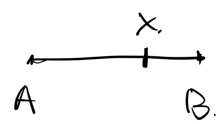
Problem 2

Let $A = (0,1,-1)$ and $B = (1,2,0)$ be two points in a plane. Let X be a point between A and B such that $AX : XB = 2:1$.

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AX}$.

(b) Hence, find the coordinate of X by finding its position vector $\overrightarrow{OX}$. (Hint: $\overrightarrow{AX} = \overrightarrow{OX} - \overrightarrow{OA}$).



$AX : XB = 2:1$.

(a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (\vec{i} + 2\vec{j}) - (\vec{j} - \vec{k})$

$= \vec{i} + \vec{j} + \vec{k}$

$|\overrightarrow{AX}| = \frac{2}{3}|\overrightarrow{AB}|$.

$$\overrightarrow{AX} = \underbrace{\tfrac{2}{3}|\overrightarrow{AB}|}_{\text{magnitude}} \times (\underbrace{\widehat{AB}}_{\text{direction}}) = \frac{2}{3}\cancel{|\overrightarrow{AB}|} \times \frac{\overrightarrow{AB}}{\cancel{|\overrightarrow{AB}|}} = \frac{2}{3}(\vec{i} + \vec{j} + \vec{k})$$

$$= \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}.$$

$$\overrightarrow{AX} = \frac{2}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}$$

(b) $\overrightarrow{AX} = \overrightarrow{OX} - \overrightarrow{OA}$.

$$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX} = (\vec{j} - \vec{k}) + (\tfrac{2}{3}\vec{i} + \tfrac{2}{3}\vec{j} + \tfrac{2}{3}\vec{k})$$

$$= \frac{2}{3}\vec{i} + \frac{5}{3}\vec{j} - \frac{1}{3}\vec{k}.$$

Problem 3

Let $\vec{a} = 2\vec{i} - 3\vec{j} + 5\vec{k}$ and $\vec{b} = \vec{i} + 3\vec{j}$ be two vectors.

(a) Find $|\vec{a}|$ and $|\vec{a} - 2\vec{b}|$.

(b) Find the unit vector of $\vec{b}$.

(c) Let $\vec{c}$ be another vector with magnitude $|2\vec{a} + \vec{b}|$ and its direction is same as that of $\vec{b}$. Find the vector $\vec{c}$.

(a). $|\vec{a}| = \sqrt{2^2 + (-3)^2 + 5^2} = \sqrt{4+9+25} = \sqrt{38}$.

$$\vec{a} - 2\vec{b} = 2\vec{i} - 3\vec{j} + 5\vec{k} - 2(\vec{i} + 3\vec{j})$$

$$= -9\vec{j} + 5\vec{k}$$

$$|\vec{a} - 2\vec{b}| = \sqrt{(-9)^2 + 5^2} = \sqrt{81+25} = \sqrt{106}.$$

(b). $\hat{b} = \dfrac{\vec{b}}{|\vec{b}|} = \dfrac{\vec{i} + 3\vec{j}}{\sqrt{1^2 + 3^2}} = \dfrac{1}{\sqrt{10}}\vec{i} + \dfrac{3}{\sqrt{10}}\vec{j}$.

(c) $\vec{c} = |2\vec{a}+\vec{b}| \times \hat{b} = \sqrt{134} \times \left(\frac{1}{\sqrt{10}}\vec{i} + \frac{3}{\sqrt{10}}\vec{j} \right) = \frac{\sqrt{134}}{\sqrt{10}}\vec{i} + \frac{3\sqrt{134}}{\sqrt{10}}\vec{j}$.

magnitude × direction

$\vec{a} = 2\vec{i} - 3\vec{j} + 5\vec{k}$ and $\vec{b} = \vec{i} + 3\vec{j}$

$2\vec{a}+\vec{b} = 2(2\vec{i}-3\vec{j}+5\vec{k}) + \vec{i} + 3\vec{j}$

$= 5\vec{i} - 3\vec{j} + 10\vec{k}$

$|2\vec{a}+\vec{b}| = \sqrt{5^2+(-3)^2+10^2} = \sqrt{134}$

Problem 4

Let $\vec{a} = \vec{\imath} + 3\vec{\jmath} - 2\vec{k}$ and $\vec{b} = -2\vec{\imath} + \vec{\jmath} + 3\vec{k}$ be two vectors.

(a) Find $\vec{a} \cdot \vec{b}$.

(b) Find the angle between the vectors $\vec{a}$ and $\vec{b}$.

(c) Let $\vec{c} = 3\vec{\imath} + x\vec{\jmath} - 2\vec{k}$ be a vector which is perpendicular to $\vec{b}$, find the value of x.

(d) Let $\vec{d} = y\vec{a} + 3\vec{b}$ be a vector which is perpendicular to $\vec{a} - \vec{b}$, find the value of y.

(a) $\vec{a}\cdot\vec{b} = 1\times(-2) + 3\times 1 + (-2)\times 3 = -2+3-6 = -5$.

