MA2001 Solutions Assignment for Chapter 1 Vector Differential calculus

1. If $\vec{r}(t) = r\cos(\omega t)\vec{i} + r\sin(\omega t)\vec{j}$ is the position vector of a point at time *t*, $\vec{v}(t)$ is the velocity vector of $\vec{r}(t)$ and $\vec{a}(t)$ is the acceleration vector of $\vec{r}(t)$, show that

   (a) $\vec{r}\cdot\vec{v} = 0$,

   (b) $\vec{r}\times\vec{v} =$ constant vector,

   (c) $\vec{a} = \omega^2\vec{r}$.

Solution:

(a)

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\left[r\cos(\omega t)\vec{i} + r\sin(\omega t)\vec{j}\right]}{dt} = \frac{d\left(r\cos\omega t\right)}{dt}\vec{i} + \frac{d\left(r\sin\omega t\right)}{dt}\vec{j} = -\omega r\sin(\omega t)\vec{i} + r\omega\cos(\omega t)\vec{j}\,.$$

$$\vec{r}\cdot\vec{v} = \left[r\cos(\omega t)\vec{i} + r\sin(\omega t)\vec{j}\right]\cdot\left[-\omega r\sin(\omega t)\vec{i} + r\omega\cos(\omega t)\vec{j}\right] = 0$$

(The velocity is in a tangential direction.)

(b)

$$\vec{r}\times\vec{v} = \left[r\cos(\omega t)\vec{i} + r\sin(\omega t)\vec{j}\right]\times\left[-\omega r\sin(\omega t)\vec{i} + r\omega\cos(\omega t)\vec{j}\right] = r^2\omega\cos^2(\omega t)\vec{k} + r^2\omega\sin^2(\omega t)\vec{k} = r^2\omega\vec{k}$$

(c)

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\left[-\omega r\sin(\omega t)\vec{i} + r\omega\cos(\omega t)\vec{j}\right]}{dt} = -\omega^2 r\cos(\omega t)\vec{i} - r\omega^2\sin(\omega t)\vec{j} = -\omega^2\vec{r}$$

(The acceleration is directed towards the origin.)

2.

(a) Compute the divergence and curl of the vector functions:

   (i) $\vec{v} = e^x\cos y\vec{i} + xy^2\vec{j} + yz^3\vec{k}$

Solution:

$$div\vec{v} = \nabla\cdot\vec{v} = \frac{\partial}{\partial x}\left(e^x\cos y\right) + \frac{\partial}{\partial y}\left(xy^2\right) + \frac{\partial}{\partial z}\left(yz^3\right) = e^x\cos y + 2xy + 3yz^2$$

$$curl\vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x\cos y & xy^2 & yz^3 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}\left(yz^3\right) - \frac{\partial}{\partial z}\left(xy^2\right)\right)\vec{i} - \left(\frac{\partial}{\partial x}\left(yz^3\right) - \frac{\partial}{\partial z}\left(e^x\cos y\right)\right)\vec{j} + \left(\frac{\partial}{\partial x}\left(xy^2\right) - \frac{\partial}{\partial y}\left(e^x\cos y\right)\right)\vec{k}$$

$$= z^3\vec{i} + \left(y^2 + e^x\sin y\right)\vec{k}$$

   (ii) $\vec{v} = yz\vec{i} + 3zx\vec{j} + z\vec{k}$

Solution:

$$div\vec{v} = \nabla\cdot\vec{v} = \frac{\partial}{\partial x}\left(yz\right) + \frac{\partial}{\partial y}\left(3zx\right) + \frac{\partial z}{\partial z} = 1$$

$$curl\vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(3zx) \right)\vec{i} - \left(\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(yz) \right)\vec{j} + \left(\frac{\partial}{\partial x}(3zx) - \frac{\partial}{\partial y}(yz) \right)\vec{k} = -3x\vec{i} + y\vec{j} + 2z\vec{k}$$

(b) (i) Find $div(grad\ f)$, for $f(x,y,z) = 1 - x^2 - 4y^2 + 2z^2$

Solution:

$$grad\ f = \frac{\partial f_x}{\partial x}\vec{i} + \frac{\partial f_y}{\partial y}\vec{j} + \frac{\partial f_z}{\partial z}\vec{k} = -2x\vec{i} - 8y\vec{j} + 4z\vec{k}$$

$$div(grad\ f) = \frac{\partial}{\partial x}(-2x) + \frac{\partial}{\partial y}(-8y) + \frac{\partial}{\partial z}(4z) = -6$$

