MA1200 CALCULUS AND BASIC LINEAR ALGEBRA I

LECTURE: CG1

Chapter 3
Polynomials and Rational Functions

Polynomials

A **polynomial function** of **degree n** is a function of the form

$$f(x) = a_n\, x^n + a_{n-1}\, x^{n-1} + \cdots + a_1\, x + a_0,$$

where the a_i's are real numbers with $a_n \neq 0$ and n is a non-negative integer. The constants a_i's are called **coefficients** of the corresponding x^i terms.

Two commonly used polynomials include:

- $f(x) = ax + b \quad (a \neq 0)$ is called a **linear** function (i.e. polynomial of degree 1).
- $f(x) = ax^2 + bx + c \quad (a \neq 0)$ is called a **quadratic** function (i.e. polynomial of degree 2)

E.g. The constant function $f(x) = k$, where $k \in \mathbb{R}$, is a polynomial of degree 0.

E.g. $3x^5 + \frac{2}{3}x^3 - \sqrt{6}\, x^2 + 2$ is a polynomial of degree 5.

E.g. $x^{\frac{1}{2}}$, x^{-1} and $x^{\cos x}$ are not polynomials.

Note: The **largest possible domain** of any polynomial is $\mathbb{R}$.

Quadratic Functions

A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c,$$

where $a, b, c \in \mathbb{R}$ and $a \neq 0$ is the coefficient of x^2 in the quadratic function.

By ***completing the square***, we can rewrite the quadratic function as

$$f(x) = ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c$$

$$= a\left[x - \underbrace{\left(-\frac{b}{2a}\right)}_{=h}\right]^2 + \underbrace{\left(c - \frac{b^2}{4a}\right)}_{=k}$$

Putting $h = -\frac{b}{2a}$ and $k = c - \frac{b^2}{4a}$, the quadratic function $f(x)$ can be written as

$$f(x) = a(x-h)^2 + k,$$

which is called the **standard form of a quadratic function**.

Let $y = a(x-h)^2 + k$. Rearranging $y = a(x-h)^2 + k$ gives

$$y = a(x-h)^2 + k \Rightarrow y - k = a(x-h)^2 \Rightarrow (x-h)^2 = \frac{1}{a}(y-k)$$

$$\Rightarrow (x-h)^2 = 4\left(\frac{1}{4a}\right)(y-k).$$

Putting $p = \frac{1}{4a}$, a quadratic function can be written in the form $(x-h)^2 = 4p(y-k)$, which represents a **parabola** (see Chapter 1).

Properties of the graph of a quadratic function $f(x) = a(x-h)^2 + k$:

- If $a > 0$, the parabola **opens upward** (∪-shaped).

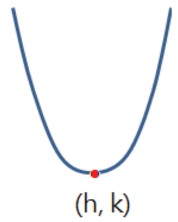
  If $a < 0$, the parabola **opens downward** (∩-shaped).

- The **vertex** of the parabola is at (h, k).

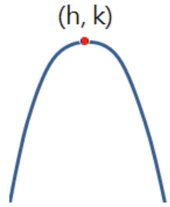
  - If $a > 0$, the **minimum** value of $f(x)$ is $f(x) = k \left(= c - \frac{b^2}{4a}\right)$ and it is attained at $x = h \left(= -\frac{b}{2a}\right)$. The largest possible range of $f(x)$ is $[k, \infty)$.

- If $\boldsymbol{a < 0}$, the **maximum** value of $f(x)$ is $f(x) = k \left(= c - \frac{b^2}{4a}\right)$ and it is attained at $x = h \left(= -\frac{b}{2a}\right)$. The largest possible range of $f(x)$ is $(-\infty, k]$.

$a > 0$



$a < 0$



- The parabola is symmetric about the vertical line $x = h$, which is the **axis of symmetry** of this parabola.
- **Intersection of parabola with the y-axis (i.e. the y-intercept)**:

  Consider the parabola $y = ax^2 + bx + c$.

  To find where its graph cuts the y-axis (i.e. the vertical line $x = 0$), we put $x = 0$ into $y = ax^2 + bx + c$ to get $y = c$.

  $\therefore$ The graph of $y = ax^2 + bx + c$ cuts the y-axis at $\boldsymbol{(0, c)}$.

- **Intersection of parabola with the x-axis (i.e. the x-intercept)**:

  Consider the parabola $y = ax^2 + bx + c = a\left[x - \left(-\frac{b}{2a}\right)\right]^2 + \left(c - \frac{b^2}{4a}\right)$, where $a \neq 0$. To find where its graph cuts the x-axis (i.e. the horizontal line $y = 0$), we put $y = 0$ into $y = a\left[x - \left(-\frac{b}{2a}\right)\right]^2 + \left(c - \frac{b^2}{4a}\right)$ and solve the equation

$$a\left[x - \left(-\frac{b}{2a}\right)\right]^2 + \left(c - \frac{b^2}{4a}\right) = 0 \Rightarrow \left[x - \left(-\frac{b}{2a}\right)\right]^2 = \frac{1}{a}\left(\frac{b^2}{4a} - c\right)$$

$$\Rightarrow \left[x - \left(-\frac{b}{2a}\right)\right]^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow x - \left(-\frac{b}{2a}\right) = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\Rightarrow x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

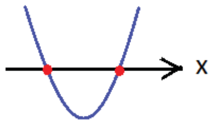
$$\Rightarrow \boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

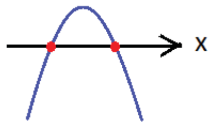
This is called the **quadratic equation formula**.

The quantity $\boxed{\Delta = b^2 - 4ac}$ is called the **discriminant** of the quadratic equation.

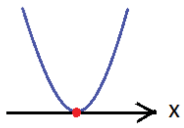
Depending on the sign of the discriminant, there are three possibilities:

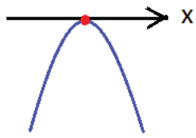
- If $b^2 - 4ac > 0$, the quadratic equation $ax^2 + bx + c = 0$ has two distinct real solutions (or roots), i.e. the graph of $y = ax^2 + bx + c$ **cuts (or intersects) the x-axis at two distinct points**.



