EE3211 Modelling Techniques

Lecture 7

Nonparametric Methods

Overview

Parametric statistical methods:

- For estimation and hypothesis testing
- We assume that the parametric form of the distribution is known

Nonparametric statistical methods

- If no assumptions about the shape of the distribution
- Cannot apply central-limit theorem due to small sample size (n)
- Make fewer assumptions about the distributional shape

Types of Data

- **Categorical variables**: two or more categories that do not have any ordering

  e.g. race and ethnicity

- **Dichotomous variables**: two possible values

  e.g. gender, death, disease status

- **Ordinal variables**: more than two ranked or ordered values

  e.g., amount of current smoking: none, <10/day, 10-20/day, 21-30/day, >30/day

Ordinal data

- Measure relative ordering of different categories

E.g. two people: A and B

→ Whether score for A is >, <, or = to the score for B

~~relative magnitude of the differences~~

Example on Dermatology

- Objective: compare the effectiveness of two ointments (A, B) in reducing excessive redness in people who cannot otherwise be exposed to sunlight
- Oinment A: randomly applied to either the left or right arm
- Oinment B: applied to the corresponding area on the other arm.
- Person is then exposed to 1 hour of sunlight
- Compare the degrees of redness of the two arms
- Make qualitative assessments:
  1. Arm A is not as red as arm B
  2. Arm B is not as red as arm A
  3. Both arms are equally red
- Of 45 people tested with the condition, 22 are better off on arm A, 18 are better off on arm B, 5 are equally well off on both arms.
- Q: How can we decide whether this evidence is enough to conclude that ointment A is better than ointment B?

Large-sample Method

- Degree of redness can be measured on a quantitative scale (higher number indicates more redness)
- Let x_i = degree of redness on arm A for the ith person

  y_i = degree of redness on arm B for the ith person

  $d_i = x_i – y_i$ = difference in redness between the A and B arms for the ith participant
- H_0: $\Delta = 0$ vs. H_1: $\Delta \neq 0$ (Δ : the population median of the d_i / the 50th percentile of the underlying distribution of the d_i)

  (i) if $\Delta = 0$: ointments are equally effective

  (ii) if $\Delta < 0$: ointment A is better (arm A is less red than arm B)

  (iii) if $\Delta > 0$: ointment B is better (arm A is redder than arm B)
- d_i cannot be observed, can on observe whether:

  (i) $d_i > 0$

  (ii) $di < 0$

  (iii) $di = 0$

The Sign Test

- Depends only on the sign of the difference, not their actual magnitude
- To test the hypothesis H_0: $\Delta = 0$ vs. H_1: $\Delta \neq 0$

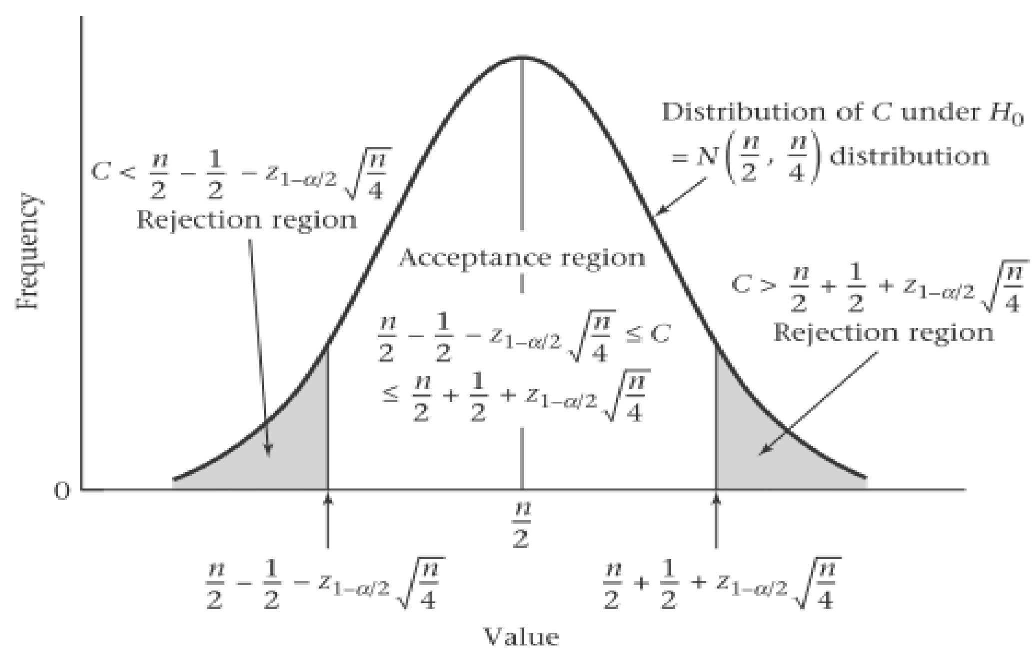
  *number of nonzero d_i's = n $\geq$ 20 and C = the number of d_i's where $d_i > 0$,

  if

$$C > c_2 = \frac{n}{2} + \frac{1}{2} + z_{1-\alpha/2}\sqrt{n/4} \quad \text{or} \quad C < c_1 = \frac{n}{2} - \frac{1}{2} - z_{1-\alpha/2}\sqrt{n/4}$$

→ Reject H_0; otherwise accept H_0

Figure 9.1 Acceptance and rejection regions for the sign test



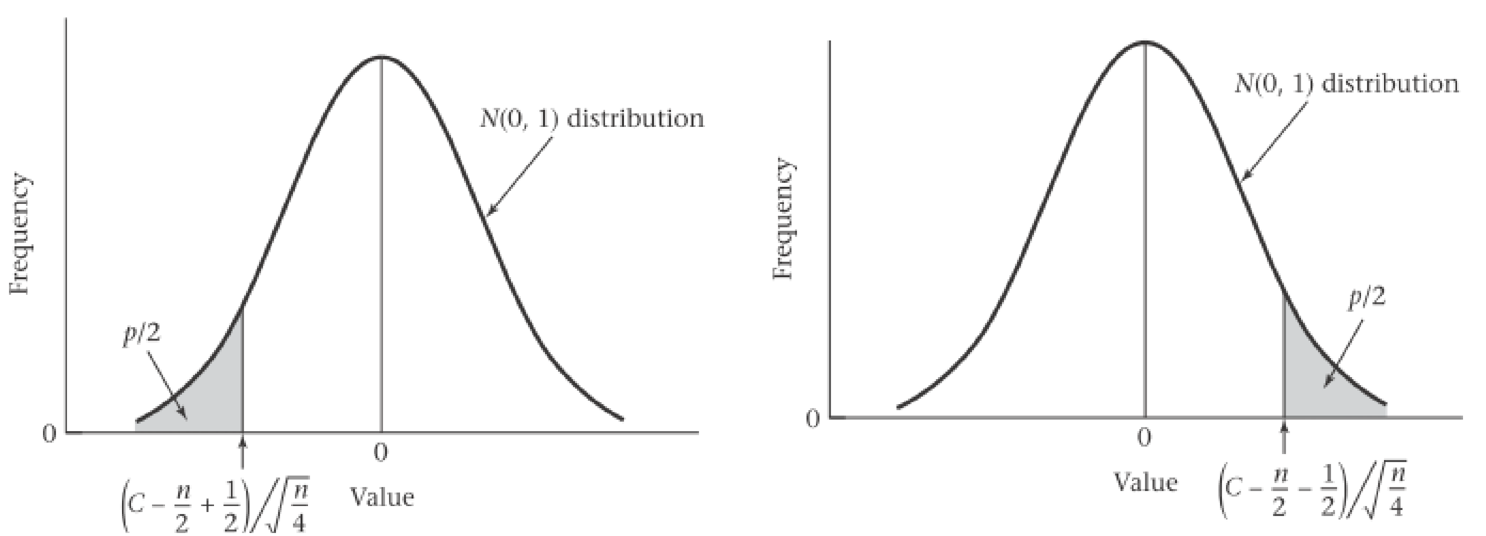
Computation of the *p*-Value for the Sign Test (Normal-Theory Method)

