MA1201 Calculus and Basic Linear Algebra II   Solution of Problem Set 6

Matrix, Determinant and System of Linear Equations

Problem 1

(a) $$AB = \begin{pmatrix} 4 & 1 & 5 \\ 2 & 1 & 7 \\ 2 & 3 & 10 \end{pmatrix} \text{ and } A^2 = \begin{pmatrix} 5 & -7 & 1 \\ 2 & -1 & 8 \\ 5 & -6 & 8 \end{pmatrix}$$

(b) $$A(B+I_3) = \begin{pmatrix} 2 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 4+1 & -1 & 2 \\ 2 & -1+1 & 1 \\ 0 & 1 & 3+1 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} 5 & -1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 6 & -1 & 6 \\ 2 & 2 & 9 \\ 3 & 2 & 13 \end{pmatrix}.$$

Problem 2

(a) $$AD = \underbrace{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}}_{1\times 3}\underbrace{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}_{3\times 1} = \underbrace{\begin{pmatrix} 1(1)+2(-1)+3(2) \end{pmatrix}}_{1\times 1} = (5).$$

$$DA = \underbrace{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}_{3\times 1}\underbrace{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}}_{1\times 3} = \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{pmatrix}}_{3\times 3}$$

(b) $$BC = \underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix}}_{2\times 3}\underbrace{\begin{pmatrix} 1 & 0 \\ -2 & 5 \end{pmatrix}}_{2\times 2} = \text{undefined.}$$

$$BD = \underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix}}_{2\times 3}\underbrace{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}_{3\times 1} = \underbrace{\begin{pmatrix} 1 \\ 5 \end{pmatrix}}_{2\times 1}$$

Problem 3

$$A^3 = \left[\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 1 & 0 & -1 \end{pmatrix}\right]\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 4 \\ 2 & 1 & -4 \\ 0 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 1 & 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 8 & 8 & 8 \\ -2 & 3 & 12 \\ 4 & -2 & 0 \end{pmatrix}.$$

Since B is diagonal matrix, we then have

$$B^4 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}^4 = \begin{pmatrix} 2^4 & 0 & 0 \\ 0 & (-3)^4 & 0 \\ 0 & 0 & 1^4 \end{pmatrix} = \begin{pmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 4

	Upper-Triangular	Lower-Triangular	Diagonal	Symmetric	Skew-symmetric
$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	Yes	Yes	Yes	Yes	No
$A = \begin{pmatrix} -2 & 1 \\ 0 & 3 \end{pmatrix}$	Yes	No	No	No	No
$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	Yes	Yes	Yes	Yes
$C = \begin{pmatrix} 2 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	Yes	No	No	No	No

$D=\begin{pmatrix}2 & -1\\ 3 & 1\end{pmatrix}^2$	No	No	No	No	No

Problem 5

(1) By expanding the determinant along the 1^{st} row, we have

$$\det\begin{pmatrix}2 & 0 & -3\\ 1 & 5 & 1\\ 0 & 0 & 3\end{pmatrix} \overset{R_1}{\cong} 2\times\det\begin{pmatrix}5 & 1\\ 0 & 3\end{pmatrix} - 0\times\det\begin{pmatrix}1 & 1\\ 0 & 3\end{pmatrix} - 3\times\det\begin{pmatrix}1 & 5\\ 0 & 0\end{pmatrix} = 30.$$

(2) By expanding the determinant along the 2^{nd} column, we have

$$\det\begin{pmatrix}2 & 0 & -3\\ 1 & 5 & 1\\ 0 & 0 & 3\end{pmatrix} \overset{C_2}{\cong} -0\times\det\begin{pmatrix}1 & 1\\ 0 & 3\end{pmatrix} + 5\times\det\begin{pmatrix}2 & -3\\ 0 & 3\end{pmatrix} - 0\times\det\begin{pmatrix}2 & -3\\ 0 & 3\end{pmatrix} = 30.$$

(3) By expanding the determinant along the 3^{rd} row, we have

$$\det\begin{pmatrix}2 & 0 & -3\\ 1 & 5 & 1\\ 0 & 0 & 3\end{pmatrix} \overset{R_3}{\cong} 0\times\det\begin{pmatrix}0 & -3\\ 5 & 1\end{pmatrix} - 0\times\det\begin{pmatrix}2 & -3\\ 1 & 1\end{pmatrix} + 3\times\det\begin{pmatrix}2 & 0\\ 1 & 5\end{pmatrix} = 30.$$

Problem 6

(a) By expanding the determinant along 2^{nd} row, we have

$$\det\begin{pmatrix}1 & -2 & 3\\ 0 & 2 & 0\\ -1 & 1 & 5\end{pmatrix} \overset{R_2}{\cong} 2\times(-1)^{2+2}\det\begin{pmatrix}1 & 3\\ -1 & 5\end{pmatrix} = 16.$$

(b) By expanding the determinant along 2^{nd} row, we have

$$\det\begin{pmatrix}3 & -1 & 2\\ 2 & 0 & 4\\ 1 & 1 & 0\end{pmatrix} \overset{R_2}{\cong} 2\times(-1)^{2+1}\det\begin{pmatrix}-1 & 2\\ 1 & 0\end{pmatrix} + 4\times(-1)^{2+3}\det\begin{pmatrix}3 & -1\\ 1 & 1\end{pmatrix} = -12.$$

(c) By expanding the determinant along 1^{st} row, we have

$$\det\begin{pmatrix}1 & a & 2\\ a & -1 & 0\\ 2 & 3 & 1\end{pmatrix}$$

$$\overset{R_1}{\cong} 1\times(-1)^{1+1}\det\begin{pmatrix}-1 & 0\\ 3 & 1\end{pmatrix} + a\times(-1)^{1+2}\det\begin{pmatrix}a & 0\\ 2 & 1\end{pmatrix} + 2\times(-1)^{1+3}\det\begin{pmatrix}a & -1\\ 2 & 3\end{pmatrix}$$

$$= -a^2 + 6a + 3.$$

(d) Note that

$$\det\begin{pmatrix}1 & 0 & -4\\ 2 & 1 & 1\\ 1 & 0 & 5\end{pmatrix} \overset{R_1}{\cong} 1\times(-1)^{1+1}\det\begin{pmatrix}1 & 1\\ 0 & 5\end{pmatrix} + (-4)\times(-1)^{1+3}\det\begin{pmatrix}2 & 1\\ 1 & 0\end{pmatrix} = 9.$$

Using the properties of determinant, we have

$$\det\begin{pmatrix}1 & 0 & -4\\ 2 & 1 & 1\\ 1 & 0 & 5\end{pmatrix}^5 = \left[\det\begin{pmatrix}1 & 0 & -4\\ 2 & 1 & 1\\ 1 & 0 & 5\end{pmatrix}\right]^5 = 9^5 = 59049.$$

(e) By expanding the determinant along 2^{nd} row, we get

$$\det\begin{pmatrix}1 & 2 & 1 & 3\\ 5 & 0 & 0 & -1\\ 2 & -1 & -1 & 0\\ 1 & 0 & 4 & 2\end{pmatrix}$$

$$\overset{R_2}{\cong} 5\times(-1)^{2+1}\det\begin{pmatrix}2 & 1 & 3\\ -1 & -1 & 0\\ 0 & 4 & 2\end{pmatrix} + (-1)\times(-1)^{2+4}\det\begin{pmatrix}1 & 2 & 1\\ 2 & -1 & -1\\ 1 & 0 & 4\end{pmatrix}$$