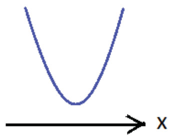


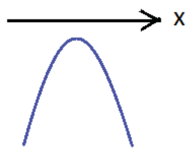
- If $b^2 - 4ac = 0$, the quadratic equation $ax^2 + bx + c = 0$ has one real solution (or root), i.e. the graph of $y = ax^2 + bx + c$ **touches the x-axis at one point**.





- If $b^2 - 4ac < 0$, the quadratic equation $ax^2 + bx + c = 0$ has no real solution, i.e. the graph of $y = ax^2 + bx + c$ **does not cut the x-axis**.





Example 1

Given the function $f(x) = 3x^2 - 12x + 9$.

(a) Express it in the standard form of quadratic function.

(b) Find the coordinates of the vertex and determine whether this is the maximum or minimum point of the function.

(c) Find the points where the graph of the function intersects the y-axis and the x-axis.

(d) Sketch its graph.

(e) Determine the largest possible domain and largest possible range of $f(x)$.

Solution

(a) $f(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x) + 9 = 3[(x-2)^2 - 2^2] + 9 = 3(x - \mathbf{2})^2 \mathbf{- 3}$.

(b) The vertex is at $(\mathbf{2}, \mathbf{-3})$. Since the coefficient of x^2 is 3 (> 0), the parabola opens upward. Therefore, the vertex $(2, -3)$ is the minimum point of the parabola.

(c) $f(0) = 3(0)^2 - 12(0) + 9 = 9$

$\therefore$ The parabola intersects the y-axis at $(0, 9)$.

To find the points (if any) where the parabola intersects the x-axis, we solve the equation $\underbrace{3}_{=a}x^2\underbrace{-12}_{=b}x+\underbrace{9}_{=c}=0$. By the quadratic equation formula, we obtain

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-12)\pm\sqrt{(-12)^2-4(3)(9)}}{2(3)}=\frac{12\pm\sqrt{36}}{6}=\frac{12\pm 6}{6}=1,3$$

$\therefore$ The parabola intersects the x-axis at $(1, 0)$ and $(3, 0)$.

(d)

(e) The largest possible domain of $f(x)$ is $\mathrm{Dom}(f)=\mathbb{R}$.

The largest possible range of $f(x)$ is $\mathrm{Ran}(f)=[-3,\infty)$.

Example 2

Given the function $g(x)=-2x^2-4x-4$.

(a) Find the coordinates of the vertex.

(b) Find the points (if any) where the parabola cuts the y-axis and the x-axis.

(c) Sketch its graph.

(d) Determine the largest possible domain and largest possible range of $g(x)$.

Solution

(a) $g(x)=-2x^2-4x-4=-2(x^2+2x)-4=-2[(x+1)^2-1^2]-4$

$$=-2(x+1)^2-2=-2[x-(\mathbf{-1})]^2\mathbf{-2}$$

$\therefore$ The vertex is at $(\mathbf{-1},\mathbf{-2})$.

(b) Since $g(0)=-4$, the parabola cuts the y-axis at $(0,-4)$.

By the quadratic equation formula,

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(-2)(-4)}}{2(-2)}=\frac{4\pm\sqrt{-16}}{-4}$$

which has no real solution. Hence the parabola does not cut the x-axis.

(c) Since the coefficient of x^2 is -2 (< 0), the parabola opens downward.

The function attains its maximum at $x = -1$ and maximum value of the function is $g(-1) = -2$.

(d) $\mathrm{Dom}(g) = \mathbb{R}$,

$\mathrm{Ran}(g) = (-\infty, -2]$.

Example 3

Recall that in **Chapter 2 Example 9**, we have determined the largest possible domains for the following functions:

(a) $f(x) = \sqrt{x^2 - 3x + 2}$ (b) $f(x) = \sqrt{3 + 2x - x^2}$

(c) $f(x) = \frac{9}{x^2+4x-5}$ (d) $f(x) = \sqrt{\frac{x+1}{x+2}}$

Now we find the largest possible range for each of the above functions.

Solution

(a) $f(x) = \sqrt{x^2 - 3x + 2}$

Recall from **Example 9 of Chapter 2** that the function $f(x)$ is well-defined when $x^2 - 3x + 2 \geq 0$, i.e. when $x \leq 1$ or $x \geq 2$.

Thus, $Dom(f) = (-\infty, 1] \cup [2, \infty)$.

To find the range of $f(x)$, since it is difficult to sketch the graph of $f(x)$, we would consider the quadratic function inside the square root first.

Consider $x^2 - 3x + 2$:

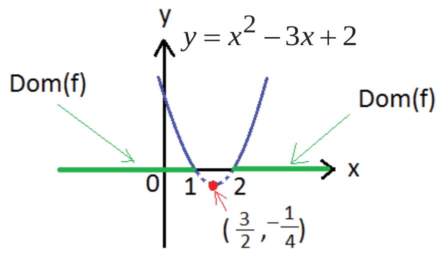
$x^2 - 3x + 2 = (x-2)(x-1)$.

$\therefore$ The parabola passes through the x-axis at $x = 1$ and $x = 2$.

Coefficient of x^2 is $1 > 0$

$\therefore$ The parabola opens upward.

A sketch of the graph of $y = x^2 - 3x + 2$ is shown on the right:



For any $x \in Dom(f) = (-\infty, 1] \cup [2, \infty)$, we observe that $x^2 - 3x + 2 \geq 0 \Rightarrow \underbrace{\sqrt{x^2 - 3x + 2}}_{=f(x)} \geq 0$.

Hence, the largest possible range of $f(x)$ is $Ran(f) = [0, \infty)$.

