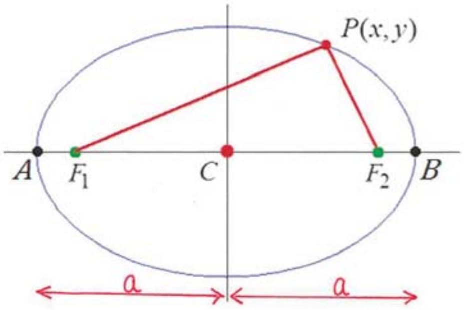
Conic Section Type 3: Ellipse

Definition: An ***ellipse*** is the set of all points P in a plane that the sum of the distances from P to two fixed points (called the ***foci***) is constant. The midpoint of the segment connecting the foci is the ***centre*** of the ellipse.



Take point B which lies on the ellipse. Then

$BF_1 + BF_2 = BF_1 + AF_1$
$= AB$
$= 2a$

- Let F_1 and F_2 be the two foci (the plural of focus).

  Furthermore, let $AC = CB = a$. Then $AB = 2a$.

  For any point P on the ellipse, $PF_1 + PF_2$ is a constant, which is equal to $2a$. That is,

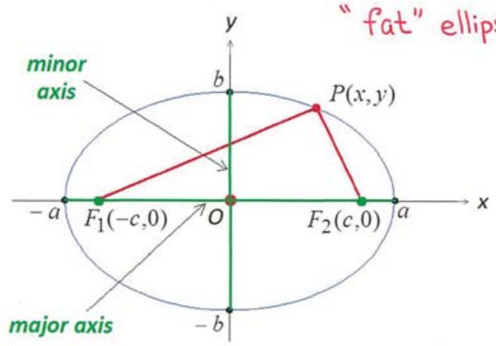
$$PF_1 + PF_2 = 2a.$$

Equation of ellipse

The equation of an ellipse with foci at the points $F_1(-c,0)$ and $F_2(c,0)$, where $c > 0$, is given by $\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$.

This is the **standard form** of the equation of an **ellipse centered at the origin**. Note that:

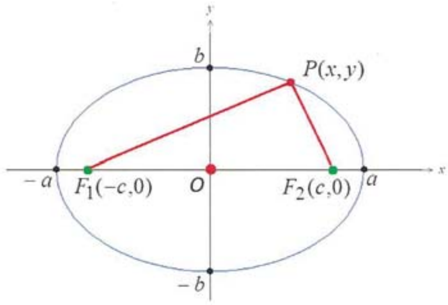
"fat" ellipse



- $a > b > 0$, $\boxed{c^2 = a^2 - b^2}$ ★ (bigger² − smaller²)
- The ***centre*** of this ellipse is at the origin $O(0,0)$.
- This ellipse is symmetrical about the x-axis and y-axis.
- The points $(a,0)$, $(-a,0)$, $(0,b)$ and $(0,-b)$ are called the ***vertices*** of the ellipse.
- The line segment joining the vertices $(a,0)$ and $(-a,0)$ is called the ***major axis***, and the line segment joining the vertices $(0,b)$ and $(0,-b)$ is called the ***minor axis***. The two foci are always on the major axis.

- The sum of the distances from any point on the ellipse to the two foci is $2a$, which is the length of the major axis.
- If $a = b$, the equation of the ellipse becomes $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \Rightarrow x^2 + y^2 = a^2$, which is the equation of a circle.

<u>*Proof of the equation*</u> $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ <u>(where $a > b$)</u>:



Let $P(x, y)$ be any point on the ellipse, which has foci at $F_1(-c, 0)$ and $F_2(c, 0)$, where $c > 0$. According to the definition of ellipse, we have

$$PF_1 + PF_2 = 2a$$

$\Rightarrow \quad \sqrt{(x-(-c))^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} = 2a$

$\Rightarrow \quad \sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$

$\Rightarrow \quad \sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$

Squaring both sides gives

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

$\Rightarrow \quad x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2$

$\Rightarrow \quad a\sqrt{(x-c)^2 + y^2} = a^2 - cx$

Squaring both sides again gives

$$a^2[(x-c)^2 + y^2] = a^4 - 2a^2cx + c^2x^2$$

$\Rightarrow \quad a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2 = a^4 - 2a^2cx + c^2x^2$

