $Ran(f) = [0, \infty)$.

(d) The function $f(x) = 3 + \frac{1}{x-5}$ is well-defined when $x - 5 \neq 0$, i.e. $x \neq 5$.

$\therefore$ The largest possible domain of f is $Dom(f) = \mathbb{R}\backslash\{5\}$. (The set of all real numbers except 5)

Since $\frac{1}{x-5} \neq 0$ for all $x \in Dom(f)$, we have $3 + \frac{1}{x-5} \neq 3 + 0$.

Therefore, $3 + \frac{1}{x-5}$ cannot be equal to 3.

$\therefore$ The largest possible range of f is $Ran(f) = \mathbb{R}\backslash\{3\}$.

(e) The function $f(x) = 5 + \sin x$ is well-defined for all $x \in \mathbb{R}$.

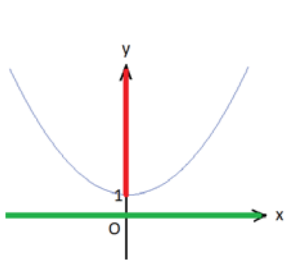
$\therefore$ The largest possible domain of f is $Dom(f) = \mathbb{R}$.

For any $x \in Dom(f)$, $-1 \leq \sin x \leq 1$ and therefore $5 - 1 \leq 5 + \sin x \leq 5 + 1$,

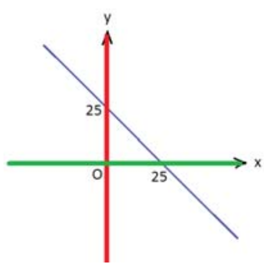
i.e. $4 \leq f(x) \leq 6$.

$\therefore$ The largest possible range of f is $Ran(f) = [4, 6]$.

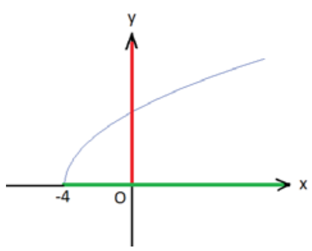
An alternative way to find the domain and range of a function is to sketch its graph first and then determine its domain and range from the graph. For example, the graphs of the first 4 functions in Example 7 are shown below (with **domain** highlighted in green and **range** highlighted in red):



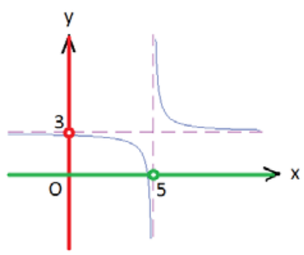
(a) $f(x) = x^2 + 1$



(b) $f(x) = 25 - x$



(c) $f(x) = \sqrt{x + 4}$



(d) $f(x) = 3 + \frac{1}{x-5}$

Example 8 (A bit harder examples)

Find the largest possible domain and largest possible range for each of the following functions:

(a) $f(x) = \frac{3x+1}{x-1}$ (b) $f(x) = 3 + \sqrt{x^2 - 16}$

(c) $f(x) = 3 + \sqrt{x^2 + 16}$ (d) $f(x) = 1 + 2x - x^2$

Solution

(a) $f(x) = \frac{3x+1}{x-1}$ is well-defined only when $x - 1 \neq 0$, i.e. $x \neq 1$.

$\therefore$ The largest possible domain of f is $Dom(f) =$

$$f(x) = \frac{3x+1}{x-1} = \frac{3(x-1+1)+1}{x-1} = \frac{3(x-1)+4}{x-1} = 3 + \frac{4}{x-1}$$

Since $\frac{4}{x-1} \neq 0$ for any $x \in Dom(f)$, it follows that $f(x) = 3 + \frac{4}{x-1}$ cannot be equal to 3.

$\therefore$ The largest possible range of f is $Ran(f) =$

Alternative method to find its range:

Let $y = \frac{3x+1}{x-1}$. Then express x in terms of y:

$$y = \frac{3x+1}{x-1} \Rightarrow y(x-1) = 3x+1 \Rightarrow x(y-3) = 1+y \Rightarrow x = \frac{1+y}{y-3}.$$

From this expression, y can be any real number except 3. Hence, $Ran(f) = \mathbb{R}\backslash\{3\}$.

(b) $f(x) = 3 + \sqrt{x^2 - 16}$ is well-defined only when $x^2 - 16 \geq 0$

$\Rightarrow x^2 \geq 16 \Rightarrow x \geq 4$ or $x \leq -4$.

$\therefore$ The largest possible domain of f is $Dom(f) =$

For any $x \in Dom(f)$, $x^2 - 16 \geq 0 \Rightarrow \sqrt{x^2-16} \geq 0 \Rightarrow 3 + \sqrt{x^2-16} \geq 3 + 0$, i.e. $f(x) \geq 3$.

$\therefore$ The largest possible range of f is $Ran(f) =$

(c) $f(x) = 3 + \sqrt{x^2 + 16}$ is well-defined only when $x^2 + 16 \geq 0$.

Clearly, $x^2 + 16 \geq 16 > 0$ for any real number x, thus the largest possible domain of f is $Dom(f) =$

Since $x^2 + 16 \geq 16$ for all $x \in Dom(f)$, we have $\sqrt{x^2 + 16} \geq \sqrt{16} = 4$ and thus $f(x) = 3 + \sqrt{x^2 + 16} \geq 3 + 4 = 7$.

$\therefore$ The largest possible range of f is $Ran(f) =$

(d) $f(x) = 1 + 2x - x^2$ is well-defined for all $x \in \mathbb{R}$.

$\therefore$ The largest possible domain of f is $Dom(f) =$

By completing the square,

$$f(x) = 1 + 2x - x^2 = -(x^2 - 2x) + 1 = -[(x-1)^2 - 1^2] + 1 = 2 - (x-1)^2.$$

For any $x \in Dom(f)$, $(x-1)^2 \geq 0 \Rightarrow -(x-1)^2 \leq 0 \Rightarrow 2 - (x-1)^2 \leq 2 + 0$, i.e. $f(x) \leq 2$. Hence the largest possible range of f is $Ran(f) =$

Example 9 **(More harder examples)**

Find the largest possible domain for each of the following functions:

(a) $f(x) = \sqrt{x^2 - 3x + 2}$ (b) $f(x) = \sqrt{3 + 2x - x^2}$

(c) $f(x) = \frac{9}{x^2+4x-5}$ (d) $f(x) = \sqrt{\frac{x+1}{x+2}}$

Solution

Two important things to remember when determining the largest possible domain of a function which involves square root or quotient:

1. We cannot take square root of a negative number.
2. We cannot divide by zero.

(a) The function $f(x) = \sqrt{x^2 - 3x + 2}$ is well-defined only when $x^2 - 3x + 2 \geq 0$, i.e. $(x-1)(x-2) \geq 0$. We want to find all those values of x which satisfy the inequality $(x-1)(x-2) \geq 0$.

One way is to draw a table like the one shown below:

	$x < 1$	$x = 1$	$1 < x < 2$	$x = 2$	$x > 2$
Sign of $(x-1)$	−	0	+	+	+
Sign of $(x-2)$	−	−	−	0	+
Sign of $(x-1)(x-2)$	+	0	−	0	+

i.e. we get $(x-1)(x-2) \geq 0$ only when $x \leq 1$ or $x \geq 2$.

$\therefore$ The largest possible domain of f is $Dom(f) =$

(b) $f(x) = \sqrt{3 + 2x - x^2}$ is well-defined only when $3 + 2x - x^2 \geq 0$, i.e. $(3-x)(1+x) \geq 0$. To solve this inequality, we draw the following table:

	$x < -1$	$x = -1$	$-1 < x < 3$	$x = 3$	$x > 3$
Sign of $(3-x)$	+	+	+	0	−
Sign of $(1+x)$	−	0	+	+	+
Sign of $(3-x)(1+x)$	−	0	+	0	−

i.e. we get $(3-x)(1+x) \geq 0$ only when $-1 \leq x \leq 3$.

$\therefore$ The largest possible domain of f is $Dom(f) =$

(c) $f(x) = \frac{9}{x^2+4x-5}$ is NOT defined when the denominator is zero.