(b) $f(x) = \sqrt{3 + 2x - x^2}$

Recall from **Example 9 of Chapter 2** that the function $f(x)$ is well-defined when $3 + 2x - x^2 \geq 0$, i.e. when $-1 \leq x \leq 3$. Thus, $Dom(f) = [-1, 3]$.

Consider $3 + 2x - x^2$:

$3 + 2x - x^2 = (3-x)(1+x)$.

$\therefore$ The parabola passes through the x-axis at $x = -1$ and $x = 3$.

Coefficient of x^2 is $-1 < 0$

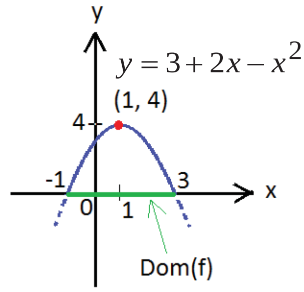
$\therefore$ The parabola opens downward.

By completing the square,

$3 + 2x - x^2 = -(x^2 - 2x) + 3 = -[(x-1)^2 - 1^2] + 3 = 4 - (x-1)^2$.

$\therefore$ The vertex is at $(1, 4)$.

A sketch of the graph of $y = 3 + 2x - x^2$ is shown on the right.



For any $x \in Dom(f) = [-1, 3]$, observe that

$$0 \leq 3 + 2x - x^2 \leq 4 \quad \Rightarrow \quad 0 \leq \underbrace{\sqrt{3 + 2x - x^2}}_{=f(x)} \leq \sqrt{4} = 2.$$

$\therefore$ The largest possible range of $f(x)$ is $Ran(f) = [0, 2]$.

(c) $f(x) = \frac{9}{x^2+4x-5}$

Recall from **Example 9 of Chapter 2** that the function $f(x)$ is well-defined when $x^2 + 4x - 5 \neq 0$, i.e. when $x \neq -5, 1$. Thus, $Dom(f) = \mathbb{R} \backslash \{-5, 1\}$.

To find the range, we let $y = \frac{9}{x^2+4x-5}$.

First note that $y \neq 0$.

(If $y = 0$, then $0 = \frac{9}{x^2+4x-5} \Rightarrow 0 = 9$ which is impossible.)

Then $y = \frac{9}{x^2+4x-5} \Rightarrow (x^2 + 4x - 5)y = 9$

$$\Rightarrow \underbrace{y}_{=a} x^2 + \underbrace{(4y)}_{=b} x + \underbrace{(-5y - 9)}_{=c} = 0 \ldots (*)$$

which is a quadratic equation provided $y \neq 0$.

$(*)$ has real roots if and only if the discriminant $b^2 - 4ac \geq 0$, that is,

$$(4y)^2 - 4y(-5y - 9) \geq 0 \Rightarrow 16y^2 + 20y^2 + 36y \geq 0 \Rightarrow 36(y^2 + y) \geq 0$$

$$\Rightarrow y(y + 1) \geq 0$$

	$y < -1$	$y = -1$	$-1 < y < 0$	~~$y = 0$~~	$y > 0$
Sign of y	–	–	–		+
Sign of $y + 1$	–	0	+		+
Sign of $y(y + 1)$	+	0	–		+

$\therefore$ The largest possible range of $f(x)$ is $Ran(f) = (-\infty, -1] \cup (0, \infty)$.

(d) $f(x) = \sqrt{\frac{x+1}{x+2}}$

Recall from **Example 9 of Chapter 2** that the function $f(x)$ is well-defined when $\frac{x+1}{x+2} \geq 0$ and $x+2 \neq 0$. Thus, $Dom(f) = (-\infty, -2) \cup [-1, \infty)$.

To find the range, let $y = \sqrt{\frac{x+1}{x+2}}$.

First note that $y \geq 0 \ldots (1)$, since the RHS ≥ 0 for all $x \in Dom(f)$.

$$y = \sqrt{\frac{x+1}{x+2}} \Rightarrow y^2 = \frac{x+1}{x+2} \Rightarrow y^2(x+2) = x+1 \Rightarrow x(y^2-1) = 1-2y^2$$

$\Rightarrow x = \frac{1-2y^2}{y^2-1}$ which has real solutions iff $y^2 - 1 \neq 0 \Rightarrow y^2 \neq 1 \Rightarrow y \neq \pm 1 \ldots (2)$

Combining conditions (1) and (2) gives

$Ran(f) = [0, \infty)\backslash\{1\}$.

Behavior of a polynomial function as x tends to ∞ or $-\infty$

Recall that a **polynomial function** of **degree n** is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$ with $a_n \neq 0$ and n is a non-negative integer. The number a_n (i.e. the coefficient of x^n) is called the **leading coefficient**.

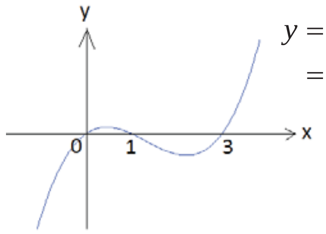
Graphs of polynomial functions are **smooth** (i.e. contain no sharp corners) and **continuous** (i.e. have no break). By observing the sign of the leading coefficient, we can deduce the behavior of a polynomial function as x tends to ∞ or $-\infty$.

We use the notation "→" to denote "tends to".

Case I: $a_n > 0$

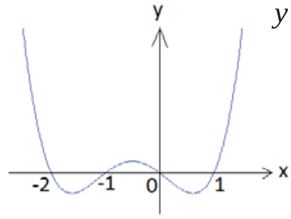
- If n is odd and $a_n > 0$, then $f(x) \to -\infty$ as $x \to -\infty$;
  and $f(x) \to \infty$ as $x \to \infty$.
- If n is even and $a_n > 0$, then $f(x) \to \infty$ as $x \to -\infty$;
  and $f(x) \to \infty$ as $x \to \infty$.

