MA1200 Calculus and Basic Linear Algebra I
Chapter 8 Applications of Derivatives

1 Dynamical applications (p.306 – p.312, p.348 – p.352)

The concept of derivatives can be applied to solve some physical problems. Consider an object which is moving along a straight line. We let

$x(t)=$ the position of the object at time t,

$v(t)=$ the velocity of the object at time t,

$a(t)=$ the acceleration of the object at time t,

then $v(t)=\dfrac{dx}{dt}$ and $a(t)=\dfrac{dv}{dt}=\dfrac{d^2x}{dt^2}$. In general, if y is a function of x, then the derivative $\dfrac{dy}{dx}$ is interpreted as the *rate of change* of y with respect to x.

Example 1

If the velocity $v(t)$ of a body varies inversely as the square root of the distance $s(t)$, prove that the acceleration $a(t)$ varies as the fourth power of the velocity.

Solution:

Now, we have $v(t)=\dfrac{k}{\sqrt{s(t)}}$, where k is a constant.

$$a(t)=\frac{dv}{dt}=\frac{dv}{ds}\frac{ds}{dt}\underset{\frac{ds}{dt}=v(t)}{=}v(t)\frac{dv}{ds}$$. Observe that $v=\dfrac{k}{\sqrt{s}}=ks^{-\frac{1}{2}}\Rightarrow\dfrac{dv}{ds}=-\dfrac{1}{2}ks^{-\frac{3}{2}}$.

So

$$a(t)=v(t)\left(-\frac{1}{2}k\left[s(t)\right]^{-\frac{3}{2}}\right)\underset{\substack{v(t)=\frac{k}{\sqrt{s(t)}}\\ \Rightarrow s(t)=\left[\frac{k}{v(t)}\right]^2}}{=}-\frac{kv(t)}{2}\left(\left[\frac{k}{v(t)}\right]^2\right)^{-\frac{3}{2}}=-\frac{kv(t)}{2}\left[\frac{k}{v(t)}\right]^{-3}=-\frac{kv(t)}{2}\frac{1}{\frac{k^3}{\left[v(t)\right]^3}}=-\frac{\left[v(t)\right]^4}{2k^2}$$

.

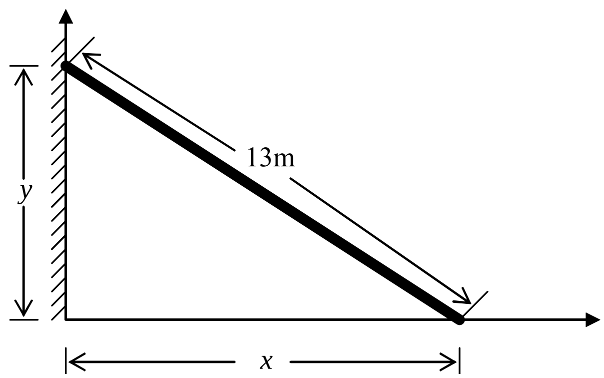
∎

Example 2

A ladder 13m long rests against a wall which is perpendicular to the level ground along which the lower end of the ladder slips at the rate 4m/s. How fast is the upper end of the ladder running down the wall when the lower end is 12m from the wall?

Solution:

The situation can be described by the following diagram:



Let $x =$ distance of the lower end of the ladder from 0 at time *t*.
$y =$ height of the upper end of the ladder above the level ground at time *t*.

$x^2 + y^2 = 169 \Rightarrow 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$, when $x = 12$, we have $2\times 12\times 4 + 2\times 5\dfrac{dy}{dt} = 0$.

Thus, $\dfrac{dy}{dt} = -\dfrac{48}{5}$ m/s . (-ve sign means that the upper end of the ladder is moving towards the ground.)

2 Local extrema of functions

It is found that the first and second derivatives of a differentiable function $y = f(x)$ can give us essential information.

2.1 Properties that are related to first derivative (p.370 – p.378)

Let $y = f(x)$ be a function which is differentiable at every x of an interval (a,b). Then we have the following result.

Theorem A

Let f be continuous on $[a,b]$ and differentiable in (a,b). Then

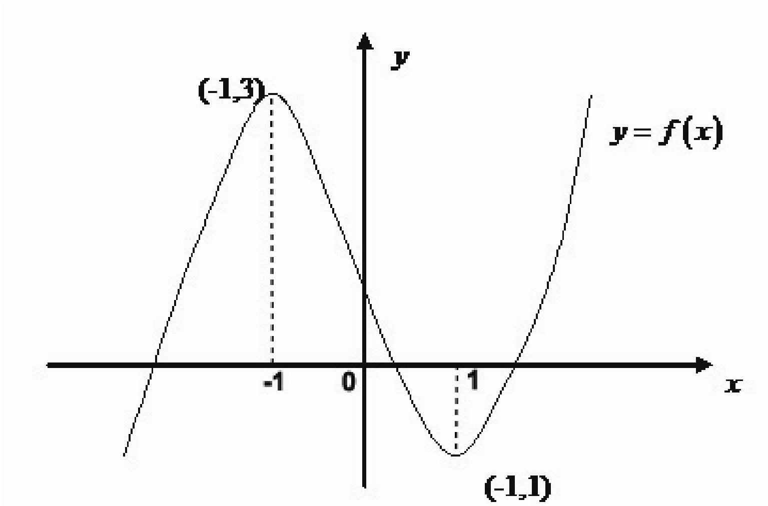
(i) f is increasing (strictly increasing) on $[a,b]$ if and only if $f'(x) \geq 0$ ($f'(x) > 0$) for all $x \in (a,b)$.