(ii) Find $\nabla \times \nabla(\nabla \cdot \vec{v})$, for $\vec{v}(x,y,z) = e^x\vec{i} + e^y\vec{j} + e^z\vec{k}$

Solution:

$$\nabla \cdot \vec{v} = div\vec{v} = \frac{\partial e^x}{\partial x} + \frac{\partial e^y}{\partial y} + \frac{\partial e^z}{\partial z} = e^x + e^y + e^z$$

$$\nabla(\nabla \cdot \vec{v}) = grad(div\vec{v}) = e^x\vec{i} + e^y\vec{j} + e^z\vec{k}$$

$$\nabla \times \nabla(\nabla \cdot \vec{v}) = \nabla \times \left(e^x\vec{i} + e^y\vec{j} + e^z\vec{k}\right) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x & e^y & e^z \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) - \vec{k}(0-0) = \vec{0}$$

Remark: $\nabla \times \nabla f = \vec{0}$

(c) Verify the formula $div(f\vec{v}) = f\ div\,\vec{v} + \vec{v} \cdot grad\ f$ for $f = e^{xyz}$ and $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$.

Solution:

LHS = $div(f\vec{v}) = div\left(e^{xyz}x\vec{i} + e^{xyz}y\vec{j} + e^{xyz}z\vec{k}\right)$

$$= \frac{\partial e^{xyz}x}{\partial x} + \frac{\partial e^{xyz}y}{\partial y} + \frac{\partial e^{xyz}z}{\partial z} = xyze^{xyz} + e^{xyz} + xyze^{xyz} + e^{xyz} + xyze^{xyz} + e^{xyz} = 3e^{xyz}(xyz+1)$$

RHS = $f\ div\,\vec{v} + \vec{v} \cdot grad\ f = e^{xyz}\left(\dfrac{\partial x}{\partial x} + \dfrac{\partial y}{\partial y} + \dfrac{\partial z}{\partial z} \right) + \left(x\vec{i} + y\vec{j} + z\vec{k}\right) \cdot \left(\dfrac{\partial e^{xyz}}{\partial x}\vec{i} + \dfrac{\partial e^{xyz}}{\partial y}\vec{j} + \dfrac{\partial e^{xyz}}{\partial z}\vec{k} \right)$

$= 3e^{xyz} + \left(xyze^{xyz} + xyze^{xyz} + xyze^{xyz}\right) = 3e^{xyz}(xyz+1)$ = LHS

(d) Prove that:

(i) $curl(\vec{v} + \vec{w}) = curl\,\vec{v} + curl\,\vec{w}$ for any vector fields $\vec{v}$ and $\vec{w}$ on $\mathbf{R}^3$.

Solution:

Let $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$

LHS = $curl(\vec{v} + \vec{w}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 + w_1 & v_2 + w_2 & v_3 + w_3 \end{vmatrix}$

$$=\left(\frac{\partial(v_3+w_3)}{\partial y}-\frac{\partial(v_2+w_2)}{\partial z}\right)\vec{i}-\left(\frac{\partial(v_3+w_3)}{\partial x}-\frac{\partial(v_1+w_1)}{\partial z}\right)\vec{j}+\left(\frac{\partial(v_2+w_2)}{\partial x}-\frac{\partial(v_1+w_1)}{\partial y}\right)\vec{k}$$

$$=\left(\frac{\partial v_3}{\partial y}+\frac{\partial w_3}{\partial y}-\frac{\partial v_2}{\partial z}-\frac{\partial w_2}{\partial z}\right)\vec{i}-\left(\frac{\partial v_3}{\partial x}+\frac{\partial w_3}{\partial x}-\frac{\partial v_1}{\partial z}-\frac{\partial w_1}{\partial z}\right)\vec{j}+\left(\frac{\partial v_2}{\partial x}+\frac{\partial w_2}{\partial x}-\frac{\partial v_1}{\partial y}-\frac{\partial w_1}{\partial y}\right)\vec{k}$$

$$=\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\vec{i}-\left(\frac{\partial v_3}{\partial x}-\frac{\partial v_1}{\partial z}\right)\vec{j}+\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\vec{k}+\left(\frac{\partial w_3}{\partial y}-\frac{\partial w_2}{\partial z}\right)\vec{i}-\left(\frac{\partial w_3}{\partial x}-\frac{\partial w_1}{\partial z}\right)\vec{j}+\left(\frac{\partial w_2}{\partial x}-\frac{\partial w_1}{\partial y}\right)\vec{k}$$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_1 & v_2 & v_3\end{vmatrix}+\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ w_1 & w_2 & w_3\end{vmatrix}$$

