(25) (a) False. We must also verify that the reversed steps are truly reversible. That is, we must also supply valid reasons for the forward steps thus generated, as in the proof of Result 2.

(b) True. "If not B then not A" is the contrapositive of "If A then B." A statement and its contrapositive are always logically equivalent.

(c) True. We saw that "A only if B" is an equivalent form for "If A then B," whose converse is "If B then A."

(d) False. "A is a necessary condition for B" is an equivalent form for "If B then A." As such, it does not include the statement "If A then B" that is included in the "A if and only if B" statement.

(e) False. "A is a necessary condition for B" and "B is a sufficient condition for A" are both equivalent forms for "If B then A." As such, neither includes the statement "If A then B" that is included in the "A if and only if B" statement.

(f) False. The inverse of a statement is the contrapositive of the converse. Hence, the converse and the inverse are logically equivalent and always have the same truth values.

(g) False. The problem describes only the inductive step. The base step of the proof is also required.

(h) True. This rule is given in the section for negating statements with quantifiers.

(i) False. The "and" must change to an "or." Thus, the negation of "A and B" is "not A or not B."

Section 1.4

(1) (a) $$\begin{bmatrix} -4 & 2 & 3 \\ 0 & 5 & -1 \\ 6 & 1 & -2 \end{bmatrix} + \begin{bmatrix} 6 & -1 & 0 \\ 2 & 2 & -4 \\ 3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} (-4+6) & (2+(-1)) & (3+0) \\ (0+2) & (5+2) & ((-1)+(-4)) \\ (6+3) & (1+(-1)) & ((-2)+1) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 3 \\ 2 & 7 & -5 \\ 9 & 0 & -1 \end{bmatrix}$$

(c) $$4\begin{bmatrix} -4 & 2 & 3 \\ 0 & 5 & -1 \\ 6 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 4(-4) & 4(2) & 4(3) \\ 4(0) & 4(5) & 4(-1) \\ 4(6) & 4(1) & 4(-2) \end{bmatrix} = \begin{bmatrix} -16 & 8 & 12 \\ 0 & 20 & -4 \\ 24 & 4 & -8 \end{bmatrix}$$

(e) Impossible. $\mathbf{C}$ is a 2×2 matrix, $\mathbf{F}$ is a 3×2 matrix, and $\mathbf{E}$ is a 3×3 matrix. Hence, $3\mathbf{F}$ is 3×2 and $-\mathbf{E}$ is 3×3. Thus, $\mathbf{C}$, $3\mathbf{F}$, and $-\mathbf{E}$ have different sizes. However, if we want to add matrices, they must be the same size.

(g) $$2\begin{bmatrix} -4 & 2 & 3 \\ 0 & 5 & -1 \\ 6 & 1 & -2 \end{bmatrix} - 3\begin{bmatrix} 3 & -3 & 5 \\ 1 & 0 & -2 \\ 6 & 7 & -2 \end{bmatrix} - \begin{bmatrix} 6 & -1 & 0 \\ 2 & 2 & -4 \\ 3 & -1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 2(-4) & 2(2) & 2(3) \\ 2(0) & 2(5) & 2(-1) \\ 2(6) & 2(1) & 2(-2) \end{bmatrix} + \begin{bmatrix} -3(3) & -3(-3) & -3(5) \\ -3(1) & -3(0) & -3(-2) \\ -3(6) & -3(7) & -3(-2) \end{bmatrix} + \begin{bmatrix} -6 & -(-1) & -0 \\ -2 & -2 & -(-4) \\ -3 & -(-1) & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 4 & 6 \\ 0 & 10 & -2 \\ 12 & 2 & -4 \end{bmatrix} + \begin{bmatrix} -9 & 9 & -15 \\ -3 & 0 & 6 \\ -18 & -21 & 6 \end{bmatrix} + \begin{bmatrix} -6 & 1 & 0 \\ -2 & -2 & 4 \\ -3 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (-8)+(-9)+(-6) & 4+9+1 & 6+(-15)+0 \\ 0+(-3)+(-2) & 10+0+(-2) & (-2)+6+4 \\ 12+(-18)+(-3) & 2+(-21)+1 & (-4)+6+(-1) \end{bmatrix} = \begin{bmatrix} -23 & 14 & -9 \\ -5 & 8 & 8 \\ -9 & -18 & 1 \end{bmatrix}$$

(i) $\begin{bmatrix} -4 & 2 & 3 \\ 0 & 5 & -1 \\ 6 & 1 & -2 \end{bmatrix}^T + \begin{bmatrix} 3 & -3 & 5 \\ 1 & 0 & -2 \\ 6 & 7 & -2 \end{bmatrix}^T = \begin{bmatrix} -4 & 0 & 6 \\ 2 & 5 & 1 \\ 3 & -1 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 6 \\ -3 & 0 & 7 \\ 5 & -2 & -2 \end{bmatrix}$

$= \begin{bmatrix} ((-4)+3) & (0+1) & (6+6) \\ (2+(-3)) & (5+0) & (1+7) \\ (3+5) & ((-1)+(-2)) & ((-2)+(-2)) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 12 \\ -1 & 5 & 8 \\ 8 & -3 & -4 \end{bmatrix}$

(l) Impossible. Since $\mathbf{C}$ is a 2×2 matrix, both $\mathbf{C}^T$ and $2\mathbf{C}^T$ are 2×2. Also, since $\mathbf{F}$ is a 3×2 matrix, $-3\mathbf{F}$ is 3×2. Thus, $2\mathbf{C}^T$ and $-3\mathbf{F}$ have different sizes. However, if we want to add matrices, they must be the same size.

(n) $(\mathbf{B}-\mathbf{A})^T = \begin{bmatrix} 10 & 2 & -3 \\ -3 & -3 & -2 \\ -3 & -3 & 3 \end{bmatrix}$ and $\mathbf{E}^T = \begin{bmatrix} 3 & 1 & 6 \\ -3 & 0 & 7 \\ 5 & -2 & -2 \end{bmatrix}$.

Thus, $(\mathbf{B}-\mathbf{A})^T + \mathbf{E}^T = \begin{bmatrix} 13 & 3 & 3 \\ -6 & -3 & 5 \\ 2 & -5 & 1 \end{bmatrix}$, and so $\left((\mathbf{B}-\mathbf{A})^T + \mathbf{E}^T\right)^T = \begin{bmatrix} 13 & -6 & 2 \\ 3 & -3 & -5 \\ 3 & 5 & 1 \end{bmatrix}$.

(2) For a matrix to be square, it must have the same number of rows as columns. A matrix $\mathbf{X}$ is diagonal if and only if it is square and $x_{ij} = 0$ for $i \neq j$. A matrix $\mathbf{X}$ is upper triangular if and only if it is square and $x_{ij} = 0$ for $i > j$. A matrix $\mathbf{X}$ is lower triangular if and only if it is square and $x_{ij} = 0$ for $i < j$. A matrix $\mathbf{X}$ is symmetric if and only if it is square and $x_{ij} = x_{ji}$ for all i, j. A matrix $\mathbf{X}$ is skew-symmetric if and only if it is square and $x_{ij} = -x_{ji}$ for all i, j. Notice that this implies that if $\mathbf{X}$ is skew-symmetric, then $x_{ii} = 0$ for all i. Finally, the transpose of an $m \times n$ matrix $\mathbf{X}$ is the $n \times m$ matrix whose (i,j)th entry is x_{ji}. We now check each matrix in turn.

$\mathbf{A}$ is a 3×2 matrix, so it is not square. Therefore, it cannot be diagonal, upper triangular, lower triangular, symmetric, or skew-symmetric. Finally, $\mathbf{A}^T = \begin{bmatrix} -1 & 0 & 6 \\ 4 & 1 & 0 \end{bmatrix}$.

$\mathbf{B}$ is a 2×2 matrix, so it is square. Because $b_{12} = b_{21} = 0$, $\mathbf{B}$ is diagonal, upper triangular, lower triangular, and symmetric. $\mathbf{B}$ is not skew-symmetric because $b_{11} \neq 0$. Finally, $\mathbf{B}^T = \mathbf{B}$.

$\mathbf{C}$ is a 2×2 matrix, so it is square. Because $c_{12} \neq 0$, $\mathbf{C}$ is neither diagonal nor lower triangular. Because $c_{21} \neq 0$, $\mathbf{C}$ is not upper triangular. $\mathbf{C}$ is not symmetric because $c_{12} \neq c_{21}$. $\mathbf{C}$ is not skew-symmetric because $c_{11} \neq 0$. Finally, $\mathbf{C}^T = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$.

$\mathbf{D}$ is a 3×1 matrix, so it is not square. Therefore, it cannot be diagonal, upper triangular, lower triangular, symmetric, or skew-symmetric. Finally, $\mathbf{D}^T = \begin{bmatrix} -1 & 4 & 2 \end{bmatrix}$.

$\mathbf{E}$ is a 3×3 matrix, so it is square. Because $e_{13} \neq 0$, $\mathbf{E}$ is neither diagonal nor lower triangular. Because $e_{31} \neq 0$, $\mathbf{E}$ is not upper triangular. $\mathbf{E}$ is not symmetric because $e_{31} \neq e_{13}$. $\mathbf{E}$ is not skew-symmetric because $e_{22} \neq 0$. Finally, $\mathbf{E}^T = \begin{bmatrix} 0 & 0 & -6 \\ 0 & -6 & 0 \\ 6 & 0 & 0 \end{bmatrix}$.

$\mathbf{F}$ is a 4×4 matrix, so it is square. Because $f_{14} \neq 0$, $\mathbf{F}$ is neither diagonal nor lower triangular. Because $f_{41} \neq 0$, $\mathbf{F}$ is not upper triangular. $\mathbf{F}$ is symmetric because $f_{ij} = f_{ji}$ for every i, j. $\mathbf{F}$ is not skew-symmetric because $f_{32} \neq -f_{23}$. Finally, $\mathbf{F}^T = \mathbf{F}$.

$\mathbf{G}$ is a 3×3 matrix, so it is square. Because $g_{ij} = 0$ whenever $i \neq j$, $\mathbf{G}$ is diagonal, lower triangular, upper triangular, and symmetric. $\mathbf{G}$ is not skew-symmetric because $g_{11} \neq 0$. Finally, $\mathbf{G}^T = \mathbf{G}$.

H is a 4×4 matrix, so it is square. Because $h_{14} \neq 0$, **H** is neither diagonal nor lower triangular. Because $h_{41} \neq 0$, **H** is not upper triangular. **H** is not symmetric because $h_{12} \neq h_{21}$. **H** is skew-symmetric because $h_{ij} = -h_{ji}$ for every i, j. Finally, $\mathbf{H}^T = -\mathbf{H}$.

J is a 4×4 matrix, so it is square. Because $j_{12} \neq 0$, **J** is neither diagonal nor lower triangular. Because $j_{21} \neq 0$, **J** is not upper triangular. **J** is symmetric because $j_{ik} = j_{ki}$ for every i, k. **J** is not skew-symmetric because $j_{32} \neq -j_{23}$. Finally, $\mathbf{J}^T = \mathbf{J}$.

K is a 4×4 matrix, so it is square. Because $k_{14} \neq 0$, **K** is neither diagonal nor lower triangular. Because $k_{41} \neq 0$, **K** is not upper triangular. **K** is not symmetric because $k_{12} \neq k_{21}$. **K** is not skew-symmetric because $k_{11} \neq 0$. Finally, $\mathbf{K}^T = \begin{bmatrix} 1 & -2 & -3 & -4 \\ 2 & 1 & -5 & -6 \\ 3 & 5 & 1 & -7 \\ 4 & 6 & 7 & 1 \end{bmatrix}$.

L is a 3×3 matrix, so it is square. Because $l_{13} \neq 0$, **L** is neither diagonal nor lower triangular. Because $l_{ij} = 0$ whenever $i > j$, **L** is upper triangular. **L** is not symmetric because $l_{31} \neq l_{13}$. **L** is not skew-symmetric because $l_{11} \neq 0$ (or because $l_{21} \neq -l_{12}$). Finally, $\mathbf{L}^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

M is a 3×3 matrix, so it is square. Because $m_{31} \neq 0$, **M** is neither diagonal nor upper triangular. Because $m_{ij} = 0$ whenever $i < j$, **M** is lower triangular. **M** is not symmetric because $m_{31} \neq m_{13}$. **M** is not skew-symmetric because $m_{31} \neq -m_{13}$. Finally, $\mathbf{M}^T = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

N is a 3×3 matrix, so it is square. Because $n_{ij} = 0$ whenever $i \neq j$, **N** is diagonal, lower triangular, upper triangular, and symmetric. **N** is not skew-symmetric because $n_{11} \neq 0$. Finally, $\mathbf{N}^T = \mathbf{N} = \mathbf{I}_3$.

P is a 2×2 matrix, so it is square. Because $p_{12} \neq 0$, **P** is neither diagonal nor lower triangular. Because $p_{21} \neq 0$, **P** is not upper triangular. **P** is symmetric because $p_{12} = p_{21}$. **P** is not skew-symmetric because $p_{12} \neq -p_{21}$. Finally, $\mathbf{P}^T = \mathbf{P}$.

Q is a 3×3 matrix, so it is square. Because $q_{31} \neq 0$, **Q** is neither diagonal nor upper triangular. Because $q_{ij} = 0$ whenever $i < j$, **Q** is lower triangular. **Q** is not symmetric because $q_{31} \neq q_{13}$. **Q** is not skew-symmetric because $q_{11} \neq 0$ (or because $q_{31} \neq -q_{13}$). Finally, $\mathbf{Q}^T = \begin{bmatrix} -2 & 4 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

R is a 3×2 matrix, so it is not square. Therefore, it cannot be diagonal, upper triangular, lower triangular, symmetric, or skew-symmetric. Finally, $\mathbf{R}^T = \begin{bmatrix} 6 & 3 & -1 \\ 2 & -2 & 0 \end{bmatrix}$.

(3) Just after Theorem 1.13 in the textbook, the desired decomposition is described as $\mathbf{A} = \mathbf{S} + \mathbf{V}$, where $\mathbf{S} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$ is symmetric and $\mathbf{V} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T)$ is skew-symmetric.

(a) $\mathbf{S} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) = \frac{1}{2}\left(\begin{bmatrix} 3 & -1 & 4 \\ 0 & 2 & 5 \\ 1 & -3 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & -3 \\ 4 & 5 & 2 \end{bmatrix}\right) =$

$\frac{1}{2}\left(\begin{bmatrix} (3+3) & ((-1)+0) & (4+1) \\ (0+(-1)) & (2+2) & (5+(-3)) \\ (1+4) & ((-3)+5) & (2+2) \end{bmatrix}\right) =$

$\frac{1}{2}\begin{bmatrix} 6 & -1 & 5 \\ -1 & 4 & 2 \\ 5 & 2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(6) & \frac{1}{2}(-1) & \frac{1}{2}(5) \\ \frac{1}{2}(-1) & \frac{1}{2}(4) & \frac{1}{2}(2) \\ \frac{1}{2}(5) & \frac{1}{2}(2) & \frac{1}{2}(4) \end{bmatrix} = \begin{bmatrix} 3 & -\frac{1}{2} & \frac{5}{2} \\ -\frac{1}{2} & 2 & 1 \\ \frac{5}{2} & 1 & 2 \end{bmatrix}$, which is symmetric, since $s_{ij} = s_{ji}$ for all i, j.

$\mathbf{V} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T) = \frac{1}{2}\left(\begin{bmatrix} 3 & -1 & 4 \\ 0 & 2 & 5 \\ 1 & -3 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & -3 \\ 4 & 5 & 2 \end{bmatrix}\right)$

$= \frac{1}{2}\left(\begin{bmatrix} (3-3) & ((-1)-0) & (4-1) \\ (0-(-1)) & (2-2) & (5-(-3)) \\ (1-4) & ((-3)-5) & (2-2) \end{bmatrix}\right) =$

$\frac{1}{2}\begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 8 \\ -3 & -8 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(0) & \frac{1}{2}(-1) & \frac{1}{2}(3) \\ \frac{1}{2}(1) & \frac{1}{2}(0) & \frac{1}{2}(8) \\ \frac{1}{2}(-3) & \frac{1}{2}(-8) & \frac{1}{2}(0) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & 0 & 4 \\ -\frac{3}{2} & -4 & 0 \end{bmatrix}$, which is skew-symmetric, since $v_{ij} = -v_{ji}$ for all i, j.

Note that $\mathbf{S} + \mathbf{V} = \begin{bmatrix} 3 & -\frac{1}{2} & \frac{5}{2} \\ -\frac{1}{2} & 2 & 1 \\ \frac{5}{2} & 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & 0 & 4 \\ -\frac{3}{2} & -4 & 0 \end{bmatrix} =$

$\begin{bmatrix} (3+0) & \left(-\frac{1}{2}+\left(-\frac{1}{2}\right)\right) & \left(\frac{5}{2}+\frac{3}{2}\right) \\ \left(-\frac{1}{2}+\frac{1}{2}\right) & (2+0) & (1+4) \\ \left(\frac{5}{2}+\left(-\frac{3}{2}\right)\right) & (1+(-4)) & (2+0) \end{bmatrix} = \begin{bmatrix} 3 & -1 & 4 \\ 0 & 2 & 5 \\ 1 & -3 & 2 \end{bmatrix} = \mathbf{A}.$

(5) (d) The matrix (call it $\mathbf{A}$) must be a square zero matrix; that is, $\mathbf{O}_n$, for some n. First, since $\mathbf{A}$ is diagonal, it must be square (by definition of a diagonal matrix). To prove that $a_{ij} = 0$ for all i, j, we consider two cases: First, if $i \neq j$, then $a_{ij} = 0$ since $\mathbf{A}$ is diagonal. Second, if $i = j$, then $a_{ij} = a_{ii} = 0$ since $\mathbf{A}$ is skew-symmetric ($a_{ii} = -a_{ii}$ for all i, implying $a_{ii} = 0$ for all i). Hence, every entry $a_{ij} = 0$, and so $\mathbf{A}$ is a zero matrix.

(11) Part (1): Let $\mathbf{B} = \mathbf{A}^T$ and $\mathbf{C} = (\mathbf{A}^T)^T$. Then $c_{ij} = b_{ji} = a_{ij}$. Since this is true for all i and j, we have $\mathbf{C} = \mathbf{A}$; that is, $(\mathbf{A}^T)^T = \mathbf{A}$.
Part (3): Let $\mathbf{B} = c(\mathbf{A}^T)$, $\mathbf{D} = c\mathbf{A}$, and $\mathbf{F} = (c\mathbf{A})^T$. Then $f_{ij} = d_{ji} = ca_{ji} = b_{ij}$. Since this is true for all i and j, we have $\mathbf{F} = \mathbf{B}$; that is, $(c\mathbf{A})^T = c(\mathbf{A}^T)$.

(14) (a) Trace $(\mathbf{B}) = b_{11} + b_{22} = 2 + (-1) = 1$, trace $(\mathbf{C}) = c_{11} + c_{22} = (-1) + 1 = 0$, trace $(\mathbf{E}) = e_{11} + e_{22} + e_{33} = 0 + (-6) + 0 = -6$, trace $(\mathbf{F}) = f_{11} + f_{22} + f_{33} + f_{44}$ $=1+0+0+1 = 2$, trace $(\mathbf{G}) = g_{11}+g_{22}+g_{33} = 6+6+6 = 18$, trace $(\mathbf{H}) = h_{11}+h_{22}+h_{33}+h_{44} =$ $0+0+0+0 = 0$, trace $(\mathbf{J}) = j_{11}+j_{22}+j_{33}+j_{44} = 0+0+1+0 = 1$, trace $(\mathbf{K}) = k_{11}+k_{22}+k_{33}+k_{44}$ $= 1+1+1+1 = 4$, trace $(\mathbf{L}) = l_{11} + l_{22} + l_{33} = 1+1+1 = 3$, trace $(\mathbf{M}) = m_{11} + m_{22} + m_{33} =$ $0+0+0 = 0$, trace $(\mathbf{N}) = n_{11} + n_{22} + n_{33} = 1+1+1 = 3$, trace $(\mathbf{P}) = p_{11} + p_{22} = 0+0 = 0$, trace $(\mathbf{Q}) = q_{11} + q_{22} + q_{33} = (-2) + 0 + 3 = 1$.

(c) Not necessarily true when $n > 1$. Consider matrices $\mathbf{L}$ and $\mathbf{N}$ in Exercise 2. From part (a) of this exercise, $\text{trace}(\mathbf{L}) = 3 = \text{trace}(\mathbf{N})$. However, $\mathbf{L} \neq \mathbf{N}$.

(15) (a) False. The main diagonal entries of a 5×6 matrix $\mathbf{A}$ are a_{11}, a_{22}, a_{33}, a_{44}, and a_{55}. Thus, there are 5 entries on the main diagonal. (There is no entry a_{66} because there is no 6th row.)

(b) True. Suppose $\mathbf{A}$ is lower triangular. Then $a_{ij} = 0$ whenever $i < j$. If $\mathbf{B} = \mathbf{A}^T$, then $b_{ij} = a_{ji} = 0$, if $i > j$, and so $\mathbf{B}$ is upper triangular.

(c) False. The square zero matrix $\mathbf{O}_n$ is both skew-symmetric and diagonal.

(d) True. This follows from the definition of a skew-symmetric matrix.

(e) True. $\left(c\left(\mathbf{A}^T + \mathbf{B}\right)\right)^T = \left(c\left(\mathbf{A}^T + \mathbf{B}\right)^T\right)$ (by part (3) of Theorem 1.12) $= \left(c\left(\left(\mathbf{A}^T\right)^T + \mathbf{B}^T\right)\right)$ (by part (2) of Theorem 1.12) $= c\left(\mathbf{A} + \mathbf{B}^T\right)$ (by part (1) of Theorem 1.12) $= c\mathbf{A} + c\mathbf{B}^T$ (by part (5) of Theorem 1.11) $= c\mathbf{B}^T + c\mathbf{A}$ (by part (1) of Theorem 1.11).

Section 1.5

(1) (b) $\mathbf{BA} = \begin{bmatrix} -5 & 3 & 6 \\ 3 & 8 & 0 \\ -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 6 & 5 \\ 1 & -4 \end{bmatrix} =$

$$\begin{bmatrix} ((-5)(-2) + (3)(6) + (6)(1)) & ((-5)(3) + (3)(5) + (6)(-4)) \\ ((3)(-2) + (8)(6) + (0)(1)) & ((3)(3) + (8)(5) + (0)(-4)) \\ ((-2)(-2) + (0)(6) + (4)(1)) & ((-2)(3) + (0)(5) + (4)(-4)) \end{bmatrix} = \begin{bmatrix} 34 & -24 \\ 42 & 49 \\ 8 & -22 \end{bmatrix}$$

(c) Impossible; (number of columns of $\mathbf{J}$) $= 1 \neq 2 =$ (number of rows of $\mathbf{M}$)

(e) $\mathbf{RJ} = \begin{bmatrix} -3 & 6 & -2 \end{bmatrix} \begin{bmatrix} 8 \\ -1 \\ 4 \end{bmatrix} = [(-3)(8) + (6)(-1) + (-2)(4)] = [-38]$

(f) $\mathbf{JR} = \begin{bmatrix} 8 \\ -1 \\ 4 \end{bmatrix} \begin{bmatrix} -3 & 6 & -2 \end{bmatrix} = \begin{bmatrix} (8)(-3) & (8)(6) & (8)(-2) \\ (-1)(-3) & (-1)(6) & (-1)(-2) \\ (4)(-3) & (4)(6) & (4)(-2) \end{bmatrix} = \begin{bmatrix} -24 & 48 & -16 \\ 3 & -6 & 2 \\ -12 & 24 & -8 \end{bmatrix}$

(g) Impossible; (number of columns of $\mathbf{R}$) $= 3 \neq 1 =$ (number of rows of $\mathbf{T}$)

(j) Impossible; (number of columns of $\mathbf{F}$) $= 2 \neq 4 =$ (number of rows of $\mathbf{F}$)

(l) $\mathbf{E}^3 = \mathbf{E}(\mathbf{E}^2)$. Let $\mathbf{W} = \mathbf{E}^2 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}^2$. Then

$w_{11} = (1)(1) + (1)(1) + (0)(0) + (1)(1) = 3$
$w_{12} = (1)(1) + (1)(0) + (0)(0) + (1)(0) = 1$
$w_{13} = (1)(0) + (1)(1) + (0)(0) + (1)(1) = 2$
$w_{14} = (1)(1) + (1)(0) + (0)(1) + (1)(0) = 1$
$w_{21} = (1)(1) + (0)(1) + (1)(0) + (0)(1) = 1$
$w_{22} = (1)(1) + (0)(0) + (1)(0) + (0)(0) = 1$
$w_{23} = (1)(0) + (0)(1) + (1)(0) + (0)(1) = 0$
$w_{24} = (1)(1) + (0)(0) + (1)(1) + (0)(0) = 2$
$w_{31} = (0)(1) + (0)(1) + (0)(0) + (1)(1) = 1$
$w_{32} = (0)(1) + (0)(0) + (0)(0) + (1)(0) = 0$

$w_{33} = (0)(0) + (0)(1) + (0)(0) + (1)(1) = 1$
$w_{34} = (0)(1) + (0)(0) + (0)(1) + (1)(0) = 0$
$w_{41} = (1)(1) + (0)(1) + (1)(0) + (0)(1) = 1$
$w_{42} = (1)(1) + (0)(0) + (1)(0) + (0)(0) = 1$
$w_{43} = (1)(0) + (0)(1) + (1)(0) + (0)(1) = 0$
$w_{44} = (1)(1) + (0)(0) + (1)(1) + (0)(0) = 2$

Let $\mathbf{Y} = \mathbf{E}^3 = \mathbf{E}\mathbf{E}^2 = \mathbf{E}\mathbf{W} =$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}. \quad \text{So,}$$

$y_{11} = (1)(3) + (1)(1) + (0)(1) + (1)(1) = 5$
$y_{12} = (1)(1) + (1)(1) + (0)(0) + (1)(1) = 3$
$y_{13} = (1)(2) + (1)(0) + (0)(1) + (1)(0) = 2$
$y_{14} = (1)(1) + (1)(2) + (0)(0) + (1)(2) = 5$
$y_{21} = (1)(3) + (0)(1) + (1)(1) + (0)(1) = 4$
$y_{22} = (1)(1) + (0)(1) + (1)(0) + (0)(1) = 1$
$y_{23} = (1)(2) + (0)(0) + (1)(1) + (0)(0) = 3$
$y_{24} = (1)(1) + (0)(2) + (1)(0) + (0)(2) = 1$
$y_{31} = (0)(3) + (0)(1) + (0)(1) + (1)(1) = 1$
$y_{32} = (0)(1) + (0)(1) + (0)(0) + (1)(1) = 1$
$y_{33} = (0)(2) + (0)(0) + (0)(1) + (1)(0) = 0$
$y_{34} = (0)(1) + (0)(2) + (0)(0) + (1)(2) = 2$
$y_{41} = (1)(3) + (0)(1) + (1)(1) + (0)(1) = 4$
$y_{42} = (1)(1) + (0)(1) + (1)(0) + (0)(1) = 1$
$y_{43} = (1)(2) + (0)(0) + (1)(1) + (0)(0) = 3$
$y_{44} = (1)(1) + (0)(2) + (1)(0) + (0)(2) = 1$

Hence, $\mathbf{E}^3 = \mathbf{Y} = \begin{bmatrix} 5 & 3 & 2 & 5 \\ 4 & 1 & 3 & 1 \\ 1 & 1 & 0 & 2 \\ 4 & 1 & 3 & 1 \end{bmatrix}$.

(n) $\mathbf{D}(\mathbf{F}\mathbf{K}) = \mathbf{D}\left(\begin{bmatrix} 9 & -3 \\ 5 & -4 \\ 2 & 0 \\ 8 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & -5 \\ 0 & 2 & 7 \end{bmatrix} \right) =$

$$\mathbf{D} \begin{bmatrix} ((9)(2) + (-3)(0)) & ((9)(1) + (-3)(2)) & ((9)(-5) + (-3)(7)) \\ ((5)(2) + (-4)(0)) & ((5)(1) + (-4)(2)) & ((5)(-5) + (-4)(7)) \\ ((2)(2) + (0)(0)) & ((2)(1) + (0)(2)) & ((2)(-5) + (0)(7)) \\ ((8)(2) + (-3)(0)) & ((8)(1) + (-3)(2)) & ((8)(-5) + (-3)(7)) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 4 & 3 & 7 \\ 2 & 1 & 7 & 5 \\ 0 & 5 & 5 & -2 \end{bmatrix} \begin{bmatrix} 18 & 3 & -66 \\ 10 & -3 & -53 \\ 4 & 2 & -10 \\ 16 & 2 & -61 \end{bmatrix} = \mathbf{Z}. \quad \text{Then,}$$

$z_{11} = (-1)(18) + (4)(10) + (3)(4) + (7)(16) = 146$
$z_{12} = (-1)(3) + (4)(-3) + (3)(2) + (7)(2) = 5$

$z_{13} = (-1)(-66) + (4)(-53) + (3)(-10) + (7)(-61) = -603$
$z_{21} = (2)(18) + (1)(10) + (7)(4) + (5)(16) = 154$
$z_{22} = (2)(3) + (1)(-3) + (7)(2) + (5)(2) = 27$
$z_{23} = (2)(-66) + (1)(-53) + (7)(-10) + (5)(-61) = -560$
$z_{31} = (0)(18) + (5)(10) + (5)(4) + (-2)(16) = 38$
$z_{32} = (0)(3) + (5)(-3) + (5)(2) + (-2)(2) = -9$
$z_{33} = (0)(-66) + (5)(-53) + (5)(-10) + (-2)(-61) = -193$

Hence, $\mathbf{D}(\mathbf{FK}) = \mathbf{Z} = \begin{bmatrix} 146 & 5 & -603 \\ 154 & 27 & -560 \\ 38 & -9 & -193 \end{bmatrix}$.

(2) (a) No. $\mathbf{LM} = \begin{bmatrix} 10 & 9 \\ 8 & 7 \end{bmatrix} \begin{bmatrix} 7 & -1 \\ 11 & 3 \end{bmatrix} = \begin{bmatrix} ((10)(7) + (9)(11)) & ((10)(-1) + (9)(3)) \\ ((8)(7) + (7)(11)) & ((8)(-1) + (7)(3)) \end{bmatrix}$

$= \begin{bmatrix} 169 & 17 \\ 133 & 13 \end{bmatrix}$, but $\mathbf{ML} = \begin{bmatrix} 7 & -1 \\ 11 & 3 \end{bmatrix} \begin{bmatrix} 10 & 9 \\ 8 & 7 \end{bmatrix}$

$= \begin{bmatrix} ((7)(10) + (-1)(8)) & ((7)(9) + (-1)(7)) \\ ((11)(10) + (3)(8)) & ((11)(9) + (3)(7)) \end{bmatrix} = \begin{bmatrix} 62 & 56 \\ 134 & 120 \end{bmatrix}$. Hence, $\mathbf{LM} \neq \mathbf{ML}$.

(c) No. Since $\mathbf{A}$ is a 3×2 matrix and $\mathbf{K}$ is a 2×3 matrix, $\mathbf{AK}$ is a 3×3 matrix, while $\mathbf{KA}$ is a 2×2 matrix. Therefore, $\mathbf{AK}$ cannot equal $\mathbf{KA}$, since they are different sizes.

(d) Yes. $\mathbf{NP} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} ((0)(3) + (0)(4)) & ((0)(-1) + (0)(7)) \\ ((0)(3) + (0)(4)) & ((0)(-1) + (0)(7)) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Also, $\mathbf{PN} = \begin{bmatrix} 3 & -1 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ((3)(0) + (-1)(0)) & ((3)(0) + (-1)(0)) \\ ((4)(0) + (7)(0)) & ((4)(0) + (7)(0)) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Hence, $\mathbf{NP} = \mathbf{PN}$, and so $\mathbf{N}$ and $\mathbf{P}$ commute.