$\Rightarrow \quad (a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$

By the triangle inequality,

$$PF_1 + PF_2 > F_1F_2 \quad \Rightarrow \quad 2a > 2c \Rightarrow a > c.$$

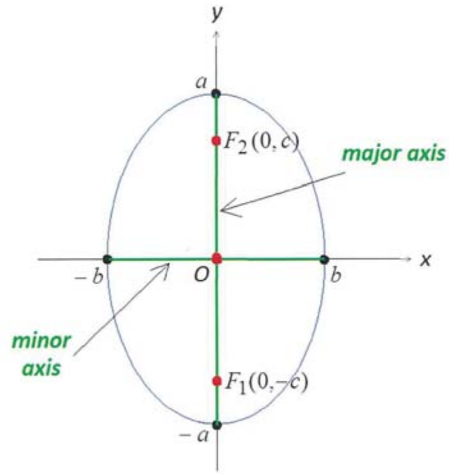
Thus, $a^2 > c^2 \Rightarrow a^2 - c^2 > 0$. Let $b^2 = a^2 - c^2 > 0$.

Then we have

$$b^2x^2 + a^2y^2 = a^2b^2 \quad \Rightarrow \quad \frac{b^2x^2}{a^2b^2} + \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2} \quad \Rightarrow \quad \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \qquad \square$$

Other type of ellipse (with centre at the origin):

Consider an ellipse with foci at the points $F_1(0,-c)$ and $F_2(0,c)$. Note that both foci lie on the y-axis (instead of the x-axis) and the centre of this ellipse is at the origin. This ellipse has the equation $\boxed{\frac{x^2}{b^2}+\frac{y^2}{a^2}=1}$, where $a>b>0$ and $c^2=a^2-b^2$. The sum of the distances from any point on the ellipse to the two foci is $2a$.



"Thin" ellipse

The results of the two types of ellipses are summarized in the following table:

Equation of ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>b>0)$	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ $(a>b>0)$
Centre	$C(0,0)$	$C(0,0)$
Foci	$F_1(-c,0)$ and $F_2(c,0)$, where $c^2=a^2-b^2$	$F_1(0,-c)$ and $F_2(0,c)$, where $c^2=a^2-b^2$
Vertices	$(a,0)$, $(-a,0)$, $(0,b)$ and $(0,-b)$	$(b,0)$, $(-b,0)$, $(0,a)$ and $(0,-a)$
Major axis	Line segment joining $(a,0)$ and $(-a,0)$ on the x-axis	Line segment joining $(0,a)$ and $(0,-a)$ on the y-axis
Minor axis	Line segment joining $(0,b)$ and $(0,-b)$ on the y-axis	Line segment joining $(b,0)$ and $(-b,0)$ on the x-axis
Shape	 "Fat"	 "Thin"

Remarks: (For ellipses)

a = distance between the centre & a vertex on major axis.

b = distance between the centre & a vertex on minor axis.

c = distance between the centre & a focus.

$$c = \sqrt{a^2 - b^2} > 0$$

Example 11

For each of the following ellipses, find the coordinates of the vertices and the foci, and sketch its graph: (a) $4x^2 + 9y^2 = 36$ (b) $8x^2 + y^2 = 8$

Solution

(a) Idea: Rearrange the equation into the standard form first. Then decide whether it is a "fat" or "thin" ellipse.

$$4x^2 + 9y^2 = 36 \quad \Rightarrow \quad \frac{4x^2}{36} + \frac{9y^2}{36} = \frac{36}{36} \quad \Rightarrow \quad \frac{x^2}{9} + \frac{y^2}{4} = 1 \quad \Rightarrow \quad \frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

(Since 3 > 2, the ellipse is a "fat" one.)

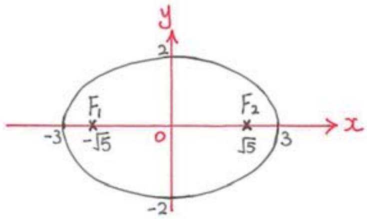
Take $a = \overset{\text{bigger number}}{3}$ and $b = \overset{\text{smaller number}}{2}$.

$\therefore$ The vertices of the ellipse are at $(3, 0)$, $(-3, 0)$, $(0, 2)$ and $(0, -2)$.

$c^2 = a^2 - b^2 = 3^2 - 2^2 = 5 \quad \Rightarrow \quad c = \pm\sqrt{5}$. Take $c = \sqrt{5}$ (since $c > 0$).