(ii) f is decreasing (strictly decreasing) on $[a,b]$ if and only if $f'(x) \leq 0$ ($f'(x) < 0$) for all $x \in (a,b)$.

Example 3

Let $f(x) = y = x^3 - 3x + 1$. We have $\dfrac{dy}{dx} = 3x^2 - 3$. We find that $\dfrac{dy}{dx} > 0$ when $x < -1$ or $x > 1$ and $\dfrac{dy}{dx} < 0$ when $-1 < x < 1$.

Therefore we know that f is strictly increasing on $(-\infty,-1]$ and $[1,+\infty)$ and strictly decreasing on $[-1,1]$. (Why?) From this information, a rough sketch can be obtained:



Apart from curve sketching, <u>theorem A</u> can be applied to prove inequalities and indicated in the following example.

□

<u>Example 4</u>

Show that $\ln(1+x) > x - \frac{x^2}{2}$ for $x > 0$.

<u>Solution</u>:

We let $f(x) = \ln(1+x) - x - \frac{x^2}{2}$. Since $f'(x) = \frac{1}{1+x} - 1 + x = \frac{x^2}{1+x} > 0$ for $x > 0$, we know that f is increasing on $(0,+\infty)$.

Thus, $f(x) > f(0)$ for any $x > 0$. It follows that $\ln(1+x) - x + \frac{x^2}{2} > f(0) = 0$ ($x > 0$) and $\ln(1+x) > x - \frac{x^2}{2}$ for $x > 0$.

□

<u>Exercise</u>

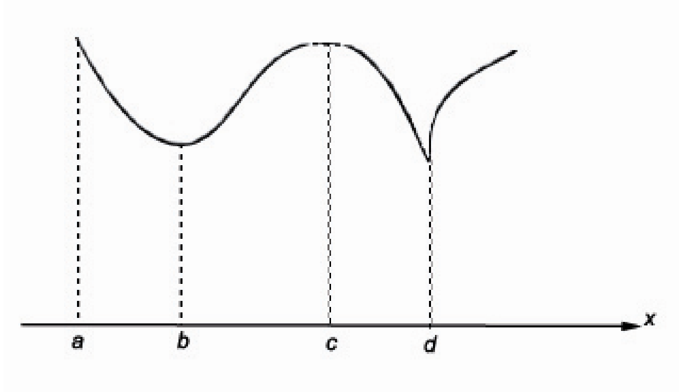
Show that $\ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ for $x > 0$.

Another important feature in the graph of $y = f(x)$ are the *local maximum* and *local minimum* value of f. They are defined as follows:

Definitions

1. A function f is said to have relative or *local minimum* at $x = c$ if there is an interval $(c-\delta, c+\delta)$ such that f is strictly decreasing on $(c-\delta, c)$ and is strictly increasing on $(c, c+\delta)$. If $f(x) \geq f(c)$ for all *x* in the domain, then c is the absolute minimum of *f*.
2. A function f is said to have relative or *local maximum* at $x = c$ if there is an interval $(c-\delta, c+\delta)$ such that f is strictly increasing on $(c-\delta, c)$ and is strictly decreasing on $(c, c+\delta)$. $f(x) \leq f(c)$ for all *x* in the domain, then c is the absolute maximum of *f*.

The following graph explains the similarity and differences between these concepts:



Location	Nature
a	absolute maximum
b	$f'(b) = 0$, local minimum.
c	$f'(c) = 0$, local maximum.
d	no $f'(d)$, local minimum, absolute minimum.

From the preceding graph, it is reasonable to have the first derivative test for local extrema:

Theorem B

If there is a neighbourhood of *c* in which $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then f has a local minimum at $x = c$.

Theorem C

If there is a neighbourhood of *c* in which $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then f has a local maximum at $x = c$.

Theorem D

Suppose f has a local maximum or minimum point at $x = c$. If $f'(c)$ exists, then $f'(c) = 0$.

Definition: $x = c$ is said to be a *stationary point* (*critical point*) of *f* if $f'(c) = 0$.

Example 5
Determine the extrema of the function $f(x) = x^3 + x^2 - x + 1$.

Solution:
$f'(x) = 3x^2 + 2x - 1 = (3x-1)(x+1)$.

Solving $f'(x) = 0$, we obtain the extremal points $x_1 = -1$ and $x_2 = \frac{1}{3}$.

