MA1201 Basic Calculus and Linear Algebra II

Chapter 6

Matrix, Determinant and System of Linear Equations

Basic Definition of Matrices

A $m \times n$ matrix, denoted by A, is a rectangular array of mn numbers arranged in the form

$$A = \overbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}^{n \text{ columns}} \left.\vphantom{\begin{pmatrix} a \\ a \\ \vdots \\ a \end{pmatrix}}\right\} m \text{ rows}$$

Example 1

$\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}$ is a 2×2 matrix. We also call this matrix as *square matrix*.

$\begin{pmatrix} 3 & 1 & 2 \end{pmatrix}$ is a 1×3 matrix. This matrix is also called a *row vector*.

$\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ is a 3×1 matrix. This matrix is also called a *column vector*.

Algebra on Matrix

1. Comparison of two matrices

Definition (Equality of matrices)

We say two matrices A and B are equal (i.e. $A = B$) if and only if BOTH the size of two matrices and ALL entries of two matrices are the same.

Example 2

$$\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \neq \begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix}, \quad (1 \quad 4) \neq \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & -3 \\ 0 & a \end{pmatrix} = \begin{pmatrix} b & -3 \\ 0 & 2 \end{pmatrix} \Rightarrow \begin{cases} a = 2 \\ b = 1 \end{cases}.$$

Remarks

Different from real number, we cannot say whether one matrix is bigger / smaller than another matrix. We cannot say

$$\cancel{A > B \text{ or } A < B}.$$

2. Addition, subtraction and multiplication matrices

Definition (Addition and subtraction of matrices)

Let A and B be two $m \times n$ matrices (must be the same size), the addition $(A+B)$ and the subtraction $(A-B)$ are defined as

$$A \pm B = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \pm \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} \pm b_{11} & \cdots & a_{1n} \pm b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & \cdots & a_{mn} \pm b_{mn} \end{pmatrix}$$

Example 3

$$\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 2+0 & 1+1 \\ -1+2 & 3+(-3) \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 2-1 & 1-2 \\ -1-(-1) & 3-0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}.$$

Definition (Multiplication of matrices)

Let A be $m \times p$ matrix and B be $p \times n$ matrix, then the product of two matrices, denoted by AB, is defined as

$$AB = \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mp} \end{pmatrix}}_{m\times p \text{ matrix}} \underbrace{\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pn} \end{pmatrix}}_{p\times n \text{ matrix}} = \underbrace{\begin{pmatrix} \sum_{k=1}^{p} a_{1k}b_{k1} & \cdots & \sum_{k=1}^{p} a_{1k}b_{kn} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{p} a_{mk}b_{k1} & \cdots & \sum_{k=1}^{p} a_{mk}b_{kn} \end{pmatrix}}_{m\times n \text{ matrix}}$$

More generally, the $(i,j)^{\text{th}}$ entry of AB is given by

$$[AB]_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}.$$

How to memorize the formula?

According to the formula, the $(i,j)^{\text{th}}$ entry of AB is formed by adding the product of the some entries in i^{th} row of A and j^{th} column of B.

$$AB = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ \vdots & \ddots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{pmatrix}}_{i^{\text{th}} \text{ row of } A} \underbrace{\begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & \ddots & b_{2j} & \ddots & b_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{p1} & \cdots & b_{pj} & \cdots & b_{pn} \end{pmatrix}}_{j^{\text{th}} \text{ column of } B}$$

$$[AB]_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} = \underbrace{a_{i1} b_{1j}}_{\substack{\text{1st term} \\ \text{of each part}}} + \underbrace{a_{i2} b_{2j}}_{\substack{\text{2nd term} \\ \text{of each part}}} + a_{i3} b_{3j} + \cdots + \underbrace{a_{ip} b_{pj}}_{\substack{p\text{th term} \\ \text{of each part}}} .$$

Remark about multiplication rule

- When doing multiplication of two matrices, the number of columns of matrix A MUST BE THE SAME as the number of rows of matrix B. You cannot find the product of the following matrices:

$$\underbrace{\begin{pmatrix} 1 & 2 \end{pmatrix}}_{1\times 2}\underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 3 \\ 2 & 0 \end{pmatrix}}_{3\times 2} \quad \text{and} \quad \underbrace{\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}}_{3\times 1}\underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}_{2\times 3}$$

- Different from real numbers, the order of multiplication of matrices cannot be reversed in general (i.e. $AB \neq BA$). As an example, we take $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, we have

$$AB = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$$
$$BA = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \neq AB.$$

Example 4

Compute the product $\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 7 \end{pmatrix}$.

☺Solution:

Let $C = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 7 \end{pmatrix}$ be the product, the entries of C are formed as follows:

$(1,1)^{\text{th}}$ entry: $C = \begin{pmatrix} 2 \times 1 + 1 \times 2 = 4 & ?? \\ ?? & ?? \end{pmatrix}$ $\quad \begin{pmatrix} \color{red}{2} & \color{red}{1} \\ -1 & 3 \end{pmatrix}\begin{pmatrix} \color{green}{1} & 0 \\ \color{green}{2} & 7 \end{pmatrix}$

$(1,2)^{\text{th}}$ entry: $C = \begin{pmatrix} ?? & 2 \times 0 + 1 \times 7 = 7 \\ ?? & ?? \end{pmatrix}$ $\quad \begin{pmatrix} \color{red}{2} & \color{red}{1} \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & \color{green}{0} \\ 2 & \color{green}{7} \end{pmatrix}$

$(2,1)^{\text{th}}$ entry: $C = \begin{pmatrix} ?? & ?? \\ -1 \times 1 + 3 \times 2 = 5 & ?? \end{pmatrix}$ $\quad \begin{pmatrix} 2 & 1 \\ \color{red}{-1} & \color{red}{3} \end{pmatrix}\begin{pmatrix} \color{green}{1} & 0 \\ \color{green}{2} & 7 \end{pmatrix}$

$(2,2)^{\text{th}}$ entry: $C = \begin{pmatrix} ?? & ?? \\ ?? & -1 \times 0 + 3 \times 7 = 21 \end{pmatrix}$ $\quad \begin{pmatrix} 2 & 1 \\ \color{red}{-1} & \color{red}{3} \end{pmatrix}\begin{pmatrix} 1 & \color{green}{0} \\ 2 & \color{green}{7} \end{pmatrix}$

Hence, $\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 7 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 5 & 21 \end{pmatrix}$.

Example 5

Compute the product $\begin{pmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 1 & -1 \end{pmatrix}$.

☺Solution:

According to the definition, the product $D = \underbrace{\begin{pmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \end{pmatrix}}_{2\times 3 \text{ matrix}} \underbrace{\begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 1 & -1 \end{pmatrix}}_{3\times 2 \text{ matrix}}$ is a 2×2 matrix. The entries of the product D can be computed as follows

$(1,1)^{\text{th}}$ entry: $D = \begin{pmatrix} 1 \times 0 + 0 \times 2 + (-1) \times 1 & ?? \\ ?? & ?? \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 1 & -1 \end{pmatrix}$

$(1,2)^{\text{th}}$ entry: $D = \begin{pmatrix} ?? & 1 \times 1 + 0 \times 3 + (-1) \times (-1) \\ ?? & ?? \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 1 & -1 \end{pmatrix}$

$(2,1)^{\text{th}}$ entry: $D = \begin{pmatrix} ?? & ?? \\ (-2) \times 0 + 1 \times 2 + 2 \times 1 & ?? \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 1 & -1 \end{pmatrix}$

$(2,2)^{\text{th}}$ entry: $D = \begin{pmatrix} ?? & ?? \\ ?? & (-2)\times 1 + 1 \times 3 + 2 \times (-1) \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 1 & -1 \end{pmatrix}$

Therefore, the product is given by

$$\begin{pmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 4 & -1 \end{pmatrix}.$$

Example 6

Compute the product $\underbrace{\begin{pmatrix}1 & 2 & 3\end{pmatrix}}_{1\times 3\ \text{matrix}}\underbrace{\begin{pmatrix}0 & -2\\ 1 & 0\\ 0 & -1\end{pmatrix}}_{3\times 2\ \text{matrix}}$.

☺Solution:

According to the definition, the product is a 1×2 matrix. The entries of the product can be computed as

$(1,1)^{\text{th}}$ entry: $D = (1\times 0 + 2\times 1 + 3\times 0 \quad ??)$ $\quad \begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}0 & -2\\ 1 & 0\\ 0 & -1\end{pmatrix}$

$(1,2)^{\text{th}}$ entry: $D = (?? \quad 1\times(-2) + 2\times 0 + 3\times(-1))$ $\quad \begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}0 & -2\\ 1 & 0\\ 0 & -1\end{pmatrix}$

Therefore the product is given by

$$\underbrace{\begin{pmatrix}1 & 2 & 3\end{pmatrix}}_{1\times 3\ \text{matrix}}\underbrace{\begin{pmatrix}0 & -2\\ 1 & 0\\ 0 & -1\end{pmatrix}}_{3\times 2\ \text{matrix}} = \underbrace{\begin{pmatrix}2 & -5\end{pmatrix}}_{1\times 2\ \text{matrix}}.$$

Example 7

Let $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, find the value of A^2 and A^3. Guess the value of A^n, where n is any positive integer.

☺Solution

$$A^2 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a \times a + 0 \times 0 & a \times 0 + 0 \times b \\ 0 \times a + b \times 0 & 0 \times 0 + b \times b \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}.$$

$$A^3 = A^2 A = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^2 \times a + 0 \times 0 & a^2 \times 0 + 0 \times b \\ 0 \times a + b^2 \times 0 & 0 \times 0 + b^2 \times b \end{pmatrix} = \begin{pmatrix} a^3 & 0 \\ 0 & b^3 \end{pmatrix}.$$

Following this pattern, we guess that

$$A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}.$$

Definition (Scalar Multiplication)

Let A be a $m \times n$ matrix and c be a scalar (or real number), then the scalar multiplication, denoted by cA, is defined as

$$cA = c\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1} & \cdots & ca_{mn} \end{pmatrix}.$$

Example 8

$$2\begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -6 & 2 \\ 2 & -2 \end{pmatrix}$$

$$0\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(*Note: The last matrix obtained is called "zero matrix").

Definition (transpose of matrix)

The transpose of a $m \times n$ matrix A, denoted by A^T or A', is a $n \times m$ matrix defined as follows:

$$A^T = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}^T = \underbrace{\begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}}_{n \times m \text{ matrix}}.$$

In other words, the i^{th} row (column) of matrix A will becomes the i^{th} column (row) of A^T.

Example 9

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Some Special Matrices

In this section, we introduce some special types of matrices which are of theoretical interest.

1. *Zero Matrix (denoted by 0)*

   A $m \times n$ matrix is a zero matrix if all entries of the matrix are 0,

$$0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

   <u>Properties of zero matrix</u>

   (1) $A \pm 0 = A$ (provided that the operation is feasible)

   (2) $0B = B0 = 0$ (provided that the operation is feasible)

   Therefore, zero matrix plays a role of "zero" in the matrix world.

2. Identity Matrix (denoted by I_n, must be a square matrix)

We say a $n \times n$ square matrix is an identity matrix (denoted by I_n) if all diagonal entries are 1 $(a_{ii} = 1)$ and all off-diagonal entries are 0 $(a_{ij} = 0,\ i \neq j)$.

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

<u>Properties of identity matrix</u>

For any $m \times n$ matrix A, we have

$$AI_n = A, \quad I_m A = A.$$

Therefore, an identity matrix plays a role of "1" in the matrix world.

3. Upper Triangular, Lower Triangular and diagonal matrix

- ✓ A square matrix is said to be an upper triangular matrix if all lower off-diagonal entries are 0.

Lower Off-Diagonal Entries

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Diagonal Entries

- ✓ A square matrix is said to be a lower triangular matrix if all upper-off-diagonal entries are 0.

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix}$$

Upper Off-Diagonal Entries

- ✓ A square matrix is said to be a diagonal matrix if all off-diagonal entries are 0 (In other words, the diagonal matrix is both upper triangular and lower triangular).

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Remark about diagonal matrix

If the given matrix A is diagaonal, there is a simple formula regarding the power of this matrix.

For any positive integer n and a $m \times m$ diagonal matrix A, we have

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_m \end{pmatrix}^n = \begin{pmatrix} (a_1)^n & 0 & \cdots & 0 \\ 0 & (a_2)^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (a_m)^n \end{pmatrix}.$$

For example:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}^4 = \begin{pmatrix} 2^4 & 0 & 0 \\ 0 & (-1)^4 & 0 \\ 0 & 0 & 3^4 \end{pmatrix}$$

- One has to be careful that the above formula can be applied to diagonal matrix ONLY!

