CITY UNIVERSITY OF HONG KONG

Course code and title	:	MA1200 Calculus and Basic Linear Algebra I
Session	:	Semester A, 2017/2018
Time allowed	:	Three hours

This paper has **SEVEN** pages (including this cover page).

A brief table of derivatives is attached on pages 6 and 7.

Instructions to candidates:

1. This paper has **EIGHT** questions.
2. Attempt **ALL** questions.
3. Each question carries 13 marks.

*This is a **closed-book** examination.*

Candidates are allowed to use the following materials/aids:

Non-programmable calculators

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorised materials or aids are found on them.

Question 1

Differentiate with respect to x :

(a) $3x^2 - 5\sqrt{x} + 1$; (2 marks)

(b) $\dfrac{2x+7}{x^2+1}$; (2 marks)

(c) $\log_{10}\left(1+\cos^2 x\right)$; (3 marks)

(d) $\sin^{-1} x + \cos^{-1} x$; (3 marks)

(e) $\left(1+x^2\right)^{-\frac{1}{2}} \cosh^{-1} x$. (3 marks)

Question 2

(a) A curve has parametric equations

$$x = 5\cos\theta \ ,$$
$$y = 3\sin\theta \ ,$$

where θ is the parameter and $0 \le \theta < 2\pi$.

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of θ . (6 marks)

(b) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when

(i) $y = x^{x^2} + 2^x$, (4 marks)

(ii) $x^2 - 4y^2 - 2x - 8y - 7 = 0$. (3 marks)

You need not simplify your answers.

Question 3

(a) Evaluate the limit

$$\lim_{x\to 0}\frac{1+2x-e^{\sin 2x}}{x^2} \quad .$$ (4 marks)

(b) Prove from first principles that

(i) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\log_{10} x)=\dfrac{\log_{10} e}{x}$, $x>0$. (5 marks)

(Hint: You may use the definition for e as $e=\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$

$=\lim\limits_{h\to 0}(1+h)^{\frac{1}{h}}$.)

(ii) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right)=e^x$. (4 marks)

(Hint: you may use the exponential theorem $e^x=1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots+\dfrac{x^r}{r!}+\ldots$ for all real values of x .)

Question 4

(a) If $A+B+C=180°$, show that

$$\frac{\sin(2A)+\sin(2B)+\sin(2C)}{\sin A+\sin B+\sin C}=8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \quad .$$ (7 marks)

(Hint: You may use the identities

$\sin x+\sin y=2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$,

$\sin x-\sin y=2\cos\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$,

$\cos x+\cos y=2\cos\dfrac{x+y}{2}\cos\dfrac{x-y}{2}$,

$\cos x-\cos y=-2\sin\dfrac{x+y}{2}\sin\dfrac{x-y}{2}$,

$\sin(180°-\theta)=\sin\theta$,

$\cos(180°-\theta)=-\cos\theta$,

$\sin(90°-\theta)=\cos\theta$,

$\cos(90°-\theta)=\sin\theta$,

$\sin(2\theta)=2\sin\theta\cos\theta$.)

(b) Find, in radians, the general solution of the equation $\sin(2x)+\cos(2x)+1=0$.

(6 marks)

Question 5

(a) The functions $f(x)$ and $g(x)$ are defined on the domain $[-1, 1]$ as follows:

$f(x) = \sin^{-1} x$;

$g(x) = \cos^{-1} x$.

(i) In each function state the largest possible range of the function. (4 marks)

(ii) Sketch separately the graphs of the curves $y = f(x)$ and $y = g(x)$. (4 marks)

(b) Let $F(x) = (x-1)^2 - 3$ for $x \in [1, \infty)$. Sketch its graph.
Find the inverse of $F(x)$ and state its largest possible domain. (5 marks)

Question 6

(a) If $y = (ax+b)^{-p}$, where a and p are positive integers, b is a constant, find the general formula for the nth derivative of y with respect to x. (3 marks)

(b) Express $\dfrac{5x^2+10x+10}{(3x+7)(x+4)^2}$ in partial fractions. (6 marks)

(c) Hence, of otherwise, find the derivative $\dfrac{\mathrm{d}^4}{\mathrm{d}x^4}\left(\dfrac{5x^2+10x+10}{(3x+7)(x+4)^2}\right)$.

You need not simplify your answer. (4 marks)

Question 7

(a) Find the equation of the tangent to the ellipse, $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$ at the point $\left(\dfrac{5}{2}, \dfrac{3\sqrt{3}}{2}\right)$.
(5 marks)

(b) A right circular cone has base radius x units and height h units. As x and h are varied, its curved surface area, $S = \pi\, x\left(x^2+h^2\right)^{\frac{1}{2}}$ units^2 , is kept constant. Show that its volume, $V = \dfrac{1}{3}\pi\, x^2 h$ units^3 , is a maximum when $h = \sqrt{2}x$. (8 marks)

Question 8

(a) Find the Maclaurin series of $\cos x$ in ascending powers of x, up to and including the term in x^6. (5 marks)

(Hint: Maclaurin's series for $f(x)$ is $f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(r)}(0)}{r!}x^r+\ldots$)

(b) For any positive integer n, the Legendre polynomial of the first kind, $P_n(x)$ is defined by $P_n(x)=\frac{1}{n!2^n}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\left(x^2-1\right)^n\right]$ for $x\in[-1,1]$.