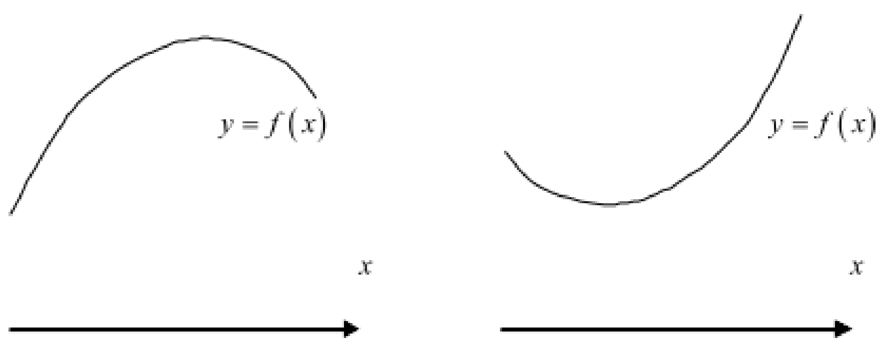
	$x < -1$	$x = -1$	$-1 < x < \frac{1}{3}$	$x = \frac{1}{3}$	$x > \frac{1}{3}$
Sign of $f'(x)$	+	0	–	0	+

The signs of the derivative of $f(x)$ indicate there is a local maximum at $x = -1$ and a local minimum at $x = \frac{1}{3}$.

□

2.2 Second derivative test for local extremum (p.393 – p.396)

We shall examine the relationship between the second derivative of f (if exists) and the graph of $y = f(x)$. First, we say the graph of a differentiable function $y = f(x)$ is *concave down* on an interval where $f'(x)$ decreases and *concave up* on an interval where $f'(x)$ increases.



concave down　　　　concave up

To test the concavity for the graph $y = f(x)$, we may consider the second derivative of *f*.

Theorem E
The graph of $y = f(x)$ is concave down on any interval where $f''(x) \le 0$, concave up on any interval where $f''(x) \ge 0$.

Illustration
The function $y = x^3 + x + 1$ is concave down on $(-\infty, 0]$ and concave up on $[0, +\infty)$. It is because $y'' = 6x \ge 0$ for $x \ge 0$, and $y'' \le 0$ for $x \le 0$. Second derivatives can also be applied to determine the local maxima and local minima of a given function.

Theorem F

Let f be a differentiable function in some neighbourhood of a point c such that $f'(c)=0$ and $f''(c)$ exists.

1. If $f''(c)<0$, then $x=c$ gives a local maximum point of the curve $y=f(x)$.
2. If $f''(c)>0$, then $x=c$ gives a local minimum.
3. If $f''(c)=0$ and $f''(x)$ is positive on one side of $x=c$ and negative on the other, then $x=c$ gives a point of inflexion (inflection).

N.B. A point $(c, f(c))$ on a curve $y=f(x)$ is a point of inflexion if $f''(c)=0$ and if the graph of $y=f(x)$ is concave upward on one side of $x=c$ and concave downward on the other.

Illustration

Let $f(x)=\frac{1}{3}x^3-x^2-3x+4$. We have

$$f'(x)=x^2-2x-3=(x+1)(x-3)$$
$$f''(x)=2x-2$$

Hence $x=-1$ and 3 are the stationary points which satisfy $f'(x)=0$. As $f''(-1)=-4<0$ and $f''(3)=4>0$, we conclude that f has a local maximum at $x=-1$ and a local minimum at $x=3$.

Example 6

Consider the following functions (i) $y=x^3$, (ii) $y=x^4$.

For (i), the curve has a point of inflexion at $x=0$ since $y''=6x$ changes sign there.

For (ii), the curve does not have a point of inflexion at $x=0$ although $y''=0$ there. (Why?)

□

2.3 Optimization problems (p.404 – p.411)

In this section, we will apply the techniques in the previous sections to solve some practical problems. Our main concern is to solve the following problem:

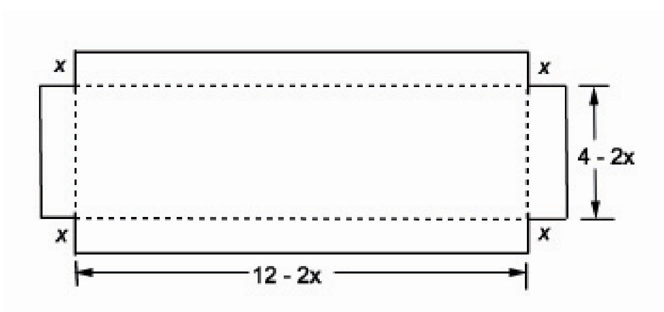
We want to determine the (absolute) maximum or minimum values of $f(x)$ where the domain f is a subset of **R** (which is normally an interval.)

We will see that many practical problems can be restated as above.

Example 7

Find the dimensions of a box without cover that will yield the largest volume, if it is to be constructed from a rectangular metal plate, 4 m by 12 m.