We want to find all those values of x which cause the denominator equal to zero, and then exclude all those numbers from the set of all real numbers.

$$x^2 + 4x - 5 = 0 \quad \Rightarrow \quad (x+5)(x-1) = 0$$
$$\Rightarrow \quad x + 5 = 0 \quad \text{or} \quad x - 1 = 0$$
$$\Rightarrow \quad x = -5 \quad \text{or} \quad x = 1$$

$\therefore$ The largest possible domain of f is $Dom(f) =$

(d) $f(x) = \sqrt{\frac{x+1}{x+2}}$ is well-defined only when $\frac{x+1}{x+2} \geq 0$ and $x + 2 \neq 0$.

For the second condition, we have $x + 2 \neq 0 \quad \Rightarrow \quad x \neq -2$.

For the first condition, we solve the inequality $\frac{x+1}{x+2} \geq 0$ by drawing the following table:

	$x < -2$	$x = -2$	$-2 < x < -1$	$x = -1$	$x > -1$
Sign of $(x+1)$	−		−	0	+
Sign of $(x+2)$	−		+	+	+
Sign of $\frac{x+1}{x+2}$	+		−	0	+

i.e. we get $\frac{x+1}{x+2} \geq 0$ and $x + 2 \neq 0$ only when $x < -2$ or $x \geq -1$.

$\therefore$ The largest possible domain of f is $Dom(f) =$

Remark: To find the largest possible ranges of the functions in Example 9, one would require a little bit more knowledge on quadratic equation, which will be discussed in Chapter 3. We will find the ranges of these functions later in Chapter 3.

Operations on functions

Given a function f with domain A and a function g with domain B.

Define 1.) $(f+g)(x) = f(x) + g(x)$ $\quad Dom(f+g) = A \cap B$

2.) $(f-g)(x) = f(x) - g(x)$ $\quad Dom(f-g) = A \cap B$

3.) $(fg)(x) = f(x) \cdot g(x)$ $\quad Dom(fg) = A \cap B$

4.) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$ $\quad Dom\left(\frac{f}{g}\right) = A \cap \{x \in B | g(x) \neq 0\}$

Example 10

Given two real-valued functions $f(x) = \sqrt{x}$ and $g(x) = x - 1$.

Determine the formulae of the following functions and state their largest possible domains:

(a) $(f+g)(x)$, (b) $(f-g)(x)$, (c) $(fg)(x)$, (d) $\left(\frac{f}{g}\right)(x)$

Solution

First note that $f(x) = \sqrt{x}$ is defined when $x \geq 0$. Also $g(x) = x - 1$ is defined for all real

numbers x. Therefore, $Dom(f) = [0, \infty)$ and $Dom(g) = \mathbb{R}$. Then

(a) $(f+g)(x) = f(x) + g(x) = \sqrt{x} + (x-1)$

$Dom(f+g) = Dom(f) \cap Dom(g) = [0, \infty)$

(b) $(f-g)(x) = f(x) - g(x) = \sqrt{x} - (x-1)$

$Dom(f-g) = Dom(f) \cap Dom(g) = [0, \infty)$

(c) $(fg)(x) = f(x) \cdot g(x) = \sqrt{x}\,(x-1)$

$Dom(fg) = Dom(f) \cap Dom(g) = [0, \infty)$

(d) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{x-1}$

$$Dom\left(\frac{f}{g}\right) = Dom(f) \cap \{x \in Dom(g) | g(x) \neq 0\} = [0, \infty) \cap \{x \in \mathbb{R} | x - 1 \neq 0\}$$

$= [0, \infty) \backslash \{1\}$ (or written as $[0,1) \cup (1, \infty)$.)

Example 11

Let $f(x) = \frac{1}{x}$ and $g(x) = x^2$ be functions. Then

$Dom(f) =$ $Ran(f) =$

$Dom(g) =$ $Ran(g) =$

$(fg)(x) =$

$Dom(fg) =$

$Ran(fg) =$

Composition of functions

Let $f: A \to B$ and $g: B \to C$ be two functions.

Then the **composite of g with f** is defined by

$$(g \circ f)(x) = g(f(x)).$$

In general, $(f \circ g)(x) \neq (g \circ f)(x)$.

Example 12

Let $f: [0, \infty) \to [0, \infty)$, $g: \mathbb{R} \to \mathbb{R}$, and $h: \mathbb{R} \to \mathbb{R}$ be functions defined by $f(x) = \sqrt{x}$, $g(x) = x - 1$, and $h(x) = 3x$. Then

(a) $(f \circ g)(x) = f\big(g(x)\big) =$

The domain of $f \circ g$ is $Dom(f \circ g) =$

(b) $(g \circ f)(x) = g\big(f(x)\big) =$

The domain of $g \circ f$ is $Dom(g \circ f) =$

(c) $(f \circ f)(x) = f\big(f(x)\big) =$

The domain of $f \circ f$ is $Dom(f \circ f) =$

(d) $(f \circ g \circ h)(x) = f\big(g(h(x))\big) =$

The domain of $f \circ g \circ h$ is $Dom(f \circ g \circ h) =$

Example 13 **(A bit tricky!)**

Let $f:\mathbb{R} \to [0,\infty)$ and $g:[0,\infty) \to [0,\infty)$ be functions defined by $f(x) = x^2$ and $g(x) = \sqrt{x}$. Then

(a) $(f \circ g)(x) = f(g(x)) =$

The domain of $f \circ g$ is $Dom(f \circ g) =$

The range of $f \circ g$ is $Ran(f \circ g) =$

(b) $(g \circ f)(x) = g(f(x)) =$

The domain of $g \circ f$ is $Dom(g \circ f) =$

The range of $g \circ f$ is $Ran(g \circ f) =$

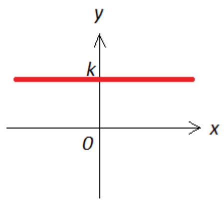
Here are some types of functions that are frequently used in this course:

1. Elementary functions

The following are examples of **elementary functions**:

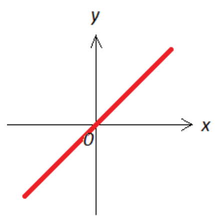
➢ ***Constant function***

A **constant function** is a function of the form $f(x) = k$, where k is a fixed real number. Its graph is a horizontal line.



➢ ***Identity function***

The **identity function** is the function $f(x) = x$. It assigns to every real number x (in the domain) the same number x (in the codomain).



➢ Polynomial functions

A **polynomial function** of **degree n** is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n \neq 0$ and the a_i's are real numbers and n is a non-negative integer. The constants a_i's are called **coefficients** of the corresponding x^i terms.

Two commonly used polynomials include:

- $f(x) = ax + b$ (where $a \neq 0$) is called a **linear** function (i.e. polynomial of degree 1).
- $f(x) = ax^2 + bx + c$ (where $a \neq 0$) is called a **quadratic** function (i.e. polynomial of degree 2)

The constant function $f(x) = k$ is a polynomial of degree 0.

E.g. $x^{\frac{1}{2}}$, x^{-1} and $x^{\cos x}$ are not polynomials.

(Details of polynomial functions will be discussed in Chapter 3.)

➢ Rational functions

A **rational function** is a quotient of two polynomials. It is of the form

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are two polynomials and $q(x) \neq 0$.

E.g. $f_1(x) = \frac{1}{3x^2+5x-2}$, $f_2(x) = \frac{4x+3}{5x-2}$, $f_3(x) = \frac{x^3+1}{x-3}$ and $f_4(x) = 1 - x^2 \left(= \frac{1-x^2}{1}\right)$ are all rational functions. *(Details of rational functions will be discussed in Chapter 3.)*

➢ Trigonometric functions

The six **trigonometric functions** that you will study in this course include **sine**, **cosine**, **tangent**, **cosecant**, **secant** and **cotangent**, written as

$$\sin x, \quad \cos x, \quad \tan x, \quad \csc x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x} \quad \text{and} \quad \cot x = \frac{1}{\tan x},$$

respectively. *(Details of trigonometric functions will be discussed in Chapter 4.)*

➢ ***Exponential functions***

An **exponential function with base a** is a function of the form $f(x) = a^x$, where $a > 0$ is a constant and $a \neq 1$.

Note that if $a = 1$, we have $f(x) = 1^x = 1$ which is the constant function.