E.g.



$$\begin{aligned} y &= x^3 - 4x^2 + 3x \\ &= x(x-1)(x-3) \end{aligned}$$

$a_3 = 1 > 0$ and n is odd



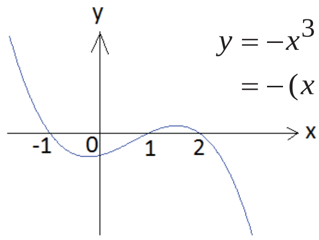
$$\begin{aligned} y &= 2x^4 + 4x^3 - 2x^2 - 4x \\ &= 2x(x-1)(x+1)(x+2) \end{aligned}$$

$a_4 = 2 > 0$ and n is even

Case II: $a_n < 0$

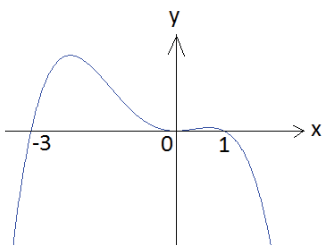
- If n is odd and $a_n < 0$, then $f(x) \to \infty$ as $x \to -\infty$;
  and $f(x) \to -\infty$ as $x \to \infty$.
- If n is even and $a_n < 0$, then $f(x) \to -\infty$ as $x \to -\infty$;
  and $f(x) \to -\infty$ as $x \to \infty$.

E.g.



$$\begin{aligned} y &= -x^3 + 2x^2 + x - 2 \\ &= -(x-2)(x-1)(x+1) \end{aligned}$$

$a_3 = -1 < 0$ and n is odd



$$\begin{aligned} y &= -\frac{1}{2}x^4 - x^3 + \frac{3}{2}x^2 \\ &= -\frac{1}{2}x^2(x-1)(x+3) \end{aligned}$$

$a_4 = -\frac{1}{2} < 0$ and n is even

Division of polynomials

Recall that when a number (called "dividend") is divided by a smaller non-zero number (called "divisor"), we obtain:

$$\text{Dividend} = \text{Quotient} \times \text{Divisor} + \text{Remainder}$$

$$\Rightarrow \quad \frac{\text{Dividend}}{\text{Divisor}} = \frac{\text{Quotient} \times \text{Divisor} + \text{Remainder}}{\text{Divisor}}$$

$$\Rightarrow \quad \frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}$$

E.g. We divide 53 by 8 using **long division** to obtain the quotient (=6) and remainder (=5).

$$\begin{array}{rrl} & 6 & \leftarrow \text{Quotient} \\ \text{Divisor} \rightarrow \; 8 & \big) \overline{53} & \leftarrow \text{Dividend} \\ -) & \underline{48} & \\ & 5 & \leftarrow \text{Remainder} \end{array}$$

$\therefore \; 53 = 6 \times 8 + 5$

E.g. We divide 56 by 7 using **long division** to obtain the quotient (=8) and remainder (=0).

$$\begin{array}{rrl} & 8 & \leftarrow \text{Quotient} \\ \text{Divisor} \rightarrow \; 7 & \big) \overline{56} & \leftarrow \text{Dividend} \\ -) & \underline{56} & \\ & & \leftarrow \text{Remainder (=0)} \end{array}$$

$\therefore \; 56 = 8 \times 7 + 0$

Similarly, when a polynomial $p(x)$ is divided by a polynomial $d(x)$ of lower degree, we have the result

$$p(x) = q(x)\,d(x) + r(x),$$

i.e.

$$\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

where the quotient $q(x)$ and remainder $r(x)$ can be found using **long division**.

$$\begin{array}{rrl} & q(x) & \leftarrow \text{Quotient} \\ \text{Divisor} \rightarrow \; d(x) & \big) \overline{\; p(x) \;} & \leftarrow \text{Dividend} \\ -) & \underline{q(x)d(x)} & \\ & r(x) & \leftarrow \text{Remainder} \end{array}$$

$$r(x) = p(x) - q(x)d(x)$$

Note that:

- degree of $q(x)$ = degree of $p(x)$ − degree of $d(x)$;
- $0 \leq$ degree of $r(x) <$ degree of $d(x)$.

An alternative method called **synthetic division** can also be used to divide polynomials if the divisor is of the form $x - c$. Consider the following examples:

Example 4

Find the quotient and remainder when the polynomial $4x^3 + 5x^2 - x + 7$ is divided by $x + 2$.

Solution

Method 1: By **long division**:

$$
\begin{array}{rr|l l}
 & & 4x^2 - 3x + 5 & \leftarrow \text{Quotient} \\
\text{Divisor} \rightarrow & x+2 & 4x^3 + 5x^2 - x + 7 & \leftarrow \text{Dividend} \\
 & & \underline{4x^3 + 8x^2} & \\
 & & \quad -3x^2 - x & \\
 & & \quad \underline{-3x^2 - 6x} & \\
 & & \qquad\quad 5x + 7 & \\
 & & \qquad\quad \underline{5x + 10} & \\
 & & \qquad\qquad -3 & \leftarrow \text{Remainder}
\end{array}
$$

$\therefore\ 4x^3 + 5x^2 - x + 7 = (4x^2 - 3x + 5)(x + 2) - 3$

The quotient is $4x^2 - 3x + 5$ and the remainder is -3.

Method 2: By **synthetic division**:

$$4x^3 + 5x^2 - x + 7 = q(x)\,(x - (-2)) + r(x).$$

$$
\begin{array}{r|rrr|rl}
-2 & 4 & 5 & -1 & 7 & \\
\hline
+) & & -8 & 6 & -10 & \\
\hline
 & 4 & -3 & 5 & -3 & \leftarrow \text{Remainder}
\end{array}
$$

↑ Coefficient of x^2 of quotient

↑ Coefficient of x^1 of quotient

↑ Coefficient of x^0 of quotient

$\therefore$ The quotient is $4x^2 - 3x + 5$ and the remainder is -3.

$$
\begin{aligned}
4x^3 + 5x^2 - x + 7 &= (4x^2 - 3x + 5)\big(x - (-2)\big) - 3 \\
&= (4x^2 - 3x + 5)(x + 2) - 3.
\end{aligned}
$$

Example 5

Find the quotient and remainder when the polynomial $2x^3 + x^2 + 8$ is divided by $2x - 3$.