$$p = 2 \times \left[1 - \Phi\left(\frac{C - \frac{n}{2} - .5}{\sqrt{n/4}}\right)\right] \quad \text{if} \quad C > \frac{n}{2}$$

$$p = 1.0 \quad \text{if} \quad C = \frac{n}{2}$$

$$p = 2 \times \Phi\left(\frac{C - \frac{n}{2} + .5}{\sqrt{n/4}}\right) \quad \text{if} \quad C < \frac{n}{2}$$

Figure 9.2 Computation of the *p*-value for the sign test



If $C < n/2$, then $p = 2 \times$ area to the left of $\left(C - \frac{n}{2} + \frac{1}{2}\right) / \sqrt{\frac{n}{4}}$ under an $N(0, 1)$ distribution

If $C > n/2$, then $p = 2 \times$ area to the right of $\left(C - \frac{n}{2} - \frac{1}{2}\right) / \sqrt{\frac{n}{4}}$ under an $N(0, 1)$ distribution

R commands to perform the sign test

#x=number of subjects with di>0

#n=total number of subjects with di not equal to 0

#codes for one-sample binomial test with p0=0.5 with continuity correction

>prop.test(x,n,p=0.5, alternative="two.sided", correct=TRUE)

Computation of the p-Value for the Sign Test (Exact Method)

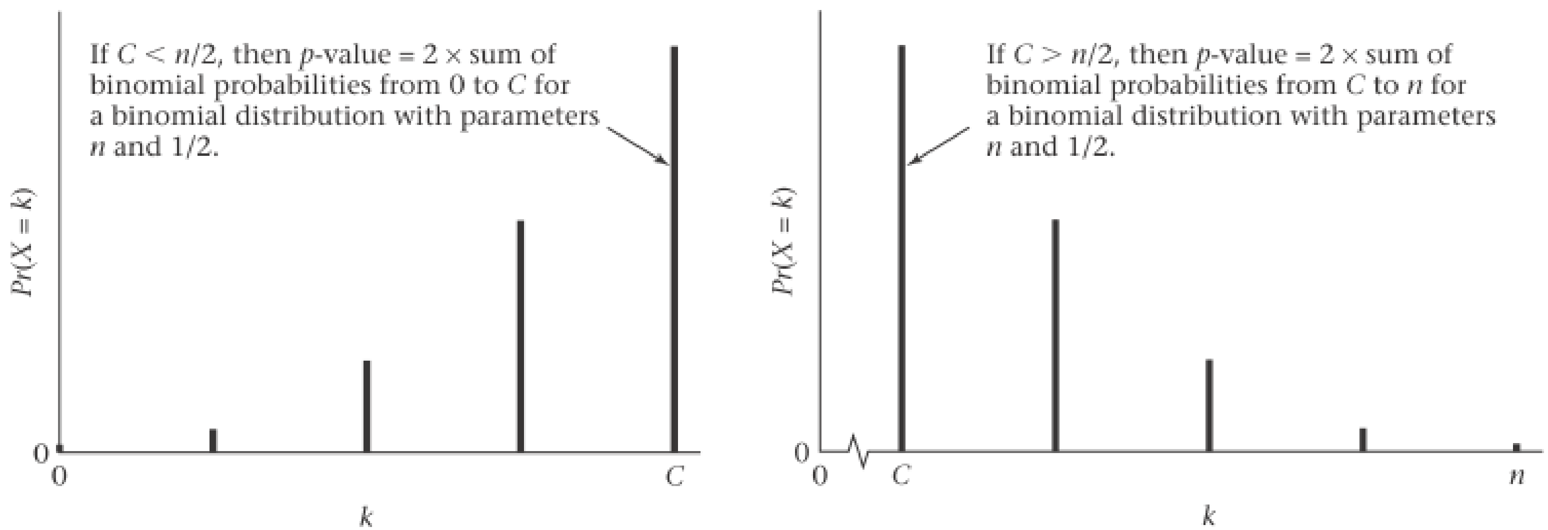
- **n < 20**: exact binomial probabilities to compute the p-value
- If C is very large or very small: reject H_0

$$\text{If } C > n/2, \quad p = 2 \times \sum_{k=C}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^{n}$$

$$\text{If } C < n/2, \quad p = 2 \times \sum_{k=0}^{C} \binom{n}{k} \left(\frac{1}{2}\right)^{n}$$

$$\text{If } C = n/2, \quad p = 1.0$$

Figure 9.3 Computation of the p-value for the sign test (exact test)



Example on Sign Test - Ophthalmology

- Suppose we wish to compare two different types of eye drops (A, B) that are intended to prevent redness in people with hay fever.

- Drug A is randomly administered to one eye and drug B to the other eye.

- The redness is noted at baseline and after 10 minutes by an observer who is unaware of which drug has been administered to which eye.

- We find that for 15 people with an equal amount of redness in each eye at baseline, after 10 minutes the drug A eye is less red than the drug B eye for 2 people (d_i < 0); the drug B eye is less red than the drug A eye for 8 people (d_i > 0); and the eyes are equally red for 5 people (d_i = 0).

- **Q: Assess the statistical significance of the results.**

Example on Sign Test - Ophthalmology

Solution:

The test is based on the 10 people who had a differential response to the two types of eye drops.