4. Symmetric matrix and skew-symmetric matrix

✓ We say a square matrix A is symmetric if and only if $A^T = A$. For example:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ -1 & 3 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ -1 & 3 & 1 \end{pmatrix}$$

✓ We say a square matrix B is skew-symmetric (anti-symmetric) if and only if $A^T = -A$. For example:

$$\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & -3 \\ 1 & 3 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -2 & 1 \\ 2 & 0 & 3 \\ -1 & -3 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & -3 \\ 1 & 3 & 0 \end{pmatrix}.$$

Example 10

Let $A = \begin{pmatrix} 1 & x \\ 3 & 2 \end{pmatrix}$. Suppose A is symmetric, find the values of x. Is A skew-symmetric?

☺Solution

Note that A is symmetric, thus $A^T = A$

$\Rightarrow \underbrace{\begin{pmatrix} 1 & 3 \\ x & 2 \end{pmatrix}}_{A^T} = \underbrace{\begin{pmatrix} 1 & x \\ 3 & 2 \end{pmatrix}}_{A}$

By comparing the entries $(1,2)^{\mathrm{th}}$, we have $x = 3$.

On the other hand, A is not skew-symmetric because

$$A^T = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} \neq -\begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix} = -A.$$

Inverse of a matrix – Motivation and Definition

- In real number world, we can solve the equation $ax = b$ for x by multiplying both sides of equations by $\frac{1}{a}$ (which is also called multiplicative inverse of a):

$$ax = b \Rightarrow \underbrace{\left(\frac{1}{a}\right) a}_{=1} x = \left(\frac{1}{a}\right) b \Rightarrow x = \frac{b}{a}.$$

- Let's consider the following system of linear equations:

$$\begin{cases} 2x - 2y + z = 3 \\ \ \ x + \ \ y - z = 1 \\ 4x + \ \ y + z = 0 \end{cases}.$$

Using the matrix multiplication, one can rewrite the above system in the following "*matrix form*":

$$\begin{cases} 2x - 2y + z = 3 \\ \ \ x + \ \ y - z = 1 \\ 4x + \ \ y + z = 0 \end{cases} \Rightarrow \begin{pmatrix} 2 & -2 & 1 \\ 1 & 1 & -1 \\ 4 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \boldsymbol{Ax} = \boldsymbol{b}$$

where $A = \begin{pmatrix} 2 & -2 & 1 \\ 1 & 1 & -1 \\ 4 & 1 & 1 \end{pmatrix}$, $\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$.

Intuitively, one may follow the method as in the single equation case and solve for $\boldsymbol{x}$ by multiplying the whole equation by the matrix A^{-1}:

$$\boldsymbol{A^{-1}(Ax) = A^{-1}b \Rightarrow \underbrace{(A^{-1}A)}_{I_n}x = A^{-1}b \Rightarrow x = A^{-1}b}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 \\ 1 & 1 & -1 \\ 4 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.$$

$$= A^{-1}B \Rightarrow \boldsymbol{x} = A^{-1}B.$$

Such the matrix A^{-1} is called (multiplicative) inverse of the matrix A. In the coming section, we shall study the inverse of a square matrix in more detail.

Definition (Inverse of a square matrix)

Let A be $n \times n$ **square matrix**, the inverse matrix of A is a $n \times n$ matrix B such that

$$BA = AB = I_n$$

where I_n is a $n \times n$ identity matrix. If such matrix B exists, we denote this matrix B by A^{-1}.

Remark:

- One has to be careful that the inverse of matrix is defined on square matrix only. The inverse of a rectangular matrix (say 5×6 matrix) is not defined.
- The inverse of some matrices may not exist (see example below). These matrices are called singular matrices (or non-invertible matrix).
- On the other hand, we say the matrix is non-singular (or invertible) if its inverse exists.

Example 11

Let $A = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix}$ be two matrices.

(a) Find AB and BA.

(b) Is A non-singular (invertible)? Explain your answer.

☺Solution

(a) By direction calculation, we find that

$$AB = \begin{pmatrix} 2 \times (-1) + 1 \times 3 & 2 \times (-1) + 1 \times 2 \\ (-3) \times (-1) + (-1) \times 3 & (-3) \times (-1) + (-1) \times 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$BA = \begin{pmatrix} (-1) \times 2 + (-1) \times (-3) & (-1) \times 1 + (-1) \times (-1) \\ 3 \times 2 + 2 \times (-3) & 3 \times 1 + 2 \times (-1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b) Since $AB = BA = I_2$, hence B is the inverse of A (i.e. $B = A^{-1}$) and the matrix A is invertible.

Example 12

Show that there is no inverse for zero matrix $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

☺Solution:

For any square matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$0M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \times a + 0 \times c & 0 \times b + 0 \times d \\ 0 \times a + 0 \times c & 0 \times b + 0 \times d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq I_2.$$

Since there is no matrix M such that $0M = I_2$, hence the inverse of $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ does not exist. In other word, $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is singular (or not invertible)

Example 13 (Harder Example)

Let A be a square matrix such that $A^3 - A + I_n = 0$, show that A is invertible and find A^{-1}.

☺Solution:

IDEA: To show the matrix is invertible, one can rewrite the equation into the form $A(??) = (??)A = I_n$. Then (??) will be the desired inverse.

Note that

$$A^3 - A + I_n = 0 \Rightarrow A^3 - A = -I_n$$

$$\Rightarrow A - A^3 = I_n$$

$$\Rightarrow A\underbrace{(I_n - A^2)}_{??} = \underbrace{(I_n - A^2)}_{??}A = I_n.$$

Therefore A is invertible and $A^{-1} = I_n - A^2$.

The following shows some important properties of inverse:

Properties of inverse matrix

Let A, B be two $n \times n$ non-singular (invertible) square matrix, then

1. $(A^{-1})^{-1} = A$
2. $(cA)^{-1} = \frac{1}{c}A^{-1}$ where c is any non-zero scalar.
3. $(AB)^{-1} = B^{-1}A^{-1}$ (NOT $A^{-1}B^{-1}$!)

   Reason:

   Since $(AB)(AB)^{-1} = (AB)(B^{-1}A^{-1}) = AI_nA^{-1} = AA^{-1} = I_n$.
4. $(A^T)^{-1} = (A^{-1})^T$ (i.e. you can reverse the order of inverse and transpose.)

Questions:

- Is there any efficient way to check whether a given square matrix A is non-singular (invertible) or not? (i.e. Does its inverse exist?)

- Is there any efficient way to find the inverse of an invertible matrix (instead of trial by error)?

To answer those questions, we need a notation called <u>determinant of matrix.</u>

Given an $n \times n$ square matrix A, then

(1) If $\det A = 0$, then the matrix A is singular (not invertible)

(2) If $\det A \neq 0$, then the matrix A is non-singular (invertible) and its inverse is given by

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A.$$

Determinant of a square matrix

We start to define the determinant of a 2×2 square matrix

Definition (Determinant of a 2×2 matrix)

The determinant of a 2×2 square matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, denoted by $\det A$ or $|A|$, is defined as follows:

$$\det A = |A| = a_{11}a_{22} - a_{12}a_{21}.$$

Example 14

$$\det\begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix} = 2(-4) - (3)(1) = -11, \qquad \det\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = 1(3) - 2(0) = 3.$$

Determinant for an $\boldsymbol{n \times n}$ matrix ($\boldsymbol{n \geq 3}$)

We need to define two quantities:

Let A be an $n \times n$ square matrix, we define

- **The *minor* of the element $\boldsymbol{a_{ij}}$,** denoted by M_{ij}, is the determinant of a $(n-1) \times (n-1)$ submatrix of A obtained by deleting i^{th} row and j^{th} column of A. As an example, we let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Then

$$M_{11} = \det\begin{pmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ \cancel{4} & 5 & 6 \\ \cancel{7} & 8 & 9 \end{pmatrix} = \det\begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} = 5(9) - 6(8) = -3,$$

$$M_{23} = \det\begin{pmatrix} 1 & 2 & \cancel{3} \\ \cancel{4} & \cancel{5} & \cancel{6} \\ 7 & 8 & \cancel{9} \end{pmatrix} = \det\begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix} = 1(8) - 2(7) = -6.$$

- **The *cofactor* of the element $\boldsymbol{a_{ij}}$,** denoted by A_{ij}, is defined as

$$A_{ij} = (-1)^{i+j} M_{ij}.$$

The determinant of an $n \times n$ matrix $(n \geq 3)$ can be defined as follows:

Definition (Determinant of an $n \times n$ matrix)

Let A be an $n \times n$ square matrix (where $n \geq 3$), the determinant of A is defined as

$$\det A \overset{R_1}{\cong} a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + \cdots + a_{1n}A_{1n}.$$

(We say that the determinant of A is expanded along the 1^{st} row)

Remark:

In general, one can compute the determinant of A by <u>expanding along any row or any column</u>, i.e.

$$\det A \overset{R_i}{\cong} a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \quad (\text{along } i^{\text{th}} \text{ row})$$

or

$$\det A \overset{C_j}{\cong} a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} \quad (\text{along } j^{\text{th}} \text{ column}).$$

Example 15

Compute the determinant of $A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 3 & -1 & 1 \end{pmatrix}$.

☺Solution

We can compute the determinant by expanding along the 1^{st} row:

$$\det A \overset{R_1}{\overbrace{=}} a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= \underbrace{1}_{a_{11}} \times \underbrace{\left[(-1)^{1+1} \det\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\right]}_{A_{11}=(-1)^{1+1}M_{11}} + \underbrace{0}_{a_{12}} \times \underbrace{\left[(-1)^{1+2} \det\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\right]}_{A_{12}=(-1)^{1+2}M_{12}} + \underbrace{2}_{a_{13}}$$

$$\times \underbrace{\left[(-1)^{1+3} \det\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}\right]}_{A_{13}=(-1)^{1+3}M_{13}}$$

$$= 1 \times \big(1(1) - 2(-1)\big) + 0 + 2 \times [1(-1) - (1)(3)] = -5.$$

Example 16

Compute the determinant

$$\det\begin{pmatrix}1 & 3 & -2\\ 0 & 0 & 3\\ 2 & -2 & 4\end{pmatrix}$$

(a) By expanding along the 1st row.

(b) By expanding along the 2nd row.

☺Solution:

(a) By expanding the determinant along the first row, we have

$$\det\begin{pmatrix}1 & 3 & -2\\ 0 & 0 & 3\\ 2 & -2 & 4\end{pmatrix} \overset{R_1}{\overbrace{=}} \underbrace{a_{11}}_{1} A_{11} + \underbrace{a_{12}}_{3} A_{12} + \underbrace{a_{13}}_{-2} A_{13}$$

$$= \underbrace{1}_{a_{11}} \underbrace{\left[(-1)^{1+1}\det\begin{pmatrix}0 & 3\\ -2 & 4\end{pmatrix}\right]}_{A_{11}=(-1)^{1+1}M_{11}} + \underbrace{3}_{a_{12}} \underbrace{\left[(-1)^{1+2}\det\begin{pmatrix}0 & 3\\ 2 & 4\end{pmatrix}\right]}_{A_{12}=(-1)^{1+2}M_{12}}$$

$$+ \underbrace{-2}_{a_{13}} \underbrace{\left[(-1)^{1+3}\det\begin{pmatrix}0 & 0\\ 2 & -2\end{pmatrix}\right]}_{A_{13}=(-1)^{1+3}M_{13}} = 1(6) + 3(6) + (-2)0 = 24.$$

(b) By expanding the determinant along the second row, we have

$$\det\begin{pmatrix} 1 & 3 & -2 \\ 0 & 0 & 3 \\ 2 & -2 & 4 \end{pmatrix} \overset{R_2}{\overbrace{=}} \underbrace{a_{21}}_{0} A_{21} + \underbrace{a_{22}}_{0} A_{22} + \underbrace{a_{23}}_{3} A_{23}$$

$$= \underbrace{3}_{a_{23}} \underbrace{\left[(-1)^{2+3} \det\begin{pmatrix} 1 & 3 \\ 2 & -2 \end{pmatrix}\right]}_{A_{23}=(-1)^{2+3}M_{23}} = 3(8) = 24.$$

Remark

In this example, we observe that the computation is relatively easier if one expands the determinant along the second row. It is simply because there are quite a number of "0" in the second row so that one does not need to compute so many confactors A_{ij}.

