EE3210 Signals & Systems
Mid-term Exam 2 (100 points in total)

Name: ______________________

This is a open-book exam. Submission due date is **9:15 pm, April. 9th, 2021**. Late submission will not be accepted. If you need more space, please feel free to attach additional papers. Once you're finished, scan and upload it to Canvas course website.

Honor Pledge

Please review the following honor code, then sign your name and write down the date.

1. I pledge that the answers in this exam are my own and that I will not seek or obtain an unfair advantage in producing these answers. Specifically,
   (a) I will not plagiarize (copy without citation) from any source;
   (b) I will not communicate or attempt to communicate with any other person during the exam;
   (c) neither will I give or attempt to give assistance to another student taking the exam; and
   (d) I will use only approved devices (e.g., calculators) and/or approved device models.
2. I understand that any act of academic dishonesty can lead to disciplinary action.

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Signature

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Date

1. (20 points) Determine the Fourier Series representation of the periodic signal $x(t)$ given by

$$x(t) = \begin{cases} t, & \text{for } 0 \le t < \pi \\ \pi, & \text{for } \pi \le t < \pi \end{cases} \quad \text{and } x(t+2\pi) = x(t).$$

typo → give full mark to any student who attempt this question

(Use the trigonometric FS form, not the complex FS form)

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2\pi} = 1$$

Sol)

$$a_0 = \frac{2}{2\pi}\int_0^{2\pi} x(\tau)\,dt = \frac{1}{\pi}\cdot\frac{3}{2}\pi^2 = \frac{3}{2}\pi$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} x(t)\cos(kt)\,dt = \frac{1}{\pi}\left[\int_0^{\pi} t\cos(kt)\,dt + \pi\int_{\pi}^{2\pi}\cos(kt)\,dt\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{1}{k^2}\cos(kt) + \frac{t}{k}\cancel{\sin(kt)}\right]_0^{\pi} + \frac{\pi}{k}\cancel{\sin(kt)}\Big|_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left(\frac{1}{k^2}(\cos(k\pi)-1)\right) = \frac{1}{k^2\pi}(\cos(k\pi)-1) = \begin{cases} \text{even } k, & \phi \\ \text{odd } k, & -\frac{2}{k^2\pi}\end{cases}$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin(kt)\,dt = \frac{1}{\pi}\left[\int_0^{\pi} t\sin(kt)\,dt + \pi\int_{\pi}^{2\pi}\sin(kt)\,dt\right]$$

$$= \frac{1}{\pi}\left[\left[\frac{1}{k^2}\cancel{\sin(kt)} - \frac{t}{k}\cos(kt)\right]_0^{\pi} + \frac{\pi}{(-k)}\cos(kt)\Big|_{\pi}^{2\pi}\right]$$

$$\Rightarrow \frac{1}{\pi}\left(-\frac{\pi}{k}\cancel{\cos(k\pi)} + \frac{\pi}{(-k)}\left(1 - \cancel{\cos(k\pi)}\right)\right)$$

$$= -\frac{1}{k}$$

$$\therefore\ x(t) = \frac{3}{4}\pi + \sum_{k=1}^{\infty}\left(\frac{\cos(k\pi)-1}{k^2\pi}\right)\cos(kt) - \sum_{k=1}^{\infty}\frac{1}{k}\sin(kt)$$

2. (20 points) Calculate the Fourier Transform of the following signals.
(*Hint.* Use the convolution property of FT)

typo

(a)

$$\int_0^t e^{-4(t-\tau)} \text{rect}\left(\frac{t-1}{2}\right) dt$$

$t \to \tau$, $dt \to d\tau$ → give full credit if the student attempt this question

Sol) Consider $x(t) = \text{rect}\left(\frac{t-1}{2}\right)u(t)$ and $y(t) = e^{-4t}u(t)$, then

$$x(t) * y(t) = \int_0^t e^{-4(t-\tau)} \text{rect}\left(\frac{\tau-1}{2}\right) d\tau$$

$$\mathcal{F}(x(t) * y(t)) = \frac{1}{4+j2\pi f} \cdot 2 \cdot e^{-j2\pi f} \cdot \text{Sinc}(2f)$$

typo

(b)

$$\int_0^t \text{rect}\left(\frac{t-1}{2}\right) dt$$

$t \to \tau$, $dt \to d\tau$ → give full credit if tried.