$\therefore$ The foci of the ellipse are at $F_1(-\sqrt{5}, 0)$ and $F_2(\sqrt{5}, 0)$.

<u>Sketch:</u>



(b) $8x^2 + y^2 = 8 \;\Rightarrow\; \frac{8x^2}{8} + \frac{y^2}{8} = \frac{8}{8} \;\Rightarrow\; \frac{x^2}{1} + \frac{y^2}{8} = 1 \;\Rightarrow\; \frac{x^2}{1^2} + \frac{y^2}{(2\sqrt{2})^2} = 1$

(Since $1 < 2\sqrt{2}$, the ellipse is a "thin" one.)

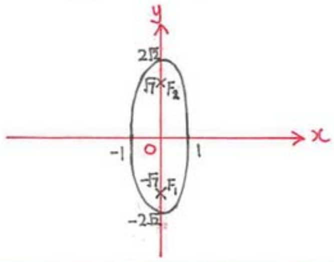
Take $a = 2\sqrt{2}$ and $b = 1$.

$\therefore$ The vertices of the ellipse are at $(1,0)$, $(-1,0)$, $(0,2\sqrt{2})$ and $(0,-2\sqrt{2})$.

$c^2 = a^2 - b^2 = (2\sqrt{2})^2 - 1^2 = 7 \;\Rightarrow\; c = \pm\sqrt{7}$. Take $c = \sqrt{7}$ (since $c > 0$).

$\therefore$ The foci of the ellipse are at $F_1(0,-\sqrt{7})$ and $F_2(0,\sqrt{7})$.

<u>Sketch:</u>



Example 12

Find the equation of the ellipse whose centre is the origin and the ellipse passes through the points $(2\sqrt{2},0)$ and $(-2,\sqrt{3})$.

<u>Solution</u>

An ellipse whose centre is at the origin has an equation of the form $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$, where $p, q > 0$. Since the ellipse passes through the point $(2\sqrt{2},0)$, we substitute $x = 2\sqrt{2}$ and $y = 0$ into the above equation and get $\frac{(2\sqrt{2})^2}{p^2} + \frac{0^2}{q^2} = 1 \Rightarrow p^2 = (2\sqrt{2})^2$.

Moreover, the ellipse also passes through the point $(-2,\sqrt{3})$, so we substitute $x = -2$ and $y = \sqrt{3}$ into the above equation and get $\frac{(-2)^2}{p^2} + \frac{(\sqrt{3})^2}{q^2} = 1$

$$\Rightarrow \frac{(-2)^2}{(2\sqrt{2})^2} + \frac{(\sqrt{3})^2}{q^2} = 1 \;\Rightarrow\; \frac{4}{8} + \frac{3}{q^2} = 1 \;\Rightarrow\; \frac{3}{q^2} = \frac{1}{2} \;\Rightarrow\; q^2 = 6 = (\sqrt{6})^2.$$

$\therefore$ The equation of the ellipse is $\frac{x^2}{(2\sqrt{2})^2} + \frac{y^2}{(\sqrt{6})^2} = 1$.

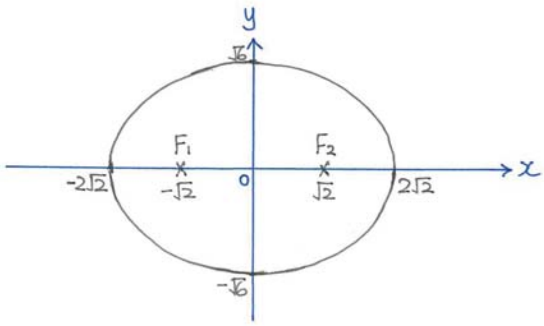
$$\frac{x^2}{(2\sqrt{2})^2} + \frac{y^2}{(\sqrt{6})^2} = 1$$

$$C = \sqrt{(2\sqrt{2})^2 - (\sqrt{6})^2} = \sqrt{2}$$

$\because \quad 2\sqrt{2} > \sqrt{6}$

$\therefore$ It's a fat ellipse.

$\therefore$ Foci : $F_1(\underbrace{-\sqrt{2}}_{-c}, 0)$ and $F_2(\underbrace{\sqrt{2}}_{c}, 0)$ ← on x-axis.