Therefore, one should expand the determinant along the row (or column) with largest number of "0" in it so that the computation cost is minimized.

Example 17

Compute the determinant of $B = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & 2 \end{pmatrix}$.

☺Solution

Although one can obtain the answer by expanding along the 1^{st} row, it may be easier if one choose to expand along the 1^{st} column.

We expand the determinant along the first column, we get

$$\det B \overset{C_1}{\cong} b_{11}A_{11} + b_{21}A_{21} + b_{31}A_{31}$$

$$= 0 \times A_{11} + 0 \times A_{21} + 0 \times A_{31}$$

$$= 0.$$

Example 18

Compute the determinant of $A=\begin{pmatrix}1&0&2&3\\0&2&1&0\\2&1&0&1\\-1&1&1&0\end{pmatrix}$.

☺Solution

We expand the determinant along the 2^{nd} row:

$$\det A \overset{R_2}{=} \overbrace{a_{21}}^{0} A_{21} + \overbrace{a_{22}}^{2} A_{22} + \overbrace{a_{23}}^{1} A_{23} + \overbrace{a_{24}}^{0} A_{24}$$

$$= 2\times\left[(-1)^{2+2}\det\begin{pmatrix}1&2&3\\2&0&1\\-1&1&0\end{pmatrix}\right] + 1\times\left[(-1)^{2+3}\det\begin{pmatrix}1&0&3\\2&1&1\\-1&1&0\end{pmatrix}\right]$$

$$= 2\times\left[2\times(-1)^{2+1}\det\begin{pmatrix}2&3\\1&0\end{pmatrix} + 1\times(-1)^{2+3}\det\begin{pmatrix}1&2\\-1&1\end{pmatrix}\right]$$

$$-1\times\left[1\times(-1)^{1+1}\det\begin{pmatrix}1&1\\1&0\end{pmatrix} + 3\times(-1)^{1+3}\det\begin{pmatrix}2&1\\-1&1\end{pmatrix}\right] = \cdots = -2.$$

Application of Determinant #1 – Finding inverse of a square matrix

Theorem 1 (Existence of inverse matrix)

Given an $n \times n$ square matrix A, if

(1) $\det A = 0$, then the matrix is singular (not invertible).

(2) $\det A \neq 0$, then the matrix is non-singular (invertible).

Theorem 2 (General formula of inverse of matrix)

Let A be an $n \times n$ invertible square matrix, then its inverse A^{-1} is given by

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}^T .$$

where A_{ij} is the **cofactor** of the entry a_{ij}.

How to memorize the inverse matrix formula?

Although the formula looks complicated, one can memorize this formula in the following way:

The entries of A (a_{ij}) are replaced by its **cofactor** (A_{ij}) in A^{-1}

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}^T$$

There are two extra terms $\frac{1}{\det A}$ and the transpose T above the matrix

Example 19

Show that the matrix $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ is invertible and find its inverse.

☺Solution

Step 1: Show that the matrix $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ is invertible.

Note that $\det\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = 1 \times 4 - 2 \times 3 = -2 \neq 0$, so $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ is invertible.

Step 2: To find the inverse, we first compute all cofactors, note that

$$A_{11} = (-1)^{1+1}(4) = 4, \qquad A_{12} = (-1)^{1+2}(2) = -2,$$
$$A_{21} = (-1)^{2+1}(3) = -3, \qquad A_{22} = (-1)^{2+2}(1) = 1.$$

Thus the inverse is given by

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1} = \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^T = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}^T = \begin{pmatrix} -2 & 3/2 \\ 1 & -1/2 \end{pmatrix}.$$

Example 20

Let $A = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$ be a 3×3 matrix. Find the inverse of A if it exists.

☺Solution:

We first check whether the inverse exists by computing the determinant of A:

By expanding the determinant along the first row, we have

$$\begin{aligned} \det A &\overset{R_1}{\cong} 1 \times \underbrace{\det\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}}_{=1\times(-1)-(1\times 0)} - (-1) \times \underbrace{\det\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{=1\times(-1)-(0\times 0)} + 2 \times \underbrace{\det\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{1\times 1-(0\times 1)} \\ &= 1 \times (-1) - (-1)(-1) + 2(1) = 0. \end{aligned}$$

Since $\det A = 0$, we can conclude that A is not invertible (or A is singular) and its inverse A^{-1} does not exist.

Example 21

Let $A = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$. Show that the matrix is invertible and find A^{-1}.

☺Solution:

<u>Step 1: Show that the matrix is invertible.</u>

Note that $\det A \overset{R_1}{\triangleq} 3\det\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} - 2\det\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} + 6\det\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = 1 \neq 0$. Hence the matrix A is invertible.

<u>Step 2: Find the inverse A^{-1}.</u>

Note that the cofactors are given by

$$A_{11} = (-1)^{1+1}\det\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = 1, \quad A_{12} = (-1)^{1+2}\det\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = -1$$

$$A_{13} = (-1)^{1+3}\det\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = 0, \qquad A_{21} = (-1)^{2+1}\det\begin{pmatrix} 2 & 6 \\ 2 & 5 \end{pmatrix} = 2$$

$$A_{22} = (-1)^{2+2} \det \begin{pmatrix} 3 & 6 \\ 2 & 5 \end{pmatrix} = 3, \qquad A_{23} = (-1)^{2+3} \det \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} = -2$$

$$A_{31} = (-1)^{3+1} \det \begin{pmatrix} 2 & 6 \\ 1 & 2 \end{pmatrix} = -2, \qquad A_{32} = (-1)^{3+2} \det \begin{pmatrix} 3 & 6 \\ 1 & 2 \end{pmatrix} = 0$$

$$A_{33} = (-1)^{3+3} \det \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = 1.$$

Hence, the inverse is given by

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T = \frac{1}{1} \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ -2 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}.$$

Example 22

Using the inverse of matrix to solve the following system of linear equations

$$\begin{cases} 3x + 2y + 6z = 1 \\ \ \ x + \ \ y + 2z = 3 \\ 2x + 2y + 5z = 1 \end{cases}.$$

☺Solution

One can rewrite the above system into the matrix form:

$$\begin{cases} 3x + 2y + 6z = 1 \\ \ \ x + \ \ y + 2z = 3 \\ 2x + 2y + 5z = 1 \end{cases} \Rightarrow \underbrace{\begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

For Example 21, we know that A is invertible and $A^{-1} = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$. Thus,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ -5 \end{pmatrix}.$$

Remark of Example 22

- This method only works for the system which the number of equations equals the number of unknowns so that the coefficient matrix A is a square matrix (say in this example, we have 3 equations and 3 unknowns).

- This method will fail when the number of unknowns does not match with the number of equations. For example

$$\begin{cases} 2x - y + z = 3 \\ \ \ x + y + z = 1 \end{cases} \Rightarrow \underbrace{\begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}}_{\text{not square matrix}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

  or the coefficient matrix is *not* invertible

$$\begin{cases} x + \ \ y = 1 \\ 3x + 3y = 2 \end{cases} \Rightarrow \underbrace{\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}}_{\text{not invertible}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

  To solve those equations, we shall apply a more general method (called Gaussian Elimination) which will be discussed later.

Example 23

Let $A = \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$.

(a) Show that $A^{-1}BA = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$.

(b) Hence, evaluate B^{100}.

☺Solution:

(a) By some calculation (Exercise!!), one can obtain

$$A^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$$

Then using multiplication rule, we have

$$A^{-1}BA = \frac{1}{5}\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} -1 & 1 \\ 8 & 12 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -5 & 0 \\ 0 & 20 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$$

(b) From (a), we observe that

$$A^{-1}BA = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} \Rightarrow A(A^{-1}BA)A^{-1} = A\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}A^{-1}$$

$$\Rightarrow B = A\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}A^{-1}$$

Then

$$B^{100}$$

$$= \underbrace{A\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}\overbrace{A^{-1}A}^{=I_2}\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}A^{-1}A\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}A^{-1} \ldots A\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}A^{-1}}_{100\ A\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}A^{-1}\prime s}$$

$$= A\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}^{100} A^{-1}$$

$$= \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} (-1)^{100} & 0 \\ 0 & 4^{100} \end{pmatrix}\left[\frac{1}{5}\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\right]$$

$$= \cdots = \frac{1}{5}\begin{pmatrix} 3 + 2 \times 4^{100} & -3 + 3 \times 4^{100} \\ -2 + 2 \times 4^{100} & 2 + 3 \times 4^{100} \end{pmatrix}.$$

Application of Determinant #2 -- Computing vector product

The determinant is also useful in computing vector product of two vectors. Given two vectors $\vec{a} = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$ and $\vec{b} = b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k}$ in a plane, one can compute the vector product $\vec{a} \times \vec{b}$ using direct expansion:

$$\vec{a} \times \vec{b} = \left(a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}\right) \times \left(b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k}\right)$$

$$= a_1b_1(\vec{\imath} \times \vec{\imath}) + a_1b_2(\vec{\imath} \times \vec{\jmath}) + a_1b_3\left(\vec{\imath} \times \vec{k}\right) + a_2b_1(\vec{\jmath} \times \vec{\imath}) + a_2b_2(\vec{\jmath} \times \vec{\jmath})$$
$$+ a_2b_3\left(\vec{\jmath} \times \vec{k}\right) + a_3b_1\left(\vec{k} \times \vec{\imath}\right) + a_3b_2\left(\vec{k} \times \vec{\jmath}\right) + a_3b_3\left(\vec{k} \times \vec{k}\right)$$

$$= (a_2b_3 - a_3b_2)\vec{\imath} - (a_1b_3 - a_3b_1)\vec{\jmath} + (a_1b_2 - a_2b_1)\vec{k}$$

$$= \det\begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix}\vec{\imath} - \det\begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix}\vec{\jmath} + \det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}\vec{k} \overset{R_1}{\triangleq} \det\begin{pmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

Hence, the vector product can be computed via determinant

$$\boxed{\vec{a} \times \vec{b} = \det\begin{pmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}}$$

Example 24

Let $\vec{a} = 5\vec{\imath} - \vec{k}$ and $\vec{b} = \vec{\imath} + 2\vec{\jmath} - \vec{k}$ be two vectors, compute $\vec{a} \times \vec{b}$.

☺Solution:

Using the fact in the previous fact, we obtain

$$\vec{a} \times \vec{b} = \det\begin{pmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 5 & 0 & -1 \\ 1 & 2 & -1 \end{pmatrix}$$

$$= \det\begin{pmatrix} 0 & -1 \\ 2 & -1 \end{pmatrix}\vec{\imath} - \det\begin{pmatrix} 5 & -1 \\ 1 & -1 \end{pmatrix}\vec{\jmath} + \det\begin{pmatrix} 5 & 0 \\ 1 & 2 \end{pmatrix}\vec{k}$$

$$= 2\vec{\imath} + 4\vec{\jmath} + 10\vec{k}.$$

Example 25 (Computation of triple scalar product using determinant)

Let $\vec{a} = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$, $\vec{b} = b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k}$ and $\vec{c} = c_1\vec{\imath} + c_2\vec{\jmath} + c_3\vec{k}$ be three vectors. Show that

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

☺Solution:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot \det\begin{pmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$$\overset{R_1}{\triangleq} = (a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}) \cdot \left[\det\begin{pmatrix} b_2 & b_3 \\ c_2 & c_3 \end{pmatrix}\vec{\imath} - \det\begin{pmatrix} b_1 & b_3 \\ c_1 & c_3 \end{pmatrix}\vec{\jmath} + \det\begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix}\vec{k}\right]$$

$$= a_1 \times \det\begin{pmatrix} b_2 & b_3 \\ c_2 & c_3 \end{pmatrix} - a_2 \times \det\begin{pmatrix} b_1 & b_3 \\ c_1 & c_3 \end{pmatrix} + a_3 \times \det\begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix}$$

$$\overset{R_1}{\triangleq} \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Shortcut in computing determinant: Properties of Determinant

The following properties are useful in computing the determinant:

1. If all the elements in a row (or column) are zero, then its determinant is 0.

$$\det\begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ b_1 & b_2 & b_3 \end{pmatrix} = 0 \quad \text{and} \quad \det\begin{pmatrix} a_1 & 0 & b_1 \\ a_2 & 0 & b_2 \\ a_3 & 0 & b_3 \end{pmatrix} = 0$$

2. The sign of determinant A is changed if one interchanges any two rows (or columns)

$$\det\begin{pmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \overset{R_1 \leftrightarrow R_2}{\triangleq} -\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

3. **(Determinant of product of matrices)** For any two $n \times n$ square matrices A, B, we have

$$\det AB = \det A \det B \quad \text{and} \quad \det A^T = \det A.$$

4. **(Determinant of inverse)** If A is invertible square matrix, we have

$$\det(A^{-1}) = \frac{1}{\det A}.$$

5. **(Powerful properties)** The value of determinant is unchanged if multiple of one row (column) is added to another row (column).

$$\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \overset{R_2+kR_1}{\triangleq} \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1+ka_1 & b_2+ka_2 & b_3+ka_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

$$\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \overset{C_2+kC_1}{\triangleq} \det\begin{pmatrix} a_1 & a_2+ka_1 & a_3 \\ b_1 & b_2+kb_1 & b_3 \\ c_1 & c_2+kc_1 & c_3 \end{pmatrix}.$$

6. For any scalar k, we have

$$\det\begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = k\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

$$\det\left[k\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}\right] = \det\begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{n1} & ka_{n2} & \cdots & ka_{nn} \end{pmatrix}$$
$$= k^n\det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Example 26

Let A be a square matrix with $\det A = 5$. Find the value of (i) $\det(A^T A)$, (ii) $\det(A^5)$ and (iii) $\det(A^{-1})$.