(3) (a) (2nd row of $\mathbf{BG}$) = (2nd row of $\mathbf{B}$)$\mathbf{G}$ = $\begin{bmatrix} 3 & 8 & 0 \end{bmatrix} \begin{bmatrix} 5 & 1 & 0 \\ 0 & -2 & -1 \\ 1 & 0 & 3 \end{bmatrix}$. The entries obtained from this are:
(1st column entry) $= ((3)(5) + (8)(0) + (0)(1)) = 15$,
(2nd column entry) $= ((3)(1) + (8)(-2) + (0)(0)) = -13$,
(3rd column entry) $= ((3)(0) + (8)(-1) + (0)(3)) = -8$.

Hence, (2nd row of $\mathbf{BG}$) $= [15, -13, -8]$.

(c) (1st column of $\mathbf{SE}$) = $\mathbf{S}$(1st column of $\mathbf{E}$) = $\begin{bmatrix} 6 & -4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

$= [(6)(1) + (-4)(1) + (3)(0) + (2)(1)] = [4]$

(4) (a) Valid, by Theorem 1.14, part (1).

(b) Invalid. The equation claims that all pairs of matrices that can be multiplied in both orders commute. However, parts (a) and (c) of Exercise 2 illustrate two different pairs of matrices that do not commute (see above).

(c) Valid, by Theorem 1.14, part (1).

(d) Valid, by Theorem 1.14, part (2).

(e) Valid, by Theorem 1.16.

(f) Invalid. For a counterexample, consider the matrices $\mathbf{L}$ and $\mathbf{M}$ from Exercises 1 through 3. Using our computation of $\mathbf{ML}$ from Exercise 2(a), above, $\mathbf{L}(\mathbf{ML}) = \begin{bmatrix} 10 & 9 \\ 8 & 7 \end{bmatrix}\begin{bmatrix} 62 & 56 \\ 134 & 120 \end{bmatrix} =$

$\begin{bmatrix} ((10)(62)+(9)(134)) & ((10)(56)+(9)(120)) \\ ((8)(62)+(7)(134)) & ((8)(56)+(7)(120)) \end{bmatrix} = \begin{bmatrix} 1826 & 1640 \\ 1434 & 1288 \end{bmatrix}$,

but $\mathbf{L}^2\mathbf{M} = (\mathbf{LL})\mathbf{M} = \left(\begin{bmatrix} 10 & 9 \\ 8 & 7 \end{bmatrix}\begin{bmatrix} 10 & 9 \\ 8 & 7 \end{bmatrix}\right)\mathbf{M}$

$= \begin{bmatrix} ((10)(10)+(9)(8)) & ((10)(9)+(9)(7)) \\ ((8)(10)+(7)(8)) & ((8)(9)+(7)(7)) \end{bmatrix}\mathbf{M} = \begin{bmatrix} 172 & 153 \\ 136 & 121 \end{bmatrix}\begin{bmatrix} 7 & -1 \\ 11 & 3 \end{bmatrix}$

$= \begin{bmatrix} ((172)(7)+(153)(11)) & ((172)(-1)+(153)(3)) \\ ((136)(7)+(121)(11)) & ((136)(-1)+(121)(3)) \end{bmatrix} = \begin{bmatrix} 2887 & 287 \\ 2283 & 227 \end{bmatrix}$.

(g) Valid, by Theorem 1.14, part (3).

(h) Valid, by Theorem 1.14, part (2).

(i) Invalid. For a counterexample, consider the matrices $\mathbf{A}$ and $\mathbf{K}$ from Exercises 1 through 3. Note that $\mathbf{A}$ is a 3×2 matrix and $\mathbf{K}$ is a 2×3 matrix. Therefore, $\mathbf{AK}$ is a 3×3 matrix. Hence, $(\mathbf{AK})^T$ is a 3×3 matrix. However, $\mathbf{A}^T$ is a 2×3 matrix and $\mathbf{K}^T$ is a 3×2 matrix. Thus, $\mathbf{A}^T\mathbf{K}^T$ is a 2×2 matrix and so cannot equal $(\mathbf{AK})^T$.
The equation is also false in general for square matrices, where it is not the sizes of the matrices that cause the problem. For example, let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, and let $\mathbf{K} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$. Then,

$(\mathbf{AK})^T = \left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right)^T = \begin{bmatrix} ((1)(1)+(1)(0)) & ((1)(-1)+(1)(0)) \\ ((0)(1)+(0)(0)) & ((0)(-1)+(0)(0)) \end{bmatrix}^T =$

$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$, while $\mathbf{A}^T\mathbf{K}^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$

$= \begin{bmatrix} ((1)(1)+(0)(-1)) & ((1)(0)+(0)(0)) \\ ((1)(1)+(0)(-1)) & ((1)(0)+(0)(0)) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

However, there are some rare instances for which the equation is true, such as when $\mathbf{A} = \mathbf{K}$, or when either $\mathbf{A}$ or $\mathbf{K}$ equals $\mathbf{I}_n$. Another example for which the equation holds is for $\mathbf{A} = \mathbf{G}$ and $\mathbf{K} = \mathbf{H}$, where $\mathbf{G}$ and $\mathbf{H}$ are the matrices used in Exercises 1, 2, and 3.

(j) Valid, by Theorem 1.14, part (3), and Theorem 1.16.

(5) To find the total in salaries paid by Outlet 1, we must compute the total amount paid to executives, the total amount paid to salespersons, and the total amount paid to others, and then add these amounts. But each of these totals is found by multiplying the number of that type of employee working at Outlet 1 by the salary for that type of employee. Hence, the total salary paid at Outlet 1 is

$$\underbrace{(3)(30000)}_{\text{Executives}} + \underbrace{(7)(22500)}_{\text{Salespersons}} + \underbrace{(8)(15000)}_{\text{Others}} = \underbrace{367500}_{\text{Salary Total}}.$$

Note that this is the $(1,1)$ entry obtained when multiplying the two given matrices. A similar analysis shows that multiplying the two given matrices gives all the desired salary and fringe benefit totals. In particular, if

$$\mathbf{W} = \begin{bmatrix} 3 & 7 & 8 \\ 2 & 4 & 5 \\ 6 & 14 & 18 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 30000 & 7500 \\ 22500 & 4500 \\ 15000 & 3000 \end{bmatrix}, \text{ then}$$

$w_{11} = (3)\,(30000) + (7)\,(22500) + (8)\,(15000) = 367500$,
$w_{12} = (3)\,(7500) + (7)\,(4500) + (8)\,(3000) = 78000$,
$w_{21} = (2)\,(30000) + (4)\,(22500) + (5)\,(15000) = 225000$,
$w_{22} = (2)\,(7500) + (4)\,(4500) + (5)\,(3000) = 48000$,
$w_{31} = (6)\,(30000) + (14)\,(22500) + (18)\,(15000) = 765000$,
$w_{32} = (6)\,(7500) + (14)\,(4500) + (18)\,(3000) = 162000$,
$w_{41} = (3)\,(30000) + (6)\,(22500) + (9)\,(15000) = 360000$,
$w_{42} = (3)\,(7500) + (6)\,(4500) + (9)\,(3000) = 76500$.

$$\text{Hence, we have: } \mathbf{W} = \begin{array}{c} \\ \text{Outlet 1} \\ \text{Outlet 2} \\ \text{Outlet 3} \\ \text{Outlet 4} \end{array} \begin{array}{c} \begin{array}{cc} \text{Salary} & \text{Fringe Benefits} \end{array} \\ \begin{bmatrix} \$367500 & \$78000 \\ \$225000 & \$48000 \\ \$765000 & \$162000 \\ \$360000 & \$76500 \end{bmatrix} \end{array}.$$

(7) To compute the tonnage of a particular chemical applied to a given field, for each type of fertilizer we must multiply the percent concentration of the chemical in the fertilizer by the number of tons of that fertilizer applied to the given field. After this is done for each fertilizer type, we add these results to get the total tonnage of the chemical that was applied. For example, to compute the number of tons of potash applied to field 2, we compute

$$\underbrace{0.05}_{\text{\% Potash in Fert. 1}} \cdot \underbrace{2}_{\text{Tons of Fert. 1}} + \underbrace{0.05}_{\text{\% Potash in Fert. 2}} \cdot \underbrace{1}_{\text{Tons of Fert. 2}}$$
$$+ \underbrace{0.20}_{\text{\% Potash in Fert. 3}} \cdot \underbrace{1}_{\text{Tons of Fert. 3}} = \underbrace{0.35}_{\text{Tons of Potash}}.$$

Note that this is the dot product (3rd column of $\mathbf{A}$)$\cdot$(2nd column of $\mathbf{B}$). Since this is the dot product of two columns, it is not in the form of an entry of a matrix product. To turn this into a form that looks like an entry of a matrix product, we consider the transpose of $\mathbf{A}^T$ and express it as (3rd row of $\mathbf{A}^T$)$\cdot$(2nd column of $\mathbf{B}$). Thus, this is the $(3, 2)$ entry of $\mathbf{A}^T\mathbf{B}$. A similar analysis shows that computing all of the entries of $\mathbf{A}^T\mathbf{B}$ produces all of the information requested in the problem. Hence, to solve the problem, we compute

$$\mathbf{Y} = \mathbf{A}^T\mathbf{B} = \begin{bmatrix} 0.10 & 0.25 & 0.00 \\ 0.10 & 0.05 & 0.10 \\ 0.05 & 0.05 & 0.20 \end{bmatrix} \begin{bmatrix} 5 & 2 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 3 \end{bmatrix}.$$

Solving for each entry produces

$y_{11} = (0.10)\,(5) + (0.25)\,(2) + (0.00)\,(3) = 1.00$,
$y_{12} = (0.10)\,(2) + (0.25)\,(1) + (0.00)\,(1) = 0.45$,
$y_{13} = (0.10)\,(4) + (0.25)\,(1) + (0.00)\,(3) = 0.65$,
$y_{21} = (0.10)\,(5) + (0.05)\,(2) + (0.10)\,(3) = 0.90$,
$y_{22} = (0.10)\,(2) + (0.05)\,(1) + (0.10)\,(1) = 0.35$,
$y_{23} = (0.10)\,(4) + (0.05)\,(1) + (0.10)\,(3) = 0.75$,

$y_{31} = (0.05)(5) + (0.05)(2) + (0.20)(3) = 0.95,$
$y_{32} = (0.05)(2) + (0.05)(1) + (0.20)(1) = 0.35,$
$y_{33} = (0.05)(4) + (0.05)(1) + (0.20)(3) = 0.85.$

$$\text{Hence,}\quad \mathbf{Y} = \begin{array}{l} \\ \text{Nitrogen} \\ \text{Phosphate} \\ \text{Potash} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Field 1} & \text{Field 2} & \text{Field 3} \end{array} \\ \left[\begin{array}{ccc} 1.00 & 0.45 & 0.65 \\ 0.90 & 0.35 & 0.75 \\ 0.95 & 0.35 & 0.85 \end{array}\right] \end{array} \text{(in tons).}$$

(9) (a) An example: $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$. Note that $\mathbf{A}^2 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} ((1)(1) + (1)(0)) & ((1)(1) + (1)(-1)) \\ ((0)(1) + (-1)(0)) & ((0)(1) + (-1)(-1)) \end{bmatrix} = \mathbf{I}_2.$

(b) One example: Suppose $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Let $\mathbf{W} = \mathbf{A}^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
Computing each entry yields

$w_{11} = (1)(1) + (1)(0) + (0)(0) = 1,$
$w_{12} = (1)(1) + (1)(-1) + (0)(0) = 0,$
$w_{13} = (1)(0) + (1)(0) + (0)(1) = 0,$
$w_{21} = (0)(1) + (-1)(0) + (0)(0) = 0,$
$w_{22} = (0)(1) + (-1)(-1) + (0)(0) = 1,$
$w_{23} = (0)(0) + (-1)(0) + (0)(1) = 0,$
$w_{31} = (0)(1) + (0)(0) + (1)(0) = 0,$
$w_{32} = (0)(1) + (0)(-1) + (1)(0) = 0,$
$w_{33} = (0)(0) + (0)(0) + (1)(1) = 1.$

Hence, $\mathbf{A}^2 = \mathbf{W} = \mathbf{I}_3$.

(c) Consider $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Let $\mathbf{Y} = \mathbf{A}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. First, we compute each entry of $\mathbf{Y}$.

$y_{11} = (0)(0) + (0)(1) + (1)(0) = 0,$
$y_{12} = (0)(0) + (0)(0) + (1)(1) = 1,$
$y_{13} = (0)(1) + (0)(0) + (1)(0) = 0,$
$y_{21} = (1)(0) + (0)(1) + (0)(0) = 0,$
$y_{22} = (1)(0) + (0)(0) + (0)(1) = 0,$
$y_{23} = (1)(1) + (0)(0) + (0)(0) = 1,$
$y_{31} = (0)(0) + (1)(1) + (0)(0) = 1,$
$y_{32} = (0)(0) + (1)(0) + (0)(1) = 0,$
$y_{33} = (0)(1) + (1)(0) + (0)(0) = 0.$

And so $\mathbf{Y} = \mathbf{A}^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Hence, $\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \mathbf{Y}\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.
Letting this product equal $\mathbf{Z}$, we compute each entry as follows:

$z_{11} = (0)(0) + (1)(1) + (0)(0) = 1,$
$z_{12} = (0)(0) + (1)(0) + (0)(1) = 0,$

$z_{13} = (0)(1) + (1)(0) + (0)(0) = 0,$
$z_{21} = (0)(0) + (0)(1) + (1)(0) = 0,$
$z_{22} = (0)(0) + (0)(0) + (1)(1) = 1,$
$z_{23} = (0)(1) + (0)(0) + (1)(0) = 0,$
$z_{31} = (1)(0) + (0)(1) + (0)(0) = 0,$
$z_{32} = (1)(0) + (0)(0) + (0)(1) = 0,$
$z_{33} = (1)(1) + (0)(0) + (0)(0) = 1.$

Therefore, $\mathbf{A}^3 = \mathbf{Z} = \mathbf{I}_3$.

(10) (a) This is the dot product of the 3rd row of $\mathbf{A}$ with the 4th column of $\mathbf{B}$. Hence, the result is the 3rd row, 4th column entry of $\mathbf{AB}$.

(c) This is (2nd column of $\mathbf{A}$)$\cdot$(3rd row of $\mathbf{B}$) = (3rd row of $\mathbf{B}$)$\cdot$(2nd column of $\mathbf{A}$) = 3rd row, 2nd column entry of $\mathbf{BA}$.

(11) (a) ((3, 2) entry of $\mathbf{AB}$) = (3rd row of $\mathbf{A}$)$\cdot$(2nd column of $\mathbf{B}$) = $\sum_{k=1}^{n} a_{3k}b_{k2}$

(12) (a) $3\mathbf{v}_1 - 2\mathbf{v}_2 + 5\mathbf{v}_3 = 3[4, 7, -2] - 2[-3, -6, 5] + 5[-9, 2, -8]$
$= [12, 21, -6] + [6, 12, -10] + [-45, 10, -40] = [-27, 43, -56]$

(b) $2\mathbf{w}_1 + 6\mathbf{w}_2 - 3\mathbf{w}_3 = 2\begin{bmatrix} 4 \\ -3 \\ -9 \end{bmatrix} + 6\begin{bmatrix} 7 \\ -6 \\ 2 \end{bmatrix} - 3\begin{bmatrix} -2 \\ 5 \\ -8 \end{bmatrix}$

$$= \begin{bmatrix} 8 \\ -6 \\ -18 \end{bmatrix} + \begin{bmatrix} 42 \\ -36 \\ 12 \end{bmatrix} + \begin{bmatrix} 6 \\ -15 \\ 24 \end{bmatrix} = \begin{bmatrix} 56 \\ -57 \\ 18 \end{bmatrix}$$

(15) Several crucial steps in the following proofs rely on Theorem 1.5 for their validity.

Proof of Part (2): The (i, j) entry of $\mathbf{A}(\mathbf{B} + \mathbf{C})$
= (ith row of $\mathbf{A}$)$\cdot$(jth column of $(\mathbf{B} + \mathbf{C})$)
= (ith row of $\mathbf{A}$)$\cdot$(jth column of $\mathbf{B}$ + jth column of $\mathbf{C}$)
= (ith row of $\mathbf{A}$)$\cdot$(jth column of $\mathbf{B}$)
+ (ith row of $\mathbf{A}$)$\cdot$(jth column of $\mathbf{C}$)
= (i, j) entry of $\mathbf{AB}$ + (i, j) entry of $\mathbf{AC}$
= (i, j) entry of $(\mathbf{AB} + \mathbf{AC})$.

Proof of Part (3): The (i, j) entry of $(\mathbf{A} + \mathbf{B})\mathbf{C}$
= (ith row of $(\mathbf{A} + \mathbf{B})$)$\cdot$($j$th column of $\mathbf{C}$)
= ((ith row of $\mathbf{A}$) + (ith row of $\mathbf{B}$)) $\cdot$ (jth column of $\mathbf{C}$)
= (ith row of $\mathbf{A}$)$\cdot$(jth column of $\mathbf{C}$)
+ (ith row of $\mathbf{B}$)$\cdot$(jth column of $\mathbf{C}$)
= (i, j) entry of $\mathbf{AC}$ + (i, j) entry of $\mathbf{BC}$
= (i, j) entry of $(\mathbf{AC} + \mathbf{BC})$.

For the first equation in part (4), the (i, j) entry of $c(\mathbf{AB})$
= c ((ith row of $\mathbf{A}$)$\cdot$(jth column of $\mathbf{B}$))
= (c(ith row of $\mathbf{A}$))$\cdot$(jth column of $\mathbf{B}$)
= (ith row of $c\mathbf{A}$)$\cdot$(jth column of $\mathbf{B}$)
= (i, j) entry of $(c\mathbf{A})\mathbf{B}$.

Similarly, the (i,j) entry of $c(\mathbf{AB})$
$= c\,((i\text{th row of }\mathbf{A})\cdot(j\text{th column of }\mathbf{B}))$
$= (i\text{th row of }\mathbf{A})\cdot(c(j\text{th column of }\mathbf{B}))$
$= (i\text{th row of }\mathbf{A})\cdot(j\text{th column of }c\mathbf{B})$
$= (i,j)$ entry of $\mathbf{A}(c\mathbf{B})$.

(20) Proof of Part (1): We use induction on the variable t. Base Step: $\mathbf{A}^{s+0} = \mathbf{A}^s = \mathbf{A}^s\mathbf{I} = \mathbf{A}^s\mathbf{A}^0$. Inductive Step: Assume $\mathbf{A}^{s+t} = \mathbf{A}^s\mathbf{A}^t$ for some $t \geq 0$. We must prove $\mathbf{A}^{s+(t+1)} = \mathbf{A}^s\mathbf{A}^{t+1}$. But $\mathbf{A}^{s+(t+1)} = \mathbf{A}^{(s+t)+1} = \mathbf{A}^{s+t}\mathbf{A} = (\mathbf{A}^s\mathbf{A}^t)\mathbf{A}$ (by the inductive hypothesis) $= \mathbf{A}^s(\mathbf{A}^t\mathbf{A}) = \mathbf{A}^s\mathbf{A}^{t+1}$.

Proof of Part (2): Again, we use induction on the variable t. Base Step: $(\mathbf{A}^s)^0 = \mathbf{I} = \mathbf{A}^0 = \mathbf{A}^{s0}$. Inductive Step: Assume $(\mathbf{A}^s)^t = \mathbf{A}^{st}$ for some integer $t \geq 0$. We must prove $(\mathbf{A}^s)^{t+1} = \mathbf{A}^{s(t+1)}$. But $(\mathbf{A}^s)^{t+1} = (\mathbf{A}^s)^t\mathbf{A}^s$ (by definition) $= \mathbf{A}^{st}\mathbf{A}^s$ (by the inductive hypothesis) $= \mathbf{A}^{st+s}$ (by part (1)) $= \mathbf{A}^{s(t+1)}$. Finally, reversing the roles of s and t in the proof above shows that $(\mathbf{A}^t)^s = \mathbf{A}^{ts}$, which equals $\mathbf{A}^{st}$.

(27) (a) Consider any matrix $\mathbf{A}$ of the form $\begin{bmatrix} 1 & 0 \\ x & 0 \end{bmatrix}$. Then $\mathbf{A}^2 = \begin{bmatrix} 1 & 0 \\ x & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ x & 0 \end{bmatrix} =$

$\begin{bmatrix} ((1)(1)+(0)(x)) & ((1)(0)+(0)(0)) \\ ((x)(1)+(0)(x)) & ((x)(0)+(0)(0)) \end{bmatrix} = \mathbf{A}$. So, for example, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is idempotent.

(28) (b) Consider $\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $\mathbf{AB} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} =$

$\begin{bmatrix} ((1)(1)+(2)(0)+(-1)(1)) & ((1)(-2)+(2)(1)+(-1)(0)) \\ ((2)(1)+(4)(0)+(-2)(1)) & ((2)(-2)+(4)(1)+(-2)(0)) \end{bmatrix} = \mathbf{O}_2$.

(29) A 2×2 matrix $\mathbf{A}$ that commutes with every other 2×2 matrix must have the form $\mathbf{A} = c\mathbf{I}_2$. To prove this, note that if $\mathbf{A}$ commutes with every other 2×2 matrix, then $\mathbf{AB} = \mathbf{BA}$, where $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. But

$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} ((a_{11})(0)+(a_{12})(0)) & ((a_{11})(1)+(a_{12})(0)) \\ ((a_{21})(0)+(a_{22})(0)) & ((a_{21})(1)+(a_{22})(0)) \end{bmatrix} = \begin{bmatrix} 0 & a_{11} \\ 0 & a_{21} \end{bmatrix}$ and

$\mathbf{BA} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ((0)(a_{11})+(1)(a_{21})) & ((0)(a_{12})+(1)(a_{22})) \\ ((0)(a_{11})+(0)(a_{21})) & ((0)(a_{12})+(0)(a_{22})) \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} \\ 0 & 0 \end{bmatrix}$.

Setting these equal shows that $a_{21} = 0$ and $a_{11} = a_{22}$. Let $c = a_{11} = a_{22}$. Then $\mathbf{A} = \begin{bmatrix} c & a_{12} \\ 0 & c \end{bmatrix}$.

Let $\mathbf{D} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $\mathbf{AD} = \begin{bmatrix} c & a_{12} \\ 0 & c \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} =$

$\begin{bmatrix} ((c)(0)+(a_{12})(1)) & ((c)(0)+(a_{12})(0)) \\ ((0)(0)+(c)(1)) & ((0)(0)+(c)(0)) \end{bmatrix} = \begin{bmatrix} a_{12} & 0 \\ c & 0 \end{bmatrix}$ and $\mathbf{DA} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c & a_{12} \\ 0 & c \end{bmatrix} =$

$\begin{bmatrix} ((0)(c)+(0)(0)) & ((0)(a_{12})+(0)(c)) \\ ((1)(c)+(0)(0)) & ((1)(a_{12})+(0)(c)) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ c & a_{12} \end{bmatrix}$.

Setting $\mathbf{AD} = \mathbf{DA}$ shows that $a_{12} = 0$, and so $\mathbf{A} = c\mathbf{I}_2$.

Finally, we must show that $c\mathbf{I}_2$ actually does commute with every 2×2 matrix. If $\mathbf{M}$ is a 2×2 matrix, then $(c\mathbf{I}_2)\mathbf{M} = c(\mathbf{I}_2\mathbf{M}) = c\mathbf{M}$. Similarly, $\mathbf{M}(c\mathbf{I}_2) = c(\mathbf{M}\mathbf{I}_2) = c\mathbf{M}$. Hence, $c\mathbf{I}_2$ commutes with $\mathbf{M}$.

(31) (a) True. This is a "boxed" fact given in the section.

(b) True. $\mathbf{D}(\mathbf{A}+\mathbf{B}) = \mathbf{DA}+\mathbf{DB}$ (by part (2) of Theorem 1.14) $= \mathbf{DB}+\mathbf{DA}$ (by part (1) of Theorem 1.1)

(c) True. This is part (4) of Theorem 1.14.

(d) False. Let $\mathbf{D} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, and let $\mathbf{E} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. Now $\mathbf{DE} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} =$

$\begin{bmatrix} ((1)(1)+(1)(0)) & ((1)(-2)+(1)(1)) \\ ((0)(1)+(-1)(0)) & ((0)(-2)+(-1)(1)) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$. Hence,

$(\mathbf{DE})^2 = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} ((1)(1)+(-1)(0)) & ((1)(-1)+(-1)(-1)) \\ ((0)(1)+(-1)(0)) & ((0)(-1)+(-1)(-1)) \end{bmatrix} = \mathbf{I}_2.$

But $\mathbf{D}^2 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} ((1)(1)+(1)(0)) & ((1)(1)+(1)(-1)) \\ ((0)(1)+(-1)(0)) & ((0)(1)+(-1)(-1)) \end{bmatrix} = \mathbf{I}_2$. Also,

$\mathbf{E}^2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ((1)(1)+(-2)(0)) & ((1)(-2)+(-2)(1)) \\ ((0)(1)+(1)(0)) & ((0)(-2)+(1)(1)) \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}.$

Hence, $\mathbf{D}^2\mathbf{E}^2 = \mathbf{I}_2\mathbf{E}^2 = \mathbf{E}^2 \neq \mathbf{I}_2$. Thus, $(\mathbf{DE})^2 \neq \mathbf{D}^2\mathbf{E}^2$. The problem here is that $\mathbf{D}$ and $\mathbf{E}$ do not commute. For, *if they did*, we would have $(\mathbf{DE})^2 = (\mathbf{DE})(\mathbf{DE}) = \mathbf{D}(\mathbf{ED})\mathbf{E} = \mathbf{D}(\mathbf{DE})\mathbf{E} = (\mathbf{DD})(\mathbf{EE}) = \mathbf{D}^2\mathbf{E}^2$.

(e) False. See the answer for part (i) of Exercise 4 for a counterexample. We know from Theorem 1.16 that $(\mathbf{DE})^T = \mathbf{E}^T\mathbf{D}^T$. So if $\mathbf{D}^T$ and $\mathbf{E}^T$ do not commute, then $(\mathbf{DE})^T \neq \mathbf{D}^T\mathbf{E}^T$.

(f) False. See the answer for part (b) of Exercise 28 for a counterexample. $\mathbf{DE} = \mathbf{O}$ whenever every row of $\mathbf{D}$ is orthogonal to every column of $\mathbf{E}$, but neither matrix is forced to be zero for $\mathbf{DE} = \mathbf{O}$ to be true.

Chapter 1 Review Exercises

(2) $\|\mathbf{x}\| = \sqrt{\left(\frac{1}{4}\right)^2 + \left(-\frac{3}{5}\right)^2 + \left(\frac{3}{4}\right)^2} = \sqrt{\frac{1}{16} + \frac{9}{25} + \frac{9}{16}} = \sqrt{\frac{394}{16\cdot 25}} = \frac{\sqrt{394}}{20} \approx 0.99247.$

Thus, $\mathbf{u} = \frac{\mathbf{x}}{\|\mathbf{x}\|} = \left[\frac{5}{\sqrt{394}}, -\frac{12}{\sqrt{394}}, \frac{15}{\sqrt{394}}\right] \approx [0.2481, -0.5955, 0.7444]$. This is slightly longer than $\mathbf{x}$ because we have *divided* $\mathbf{x}$ by a scalar with absolute value less than 1, which amounts to multiplying $\mathbf{x}$ by a scalar having absolute value greater than 1.

(4) We will use Newton's Second Law, which states that $\mathbf{f} = m\mathbf{a}$, or equivalently, $\mathbf{a} = \frac{1}{m}\mathbf{f}$. For the force $\mathbf{f}_1$, $\mathbf{a}_1 = \frac{1}{7}\left(133\left(\frac{[6,17,-6]}{\|[6,17,-6]\|}\right)\right) = 19\left(\frac{[6,17,-6]}{\sqrt{6^2+17^2+(-6)^2}}\right) = 19\left(\frac{[6,17,-6]}{\sqrt{361}}\right) = [6,17,-6]$. Similarly, for the force $\mathbf{f}_2$, $\mathbf{a}_2 = \frac{1}{7}\left(168\left(\frac{[-8,-4,8]}{\|[-8,-4,8]\|}\right)\right) = 24\left(\frac{[-8,-4,8]}{\sqrt{(-8)^2+(-4)^2+8^2}}\right) = 24\left(\frac{[-8,-4,8]}{\sqrt{144}}\right) = [-16,-8,16]$. Hence, $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2 = [6,17,-6] + [-16,-8,16] = [-10,9,10]$, in m/sec^2.

(6) The angle $\theta = \cos^{-1}\left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}\right) \approx \cos^{-1}\left(\frac{-69}{(10.05)(9.49)}\right) \approx \cos^{-1}\left(\frac{-69}{95.34}\right) \approx \cos^{-1}(-0.7237) \approx 136°.$

Chapter 2

Section 2.1

(1) In each part, we first set up the augmented matrix corresponding to the given system of linear equations. Then we perform the row operations designated by the Gaussian elimination method. Finally, we use the final augmented matrix obtained to give the solution set for the original system.

(a) First augmented matrix: $\left[\begin{array}{rrr|r} -5 & -2 & 2 & 14 \\ 3 & 1 & -1 & -8 \\ 2 & 2 & -1 & -3 \end{array}\right]$. The first pivot is the (1,1) entry. We make this a "1" by performing the following type (I) row operation:

Type (I) operation: $\langle 1\rangle \leftarrow -\frac{1}{5}\langle 1\rangle$: $\left[\begin{array}{rrr|r} 1 & \frac{2}{5} & -\frac{2}{5} & -\frac{14}{5} \\ 3 & 1 & -1 & -8 \\ 2 & 2 & -1 & -3 \end{array}\right]$

We now target the (2,1) entry and make it "0." We do this using the following type (II) row operation:

Type (II) operation: $\langle 2\rangle \leftarrow (-3)\times\langle 1\rangle + \langle 2\rangle$:

Side Calculation

$$\begin{array}{rrrr|r} (-3)\times\ (\text{row } 1) & -3 & -\frac{6}{5} & \frac{6}{5} & \frac{42}{5} \\ (\text{row } 2) & 3 & 1 & -1 & -8 \\ \hline (\text{sum}) & 0 & -\frac{1}{5} & \frac{1}{5} & \frac{2}{5} \end{array}$$

Resultant Matrix

$$\left[\begin{array}{rrr|r} 1 & \frac{2}{5} & -\frac{2}{5} & -\frac{14}{5} \\ 0 & -\frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ 2 & 2 & -1 & -3 \end{array}\right]$$

Next, we target the (3,1) entry.

Type (II) operation: $\langle 3\rangle \leftarrow (-2)\times\langle 1\rangle + \langle 3\rangle$:

Side Calculation

$$\begin{array}{rrrr|r} (-2)\times\ (\text{row } 1) & -2 & -\frac{4}{5} & \frac{4}{5} & \frac{28}{5} \\ (\text{row } 3) & 2 & 2 & -1 & -3 \\ \hline (\text{sum}) & 0 & \frac{6}{5} & -\frac{1}{5} & \frac{13}{5} \end{array}$$

Resultant Matrix

$$\left[\begin{array}{rrr|r} 1 & \frac{2}{5} & -\frac{2}{5} & -\frac{14}{5} \\ 0 & -\frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ 0 & \frac{6}{5} & -\frac{1}{5} & \frac{13}{5} \end{array}\right]$$

Now we move the pivot to the (2,2) entry. We make this entry a "1" using a type (I) row operation.

