Exam 16/17B ①

Q1.

(a) $\displaystyle\int \frac{2x+\sqrt{x}-\frac{3}{x^{2/3}}}{\sqrt[3]{x}}\,dx = \int \frac{2x + x^{\frac{1}{2}} - 3x^{-2/3}}{x^{\frac{1}{3}}}\,dx = \int \left(2x^{\underbrace{1-\frac{1}{3}}_{2/3}} + x^{\underbrace{\frac{1}{2}-\frac{1}{3}}_{\frac{1}{6}}} - 3x^{\underbrace{-2/3-\frac{1}{3}}_{-1}}\right)dx$

$\displaystyle = \int 2\frac{x^{5/3}}{5/3} + \frac{x^{7/6}}{7/6} - 3\ln|x| + C$

(b) $\displaystyle\int \frac{x^2+2}{x+2}\,dx = \int \left[(x-2) + \frac{6}{x+2}\right]dx = \frac{x^2}{2} - 2x + 6\ln|x+2| + C$

$$\begin{array}{r|l} & x-2 \\ \hline x+2 & x^2+2 \\ & x^2+2x \\ \hline & -2x+2 \\ & -2x-4 \\ \hline & 6 \end{array}$$

(c) $\displaystyle\int_0^{\frac{\pi}{3}} \sin^3 x\,dx = \int^{\frac{1}{2}} \sin^3 x\left(\frac{dy}{-\sin x}\right)$

$\displaystyle = -\int_1^{\frac{1}{2}} \underbrace{\sin^2 x}_{1-\cos^2 x = 1-y^2}\,dy$

$\displaystyle = \int_{\frac{1}{2}}^1 (1-y^2)\,dy = \left[y - \frac{y^3}{3}\right]_{\frac{1}{2}}^1 = \frac{5}{24}$ //

$y = \cos x$

$\frac{dy}{dx} = -\sin x$

$x = 0 \Rightarrow y = \cos 0 = 1$

$x = \frac{\pi}{3} \Rightarrow y = \cos\frac{\pi}{3} = \frac{1}{2}$

Q2 (a) $\displaystyle\int \frac{1}{(x^2-4)^{3/2}}\,dx = \int \frac{1}{(4\tan^2\theta)^{3/2}}\left(2\sec\theta\tan\theta\,d\theta\right)$

$\boxed{x = 2\sec\theta} \Rightarrow \frac{dx}{d\theta} = 2\sec\theta\tan\theta$

$x^2 - 4 = 4\sec^2\theta - 4 = 4(\sec^2\theta - 1)$

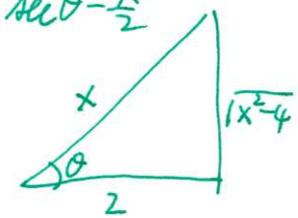
$= 4\tan^2\theta$

$\displaystyle = \int \frac{1}{8\tan^3\theta}\left(2\sec\theta\tan\theta\,d\theta\right) = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$

$\displaystyle = \frac{1}{4}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cos^2\theta}}\,d\theta = \cancel{\int \frac{1}{\cos^3\theta\,\sin\theta}}\,d\theta = \frac{1}{4}\int \frac{1}{\cos\theta}\,\frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$

$$= \frac{1}{4}\int \frac{d(\sin\theta)}{\sin^2\theta} = \frac{1}{4}\int [\sin\theta]^{-2} d(\sin\theta) = \frac{1}{4}\frac{[\sin\theta]^{-1}}{-1} = \frac{-1}{4}\frac{1}{\sin\theta} + C$$

$\sec\theta = \frac{x}{2}$



$$= \frac{-1}{4}\frac{1}{\frac{\sqrt{x^2-4}}{x}} + C = \frac{-1}{4}\frac{x}{\sqrt{x^2-4}} + C \quad //$$

2(b) $\int \frac{1}{x^2}\ln x\, dx = \int \underbrace{\ln x}_{u} \underbrace{\left(\frac{1}{x^2}dx\right)}_{dv \Rightarrow v = \int \frac{1}{x^2}dx = \frac{-1}{x}} \overset{IBS}{=} \frac{-1}{x}\ln x - \int\left(\frac{-1}{x}\right)\underbrace{d(\ln x)}_{\frac{1}{x}dx}$

$$= \frac{-1}{x}\ln x + \underbrace{\int \frac{1}{x^2}dx}_{-\frac{1}{x}+C}$$

(c) $I = \int \frac{9x^2}{\underbrace{(x-2)(x^2+2x+10)}_{\text{proper}}} dx$

$$= \int\left[\frac{2}{x-2} + \frac{7x+10}{x^2+2x+10}\right]dx$$

$$= 2\ln|x-2| + \underbrace{\int \frac{7x+10}{x^2+2x+10}dx}_{I_1}$$

Partial fractions

$$\frac{9x^2}{(x-2)(x^2+2x+10)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+10}$$

$\Rightarrow 9x^2 = A(x^2+2x+10) + (Bx+C)(x-2)$

$x=2: \quad 36 = 18A \Rightarrow A = 2$

compare the coefficient of x^2: $9 = A + B \Rightarrow B = 9-2 = 7$

compare the constant term: $0 = 10A - 2C \Rightarrow C = 10$

$\frac{d}{dx}(x^2+2x+10) = 2x+2$

express $7x+10 = a(2x+2)+b = 2ax + (2a+b)$

$\Rightarrow \begin{cases} 2a = 7 \Rightarrow a = \frac{7}{2} \\ 2a+b = 10 \Rightarrow b = 10-7 = 3 \end{cases}$

$$I_1 = \int \frac{7x+10}{x^2+2x+10}\,dx = \frac{7}{2}\underbrace{\int \frac{2x+2}{x^2+2x+10}\,dx}_{\ln|x^2+2x+10|} + 3\underbrace{\int \frac{1}{x^2+2x+10}\,dx}$$

$$= \int \frac{1}{(x+1)^2+9}\,dx = \frac{1}{9}\int \frac{1}{\left(\frac{x+1}{3}\right)^2+1}\,dx = \frac{1}{9}\tan^{-1}\left(\underbrace{\frac{x+1}{3}}_{\frac{1}{3}}\right)$$

$$\therefore I = 2\ln|x+2| + \frac{7}{2}\ln|x^2+2x+10| + \tan^{-1}\left(\frac{x+1}{3}\right) + C$$

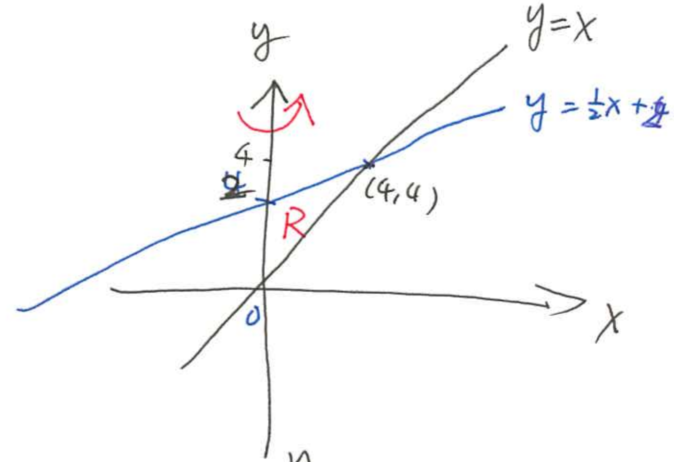
Q 3(a)

$$R: \begin{cases} 2y = x+4 \Rightarrow y = \frac{1}{2}x+2 \\ y = x \\ x = 0 \end{cases} \quad \text{about } y\text{-axis}$$

$x = \frac{1}{2}x+2$

$\Rightarrow \frac{1}{2}x = 2 \Rightarrow x = 4$

$\Rightarrow y = x = 4$



$$V_y = V_{outer} - V_{inner}$$

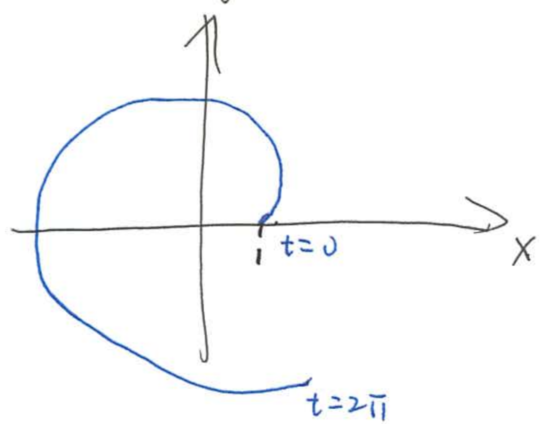
$$= \int_0^4 \pi \underbrace{x_{outer}^2}_{y^2}\,dy - \int_2^4 \pi \underbrace{x_{inner}^2}_{(2y-4)^2}\,dy = \frac{32}{3}\pi$$

(b)

$$\begin{cases} x = \cos t + t\sin t \\ y = \sin t - t\cos t \end{cases} \quad 0 \le t \le 2\pi$$

$$\frac{dx}{dt} = -\sin t + (t\cos t + \sin t) = t\cos t$$

$$\frac{dy}{dt} = \cos t - (-t\sin t + \cos t) = t\sin t$$



$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = |t|$$

$$s = \int_0^{2\pi} \frac{ds}{dt}\,dt = \int_0^{2\pi} |t|\,dt = \frac{t^2}{2}\Big|_0^{2\pi} = 2\pi^2 \,//$$

④

4(a)' $\begin{cases} \vec{a} = \vec{i} - 2\vec{j} + 3\vec{k} \\ \vec{b} = -2\vec{i} + 5\vec{k} \\ \vec{c} = 3\vec{j} - 4\vec{k} \end{cases}$ $\quad \vec{a}\cdot\vec{b}\times\vec{c} = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 0 & 5 \\ 0 & 3 & -4 \end{vmatrix} = -17$

Volume of parallelepiped $V = |\vec{a}\cdot\vec{b}\times\vec{c}| = |-17| = 17$ //

Q Volume of tetrahedron $\frac{1}{6}|\vec{a}\cdot\vec{b}\times\vec{c}| = \frac{17}{6}$

(b) $A(0,1,-2)$, $B(2,-3,1)$, $C(3,-2,0)$, $P(1,2,-4)$

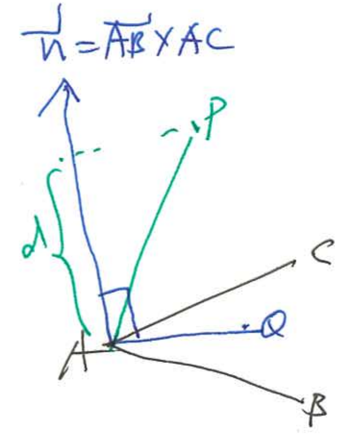
$\vec{AB} = \vec{OB} - \vec{OA} = 2\vec{i} - 4\vec{j} + 3\vec{k}$

$\vec{AC} = \vec{OC} - \vec{OA} = 3\vec{i} - 3\vec{j} + 2\vec{k}$

$\vec{n} = \vec{AB}\times\vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & 3 \\ 3 & -3 & 2 \end{vmatrix} = \vec{i} + 5\vec{j} + 6\vec{k}$

shortest distance $\vec{AP} = \vec{OP} - \vec{OA} = \vec{i} + \vec{j} + 2\vec{k}$

$d = |\mathrm{proj}_{\vec{n}}\vec{AP}| = \left|\vec{AP}\cdot\frac{\vec{n}}{|\vec{n}|}\right| = \frac{|-6|}{\sqrt{62}} = \frac{6}{\sqrt{62}}$ //



Q Find the plane equation containing A, B, C

let $Q = x\vec{i} + y\vec{j} + z\vec{k}$ in the plane.

$\vec{AQ} = \vec{OQ} - \vec{OA} = x\vec{i} + (y-1)\vec{j} + (z+2)\vec{k}$

$0 = \vec{AQ}\cdot\vec{n} = x + 5(y-1) + 6(z+2) \Rightarrow x + 5y - 5 + 6z + 12 = 0$

$\Rightarrow x + 5y + 6z = 5 - 12 = -7$ ← plane equation

Q 5(a) Simplify the complex number into Cartesian form $(a+bi)$

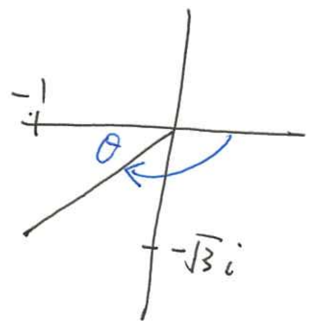
$$\frac{1+2i}{3-4i} - \frac{3-2i}{5i} = \underbrace{\frac{1+2i}{3-4i}\,\frac{3+4i}{3+4i}}_{\frac{(3-8)+(6+4)i}{9+16} = \frac{-5+10i}{25} = \frac{-1+2i}{5}} - \underbrace{\frac{3-2i}{5i}\,\frac{-i}{-i}}_{\frac{2-3i}{5}} = \frac{-1+2i}{5} - \frac{-2-3i}{5} = \cancel{\frac{4-i}{5}} =$$