Solution

Method 1: By **long division**,

$$
\begin{array}{rrl}
 & x^2 + 2x + 3 & \leftarrow \text{Quotient} \\
\text{Divisor} \rightarrow \quad 2x - 3 & \overline{)\,2x^3 + x^2 \qquad + 8} & \leftarrow \text{Dividend} \\
 & \underline{2x^3 - 3x^2} \qquad\qquad & \\
 & 4x^2 \qquad\quad & \\
 & \underline{4x^2 - 6x} \quad & \\
 & 6x + 8 & \\
 & \underline{6x - 9} & \\
 & 17 & \leftarrow \text{Remainder}
\end{array}
$$

$\therefore\ 2x^3 + x^2 + 8 = (x^2 + 2x + 3)(2x - 3) + 17$

The quotient is $x^2 + 2x + 3$ and the remainder is 17.

Method 2: By **synthetic division**,

$$2x^3 + x^2 + 8 = q(x) \cdot (2x - 3) + r(x) = 2\, q(x) \cdot \left(x - \frac{3}{2}\right) + r(x)$$

$$
\begin{array}{c|cccc}
\frac{3}{2} & 2 & 1 & 0 & 8 \\
 & & & & \\
+) & & 3 & 6 & 9 \\
\hline
 & 2 & 4 & 6 & 17
\end{array}
$$

$\therefore\ 2x^3 + x^2 + 8 = (2x^2 + 4x + 6)\left(x - \frac{3}{2}\right) + 17 = (x^2 + 2x + 3)(2x - 3) + 17$

That is, the quotient is $x^2 + 2x + 3$ and the remainder is 17.

Note that when a polynomial $p(x)$ of degree n is divided by a linear divisor $ax - b$ $(a \neq 0)$, the degree of the quotient $q(x)$ is $n - 1$ and the degree of the remainder r is 0 (i.e. r is a constant).

Remainder Theorem

When a polynomial $f(x)$ is divided by $ax - b$ (where $a \neq 0$), the remainder is $f\left(\frac{b}{a}\right)$.

Proof:

We have $f(x) = q(x) \cdot (ax - b) + r$. Putting $x = \frac{b}{a}$ into this expression, we obtain

$$f\left(\frac{b}{a}\right) = q\left(\frac{b}{a}\right) \cdot \underbrace{\left[a\left(\frac{b}{a}\right) - b\right]}_{=0} + r = r.$$

Example 6

By using the **Remainder Theorem**,

(a) find the remainder when $f(x) = 4x^3 + 5x^2 - x + 7$ is divided by $x + 2$. (*Example 4*)

(b) find the remainder when $f(x) = 2x^3 + x^2 + 8$ is divided by $2x - 3$. (*Example 5*)

Solution

(a) $a = 1$, $b = -2$. By the **Remainder Theorem**, the remainder is

$$f\left(\frac{-2}{1}\right) = 4\left(\frac{-2}{1}\right)^3 + 5\left(\frac{-2}{1}\right)^2 - \left(\frac{-2}{1}\right) + 7 = -3.$$

(b) $a = 2$, $b = 3$. By the **Remainder Theorem**, the remainder is

$$f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^2 + 8 = 17.$$

Zeros of polynomial functions

The values of x for which the polynomial function $f(x) = 0$ are called the **zeros**. They are also called the **roots** (or **solutions**) of the equation $f(x) = 0$. The **real roots** are the x-coordinates of the points where the graph of $y = f(x)$ meets the x-axis, and these points are called the x-intercepts.

The following theorem is useful in finding the linear factors of a polynomial function:

Factor Theorem

The linear term $ax - b$ (where $a \neq 0$) is a factor of a polynomial $f(x)$ if and only if $f\left(\frac{b}{a}\right) = 0$.

Example 7

Determine whether $2x-1$ and $x+1$ are factors of $f(x)=2x^3+3x^2-8x+3$. Then factorize $f(x)=2x^3+3x^2-8x+3$.

Solution

$$f\left(\frac{1}{2}\right)=2\left(\frac{1}{2}\right)^3+3\left(\frac{1}{2}\right)^2-8\left(\frac{1}{2}\right)+3=0$$

$\therefore$ By the **Factor Theorem**, $2x-1$ is a factor of $f(x)$.

$$f\left(\frac{-1}{1}\right)=f(-1)=2(-1)^3+3(-1)^2-8(-1)+3=12\neq \mathbf{0}$$

$\therefore$ By the **Factor Theorem**, $x+1$ is **NOT** a factor of $f(x)$.

Since $2x-1$ is a factor of $f(x)=2x^3+3x^2-8x+3$, we use **long division** to divide $f(x)$ by $2x-1$.

$$\begin{array}{r|l} & x^2+2x-3 \\ \hline 2x-1 & 2x^3+3x^2-8x+3 \\ & \underline{2x^3-x^2} \\ & \quad 4x^2-8x \\ & \quad \underline{4x^2-2x} \\ & \qquad -6x+3 \\ & \qquad \underline{-6x+3} \end{array}$$

$$\begin{aligned}\therefore\ f(x)&=2x^3+3x^2-8x+3\\ &=(2x-1)(x^2+2x-3)\\ &=(2x-1)(x+3)(x-1)\end{aligned}$$

Example 8

Factorize $f(x)=2x^6-8x^4-2x^2+8$.

Solution

Since the polynomial $f(x)=2x^6-8x^4-2x^2+8$ only consists of even powers of x, we let $u=x^2$ and the function becomes $g(u)=2u^3-8u^2-2u+8$. This polynomial is simpler and has lower degree than $f(x)$, so it should be easier to factorize.

Consider the **constant term** (which is 8) in $g(u)$.