Because $n = 10 < 20$ →cannot use normal-theory method

→ exact method due to:

$C = 8 > \frac{10}{2} = 5, p = 2 \times \sum_{k=8}^{10} \binom{10}{k} (1/2)^{10}$

- binomial tables using

$n = 10, p = 0.5$, and note that $\Pr(X = 8) = .0439,$

$\Pr(X = 9) = .0098, \Pr(X = 10) = .001$

$p = 2 \times \Pr(X \geq 8) = 2(.0439 + 0.0098 + 0.0010)$

$= 2 \times 0.0547 = 0.109$ (not statistically significant)

- accept H_0: two types of eye drops are equally effective in reducing redness in people with hay fever

TABLE 1 Exact binomial probabilities $Pr(X = k) = \binom{n}{k} p^k q^{n-k}$ *(continued)*

n	k	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
	4	.0004	.0046	.0185	.0459	.0865	.1361	.1875	.2322	.2627	.2734
	5	.0000	.0004	.0026	.0092	.0231	.0467	.0808	.1239	.1719	.2188
	6	.0000	.0000	.0002	.0011	.0038	.0100	.0217	.0413	.0703	.1094
	7	.0000	.0000	.0000	.0001	.0004	.0012	.0033	.0079	.0164	.0313
	8	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0007	.0017	.0039
9	0	.6302	.3874	.2316	.1342	.0751	.0404	.0207	.0101	.0046	.0020
	1	.2985	.3874	.3679	.3020	.2253	.1556	.1004	.0605	.0339	.0176
	2	.0629	.1722	.2597	.3020	.3003	.2668	.2162	.1612	.1110	.0703
	3	.0077	.0446	.1069	.1762	.2336	.2668	.2716	.2508	.2119	.1641
	4	.0006	.0074	.0283	.0661	.1168	.1715	.2194	.2508	.2600	.2461
	5	.0000	.0008	.0050	.0165	.0389	.0735	.1181	.1672	.2128	.2461
	6	.0000	.0001	.0006	.0028	.0087	.0210	.0424	.0743	.1160	.1641
	7	.0000	.0000	.0000	.0003	.0012	.0039	.0098	.0212	.0407	.0703
	8	.0000	.0000	.0000	.0000	.0001	.0004	.0013	.0035	.0083	.0176
	9	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0003	.0008	.0020
10	0	.5987	.3487	.1969	.1074	.0563	.0282	.0135	.0060	.0025	.0010
	1	.3151	.3874	.3474	.2684	.1877	.1211	.0725	.0403	.0207	.0098
	2	.0746	.1937	.2759	.3020	.2816	.2335	.1757	.1209	.0763	.0439
	3	.0105	.0574	.1298	.2013	.2503	.2668	.2522	.2150	.1665	.1172
	4	.0010	.0112	.0401	.0881	.1460	.2001	.2377	.2508	.2384	.2051
	5	.0001	.0015	.0085	.0264	.0584	.1029	.1536	.2007	.2340	.2461
	6	.0000	.0001	.0012	.0055	.0162	.0368	.0689	.1115	.1596	.2051
	7	.0000	.0000	.0001	.0008	.0031	.0090	.0212	.0425	.0746	.1172
	8	.0000	.0000	.0000	.0001	.0004	.0014	.0043	.0106	.0229	.0439
	9	.0000	.0000	.0000	.0000	.0000	.0001	.0005	.0016	.0042	.0098
	10	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0003	.0010
11	0	.5688	.3138	.1673	.0859	.0422	.0198	.0088	.0036	.0014	.0005
	1	.3293	.3835	.3248	.2362	.1549	.0932	.0518	.0266	.0125	.0054
	2	.0867	.2131	.2866	.2953	.2581	.1998	.1395	.0887	.0513	.0269
	3	.0137	.0710	.1517	.2215	.2581	.2568	.2254	.1774	.1259	.0806
	4	.0014	.0158	.0536	.1107	.1721	.2201	.2428	.2365	.2060	.1611
	5	.0001	.0025	.0132	.0388	.0803	.1321	.1830	.2207	.2360	.2256
	6	.0000	.0003	.0023	.0097	.0268	.0566	.0985	.1471	.1931	.2256
	7	.0000	.0000	.0003	.0017	.0064	.0173	.0379	.0701	.1128	.1611
	8	.0000	.0000	.0000	.0002	.0011	.0037	.0102	.0234	.0462	.0806
	9	.0000	.0000	.0000	.0000	.0001	.0005	.0018	.0052	.0126	.0269
	10	.0000	.0000	.0000	.0000	.0000	.0000	.0002	.0007	.0021	.0054
	11	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0002	.0005
12	0	.5404	.2824	.1422	.0687	.0317	.0138	.0057	.0022	.0008	.0002
	1	.3413	.3766	.3012	.2062	.1267	.0712	.0368	.0174	.0075	.0029
	2	.0988	.2301	.2924	.2835	.2323	.1678	.1088	.0639	.0339	.0161
	3	.0173	.0852	.1720	.2362	.2581	.2397	.1954	.1419	.0923	.0537
	4	.0021	.0213	.0683	.1329	.1936	.2311	.2367	.2128	.1700	.1208
	5	.0002	.0038	.0193	.0532	.1032	.1585	.2039	.2270	.2225	.1934
	6	.0000	.0005	.0040	.0155	.0401	.0792	.1281	.1766	.2124	.2256
	7	.0000	.0000	.0006	.0033	.0115	.0291	.0591	.1009	.1489	.1934
	8	.0000	.0000	.0001	.0005	.0024	.0078	.0199	.0420	.0762	.1208
	9	.0000	.0000	.0000	.0001	.0004	.0015	.0048	.0125	.0277	.0537
	10	.0000	.0000	.0000	.0000	.0000	.0002	.0008	.0025	.0068	.0161
	11	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0003	.0010	.0029
	12	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002
13	0	.5133	.2542	.1209	.0550	.0238	.0097	.0037	.0013	.0004	.0001
	1	.3512	.3672	.2774	.1787	.1029	.0540	.0259	.0113	.0045	.0016
	2	.1109	.2448	.2937	.2680	.2059	.1388	.0836	.0453	.0220	.0095
	3	.0214	.0997	.1900	.2457	.2517	.2181	.1651	.1107	.0660	.0349
	4	.0028	.0277	.0838	.1535	.2097	.2337	.2222	.1845	.1350	.0873
	5	.0003	.0055	.0266	.0691	.1258	.1803	.2154	.2214	.1989	.1571

Modified Scenario on Dermatology

- New assumption: degree of burn can be quantified on a 10-point scale (10: worst burn; 1 no burn)
- Compute $d_i = x_i - y_i$ and x_i = degree of burn for ointment A and y_i = degree of burn for ointment B

If $d_i > 0$: ointment B is doing better than ointment A

If $d_i < 0$: ointment A is doing better than ointment B

E.g. $d_i = +5$: degree of redness is 5 units > on the ointment A arm than on the ointment B arm

$d_i = -3$: degree of redness is 3 units < on the ointment A arm than on the ointment B arm

- **Q: how can this additional information be used to test if the ointments are equally effectively?** **Wilcoxon Signed-Rank**

Wilcoxon Signed-Rank test's ranking procedure:

1.Arrange d_i (the differences) in order of *absolute value*

2.Count the number of differences with the same absolute value

3. Rank d_i from 1 (observation with the lowest absolute value) to n (highest absolute value)

4. Group of several observations with the same absolute value

- find the lowest rank in the range = 1+R and the highest rank in the range = G +R
- R = highest rank used prior to considering this group and G = the number of differences in the *range of ranks* for the group
- Assign the *average rank* = (lowest rank in the range + highest rank in the range)/2 as the rank for each difference in the group

The Wilcoxon Signed-Rank Test

H0: $\Delta = 0$ vs. H1: $\Delta \neq 0$

- Δ = median score difference between ointment A and B
- If $\Delta < 0$, then ointment A is better
- If $\Delta > 0$, then ointment B is better

Table 9.1 Difference in degree of redness between ointment A and ointment B arms after 10 minutes of exposure to sunlight

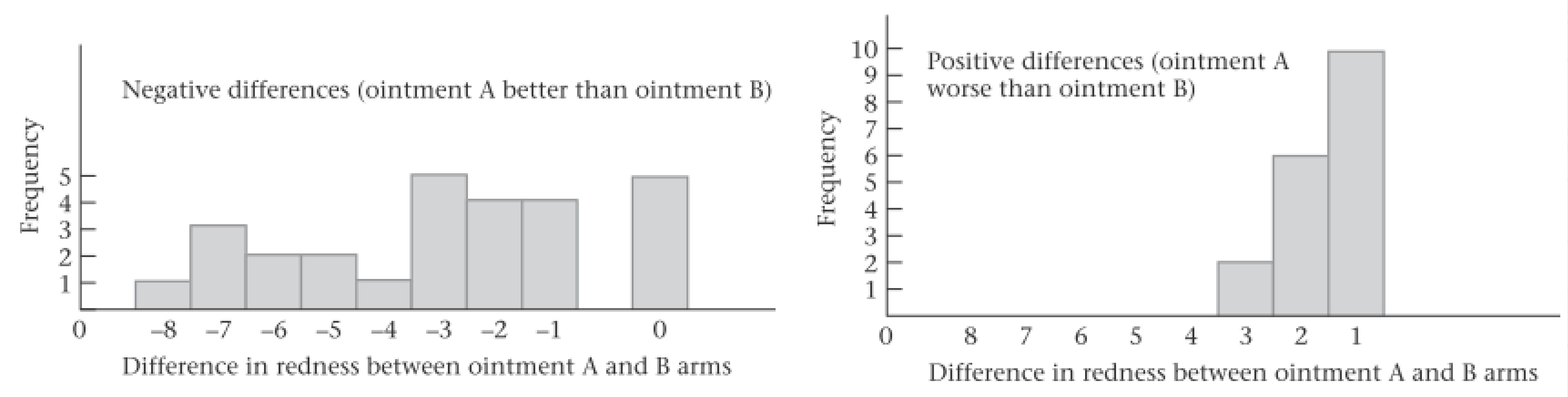
	Negative		Positive		Number of people with same absolute value	Range of ranks	Average rank
$\lvert d_i \rvert$	d_i	f_i	d_i	f_i			
10	−10	0	10	0		—	—
9	−9	0	9	0	0	—	—
8	−8	1	8	0	1	40	40.0
7	−7	3	7	0	3	37–39	38.0
6	−6	2	6	0	2	35–36	35.5
5	−5	2	5	0	2	33–34	33.5
4	−4	1	4	0	1	32	32.0
3	−3	5	3	2	7	25–31	28.0
2	−2	4	2	6	10	15–24	19.5
1	−1	4	1	10	14	1–14	7.5
		22		18			
0	0	5					

Question: Compute the ranks for the skin-ointment data in Table 9.1.

Solution:

- First collect the differences with the same absolute value
- Fourteen people have absolute value 1; this group has a rank range from 1 to 14 and an average rank of (1 + 14)/2 = 7.5.
- The group of 10 people with absolute value 2 has a rank range from (1 + 14) to (10 + 14) = 15 to 24 and an average rank = (15 + 24)/2 = 19.5, . . . , and so on. T
- The column 'Average rank' in table 9.1 includes the ranks for each row of absolute value.

Bar graph of the differences in redness between the ointment A and ointment B arms for the data in Example 9.10



Wilcoxon Signed-Rank test:

- nonparametric test that is analogous to the paired *t* test
- based on the ***ranks*** of the observations rather than on their actual values

Wilcoxon Signed-Rank Test (Normal Approximation Method for Two-Sided Level α Test)

***number of nonzero di's ≥16* (normal approximation → sampling distribution of R1)**

1. Rank the differences
2. Compute the rank sum R1 of the positive differences
3. If R1 $\neq$ [n(n+1)]/4 and there are no ties (no groups of differences with the same absolute value), then

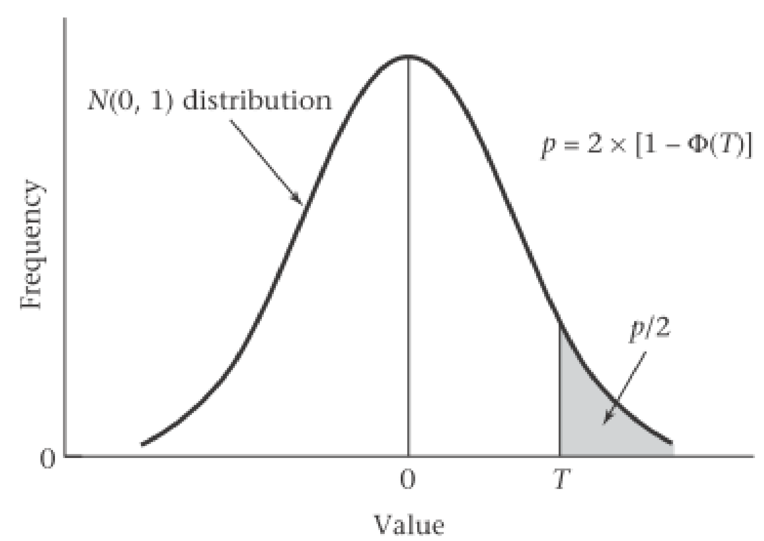
$$T = \left[\left|R_1 - \frac{n(n+1)}{4}\right| - \frac{1}{2}\right] \Big/ \sqrt{n(n+1)(2n+1)/24}$$

4. If R1 $\neq$ [n(n+1)]/4 and there are ties, where t_i refers to the number of differences with the same absolute value in the *i*th tied group and g is the number of tied groups, then

$$T = \left[\left|R_1 - \frac{n(n+1)}{4}\right| - \frac{1}{2}\right] \Big/ \sqrt{n(n+1)(2n+1)\big/24 - \sum_{i=1}^{g}(t_i^3 - t_i)/48}$$

5. If $T > z_{1-\alpha/2}$ then reject H_0. Otherwise, accept H_0.
6. The *p*-value for the test is given by $p = 2 \times [1-\Phi(T)]$
7. This test should be used only if the number of nonzero differences is ≥ 16 and if the difference scores have an underlying continuous symmetric distribution.

Figure 9.5 Computation of the *p*-value for the Wilcoxon signed-rank test



R Commands to perform the Wilcoxon Signed Rank Test

#For large sample method with one set of difference scores in variable x

>wilcox.test(x, y=NULL, alternative="two.sided", mu=0, paired=FALSE, exact=NULL, correct=TRUE, conf.int=FALSE)

#Two sets of paired scores in variables x and y

>wilcox.test(x, y, alternative="two.sided", mu=0, paired=TRUE, exact=NULL, correct=TRUE, conf.int=FALSE)

Example on Wilcoxon Signed-rank test - Dermatology

- **Question: Perform the Wilcoxon signed-rank test for the data below.**

Table 9.1 Difference in degree of redness between ointment A and ointment B arms after 10 minutes of exposure to sunlight

	Negative		Positive		Number of people with same absolute value	Range of ranks	Average rank
$\lvert d_i \rvert$	d_i	f_i	d_i	f_i			
10	−10	0	10	0		—	—
9	−9	0	9	0	0	—	—
8	−8	1	8	0	1	40	40.0
7	−7	3	7	0	3	37–39	38.0
6	−6	2	6	0	2	35–36	35.5
5	−5	2	5	0	2	33–34	33.5
4	−4	1	4	0	1	32	32.0
3	−3	5	3	2	7	25–31	28.0
2	−2	4	2	6	10	15–24	19.5
1	−1	4	1	10	14	1–14	7.5
		22		18			
0	0	5					

Example on Wilcoxon Signed-rank test - Dermatology

Solution: Because the number of nonzero differences $(22 + 18 = 40) \geq 16$, the normal approximation method can be used.

Compute the rank sum for the people with positive d_i—that is, where ointment B performs better than ointment A, as follows:

$$R_1 = 10(7.5) + 6(19.5) + 2(28.0) = 75 + 117 + 56 = 248$$

The expected rank sum is given by

$$E(R_1) = 40(41)/4 = 410$$

[n(n+1)]/4

The variance of the rank sum corrected for ties is given by

$$Var(R_1) = 40(41)(81)/24 - [(14^3 - 14) + (10^3 - 10) + (7^3 - 7) + (1^3 - 1) + (2^3 - 2) + (2^3 - 2) + (3^3 - 3) + (1^3 - 1)]/48 = 5535 - (2730 + 990 + 336 + 0 + 6 + 6 + 24 + 0) / 48 = 5535 - 4092 / 48 = 5449.75$$

$$T = \left[\left|R_1 - \frac{n(n+1)}{4}\right| - \frac{1}{2}\right] \Big/ \boxed{\sqrt{n(n+1)(2n+1)/24 - \sum_{i=1}^{g}(t_i^3 - t_i)/48}}$$

Example on Wilcoxon Signed-rank test - Dermatology

Thus, $sd(R_1) = \sqrt{5449.75} = 73.82$.

Therefore, the test statistic T is given by

$$T = \left[\left|R_1 - \frac{n(n+1)}{4}\right| - \frac{1}{2}\right] \Big/ \sqrt{n(n+1)(2n+1)\Big/24 - \sum_{i=1}^{g}(t_i^3 - t_i)/48}$$

$$T = \frac{|248 - 410| - 0.5}{73.82} = \frac{161.5}{73.82} = 2.19$$

The p-value of the test is given by $p = 2[1 - \Phi(2.19)] = 2 \times (1 - 0.9857) = 0.029$

$\boxed{p = 2 \times [1 - \Phi(T)]}$

Conclusion: there is a significant difference between ointments, with ointment A doing better than ointment B because the observed rank sum (248) is smaller than the expected rank sum (410).

TABLE 3 The normal distribution

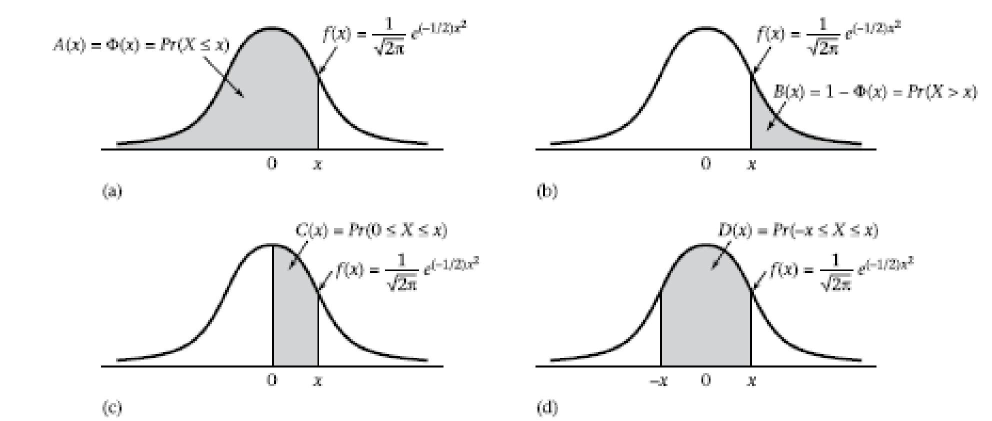


TABLE 3 The normal distribution *(continued)*

x	A[a]	B[b]	C[c]	D[d]
1,82	,9656	,0344	,4656	,9312
1,83	,9664	,0336	,4664	,9327
1,84	,9671	,0329	,4671	,9342
1,85	,9678	,0322	,4678	,9357
1,86	,9686	,0314	,4686	,9371
1,87	,9693	,0307	,4693	,9385
1,88	,9699	,0301	,4699	,9399
1,89	,9706	,0294	,4706	,9412
1,90	,9713	,0287	,4713	,9426
1,91	,9719	,0281	,4719	,9439
1,92	,9726	,0274	,4726	,9451
1,93	,9732	,0268	,4732	,9464
1,94	,9738	,0262	,4738	,9476
1,95	,9744	,0256	,4744	,9488
1,96	,9750	,0250	,4750	,9500
1,97	,9756	,0244	,4756	,9512
1,98	,9761	,0239	,4761	,9523
1,99	,9767	,0233	,4767	,9534
2.00	,9772	,0228	,4772	,9545
2,01	,9778	,0222	,4778	,9556
2.02	,9783	,0217	,4783	,9566
2,03	,9788	,0212	,4788	,9576
2,04	,9793	,0207	,4793	,9586
2,05	,9798	,0202	,4798	,9596
2,06	,9803	,0197	,4803	,9606
2,07	,9808	,0192	,4808	,9615
2,08	,9812	,0188	,4812	,9625
2,09	,9817	,0183	,4817	,9634
2,10	,9821	,0179	,4821	,9643
2,11	,9826	,0174	,4826	,9651
2,12	,9830	,0170	,4830	,9660
2,13	,9834	,0166	,4834	,9668
2,14	,9838	,0162	,4838	,9676
2,15	,9842	,0158	,4842	,9684
2,16	,9846	,0154	,4846	,9692
2,17	,9850	,0150	,4850	,9700
2,18	,9854	,0146	,4854	,9707
2,19	,9857	,0143	,4857	,9715
2,20	,9861	,0139	,4861	,9722
2,21	,9864	,0136	,4864	,9729
2,22	,9868	,0132	,4868	,9736
2,23	,9871	,0129	,4871	,9743
2,24	,9875	,0125	,4875	,9749
2,25	,9878	,0122	,4878	,9756
2,26	,9881	,0119	,4881	,9762
2,27	,9884	,0116	,4884	,9768
2,28	,9887	,0113	,4887	,9774
2,29	,9890	,0110	,4890	,9780
2,30	,9893	,0107	,4893	,9786
2,31	,9896	,0104	,4896	,9791
2,32	,9898	,0102	,4898	,9797
2,33	,9901	,0099	,4901	,9802
2,34	,9904	,0096	,4904	,9807
2,35	,9906	,0094	,4906	,9812
2,36	,9909	,0091	,4909	,9817
2,37	,9911	,0089	,4911	,9822
2,38	,9913	,0087	,4913	,9827

Scenario on Opthalmology

- Different genetic types of the disease retinitis pigmentosa (RP) are thought to have different rates of progression
  - Dominant form of the disease: slowest progression
  - Recessive form: next slowest
  - Gender-linked form: fastest progression
- Test the hypothesis by comparing the visual acuity of people who have different genetic types of RP
- Scenario: 25 people aged 10-19 with dominant disease and 30 people with gender-linked disease
- The best-corrected visual acuities (i.e. with appropriate glasses) in the better eye of these people are shown in the next slide.
- **Q: how can these data be used to test if the distribution of visual acuity is different in the two groups?**

Wilcoxon Rank Sum Test

- Nonparametric analog to **the *t* test for two independent samples**
- Hypothesis H_0: $F_D = F_{SL}$ vs. H_1: $F_D(x) = F_{SL}(x - \Delta)$ where $\Delta \neq 0$
- F_D = cdf of visual acuity for dominant grp
- F_{SL} = cdf of visual acuity for sex-linked grp
- Δ = location shift of cdf for the sex linked grp relative to dominant group

Table 9.3 Comparison of visual acuity in people ages 10–19 with dominant and sex-linked RP

Visual acuity	Dominant	Sex-linked	Combined sample	Range of ranks	Average rank
20–20	5	1	6	1–6	3.5
20–25	9	5	14	7–20	13.5
20–30	6	4	10	21–30	25.5
20–40	3	4	7	31–37	34.0
20–50	2	8	10	38–47	42.5
20–60	0	5	5	48–52	50.0
20–70	0	2	2	53–54	53.5
20–80	0	1	1	55	55.0
	25	30	55		

Ranking Procedure for the Wilcoxon Rank-Sum Test

1.Combine data from the two groups, order the values smallest to largest (or e.g. visual acuity from best [20-20] to worst [20-80])

2.Assign ranks to the individual values
e.g. lowest rank: with the best visual acuity (20-20)
highest rank: with worst visual acuity (20-80)
or vice versa

3. Group of observations has the same value
- compute the range of ranks for the group (similar to signed-rank test)
- assign the average rank for each observation in the group

Wilcoxon Rank-Sum Test (Normal Approximation Method for Two-Sided Level α Test)

1. Rank the observations
2. Compute the rank sum R1 in the first sample (the choice of sample is arbitrary)
3. If (i) $R_1 \neq n_1(n_1+n_2+1)/2$ and there are no ties, then compute

$$T = \left[\left|R_1 - \frac{n_1(n_1+n_2+1)}{2}\right| - \frac{1}{2}\right] \Big/ \sqrt{\left(\frac{n_1 n_2}{12}\right)(n_1+n_2+1)}$$

(ii) $R_1 \neq n_1(n_1+n_2+1)/2$ and there are ties, then compute

$$T = \left[\left|R_1 - \frac{n_1(n_1+n_2+1)}{2}\right| - \frac{1}{2}\right] \Bigg/ \sqrt{\left(\frac{n_1 n_2}{12}\right)\left[n_1+n_2+1 - \frac{\sum_{i=1}^{g} t_i(t_i^2-1)}{(n_1+n_2)(n_1+n_2-1)}\right]}$$

where t_i refers to the number of observations with the same value in the ith tied group and g is the number of tied groups

(iii) If $R1 = n1(n1+n2+1)/2$, then $T = 0$

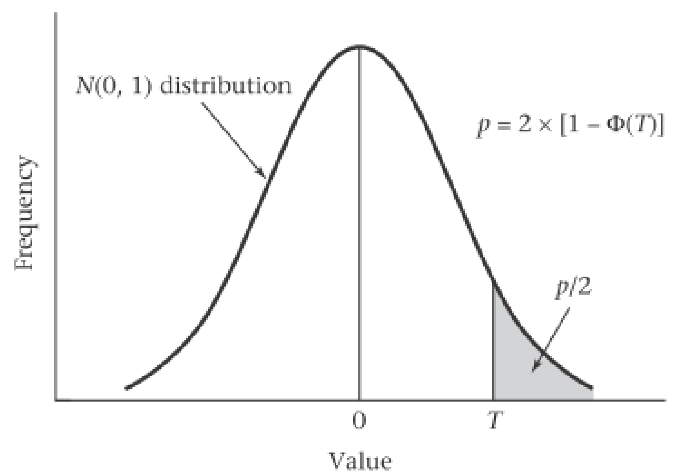
4.If $T > z_{1-\alpha/2}$ then reject H0. Otherwise, accept H0.

5.Compute the exact p-value by $p = 2 \times [1-\Phi(T)]$

5.Conditions:

- both **n1** and **n2** are $\geqslant 10$
- there is an underlying continuous distribution

Figure 9.6 Computation of the *p*-value for the Wilcoxon rank-sum test



Example on Wilcoxon Rank-sum Test - Ophthalmology

- **Question: Compute the ranks for the visual-acuity data in Table 9.3.**

Table 9.3 Comparison of visual acuity in people ages 10–19 with dominant and sex-linked RP

Visual acuity	Dominant	Sex-linked	Combined sample	Range of ranks	Average rank
20–20	5	1	6	1–6	3.5
20–25	9	5	14	7–20	13.5
20–30	6	4	10	21–30	25.5
20–40	3	4	7	31–37	34.0
20–50	2	8	10	38–47	42.5
20–60	0	5	5	48–52	50.0
20–70	0	2	2	53–54	53.5
20–80	0	1	1	55	55.0
	25	30	55		

Example on Wilcoxon Rank-sum Test
- Ophthalmology

Solution:

- First collect all people with the same visual acuity over the two groups, as shown in Table 9.3.

- There are 6 people with visual acuity 20–20 who have a rank range of 1–6 and are assigned an average rank of (1 + 6)/2 = 3.5.

- There are 14 people for the two groups combined with visual acuity 20–25. The rank range for this group is from (1 + 6) to (14 + 6) = 7 to 20.

- All people in this group are assigned the average rank = (7 + 20)/2 = 13.5, and similarly for the other groups. The column ‘Average rank’ in table 9.3 depicts the ranks as required.

Example on Wilcoxon Rank-sum Test - Ophthalmology

Question: Perform the Wilcoxon rank-sum test for the data in Table 9.3.

Solution:

Because the minimum sample size in the two samples is 25 ≥ 10,the normal approximation can be used.

The rank sum in the dominant group is given by

$R_1 = 5(3.5) + 9(13.5) + 6(25.5) + 3(34) + 2(42.5)$
$= 17.5 + 121.5 + 153 + 102 + 85 = 479$

Furthermore, $E(R_1) = 25(56)/2 = \frac{1400}{2} = 700$ $\boxed{n_1(n_1+n_2+1)/2}$

and $Var(R_1)$ corrected for ties is given by

$$[25(30)/12]\{56 - [6(6^2 - 1) + 14(14^2 - 1) + 10(10^2 - 1) + 7(7^2 - 1) + 10(10^2 - 1) + 5(5^2 - 1) + 2(2^2 - 1) + 1(1^2 - 1)]/[55(54)]\}$$
$$= 62.5(56 - 5382 / 2970) = 3386.74$$

$$T = \left[\left|R_1 - \frac{n_1(n_1+n_2+1)}{2}\right| - \frac{1}{2}\right] \Big/ \sqrt{\boxed{\left(\frac{n_1 n_2}{12}\right)\left[n_1 + n_2 + 1 - \frac{\sum_{i=1}^{g} t_i(t_i^2 - 1)}{(n_1+n_2)(n_1+n_2-1)}\right]}}$$

Wilcoxon Rank-sum Test - Ophthalmology

Therefore, the test statistic T is given by

$$T = \left[\left| R_1 - \frac{n_1(n_1+n_2+1)}{2} \right| - \frac{1}{2} \right] \Bigg/ \sqrt{\left(\frac{n_1 n_2}{12}\right)\left[n_1 + n_2 + 1 - \frac{\sum_{i=1}^{g} t_i(t_i^2-1)}{(n_1+n_2)(n_1+n_2-1)} \right]}$$

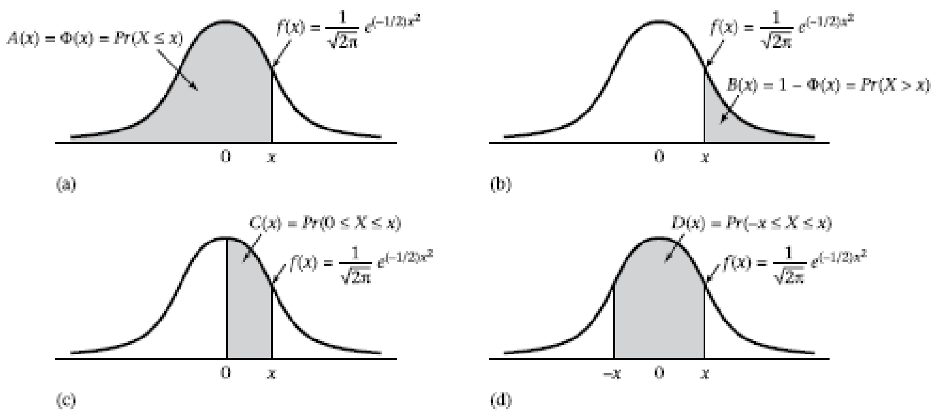
$$T = \frac{|479 - 700| - 0.5}{\sqrt{3386.74}} = \frac{220.5}{58.2} = 3.79$$

which follows an $N(0,1)$ distribution under H_0.

The p-value of the test is $p = 2[1 - \Phi(3.79)] < 0.001$

Conclusion: the visual acuities of the two groups are significantly different. Because the observed rank sum in the dominant group (479) is lower than the expected rank sum (700), the dominant group has better visual acuity than the sex-linked group.

TABLE 3 The normal distribution



x	A	B	C	D
3,49	,9998	,0002	,4998	,9995
3,50	,9998	,0002	,4998	,9995
3,51	,9998	,0002	,4998	,9996
3,52	,9998	,0002	,4998	,9996
3,53	,9998	,0002	,4998	,9996
3,54	,9998	,0002	,4998	,9996
3,55	,9998	,0002	,4998	,9996
3,56	,9998	,0002	,4998	,9996
3,57	,9998	,0002	,4998	,9996
3,58	,9998	,0002	,4998	,9997
3,59	,9998	,0002	,4998	,9997
3,60	,9998	,0002	,4998	,9997
3,61	,9998	,0002	,4998	,9997
3,62	,9999	,0001	,4999	,9997
3,63	,9999	,0001	,4999	,9997
3,64	,9999	,0001	,4999	,9997
3,65	,9999	,0001	,4999	,9997
3,66	,9999	,0001	,4999	,9997
3,67	,9999	,0001	,4999	,9998
3,68	,9999	,0001	,4999	,9998
3,69	,9999	,0001	,4999	,9998
3,70	,9999	,0001	,4999	,9998
3,71	,9999	,0001	,4999	,9998
3,72	,9999	,0001	,4999	,9998
3,73	,9999	,0001	,4999	,9998
3,74	,9999	,0001	,4999	,9998
3,75	,9999	,0001	,4999	,9998
3,76	,9999	,0001	,4999	,9998
3,77	,9999	,0001	,4999	,9998
3,78	,9999	,0001	,4999	,9998
3,79	,9999	,0001	,4999	,9998
3,80	,9999	,0001	,4999	,9999
3,81	,9999	,0001	,4999	,9999
3,82	,9999	,0001	,4999	,9999
3,83	,9999	,0001	,4999	,9999
3,84	,9999	,0001	,4999	,9999
3,85	,9999	,0001	,4999	,9999
3,86	,9999	,0001	,4999	,9999
3,87	,9999	,0001	,4999	,9999
3,88	,9999	,0001	,4999	,9999
3,89	,9999	,0001	,4999	,9999
3,90	1,0000	,0000	,5000	,9999
3,91	1,0000	,0000	,5000	,9999
3,92	1,0000	,0000	,5000	,9999
3,93	1,0000	,0000	,5000	,9999
3,94	1,0000	,0000	,5000	,9999
3,95	1,0000	,0000	,5000	,9999
3,96	1,0000	,0000	,5000	,9999
3,97	1,0000	,0000	,5000	,9999
3,98	1,0000	,0000	,5000	,9999
3,99	1,0000	,0000	,5000	,9999

