MA1201 Calculus and Basic Linear Algebra II

Solution of Problem Set 1 Basic Concept in Integration

Problem 1

(a) $\int \cos(3x+1)\,dx = \frac{1}{3}\sin(3x+1) + C$

(b) $\int \left(\frac{1}{x^3} - \sqrt{x}\right) dx = \int \left(x^{-3} - x^{\frac{1}{2}}\right) dx = \frac{x^{-3+1}}{-3+1} - \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = -\frac{1}{2x^2} - \frac{2}{3}x^{\frac{3}{2}} + C.$

(c) $\int e^{1-x} dx = \frac{1}{(-1)}e^{1-x} + C = -e^{1-x} + C.$

(d) $\int \frac{1}{1+16x^2} dx = \int \frac{1}{1+(4x)^2} dx = \frac{1}{4}\tan^{-1} 4x + C$

(e) $\int \frac{1}{2x+1} dx = \frac{1}{2}\ln|2x+1| + C$

(f) $\int \frac{1}{(2x+1)^2} dx = \int (2x+1)^{-2} dx = \frac{1}{2}\frac{(2x+1)^{-2+1}}{-2+1} + C = -\frac{1}{2(2x+1)} + C.$

Problem 2

(a) $\int \frac{x^2 - x + 1}{x^2} dx = \int 1dx - \int \frac{1}{x} dx + \int x^{-2} dx = x - \ln|x| + \frac{x^{-2+1}}{-2+1} + C = x - \ln|x| - \frac{1}{x} + C$

(b) $\int \frac{2x^2}{x^2+1} dx = \int \frac{2x^2+2}{x^2+1} dx - \int \frac{2}{x^2+1} dx = \int 2dx - 2\int \frac{1}{x^2+1} dx = 2x - 2\tan^{-1} x + C$

(c) $\int \frac{e^{2x} + e^{x-3} + 1}{e^{x+1}} dx = \int e^{x-1} dx + \int e^{-4} dx + \int e^{-1-x} dx = e^{x-1} + e^{-4}x - e^{-1-x} + C.$

(d)
$$\int \sin 3x \sin 2x\,dx = \int -\frac{1}{2}[\cos(3x+2x) - \cos(3x-2x)]dx = -\frac{1}{2}\int \cos 5x\,dx + \frac{1}{2}\int \cos x\,dx$$
$$= -\frac{1}{10}\sin 5x + \frac{1}{2}\sin x + C$$

(e)
$$\int \cos^3 2x\,dx = \int (\cos 2x)(\cos 2x)(\cos 2x)dx = \int \frac{1}{2}(\cos 4x + 1)(\cos 2x)dx$$
$$= \frac{1}{2}\int \cos 4x \cos 2x\,dx + \frac{1}{2}\int \cos 2x\,dx$$
$$= \frac{1}{2}\int \frac{1}{2}(\cos 6x + \cos 2x)dx + \frac{1}{2}\int \cos 2x\,dx = \frac{1}{24}\sin 6x + \frac{1}{8}\sin 2x + \frac{1}{4}\sin 2x + C$$
$$= \frac{1}{24}\sin 6x + \frac{3}{8}\sin 2x + C$$

(f) We use method of partial fraction and decompose the integrand as
$$\frac{1}{(x-1)(2x-3)} = \frac{A}{x-1} + \frac{B}{2x-3}$$
$$\Rightarrow 1 = A(2x-3) + B(x-1)$$

- Put $x = \frac{3}{2}$, we get $\frac{B}{2} = 1 \Rightarrow B = 2$.
- Put $x = 1$, we get $1 = -A \Rightarrow A = -1$.

$$\int \frac{1}{(x-1)(2x-3)}dx = -\int \frac{1}{x-1}dx + 2\int \frac{1}{2x-3}dx = -\ln|x-1| + \ln|2x-3| + C.$$

(g)
$$\int \frac{3}{x^2-2x+5}dx = 3\int \frac{1}{(x-1)^2+4}dx = \frac{3}{4}\int \frac{1}{\left(\frac{x-1}{2}\right)^2+1}dx = \frac{3}{4}\left(\frac{1}{\frac{1}{2}}\tan^{-1}\frac{x-1}{2}\right) + C$$
$$= \frac{3}{2}\tan^{-1}\frac{x-1}{2} + C.$$

(h)
$$\int \frac{1}{2x^2-4x+9}dx = \int \frac{1}{2(x-1)^2+7}dx = \frac{1}{7}\int \frac{1}{\left(\sqrt{\frac{2}{7}}(x-1)\right)^2+1}dx$$
$$= \frac{1}{7}\left(\frac{1}{\sqrt{\frac{2}{7}}}\tan^{-1}\left(\sqrt{\frac{2}{7}}x - \sqrt{\frac{2}{7}}\right)\right) + C = \frac{1}{\sqrt{14}}\tan^{-1}\left(\sqrt{\frac{2}{7}}x - \sqrt{\frac{2}{7}}\right) + C.$$

(i)
$$\int \frac{x+6}{(2x-1)^3}dx = \int \frac{x-\frac{1}{2}}{(2x-1)^3}dx + \int \frac{\frac{13}{2}}{(2x-1)^3}dx = \frac{1}{2}\int \frac{1}{(2x-1)^2}dx + \frac{13}{2}\int \frac{1}{(2x-1)^3}dx$$
$$= \frac{1}{2}\int (2x-1)^{-2}dx + \frac{13}{2}\int (2x-1)^{-3}dx$$
$$= \frac{1}{2}\left(\frac{1}{2}\frac{(2x-1)^{-2+1}}{-2+1}\right) + \frac{13}{2}\left(\frac{1}{2}\frac{(2x-1)^{-3+1}}{-3+1}\right) + C$$
$$= -\frac{1}{4(2x-1)} - \frac{13}{8(2x-1)^2} + C$$

(j)
$$\int \tan^2 x\, dx = \int (\sec^2 x - 1)dx = \tan x - x + C$$

Problem 3

(a)
$$\int_1^2 \frac{x-1}{3x^2}dx = \frac{1}{3}\int_1^2 \frac{1}{x}dx - \frac{1}{3}\int_1^2 \frac{1}{x^2}dx = \frac{1}{3}\ln x\,|_1^2 + \frac{1}{3x}|_1^2 = \frac{1}{3}\ln 2 - \frac{1}{6}.$$

(b)
$$\int_{-1}^1 \cos(3x+1)dx = \frac{1}{3}\sin(3x+1)\,|_{-1}^1 = \frac{1}{3}(\sin 4 + \sin 2).$$

(c)
$$\int_0^1 (e^{2x+1} - e^{2x-1})dx = \frac{1}{2}e^{2x+1}|_0^1 - \frac{1}{2}e^{2x-1}|_0^1 = \frac{e^3 - 2e + e^{-1}}{2}.$$

(d)
$$\int_0^{\frac{\pi}{2}} \sin^2 x\, dx = \int_0^{\frac{\pi}{2}} -\frac{1}{2}[\cos(x+x) - \cos(x-x)]dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} 1dx - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2x\, dx$$
$$= \frac{1}{2}x|_0^{\frac{\pi}{2}} - \frac{1}{4}\sin 2x\,|_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

(e) Note that

$$|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1 \geq 0 \\ -(2x-1) & \text{if } 2x-1 < 0 \end{cases} = \begin{cases} 2x-1 & \text{if } x \geq \frac{1}{2} \\ 1-2x & \text{if } x < \frac{1}{2} \end{cases}.$$

Then the integral is found to be

$$\int_0^1 |2x-1|dx = \int_0^{\frac{1}{2}} |2x-1|dx + \int_{\frac{1}{2}}^1 |2x-1|dx = \int_0^{\frac{1}{2}} (1-2x)dx + \int_{\frac{1}{2}}^1 (2x-1)dx$$
$$= (x-x^2)|_0^{\frac{1}{2}} + (x^2-x)|_{\frac{1}{2}}^1 = \frac{1}{2}.$$

(f) Note that for $-\pi \leq x \leq x$

$$|\sin x| = \begin{cases} \sin x & \text{if } \sin x \geq 0 \\ -\sin x & \text{if } \sin x < 0 \end{cases} = \begin{cases} \sin x & \text{if } \ 0 \leq x \leq \pi \\ -\sin x & \text{if } \ -\pi \leq x \leq 0 \end{cases}.$$

Then the integral is found to be

$$\int_{-\pi}^{\pi} |\sin x|dx = \int_0^{\pi} |\sin x|dx + \int_{-\pi}^{0} |\sin x|dx = \int_0^{\pi} \sin x\, dx - \int_{-\pi}^{0} \sin x\, dx$$
$$= -\cos x\,|_0^{\pi} - (-\cos x)|_{-\pi}^{0} = 4.$$

(g) Note that

$$|x-1| = \begin{cases} x-1 & \text{if } x-1 \geq 0 \\ -(x-1) & \text{if } x-1 < 0 \end{cases} = \begin{cases} x-1 & \text{if } x \geq 1 \\ 1-x & \text{if } x < 1 \end{cases}.$$

Then the integral is found to be

$$\int_0^2 e^{1+|x-1|}dx = \int_0^1 e^{1+|x-1|}dx + \int_1^2 e^{1+|x-1|}dx = \int_0^1 e^{1+1-x}dx + \int_1^2 e^{1+x-1}dx$$
$$= \int_0^1 e^{2-x}dx + \int_1^2 e^{x}dx = -e^{2-x}|_0^1 + e^{x}|_1^2 = 2e^2 - 2e^1.$$

(h) Let $f(x) = x^4 \sin^9 x$, one can verify that

$$f(-x) = (-x)^4 \sin^9(-x) \overset{\sin(-x)=-\sin x}{\cong} x^4(-\sin x)^9 = -x^4 \sin^9 x = -f(x).$$

Then $f(x)$ is an *odd* function and

$$\int_{-1}^{1} x^4 \sin^9 x\, dx = 0.$$

(i) Let $f(x) = \frac{x^2 \sin^3 x}{1+\cos^5 x}$, then we can verify that

$$f(-x) = \frac{(-x)^2 \sin^3(-x)}{1+\cos^5(-x)} = \frac{x^2(-\sin x)^3}{1+(\cos x)^5} = \frac{-x^2 \sin^3 x}{1+\cos^5 x} = -f(x).$$

Then $f(x)$ is an *odd* function and

$$\int_{-\pi}^{\pi} \frac{x^2 \sin^3 x}{1+\cos^5 x} dx = 0.$$

(j) Take $g(x) = \frac{\sin^3 x}{x^2+1}$, one can show that

$$g(-x) = \frac{\sin^3(-x)}{(-x)^2+1} = \frac{(-\sin x)^3}{x^2+1} = -\frac{\sin^3 x}{x^2+1} = -g(x).$$

This shows $g(x)$ is an odd function, we can compute the integral as

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x^2 \cos x + \cos x + \sin^3 x}{x^2+1} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos x\,(1+x^2)}{x^2+1} dx + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^3 x}{x^2+1} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x\, dx + 0$$
$$= \sin x\,|_{-\pi/4}^{\pi/4} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Problem 4

(a) We let $F(y) = \int e^{2y^2+1}dy$, one can use fundamental theorem of calculus and obtain

$$\int_3^x e^{2y^2+1}dy = F(x) - F(3).$$

Then the derivative can be computed as

$$\frac{d}{dx}\int_3^x e^{2y^2+1}dy = \frac{d}{dx}F(x) - \frac{d}{dx}\underbrace{F(3)}_{\text{number}} \overset{\frac{d}{dy}F(y)=\frac{d}{dy}\int e^{2y^2+1}dy=e^{2y^2+1}}{\cong} e^{2x^2+1} - 0 = e^{2x^2+1}.$$

(b) We let $G(y) = \int \cos(y^2)\,dy$, one can use fundamental theorem of calculus and obtain

$$\int_{2x}^{x^2} \cos(y^2)\,dy = G(x^2) - G(2x).$$

Using the fact that $\frac{d}{dy}G(y) = \frac{d}{dy}\int \cos(y^2)\,dy = \cos(y^2)$, the derivative can be computed as

$$\frac{d}{dx}\int_{2x}^{x^2} \cos(y^2)\,dy = \frac{d}{dx}G(x^2) - \frac{d}{dx}G(2x) = \frac{dG(x^2)}{d(x^2)}\frac{d(x^2)}{dx} - \frac{dG(2x)}{d(2x)}\frac{d(2x)}{dx}$$

$$\overset{\text{take } y=x^2,\ y=2x}{\cong} 2x\cos(x^4) - 2\cos(4x^2).$$

Problem 5

(a) Let $F(x) = \int f(x)dx$, then we have

$$\int_0^a f(x)dx = F(a) - F(0) \dots\dots (1).$$

On the other hand,

$$\int_0^a f(a-x)dx = -F(a-x)|_0^a = -F(0) + F(a) = \int_0^a f(x)dx.$$

(b) Let $G(x) = \int g(x)dx$.

(i)
$$\int_0^4 g(x)dx = \int_3^4 g(x)dx + \int_2^3 g(x)dx + \int_1^2 g(x)dx + \int_0^1 g(x)dx$$
$$= \int_3^4 g(x-3)dx + \int_2^3 g(x-2)dx + \int_1^2 g(x-1)dx + \int_0^1 g(x)dx$$
$$= G(x-3)|_3^4 + G(x-2)|_2^3 + G(x-1)|_1^2 + G(x)|_0^1$$
$$= G(1) - G(0) + G(1) - G(0) + G(1) - G(0) + G(1) - G(0)$$
$$= 4\big(G(1) - G(0)\big) \overset{\int_0^1 g(x)dx=G(1)-G(0)}{\cong} 4\int_0^1 g(x)dx.$$

(ii)
$$\int_0^1 g(3x)dx = \frac{1}{3}G(3x)|_0^1 = \frac{1}{3}[G(3) - G(0)] = \frac{1}{3}\int_0^3 g(x)dx.$$