MA1200 Calculus and Basic Linear Algebra I
Chapter 6 Limits, Continuity and Differentiability

1 <u>The concept of limits</u> (p.217 – p.229, p.242 – p.249, p.255 – p.256)

1.1 Limits

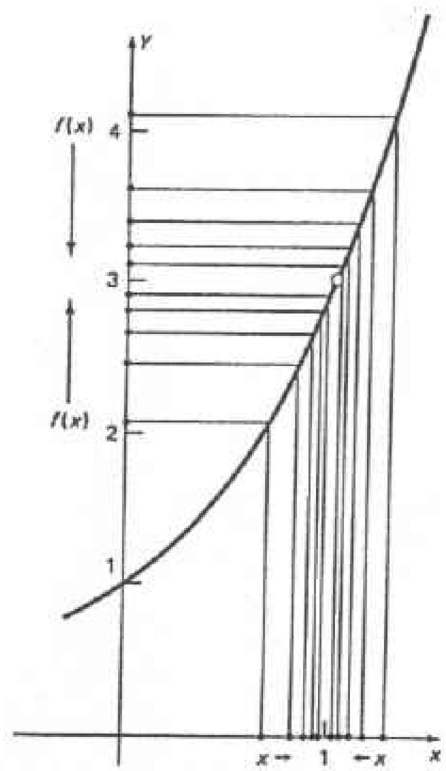
Consider the function determined by the formula

$$f(x)=\frac{x^3-1}{x-1}$$

Note that it is not defined at $x=1$ since at this point $f(x)$ has the form $\frac{0}{0}$, which is meaningless. We can, however, still ask what is happening to $f(x)$ as *x* approaches 1. More precisely, is $f(x)$ approaching some specific number as *x* approaches 1? To answer this question, we do two things: we calculate some values of $f(x)$ for *x* near 1 and sketch the graph of $y=f(x)$ as follows.

x	$y=\frac{x^3-1}{x-1}$
1.25	3.813
1.1	3.310
1.01	3.030
1.001	3.003
↓	↓
1.000	?
↑	↑
0.999	2.997
0.99	2.970
0.9	2.710
0.75	2.313

Table of values



From the figure we see that $f(x)$ approaches 3 as *x* approaches 1. In mathematical symbols, we write

$$\lim_{x\to 1}\frac{x^3-1}{x-1}=3$$

In fact,

$$\lim_{x\to 1}\frac{x^3-1}{x-1}=\lim_{x\to 1}\frac{(x-1)\left(x^2+x+1\right)}{x-1}=\lim_{x\to 1}\left(x^2+x+1\right)=1^2+1+1=3$$

(* Note that $\frac{x-1}{x-1}=1$ as long as $x\neq 1$).

In general, to say that $\lim_{x\to a} f(x)=l$ means that the difference between $f(x)$ and l can be made arbitrarily small by requiring that *x* be *sufficiently close* to but *different from a*.

<u>Example 1</u>

$\lim\limits_{x\to 1}\dfrac{x-1}{\sqrt{x}-1}$ can be calculated as follows:

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}=\lim_{x\to 1}\frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}{\sqrt{x}-1}=\lim_{x\to 1}\left(\sqrt{x}+1\right)=\sqrt{1}+1=2\,.$$

$\square$

<u>Example 2</u>

If n is a positive integer, we have $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$.

It is because, by long division, we have

$$\frac{x^n-a^n}{x-a}=x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ldots+a^{n-1},$$

which tends to na^{n-1} as x tends to a.

$\square$

<u>Example 3</u>

$\lim_{x\to 2}\left[x\right]$ does not exist. It is because

$$\left[x\right]=\begin{cases}1 & \text{if } 1\le x<2\\ 2 & \text{if } 2\le x<3\end{cases}$$

Therefore as x approaches 2 from the left, the values of $\left[x\right]$ approach the number 1. However, if x approaches 2 from the right, the values of $\left[x\right]$ approach the number 2. So limit does not exist in this case.

