CITY UNIVERSITY OF HONG KONG

Course code and title : MA1200 Calculus and Basic Linear Algebra I

Session : Semester B, 2016/2017

Time allowed : Three hours

This paper has **SIX** pages (including this cover page).

A brief table of derivatives is attached on pages 5 and 6.

Instructions to candidates:

1. This paper has **TEN** questions.
2. Attempt **ALL** questions.
3. Each question carries 10 marks.

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*This is a **closed-book** examination.*

Candidates are allowed to use the following materials/aids:

Non-programmable calculators

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorised materials or aids are found on them.

Question 1

The functions $f(x)$ and $g(x)$ are defined by

$f(x) = x^2 - 1$, for $x \geq 0$,

$g(x) = \sqrt{x}$, for $x \geq 0$.

Find, in a similar form

(a) the inverse function $f^{-1}(x)$, (5 marks)

(b) the composite function $(g \circ f)(x)$. (5 marks)

In each case state the largest possible domain and the range of the function.

Question 2

Evaluate the following limits:

(a) $\lim\limits_{x \to 0} \dfrac{1 - \cos(3x)}{x^2}$, (3 marks)

(b) $\lim\limits_{x \to \infty} \dfrac{1 - x^2}{1 + x^2}$, (3 marks)

(c) $\lim\limits_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{3}{3x + 2x^2} \right)$. (4 marks)

Question 3

(a) A circle has equation $x^2 + y^2 - 2x - 4y - 20 = 0$.
Find the equation of the tangent to the circle at the point $P(-3, 5)$. (5 marks)

(b) Let $Q(x_1, y_1)$ be a point outside a circle, $x^2 + y^2 = r^2$, and let $y = mx + c$ be the equation of a tangent drawn from Q to the circle.

Show that $(r^2 - x_1{}^2)m^2 + 2x_1y_1m + (r^2 - y_1{}^2) = 0$. (5 marks)

Question 4

(a) A Greek Mathematician, Archimedes (287-212 B.C.) proposed a method to compute an approximation to the value of π.

Given a circle of radius r units, he calculated the perimeter of inscribed and circumscribed regular hexagons as shown in Figure 1, thus obtaining lower and upper bounds for the circumference of the circle ($= 2\pi r$ units). Find the perimeters of hexagon ABCDEF and hexagon PQRSTU, in terms of r.

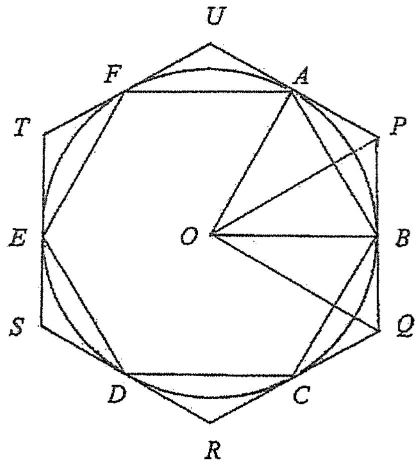

Figure 1 (5 marks)

(b) Find the smallest positive value of x which satisfies the equation $2\cos^2 x + 3\cos x + 1 = 0$. (5 marks)

Question 5

Differentiate with respect to x :

(a) $3(5x-1)^2 + \sqrt{x}$, $x > 0$; (3 marks)

(b) $e^{-x}\cos(3x)$; (3 marks)

(c) $\dfrac{\sin x}{x} + (\sin x)^x$, $x \neq 0$. (4 marks)

Question 6

(a) A curve has parametric equations $x = t + \sin t$, $y = 1 + \cos t$, where t is the parameter. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of t . (5 marks)

(b) Find the seventh derivative of $(x^2 - x + 3)e^{-2x}$ with respect to x. (5 marks)

(Hint: Leibnitz' rule: For any functions u and v whose derivatives up to the nth order exist,

$(uv)^{(n)} = {}_nC_0 u^{(n)}v^{(0)} + {}_nC_1 u^{(n-1)}v^{(1)} + {}_nC_2 u^{(n-2)}v^{(2)} + \ldots + {}_nC_r u^{(n-r)}v^{(r)} + \ldots + {}_nC_n u^{(0)}v^{(n)}$, where

${}_nC_r = \dfrac{n!}{(n-r)!\,r!}$, $u^{(0)} = u$, $v^{(0)} = v$ and $u^{(r)}$, $v^{(r)}$ are the rth derivatives of u and v,

respectively, for $r = 1, 2, 3, \ldots, n$.)

Question 7

Express $\dfrac{x^3-3x^2+3x-4}{\left(x^2+1\right)^2}$ in partial fractions. (10 marks)

Question 8

(a) If $y=\cos^{-1}x$, show that $\left(1-x^2\right)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=0$. By repeated differentiation of this result and use the Maclaurin series of $y=(x)$, find the series of $\cos^{-1}x$ in ascending powers of x as far as the term in x^7 . (7 marks)

(b) Using the result in part (a), find an approximation to the value of π, giving 5 decimal places in your answer. (3 marks)

Question 9

A right circular cylinder is inscribed in a sphere of radius R cm as shown in Figure 2.

Find the dimensions of the cylinder if it is to have maximum volume.

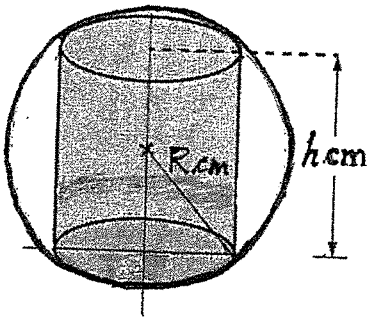

Figure 2

(10 marks)

Question 10

(a) Prove from first principles that $\dfrac{d}{dx}\left(x^3\right)=3x^2$. (4 marks)

(b) Let $F(x)=|\cos x|$, for $x\in R$.
Determine whether $F(x)$ is differentiable at $x=0$. Give your reason. (6 marks)
(Hint: You may use $\cos 2\theta=1-2\sin^2\theta$.)

Short Table of Derivatives of $y = f(u)$ with respect to x, where u is a function of x

Functions, $y = f(u)$	**Derivative of y with respect to x**
$y = c$, where c is a constant.	$\frac{dy}{dx} = 0$
$y = cu$, where c is a constant.	$\frac{dy}{dx} = c\frac{du}{dx}$
$y = u^p$, where p is a constant.	$\frac{dy}{dx} = pu^{p-1}\frac{du}{dx}$
$y = u + v$	$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = f(u)$, where u is a function of x.	$\frac{dy}{dx} = \frac{d f(u)}{du} \cdot \frac{du}{dx}$, the chain rule
$y = \log_a u$, $a > 0$.	$\frac{dy}{dx} = \frac{1}{u}\log_a e\frac{du}{dx}$
$y = a^u$, $a > 0$.	$\frac{dy}{dx} = a^u \log_e a\frac{du}{dx}$
$y = e^u$	$\frac{dy}{dx} = e^u\frac{du}{dx}$
$y = u^v$	$\frac{dy}{dx} = vu^{v-1}\frac{du}{dx} + u^v \log_e u\frac{dv}{dx}$
$y = \sin u$	$\frac{dy}{dx} = \cos u\frac{du}{dx}$
$y = \cos u$	$\frac{dy}{dx} = -\sin u\frac{du}{dx}$
$y = \tan u$	$\frac{dy}{dx} = \sec^2 u\frac{du}{dx}$
$y = \cot u$	$\frac{dy}{dx} = -\mathrm{cosec}^2 u\frac{du}{dx}$
$y = \sec u$	$\frac{dy}{dx} = \sec u \tan u\frac{du}{dx}$
$y = \mathrm{cosec}\, u$	$\frac{dy}{dx} = -\mathrm{cosec}\, u \cot u\frac{du}{dx}$
$y = \sin^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y = \cos^{-1} u$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y = \tan^{-1} u$	$\frac{dy}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$

Functions, $y = f(u)$	**Derivative of** y **with respect to** x
$y = \cot^{-1} u$	$\frac{dy}{dx} = -\frac{1}{1+u^2}\frac{du}{dx}$
$y = \sec^{-1} u$	$\frac{dy}{dx} = \frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$
$y = \operatorname{cosec}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$
$y = \sinh u$	$\frac{dy}{dx} = \cosh u\frac{du}{dx}$
$y = \cosh u$	$\frac{dy}{dx} = \sinh u\frac{du}{dx}$
$y = \tanh u$	$\frac{dy}{dx} = \operatorname{sech}^2 u\frac{du}{dx}$
$y = \coth u$	$\frac{dy}{dx} = -\operatorname{cosech}^2 u\frac{du}{dx}$
$y = \operatorname{sech} u$	$\frac{dy}{dx} = -\operatorname{sech} u \tanh u\frac{du}{dx}$
$y = \operatorname{cosech} u$	$\frac{dy}{dx} = -\operatorname{cosech} u \coth u\frac{du}{dx}$
$y = \sinh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$
$y = \cosh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$
$y = \tanh^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$
$y = \coth^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$
$y = \operatorname{sech}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$
$y = \operatorname{cosech}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{\lvert u\rvert\sqrt{u^2+1}}\frac{du}{dx}$

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