Additional Exercise

It is given that $\sin A = -\frac{12}{13}$ where $-90^{\circ} < A < 0^{\circ}$, and that $\cos B = -\frac{4}{5}$ where $180^{\circ} < B < 270^{\circ}$. Without using calculator,

(i) find the value of $\sin(A+B)$,

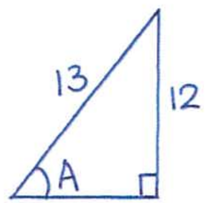
(ii) find the value of $\cos(A+B)$,

(iii) deduce that $90^{\circ} < A+B < 180^{\circ}$,

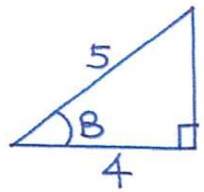
(iv) find the value of $\cos\left(\frac{B}{2}\right)$.

Solution:

Consider the following right-angled triangles:



$\sqrt{13^2-12^2} = \sqrt{25} = 5$



$\sqrt{5^2-4^2} = \sqrt{9} = 3$

$\boxed{\sin A = -\frac{12}{13}}$

$\boxed{\cos B = -\frac{4}{5}}$

$\because \; -90^\circ < A < 0^\circ$ (Quad. IV)

$\therefore \; \cos A = \frac{5}{13}$ (cosine is positive in Quad. IV)

S	A
T	C

$\because \; 180^\circ < B < 270^\circ$ (Quad. III)

$\therefore \; \sin B = -\frac{3}{5}$ (sine is negative in Quad. III)

S	A
T	C

(i) $\sin(A+B) = \sin A \cos B + \cos A \sin B$ (compound angle formula)

$$= \left(-\frac{12}{13}\right)\left(-\frac{4}{5}\right) + \left(\frac{5}{13}\right)\left(-\frac{3}{5}\right)$$

$$= \frac{48}{65} - \frac{15}{65}$$

$$= \frac{33}{65}$$

(ii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$ (compound angle formula)

$$= \left(\frac{5}{13}\right)\left(-\frac{4}{5}\right) - \left(-\frac{12}{13}\right)\left(-\frac{3}{5}\right)$$

$$= -\frac{20}{65} - \frac{36}{65}$$

$$= -\frac{56}{65}$$

(iii) $\because$ $-90^\circ < A < 0^\circ$ and $180^\circ < B < 270^\circ$

$\therefore$ $90^\circ < A+B < 270^\circ$

II	I
III	IV

In Quad. II, $90^\circ < \theta < 180^\circ$,

the sine ratio is positive

but the cosine ratio is negative.

S	A
T	C

In Quad. III, $180^\circ < \theta < 270^\circ$,

both sine and cosine ratios are negative.

S	A
T	C

$\because$ $\sin(A+B) > 0$ & $\cos(A+B) < 0$ from (i) and (ii),

$\therefore$ $90^\circ < A+B < 180^\circ$ (A+B is in Quad. II)

(iv) $\cos^2\left(\frac{B}{2}\right) = \frac{1}{2}(1+\cos B)$ (Half-angle formula)

$$= \frac{1}{2}\left[1+\left(-\frac{4}{5}\right)\right]$$

$$= \frac{1}{10}$$

$$\Rightarrow \cos\left(\frac{B}{2}\right) = \pm\frac{1}{\sqrt{10}}$$

$\because$ $180^\circ < B < 270^\circ$

$\therefore$ $90^\circ < \frac{B}{2} < 135^\circ$ (Quad. II)

S	A
T	C

$\therefore$ $\cos\left(\frac{B}{2}\right) < 0$ in Quad. II

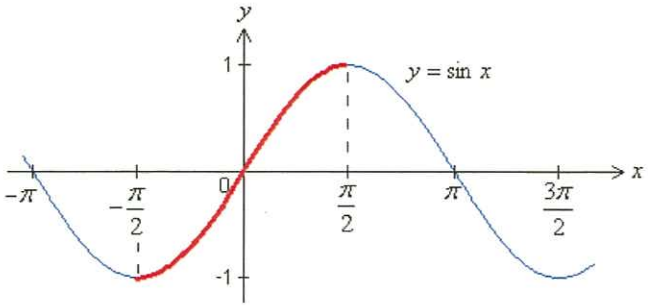
$\therefore$ $\cos\left(\frac{B}{2}\right) = -\frac{1}{\sqrt{10}}$

Inverse Trigonometric Functions

In this section, we will study the inverse functions of $\sin x$, $\cos x$ and $\tan x$.

➢ Inverse function of $\sin x$:

Consider the graph of $y = \sin x$.



The function $g(x) = \sin x$, where $x \in \mathbb{R}$, is not one-to-one, so $g(x)$ has no inverse.

The **principal part** of sine function is defined as $f(x) = \sin x$, where $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then $f(x)$ is one-to-one and therefore its inverse $f^{-1}(x)$ exists.

restricted domain

$$f^{-1}(x) = \sin^{-1} x \ , \quad \text{for} \quad x \in [-1, 1].$$

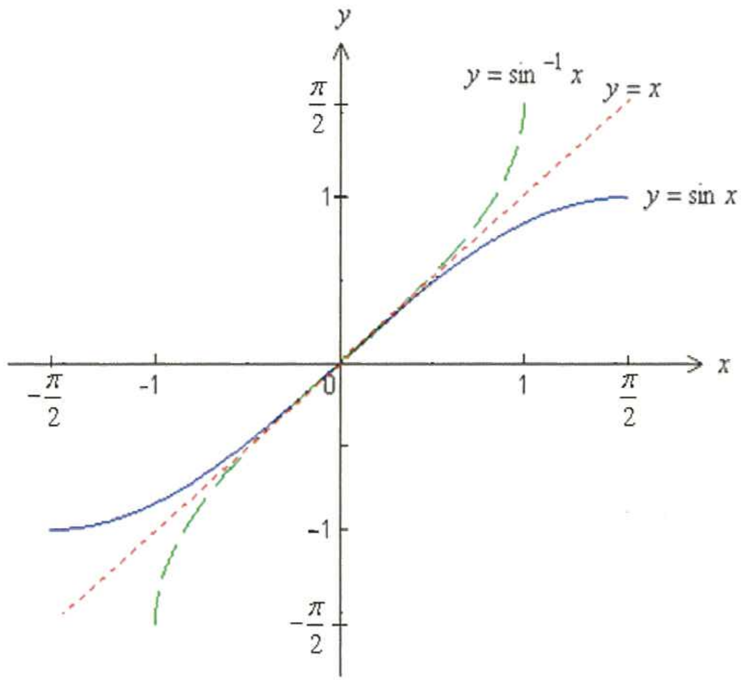
This is called the **inverse sine** (or **arcsine**) function, denoted by $\sin^{-1}$ (or arcsin).

$$\boxed{y = \sin^{-1} x \quad \Leftrightarrow \quad x = \sin y \quad \text{for } -1 \le x \le 1 \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}.}$$

Thus, (i) $\sin(\sin^{-1} x) = x$ for $-1 \le x \le 1$.

(ii) $\sin^{-1}(\sin y) = y$ for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Graphs of $y = \sin x$ and its inverse $y = \sin^{-1} x$:



$$f(x) = \sin x$$
$$f^{-1}(x) = \sin^{-1} x$$

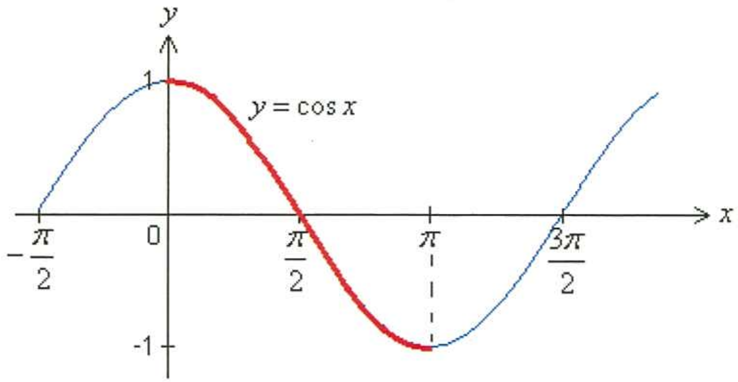
Note: $Dom(f) = Ran(f^{-1}) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ← principal range

$Ran(f) = Dom(f^{-1}) = [-1, 1]$

Is $f^{-1}(x) = \sin^{-1} x$ an odd function, even function, or neither of them? Odd

➢ Inverse function of $\cos x$:

Consider the graph of $y = \cos x$.



The function $g(x) = \cos x$, where $x \in \mathbb{R}$, is not one-to-one, so $g(x)$ has no inverse.

The **principal part** of cosine function is defined as $f(x) = \cos x$, where $x \in \underbrace{[0, \pi]}_{\text{restricted domain}}$. Then $f(x)$ is one-to-one and therefore its inverse $f^{-1}(x)$ exists.

$$f^{-1}(x) = \cos^{-1} x \ , \quad \text{for} \quad x \in [-1, 1].$$

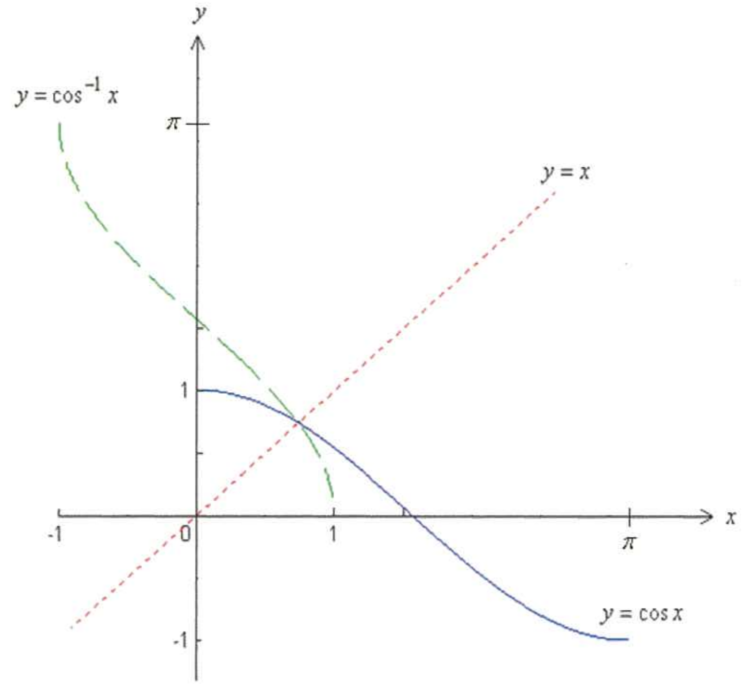
This is called the **inverse cosine** (or **arccosine**) function, denoted by $\cos^{-1}$ (or arccos).

$$\boxed{y = \cos^{-1} x \quad \Leftrightarrow \quad x = \cos y \quad \text{for } -1 \le x \le 1 \text{ and } 0 \le y \le \pi.}$$

Thus, (i) $\cos(\cos^{-1} x) = x$ for $-1 \le x \le 1$.

(ii) $\cos^{-1}(\cos y) = y$ for $0 \le y \le \pi$.

Graphs of $y = \cos x$ and its inverse $y = \cos^{-1} x$:



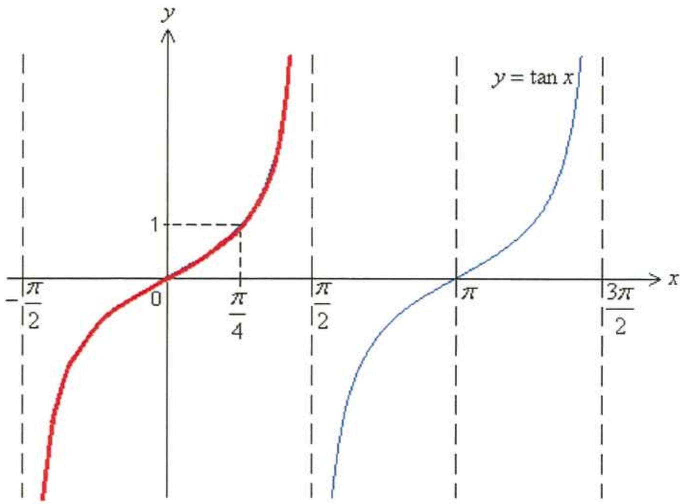
$$f(x) = \cos x$$
$$f^{-1}(x) = \cos^{-1} x$$

Note: $Dom(f) = Ran(f^{-1}) = [0, \pi]$ ← principal range

$Ran(f) = Dom(f^{-1}) = [-1, 1]$

Is $f^{-1}(x) = \cos^{-1} x$ an odd function, even function, or neither of them? Neither even nor odd

➢ Inverse function of $\tan x$:



The function $g(x) = \tan x$, where $x \in \mathbb{R}\backslash\left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots\right\}$, is not one-to-one, so $g(x)$ has no inverse.

The **principal part** of tangent function is defined as $f(x) = \tan x$, where $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $f(x)$ is one-to-one and therefore its inverse $f^{-1}(x)$ exists.

endpoints are not included

$$f^{-1}(x) = \tan^{-1} x \;, \quad \text{for} \quad x \in \mathbb{R}.$$

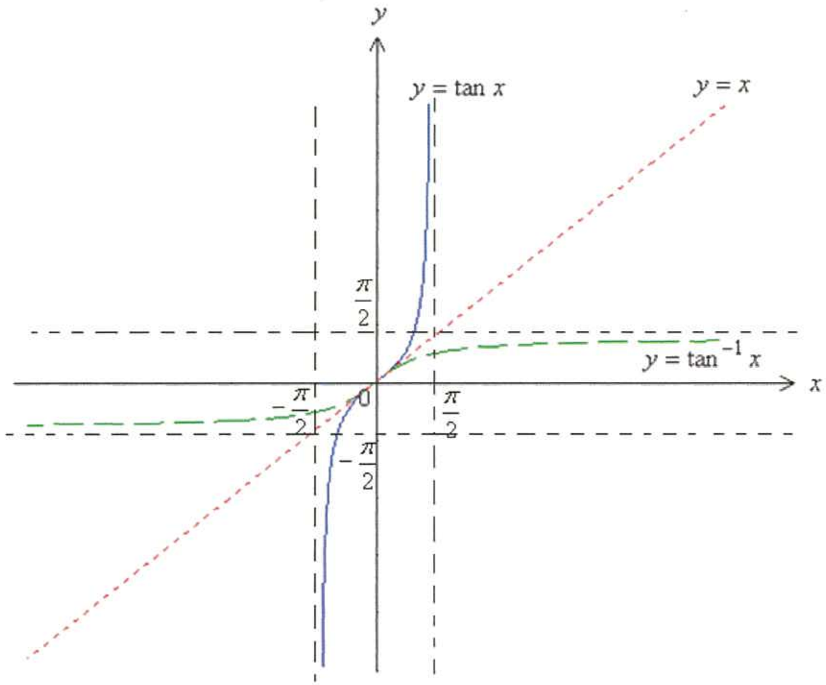
This is called the **inverse tangent** (or **arctangent**) function, denoted by $\tan^{-1}$ (or arctan).

$$\boxed{y = \tan^{-1} x \quad \Leftrightarrow \quad x = \tan y \quad \text{for every real number } x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}.}$$

Thus, (i) $\tan(\tan^{-1} x) = x$ for $x \in \mathbb{R}$.

(ii) $\tan^{-1}(\tan y) = y$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

<u>*Graphs of $y = \tan x$ and its inverse $y = \tan^{-1} x$*</u>:



$$f(x) = \tan x$$
$$f^{-1}(x) = \tan^{-1} x$$

<u>Note</u>: $Dom(f) = Ran(f^{-1}) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ← principal range

$Ran(f) = Dom(f^{-1}) = \mathbb{R}$

Is $f^{-1}(x) = \tan^{-1} x$ an odd function, even function, or neither of them? Odd

Remarks:

1. $\sin^2 x = (\sin x)^2$, $\sin^3 x = (\sin x)^3$, etc.

   However, $\sin^{-1} x \neq (\sin x)^{-1} = \frac{1}{\sin x}$. (Similarly for $\cos^{-1} x$ and $\tan^{-1} x$.)

2. The ranges of the inverse trigonometric functions are known as the **principal ranges**.

Inverse functions of cosecant, secant and cotangent

Similarly, we use the notations $\csc^{-1}$, $\sec^{-1}$ and $\cot^{-1}$ to denote the inverse functions of cosecant, secant and cotangent, respectively.

in rad. in deg.

Example 15

Recall: Principal range of $\sin^{-1}x$: $[-\frac{\pi}{2}, \frac{\pi}{2}]$ $[-90°, 90°]$

$\cos^{-1}x$: $[0, \pi]$ $[0°, 180°]$

$\tan^{-1}x$: $(-\frac{\pi}{2}, \frac{\pi}{2})$ $(-90°, 90°)$

Find the value of each of the following.

(a) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$	(b) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$	(c) $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$
(d) $\cos\left(\cos^{-1}\left(\frac{1}{4}\right)\right)$	(e) $\sin^{-1}(\sin 10°)$	(f) $\sin^{-1}(\sin 380°)$
(g) $\sin^{-1}\left(\sin\left(-\frac{5\pi}{6}\right)\right)$	(h) $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$	(i) $\cos^{-1}(\cos 300°)$
(j) $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$	(k) $\sin^{-1}(\cos 390°)$	(l) $\cos^{-1}\left(\sin\frac{5\pi}{4}\right)$

Solution

(a) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$ (in radians) or $45°$ (in degrees)

(b) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$ (in radians) or $150°$ (in degrees)

(c) $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{6}$ (in radians) or $-30°$ (in degrees)

(d) $\cos\left(\cos^{-1}\left(\frac{1}{4}\right)\right) = \frac{1}{4}$ $\because \frac{1}{4} \in [-1, 1]$

(e) $\sin^{-1}(\sin 10°) = 10°$ (since $10°$ lies in the principal range $[-90°, 90°]$.)

outside $[-90°, 90°]$

(f) $\sin^{-1}(\sin 380°) = \sin^{-1}(\sin(360° + 20°)) = \sin^{-1}(\sin(20°)) = 20°$

(which lies in the principal range $[-90°, 90°]$.)

outside $[-\frac{\pi}{2}, \frac{\pi}{2}]$

(g) $\sin^{-1}\left(\sin\left(-\frac{5\pi}{6}\right)\right) = \sin^{-1}\left(\sin\left(\frac{\pi}{6} - \pi\right)\right) = \sin^{-1}\left(-\sin\left(\overset{180°}{\boxed{\pi}} - \frac{\pi}{6}\right)\right)$

S	A
T	C

$= \sin^{-1}\left(-\sin\frac{\pi}{6}\right) = \sin^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right) = -\frac{\pi}{6}$

∵ sin is odd

(which lies in the principal range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.)

outside $[0, \pi]$

(h) $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right) = \cos^{-1}\left(\cos\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$ (which lies in the principal range $[0, \pi]$.)

∵ cos is even

(i) $\cos^{-1}(\cos 300°) = \cos^{-1}(\cos(360° - 60°)) = \cos^{-1}(\cos 60°) = 60°$

S	A
T	C

(which lies in the principal range $[0°, 180°]$.)

(j) $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)=\tan^{-1}\left(\tan\left(\pi+\frac{\pi}{6}\right)\right)=\tan^{-1}\left(\tan\left(\frac{\pi}{6}\right)\right)=\frac{\pi}{6}$

outside $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(which lies in the principal range $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.)

(k) $\sin^{-1}(\cos 390°)=\sin^{-1}(\cos(360°+30°))=\sin^{-1}(\cos(30°))$

$$=\sin^{-1}(\cos(90°-60°))=\sin^{-1}(\sin(60°))=60°$$

odd changed

(which lies in the principal range $[-90°,90°]$.)

S | A
T | C

within $[0,\pi]$

(l) $\cos^{-1}\left(\sin\frac{5\pi}{4}\right)=\cos^{-1}\left(\sin\left(\boxed{\frac{\pi}{2}}+\frac{3\pi}{4}\right)\right)=\cos^{-1}\left(\cos\left(\frac{3\pi}{4}\right)\right)=\frac{3\pi}{4}$

(which lies in the principal range $[0,\pi]$.)

$\frac{\pi}{2} = \boxed{9}0°$

odd changed

Exercise :

Find the following, if they exist. (Note: In this example, the angles are measured in radians.)

(a) $\cos^{-1}(\cos 4)$

(b) $\cos(\cos^{-1} 4)$

(c) $\sin^{-1}(\sin 4)$

(d) $\sin(\sin^{-1} 4)$

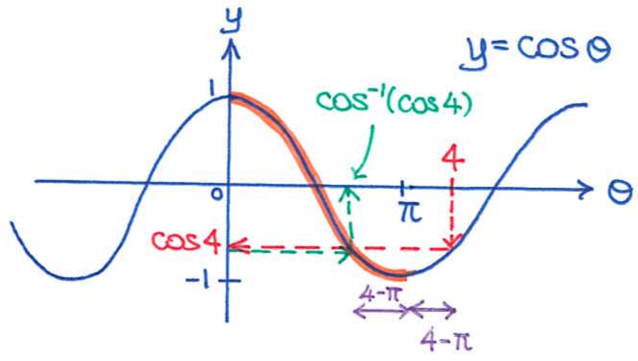
(e) $\tan^{-1}(\tan 4)$

(f) $\tan(\tan^{-1} 4)$

Solution:

(a) $\cos^{-1}(\cos 4) = \pi - (4-\pi)$

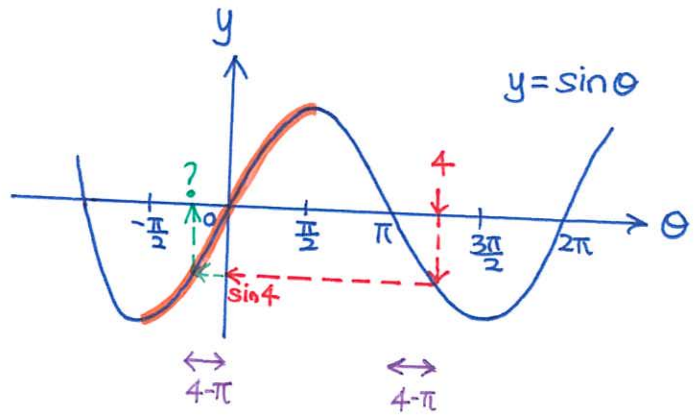
$= 2\pi - 4$



(b) $\cos(\cos^{-1} 4)$ does not exist, since the domain of $\cos^{-1} x$ is $[-1, 1]$ and $4 \notin [-1, 1]$.

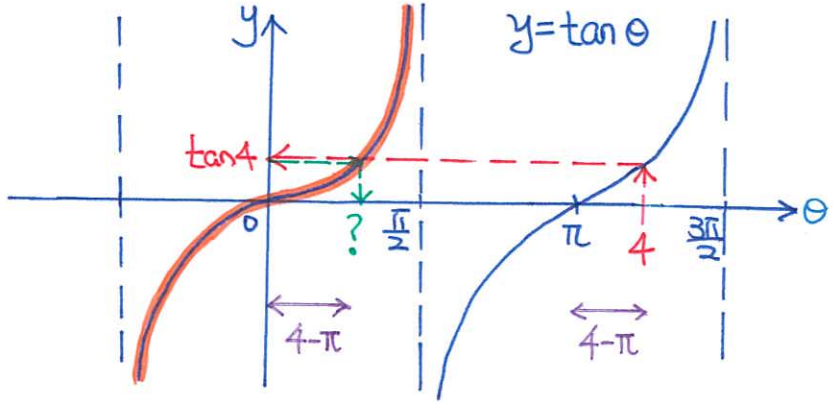
(c) $\sin^{-1}(\sin 4) = -(4-\pi)$

$= \pi - 4$



(d) $\sin(\sin^{-1} 4)$ does not exist, since the domain of $\sin^{-1} x$ is $[-1, 1]$ and $4 \notin [-1, 1]$.

(e) $\tan^{-1}(\tan 4) = 4-\pi$



(f) $\tan(\tan^{-1} 4) = 4$

Additional Ex: Find the following.

(i) $\cos^{-1}(\cos 2)$

(ii) $\sin^{-1}(\sin 2)$

(iii) $\tan^{-1}(\tan 2)$

Ans: (i) $\cos^{-1}(\cos 2) = 2$

(ii) $\sin^{-1}(\sin 2) = \pi-2$

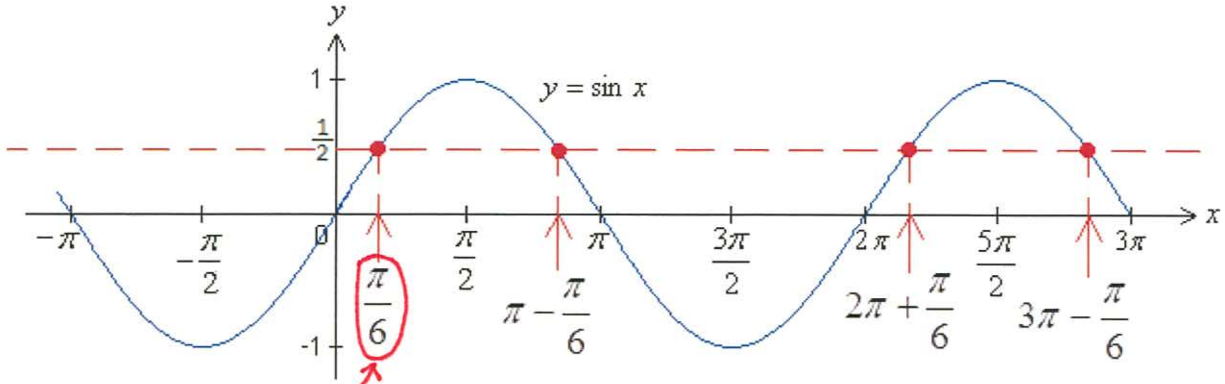
(iii) $\tan^{-1}(\tan 2) = 2-\pi$

General Solutions of Trigonometric Equations

Sine function:

all possible solutions

Find the general solution of $\sin x = \frac{1}{2}$.



$\alpha = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ ($\because$ the principal range of $\alpha = \sin^{-1}(x)$ is $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$.)

The solutions of $\sin x = \frac{1}{2}$ in $[0, 2\pi)$ are Consider an interval with length 2π (period of $\sin x$)

$$x = \alpha = \frac{\pi}{6} \quad \text{and} \quad x = \pi - \alpha = \pi - \frac{\pi}{6} = \frac{5\pi}{6}.$$

Since $y = \sin x$ is periodic with period 2π, scalar multiple of period

$$x = \frac{\pi}{6} + (2\pi)m \quad \text{and} \quad x = \pi - \frac{\pi}{6} + (2\pi)m,$$

where $m \in \mathbb{Z}$, are also solutions of $\sin x = \frac{1}{2}$.

+ when even

That is, $x = \underbrace{(2m)}_{\text{even no.}}\pi + \frac{\pi}{6}$ and $x = \underbrace{(2m+1)}_{\text{odd no.}}\pi - \frac{\pi}{6}$, where $m \in \mathbb{Z}$.

– when odd

$\therefore$ The **general solution** of the equation $\sin x = \frac{1}{2}$ is

Combine 2 cases

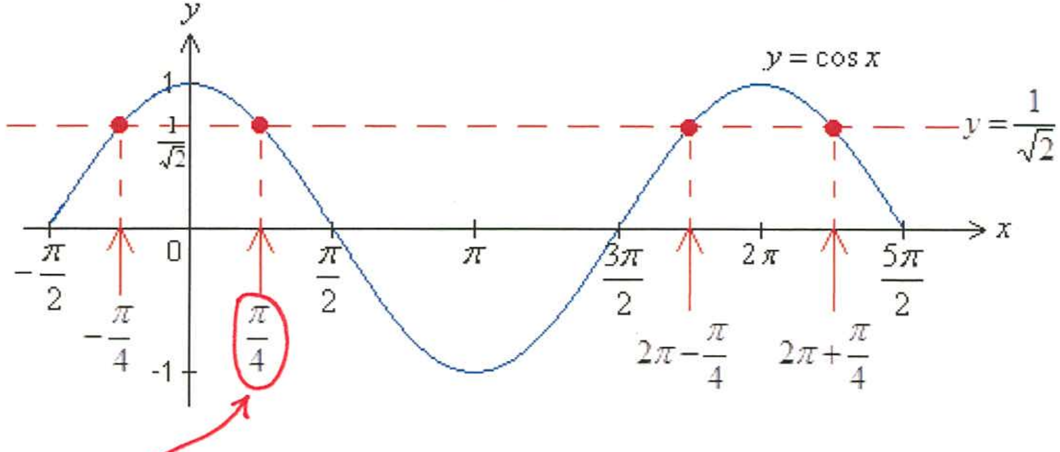
$\boxed{x = n\pi + (-1)^n \cdot \alpha}$, where $n \in \mathbb{Z}$ and $\boxed{\alpha = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}}$.

That is, $x = n\pi + (-1)^n \cdot \frac{\pi}{6}$, where $n \in \mathbb{Z}$.

$(-1)^n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$

Cosine function:

Find the general solution of $\cos x = \frac{1}{\sqrt{2}}$.



$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ ($\because$ the principal range of $\alpha = \cos^{-1}(x)$ is $0 \le \alpha \le \pi$.)

Consider an interval with length 2π (period of $\cos x$)

The solutions of $\cos x = \frac{1}{\sqrt{2}}$ in $(-\pi, \pi]$ are

$$x = \alpha = \frac{\pi}{4} \quad \text{and} \quad x = -\alpha = -\frac{\pi}{4} \quad (\because \cos(-x) = \cos x).$$

Since $y = \cos x$ is periodic with period 2π,

scalar multiple of period

$$x = \frac{\pi}{4} + (2\pi)n = 2n\pi + \frac{\pi}{4} \quad \text{and} \quad x = -\frac{\pi}{4} + (2\pi)n = 2n\pi - \frac{\pi}{4},$$

where $n \in \mathbb{Z}$, are also solutions of $\cos x = \frac{1}{\sqrt{2}}$.

Combine 2 cases

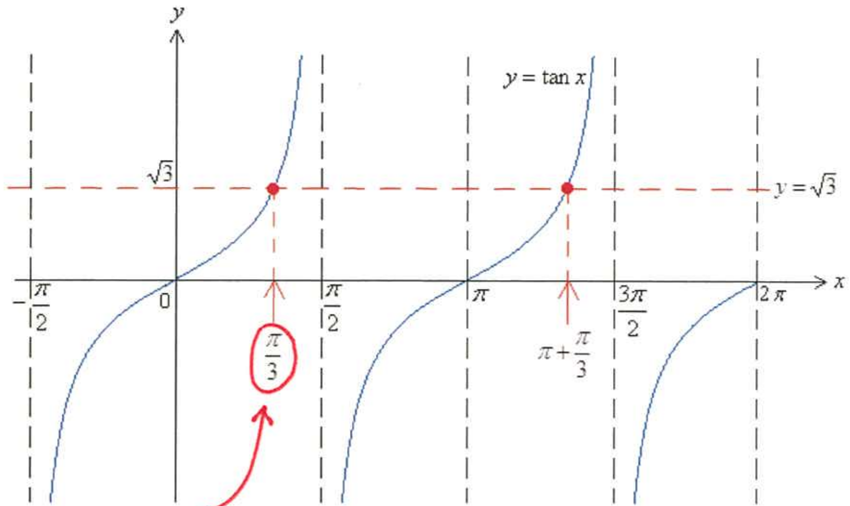
$\therefore$ The **general solution** of the equation $\cos x = \frac{1}{\sqrt{2}}$ is

$\boxed{x = 2n\pi \pm \alpha}$, where $n \in \mathbb{Z}$ and $\boxed{\alpha = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}}$ (since $0 \le \alpha \le \pi$).

That is, $x = 2n\pi \pm \frac{\pi}{4}$, where $n \in \mathbb{Z}$.

Tangent function:

Find the general solution of $\tan x = \sqrt{3}$.



$\alpha = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ ($\because$ the principal range of $\alpha = \tan^{-1}(x)$ is $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.)

The solution of $\tan x = \sqrt{3}$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is Consider an interval with length π (period of $\tan x$)

$$x = \alpha = \frac{\pi}{3}.$$

Since $y = \tan x$ is periodic with period π,

$$x = n\pi + \frac{\pi}{3},$$

where $n \in \mathbb{Z}$, are also solutions of $\tan x = \sqrt{3}$.

$\therefore$ The **general solution** of the equation $\tan x = \sqrt{3}$ is

$$\boxed{x = n\pi + \alpha} \text{ , where } n \in \mathbb{Z} \text{ and } \boxed{\alpha = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}} \text{ (since } -\frac{\pi}{2} < \alpha < \frac{\pi}{2}).$$

(since $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$).

That is, $x = n\pi + \frac{\pi}{3}$, where $n \in \mathbb{Z}$.

The results are summarized on the next page.

Summary ☆ Memorize ☆

- The **general solution** of $\sin x = k$ (where $-1 \le k \le 1$) is

$$x = n\pi + (-1)^n \alpha,$$

for $n \in \mathbb{Z}$, where $\alpha = \sin^{-1} k$ and $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$.

- The **general solution** of $\cos x = k$ (where $-1 \le k \le 1$) is

$$x = 2n\pi \pm \alpha,$$

for $n \in \mathbb{Z}$, where $\alpha = \cos^{-1} k$ and $0 \le \alpha \le \pi$.

- The **general solution** of $\tan x = k$ (where $k \in \mathbb{R}$) is

$$x = n\pi + \alpha,$$

for $n \in \mathbb{Z}$, where $\alpha = \tan^{-1} k$ and $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.

Example 16

all possible solutions

express this equation in terms of one trigo. function

Find, in radians, the general solution of the equation $\sin\theta + \cos\theta = 0$, and give all the values of θ which lie between 0 and 2π.

Solution

$$\sin\theta + \cos\theta = 0 \quad\Rightarrow\quad \sin\theta = -\cos\theta \quad\Rightarrow\quad \frac{\sin\theta}{\cos\theta} = -1 \quad\Rightarrow\quad \tan\theta = -1$$

$\therefore$ The general solution of the equation is

$$\theta = n\pi + \alpha,$$

where $\alpha = \tan^{-1}(-1) = -\frac{\pi}{4}$ and $n \in \mathbb{Z}$,

i.e. $\boxed{\theta = n\pi - \frac{\pi}{4}}$ for $n \in \mathbb{Z}$.

When $n = 1$, $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$.

When $n = 2$, $\theta = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$.

$\therefore$ The solutions of the equation which lie between 0 and 2π are

$$\boxed{\theta = \frac{3\pi}{4}} \text{ and } \boxed{\theta = \frac{7\pi}{4}}.$$

Example 17

Find, in radians, the general solution of the equation $2\sin 5x = -1$.

Solution

$2\sin 5x = -1 \quad \Rightarrow \quad \sin \underline{5x} = -\frac{1}{2}$

$\therefore$ The general solution of the equation is

$$\underline{5x} = n\pi + (-1)^n\,\alpha,$$

where $\alpha = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ and $n \in \mathbb{Z}$.

That is, $\boxed{x = \frac{n\pi}{5} + \frac{(-1)^n\left(-\frac{\pi}{6}\right)}{5} = \frac{n\pi}{5} + (-1)^n\left(-\frac{\pi}{30}\right)}$ for $n \in \mathbb{Z}$

Example 18

Find the general solution of the equation $\sin x = \cos 2x$.

Solution By using the **Double angle formula**, we have

$$\cos 2x = \cos^2 x - \sin^2 x = (1 - \sin^2 x) - \sin^2 x = 1 - 2\sin^2 x.$$

Then $\sin x = \cos 2x \quad \Rightarrow \quad \sin x = 1 - 2\sin^2 x$

$$\Rightarrow \quad 2\sin^2 x + \sin x - 1 = 0 \quad \leftarrow \text{quadratic equation in } \sin x$$

$$\Rightarrow \quad (2\sin x - 1)(\sin x + 1) = 0$$

$$\Rightarrow \quad 2\sin x - 1 = 0 \quad \text{or} \quad \sin x + 1 = 0$$

$$\Rightarrow \quad \sin x = \frac{1}{2} \quad \text{or} \quad \sin x = -1$$

$\therefore$ The general solution of the equation is

$$x = n\pi + (-1)^n \alpha_1, \quad \text{where } \alpha_1 = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}, \quad \text{for } n \in \mathbb{Z},$$

and $$x = n\pi + (-1)^n \alpha_2, \quad \text{where } \alpha_2 = \sin^{-1}(-1) = -\frac{\pi}{2}, \quad \text{for } n \in \mathbb{Z}.$$

That is, $\boxed{x = n\pi + (-1)^n \left(\frac{\pi}{6}\right)}$ or $\boxed{x = n\pi + (-1)^n \left(-\frac{\pi}{2}\right)}$, for $n \in \mathbb{Z}$.

Example 19

Find the general solution of the equation $2\sin^2 4x + 3\cos 4x = 3$.

Solution

$2\sin^2 4x + 3\cos 4x = 3$

$$\Rightarrow \quad 2(1-\cos^2 4x) + 3\cos 4x = 3$$

$$\Rightarrow \quad 2\cos^2 4x - 3\cos 4x + 1 = 0$$

$$\Rightarrow \quad (2\cos 4x - 1)(\cos 4x - 1) = 0$$

$$\Rightarrow \quad 2\cos 4x - 1 = 0 \quad \text{or} \quad \cos 4x - 1 = 0$$

$$\Rightarrow \quad \cos 4x = \frac{1}{2} \quad \text{or} \quad \cos 4x = 1$$

$\therefore$ The general solution of the equation is

$$4x = 2n\pi \pm \alpha_1, \quad \text{where } \alpha_1 = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}, \quad \text{for } n \in \mathbb{Z},$$

and $$4x = 2n\pi \pm \alpha_2, \quad \text{where } \alpha_2 = \cos^{-1}(1) = 0, \quad \text{for } n \in \mathbb{Z}.$$

That is, $\boxed{x = \frac{2n\pi \pm \frac{\pi}{3}}{4} = \frac{n\pi}{2} \pm \frac{\pi}{12}}$ or $\boxed{x = \frac{2n\pi \pm 0}{4} = \frac{n\pi}{2}}$, for $n \in \mathbb{Z}$.