☺Solution:

Using the properties of determinant (3 and 4), we obtain

$$\det(A^T A) = \det(A^T)\det A = (\det A)\det A = 5(5) = 25.$$

$$\begin{aligned}\det(A^5) &= \det(A \cdot A \cdot A \cdot A \cdot A) = (\det A)(\det A)(\det A)(\det A)(\det A) = (\det A)^5 \\ &= 5^5 = 3125.\end{aligned}$$

$$\det(A^{-1}) = \frac{1}{\det A} = \frac{1}{5}.$$

Example 27

Compute $\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & -1 & 3 \end{pmatrix}$

☺IDEA:
"Generate" some "0" in a particular row/ column so that the computation of determinant can be simplified.

☺Solution:

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & -1 & 3 \end{pmatrix} \overset{\substack{R_2-4R_1 \\ R_3-2R_1}}{\triangleq} \det\begin{pmatrix} 1 & 2 & 3 \\ 4-4(1) & 5-4(2) & 6-4(3) \\ 2-2(1) & -1-2(2) & 3-2(3) \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -5 & -3 \end{pmatrix}$$

$$\overset{C_1}{\triangleq} 1 \times \left[(-1)^{1+1}\det\begin{pmatrix} -3 & -6 \\ -5 & -3 \end{pmatrix}\right] + 0 \times \left[(-1)^{2+1}\det\begin{pmatrix} 2 & 3 \\ -5 & -3 \end{pmatrix}\right] + 0 \times \left[(-1)^{3+1}\det\begin{pmatrix} 2 & 3 \\ -3 & -6 \end{pmatrix}\right]$$

$$= 1 \times \big(-3 \times (-3) - (-5) \times (-6)\big) = 9 - 30 = -21.$$

Example 28

Compute the determinant $\det\begin{pmatrix} 1 & -1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ 1 & -1 & 1 & 2 \\ 2 & 7 & 1 & 0 \end{pmatrix}$

☺IDEA:
"Generate" some "0" in first row

☺Solution:

$$\det\begin{pmatrix} 1 & -1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ 1 & -1 & 1 & 2 \\ 2 & 7 & 1 & 0 \end{pmatrix} \overset{\substack{C_2+C_1 \\ C_3-2C_1 \\ C_4-C_1}}{\triangleq} \det\begin{pmatrix} 1 & -1+1 & 2-2 & 1-1 \\ 3 & 0+3 & 1-6 & 1-3 \\ 1 & -1+1 & 1-2 & 2-1 \\ 2 & 7+2 & 1-4 & 0-2 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 3 & -5 & -2 \\ 1 & 0 & -1 & 1 \\ 2 & 9 & -3 & -2 \end{pmatrix} \overset{R_1}{\triangleq} 1 \times \det\begin{pmatrix} 3 & -5 & -2 \\ 0 & -1 & 1 \\ 9 & -3 & -2 \end{pmatrix}$$

$$\overset{C_1}{\triangleq} 3 \times \det\begin{pmatrix} -1 & 1 \\ -3 & -2 \end{pmatrix} + 9 \times \det\begin{pmatrix} -5 & -2 \\ -1 & 1 \end{pmatrix} = 15 - 63 = -48.$$

Example 29

Factorize $\det\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix}$

☺Solution:

$$\det\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} \overset{\substack{C_2-C_1 \\ C_3-C_1}}{\triangleq} \det\begin{pmatrix} 1 & 1-1 & 1-1 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{pmatrix} \overset{R_1}{\triangleq} 1 \times \left[(-1)^{1+1}\det\begin{pmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{pmatrix}\right]$$

$$= \det\begin{pmatrix} b-a & c-a \\ (b-a)(b+a) & (c-a)(c+a) \end{pmatrix} = (b-a)(c-a)\det\begin{pmatrix} 1 & 1 \\ a+b & c+a \end{pmatrix}$$

$$= (b-a)(c-a)[(1)(c+a)-(a+b)(1)]$$

$$= (b-a)(c-a)(c-b) = (a-b)(b-c)(c-a).$$

Example 30

Factorize $\det\begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix}$.

☺Solution

$$\det\begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix} \overset{R_1 \leftrightarrow R_3}{\cong} -\det\begin{pmatrix} 1 & 1 & a \\ 1 & a & 1 \\ a & 1 & 1 \end{pmatrix}$$

$$\overset{\substack{R_2 - R_1 \\ R_3 - aR_1}}{\cong} -\det\begin{pmatrix} 1 & 1 & a \\ 1-1 & a-1 & 1-a \\ a-a(1) & 1-a(1) & 1-a(a) \end{pmatrix} = -\det\begin{pmatrix} 1 & 1 & a \\ 0 & a-1 & -(a-1) \\ 0 & 1-a & 1-a^2 \end{pmatrix}$$

$$\overset{C_1}{\cong} -1 \times (-1)^{1+1}\det\begin{pmatrix} a-1 & -(a-1) \\ 1-a & 1-a^2 \end{pmatrix} = -(a-1)\det\begin{pmatrix} 1 & -1 \\ 1-a & (1-a)(1+a) \end{pmatrix}$$

$$= -(a-1)(1-a)\det\begin{pmatrix} 1 & -1 \\ 1 & 1+a \end{pmatrix} = (a-1)^2[(1)(1+a) - (1)(-1)]$$

$$= (a-1)^2(a+2).$$

Example 31

Show that $\det\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} = -\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$.

☺Solution

$$\det\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix} \overset{R_1+R_2+R_3}{\triangleq} \det\begin{pmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{pmatrix}$$

$$= (a+b+c)\det\begin{pmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{pmatrix} \overset{\substack{C_2-C_1 \\ C_3-C_1}}{\triangleq} (a+b+c)\det\begin{pmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{pmatrix}$$

$$\overset{R_1}{\triangleq} (a+b+c)\left[1\times\det\begin{pmatrix} c-b & a-b \\ a-c & b-c \end{pmatrix}\right]$$

$$= -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$= -\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$$

System of Linear Equations

A system of m linear equations in n unknowns, x_1, x_2, ... , x_n is a set of equations of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}.$$

Here, $a_{ij}'s$ are the coefficients of the system and $b_i's$ are called the non-homogeneous terms of the system.

- If at least one of $b_i's$ is non-zero, the system is called a *non-homogeneous system* of linear equations.
- If all $b_i's$ are all zero, the system is called a *homogeneous system* of linear equations. Note that a homogenous system always has zeros as its solution called the trivial solution. We are interested to find whether there are non-trivial solutions of the homogenous system.

Some examples of system of linear equations and its solutions

Example 32 (Unique Solution)

$$\begin{cases} 2x - y = 3 \\ \quad x + y = 3 \end{cases}$$

One can use the method of elimination by adding the first equation to the second equation. We obtain $3x = 6 \Rightarrow x = 2$ and $y = 3 - x = 1$. This is the only solution of the system.

Example 33 (No Solution)

$$\begin{cases} x \;\; + \;\; y + \;\; z = 2 \\ 2x + 2y + 2z = 5 \end{cases}$$

One can obtain from first equation that $2x + 2y + 2z = 4$. This contradicts to the second equation.

Hence, the system has no solution.

Example 34 (Infinite many solutions)

$$\begin{cases} x + \quad y = 2 \\ 2x + 2y = 4 \end{cases}$$

The second equation is redundant and we only have one equation governing x and y, i.e. $x + y = 2$.

In fact, this system has infinitely many solutions, i.e.

$$x = 2 - t, \quad y = t$$

where t is any real number.

(For example, $(x, y) = (2,0)$, $(1,1)$, $(3, -1)$, $(-3,5)$ are all possible solutions).

A system of linear equations may have either no solutions, unique solutions or infinitely many solutions.

- We say a system is **consistent** if the system has at least one solution (one or infinitely many).
- We say a system is **inconsistent** if the system has no solution.

Question

- How do we determine whether a given system has no solution, a unique solution or infinitely many solutions?
- How do we obtain a solution of a given system efficiently?

Answer: Gaussian Elimination.

Matrix Representation of the system

Using the fact that $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$ (can be verified by direct matrix multiplication) , we can express the system in the following form:

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{\boldsymbol{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{\boldsymbol{b}}$$

$\boldsymbol{A}$ is called the *coefficient matrix*, $\boldsymbol{x}$ is called an unknown vector and $\boldsymbol{b}$ is called the known vector.

Example 35

The matrix representation of the system

$$\begin{cases} 2x + 5y = 3 \\ \ \ x - \ \ y = 1 \end{cases}$$

is given by

$$\begin{pmatrix} 2 & 5 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

The matrix representation of the system

$$\begin{cases} x - 2y + z = 0 \\ 3x \qquad\ \ - z = 1 \end{cases}$$

is given by

$$\begin{pmatrix} 1 & -2 & 1 \\ 3 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Given a matrix representation of the system $\boldsymbol{Ax} = \boldsymbol{b},$ we can define the corresponding augmented matrix as

$$(\boldsymbol{A} \mid \boldsymbol{b})$$

or

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right).$$

This matrix plays a crucial role in analyzing the solutions of a given system.

Gaussian Elimination

Motivation

Suppose we would like to solve the following equations:

$$\begin{cases} x + \quad y = 1 \ldots\ldots(1) \\ 2x + 3y = 2 \ldots\ldots(2) \end{cases}$$

One can adopt the *method of elimination* by multiplying equation (1) by a factor of 2 and subtract from the equation (2):

$$\begin{cases} x + \quad y = 1 \\ 2x + 3y = 2 \end{cases} \Rightarrow \begin{cases} x + y = 1 \\ \qquad y = 0 \end{cases}.$$

One can obtain from last equation that $y = 0$. Then one obtains $x = 1$ using the first equation.

Consider another system of linear equations

$$\begin{cases} x + \ \ y + \ \ z = \ \ 4 \\ 2x - 4y + \ \ z = -5 \\ 3x - 4y + 5z = \ \ 0 \end{cases}.$$

One can use the same method to solve the system. There are a lot of ways doing this:

(Approach 1: Eliminate z first)

$$\begin{cases} x + \ \ y + \ \ z = \ \ 4 \\ 2x - 4y + \ \ z = -5 \\ 3x - 4y + 5z = \ \ 0 \end{cases} \Rightarrow \begin{cases} x + \ \ y + z = \ \ 4 \\ x - 5y = \ -9 \\ -2x - 9y = -20 \end{cases} \Rightarrow \begin{cases} x + \ \ y + z = \ \ 4 \\ x - \ 5y = \ \ -9 \\ -19y = -38 \end{cases} \Rightarrow \begin{cases} y = 2 \\ x = 1 \\ z = 1 \end{cases}$$

(Approach 2: Eliminate y first)

$$\begin{cases} x + \ \ y + \ \ z = \ \ 4 \\ 2x - 4y + \ \ z = -5 \\ 3x - 4y + 5z = \ \ 0 \end{cases} \Rightarrow \begin{cases} x + y + \ \ z = \ 4 \\ 6x \quad + 5z = 11 \\ 7x \quad + 9z = 16 \end{cases} \Rightarrow \begin{cases} x + y + \ \ z = \ 4 \\ 6x \quad + \ \ 5z = \ 11 \\ \dfrac{19}{6}z = \dfrac{19}{6} \end{cases} \Rightarrow \begin{cases} z = 1 \\ x = 1. \\ y = 2 \end{cases}$$

When we encounter a bigger system (more unknown and more equations) or "irregular" system (infinite many solutions or no solution), we need a more systematic elimination procedure so that the solution of a system can be obtained easily.