E.g. $f(x) = 2^x$, $f(x) = 10^x$ and $f(x) = e^x$ (where $e = 2.7182818284\ldots$) are all exponential functions. *(Details of exponential functions will be discussed in Chapter 5.)*

➢ ***Logarithmic functions***

The **logarithmic function with base a** is a function of the form $f(x) = \log_a x$, where $a > 0$ and $a \neq 1$. It is a function such that if $y = \log_a x$, then this implies $x = a^y$.

$f(x) = \log_a x$ is only defined when $x > 0$.

E.g. $f(x) = \log_2 x$, $f(x) = \log_{10} x$ and $f(x) = \log_e x = \ln x$ are logarithmic functions. *(Details of logarithmic functions will be discussed in Chapter 5.)*

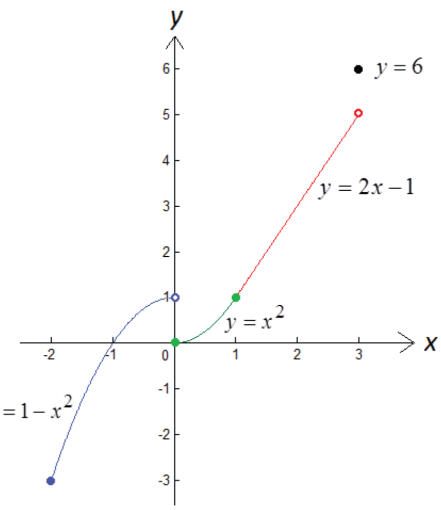
2. Piecewise-defined function

A **piecewise-defined function** is a function whose domain is divided into different intervals and within each interval the function is defined by a different formula.

Example 14

Let f be a function which has domain $\{x \in \mathbb{R} | -2 \leq x \leq 3\}$, i.e. $Dom(f) = [-2,3]$, and is defined by

$$f(x) = \begin{cases} 1 - x^2 & \text{if } -2 \leq x < 0 \\ x^2 & \text{if } 0 \leq x \leq 1 \\ 2x - 1 & \text{if } 1 < x < 3 \\ 6 & \text{if } x = 3 \end{cases}$$



This is an example of **piecewise-defined function**.

Its graph is shown on the right.

Its largest possible range is $Ran(f) =$

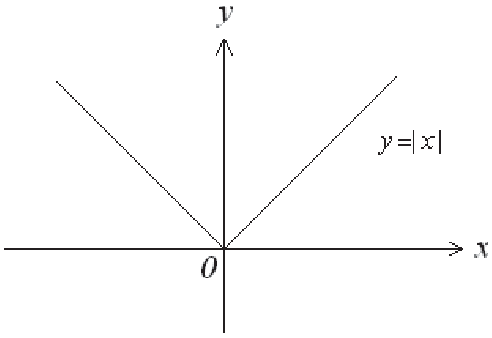
The following functions are also examples of piecewise-defined functions:

➢ ***Absolute value function***

The **absolute value function** is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

E.g. $|3.2| = 3.2, \quad |-4.6| = 4.6, \quad |0| = 0.$



Domain of $|x| = \mathbb{R}$

Range of $|x| = [0, \infty)$ (the set of all real numbers which are greater than or equal to zero.)

Properties: For any real numbers a and b,

1. $|ab| = |a||b|$
2. $|a + b| \leq |a| + |b|$
3. $\sqrt{a^2} = |a|$

Example 15

Sketch the graph of $y = \frac{2x}{|x|} - 1$ for $x \neq 0$.

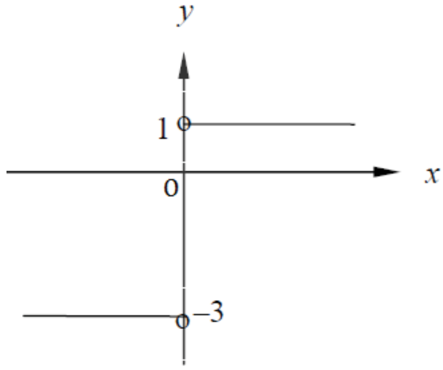
Solution

First note that $y = \frac{2x}{|x|} - 1$ is not defined when $x = 0$.

For $x \neq 0$, $y = \frac{2x}{|x|} - 1 = \begin{cases} \frac{2x}{-x} - 1, & \text{if } x < 0 \\ \frac{2x}{x} - 1, & \text{if } x > 0 \end{cases}$

$$= \begin{cases} -3, & \text{if } x < 0 \\ 1, & \text{if } x > 0 \end{cases}$$

The graph of $y = \frac{2x}{|x|} - 1$ is shown on the right:



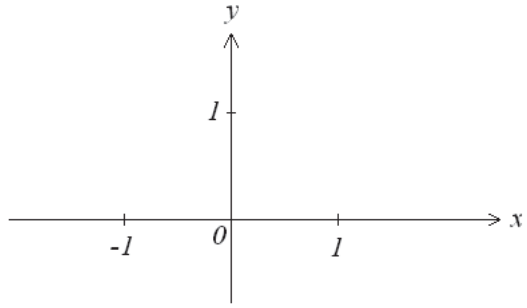
Example 16

Sketch the graph of the function $f(x)=\left|1-x^2\right|$. Then state the largest possible domain and range of f.

Solution

$$\left|1-x^2\right| = \begin{cases} 1-x^2 & if \;\; 1-x^2 \geq 0 \\ -(1-x^2) & if \;\; 1-x^2 < 0 \end{cases}$$

$$=$$



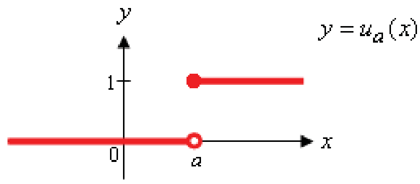
$Domain =$

$Range =$

➢ ***Unit step function***

The **unit step function** at $x=a$ (where $a \geq 0$) is defined as

$$u_a(x)=\begin{cases} 0 & \text{if } x<a \\ 1 & \text{if } x \geq a \end{cases}.$$



$Domain = \mathbb{R}$ $Range = \{0,1\}$ (the set containing the numbers 0 and 1)

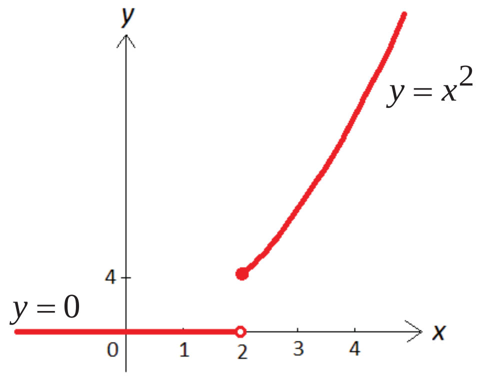
Example 17

Sketch the graph of the function $f(x)=x^2 \cdot u_2(x)$. Then state the largest possible domain and range of f.

Solution

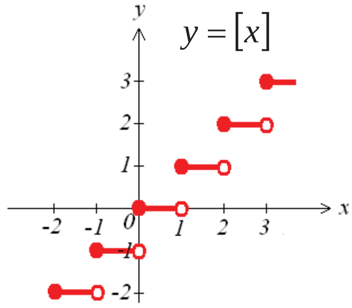
$$f(x)=x^2 \cdot u_2(x)=\begin{cases} x^2 \cdot 0 & \text{if } x<2 \\ x^2 \cdot 1 & \text{if } x \geq 2 \end{cases}$$

$$=\begin{cases} 0 & \text{if } x<2 \\ x^2 & \text{if } x \geq 2 \end{cases}$$



$Dom(f) =$ $Ran(f) =$

➢ Greatest integer function / Least integer function



The **greatest integer function** is defined as

$$f(x) = [x] = \text{“the greatest integer } \leq x\text{”.}$$

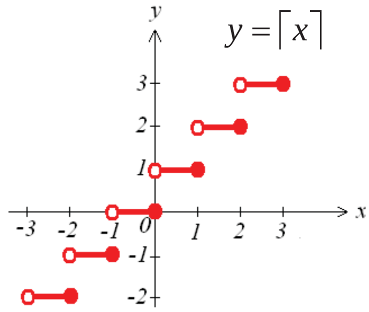
It is also denoted as $f(x) = \lfloor x \rfloor$.