Type (I) operation: $\langle 2\rangle \leftarrow -5\,\langle 2\rangle$: $\left[\begin{array}{rrr|r} 1 & \frac{2}{5} & -\frac{2}{5} & -\frac{14}{5} \\ 0 & 1 & -1 & -2 \\ 0 & \frac{6}{5} & -\frac{1}{5} & \frac{13}{5} \end{array}\right]$

Now we target the (3,2) entry.

Type (II) operation: $\langle 3\rangle \leftarrow (-\frac{6}{5})\times\langle 2\rangle + \langle 3\rangle$:

Side Calculation

$$\begin{array}{rrrr|r} (-\frac{6}{5})\times\ (\text{row } 2) & 0 & -\frac{6}{5} & \frac{6}{5} & \frac{12}{5} \\ (\text{row } 3) & 0 & \frac{6}{5} & -\frac{1}{5} & \frac{13}{5} \\ \hline (\text{sum}) & 0 & 0 & 1 & 5 \end{array}$$

Resultant Matrix

$$\left[\begin{array}{rrr|r} 1 & \frac{2}{5} & -\frac{2}{5} & -\frac{14}{5} \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

The pivot moves to the (3,3) entry. However, the (3,3) entry already equals 1. Since there are no rows below the third row, we are finished performing row operations. The final augmented matrix corresponds to the following linear system:

$$\begin{cases} x_1 + \frac{2}{5}x_2 - \frac{2}{5}x_3 = -\frac{14}{5} \\ \qquad\quad x_2 - x_3 = -2 \\ \qquad\qquad\quad x_3 = 5 \end{cases}$$

The last equation clearly gives $x_3 = 5$. Substituting this value into the second equation yields $x_2 - 5 = -2$, or $x_2 = 3$. Substituting both values into the first equation produces $x_1 + \frac{2}{5}(3) - \frac{2}{5}(5) = -\frac{14}{5}$, which leads to $x_1 = -2$. Hence, the solution set for the system is $\{(-2, 3, 5)\}$.

(c) First augmented matrix: $\left[\begin{array}{rrr|r} 3 & -2 & 4 & -54 \\ -1 & 1 & -2 & 20 \\ 5 & -4 & 8 & -83 \end{array}\right]$. The first pivot is the (1,1) entry. We make this a "1" by performing the following type (I) row operation:

Type (I) operation: $\langle 1\rangle \leftarrow \frac{1}{3}\langle 1\rangle$: $\left[\begin{array}{rrr|r} 1 & -\frac{2}{3} & \frac{4}{3} & -18 \\ -1 & 1 & -2 & 20 \\ 5 & -4 & 8 & -83 \end{array}\right]$

We now target the (2,1) entry and make it "0." We do this using the following type (II) row operation:

Type (II) operation: $\langle 2\rangle \leftarrow (1) \times \langle 1\rangle + \langle 2\rangle$:

Side Calculation

$$\begin{array}{rrrr|r} (1)\times\ (\text{row } 1) & 1 & -\frac{2}{3} & \frac{4}{3} & -18 \\ (\text{row } 2) & -1 & 1 & -2 & 20 \\ \hline (\text{sum}) & 0 & \frac{1}{3} & -\frac{2}{3} & 2 \end{array}$$

Resultant Matrix

$$\left[\begin{array}{rrr|r} 1 & -\frac{2}{3} & \frac{4}{3} & -18 \\ 0 & \frac{1}{3} & -\frac{2}{3} & 2 \\ 5 & -4 & 8 & -83 \end{array}\right]$$

Next, we target the (3,1) entry.

Type (II) operation: $\langle 3\rangle \leftarrow (-5) \times \langle 1\rangle + \langle 3\rangle$:

Side Calculation

$$\begin{array}{rrrr|r} (-5)\times\ (\text{row } 1) & -5 & \frac{10}{3} & -\frac{20}{3} & 90 \\ (\text{row } 3) & 5 & -4 & 8 & -83 \\ \hline (\text{sum}) & 0 & -\frac{2}{3} & \frac{4}{3} & 7 \end{array}$$

Resultant Matrix

$$\left[\begin{array}{rrr|r} 1 & -\frac{2}{3} & \frac{4}{3} & -18 \\ 0 & \frac{1}{3} & -\frac{2}{3} & 2 \\ 0 & -\frac{2}{3} & \frac{4}{3} & 7 \end{array}\right]$$

Now we move the pivot to the (2,2) entry. We make this entry a "1" using a type (I) row operation.

Type (I) operation: $\langle 2\rangle \leftarrow 3\,\langle 2\rangle$: $\left[\begin{array}{rrr|r} 1 & -\frac{2}{3} & \frac{4}{3} & -18 \\ 0 & 1 & -2 & 6 \\ 0 & -\frac{2}{3} & \frac{4}{3} & 7 \end{array}\right]$

Now we target the (3,2) entry.

Type (II) operation: $\langle 3\rangle \leftarrow (\frac{2}{3}) \times \langle 2\rangle + \langle 3\rangle$:

Side Calculation

$$\begin{array}{rrrr|r} (\frac{2}{3})\times\ (\text{row } 2) & 0 & \frac{2}{3} & -\frac{4}{3} & 4 \\ (\text{row } 3) & 0 & -\frac{2}{3} & \frac{4}{3} & 7 \\ \hline (\text{sum}) & 0 & 0 & 0 & 11 \end{array}$$

Resultant Matrix

$$\left[\begin{array}{rrr|r} 1 & -\frac{2}{3} & \frac{4}{3} & -18 \\ 0 & 1 & -2 & 6 \\ 0 & 0 & 0 & 11 \end{array}\right]$$

The pivot moves to the (3,3) entry. However, the (3,3) entry equals 0. Because there are no rows below the third row, we cannot switch with a row below the third row to produce a nonzero pivot. Also, since there are no more columns before the augmentation bar, we are finished performing row operations. The final augmented matrix corresponds to the following linear system:

$$\begin{cases} x_1 - \frac{2}{3}x_2 + \frac{4}{3}x_3 = -18 \\ \qquad x_2 - 2x_3 = 6 \\ \qquad\qquad 0 = 11 \end{cases}$$

The last equation states that $0 = 11$. Since this equation cannot be satisfied, the original linear system has no solutions. The solution set is empty.

(e) First augmented matrix: $\left[\begin{array}{ccccc|c} 6 & -12 & -5 & 16 & -2 & -53 \\ -3 & 6 & 3 & -9 & 1 & 29 \\ -4 & 8 & 3 & -10 & 1 & 33 \end{array}\right]$. The first pivot is the (1,1) entry. We make this a "1" by performing the following type (I) row operation:

Type (I) operation: $\langle 1 \rangle \leftarrow \frac{1}{6}\langle 1 \rangle$: $\left[\begin{array}{ccccc|c} 1 & -2 & -\frac{5}{6} & \frac{8}{3} & -\frac{1}{3} & -\frac{53}{6} \\ -3 & 6 & 3 & -9 & 1 & 29 \\ -4 & 8 & 3 & -10 & 1 & 33 \end{array}\right]$

We now target the (2,1) entry and make it "0." We do this using the following type (II) row operation:
Type (II) operation: $\langle 2 \rangle \leftarrow (3) \times \langle 1 \rangle + \langle 2 \rangle$:

Side Calculation:

$$\begin{array}{rcccccc|c} (3)\times & (\text{row } 1) & 3 & -6 & -\frac{5}{2} & 8 & -1 & -\frac{53}{2} \\ & (\text{row } 2) & -3 & 6 & 3 & -9 & 1 & 29 \\ \hline & (\text{sum}) & 0 & 0 & \frac{1}{2} & -1 & 0 & \frac{5}{2} \end{array}$$

Resultant Matrix: $\left[\begin{array}{ccccc|c} 1 & -2 & -\frac{5}{6} & \frac{8}{3} & -\frac{1}{3} & -\frac{53}{6} \\ 0 & 0 & \frac{1}{2} & -1 & 0 & \frac{5}{2} \\ -4 & 8 & 3 & -10 & 1 & 33 \end{array}\right]$

Next, we target the (3,1) entry.
Type (II) operation: $\langle 3 \rangle \leftarrow (4) \times \langle 1 \rangle + \langle 3 \rangle$:

Side Calculation:

$$\begin{array}{rcccccc|c} (4)\times & (\text{row } 1) & 4 & -8 & -\frac{10}{3} & \frac{32}{3} & -\frac{4}{3} & -\frac{106}{3} \\ & (\text{row } 3) & -4 & 8 & 3 & -10 & 1 & 33 \\ \hline & (\text{sum}) & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{7}{3} \end{array}$$

Resultant Matrix: $\left[\begin{array}{ccccc|c} 1 & -2 & -\frac{5}{6} & \frac{8}{3} & -\frac{1}{3} & -\frac{53}{6} \\ 0 & 0 & \frac{1}{2} & -1 & 0 & \frac{5}{2} \\ 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{7}{3} \end{array}\right]$

We attempt to move the pivot to the (2,2) entry; however, there is a "0" there. Also, the entry below this is also 0, so we cannot switch rows to make the (2,2) entry nonzero. Thus, we move the pivot horizontally to the next column – to the (2,3) entry. We make this pivot a "1" using the following type (I) row operation:

Type (I) operation: $\langle 2 \rangle \leftarrow 2\,\langle 2 \rangle$: $\left[\begin{array}{ccccc|c} 1 & -2 & -\frac{5}{6} & \frac{8}{3} & -\frac{1}{3} & -\frac{53}{6} \\ 0 & 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{7}{3} \end{array}\right]$

Now we target the (3,3) entry.

Type (II) operation: $\langle 3\rangle \leftarrow (\frac{1}{3}) \times \langle 2\rangle + \langle 3\rangle$:

Side Calculation:
$$\begin{array}{rrrrrrr|r} (\frac{1}{3})\times & \text{(row 2)} & 0 & 0 & \frac{1}{3} & -\frac{2}{3} & 0 & \frac{5}{3} \\ & \text{(row 3)} & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{7}{3} \\ \hline & \text{(sum)} & 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \end{array}$$

Resultant Matrix:
$$\left[\begin{array}{rrrrr|r} 1 & -2 & -\frac{5}{6} & \frac{8}{3} & -\frac{1}{3} & -\frac{53}{6} \\ 0 & 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \end{array}\right]$$

We attempt to move the pivot diagonally to the (3,4) entry; however, that entry is also 0. Since there are no rows below with which to swap, we must move the pivot horizontally to the (3,5) entry. We make this pivot a "1" by performing the following type (I) row operation:

Type (I) operation: $\langle 3\rangle \leftarrow -3\,\langle 3\rangle$:
$$\left[\begin{array}{rrrrr|r} 1 & -2 & -\frac{5}{6} & \frac{8}{3} & -\frac{1}{3} & -\frac{53}{6} \\ 0 & 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

At this point, we are finished performing row operations. The linear system corresponding to the final augmented matrix is

$$\begin{cases} x_1 - 2x_2 - \frac{5}{6}x_3 + \frac{8}{3}x_4 - \frac{1}{3}x_5 = -\frac{53}{6} \\ \qquad\qquad x_3 - 2x_4 = 5 \\ \qquad\qquad\qquad\qquad x_5 = 2 \end{cases}$$

The last equation clearly gives us $x_5 = 2$. The variable x_4 corresponds to a nonpivot column and hence is an independent variable. Let $x_4 = d$. Then the second equation yields $x_3 - 2d = 5$, or $x_3 = 2d + 5$. The variable x_2 also corresponds to a nonpivot column, making it an independent variable as well. We let $x_2 = b$. Finally, plugging all these values into the first equation produces $x_1 - 2b - \frac{5}{6}(2d+5) + \frac{8}{3}d - \frac{1}{3}(2) = -\frac{53}{6}$. Solving this equation for x_1 gives $x_1 = 2b - d - 4$. Hence, the general solution set for the original system is $\{(2b-d-4, b, 2d+5, d, 2)\,|\, b, d \in \mathbb{R}\}$. To find three particular solutions, we plug in three different sets of values for b and d. Using $b = 0$, $d = 0$ yields the solution $(-4, 0, 5, 0, 2)$. Letting $b = 1$, $d = 0$ gives us $(-2, 1, 5, 0, 2)$. Finally, setting $b = 0$, $d = 1$ produces $(-5, 0, 7, 1, 2)$.

(g) First augmented matrix: $\left[\begin{array}{rrr|r} 4 & -2 & -7 & 5 \\ -6 & 5 & 10 & -11 \\ -2 & 3 & 4 & -3 \\ -3 & 2 & 5 & -5 \end{array}\right]$. The first pivot is the (1,1) entry. We make this a "1" by performing the following type (I) row operation:

Type (I) operation: $\langle 1\rangle \leftarrow \frac{1}{4}\,\langle 1\rangle$:
$$\left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & -\frac{7}{4} & \frac{5}{4} \\ -6 & 5 & 10 & -11 \\ -2 & 3 & 4 & -3 \\ -3 & 2 & 5 & -5 \end{array}\right]$$

We now target the (2,1) entry and make it "0." We do this using the following type (II) row operation:
Type (II) operation: $\langle 2\rangle \leftarrow (6) \times \langle 1\rangle + \langle 2\rangle$:

Side Calculation

$$\begin{array}{rrrr|r} (6)\times\ (\text{row } 1) & 6 & -3 & -\frac{21}{2} & \frac{15}{2} \\ (\text{row } 2) & -6 & 5 & 10 & -11 \\ \hline (\text{sum}) & 0 & 2 & -\frac{1}{2} & -\frac{7}{2} \end{array}$$

Resultant Matrix

$$\left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & -\frac{7}{4} & \frac{5}{4} \\ 0 & 2 & -\frac{1}{2} & -\frac{7}{2} \\ -2 & 3 & 4 & -3 \\ -3 & 2 & 5 & -5 \end{array}\right]$$

Next, we target the (3,1) entry.
Type (II) operation: $\langle 3\rangle \leftarrow (2) \times \langle 1\rangle + \langle 3\rangle$:

Side Calculation

$$\begin{array}{rrrr|r} (2)\times\ (\text{row } 1) & 2 & -1 & -\frac{7}{2} & \frac{5}{2} \\ (\text{row } 3) & -2 & 3 & 4 & -3 \\ \hline (\text{sum}) & 0 & 2 & \frac{1}{2} & -\frac{1}{2} \end{array}$$

Resultant Matrix

$$\left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & -\frac{7}{4} & \frac{5}{4} \\ 0 & 2 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 2 & \frac{1}{2} & -\frac{1}{2} \\ -3 & 2 & 5 & -5 \end{array}\right]$$

To finish the first column, we now target the (4,1) entry.
Type (II) operation: $\langle 4\rangle \leftarrow (3) \times \langle 1\rangle + \langle 4\rangle$:

Side Calculation

$$\begin{array}{rrrr|r} (3)\times\ (\text{row } 1) & 3 & -\frac{3}{2} & -\frac{21}{4} & \frac{15}{4} \\ (\text{row } 4) & -3 & 2 & 5 & -5 \\ \hline (\text{sum}) & 0 & \frac{1}{2} & -\frac{1}{4} & -\frac{5}{4} \end{array}$$

Resultant Matrix

$$\left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & -\frac{7}{4} & \frac{5}{4} \\ 0 & 2 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 2 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{4} & -\frac{5}{4} \end{array}\right]$$

Now we move the pivot to the (2,2) entry. We make this entry a "1" using a type (I) row operation.

Type (I) operation: $\langle 2\rangle \leftarrow \frac{1}{2}\,\langle 2\rangle$: $\left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & -\frac{7}{4} & \frac{5}{4} \\ 0 & 1 & -\frac{1}{4} & -\frac{7}{4} \\ 0 & 2 & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{4} & -\frac{5}{4} \end{array}\right]$

Now we target the (3,2) entry.
Type (II) operation: $\langle 3\rangle \leftarrow (-2) \times \langle 2\rangle + \langle 3\rangle$:

Side Calculation

$$\begin{array}{rrrr|r} (-2)\times\ (\text{row } 2) & 0 & -2 & \frac{1}{2} & \frac{7}{2} \\ (\text{row } 3) & 0 & 2 & \frac{1}{2} & -\frac{1}{2} \\ \hline (\text{sum}) & 0 & 0 & 1 & 3 \end{array}$$

Resultant Matrix

$$\left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & -\frac{7}{4} & \frac{5}{4} \\ 0 & 1 & -\frac{1}{4} & -\frac{7}{4} \\ 0 & 0 & 1 & 3 \\ 0 & \frac{1}{2} & -\frac{1}{4} & -\frac{5}{4} \end{array}\right]$$

To finish the second column, we target the (4,2) entry.
Type (II) operation: $\langle 4\rangle \leftarrow (-\frac{1}{2}) \times \langle 2\rangle + \langle 4\rangle$:

Side Calculation

$$\begin{array}{rrrr|r} (-\frac{1}{2})\times\ (\text{row } 2) & 0 & -\frac{1}{2} & \frac{1}{8} & \frac{7}{8} \\ (\text{row } 4) & 0 & \frac{1}{2} & -\frac{1}{4} & -\frac{5}{4} \\ \hline (\text{sum}) & 0 & 0 & -\frac{1}{8} & -\frac{3}{8} \end{array}$$

Resultant Matrix

$$\left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & -\frac{7}{4} & \frac{5}{4} \\ 0 & 1 & -\frac{1}{4} & -\frac{7}{4} \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -\frac{1}{8} & -\frac{3}{8} \end{array}\right]$$

The pivot now moves to the (3,3) entry. Luckily, the pivot already equals 1. Therefore, we next target the (4,3) entry.
Type (II) operation: $\langle 4\rangle \leftarrow (\frac{1}{8}) \times \langle 3\rangle + \langle 4\rangle$:

Side Calculation

$$\begin{array}{rrrr|r} (\frac{1}{8})\times \text{ (row 3)} & 0 & 0 & \frac{1}{8} & \frac{3}{8} \\ \text{(row 4)} & 0 & 0 & -\frac{1}{8} & -\frac{3}{8} \\ \hline \text{(sum)} & 0 & 0 & 0 & 0 \end{array}$$

Resultant Matrix

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{7}{4} & \frac{5}{4} \\ 0 & 1 & -\frac{1}{4} & -\frac{7}{4} \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This finishes the third column. Since there are no more columns before the augmentation bar, we are finished performing row operations. The linear system corresponding to the final matrix is

$$\begin{cases} x_1 - \frac{1}{2}x_2 - \frac{7}{4}x_3 = \frac{5}{4} \\ \quad\quad x_2 - \frac{1}{4}x_3 = -\frac{7}{4} \\ \quad\quad\quad\quad x_3 = 3 \\ \quad\quad\quad\quad 0 = 0 \end{cases}$$

The last equation, $0 = 0$, provides no information regarding the solution set for the system because it is true for every value of x_1, x_2, and x_3. Thus, we can ignore it. The third equation clearly gives $x_3 = 3$. Plugging this value into the second equation yields $x_2 - \frac{1}{4}(3) = -\frac{7}{4}$, or $x_2 = -1$. Substituting the values we have for x_2 and x_3 into the first equation produces $x_1 - \frac{1}{2}(-1) - \frac{7}{4}(3) = \frac{5}{4}$, or $x_1 = 6$. Hence, the original linear system has a unique solution. The full solution set is $\{(6, -1, 3)\}$.

(2) (a) The system of linear equations corresponding to the given augmented matrix is

$$\begin{cases} x_1 - 5x_2 + 2x_3 + 3x_4 - 2x_5 = -4 \\ \quad\quad x_2 - x_3 - 3x_4 - 7x_5 = -2 \\ \quad\quad\quad\quad x_4 + 2x_5 = 5 \\ \quad\quad\quad\quad\quad\quad 0 = 0 \end{cases}$$

The last equation, $0 = 0$, provides no information regarding the solution set for the system because it is true for every value of x_1, x_2, x_3, x_4, and x_5. So, we can ignore it. The column corresponding to the variable x_5 is not a pivot column, and so x_5 is an independent variable. Let $x_5 = e$. Substituting e for x_5 into the third equation yields $x_4 + 2e = 5$, or $x_4 = -2e + 5$. The column corresponding to the variable x_3 is not a pivot column, and so x_3 is another independent variable. Let $x_3 = c$. Substituting the expressions we have for x_3, x_4, and x_5 into the second equation produces $x_2 - c - 3(-2e + 5) - 7e = -2$. Solving for x_2 and simplifying gives $x_2 = c + e + 13$. Finally, we plug all these expressions into the first equation to get
$x_1 - 5(c + e + 13) + 2c + 3(-2e + 5) - 2e = -4$, which simplifies to $x_1 = 3c + 13e + 46$. Hence, the complete solution set is $\{(3c + 13e + 46, c + e + 13, c, -2e + 5, e) \mid c, e \in \mathbb{R}\}$.

(c) The system of linear equations corresponding to the given augmented matrix is

$$\begin{cases} x_1 + 4x_2 - 8x_3 - x_4 + 2x_5 - 3x_6 = -4 \\ \quad\quad x_2 - 7x_3 + 2x_4 - 9x_5 - x_6 = -3 \\ \quad\quad\quad\quad x_5 - 4x_6 = 2 \\ \quad\quad\quad\quad\quad\quad 0 = 0 \end{cases}.$$

The last equation, $0 = 0$, provides no information regarding the solution set for the system because it is true for every value of x_1, x_2, x_3, x_4, x_5, and x_6. So, we can ignore it. The column corresponding to the variable x_6 is not a pivot column, and so x_6 is an independent variable. Let $x_6 = f$. Substituting f for x_6 into the third equation yields $x_5 - 4f = 2$, or $x_5 = 4f + 2$. The columns corresponding to the variables x_3 and x_4 are not pivot columns, and so x_3 and x_4 are also independent variables. Let $x_3 = c$ and $x_4 = d$. Substituting the expressions we have for x_3, x_4, x_5, and x_6 into the second equation produces $x_2 - 7c + 2d - 9(4f + 2) - f = -3$. Solving for x_2 and simplifying gives $x_2 = 7c - 2d + 37f + 15$. Finally, we plug all these expressions into the first equation to get $x_1 + 4(7c - 2d + 37f + 15) - 8c - d + 2(4f + 2) - 3f = -4$, which simplifies to $x_1 = -20c + 9d - 153f - 68$. Hence, the complete solution set is
$\{(-20c + 9d - 153f - 68, 7c - 2d + 37f + 15, c, d, 4f + 2, f) \mid c, d, f \in \mathbb{R}\}$.

(3) Let x represent the number of nickels, y represent the number of dimes, and z represent the number of quarters. The fact that the total value of all of the coins is \$16.50 gives us the following equation: $0.05x + 0.10y + 0.25z = 16.50$, or $5x + 10y + 25z = 1650$.
Next, using the fact that there are twice as many dimes as quarters gives us $y = 2z$, or $y - 2z = 0$.
Finally, given that the total number of nickels and quarters is 20 more than the number of dimes produces $x + z = y + 20$, or $x - y + z = 20$. Putting these three equations into a single system yields

$$\begin{cases} x - y + z = 20 \\ \phantom{x - {}} y - 2z = 0 \\ 5x + 10y + 25z = 1650 \end{cases},$$

where we have arranged the equations in an order that will make the row operations easier. The augmented matrix for this system is

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 20 \\ 0 & 1 & -2 & 0 \\ 5 & 10 & 25 & 1650 \end{array}\right].$$

The (1,1) entry is the first pivot. Since this already equals 1, we continue by targeting nonzero entries in the first column. We get a zero in the (3,1) entry by performing a type (II) operation:
Type (II) operation: $\langle 3 \rangle \leftarrow (-5) \times \langle 1 \rangle + \langle 3 \rangle$:

Side Calculation

$$\begin{array}{r|rrr|r} (-5)\times \text{ (row 1)} & -5 & 5 & -5 & -100 \\ \text{(row 3)} & 5 & 10 & 25 & 1650 \\ \hline \text{(sum)} & 0 & 15 & 20 & 1550 \end{array}$$

Resultant Matrix

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 20 \\ 0 & 1 & -2 & 0 \\ 0 & 15 & 20 & 1550 \end{array}\right]$$

Next, the pivot moves to the (2,2) entry. Since this already equals 1, we continue with the second column by targeting the (3,2) entry.
Type (II) operation: $\langle 3 \rangle \leftarrow (-15) \times \langle 2 \rangle + \langle 3 \rangle$:

Side Calculation

$$\begin{array}{r|rrr|r} (-15)\times \text{ (row 2)} & 0 & -15 & 30 & 0 \\ \text{(row 3)} & 0 & 15 & 20 & 1550 \\ \hline \text{(sum)} & 0 & 0 & 50 & 1550 \end{array}$$

Resultant Matrix

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 20 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 50 & 1550 \end{array}\right]$$

The pivot now moves to the (3,3) entry. We must perform a type (I) operation to turn the pivot into a "1."

Type (I) operation: $\langle 3 \rangle \leftarrow \frac{1}{50} \langle 3 \rangle$: $\left[\begin{array}{rrr|r} 1 & -1 & 1 & 20 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 31 \end{array}\right]$

This ends the row operations. The final matrix corresponds to the linear system

$$\left\{\begin{array}{rcrcrcr} x & - & y & + & z & = & 20 \\ & & y & - & 2z & = & 0 \\ & & & & z & = & 31 \end{array}\right. .$$

The third equation in this system yields $z = 31$. Plugging this value into the second equation produces $y - 2(31) = 0$, or $y = 62$. Finally, we substitute these values into the first equation to get $x - 62 + 31 = 20$, or $x = 51$. Hence, there are 51 nickels, 62 dimes, and 31 quarters.

(4) First, plug each of the three given points into the equation $y = ax^2 + bx + c$. This yields the following linear system:

$$\left\{\begin{array}{rcrcrcrcl} 18 & = & 9a & + & 3b & + & c & \longleftarrow & \text{using } (3,18) \\ 9 & = & 4a & + & 2b & + & c & \longleftarrow & \text{using } (2,9) \\ 13 & = & 4a & - & 2b & + & c & \longleftarrow & \text{using } (-2,13) \end{array}\right. .$$

The augmented matrix for this system is $\left[\begin{array}{rrr|r} 9 & 3 & 1 & 18 \\ 4 & 2 & 1 & 9 \\ 4 & -2 & 1 & 13 \end{array}\right]$. The first pivot is the (1,1) entry. We make this a "1" by performing the following type (I) row operation:

Type (I) operation: $\langle 1\rangle \leftarrow \frac{1}{9}\langle 1\rangle$: $\left[\begin{array}{rrr|r} 1 & \frac{1}{3} & \frac{1}{9} & 2 \\ 4 & 2 & 1 & 9 \\ 4 & -2 & 1 & 13 \end{array}\right]$

We now target the (2,1) entry and make it "0." We do this using the following type (II) row operation:
Type (II) operation: $\langle 2\rangle \leftarrow (-4)\times\langle 1\rangle + \langle 2\rangle$:

Side Calculation

$$\begin{array}{rrrr|r} (-4)\times\ (\text{row } 1) & -4 & -\frac{4}{3} & -\frac{4}{9} & -8 \\ (\text{row } 2) & 4 & 2 & 1 & 9 \\ \hline (\text{sum}) & 0 & \frac{2}{3} & \frac{5}{9} & 1 \end{array}$$

Resultant Matrix

$$\left[\begin{array}{rrr|r} 1 & \frac{1}{3} & \frac{1}{9} & 2 \\ 0 & \frac{2}{3} & \frac{5}{9} & 1 \\ 4 & -2 & 1 & 13 \end{array}\right]$$

Next, we target the (3,1) entry.
Type (II) operation: $\langle 3\rangle \leftarrow (-4)\times\langle 1\rangle + \langle 3\rangle$:

Side Calculation

$$\begin{array}{rrrr|r} (-4)\times\ (\text{row } 1) & -4 & -\frac{4}{3} & -\frac{4}{9} & -8 \\ (\text{row } 3) & 4 & -2 & 1 & 13 \\ \hline (\text{sum}) & 0 & -\frac{10}{3} & \frac{5}{9} & 5 \end{array}$$

Resultant Matrix

$$\left[\begin{array}{rrr|r} 1 & \frac{1}{3} & \frac{1}{9} & 2 \\ 0 & \frac{2}{3} & \frac{5}{9} & 1 \\ 0 & -\frac{10}{3} & \frac{5}{9} & 5 \end{array}\right]$$

Now we move the pivot to the (2,2) entry. We make this entry a "1" using a type (I) row operation.

Type (I) operation: $\langle 2\rangle \leftarrow \frac{3}{2}\langle 2\rangle$: $\left[\begin{array}{rrr|r} 1 & \frac{1}{3} & \frac{1}{9} & 2 \\ 0 & 1 & \frac{5}{6} & \frac{3}{2} \\ 0 & -\frac{10}{3} & \frac{5}{9} & 5 \end{array}\right]$

Now we target the (3,2) entry.
Type (II) operation: $\langle 3\rangle \leftarrow (\frac{10}{3})\times\langle 2\rangle + \langle 3\rangle$:

Side Calculation					Resultant Matrix
$(\frac{10}{3})\times$ (row 2)	0	$\frac{10}{3}$	$\frac{25}{9}$	5	$\left[\begin{array}{ccc\|c} 1 & \frac{1}{3} & \frac{1}{9} & 2 \\ 0 & 1 & \frac{5}{6} & \frac{3}{2} \\ 0 & 0 & \frac{10}{3} & 10 \end{array}\right]$
(row 3)	0	$-\frac{10}{3}$	$\frac{5}{9}$	5	
(sum)	0	0	$\frac{10}{3}$	10	

Finally, we move the pivot to the (3,3) entry and perform the following row operation to turn that entry into a "1."

Type (I) operation: $\langle 3\rangle \leftarrow \frac{3}{10}\langle 3\rangle$: $\left[\begin{array}{ccc|c} 1 & \frac{1}{3} & \frac{1}{9} & 2 \\ 0 & 1 & \frac{5}{6} & \frac{3}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$

This matrix corresponds to the following linear system:

$$\begin{cases} a + \frac{1}{3}b + \frac{1}{9}c = 2 \\ \qquad b + \frac{5}{6}c = \frac{3}{2} \\ \qquad\qquad c = 3 \end{cases}.$$

The last equation clearly gives $c = 3$. Plugging $c = 3$ into the second equation yields $b + \frac{5}{6}(3) = \frac{3}{2}$, or $b = -1$. Substituting these values into the first equation produces $a + \frac{1}{3}(-1) + \frac{1}{9}(3) = 2$, or $a = 2$. Hence, the desired quadratic equation is $y = 2x^2 - x + 3$.

(6) First, plug each of the three given points into the equation $x^2 + y^2 + ax + by = c$. This yields the following linear system:

$$\begin{cases} (6^2+8^2) + 6a + 8b = c \longleftarrow \text{ using } (6,8) \\ (8^2+4^2) + 8a + 4b = c \longleftarrow \text{ using } (8,4) \\ (3^2+9^2) + 3a + 9b = c \longleftarrow \text{ using } (3,9) \end{cases}.$$

Rearranging terms leads to the following augmented matrix:

$$\left[\begin{array}{ccc|c} 6 & 8 & -1 & -100 \\ 8 & 4 & -1 & -80 \\ 3 & 9 & -1 & -90 \end{array}\right].$$

The first pivot is the (1,1) entry. We make this a "1" by performing the following type (I) row operation:

Type (I) operation: $\langle 1\rangle \leftarrow \frac{1}{6}\langle 1\rangle$: $\left[\begin{array}{ccc|c} 1 & \frac{4}{3} & -\frac{1}{6} & -\frac{50}{3} \\ 8 & 4 & -1 & -80 \\ 3 & 9 & -1 & -90 \end{array}\right]$

We now must target the (2,1) entry and make it "0." We do this using the following type (II) row operation:

Type (II) operation: $\langle 2\rangle \leftarrow (-8)\times\langle 1\rangle + \langle 2\rangle$:

Side Calculation					Resultant Matrix
$(-8)\times$ (row 1)	-8	$-\frac{32}{3}$	$\frac{4}{3}$	$\frac{400}{3}$	$\left[\begin{array}{ccc\|c} 1 & \frac{4}{3} & -\frac{1}{6} & -\frac{50}{3} \\ 0 & -\frac{20}{3} & \frac{1}{3} & \frac{160}{3} \\ 3 & 9 & -1 & -90 \end{array}\right]$
(row 2)	8	4	-1	-80	
(sum)	0	$-\frac{20}{3}$	$\frac{1}{3}$	$\frac{160}{3}$	

Next, we target the (3,1) entry.