Solution:



Let x m be the height of the box. Then $(12-2x)$m is the length of the box and $(4-2x)$m is the width of the box. The volume of the box is given by

$$\begin{aligned} V(x) &= x(4-2x)(12-2x) \text{ m}^3 \\ &= 4x(2-x)(6-x) \text{ m}^3 \\ &= 4\left(12x-8x^2+x^3\right) \text{ m}^3 \end{aligned}$$

Moreover, $V'(x)=4\left(12-16x+3x^2\right)$. If $V'(x)=0$, then $x=0.903,\ 4.43$ (to 3 sig. fig.).

Because the dimensions are non-negative, x can only take values in $(0,2)$. However, 4.43 is not in $(0,2)$.

$V''(x)=4(-16+6x) \Rightarrow V''(0.903)<0 \Rightarrow V(x)$ has a local maximum at $x=0.903$

The values $V(0)=0=V(2)$ and $V(0.903)\approx 20.2$ reveal that the absolute maximum is at $x=0.903$.

$\therefore$ The dimensions of the largest box are 10.2 m by 2.2 m by 0.903 m.

□

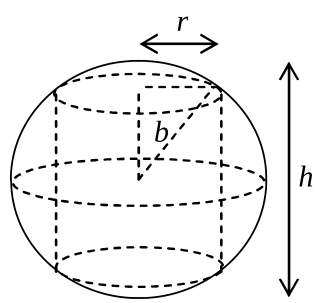
Example 8

Find the greatest volume of a cylinder that can be inscribed in a sphere of radius *b*.

Solution:

Let *h* and *r* be the height and radius of the cylinder respectively.

Step 1: It is usually useful to represent to situation by a figure first:



Step 2: In order to apply the techniques that we have learnt, we express the problem as:

We want to find the maximum value of *V* (volume), $V=\pi r^2 h$.

As $b^2=r^2+\left(\frac{h}{2}\right)^2$, we have $V=\pi\left(b^2h-\frac{h^3}{4}\right)$. Now the value *V* depends on the variable h only and the range of h is $0\le h\le 2b$.

Step 3: We shall sketch the graph of $V=\pi\left(b^2h-\frac{h^3}{4}\right)$, $0\le h\le 2b$ and hence determine the absolute maximum of *V* over the range $0\le h\le 2b$.

Now, $V'=\pi\left(b^2-\frac{3h^2}{4}\right)$. $V'=0$ when $h=\frac{2b}{\sqrt{3}}$ (since $h\ge 0$).

As $V'>0$ when $0\le h<\frac{2b}{\sqrt{3}}$ and $V'<0$ when $\frac{2b}{\sqrt{3}}<h\le 2b$, we conclude that the greatest volume of cylinder that can be cut is $V_{\max}=V\left(\frac{2b}{\sqrt{3}}\right)=\pi\left(\frac{2b^3}{\sqrt{3}}-\frac{8b^3}{12\sqrt{3}}\right)$. □

Remark
For optimization problems, a simpler criteria can be used in some situations as suggested by the following result:
If f is a twice differentiable function and $f'(x)$ is zero at only one point $x=c$ and $f''(c)\neq 0$ then this point is either an absolute maximum or an absolute minimum.

3 Taylor's Theorem (p.445 – p.459)

For a function $f(x)$ of one variable with continuous derivatives $f'(x), f''(x),\ldots,$ up to and including $f^{(N-1)}(x)$ in an interval $a\le x\le b$ and if $f^{(N)}(x)$ exists in $a<x<b$ then

$$f(a+h)=f(a)+\frac{f'(a)}{1!}h+\frac{f''(a)}{2!}h^2+\cdots+\frac{f^{(N-1)}(a)}{(N-1)!}h^{N-1}+R_N(h),\ \text{where } R_N(h)=\frac{f^{(N)}(a+\theta h)}{N!}h^N$$

for some $0<\theta<1$.

If $\lim_{N\to\infty}R_N(h)=0$, then $f(x)$ may be represented by the Taylor series expanded about a (assuming all derivatives exist at a): $f(a+h)=f(a)+\frac{f'(a)}{1!}h+\frac{f''(a)}{2!}h^2+\cdots+\frac{f^{(n)}(a)}{n!}h^n+\cdots=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}h^n$.

If $a=0$ we have Maclaurin's Theorem:

$$f(h)=f(0)+\frac{f'(0)}{1!}h+\frac{f''(0)}{2!}h^2+\cdots+\frac{f^{(N-1)}(0)}{(N-1)!}h^{N-1}+R_N(h)\ \text{ with } R_N(h)=\frac{f^{(N)}(\theta h)}{N!}h^N \text{ for some}$$

$0<\theta<1$.
Again, if $\lim_{N\to\infty}R_N(h)=0$, we have the Maclaurin series:

$$f(h)=f(0)+\frac{f'(0)}{1!}h+\frac{f''(0)}{2!}h^2+\cdots+\frac{f^{(n)}(0)}{n!}h^n+\cdots=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}h^n$$

Let $x=a+h$, alternatively, we have the equivalent form of Taylor series of $f(x)$ about a, that is,

$$f(x)=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(N-1)}(a)}{(N-1)!}(x-a)^{N-1}+R_N(x-a)$$

and Maclaurin series when $a=0$, that is, $f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$.