Sol) if $x(t) = \text{rect}\left(\frac{t-1}{2}\right)u(t)$, then

$$x(t) * u(t) = \int_0^t \text{rect}\left(\frac{\tau-1}{2}\right) d\tau$$

$$\mathcal{F}(x(t) * u(t)) = 2e^{-j2\pi f} \cdot \text{Sinc}(2f)\left\{\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right\}$$

$$= \delta(f) + \frac{2e^{-j2\pi f}}{j2\pi f}\text{Sinc}(2f)$$

(c)

$$\cos(4t)\,\text{sgn}(t), \quad \text{where } \text{sgn}(t) = 2U(t) - 1$$

Sol) $\mathcal{F}(\text{Cos}(4t)\,\text{Sgn}(t))$

$$= \mathcal{F}(2\text{Cos}(4t)\,u(t)) - \mathcal{F}(\text{Cos}(4t))$$

$$\Rightarrow \frac{1}{2}\left[\cancel{\delta\left(f-\frac{2}{\pi}\right) + \delta\left(f+\frac{2}{\pi}\right)}\right] + \frac{j4\pi f}{16-(2\pi f)^2} - \frac{1}{2}\left[\cancel{\delta\left(f+\frac{2}{\pi}\right) + \delta\left(f-\frac{2}{\pi}\right)}\right]$$

$$= \frac{j\pi f}{4-(\pi f)^2}$$

3. (20 points) Let us consider a periodic signal $x(t)$ given by

$$x(t) = |t| \quad \text{for } -\pi \le t < \pi \quad \text{and } x(t+2\pi) = x(t).$$

(a) Find the Fourier Series representation of $x(t)$.
(Use the trigonometric FS form, not the complex FS form) $T_0 = 2\pi, \quad \omega_0 = 1$

Sol) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |t|\, dt = \frac{2}{\pi}\int_0^{\pi} t\, dt = \pi$, Since this is an even function, $b_k = \phi$

$$a_k = \frac{2}{\pi}\int_0^{\pi} t \cos(kt)\, dt = \frac{2}{\pi}\left[\frac{1}{k^2}\cos(kt) + \cancel{\frac{t}{k}\sin(kt)}^{\,\phi} \Big|_0^{\pi} \right]$$

$$= \frac{2}{k^2\pi}\left(\cos(k\pi) - 1\right) = \begin{cases} \text{even } k, & \phi \\ \text{odd } k, & -\frac{4}{k^2\pi} \end{cases}$$

$$x(t) = \frac{\pi}{2} + \underbrace{\sum_{k=1}^{\infty} a_k \cos(kt) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(-\frac{4}{(2n-1)^2\pi}\right)\cos(kt)}_{(\,k = 2n-1 \text{ only consider odd } k\,)} \quad \text{—— (1)}$$

(b) Prove the following equality using the FS form derived in Q3-(a).

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Sol) let's consider $t = 0$ in eq (1), then

$$\Rightarrow \quad 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

4. (20 points) Let $x(t)$ be a continuous-time signal with Fourier transform $X(f) = \mathcal{F}(x(t))$. Although we do not know the functional form of $x(t)$, the following conditions are given.

   (a) $x(t)$ is a real signal,

   (b) $x(t) = 0$ for $t \le 0$,

   (c) $\mathcal{F}\left(|t|e^{-|t|}\right) = \frac{X(f)+X(f)^*}{2} = \mathrm{Re}\{X(f)\}$, where $\mathrm{Re}\{X(f)\}$ is the real part of $X(f)$.

Find $x(t)$ signal that satisfies these three conditions.

Sol) if $x(t)$ is a real signal, $X^*(f) = X(-f)$

Then,

$$|t|e^{-|t|} = \mathcal{F}^{-1}\left\{\frac{1}{2}\left(X(f) + X(-f)\right)\right\}$$

$$= \frac{1}{2}\mathcal{F}^{-1}(X(f)) + \frac{1}{2}\mathcal{F}^{-1}(X(-f))$$

$$= \frac{1}{2}\left(x(t) + x(-t)\right) \qquad \text{——— } (1)$$

Since $x(t)$ is a right-sided signal, i.e, $x(t) = 0$ for $t \le 0$,

$x(-t) = 0$ for $t \ge 0$.

For $t \ge 0$, (1) becomes

$$te^{-t} = \frac{1}{2}x(t).$$

Hence,

$$x(t) = 2te^{-t}u(t)$$

5. (20 points) Consider a LTI system described by the following differential equations.

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 5y(t) = 2x(t)$$

(a) Derive the frequency response $H(f)$ and the corresponding impulse response $h(t)$.

Sol)

$$H(f) = \frac{2}{(j2\pi f)^2 + 6(j2\pi f) + 5} = \frac{2}{(5+j2\pi f)(1+j2\pi f)}$$

$$= \frac{-\frac{1}{2}}{5+j2\pi f} + \frac{\frac{1}{2}}{1+j2\pi f}$$

$$h(t) = \frac{1}{2}\left(e^{-t}u(t) - e^{-5t}u(t)\right)$$

$$= \frac{1}{2}\left(e^{-t} - e^{-5t}\right)u(t)$$

(b) Derive the system output $y(t)$ when the input signal is $x(t) = e^{-t}u(t)$.

Sol) $$X(f) = \frac{1}{1+j2\pi f}$$

$$Y(f) = H(f)X(f) = \frac{2}{(5+j2\pi f)(1+j2\pi f)^2}$$

$$= \frac{1/8}{5+j2\pi f} + \frac{1/2}{(1+j2\pi f)^2} + \frac{-1/8}{(1+j2\pi f)}$$

$$= \frac{1}{8}e^{-5t}u(t) + \frac{1}{2}te^{-t}u(t) - \frac{1}{8}e^{-t}u(t)$$