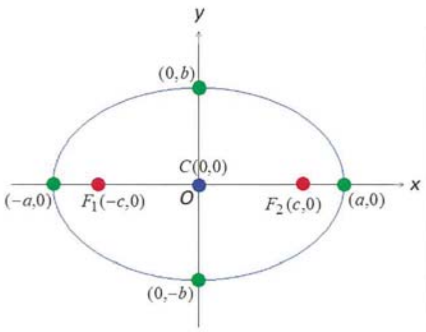
Translation of ellipse

If the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is translated h units to the right and k units upward, then we get **an ellipse with centre at $C(h, k)$**, and the equation of the new ellipse becomes

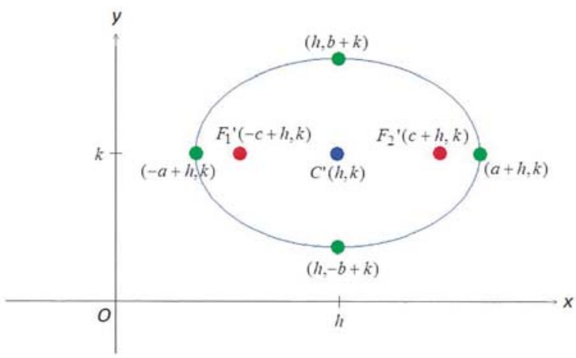
$$\boxed{\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.}$$

That is, we replace "x" with "$x - h$", and "y" with "$y - k$".

Consider the translation of a "fat" ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b > 0)$:



Translate $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ h units to the right and k units upward



	Before translation		After translation	
Equation	$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$	Translate $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ h units to the right and k units upward $\Longrightarrow$	$\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1.$	*
Centre	$C(0,0)$		$C'(h,k)$	**
Foci	$F_1(-c,0)$ and $F_2(c,0)$, where $c^2=a^2-b^2$		$F_1{}'(-c+h,k)$ and $F_2{}'(c+h,k)$, where $c^2=a^2-b^2$	**
Vertices	$(a,0)$, $(-a,0)$, $(0,b)$ and $(0,-b)$		$(a+h,k)$, $(-a+h,k)$, $(h,b+k)$ and $(h,-b+k)$	**

<u>Remark:</u> Similar method could be used to translate "thin" ellipses in horizontal/vertical directions.

* For equations, replace x with x-h, y with y-k.

** For points, +h to x-coordinate, +k to y-coordinate.

<u>Example 13</u>

Consider the equation $4y^2+25x^2-24y+200x+336=0$.

(a) Show that this equation represents an ellipse by rewriting the equation into the standard form of an ellipse.

(b) Find the coordinates of the centre, vertices and foci of this ellipse.

(c) Sketch the graph of this ellipse with the centre, vertices and foci clearly shown.

<u>Solution</u>

(a) Idea: Rewrite the equation into the standard form by "completing the square" first.

$$4y^2+25x^2-24y+200x+336=0$$

$$\Rightarrow\quad 4(y^2-6y)+25(x^2+8x)+336=0$$

$$\Rightarrow\quad 4[(y-3)^2-3^2]+25[(x+4)^2-4^2]+336=0$$

$$\Rightarrow\quad 4(y-3)^2-36+25(x+4)^2-400+336=0$$

$$\Rightarrow\quad 4(y-3)^2+25(x+4)^2=100$$

$$\Rightarrow \quad \frac{4(y-3)^2}{100}+\frac{25(x+4)^2}{100}=\frac{100}{100}$$

$$\Rightarrow \quad \frac{(y-3)^2}{25}+\frac{(x+4)^2}{4}=1$$

$$\Rightarrow \quad \frac{(y-3)^2}{5^2}+\frac{(x-(-4))^2}{2^2}=1,$$

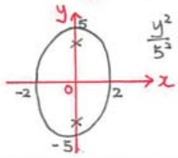
which represents an ellipse.

(b) Idea: Consider the coordinates of the centre, vertices and foci for the ellipse $\frac{y^2}{5^2}+\frac{x^2}{2^2}=1$. Then apply appropriate translations to obtain the required ellipse.

Take $a=5,\ b=2$. Then $c=\sqrt{a^2-b^2}=\sqrt{5^2-2^2}=\sqrt{21}$.

For the ellipse $\frac{y^2}{5^2}+\frac{x^2}{2^2}=1$, its centre is at $(0,0)$;
its vertices are at $(-2,0)$, $(2,0)$, $(0,-5)$ and $(0,5)$;
and its foci are at $(0,-\sqrt{21})$ and $(0,\sqrt{21})$.

$5>2$ $\therefore$ Thin ellipse.
$\therefore$ Foci on y-axis.