Step 1: Eliminate x (first variable) in the 2nd and 3rd equation

$$\begin{cases} x + y + z = 4 \\ 2x - 4y + z = -5 \\ 3x - 4y + 5z = 0 \end{cases} \xrightarrow[(3)-3\times(1)]{(2)-2\times(1)} \begin{cases} x + y + z = 4 \\ -6y - z = -13 \\ -7y + 2z = -12 \end{cases}$$

Step 2: Eliminate y (second variable) in the 3rd equation

$$\begin{cases} x + y + z = 4 \\ -6y - z = -13 \\ -7y + 2z = -12 \end{cases} \xrightarrow{6\times(3)} \begin{cases} x + y + z = 4 \\ -6y - z = -13 \\ -42y + 12z = -72 \end{cases} \xrightarrow{(3)-7\times(2)} \begin{cases} x + y + z = 4 \\ -6y - z = -13. \\ 19z = 19 \end{cases}$$

Then we can solve backward and obtain $z = 1$, $y = 2$ and $x = 1$.

Such process is called the *Gaussian Elimination Method*.

Roughly speaking, the Gaussian Elimination is a solution technique which uses the method of elimination (also called elementary transformation or elementary row operation) and transforms a given system into a *specific* form (known as row echelon form in matrix form) which is easy to obtain, i.e.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \Rightarrow \begin{cases} a'_{11}x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n = b'_1 \\ \qquad\quad a'_{22}x_2 + \cdots + a'_{2n}x_n = b'_2 \\ \qquad\qquad\vdots \\ \qquad\qquad\qquad\quad a'_{mn}x_n = b'_m \end{cases}$$

or in augmented form

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right) \Rightarrow \left(\begin{array}{cccccc|c} a'_{11} & a'_{12} & a'_{13} & \cdots & a'_{1,n-1} & a'_{1n} & b'_1 \\ 0 & a'_{22} & a'_{23} & \cdots & a'_{2,n-1} & a'_{2n} & b'_2 \\ 0 & 0 & a'_{33} & \cdots & a'_{3,n-1} & a'_{3n} & b'_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a'_{mn} & b'_m \end{array}\right).$$

Definition (Row Echelon Form & Reduced Row Echelon Form)

A (augmented) matrix representation of the <u>row echelon form</u> has the following two properties:

$$\begin{pmatrix} a & * & * & * & * & * & * & * \\ 0 & b & * & * & * & * & * & * \\ 0 & 0 & 0 & c & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & d & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & e & * \end{pmatrix}$$

All entries below each pivot (the first non-zero entry of each row) are all zero.

Each pivot lies to the right of the pivot in the row above.

Remark:

The number of pivots is also called the *rank* of the matrix.

As an example, we suppose that the augmented matrix of a linear system is transformed into echelon form as follows:

$$\left(\begin{array}{ccc|c} 2 & 6 & 1 & 1 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

It is equivalent to the following linear system of equations

$$\begin{cases} 2x + 6y + z = 1 \\ \quad\quad 3y - z = 2. \\ \quad\quad\quad\quad z = 3 \end{cases}$$

One can solve the equation backwards and get $z = 3,\ y = \frac{5}{3}$ and $x = -6$.

The matrix is of reduced row echelon form if it is in row echelon form, the pivot is 1 in each non-zero row and all entries above each pivot are all zero.

Remark:

The reduced row echelon form of a matrix is unique.

Some Notions: Elementary transformation & elementary row operations

Roughly speaking, *elementary transformations* are simply the operations that we carried out in solving the system. The corresponding action in augmented matrix form is called *elementary row operations*.

Elementary Transformation	Elementary Row Operation
(1) Interchange two equations $\begin{cases} 2x - y = 2 \\ x + y = 1 \end{cases} \xrightarrow{(1)\leftrightarrow(2)} \begin{cases} x + y = 1 \\ 2x - y = 2 \end{cases}$	**(1) Interchange two rows** $\left(\begin{array}{cc\|c} 2 & -1 & 2 \\ 1 & 1 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc\|c} 1 & 1 & 1 \\ 2 & -1 & 2 \end{array}\right)$
(2) Multiply an equation by $c \neq 0$ $\begin{cases} 2x - y = 2 \\ x + y = 1 \end{cases} \xrightarrow{2\times(2)} \begin{cases} 2x - y = 2 \\ 2x + 2y = 2 \end{cases}$.	**(2) Multiply a row by $c \neq 0$** $\left(\begin{array}{cc\|c} 2 & -1 & 2 \\ 1 & 1 & 1 \end{array}\right) \xrightarrow{2R_2} \left(\begin{array}{cc\|c} 2 & -1 & 2 \\ 2 & 2 & 2 \end{array}\right)$
(3) Addition of a scalar multiple of any equation to another equation. e.g. $\begin{cases} 2x - y = 2 \\ x + y = 1 \end{cases} \xrightarrow{(2)+(1)} \begin{cases} 2x - y = 2 \\ 3x \quad\;\; = 3 \end{cases}$.	**(3) Addition of a scalar multiple of any row to another row.** $\left(\begin{array}{cc\|c} 2 & -1 & 2 \\ 1 & 1 & 1 \end{array}\right) \xrightarrow{R_2 + R_1} \left(\begin{array}{cc\|c} 2 & -1 & 2 \\ 3 & 0 & 3 \end{array}\right)$

Example 36

Using Gaussian elimination, solve the following system of linear equations

$$\begin{cases} x + 2y - 4z = -4 \\ 5x - 3y - 7z = 6 \\ 3x - 2y + 3z = 11 \end{cases}.$$

☺Solution:

We first express the system in augmented form

$$\left(\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 5 & -3 & -7 & 6 \\ 3 & -2 & 3 & 11 \end{array}\right)$$

We perform the Gaussian Elimination as follows:

Standard Form

$$\begin{cases} x + 2y - 4z = -4 \\ 5x - 3y - 7z = 6 \\ 3x - 2y + 3z = 11 \end{cases}$$

$$\xrightarrow[(3)-3\times(1)]{(2)-5\times(1)} \begin{cases} x + 2y - 4z = -4 \\ -13y + 13z = 26 \\ -8y + 15z = 23 \end{cases}$$

$$\xrightarrow{(2)\div -13} \begin{cases} x + 2y - 4z = -4 \\ y - z = -2 \\ -8y + 15z = 23 \end{cases}$$

$$\xrightarrow{(3)+8\times(2)} \begin{cases} x + 2y - 4z = -4 \\ y - z = -2 \\ 7z = 7 \end{cases}$$

Augmented Matrix Form

$$\left(\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 5 & -3 & -7 & 6 \\ 3 & -2 & 3 & 11 \end{array}\right)$$

$$\xrightarrow[R_3-3R_1]{R_2-5R_1} \left(\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 0 & -13 & 13 & 26 \\ 0 & -8 & 15 & 23 \end{array}\right)$$

$$\xrightarrow{R_2/-13} \left(\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 0 & 1 & -1 & -2 \\ 0 & -8 & 15 & 23 \end{array}\right)$$

$$\xrightarrow{R_3+8R_2} \left(\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 7 & 7 \end{array}\right)$$

By solving backward, we get $z = 1,\ y = -1,\ x = 2$.

Example 37

Solve $\begin{cases} 4y - z = 5 \\ 2x + y + 3z = 13 \\ x - 2y + 2z = 3 \end{cases}$ using the Gaussian Elimination.

☺Solution

We first express the system in augmented form: $\left(\begin{array}{ccc|c} 0 & 4 & -1 & 5 \\ 2 & 1 & 3 & 13 \\ 1 & -2 & 2 & 3 \end{array}\right)$

We perform the Gaussian Elimination as follows:

Standard Form	**Augmented Matrix Form**
$\begin{cases} 4y - z = 5 \\ 2x + y + 3z = 13 \\ x - 2y + 2z = 3 \end{cases}$	$\left(\begin{array}{ccc\|c} 0 & 4 & -1 & 5 \\ 2 & 1 & 3 & 13 \\ 1 & -2 & 2 & 3 \end{array}\right)$
$\xrightarrow{(1)\leftrightarrow(3)} \begin{cases} x - 2y + 2z = 3 \\ {\color{red}2x} + y + 3z = 13 \\ 4y - z = 5 \end{cases}$	$\xrightarrow{R_1\leftrightarrow R_3} \left(\begin{array}{ccc\|c} 1 & -2 & 2 & 3 \\ {\color{red}2} & 1 & 3 & 13 \\ 0 & 4 & -1 & 5 \end{array}\right)$

$$\xrightarrow{(2)-2\times(1)} \begin{cases} x - 2y + 2z = 3 \\ 5y - \ \ z = 7 \\ {\color{red}4y} - \ \ z = 5 \end{cases}$$

$$\xrightarrow{R_2-2R_1} \left(\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 0 & 5 & -1 & 7 \\ 0 & {\color{red}4} & -1 & 5 \end{array}\right)$$

$$\xrightarrow{(2)\div 5} \begin{cases} x - 2y + 2z = 3 \\ y - \frac{1}{5}z = \frac{7}{5} \\ {\color{red}4y} - \ \ z = 5 \end{cases}$$

$$\xrightarrow{R_2/5} \left(\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 0 & 1 & -\frac{1}{5} & \frac{7}{5} \\ 0 & {\color{red}4} & -1 & 5 \end{array}\right)$$

$$\xrightarrow{(3)-4\times(2)} \begin{cases} x - 2y + \ 2z = \ \ 3 \\ y - \frac{1}{5}z = \ \ \frac{7}{5} \\ -\frac{1}{5}z = -\frac{3}{5} \end{cases}$$

$$\xrightarrow{R_3-4R_2} \left(\begin{array}{ccc|c} 1 & -2 & 2 & 3 \\ 0 & 1 & -\frac{1}{5} & \frac{7}{5} \\ 0 & 0 & -\frac{1}{5} & -\frac{3}{5} \end{array}\right)$$

By solving backward, we get $z = 3,\ y = 2,\ x = 1.$

Example 38 (The system has infinitely many solutions)

Solve $\begin{cases} x + y - 2z = 2 \\ 2x + 2y - 3z = 1 \end{cases}$.

☺Solution

The augmented matrix is given by $\left(\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 2 & 2 & -3 & 1 \end{array}\right)$.

Using the Gaussian Elimination, we have

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ \color{red}{2} & 2 & -3 & 1 \end{array}\right) \xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Solving backward, we get $z = -3$ and $x + y - 2z = 2 \Rightarrow x + y = -4$.

We only have one equation governing x and y. So there are infinite many solutions of the system and the general solution is given by

$$x = -4 - t, \qquad y = t, \quad t \text{ is any real number.}$$

In Example 38, there is a column without a pivot (column 2) in the echelon form.

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

We observe that the corresponding variable y can take any values, y is also called "free variable".

In general, if there is column which does not have any pivots, then the corresponding unknown will be the *free variable* (if the system is consistent). Hence, the system has *infinite number of solutions*.

As an example, if the echelon form of the system is

$$\left(\begin{array}{ccccc|c} 1 & 3 & 4 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 3 & 3 \end{array}\right) \quad \text{or} \quad \begin{cases} x_1 + 3x_2 + 4x_3 + x_4 + 2x_5 = 1 \\ \qquad\qquad\quad x_3 + 2x_4 + 3x_5 = 2. \\ \qquad\qquad\qquad\qquad\quad 3x_5 = 3 \end{cases}$$

So x_2, x_4 are the "free variables" and the general solution is given by

$$x_5 = 1, \qquad x_4 = t, \qquad x_3 = -1 - 2t, \qquad x_2 = s, \qquad x_1 = 3 - 3s + 7t.$$

Example 39 (The system has no solutions)

Solve the following system

$$\begin{cases} \ \ x - 2y + 3z = 2 \\ 2x + 3y - 2z = 5. \\ 4x - \ \ y + 4z = 1 \end{cases}$$

☺Solution:

The augmented matrix is $\left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 2 & 3 & -2 & 5 \\ 4 & -1 & 4 & 1 \end{array}\right)$. Then using the Gaussian elimination, we have

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ \color{red}{2} & 3 & -2 & 5 \\ \color{red}{4} & -1 & 4 & 1 \end{array}\right) \xrightarrow[R_3-4R_1]{R_2-2R_1} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 7 & -8 & 1 \\ 0 & \color{red}{7} & -8 & -7 \end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 0 & 7 & -8 & 1 \\ \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{-8} \end{array}\right).$$

Since the last row is of the type $(0\ 0\ 0 \mid b)$ with $b = -8 \neq 0$, hence the system has no solution.