$= \cdots = -5(-14) + (-1)(-21) = 91.$

(f) By expanding the determinant along 2nd column

$$\det\begin{pmatrix} 2 & 1 & 1 & 3 \\ 1 & 0 & 1 & -1 \\ -3 & 2 & -1 & 0 \\ 5 & 0 & 4 & 2 \end{pmatrix}$$
$$\overset{C_2}{\cong} 1 \times (-1)^{1+2} \det\begin{pmatrix} 1 & 1 & -1 \\ -3 & -1 & 0 \\ 5 & 4 & 2 \end{pmatrix} + 2 \times (-1)^{3+2} \det\begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & -1 \\ 5 & 4 & 2 \end{pmatrix}$$
$$= \cdots = -1(11) - 2(2) = -15.$$

Problem 7

We let $D = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$. By expanding along 1st row, we get

$$\det D \overset{R_1}{\cong} a \times \det\begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix} = abc.$$

Problem 8

(a) $\det(A^3) = (\det A)^3 = 3^3 = 27.$

(b) $\det(A^{-1}) = \dfrac{1}{\det A} = \dfrac{1}{3}.$

(c) $\det(A^{-1}B) = \det(A^{-1}) \det B = \dfrac{1}{3}.$

(d) $\det(B^T A) = \det(B^T) \det A = \det B \det A = 3.$

(e) $\det(2A) = 2^4 \det A = 48,$
$\det(3A^T B) = 3^4 \det(A^T B) = 81 \det A^T \det B = 81 \det A \det B = 243.$

(f) $\det(2C^2) = 2^2 \det(C^2) = 4(\det C)^2 = 36.$

Problem 9

(a) Note that

$$\det A = \det\begin{pmatrix} 2 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix} = \cdots = 9 \neq 0.$$

The matrix A is invertible and its inverse is given by

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T = \frac{1}{9}\begin{pmatrix} 5 & -3 & 1 \\ -1 & 6 & -2 \\ 2 & -3 & 4 \end{pmatrix}^T = \begin{pmatrix} 5/9 & -1/9 & 2/9 \\ -1/3 & 2/3 & -1/3 \\ 1/9 & -2/9 & 4/9 \end{pmatrix}$$

$$\begin{pmatrix} A_{11} = (-1)^{1+1}\det\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} & A_{12} = (-1)^{1+2}\det\begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} & A_{13} = (-1)^{1+3}\det\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ A_{21} = (-1)^{2+1}\det\begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix} & A_{22} = (-1)^{2+2}\det\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} & A_{23} = (-1)^{2+3}\det\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \\ A_{31} = (-1)^{3+1}\det\begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix} & A_{32} = (-1)^{3+2}\det\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} & A_{33} = (-1)^{3+3}\det\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \end{pmatrix}$$

(b) Note that

$$\det B = \det\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \cdots = 4 \neq 0.$$

The matrix B is invertible and its inverse is given by

$$B^{-1} = \frac{1}{\det B}\begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}^T = \frac{1}{4}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}^T = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} B_{11}=(-1)^{1+1}\det\begin{pmatrix}2&0\\0&1\end{pmatrix} & B_{12}=(-1)^{1+2}\det\begin{pmatrix}0&0\\0&1\end{pmatrix} & B_{13}=(-1)^{1+3}\det\begin{pmatrix}0&2\\0&0\end{pmatrix}\\ B_{21}=(-1)^{2+1}\det\begin{pmatrix}0&0\\0&1\end{pmatrix} & B_{22}=(-1)^{2+2}\det\begin{pmatrix}2&0\\0&1\end{pmatrix} & B_{23}=(-1)^{2+3}\det\begin{pmatrix}2&0\\0&0\end{pmatrix}\\ B_{31}=(-1)^{3+1}\det\begin{pmatrix}0&0\\2&0\end{pmatrix} & B_{32}=(-1)^{3+2}\det\begin{pmatrix}2&0\\0&0\end{pmatrix} & B_{33}=(-1)^{3+3}\det\begin{pmatrix}2&0\\0&2\end{pmatrix}\end{pmatrix}$$

(c) Note that

$$\det C=\det\begin{pmatrix}1&1&1\\0&2&1\\0&0&3\end{pmatrix}=\cdots=6\neq 0.$$

The matrix C is invertible and its inverse is given by

$$C^{-1}=\frac{1}{\det C}\begin{pmatrix}C_{11}&C_{12}&C_{13}\\C_{21}&C_{22}&C_{23}\\C_{31}&C_{32}&C_{33}\end{pmatrix}^T=\frac{1}{6}\begin{pmatrix}6&0&0\\-3&3&0\\-1&-1&2\end{pmatrix}^T=\begin{pmatrix}1&-1/2&-1/6\\0&1/2&-1/6\\0&0&1/3\end{pmatrix}.$$

$$\begin{pmatrix} C_{11}=(-1)^{1+1}\det\begin{pmatrix}2&1\\0&3\end{pmatrix} & C_{12}=(-1)^{1+2}\det\begin{pmatrix}0&1\\0&3\end{pmatrix} & C_{13}=(-1)^{1+3}\det\begin{pmatrix}0&2\\0&0\end{pmatrix}\\ C_{21}=(-1)^{2+1}\det\begin{pmatrix}1&1\\0&3\end{pmatrix} & C_{22}=(-1)^{2+2}\det\begin{pmatrix}1&1\\0&3\end{pmatrix} & C_{23}=(-1)^{2+3}\det\begin{pmatrix}1&1\\0&0\end{pmatrix}\\ C_{31}=(-1)^{3+1}\det\begin{pmatrix}1&1\\2&1\end{pmatrix} & C_{32}=(-1)^{3+2}\det\begin{pmatrix}1&1\\0&1\end{pmatrix} & C_{33}=(-1)^{3+3}\det\begin{pmatrix}1&1\\0&2\end{pmatrix}\end{pmatrix}$$

(d) Note that

$$\det D=\det\begin{pmatrix}2&-3&0\\1&0&1\\1&0&1\end{pmatrix}=\cdots=0.$$

Thus the matrix D is not invertible and its inverse D^{-1} does not exist.

Problem 10

(a) If the matrix is singular (non-invertible), then we must have

$\det C \overset{R_1}{=} \det\begin{pmatrix}2&0\\0&a\end{pmatrix}-a\times\det\begin{pmatrix}a&0\\2a&a\end{pmatrix}+\det\begin{pmatrix}a&2\\2a&0\end{pmatrix}=0$

$\Rightarrow -a^3-2a=0\Rightarrow a(a^2+2)=0$

$\Rightarrow a=0.$

(b) Take $a=2$, we have $C=\begin{pmatrix}1&2&1\\2&2&0\\4&0&2\end{pmatrix}$. Since $\det C=-2^3-2(2)=-12\neq 0$, C is invertible and its inverse is found to be

$$C^{-1}=\frac{1}{\det C}\begin{pmatrix}C_{11}&C_{12}&C_{13}\\C_{21}&C_{22}&C_{23}\\C_{31}&C_{32}&C_{33}\end{pmatrix}^T=\frac{1}{-12}\begin{pmatrix}4&-4&-8\\-4&-2&8\\-2&2&-2\end{pmatrix}^T=\begin{pmatrix}-1/3&1/3&1/6\\1/3&1/6&-1/6\\2/3&-2/3&1/6\end{pmatrix}$$

$$\begin{pmatrix} C_{11}=(-1)^{1+1}\det\begin{pmatrix}2&0\\0&2\end{pmatrix} & C_{12}=(-1)^{1+2}\det\begin{pmatrix}2&0\\4&2\end{pmatrix} & C_{13}=(-1)^{1+3}\det\begin{pmatrix}2&2\\4&0\end{pmatrix}\\ C_{21}=(-1)^{2+1}\det\begin{pmatrix}2&1\\0&2\end{pmatrix} & C_{22}=(-1)^{2+2}\det\begin{pmatrix}1&1\\4&2\end{pmatrix} & C_{23}=(-1)^{2+3}\det\begin{pmatrix}1&2\\4&0\end{pmatrix}\\ C_{31}=(-1)^{3+1}\det\begin{pmatrix}2&1\\2&0\end{pmatrix} & C_{32}=(-1)^{3+2}\det\begin{pmatrix}1&1\\2&0\end{pmatrix} & C_{33}=(-1)^{3+3}\det\begin{pmatrix}1&2\\2&2\end{pmatrix}\end{pmatrix}$$

Problem 11

(a) $AB^T=\begin{pmatrix}0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}1&2\\0&-3\\2&1\end{pmatrix}=\begin{pmatrix}0&-3\\3&3\end{pmatrix}$

$$BA^T=\begin{pmatrix}1&0&2\\2&-3&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\\0&1\end{pmatrix}=\begin{pmatrix}0&3\\-3&3\end{pmatrix}$$

(b) Note that $\det(AB^T) = \det\begin{pmatrix}0 & -3\\ 3 & 3\end{pmatrix} = 9 \neq 0$ and $\det(BA^T) = \det\begin{pmatrix}0 & 3\\ -3 & 3\end{pmatrix} = 9 \neq 0$, this implies that both matrices are invertible and their inverse is found to be

$$(AB^T)^{-1} = \frac{1}{\det(AB^T)}\begin{pmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{pmatrix}^T = \frac{1}{9}\begin{pmatrix}3 & -3\\ 3 & 0\end{pmatrix}^T = \begin{pmatrix}1/3 & 1/3\\ -1/3 & 0\end{pmatrix}$$

(Here, C_{ij} represents the cofactor of $(i,j)^{\text{th}}$ element of AB^T)

$$(BA^T)^{-1} = \frac{1}{\det(BA^T)}\begin{pmatrix}D_{11} & D_{12}\\ D_{21} & D_{22}\end{pmatrix}^T = \frac{1}{9}\begin{pmatrix}3 & 3\\ -3 & 0\end{pmatrix}^T = \begin{pmatrix}1/3 & -1/3\\ 1/3 & 0\end{pmatrix}.$$

(Here, D_{ij} represents the cofactor of $(i,j)^{\text{th}}$ element of BA^T.)

Problem 12

Note that $\det E \overset{R_1}{=} \det\begin{pmatrix}2 & 5\\ 2 & 6\end{pmatrix} - \det\begin{pmatrix}2 & 5\\ 3 & 6\end{pmatrix} + 2\times\det\begin{pmatrix}2 & 2\\ 3 & 2\end{pmatrix} = 1 \neq 0$, then E is invertible and its inverse is given by

$$E^{-1} = \frac{1}{\det E}\begin{pmatrix}E_{11} & E_{12} & E_{13}\\ E_{21} & E_{22} & E_{23}\\ E_{31} & E_{32} & E_{33}\end{pmatrix}^T = \frac{1}{1}\begin{pmatrix}2 & 3 & -2\\ -2 & 0 & 1\\ 1 & -1 & 0\end{pmatrix}^T = \begin{pmatrix}2 & -2 & 1\\ 3 & 0 & -1\\ -2 & 1 & 0\end{pmatrix}$$

$$\begin{pmatrix}E_{11} = (-1)^{1+1}\det\begin{pmatrix}2 & 5\\ 2 & 6\end{pmatrix} & E_{12} = (-1)^{1+2}\det\begin{pmatrix}2 & 5\\ 3 & 6\end{pmatrix} & E_{13} = (-1)^{1+3}\det\begin{pmatrix}2 & 2\\ 3 & 2\end{pmatrix}\\ E_{21} = (-1)^{2+1}\det\begin{pmatrix}1 & 2\\ 2 & 6\end{pmatrix} & E_{22} = (-1)^{2+2}\det\begin{pmatrix}1 & 2\\ 3 & 6\end{pmatrix} & E_{23} = (-1)^{2+3}\det\begin{pmatrix}1 & 1\\ 3 & 2\end{pmatrix}\\ E_{31} = (-1)^{3+1}\det\begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix} & E_{32} = (-1)^{3+2}\det\begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix} & E_{33} = (-1)^{3+3}\det\begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix}\end{pmatrix}$$

Thus the matrix X is found to be

$$X = E^{-1}\begin{pmatrix}0 & 1 & 2\\ -1 & 3 & 5\\ 6 & 0 & 1\end{pmatrix} = \begin{pmatrix}2 & -2 & 1\\ 3 & 0 & -1\\ -2 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 2\\ -1 & 3 & 5\\ 6 & 0 & 1\end{pmatrix} = \begin{pmatrix}8 & -4 & -5\\ -6 & 3 & 5\\ -1 & 1 & 1\end{pmatrix}.$$

Problem 13

The system can be rewritten as

$$\begin{cases} x - 2y + z = 0\\ 2x + y - 3z = -5\\ -x + 4z = 11\end{cases} \Rightarrow \underbrace{\begin{pmatrix}1 & -2 & 1\\ 2 & 1 & -3\\ -1 & 0 & 4\end{pmatrix}}_{A}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ -5\\ 11\end{pmatrix} \dots\dots (*)$$

Note that $\det A \overset{R_1}{=} \det\begin{pmatrix}1 & -3\\ 0 & 4\end{pmatrix} - (-2)\times\det\begin{pmatrix}2 & -3\\ -1 & 4\end{pmatrix} + \det\begin{pmatrix}2 & 1\\ -1 & 0\end{pmatrix} = 15 \neq 0$, then A is invertible and its inverse is given by

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}^T = \frac{1}{15}\begin{pmatrix}4 & -5 & 1\\ 8 & 5 & 2\\ 5 & 5 & 5\end{pmatrix}^T = \begin{pmatrix}4/15 & 8/15 & 1/3\\ -1/3 & 1/3 & 1/3\\ 1/15 & 2/15 & 1/3\end{pmatrix}$$

$$\begin{pmatrix}A_{11} = (-1)^{1+1}\det\begin{pmatrix}1 & -3\\ 0 & 4\end{pmatrix} & A_{12} = (-1)^{1+2}\det\begin{pmatrix}2 & -3\\ -1 & 4\end{pmatrix} & A_{13} = (-1)^{1+3}\det\begin{pmatrix}2 & 1\\ -1 & 0\end{pmatrix}\\ A_{21} = (-1)^{2+1}\det\begin{pmatrix}-2 & 1\\ 0 & 4\end{pmatrix} & A_{22} = (-1)^{2+2}\det\begin{pmatrix}1 & 1\\ -1 & 4\end{pmatrix} & A_{23} = (-1)^{2+3}\det\begin{pmatrix}1 & -2\\ -1 & 0\end{pmatrix}\\ A_{31} = (-1)^{3+1}\det\begin{pmatrix}-2 & 1\\ 1 & -3\end{pmatrix} & A_{32} = (-1)^{3+2}\det\begin{pmatrix}1 & 1\\ 2 & -3\end{pmatrix} & A_{33} = (-1)^{3+3}\det\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}\end{pmatrix}$$

Therefore, the solution of $(*)$ is found to be

$$\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}1 & -2 & 1\\ 2 & 1 & -3\\ -1 & 0 & 4\end{pmatrix}^{-1}\begin{pmatrix}0\\ -5\\ 11\end{pmatrix} = \begin{pmatrix}4/15 & 8/15 & 1/3\\ -1/3 & 1/3 & 1/3\\ 1/15 & 2/15 & 1/3\end{pmatrix}\begin{pmatrix}0\\ -5\\ 11\end{pmatrix} = \begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$$

$$\Rightarrow x = 1, \qquad y = 2, \qquad z = 3.$$

Problem 14

(a) $\left(\begin{array}{ccc|c} 1 & -1 & 3 & 15 \\ -3 & 2 & 1 & 4 \\ 2 & -3 & 2 & 9 \end{array}\right) \xrightarrow[R_3-2R_1]{R_2+3R_1} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 15 \\ 0 & -1 & 10 & 49 \\ 0 & -1 & -4 & -21 \end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 15 \\ 0 & -1 & 10 & 49 \\ 0 & 0 & -14 & -70 \end{array}\right).$

Since there is no column without pivot, the system has a unique solution.

The system can be expressed as $\begin{cases} x-y+\ 3z = \quad 15 \\ \quad -y+10z = \quad 49 \\ \qquad\ -14z = -70 \end{cases}$. Solving the equations backward, we obtain $z=5,\ y=1,\ x=1.$

(b) $\left(\begin{array}{ccc|c} 2 & 1 & -3 & 12 \\ 4 & 0 & 1 & 5 \\ 3 & -1 & 2 & 1 \end{array}\right) \xrightarrow{R_1\div 2} \left(\begin{array}{ccc|c} 1 & 1/2 & -3/2 & 6 \\ 4 & 0 & 1 & 5 \\ 3 & -1 & 2 & 1 \end{array}\right) \xrightarrow[R_3-3R_2]{R_2-4R_1} \left(\begin{array}{ccc|c} 1 & 1/2 & -3/2 & 6 \\ 0 & -2 & 7 & -19 \\ 0 & -5/2 & 13/2 & -17 \end{array}\right)$

$\xrightarrow{R_2\div(-2)} \left(\begin{array}{ccc|c} 1 & 1/2 & -3/2 & 6 \\ 0 & 1 & -7/2 & 19/2 \\ 0 & -5/2 & 13/2 & -17 \end{array}\right) \xrightarrow{R_2+\frac{5}{2}R_2} \left(\begin{array}{ccc|c} 1 & 1/2 & -3/2 & 6 \\ 0 & 1 & -7/2 & 19/2 \\ 0 & 0 & 61/4 & -163/4 \end{array}\right).$

Since there is no column without pivot, the system has a unique solution.

The system can be expressed as $\begin{cases} x+\frac{1}{2}y-\frac{3}{2}z = \quad 6 \\ \qquad y-\frac{7}{2}z = \quad \frac{19}{2} \\ \qquad\quad \frac{61}{4}z = -\frac{163}{4} \end{cases}$. Solving the equations backward, we obtain $z=-3,\ y=-1,\ x=2.$

(c) $\left(\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 3 & -5 & 1 & 4 \end{array}\right) \xrightarrow{R_2-3R_1} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 1 & -8 & -5 \end{array}\right)$

Since the 3rd column has no pivot, the system has infinitely many solutions and z is a free variable. The corresponding system is given by $\begin{cases} x-2y+3z = \quad 3 \\ \qquad y-8z = -5 \end{cases}$.

Let $z=t,\ t$ is real and solve the equations backward, we get $y=-5+8t$ and $x=-7+13t$.

(d) $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ -3 & 4 & 1 & 0 \\ -2 & 5 & 3 & 0 \end{array}\right) \xrightarrow[R_3+2R_1]{R_2+3R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 7 & 7 & 0 \\ 0 & 7 & 7 & 0 \end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 7 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Since the 3rd column has no pivot, the system has infinitely many solutions and z is a free variable. The corresponding system is given by $\begin{cases} x+\ y+2z = 0 \\ \quad 7y+7z = 0 \end{cases}$.

Take $z=t,\ t$ is real. By solving the equations backward, we get $y=-t$ and $x=-t$.

(e) $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \end{array}\right) \xrightarrow[R_3-2R_1]{R_2-2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & -1 & -3 & -18 \end{array}\right) \xrightarrow{R_2\div 3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 5/3 & 34/3 \\ 0 & -1 & -3 & -18 \end{array}\right)$

$\xrightarrow{R_3+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 5/3 & 34/3 \\ 0 & 0 & -4/3 & -20/3 \end{array}\right)$

The system can be expressed as $\begin{cases} x+y+\ \ z = \quad 9 \\ \quad y+\frac{5}{3}z = \quad \frac{34}{3} \\ \quad -\frac{4}{3}z = -\frac{20}{3} \end{cases}$. Solving the equations backward, we obtain $z=5,\ y=3,\ x=1.$

(f) $\left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 1 \\ 2 & 5 & -1 & 3 & 2 \\ -1 & -1 & -3 & 2 & -3 \end{array}\right) \xrightarrow[R_3+R_1]{R_2-2R_1} \left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 1 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 2 & -5 & 1 & -2 \end{array}\right) \xrightarrow{R_3+2R_2} \left(\begin{array}{cccc|c} 1 & 3 & -2 & -1 & 1 \\ 0 & -1 & 3 & 5 & 0 \\ 0 & 0 & 1 & 11 & -2 \end{array}\right)$

Since the fourth column has no pivot, the system has infinitely many solutions and w is the free variable. The corresponding system is $\begin{cases} x+3y-2w-\ \ z = \ \ 1 \\ \quad -y+3z+5w = \ \ 0. \\ \qquad\quad z+11w = -2 \end{cases}$

Take $w=t,\ t$ is real. By solving the equation backward, we obtain $z=-2-11t,\ y=-6-28t,$ $x=15+63t.$

(g) $\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 2 & 4 & 5 & 9 & 1 \\ -3 & -6 & 0 & 1 & 5 \end{array}\right) \xrightarrow[R_3+3R_1]{R_2-2R_1} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 9 & 13 & -1 \end{array}\right) \xrightarrow{R_3+9R_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & 22 & 44 \end{array}\right).$

Since the second column has no pivot, the system has infinitely many solutions and y is the free variable. The corresponding system is $\begin{cases} x+2y+3z+4w = -2 \\ \qquad\quad -z+\ \ \ w = \ \ 5 \\ \qquad\qquad\quad 22w = 44 \end{cases}$

Take $y=t,\ t$ is real. By solving the equation backward, we obtain $w=2,\ \ z=-3,\ x=-1-2t.$

(h) $\left(\begin{array}{cccc|c} 1 & -3 & 4 & 7 & 1 \\ 2 & -6 & -3 & 5 & 2 \\ 4 & -12 & -17 & 1 & 4 \end{array}\right) \xrightarrow[R_3-4R_1]{R_2-2R_1} \left(\begin{array}{cccc|c} 1 & -3 & 4 & 7 & 1 \\ 0 & 0 & -11 & -9 & 0 \\ 0 & 0 & -33 & -27 & 0 \end{array}\right) \xrightarrow{R_3-3R_2} \left(\begin{array}{cccc|c} 1 & -3 & 4 & 7 & 1 \\ 0 & 0 & -11 & -9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$

Since the second column and fourth column have no pivots, the system has infinitely many solutions. Here, y and w are both free variables.

The corresponding system is $\begin{cases} x-3y+\quad 4z+7w = 1 \\ \qquad\quad -11z-9w = 0 \end{cases}.$

Take $y=s$ and $w=t,\ s,\ t$ are real. Solving the equation backward, we get $z=-\frac{9}{11}t$ and $x=1+3s-\frac{41}{11}t.$

Problem 15

Note that

$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 2 & -4 & -5 & 3 \\ 3 & -6 & 24 & a \end{array}\right) \xrightarrow[R_3-3R_1]{R_2-2R_1} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 0 & -7 & 1 \\ 0 & 0 & 21 & a-3 \end{array}\right) \xrightarrow{R_3+3R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 0 & -7 & 1 \\ 0 & 0 & 0 & a \end{array}\right)$

(a) If $a=3,$ the system becomes $\left(\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 0 & -7 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right).$

Since the last row is of the type $(0,0,0|b),\ b\neq 0,$ thus the system has no solution.

(b) The system is consistent when there is no row with $(0,0,0|b),\ b\neq 0.$ This happens when $a=0.$

Problem 16

$\left(\begin{array}{ccc|c} 3 & -1 & -1 & 1 \\ 2 & -4 & 5 & 1 \\ 4 & 2 & -7 & c \end{array}\right) \xrightarrow{R_1\div 3} \left(\begin{array}{ccc|c} 1 & -1/3 & -1/3 & 1/3 \\ 2 & -4 & 5 & 1 \\ 4 & 2 & -7 & c \end{array}\right) \xrightarrow[R_3-4R_1]{R_2-2R_1} \left(\begin{array}{ccc|c} 1 & -1/3 & -1/3 & 1/3 \\ 0 & -10/3 & 17/3 & 1/3 \\ 0 & 10/3 & -17/3 & c-4/3 \end{array}\right)$

$\xrightarrow{R_3+R_2} \left(\begin{array}{ccc|c} 1 & -1/3 & -1/3 & 1/3 \\ 0 & -10/3 & 17/3 & 1/3 \\ 0 & 0 & 0 & c-1 \end{array}\right).$

The system is consistent only when there is no row with $(0,0,0|b),\ b\neq 0.$ This implies that $c-1=0\Rightarrow$ $c=1.$

Remark: Since the last column has no pivot, the system has infinitely many solutions if the system is consistent.

Problem 17

$$\left(\begin{array}{ccc|c}1 & 2 & -1 & c\\ -1 & 4 & 1 & c^2\\ 1 & 8 & -1 & c^3\end{array}\right) \xrightarrow[R_3-R_1]{R_2+R_1} \left(\begin{array}{ccc|c}1 & 2 & -1 & c\\ 0 & 6 & 0 & c^2+c\\ 0 & 6 & 0 & c^3-c\end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c}1 & 2 & -1 & c\\ 0 & 6 & 0 & c^2+c\\ 0 & 0 & 0 & c^3-c^2-2c\end{array}\right)$$

(a) Since the third column has no pivot, thus the system cannot have unique solution for any value of c.

(b) The system has infinitely many solution if there is no row with $(0,0,0|b)$, $b \neq 0$. This happens when $c^3 - c^2 - 2c = 0 \Rightarrow c(c-2)(c+1) = 0 \Rightarrow c = 0$ or $c = 2$ or $c = -1$.

(c) The system has no solution if there is a row with $(0,0,0|b)$, $b \neq 0$. This happens when $c^3 - c^2 - 2c \neq 0 \Rightarrow c \neq 0$ and $c \neq 2$ and $c \neq -1$.

Problem 18

$$\left(\begin{array}{ccc|c}1 & -2 & 1 & 1\\ 1 & -1 & 2 & 2\\ 0 & 1 & c^2 & c\end{array}\right) \xrightarrow{R_2-R_1} \left(\begin{array}{ccc|c}1 & -2 & 1 & 1\\ 0 & 1 & 1 & 1\\ 0 & 1 & c^2 & c\end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c}1 & -2 & 1 & 1\\ 0 & 1 & 1 & 1\\ 0 & 0 & c^2-1 & c-1\end{array}\right)$$

(a) The system has a unique solution if there is no column without pivot, this happens when $c^2 - 1 \neq 0 \Rightarrow c \neq \pm 1$.

(b) The system has infinitely many solutions if there is a column without pivot and there is no row with $(0,0,0|b)$, $b \neq 0$. This happens when $c^2 - 1 = 0$ and $c - 1 = 0$. Solving the equations, we get $c = 1$.

(c) The system has no solutions if there is a row with $(0,0,0|b)$, $b \neq 0$. This happens when $c^2 - 1 = 0$ and $c - 1 \neq 0$. Solving the equations, we get $c = -1$.

Problem 19

$$\left(\begin{array}{ccc|c}2 & 1 & -b & 3\\ 0 & a & -1 & 2\\ -2 & 5 & 0 & 1\end{array}\right) \xrightarrow{R_3+R_1} \left(\begin{array}{ccc|c}2 & 1 & -b & 3\\ 0 & a & -1 & 2\\ 0 & 6 & -b & 4\end{array}\right) \xrightarrow{R_3 \leftrightarrow R_2} \left(\begin{array}{ccc|c}2 & 1 & -b & 3\\ 0 & 6 & -b & 4\\ 0 & a & -1 & 2\end{array}\right)$$

$$\xrightarrow{R_2 \div 6} \left(\begin{array}{ccc|c}2 & 1 & -b & 3\\ 0 & 1 & -b/6 & 2/3\\ 0 & a & -1 & 2\end{array}\right) \xrightarrow{R_3-aR_2} \left(\begin{array}{ccc|c}2 & 1 & -b & 3\\ 0 & 1 & -b/6 & 2/3\\ 0 & 0 & ab/6-1 & 2-2a/3\end{array}\right)$$

(a) The system has a unique solution if there is no column without pivot. This happens when $\frac{ab}{6} - 1 \neq 0 \Rightarrow ab \neq 6$.

(b) The system has infinitely many solutions if there is a column without pivot and there is no row with $(0,0,0|b)$, $b \neq 0$. This happens when $\begin{cases}\frac{ab}{6} - 1 = 0\\ 2 - \frac{2a}{3} = 0\end{cases} \Rightarrow a = 3,\ b = 2$.

(c) The system has no solution if there is a row with $(0,0,0|b)$, $b \neq 0$. This happens when $\begin{cases}\frac{ab}{6} - 1 = 0\\ 2 - \frac{2a}{3} \neq 0\end{cases} \Rightarrow ab = 6,\ a \neq 3$.

Problem 20

$$\left(\begin{array}{cccc|c}1 & 3 & -2 & -1 & 1\\ 2 & 5 & -1 & 3 & 2\\ -1 & -1 & a-3 & a^2-10 & b\end{array}\right) \xrightarrow[R_3+R_1]{R_2-2R_1} \left(\begin{array}{cccc|c}1 & 3 & -2 & -1 & 1\\ 0 & -1 & 3 & 5 & 0\\ 0 & 2 & a-5 & a^2-11 & b+1\end{array}\right)$$

$$\xrightarrow{R_3+2R_2} \left(\begin{array}{cccc|c}1 & 3 & -2 & -1 & 1\\ 0 & -1 & 3 & 5 & 0\\ 0 & 0 & a+1 & a^2-1 & b+1\end{array}\right)$$

(a) Since the matrix has 3 rows only, there are at most 3 pivots in the matrix. Thus at least one of the 3rd column and 4th column has no pivot, the system cannot have a unique solution for any values of a and b.

(b) The system has infinitely many solutions when there is a column without pivot provided that the system is consistent. As discussed in (a), at least one of the 3rd column and 4th column has no pivot and the system is consistent if case (c) doesn't happen, i.e., when $a = -1$ and $b = -1$ or when $a \neq -1$. We can conclude that the system has infinitely many solutions if $a = b = -1$ or $a \neq -1$ and b has no restriction.

(c) The system has no solution if there is a row with $(0,0,0,0|b)$, $b \neq 0$. This happens when $a + 1 = 0$ and $a^2 - 1 = 0$ and $b + 1 \neq 0$. Solving these equations, we get $a = -1$ and $b \neq -1$.

Problem 21

The given system can be expressed as $\begin{pmatrix} 1 & -2 & 3 \\ 4 & 1 & -2 \\ 2 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}$.

Using Cramer's rule, the solution of the system is found to be

$$x = \frac{\det\begin{pmatrix} 1 & -2 & 3 \\ 5 & 1 & -2 \\ 6 & -1 & 3 \end{pmatrix}}{\det\begin{pmatrix} 1 & -2 & 3 \\ 4 & 1 & -2 \\ 2 & -1 & 3 \end{pmatrix}} = \frac{22}{15}, \quad y = \frac{\det\begin{pmatrix} 1 & 1 & 3 \\ 4 & 5 & -2 \\ 2 & 6 & 3 \end{pmatrix}}{\det\begin{pmatrix} 1 & -2 & 3 \\ 4 & 1 & -2 \\ 2 & -1 & 3 \end{pmatrix}} = \frac{53}{15}, \quad z = \frac{\det\begin{pmatrix} 1 & -2 & 1 \\ 4 & 1 & 5 \\ 2 & -1 & 6 \end{pmatrix}}{\det\begin{pmatrix} 1 & -2 & 3 \\ 4 & 1 & -2 \\ 2 & -1 & 3 \end{pmatrix}} = \frac{11}{5}.$$

Problem 22

(a) $$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -3 & 4 & 0 & 1 \end{array}\right) \xrightarrow{R_2+3R_1} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 10 & 3 & 1 \end{array}\right) \xrightarrow{R_2 \div 10} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/10 & 1/10 \end{array}\right)$$
$$\xrightarrow{R_1-2R_2} \left(\begin{array}{cc|cc} 1 & 0 & 4/10 & -2/10 \\ 0 & 1 & 3/10 & 1/10 \end{array}\right)$$

Thus we conclude that $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{4}{10} & -\frac{2}{10} \\ \frac{3}{10} & \frac{1}{10} \end{pmatrix}$.

(b) $$\left(\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ -6 & 4 & 0 & 1 \end{array}\right) \xrightarrow{R_2+2R_1} \left(\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 0 & 6 & 2 & 1 \end{array}\right) \xrightarrow[R_2 \div 6]{R_1 \div 3} \left(\begin{array}{cc|cc} 1 & 1/3 & 1/3 & 0 \\ 0 & 1 & 1/3 & 1/6 \end{array}\right)$$
$$\xrightarrow{R_1-\frac{1}{3}R_2} \left(\begin{array}{cc|cc} 1 & 0 & 2/9 & -1/18 \\ 0 & 1 & 1/3 & 1/6 \end{array}\right)$$

Thus we deduce that $\begin{pmatrix} 3 & 1 \\ -6 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{2}{9} & -\frac{1}{18} \\ \frac{1}{3} & \frac{1}{6} \end{pmatrix}$.

(c) $$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 4 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3-4R_1]{R_2-3R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -3 & -3 & 1 & 0 \\ 0 & -2 & -7 & -4 & 0 & 1 \end{array}\right)$$
$$\xrightarrow{R_3-2R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -3 & -3 & 1 & 0 \\ 0 & 0 & -1 & 2 & -2 & 1 \end{array}\right) \xrightarrow[R_3 \div (-1)]{R_2 \div (-1)} \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & -1 & 0 \\ 0 & 0 & 1 & -2 & 2 & -1 \end{array}\right)$$
$$\xrightarrow[R_2-3R_3]{R_1-2R_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 5 & -4 & 2 \\ 0 & 1 & 0 & 9 & -7 & 3 \\ 0 & 0 & 1 & -2 & 2 & -1 \end{array}\right) \xrightarrow{R_1-R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 3 & -1 \\ 0 & 1 & 0 & 9 & -7 & 3 \\ 0 & 0 & 1 & -2 & 2 & -1 \end{array}\right)$$

Thus we conclude that $\begin{pmatrix} 1 & 1 & 2 \\ 3 & 2 & 3 \\ 4 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -4 & 3 & -1 \\ 9 & -7 & 3 \\ -2 & 2 & -1 \end{pmatrix}$.

(d) $$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3-5R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3\div(-4)]{R_2\div 2}\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&3/2&0&1/2&0\\0&0&1&5/4&0&-1/4\end{array}\right)\xrightarrow[R_2-\frac{3}{2}R_3]{R_1-R_3}\left(\begin{array}{ccc|ccc}1&1&0&-1/4&0&1/4\\0&1&0&-15/8&1/2&3/8\\0&0&1&5/4&0&-1/4\end{array}\right)$$

$$\xrightarrow{R_1-R_2}\left(\begin{array}{ccc|ccc}1&0&0&13/8&-1/2&-1/8\\0&1&0&-15/8&1/2&3/8\\0&0&1&5/4&0&-1/4\end{array}\right).$$

Thus we conclude that $\begin{pmatrix}1&1&1\\0&2&3\\5&5&1\end{pmatrix}^{-1}=\begin{pmatrix}13/8&-1/2&-1/8\\-15/8&1/2&3/8\\5/4&0&-1/4\end{pmatrix}$.

(e) $$\left(\begin{array}{ccc|ccc}0&-1&2&1&0&0\\2&1&4&0&1&0\\1&-1&5&0&0&1\end{array}\right)\xrightarrow{R_3\leftrightarrow R_1}\left(\begin{array}{ccc|ccc}1&-1&5&0&0&1\\2&1&4&0&1&0\\0&-1&2&1&0&0\end{array}\right)$$

$$\xrightarrow{R_2-2R_1}\left(\begin{array}{ccc|ccc}1&-1&5&0&0&1\\0&3&-6&0&1&-2\\0&-1&2&1&0&0\end{array}\right)\xrightarrow{R_2\div 3}\left(\begin{array}{ccc|ccc}1&-1&5&0&0&1\\0&1&-2&0&1/3&-2/3\\0&-1&2&1&0&0\end{array}\right)$$

$$\xrightarrow{R_3+R_2}\left(\begin{array}{ccc|ccc}1&-1&5&0&0&1\\0&1&-2&0&1/3&-2/3\\0&0&0&1&1/3&-2/3\end{array}\right)$$

Since the last row is of the form $(0,0,0|a,b,c)$, a, b, c not all zero, thus the matrix is not invertible.

(f) $$\left(\begin{array}{ccc|ccc}-2&3&4&1&0&0\\1&0&-1&0&1&0\\-1&5&8&0&0&1\end{array}\right)\xrightarrow{R_2\leftrightarrow R_1}\left(\begin{array}{ccc|ccc}1&0&-1&0&1&0\\-2&3&4&1&0&0\\-1&5&8&0&0&1\end{array}\right)\xrightarrow[R_3+R_1]{R_2+2R_1}\left(\begin{array}{ccc|ccc}1&0&-1&0&1&0\\0&3&2&1&2&0\\0&5&7&0&1&1\end{array}\right)$$

$$\xrightarrow{R_2\div 3}\left(\begin{array}{ccc|ccc}1&0&-1&0&1&0\\0&1&\frac{2}{3}&\frac{1}{3}&\frac{2}{3}&0\\0&5&7&0&1&1\end{array}\right)\xrightarrow{R_3-5R_2}\left(\begin{array}{ccc|ccc}1&0&-1&0&1&0\\0&1&\frac{2}{3}&\frac{1}{3}&\frac{2}{3}&0\\0&0&\frac{11}{3}&-\frac{5}{3}&-\frac{7}{3}&1\end{array}\right)$$

$$\xrightarrow{R_3\div\frac{11}{3}}\left(\begin{array}{ccc|ccc}1&0&-1&0&1&0\\0&1&2/3&1/3&2/3&0\\0&0&1&-5/11&-7/11&3/11\end{array}\right)\xrightarrow[R_2-\frac{2}{3}R_3]{R_1+R_3}\left(\begin{array}{ccc|ccc}1&0&0&-5/11&4/11&3/11\\0&1&0&7/11&12/11&-2/11\\0&0&1&-5/11&-7/11&3/11\end{array}\right)$$

Thus we conclude that $\begin{pmatrix}-2&3&4\\1&0&-1\\-1&5&8\end{pmatrix}^{-1}=\begin{pmatrix}-5/11&4/11&3/11\\7/11&12/11&-2/11\\-5/11&-7/11&3/11\end{pmatrix}$.

(g) $$\left(\begin{array}{cccc|cccc}1&1&2&-3&1&0&0&0\\0&2&1&0&0&1&0&0\\0&0&3&0&0&0&1&0\\1&-1&1&1&0&0&0&1\end{array}\right)\xrightarrow{R_4-R_1}\left(\begin{array}{cccc|cccc}1&1&2&-3&1&0&0&0\\0&2&1&0&0&1&0&0\\0&0&3&0&0&0&1&0\\0&-2&-1&4&-1&0&0&1\end{array}\right)$$

$$\xrightarrow{R_4+R_2}\left(\begin{array}{cccc|cccc}1&1&2&-3&1&0&0&0\\0&2&1&0&0&1&0&0\\0&0&3&0&0&0&1&0\\0&0&0&4&-1&1&0&1\end{array}\right)\xrightarrow{\substack{R_2\div 2\\R_3\div 3\\R_4\div 4}}\left(\begin{array}{cccc|cccc}1&1&2&-3&1&0&0&0\\0&1&1/2&0&0&1/2&0&0\\0&0&1&0&0&0&1/3&0\\0&0&0&1&-1/4&1/4&0&1/4\end{array}\right)$$

$$\xrightarrow{R_1+3R_4}\left(\begin{array}{cccc|cccc}1&1&2&0&1/4&3/4&0&3/4\\0&1&1/2&0&0&1/2&0&0\\0&0&1&0&0&0&1/3&0\\0&0&0&1&-1/4&1/4&0&1/4\end{array}\right)\xrightarrow[R_2-\frac{1}{2}R_3]{R_1-2R_3}\left(\begin{array}{cccc|cccc}1&1&0&0&1/4&3/4&-2/3&3/4\\0&1&0&0&0&1/2&-1/6&0\\0&0&1&0&0&0&1/3&0\\0&0&0&1&-1/4&1/4&0&1/4\end{array}\right)$$

$$\xrightarrow{R_1-R_2}\left(\begin{array}{cccc|cccc}1&0&0&0&1/4&1/4&-1/2&3/4\\0&1&0&0&0&1/2&-1/6&0\\0&0&1&0&0&0&1/3&0\\0&0&0&1&-1/4&1/4&0&1/4\end{array}\right)$$

Thus the inverse is given by $\begin{pmatrix}1&1&2&-3\\0&2&1&0\\0&0&3&0\\1&-1&1&1\end{pmatrix}^{-1}=\begin{pmatrix}1/4&1/4&-1/2&3/4\\0&1/2&-1/6&0\\0&0&1/3&0\\-1/4&1/4&0&1/4\end{pmatrix}$.

(h) $$\left(\begin{array}{cccc|cccc}1&3&-1&1&1&0&0&0\\0&1&1&3&0&1&0&0\\1&1&2&-2&0&0&1&0\\2&0&-1&3&0&0&0&1\end{array}\right)\xrightarrow[R_4-2R_1]{R_3-R_1}\left(\begin{array}{cccc|cccc}1&3&-1&1&1&0&0&0\\0&1&1&3&0&1&0&0\\0&-2&3&-3&-1&0&1&0\\0&-6&1&1&-2&0&0&1\end{array}\right)$$

$$\xrightarrow[R_4+6R_2]{R_3+2R_2}\left(\begin{array}{cccc|cccc}1&3&-1&1&1&0&0&0\\0&1&1&3&0&1&0&0\\0&0&5&3&-1&2&1&0\\0&0&7&19&-2&6&0&1\end{array}\right)\xrightarrow{R_3\div 5}\left(\begin{array}{cccc|cccc}1&3&-1&1&1&0&0&0\\0&1&1&3&0&1&0&0\\0&0&1&3/5&-1/5&2/5&1/5&0\\0&0&7&19&-2&6&0&1\end{array}\right)$$

$$\xrightarrow{R_4-7R_3}\left(\begin{array}{cccc|cccc}1&3&-1&1&1&0&0&0\\0&1&1&3&0&1&0&0\\0&0&1&3/5&-1/5&2/5&1/5&0\\0&0&0&74/5&-3/5&16/5&-7/5&1\end{array}\right)$$

$$\xrightarrow{R_4\div\frac{74}{5}R_3}\left(\begin{array}{cccc|cccc}1&3&-1&1&1&0&0&0\\0&1&1&3&0&1&0&0\\0&0&1&3/5&-1/5&2/5&1/5&0\\0&0&0&1&-3/74&8/37&-7/74&5/74\end{array}\right)$$

$$\xrightarrow[R_3-\frac{3}{5}R_4]{\substack{R_1-R_4\\R_2-3R_4}}\left(\begin{array}{cccc|cccc}1&3&-1&0&77/74&-8/37&7/74&-5/74\\0&1&1&0&9/74&13/37&21/74&-15/74\\0&0&1&0&-13/74&10/37&19/74&-3/74\\0&0&0&1&-3/74&8/37&-7/74&5/74\end{array}\right)$$

$$\xrightarrow[R_2-R_3]{R_1+R_3}\left(\begin{array}{cccc|cccc}1&3&0&0&32/37&2/37&13/37&-4/37\\0&1&0&0&11/37&3/37&1/37&-6/37\\0&0&1&0&-13/74&10/37&19/74&-3/74\\0&0&0&1&-3/74&8/37&-7/74&5/74\end{array}\right)$$

$$\xrightarrow{R_1-3R_2}\left(\begin{array}{cccc|cccc}1&0&0&0&-1/37&-7/37&10/37&14/37\\0&1&0&0&11/37&3/37&1/37&-6/37\\0&0&1&0&-13/74&10/37&19/74&-3/74\\0&0&0&1&-3/74&8/37&-7/74&5/74\end{array}\right)$$

The inverse is found to be $\begin{pmatrix}1&3&-1&1\\0&1&1&3\\1&1&2&-2\\2&0&-1&3\end{pmatrix}^{-1}=\begin{pmatrix}-1/37&-7/37&10/37&14/37\\11/37&3/37&1/37&-6/37\\-13/74&10/37&19/74&-3/74\\-3/74&8/37&-7/74&5/74\end{pmatrix}$.

Problem 23

(a) We let $\vec{a}=\begin{pmatrix}0\\1\\1\end{pmatrix}$, $\vec{b}=\begin{pmatrix}1\\2\\0\end{pmatrix}$, $\vec{c}=\begin{pmatrix}-1\\1\\1\end{pmatrix}$. Then we consider the following equation:

$x_1\vec{a}+x_2\vec{b}+x_3\vec{c}=\vec{0}$

$\Rightarrow x_1\begin{pmatrix}0\\1\\1\end{pmatrix}+x_2\begin{pmatrix}1\\2\\0\end{pmatrix}+x_3\begin{pmatrix}-1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}x_2-x_3\\x_1+2x_2+x_3\\x_1+x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$.

Comparing the components, we obtain $\begin{cases}x_2-x_3=0\\x_1+2x_2+x_3=0\\x_1\qquad+x_3=0\end{cases}$.

Using Gaussian Elimination, we have

$$\left(\begin{array}{ccc|c}0&1&-1&0\\1&2&1&0\\1&0&1&0\end{array}\right)\xrightarrow{R_1\leftrightarrow R_2}\left(\begin{array}{ccc|c}1&2&1&0\\0&1&-1&0\\1&0&1&0\end{array}\right)\xrightarrow{R_3-R_1}\left(\begin{array}{ccc|c}1&2&1&0\\0&1&-1&0\\0&-2&0&0\end{array}\right)\xrightarrow{R_3+2R_2}\left(\begin{array}{ccc|c}1&2&1&0\\0&1&-1&0\\0&0&-2&0\end{array}\right)$$

The system corresponds to $\begin{cases} x_1 + 2x_2 + \ \ x_3 = 0 \\ \qquad\ \ x_2 - \ \ x_3 = 0 \\ \qquad\quad -2x_3 = 0 \end{cases}$. Solving the equations backward, we have $x_3 = 0,\ x_2 = 0,\ x_1 = 0$.

Since $(x_1, x_2, x_3) = (0,0,0)$ is the only solution, thus the vectors are linearly independent.

(b)

We let $\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. Then we consider the following equation:

$x_1\vec{a} + x_2\vec{b} + x_3\vec{c} = \vec{0}$

$\Rightarrow x_1\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + x_2\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 - 2x_2 + 2x_3 \\ 2x_1 + x_2 + x_3 \\ x_1 + x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Comparing the components, we obtain $\begin{cases} x_1 - 2x_2 + 2x_3 = 0 \\ 2x_1 + \ \ x_2 + \ \ x_3 = 0 \\ x_1 \qquad\quad + \ \ x_3 = 0 \end{cases}$.

Using Gaussian Elimination, we have

$$\left(\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow[R_3-R_1]{R_2-2R_1} \left(\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 0 & 5 & -3 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right) \xrightarrow{R_2\div 5} \left(\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 0 & 1 & -3/5 & 0 \\ 0 & 2 & -1 & 0 \end{array}\right)$$

$$\xrightarrow{R_3-2R_2} \left(\begin{array}{ccc|c} 1 & -2 & 2 & 0 \\ 0 & 1 & -3/5 & 0 \\ 0 & 0 & 1/5 & 0 \end{array}\right)$$

The system corresponds to $\begin{cases} x_1 - 2x_2 + 2x_3 = 0 \\ \qquad\quad x_2 - \frac{3}{5}x_3 = 0 \\ \qquad\qquad\quad \frac{1}{5}x_3 = 0 \end{cases}$. Solving the equations backward, we have

$x_3 = 0,\ x_2 = 0,\ x_1 = 0$.

Since $(x_1, x_2, x_3) = (0,0,0)$ is the only solution, thus the vectors are linearly independent.

(c)

We let $\vec{a} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 4 \\ 6 \\ -3 \end{pmatrix}$. Then we consider the following equation:

$x_1\vec{a} + x_2\vec{b} + x_3\vec{c} = \vec{0}$

$\Rightarrow x_1\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + x_3\begin{pmatrix} 4 \\ 6 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 + 4x_3 \\ 3x_1 + 2x_2 + 6x_3 \\ x_2 - 3x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Comparing the components, we obtain $\begin{cases} x_1 \qquad\quad + 4x_3 = 0 \\ 3x_1 + 2x_2 + 6x_3 = 0 \\ \qquad\quad x_2 - 3x_3 = 0 \end{cases}$.

Using Gaussian Elimination, we have

$$\left(\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 3 & 2 & 6 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right) \xrightarrow{R_2-3R_1} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 2 & -6 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right) \xrightarrow{R_2\div 2} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The 3rd column has no pivot and the corresponding variable x_3 is a free variable. Take $x_3 = t$, t is real. By solving the corresponding system $\begin{cases} x_1 \qquad\quad + 4x_3 = 0 \\ \qquad\quad x_2 - 3x_3 = 0 \end{cases}$ backward, we have $x_1 = -4t$ and $x_2 = 3t$.

Since there is non-trivial solutions for the equation (for example, take $t = 1$, we get $x_1 = -4,\ x_2 = 3,\ x_3 = 1$), the vectors are linearly dependent.

(d)

We let $\vec{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{d} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Then we consider the following equation:

$x_1\vec{a} + x_2\vec{b} + x_3\vec{c} + x_4\vec{d} = \vec{0}$

$$\Rightarrow x_1\begin{pmatrix}1\\0\\1\end{pmatrix}+x_2\begin{pmatrix}0\\1\\1\end{pmatrix}+x_3\begin{pmatrix}1\\1\\0\end{pmatrix}+x_4\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}x_1+x_3+x_4\\x_2+2x_3+2x_4\\x_1+x_2+3x_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Comparing the components, we obtain $\begin{cases}x_1 \quad + \quad x_3 + \quad x_4 = 0\\ \qquad x_2 + 2x_3 + 2x_4 = 0\\ x_1 + x_2 \qquad + 3x_4 = 0\end{cases}.$

Using Gaussian Elimination, we have

$$\left(\begin{array}{cccc|c}1&0&1&1&0\\0&1&2&2&0\\1&1&0&3&0\end{array}\right)\xrightarrow{R_3-R_1}\left(\begin{array}{cccc|c}1&0&1&1&0\\0&1&2&2&0\\0&1&-1&2&0\end{array}\right)\xrightarrow{R_3-R_2}\left(\begin{array}{cccc|c}1&0&1&1&0\\0&1&2&2&0\\0&0&-3&0&0\end{array}\right)$$

The 4^{th} column has no pivot and the corresponding variable x_4 is a free variable. Take $x_4=t$, t is real. By solving the corresponding system $\begin{cases}x_1 \quad + \quad x_3 + x_4 = 0\\ \qquad x_2 + 2x_3 + 2x_4 = 0\\ \qquad\qquad -3x_3 = 0\end{cases}$ backward, we have $x_3=0$, $x_2=-2t$ and $x_1=-t$.

Since there is a non-trivial solution for the equation (for example, take $t=1$, we get $x_1=-1$, $x_2=-2$, $x_3=0$, $x_4=1$), the vectors are linearly dependent.

Problem 24

(a) $\begin{pmatrix}1&0&-1\\2&1&0\end{pmatrix}\xrightarrow{R_2-2R_1}\begin{pmatrix}1&0&-1\\0&1&2\end{pmatrix}$

Since the number of pivots is 2, thus the rank of the matrix is 2.

(b) $\begin{pmatrix}1&2&3\\3&2&1\\-1&1&1\end{pmatrix}\xrightarrow[R_3+R_1]{R_2-3R_1}\begin{pmatrix}1&2&3\\0&-4&-8\\0&3&4\end{pmatrix}\xrightarrow{R_2\div(-4)}\begin{pmatrix}1&2&3\\0&1&2\\0&3&4\end{pmatrix}\xrightarrow{R_3-3R_2}\begin{pmatrix}1&2&3\\0&1&2\\0&0&-2\end{pmatrix}.$

Since the number of pivots is 3, thus the rank of the matrix is 3.

(c) $\begin{pmatrix}1&1&2\\2&2&4\\3&3&6\end{pmatrix}\xrightarrow[R_3-3R_1]{R_2-2R_1}\begin{pmatrix}1&1&2\\0&0&0\\0&0&0\end{pmatrix}$

Since the number of pivots is 1, thus the rank of the matrix is 1.