single integral $\int_a^b f(x)\,dx = F(b) - F(a)$

double integral $\iint_D f(x,y)\,dx\,dy = \int_{y=c}^{y=d}\int_{x=x_1(y)}^{x=x_2(y)} f(x,y)\,dx\,dy$

$= \int_{x=a}^{x=b}\int_{y=y_1(x)}^{y=y_2(x)} f(x,y)\,dy\,dx$

triple integral $\iiint_V f(x,y,z)\,dx\,dy\,dz$

$= \int_{z=a}^{b}\int_{y=y_1(z)}^{y=y_2(z)}\int_{x=x_1(y,z)}^{x=x_2(y,z)} f(x,y,z)\,dx\,dy\,dz$

Assignment 3 — Q 1.

$= \ldots \quad dx\,dz\,dy$

$dy\,dx\,dz$

line integral

$\int_C f(x,y,z)\,dC \overset{\vec r(t),\ a\le t\le b}{=\!=} \int_a^b \underbrace{f(\vec r(t))}_{\text{single}} \cdot \underbrace{|\vec r\,'(t)|}_{\text{tangent vector. length}}\,dt$

$\int_C \vec F(x,y,z)\,d\vec C \overset{\vec r(t)}{=\!=} \int_a^b \underbrace{\vec F(\vec r(t)) \cdot \vec r\,'(t)}_{\text{single}}\,dt$ ✓

particularly, if $\vec F$ is conservative.

$\int_{\overrightarrow{AB}} \vec F\,d\vec C = \varphi(B) - \varphi(A)$.

φ is potential of $\vec F$ i.e $\vec F = \nabla\varphi$

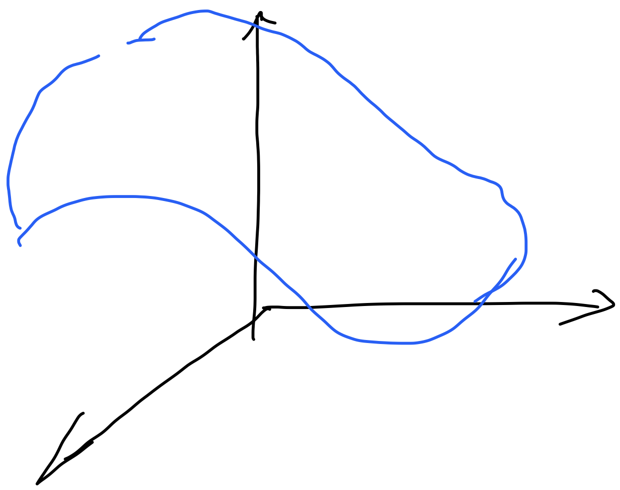
Chapter 5. Line and Surface Integrals

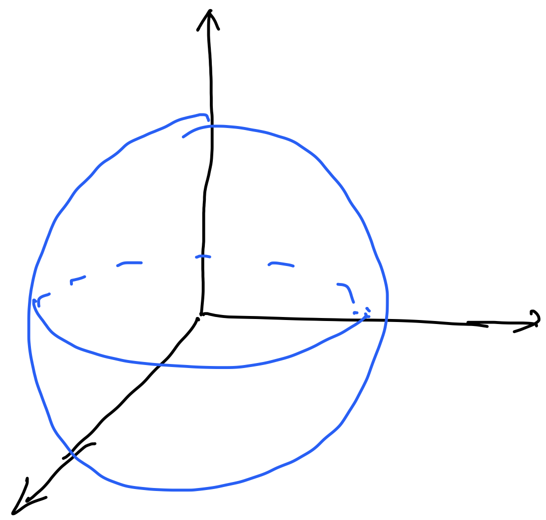
1st kind
2nd kind.

1 Mathematical Representation of Surfaces

Question 1: What is a surface in 3-dimensional (~~2-dimensional~~) space?

Question 2: How to represent it in mathematics?





representation of a surface

① word description

eg. unit sphere, a plane through $(1,2,3)$ perpendicular to $(-1,3,2)$.

cone.

② using xyz-equation.

eg. $-x+1+3y-6+2z-6=0$

$x-3y-2z=-11$

eg. $x^2+y^2+z^2=1 \to$ unit sphere.

eg. $z=x^2+y^2 \to$ cone.