Type (II) operation: $\langle 3\rangle \leftarrow (-3) \times \langle 1\rangle + \langle 3\rangle$:

$$\begin{array}{rrrr|r}
\multicolumn{5}{c}{\textbf{Side Calculation}} \\
(-3)\times\ (\text{row } 1) & -3 & -4 & \frac{1}{2} & 50 \\
(\text{row } 3) & 3 & 9 & -1 & -90 \\
\hline
(\text{sum}) & 0 & 5 & -\frac{1}{2} & -40
\end{array}
\qquad
\begin{array}{c}
\textbf{Resultant Matrix} \\
\left[\begin{array}{rrr|r}
1 & \frac{4}{3} & -\frac{1}{6} & -\frac{50}{3} \\
0 & -\frac{20}{3} & \frac{1}{3} & \frac{160}{3} \\
0 & 5 & -\frac{1}{2} & -40
\end{array}\right]
\end{array}$$

Now we move the pivot to the (2,2) entry. We make this entry a "1" using a type (I) row operation.

Type (I) operation: $\langle 2\rangle \leftarrow -\frac{3}{20}\,\langle 2\rangle$: $\left[\begin{array}{rrr|r} 1 & \frac{4}{3} & -\frac{1}{6} & -\frac{50}{3} \\ 0 & 1 & -\frac{1}{20} & -8 \\ 0 & 5 & -\frac{1}{2} & -40 \end{array}\right]$

Now we target the (3,2) entry.
Type (II) operation: $\langle 3\rangle \leftarrow (-5) \times \langle 2\rangle + \langle 3\rangle$:

$$\begin{array}{rrrr|r}
\multicolumn{5}{c}{\textbf{Side Calculation}} \\
(-5)\times\ (\text{row } 2) & 0 & -5 & \frac{1}{4} & 40 \\
(\text{row } 3) & 0 & 5 & -\frac{1}{2} & -40 \\
\hline
(\text{sum}) & 0 & 0 & -\frac{1}{4} & 0
\end{array}
\qquad
\begin{array}{c}
\textbf{Resultant Matrix} \\
\left[\begin{array}{rrr|r}
1 & \frac{4}{3} & -\frac{1}{6} & -\frac{50}{3} \\
0 & 1 & -\frac{1}{20} & -8 \\
0 & 0 & -\frac{1}{4} & 0
\end{array}\right]
\end{array}$$

Finally, we move the pivot to the (3,3) entry and perform the following row operation to turn that entry into a "1."

Type (I) operation: $\langle 3\rangle \leftarrow (-4)\,\langle 3\rangle$: $\left[\begin{array}{rrr|r} 1 & \frac{4}{3} & -\frac{1}{6} & -\frac{50}{3} \\ 0 & 1 & -\frac{1}{20} & -8 \\ 0 & 0 & 1 & 0 \end{array}\right]$

This matrix corresponds to the following linear system:

$$\left\{\begin{array}{rcrcrcr}
a & + & \frac{4}{3}b & - & \frac{1}{6}c & = & -\frac{50}{3} \\
 & & b & - & \frac{1}{20}c & = & -8 \\
 & & & & c & = & 0
\end{array}\right. .$$

The last equation clearly gives $c = 0$. Plugging $c = 0$ into the second equation yields $b - \frac{1}{20}(0) = -8$, or $b = -8$. Substituting these values into the first equation produces $a + \frac{4}{3}(-8) - \frac{1}{6}(0) = -\frac{50}{3}$, or $a = -6$. Hence, the desired equation of the circle is $x^2 + y^2 - 6x - 8y = 0$, or $(x-3)^2 + (y-4)^2 = 25$.

(7) (a) First, we compute $\mathbf{AB} = \begin{bmatrix} 26 & 15 & -6 \\ 6 & 4 & 1 \\ 18 & 6 & 15 \\ 10 & 4 & -14 \end{bmatrix}$. To find $R(\mathbf{AB})$ we perform the type (II) row operation $R : \langle 3\rangle \leftarrow (-3) \times \langle 2\rangle + \langle 3\rangle$:

$$\begin{array}{rrrr|}
\multicolumn{4}{c}{\textbf{Side Calculation}} \\
(-3)\times\ (\text{row } 2) & -18 & -12 & -3 \\
(\text{row } 3) & 18 & 6 & 15 \\
\hline
(\text{sum}) & 0 & -6 & 12
\end{array}
\qquad
\begin{array}{c}
\textbf{Resultant Matrix} \\
R(\mathbf{AB}) = \begin{bmatrix} 26 & 15 & -6 \\ 6 & 4 & 1 \\ 0 & -6 & 12 \\ 10 & 4 & -14 \end{bmatrix}
\end{array}$$

Next we compute $R(\mathbf{A})$. Using $\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 1 \\ -2 & 1 & 5 \\ 3 & 0 & 1 \end{bmatrix}$ and $R : \langle 3 \rangle \leftarrow (-3) \times \langle 2 \rangle + \langle 3 \rangle$:

Side Calculation				**Resultant Matrix**
$(-3)\times$ (row 2)	0	-3	-3	$R(\mathbf{A}) = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 1 \\ -2 & -2 & 2 \\ 3 & 0 & 1 \end{bmatrix}$
(row 3)	-2	1	5	
(sum)	-2	-2	2	

Multiplying $R(\mathbf{A})$ by the given matrix $\mathbf{B}$ produces the same result computed above for $R(\mathbf{AB})$.

(8) (a) To save space, we use the notation $\langle \mathbf{C} \rangle_i$ for the ith row of a matrix $\mathbf{C}$. Also, recall the following hint from the textbook, which we can rewrite as follows in the notation we have just defined: (Hint: Use the fact from Section 1.5 that $\langle \mathbf{AB} \rangle_k = \langle \mathbf{A} \rangle_k \mathbf{B}$.)

For the Type (I) operation $R : \langle i \rangle \longleftarrow c\langle i \rangle$: Now, $\langle R(\mathbf{AB})\rangle_i = c\langle \mathbf{AB}\rangle_i = c\langle \mathbf{A}\rangle_i \mathbf{B}$ (by the hint) $= \langle R(\mathbf{A})\rangle_i \mathbf{B} = \langle R(\mathbf{A})\mathbf{B}\rangle_i$. But, if $k \neq i$, $\langle R(\mathbf{AB})\rangle_k = \langle \mathbf{AB}\rangle_k = \langle \mathbf{A}\rangle_k \mathbf{B}$ (by the hint) $= \langle R(\mathbf{A})\rangle_k \mathbf{B}$ $= \langle R(\mathbf{A})\mathbf{B}\rangle_k$.

For the Type (II) operation $R : \langle i \rangle \longleftarrow c\langle j \rangle + \langle i \rangle$: Now, $\langle R(\mathbf{AB})\rangle_i = c\langle \mathbf{AB}\rangle_j + \langle \mathbf{AB}\rangle_i = c\langle \mathbf{A}\rangle_j \mathbf{B} + \langle \mathbf{A}\rangle_i \mathbf{B}$ (by the hint) $= (c\langle \mathbf{A}\rangle_j + \langle \mathbf{A}\rangle_i)\mathbf{B} = \langle R(\mathbf{A})\rangle_i \mathbf{B} = \langle R(\mathbf{A})\mathbf{B}\rangle_i$. But, if $k \neq i$, $\langle R(\mathbf{AB})\rangle_k = \langle \mathbf{AB}\rangle_k = \langle \mathbf{A}\rangle_k \mathbf{B}$ (by the hint) $= \langle R(\mathbf{A})\rangle_k \mathbf{B} = \langle R(\mathbf{A})\mathbf{B}\rangle_k$.

For the Type (III) operation $R : \langle i \rangle \longleftrightarrow \langle j \rangle$: Now, $\langle R(\mathbf{AB})\rangle_i = \langle \mathbf{AB}\rangle_j = \langle \mathbf{A}\rangle_j \mathbf{B}$ (by the hint) $= \langle R(\mathbf{A})\rangle_i \mathbf{B} = \langle R(\mathbf{A})\mathbf{B}\rangle_i$. Similarly, $\langle R(\mathbf{AB})\rangle_j = \langle \mathbf{AB}\rangle_i = \langle \mathbf{A}\rangle_i \mathbf{B}$ (by the hint) $= \langle R(\mathbf{A})\rangle_j \mathbf{B}$ $= \langle R(\mathbf{A})\mathbf{B}\rangle_j$. And, if $k \neq i$ and $k \neq j$, $\langle R(\mathbf{AB})\rangle_k = \langle \mathbf{AB}\rangle_k = \langle \mathbf{A}\rangle_k \mathbf{B}$ (by the hint) $= \langle R(\mathbf{A})\rangle_k \mathbf{B}$ $= \langle R(\mathbf{A})\mathbf{B}\rangle_k$.

(11) (a) True. The augmented matrix contains all of the information necessary to solve the system completely.

(b) False. A linear system has either no solutions, one solution, or infinitely many solutions. Having exactly three solutions is not one of the possibilities.

(c) False. A linear system is consistent if it has at least one solution. It could have either just one or infinitely many solutions.

(d) False. Type (I) operations are used to convert nonzero pivot entries to 1. When following the standard algorithm for Gaussian elimination, type (II) row operations are never used for this purpose. Type (II) row operations are used only to convert a nonzero target to a zero.

(e) True. In the standard algorithm for Gaussian elimination, this is the only instance in which a type (III) row operation is used.

(f) True. This is the statement of part (1) of Theorem 2.1 expressed in words rather than symbols.

Section 2.2

(1) The matrix in (e) is in reduced row echelon form, as it satisfies all four conditions. The matrices in (a), (b), (c), (d), and (f) are not in reduced row echelon form.
The matrix in (a) fails condition 2 of the definition, since the first nonzero entry in row 2 is in the third column, while the first nonzero entry in row 3 is only in the second column.

The matrix in (b) fails condition 4 of the definition, as the second row contains all zeroes, but later rows do not.
The matrix in (c) fails condition 1 of the definition because the first nonzero entry in the second row equals 2, not 1.
The matrix in (d) fails conditions 1, 2, and 3 of the definition. Condition 1 fails because the first nonzero entry in row 3 equals 2, not 1. Condition 2 fails because the first nonzero entries in rows 2 and 3 appear in the same column. Condition 3 fails because the (3,3) entry below the first nonzero entry of the second column is nonzero.
The matrix in (f) fails condition 3 of the definition since the entries in the fifth column above the 1 in the third row do not equal zero.

(2) (a) First, convert the pivot (the (1,1) entry) to "1."

$$\langle 1\rangle \leftarrow \left(\tfrac{1}{5}\right)\langle 1\rangle \qquad \left[\begin{array}{ccc|c} 1 & 4 & -\frac{18}{5} & -\frac{11}{5} \\ 3 & 12 & -14 & 3 \\ -4 & -16 & 13 & 13 \end{array}\right]$$

Next, target the (2,1) and (3,1) entries.

$$\begin{array}{l}\langle 2\rangle \leftarrow (-3)\times\langle 1\rangle + \langle 2\rangle \\ \langle 3\rangle \leftarrow (4)\times\langle 1\rangle + \langle 3\rangle\end{array} \qquad \left[\begin{array}{ccc|c} 1 & 4 & -\frac{18}{5} & -\frac{11}{5} \\ 0 & 0 & -\frac{16}{5} & \frac{48}{5} \\ 0 & 0 & -\frac{7}{5} & \frac{21}{5} \end{array}\right]$$

We try to move the pivot to the second column but cannot due to the zeroes in the (2,2) and (3,2) entries. Hence, the pivot moves to the (2,3) entry. We set that equal to 1.

$$\langle 2\rangle \leftarrow \left(-\tfrac{5}{16}\right)\langle 2\rangle \qquad \left[\begin{array}{ccc|c} 1 & 4 & -\frac{18}{5} & -\frac{11}{5} \\ 0 & 0 & 1 & -3 \\ 0 & 0 & -\frac{7}{5} & \frac{21}{5} \end{array}\right]$$

Finally, we target the (1,3) and (3,3) entries.

$$\begin{array}{l}\langle 1\rangle \leftarrow \left(\tfrac{18}{5}\right)\times\langle 2\rangle + \langle 1\rangle \\ \langle 3\rangle \leftarrow \left(\tfrac{7}{5}\right)\times\langle 2\rangle + \langle 3\rangle\end{array} \qquad \left[\begin{array}{ccc|c} 1 & 4 & 0 & -13 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is the desired reduced row echelon form matrix.

(b) First, convert the pivot (the (1,1) entry) to "1."

$$\langle 1\rangle \leftarrow \left(-\tfrac{1}{2}\right)\langle 1\rangle \qquad \left[\begin{array}{cccc} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{15}{2} \\ 6 & -1 & -2 & -36 \\ 1 & -1 & -1 & -11 \\ -5 & -5 & -5 & -14 \end{array}\right]$$

Next, target the (2,1), (3,1), and (4,1) entries.

$$\begin{array}{l}\langle 2\rangle \leftarrow (-6)\times\langle 1\rangle + \langle 2\rangle \\ \langle 3\rangle \leftarrow (-1)\times\langle 1\rangle + \langle 3\rangle \\ \langle 4\rangle \leftarrow (5)\times\langle 1\rangle + \langle 4\rangle\end{array} \qquad \left[\begin{array}{cccc} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{15}{2} \\ 0 & 2 & 1 & 9 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{7}{2} \\ 0 & -\frac{15}{2} & -\frac{15}{2} & -\frac{103}{2} \end{array}\right]$$

The pivot now moves to the (2,2) entry.

$$\langle 2\rangle \leftarrow \left(\tfrac{1}{2}\right)\langle 2\rangle \qquad \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{15}{2} \\ 0 & 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{7}{2} \\ 0 & -\frac{15}{2} & -\frac{15}{2} & -\frac{103}{2} \end{bmatrix}$$

Next, target the (1,2), (3,2), and (4,2) entries.

$$\begin{array}{l} \langle 1\rangle \leftarrow \left(\frac{1}{2}\right)\times\langle 2\rangle + \langle 1\rangle \\ \langle 3\rangle \leftarrow \left(\frac{1}{2}\right)\times\langle 2\rangle + \langle 3\rangle \\ \langle 4\rangle \leftarrow \left(\frac{15}{2}\right)\times\langle 2\rangle + \langle 4\rangle \end{array} \qquad \begin{bmatrix} 1 & 0 & -\frac{1}{4} & -\frac{21}{4} \\ 0 & 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & 0 & -\frac{1}{4} & -\frac{5}{4} \\ 0 & 0 & -\frac{15}{4} & -\frac{71}{4} \end{bmatrix}$$

The pivot now moves to the (3,3) entry.

$$\langle 3\rangle \leftarrow (-4)\langle 3\rangle \qquad \begin{bmatrix} 1 & 0 & -\frac{1}{4} & -\frac{21}{4} \\ 0 & 1 & \frac{1}{2} & \frac{9}{2} \\ 0 & 0 & 1 & 5 \\ 0 & 0 & -\frac{15}{4} & -\frac{71}{4} \end{bmatrix}$$

Next, target the (1,3), (2,3), and (4,3) entries.

$$\begin{array}{l} \langle 1\rangle \leftarrow \left(\frac{1}{4}\right)\times\langle 3\rangle + \langle 1\rangle \\ \langle 2\rangle \leftarrow \left(-\frac{1}{2}\right)\times\langle 3\rangle + \langle 2\rangle \\ \langle 4\rangle \leftarrow \left(\frac{15}{4}\right)\times\langle 3\rangle + \langle 4\rangle \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The pivot moves to the (4,4) entry. Since this already equals 1, we now need to target only the (1,4), (2,4), and (3,4) entries.

$$\begin{array}{l} \langle 1\rangle \leftarrow (4)\times\langle 4\rangle + \langle 1\rangle \\ \langle 2\rangle \leftarrow (-2)\times\langle 4\rangle + \langle 2\rangle \\ \langle 3\rangle \leftarrow (-5)\times\langle 4\rangle + \langle 3\rangle \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, $\mathbf{I}_4$ is the desired reduced row echelon form matrix.

(c) First, convert the pivot (the (1,1) entry) to "1."

$$\langle 1\rangle \leftarrow \left(-\tfrac{1}{5}\right)\langle 1\rangle \qquad \left[\begin{array}{cccc|c} 1 & -2 & \frac{19}{5} & \frac{17}{5} & -4 \\ -3 & 6 & -11 & -11 & 14 \\ -7 & 14 & -26 & -25 & 31 \\ 9 & -18 & 34 & 31 & -37 \end{array}\right]$$

Next, target the (2,1), (3,1), and (4,1) entries.

$$\begin{array}{l} \langle 2\rangle \leftarrow (3)\times\langle 1\rangle + \langle 2\rangle \\ \langle 3\rangle \leftarrow (7)\times\langle 1\rangle + \langle 3\rangle \\ \langle 4\rangle \leftarrow (-9)\times\langle 1\rangle + \langle 4\rangle \end{array} \qquad \left[\begin{array}{cccc|c} 1 & -2 & \frac{19}{5} & \frac{17}{5} & -4 \\ 0 & 0 & \frac{2}{5} & -\frac{4}{5} & 2 \\ 0 & 0 & \frac{3}{5} & -\frac{6}{5} & 3 \\ 0 & 0 & -\frac{1}{5} & \frac{2}{5} & -1 \end{array}\right]$$

We try to move the pivot to the second column but cannot due to the zeroes in the (2,2), (3,2), and (4,2) entries. Hence, the pivot moves to the (2,3) entry. We set that equal to 1.

$\langle 2\rangle \leftarrow (\frac{5}{2})\,\langle 2\rangle$ $\left[\begin{array}{cccc|c} 1 & -2 & \frac{19}{5} & \frac{17}{5} & -4 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & \frac{3}{5} & -\frac{6}{5} & 3 \\ 0 & 0 & -\frac{1}{5} & \frac{2}{5} & -1 \end{array}\right]$

Next, target the (1,3), (3,3), and (4,3) entries.

$\langle 1\rangle \leftarrow (-\frac{19}{5}) \times \langle 2\rangle + \langle 1\rangle$
$\langle 3\rangle \leftarrow (-\frac{3}{5}) \times \langle 2\rangle + \langle 3\rangle$
$\langle 4\rangle \leftarrow (\frac{1}{5}) \times \langle 2\rangle + \langle 4\rangle$ $\left[\begin{array}{cccc|c} 1 & -2 & 0 & 11 & -23 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

This is the desired reduced row echelon form matrix.

(e) First, convert the pivot (the (1,1) entry) to "1."

$\langle 1\rangle \leftarrow (-\frac{1}{3})\,\langle 1\rangle$ $\left[\begin{array}{ccccc|c} 1 & -2 & \frac{1}{3} & \frac{5}{3} & 0 & \frac{5}{3} \\ -1 & 2 & 3 & -5 & 10 & 5 \end{array}\right]$

Next, target the (2,1) entry.

$\langle 2\rangle \leftarrow (1) \times \langle 1\rangle + \langle 2\rangle$ $\left[\begin{array}{ccccc|c} 1 & -2 & \frac{1}{3} & \frac{5}{3} & 0 & \frac{5}{3} \\ 0 & 0 & \frac{10}{3} & -\frac{10}{3} & 10 & \frac{20}{3} \end{array}\right]$

We try to move the pivot to the (2,2) entry, but it equals zero. There are no rows below it to provide a nonzero pivot to be swapped up, and so the pivot moves to the (2,3) entry. We set that equal to 1.

$\langle 2\rangle \leftarrow (\frac{3}{10})\,\langle 2\rangle$ $\left[\begin{array}{ccccc|c} 1 & -2 & \frac{1}{3} & \frac{5}{3} & 0 & \frac{5}{3} \\ 0 & 0 & 1 & -1 & 3 & 2 \end{array}\right]$

Finally, we target the (1,3) entry to obtain the desired reduced row echelon form matrix.

$\langle 1\rangle \leftarrow (-\frac{1}{3}) \times \langle 2\rangle + \langle 1\rangle$ $\left[\begin{array}{ccccc|c} 1 & -2 & 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & -1 & 3 & 2 \end{array}\right].$

(3) (a) The final matrix obtained in Exercise 1(a) in Section 2.1 is

$$\left[\begin{array}{ccc|c} 1 & \frac{2}{5} & -\frac{2}{5} & -\frac{14}{5} \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right].$$

We target the entries above the pivots to obtain the reduced row echelon form matrix. First, we use the (2,2) entry as the pivot and target the (1,2) entry.

$\langle 1\rangle \leftarrow (-\frac{2}{5}) \times \langle 2\rangle + \langle 1\rangle$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right]$

Next, we pivot at the (3,3) entry and target the (2,3) entry. (Note that the (1,3) entry is already zero.)

$\langle 2\rangle \leftarrow (1) \times \langle 3\rangle + \langle 2\rangle$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$

This matrix corresponds to the linear system

$$\begin{cases} x_1 & & & = & -2 \\ & x_2 & & = & 3 \\ & & x_3 & = & 5 \end{cases}.$$

Hence, the unique solution to the system is $(-2, 3, 5)$.

(e) The final matrix obtained in Exercise 1(e) in Section 2.1 is

$$\left[\begin{array}{ccccc|c} 1 & -2 & -\frac{5}{6} & \frac{8}{3} & -\frac{1}{3} & -\frac{53}{6} \\ 0 & 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right].$$

We target the entries above the pivots to obtain the reduced row echelon form matrix. First, we use the (2,3) entry as the pivot and target the (1,3) entry.

$$\langle 1\rangle \leftarrow (\tfrac{5}{6}) \times \langle 2\rangle + \langle 1\rangle \qquad \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 1 & -\frac{1}{3} & -\frac{14}{3} \\ 0 & 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

Next, we pivot at the (3,5) entry and target the (1,5) entry. (Note that the (2,5) entry is already zero.)

$$\langle 1\rangle \leftarrow (\tfrac{1}{3}) \times \langle 3\rangle + \langle 1\rangle \qquad \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -2 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

This matrix corresponds to the linear system

$$\begin{cases} x_1 - 2x_2 + x_4 = -4 \\ x_3 - 2x_4 = 5 \\ x_5 = 2 \end{cases}.$$

Columns 2 and 4 are not pivot columns, so x_2 and x_4 are independent variables. Let $x_2 = b$ and $x_4 = d$. Then the first equation gives $x_1 - 2b + d = -4$, or $x_1 = 2b - d - 4$. The second equation yields $x_3 - 2d = 5$, or $x_3 = 2d + 5$. The last equation states that $x_5 = 2$. Hence, the complete solution set is $\{(2b - d - 4, b, 2d + 5, d, 2) \,|\, b, d \in \mathbb{R}\}$.

(g) The final matrix obtained in Exercise 1(g) in Section 2.1 is

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{7}{4} & \frac{5}{4} \\ 0 & 1 & -\frac{1}{4} & -\frac{7}{4} \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We target the entries above the pivots to obtain the reduced row echelon form matrix. First, we use the (2,2) entry as the pivot and target the (1,2) entry.

$$\langle 1\rangle \leftarrow (\tfrac{1}{2}) \times \langle 2\rangle + \langle 1\rangle \qquad \left[\begin{array}{ccc|c} 1 & 0 & -\frac{15}{8} & \frac{3}{8} \\ 0 & 1 & -\frac{1}{4} & -\frac{7}{4} \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Next, we use the (3,3) entry as a pivot and target the (1,3) and (2,3) entries.

$$\begin{array}{l}\langle 1\rangle \leftarrow (\frac{15}{8}) \times \langle 3\rangle + \langle 1\rangle \\ \langle 2\rangle \leftarrow (\frac{1}{4}) \times \langle 3\rangle + \langle 2\rangle\end{array} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Ignoring the last row, which gives the equation $0 = 0$, this matrix yields the linear system

$$\left\{\begin{array}{rcrcrcr} x_1 & & & & & = & 6 \\ & & x_2 & & & = & -1 \\ & & & & x_3 & = & 3 \end{array}\right. ,$$

producing the unique solution $(6, -1, 3)$.

(4) (a) The augmented matrix for the system is $\left[\begin{array}{rrrr|r} -2 & -3 & 2 & -13 & 0 \\ -4 & -7 & 4 & -29 & 0 \\ 1 & 2 & -1 & 8 & 0 \end{array}\right]$.

The (1,1) entry is the first pivot. We convert it to 1.

$$\langle 1\rangle \leftarrow (-\tfrac{1}{2})\,\langle 1\rangle \qquad \left[\begin{array}{rrrr|r} 1 & \frac{3}{2} & -1 & \frac{13}{2} & 0 \\ -4 & -7 & 4 & -29 & 0 \\ 1 & 2 & -1 & 8 & 0 \end{array}\right]$$

Next, we target the (2,1) and (3,1) entries.

$$\begin{array}{l}\langle 2\rangle \leftarrow (4) \times \langle 1\rangle + \langle 2\rangle \\ \langle 3\rangle \leftarrow (-1) \times \langle 1\rangle + \langle 3\rangle\end{array} \qquad \left[\begin{array}{rrrr|r} 1 & \frac{3}{2} & -1 & \frac{13}{2} & 0 \\ 0 & -1 & 0 & -3 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 \end{array}\right]$$

The (2,2) entry is the second pivot. We convert it to 1.

$$\langle 2\rangle \leftarrow (-1)\,\langle 2\rangle \qquad \left[\begin{array}{rrrr|r} 1 & \frac{3}{2} & -1 & \frac{13}{2} & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 \end{array}\right]$$

Next, we target the (1,2) and (3,2) entries.

$$\begin{array}{l}\langle 1\rangle \leftarrow (-\frac{3}{2}) \times \langle 2\rangle + \langle 1\rangle \\ \langle 3\rangle \leftarrow (-\frac{1}{2}) \times \langle 2\rangle + \langle 3\rangle\end{array} \qquad \left[\begin{array}{rrrr|r} 1 & 0 & -1 & 2 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is in reduced row echelon form. It corresponds to the linear system

$$\left\{\begin{array}{rcrcrcrcr} x_1 & & & - & x_3 & + & 2x_4 & = & 0 \\ & & x_2 & & & + & 3x_4 & = & 0 \\ & & & & & & 0 & = & 0 \end{array}\right. .$$

We can ignore the last equation, since it provides us with no information regarding the values of the variables. Columns 3 and 4 of the matrix are nonpivot columns, and so x_3 and x_4 are independent variables. Let $x_3 = c$ and $x_4 = d$. Substituting these into the first equation yields $x_1 - c + 2d = 0$, or $x_1 = c - 2d$. The second equation gives us $x_2 + 3d = 0$, or $x_2 = -3d$. Therefore. the complete solution set is $\{(c - 2d, -3d, c, d) \,|\, c, d \in \mathbb{R}\}$. Letting $c = 1$ and $d = 2$ produces the particular nontrivial solution $(-3, -6, 1, 2)$.

(c) The augmented matrix for the system is

$$\left[\begin{array}{rrrrrr|r} 7 & 28 & 4 & -2 & 10 & 19 & 0 \\ -9 & -36 & -5 & 3 & -15 & -29 & 0 \\ 3 & 12 & 2 & 0 & 6 & 11 & 0 \\ 6 & 24 & 3 & -3 & 10 & 20 & 0 \end{array}\right].$$

The (1,1) entry is the first pivot. We convert it to 1.

$$\langle 1\rangle \leftarrow \left(\tfrac{1}{7}\right)\langle 1\rangle \qquad \left[\begin{array}{rrrrrr|r} 1 & 4 & \frac{4}{7} & -\frac{2}{7} & \frac{10}{7} & \frac{19}{7} & 0 \\ -9 & -36 & -5 & 3 & -15 & -29 & 0 \\ 3 & 12 & 2 & 0 & 6 & 11 & 0 \\ 6 & 24 & 3 & -3 & 10 & 20 & 0 \end{array}\right]$$

Next, we target the (2,1), (3,1), and (4,1) entries.

$$\begin{array}{l} \langle 2\rangle \leftarrow (9)\times\langle 1\rangle + \langle 2\rangle \\ \langle 3\rangle \leftarrow (-3)\times\langle 1\rangle + \langle 3\rangle \\ \langle 4\rangle \leftarrow (-6)\times\langle 1\rangle + \langle 4\rangle \end{array} \qquad \left[\begin{array}{rrrrrr|r} 1 & 4 & \frac{4}{7} & -\frac{2}{7} & \frac{10}{7} & \frac{19}{7} & 0 \\ 0 & 0 & \frac{1}{7} & \frac{3}{7} & -\frac{15}{7} & -\frac{32}{7} & 0 \\ 0 & 0 & \frac{2}{7} & \frac{6}{7} & \frac{12}{7} & \frac{20}{7} & 0 \\ 0 & 0 & -\frac{3}{7} & -\frac{9}{7} & \frac{10}{7} & \frac{26}{7} & 0 \end{array}\right]$$

We try to move the pivot to the second column but cannot due to the zeroes in the (2,2), (3,2), and (4,2) entries. Hence, the pivot moves to the (2,3) entry. We convert that to 1.

$$\langle 2\rangle \leftarrow (7)\,\langle 2\rangle \qquad \left[\begin{array}{rrrrrr|r} 1 & 4 & \frac{4}{7} & -\frac{2}{7} & \frac{10}{7} & \frac{19}{7} & 0 \\ 0 & 0 & 1 & 3 & -15 & -32 & 0 \\ 0 & 0 & \frac{2}{7} & \frac{6}{7} & \frac{12}{7} & \frac{20}{7} & 0 \\ 0 & 0 & -\frac{3}{7} & -\frac{9}{7} & \frac{10}{7} & \frac{26}{7} & 0 \end{array}\right]$$

Next, we target the (1,3), (3,3), and (4,3) entries.

$$\begin{array}{l} \langle 1\rangle \leftarrow (-\frac{4}{7})\times\langle 2\rangle + \langle 1\rangle \\ \langle 3\rangle \leftarrow (-\frac{2}{7})\times\langle 2\rangle + \langle 3\rangle \\ \langle 4\rangle \leftarrow (\frac{3}{7})\times\langle 2\rangle + \langle 4\rangle \end{array} \qquad \left[\begin{array}{rrrrrr|r} 1 & 4 & 0 & -2 & 10 & 21 & 0 \\ 0 & 0 & 1 & 3 & -15 & -32 & 0 \\ 0 & 0 & 0 & 0 & 6 & 12 & 0 \\ 0 & 0 & 0 & 0 & -5 & -10 & 0 \end{array}\right]$$

We try to move the pivot to the third column but cannot due to the zeroes in the (3,4) and (4,4) entries. Hence, the pivot moves to the (3,5) entry. We convert that to 1.

$$\langle 3\rangle \leftarrow \left(\tfrac{1}{6}\right)\langle 3\rangle \qquad \left[\begin{array}{rrrrrr|r} 1 & 4 & 0 & -2 & 10 & 21 & 0 \\ 0 & 0 & 1 & 3 & -15 & -32 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -5 & -10 & 0 \end{array}\right]$$

Next, we target the (1,5), (2,5), and (4,5) entries.

$$\begin{array}{l} \langle 1\rangle \leftarrow (-10)\times\langle 3\rangle + \langle 1\rangle \\ \langle 2\rangle \leftarrow (15)\times\langle 3\rangle + \langle 2\rangle \\ \langle 4\rangle \leftarrow (5)\times\langle 3\rangle + \langle 4\rangle \end{array} \qquad \left[\begin{array}{rrrrrr|r} 1 & 4 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is in reduced row echelon form. It corresponds to the linear system

$$\begin{cases} x_1 + 4x_2 \qquad - 2x_4 \qquad + x_6 = 0 \\ \qquad\qquad x_3 + 3x_4 \qquad - 2x_6 = 0 \\ \qquad\qquad\qquad\qquad x_5 + 2x_6 = 0 \\ \qquad\qquad\qquad\qquad\qquad 0 = 0 \end{cases}.$$

We can ignore the last equation, since it provides us with no information regarding the values of the variables. Columns 2, 4, and 6 of the matrix are nonpivot columns, and so x_2, x_4, and x_6 are independent variables. Let $x_2 = b$, $x_4 = d$, and $x_6 = f$. Substituting these into the first equation yields $x_1+4b-2d+f = 0$, or $x_1 = -4b+2d-f$. The second equation gives us $x_3+3d-2f = 0$, or $x_3 = -3d+2f$. The third equation produces $x_5 + 2f = 0$, or $x_5 = -2f$. Therefore, the complete solution set is $\{(-4b+2d-f, b, -3d+2f, d, -2f, f) \,|\, b, d, f \in \mathbb{R}\}$. Letting $b = 1$, $d = 2$, and $f = 3$ produces the particular nontrivial solution $(-3, 1, 0, 2, -6, 3)$.