$\square$

From the example we see that the function $\left[x\right]$ tends to different values as x approaches the number 2 from different sides. In fact, given a function $f\left(x\right)$, we can write

$$\lim_{x\to a^-}f\left(x\right)=l_1 \quad \text{(resp.} \quad \lim_{x\to a^+}f\left(x\right)=l_2\text{).}$$

[read as "the left (resp. right) hand limit of $f\left(x\right)$ is l_1 (resp. l_2)"], if the values of $f\left(x\right)$ tends to l_1 (resp. l_2) as x approaches a from the left (resp. right). By using this notation, we can write

$$\lim_{x\to 2^-}\left[x\right]=1 \quad \text{and} \quad \lim_{x\to 2^+}\left[x\right]=2\,.$$

Finally, we will use the symbol

$$\lim_{x\to +\infty}f\left(x\right)=l$$

to represent the situation that when x is increasing indefinitely, the values of $f\left(x\right)$ tends to the value l, the limit

$$\lim_{x\to -\infty}f\left(x\right)=l$$

is defined similarly.

Example 4

$\lim_{x \to +\infty} \frac{x}{1+x} = 1$. It is because

$$\lim_{x \to +\infty} \frac{x}{1+x} = \lim_{x \to +\infty} \frac{1}{1+\frac{1}{x}} = \frac{1}{1} = 1$$

(* Note that the values of $\frac{1}{x}$ tends to 0 as x tends to positive infinity).

∎

1.2 Theorems on limits

In the previous section, we have introduced the notion of limits. However in practical calculations, it is difficult to apply the definition directly. In this section, a lot of 'tools' will be introduced to help us calculate the limits. The following theorems are typical examples.

Theorem 1A

(Main Limit Theorem) Let n be a positive integer, k be a constant, and f and g be functions which have limits at c. Then

1. $\lim_{x \to c} k = k$;
2. $\lim_{x \to c} x = c$;
3. $\lim_{x \to c} kf(x) = k \lim_{x \to c} f(x)$;
4. $\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$;
5. $\lim_{x \to c} [f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$;
6. $\lim_{x \to c} [f(x) g(x)] = \lim_{x \to c} f(x) \lim_{x \to c} g(x)$;
7. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$, provided $\lim_{x \to c} g(x) \neq 0$;
8. $\lim_{x \to c} \left[f(x) \right]^n = \left[\lim_{x \to c} f(x) \right]^n$;
9. $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$, provided $\lim_{x \to c} f(x) \geq 0$ when n is even.

Theorem 1B

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ and $g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$ be 2 polynomials as shown. Then we have:

(i) $\lim_{x \to a} f(x) = f(a) = a_n \times a^n + a_{n-1} \times a^{n-1} + \cdots + a_1 \times a + a_0$.

(ii) If $g(a) \neq 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}$.

Example 5

(i) Find $\lim_{t\to 2}\dfrac{t^2+2t+4}{t+2}$.

Solution:

The function whose limit we are to find is the quotient of two polynomials. The denominator $t+2$ is not 0 when $t=2$. Therefore, the limit is the value of the quotient at $t=2$:

$$\lim_{t\to 2}\frac{t^2+2t+4}{t+2}=\frac{(2)^2+2(2)+4}{2+2}=\frac{12}{4}=3$$

(ii) Find $\lim_{t\to 2}\dfrac{t^3-8}{t^2-4}$.

Solution:

The denominator is 0 at $t=2$ and we cannot calculate the limit by direct substitution. However, if we factor the numerator and denominator we find that

$$\frac{t^3-8}{t^2-4}=\frac{(t-2)(t^2+2t+4)}{(t-2)(t+2)}$$

For $t\neq 2$,

$$\frac{t-2}{t-2}=1$$

Therefore, for all values of *t* different from 2 (the values that really determine the limit as $t\to 2$),

$$\lim_{t\to 2}\frac{t^3-8}{t^2-4}=\lim_{t\to 2}\frac{t^2+2t+4}{t+2}=\frac{2^2+2\times 2+4}{2+2}=\frac{12}{4}=3$$

To calculate this limit, we divided the numerator and denominator by a common factor and evaluated the result at $t=2$.