Example 9
Find the Maclaurin series for $\sin x$ and show that it converges to $\sin x$ for all x.

Solution:
If $f(x)=\sin x$, then

$f(0)=\sin x\big|_{x=0}=0;\ f'(0)=\cos x\big|_{x=0}=1;\ f''(0)=-\sin x\big|_{x=0}=0;\ f'''(x)=-\cos x\big|_{x=0}=-1,$

$f''''(x)=\sin x\big|_{x=0}=0$, etc.

Hence

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+R_n,\ \text{where } R_n=\frac{d^n}{dx^n}(\sin x)\Big|_{x=\theta_n}\frac{x^n}{n!}$$ and θ_n is between x and 0.

The n^{th} derivative of $\sin x$ is $\pm\sin x$ or $\pm\cos x$, so that $\left|\frac{d^n}{dx^n}(\sin x)\Big|_{x=\theta_n}\right| \le 1$.

Therefore, $\left|\frac{d^n}{dx^n}(\sin x)\Big|_{x=\theta_n}\frac{x^n}{n!}\right| \le \frac{|x|^n}{n!}$.

Since $\lim\limits_{n\to\infty}\frac{|x|^n}{n!} = 0$ for any *x*, we have $\lim\limits_{n\to\infty} R_n = 0$.

The Maclaurin series for $\sin x$ therefore converges to $\sin x$ for all *x*, and we may write

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} \quad \text{for } -\infty < x < \infty.$$

□

Example 10

Find the Maclaurin series for $\cos x$ and show that it converges to $\cos x$ for all *x*.

Solution:

If $f(x) = \cos x$, then

$f(0) = \cos x\big|_{x=0} = 1,\ f'(0) = -\sin x\big|_{x=0} = 0,\ f''(0) = -\cos x\big|_{x=0} = -1,\ f'''(0) = \sin x\big|_{x=0} = 0$,

$f''''(0) = \cos x\big|_{x=0} = 1$, etc. Hence

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + R_n$, where $R_n = \frac{d^n}{dx^n}(\cos x)\Big|_{x=\theta_n}\frac{x^n}{n!}$ and θ_n is between *x* and 0.

The n^{th} derivative of $\cos x$ is $\pm\sin x$ or $\pm\cos x$, so that $\left|\frac{d^n}{dx^n}(\cos x)\Big|_{x=\theta_n}\right| \le 1$.

Therefore, $\left|\frac{d^n}{dx^n}(\cos x)\Big|_{x=\theta_n}\frac{x^n}{n!}\right| \le \frac{|x|^n}{n!}$. Since $\lim\limits_{n\to\infty}\frac{|x|^n}{n!} = 0$ for any *x*, we have $\lim\limits_{n\to\infty} R_n = 0$.

The Maclaurin series for $\cos x$ therefore converges to $\cos x$ for all *x*, and we may write

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} \quad \text{for } -\infty < x < \infty.$$

□

Example 11

The Maclaurin series for e^x is $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ (exercise).

□

The first few terms of the Maclaurin series for some elementary functions:

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ for all x
2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ for all x
3. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$
4. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^r}{r!} + \cdots$ for all x
5. $\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ for $-1 < x \le 1$
6. $\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$ for $-1 \le x < 1$
7. $\log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right)$ for $-1 < x < 1$
8. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ for $-1 \le x \le 1$
9. $\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \cdots$ for $-1 < x < 1$

<u>Example 12</u>

(a) Put $x = \frac{1}{2}$ into $\sin^{-1} x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \cdots$, we have

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{1}{48} + \frac{3}{1280} + \frac{5}{112 \times 128} + \cdots$$

$$= \frac{\pi}{6}, \text{ it is a useful formula for the calculation of } \pi.$$

(b) Put $x = \frac{1}{2}$ and $x = \frac{1}{3}$ respectively into $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$. Upon adding, we have

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{1}{2} - \frac{1}{3\times 8} + \frac{1}{5\times 32} - \frac{1}{7\times 128} + \cdots$$
$$+ \frac{1}{3} - \frac{1}{3\times 27} + \frac{1}{5\times 243} - \frac{1}{7\times 2187} + \cdots$$

Moreover,

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\therefore\ \pi \approx 4\left[\left(0.5 - 0.041\dot{6} + 0.00625 - 0.001116\right) + \left(0.\dot{3} - 0.0123456 + 0.000823 - 0.0000653\right)\right] \approx 3.1409$$

Example 13

Find the Taylor series expansion of $\cos x$ about $x = \frac{\pi}{3}$. Hence find an approximation to $\cos 61^\circ$.

Solution

Let $f(x) = \cos x$, then $f^{(n)}(x) = \cos\left(x + \frac{n\pi}{2}\right)$.