(5) (a) The augmented matrix for the system is $\left[\begin{array}{rrr|r} -2 & 1 & 8 & 0 \\ 7 & -2 & -22 & 0 \\ 3 & -1 & -10 & 0 \end{array}\right]$.

The (1,1) entry is the first pivot. We convert it to 1.

$$\langle 1\rangle \leftarrow \left(-\tfrac{1}{2}\right)\langle 1\rangle \qquad \left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & -4 & 0 \\ 7 & -2 & -22 & 0 \\ 3 & -1 & -10 & 0 \end{array}\right]$$

Next, we target the (2,1) and (3,1) entries.

$$\begin{array}{l} \langle 2\rangle \leftarrow (-7)\times\langle 1\rangle + \langle 2\rangle \\ \langle 3\rangle \leftarrow (-3)\times\langle 1\rangle + \langle 3\rangle \end{array} \qquad \left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & -4 & 0 \\ 0 & \frac{3}{2} & 6 & 0 \\ 0 & \frac{1}{2} & 2 & 0 \end{array}\right]$$

The (2,2) entry is the second pivot. We convert it to 1.

$$\langle 2\rangle \leftarrow \left(\tfrac{2}{3}\right)\langle 2\rangle \qquad \left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & -4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & \frac{1}{2} & 2 & 0 \end{array}\right]$$

Next, we target the (1,2) and (3,2) entries.

$$\begin{array}{l} \langle 1\rangle \leftarrow \left(\tfrac{1}{2}\right)\times\langle 2\rangle + \langle 1\rangle \\ \langle 3\rangle \leftarrow \left(-\tfrac{1}{2}\right)\times\langle 2\rangle + \langle 3\rangle \end{array} \qquad \left[\begin{array}{rrr|r} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is in reduced row echelon form. It corresponds to the linear system

$$\begin{cases} x_1 \qquad - 2x_3 = 0 \\ \qquad x_2 + 4x_3 = 0 \\ \qquad\qquad 0 = 0 \end{cases}.$$

We can ignore the last equation, since it provides us with no information regarding the values of the variables. Column 3 of the matrix is a nonpivot column, and so x_3 is an independent variable. Let $x_3 = c$. Substituting this into the first equation yields $x_1 - 2c = 0$, or $x_1 = 2c$. The second equation gives us $x_2 + 4c = 0$, or $x_2 = -4c$. Therefore, the complete solution set is $\{(2c, -4c, c) \,|\, c \in \mathbb{R}\} = \{c(2, -4, 1) \,|\, c \in \mathbb{R}\}$.

(c) The augmented matrix for the system is $\left[\begin{array}{cccc|c} 2 & 6 & 13 & 1 & 0 \\ 1 & 4 & 10 & 1 & 0 \\ 2 & 8 & 20 & 1 & 0 \\ 3 & 10 & 21 & 2 & 0 \end{array}\right]$.

The (1,1) entry is the first pivot. We convert it to 1.

$\langle 1\rangle \leftarrow \left(\frac{1}{2}\right)\langle 1\rangle \qquad \left[\begin{array}{cccc|c} 1 & 3 & \frac{13}{2} & \frac{1}{2} & 0 \\ 1 & 4 & 10 & 1 & 0 \\ 2 & 8 & 20 & 1 & 0 \\ 3 & 10 & 21 & 2 & 0 \end{array}\right]$

Next, we target the (2,1), (3,1), and (4,1) entries.

$\begin{array}{l} \langle 2\rangle \leftarrow (-1)\times\langle 1\rangle + \langle 2\rangle \\ \langle 3\rangle \leftarrow (-2)\times\langle 1\rangle + \langle 3\rangle \\ \langle 4\rangle \leftarrow (-3)\times\langle 1\rangle + \langle 4\rangle \end{array} \qquad \left[\begin{array}{cccc|c} 1 & 3 & \frac{13}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{7}{2} & \frac{1}{2} & 0 \\ 0 & 2 & 7 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} & 0 \end{array}\right]$

The (2,2) entry is the second pivot. It already equals 1, so now we target the (1,2), (3,2), and (4,2) entries.

$\begin{array}{l} \langle 1\rangle \leftarrow (-3)\times\langle 2\rangle + \langle 1\rangle \\ \langle 3\rangle \leftarrow (-2)\times\langle 2\rangle + \langle 3\rangle \\ \langle 4\rangle \leftarrow (-1)\times\langle 2\rangle + \langle 4\rangle \end{array} \qquad \left[\begin{array}{cccc|c} 1 & 0 & -4 & -1 & 0 \\ 0 & 1 & \frac{7}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -2 & 0 & 0 \end{array}\right]$

The (3,3) entry equals 0. However, we can switch the 3rd and 4th rows to bring a nonzero number into the (3,3) position.

$\langle 3\rangle \longleftrightarrow \langle 4\rangle \qquad \left[\begin{array}{cccc|c} 1 & 0 & -4 & -1 & 0 \\ 0 & 1 & \frac{7}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$

We now convert the pivot (the (3,3) entry) to 1.

$\langle 3\rangle \leftarrow \left(-\frac{1}{2}\right)\langle 3\rangle \qquad \left[\begin{array}{cccc|c} 1 & 0 & -4 & -1 & 0 \\ 0 & 1 & \frac{7}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$

Next, we target the (1,3) and (2,3) entries. The (4,3) entry already equals 0.

$\begin{array}{l} \langle 1\rangle \leftarrow (4)\times\langle 3\rangle + \langle 1\rangle \\ \langle 2\rangle \leftarrow \left(-\frac{7}{2}\right)\times\langle 3\rangle + \langle 2\rangle \end{array} \qquad \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right]$

The (4,4) entry is the last pivot. We convert it to 1.

$\langle 4\rangle \leftarrow (-1)\,\langle 4\rangle \qquad \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

Finally, we target the (1,4) and (2,4) entries. The (3,4) entry is already 0.

$\begin{array}{l} \langle 1\rangle \leftarrow (1)\times\langle 4\rangle + \langle 1\rangle \\ \langle 2\rangle \leftarrow (-\frac{1}{2})\times\langle 4\rangle + \langle 2\rangle \end{array} \qquad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

This matrix is in reduced row echelon form. It corresponds to the linear system

$$\left\{\begin{array}{ccccl} x_1 & & & & = 0 \\ & x_2 & & & = 0 \\ & & x_3 & & = 0 \\ & & & x_4 & = 0 \end{array}\right. .$$

Clearly, this system has only the trivial solution. The solution set is $\{(0,0,0,0)\}$.

(6) (a) First, we find the system of linear equations corresponding to the given chemical equation by considering each element separately. This produces

$$\left\{\begin{array}{ccccccl} 6a & & = & c & & & \leftarrow \text{Carbon equation} \\ 6a & & = & & & 2d & \leftarrow \text{Hydrogen equation} \\ & 2b & = & 2c & + & d & \leftarrow \text{Oxygen equation} \end{array}\right. .$$

Moving all variables to the left side and then creating an augmented matrix yields

$$\left[\begin{array}{cccc|c} 6 & 0 & -1 & 0 & 0 \\ 6 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right].$$

The (1,1) entry is the first pivot. We convert it to 1.

$\langle 1\rangle \leftarrow (\frac{1}{6})\,\langle 1\rangle \qquad \left[\begin{array}{cccc|c} 1 & 0 & -\frac{1}{6} & 0 & 0 \\ 6 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right]$

Next, we target the (2,1) entry. (The (3,1) entry is already 0.)

$\langle 2\rangle \leftarrow (-6)\times\langle 1\rangle + \langle 2\rangle \qquad \left[\begin{array}{cccc|c} 1 & 0 & -\frac{1}{6} & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right]$

The pivot moves to the (2,2) entry, which is 0. So, we switch the 2nd and 3rd rows to move a nonzero number into the pivot position.

$\langle 2\rangle \longleftrightarrow \langle 3\rangle \qquad \left[\begin{array}{cccc|c} 1 & 0 & -\frac{1}{6} & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right]$

Now we convert the (2,2) entry to 1.

$\langle 2\rangle \leftarrow \left(\frac{1}{2}\right)\langle 2\rangle$ $\quad\left[\begin{array}{cccc|c} 1 & 0 & -\frac{1}{6} & 0 & 0 \\ 0 & 1 & -1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right]$

The pivot moves to the (3,3) entry. Since this already equals 1, we target the (1,3) and (2,3) entries.

$\langle 1\rangle \leftarrow \left(\frac{1}{6}\right)\times\langle 3\rangle + \langle 1\rangle$
$\langle 2\rangle \leftarrow (1)\times\langle 3\rangle + \langle 2\rangle$ $\quad\left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{5}{2} & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right]$

The matrix is now in reduced row echelon form. It corresponds to the following linear system:

$$\begin{cases} a \quad\quad\quad - \frac{1}{3}d = 0 \\ \quad b \quad\quad - \frac{5}{2}d = 0 \\ \quad\quad c - 2d = 0 \end{cases},$$

or $a = \frac{1}{3}d$, $b = \frac{5}{2}d$, and $c = 2d$. Setting $d = 6$ eliminates all fractions and provides the smallest solution containing all positive integers. Hence, the desired solution is $a = 2$, $b = 15$, $c = 12$, $d = 6$.

(c) First, we find the system of linear equations corresponding to the given chemical equation by considering each element separately. This produces

$$\begin{cases} a = c & \leftarrow \text{Silver equation} \\ a = e & \leftarrow \text{Nitrogen equation} \\ 3a + b = 2d + 3e & \leftarrow \text{Oxygen equation} \\ 2b = e & \leftarrow \text{Hydrogen equation} \end{cases},$$

Moving all variables to the left side and then creating an augmented matrix yields

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 3 & 1 & 0 & -2 & -3 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 \end{array}\right].$$

The (1,1) entry is the first pivot. Since it already equals 1, we target the (2,1) and (3,1) entries. (The (4,1) entry is already 0.)

$\langle 2\rangle \leftarrow (-1)\times\langle 1\rangle + \langle 2\rangle$
$\langle 3\rangle \leftarrow (-3)\times\langle 1\rangle + \langle 3\rangle$ $\quad\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 3 & -2 & -3 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 \end{array}\right]$

The pivot moves to the (2,2) entry. However, it equals 0. So, we switch the 2nd and 3rd rows to move a nonzero number into the pivot position.

$\langle 2\rangle \longleftrightarrow \langle 3\rangle$ $\quad\left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 & -3 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 \end{array}\right]$

The (2,2) entry already equals 1, so we target the (4,2) entry. (The (1,2) and (3,2) entries already equal 0, and so do not need to be targeted.)

$\langle 4\rangle \leftarrow (-2) \times \langle 2\rangle + \langle 4\rangle$ $\quad \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 & -3 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -6 & 4 & 5 & 0 \end{array}\right]$

The pivot moves to the (3,3) entry. Since this already equals 1, we target the (1,3), (2,3), and (4,3) entries.

$\langle 1\rangle \leftarrow (1) \times \langle 3\rangle + \langle 1\rangle$
$\langle 2\rangle \leftarrow (-3) \times \langle 3\rangle + \langle 2\rangle$
$\langle 4\rangle \leftarrow (6) \times \langle 3\rangle + \langle 4\rangle$ $\quad \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 & -1 & 0 \end{array}\right]$

The pivot moves to the (4,4) entry. We convert this to 1.

$\langle 4\rangle \leftarrow (\frac{1}{4}) \langle 4\rangle$ $\quad \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{4} & 0 \end{array}\right]$

Finally, we target the (2,4) entry.

$\langle 2\rangle \leftarrow (2) \times \langle 4\rangle + \langle 2\rangle$ $\quad \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{4} & 0 \end{array}\right]$

The matrix is now in reduced row echelon form. It corresponds to the following linear system:

$$\begin{cases} a & & & & - & e & = & 0 \\ & b & & & - & \frac{1}{2}e & = & 0 \\ & & c & & - & e & = & 0 \\ & & & d & - & \frac{1}{4}e & = & 0 \end{cases},$$

or $a = e$, $b = \frac{1}{2}e$, $c = e$, and $d = \frac{1}{4}e$. Setting $e = 4$ eliminates all fractions and provides the smallest solution containing all positive integers. Hence, the desired solution is $a = 4$, $b = 2$, $c = 4$, $d = 1$, $e = 4$.

(7) (a) Combining the fractions on the right side of the given equation over a common denominator produces

$$\frac{A}{(x-1)} + \frac{B}{(x-3)} + \frac{C}{(x+4)} = \frac{A(x-3)(x+4) + B(x-1)(x+4) + C(x-1)(x-3)}{(x-1)(x-3)(x+4)}$$
$$= \frac{(A+B+C)\,x^2 + (A+3B-4C)\,x + (-12A-4B+3C)}{(x-1)(x-3)(x+4)}.$$

Now we set the coefficients of x^2 and x and the constant term equal to the corresponding coefficients in the numerator of the left side of the equation given in the problem. Doing this yields the following linear system:

$$\begin{cases} A & + & B & + & C & = & 5 & \leftarrow \text{coefficients of } x^2 \\ A & + & 3B & - & 4C & = & 23 & \leftarrow \text{coefficients of } x \\ -12A & - & 4B & + & 3C & = & -58 & \leftarrow \text{constant terms} \end{cases}.$$

The augmented matrix for the system is $\left[\begin{array}{rrr|r} 1 & 1 & 1 & 5 \\ 1 & 3 & -4 & 23 \\ -12 & -4 & 3 & -58 \end{array}\right]$.

The (1,1) entry is the first pivot. Since it already equals 1, we target the (2,1) and (3,1) entries.

$\langle 2\rangle \leftarrow (-1) \times \langle 1\rangle + \langle 2\rangle$
$\langle 3\rangle \leftarrow (12) \times \langle 1\rangle + \langle 3\rangle$
$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 5 \\ 0 & 2 & -5 & 18 \\ 0 & 8 & 15 & 2 \end{array}\right]$

The (2,2) entry is the second pivot. We convert it to 1.

$\langle 2\rangle \leftarrow (\frac{1}{2})\,\langle 2\rangle$
$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 5 \\ 0 & 1 & -\frac{5}{2} & 9 \\ 0 & 8 & 15 & 2 \end{array}\right]$

Next, we target the (1,2) and (3,2) entries.

$\langle 1\rangle \leftarrow (-1) \times \langle 2\rangle + \langle 1\rangle$
$\langle 3\rangle \leftarrow (-8) \times \langle 2\rangle + \langle 3\rangle$
$\left[\begin{array}{rrr|r} 1 & 0 & \frac{7}{2} & -4 \\ 0 & 1 & -\frac{5}{2} & 9 \\ 0 & 0 & 35 & -70 \end{array}\right]$

The (3,3) entry is the last pivot. We convert it to 1.

$\langle 3\rangle \leftarrow (\frac{1}{35})\,\langle 3\rangle$
$\left[\begin{array}{rrr|r} 1 & 0 & \frac{7}{2} & -4 \\ 0 & 1 & -\frac{5}{2} & 9 \\ 0 & 0 & 1 & -2 \end{array}\right]$

Finally, we target the (1,3) and (2,3) entries.

$\langle 1\rangle \leftarrow (-\frac{7}{2}) \times \langle 3\rangle + \langle 1\rangle$
$\langle 2\rangle \leftarrow (\frac{5}{2}) \times \langle 3\rangle + \langle 2\rangle$
$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$

This matrix is in reduced row echelon form. It corresponds to the linear system

$$\left\{\begin{array}{rcrcrcr} A & & & & & = & 3 \\ & & B & & & = & 4 \\ & & & & C & = & -2 \end{array}\right.,$$

which gives the unique solution to the problem.

(8) We set up the augmented matrix having two columns to the right of the augmentation bar:

$$\left[\begin{array}{rrr|rr} 9 & 2 & 2 & -6 & -12 \\ 3 & 2 & 4 & 0 & -3 \\ 27 & 12 & 22 & 12 & 8 \end{array}\right].$$

The (1,1) entry is the first pivot. We convert it to 1.

$\langle 1\rangle \leftarrow (\frac{1}{9})\,\langle 1\rangle$
$\left[\begin{array}{rrr|rr} 1 & \frac{2}{9} & \frac{2}{9} & -\frac{2}{3} & -\frac{4}{3} \\ 3 & 2 & 4 & 0 & -3 \\ 27 & 12 & 22 & 12 & 8 \end{array}\right]$

We next target the (2,1) and (3,1) entries.

$\langle 2\rangle \leftarrow (-3)\times\langle 1\rangle + \langle 2\rangle$
$\langle 3\rangle \leftarrow (-27)\times\langle 1\rangle + \langle 3\rangle$
$$\left[\begin{array}{ccc|cc} 1 & \frac{2}{9} & \frac{2}{9} & -\frac{2}{3} & -\frac{4}{3} \\ 0 & \frac{4}{3} & \frac{10}{3} & 2 & 1 \\ 0 & 6 & 16 & 30 & 44 \end{array}\right]$$

The (2,2) entry is the second pivot. We convert it to 1.

$\langle 2\rangle \leftarrow (\frac{3}{4})\,\langle 2\rangle$
$$\left[\begin{array}{ccc|cc} 1 & \frac{2}{9} & \frac{2}{9} & -\frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & \frac{5}{2} & \frac{3}{2} & \frac{3}{4} \\ 0 & 6 & 16 & 30 & 44 \end{array}\right]$$

Next, we target the (1,2) and (3,2) entries.

$\langle 1\rangle \leftarrow (-\frac{2}{9})\times\langle 2\rangle + \langle 1\rangle$
$\langle 3\rangle \leftarrow (-6)\times\langle 2\rangle + \langle 3\rangle$
$$\left[\begin{array}{ccc|cc} 1 & 0 & -\frac{1}{3} & -1 & -\frac{3}{2} \\ 0 & 1 & \frac{5}{2} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & 1 & 21 & \frac{79}{2} \end{array}\right]$$

The (3,3) entry is the last pivot. Since it already equals 1, we target the (1,3) and (2,3) entries.

$\langle 1\rangle \leftarrow (\frac{1}{3})\times\langle 3\rangle + \langle 1\rangle$
$\langle 2\rangle \leftarrow (-\frac{5}{2})\times\langle 3\rangle + \langle 2\rangle$
$$\left[\begin{array}{ccc|cc} 1 & 0 & 0 & 6 & \frac{35}{3} \\ 0 & 1 & 0 & -51 & -98 \\ 0 & 0 & 1 & 21 & \frac{79}{2} \end{array}\right]$$

This matrix is in reduced row echelon form. It gives the unique solution for each of the two original linear systems. In particular, the solution for the system $\mathbf{AX} = \mathbf{B}_1$ is $(6, -51, 21)$, and the solution for the system $\mathbf{AX} = \mathbf{B}_2$ is $(\frac{35}{3}, -98, \frac{79}{2})$.

(11) (b) Any nonhomogeneous system with two equations and two unknowns that has a unique solution will serve as a counterexample. For instance, consider

$$\begin{cases} x + y = 1 \\ x - y = 1 \end{cases}.$$

This system has a unique solution: $(1, 0)$. Let (s_1, s_2) and (t_1, t_2) both equal $(1, 0)$. Then the sum of solutions is not a solution in this case. Also, if $c \neq 1$, then the scalar multiple of a solution by c is not a solution.

(14) (a) True. The Gaussian elimination method puts the augmented matrix for a system into row echelon form. The statement gives a verbal description of the definition of row echelon form given in the section.

(b) True. The Gaussian elimination method puts the augmented matrix for a system into row echelon form, while the Gauss-Jordan method puts the matrix into reduced row echelon form. The statement describes the difference between those two forms for a matrix.

(c) False. A column is skipped if the pivot entry that would usually be used is zero, and all entries below that position in that column are zero. For example, the reduced row echelon form matrix $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has no pivot in the second column. Rows, however, are never skipped over, although rows of zeroes may occur at the bottom of the matrix.

(d) True. A homogeneous system always has the trivial solution.

(e) False. This statement reverses the roles of the "independent (nonpivot column)" and "dependent (pivot column)" variables. Note, however, that the statement does correctly define the terms "independent" and "dependent."

(f) False. For a counterexample, consider the system

$$\begin{cases} x & & = 0 \\ & y & = 0 \\ x & + \ y & = 0 \end{cases},$$

which clearly has only the trivial solution. The given statement is the reverse of a related true statement, which says: If a homogeneous system has more variables than equations, then the system has a nontrivial solution.

Section 2.3

(1) Recall that a matrix $\mathbf{A}$ is row equivalent to a matrix $\mathbf{B}$ if a finite sequence of row operations will convert $\mathbf{A}$ into $\mathbf{B}$. In parts (a) and (c), below, only one row operation is necessary, although longer finite sequences could be used.

(a) A row operation of type (I) converts $\mathbf{A}$ to $\mathbf{B}$: $\langle 2\rangle \longleftarrow -5\langle 2\rangle$.

(c) A row operation of type (II) converts $\mathbf{A}$ to $\mathbf{B}$: $\langle 2\rangle \longleftarrow \langle 3\rangle + \langle 2\rangle$.

(2) (b) We must first compute $\mathbf{B}$ by putting $\mathbf{A}$ into reduced row echelon form. Performing the following sequence of row operations converts $\mathbf{A}$ to $\mathbf{B} = \mathbf{I}_3$:

(I): $\langle 1\rangle \leftarrow \frac{1}{4}\langle 1\rangle$
(II): $\langle 2\rangle \leftarrow 2\langle 1\rangle + \langle 2\rangle$
(II): $\langle 3\rangle \leftarrow -3\langle 1\rangle + \langle 3\rangle$
(III): $\langle 2\rangle \leftrightarrow \langle 3\rangle$
(II): $\langle 1\rangle \leftarrow 5\langle 3\rangle + \langle 1\rangle$

To convert $\mathbf{B}$ back to $\mathbf{A}$, we use the inverses of these row operations in the reverse order:

(II): $\langle 1\rangle \leftarrow -5\langle 3\rangle + \langle 1\rangle$	the inverse of (II): $\langle 1\rangle \leftarrow 5\langle 3\rangle + \langle 1\rangle$
(III): $\langle 2\rangle \leftrightarrow \langle 3\rangle$	the inverse of (III): $\langle 2\rangle \leftrightarrow \langle 3\rangle$
(II): $\langle 3\rangle \leftarrow 3\langle 1\rangle + \langle 3\rangle$	the inverse of (II): $\langle 3\rangle \leftarrow -3\langle 1\rangle + \langle 3\rangle$
(II): $\langle 2\rangle \leftarrow -2\langle 1\rangle + \langle 2\rangle$	the inverse of (II): $\langle 2\rangle \leftarrow 2\langle 1\rangle + \langle 2\rangle$
(I): $\langle 1\rangle \leftarrow 4\langle 1\rangle$	the inverse of (I): $\langle 1\rangle \leftarrow \frac{1}{4}\langle 1\rangle$

(3) (a) Both $\mathbf{A}$ and $\mathbf{B}$ row reduce to $\mathbf{I}_3$. To convert $\mathbf{A}$ to $\mathbf{I}_3$, use the following sequence of row operations:

(II): $\langle 3\rangle \longleftarrow 2\langle 2\rangle + \langle 3\rangle$
(I): $\langle 3\rangle \longleftarrow -1\langle 3\rangle$
(II): $\langle 1\rangle \longleftarrow -9\langle 3\rangle + \langle 1\rangle$
(II): $\langle 2\rangle \longleftarrow 3\langle 3\rangle + \langle 2\rangle$

To convert $\mathbf{B}$ to $\mathbf{I}_3$, use these row operations:

(I): $\langle 1\rangle \longleftarrow -\frac{1}{5}\langle 1\rangle$
(II): $\langle 2\rangle \longleftarrow 2\langle 1\rangle + \langle 2\rangle$
(II): $\langle 3\rangle \longleftarrow 3\langle 1\rangle + \langle 3\rangle$
(I): $\langle 2\rangle \longleftarrow -5\langle 2\rangle$
(II): $\langle 1\rangle \longleftarrow \frac{3}{5}\langle 2\rangle + \langle 1\rangle$
(II): $\langle 3\rangle \longleftarrow \frac{9}{5}\langle 2\rangle + \langle 3\rangle$

(b) First, we convert $\mathbf{A}$ to $\mathbf{I}_3$ using the sequence of row operations we computed in part (a):

(II): $\langle 3\rangle \longleftarrow 2\langle 2\rangle + \langle 3\rangle$

(b) True. In general, if two matrices are row equivalent, then they are both row equivalent to the same reduced row echelon form matrix. Hence, **A** and **B** are both row equivalent to the same matrix **C** in reduced row echelon form. Since the ranks of **A** and **B** are both the number of nonzero rows in their common reduced echelon form matrix **C**, **A** and **B** must have the same rank.

(c) False. The inverse of the type (I) row operation $\langle i\rangle \leftarrow c\,\langle i\rangle$ is the type (I) row operation $\langle i\rangle \leftarrow \frac{1}{c}\,\langle i\rangle$.

(d) False. The statement is true for homogeneous systems but is false in general. If the system in question is not homogeneous, it could be inconsistent and thus have no solutions at all, and hence, no nontrivial solutions. For a particular example, consider the system $\left\{\begin{array}{l} x+y+z=1 \\ x+y+z=2 \end{array}\right.$. This system has three variables, but the rank of its augmented matrix is ≤ 2, since the matrix has only two rows. However, the system has no solutions because it is impossible for $x+y+z$ to equal both 1 and 2.

(e) False. This statement directly contradicts part (2) of Theorem 2.5.

(f) True. By the definition of row space, **x** is clearly in the row space of **A**. But Theorem 2.8 shows that **A** and **B** have the same row space. Hence, **x** is also in the row space of **B**.

Section 2.4

(2) To find the rank of a matrix, find its corresponding reduced row echelon form matrix and count its nonzero rows. An $n\times n$ matrix is nonsingular if and only if its rank equals n.

(a) The given 2×2 matrix row reduces to $\mathbf{I}_2$, which has two nonzero rows. Hence, the original matrix has rank $= 2$. Therefore, the matrix is nonsingular.

(c) The given 3×3 matrix row reduces to $\mathbf{I}_3$, which has three nonzero rows. Hence, the original matrix has rank $= 3$. Therefore, the matrix is nonsingular.

(e) The given 4×4 matrix row reduces to $\left[\begin{array}{rrrr} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$,

which has three nonzero rows. Hence, the rank of the original matrix is 3. But this is less than the number of rows in the matrix, and so the original matrix is singular.

(3) Use the formula for the inverse of a 2×2 matrix given in Theorem 2.13.

(a) First, we compute $\delta = (4)(-3) - (2)(9) = -30$. Since $\delta \neq 0$, the given matrix is nonsingular, and its inverse is $\frac{1}{-30}\left[\begin{array}{rr} -3 & -2 \\ -9 & 4 \end{array}\right] = \left[\begin{array}{rr} \frac{1}{10} & \frac{1}{15} \\ \frac{3}{10} & -\frac{2}{15} \end{array}\right]$.

(c) First, we compute $\delta = (-3)(-8) - (5)(-12) = 84$. Since $\delta \neq 0$, the given matrix is nonsingular, and its inverse is $\frac{1}{84}\left[\begin{array}{rr} -8 & -5 \\ 12 & -3 \end{array}\right] = \left[\begin{array}{rr} -\frac{2}{21} & -\frac{5}{84} \\ \frac{1}{7} & -\frac{1}{28} \end{array}\right]$.

(e) First, we compute $\delta = (-6)(-8) - (12)(4) = 0$. Since $\delta = 0$, the given matrix is singular; it has no inverse.

(4) To find the inverse of an $n \times n$ matrix $\mathbf{A}$, row reduce $[\mathbf{A}|\mathbf{I}_n]$ to obtain $[\mathbf{I}_n|\mathbf{A}^{-1}]$. If row reduction does not produce $\mathbf{I}_n$ to the left of the augmentation bar, then $\mathbf{A}$ does not have an inverse.

(a) We row reduce $\left[\begin{array}{ccc|ccc} -4 & 7 & 6 & 1 & 0 & 0 \\ 3 & -5 & -4 & 0 & 1 & 0 \\ -2 & 4 & 3 & 0 & 0 & 1 \end{array}\right]$ to obtain $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 2 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 2 & 2 & -1 \end{array}\right]$. Therefore, $\begin{bmatrix} 1 & 3 & 2 \\ -1 & 0 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ is the inverse of the original matrix.

(c) We row reduce $\left[\begin{array}{ccc|ccc} 2 & -2 & 3 & 1 & 0 & 0 \\ 8 & -4 & 9 & 0 & 1 & 0 \\ -4 & 6 & -9 & 0 & 0 & 1 \end{array}\right]$ to obtain $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -3 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{8}{3} & \frac{1}{3} & -\frac{2}{3} \end{array}\right]$. Therefore, $\begin{bmatrix} \frac{3}{2} & 0 & \frac{1}{2} \\ -3 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{8}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$ is the inverse of the original matrix.

(e) We row reduce $\left[\begin{array}{cccc|cccc} 2 & 0 & -1 & 3 & 1 & 0 & 0 & 0 \\ 1 & -2 & 3 & 1 & 0 & 1 & 0 & 0 \\ 4 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 3 & -2 & -5 & 0 & 0 & 0 & 1 \end{array}\right]$ to $\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{8}{15} & \frac{1}{5} & \frac{1}{15} & \frac{2}{15} & 0 \\ 0 & 1 & 0 & -\frac{47}{15} & -\frac{4}{5} & -\frac{4}{15} & \frac{7}{15} & 0 \\ 0 & 0 & 1 & -\frac{29}{15} & -\frac{3}{5} & \frac{2}{15} & \frac{4}{15} & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$.

Therefore, since $\mathbf{I}_4$ is not obtained to the left of the augmentation bar, the original matrix does not have an inverse. (Note: If you use a calculator to perform the row reduction above, it might continue to row reduce beyond the augmentation bar.)

(5) (c) To find the inverse of a diagonal matrix, merely find the diagonal matrix whose main diagonal entries are the reciprocals of the main diagonal entries of the original matrix. Hence, when this matrix is multiplied by the original, the product will be a diagonal matrix with all 1's on the main diagonal – that is, the identity matrix. Therefore, the inverse matrix we need is $\begin{bmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{bmatrix}$.