□

Theorem 1C (Sandwich Theorem or Squeeze Theorem)

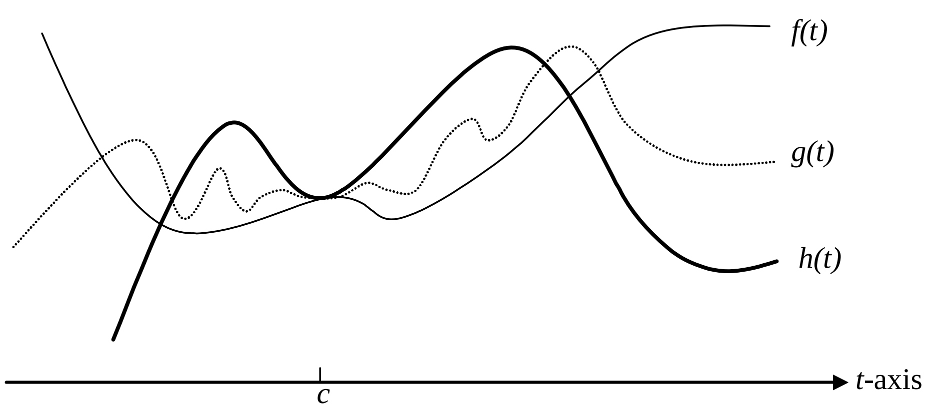
Suppose that

$$f(t)\leq g(t)\leq h(t)$$

for all $t\neq c$ in some interval about *c* and that $f(t)$ and $h(t)$ approach the same limit l as *t* approaches *c* (that is, $\lim_{t\to c} f(t)=l=\lim_{t\to c} h(t)$), then

$$\lim_{t\to c} g(t)=l.$$

The following figure illustrates the idea behind the theorem.



Can you see the sandwich?

Example 6

Show that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

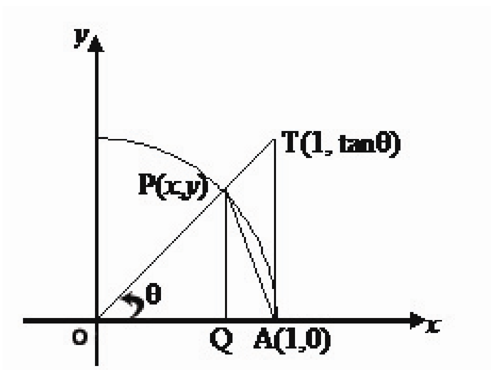
Solution:

If we can show

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}=1=\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta},$$

then we are done.

For the limit $\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}$, we consider the following diagram ($0<\theta<\pi/2$)



We see that

$$\text{Area } \Delta\text{OAP} < \text{Area sector OAP} < \text{Area } \Delta\text{OAT}.$$

This implies

$$\frac{1}{2}\sin\theta<\frac{\theta}{2}<\frac{1}{2}\tan\theta.$$

Further simplification gives

$$\cos\theta<\frac{\sin\theta}{\theta}<1.$$

As $\lim_{\theta\to 0^+}\cos\theta=1$, by sandwich theorem, we have

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}=1.$$

This is a right hand limit since we only consider $0<\theta<\pi/2$. For $\theta=-\alpha$ where α is positive, we see that

$$\frac{\sin\theta}{\theta}=\frac{\sin(-\alpha)}{-\alpha}=\frac{-\sin\alpha}{-\alpha}=\frac{\sin\alpha}{\alpha}.$$

Hence we have

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta}=\lim_{\alpha\to 0^+}\frac{\sin\alpha}{\alpha}=1.$$

Thus we can conclude that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1.$$

Example 7

Does $\lim\limits_{x\to 0}\dfrac{\sin x}{|x|}$ exist?

Solution:

Observe that $\dfrac{\sin x}{|x|}=\begin{cases}\dfrac{\sin x}{x} & x>0\\ \dfrac{\sin x}{-x} & x<0\end{cases}$. To the right of 0 and to the left of 0, the formulas are different so we have to consider $\lim\limits_{x\to 0^+}\dfrac{\sin x}{x}$, $\lim\limits_{x\to 0^-}\dfrac{\sin x}{-x}$, separately. But $\lim\limits_{x\to 0^+}\dfrac{\sin x}{x}=1$, $\lim\limits_{x\to 0^-}\dfrac{\sin x}{-x}=-\lim\limits_{x\to 0^-}\dfrac{\sin x}{x}=-1$.

We conclude that $\lim\limits_{x\to 0}\dfrac{\sin x}{|x|}$ does not exist.

□

Example 8

Find the limit $$\lim_{x\to 0}\frac{2\sin x\cos x}{x}$$

Solution:

First, $$\frac{2\sin x\cos x}{x}=\frac{2\sin 2x}{2x}.$$

Let $t=2x$. Observe that t tends to 0 as x tends to 0.