Solution of System of Linear Equations: A brief summary

Given a system of linear equations, one can adopt the Gaussian Elimination to transform the system into Row Echelon form. The solution of the system can be studied easily by investigating the row echelon form of the system.

Case 1: There is a row with $\begin{pmatrix} 0 & 0 & \cdots & 0 & |b \end{pmatrix}$ (where $b \neq 0$) in the echelon form of the augmented matrix:

- This row corresponds to the equation:

$$0x_1 + 0x_2 + \cdots + 0x_n = b \Rightarrow 0 = b.$$

- Since $b \neq 0$, this implies the system has no solutions (or inconsistent).

If case 1 does not happen, then the system has a solution (consistent). It remains to check whether the system has a unique solution or infinitely many solutions. To check this, we check whether there is column without pivots.

Case 2A: There is a column without a pivot:

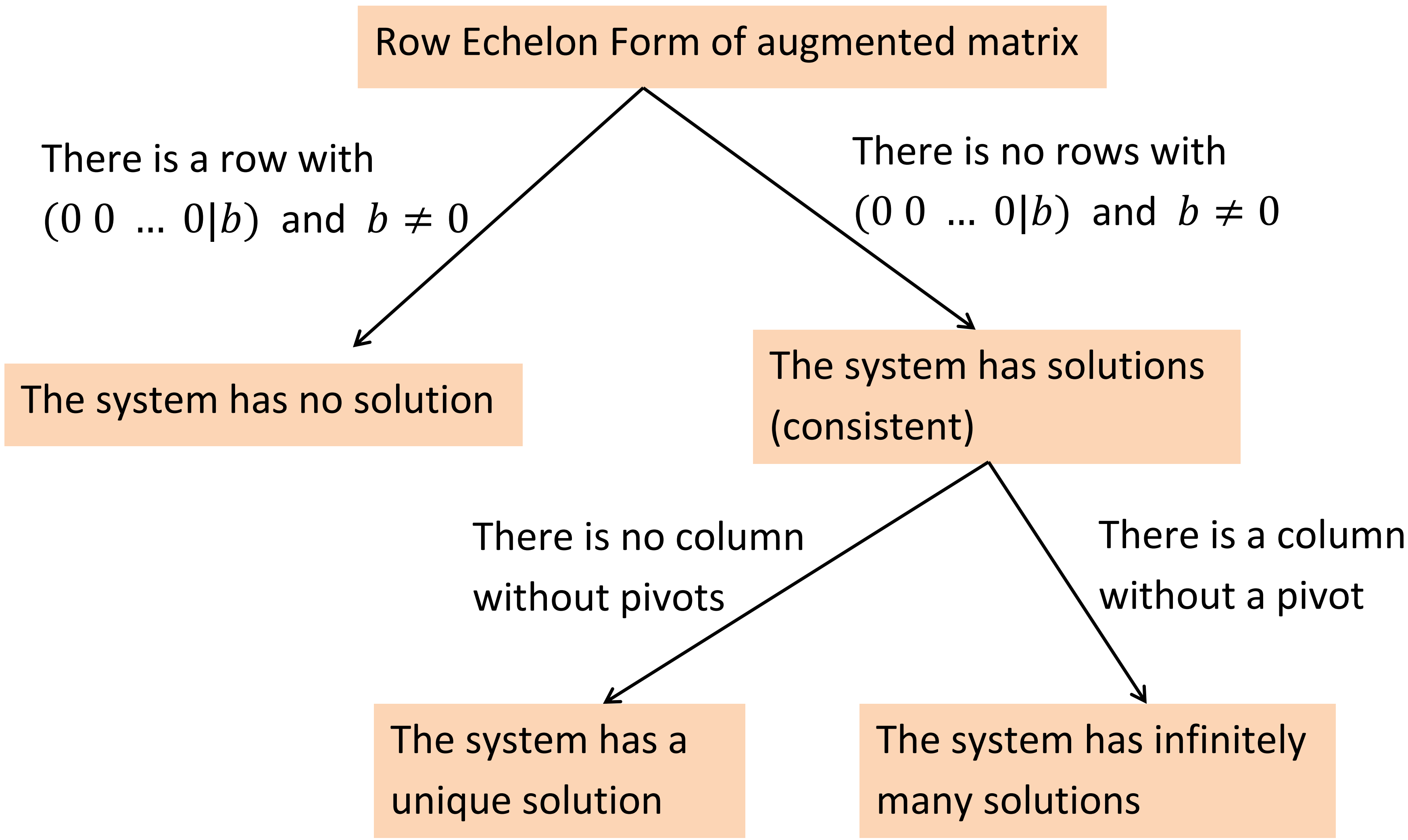
- The corresponding variables in such columns (without pivots) are “free variables” (see Example 38).
- The system has infinitely many solutions.

Case 2B: There are no columns without pivots:

- There is no free variables and the echelon form of augmented matrix is given below

$$\left(\begin{array}{cccc|c} a & * & * & * & * \\ 0 & b & * & * & * \\ 0 & 0 & c & * & * \\ 0 & 0 & 0 & d & * \end{array}\right)$$

- Every unknown is determined uniquely by solving backward.
- The system has a unique solution.


Row Echelon Form of augmented matrix
There is a row with
$(0\ 0\ \ldots\ 0|b)$ and $b \neq 0$
There is no rows with
$(0\ 0\ \ldots\ 0|b)$ and $b \neq 0$
The system has no solution
The system has solutions
(consistent)
There is no column
without pivots
There is a column
without a pivot
The system has a
unique solution
The system has infinitely
many solutions

Example 40

Solve the following system

$$\begin{cases} x_1 + 2x_2 - 3x_3 + x_5 = 2 \\ 2x_1 + 4x_2 + x_4 - 2x_5 = 3 \\ 3x_1 + 6x_2 + 5x_3 + x_4 - x_5 = 1 \end{cases}.$$

☺Solution

The corresponding augmented matrix is $\left(\begin{array}{ccccc|c} 1 & 2 & -3 & 0 & 1 & 2 \\ 2 & 4 & 0 & 1 & -2 & 3 \\ 3 & 6 & 5 & 1 & -1 & 1 \end{array}\right)$. Using the Gaussian Elimination, we reduce the system into row echelon form:

$$\left(\begin{array}{ccccc|c} 1 & 2 & -3 & 0 & 1 & 2 \\ 2 & 4 & 0 & 1 & -2 & 3 \\ 3 & 6 & 5 & 1 & -1 & 1 \end{array}\right) \xrightarrow[R_3-3R_1]{R_2-2R_1} \left(\begin{array}{ccccc|c} 1 & 2 & -3 & 0 & 1 & 2 \\ 0 & 0 & 6 & 1 & -4 & -1 \\ 0 & 0 & 14 & 1 & -4 & -5 \end{array}\right)$$

$$\xrightarrow{R_2/6} \left(\begin{array}{ccccc|c} 1 & 2 & -3 & 0 & 1 & 2 \\ 0 & 0 & 1 & \frac{1}{6} & -\frac{4}{6} & -\frac{1}{6} \\ 0 & 0 & 14 & 1 & -4 & -5 \end{array}\right) \xrightarrow{R_3-14R_2} \left(\begin{array}{ccccc|c} 1 & 2 & -3 & 0 & 1 & 2 \\ 0 & 0 & 1 & \frac{1}{6} & -\frac{4}{6} & -\frac{1}{6} \\ 0 & 0 & 0 & -\frac{8}{6} & \frac{32}{6} & -\frac{16}{6} \end{array}\right).$$

Since there are columns (2nd column and 5th column) without pivots, the system should have infinitely many solutions.

The corresponding unknowns x_2 and x_5 will be *free variables*. We take $x_2 = s$ and $x_5 = t$, where s, t are real numbers. The corresponding system is

$$\begin{cases} x_1 + 2x_2 - 3x_3 \qquad\qquad + \quad x_5 = \quad 2 \\ \qquad\qquad x_3 + \frac{1}{6}x_4 - \frac{4}{6}x_5 = -\frac{1}{6} \\ \qquad\qquad -\frac{8}{6}x_4 + \frac{32}{6}x_5 = -\frac{16}{6} \end{cases}.$$

Solving the equations backwards, we obtain $x_4 = 2 + 4t,\ x_3 = -\frac{1}{2},\ x_1 = \frac{1}{2} - 2s - t$ and the general solution vector becomes:

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ 2 \\ 0 \end{pmatrix}}_{\substack{\text{particular} \\ \text{solution}}} + \underbrace{s\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \\ 4 \\ 1 \end{pmatrix}}_{\substack{\text{general solution of the} \\ \text{homogeneous system}}}.$$

Note that $\boldsymbol{x_p} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \\ 2 \\ 0 \end{pmatrix}$ is a particular solution of the nonhomogeneous system ($s = t = 0$), so that $\boldsymbol{A}(\boldsymbol{x} - \boldsymbol{x_p}) = \boldsymbol{b} - \boldsymbol{b} = \boldsymbol{0}$ and $\boldsymbol{x} - \boldsymbol{x_p}$ is the general solution of the corresponding homogeneous system:

$$\begin{cases} x_1 + 2x_2 - 3x_3 \quad\quad + x_5 = 0 \\ 2x_1 + 4x_2 + \quad\quad x_4 - 2x_5 = 0 \\ 3x_1 + 6x_2 + 5x_3 + x_4 - x_5 = 0 \end{cases}.$$

Thus $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 0 \\ 4 \\ 1 \end{pmatrix}$ are two linearly independent solutions of the above homogeneous system.

Example 41

Find all values of c for which the following system is consistent.

$$\begin{cases} x+2y-z=c \\ -x+4y+z=c^2 \\ x+8y-z=c^3 \end{cases}$$

☺Solution:

To start with, we first transform the augmented matrix $\left(\begin{array}{ccc|c} 1 & 2 & -1 & c \\ -1 & 4 & 1 & c^2 \\ 1 & 8 & -1 & c^3 \end{array}\right)$ into echelon form using the Gaussian Elimination.

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & c \\ -1 & 4 & 1 & c^2 \\ 1 & 8 & -1 & c^3 \end{array}\right) \xrightarrow[R_3-R_1]{R_2+R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & c \\ 0 & 6 & 0 & c^2+c \\ 0 & 6 & 0 & c^3-c \end{array}\right)$$

$$\xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & c \\ 0 & 6 & 0 & c^2+c \\ 0 & 0 & 0 & c^3-c^2-2c \end{array}\right).$$

The system is consistent if there is no row with $(0\ 0\ 0|b),\ b \neq 0$. This implies that

$c^3 - c^2 - 2c = 0 \Leftrightarrow c(c^2 - c - 2) = 0 \Leftrightarrow c(c-2)(c+1) = 0$

$\Rightarrow c = 0 \text{ or } c = 2 \text{ or } c = -1.$

☺Remark: Since the column 3 has no pivot, the system has infinitely many solutions (with z as the free variable) if the system is consistent.

Example 42

We consider the following system of linear equations

$$\begin{cases} x - 2y + \quad z = 1 \\ x - \quad y + \ 2z = 2 \\ \qquad\quad y + c^2 z = c \end{cases}.$$

Determine all values of c for each of the following cases:

(a) The system has unique solution.
(b) The system has no solution.
(c) The system has infinitely many solutions.

☺Solution:

The corresponding augmented matrix is given by $\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 1 & -1 & 2 & 2 \\ 0 & 1 & c^2 & c \end{array}\right)$. We first reduce the system into row echelon form using the Gaussian Elimination.

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 1 & -1 & 2 & 2 \\ 0 & 1 & c^2 & c \end{array}\right) \xrightarrow{R_2-R_1} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & c^2 & c \end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & c^2-1 & c-1 \end{array}\right)$$

(a) The system has a unique solution only when there is no columns without pivots. This implies $c^2 - 1 \neq 0 \Rightarrow c \neq \pm 1$.

(b) The system has no solutions only when there is a row of the type $(0\ 0\ 0|b)$ with $b \neq 0$. This implies $c^2 - 1 = 0$ and $c - 1 \neq 0$.
So we get $c = -1$.