Factors of 8 are ±1, ±2, ±4 and ±8.

We use the "trial and error" method to find some roots first.

$g(1)=2(1)^3-8(1)^2-2(1)+8=0$ ✓ $\Rightarrow$ **$(u-1)$ is a factor of $g(u)$.**

$g(2)=2(2)^3-8(2)^2-2(2)+8=-12\ (\neq 0)$

$g(-1)=2(-1)^3-8(-1)^2-2(-1)+8=0$ ✓ $\Rightarrow$ **$(u+1)$ is a factor of $g(u)$.**

Since $(u-1)$ and $(u+1)$ are factors of $g(u)$, then $g(u)$ must be divisible by $(u-1)(u+1)=u^2-1$.

By long division, $g(u)=2u^3-8u^2-2u+8$

$$= (u^2-1)(2u-8)$$

$$= 2(u-1)(u+1)(u-4)$$

$$\begin{array}{r|rrrr} & 2u & -8 & & \\ \hline u^2-1 & 2u^3 & -8u^2 & -2u & +8 \\ & 2u^3 & & -2u & \\ \hline & & -8u^2 & & +8 \\ & & -8u^2 & & +8 \\ \hline \end{array}$$

Replacing $u=x^2$, we get $f(x)=2(x^2-1)(x^2+1)(x^2-4)$

$$= 2(x-1)(x+1)(x-2)(x+2)(x^2+1).$$

Note: The quadratic term x^2+1 cannot be further factorized.

Rational functions

A **rational function** is a quotient of two polynomials. It is of the form

$$f(x)=\frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials and $q(x)\neq 0$.

Note that the largest possible domain of $f(x)=\frac{p(x)}{q(x)}$ is $\boxed{\mathbb{R}\backslash\{x\in\mathbb{R}|q(x)=0\}}$, i.e. the set of all real numbers except the value(s) of x such that $q(x)=0$.

- $f(x)=\frac{p(x)}{q(x)}$ is called a **proper rational function** if $\boxed{\text{degree of } p(x) < \text{degree of } q(x)}$.

  For example, $f(x)=\frac{x^2+3}{x^3+2x-4}$ is a proper rational function.

- $f(x)=\frac{p(x)}{q(x)}$ is called an **improper rational function** if $\boxed{\text{degree of } p(x) \geq \text{degree of } q(x)}$.

  For example, $g(x)=\frac{x^2+3}{x^2+5x-7}$ and $h(x)=\frac{x^3+x-4}{x^2+5x-7}$ are improper rational functions.

If $f(x) = \frac{p(x)}{q(x)}$ is an **improper** rational function, use **long division** / **synthetic division** to write $f(x)$ as

$$\boxed{f(x) = \text{a polynomial} + \text{a proper rational function}}.$$

This is because $$p(x) = \underbrace{s(x)}_{Quotient}\ \underbrace{q(x)}_{Divisor} + \underbrace{r(x)}_{Remainder}$$

where $0 \le$ degree of remainder $r(x) <$ degree of divisor $q(x)$.

$$\therefore\ f(x) = \frac{p(x)}{q(x)} = \frac{s(x)\,q(x)+r(x)}{q(x)} = \underbrace{s(x)}_{\text{polynomial}} + \underbrace{\frac{r(x)}{q(x)}}_{\text{proper rational function}}$$

For example, $g(x) = \frac{x^2+3}{x^2+5x-7} = 1 + \frac{-5x+10}{x^2+5x-7}$ and $h(x) = \frac{x^3+x-4}{x^2+5x-7} = x - 5 + \frac{33x-39}{x^2+5x-7}$.

Example 9

Find the largest possible domain of the function $f(x) = \frac{x-7}{x^3+2x^2+5x+10}$.

Solution

Let $g(x) = x^3 + 2x^2 + 5x + 10$.

Factors of 10 (the constant term) are $\pm 1, \pm 2, \pm 5, \pm 10$.

Use the "trial and error" method to find a root of the equation $g(x) = 0$:

$g(-1) = (-1)^3 + 2(-1)^2 + 5(-1) + 10 = 6\ (\neq 0)$

$g(-2) = (-2)^3 + 2(-2)^2 + 5(-2) + 10 = 0$ ✓ $\therefore (x-(-2)) = x + 2$ **is a factor of** $g(x)$

$$\begin{array}{r|l} & x^2 + 5 \\ \hline x+2 & x^3 + 2x^2 + 5x + 10 \\ & x^3 + 2x^2 \\ \hline & 5x + 10 \\ & 5x + 10 \\ \hline \end{array}$$

By long division,

$g(x) = x^3 + 2x^2 + 5x + 10 = (x + 2)(x^2 + 5)$

The function $f(x)$ is NOT well-defined when $g(x) = 0$,

i.e. when $(x + 2)(x^2 + 5) = 0 \Rightarrow x + 2 = 0$ or $\underbrace{x^2 + 5 = 0}_{\text{has no real solution}} \Rightarrow x = -2$.

$\therefore$ The largest possible domain of $f(x)$ is $\mathbb{R}\backslash\{-2\}$.

Example 10

Evaluate $\dfrac{5}{2x-1}-\dfrac{2}{x+3}$

$$=\frac{5(x+3)-2(2x-1)}{(2x-1)(x+3)}$$

$$=\frac{x+17}{(2x-1)(x+3)}$$

Easy!

Difficult!

(use **partial fractions**)

Partial Fractions

Partial fraction is a technique used in writing a complicated fraction as a sum of simpler fractions.

The following table lists some typical cases we shall mostly encounter and how we should resolve them into partial fractions (Note: We assume the expressions are already ***proper*** rational functions.)