Therefore $$\lim_{x\to 0}\frac{2\sin x\cos x}{x}=\lim_{x\to 0}\frac{2\sin 2x}{2x}=2\lim_{t\to 0}\frac{\sin t}{t}=2.$$

□

Example 9

Evaluate $\lim\limits_{x\to 0}\dfrac{(\sin 3x)^2}{x^2\cos x}$

Solution:

$$\lim_{x\to 0}\frac{(\sin 3x)^2}{x^2\cos x}=\lim_{x\to 0}\left(\frac{\sin 3x}{x}\frac{\sin 3x}{x}\frac{1}{\cos x}\right)=\lim_{x\to 0}\left(\frac{3\sin 3x}{3x}\frac{3\sin 3x}{3x}\frac{1}{\cos x}\right)$$
$$=\left(\lim_{x\to 0}\frac{3\sin 3x}{3x}\right)\left(\lim_{x\to 0}\frac{3\sin 3x}{3x}\right)\left(\lim_{x\to 0}\frac{1}{\cos x}\right)=3\left(\lim_{3x\to 0}\frac{\sin 3x}{3x}\right)3\left(\lim_{3x\to 0}\frac{\sin 3x}{3x}\right)\left(\lim_{x\to 0}\frac{1}{\cos x}\right)$$
$$=3\times 1\times 3\times 1\times 1=9.$$

□

Example 10

Evaluate $\lim\limits_{\theta\to 0}\dfrac{\sin\theta^o}{\theta}$, ($\theta^o$ is the value of an angle measured in degree).

Solution:

$\because\ \theta^\circ=\theta\times\dfrac{\pi}{180}$ radians

It follows that

$$\lim_{\theta\to 0}\frac{\sin\theta^o}{\theta}=\lim_{\theta\to 0}\frac{\sin\frac{\pi\theta}{180}}{\theta}=\lim_{\theta\to 0}\frac{\pi}{180}\frac{\sin\frac{\pi\theta}{180}}{\frac{\pi\theta}{180}}=\frac{\pi}{180}\lim_{\theta\to 0}\frac{\sin\frac{\pi\theta}{180}}{\frac{\pi\theta}{180}}\underset{\frac{\pi\theta}{180}=t}{=}\frac{\pi}{180}\lim_{t\to 0}\frac{\sin t}{t}=\frac{\pi}{180}.$$

□

Example 11

Evaluate $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x\sin x}$.

Solution:

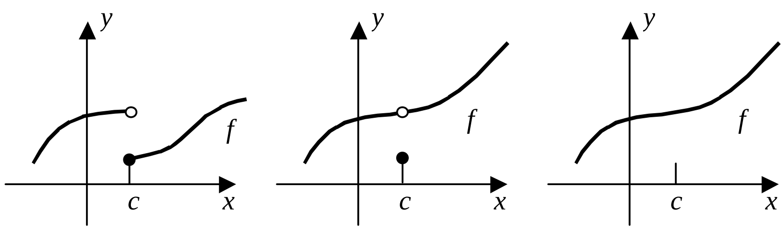
$$\lim_{x\to 0}\frac{1-\cos x}{x\sin x}=\lim_{x\to 0}\frac{1-\cos 2\frac{x}{2}}{x\sin 2\frac{x}{2}}=\lim_{x\to 0}\frac{1-\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}{2x\sin\frac{x}{2}\cos\frac{x}{2}}=\lim_{x\to 0}\frac{1-\left(1-\sin^2\frac{x}{2}\right)+\sin^2\frac{x}{2}}{2x\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{2x\sin\frac{x}{2}\cos\frac{x}{2}}=\lim_{x\to 0}\frac{\sin\frac{x}{2}}{2\frac{x}{2}}\frac{1}{\cos\frac{x}{2}}=\frac{1}{2}\left(\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)\left(\lim_{x\to 0}\frac{1}{\cos\frac{x}{2}}\right)=\frac{1}{2}\left(\lim_{t\to 0}\frac{\sin t}{t}\right)\left(\lim_{t\to 0}\frac{1}{\cos t}\right)=\frac{1}{2}$$

□

2 Continuity of functions (p.264 – p.272)

In ordinary language, we use the word continuous to describe a process that goes on without abrupt changes. It is this notion as it pertains to functions that we now want to make precise. Consider the three graphs shown in the following figure. Only the third graph exhibits continuity at *c*. Here is the formal definition.