(c) Finally, the system has infinitely many solutions only when (i) there is a column without a pivot and (ii) no row with $(0\ 0\ 0|b)$, $b \neq 0$. These imply $c^2 - 1 = 0$ and $c - 1 = 0$. So we get $c = 1$.

Cramer's Rule

This method provides an alternative to find the solution of a linear system when the number of equations equal to number of unknowns.

We consider the system of linear equations with n equations and n unknowns,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \quad \dots (*)$$

In matrix form, we have $\boldsymbol{Ax} = \boldsymbol{b}$.

The coefficient matrix $\boldsymbol{A}$ is now a $n \times n$ square matrix.

If $\det \boldsymbol{A} \neq 0$, then $\boldsymbol{A}$ is invertible (non-singular) and $\boldsymbol{A}^{-1}$ exists.

Then one can obtain the solution vector $\boldsymbol{x}$ by multiplying both sides by $\boldsymbol{A}^{-1}$, i.e.

$$\boldsymbol{A}^{-1}\boldsymbol{Ax} = \boldsymbol{A}^{-1}\boldsymbol{b} \;\Rightarrow\; \boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}.$$

Using the formula of $\boldsymbol{A}^{-1}$, we get

$$\boldsymbol{x} = \frac{1}{\det \boldsymbol{A}} \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix}^T \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \frac{1}{\det \boldsymbol{A}} \begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$= \frac{1}{\det \boldsymbol{A}} \begin{pmatrix} A_{11}b_1 + A_{21}b_2 + \cdots + A_{n1}b_n \\ A_{12}b_1 + A_{22}b_2 + \cdots + A_{n2}b_n \\ \vdots \\ A_{1n}b_1 + A_{2n}b_2 + \cdots + A_{nn}b_n \end{pmatrix}$$

where A_{ij} is the cofactor of the element a_{ij}.

Theorem (Cramer's Rule)

Let $\boldsymbol{A}$ be the coefficient matrix of the system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$.

If $\det \boldsymbol{A} \neq 0$, then the system has a unique solution given by

$$x_i = \frac{A_{1i}b_1 + A_{2i}b_2 + \cdots + A_{ni}b_n}{\det \boldsymbol{A}}, \quad i = 1, 2, \dots, n$$

where A_{ij} is the cofactor of a_{ij}.

Remark about Cramer's Rule:

By the definition of determinant, one can express the numerator $A_{1i}b_1 + A_{2i}b_2 + \cdots + A_{ni}b_n$ as

$$A_{1i}b_1 + A_{2i}b_2 + \cdots + A_{ni}b_n = \det\begin{pmatrix} a_{11} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{n1} \\ a_{21} & \cdots & a_{2,i-1} & b_2 & a_{2,i+1} & \cdots & a_{n2} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{nn} \end{pmatrix}$$

which is the determinant of the matrix obtained by replacing the elements in i^{th} column of matrix $\boldsymbol{A}$ by the column vector $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$.

Example 43

Solve the following system of linear equations by Cramer's Rule

$$\begin{cases} 5x + 2y = 13 \\ 2x - 3y = \ \ 9 \end{cases}.$$

☺Solution:

Note that $\boldsymbol{A} = \begin{pmatrix} 5 & 2 \\ 2 & -3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 13 \\ 9 \end{pmatrix}$. Using Cramer's rule, we have

$$x = \frac{\det \begin{pmatrix} 13 & 2 \\ 9 & -3 \end{pmatrix}}{\det \boldsymbol{A}} = \frac{13 \times (-3) - 9 \times 2}{5 \times (-3) - 2 \times 2} = \frac{-57}{-19} = 3,$$

$$y = \frac{\det \begin{pmatrix} 5 & 13 \\ 2 & 9 \end{pmatrix}}{\det \boldsymbol{A}} = \frac{5 \times 9 - 2 \times 13}{5 \times (-3) - 2 \times 2} = \frac{19}{-19} = -1.$$

Application of the Gaussian Elimination #1 – Finding the inverse of matrix (the Gauss Jordan Elimination)

The method of the Gauss-Jordan Elimination provides an alternative of finding the inverse of an invertible matrix.

As an example, we would like to find the inverse of $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. We let $\boldsymbol{X} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$ be the inverse of $\boldsymbol{A}$. According to the definition of inverse, $\boldsymbol{X}$ must satisfy

$$\boldsymbol{AX} = \boldsymbol{I_3} \quad \text{or}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This system can be separated into three systems:

$$\begin{pmatrix}1 & 1 & 2\\ 1 & 2 & 0\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}1\\ 0\\ 0\end{pmatrix},\ \begin{pmatrix}1 & 1 & 2\\ 1 & 2 & 0\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}=\begin{pmatrix}0\\ 1\\ 0\end{pmatrix},\begin{pmatrix}1 & 1 & 2\\ 1 & 2 & 0\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}z_1\\ z_2\\ z_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}$$

In augmented matrix form, we have

$$\left(\begin{array}{ccc|c}1 & 1 & 2 & 1\\ 1 & 2 & 0 & 0\\ 1 & 1 & 1 & 0\end{array}\right),\left(\begin{array}{ccc|c}1 & 1 & 2 & 0\\ 1 & 2 & 0 & 1\\ 1 & 1 & 1 & 0\end{array}\right),\left(\begin{array}{ccc|c}1 & 1 & 2 & 0\\ 1 & 2 & 0 & 0\\ 1 & 1 & 1 & 1\end{array}\right).$$

We can apply Gaussian Elimination to solve each system above.

Since we handle the same coefficient matrix $\begin{pmatrix}1 & 1 & 2\\ 1 & 2 & 0\\ 1 & 1 & 1\end{pmatrix}$ for each system, it would be easier if we put three augmented matrices into a single augmented matrix, i.e.

$$\left(\begin{array}{ccc|ccc}1 & 1 & 2 & 1 & 0 & 0\\ 1 & 2 & 0 & 0 & 1 & 0\\ 1 & 1 & 1 & 0 & 0 & 1\end{array}\right).$$

Using the Gauss-Jordan Elimination, we get

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3-R_1]{R_2-R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

$$\xrightarrow[R_2+2R_3]{R_1-2R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

$$\xrightarrow{R_1-R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 4 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \ldots\ldots (**)$$

Then $\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$ gives $x_1 = -2,\ x_2 = 1,\ x_3 = 1.$

$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$ gives $y_1 = -1,\ y_2 = 1,\ y_3 = 0.$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right) \text{ gives } z_1 = 4,\ z_2 = -2,\ z_3 = -1.$$

So the inverse is given by

$$\boldsymbol{X} = \boldsymbol{A}^{-1} = \begin{pmatrix} -2 & -1 & 4 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{pmatrix}$$

which also equals the matrix on the R.H.S. of augmented matrix $(**)$.

General Procedure in finding the inverse of $\boldsymbol{A}$ *using the Gauss-Jordon Elimination:*

Step 1: We first write the augmented matrix

$$\left(\begin{array}{c|c} \underbrace{\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix}}_{\boldsymbol{A}} & \underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{matrix}}_{\boldsymbol{I_n}} \end{array}\right)$$

Step 2: Use elementary row operations to transform the matrix into row echelon form

$$\left(\begin{array}{cccc|cccc} 1 & * & \cdots & * & * & * & * & * \\ 0 & 1 & \cdots & * & * & * & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & * & * & * & * \end{array}\right).$$

Step 3: Finally, we use elementary row operations to transform the augmented matrix into the form which the matrix on the L.H.S. is an identity matrix I_n.

Then the matrix on the R.H.S. will be the inverse $\boldsymbol{A^{-1}}$.

$$\left(\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{matrix}}_{\boldsymbol{I_n}} & \underbrace{\begin{matrix} * & * & * & * \\ * & * & * & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & * & * \end{matrix}}_{\boldsymbol{A^{-1}}} \end{array}\right)$$

This method is called the **Gauss-Jordan Method.**

Example 44

Find the inverse of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}.$$

☺Solution

We first write down the augmented matrix

$$\left(\begin{array}{ccc|ccc} 3 & 2 & 6 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 5 & 0 & 0 & 1 \end{array}\right)$$

Using elementary row operations, we have

$$\left(\begin{array}{ccc|ccc} 3 & 2 & 6 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 5 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 3 & 2 & 6 & 1 & 0 & 0 \\ 2 & 2 & 5 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 - 2R_1]{R_2 - 3R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{-R_2}\left(\begin{array}{ccc|ccc}1 & 1 & 2 & 0 & 1 & 0\\ 0 & 1 & 0 & -1 & 3 & 0\\ 0 & 0 & 1 & 0 & -2 & 1\end{array}\right)$$

$$\xrightarrow{R_1-2R_3}\left(\begin{array}{ccc|ccc}1 & 1 & 0 & 0 & 5 & -2\\ 0 & 1 & 0 & -1 & 3 & 0\\ 0 & 0 & 1 & 0 & -2 & 1\end{array}\right)$$

$$\xrightarrow{R_1-R_2}\left(\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 2 & -2\\ 0 & 1 & 0 & -1 & 3 & 0\\ 0 & 0 & 1 & 0 & -2 & 1\end{array}\right).$$

Since the L.H.S. becomes an identity matrix, therefore the inverse of $\boldsymbol{A}$ is given by

$$\boldsymbol{A}^{-1}=\begin{pmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{pmatrix}.$$

Example 45

Find the inverse of the matrix

$$B=\begin{pmatrix}1 & -2 & 0 & 0\\ 2 & -3 & 3 & 1\\ 0 & 1 & 2 & -2\\ -1 & 1 & 1 & 3\end{pmatrix}$$

☺Solution:

$$\left(\begin{array}{cccc|cccc}1 & -2 & 0 & 0 & 1 & 0 & 0 & 0\\ 2 & -3 & 3 & 1 & 0 & 1 & 0 & 0\\ 0 & 1 & 2 & -2 & 0 & 0 & 1 & 0\\ -1 & 1 & 1 & 3 & 0 & 0 & 0 & 1\end{array}\right)\xrightarrow[R_4+R_1]{R_2-2R_1}\left(\begin{array}{cccc|cccc}1 & -2 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 3 & 1 & -2 & 1 & 0 & 0\\ 0 & 1 & 2 & -2 & 0 & 0 & 1 & 0\\ 0 & -1 & 1 & 3 & 1 & 0 & 0 & 1\end{array}\right)$$

$$\xrightarrow[R_4+R_2]{R_3-R_2}\left(\begin{array}{cccc|cccc}1 & -2 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 3 & 1 & -2 & 1 & 0 & 0\\ 0 & 0 & -1 & -3 & 2 & -1 & 1 & 0\\ 0 & 0 & 4 & 4 & -1 & 1 & 0 & 1\end{array}\right)$$

$$\xrightarrow{R_4+4R_3}\left(\begin{array}{cccc|cccc} 1 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & -3 & 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -8 & 7 & -3 & 4 & 1 \end{array}\right)$$

$$\xrightarrow[R_4/(-8)]{-R_3}\left(\begin{array}{cccc|cccc} 1 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & \color{red}{1} & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \color{red}{3} & -2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -7/8 & 3/8 & -1/2 & -1/8 \end{array}\right)$$

$$\xrightarrow[R_3-3R_4]{R_2-R_4}\left(\begin{array}{cccc|cccc} 1 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & \color{red}{3} & 0 & -9/8 & 5/8 & 1/2 & 1/8 \\ 0 & 0 & 1 & 0 & 5/8 & -1/8 & 1/2 & 3/8 \\ 0 & 0 & 0 & 1 & -7/8 & 3/8 & -1/2 & -1/8 \end{array}\right)$$

$$\xrightarrow{R_2-3R_3}\left(\begin{array}{cccc|cccc} 1 & \color{red}{-2} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -3 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 5/8 & -1/8 & 1/2 & 3/8 \\ 0 & 0 & 0 & 1 & -7/8 & 3/8 & -1/2 & -1/8 \end{array}\right)$$

$$\xrightarrow{R_1+2R_2}\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -5 & 2 & -2 & -2 \\ 0 & 1 & 0 & 0 & -3 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 5/8 & -1/8 & 1/2 & 3/8 \\ 0 & 0 & 0 & 1 & -7/8 & 3/8 & -1/2 & -1/8 \end{array}\right).$$

Thus the inverse of the matrix B is found to be

$$B^{-1} = \begin{pmatrix} -5 & 2 & -2 & -2 \\ -3 & 1 & -1 & -1 \\ 5/8 & -1/8 & 1/2 & 3/8 \\ -7/8 & 3/8 & -1/2 & -1/8 \end{pmatrix}.$$

Remark:

Comparing with the cofactor method that we used in previous section, the Gauss-Jordan method provides a simpler way in finding the inverse of matrices with larger size. If one wishes to compute B^{-1} using cofactor method in the previous example, one needs to compute 16 cofactors (determinant of 3×3 sub-matrices) in order to obtain the answer!