③ using uv-parametric equations

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases} \quad (u,v) \in D.$$

eg

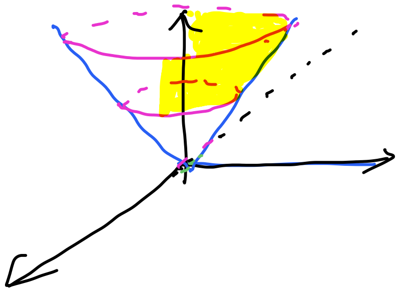
$$\begin{cases} x = 3\cos\theta\sin\varphi \\ y = 3\sin\theta\sin\varphi \\ z = 3\cos\varphi \end{cases} \qquad \theta \in [0, 2\pi] \quad \varphi \in [0, \pi]$$

Sphere with radius 3.

$x^2+y^2+z^2 = 9$

eg.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r \end{cases} \qquad 0 \le \theta \le \pi \quad 1 \le r \le 2$$



1.1 Parameterization of Surfaces

Given a surface with $F(x,y,z)=0$,

introduce two free parameters u, v,

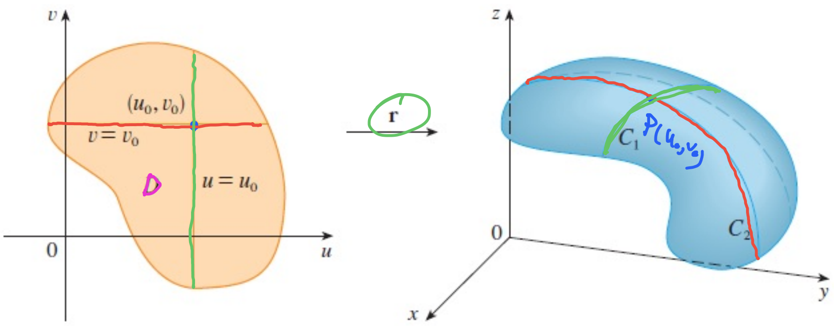
$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases} \qquad (u,v)\in D.$$

equivalently,

$$\underline{\vec{r}(u,v)} = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

$$(u,v)\in D.$$

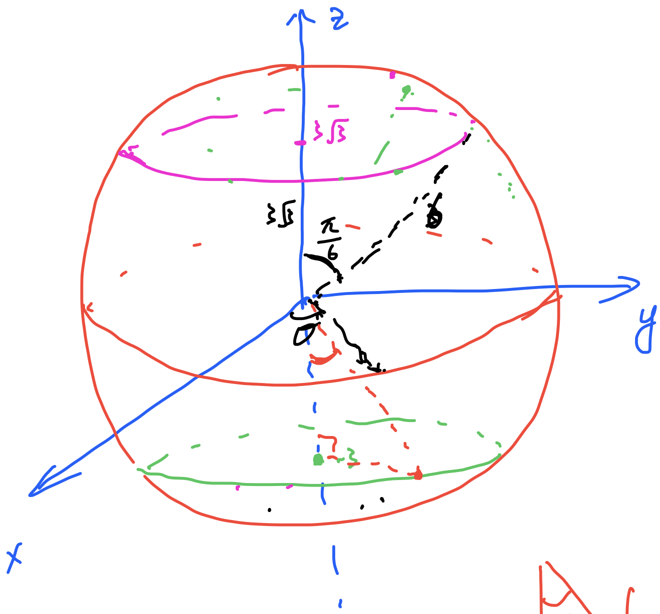
$\vec{r}(u,v)$ is a <u>vector function</u> from $\mathbb{R}^2 \to \mathbb{R}^3$.