Definition (Continuity at a point)

Let f be defined on an open interval containing *c*. We say that f is continuous at c if

$$\lim_{x\to c}f(x)=f(c)$$

By this definition we require three things:

(1) $\lim_{x\to c}f(x)$ exists,

(2) $f(c)$ exists (that is, *c* is in the domain of f), and

(3) $\lim_{x\to c}f(x)=f(c)$.

If any of these three fails, then f is discontinuous at *c*.

Example 12

The function

$$f(x)=\frac{x^2-3x-10}{x-5}$$

is undefined at $x=5$. However, if we define

$$g(x)=\begin{cases}\dfrac{x^2-3x-10}{x-5} & \text{if } x\neq 5\\ 7 & \text{if } x=5\end{cases}$$

then $g(x)$ will be continuous at $x=5$.

It is because

$$\lim_{x\to 5} g(x)=\lim_{x\to 5}\frac{x^2-3x-10}{x-5}=\lim_{x\to 5}\frac{(x-5)(x+2)}{x-5}=\lim_{x\to 5}(x+2)=7=g(5).$$

Thus $g(x)$ becomes continuous at $x=5$.

□

Most of the functions that we here encountered are continuous. To name a few, we have

- All polynomials are continuous at every real number *c*.
- The absolute function is continuous at every real number *c*.
- $\sin x$ and $\cos x$ are continuous at every real number *c*.
- A rational function

$$\frac{f(x)}{g(x)}$$

is continuous at every real number c with $g(c)\neq 0$ if $f(x)$ and $g(x)$ are both continuous at c.

The following theorem will also help us to identify those continuous functions.

Theorem 2A

If f and g are continuous at c, then so are kf, $f+g$, $f-g$, fg, f/g $\left(g(c)\neq 0\right)$ and f^n, where k is any real number and n is any positive integer.

Theorem 2B

If $\lim\limits_{x\to c} g(x)=l$ and if f is continuous at l, then

$$\lim_{x\to c} f\left(g(x)\right)=f\left(\lim_{x\to c} g(x)\right)=f(l)$$

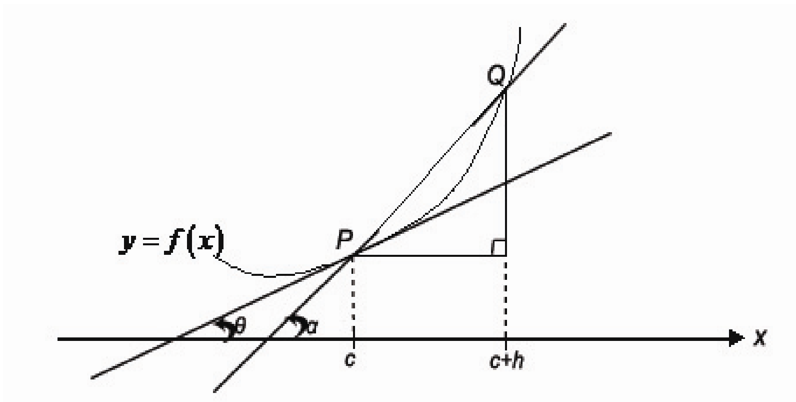
In particular, if g is continuous at c and f is continuous at $g(c)$, then the composite $f\circ g$ is continuous at c.

So far, we have discussed continuity at a point. We can discuss continuity on an interval. We say f is continuous on an open interval if it is continuous at each point of that interval. It is continuous on the closed interval $[a,b]$ if it is continuous on (a,b), right continuous at a (that is, $\lim\limits_{x\to a^+} f(x)=f(a)$), and left continuous at b (that is, $\lim\limits_{x\to b^-} f(x)=f(b)$).

Continuous functions have a lot of nice properties. One of them is:

Theorem 2C (Intermediate Value Theorem)
If f is continuous on $[a,b]$ and if l is a number between $f(a)$ and $f(b)$, then there is a number c between a and b such that $f(c)=l$.
This means that a continuous function must assume every value between any two of its values.

3 Differentiability of functions (p.274 – p.279, p.282 – p.290)
Consider the graph of the function $y=f(x)$ as shown below:



The points P and Q have coordinates $(c, f(c))$ and $(c+h, f(c+h))$ respectively. The slope of the secant line PQ is given by

$$\text{slope of } PQ = \frac{f(c+h)-f(c)}{h}$$

As h tends to 0, we have the slope of the tangent at P is given by

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} = \tan\theta,$$

if such a limit exists. This leads to the concept of differentiability. A function f is said to be differentiable at $x=c$ if the limit

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} \quad \left(\text{or } \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}\right)$$

exists. If so, it is called the derivative of $f(x)$ at c and is denoted by $f'(c)$. Again, we say that f is differentiable in an open interval I if it is differentiable at each point of I.

