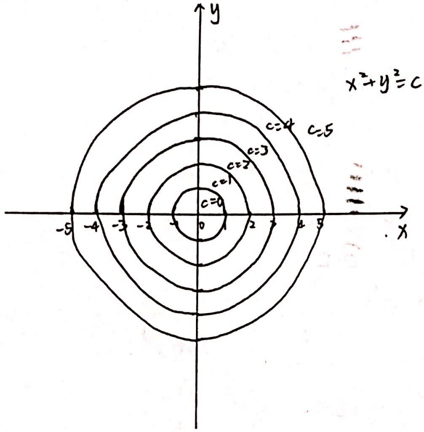
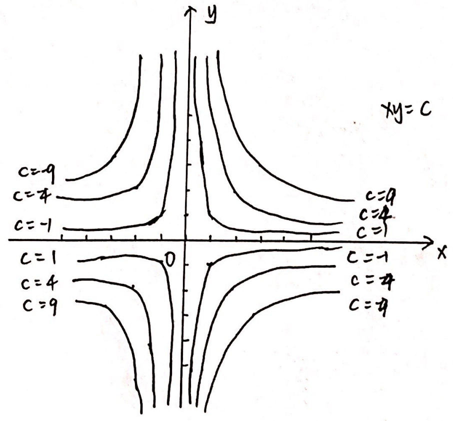
Take Home Assignment 2

1. (a) $f(x,y) = x^2 + y^2$, $c = 0, 1, 4, 9, 16, 25$



(b)



2. (a) For any fixed $x \neq 0$

$$\frac{\partial f}{\partial y}(x,0) = \lim_{h\to 0} \frac{f(x,h) - f(x,0)}{h}$$

$$= \lim_{h\to 0} \frac{xh \cdot \frac{x^2-h^2}{x^2+h^2} - 0}{h}$$

$$= \lim_{h\to 0} x \cdot \frac{x^2-h^2}{x^2+h^2} = x \cdot \lim_{h\to 0} \frac{x^2-h^2}{x^2+h^2} = x \cdot 1 = x$$

For $x = 0$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

Above all, $\frac{\partial f}{\partial y}(x,0) = x$ for any x.

Similarly, for $\frac{\partial f}{\partial x}(0,y) = -y$.

(b) $$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0)$$

$$= \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}(0,y)\right]\Big|_{y=0} = \frac{\partial}{\partial y}(-y)\Big|_{y=0} = -1$$

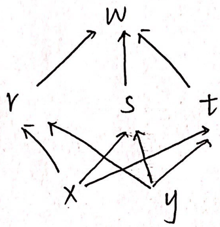
$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0)$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}(x,0)\right]\Big|_{x=0} = \frac{\partial}{\partial x}(x)\Big|_{x=0} = 1$$

Hence, $\frac{\partial^2 f}{\partial y \partial x}(0,0) \neq \frac{\partial^2 f}{\partial x \partial y}(0,0)$

3. Since $w = f(r, s, t)$, with $r = g(x, y)$, $s = h(x, y)$, $t = k(x, y)$

Diagram:



$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial w}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial w}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial w}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial w}{\partial t} \cdot \frac{\partial t}{\partial y}$$

4. Since $u = \frac{x^2 - y^2}{2}$, $v = xy$, we have

$$\frac{\partial u}{\partial x} = x \quad \frac{\partial u}{\partial y} = -y \quad \frac{\partial v}{\partial x} = y \quad \frac{\partial v}{\partial y} = x$$

$$W_x = \frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = f_u \cdot x + f_v \cdot y$$

$$W_{xx} = f_u + x\left(f_{uu} \cdot u_x + f_{uv} \cdot v_x\right) + y\left(f_{vu} \cdot u_x + f_{vv} \cdot v_x\right)$$

$$= f_u + x^2 f_{uu} + xy f_{uv} + xy f_{vu} + y^2 f_{vv} \qquad ①$$

$$W_y = \frac{\partial w}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} = f_u \cdot (-y) + f_v \cdot x$$

$$W_{yy} = -f_u - y\left(f_{uu} \cdot u_y + f_{uv} \cdot v_y\right) + x\left(f_{vu} \cdot u_y + f_{vv} \cdot v_y\right)$$

$$= -f_u + y^2 f_{uu} - xy f_{uv} - xy f_{vu} + x^2 f_{vv} \qquad ②$$

By adding ① & ②, we get

$$W_{xx} + W_{yy} = f_u + x^2 f_{uu} + xy f_{uv} + xy f_{vu} + y^2 f_{vv} - f_u$$
$$+ y^2 f_{uu} - xy f_{uv} - xy f_{vu} + x^2 f_{vv}$$
$$= (x^2 + y^2)(f_{uu} + f_{vv})$$