$0, \pi$

Example Parameterize the following surfaces:

1. Spherical band: the portion of the sphere $x^2 + y^2 + z^2 = 36$ between the plane $z = -3$ and $z = 3\sqrt{3}$.



$$\begin{cases} x = 6\cos\theta \sin\varphi \\ y = 6 \sin\theta \sin\varphi \\ z = 6\cos\varphi \end{cases}$$

$0 \le \theta \le 2\pi$

$\frac{\pi}{6} \le \varphi \le \pi - \frac{\pi}{3}$

$\frac{\pi}{6} \le \varphi \le \frac{2\pi}{3}$

$-3 = 6\cos\varphi \Rightarrow \cos\varphi = -\frac{1}{2}$

$\Rightarrow \varphi = \frac{2\pi}{3}$

$3\sqrt{3} = 6\cos\varphi \Rightarrow \cos\varphi = \frac{\sqrt{3}}{2}$

$\Rightarrow \varphi = \frac{\pi}{6}$

2. The surface $z = x + y^2, 0 \le x \le 1, 0 \le y \le 2$.

$$\begin{cases} x = u \\ y = v \\ z = u + v^2 \end{cases} \qquad 0 \le u \le 1 \qquad 0 \le v \le 2.$$

alternative since $x = y^2 - z$

$$\begin{cases} x = u^2 - v \\ y = u \\ z = v \end{cases} \qquad 0 \le u \le 2, \qquad 0 \le v \le 5$$

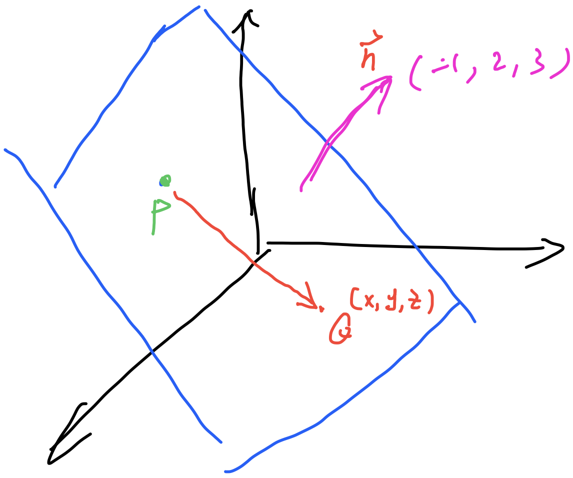
3. Find a parametrization of the cone $z = \sqrt{x^2 + y^2}$, $1 \le z \le 2$.

$$\begin{cases} x = u \\ y = v \\ z = \sqrt{u^2+v^2} \end{cases} \qquad 1 \le \sqrt{u^2+v^2} \le 2$$

alternatively

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = r \end{cases} \qquad 1 \le r \le 2, \qquad 0 \le \theta \le 2\pi .$$

4. The plane with norm vector $(-1, 2, 3)$ passes through $(2, 3, 1)$. P.



$\vec{n} \perp (\overrightarrow{PQ})$.

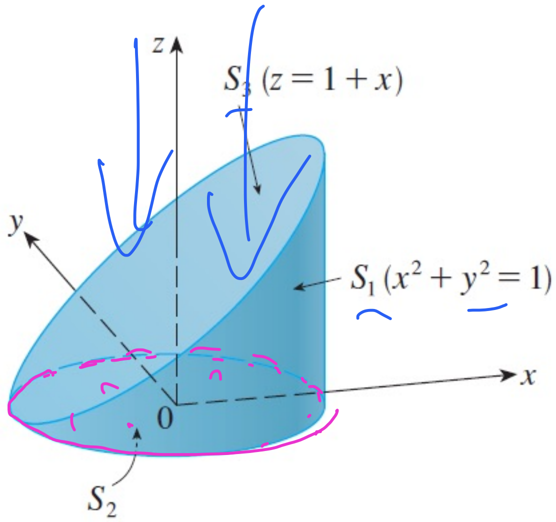
$\Rightarrow (-1,2,3)\cdot(x-2,\ y-3,\ z-1) = 0$

$\Rightarrow -x+2+2y-6+3z-3 = 0.$

$\Rightarrow x = 2y+3z-7$

$$\Rightarrow \begin{cases} x = 2u+3v-7 \\ y = u \\ z = v \end{cases}$$

5. S is the surface whose sides S_1 is given by $x^2 + y^2 = 1$, bottom S_2 $x^2 + y^2 \leq 1$ in the plane $z = 0$, and top S_3 $z = 1 + x$ lies above S_2.