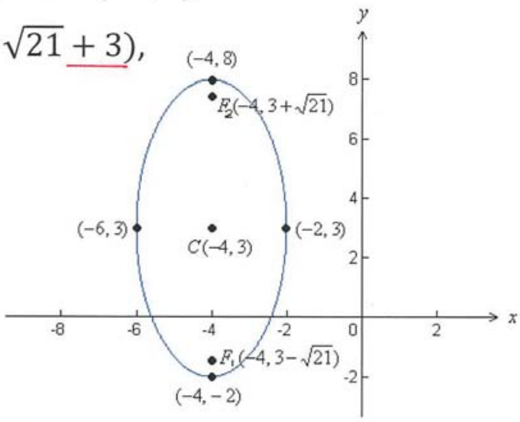
$\frac{y^2}{5^2}+\frac{x^2}{2^2}=1$

The graph of the ellipse $\frac{(y-3)^2}{5^2}+\frac{(x-(-4))^2}{2^2}=1$ is obtained by translating the graph of $\frac{y^2}{5^2}+\frac{x^2}{2^2}=1$ to the left by 4 units and upward by 3 units.

$\therefore$ For the ellipse $\frac{(y-3)^2}{5^2}+\frac{(x-(-4))^2}{2^2}=1$:

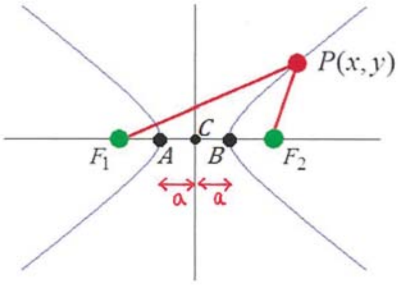
- its **centre** is at $(-4,3)$.
- its **vertices** are at $(-2-4,0+3)$, $(2-4,0+3)$, $(0-4,-5+3)$ and $(0-4,5+3)$, i.e. $(-6,3)$, $(-2,3)$, $(-4,-2)$ and $(-4,8)$.
- its **foci** are at $(0-4,-\sqrt{21}+3)$ and $(0-4,\ \sqrt{21}+3)$, i.e. $(-4,3-\sqrt{21})$ and $(-4,3+\sqrt{21})$.

(c) The graph of the ellipse $\frac{(y-3)^2}{5^2}+\frac{(x-(-4))^2}{2^2}=1$ is shown on the right.



Conic Section Type 4: Hyperbola

Definition: A ***hyperbola*** is the set of all points P in a plane that the *difference* of the distances from P to two fixed points (the ***foci***) is a constant.



Take the point B.
Then $BF_1 - BF_2$ is a constant.
Now $BF_1 - BF_2$
$= (AF_1 + AB) - AF_1$
$= AB$
$= 2a$

➢ Let F_1 and F_2 be the two foci.

Furthermore, let $AC = CB = a$. Then $AB = 2a$.

For any point P on the hyperbola, $|PF_1 - PF_2|$ is a constant, which is equal to $\underline{2a}$.

That is,

$$|PF_1 - PF_2| = 2a.$$

Equation of hyperbola

The equation of a hyperbola with foci at the points $F_1(-c,0)$ and $F_2(c,0)$, where $c>0$, is

a^2 is in the first term,
b^2 is in the second term.

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}.$$

x^2 is the first term
$\therefore$)(shape ("East-West openings")

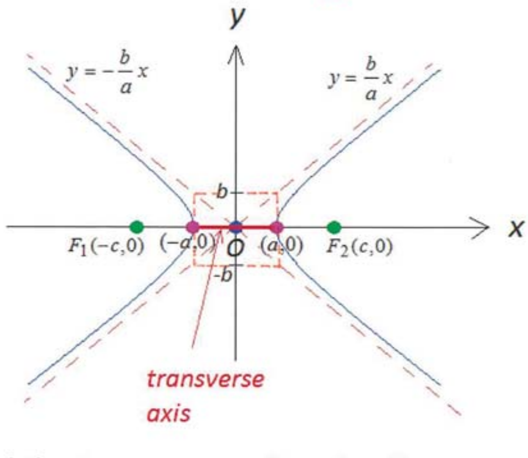
This is the **standard form of the equation of a hyperbola centered at the origin**. Note that:

➢ $a, b > 0$, $\boxed{c^2 = a^2 + b^2}$ ☆

➢ This hyperbola is symmetrical about the x-axis and y-axis.