Since $f_{uu} + f_{vv} = 0$, we have $W_{xx} + W_{yy} = 0$

5. $xe^y + \sin(xy) + y - \ln 2 = 0$

Differentiate with respect to x at both sides.

$$\frac{d}{dx}(xe^y) + \frac{d}{dx}(\sin(xy)) + \frac{dy}{dx} - \frac{d(\ln 2)}{dx} = 0$$

$$e^y + xe^y \cdot \frac{dy}{dx} + \cos(xy) \cdot \frac{d}{dx}(xy) + \frac{dy}{dx} = 0$$

$$e^y + xe^y \cdot \frac{dy}{dx} + \cos(xy)(y + x\frac{dy}{dx}) + \frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{e^y + y\cos(xy)}{xe^y + x\cos(xy) + 1} \qquad ①$$

We put $x=0$, $y=\ln 2$ into ①, we get

$$\frac{dy}{dx} = -\frac{2+\ln 2}{1} = -2-\ln 2$$

6.1(a) $f(2,2) = \frac{1}{2}\cdot 2^2 + 2\cdot 2 + \frac{1}{4}\cdot 2^2 + 3\cdot 2 - 3\cdot 2 + 4$

$= 11$

$f_x(2,2) = (x+y+3)\Big|_{(x,y)=(2,2)} = 7$

$f_y(2,2) = (x+\frac{1}{2}y-3)\Big|_{(x,y)=(2,2)} = 0$

The linearization $L(x,y)$ of the function at $P_0(2,2)$ is

$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$= 11 + 7(x-2) + 0(y-2)$

$= 7x - 3$

(b) $f_{xx} = 1$

$f_{yy} = \frac{1}{2}$

$f_{xy} = 1$

An upper bound for the second order partial derivatives in the region is $M=1$. Thus

$|E| \le \frac{1}{2} M (|x-x_0| + |y-y_0|)^2$

$= \frac{1}{2}\cdot 1\cdot (0.1+0.1)^2$

$= 0.02$

7. (a) $f(x,y) = x^2 - 2xy + 2y^2 - 2x + 2y + 1$

$f_x = 2x - 2y - 2$

$f_y = -2x + 4y + 2$

For critical point, we can put $f_x = f_y = 0$, we get

$$\begin{cases} 2x - 2y - 2 = 0 \\ -2x + 4y + 2 = 0 \end{cases} \Rightarrow x = 1,\ y = 0$$

The critical point is $(1,0)$

$f_{xx} = 2$

$f_{yy} = 4$

$f_{xy} = -2$

$D = f_{xx} \cdot f_{yy} - f_{xy}^2 = 2 \times 4 - (-2)^2 = 4 > 0$

Since $f_{xx} > 0$ and $D(1,0) > 0$, f has a local minimum at $(1,0)$

and minimum value is 0.

7. (b). $f(x,y) = e^y - ye^x$

$f_x = -ye^x$

$f_y = e^y - e^x$

For critical point, we can put $f_x = f_y = 0$, we get

$$\begin{cases} -ye^x = 0 \\ e^y - e^x = 0 \end{cases} \Rightarrow x = 0, y = 0$$

The critical point is $(0,0)$

$f_{xx} = -ye^x = 0$

$f_{yy} = e^y = 1$

$f_{xy} = -e^x = -1$

$f_{xx} \cdot f_{yy} - f_{xy}^2 = 0 \cdot 1 - (-1)^2 = -1 < 0$

Since $D(0,0) < 0$, f has a saddle point $(0,0)$ and the value of f at this point is $f(0,0) = 1$.

8. $T(x,y) = x^2 + xy + y^2 - 6x$

$T_x = 2x + y - 6$

$T_y = x + 2y$

For critical point, we can put $T_x = T_y = 0$, we get

$T_x = 2x + y - 6 = 0$

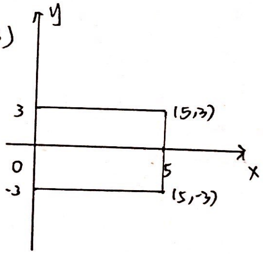
$T_y = x + 2y = 0 \quad \Rightarrow \quad x = 4, \ y = -2$