(6) (a) Using Theorem 2.13, $\delta = (\cos\theta)(\cos\theta) - (\sin\theta)(-\sin\theta) = \cos^2\theta + \sin^2\theta = 1$. Therefore, the general inverse is $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Plugging in the given values for θ yields

θ	Original Matrix	Inverse Matrix
$\frac{\pi}{6}$	$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$
$\frac{\pi}{4}$	$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$
$\frac{\pi}{2}$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(b) It is easy to see that if $\mathbf{A}$ and $\mathbf{B}$ are 2×2 matrices with $\mathbf{AB} = \mathbf{C}$, then $\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (Check it out!) Therefore, to find the inverse of the matrix given in part (b), we merely use the associated 2×2 inverses we found in part (a), glued into the upper corner of a 3×3 matrix, as illustrated. Hence, the general inverse is $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Substituting the given values of θ produces

θ	Original Matrix	Inverse Matrix
$\frac{\pi}{6}$	$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\frac{\pi}{4}$	$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$\frac{\pi}{2}$	$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(7) By Theorem 2.15, if $\mathbf{A}$ is a nonsingular matrix, then the unique solution to $\mathbf{AX} = \mathbf{B}$ is $\mathbf{A}^{-1}\mathbf{B}$.

(a) The given system corresponds to the matrix equation $\begin{bmatrix} 5 & -1 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ -31 \end{bmatrix}$. We use Theorem 2.13 to compute the inverse of $\begin{bmatrix} 5 & -1 \\ -7 & 2 \end{bmatrix}$. First, $\delta = (5)(2) - (-7)(-1) = 3$. Hence, the inverse $= \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{7}{3} & \frac{5}{3} \end{bmatrix}$. The unique solution to the system is thus $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{7}{3} & \frac{5}{3} \end{bmatrix} \begin{bmatrix} 20 \\ -31 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. That is, the solution set for the system is $\{(3, -5)\}$.

(c) The system corresponds to the matrix equation $\begin{bmatrix} 0 & -2 & 5 & 1 \\ -7 & -4 & 5 & 22 \\ 5 & 3 & -4 & -16 \\ -3 & -1 & 0 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 25 \\ -15 \\ 9 \\ -16 \end{bmatrix}$.

We find the inverse of the coefficient matrix by row reducing $\left[\begin{array}{rrrr|rrrr} 0 & -2 & 5 & 1 & 1 & 0 & 0 & 0 \\ -7 & -4 & 5 & 22 & 0 & 1 & 0 & 0 \\ 5 & 3 & -4 & -16 & 0 & 0 & 1 & 0 \\ -3 & -1 & 0 & 9 & 0 & 0 & 0 & 1 \end{array}\right]$ to obtain $\left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 1 & -13 & -15 & 5 \\ 0 & 1 & 0 & 0 & -3 & 3 & 0 & -7 \\ 0 & 0 & 1 & 0 & -1 & 2 & 1 & -3 \\ 0 & 0 & 0 & 1 & 0 & -4 & -5 & 1 \end{array}\right]$. The inverse of the coefficient matrix is the 4×4 matrix to the right of the augmentation bar in the row reduced matrix. Hence, the unique solution to the system is

$$\begin{bmatrix} 1 & -13 & -15 & 5 \\ -3 & 3 & 0 & -7 \\ -1 & 2 & 1 & -3 \\ 0 & -4 & -5 & 1 \end{bmatrix} \begin{bmatrix} 25 \\ -15 \\ 9 \\ -16 \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ 2 \\ -1 \end{bmatrix}.$$

That is, the solution set for the linear system is $\{(5, -8, 2, -1)\}$.

(8) (a) Through trial and error, we find the involutory matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(b) Using the answer to part (a), the comments at the beginning of the answer to Section 2.4, Exercise 6(b) in this manual, suggest that we consider the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Squaring this matrix verifies that it is, in fact, involutory.

(c) If $\mathbf{A}$ is involutory, then $\mathbf{A}^2 = \mathbf{AA} = \mathbf{I}_n$. Hence, $\mathbf{A}$ itself satisfies the definition of an inverse for $\mathbf{A}$, and so $\mathbf{A}^{-1} = \mathbf{A}$.

(10) (a) Since $\mathbf{A}$ is nonsingular, $\mathbf{A}^{-1}$ exists. Hence, $\mathbf{AB} = \mathbf{O}_n \Rightarrow \mathbf{A}^{-1}(\mathbf{AB}) = \mathbf{A}^{-1}\mathbf{O}_n \Rightarrow (\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{O}_n \Rightarrow \mathbf{I}_n\mathbf{B} = \mathbf{O}_n \Rightarrow \mathbf{B} = \mathbf{O}_n$. Thus, $\mathbf{B}$ must be the zero matrix.

(b) No. $\mathbf{AB} = \mathbf{I}_n$ implies $\mathbf{B} = \mathbf{A}^{-1}$, and so $\mathbf{A}$ is nonsingular. We can now apply part (a), substituting $\mathbf{C}$ where $\mathbf{B}$ appears, proving that $\mathbf{AC} = \mathbf{O}_n \Rightarrow \mathbf{C} = \mathbf{O}_n$.

(11) Now $\mathbf{A}^4 = \mathbf{I}_n \Rightarrow \mathbf{A}^3\mathbf{A} = \mathbf{I}_n \Rightarrow \mathbf{A}^{-1} = \mathbf{A}^3$. Also, since $\mathbf{I}_n^k = \mathbf{I}_n$ for every integer k, we see that $\mathbf{A}^{4k} = (\mathbf{A}^4)^k = \mathbf{I}_n^k = \mathbf{I}_n$ for every integer k. Hence, for every integer k, $\mathbf{A}^{4k+3} = \mathbf{A}^{4k}\mathbf{A}^3 = \mathbf{I}_n\mathbf{A}^3 = \mathbf{A}^3 = \mathbf{A}^{-1}$. Thus, all $n \times n$ matrices of the form $\mathbf{A}^{4k+3}$ equal $\mathbf{A}^{-1}$. These are $\ldots, \mathbf{A}^{-9}, \mathbf{A}^{-5}, \mathbf{A}^{-1}, \mathbf{A}^3, \mathbf{A}^7, \mathbf{A}^{11}, \ldots$. Now, could any other powers of $\mathbf{A}$ also equal $\mathbf{A}^{-1}$? Suppose that $\mathbf{A}^m = \mathbf{A}^{-1}$. Then, $\mathbf{A}^{m+1} = \mathbf{A}^m\mathbf{A} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$. Now, if $m + 1 = 4l$ for some integer l, then $m = 4(l-1) + 3$, and so m is already on our list of powers of $\mathbf{A}$ such that $\mathbf{A}^m = \mathbf{A}^{-1}$. On the other hand, if $m + 1$ is not a multiple of 4, then, dividing $m + 1$ by 4 yields an integer k, with a remainder r of either 1, 2, or 3. Then $m + 1 = 4k + r$, and so $\mathbf{A}^{4k+r} = \mathbf{I}_n$ implying $\mathbf{A}^{4k}\mathbf{A}^r = \mathbf{I}_n$. But $\mathbf{A}^{4k} = \mathbf{I}_n$ from above, and so $\mathbf{A}^r = \mathbf{I}_n$. But, since r equals either 1, 2, or 3, this contradicts the given conditions of the problem, and so $m+1$ cannot equal $4k + r$. Therefore, the only powers of $\mathbf{A}$ that equal $\mathbf{A}^{-1}$ are of the form $\mathbf{A}^{4k+3}$.

(12) By parts (1) and (3) of Theorem 2.11, $\mathbf{B}^{-1}\mathbf{A}$ is the inverse of $\mathbf{A}^{-1}\mathbf{B}$. (Proof: $(\mathbf{A}^{-1}\mathbf{B})^{-1} = \mathbf{B}^{-1}(\mathbf{A}^{-1})^{-1} = \mathbf{B}^{-1}\mathbf{A}$.) Thus, if $\mathbf{A}^{-1}\mathbf{B}$ is known, simply compute its inverse to find $\mathbf{B}^{-1}\mathbf{A}$.

(14) (a) No step in the row reduction process will alter the column of zeroes, and so the unique reduced row echelon form for the matrix must contain a column of zeroes and so cannot equal $\mathbf{I}_n$.

(15) (a) Part (1): Since $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, we must have $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.
Part (2): For $k > 0$, to show $(\mathbf{A}^k)^{-1} = (\mathbf{A}^{-1})^k$, we must show that $\mathbf{A}^k(\mathbf{A}^{-1})^k = \mathbf{I}$. Proceed by induction on k.
Base Step: For $k = 1$, clearly $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.
Inductive Step: Assume $\mathbf{A}^k(\mathbf{A}^{-1})^k = \mathbf{I}$. Prove $\mathbf{A}^{k+1}(\mathbf{A}^{-1})^{k+1} = \mathbf{I}$.
Now, $\mathbf{A}^{k+1}(\mathbf{A}^{-1})^{k+1} = \mathbf{A}\mathbf{A}^k(\mathbf{A}^{-1})^k\mathbf{A}^{-1} = \mathbf{A}\mathbf{I}\mathbf{A}^{-1} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. This concludes the proof for $k > 0$.
We now show $\mathbf{A}^k(\mathbf{A}^{-1})^k = \mathbf{I}$ for $k \leq 0$.
For $k = 0$, clearly $\mathbf{A}^0(\mathbf{A}^{-1})^0 = \mathbf{I}\,\mathbf{I} = \mathbf{I}$. The case $k = -1$ is covered by part (1) of the theorem.
For $k \leq -2$, $(\mathbf{A}^k)^{-1} = ((\mathbf{A}^{-1})^{-k})^{-1}$ (by definition) $= ((\mathbf{A}^{-k})^{-1})^{-1}$ (by the $k > 0$ case) $= \mathbf{A}^{-k}$ (by part (1)).

(17) (a) Let $p = -s$, $q = -t$. Then $p, q > 0$. Now, $\mathbf{A}^{s+t} = \mathbf{A}^{-(p+q)} = (\mathbf{A}^{-1})^{p+q} = (\mathbf{A}^{-1})^p(\mathbf{A}^{-1})^q$ (by Theorem 1.15) $= \mathbf{A}^{-p}\mathbf{A}^{-q} = \mathbf{A}^s\mathbf{A}^t$.

(21) (a) Suppose that $n > k$. Then, by Corollary 2.6, there is a nontrivial $\mathbf{X}$ such that $\mathbf{B}\mathbf{X} = \mathbf{O}$. Hence, $(\mathbf{A}\mathbf{B})\mathbf{X} = \mathbf{A}(\mathbf{B}\mathbf{X}) = \mathbf{A}\mathbf{O} = \mathbf{O}$. But $(\mathbf{A}\mathbf{B})\mathbf{X} = \mathbf{I}_n\mathbf{X} = \mathbf{X}$. Therefore, $\mathbf{X} = \mathbf{O}$, which gives a contradiction.

(b) Suppose that $k > n$. Then, by Corollary 2.6, there is a nontrivial $\mathbf{Y}$ such that $\mathbf{A}\mathbf{Y} = \mathbf{O}$. Hence, $(\mathbf{B}\mathbf{A})\mathbf{Y} = \mathbf{B}(\mathbf{A}\mathbf{Y}) = \mathbf{B}\mathbf{O} = \mathbf{O}$. But $(\mathbf{B}\mathbf{A})\mathbf{Y} = \mathbf{I}_k\mathbf{Y} = \mathbf{X}$. Therefore, $\mathbf{Y} = \mathbf{O}$, which gives a contradiction.

(c) Parts (a) and (b) combine to prove that $n = k$. Hence, $\mathbf{A}$ and $\mathbf{B}$ are both square matrices. They are nonsingular with $\mathbf{A}^{-1} = \mathbf{B}$ by the definition of a nonsingular matrix, since $\mathbf{A}\mathbf{B} = \mathbf{I}_n = \mathbf{B}\mathbf{A}$.

(22) (a) False. Many $n \times n$ matrices are singular, having no inverse. For example, the 2×2 matrix $\begin{bmatrix} 6 & 3 \\ 8 & 4 \end{bmatrix}$ has no inverse, by Theorem 2.13, because $\delta = (6)(4) - (3)(8) = 0$.

(b) True. This follows directly from Theorem 2.9.

(c) True. $((\mathbf{A}\mathbf{B})^T)^{-1} = ((\mathbf{A}\mathbf{B})^{-1})^T$ (by part (4) of Theorem 2.11) $= (\mathbf{B}^{-1}\mathbf{A}^{-1})^T$ (by part (3) of Theorem 2.11) $= (\mathbf{A}^{-1})^T(\mathbf{B}^{-1})^T$ (by Theorem 1.16).

(d) False. This statement contradicts Theorem 2.13, which states that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is *non*singular if and only if $\delta = ad - bc \neq 0$.

(e) False. This statement contradicts the Inverse Method given in Section 2.4 as well as Theorem 2.14. To correct the statement, change the word "nonsingular" to "singular."

(f) True. This follows directly from combining Theorems 2.14 and 2.15.

Chapter 3

Section 3.1

(1) (a) $\begin{vmatrix} -2 & 5 \\ 3 & 1 \end{vmatrix} = (-2)(1) - (5)(3) = -17$

(c) $\begin{vmatrix} 6 & -12 \\ -4 & 8 \end{vmatrix} = (6)(8) - (-12)(-4) = 0$

(e) To use basketweaving, we form a new array by taking the given matrix and then adding a second copy of columns 1 and 2 as the new columns 4 and 5. We then form terms using the basketweaving pattern.

$$\begin{array}{ccccccccccc}
\searrow & & \searrow & & \searrow & & \swarrow & & \swarrow & & \swarrow \\
 & 2 & & 0 & & 5 & & 2 & & 0 & \\
 & & \times & & \times & & \times & & \times & & \\
 & -4 & & 1 & & 7 & & -4 & & 1 & \\
 & & \times & & \times & & \times & & \times & & \\
 & 0 & & 3 & & -3 & & 0 & & 3 & \\
\swarrow & & \swarrow & & \swarrow & & \searrow & & \searrow & & \searrow
\end{array}$$

Hence, the determinant equals $(2)(1)(-3) + (0)(7)(0) + (5)(-4)(3) - (5)(1)(0) - (2)(7)(3) - (0)(-4)(-3) = -108$.

(g) To use basketweaving, we form a new array by taking the given matrix and then adding a second copy of columns 1 and 2 as the new columns 4 and 5. We then form terms using the basketweaving pattern.

$$\begin{array}{ccccccccccc}
\searrow & & \searrow & & \searrow & & \swarrow & & \swarrow & & \swarrow \\
 & 5 & & 0 & & 0 & & 5 & & 0 & \\
 & & \times & & \times & & \times & & \times & & \\
 & 3 & & -2 & & 0 & & 3 & & -2 & \\
 & & \times & & \times & & \times & & \times & & \\
 & -1 & & 8 & & 4 & & -1 & & 8 & \\
\swarrow & & \swarrow & & \swarrow & & \searrow & & \searrow & & \searrow
\end{array}$$

Hence, the determinant equals $(5)(-2)(4) + (0)(0)(-1) + (0)(3)(8) - (0)(-2)(-1) - (5)(0)(8) - (0)(3)(4) = -40$.

(i) To use basketweaving, we form a new array by taking the given matrix and then adding a second copy of columns 1 and 2 as the new columns 4 and 5. We then form terms using the basketweaving pattern.

$$\begin{array}{ccccccccccc}
\searrow & & \searrow & & \searrow & & \swarrow & & \swarrow & & \swarrow \\
 & 3 & & 1 & & -2 & & 3 & & 1 & \\
 & & \times & & \times & & \times & & \times & & \\
 & -1 & & 4 & & 5 & & -1 & & 4 & \\
 & & \times & & \times & & \times & & \times & & \\
 & 3 & & 1 & & -2 & & 3 & & 1 & \\
\swarrow & & \swarrow & & \swarrow & & \searrow & & \searrow & & \searrow
\end{array}$$

Hence, the determinant equals $(3)(4)(-2) + (1)(5)(3) + (-2)(-1)(1) - (-2)(4)(3) - (3)(5)(1) - (1)(-1)(-2) = 0$.

(j) The determinant of a 1×1 matrix is defined to be the (1,1) entry of the matrix. Therefore, the determinant equals -3.

(2) Recall that the submatrix $\mathbf{A}_{ij}$ is found by deleting the ith row and the jth column from $\mathbf{A}$. The (i, j) minor is the determinant of this submatrix.

(a) $|\mathbf{A}_{21}| = \begin{vmatrix} 4 & 3 \\ -2 & 4 \end{vmatrix} = (4)(4) - (3)(-2) = 22$.

(c) $|\mathbf{C}_{42}| = \begin{vmatrix} -3 & 0 & 5 \\ 2 & -1 & 4 \\ 6 & 4 & 0 \end{vmatrix}$. We will use basketweaving to compute this determinant. To do this, we form a new array by taking $\mathbf{C}_{42}$ and then adding a second copy of columns 1 and 2 as the new columns 4 and 5. We then form terms using the basketweaving pattern.

$$\begin{array}{ccccccccccc}
\searrow & & \searrow & & \searrow & & \swarrow & & \swarrow & & \swarrow \\
 & -3 & & 0 & & 5 & & -3 & & 0 & \\
 & & \times & & \times & & \times & & \times & & \\
 & 2 & & -1 & & 4 & & 2 & & -1 & \\
 & & \times & & \times & & \times & & \times & & \\
 & 6 & & 4 & & 0 & & 6 & & 4 & \\
\swarrow & & \swarrow & & \swarrow & & \searrow & & \searrow & & \searrow
\end{array}$$

Hence, $|\mathbf{C}_{42}| = (-3)(-1)(0) + (0)(4)(6) + (5)(2)(4) - (5)(-1)(6) - (-3)(4)(4) - (0)(2)(0) = 118$.

(3) The cofactor $\mathcal{A}_{ij}$ is defined to be $(-1)^{i+j}|\mathbf{A}_{ij}|$, where $|\mathbf{A}_{ij}|$ is the (i, j) minor – that is, the determinant of the (i, j) submatrix, obtained by deleting the ith row and the jth column from $\mathbf{A}$.

(a) $\mathcal{A}_{22} = (-1)^{2+2}|\mathbf{A}_{22}| = (-1)^4 \begin{vmatrix} 4 & -3 \\ 9 & -7 \end{vmatrix} = (1)\Big((4)(-7) - (-3)(9)\Big) = -1$.

(c) $\mathcal{C}_{43} = (-1)^{4+3}|\mathbf{C}_{43}| = (-1)^7 \begin{vmatrix} -5 & 2 & 13 \\ -8 & 2 & 22 \\ -6 & -3 & -16 \end{vmatrix} = (-1)(-222) = 222$, where we have computed the 3×3 determinant $|\mathbf{C}_{43}|$ to be -222 using basketweaving as follows:

To compute $|\mathbf{C}_{43}|$, we form a new array by taking $\mathbf{C}_{43}$ and then adding a second copy of columns 1 and 2 as the new columns 4 and 5. We then form terms using the basketweaving pattern.

$$\begin{array}{ccccccccccc}
\searrow & & \searrow & & \searrow & & \swarrow & & \swarrow & & \swarrow \\
& -5 & & 2 & & 13 & & -5 & & 2 & \\
& & \times & & \times & & \times & & \times & & \\
& -8 & & 2 & & 22 & & -8 & & 2 & \\
& & \times & & \times & & \times & & \times & & \\
& -6 & & -3 & & -16 & & -6 & & -3 & \\
\swarrow & & \swarrow & & \swarrow & & \searrow & & \searrow & & \searrow
\end{array}$$

Hence, $|\mathbf{C}_{43}| = (-5)(2)(-16) + (2)(22)(-6) + (13)(-8)(-3) - (13)(2)(-6) - (-5)(22)(-3) - (2)(-8)(-16) = -222$.

(d) $\mathcal{D}_{12} = (-1)^{1+2}|\mathbf{D}_{12}| = (-1)^3 \begin{vmatrix} x-4 & x-3 \\ x-1 & x+2 \end{vmatrix} = (-1)\Big((x-4)(x+2) - (x-3)(x-1)\Big)$
$= (-1)\Big(\big(x^2-2x-8\big) - \big(x^2-4x+3\big)\Big) = -2x+11$.

(4) In this problem, we are asked to use only the formal definition (cofactor expansion) to compute determinants. Therefore, we will not use basketweaving for 3×3 determinants or the simple $a_{11}a_{22} - a_{12}a_{21}$ formula for 2×2 determinants. Of course, the cofactor expansion method still produces the same results obtained in Exercise 1.

(a) Let $\mathbf{A} = \begin{bmatrix} -2 & 5 \\ 3 & 1 \end{bmatrix}$. Then $|\mathbf{A}| = a_{21}\mathcal{A}_{21} + a_{22}\mathcal{A}_{22} = a_{21}(-1)^{2+1}(a_{12}) + a_{22}(-1)^{2+2}(a_{11})$
$= (3)(-1)(5) + (1)(1)(-2) = -17$.

(c) Let $\mathbf{A} = \begin{bmatrix} 6 & -12 \\ -4 & 8 \end{bmatrix}$. Then $|\mathbf{A}| = a_{21}\mathcal{A}_{21} + a_{22}\mathcal{A}_{22} = a_{21}(-1)^{2+1}(a_{12}) + a_{22}(-1)^{2+2}(a_{11})$
$= (-4)(-1)(-12) + (8)(1)(6) = 0$.

(e) Let $\mathbf{A} = \begin{bmatrix} 2 & 0 & 5 \\ -4 & 1 & 7 \\ 0 & 3 & -3 \end{bmatrix}$. Then $|\mathbf{A}| = a_{31}\mathcal{A}_{31} + a_{32}\mathcal{A}_{32} + a_{33}\mathcal{A}_{33}$

$= (0)(-1)^{3+1}\begin{vmatrix} 0 & 5 \\ 1 & 7 \end{vmatrix} + (3)(-1)^{3+2}\begin{vmatrix} 2 & 5 \\ -4 & 7 \end{vmatrix} + (-3)(-1)^{3+3}\begin{vmatrix} 2 & 0 \\ -4 & 1 \end{vmatrix}$. Now, for a 2×2 matrix $\mathbf{B}$, $|\mathbf{B}| = b_{21}\mathcal{B}_{21} + b_{22}\mathcal{B}_{22} = b_{21}(-1)^{2+1}(b_{12}) + b_{22}(-1)^{2+2}(b_{11})$.

Hence, $\begin{vmatrix} 2 & 5 \\ -4 & 7 \end{vmatrix} = (-4)(-1)(5) + (7)(1)(2) = 34$ and $\begin{vmatrix} 2 & 0 \\ -4 & 1 \end{vmatrix} = (-4)(-1)(0) + (1)(1)(2) = 2$.

It is not necessary to compute $\begin{vmatrix} 0 & 5 \\ 1 & 7 \end{vmatrix}$ because its coefficient in the determinant formula is zero.

Hence, $|\mathbf{A}| = 0 + (3)(-1)(34) + (-3)(1)(2) = -108$.

(g) Let $\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 \\ 3 & -2 & 0 \\ -1 & 8 & 4 \end{bmatrix}$. Then $|\mathbf{A}| = a_{31}\mathcal{A}_{31} + a_{32}\mathcal{A}_{32} + a_{33}\mathcal{A}_{33}$

$= (-1)(-1)^{3+1}\begin{vmatrix} 0 & 0 \\ -2 & 0 \end{vmatrix} + (8)(-1)^{3+2}\begin{vmatrix} 5 & 0 \\ 3 & 0 \end{vmatrix} + (4)(-1)^{3+3}\begin{vmatrix} 5 & 0 \\ 3 & -2 \end{vmatrix}$. Now, for a 2×2 matrix $\mathbf{B}$, $|\mathbf{B}| = b_{21}\mathcal{B}_{21} + b_{22}\mathcal{B}_{22} = b_{21}(-1)^{2+1}(b_{12}) + b_{22}(-1)^{2+2}(b_{11})$.

Hence, $\begin{vmatrix} 0 & 0 \\ -2 & 0 \end{vmatrix} = (-2)(-1)(0) + (0)(1)(0) = 0$, $\begin{vmatrix} 5 & 0 \\ 3 & 0 \end{vmatrix} = (3)(-1)(0) + (0)(1)(5) = 0$ and $\begin{vmatrix} 5 & 0 \\ 3 & -2 \end{vmatrix} = (3)(-1)(0) + (-2)(1)(5) = -10$. Hence, $|\mathbf{A}| = (-1)(1)(0) + (8)(-1)(0) + (4)(1)(-10) = -40$.

(i) Let $\mathbf{A} = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 4 & 5 \\ 3 & 1 & -2 \end{bmatrix}$. Then $|\mathbf{A}| = a_{31}\mathcal{A}_{31} + a_{32}\mathcal{A}_{32} + a_{33}\mathcal{A}_{33}$

$= (3)(-1)^{3+1}\begin{vmatrix} 1 & -2 \\ 4 & 5 \end{vmatrix} + (1)(-1)^{3+2}\begin{vmatrix} 3 & -2 \\ -1 & 5 \end{vmatrix} + (-2)(-1)^{3+3}\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix}$. Now, for a 2×2 matrix $\mathbf{B}$, $|\mathbf{B}| = b_{21}\mathcal{B}_{21} + b_{22}\mathcal{B}_{22} = b_{21}(-1)^{2+1}(b_{12}) + b_{22}(-1)^{2+2}(b_{11})$.

Hence, $\begin{vmatrix} 1 & -2 \\ 4 & 5 \end{vmatrix} = (4)(-1)(-2) + (5)(1)(1) = 13$, $\begin{vmatrix} 3 & -2 \\ -1 & 5 \end{vmatrix} = (-1)(-1)(-2) + (5)(1)(3) = 13$, and $\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = (-1)(-1)(1) + (4)(1)(3) = 13$. Hence, $|\mathbf{A}| = (3)(1)(13) + (1)(-1)(13) + (-2)(1)(13) = 0$.

(j) By definition, the determinant of a 1×1 matrix is defined to be the (1,1) entry of the matrix. Thus, the determinant of $[-3]$ equals -3.

(5) (a) Let $\mathbf{A} = \begin{bmatrix} 5 & 2 & 1 & 0 \\ -1 & 3 & 5 & 2 \\ 4 & 1 & 0 & 2 \\ 0 & 2 & 3 & 0 \end{bmatrix}$. Then $|\mathbf{A}| = a_{41}\mathcal{A}_{41} + a_{42}\mathcal{A}_{42} + a_{43}\mathcal{A}_{43} + a_{44}\mathcal{A}_{44}$

$= (0)(-1)^{4+1}|\mathbf{A}_{41}| + (2)(-1)^{4+2}|\mathbf{A}_{42}| + (3)(-1)^{4+3}|\mathbf{A}_{43}| + (0)(-1)^{4+4}|\mathbf{A}_{44}|$

$= 0 + 2|\mathbf{A}_{42}| - 3|\mathbf{A}_{43}| + 0$. Now, let $\mathbf{B} = \mathbf{A}_{42} = \begin{bmatrix} 5 & 1 & 0 \\ -1 & 5 & 2 \\ 4 & 0 & 2 \end{bmatrix}$. Then $|\mathbf{A}_{42}| = |\mathbf{B}| = b_{31}\mathcal{B}_{31} + b_{32}\mathcal{B}_{32} + b_{33}\mathcal{B}_{33} = (4)(-1)^{3+1}\begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} + (0)(-1)^{3+2}\begin{vmatrix} 5 & 0 \\ -1 & 2 \end{vmatrix} + (2)(-1)^{3+3}\begin{vmatrix} 5 & 1 \\ -1 & 5 \end{vmatrix}$

$= (4)(1)\Big((1)(2) - (0)(5)\Big) + 0 + 2(1)\Big((5)(5) - (1)(-1)\Big) = 60$. Also, if $\mathbf{C} = \mathbf{A}_{43} = \begin{bmatrix} 5 & 2 & 0 \\ -1 & 3 & 2 \\ 4 & 1 & 2 \end{bmatrix}$.

Then $|\mathbf{A}_{43}| = |\mathbf{C}| = c_{31}\mathcal{C}_{31} + c_{32}\mathcal{C}_{32} + c_{33}\mathcal{C}_{33}$

$= (4)(-1)^{3+1}\begin{vmatrix} 2 & 0 \\ 3 & 2 \end{vmatrix} + (1)(-1)^{3+2}\begin{vmatrix} 5 & 0 \\ -1 & 2 \end{vmatrix} + (2)(-1)^{3+3}\begin{vmatrix} 5 & 2 \\ -1 & 3 \end{vmatrix}$

$= (4)(1)\Big((2)(2) - (3)(0)\Big) + (1)(-1)\Big((5)(2) - (0)(-1)\Big) + 2(1)\Big((5)(3) - (2)(-1)\Big) = 40$. Hence, $|\mathbf{A}| = 2|\mathbf{A}_{42}| - 3|\mathbf{A}_{43}| = 2(60) - 3(40) = 0$.

(d) Let $\mathbf{A}$ be the given 5×5 matrix. Then $|\mathbf{A}| = a_{51}\mathcal{A}_{51} + a_{52}\mathcal{A}_{52} + a_{53}\mathcal{A}_{53} + a_{54}\mathcal{A}_{54} + a_{55}\mathcal{A}_{55}$ $= (0)(-1)^{5+1}|\mathbf{A}_{51}| + (3)(-1)^{5+2}|\mathbf{A}_{52}| + (0)(-1)^{5+3}|\mathbf{A}_{53}| + (0)(-1)^{5+4}|\mathbf{A}_{54}| + (2)(-1)^{5+5}|\mathbf{A}_{55}|$ $= (0) + (3)(-1)|\mathbf{A}_{52}| + (0) + (0) + (2)(1)|\mathbf{A}_{55}|$.