➢ The points $(a,0)$ and $(-a,0)$ are called the ***vertices*** of the hyperbola.

➢ The ***centre*** of this hyperbola (the midpoint of two foci) is the origin $O(0,0)$.



➢ The line segment joining the two vertices is called the ***transverse axis***. ← length = 2a

➢ As x gets further away from the origin O, the two branches of the graph approach a pair of intersecting straight lines $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$, which are called the ***asymptotes*** of the hyperbola.

Solve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \boxed{0}$ to get

the asymptotes intersect at the centre of the hyperbola.

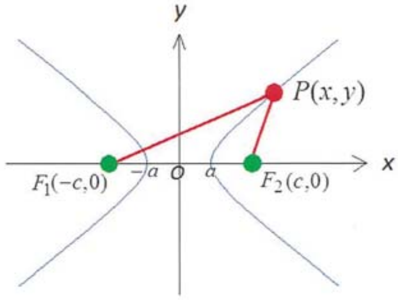
➢ The difference of the distances from any point on the hyperbola to the two foci is $2a$, which is the distance between the two vertices.

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow y = \pm\sqrt{b^2\left(\boxed{\frac{x^2}{a^2}-1}\right)} \approx \pm\frac{b}{a}x$ when x is large positive or large negative

$\approx \frac{x^2}{a^2}$ when x is large positive or large negative

<u>*Proof of the equation*</u> $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$:

Let $P(x,y)$ be any point on the hyperbola, which has foci at $F_1(-c,0)$ and $F_2(c,0)$, where $c > 0$. According to the definition of hyperbola, we have



$|PF_1 - PF_2| = 2a$

$\Rightarrow \quad |\sqrt{(x-(-c))^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2}| = 2a$

$\Rightarrow \quad |\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}| = 2a$

Squaring both sides gives

$$(x+c)^2 + y^2 - 2\sqrt{(x+c)^2 + y^2}\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2 = 4a^2$$

After some calculations, we get $(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2)$.

Clearly from the graph, $a < c$. Thus, $a^2 < c^2 \Rightarrow c^2 - a^2 > 0$.

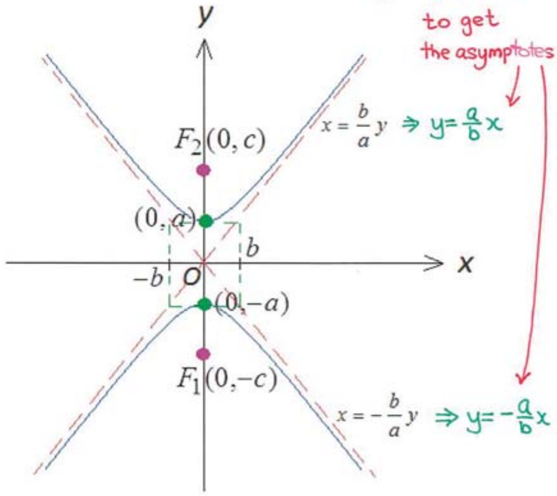
Let $b^2 = c^2 - a^2 > 0$. Then we have

$$b^2x^2 - a^2y^2 = a^2b^2 \Rightarrow \frac{b^2x^2}{a^2b^2} - \frac{a^2y^2}{a^2b^2} = \frac{a^2b^2}{a^2b^2} \Rightarrow \boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1} \qquad \square$$

<u>Other type of hyperbola (with centre at the origin):</u>

Solve $\frac{y^2}{a^2} - \frac{x^2}{b^2} = \boxed{0}$ to get the asymptotes

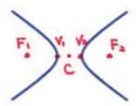
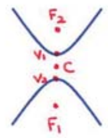
Consider a hyperbola with foci at the points $F_1(0,-c)$ and $F_2(0,c)$. Note that both foci lie on the y-axis (instead of the x-axis) and the centre of this hyperbola is at the origin. This hyperbola has the equation $\boxed{\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1}$, where $a, b > 0$ and $c^2 = a^2 + b^2$. The difference of the distances from any point on the hyperbola to the two foci is $2a$.



y^2 is in the first term

$\therefore$ "North-South openings" hyperbola

The two types of hyperbolae are summarized in the following table:

Equation of hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$
Centre	$C(0,0)$	$C(0,0)$
Foci	$F_1(-c,0)$ and $F_2(c,0)$, ← on x-axis where $c^2=a^2+b^2$	$F_1(0,-c)$ and $F_2(0,c)$, ← on y-axis where $c^2=a^2+b^2$
Vertices	$(a,0)$ and $(-a,0)$	$(0,a)$ and $(0,-a)$
Asymptotes	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\boxed{0} \Rightarrow y=\pm\frac{b}{a}x$	$\frac{y^2}{a^2}-\frac{x^2}{b^2}=\boxed{0} \Rightarrow y=\pm\frac{a}{b}x$
Shape	 "East-West openings"	 "North-South openings"

Note: The centre, foci and vertices are lying on the same straight line.