E.g. $f(2.8) = [2.8] = 2$, $f(1) = [1] = 1$,

$f(-2.8) = [-2.8] = -3$.

$Domain = \mathbb{R}$ (the set of all real numbers)

$Range = \mathbb{Z}$ (the set of all integers)



The **least integer function** is defined as

$$f(x) = \lceil x \rceil = \text{“the least integer } \geq x\text{”.}$$

E.g. $f(2.8) = \lceil 2.8 \rceil = 3$, $f(1) = \lceil 1 \rceil = 1$,

$f(-2.8) = \lceil -2.8 \rceil = -2$.

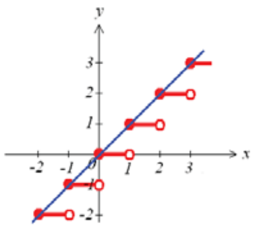
$Domain = \mathbb{R}$, $Range = \mathbb{Z}$.

Example 18

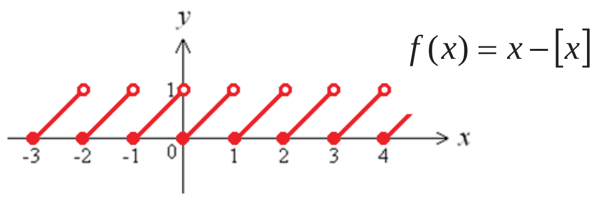
Sketch the graph of the function $f(x) = x - [x]$. Then state the largest possible domain and range of f.

Solution:

Consider the graphs of $y = x$ and $y = [x]$ first:



The graph of $f(x) = x - [x]$ is sketched below:



$Dom(f) =$ $Ran(f) =$

3. Periodic functions

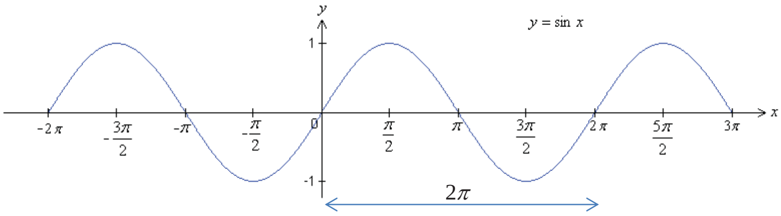
A function $f(x)$ is called a **periodic function** with **period** $\boldsymbol{T}$ (> 0) if

$$f(x+T) = f(x) \text{ for all } x \in Dom(f).$$

The graph of a periodic function **repeats** itself at regular intervals of length T.

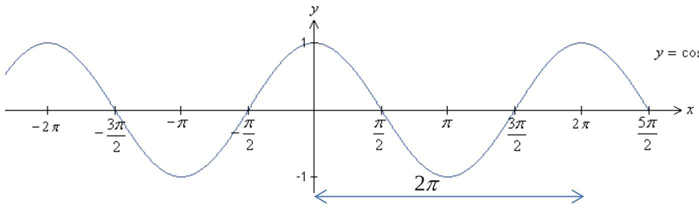
For example,

1. $f(x) = \sin x$ is **periodic** with period 2π.



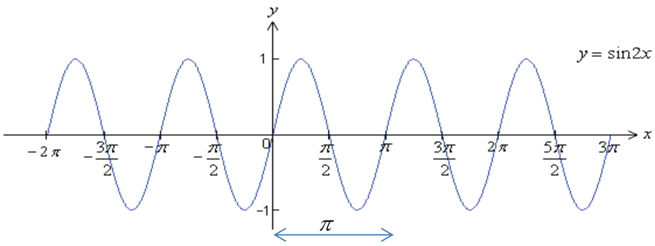
$$\boxed{\sin(x+2\pi) = \sin x}$$

2. $f(x) = \cos x$ is **periodic** with period 2π.



$$\boxed{\cos(x+2\pi) = \cos x}$$

3. $f(x) = \sin 2x$ is **periodic** with period π.



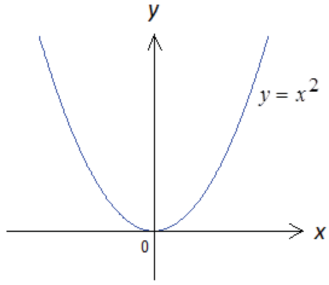
$$\sin(2x+2\pi) = \sin 2x$$

i.e. $\sin[2(x+\pi)] = \sin 2x$

4. Even and Odd functions

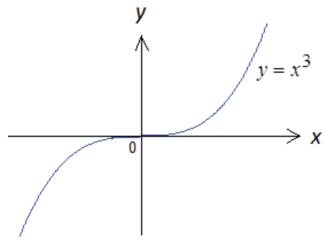
➢ The function f is called an **even function** if $\boxed{f(-x) = f(x)}$ for all x in the domain of f. The graph of an even function is **symmetric with respect to the y-axis**.

For example, 1, x^2, x^4, $\cos x$ are even functions.



➢ The function f is called an **odd function** if $\boxed{f(-x) = -f(x)}$ for all x in the domain of f. The graph of an odd function is **symmetric with respect to the origin**.

For example, x, x^3, $\sin x$, $\tan x$ are odd functions.



Note that a function could be neither even nor odd. For example, $f(x) = 3 + 2x^5$ is neither odd nor even, since $f(-x) = 3 + 2(-x)^5 = 3 - 2x^5 \neq f(x)$ and $f(-x) \neq -f(x)$.

Example 19

For each of the following functions, determine whether it is even, odd or neither of them.

(a) $f(x) = 2x^5 \cos x + \sin x$ (b) $f(x) = \sin(x^2 + 1)$ (c) $f(x) = \frac{x-1}{x+1}$

(d) $f(x) = |x^3|$ (e) $f(x) = \frac{x^4 \sin^3 x}{1 + \cos^4 x}$

Solution

(a) $f(-x) = 2(-x)^5 \cos(-x) + \sin(-x) = -2x^5 \cos x - \sin x = -(2x^5 \cos x + \sin x)$

$= -f(x)$

$\therefore$ $f(x)$ is an **odd** function.

(b) $f(-x) = \sin((-x)^2 + 1) = \sin(x^2 + 1) = f(x)$

$\therefore$ $f(x)$ is an **even** function.

(c) $f(-x) = \frac{(-x)-1}{(-x)+1} = \frac{-x-1}{-x+1}$ which is neither $f(x)$ nor $-f(x)$.

$\therefore$ $f(x)$ is **neither even nor odd**.

(d) $f(-x) = |(-x)^3| = |-x^3| = |-1| \cdot |x^3| = 1 \cdot |x^3| = |x^3| = f(x)$

$\therefore$ $f(x)$ is an **even** function.

(e) $f(-x) = \frac{(-x)^4 \sin^3(-x)}{1+\cos^4(-x)} = \frac{(-x)^4 \ [\sin(-x)]^3}{1+[\cos(-x)]^4} = \frac{x^4 \ [-\sin x]^3}{1+[\cos x]^4} = \frac{x^4 \ [-(\sin x)^3]}{1+[\cos x]^4} = -\frac{x^4 \sin^3 x}{1+\cos^4 x}$

$= -f(x)$

$\therefore$ $f(x)$ is an **odd** function.

Note: In the above example, we use $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$, since $\sin x$ is an odd function and $\cos x$ is an even function.

Some useful results:

Let O be an odd function, and E be an even function. Then we have the following results:

$$\boxed{\begin{array}{c} O \times E = O \\ O \times O = E \\ E \times E = E \end{array}}$$

(To remember the above results, you may treat E as "+" and O as "–".

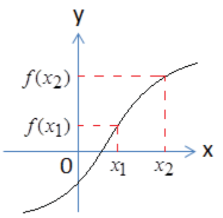
DON'T treat E as "even number" and O as "odd number".)

These results can be proved by using the definitions of odd and even functions.

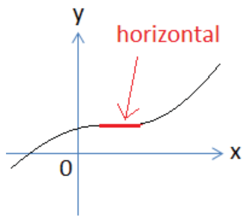
5. Monotonic functions

Let f be a function. It is said to be a

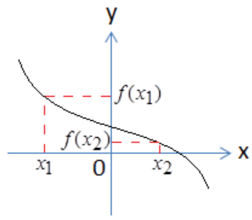
- **monotonic increasing** function if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$.
- **monotonic decreasing** function if $f(x_1) \geq f(x_2)$ whenever $x_1 < x_2$.
- **strictly increasing** function if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.
- **strictly decreasing** function if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.