Hence

$$f\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2},\ \ f'\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3} + \frac{\pi}{2}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2},$$

$$f''\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3} + \pi\right) = -\cos\frac{\pi}{3} = -\frac{1}{2},$$

$$f'''\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3} + \frac{3\pi}{2}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2},\ \text{ and so on.}$$

Using the Taylor series for $f(x)$ about the point $x = \frac{\pi}{3}$,

$$f(x) = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x - \frac{\pi}{3}\right)^2 + \cdots + \frac{f^{(r)}\left(\frac{\pi}{3}\right)}{r!}\left(x - \frac{\pi}{3}\right)^r + \cdots,$$

we obtain

$$\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{2(2!)}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{2(3!)}\left(x - \frac{\pi}{3}\right)^3 + \cdots + \frac{\cos\left(\frac{\pi}{3} + \frac{r\pi}{2}\right)}{r!}\left(x - \frac{\pi}{3}\right)^r + \cdots.$$

In particular,

$$\begin{aligned}
\cos 61^\circ &= \cos\frac{61\pi}{180} \\
&= \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{180}\right) - \frac{1}{2(2!)}\left(\frac{\pi}{180}\right)^2 + \frac{\sqrt{3}}{2(3!)}\left(\frac{\pi}{180}\right)^3 + \cdots \\
&\approx 0.5 - 0.015115 - 0.000076, \ \text{ with the first three terms} \\
&\approx 0.48481.
\end{aligned}$$

□

3.1 Numerical solution of nonlinear equation, $f(x)=0$, using Newton's (or Newton-Raphson) method (p.425 – p.430) (OPTIONAL)

Suppose we have an approximation x_k to a simple real root $x^* \ (= x_k + h)$ of the equation $f(x)=0$. Then

$$\begin{aligned} f(x^*) &= f(x_k+h) = f(x_k) + hf'(x_k) + \frac{h^2}{2!} f''(x_k) + \cdots \\ &= 0 \quad \text{by Taylor's expansion for } f(x_k+h). \end{aligned}$$

If h is small, then $h^2 \ll 1$, we may neglect the second and higher order terms in h,

$$f(x_k) + hf'(x_k) \approx 0$$

$$\Rightarrow \qquad h \approx -\frac{f(x_k)}{f'(x_k)}, \quad \text{provided } f'(x_k) \neq 0,$$

and so we take the next approximation to x^* as

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Using this formula for $k = 0, 1, 2, \ldots,$ we have a process known as the *Newton* (or *Newton-Raphson*) *method.*

Example 14

Perform 4 iterations of Newton's method to compute the root of $xe^x - 1 = 0$ with an initial approximation $x_0 = 0.5$.

Solution:

Let $f(x) := xe^x - 1$.

Using Newton's method,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k e^{x_k} - 1}{(x_k+1)e^{x_k}}, \quad k = 0, 1, 2, \ldots.$$

With an initial approximation, $x_0 = 0.5$, we have

$$x_1 = 0.5 - \frac{0.5e^{0.5} - 1}{(0.5+1)e^{0.5}} \approx 0.571020,$$

$$x_2 = 0.57102 - \frac{0.57102e^{0.57102} - 1}{(0.57102+1)e^{0.57102}} \approx 0.567156,$$

$$x_3 = 0.567156 - \frac{0.567156e^{0.567156} - 1}{(0.567156+1)e^{0.567156}} \approx 0.567143,$$

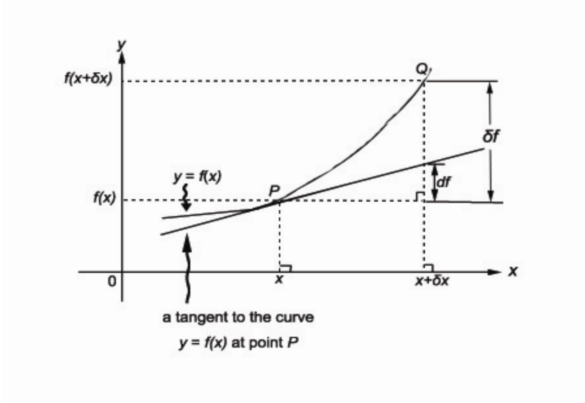
$$x_4 = 0.567143 - \frac{0.567143e^{0.567143} - 1}{(0.567143+1)e^{0.567143}} \approx 0.567143.$$

$\square$

3.2 Differential of a function (p.356 – p.366)
Let f be a differentiable function. Then df, the *differential* of f, which is a function of x and δx, is defined by

$$df = f'(x)\delta x$$

Let $y = f(x)$, then $y+\delta y = f(x+\delta x)$. Thus, $f(x+\delta x) = f(x)+\delta f \approx f(x)+df$.



4 Indeterminate Forms (p.418 – p.423)

Differentiation can be used to evaluate limits.