Example 14

Arrange the equation $9x^2-4y^2=144$ into the standard form of hyperbola. Find the coordinates of the centre, vertices and foci of this hyperbola, and sketch its graph.

Solution

$$9x^2-4y^2=144 \Rightarrow \frac{9x^2}{144}-\frac{4y^2}{144}=\frac{144}{144} \Rightarrow \frac{x^2}{16}-\frac{y^2}{36}=1 \Rightarrow \frac{x^2}{4^2}-\frac{y^2}{6^2}=1$$

This is a hyperbola with "East-West openings" and is centred at the origin $(0,0)$.

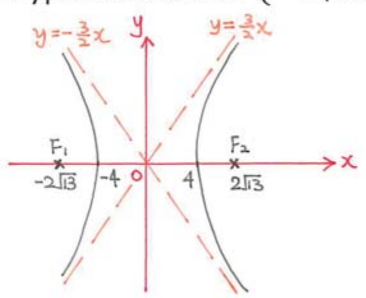
∵ the first term is the x^2 term

Its vertices are at $(4,0)$ and $(-4,0)$.

$c^2=a^2+b^2=4^2+6^2=52 \Rightarrow c=\sqrt{52}=2\sqrt{13}$ (Take positive value of c.)

∴ The foci of this hyperbola are at $(-2\sqrt{13},0)$ and $(2\sqrt{13},0)$.

Sketch:



Solve $\frac{x^2}{4^2}-\frac{y^2}{6^2}=\boxed{0}$

$\Rightarrow y=\pm\frac{6}{4}x=\pm\frac{3}{2}x$

∴ The asymptotes are $y=\pm\frac{3}{2}x$.

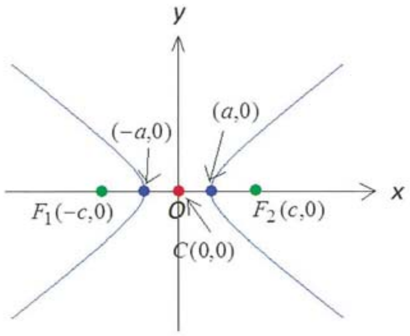
Translation of Hyperbola

Consider the translation of the hyperbola with "East-West openings" $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$:

If the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is translated h units to the right and k units upward, then we get a **hyperbola with centre at $C(h,k)$**, and the equation of the new hyperbola becomes

$$\boxed{\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1}.$$

That is, we replace "x" with "$x-h$", and "y" with "$y-k$".



Translate

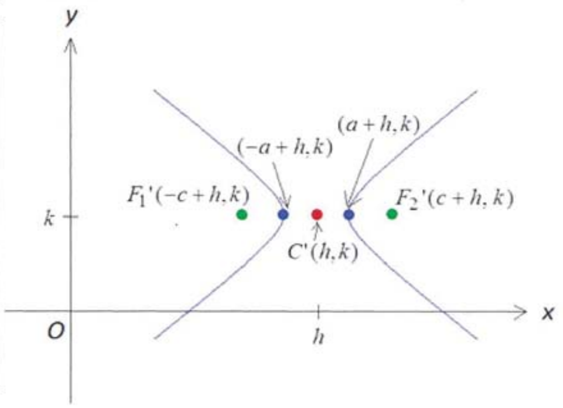
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

h units to the right

and

k units upward

$\Longrightarrow$



	Before translation		**After translation**
Equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	Translate $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ h units to the right and k units upward $\Longrightarrow$	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1.$
Centre	$C(0,0)$		$C'(h,k)$
Foci	$F_1(-c,0)$ and $F_2(c,0)$, where $c^2=a^2+b^2$		$F_1'(-c+h,k)$ and $F_2'(c+h,k)$, where $c^2=a^2+b^2$
Vertices	$(a,0)$, $(-a,0)$		$(a+h,k)$, $(-a+h,k)$
Asymptotes	$y=\pm\frac{b}{a}x$		$(y-k)=\pm\frac{b}{a}(x-h)$

Remark: Similar method could be done to translate hyperbola with "North-South openings" in horizontal/vertical directions. The equation of the new hyperbola becomes

$$\boxed{\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1}.$$

Remark: (For hyperbola)

$a =$ distance between vertex & centre.