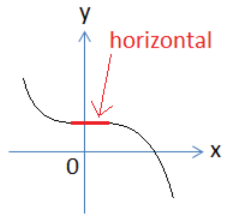
Strictly increasing



Monotonic increasing



Strictly decreasing



Monotonic decreasing

Examples of strictly increasing function: x, x^3, e^x, $\ln x$, etc.

Examples of strictly decreasing function: $-x$, e^{-x}, $-\ln x$, etc.

Example 20

Show that $f(x) = x^3$ is a strictly increasing function over $\mathbb{R}$.

Solution

For any $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$, we consider

$$\begin{aligned}
f(x_2) - f(x_1) &= x_2{}^3 - x_1{}^3 \\
&= (x_2 - x_1)(x_2{}^2 + x_1x_2 + x_1{}^2) \\
&= (x_2 - x_1) \cdot \frac{1}{2}(2x_2{}^2 + 2x_1x_2 + 2x_1{}^2) \\
&= (x_2 - x_1) \cdot \frac{1}{2}[(x_2{}^2 + 2x_1x_2 + x_1{}^2) + x_2{}^2 + x_1{}^2] \\
&= \frac{1}{2}\underbrace{(x_2 - x_1)}_{\substack{>0 \\ \because x_2 > x_1}}\underbrace{[(x_2 + x_1)^2 + x_2{}^2 + x_1{}^2]}_{>0} \\
&> 0
\end{aligned}$$

That is, $f(x_2) > f(x_1)$ whenever $x_2 > x_1$.

Hence, f is a strictly increasing function over $\mathbb{R}$.

6. Inverse functions

A function f takes a number x from its domain $Dom(f)$ and assigns to it a single value y from its range $Ran(f)$. For some (but not all) functions, we can reverse f. That is, for any given y in $Ran(f)$, we can go back and find the value of x which gives this value of y. This new function (which takes y and assigns an x to it) is denoted by f^{-1} and is called the **inverse** of f.

A function $f: A \to B$ (whose domain is A and codomain is B) is called **injective** (or **one-to-one**) if for every y in $Ran(f)$, there is **exactly one x** in domain A for which $y = f(x)$. Equivalently,

f is **one-to-one** if and only if $\boldsymbol{f(x_1) = f(x_2) \Rightarrow x_1 = x_2}$ **is the only solution**, where $x_1, x_2 \in A$.

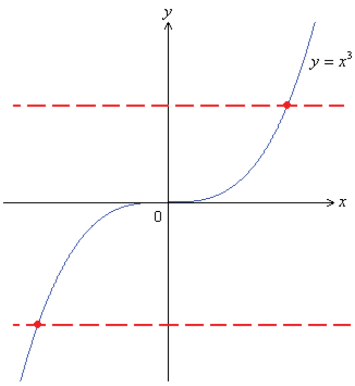
Another equivalent definition is that f is **one-to-one** if

$$\boldsymbol{f(x_1) \neq f(x_2)} \text{ whenever } \boldsymbol{x_1 \neq x_2}.$$

(Graphically, if you draw a horizontal line at **every $y \in Ran(f)$** and all horizontal lines cross the curve of the function at **exactly one point**, then f is one-to-one.)

Example 21

The function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is **one-to-one** (or **injective**), since every horizontal line $y = a$ (where $a \in \mathbb{R}$) cuts the graph of f at exactly one point, i.e. different values of x always give different values of $f(x)$.

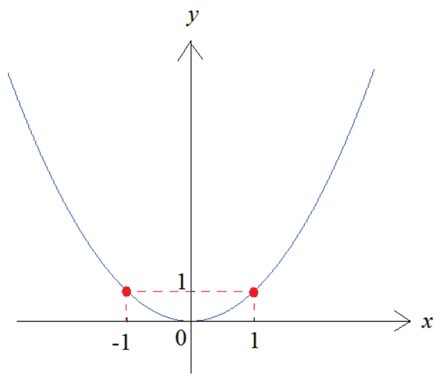


OR If $f(x_1) = f(x_2)$, then $x_1{}^3 = x_2{}^3 \Rightarrow x_1 = x_2$ is the only solution.

Example 22

- Let $f: \underbrace{\mathbb{R}}_{\text{Domain}} \rightarrow \underbrace{[0, \infty)}_{\text{Codomain}}$ be defined by $f(x) = x^2$.

  Then $f(x)$ is **not one-to-one** since for example, 1 and -1 are both elements of $\mathrm{Dom}(f) = \mathbb{R}$ but they correspond to the same value of $f(x)$ in $[0, \infty)$, the codomain of f, i.e. $f(1) = 1^2 = (-1)^2 = f(-1)$.



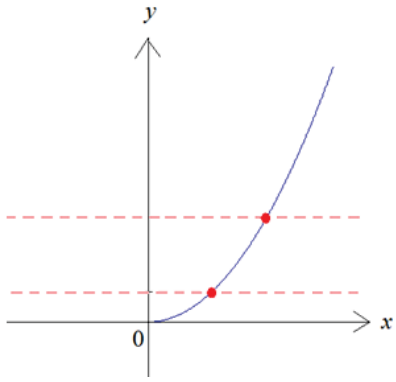
- Let $g: [0, \infty) \rightarrow [0, \infty)$ be defined by $g(x) = x^2$.

  If $g(x_1) = g(x_2)$, then we have

  $x_1{}^2 = x_2{}^2 \Rightarrow x_1 = x_2$ is the only solution,

  since $g(x)$ is only defined for all non-negative values of x, i.e. $\mathrm{Dom}(g) = [0, \infty)$.

  Thus, $g(x)$ is **one-to-one** (or **injective**).



Important result #1:

A function f has an **inverse** if and only if it is **one-to-one** (**injective**). We denote the inverse function of f by f^{-1}.

Important result #2:

If f is **either a strictly increasing** or **strictly decreasing function** over the domain of f, then f is **one-to-one** and thus its **inverse** f^{-1} **exists**.

Methods for determining whether a function f is one-to-one:

You may use one of the following:

1. If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ is the only solution (i.e. no two distinct values of x giving the same value of $f(x)$), then the function f is one-to-one.
2. Sketch its graph first. If you draw a horizontal line at every $y \in Ran(f)$ and all horizontal lines cross the graph of the function at exactly one point, then the function f is one-to-one.
3. If f is strictly increasing or strictly decreasing over $Dom(f)$, then f is one-to-one.

Procedure for finding the inverse function f^{-1} of f:

Step 1: Check that f is one-to-one.

Step 2: Let $y = f(x)$. Then express x in terms of y.

Step 3: To express f^{-1} as a function of x, replace x with $f^{-1}(x)$ and replace y with x.

Example 23

Show that $f(x) = \sqrt{3x+2}$ is one-to-one and find its inverse.

Solution:

If $f(x_1) = f(x_2)$, then $\sqrt{3x_1+2} = \sqrt{3x_2+2} \Rightarrow 3x_1 + 2 = 3x_2 + 2$

$\Rightarrow x_1 = x_2$ is the only solution.

$\therefore$ $f(x)$ is one-to-one.

Let $y = \sqrt{3x+2}$. (Note that $y \geq 0$.)

Then $y^2 = 3x + 2 \Rightarrow x = \frac{y^2-2}{3}$.

$\therefore$ The inverse function of $f(x) = \sqrt{3x+2}$ is given by

$$f^{-1}(x) = \frac{x^2-2}{3}.$$

Note that $\boxed{f^{-1}(x) \neq [f(x)]^{-1}}$.

$f^{-1}(x)$ is the inverse function of $f(x)$, whereas $[f(x)]^{-1} = \frac{1}{f(x)}$ is the reciprocal of $f(x)$.