4.1 Type $\frac{0}{0}$

Here are two familiar limit problems: $\lim\limits_{x\to 0}\frac{\sin x}{x}$, $\lim\limits_{x\to 3}\frac{x^2-9}{x^2-x-6}$. The two limits have a common feature. In each case, a quotient is involved and, in each case, both numerator and denominator have 0 as their limits. An attempt to apply the quotient rule for limits leads to the nonsensical answer $\frac{0}{0}$. In fact, the quotient rule does not apply since it requires that the limit of the denominator be different from 0.
We are not saying that these limits do not exist, only that the quotient rule will not determine them.
Recall then an intricate geometric argument led us to the conclusion $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$. On the other hand, the algebraic techniques of factorizing yields $\lim\limits_{x\to 3}\frac{x^2-9}{x^2-x-6}=\lim\limits_{x\to 3}\frac{(x-3)(x+3)}{(x-3)(x+2)}=\lim\limits_{x\to 3}\frac{x+3}{x+2}=\frac{6}{5}$.

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However, there is a simple rule that works beautifully on a wide variety of such problems. It is known as l'Hôpital's rule (pronounced low-pee-tal).

Theorem F (l'Hôpital's rule for forms of type $\frac{0}{0}$)

Let $\lim\limits_{x\to u} f(x)=\lim\limits_{x\to u} g(x)=0$. If $\lim\limits_{x\to u}\frac{f'(x)}{g'(x)}$ exists in either the finite or infinite sense (that is, if this limit is a finite number or $-\infty$ or $+\infty$), then $\lim\limits_{x\to u}\frac{f(x)}{g(x)}=\lim\limits_{x\to u}\frac{f'(x)}{g'(x)}$.

Here u may stand for any of the symbols a, a^-, a^+, $-\infty$ or $+\infty$.
Proof: Omitted.

Example 15

Use l'Hôpital's rule to show that $\lim_{x\to 0}\frac{\sin x}{x}=1$, $\lim_{x\to 3}\frac{x^2-9}{x^2-x-6}=\frac{6}{5}$.

Proof:

Both limits have the $\frac{0}{0}$ form, so by l'Hôpital's rule, $\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=1$;

$\lim_{x\to 3}\frac{x^2-9}{x^2-x-6}=\lim_{x\to 3}\frac{2x}{2x-1}=\frac{6}{5}$.

□

Example 16

Find $\lim_{x\to 0}\frac{\tan 2x}{\ln(1+x)}$.

Solution:

Both numerator and denominator have limits 0.

Hence, $\lim_{x\to 0}\frac{\tan 2x}{\ln(1+x)}=\lim_{x\to 0}\frac{2\sec^2 2x}{\frac{1}{1+x}}=\frac{2}{1}=2$.

□

Sometimes, $\lim_{x\to u}\frac{f'(x)}{g'(x)}$ also has the indeterminate form $\frac{0}{0}$. Then we may apply l'Hôpital's rule again, as we now illustrate.

Example 17

Find $\lim_{x\to 0}\frac{\sin x-x}{x^3}$.

Solution:

By l'Hôpital's rule applied three rimes in succession,

$\lim_{x\to 0}\frac{\sin x-x}{x^3}=\lim_{x\to 0}\frac{\cos x-1}{3x^2}=\lim_{x\to 0}\frac{-\sin x}{6x}=\lim_{x\to 0}\frac{-\cos x}{6}=-\frac{1}{6}$.

□

Just because we have an elegant rule does not mean we could use it indiscriminately. In particular, we must always make sure that it applies. Otherwise, we will be led into all kinds of errors, as we now illustrate.

Example 18

Find $\lim_{x\to 0}\frac{1-\cos x}{x^2+3x}$.

Solution:

$\lim_{x\to 0}\frac{1-\cos x}{x^2+3x}=\lim_{x\to 0}\frac{\sin x}{2x+3}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$, WRONG!!

The first application of l'Hôpital's rule was correct; the second was not since, at that stage, the limit did not have the $\frac{0}{0}$ form.

$\lim_{x\to 0}\frac{1-\cos x}{x^2+3x}=\lim_{x\to 0}\frac{\sin x}{2x+3}=0$, CORRECT.

We stop differentiating as soon as either the numerator or denominator has a non-zero limit.

□

4.2 Type $\frac{\infty}{\infty}$

Consider $\lim_{x\to\infty}\frac{x}{e^x}$. This is typical of a class of problems of the form $\lim_{x\to u}\frac{f(x)}{g(x)}$, where both numerator and denominator are growing indefinitely large. It can be shown that l'Hôpital's rule also applies in this situation.

<u>Theorem G</u> (l'Hôpital's rule for forms of type $\frac{\infty}{\infty}$)

Let $\lim_{x\to u}|f(x)|=\lim_{x\to u}|g(x)|=\infty$. If $\lim_{x\to u}\frac{f'(x)}{g'(x)}$ exists in either the finite or infinite sense, then $\lim_{x\to u}\frac{f(x)}{g(x)}=\lim_{x\to u}\frac{f'(x)}{g'(x)}$. Here u may stand for any of the symbols a, a^-, a^+, $-\infty$ or $+\infty$.

Proof: Omitted.

<u>Example 19</u>

Find $\lim_{x\to\infty}\frac{x}{e^x}$.

<u>Solution</u>:

Both x and e^x tend to ∞ as $x\to\infty$. Hence, by l'Hôpital's rule, $\lim_{x\to\infty}\frac{x}{e^x}=\lim_{x\to\infty}\frac{1}{e^x}=0$.