TABLE 11 Two-tailed critical values for the Wilcoxon rank-sum test

	$\alpha = .10$ n_1[a]						$\alpha = .05$ n_1					
n_2[b]	4	5	6	7	8	9	4	5	6	7	8	9
	T_l[c] T_r[d]	T_l T_r	T_l T_r	T_l T_r	T_l T_r	T_l T_r	T_l T_r	T_l T_r	T_l T_r	T_l T_r	T_l T_r	T_l T_r
4	11–25	17–33	24–42	32–52	41–63	51–75	10–26	16–34	23–43	31–53	40–64	49–77
5	12–28	19–36	26–46	34–57	44–68	54–81	11–29	17–38	24–48	33–58	42–70	52–83
6	13–31	20–40	28–50	36–62	46–74	57–87	12–32	18–42	26–52	34–64	44–76	55–89
7	14–34	21–44	29–55	39–66	49–79	60–93	13–35	20–45	27–57	36–69	46–82	57–96
8	15–37	23–47	31–59	41–71	51–85	63–99	14–38	21–49	29–61	38–74	49–87	60–102
9	16–40	24–51	33–63	43–76	54–90	66–105	14–42	22–53	31–65	40–79	51–93	62–109
10	17–43	26–54	35–67	45–81	56–96	69–111	15–45	23–57	32–70	42–84	53–99	65–115
11	18–46	27–58	37–71	47–86	59–101	72–117	16–48	24–61	34–74	44–89	55–105	68–121
12	19–49	28–62	38–76	49–91	62–106	75–123	17–51	26–64	35–79	46–94	58–110	71–127
13	20–52	30–65	40–80	52–95	64–112	78–129	18–54	27–68	37–83	48–99	60–116	73–134
14	21–55	31–69	42–84	54–100	67–117	81–135	19–57	28–72	38–88	50–104	62–122	76–140
15	22–58	33–72	44–88	56–105	69–123	84–141	20–60	29–76	40–92	52–109	65–127	79–146
16	24–60	34–76	46–92	58–110	72–128	87–147	21–63	30–80	42–96	54–114	67–133	82–152
17	25–63	35–80	47–97	61–114	75–133	90–153	21–67	32–83	43–101	56–119	70–138	84–159
18	26–66	37–83	49–101	63–119	77–139	93–159	22–70	33–87	45–105	58–124	72–144	87–165
19	27–69	38–87	51–105	65–124	80–144	96–165	23–73	34–91	46–110	60–129	74–150	90–171
20	28–72	40–90	53–109	67–129	83–149	99–171	24–76	35–95	48–114	62–134	77–155	93–177
21	29–75	41–94	55–113	69–134	85–155	102–177	25–79	37–98	50–118	64–139	79–161	95–184
22	30–78	43–97	57–117	72–138	88–160	105–183	26–82	38–102	51–123	66–144	81–167	98–190
23	31–81	44–101	58–122	74–143	90–166	108–189	27–85	39–106	53–127	68–149	84–172	101–196
24	32–84	45–105	60–126	76–148	93–171	111–195	27–89	40–110	54–132	70–154	86–178	104–202
25	33–87	47–108	62–130	78–153	96–176	114–201	28–92	42–113	56–136	72–159	89–183	107–208
26	34–90	48–112	64–134	81–157	98–182	117–207	29–95	43–117	58–140	74–164	91–189	109–215
27	35–93	50–115	66–138	83–162	101–187	120–213	30–98	44–121	59–145	76–169	93–195	112–221
28	36–96	51–119	67–143	85–167	103–193	123–219	31–101	45–125	61–149	78–174	96–200	115–227
29	37–99	53–122	69–147	87–172	106–198	126–225	32–104	47–128	63–153	80–179	98–206	118–233
30	38–102	54–126	71–151	89–177	109–203	129–231	33–107	48–132	64–158	82–184	101–211	121–239
31	39–105	55–130	73–155	92–181	111–209	132–237	34–110	49–136	66–162	84–189	103–217	123–246
32	40–108	57–133	75–159	94–186	114–214	135–243	34–114	50–140	67–167	86–194	106–222	126–252
33	41–111	58–137	77–163	96–191	117–219	138–249	35–117	52–143	69–171	88–199	108–228	129–258
34	42–114	60–140	78–168	98–196	119–225	141–255	36–120	53–147	71–175	90–204	110–234	132–264
35	43–117	61–144	80–172	100–201	122–230	144–261	37–123	54–151	72–180	92–209	113–239	135–270
36	44–120	62–148	82–176	102–206	124–236	148–266	38–126	55–155	74–184	94–214	115–245	137–277
37	45–123	64–151	84–180	105–210	127–241	151–272	39–129	57–158	76–188	96–219	117–251	140–283
38	46–126	65–155	85–185	107–215	130–246	154–278	40–132	58–162	77–193	98–224	120–256	143–289
39	47–129	67–158	87–189	109–220	132–252	157–284	41–135	59–166	79–197	100–229	122–262	146–295
40	48–132	68–162	89–193	111–225	135–257	160–290	41–139	60–170	80–202	102–234	125–267	149–301
41	49–135	69–166	91–197	114–229	138–262	163–296	42–142	61–174	82–206	104–239	127–273	151–308
42	50–138	71–169	93–201	116–234	140–268	166–302	43–145	63–177	84–210	106–244	129–279	154–314
43	51–141	72–173	95–205	118–239	143–273	169–308	44–148	64–181	85–215	108–249	132–284	157–320
44	52–144	74–176	96–210	120–244	146–278	172–314	45–151	65–185	87–219	110–254	134–290	160–326
45	53–147	75–180	98–214	123–248	148–284	175–320	46–154	66–189	88–224	112–259	137–295	163–332
46	55–149	77–183	100–218	125–253	151–289	178–326	47–157	68–192	90–228	114–264	139–301	165–339
47	56–152	78–187	102–222	127–258	154–294	181–332	48–160	69–196	92–232	116–269	141–307	168–345
48	57–155	79–191	104–226	129–263	156–300	184–338	48–164	70–200	93–237	118–274	144–312	171–351
49	58–158	81–194	106–230	132–267	159–305	187–344	49–167	71–204	95–241	120–279	146–318	174–357
50	59–161	82–198	107–235	134–272	162–310	190–350	50–170	73–207	97–245	122–284	149–323	177–363

[a] n_1 = minimum of the two sample sizes.
[b] n_2 = maximum of the two sample sizes.
[c] T_l = lower critical value for the rank sum in the first sample.
[d] T_r = upper critical value for the rank sum in the first sample.

R commands to perform the Wilcoxon Rank Sum Test

#scores for the two groups

#x=values for one group and y=values for the other group

```
>wilcox.test(x,y,alternative="two.sided", mu=0, paired=FALSE, exact=NULL,
Correct=TRUE, conf.int=FALSE)
```

Summary

1. Advantage of nonparametric methods: assumption of normality can be relaxed when such assumptions are unreasonable

2. Disadvantage of nonparameteric procedures: some power is lost relative to using a parametric procedure (such as a t test) if the data truly follow a normal distribution (or central-limit theorem is applicable)

3. The sign test and the signed-rank test are nonparametric analogs to the **paired t test**.

- Sign test: to determine whether one member of a matched pair has a higher or lower score than the other member of the pair
- For the signed-rank test: the magnitude of the absolute value of the difference score and its sign is used in performing the significance test.

4. The Wilcoxon rank-sum test is an analog to the **two-sample t test for independent samples** in which the actual values are replaced by rank scores.

Parametric tests and analogous nonparametric procedures

Analysis Type	Example	Parametric Procedure	Nonparametric Procedure
Compare means between two distinct/independent groups	Is the mean systolic blood pressure (at baseline) for patients assigned to placebo different from the mean for patients assigned to the treatment group?	Two-sample t-test	Wilcoxon rank-sum test
Compare two quantitative measurements taken from the same individual	Was there a significant change in systolic blood pressure between baseline and the six-month follow-up measurement in the treatment group?	Paired t-test	Wilcoxon signed-rank test
Estimate the degree of association between two quantitative variables	Is systolic blood pressure associated with the patient's age?	Pearson coefficient of correlation	Spearman's rank correlation