Example 46 (Gauss-Jordan Elimination on non-invertible matrix)

Find the inverse of $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

☺Solution:

Note that

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right) \overset{R_2-2R_1}{\widetilde{\sim}} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right)$$

We see that the last row is of the form $(0\ 0|b_1\ b_2)$, $b_1,\ b_2 \neq 0$. There is no solution for the system.

Hence the inverse does not exist!

Application of Gauss Elimination #2: Checking the Linear Independency of vectors

One can use the following fact to check the linear independency of vectors:

Theorem (General procedure of checking linear independency)

The vectors $\overrightarrow{a_1}, \overrightarrow{a_2}, \cdots, \overrightarrow{a_n}$ are linearly independent if and only if the equation $x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + \cdots + x_n\overrightarrow{a_n} = \vec{0}$ has only *trivial solution* (i.e. $x_1 = x_2 = \cdots = x_n = 0$).

On the other hand, if the equation $x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + \cdots + x_n\overrightarrow{a_n} = \vec{0}$ has other solutions $x_1, x_2, \dots, x_n$ (other than the trivial solution). Then the vectors $\overrightarrow{a_1}, \overrightarrow{a_2}, \cdots, \overrightarrow{a_n}$ are linearly dependent.

One can use the Gaussian Elimination to solve the equation $x_1\overrightarrow{a_1} + x_2\overrightarrow{a_2} + \cdots + x_n\overrightarrow{a_n} = \vec{0}$.

Example 47

Let $\vec{a} = 3\vec{\imath} + 2\vec{\jmath}$, $\vec{b} = \vec{\imath} - \vec{k}$ and $\vec{c} = 2\vec{\imath} + 3\vec{\jmath} + 4\vec{k}$ be three vectors, determine whether these three vectors are linearly independent.

☺Solution:

We need to solve the following equation for x_1, x_2, x_3.

$$x_1\vec{a} + x_2\vec{b} + x_3\vec{c} = \vec{0}$$

$$\Rightarrow (3x_1 + x_2 + 2x_3)\vec{\imath} + (2x_1 + 3x_3)\vec{\jmath} + (-x_2 + 4x_3)\vec{k} = 0\vec{\imath} + 0\vec{\jmath} + 0\vec{k}$$

By comparing the coefficients, we get

$$\begin{cases} 3x_1 + x_2 + 2x_3 = 0 \\ 2x_1 \qquad + 3x_3 = 0. \\ \qquad -x_2 + 4x_3 = 0 \end{cases}$$

Using the Gaussian Elimination, we get

$$\left(\begin{array}{ccc|c} 3 & 1 & 2 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right) \xrightarrow{R_1/3} \left(\begin{array}{ccc|c} 1 & 1/3 & 2/3 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right) \xrightarrow{R_2-2R_1} \left(\begin{array}{ccc|c} 1 & 1/3 & 2/3 & 0 \\ 0 & -2/3 & 5/3 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right)$$

$$\xrightarrow{R_2/(-\frac{2}{3})} \left(\begin{array}{ccc|c} 1 & 1/3 & 2/3 & 0 \\ 0 & 1 & -5/2 & 0 \\ 0 & -1 & 4 & 0 \end{array}\right) \xrightarrow{R_3+R_2} \left(\begin{array}{ccc|c} 1 & 1/3 & 2/3 & 0 \\ 0 & 1 & -5/2 & 0 \\ 0 & 0 & 3/2 & 0 \end{array}\right)$$

The corresponding system is

$$\begin{cases} x_1 + \dfrac{1}{3}x_2 + \dfrac{2}{3}x_3 = 0 \\ \qquad\quad x_2 - \dfrac{5}{2}x_3 = 0 \\ \qquad\qquad\quad \dfrac{3}{2}x_3 = 0 \end{cases} \Rightarrow x_3 = 0,\ x_2 = 0,\ x_1 = 0.$$

Since the system has only trivial solution, the vectors are linearly independent.

Example 48

Determine if the vectors $\vec{a} = \vec{i} + 5\vec{j} - 2\vec{k},\ \vec{b} = \vec{i} - \vec{k},\ \vec{c} = -2\vec{i} + 10\vec{j}$ are linearly independent.

☺Solution:

We consider the following equations:

$$x_1\vec{a} + x_2\,\vec{b} + x_3\vec{c} = \vec{0}$$

$$\Rightarrow (x_1 + x_2 - 2x_3)\vec{i} + (5x_1 + 10x_3)\vec{j} + (-2x_1 - x_2)\vec{k} = \vec{0}$$

$$\Rightarrow \begin{cases} x_1 + x_2 - 2x_3 = 0 \\ 5x_1 + 10x_3 = 0 \\ -2x_1 - x_2 = 0 \end{cases}$$

Using the Gaussian Elimination, we have

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 5 & 0 & 10 & 0 \\ -2 & -1 & 0 & 0 \end{array}\right) \xrightarrow[R_3+2R_1]{R_2-5R_1} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -5 & 20 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right) \xrightarrow{R_2/(-5)} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right)$$

$$\xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since the 3rd column does not have any pivots, the corresponding variable x_3 is a free variable, we take $x_3 = t$, t is real. Then we have $x_2 = 4t$ and $x_1 = -2t$.

Since the equation $x_1\vec{a} + x_2\,\vec{b} + x_3\vec{c} = \vec{0}$ has non-trivial solutions (say when $t = 1$, we have $x_1 = -2$, $x_2 = 4$ and $x_3 = 1$). Thus the vectors $\vec{a}$, $\overrightarrow{b}$ and $\vec{c}$ are linearly dependent.

Example 49

Let $\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ be three vectors in 4-D space. Determine if the vectors are linearly independent.

☺Solution

We consider the following equations:

$$x_1\vec{a} + x_2\,\vec{b} + x_3\vec{c} = \vec{0} \Rightarrow x_1\begin{pmatrix} 1 \\ 2 \\ 0 \\ -3 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + x_3\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 + x_2 - 3x_3 \\ 2x_1 + x_2 \\ x_2 + x_3 \\ -3x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + x_2 - 3x_3 = 0 \\ 2x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \\ -3x_1 + x_2 = 0 \end{cases}$$

Using the Gaussian Elimination, we get

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -3 & 1 & 0 & 0 \end{array}\right) \xrightarrow[R_4+3R_1]{R_2-2R_1} \left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & -9 & 0 \end{array}\right) \xrightarrow[R_4+4R_2]{R_3+R_2} \left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 15 & 0 \end{array}\right)$$

$$\xrightarrow{R_3/7} \left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 15 & 0 \end{array}\right) \xrightarrow{R_4-15R_3} \left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The corresponding system is $\begin{cases} x_1 + x_2 - 3x_3 = 0 \\ \quad\; - x_2 + 6x_3 = 0 \\ \qquad\qquad\;\; x_3 = 0 \end{cases}$. Solving the equations backward,

we get $x_3 = 0$, $x_2 = 0$ and $x_1 = 0$.

Since the system has only trivial solution, the vectors are linearly independent.

Review Example 1

Solve the following system

$$\begin{cases} 2x + \ y - 3z = \ \ 8 \\ 5x + 3y + \ \ z = \ 12 \\ 3x - 2y + 5z = -1 \end{cases}.$$

☺Solution: Using the Gaussian Elimination, we have

$$\left(\begin{array}{ccc|c} 2 & 1 & -3 & 8 \\ 5 & 3 & 1 & 12 \\ 3 & -2 & 5 & -1 \end{array}\right) \xrightarrow{R_1/2} \left(\begin{array}{ccc|c} 1 & 1/2 & -3/2 & 4 \\ 5 & 3 & 1 & 12 \\ 3 & -2 & 5 & -1 \end{array}\right)$$

$$\xrightarrow[R_3-3R_1]{R_2-5R_1} \left(\begin{array}{ccc|c} 1 & 1/2 & -3/2 & 4 \\ 0 & 1/2 & 17/2 & -8 \\ 0 & -7/2 & 19/2 & -13 \end{array}\right) \xrightarrow{R_3+7R_2} \left(\begin{array}{ccc|c} 1 & 1/2 & -3/2 & 4 \\ 0 & 1/2 & 17/2 & -8 \\ 0 & 0 & 69 & -69 \end{array}\right)$$

Solving backward, we get $69z = -69 \Rightarrow z = -1$; $\frac{1}{2}y + \frac{17}{2}z = -8 \Rightarrow y = 1$;

$x + \frac{1}{2}y - \frac{3}{2}z = 4 \Rightarrow x = 2$.

Review Example 2

Solve the system

$$\begin{cases} x - y - z + w = 1 \\ 3x + 2y \phantom{{}-5z} - w = 0 \\ -x - 9y - 5z + 7w = 5 \end{cases}.$$

☺Solution:

Using the Gaussian Elimination method again, we get

$$\left(\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 1 \\ \color{red}{3} & 2 & 0 & -1 & 0 \\ \color{red}{-1} & -9 & -5 & 7 & 5 \end{array}\right) \xrightarrow[R_3+R_1]{R_2-3R_1} \left(\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 1 \\ 0 & 5 & 3 & -4 & -3 \\ 0 & \color{red}{-10} & -6 & 8 & 6 \end{array}\right)$$

$$\xrightarrow{R_3+2R_2} \left(\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 1 \\ 0 & 5 & 3 & -4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Note that the 3rd column and 4th column have no pivots, the corresponding variables z and w are *free variables*.

To obtain the full solution, we write $z = s$ and $w = t$ where s, t are two real numbers. From the second equation (second row), we have

$$5y + 3z - 4w = -3 \Rightarrow y = \frac{-3 - 3z + 4w}{5} = \frac{-3 - 3s + 4t}{5}.$$

From the first equation, we get

$$x - y - z + w = 1 \Rightarrow x = 1 + y + z - w = \frac{2 + 2s - t}{5}.$$

The general solution vector: $$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \underbrace{\begin{pmatrix} 2/5 \\ -3/5 \\ 0 \\ 0 \end{pmatrix}}_{\substack{\text{particular} \\ \text{solution}}} + \underbrace{s \begin{pmatrix} 2/5 \\ -3/5 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/5 \\ 4/5 \\ 0 \\ 1 \end{pmatrix} t}_{\substack{\text{two linearly independent} \\ \text{solutions of the corresponding} \\ \text{homogeneous system}}}.$$

Review Example 3

Using the Gauss-Jordan Method to find the inverse of $A=\begin{pmatrix}1 & -2 & 0\\ 0 & 2 & 3\\ 4 & 2 & 3\end{pmatrix}$.

☺Solution

$$\left(\begin{array}{ccc|ccc}1 & -2 & 0 & 1 & 0 & 0\\ 0 & 2 & 3 & 0 & 1 & 0\\ 4 & 2 & 3 & 0 & 0 & 1\end{array}\right)\xrightarrow{R_3-4R_1}\left(\begin{array}{ccc|ccc}1 & -2 & 0 & 1 & 0 & 0\\ 0 & 2 & 3 & 0 & 1 & 0\\ 0 & 10 & 3 & -4 & 0 & 1\end{array}\right)$$

$$\xrightarrow{R_3-5R_2}\left(\begin{array}{ccc|ccc}1 & -2 & 0 & 1 & 0 & 0\\ 0 & 2 & 3 & 0 & 1 & 0\\ 0 & 0 & -12 & -4 & -5 & 1\end{array}\right)\xrightarrow[R_3/(-12)]{R_2/2}\left(\begin{array}{ccc|ccc}1 & -2 & 0 & 1 & 0 & 0\\ 0 & 1 & 3/2 & 0 & 1/2 & 0\\ 0 & 0 & 1 & 1/3 & 5/12 & -1/12\end{array}\right)$$

$$\xrightarrow{R_2-\frac{3}{2}R_3}\left(\begin{array}{ccc|ccc}1 & -2 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & -1/2 & -1/8 & 1/8\\ 0 & 0 & 1 & 1/3 & 5/12 & -1/12\end{array}\right)\xrightarrow{R_1+2R_2}\left(\begin{array}{ccc|c}1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{array}{ccc}0 & -1/4 & 1/4\\ -1/2 & -1/8 & 1/8\\ 1/3 & 5/12 & -1/12\end{array}}_{A^{-1}}\\ 0 & 0 & 1 & \end{array}\right)$$