Properties of inverse function

1. $y = f^{-1}(x) \iff x = f(y)$
2. The domain of f^{-1} is the range of f, i.e. $Dom(f^{-1}) = Ran(f)$.
3. The range of f^{-1} is the domain of f, i.e. $Ran(f^{-1}) = Dom(f)$.
4. $f^{-1}(f(x)) = x$ for all $x \in Dom(f)$.
5. $f(f^{-1}(x)) = x$ for all $x \in Dom(f^{-1})$.
6. $(f^{-1})^{-1}(x) = f(x)$ for all $x \in Dom(f)$, i.e. the inverse of f^{-1} is f.
7. The graph of f^{-1} is the reflection of the graph of f in the straight line $y = x$.

Example 24

For the function $f(x) = \sqrt{3x+2}$ in Example 23,

$f(x) = \sqrt{3x+2}$ is defined only when $3x + 2 \geq 0 \Rightarrow x \geq -\frac{2}{3}$.

$\therefore\ Dom(f) = \left[-\frac{2}{3}, \infty\right)$.

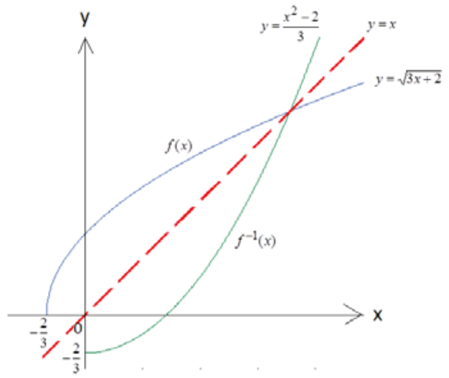
Moreover, for any $x \in Dom(f)$, $3x + 2 \geq 0 \Rightarrow \sqrt{3x+2} \geq 0$.

$\therefore\ Ran(f) = [0, \infty)$.

Since $Dom(f^{-1}) = Ran(f)$ and $Ran(f^{-1}) = Dom(f)$, we get

$$Dom(f^{-1}) = [0, \infty) \quad \text{and} \quad Ran(f^{-1}) = \left[-\frac{2}{3}, \infty\right).$$

Sketch:



Example 25

Determine whether $f(x)=\frac{x}{1+x}$ is one-to-one. Find f^{-1} and the largest possible domain and range of f^{-1} if f is a one-to-one function.

Solution

$f(x_1)=f(x_2) \Rightarrow \frac{x_1}{1+x_1}=\frac{x_2}{1+x_2} \Rightarrow x_1(1+x_2)=x_2(1+x_1)$

$\Rightarrow x_1+x_1x_2=x_2+x_1x_2 \Rightarrow x_1=x_2$ is the only solution.

$\therefore$ f is one-to-one.

Let $y=\frac{x}{1+x}$. Then $y(1+x)=x \Rightarrow y+xy=x \Rightarrow x(1-y)=y \Rightarrow x=\frac{y}{1-y}$.

$\therefore$ The inverse of f is $f^{-1}(x)=\frac{x}{1-x}$.

The function $f^{-1}(x)=\frac{x}{1-x}$ is defined only when $1-x\neq 0 \Rightarrow x\neq 1$.

$\therefore$ $\mathrm{Dom}(f^{-1})=\mathbb{R}\backslash\{1\}$.

The function $f(x)=\frac{x}{1+x}$ is defined only when $1+x\neq 0 \Rightarrow x\neq -1$.

$\therefore$ $\mathrm{Dom}(f)=\mathbb{R}\backslash\{-1\}$.

Thus, $\mathrm{Ran}(f^{-1})=\mathrm{Dom}(f)=\mathbb{R}\backslash\{-1\}$.

Exercise

1. The function f is defined by $f(x)=x^2-2x+3$, for $x\in\mathbb{R}$.

   Sketch the graph of f and say whether or not it is a one-to-one function.

2. Consider the function $g(x)=x^2-2x+3$, for $x\in[1,\infty)$.

   (a) Find the largest possible domain and largest possible range of g.

   (b) Is the function g a one-to-one function?

   (c) Find g^{-1} and state its largest possible domain and largest possible range if g is a one-to-one function.

   (d) Sketch the graphs of g and g^{-1} on the same graph.

Here are some common examples of inverse functions, which will be discussed in later chapters:

$f(x)$	Inverse of $f(x)$	
$f:\mathbb{R}\to(0,\infty),\quad f(x)=10^x$	$f^{-1}(x)=\log_{10}x$	Chapter 5
$f:\mathbb{R}\to(0,\infty),\quad f(x)=e^x$	$f^{-1}(x)=\log_e x\ \ (=\ln x)$	
$f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to[-1,1],\quad f(x)=\sin x$	$f^{-1}(x)=\sin^{-1}x$	Chapter 4
$f:[0,\pi]\to[-1,1],\quad f(x)=\cos x$	$f^{-1}(x)=\cos^{-1}x$	
$f:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to\mathbb{R},\quad f(x)=\tan x$	$f^{-1}(x)=\tan^{-1}x$	

Transformation of functions (***For your reference***)

Consider the function $y=f(x)$. Let c be a positive constant. Then we can transform the function in the following ways:

➢ ***Vertical translation***

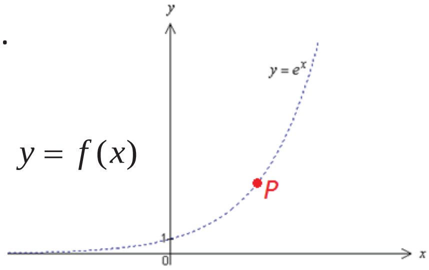
- If the graph of $y=f(x)$ is **shifted (or translated) c units upward**, we obtain the graph of $y=f(x)+c$. (That is, we replace "y" with "$y-c$" and the y-coordinate of each point of $y=f(x)$ increases by c units.)

- If the graph of $y=f(x)$ is **shifted (or translated) c units downward**, we obtain the graph of $y=f(x)-c$. (That is, we replace "y" with "$y+c$" and the y-coordinate of each point of $y=f(x)$ decreases by c units.)

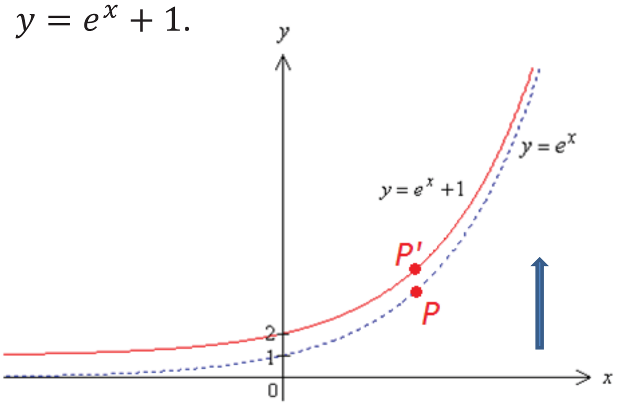
Example 26 (i): Consider the graph of the function $y=e^{x}$.

(This is the exponential function, where $e\approx 2.71828$.)

Its graph is shown on the right.

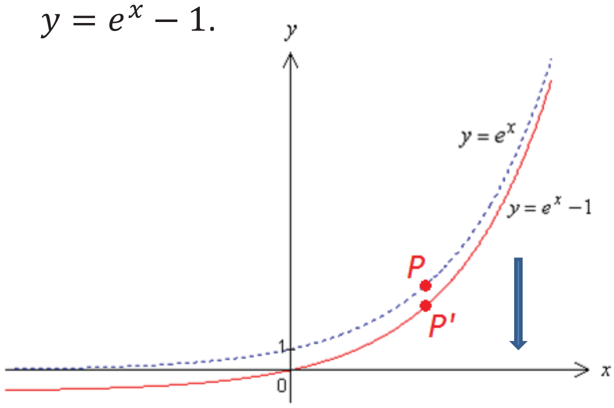


◆ If the graph of $y=e^{x}$ is translated 1 unit upward, we obtain the graph of $y=e^{x}+1$.



$y=f(x)\ \rightarrow\ y=f(x)+1$

◆ If the graph of $y=e^{x}$ is translated 1 unit downward, we obtain the graph of $y=e^{x}-1$.



$y=f(x)\ \rightarrow\ y=f(x)-1$

➢ ***Horizontal translation***

- If the graph of $y=f(x)$ is **shifted c units to the right**, we obtain the graph of $y=f(x-c)$.