$= curl\,\vec{v} + curl\,\vec{w}$ = RHS

(ii) $div(curl\,\vec{v}) = 0$

Solution:

Let $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$

LHS = $div(curl\,\vec{v}) = div\left(\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_1 & v_2 & v_3\end{vmatrix}\right)$

$$=div\left(\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\vec{i}-\left(\frac{\partial v_3}{\partial x}-\frac{\partial v_1}{\partial z}\right)\vec{j}+\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\vec{k}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)-\frac{\partial}{\partial y}\left(\frac{\partial v_3}{\partial x}-\frac{\partial v_1}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)$$

$$=\frac{\partial^2 v_3}{\partial x\partial y}-\frac{\partial^2 v_2}{\partial x\partial z}-\frac{\partial^2 v_3}{\partial y\partial x}+\frac{\partial^2 v_1}{\partial y\partial z}+\frac{\partial^2 v_2}{\partial z\partial x}-\frac{\partial^2 v_1}{\partial z\partial y}=0$$

3. It is given that $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{p} = a\vec{i} + b\vec{j} + c\vec{k}$ is a constant vector and $\vec{u} = (\vec{p}\cdot\vec{r})\vec{r}$.

(a) Evaluate $\vec{u} = (\vec{p}\cdot\vec{r})\vec{r}$.

(b) Show that

(i) $\nabla\cdot\vec{u} = 4\vec{p}\cdot\vec{r}$,

(ii) $\nabla\times\vec{u} = \vec{p}\times\vec{r}$,

(iii) $\nabla\times(\vec{p}\times\vec{r}) = 2\vec{p}$.

Solution:

(a)

$$\vec{u} = (\vec{p}\cdot\vec{r})\vec{r} = \left[(a\vec{i}+b\vec{j}+c\vec{k})\cdot(x\vec{i}+y\vec{j}+z\vec{k})\right](x\vec{i}+y\vec{j}+z\vec{k}) = (ax+by+cz)(xi+yj+zk)$$

(b)(i)

$$\nabla \cdot \vec{u} = \nabla \cdot \left(\vec{p} \cdot \vec{r}\right)\vec{r} = \nabla \cdot \left[\left(ax+by+cz\right)(xi+yj+zk)\right]$$

$$= \frac{d\left[\left(ax+by+cz\right)x\right]}{dx} + \frac{d\left[\left(ax+by+cz\right)y\right]}{dy} + \frac{d\left[\left(ax+by+cz\right)z\right]}{dz}$$

$$= \left(ax+by+cz\right) + ax + \left(ax+by+cz\right) + by + \left(ax+by+cz\right) + cz$$

$$= 4\left(ax+by+cz\right) = 4\vec{p} \cdot \vec{r}$$

(b)(ii)

$$\nabla \times \vec{u} = \nabla \times \left[\left(\vec{p} \cdot \vec{r}\right)\vec{r}\right] = \nabla \times \left[\left(ax+by+cz\right)x\vec{i} + \left(ax+by+cz\right)y\vec{j} + \left(ax+by+cz\right)z\vec{k}\right]$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \left(ax+by+cz\right)x & \left(ax+by+cz\right)y & \left(ax+by+cz\right)z \end{vmatrix} = \left(bz-cy\right)\vec{i} + \left(cx-az\right)\vec{j} + \left(ay-bx\right)\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a & b & c \\ x & y & z \end{vmatrix} = \vec{p} \times \vec{r}$$

(b)(iii)

$$\nabla \times \left(\vec{p} \times \vec{r}\right) = \nabla \times \left[\left(bz-cy\right)\vec{i} + \left(cx-az\right)\vec{j} + \left(ay-bx\right)\vec{k}\right] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ bz-cy & cx-az & ay-bx \end{vmatrix} = 2a\vec{i} + 2b\vec{j} + 2c\vec{k} = 2\vec{p}$$

4. Let $\vec{F}(x,y,z) = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ be a vector field on $\mathbf{R}^3$, where a, b and c are real constants.

(a) Find the values of a, b and c such that $\vec{F}$ is irrotational.

(b) With the values of a, b and c obtained in (a), determine a potential function φ on $\mathbf{R}^3$ for which $\nabla\varphi = \vec{F}$.