Example 13
If $f(x)=\frac{1}{x}$, then

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\lim_{h\to 0}\left[\frac{x-(x+h)}{(x+h)x}\frac{1}{h}\right]$$

$$=\lim_{h\to 0}\left[\frac{-h}{(x+h)x}\frac{1}{h}\right]=\lim_{h\to 0}\frac{-1}{(x+h)x}=\frac{-1}{x^2}$$

So the function is differentiable at every real number x except at $x=0$.

Theorem 3A

If f is differentiable at c, then f is continuous at c. The above theorem describes the relationship between continuity and differentiability of a function. It should be noted that a continuous function may not be differentiable.

Example 14

Show that $f(x) = |x|, x \in \mathbf{R}$ is not differentiable at $x = 0$. (Note that $f(x)$ is continuous for all $x \in \mathbf{R}$).

Solution:

$$f'(0) = \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

This limit depends on the sign of *h*.

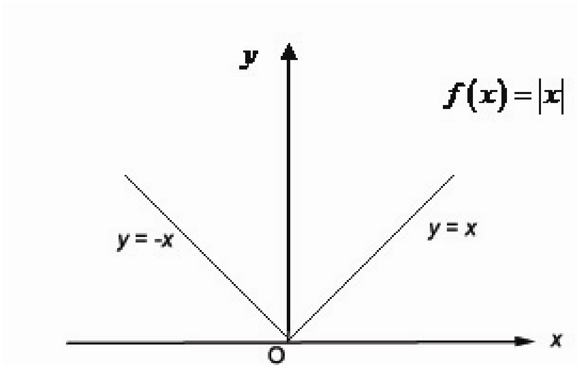
$$\lim_{\substack{h \to 0^+ \\ (h>0)}} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

$$\lim_{\substack{h \to 0^- \\ (h<0)}} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

Hence,

$$\lim_{h \to 0^+} \frac{|h|}{h} \neq \lim_{h \to 0^-} \frac{|h|}{h}.$$

$$\Rightarrow \text{ The limit } \lim_{h \to 0} \frac{|h|}{h} \text{ does not exist.}$$



Therefore, $f(x)$ is not differentiable at $x = 0$. The geometric interpretation of this can be seen from the graph of $y = f(x) = |x|$.

For $x > 0$, the graph is the line $y = x$ with slope 1.

For $x < 0$, the graph is the line $y = -x$ with slope –1.

At point O, where these two sections of the graph meet, the graph has no well-defined slope. That is, the derivative is not defined at $x = 0$.

□

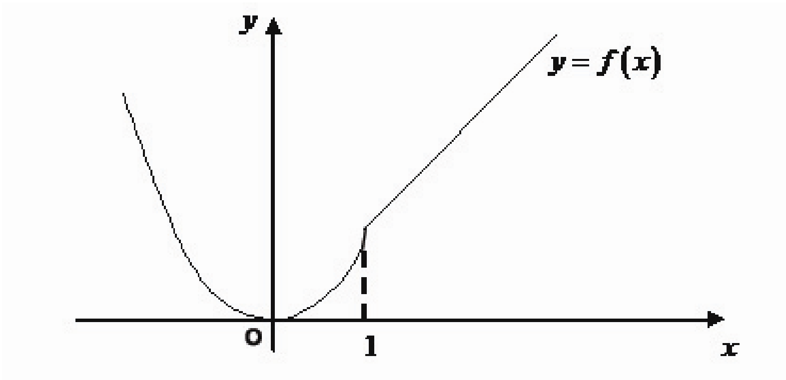
Example 15

Let

$$f(x)=\begin{cases} x & \text{for} \quad x \ge 1 \\ x^2 & \text{for} \quad x < 1. \end{cases}$$

Show that $f(x)$ is not differentiable at $x=1$.

Proof:



$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^+}\frac{(1+h)-1}{h}=1$$

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{(1+h)^2-1}{h}=\lim_{h\to 0^-}\frac{1+2h+h^2-1}{h}=\lim_{h\to 0^-}(2+h)=2$$

$$\Rightarrow \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} \text{ does not exist.}$$

$\therefore$ $f(x)$ is not differentiable at $x=1$.

$\square$