Let $\mathbf{B} = \mathbf{A}_{52} = \begin{bmatrix} 0 & 1 & 3 & -2 \\ 2 & 3 & -1 & 0 \\ 3 & 2 & -5 & 1 \\ 1 & -4 & 0 & 0 \end{bmatrix}$. Then, $|\mathbf{A}_{52}| = |\mathbf{B}| = b_{41}\mathcal{B}_{41} + b_{42}\mathcal{B}_{42} + b_{43}\mathcal{B}_{43} + b_{44}\mathcal{B}_{44}$

$= (1)(-1)^{4+1}|\mathbf{B}_{41}| + (-4)(-1)^{4+2}|\mathbf{B}_{42}| + (0)(-1)^{4+3}|\mathbf{B}_{43}| + (0)(-1)^{4+4}|\mathbf{B}_{44}| =$ $(1)(-1)|\mathbf{B}_{41}| + (-4)(1)|\mathbf{B}_{42}| + (0) + (0)$. Now, $\mathbf{B}_{41} = \begin{bmatrix} 1 & 3 & -2 \\ 3 & -1 & 0 \\ 2 & -5 & 1 \end{bmatrix}$. We compute $|\mathbf{B}_{41}|$ using basketweaving.

$$\begin{array}{ccccccccc}
\searrow & \searrow & \searrow & & & \swarrow & \swarrow & \swarrow & \\
1 & & 3 & & -2 & & 1 & & 3 \\
& \times & & \times & & \times & & \times & \\
3 & & -1 & & 0 & & 3 & & -1 \\
& \times & & \times & & \times & & \times & \\
2 & & -5 & & 1 & & 2 & & -5 \\
\swarrow & \swarrow & \swarrow & & & \searrow & \searrow & \searrow &
\end{array}$$

This yields $|\mathbf{B}_{41}| = (1)(-1)(1) + (3)(0)(2) + (-2)(3)(-5) - (-2)(-1)(2) - (1)(0)(-5) - (3)(3)(1) =$ 16. Next, $\mathbf{B}_{42} = \begin{bmatrix} 0 & 3 & -2 \\ 2 & -1 & 0 \\ 3 & -5 & 1 \end{bmatrix}$. We compute $|\mathbf{B}_{42}|$ using basketweaving.

$$\begin{array}{ccccccccc}
\searrow & \searrow & \searrow & & & \swarrow & \swarrow & \swarrow & \\
0 & & 3 & & -2 & & 0 & & 3 \\
& \times & & \times & & \times & & \times & \\
2 & & -1 & & 0 & & 2 & & -1 \\
& \times & & \times & & \times & & \times & \\
3 & & -5 & & 1 & & 3 & & -5 \\
\swarrow & \swarrow & \swarrow & & & \searrow & \searrow & \searrow &
\end{array}$$

This yields $|\mathbf{B}_{42}| = (0)(-1)(1) + (3)(0)(3) + (-2)(2)(-5) - (-2)(-1)(3) - (0)(0)(-5) - (3)(2)(1) =$ 8. Therefore, $|\mathbf{A}_{52}| = (1)(-1)|\mathbf{B}_{41}| + (-4)(1)|\mathbf{B}_{42}| = (1)(-1)(16) + (-4)(1)(8) = -48$. Now, let $\mathbf{C} = \mathbf{A}_{55} = \begin{bmatrix} 0 & 4 & 1 & 3 \\ 2 & 2 & 3 & -1 \\ 3 & 1 & 2 & -5 \\ 1 & 0 & -4 & 0 \end{bmatrix}$. Then $|\mathbf{A}_{55}| = |\mathbf{C}| = c_{41}\mathcal{C}_{41} + c_{42}\mathcal{C}_{42} + c_{43}\mathcal{C}_{43} + c_{44}\mathcal{C}_{44}$ $= (1)(-1)^{4+1}|\mathbf{C}_{41}| + (0)(-1)^{4+2}|\mathbf{C}_{42}| + (-4)(-1)^{4+3}|\mathbf{C}_{43}| + (0)(-1)^{4+4}|\mathbf{C}_{44}|$

$= (1)(-1)^{4+1}|\mathbf{C}_{41}| + (0) + (-4)(-1)^{4+3}|\mathbf{C}_{43}| + (0)$. Now, $\mathbf{C}_{41} = \begin{bmatrix} 4 & 1 & 3 \\ 2 & 3 & -1 \\ 1 & 2 & -5 \end{bmatrix}$. We compute $|\mathbf{C}_{41}|$ using basketweaving.

$$\begin{array}{ccccccccccc} \searrow & & \searrow & & \searrow & & \swarrow & & \swarrow & & \swarrow \\ & 4 & & 1 & & 3 & & 4 & & 1 & \\ & & \times & & \times & & \times & & \times & & \\ & 2 & & 3 & & -1 & & 2 & & 3 & \\ & & \times & & \times & & \times & & \times & & \\ & 1 & & 2 & & -5 & & 1 & & 2 & \\ \swarrow & & \swarrow & & \swarrow & & \searrow & & \searrow & & \searrow \end{array}$$

This yields $|\mathbf{C}_{41}| = (4)(3)(-5)+(1)(-1)(1)+(3)(2)(2)-(3)(3)(1)-(4)(-1)(2)-(1)(2)(-5) = -40$.
Similarly, $\mathbf{C}_{43} = \begin{bmatrix} 0 & 4 & 3 \\ 2 & 2 & -1 \\ 3 & 1 & -5 \end{bmatrix}$. We compute $|\mathbf{C}_{43}|$ using basketweaving.

$$\begin{array}{ccccccccccc} \searrow & & \searrow & & \searrow & & \swarrow & & \swarrow & & \swarrow \\ & 0 & & 4 & & 3 & & 0 & & 4 & \\ & & \times & & \times & & \times & & \times & & \\ & 2 & & 2 & & -1 & & 2 & & 2 & \\ & & \times & & \times & & \times & & \times & & \\ & 3 & & 1 & & -5 & & 3 & & 1 & \\ \swarrow & & \swarrow & & \swarrow & & \searrow & & \searrow & & \searrow \end{array}$$

This yields $|\mathbf{C}_{43}| = (0)(2)(-5)+(4)(-1)(3)+(3)(2)(1)-(3)(2)(3)-(0)(-1)(1)-(4)(2)(-5) = 16$.
Therefore, $|\mathbf{A}_{55}| = (1)(-1)^{4+1}|\mathbf{C}_{41}| + (-4)(-1)^{4+3}|\mathbf{C}_{43}| = (1)(-1)(-40) + (-4)(-1)(16) = 104$.
Finally, $|\mathbf{A}| = (0) + (3)(-1)|\mathbf{A}_{52}| + (2)(1)|\mathbf{A}_{55}| = (3)(-1)(-48) + (2)(1)(104) = 352$.

(7) Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and let $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $|\mathbf{A}| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = (1)(1) - (1)(1) = 0$ and $|\mathbf{B}| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = (1)(1) - (0)(0) = 1$. Hence, $|\mathbf{A}| + |\mathbf{B}| = 1$. But, $|\mathbf{A}+\mathbf{B}| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = (2)(2) - (1)(1) = 3$. Hence, $|\mathbf{A}+\mathbf{B}| \neq |\mathbf{A}| + |\mathbf{B}|$.

(9) According to part (1) of Theorem 3.1, the area of the parallelogram is the absolute value of the determinant of the matrix whose rows are the two given vectors. Be careful! If the determinant is negative, we must take the absolute value to find the area.

(a) The area of the parallelogram is the absolute value of the determinant $\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix}$. This determinant equals $(3)(5) - (2)(4) = 7$. Hence, the area is 7.

(c) The area of the parallelogram is the absolute value of the determinant $\begin{vmatrix} 5 & -1 \\ -3 & 3 \end{vmatrix}$. This determinant equals $(5)(3) - (-1)(-3) = 12$. Hence, the area is 12.

(10) Let $\mathbf{x} = [x_1, x_2]$ and $\mathbf{y} = [y_1, y_2]$. Consider the hint in the text. Then the area of the parallelogram is $\|\mathbf{x}\|\,\|\mathbf{h}\|$, where $\mathbf{h} = \mathbf{y} - \mathbf{proj_x y}$. Now, $\mathbf{proj_x y} = \left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|^2}\right)\mathbf{x} =$
$\frac{1}{\|\mathbf{x}\|^2}[(x_1y_1 + x_2y_2)x_1, (x_1y_1 + x_2y_2)x_2]$. Hence, $\mathbf{h} = \mathbf{y} - \mathbf{proj_x y} = \frac{1}{\|\mathbf{x}\|^2}(\|\mathbf{x}\|^2\mathbf{y}) - \left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|^2}\right)\mathbf{x} =$
$\frac{1}{\|\mathbf{x}\|^2}[(x_1^2 + x_2^2)y_1, (x_1^2 + x_2^2)y_2] - \frac{1}{\|\mathbf{x}\|^2}[(x_1y_1 + x_2y_2)x_1, (x_1y_1 + x_2y_2)x_2]$
$= \frac{1}{\|\mathbf{x}\|^2}[x_1^2y_1 + x_2^2y_1 - x_1^2y_1 - x_1x_2y_2, x_1^2y_2 + x_2^2y_2 - x_1x_2y_1 - x_2^2y_2] =$
$\frac{1}{\|\mathbf{x}\|^2}[x_2^2y_1 - x_1x_2y_2, x_1^2y_2 - x_1x_2y_1] = \frac{1}{\|\mathbf{x}\|^2}[x_2(x_2y_1 - x_1y_2), x_1(x_1y_2 - x_2y_1)] = \frac{x_1y_2 - x_2y_1}{\|\mathbf{x}\|^2}[-x_2, x_1]$. Thus, $\|\mathbf{x}\|\,\|\mathbf{y} - \mathbf{proj_x y}\| = \|\mathbf{x}\|\frac{|x_1y_2 - x_2y_1|}{\|\mathbf{x}\|^2}\sqrt{x_2^2 + x_1^2} = |x_1y_2 - x_2y_1| =$ absolute value of $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$.

(11) According to part (2) of Theorem 3.1, the volume of the parallelepiped is the absolute value of the determinant of the matrix whose rows are the three given vectors. Be careful! If the determinant is negative, we must take the absolute value to find the volume.

(a) The volume of the parallelepiped is the absolute value of the determinant $\begin{vmatrix} -2 & 3 & 1 \\ 4 & 2 & 0 \\ -1 & 3 & 2 \end{vmatrix}$. We use basketweaving to compute this.

$$\begin{array}{ccccccccccc}
\searrow & & \searrow & & \searrow & & \swarrow & & \swarrow & & \swarrow \\
 & -2 & & 3 & & 1 & & -2 & & 3 & \\
 & & \times & & \times & & \times & & \times & & \\
 & 4 & & 2 & & 0 & & 4 & & 2 & \\
 & & \times & & \times & & \times & & \times & & \\
 & -1 & & 3 & & 2 & & -1 & & 3 & \\
\swarrow & & \swarrow & & \swarrow & & \searrow & & \searrow & & \searrow
\end{array}$$

The determinant equals $(-2)(2)(2)+(3)(0)(-1)+(1)(4)(3)-(1)(2)(-1)-(-2)(0)(3)-(3)(4)(2) = -18$. The volume is the absolute value of this result. Hence, the volume equals 18.

(c) The volume of the parallelepiped is the absolute value of the determinant $\begin{vmatrix} -3 & 4 & 0 \\ 6 & -2 & 1 \\ 0 & -3 & 3 \end{vmatrix}$. We use basketweaving to compute this.

$$\begin{array}{ccccccccccc}
\searrow & & \searrow & & \searrow & & \swarrow & & \swarrow & & \swarrow \\
 & -3 & & 4 & & 0 & & -3 & & 4 & \\
 & & \times & & \times & & \times & & \times & & \\
 & 6 & & -2 & & 1 & & 6 & & -2 & \\
 & & \times & & \times & & \times & & \times & & \\
 & 0 & & -3 & & 3 & & 0 & & -3 & \\
\swarrow & & \swarrow & & \swarrow & & \searrow & & \searrow & & \searrow
\end{array}$$

The determinant equals $(-3)(-2)(3) + (4)(1)(0) + (0)(6)(-3) - (0)(-2)(0) - (-3)(1)(-3) - (4)(6)(3) = -63$. The volume is the absolute value of this result. Hence, the volume equals 63.

(12) First, read the hint in the textbook. Then note that

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_3y_2z_1 - x_1y_3z_2 - x_2y_1z_3 =$$

$(x_2y_3 - x_3y_2)z_1 + (x_3y_1 - x_1y_3)z_2 + (x_1y_2 - x_2y_1)z_3 = (\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z}$ (from the definition in Exercise 8). Also note that the formula $\sqrt{(x_2y_3 - x_3y_2)^2 + (x_1y_3 - x_3y_1)^2 + (x_1y_2 - x_2y_1)^2}$ given in the hint for this exercise for the area of the parallelogram (verified below) equals $\|\mathbf{x} \times \mathbf{y}\|$. Hence, the volume of the parallelepiped equals $\|\mathbf{proj}_{(\mathbf{x}\times\mathbf{y})}\mathbf{z}\| \, \|\mathbf{x} \times \mathbf{y}\| = \left|\frac{\mathbf{z}\cdot(\mathbf{x}\times\mathbf{y})}{\|\mathbf{x}\times\mathbf{y}\|}\right| \|\mathbf{x} \times \mathbf{y}\| = |\mathbf{z}\cdot(\mathbf{x} \times \mathbf{y})| =$ absolute value of $\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$.

Now let us verify the formula $A = \sqrt{(x_2y_3 - x_3y_2)^2 + (x_1y_3 - x_3y_1)^2 + (x_1y_2 - x_2y_1)^2}$ for the area of the parallelogram determined by $\mathbf{x}$ and $\mathbf{y}$. As in the solution for Exercise 10, above, the area of this parallelogram equals $\|\mathbf{x}\| \, \|\mathbf{y} - \mathbf{proj}_{\mathbf{x}}\mathbf{y}\|$. We must show that $A = \|\mathbf{x}\| \, \|\mathbf{y} - \mathbf{proj}_{\mathbf{x}}\mathbf{y}\|$. We can verify this by a tedious, brute force, argument. (Algebraically expand and simplify $\|\mathbf{x}\|^2 \, \|\mathbf{y} - \mathbf{proj}_{\mathbf{x}}\mathbf{y}\|^2$ to get $(x_2y_3 - x_3y_2)^2 + (x_1y_3 - x_3y_1)^2 + (x_1y_2 - x_2y_1)^2$.) An alternate approach is the following:

Now, $A^2 = (x_2y_3 - x_3y_2)^2 + (x_1y_3 - x_3y_1)^2 + (x_1y_2 - x_2y_1)^2$
$= x_2^2y_3^2 - 2x_2x_3y_2y_3 + x_3^2y_2^2 + x_1^2y_3^2 - 2x_1x_3y_1y_3 + x_3^2y_1^2 + x_1^2y_2^2 - 2x_1x_2y_1y_2 + x_2^2y_1^2$.
Using some algebraic manipulation, we can express this as
$A^2 = (x_1^2+x_2^2+x_3^2)(y_1^2+y_2^2+y_3^2) - (x_1y_1+x_2y_2+x_3y_3)$. (Verify this!) Therefore, $A^2 = \|\mathbf{x}\|^2 \, \|\mathbf{y}\|^2 - (\mathbf{x}\cdot\mathbf{y})^2$.
Now,

$$\begin{aligned}
\|\mathbf{x}\|^2\|\mathbf{y} - \mathbf{proj}_{\mathbf{x}}\mathbf{y}\|^2 &= \|\mathbf{x}\|^2 \left\|\mathbf{y} - \frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|^2}\mathbf{x}\right\|^2 \\
&= \|\mathbf{x}\|^2 \left(\mathbf{y} - \frac{(\mathbf{x}\cdot\mathbf{y})\,\mathbf{x}}{\|\mathbf{x}\|^2}\right) \cdot \left(\mathbf{y} - \frac{(\mathbf{x}\cdot\mathbf{y})\,\mathbf{x}}{\|\mathbf{x}\|^2}\right) \\
&= \|\mathbf{x}\|^2 \left((\mathbf{y}\cdot\mathbf{y}) - 2\left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|^2}\right)(\mathbf{x}\cdot\mathbf{y}) + \left(\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|^2}\right)^2 (\mathbf{x}\cdot\mathbf{x})\right) \\
&= \|\mathbf{x}\|^2 \left(\frac{\|\mathbf{x}\|^2(\mathbf{y}\cdot\mathbf{y})}{\|\mathbf{x}\|^2} - 2\left(\frac{(\mathbf{x}\cdot\mathbf{y})^2}{\|\mathbf{x}\|^2}\right) + \frac{(\mathbf{x}\cdot\mathbf{y})^2}{\|\mathbf{x}\|^2}\right) \\
&= \|\mathbf{x}\|^2\,\|\mathbf{y}\|^2 - (\mathbf{x}\cdot\mathbf{y})^2 = A^2.
\end{aligned}$$

Hence, $A = \|\mathbf{x}\| \, \|\mathbf{y} - \mathbf{proj}_{\mathbf{x}}\mathbf{y}\|$.

(15) (a) $\begin{vmatrix} x & 2 \\ 5 & x+3 \end{vmatrix} = 0 \Rightarrow x(x+3) - (2)(5) = 0 \Rightarrow x^2 + 3x - 10 = 0 \Rightarrow (x+5)(x-2) = 0 \Rightarrow x = -5$ or $x = 2$.

(c) First, we compute the given determinant using basketweaving.

$$
\begin{array}{ccccccccccc}
\searrow & & \searrow & & \searrow & & \swarrow & & \swarrow & & \swarrow \\
& x-3 & & 5 & & -19 & & x-3 & & 5 & \\
& & \times & & \times & & \times & & \times & & \\
& 0 & & x-1 & & 6 & & 0 & & x-1 & \\
& & \times & & \times & & \times & & \times & & \\
& 0 & & 0 & & x-2 & & 0 & & 0 & \\
\swarrow & & \swarrow & & \swarrow & & \searrow & & \searrow & & \searrow
\end{array}
$$

The determinant equals $(x-3)(x-1)(x-2) + (5)(6)(0) + (-19)(0)(0) - (-19)(x-1)(0)$ $-\,(x-3)(6)(0) - (5)(0)(x-2) = (x-3)(x-1)(x-2)$. Setting this determinant equal to zero yields $x = 3$, $x = 1$, or $x = 2$.

(16) (b) The given matrix is a 3×3 Vandermonde matrix with $a = 2$, $b = 3$, and $c = -2$. Thus, when we use part (a), the determinant is $(a-b)(b-c)(c-a) = \Big(2-3\Big)\Big(3-(-2)\Big)\Big((-2)-2\Big) = 20$.

(18) (a) False. The basketweaving technique only works to find determinants of 3×3 matrices. For larger matrices, the only method we have learned at this point is the cofactor expansion along the last row, although alternate methods are given in Sections 3.2 and 3.3.

(b) True. Note that $\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1y_2 - x_2y_1$. Part (1) of Theorem 3.1 assures us that the absolute value of this determinant gives the area of the desired parallelogram.

(c) False in general. (True for $n = 2$.) An $n \times n$ matrix has n^2 cofactors – one corresponding to each entry in the matrix.

(d) False in general. (True in the special case in which $\mathcal{B}_{23} = 0$.). By definition, $\mathcal{B}_{23} = (-1)^{2+3}|\mathbf{B}_{23}| = -|\mathbf{B}_{23}|$, *not* $|\mathbf{B}_{23}|$.

(e) True. The given formula is the cofactor expansion of $\mathbf{A}$ along the last row. By definition, this equals the determinant of $\mathbf{A}$.

Section 3.2

(1) (a) The row operation used is (II): $\langle 1\rangle \leftarrow -3\,\langle 2\rangle + \langle 1\rangle$. By Theorem 3.3, performing a type (II) operation does not change the determinant of a matrix. Thus, since $|\mathbf{I}_3| = 1$, the determinant of the given matrix also equals 1.

(c) The row operation used is (I): $\langle 3\rangle \leftarrow -4\,\langle 3\rangle$. By Theorem 3.3, performing a type (I) operation multiplies the determinant by the constant used in the row operation. Thus, since $|\mathbf{I}_3| = 1$, the determinant of the given matrix equals $(-4)(1) = -4$.

(f) The row operation used is (III): $\langle 1\rangle \leftrightarrow \langle 2\rangle$. By Theorem 3.3, performing a type (III) row operation changes the sign of the determinant. Thus, since $|\mathbf{I}_3| = 1$, the determinant of the given matrix equals -1.

(2) In each part, we use row operations to put the given matrix into upper triangular form. Notice in the solutions, below, that we stop performing row operations as soon as we obtain an upper triangular

matrix. We give a chart indicating the row operations used, keeping track of the variable P, as described in Example 5 in Section 3.2 of the textbook. Recall that $P = 1$ at the beginning of the process. We then use the final upper triangular matrix obtained and the value of P to compute the desired determinant.

(a)

Row Operations	**Effect**	P
(I): $\langle 1\rangle \leftarrow \frac{1}{10}\langle 1\rangle$	Multiply P by $\frac{1}{10}$	$\frac{1}{10}$
(II): $\langle 3\rangle \leftarrow 5\langle 1\rangle + \langle 3\rangle$	No change	$\frac{1}{10}$
(I): $\langle 2\rangle \leftarrow -\frac{1}{4}\langle 2\rangle$	Multiply P by $-\frac{1}{4}$	$-\frac{1}{40}$
(II): $\langle 3\rangle \leftarrow (-1)\langle 2\rangle + \langle 3\rangle$	No change	$-\frac{1}{40}$

The upper triangular matrix obtained from these operations is $\mathbf{B} = \begin{bmatrix} 1 & \frac{2}{5} & \frac{21}{10} \\ 0 & 1 & -\frac{3}{4} \\ 0 & 0 & -\frac{3}{4} \end{bmatrix}$. By Theorem 3.2, $|\mathbf{B}| = (1)(1)(-\frac{3}{4}) = -\frac{3}{4}$. Hence, as in Example 5 in the textbook, the determinant of the original matrix is $\frac{1}{P} \times |\mathbf{B}| = (-40)(-\frac{3}{4}) = 30$.

(c)

Row Operations	**Effect**	P
(II): $\langle 2\rangle \leftarrow 2\langle 1\rangle + \langle 2\rangle$	No change	1
(II): $\langle 3\rangle \leftarrow 3\langle 1\rangle + \langle 3\rangle$	No change	1
(II): $\langle 4\rangle \leftarrow (-2)\langle 1\rangle + \langle 4\rangle$	No change	1
(I): $\langle 2\rangle \leftarrow (-1)\langle 2\rangle$	Multiply P by -1	-1
(II): $\langle 3\rangle \leftarrow \langle 2\rangle + \langle 3\rangle$	No change	-1
(II): $\langle 4\rangle \leftarrow (-1)\langle 2\rangle + \langle 4\rangle$	No change	-1
(III): $\langle 3\rangle \leftrightarrow \langle 4\rangle$	Multiply P by -1	1

The upper triangular matrix obtained from these operations is $\mathbf{B} = \begin{bmatrix} 1 & -1 & 5 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$. By Theorem 3.2, $|\mathbf{B}| = (1)(1)(2)(-2) = -4$. Hence, as in Example 5 in the textbook, the determinant of the original matrix is $\frac{1}{P} \times |\mathbf{B}| = 1(-4) = -4$.

(e)

Row Operations	**Effect**	P
(I): $\langle 1\rangle \leftarrow \frac{1}{5}\langle 1\rangle$	Multiply P by $\frac{1}{5}$	$\frac{1}{5}$
(II): $\langle 2\rangle \leftarrow (-\frac{15}{2})\langle 1\rangle + \langle 2\rangle$	No change	$\frac{1}{5}$
(II): $\langle 3\rangle \leftarrow \frac{5}{2}\langle 1\rangle + \langle 3\rangle$	No change	$\frac{1}{5}$
(II): $\langle 4\rangle \leftarrow (-10)\langle 1\rangle + \langle 4\rangle$	No change	$\frac{1}{5}$
(I): $\langle 2\rangle \leftarrow (-\frac{1}{4})\langle 2\rangle$	Multiply P by $-\frac{1}{4}$	$-\frac{1}{20}$
(II): $\langle 3\rangle \leftarrow (-3)\langle 2\rangle + \langle 3\rangle$	No change	$-\frac{1}{20}$
(II): $\langle 4\rangle \leftarrow 9\langle 2\rangle + \langle 4\rangle$	No change	$-\frac{1}{20}$
(I): $\langle 3\rangle \leftarrow 4\langle 3\rangle$	Multiply P by 4	$-\frac{1}{5}$
(II): $\langle 4\rangle \leftarrow \frac{3}{4}\langle 3\rangle + \langle 4\rangle$	No change	$-\frac{1}{5}$

The upper triangular matrix obtained from these operations is $\mathbf{B} = \begin{bmatrix} 1 & \frac{3}{5} & -\frac{8}{5} & \frac{4}{5} \\ 0 & 1 & -\frac{11}{4} & \frac{13}{4} \\ 0 & 0 & 1 & -27 \\ 0 & 0 & 0 & -7 \end{bmatrix}$. By Theorem 3.2, $|\mathbf{B}| = (1)(1)(1)(-7) = -7$. Hence, as in Example 5 in the textbook, the determinant of the original matrix is $\frac{1}{P} \times |\mathbf{B}| = (-5)(-7) = 35$.

(3) In this problem, first compute the determinant by any convenient method.

(a) $\begin{vmatrix} 5 & 6 \\ -3 & -4 \end{vmatrix} = (5)(-4) - (6)(-3) = -2$. Since the determinant is nonzero, Theorem 3.5 implies that the given matrix is nonsingular.

(c) Using basketweaving on the given matrix produces the value $(-12)(-1)(-8) + (7)(2)(3) + (-27)(4)(2) - (-27)(-1)(3) - (-12)(2)(2) - (7)(4)(-8) = -79$ for the determinant. Since the determinant is nonzero, Theorem 3.5 implies that the given matrix is nonsingular.

(4) (a) The coefficient matrix for the given system is $\mathbf{A} = \begin{bmatrix} -6 & 3 & -22 \\ -7 & 4 & -31 \\ 11 & -6 & 46 \end{bmatrix}$. Using basketweaving yields $|\mathbf{A}| = (-6)(4)(46) + (3)(-31)(11) + (-22)(-7)(-6) - (-22)(4)(11) - (-6)(-31)(-6) - (3)(-7)(46) = -1$. Because $|\mathbf{A}| \neq 0$, Corollary 3.6 shows that $\text{rank}(\mathbf{A}) = 3$. Hence, by Theorem 2.5, the system has only the trivial solution.

(6) Perform the following type (III) row operations on $\mathbf{A}$:

$\begin{cases} \text{(III)}: \langle 1 \rangle \leftrightarrow \langle 6 \rangle \\ \text{(III)}: \langle 2 \rangle \leftrightarrow \langle 5 \rangle \\ \text{(III)}: \langle 3 \rangle \leftrightarrow \langle 4 \rangle \end{cases}$. This produces the matrix $\mathbf{B} = \begin{bmatrix} a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \\ 0 & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ 0 & 0 & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & 0 & 0 & a_{34} & a_{35} & a_{36} \\ 0 & 0 & 0 & 0 & a_{25} & a_{26} \\ 0 & 0 & 0 & 0 & 0 & a_{16} \end{bmatrix}$.

Since $\mathbf{B}$ is upper triangular, Theorem 3.2 gives $|\mathbf{B}| = a_{61}a_{52}a_{43}a_{34}a_{25}a_{16}$. But applying a type (III) row operation to a matrix changes the sign of its determinant. Since we performed three type (III) row operations, $|\mathbf{B}| = -(-(-|\mathbf{A}|))$, or $|\mathbf{A}| = -|\mathbf{B}| = -a_{61}a_{52}a_{43}a_{34}a_{25}a_{16}$.

(16) (a) False in general, although Theorem 3.2 shows that it is true for upper triangular matrices. For a counterexample to the general statement, consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ whose determinant is $(1)(1) - (1)(1) = 0$. However, the product of the main diagonal entries of $\mathbf{A}$ is $(1)(1) = 1$.

(b) True. Part (3) of Theorem 3.3 shows that performing a type (III) row operation changes the sign of the determinant. Therefore, performing two type (III) row operations in succession changes the sign twice. Hence, there is no overall effect on the determinant.

(c) False in general. If $\mathbf{A}$ is a 4×4 matrix, Corollary 3.4 shows that $|3\mathbf{A}| = 3^4|\mathbf{A}| = 81|\mathbf{A}|$. This equals $3|\mathbf{A}|$ only in the exceptional case in which $|\mathbf{A}| = 0$. The matrix $\mathbf{A} = \mathbf{I}_4$ provides a specific counterexample to the original statement.

(d) False. If $\mathbf{A}$ is a matrix having (row i) = (row j), then $|\mathbf{A}| = 0 \neq 1$. To prove this, note that performing the type (III) row operation $\langle i\rangle \leftrightarrow \langle j\rangle$ on $\mathbf{A}$ results in $\mathbf{A}$. Thus, by part (3) of Theorem 3.3, $|\mathbf{A}| = -|\mathbf{A}|$. Hence, $|\mathbf{A}| = 0$. For a specific counterexample to the original statement, consider $\mathbf{O}_n$, for any $n \geq 2$.

(e) False. This statement contradicts Theorem 3.5.

(f) True. This statement is the contrapositive of Corollary 3.6.

Section 3.3

(1) (a) $|\mathbf{A}| = a_{31}\mathcal{A}_{31} + a_{32}\mathcal{A}_{32} + a_{33}\mathcal{A}_{33} + a_{34}\mathcal{A}_{34}$
$= a_{31}(-1)^{3+1}|\mathbf{A}_{31}| + a_{32}(-1)^{3+2}|\mathbf{A}_{32}| + a_{33}(-1)^{3+3}|\mathbf{A}_{33}| + a_{34}(-1)^{3+4}|\mathbf{A}_{34}|$
$= a_{31}|\mathbf{A}_{31}| - a_{32}|\mathbf{A}_{32}| + a_{33}|\mathbf{A}_{33}| - a_{34}|\mathbf{A}_{34}|$.

(c) $|\mathbf{A}| = a_{14}\mathcal{A}_{14} + a_{24}\mathcal{A}_{24} + a_{34}\mathcal{A}_{34} + a_{44}\mathcal{A}_{44}$
$= a_{14}(-1)^{1+4}|\mathbf{A}_{14}| + a_{24}(-1)^{2+4}|\mathbf{A}_{24}| + a_{34}(-1)^{3+4}|\mathbf{A}_{34}| + a_{44}(-1)^{4+4}|\mathbf{A}_{44}|$
$= -a_{14}|\mathbf{A}_{14}| + a_{24}|\mathbf{A}_{24}| - a_{34}|\mathbf{A}_{34}| + a_{44}|\mathbf{A}_{44}|$.

(2) (a) Let $\mathbf{A}$ be the given matrix. Then $|\mathbf{A}| = a_{21}\mathcal{A}_{21} + a_{22}\mathcal{A}_{22} + a_{23}\mathcal{A}_{23} = a_{21}(-1)^{2+1}|\mathbf{A}_{21}| + a_{22}(-1)^{2+2}|\mathbf{A}_{22}| + a_{23}(-1)^{2+3}|\mathbf{A}_{23}| = -(0)\begin{vmatrix} -1 & 4 \\ -2 & -3 \end{vmatrix} + (3)\begin{vmatrix} 2 & 4 \\ 5 & -3 \end{vmatrix} - (-2)\begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} = 0 + (3)\Big((2)(-3) - (4)(5)\Big) - (-2)\Big((2)(-2) - (-1)(5)\Big) = -76$.

(c) Let $\mathbf{C}$ be the given matrix. Then $|\mathbf{C}| = c_{11}\mathcal{C}_{11} + c_{21}\mathcal{C}_{21} + c_{31}\mathcal{C}_{31} = c_{11}(-1)^{1+1}|\mathbf{C}_{11}| + c_{21}(-1)^{2+1}|\mathbf{C}_{21}| + c_{31}(-1)^{3+1}|\mathbf{C}_{31}| = (4)\begin{vmatrix} -1 & -2 \\ 3 & 2 \end{vmatrix} - (5)\begin{vmatrix} -2 & 3 \\ 3 & 2 \end{vmatrix} + (3)\begin{vmatrix} -2 & 3 \\ -1 & -2 \end{vmatrix} = (4)\Big((-1)(2) - (-2)(3)\Big) - (5)\Big((-2)(2) - (3)(3)\Big) + (3)\Big((-2)(-2) - (3)(-1)\Big) = 102$.

(3) (a) Let $\mathbf{A}$ represent the given matrix. First, we compute each cofactor:

$$\mathcal{A}_{11} = (-1)^{1+1}\begin{vmatrix} 0 & -3 \\ -2 & -33 \end{vmatrix} = -6, \qquad \mathcal{A}_{12} = (-1)^{1+2}\begin{vmatrix} 2 & -3 \\ 20 & -33 \end{vmatrix} = 6,$$

$$\mathcal{A}_{21} = (-1)^{2+1}\begin{vmatrix} -1 & -21 \\ -2 & -33 \end{vmatrix} = 9, \qquad \mathcal{A}_{22} = (-1)^{2+2}\begin{vmatrix} 14 & -21 \\ 20 & -33 \end{vmatrix} = -42,$$

$$\mathcal{A}_{31} = (-1)^{3+1}\begin{vmatrix} -1 & -21 \\ 0 & -3 \end{vmatrix} = 3, \qquad \mathcal{A}_{32} = (-1)^{3+2}\begin{vmatrix} 14 & -21 \\ 2 & -3 \end{vmatrix} = 0,$$

$$\mathcal{A}_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 0 \\ 20 & -2 \end{vmatrix} = -4, \qquad \mathcal{A}_{23} = (-1)^{2+3}\begin{vmatrix} 14 & -1 \\ 20 & -2 \end{vmatrix} = 8,$$

$$\mathcal{A}_{33} = (-1)^{3+3}\begin{vmatrix} 14 & -1 \\ 2 & 0 \end{vmatrix} = 2.$$

The adjoint matrix $\mathcal{A}$ is the 3×3 matrix whose (i,j) entry is $\mathcal{A}_{ji}$. Hence, $\mathcal{A} = \begin{bmatrix} -6 & 9 & 3 \\ 6 & -42 & 0 \\ -4 & 8 & 2 \end{bmatrix}$.