Solution:

(a) Since

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= \left(\frac{\partial(4x+cy+2z)}{\partial y} - \frac{\partial(bx-3y-z)}{\partial z}\right)\vec{i} - \left(\frac{\partial(4x+cy+2z)}{\partial x} - \frac{\partial(x+2y+az)}{\partial z}\right)\vec{j}$$

$$+ \left(\frac{\partial(bx-3y-z)}{\partial x} - \frac{\partial(x+2y+az)}{\partial y}\right)\vec{k}$$

$$= (c+1)\vec{i} - (4-a)\vec{j} + (b-2)\vec{k}$$

The vector field $\vec{F}$ is irrotational provided $\nabla \times \vec{F} = \vec{0}$

$\therefore \quad c=-1, a=4, b=2$

(b) Let φ be a scalar field on $\mathbf{R}^3$ such that $\nabla\varphi=\vec{F}$, i.e.

$$\frac{\partial\varphi}{\partial x}=x+2y+4z,\ \frac{\partial\varphi}{\partial y}=2x-3y-z,\ \frac{\partial\varphi}{\partial z}=4x-y+2z$$

From the first equality,

$$\varphi(x,y,z)=\int(x+2y+4z)dx=\frac{x^2}{2}+2xy+4xz+f(y,z),$$

where f is a function to be determined. Then $\dfrac{\partial\varphi}{\partial y}=2x+\dfrac{\partial f(y,z)}{\partial y}$.

Equating this with the equality $\dfrac{\partial\varphi}{\partial y}=2x-3y-z$ gives

$$\frac{\partial f(y,z)}{\partial y}=-3y-z$$

It follows that $f(y,z)=\int(-3y-z)dy=-\dfrac{3y^2}{2}-yz+g(z)$ for a function g.

Thus, $\quad \varphi(x,y,z)=\dfrac{x^2}{2}+2xy+4xz-\dfrac{3y^2}{2}-yz+g(z)$

and $\quad \dfrac{\partial\varphi}{\partial z}=4x-y+g'(z)$

Equating with $\dfrac{\partial\varphi}{\partial z}=4x-y+2z$, we have

$g'(z)=2z$, so that $g(z)=\int 2zdz=z^2$ (we set the constant of integration to be 0.)

$\therefore$ A potential function φ is given by $\varphi(x,y,z)=\dfrac{x^2}{2}-\dfrac{3y^2}{2}+z^2+2xy-yz+4xz$

5. Let $\vec{G}(x,y,z)=3yz\vec{i}+x^2\vec{j}+x\cos y\vec{k}$ be a vector field on $\mathbf{R}^3$.

(a) Show that $\vec{G}$ is solenoidal.

(b) Find a vector field $\vec{F}(x,y,z)=f_1(x,y,z)\vec{i}+f_2(x,y,z)\vec{j}$ on $\mathbf{R}^3$ such that $\nabla\times\vec{F}=\vec{G}$.

Solution:

(a) The vector field $\vec{G}$ is solenoidal, since

$$div\vec{G}=\frac{\partial 3yz}{\partial x}+\frac{\partial x^2}{\partial y}+\frac{\partial x\cos y}{\partial z}=0$$

(b) Let $\vec{F}(x,y,z)=f_1(x,y,z)\vec{i}+f_2(x,y,z)\vec{j}$ be a vector field on R^3. Then

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ f_1 & f_2 & 0\end{vmatrix}=\left(-\frac{\partial f_2}{\partial z}\right)\vec{i}-\left(-\frac{\partial f_1}{\partial z}\right)\vec{j}+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)\vec{k}$$

To find f_1 and f_2 such that

$$\frac{\partial f_1}{\partial z}=x^2,\ \frac{\partial f_2}{\partial z}=-3yz \text{ and } \frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}=x\cos y$$

From the first two equalities,

$$f_1(x,y,z)=\int x^2 dz = x^2 z+\varphi(x,y)$$

and $$f_2(x,y,z)=-\int 3yzdz=-\frac{3yz^2}{2}+\psi(x,y)$$

for two functions φ and ψ. Then $\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}=\frac{\partial \psi}{\partial x}-\frac{\partial \varphi}{\partial y}$

Equating this with the equality $\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}=x\cos y$ gives

$$\frac{\partial \psi}{\partial x}-\frac{\partial \varphi}{\partial y}=x\cos y$$

By setting φ = 0, we have

$$\frac{\partial \psi}{\partial x}=x\cos y\text{, so that}\qquad \psi(x,y)=\int x\cos y dx=\frac{x^2\cos y}{2}+h(y)$$

By choosing $h(y)$ = 0, we have $\psi(x,y)=\frac{x^2\cos y}{2}$.