S_2 : $z = 0$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 0 \end{cases} \qquad \begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 1 \end{cases}$$

S_1 :

$$\begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = z \end{cases} \qquad \begin{array}{l} 0 \leq z \leq 1 + \cos\theta \\ 0 \leq \theta \leq 2\pi \end{array}$$

S_3 :

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = 1 + r\cos\theta \end{cases} \qquad \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

6. Find a parametrization of the cylinder

$$x^2 + (y - 3)^2 = 9, \quad 0 \leq z \leq 5$$

$$\begin{cases} x = 3\cos\theta \\ y = 3 + 3\sin\theta \\ z = z \end{cases}$$

1.2 Tangent Plane of ~~Curves~~ Surface.

Definition Given a surface S with a parameterization $\vec{r}(u, v)$, the tangent plane of S at point $P = \vec{r}(u_0, v_0)$ is the one with equation

$$[(x, y, z) - P] \cdot \left[\partial_u \vec{r}(u_0, v_0) \times \partial_v \vec{r}(u_0, v_0)\right] = 0,$$

passing through.

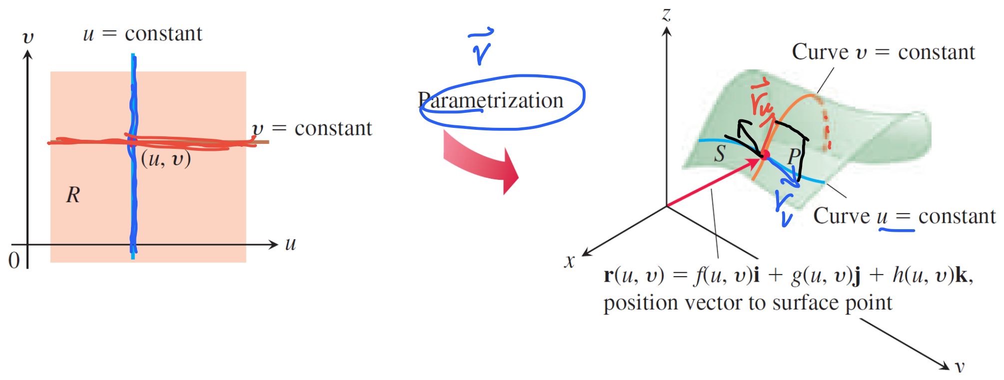
where

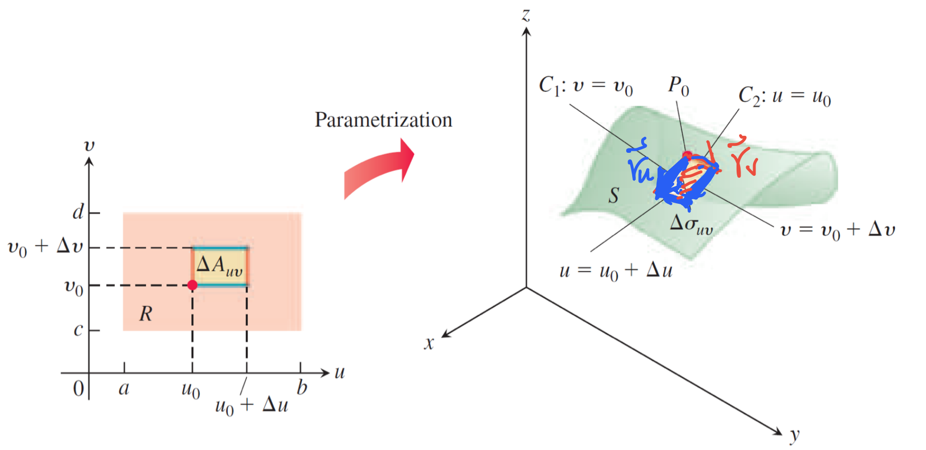
$$\partial_u \vec{r}(u_0, v_0) \times \partial_v \vec{r}(u_0, v_0)$$

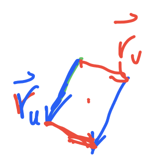
is the norm vector of the tangent plane at $P = \vec{r}(u_0, v_0)$.