$$= \frac{1+5i}{5} = \frac{1}{5} + i \;//$$

(b) Solve $iz^3 = \sqrt{3} - i$ in Euler form with principal argument

Key Step: Express z^3 into a complex number

$$\Rightarrow z^3 = \frac{\sqrt{3}-i}{i}\,\frac{-i}{-i} = \frac{-1-\sqrt{3}i}{1} = -1-\sqrt{3}i$$



$$|-1-\sqrt{3}i| = \sqrt{(-1)^2+(-\sqrt{3})^2} = \sqrt{1+3} = 2$$

$$\arg(-1-\sqrt{3}i) = -(\pi - \theta) = -\left(\pi - \tan^{-1}\left|\frac{-\sqrt{3}}{-1}\right|\right) = -(\pi - \underbrace{\tan^{-1}\sqrt{3}}_{\frac{\pi}{3}}) = -\frac{2\pi}{3}$$

$$z^3 = 2e^{i(-\frac{2\pi}{3})} = 2e^{i(-\frac{2\pi}{3}+2k\pi)}$$

$$\Rightarrow z_k = 2^{\frac{1}{3}} e^{i(\frac{-2\pi}{3}+2k\pi)/3}, \quad k=0,1,2.$$

$$z_0 = 2^{1/3} e^{-i2\pi/9}$$

$$z_1 = 2^{1/3} e^{i(-\frac{2\pi}{3}+2\pi)/3} = 2^{1/3} e^{i\frac{4\pi}{9}}$$

$$z_2 = 2^{1/3} e^{i(-\frac{2\pi}{3}+4\pi)/3} = 2^{1/3} e^{i\frac{10\pi}{9}} = 2^{1/3} e^{i(\frac{10\pi}{9}-2\pi)} = 2^{1/3} e^{-i\frac{8\pi}{9}} \;//$$

not principal

Q5 (a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \dfrac{1}{|A|}\,\mathrm{adj}\,A = \dfrac{1}{4-6}\begin{pmatrix} 4 & -3 \\ -2 & 1 \end{pmatrix}^{T} = \dfrac{-1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$

Gauss elimination

(b)(i) $(A|\vec{b}) = \left(\begin{array}{cccc|c} 2 & -1 & 0 & -5 & 1 \\ 1 & -1 & 2 & 0 & 2 \\ -3 & 1 & 2 & 10 & 0 \end{array}\right) \xrightarrow{R_1 \Leftrightarrow R_1} \left(\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 2 \\ 2 & -1 & 0 & -5 & 1 \\ -3 & 1 & 2 & 10 & 0 \end{array}\right)$

$\xrightarrow[R_3+3R_1]{R_2-2R_1} \left(\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 2 \\ 0 & 1 & -4 & -5 & -3 \\ 0 & -2 & 8 & 10 & 6 \end{array}\right) \xrightarrow[R_3+2R_2]{} \left(\begin{array}{cccc|c} 1 & -1 & 2 & 0 & 2 \\ 0 & 1 & -4 & -5 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

R_3 : $0 = 0$ consistent

C_3 and C_4 have no pivots $\Rightarrow$ $z = s$, $w = t$ are free variables

R_2 : $y - 4z - 5w = -3 \Rightarrow y = -3 + 4\underbrace{z}_{s} + 5\underbrace{w}_{t} = -3 + 4s + 5t$

R_1 : $x - y + 2z = 2 \Rightarrow x = 2 + y - 2\underbrace{z}_{s} = 2 + (-3 + 4s + 5t) - 2s$

$= -1 + 2s + 5t$

vector solution $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -1 + 2s + 5t \\ -3 + 4s + 5t \\ s \\ t \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ 4 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 5 \\ 5 \\ 0 \\ 1 \end{pmatrix}$

b(ii) Corresponding homogeneous system

$$\begin{cases} 2x - y - 5w = 0 \\ x - y + 2z = 0 \\ -3x + y + 2z + 10w = 0 \end{cases}$$

Two linearly independent homogeneous solutions $\begin{pmatrix} 2 \\ 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 5 \\ 0 \\ 1 \end{pmatrix}$