Three common types of factors in the denominator:

Type	Expression	Form of Partial Fraction
Distinct Linear Factors	E.g. $\dfrac{f(x)}{(x+a)(x+b)(x+c)}$	$\dfrac{A}{(x+a)}+\dfrac{B}{(x+b)}+\dfrac{C}{(x+c)}$
Repeated Linear Factors	E.g. $\dfrac{f(x)}{(x+a)^3}$	$\dfrac{A}{(x+a)}+\dfrac{B}{(x+a)^2}+\dfrac{C}{(x+a)^3}$
Quadratic Factors	E.g. $\dfrac{f(x)}{(ax^2+bx+c)(x+d)}$ where ax^2+bx+c cannot be further factorized	$\dfrac{Ax+B}{(ax^2+bx+c)}+\dfrac{C}{(x+d)}$

Here, A, B, C are unknown constants to be found.

Note:

In general, if a linear factor $(ax+b)$ is repeated n times, we would have n terms in the decomposition of the form $\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$. Here, $A_1, A_2, \dots, A_n$ are unknown constants to be found.

Similarly, if a quadratic factor (ax^2+bx+c) is repeated n times, where (ax^2+bx+c) cannot be further factorized, we would have n terms in the decomposition of the form $\frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$.

Here, $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_n$ are unknown constants to be found.

Procedure for resolving a rational function into partial fractions

Consider the rational function $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials:

Step 1: Check whether $\frac{p(x)}{q(x)}$ is a proper rational function or not. If it is improper, use long division to express $\frac{p(x)}{q(x)}$ as "a polynomial + a proper rational function".

Step 2: For the proper rational function, factorize its denominator.

Step 3: Write down the form of the partial fractions.

Step 4: Find the unknowns.

Example 11 (**Distinct linear factors**)

Express $\frac{x+17}{2x^2+5x-3}$ into partial fractions.

Solution

First note that this is a **proper** rational function. Then notice that the denominator can be factorized as $2x^2+5x-3=(2x-1)(x+3)$. Thus $\frac{x+17}{2x^2+5x-3}=\frac{x+17}{(2x-1)(x+3)}$.

Let $\frac{x+17}{(2x-1)(x+3)}=\frac{A}{2x-1}+\frac{B}{x+3}$, where A and B are constants to be determined.

Multiplying both sides by $(2x-1)(x+3)$, we get

$$x+17=A(x+3)+B(2x-1)$$

Put $x=-3$: $-3+17=A\underbrace{(-3+3)}_{=0}+B[2\cdot(-3)-1] \Rightarrow 14=-7B \Rightarrow B=-2$

Put $x=\frac{1}{2}$: $\frac{1}{2}+17=A\left(\frac{1}{2}+3\right)+B\underbrace{\left[2\cdot\left(\frac{1}{2}\right)-1\right]}_{=0} \Rightarrow \frac{35}{2}=\frac{7}{2}A \Rightarrow A=5$

$\therefore \frac{x+17}{(2x-1)(x+3)}=\frac{5}{2x-1}-\frac{2}{x+3}$.

Example 12 (**Three distinct linear factors**)

Resolve $\frac{4x^2+12x+18}{x^3-9x}$ into partial fractions.

Solution

First note that this is a **proper** rational function. Also, note that the denominator can be factorized as $x^3-9x=x(x^2-9)=x(x-3)(x+3)$.

Then we have $\frac{4x^2+12x+18}{x^3-9x}=\frac{4x^2+12x+18}{x(x-3)(x+3)}=\frac{A}{x}+\frac{B}{x-3}+\frac{C}{x+3}$.

Multiplying both sides by $x(x-3)(x+3)$, we get

$$4x^2+12x+18=A(x-3)(x+3)+Bx(x+3)+Cx(x-3).$$

Put $x=0$: $18=-9A \Rightarrow A=-2$

Put $x=3$: $90=18B \Rightarrow B=5$

Put $x=-3$: $18=18C \Rightarrow C=1$

$\therefore \frac{4x^2+12x+18}{x^3-9x}=\frac{-2}{x}+\frac{5}{x-3}+\frac{1}{x+3}$

Example 13 (Improper rational function)

Express $\frac{2x^3-x^2-9x-10}{x^2-4}$ into partial fractions.

Solution

First note that the degree of the numerator is greater than the degree of the denominator, i.e. it is an improper rational function.

By long division,

$$\begin{array}{r} 2x - 1 \\ x^2-4 \overline{\smash{)}\,2x^3 - x^2 - 9x - 10} \\ \underline{2x^3 \qquad - 8x} \\ -x^2 - x - 10 \\ \underline{-x^2 \qquad + 4} \\ -x - 14 \end{array}$$

$$\underbrace{\frac{2x^3-x^2-9x-10}{x^2-4}}_{\text{improper rational function}} = \underbrace{2x-1}_{\text{polynomial}} + \underbrace{\frac{-x-14}{x^2-4}}_{\text{proper rational function}}$$

Consider $\frac{-x-14}{x^2-4} = \frac{-x-14}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$.

Multiplying both sides by $(x-2)(x+2)$, we get

$$-x-14 = A(x+2) + B(x-2)$$

Put $x=-2$: $-(-2)-14 = 0 + B(-4) \Rightarrow -12 = -4B \Rightarrow B = 3$

Put $x=2$: $-2-14 = A(4)+0 \Rightarrow -16 = 4A \Rightarrow A = -4$

$\therefore \frac{2x^3-x^2-9x-10}{x^2-4} = 2x - 1 - \frac{4}{x-2} + \frac{3}{x+2}$

Example 14 (**Repeated linear factors**)

Express $\frac{-7x^2+11x-3}{x(x-1)^3}$ into partial fractions.