Particularly, $|\vec{r}_u\,(u_0, v_0) \times \vec{r}_v\,(u_0, v_0)|$ $\Delta u\ \Delta v$ represents the area on the tangent plane over unit square on uv-plane.

surface.png







$|\vec{r}_u \times \vec{r}_v|$

Example Let S as the surface of the cone $z = 1 + \sqrt{x^2 + y^2}$, for $2 \leq z \leq 8$. Find its tangent plane and norm vector at $P(3, 4, 6)$. $= P(a_0, \theta_0)$

$z = 6 \Rightarrow a_0 = 5$.
$x = 5\cos\theta_0 = 3 \Rightarrow \cos\theta_0 = \frac{3}{5}$
$y = 5\sin\theta_0 = 4 \Rightarrow \sin\theta_0 = \frac{4}{5}$.

Sol: parameterize S.

$$\begin{cases} x = a\cos\theta \\ y = a\sin\theta \\ z = 1 + a \end{cases} \qquad \begin{array}{l} 1 \leq a \leq 7 \\ 0 \leq \theta \leq 2\pi \end{array}$$

$$\partial_a \vec{r} = (\cos\theta, \sin\theta, 1)$$

$$\partial_\theta \vec{r} = (-a\sin\theta, a\cos\theta, 0)$$

$$\partial_a \vec{r}(P) = (\cos\theta_0, \sin\theta_0, 1) = \left(\frac{3}{5}, \frac{4}{5}, 1\right)$$

$$\partial_\theta \vec{r}(P) = \left(-5\,\frac{4}{5}, 5\,\frac{3}{5}, 0\right) = (-4, 3, 0).$$

$$\partial_a \vec{r}(P) \times \partial_\theta \vec{r}(P) = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3/5 & 4/5 & 1 \\ -4 & 3 & 0 \end{pmatrix} = \vec{n}.$$

$$\vec{n} \cdot \big((x, y, z) - (3, 4, 6)\big) = 0$$

2 Surface Integral of 1st kind

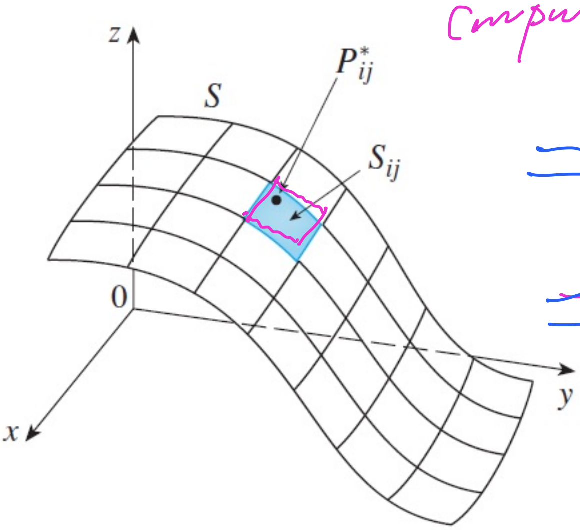
Definition: Given a surface S and scalar function $w = f(x,y,z)$,

$$\iint_S f(x,y,z)\,dS = \lim_{n\to\infty}\lim_{m\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} f(P_{ij}^*)\,\underbrace{|S_{ij}|}_{\text{Area of } ij \text{ patch}}$$

Computation: parameterize S as $\vec{r}(u,v)$

$$= \lim_{n\to\infty}\lim_{m\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} f(\vec{r}(u_i^*, v_j^*))\,|\partial_u\vec{r}\times\partial_v\vec{r}|\,\Delta u\,\Delta v$$

$$= \iint_D f(\vec{r}(u,v))\,|\vec{r}_u\times\vec{r}_v|\,du\,dv$$



Remark: If S is a region D on xy-plan, $\int_S f dS$ is same as the double integral $\int_D f(x,y,0)dxdy$.

Physical Interpretation Let $f(x,y,z)$ be the point density of a thin sheet shaped of S. Then

- $\int_S f(x,y,z)dS$ is the mass of the sheet.
- $(\int_S xf(x,y,z)dS, \int_S yf(x,y,z)dS, \int_S zf(x,y,z)dS)$ is the center of mass of the wire.
- $\int_S 1\ dS$ is the area of the surface S.