□

<u>Example 20</u>

Show that if a is any positive real number, then $\lim_{x\to\infty}\frac{x^a}{e^x}=0$.

Proof:

Suppose first that $a=2.5$. Then three applications of l'Hôpital's rule give

$$\lim_{x\to\infty}\frac{x^{2.5}}{e^x}=\lim_{x\to\infty}\frac{2.5x^{2.5}}{e^x}=\lim_{x\to\infty}\frac{(2.5)(1.5)x^{0.5}}{e^x}=\lim_{x\to\infty}\frac{2.5\times1.5\times0.5}{x^{0.5}e^x}=0.$$

A similar argument works for any $a>0$.

□

Example 21

Show that if a is any positive real number, then $\lim_{x\to\infty}\frac{\ln x}{x^a}=a$.

Proof:

Both $\ln x$ and x^a tend to ∞ as $x\to\infty$. Hence, by one application of l'Hôpital's rule,

$\lim_{x\to\infty}\frac{\ln x}{x^a}=\lim_{x\to\infty}\frac{\frac{1}{x}}{ax^{a-1}}=\lim_{x\to\infty}\frac{1}{ax^a}=0$.

□

Example 22

Find $\lim_{x\to0^+}\frac{\ln x}{\cot x}$.

Solution:

As $x\to0^+$, $\ln x\to-\infty$ and $\cot x\to\infty$, so l'Hôpital's rule applies.

$\lim_{x\to0^+}\frac{\ln x}{\cot x}=\lim_{x\to0+}\left(\frac{\frac{1}{x}}{-\operatorname{cosec}^2 x}\right)$. This is still indeterminate form $\frac{\infty}{\infty}$. Rather than applying l'Hôpital's rule again (which only makes things worse), we rewrite the bracketed impression as

$\frac{1}{x\left(-\operatorname{cosec}^2 x\right)}=-\frac{\sin^2 x}{x}=\left(-\sin x\right)\frac{\sin x}{x}$. Thus, $\lim_{x\to0^+}\frac{\ln x}{\cot x}=\lim_{x\to0^+}\left[\left(-\sin x\right)\frac{\sin x}{x}\right]=0$.

□

4.3 Type $0\times\infty$ and $\infty-\infty$

L'Hôpital's rule will determine the result, but only after we rewrite the problem to a $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

Example 23

Find $\lim_{x\to\frac{\pi}{2}}\left(\tan x\ln\sin x\right)$.

Solution:

Since $\lim_{x\to\frac{\pi}{2}}\left|\tan x\right|=\infty$ and $\lim_{x\to\frac{\pi}{2}}\ln\sin x=0$, this is a $0\times\infty$ indeterminate form.

We can rewrite it as a $\frac{0}{0}$ form by changing $\tan x$ to $\frac{1}{\cot x}$.

Thus, $\lim_{x\to\frac{\pi}{2}}\left(\tan x\ln\sin x\right)=\lim_{x\to\frac{\pi}{2}}\frac{\ln\sin x}{\cot x}=\lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{\sin x}\cos x}{-\operatorname{cosec}^2 x}=\lim_{x\to\frac{\pi}{2}}\left(-\cos x\sin x\right)=0.$

□

Example 24

Find $\lim_{x\to1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$.

Solution:

The first term is growing without bound, so is the second. This is an $\infty-\infty$ indeterminate form. We can rewrite it as a $\frac{0}{0}$ form by combining the two fractions. Thus,

$$\lim_{x\to1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to1^+}\frac{x\ln x-x+1}{(x-1)\ln x}=\lim_{x\to1^+}\frac{x\frac{1}{x}+\ln x-1}{(x-1)\frac{1}{x}+\ln x}=\lim_{x\to1^+}\frac{x\ln x}{x-1+x\ln x}=\lim_{x\to1^+}\frac{1+\ln x}{2+\ln x}=\frac{1}{2}.$$

□

4.4 Type $0^0,\infty^0,1^\infty$

To consider not the original expression, but rather its logarithm. Usually l'Hôpital's rule will apply to the logarithm.

Example 25

Find $\lim_{x\to0^+}(x+1)^{\cot x}$.

Solution:

This is the indeterminate form 1^∞. Let $y=(x+1)^{\cot x}$.

Then $y=(x+1)^{\cot x}\Rightarrow\ln y=\cot x\ln(x+1)=\frac{\ln(x+1)}{\tan x}$. By l'Hôpital's rule for $\frac{0}{0}$ forms,

$$\lim_{x\to0^+}\ln y=\lim_{x\to0^+}\frac{\ln(x+1)}{\tan x}=\lim_{x\to0^+}\frac{\frac{1}{x+1}}{\sec^2 x}=1.$$

Now, $y=e^{\ln y}$, so $\lim_{x\to0^+}y=\lim_{x\to0^+}e^{\ln y}=e^{\lim_{x\to0^+}\ln y}=e^1=e.$

□