Solution

First note that it's a proper rational function.

$$\frac{-7x^2+11x-3}{x(x-1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

Multiplying both sides by $x(x-1)^3$, we get

$$-7x^2 + 11x - 3 = A(x-1)^3 + Bx(x-1)^2 + Cx(x-1) + Dx$$

Put $x = 1$: $-7(1)^2 + 11(1) - 3 = 0 + 0 + 0 + D(1) \Rightarrow D = 1$

Put $x = 0$: $-7(0)^2 + 11(0) - 3 = A(0-1)^3 + 0 + 0 + 0 \Rightarrow -3 = -A \Rightarrow A = 3$

Equating coefficients of x^3: $0 = A + B \Rightarrow B = -A = -3$

Put $x = -1$: $-7(-1)^2 + 11(-1) - 3 = A(-1-1)^3 + B(-1)(-1-1)^2$

$$+C(-1)(-1-1) + D(-1)$$

$$\Rightarrow -21 = -8A - 4B + 2C - D$$

$$\Rightarrow C = \frac{1}{2}(-21 + 8A + 4B + D) = \frac{1}{2}[-21 + 8(3) + 4(-3) + 1] = -4$$

$$\therefore \frac{-7x^2+11x-3}{x(x-1)^3} = \frac{3}{x} - \frac{3}{x-1} - \frac{4}{(x-1)^2} + \frac{1}{(x-1)^3}$$

Example 15 (Linear and Quadratic Factors)

Resolve $\frac{9x^2-12x-2}{(2x+1)(x^2-2x+5)}$ into partial fractions.

Solution

First note that $\frac{9x^2-12x-2}{(2x+1)(x^2-2x+5)}$ is a proper rational function.

Also, note that $x^2 - 2x + 5 = 0$ has no real solution (since the discriminant $b^2 - 4ac = (-2)^2 - 4(1)(5) = -16 < 0$), which means that $x^2 - 2x + 5$ cannot be further factorized.

Let $\frac{9x^2-12x-2}{(2x+1)(x^2-2x+5)} = \frac{A}{2x+1} + \frac{Bx+C}{x^2-2x+5}$

Multiplying both sides by $(2x+1)(x^2-2x+5)$, we get

$$9x^2 - 12x - 2 = A(x^2 - 2x + 5) + (Bx + C)(2x + 1)$$

Put $x = -\frac{1}{2}$: $9\left(-\frac{1}{2}\right)^2 - 12\left(-\frac{1}{2}\right) - 2 = A\left[\left(-\frac{1}{2}\right)^2 - 2\left(-\frac{1}{2}\right) + 5\right] + 0$

$$\Rightarrow \frac{25}{4} = \frac{25}{4}A \Rightarrow A = 1$$

Equating coefficients of x^2: $9 = A + 2B \Rightarrow 9 = 1 + 2B \Rightarrow B = 4$

Equating constant term: $-2 = 5A + C \Rightarrow -2 = 5(1) + C \Rightarrow C = -7$

$$\therefore \frac{9x^2-12x-2}{(2x+1)(x^2-2x+5)} = \frac{1}{2x+1} + \frac{4x-7}{x^2-2x+5}$$

Example 16 (Repeated Quadratic Factors) – *It's a bit complicated!*

Express $\frac{8x-1}{(x+1)(x^2+2)^2}$ into partial fractions.

Solution

Note that $\frac{8x-1}{(x+1)(x^2+2)^2}$ is a proper rational function, and also x^2+2 cannot be further factorized.

Let

$$\frac{8x-1}{(x+1)(x^2+2)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2} + \frac{Dx+E}{(x^2+2)^2}.$$

Multiplying both sides by $(x+1)(x^2+2)^2$, we get

$$8x-1 = A(x^2+2)^2 + (Bx+C)(x+1)(x^2+2) + (Dx+E)(x+1) \quad \text{......(*)}$$

Put $x=-1$: $-9 = 9A \quad \Rightarrow \quad A=-1$

Equating coefficients of x^4: $0 = A+B \quad \Rightarrow \quad B=-A=1$

Substitute $A=-1$ and $B=1$ into (*):

$$\begin{aligned} 8x-1 &= -(x^2+2)^2 + (x+C)(x+1)(x^2+2) + (Dx+E)(x+1) \\ &= -(x^2+2)^2 + x(x+1)(x^2+2) + C(x+1)(x^2+2) + (Dx+E)(x+1) \end{aligned}$$

$$\Rightarrow \underbrace{8x-1+(x^2+2)^2 - x(x+1)(x^2+2)}_{\substack{=8x-1+(x^2+2)[x^2+2-x(x+1)] \\ =8x-1+(x^2+2)[x^2+2-x^2-x] \\ =8x-1+(x^2+2)(2-x) \\ =8x-1+(2x^2-x^3+4-2x) \\ =-x^3+2x^2+6x+3}} = C(x+1)(x^2+2) + (Dx+E)(x+1)$$

Since $(x+1)$ is a factor of the RHS, it must be a factor of the LHS as well. Using long division to divide the LHS $(-x^3+2x^2+6x+3)$ by $(x+1)$, we get

$$\begin{array}{r} -x^2+3x+3 \\ x+1 \overline{) -x^3+2x^2+6x+3} \\ \underline{-x^3-x^2 \quad\quad\quad} \\ 3x^2+6x \quad \\ \underline{3x^2+3x \quad} \\ 3x+3 \\ \underline{3x+3} \end{array}$$

$$LHS = -x^3+2x^2+6x+3 = (x+1)(-x^2+3x+3)$$

Hence, $(x+1)(-x^2+3x+3) = C(x+1)(x^2+2) + (Dx+E)(x+1)$

$$\Rightarrow -x^2+3x+3 = C(x^2+2) + (Dx+E)$$

Equating coefficients of x^2: $-1 = C \quad \Rightarrow \quad C = -1$

Equating coefficients of x: $3 = D \quad \Rightarrow \quad D = 3$

Equating constant term: $3 = 2C + E \quad \Rightarrow \quad E = 3 - 2\mathrm{C} = 3 - 2(-1) = 5$

Hence, $\frac{8x-1}{(x+1)(x^2+2)^2} = \frac{-1}{x+1} + \frac{x-1}{x^2+2} + \frac{3x+5}{(x^2+2)^2}$

Class Exercise

Express $\frac{5x^3-16x^2+14x-11}{(x-1)^2\,(x^2+3)}$ in partial fractions.