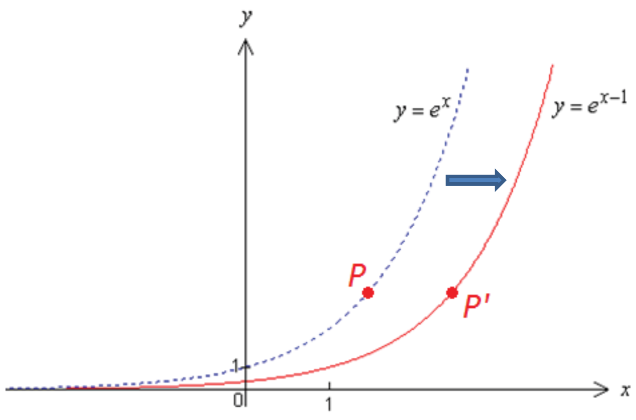
  (That is, we replace "x" with "$x-c$" and the x-coordinate of each point of $y=f(x)$ increases by c units.)

- If the graph of $y=f(x)$ is **shifted c units to the left**, (i.e. $-c$ units to the right), we obtain the graph of $y=f(x+c)$.

  (That is, we replace "x" with "$x+c$" and the x-coordinate of each point of $y=f(x)$ decreases by c units.)

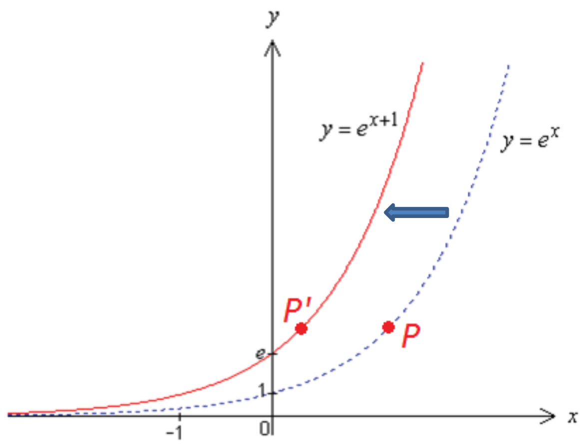
Example 26 (ii): For the previous example $y = e^x$:

- If the graph of $y = e^x$ is translated 1 unit to the right, we obtain the graph of $y = e^{x-1}$.



$y = f(x) \rightarrow y = f(x-1)$

- If the graph of $y = e^x$ is translated 1 unit to the left, we obtain the graph of $y = e^{x+1}$.



$y = f(x) \rightarrow y = f(x+1)$

➢ ***Reflection about x-axis***

- If the graph of $y = f(x)$ is **reflected about the x-axis**, we obtain the graph of $y = -f(x)$.

  (That is, we replace "y" with "$-y$" so that the sign of y is reversed.)

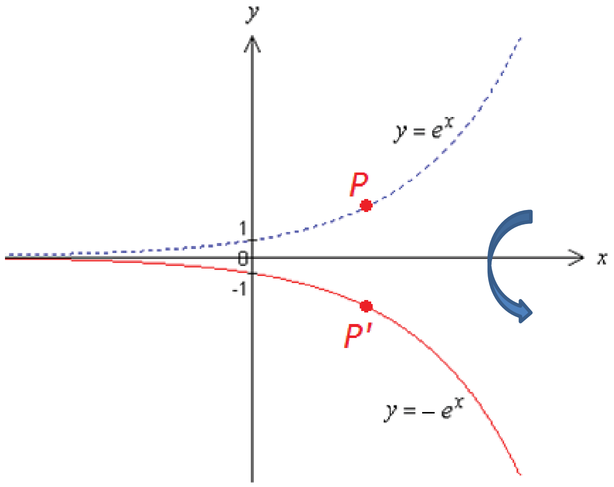
➢ ***Reflection about y-axis***

- If the graph of $y = f(x)$ is **reflected about the y-axis**, we obtain the graph of $y = f(-x)$.

  (That is, we replace "x" with "$-x$" so that the sign of x is reversed.)

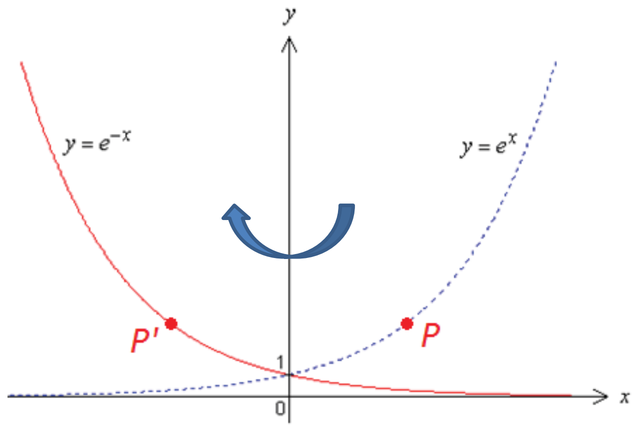
Example 26 (iii): For the previous example $y = e^x$:

♦ If the graph of $y = e^x$ is reflected about the x-axis, we obtain the graph of $y = -e^x$.



$y = f(x) \to y = -f(x)$

♦ If the graph of $y = e^x$ is reflected about the y-axis, we obtain the graph of $y = e^{-x}$.



$y = f(x) \to y = f(-x)$

➢ *Vertical Stretch / Shrink*

- **Vertical stretch ($c > 1$)**

  If the graph of $y = f(x)$ is stretched vertically by a factor of c (where $c > 1$) from the x-axis, we obtain the graph of $y = c\,f(x)$.

  (That is, we multiply the y-coordinate of each point of $y = f(x)$ by c.)

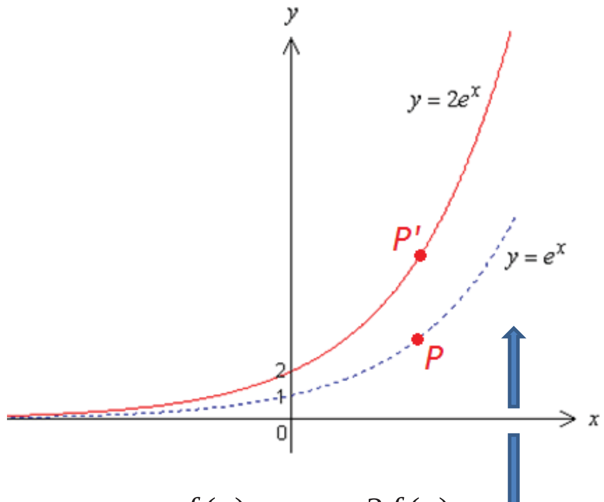
- **Vertical shrink ($0 < c < 1$)**

  If the graph of $y = f(x)$ is stretched vertically by a factor of c (where $0 < c < 1$) from the x-axis, then this is the same as $y = f(x)$ being compressed (or shrunk) vertically by a factor of $\frac{1}{c}$ (> 1) towards the x-axis and we obtain the graph of $y = c\,f(x)$.

  (That is, we multiply the y-coordinate of each point of $y = f(x)$ by c, where $0 < c < 1$.)

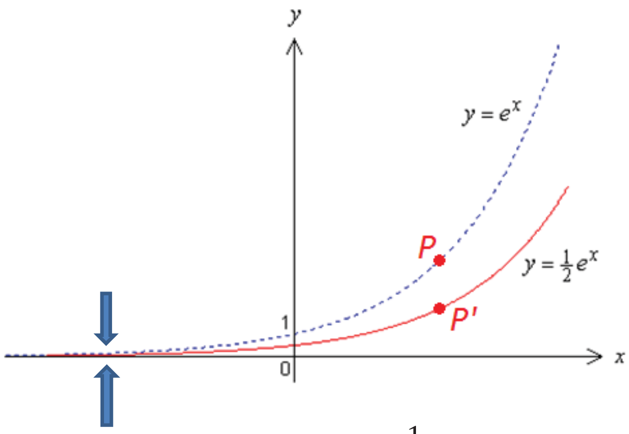
<u>Example 26 (iv):</u> For the previous example $y = e^x$:

♦ If the graph of $y = e^x$ is stretched vertically by a factor of 2, we obtain the graph of $y = 2e^x$.



$y = f(x) \rightarrow y = 2f(x)$

♦ If the graph of $y = e^x$ is stretched vertically by a factor of $\frac{1}{2}$ (i.e. it is compressed vertically by a factor of 2), we obtain the graph of $y = \frac{1}{2}e^x$.