$T(4,-2) = 4^2 + 4\cdot(-2) + (-2)^2 - 6\times 4 = -12$

Corner points: $(0,3), (0,-3), (5,3), (5,-3)$

$\Rightarrow T(0,3) = 9 \quad T(0,-3) = 9$

$T(5,3) = 19 \quad T(5,-3) = -11$



Boundary points:

(i) Put $x = 0$

$T(0,y) = y^2 \quad T_y = 2y = 0 \Rightarrow y = 0$

$T(0,0) = 0$

(ii) Put $x = 5$

$T(5,y) = 5^2 + 5y + y^2 - 30 = y^2 + 5y - 5$

$T_y = 0 \Rightarrow \quad 2y + 5 = 0 \quad \Rightarrow \quad y = -\frac{5}{2}$

$T(5, -\frac{5}{2}) = -\frac{45}{4}$

(iii) Put $y = 3$

$T(x,3) = x^2 + 3x + 9 - 6x = x^2 - 3x + 9$

$T_x = 0 \Rightarrow \quad 2x - 3 = 0 \quad \Rightarrow \quad x = \cancel{\frac{5}{2}}\ \frac{3}{2}$

$T(\frac{3}{2}, 3) = \frac{27}{4}$

(iv) Put $y=-3$

$$T(x,-3) = x^2-3x+(-3)^2-6x = x^2-9x+9$$

$$T_x=0 \Rightarrow 2x-9=0 \Rightarrow x=\frac{9}{2}$$

$$T(\tfrac{9}{2},-3) = -\frac{45}{4}$$

Above all, absolute maxima is 19 occur at $(5,3)$

absolute minima is -12 occur at $(4,-2)$

9. Consider $f(x,y) = \frac{x-y}{x+y}$

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

$$= \left[\frac{(x+y)-(x-y)}{(x+y)^2}\right]\vec{i} + \left[\frac{(x+y)(-1)-(x-y)}{(x+y)^2}\right]\vec{j}$$

$$= \left[\frac{2y}{(x+y)^2}\right]\vec{i} - \left[\frac{2x}{(x+y)^2}\right]\vec{j}$$

$\nabla f(-\frac{1}{2},\frac{3}{2}) = 3\vec{i}+\vec{j}$ $\qquad \vec{u} = (1,2)$ $\quad \|\vec{u}\| = \sqrt{5}$ $\quad \frac{\vec{u}}{\|\vec{u}\|} = (\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$

$D_u f(-\frac{1}{2},\frac{3}{2}) = (3,1)\cdot(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$

$= \sqrt{5}$

(a) Since $\nabla f(-\frac{1}{2},\frac{3}{2}) = (3,1)$,

the unit vector in this direction is $\vec{u}_1 = \frac{1}{\sqrt{3^2+1^2}}(3,1) = (\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}})$

$D_u f(-\frac{1}{2},\frac{3}{2})$ is maximum in direction of unit vector $(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}})$
where $D_u f(-\frac{1}{2},\frac{3}{2}) = (3,1)\cdot(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}) = \sqrt{10}$

(b) $D_u f(-\frac{1}{2},\frac{3}{2})$ is minimum is in direction of unit vector $(\frac{-3}{\sqrt{10}}, -\frac{1}{\sqrt{10}})$
where $D_u f(-\frac{1}{2},\frac{3}{2}) = (3,1)\cdot(\frac{-3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}) = -\sqrt{10}$

(c) Assume $\vec{u} = (a,b)$

$D_u f(-\frac{1}{2},\frac{3}{2}) = \nabla f(-\frac{1}{2},\frac{3}{2})\cdot\vec{u} = (3,1)\cdot(a,b) = 3a+b=0$ ①

$\vec{u} = (a,b) \Rightarrow a^2+b^2=1$ ② $\qquad$ By ① and ②, we get

$\vec{u} = (-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}})$ or $\vec{u} = (\frac{1}{\sqrt{10}}, \frac{-3}{\sqrt{10}})$

(d) Assume $\vec{u} = (a,b)$

$D_u f(-\frac{1}{2},\frac{3}{2}) = \nabla f(-\frac{1}{2},\frac{3}{2})\cdot\vec{u} = (3,1)(a,b) = 3a+b=-2$ ①

$\vec{u} = (a,b) \Rightarrow a^2+b^2=1$ ② $\qquad$ By ① and ②, we get $\vec{u} = (-\frac{6-\sqrt{6}}{10}, \frac{-2+3\sqrt{6}}{10})$ or $(\frac{-6+\sqrt{6}}{10}, \frac{-2-3\sqrt{6}}{10})$