$c =$ distance between focus & centre.

$b > 0$ is the value such that $c^2 = a^2 + b^2$.

Example 15

Consider the equation $16x^2 - 9y^2 - 160x - 18y + 247 = 0$.

(a) Show that this equation describes a hyperbola by writing the equation into the standard form of hyperbola.

(b) Find the coordinates of the centre, vertices and foci of this hyperbola.

(c) Sketch the graph of this hyperbola with the centre, vertices and foci clearly shown.

Solution

Solution:

(a) $16x^2 - 9y^2 - 160x - 18y + 247 = 0$

$$\Rightarrow 16(x^2 - 10x) - 9(y^2 + 2y) + 247 = 0$$

$$\Rightarrow 16[(x-5)^2 - 25] - 9[(y+1)^2 - 1] + 247 = 0$$

$$\Rightarrow 16(x-5)^2 - 9(y+1)^2 = 144$$

$$\Rightarrow \frac{(x-5)^2}{9} - \frac{(y+1)^2}{16} = 1$$

$$\Rightarrow \frac{(x-5)^2}{\underset{\uparrow\, a=3}{3^2}} - \frac{[y-(-1)]^2}{\underset{\uparrow\, b=4}{4^2}} = 1$$

which represents a hyperbola, centred at $(5, -1)$.

[Note: $\frac{(x-5)^2}{3^2}$ is the first term

$\therefore$ East - west openings)( ]

(b) For the hyperbola $\dfrac{(x-5)^2}{3^2} - \dfrac{[y-(-1)]^2}{4^2} = 1$,

its centre is at $(0+5, 0+(-1)) = (5,-1)$,

its vertices are at $(-3+5, 0-1) = (2,-1)$

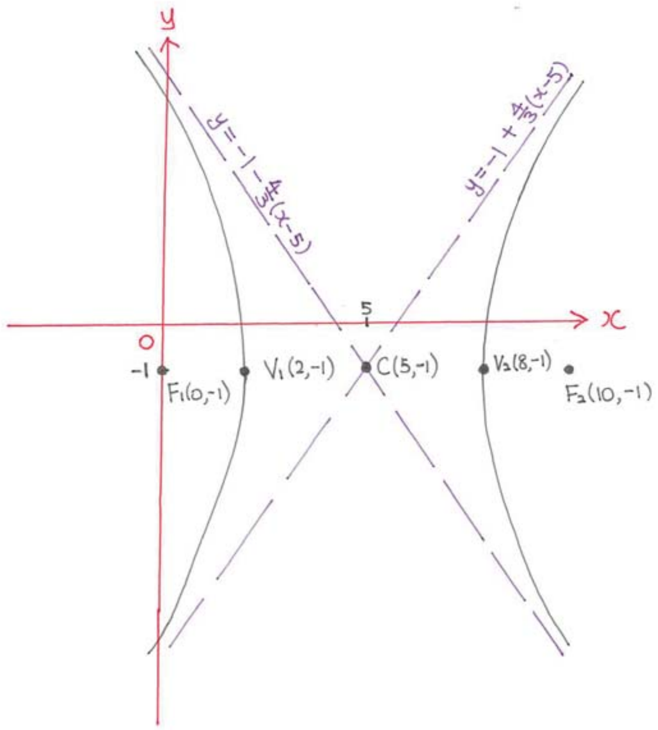
and $(3+5, 0-1) = (8,-1)$.

$c = \sqrt{a^2+b^2} = \sqrt{3^2+4^2} = 5$.

Its foci are at $(-5+5, 0-1) = (0,-1)$

and $(5+5, 0-1) = (10,-1)$.

(c)



To find the asymptotes, we solve

$$\frac{(x-5)^2}{3^2} - \frac{(y+1)^2}{4^2} = \boxed{0}$$

$$\Rightarrow y+1 = \pm \frac{4}{3}(x-5)$$

$$\Rightarrow y = -1 \pm \frac{4}{3}(x-5)$$