$y = f(x) \rightarrow y = \frac{1}{2} f(x)$

➢ ***<u>Horizontal Shrink / Stretch</u>***

- **Horizontal shrink** ($c > 1$)

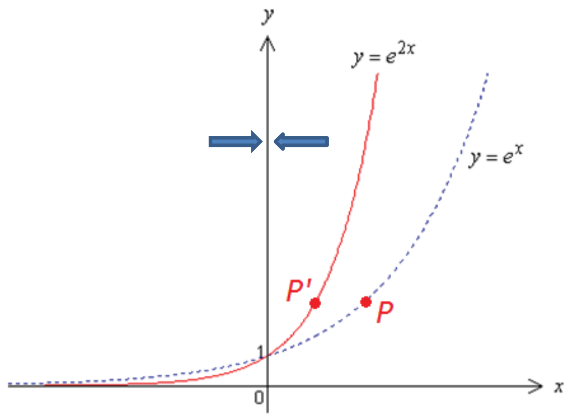
  If the graph of $y = f(x)$ is compressed (or shrunk) horizontally by a factor of c (where $c > 1$) towards the y-axis, we obtain the graph of $y = f(cx)$.
  (That is, we replace "x" with "cx". In other words, we divide the x-coordinate of each point of $y = f(x)$ by c, and the graph of $y = f(x)$ is stretched horizontally by a factor of $\frac{1}{c}$.)

- **Horizontal stretch** ($0 < c < 1$)

  If the graph of $y = f(x)$ is compressed (or shrunk) horizontally by a factor of c (where $0 < c < 1$) towards the y-axis, then this is the same as $y = f(x)$ being stretched horizontally by a factor of $\frac{1}{c}$ (> 1) and we obtain the graph of $y = f(cx)$.
  (That is, we replace "x" with "cx". In other words, we divide the x-coordinate of each point of $y = f(x)$ by c, where $0 < c < 1$.)

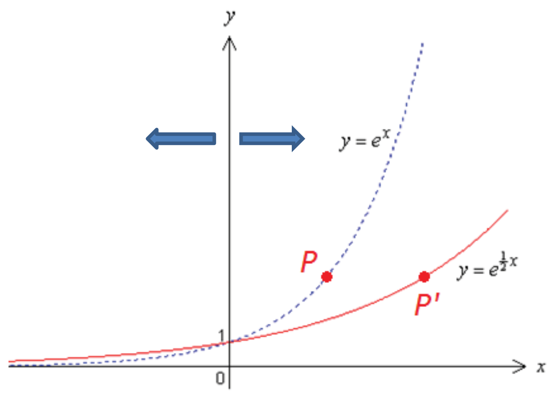
Example 26 (v): For the previous example $y = e^x$:

- If the graph of $y = e^x$ is compressed horizontally by a factor of 2 towards the y-axis, we obtain the graph of $y = e^{2x}$.



$y = f(x) \to y = f(2x)$

- If the graph of $y = e^x$ is compressed horizontally by a factor of $\frac{1}{2}$ (i.e. it is being stretched horizontally by a factor of 2 from the y-axis), we obtain the graph of $y = e^{\frac{1}{2}x}$.



$y = f(x) \to y = f(\frac{1}{2}x)$

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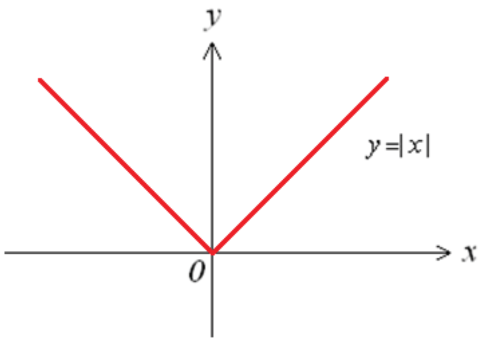
Sometimes we may obtain the graph of a required function by performing a sequence of transformations.

Example 27

Given the function $f(x) = |x|$. By performing a sequence of transformations, sketch the graph of $g(x) = 3 - |2x + 1|$.

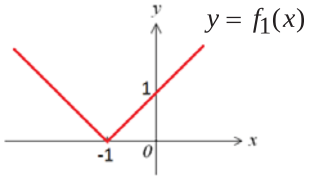
Solution

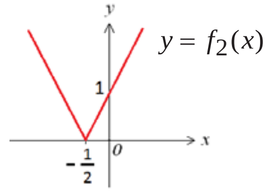
We start with the function $f(x) = |x|$, whose graph is shown below.

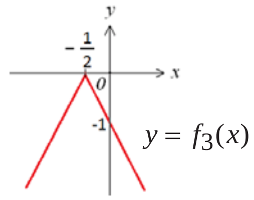


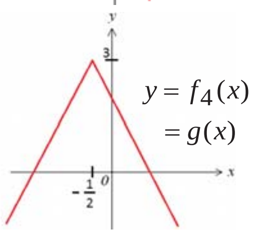
$y = f(x)$

	Function obtained	Transformation
Step 1:	$f_1(x) = f(x+1)$ $= \lvert x+1 \rvert$	The graph of $y = f(x)$ is shifted 1 unit to the left.
Step 2:	$f_2(x) = f_1(2x)$ $= \lvert 2x+1 \rvert$	The graph of $y = f_1(x)$ is compressed horizontally by a factor of 2 towards the y-axis.
Step 3:	$f_3(x) = -f_2(x)$ $= -\lvert 2x+1 \rvert$	The graph of $y = f_2(x)$ is reflected about the x-axis.
Step 4:	$f_4(x) = f_3(x) + 3$ $= -\lvert 2x+1 \rvert + 3$ $= g(x)$	The graph of $y = f_3(x)$ is shifted 3 units upward.

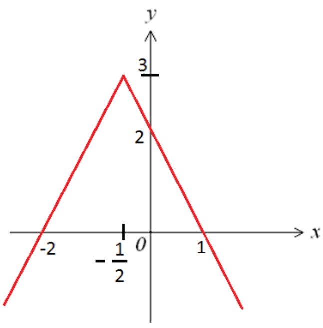








The graph of $g(x) = 3 - |2x+1|$ is shown below:



Example 28

Given the function $f(x) = x^2$. By completing the square and then performing a sequence of transformations, sketch the graph of

$$h(x) = 3x^2 - 6x - 2.$$

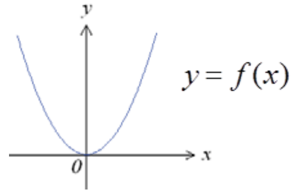
Solution

By completing the square,

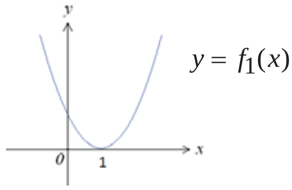
$$h(x) = 3x^2 - 6x - 2 = 3(x^2 - 2x) - 2 = 3[(x-1)^2 - 1^2] - 2 = 3(x-1)^2 - 5.$$

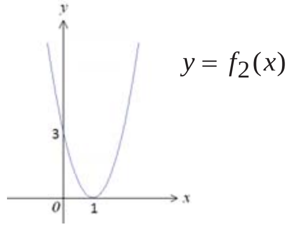
We start with the function $f(x) = x^2$,

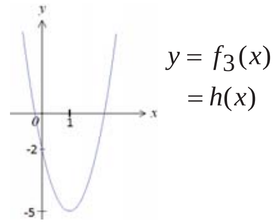
whose graph is shown on the right.



	Function obtained	Transformation
Step 1:	$f_1(x) = f(x-1)$ $= (x-1)^2$	The graph of $y = f(x)$ is shifted 1 unit to the right.
Step 2:	$f_2(x) = 3\,f_1(x)$ $= 3(x-1)^2$	The graph of $y = f_1(x)$ is stretched vertically by a factor of 3 from the x-axis.
Step 3:	$f_3(x) = f_2(x) - 5$ $= 3(x-1)^2 - 5$ $= h(x)$	The graph of $y = f_2(x)$ is shifted 5 units downward.







The graph of $h(x) = 3x^2 - 6x - 2 = 3(x-1)^2 - 5$ is shown below: