## Partial Fraction Expansion

 $\square$  Assuming the poles are simple, the partial fraction expansion of rational X(s) with M < N is given by

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_0 s^M + b_1 s^{M-1} + \dots + b_M}{(s - s_{p_1})(s - s_{p_2}) \dots (s - s_{p_N})}$$

$$= \frac{A_1}{s - s_{p_1}} + \frac{A_2}{s - s_{p_2}} + \dots + \frac{A_N}{s - s_{p_N}}$$

 $\Box$  The coefficients  $\{A_l\}$  are computed as

$$A_l = \underbrace{\left(s - s_{pl}\right)}_{s = s_{pl}} X(s) \Big|_{s = s_{pl}}$$

M < N

 $\square$  For M < N, X(s) is a proper rational function.

Differential Equations & Laplace Transform

Sp! roots of Das) / Si N

## Example

☐ Find the partial-fraction expansion of

$$X(s) = \frac{s + 0.5}{(s+1)(s+2)}$$

☐ Partial-fraction expansion.

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

where

$$A_1 = (s+1)X(s)|_{s=-1} = (s+1)\frac{s+0.5}{(s+1)(s+2)}|_{s=-1} = \frac{s+0.5}{(s+2)}|_{s=-1} = \frac{-0.5}{1} = -0.5$$

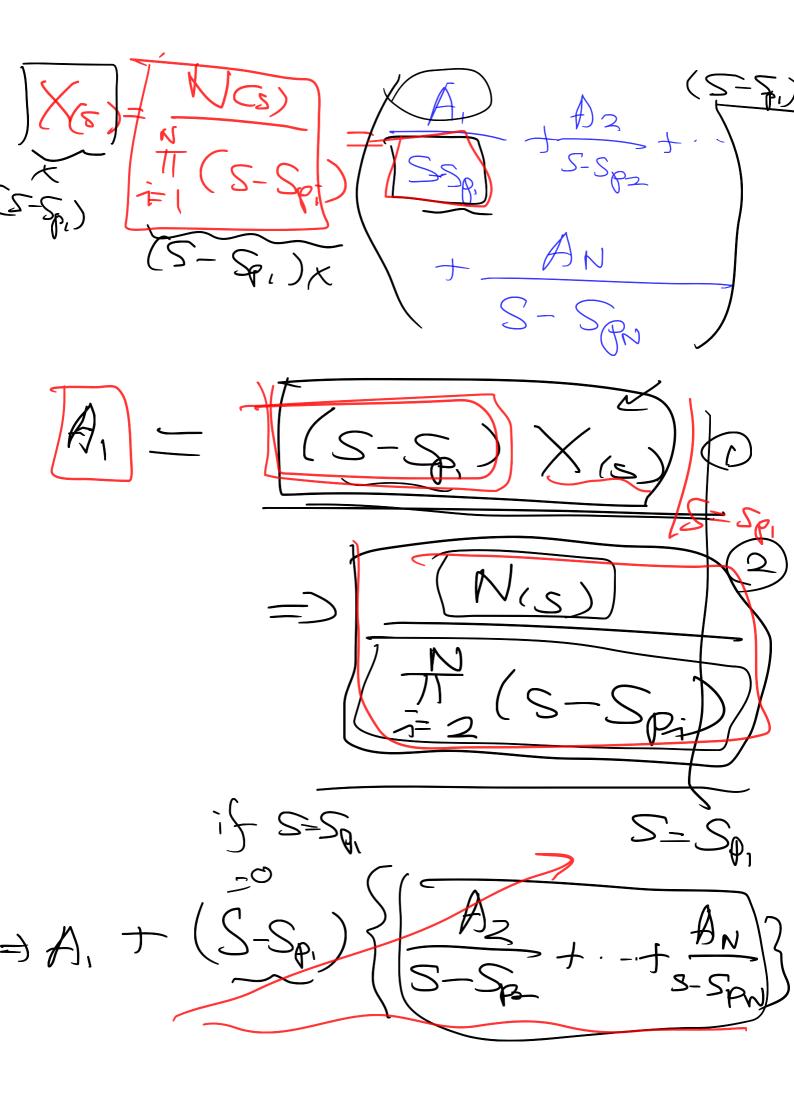
$$A_1 = (s+2)X(s)|_{s=-1} = (s+2)\frac{s+0.5}{(s+2)(s+2)}|_{s=-1} = \frac{s+0.5}{1}|_{s=-1} = \frac{-0.5}{1} = -0.5$$

$$A_2 = (s+2)X(s)|_{s=-2} = \underbrace{(s+2)}_{(s+2)} \underbrace{\frac{s+0.5}{(s+1)(s+2)}}_{|s=-2} |_{s=-2} = \underbrace{\frac{s+0.5}{(s+1)}}_{|s=-2} |_{s=-2} = \frac{-1.5}{-1} = 1.5$$

We have  $X(s) = \frac{-0.5}{s+1} + \frac{1.5}{s+2}$ 

Differential Equations & Laplace Transform

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$$X(s) = \left(\frac{S+0.5}{(s+1)(s+2)}\right) \times (s+2)$$

$$= \frac{C_1}{(s+1)(s+2)} \times (s+2)$$

$$= \frac{C_1}{(s+1)} + \frac{C_2}{s+2} = \frac{1.5}{s+2} - \frac{0.5}{s+1}$$

$$C_1 = (s+1) \times (s) = \frac{S+0.5}{s+2}$$

$$= \frac{-0.5}{1} = -0.5$$

$$C_2 = (s+2) \times (s) = \frac{(s+0.5)}{(s+1)}$$

$$= \frac{-1.5}{1} = 1.5$$



## Partial Fraction Expansion Cont.

- $\square$  If M > N, we need to use long division to express the rational function as the sum of quotient plus a proper rational function.
- Example of long division:

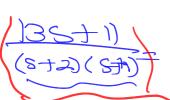
$$X(s) = \frac{2s^3 + s^2 + 2s + 1}{(s^2 + 3s + 2)}$$

Use the long division to convert the rational function to a proper one:

$$\begin{array}{r}
(2s - 5) \\
\underline{s^2 + 3s + 2\sqrt{2s^3 + s^2 + 2s + 1}} \\
2s^3 + 6s^2 + 4s \\
\underline{-5s^2 - 2s + 1} \\
-5s^2 - 15s - 10 \\
\underline{13s + 11}
\end{array}$$

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$$A_{1} = \frac{35+11}{5+1} = 15$$

$$A_{2} = \frac{35+11}{5+2} = 15$$

$$A_{2} = \frac{35+11}{5+2} = -15$$

## Cont.

After long division, 
$$X(s) = 2s - 5 + \frac{13s+11}{s^2+3s+2} = 2s - 5 + \frac{13s+11}{(s+1)(s+2)}$$

Taking partial fraction expansion of the second term,

$$X(s) = 2s - 5 + \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

where

$$A_1 = (s+1)X(s)|_{s=-1} = \frac{13s+11}{s+2}|_{s=-1} = -2$$

$$A_2 = (s+2)X(s)|_{s=-2} = \frac{13s+11}{s+1}|_{s=-2} = 15.$$

Finally, the partial fraction expansion of X(s) is

$$X(s) = 2s - 5 + \frac{-2}{s+1} + \frac{15}{s+2}$$

Differential Equations & Laplace Transform