(i) Find $P_1(x)$, $P_2(x)$ and $P_3(x)$. (3 marks)

(ii) If $w=\frac{1}{n!2^n}\left(x^2-1\right)^n$, show that $\left(1-x^2\right)\frac{\mathrm{d}w}{\mathrm{d}x}+2nxw=0$. ----- (1) (2 marks)

(iii) Using Leibnitz' rule, show that $y=P_n(x)$ satisfies the differential equation
$\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2x\frac{\mathrm{d}y}{\mathrm{d}x}+n(n+1)y=0$. (3 marks)

(Hint: Leibnitz' rule: For any functions u and v whose derivatives up to the nth order exist, $(uv)^{(n)}={}_nC_0u^{(n)}v^{(0)}+{}_nC_1u^{(n-1)}v^{(1)}+{}_nC_2u^{(n-2)}v^{(2)}+\ldots+{}_nC_ru^{(n-r)}v^{(r)}+\ldots+{}_nC_nu^{(0)}v^{(n)}$, where

${}_nC_r=\frac{n!}{(n-r)!\,r!}$, $u^{(0)}=u$, $v^{(0)}=v$ and $u^{(r)}$, $v^{(r)}$ are the rth derivatives of u and v, respectively, for $r=1,2,3,\ldots,n$.)

Short Table of Derivatives of $y=f(u)$ with respect to x, where u is a function of x

Functions, $y=f(u)$	**Derivative of y with respect to x**
$y=c$, where c is a constant.	$\frac{dy}{dx}=0$
$y=cu$, where c is a constant.	$\frac{dy}{dx}=c\frac{du}{dx}$
$y=u^p$, where p is a constant.	$\frac{dy}{dx}=pu^{p-1}\frac{du}{dx}$
$y=u+v$	$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$
$y=uv$	$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}$
$y=\frac{u}{v}$	$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
$y=f(u)$, where u is a function of x.	$\frac{dy}{dx}=\frac{d\,f(u)}{du}\cdot\frac{du}{dx}$, the chain rule
$y=\log_a u$, $a>0$.	$\frac{dy}{dx}=\frac{1}{u}\log_a e\frac{du}{dx}$
$y=a^u$, $a>0$.	$\frac{dy}{dx}=a^u\log_e a\frac{du}{dx}$
$y=e^u$	$\frac{dy}{dx}=e^u\frac{du}{dx}$
$y=u^v$	$\frac{dy}{dx}=vu^{v-1}\frac{du}{dx}+u^v\log_e u\frac{dv}{dx}$
$y=\sin u$	$\frac{dy}{dx}=\cos u\frac{du}{dx}$
$y=\cos u$	$\frac{dy}{dx}=-\sin u\frac{du}{dx}$
$y=\tan u$	$\frac{dy}{dx}=\sec^2 u\frac{du}{dx}$
$y=\cot u$	$\frac{dy}{dx}=-\mathrm{cosec}^2 u\frac{du}{dx}$
$y=\sec u$	$\frac{dy}{dx}=\sec u\tan u\frac{du}{dx}$
$y=\mathrm{cosec}\, u$	$\frac{dy}{dx}=-\mathrm{cosec}\, u\cot u\frac{du}{dx}$
$y=\sin^{-1}u$	$\frac{dy}{dx}=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y=\cos^{-1}u$	$\frac{dy}{dx}=\frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$
$y=\tan^{-1}u$	$\frac{dy}{dx}=\frac{1}{1+u^2}\frac{du}{dx}$

Functions, $y = f(u)$	**Derivative of y with respect to x**
$y = \cot^{-1} u$	$\dfrac{dy}{dx} = -\dfrac{1}{1+u^2}\dfrac{du}{dx}$
$y = \sec^{-1} u$	$\dfrac{dy}{dx} = \dfrac{1}{\lvert u\rvert\sqrt{u^2-1}}\dfrac{du}{dx}$
$y = \operatorname{cosec}^{-1} u$	$\dfrac{dy}{dx} = -\dfrac{1}{\lvert u\rvert\sqrt{u^2-1}}\dfrac{du}{dx}$
$y = \sinh u$	$\dfrac{dy}{dx} = \cosh u\dfrac{du}{dx}$
$y = \cosh u$	$\dfrac{dy}{dx} = \sinh u\dfrac{du}{dx}$
$y = \tanh u$	$\dfrac{dy}{dx} = \operatorname{sech}^2 u\dfrac{du}{dx}$
$y = \coth u$	$\dfrac{dy}{dx} = -\operatorname{cosech}^2 u\dfrac{du}{dx}$
$y = \operatorname{sech} u$	$\dfrac{dy}{dx} = -\operatorname{sech} u \tanh u\dfrac{du}{dx}$
$y = \operatorname{cosech} u$	$\dfrac{dy}{dx} = -\operatorname{cosech} u \coth u\dfrac{du}{dx}$
$y = \sinh^{-1} u$	$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+u^2}}\dfrac{du}{dx}$
$y = \cosh^{-1} u$	$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{u^2-1}}\dfrac{du}{dx}$
$y = \tanh^{-1} u$	$\dfrac{dy}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx}$
$y = \coth^{-1} u$	$\dfrac{dy}{dx} = \dfrac{1}{1-u^2}\dfrac{du}{dx}$
$y = \operatorname{sech}^{-1} u$	$\dfrac{dy}{dx} = -\dfrac{1}{u\sqrt{1-u^2}}\dfrac{du}{dx}$
$y = \operatorname{cosech}^{-1} u$	$\dfrac{dy}{dx} = -\dfrac{1}{\lvert u\rvert\sqrt{u^2+1}}\dfrac{du}{dx}$

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