1. If $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ is an eigen~~value~~ vector of $A = \begin{bmatrix} a & 2 & -2 \\ 2 & b & 0 \\ -2 & 0 & 7 \end{bmatrix}$, find the value for a, b.

Sol: $A\vec{v} = \lambda \vec{v}$ if $\vec{v}$ is an eigenvector of A.

$\Rightarrow$ left $= \begin{pmatrix} a & 2 & -2 \\ 2 & b & 0 \\ -2 & 0 & 7 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2a-4-2 \\ 4-2b+0 \\ -4+0+7 \end{pmatrix}$

Right $= \begin{pmatrix} 2\lambda \\ -2\lambda \\ \lambda \end{pmatrix}$

$$\Rightarrow \begin{cases} 2a-6 = 2\lambda \\ 4-2b = -2\lambda \\ -4+7 = \lambda \end{cases} \Rightarrow \begin{cases} 2a-6 = 2\cdot 3 \\ 4-2b = -2\cdot 3 \\ \lambda = 3 \end{cases} \Rightarrow \begin{cases} a = 6 \\ b = 5 \\ \lambda = 3 \end{cases}$$

To find all eigenvalues & eigenvectors of A.

$$\det(A - \lambda I) = 0$$

$$= \det\begin{pmatrix} 6-\lambda & 2 & -2 \\ 2 & 5-\lambda & 0 \\ -2 & 0 & 7-\lambda \end{pmatrix} = -2\begin{vmatrix} 2 & -2 \\ 5-\lambda & 0 \end{vmatrix} + (7-\lambda)\begin{vmatrix} 6-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix}$$

$$= -2\left(0 + 2(5-\lambda)\right) + (7-\lambda)\left[(6-\lambda)(5-\lambda) - 4\right].$$

$$= -\lambda^3 + 18\lambda^2 - 99\lambda + 9\cdot 18$$

$$= (\lambda - 3)(-\lambda^2 + 15\lambda - 54)$$

$$= (\lambda - 3)(-(\lambda-6)(\lambda-9)) \longrightarrow \begin{cases} \lambda_1 = 3 \\ \lambda_2 = 6 \\ \lambda_3 = 9. \end{cases}$$

$$
\begin{array}{r|l}
 & -\lambda^2 + 15\lambda - 54 \\
\hline
\lambda - 3 & -\lambda^3 + 18\lambda^2 - 99\lambda + 162 \\
 & -\lambda^3 + 3\lambda^2 \\
\hline
 & 15\lambda^2 - 99\lambda \\
 & 15\lambda^2 - 45\lambda \\
\hline
 & -54\lambda + 162 \\
 & -54\lambda + 162 \\
\hline
 & 0
\end{array}
$$

A is orthogonally diagonalizable. $A = PDP^T$.

$$\tilde{P} = \begin{pmatrix} 2 & \frac{1}{2} & -1 \\ -2 & 1 & -\frac{1}{2} \\ 1 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & & \\ & 6 & \\ & & 9 \end{pmatrix}$$

$\sqrt{2^2+2^2+1}$

$\sqrt{1+\left(\frac{1}{2}\right)^2+1} = \sqrt{\frac{9}{4}}$

$A = \tilde{P} D \tilde{P}^{-1}$

orthogonal matrix.

$$P = \begin{pmatrix} 2/3 & 1/3 & -2/3 \\ -2/3 & 2/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \end{pmatrix}$$

$A = PDP^T$.

$P^T = P^{-1}$

2. It is given the symmetric matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$. =

(a) find the eigenvalues of A;

(b) find the eigenvectors corresponding to each of these eigenvalues; $A = PDP^T$

(c) find an orthogonal matrix P such that $P^\top AP$ gives a diagonal matrix D and calculates P^{-1};

(d) Determine the eigenvalues of the matrix $B = A^5 + (A^2)^\top$.

$$B = A^5 + (A^2)^T$$

$$= (PDP^T)^5 + \left((PDP^T)^2\right)^T$$

$$= (PDP^TPDP^TP\cdots) + (PDP^TPDP^T)^T$$

$$= PD^5P^T + (PD^2P^T)^T$$

$$= PD^5P^T + PD^2P^T$$

$$= P(D^5 + D^2)P^T$$

$$= P\left[\begin{pmatrix} 1^5 & & \\ & 2^5 & \\ & & 3^5 \end{pmatrix} + \begin{pmatrix} 1^2 & & \\ & 2^2 & \\ & & 3^2 \end{pmatrix}\right]P^T$$

$$= P\begin{bmatrix} 2 & & \\ & 12 & \\ & & 3^5+3^2 \end{bmatrix}P^T.$$

eigenvalues of B are 2, 12, 3^5+3^2.

3. A quadratic form Q in the components $x_1, \ldots, x_n$ of a vector $\vec{x} = [x_1, \ldots, x_n]^\top$ with symmetric coefficient matrix $A = (a_{ij})_{1 \le i,j \le n}$ is defined to be

$$Q(\vec{x}) := \vec{x}^\top A \vec{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j.$$

Determine whether each of the following quadratic forms in two variables is positive or negative definite or semidefinite, or indefinite.

eigenvalues

LPM : $LPM_1 = a_{11} > 0$ $LPM_2 = \det \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$ positive

$LPM_1 < 0$ $LPM_2 > 0 \Rightarrow$ negative

echelon form.

(a) $3x_1^2 + 8x_1x_2 - 3x_2^2$.

(b) $9x_1^2 + 6x_1x_2 + x_2^2$. $= (x_1, x_2)\begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ positive semidefinite

$\lambda_1 > 0$

$\lambda_2 = 0$.

(c) $4x_1^2 + 12x_1x_2 + 13x_2^2$.

4. Determine the values of a for which the quadratic form $x^2 + 2xz + y^2 + 2ayz + 2z^2$ is positive definite.

(1)

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & a \\ 1 & a & 2 \end{pmatrix}$$

$Q(x,y,z) = (x, y, z)\begin{pmatrix} & & a_{31} \\ & & \\ a_{13} & & \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$= x^2 + 2xz + y^2 + \cdots$

$= a_{13}xz + a_{31}xz$

(2) $LPM_1 = a_{11} = 1 > 0$

$LPM_2 = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 > 0$

$LPM_3 = \det A = 1 \begin{vmatrix} 1 & a \\ a & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & a \end{vmatrix}$

$= 2 - a^2 + (-1) = 1 - a^2 > 0$

$\Downarrow$

$a^2 < 1$

$\Downarrow$

$|a| < 1 \Rightarrow -1 < a < 1$

$\Rightarrow \boxed{-1 < a < 1}$

5. Find the limit of f as $(x, y) \to (0,0)$ or show that the limit $\boxed{\text{does not exist.}}$

$$f(x,y) = \frac{2x}{x^2 + x + y^2}$$

$\ell_1 = \{(x,0),\ x>0\}$ x-axis

$\ell_2 = \{(0,y),\ y>0\}$ y-axis.

$$\lim_{(x,y) \xrightarrow{\ell_1} (0,0)} \frac{2x}{x^2+x+y^2} = \lim_{x \to 0^+} \frac{2x}{x^2+x+0} = \lim_{x\to 0^+} \frac{2}{x+1+0} = 2$$

$$\lim_{(x,y) \xrightarrow{\ell_2} (0,0)} \frac{2x}{x^2+x+y^2} = \lim_{y \to 0^+} \frac{2\cdot 0}{0^2+0+y^2} = 0$$

6. Let

$$f(x,y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases} \Rightarrow \text{one of inputs is } 0.$$

(a) Find the limit of f as (x, y) approaches $(0, 0)$ along the line $y = x$.

(b) Prove that f is not continuous at the origin.

(c) Show that both partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist at the origin.

a) $$\lim_{(x,y) \xrightarrow{\ell} (0,0)} f(x,y) = \lim_{\substack{x=y \\ x\to 0}} f(x,x) = \lim_{x\to 0} 0 = 0$$

$xy \neq 0,\ x \neq 0$

b) $$f(0,0) = 1 \neq \lim_{(x,y)\to(0,0)} f(x,y) \Rightarrow f \text{ is not continuous.}$$

c). $$\frac{\partial f}{\partial x}_{(0,0)} = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x \to 0} \frac{1-1}{x} = 0$$

7. It is given that $f(x,y) = x\cos y + ye^x$. Find all the first and second order partial derivatives of f,

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

8. Suppose $2z^3 - 2yz + x^2 = 1$ determines the function $z = z(x,y)$ as a function of x, y locally at $(x,y,z) = (1,1,1)$.

— output. input $z^3 \leftarrow z \leftarrow x, y$

$\frac{\partial z}{\partial x}$ $\frac{\partial z}{\partial y}$

   (a) Find the linear approximation of z at $(x,y,z) = (1,1,1)$.

   (b) Find the quadratic surface approximation of z at $(x,y,z) = (1,1,1)$.

Take $\frac{\partial}{\partial x}$ for both sides, treating y as constant, treating z as function in x

$$\frac{\partial}{\partial x}\left(2z^3 - 2yz + x^2\right) = \frac{\partial}{\partial x} 1$$

$$\Rightarrow 2\cdot 3z^2 \cdot \frac{\partial z}{\partial x} - 2y\frac{\partial z}{\partial x} + 2x = 0$$

$$\underset{y=1,\ z=1}{\overset{x=1}{\Rightarrow}} \quad 6\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial x} + 2 = 0 \Rightarrow \frac{\partial z}{\partial x}(1,1) = -\frac{2}{4}$$

Take $\frac{\partial}{\partial y}$ for both sides, treating x as constant, ... z as function in y.

$$\frac{\partial}{\partial y}\left(2z^3 - 2y\cdot z + x^2\right) = \frac{\partial}{\partial y} 1$$

$$\Rightarrow 2\cdot 3z^2\cdot\frac{\partial z}{\partial y} - 2\cdot\left(\frac{\partial y}{\partial y}\cdot z + y\cdot\frac{\partial z}{\partial y}\right) + 0 = 0$$

$$\underset{y=1,\ z=1}{\overset{x=1}{\Rightarrow}} \quad 6\frac{\partial z}{\partial y} - 2\cdot 1 - 2\cdot 1\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow \frac{\partial z}{\partial y}(1,1) = \frac{2}{4}$$

$$L(x,y) = z(1,1) + \frac{\partial z}{\partial x}(x-1) + \frac{\partial z}{\partial y}(y-1)$$
$$= 1 + \frac{-1}{2}(x-1) + \frac{1}{2}(y-1)$$

9. It is given that $f(x, y) = e^{2x} \sin 2y$.

   (a) Use Taylor's formula to find a linear approximation of $f(x, y)$ at the origin.

   (b) Estimate the error in the linear approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$.

10. Find the stationary points of the function $f(x, y) = xye^{-(x^2+2y^2)}$ and determine their nature.

Solution. From,

$$f_x = y(1 - 2x^2)e^{-x^2-2y^2} = 0, \quad f_y = x(1 - 4y^2)e^{-x^2-2y^2} = 0.$$

$f_y = \frac{1}{\sqrt{2}}(1-0) \neq 0$

which is equivalent to solving $y(1 - 2x^2) = 0$ and $x(1 - 4y^2) = 0$. We get

$$\begin{cases} y = 0 \text{ or } x = \pm\frac{1}{\sqrt{2}}, \\ x = 0 \text{ or } y = \pm\frac{1}{2}. \end{cases}$$

Hence, stationary points are $(0, 0), (\frac{1}{\sqrt{2}}, \frac{1}{2}), (\frac{1}{\sqrt{2}}, -\frac{1}{2}), (-\frac{1}{\sqrt{2}}, \frac{1}{2}), (-\frac{1}{\sqrt{2}}, -\frac{1}{2})$.

Note that

$$\begin{aligned} f_{xx} &= 2xy(2x^2 - 3)e^{-x^2-2y^2}, \\ f_{yy} &= 4xy(4y^2 - 3)e^{-x^2-2y^2}, \\ f_{xy} &= (1 - 2x^2)(1 - 4y^2)e^{-x^2-2y^2}. \end{aligned}$$

Then,

$$D = f_{xx}f_{yy} - f_{xy}^2 = e^{-2x^2-4y^2}[8x^2y^2(2x^2 - 3)(4y^2 - 3) - (1 - 2x^2)^2(1 - 4y^2)^2].$$

We have Table 1 showing the nature of the stationary points.

point	f_{xx}	f_{yy}	f_{xy}	$D = f_{xx}f_{yy} - f_{xy}^2$	Nature
$(0, 0)$	0	0	1	$-1 < 0$	saddle point
$(\frac{1}{\sqrt{2}}, \frac{1}{2})$	$\frac{-\sqrt{2}}{e} < 0$	$\frac{-2\sqrt{2}}{e}$	0	$\frac{4}{e^2}$	local max.
$(\frac{1}{\sqrt{2}}, -\frac{1}{2})$	$\frac{\sqrt{2}}{e} > 0$	$\frac{2\sqrt{2}}{e}$	0	$\frac{4}{e^2}$	local min.
$(-\frac{1}{\sqrt{2}}, \frac{1}{2})$	$\frac{\sqrt{2}}{e} > 0$	$\frac{2\sqrt{2}}{e}$	0	$\frac{4}{e^2}$	local min.
$(-\frac{1}{\sqrt{2}}, -\frac{1}{2})$	$\frac{-\sqrt{2}}{e} < 0$	$\frac{-2\sqrt{2}}{e}$	0	$\frac{4}{e^2}$	local max.

Table 1: Table for Q3

11. Let $f(x,y) = x^2 - xy + y^2 - y$. Find the directions $\vec{u}$ and the values of $D_{\vec{u}} f(1,-1)$ for which

(a) $D_{\vec{u}} f(1,-1)$ is the largest; $\Leftrightarrow$ along $\vec{u}$ f increases most rapidly $\Leftrightarrow$ $\vec{u} \parallel \nabla f$.

$\Leftrightarrow \vec{u} = \frac{\nabla f(1,-1)}{\|\nabla f(1,-1)\|} = \frac{(3,-4)}{5}$.

(b) $D_{\vec{u}} f(1,-1)$ is the smallest; $\Leftrightarrow$ along $\vec{u}$ f decreases most rapidly

$\Leftrightarrow \vec{u} = -\frac{\nabla f(1,-1)}{\|\nabla f(1,-1)\|} = \frac{(-3,4)}{5}$

(c) $D_{\vec{u}} f(1,-1) = 0$;

(d) $D_{\vec{u}} f(1,-1) = 4$;

(e) $D_{\vec{u}} f(1,-1) = -3$.

$D_{\vec{u}} f(1,-1) = \nabla f(1,-1) \cdot \frac{\vec{u}}{\|\vec{u}\|}$

inner product.

$\nabla f(1,-1) = (f_x, f_y)\Big|_{\substack{x=1\\y=-1}}$

$= (2x-y, -x+2y-1)\Big|_{\substack{x=1\\y=-1}}$

$= (3, -4)$.

c). $\vec{u} = \frac{(-4, 3)}{5} \perp \nabla f$

12. Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 13 & 5 & 2 \\ 2 & 7 & -8 \\ 5 & 4 & 7 \end{bmatrix}$.

13. If $\begin{pmatrix} 3 \\ 4 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvalue of $A = \begin{bmatrix} -1 & 0 & 12 & 0 \\ 0 & c & 0 & 12 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & -4 & -1 \end{bmatrix}$. Determine the value for c and find eigenvalues and eigenvectors of A.