Hence, one of the vector field $\vec{F}$ is

$$\vec{F}(x,y,z)=x^2 z\vec{i}+\frac{x^2\cos y-3yz^2}{2}\vec{j}$$

Remark: The vector field $\vec{F}$ is not unique.

6. (a) A vector field $\vec{F}$ is said to be solenoidal if $\nabla\cdot\vec{F}=0$. Let $\vec{F}=(y+z)\vec{i}+(x+z)\vec{j}+(x+y)\vec{k}$. Show that $\vec{F}$ is solenoidal.

(b) As a consequence of $\vec{F}$ being solenoidal, there exists a vector field $\vec{H}$ such that $\vec{F}=\nabla\times\vec{H}$. Find a vector field $\vec{H}=h_1(x,y,z)\vec{i}+h_2(x,y,z)\vec{j}+h_3(x,y,z)\vec{k}$ with $h_2(x,y,z)\equiv 0$ such that $\vec{F}=\nabla\times\vec{H}$.

(c) Observe that if φ is a scalar field and $\vec{H},\vec{F}$ are vector fields such that $\vec{F}=\nabla\times\vec{H}$, then we have $\nabla\times\left(\vec{H}+\nabla\varphi\right)=\nabla\times\vec{H}+\nabla\times\nabla\varphi=\nabla\times\vec{H}=\vec{F}\cdots\cdots$(I).

Using (b) and observation (I), find a vector field $\vec{G}=g_1(x,y,z)\vec{i}+g_2(x,y,z)\vec{j}+g_3(x,y,z)\vec{k}$ such that $\vec{F}=\nabla\times\vec{G}$ and $g_2(x,y,z)=2y$.

Solution:

(a)

$$\nabla\cdot\vec{F}=\frac{\partial(y+z)}{\partial x}+\frac{\partial(x+z)}{\partial y}+\frac{\partial(x+y)}{\partial z}=0.$$

(b)

$$\nabla\times\vec{H}=\det\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ h_1 & h_2 & h_3\end{pmatrix}=\left(\frac{\partial h_3}{\partial y}-\frac{\partial h_2}{\partial z}\right)\vec{i}+\left(\frac{\partial h_1}{\partial z}-\frac{\partial h_3}{\partial x}\right)\vec{j}+\left(\frac{\partial h_2}{\partial x}-\frac{\partial h_1}{\partial y}\right)\vec{k}$$

$$\vec{F}=\nabla\times\vec{H}\text{ and }h_2\equiv 0\Rightarrow\vec{F}=(y+z)\vec{i}+(x+z)\vec{j}+(x+y)\vec{k}=\frac{\partial h_3}{\partial y}\vec{i}+\left(\frac{\partial h_1}{\partial z}-\frac{\partial h_3}{\partial x}\right)\vec{j}-\frac{\partial h_1}{\partial y}\vec{k}.$$

That is, $\begin{cases} y+z=\dfrac{\partial h_3}{\partial y} \\ x+z=\dfrac{\partial h_1}{\partial z}-\dfrac{\partial h_3}{\partial x} \\ x+y=-\dfrac{\partial h_1}{\partial y} \end{cases}$.

$x+y=-\dfrac{\partial h_1}{\partial y} \Rightarrow h_1=-y-\dfrac{y^2}{2}$, $y+z=\dfrac{\partial h_3}{\partial y} \Rightarrow h_3=\dfrac{y^2}{2}+zy+C(x,z)$.

$x+z=\dfrac{\partial h_1}{\partial z}-\dfrac{\partial h_3}{\partial x}=-\dfrac{\partial C}{\partial x} \Rightarrow C(y,z)=-\dfrac{x^2}{2}-zx$. Thus, $h_3=\dfrac{y^2}{2}+zy-\dfrac{x^2}{2}-zx=\dfrac{y^2-x^2}{2}+z(y-x)$.

So, $\overrightarrow{H}=\left(-xy-\dfrac{y^2}{2}\right)\vec{i}+\left[\dfrac{y^2-x^2}{2}+z(y-x)\right]\vec{k}$.

(c)

Let $\varphi(x,y,z)=y^2 \Rightarrow \nabla\varphi=2y\vec{j}$.

$\overrightarrow{G}=\overrightarrow{H}+\nabla\varphi=\left(-xy-\dfrac{y^2}{2}\right)\vec{i}+2y\vec{j}+\left[\dfrac{y^2-x^2}{2}+z(y-x)\right]\vec{k}$.

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