(b). $\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\theta \rightarrow \theta$ is angle between $\vec{a}, \vec{b}$.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-5}{\sqrt{1^2+3^2+(-2)^2}\cdot\sqrt{(-2)^2+1^2+3^2}} = \frac{-5}{\sqrt{14}\cdot\sqrt{14}} = \frac{-5}{14}.$$

$\cos\theta < 0 \rightarrow \theta > 90°$.

$\theta \approx 110.92°$.

(c). $\vec{b}\cdot\vec{c} = |\vec{c}||\vec{b}|\cdot\cos 90° = 0$.

$\vec{a} = \vec{\imath} + 3\vec{\jmath} - 2\vec{k}$ and $\vec{b} = -2\vec{\imath} + \vec{\jmath} + 3\vec{k}$

$\vec{b}\cdot\vec{c} = (-2)\cdot 3 + 1\cdot x + 3\cdot(-2) = 0$.

$-6 + x - 6 = 0$

$x = 12$.

(d) $(y\vec{a}+3\vec{b})\cdot(\vec{a}-\vec{b}) = 0$.

$y\vec{a}\cdot\vec{a} - y\vec{a}\cdot\vec{b} + 3\vec{b}\cdot\vec{a} - 3\vec{b}\cdot\vec{b} = 0$

$y(\vec{a}\cdot\vec{a}) + (3-y)(\vec{a}\cdot\vec{b}) - 3(\vec{b}\cdot\vec{b}) = 0$

$$y\,|\vec{a}|^2 + (3-y)(-5) - 3|\vec{b}|^2 = 0.$$

$$14y - 15 + 5y - 3\times 14 = 0.$$

$$19y = 57$$

$$y = 3.$$

Problem 8

Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{a}| = 1$, $|\vec{b}| = 2$ and $\vec{a}\cdot\vec{b} = 1$.

(a) Find the angle between the vectors $\vec{a}$ and $\vec{b}$.

(b) Find the value of $(3\vec{a} - 2\vec{b})\cdot(\vec{a} + 3\vec{b})$ and $|\vec{a} - 2\vec{b}|$.

(c) Find the angle between two vectors $\vec{a} - 2\vec{b}$ and $2\vec{a} + 3\vec{b}$.

(a) $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{1}{1\cdot 2} = \dfrac{1}{2} \Rightarrow \theta = 60^\circ.$

(b). $(3\vec{a} - 2\vec{b})\cdot(\vec{a} + 3\vec{b})$

$= 3(\vec{a}\cdot\vec{a}) + 9(\vec{a}\cdot\vec{b}) - 2(\vec{a}\cdot\vec{b}) - 6(\vec{b}\cdot\vec{b})$

$= 3|\vec{a}|^2 + 7(\vec{a}\cdot\vec{b}) - 6|\vec{b}|^2.$

$= 3\cdot 1 + 7\cdot 1 - 6\cdot 4 = -14.$

$|\vec{a} - 2\vec{b}| = \sqrt{(\vec{a} - 2\vec{b})\cdot(\vec{a} - 2\vec{b})}$

$= \sqrt{(\vec{a}\cdot\vec{a}) - 2(\vec{a}\cdot\vec{b}) - 2(\vec{a}\cdot\vec{b}) + 4(\vec{b}\cdot\vec{b})}$

$= \sqrt{|\vec{a}|^2 - 4(\vec{a}\cdot\vec{b}) + 4|\vec{b}|^2}.$

$= \sqrt{1 - 4 + 4\times 4} = \sqrt{13}.$

$\vec{a} - 2\vec{b}$ and $2\vec{a} + 3\vec{b}$.

(c) $\cos\theta = \dfrac{(\vec{a} - 2\vec{b})\cdot(2\vec{a} + 3\vec{b})}{|\vec{a} - 2\vec{b}|\times|2\vec{a} + 3\vec{b}|}$ $(*)$

$(\vec{a} - 2\vec{b})\cdot(2\vec{a} + 3\vec{b}) = 2|\vec{a}|^2 + 3(\vec{a}\cdot\vec{b}) - 4(\vec{a}\cdot\vec{b}) - 6|\vec{b}|^2$

$= 2 + 3 - 4 - 6\times 4 = -23.$

$|2\vec{a} + 3\vec{b}| = \sqrt{(2\vec{a} + 3\vec{b})\cdot(2\vec{a} + 3\vec{b})}$

$$= \sqrt{4|\vec{a}|^2 + 12(\vec{a}\cdot\vec{b}) + 9|\vec{b}|^2} = \sqrt{4+12+36} = \sqrt{52}$$

$$\cos\theta = \frac{(\vec{a}+2\vec{b})\cdot(2\vec{a}+3\vec{b})}{|\vec{a}-2\vec{b}|\times|2\vec{a}+3\vec{b}|}$$

$$= \frac{-23}{\sqrt{52}\cdot\sqrt{13}} = \frac{-23}{26} \Rightarrow \quad \theta = 152.2^\circ.$$