Chapter 3. Multiple Integral

Two-Variable Case (Double Integral):

Computation of Double Integrals:

Case 4. Substitution needed first.

Necessity: For some double integrals (with complicated region or complicated integral function), only with iterated integrals it is NOT sufficient to solve out.

Example: Solve

$$\iint_R \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dxdy$$

with R enclosed by $y = x$, $y = 2$ and $xy = 1$.

Example: Evaluate

$$\iint_Q e^{\frac{y-x}{x+y}} dxdy$$

where Q is bounded by $x + y = 2, x = 0, y = 0$.

Recall Substitution for single integral.

Example Evaluate $\int_0^2 x \cos(x^2)\, dx$.

Substitution for double integral: With $x = x(u, v), y = y(u, v)$, we have

$$\iint_R f(x, y) dx dy = \iint_{R^*} f\left(x(u, v), y(u, v)\right) \cdot \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du dv,$$

where

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix},$$

called the Jacobian of substitution $\begin{cases} x = x(u, v) \\ y = y(u, v). \end{cases}$

Example. Find the Jacobian determinant of the following substitutions.

1) $\begin{cases} x = 5u - v \\ y = u + 3v \end{cases}$ 2) $\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$ 3) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

Example: Solve

$$\iint_R \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dx dy$$

with R enclosed by $y = x$, $y = 2$ and $xy = 1$.

Example: Evaluate

$$\iint_Q e^{\frac{y-x}{x+y}} dx dy$$

where Q is bounded by $x + y = 2, x = 0, y = 0$.

Example Evaluate

$$\iint_R e^{-\left(x^2+y^2\right)} dxdy,$$

where $R = \{(x, y) : \quad x \geq 0, y \geq 0\}$.

Example Evaluate

$$\iint_R \sqrt{x^2+y^2}\,dR,$$

where R is enclosed by $x^2+y^2 = x+\sqrt{x^2+y^2}$ and $x^2+y^2=1$.

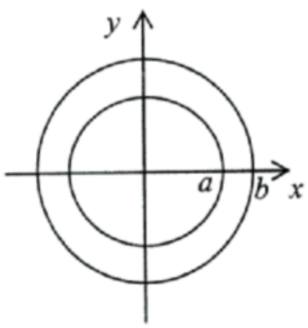
Example Evaluate $\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dydx$.

Polar coordination substitution

Remark: A particular and popularly used substitution for double integrals.

Example. Find the area enclosed by the lemniscate $r^2 = 1 + \cos\theta$ with (r, θ) polar coordinates.

Example. Find the moment of inertia of a hollow circular cylinder of inner radius a, outer radius b, height h and constant density ρ about the axis of the cylinder. The formula for the moment of inertia of the hollow circular cylinder is given as $MI = \iint_S \rho h \left(x^2 + y^2\right) dxdy$.



Exercises for figuring out images under substitution.

Example Find the image of S under the substitution $\begin{cases} x = 2u + 3v \\ y = u - v \end{cases}$, where S is bounded by $x + 3y = 0, x + 3y = 15, x - 2y = 0, x - 2y = 10$.

Example Find the $r\theta-$ image of R under polar coordinate transformation.