Next, to find the determinant, we perform a cofactor expansion along the second column, since $a_{22} = 0$. Using the cofactors we have already computed, $|\mathbf{A}| = a_{12}\mathcal{A}_{12} + a_{22}\mathcal{A}_{22} + a_{32}\mathcal{A}_{32} =$

$(-1)(6)+(0)(-42)+(-2)(0)=-6$. Finally, by Corollary 3.12,

$$\mathbf{A}^{-1}=\frac{1}{|\mathbf{A}|}\mathcal{A}=\frac{1}{(-6)}\begin{bmatrix}-6&9&3\\6&-42&0\\-4&8&2\end{bmatrix}=\begin{bmatrix}1&-\frac{3}{2}&-\frac{1}{2}\\-1&7&0\\\frac{2}{3}&-\frac{4}{3}&-\frac{1}{3}\end{bmatrix}.$$

(c) Let $\mathbf{A}$ represent the given matrix. First, we compute each cofactor. Each of the 3×3 determinants can be calculated using basketweaving.

$$\mathcal{A}_{11}=(-1)^{1+1}\begin{vmatrix}-4&1&4\\11&-2&-8\\10&-2&-7\end{vmatrix}=-3,\qquad \mathcal{A}_{21}=(-1)^{2+1}\begin{vmatrix}1&0&-1\\11&-2&-8\\10&-2&-7\end{vmatrix}=0,$$

$$\mathcal{A}_{31}=(-1)^{3+1}\begin{vmatrix}1&0&-1\\-4&1&4\\10&-2&-7\end{vmatrix}=3,\qquad \mathcal{A}_{41}=(-1)^{4+1}\begin{vmatrix}1&0&-1\\-4&1&4\\11&-2&-8\end{vmatrix}=-3,$$

$$\mathcal{A}_{12}=(-1)^{1+2}\begin{vmatrix}7&1&4\\-14&-2&-8\\-12&-2&-7\end{vmatrix}=0,\qquad \mathcal{A}_{22}=(-1)^{2+2}\begin{vmatrix}-2&0&-1\\-14&-2&-8\\-12&-2&-7\end{vmatrix}=0,$$

$$\mathcal{A}_{32}=(-1)^{3+2}\begin{vmatrix}-2&0&-1\\7&1&4\\-12&-2&-7\end{vmatrix}=0,\qquad \mathcal{A}_{42}=(-1)^{4+2}\begin{vmatrix}-2&0&-1\\7&1&4\\-14&-2&-8\end{vmatrix}=0,$$

$$\mathcal{A}_{13}=(-1)^{1+3}\begin{vmatrix}7&-4&4\\-14&11&-8\\-12&10&-7\end{vmatrix}=-3,\qquad \mathcal{A}_{23}=(-1)^{2+3}\begin{vmatrix}-2&1&-1\\-14&11&-8\\-12&10&-7\end{vmatrix}=0,$$

$$\mathcal{A}_{33}=(-1)^{3+3}\begin{vmatrix}-2&1&-1\\7&-4&4\\-12&10&-7\end{vmatrix}=3,\qquad \mathcal{A}_{43}=(-1)^{4+3}\begin{vmatrix}-2&1&-1\\7&-4&4\\-14&11&-8\end{vmatrix}=-3,$$

$$\mathcal{A}_{14}=(-1)^{1+4}\begin{vmatrix}7&-4&1\\-14&11&-2\\-12&10&-2\end{vmatrix}=6,\qquad \mathcal{A}_{24}=(-1)^{2+4}\begin{vmatrix}-2&1&0\\-14&11&-2\\-12&10&-2\end{vmatrix}=0,$$

$$\mathcal{A}_{34}=(-1)^{3+4}\begin{vmatrix}-2&1&0\\7&-4&1\\-12&10&-2\end{vmatrix}=-6,\qquad \mathcal{A}_{44}=(-1)^{4+4}\begin{vmatrix}-2&1&0\\7&-4&1\\-14&11&-2\end{vmatrix}=6.$$

The adjoint matrix $\mathcal{A}$ is the 4×4 matrix whose (i,j) entry is $\mathcal{A}_{ji}$. So, $\mathcal{A}=\begin{bmatrix}-3&0&3&-3\\0&0&0&0\\-3&0&3&-3\\6&0&-6&6\end{bmatrix}$.

Next, we use a cofactor expansion along the first row to compute the determinant of $\mathbf{A}$ because $a_{13}=0$. Using the cofactors we calculated above:

$$|\mathbf{A}|=a_{11}\mathcal{A}_{11}+a_{12}\mathcal{A}_{12}+a_{13}\mathcal{A}_{13}+a_{14}\mathcal{A}_{14}=(-2)(-3)+(1)(0)+(0)(-3)+(-1)(6)=0.$$

Finally, since $|\mathbf{A}|=0$, $\mathbf{A}$ has no inverse, by Theorem 3.5.

(e) Let $\mathbf{A}$ represent the given matrix. First, we compute each cofactor:

$$\mathcal{A}_{11} = (-1)^{1+1}\begin{vmatrix} -3 & 2 \\ 0 & -1 \end{vmatrix} = 3, \qquad \mathcal{A}_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 2 \\ 0 & -1 \end{vmatrix} = 0,$$

$$\mathcal{A}_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = -1, \qquad \mathcal{A}_{22} = (-1)^{2+2}\begin{vmatrix} 3 & 0 \\ 0 & -1 \end{vmatrix} = -3,$$

$$\mathcal{A}_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 0 \\ -3 & 2 \end{vmatrix} = -2, \qquad \mathcal{A}_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = -6,$$

$$\mathcal{A}_{13} = (-1)^{1+3}\begin{vmatrix} 0 & -3 \\ 0 & 0 \end{vmatrix} = 0, \qquad \mathcal{A}_{23} = (-1)^{2+3}\begin{vmatrix} 3 & -1 \\ 0 & 0 \end{vmatrix} = 0,$$

$$\mathcal{A}_{33} = (-1)^{3+3}\begin{vmatrix} 3 & -1 \\ 0 & -3 \end{vmatrix} = -9.$$

The adjoint matrix $\mathcal{A}$ is the 3×3 matrix whose (i,j) entry is $\mathcal{A}_{ji}$. Hence, $\mathcal{A} = \begin{bmatrix} 3 & -1 & -2 \\ 0 & -3 & -6 \\ 0 & 0 & -9 \end{bmatrix}$.

Next, since $\mathbf{A}$ is upper triangular, $|\mathbf{A}| = (3)(-3)(-1) = 9$. Finally, by Corollary 3.12,

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathcal{A} = \frac{1}{9}\begin{bmatrix} 3 & -1 & -2 \\ 0 & -3 & -6 \\ 0 & 0 & -9 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{9} & -\frac{2}{9} \\ 0 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & -1 \end{bmatrix}.$$

(4) (a) Let $\mathbf{A}$ be the coefficient matrix $\begin{bmatrix} 3 & -1 & -1 \\ 2 & -1 & -2 \\ -9 & 1 & 0 \end{bmatrix}$. The matrices $\mathbf{A}_1$, $\mathbf{A}_2$, and $\mathbf{A}_3$ are formed by removing, respectively, the first, second, and third columns, from $\mathbf{A}$, and replacing them with $[-8, 3, 39]$ (as a column vector). Hence, $\mathbf{A}_1 = \begin{bmatrix} -8 & -1 & -1 \\ 3 & -1 & -2 \\ 39 & 1 & 0 \end{bmatrix}$, $\mathbf{A}_2 = \begin{bmatrix} 3 & -8 & -1 \\ 2 & 3 & -2 \\ -9 & 39 & 0 \end{bmatrix}$, and $\mathbf{A}_3 = \begin{bmatrix} 3 & -1 & -8 \\ 2 & -1 & 3 \\ -9 & 1 & 39 \end{bmatrix}$. Basketweaving, or any other convenient method, can be used to compute the determinant of each of these matrices, yielding $|\mathbf{A}| = -5$, $|\mathbf{A}_1| = 20$, $|\mathbf{A}_2| = -15$, and $|\mathbf{A}_3| = 35$. Therefore, Cramer's Rule states that the unique solution to the system is given by $x_1 = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \frac{20}{(-5)} = -4$, $x_2 = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{(-15)}{(-5)} = 3$, and $x_3 = \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = \frac{35}{(-5)} = -7$. The full solution set is $\{(-4, 3, -7)\}$.

(d) Let $\mathbf{A}$ be the coefficient matrix $\begin{bmatrix} -5 & 2 & -2 & 1 \\ 2 & -1 & 2 & -2 \\ 5 & -2 & 3 & -1 \\ -6 & 2 & -2 & 1 \end{bmatrix}$. The matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, and $\mathbf{A}_4$ are formed by removing, respectively, the first, second, third, and fourth columns, from $\mathbf{A}$, and replacing them with $[-10, -9, 7, -14]$ (as a column vector). Hence,

$$\mathbf{A}_1 = \begin{bmatrix} -10 & 2 & -2 & 1 \\ -9 & -1 & 2 & -2 \\ 7 & -2 & 3 & -1 \\ -14 & 2 & -2 & 1 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} -5 & -10 & -2 & 1 \\ 2 & -9 & 2 & -2 \\ 5 & 7 & 3 & -1 \\ -6 & -14 & -2 & 1 \end{bmatrix}, \mathbf{A}_3 = \begin{bmatrix} -5 & 2 & -10 & 1 \\ 2 & -1 & -9 & -2 \\ 5 & -2 & 7 & -1 \\ -6 & 2 & -14 & 1 \end{bmatrix}$$

and $\mathbf{A}_4 = \begin{bmatrix} -5 & 2 & -2 & -10 \\ 2 & -1 & 2 & -9 \\ 5 & -2 & 3 & 7 \\ -6 & 2 & -2 & -14 \end{bmatrix}$. Row reduction or cofactor expansion can be used to compute the determinant of each of these matrices, yielding $|\mathbf{A}| = 3$, $|\mathbf{A}_1| = 12$, $|\mathbf{A}_2| = -3$, $|\mathbf{A}_3| = -9$, and $|\mathbf{A}_4| = 18$. Therefore, Cramer's Rule states that the unique solution to the system is given by $x_1 = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \frac{12}{3} = 4$, $x_2 = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{(-3)}{3} = -1$, $x_3 = \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = \frac{(-9)}{3} = -3$, and $x_4 = \frac{|\mathbf{A}_4|}{|\mathbf{A}|} = \frac{18}{3} = 6$. The full solution set is $\{(4, -1, -3, 6)\}$.

(8) (b) Try $n = 2$. All 2×2 skew-symmetric matrices are of the form $\begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}$. Letting $a = 1$ yields $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Note that $|\mathbf{A}| = 1 \neq 0$.

(9) (b) Use trial and error, rearranging the rows of $\mathbf{I}_3$. For example, $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ works, since $\mathbf{A}\mathbf{A}^T = \mathbf{A}\mathbf{A}$ (because $\mathbf{A}$ is symmetric) $= \mathbf{I}_3$. (Note: All of the four other matrices obtained by rearranging the rows of $\mathbf{I}_3$ are also orthogonal matrices!)

(13) (b) Simply choose any nonsingular 2×2 matrix $\mathbf{P}$ and compute $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$. For example, using $\mathbf{P} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, we get $\mathbf{P}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ (see Theorem 2.13), yielding

$\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -6 & -4 \\ 16 & 11 \end{bmatrix}$. Similarly, letting $\mathbf{P} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ produces $\mathbf{B} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 10 & -12 \\ 4 & -5 \end{bmatrix}$.

(14) By Corollary 3.12, $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathcal{A}$, and $\mathbf{B}^{-1} = \frac{1}{|\mathbf{B}|}\mathcal{B}$. Hence, by part (3) of Theorem 2.11, $(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} = \left(\frac{1}{|\mathbf{B}|}\mathcal{B}\right)\left(\frac{1}{|\mathbf{A}|}\mathcal{A}\right) = \mathcal{B}\mathcal{A}/(|\mathbf{A}|\,|\mathbf{B}|)$.

(18) (b) If we try a skew-symmetric 2×2 matrix as a possible example, we find that the adjoint *is* skew-symmetric. So we consider a skew-symmetric 3×3 matrix, such as $\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$. Then each of the nine minors $|\mathbf{A}_{ij}|$ is easily seen to equal 1. Hence, the (i, j) entry of $\mathcal{A}$ is $\mathcal{A}_{ji} = (-1)^{j+i}|\mathbf{A}_{ji}| = (-1)^{j+i}(1) = (-1)^{j+i}$. Therefore, $\mathcal{A} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$, which is not skew-symmetric.

(22) (a) True. Corollary 3.8 shows that $|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|}$. But Theorem 3.9 asserts that $|\mathbf{A}| = |\mathbf{A}^T|$. Combining these two equations produces the desired result.

(b) True. According to Theorem 3.10, both cofactor expansions produce $|\mathbf{A}|$ as an answer. (The size of the square matrix is not important.)

(c) False. Type (III) column operations applied to square matrices change the sign of the determinant, just as type (III) row operations do. Hence, $|\mathbf{B}| = -|\mathbf{A}|$. The equation $|\mathbf{B}| = |\mathbf{A}|$ can be true only in the special case in which $|\mathbf{A}| = 0$. For a particular counterexample to $|\mathbf{B}| = |\mathbf{A}|$, consider $\mathbf{A} = \mathbf{I}_2$ and use the column operation $\langle \text{col.}\ 1\rangle \leftrightarrow \langle \text{col.}\ 2\rangle$. Then $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Note that $|\mathbf{B}| = -1$, but $|\mathbf{A}| = 1$.

(d) True. The (i, j) entry of the adjoint is defined to be $\mathcal{A}_{ji} = (-1)^{j+i}|\mathbf{A}_{ji}|$, which clearly equals $(-1)^{i+j}|\mathbf{A}_{ji}|$.

(e) False. Theorem 3.11 states that $\mathbf{A}\mathcal{A} = |\mathbf{A}|\,\mathbf{I}$. Hence, the equation $\mathbf{A}\mathcal{A} = \mathbf{I}$ can only be true when $|\mathbf{A}| = 1$. For a specific counterexample, consider $\mathbf{A} = 2\mathbf{I}_2$. A very short computation shows that $\mathcal{A} = 2\mathbf{I}_2$ as well. Hence, $\mathbf{A}\mathcal{A} = (2\mathbf{I}_2)(2\mathbf{I}_2) = 4\mathbf{I}_2 \neq \mathbf{I}_2$.

(f) True. This follows directly from Cramer's Rule. The coefficient matrix $\mathbf{A}$ for the given system is upper triangular, and so $|\mathbf{A}|$ is easily found to be $(4)(-3)(1) = -12$ (see Theorem 3.2). Notice that $\mathbf{A}_2 = \begin{bmatrix} 4 & -6 & -1 \\ 0 & 5 & 4 \\ 0 & 3 & 1 \end{bmatrix}$. Cramer's Rule then states that $x_2 = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = -\frac{1}{12}|\mathbf{A}_2|$.

(23) (a) Let R be the type (III) row operation $\langle k\rangle \longleftrightarrow \langle k-1\rangle$, let $\mathbf{A}$ be an $n \times n$ matrix, and let $\mathbf{B} = R(\mathbf{A})$. Then, by part (3) of Theorem 3.3, $|\mathbf{B}| = (-1)|\mathbf{A}|$. Next, notice that the submatrix $\mathbf{A}_{(k-1)j} = \mathbf{B}_{kj}$ because the $(k-1)$st row of $\mathbf{A}$ becomes the kth row of $\mathbf{B}$, implying the same row is being eliminated from both matrices, and since all other rows maintain their original relative positions (notice the same column is being eliminated in both cases). Hence, $a_{(k-1)1}\mathcal{A}_{(k-1)1} + a_{(k-1)2}\mathcal{A}_{(k-1)2} + \cdots + a_{(k-1)n}\mathcal{A}_{(k-1)n} = b_{k1}(-1)^{(k-1)+1}|\mathbf{A}_{(k-1)1}| + b_{k2}(-1)^{(k-1)+2}|\mathbf{A}_{(k-1)2}| + \cdots + b_{kn}(-1)^{(k-1)+n}|\mathbf{A}_{(k-1)n}|$ (because $a_{(k-1)j} = b_{kj}$ for $1 \leq j \leq n$) $= b_{k1}(-1)^{k}|\mathbf{B}_{k1}| + b_{k2}(-1)^{k+1}|\mathbf{B}_{k2}| + \cdots + b_{kn}(-1)^{k+n-1}|\mathbf{B}_{kn}| = (-1)(b_{k1}(-1)^{k+1}|\mathbf{B}_{k1}| + b_{k2}(-1)^{k+2}|\mathbf{B}_{k2}| + \cdots + b_{kk}(-1)^{k+n}|\mathbf{B}_{kn}|) = (-1)|\mathbf{B}|$ (by applying Theorem 3.10 along the kth row of $\mathbf{B}$) $= (-1)|R(\mathbf{A})|$ $= (-1)(-1)|\mathbf{A}| = |\mathbf{A}|$, finishing the proof.

(b) The definition of the determinant is the Base Step for an induction proof on the row number, counting down from n to 1. Part (a) is the Inductive Step.

(24) Suppose row i of $\mathbf{A}$ equals row j of $\mathbf{A}$. Let R be the type (III) row operation $\langle i\rangle \longleftrightarrow \langle j\rangle$: Then, clearly, $\mathbf{A} = R(\mathbf{A})$. Thus, by part (3) of Theorem 3.3, $|\mathbf{A}| = -|\mathbf{A}|$, or $2|\mathbf{A}| = 0$, implying $|\mathbf{A}| = 0$.

(25) Let $\mathbf{A}$, i, j, and $\mathbf{B}$ be as given in the exercise and its hint. Then by Exercise 24, $|\mathbf{B}| = 0$, since its ith and jth rows are the same. Also, since every row of $\mathbf{A}$ equals the corresponding row of $\mathbf{B}$, with the exception of the jth row, the submatrices $\mathbf{A}_{jk}$ and $\mathbf{B}_{jk}$ are equal for $1 \leq k \leq n$. Hence, $\mathcal{A}_{jk} = \mathcal{B}_{jk}$ for $1 \leq k \leq n$. Now, computing the determinant of $\mathbf{B}$ using a cofactor expansion along the jth row (allowed by Exercise 23) yields $0 = |\mathbf{B}| = b_{j1}\mathcal{B}_{j1} + b_{j2}\mathcal{B}_{j2} + \cdots + b_{jn}\mathcal{B}_{jn} = a_{i1}\mathcal{B}_{j1} + a_{i2}\mathcal{B}_{j2} + \cdots + a_{in}\mathcal{B}_{jn}$ (because the jth row of $\mathbf{B}$ equals the ith row of $\mathbf{A}$) $= a_{i1}\mathcal{A}_{j1} + a_{i2}\mathcal{A}_{j2} + \cdots + a_{in}\mathcal{A}_{jn}$, completing the proof.

(26) Using the fact that the general (k, m) entry of $\mathcal{A}$ is $\mathcal{A}_{mk}$, we see that the (i, j) entry of $\mathbf{A}\mathcal{A}$ equals $a_{i1}\mathcal{A}_{j1} + a_{i2}\mathcal{A}_{j2} + \cdots + a_{in}\mathcal{A}_{jn}$. If $i = j$, Exercise 23 implies that this sum equals $|\mathbf{A}|$, while if $i \neq j$,

Exercise 25 implies that the sum equals 0. Hence, $\mathbf{A}\mathcal{A}$ equals $|\mathbf{A}|$ on the main diagonal and 0 off the main diagonal, yielding $(|\mathbf{A}|)\mathbf{I}_n$.

(27) Suppose $\mathbf{A}$ is nonsingular. Then $|\mathbf{A}| \neq 0$ by Theorem 3.5. Thus, by Exercise 26, $\mathbf{A}\left(\frac{1}{|\mathbf{A}|}\mathcal{A}\right) = \mathbf{I}_n$. Therefore, by Theorem 2.9, $\left(\frac{1}{|\mathbf{A}|}\mathcal{A}\right)\mathbf{A} = \mathbf{I}_n$, implying $\mathcal{A}\mathbf{A} = (|\mathbf{A}|)\mathbf{I}_n$.

(28) Using the fact that the general (k,m) entry of $\mathcal{A}$ is $\mathcal{A}_{mk}$, we see that the (j,j) entry of $\mathcal{A}\mathbf{A}$ equals $a_{1j}\mathcal{A}_{1j} + a_{2j}\mathcal{A}_{2j} + \cdots + a_{nj}\mathcal{A}_{nj}$. But all main diagonal entries of $\mathcal{A}\mathbf{A}$ equal $|\mathbf{A}|$ by Exercise 27.

(29) Let $\mathbf{A}$ be a singular $n \times n$ matrix. Then $|\mathbf{A}| = 0$ by Theorem 3.5. By part (4) of Theorem 2.11, $\mathbf{A}^T$ is also singular (or else $\mathbf{A} = (\mathbf{A}^T)^T$ is nonsingular). Hence, $|\mathbf{A}^T| = 0$ by Theorem 3.5, and so $|\mathbf{A}| = |\mathbf{A}^T|$ in this case.

(30) Case 1: Assume $1 \le k < j$ and $1 \le i < m$. Then the (i,k) entry of $(\mathbf{A}_{jm})^T = (k,i)$ entry of $\mathbf{A}_{jm}$ $= (k,i)$ entry of $\mathbf{A} = (i,k)$ entry of $\mathbf{A}^T = (i,k)$ entry of $(\mathbf{A}^T)_{mj}$. Case 2: Assume $j \le k < n$ and $1 \le i < m$. Then the (i,k) entry of $(\mathbf{A}_{jm})^T = (k,i)$ entry of $\mathbf{A}_{jm} = (k+1,i)$ entry of $\mathbf{A} = (i,k+1)$ entry of $\mathbf{A}^T = (i,k)$ entry of $(\mathbf{A}^T)_{mj}$. Case 3: Assume $1 \le k < j$ and $m < i \le n$. Then the (i,k) entry of $(\mathbf{A}_{jm})^T = (k,i)$ entry of $\mathbf{A}_{jm} = (k,i+1)$ entry of $\mathbf{A} = (i+1,k)$ entry of $\mathbf{A}^T = (i,k)$ entry of $(\mathbf{A}^T)_{mj}$. Case 4: Assume $j \le k < n$ and $m < i \le n$. Then the (i,k) entry of $(\mathbf{A}_{jm})^T = (k,i)$ entry of $\mathbf{A}_{jm} = (k+1,i+1)$ entry of $\mathbf{A} = (i+1,k+1)$ entry of $\mathbf{A}^T = (i,k)$ entry of $(\mathbf{A}^T)_{mj}$. Hence, the corresponding entries of $(\mathbf{A}_{jm})^T$ and $(\mathbf{A}^T)_{mj}$ are all equal, proving that the matrices themselves are equal.

(31) Suppose $\mathbf{A}$ is a nonsingular $n \times n$ matrix. Then, by part (4) of Theorem 2.11, $\mathbf{A}^T$ is also nonsingular. We prove $|\mathbf{A}| = |\mathbf{A}^T|$ by induction on n.
Base Step: Assume $n = 1$. Then $\mathbf{A} = [a_{11}] = \mathbf{A}^T$, and so their determinants must be equal.
Inductive Step: Assume that $|\mathbf{B}| = |\mathbf{B}^T|$ for any $(n-1) \times (n-1)$ nonsingular matrix $\mathbf{B}$, and prove that $|\mathbf{A}| = |\mathbf{A}^T|$ for any $n \times n$ nonsingular matrix $\mathbf{A}$. Now, first note that, by Exercise 30, $|(\mathbf{A}^T)_{ni}| = |(\mathbf{A}_{in})^T|$. But, $|(\mathbf{A}_{in})^T| = |\mathbf{A}_{in}|$, either by the inductive hypothesis, if $\mathbf{A}_{in}$ is nonsingular, or by Exercise 29, if $\mathbf{A}_{in}$ is singular. Hence, $|(\mathbf{A}^T)_{ni}| = |\mathbf{A}_{in}|$. Let $\mathbf{D} = \mathbf{A}^T$. So, $|\mathbf{D}_{ni}| = |\mathbf{A}_{in}|$. Then, by Exercise 28, $|\mathbf{A}| = a_{1n}\mathcal{A}_{1n} + a_{2n}\mathcal{A}_{2n} + \cdots + a_{nn}\mathcal{A}_{nn} = d_{n1}(-1)^{1+n}|\mathbf{A}_{1n}| + d_{n2}(-1)^{2+n}|\mathbf{A}_{2n}| + \cdots + d_{nn}(-1)^{n+n}|\mathbf{A}_{nn}| = d_{n1}(-1)^{1+n}|\mathbf{D}_{n1}| + d_{n2}(-1)^{2+n}|\mathbf{D}_{n2}| + \cdots + d_{nn}(-1)^{n+n}|\mathbf{D}_{nn}| = |\mathbf{D}|$, since this is the cofactor expansion for $|\mathbf{D}|$ along the last row of $\mathbf{D}$. This completes the proof.

(32) Let $\mathbf{A}$ be an $n \times n$ singular matrix. Let $\mathbf{D} = \mathbf{A}^T$. $\mathbf{D}$ is also singular by part (4) of Theorem 2.11 (or else $\mathbf{A} = \mathbf{D}^T$ is nonsingular). Now, Exercise 30 shows that $|\mathbf{D}_{jk}| = |(\mathbf{A}_{kj})^T|$, which equals $|\mathbf{A}_{kj}|$, by Exercise 29 if $\mathbf{A}_{kj}$ is singular, or by Exercise 31 if $\mathbf{A}_{kj}$ is nonsingular. So, $|\mathbf{A}| = |\mathbf{D}|$ (by Exercise 29) $= d_{j1}\mathcal{D}_{j1} + d_{j2}\mathcal{D}_{j2} + \cdots + d_{jn}\mathcal{D}_{jn}$ (by Exercise 23) $= a_{1j}(-1)^{j+1}|\mathbf{D}_{j1}| + a_{2j}(-1)^{j+2}|\mathbf{D}_{j2}| + \cdots + a_{nj}(-1)^{j+n}|\mathbf{D}_{jn}| = a_{1j}(-1)^{1+j}|\mathbf{A}_{1j}| + a_{2j}(-1)^{2+j}|\mathbf{A}_{2j}| + \cdots + a_{nj}(-1)^{n+j}|\mathbf{A}_{nj}|$
$= a_{1j}\mathcal{A}_{1j} + a_{2j}\mathcal{A}_{2j} + \cdots + a_{nj}\mathcal{A}_{nj}$, and we are finished.

(33) If $\mathbf{A}$ has two identical columns, then $\mathbf{A}^T$ has two identical rows. Hence, $|\mathbf{A}^T| = 0$ by Exercise 24. Thus, $|\mathbf{A}| = 0$, by Theorem 3.9 (which is proven in Exercises 29 and 31).

(34) Let $\mathbf{B}$ be the $n \times n$ matrix, all of whose entries are equal to the corresponding entries in $\mathbf{A}$, except that the jth column of $\mathbf{B}$ equals the ith column of $\mathbf{A}$ (as does the ith column of $\mathbf{B}$). Then $|\mathbf{B}| = 0$ by Exercise 33. Also, since every column of $\mathbf{A}$ equals the corresponding column of $\mathbf{B}$, with the exception of the jth column, the submatrices $\mathbf{A}_{kj}$ and $\mathbf{B}_{kj}$ are equal for $1 \le k \le n$. Hence, $\mathcal{A}_{kj} = \mathcal{B}_{kj}$ for $1 \le k \le n$. Now, computing the determinant of $\mathbf{B}$ using a cofactor expansion along the jth column (allowed by part (2) of Theorem 3.10, proven in Exercises 28 and 32) yields $0 = |\mathbf{B}| = b_{1j}\mathcal{B}_{1j} + b_{2j}\mathcal{B}_{2j} + \cdots + b_{nj}\mathcal{B}_{nj}$

$= a_{1i}\mathcal{B}_{1j} + a_{2i}\mathcal{B}_{2j} + \cdots + a_{ni}\mathcal{B}_{nj}$ (because the jth column of $\mathbf{B}$ equals the ith column of $\mathbf{A}$) $=$ $a_{1i}\mathcal{A}_{1j} + a_{2i}\mathcal{A}_{2j} + \cdots + a_{ni}\mathcal{A}_{nj}$, completing the proof.

(35) Using the fact that the general (k,m) entry of $\mathcal{A}$ is $\mathcal{A}_{mk}$, we see that the (i,j) entry of $\mathcal{A}\mathbf{A}$ equals $a_{1j}\mathcal{A}_{1i} + a_{2j}\mathcal{A}_{2i} + \cdots + a_{nj}\mathcal{A}_{ni}$. If $i = j$, Exercise 32 implies that this sum equals $|\mathbf{A}|$, while if $i \neq j$, Exercise 34 implies that the sum equals 0. Hence, $\mathbf{A}\mathcal{A}$ equals $|\mathbf{A}|$ on the main diagonal and 0 off the main diagonal, and so $\mathbf{A}\mathcal{A} = (|\mathbf{A}|)\mathbf{I}_n$. (Note that, since $\mathbf{A}$ is singular, $|\mathbf{A}| = 0$, and so, in fact, $\mathcal{A}\mathbf{A} = \mathbf{O}_n$.)

(36) (a) By Exercise 27, $\frac{1}{|\mathbf{A}|}\mathcal{A}\mathbf{A} = \mathbf{I}_n$. Hence, $\mathbf{AX} = \mathbf{B} \Rightarrow \frac{1}{|\mathbf{A}|}\mathcal{A}\mathbf{AX} = \frac{1}{|\mathbf{A}|}\mathcal{A}\mathbf{B} \Rightarrow \mathbf{I}_n\mathbf{X} = \frac{1}{|\mathbf{A}|}\mathcal{A}\mathbf{B} \Rightarrow \mathbf{X} = \frac{1}{|\mathbf{A}|}\mathcal{A}\mathbf{B}$.

(b) Using part (a) and the fact that the general (k,m)th entry of $\mathcal{A}$ is $\mathcal{A}_{mk}$, we see that the kth entry of $\mathbf{X}$ equals the kth entry of $\frac{1}{|\mathbf{A}|}\mathcal{A}\mathbf{B} = \frac{1}{|\mathbf{A}|}(b_1\mathcal{A}_{1k} + \cdots + b_n\mathcal{A}_{nk})$.

(c) The matrix $\mathbf{A}_k$ is defined to equal $\mathbf{A}$ in every entry, except in the kth column, which equals the column vector $\mathbf{B}$. Thus, since this kth column is ignored when computing $(\mathbf{A}_k)_{ik}$, the (i,k) submatrix of $\mathbf{A}_k$, we see that $(\mathbf{A}_k)_{ik} = \mathbf{A}_{ik}$ for each $1 \leq i \leq n$. Hence, computing $|\mathbf{A}_k|$ by performing a cofactor expansion down the kth column of $\mathbf{A}_k$ (by part (2) of Theorem 3.10) yields $|\mathbf{A}_k| = b_1(-1)^{1+k}|(\mathbf{A}_k)_{1k}| + b_2(-1)^{2+k}|(\mathbf{A}_k)_{2k}| + \cdots + b_n(-1)^{n+k}|(\mathbf{A}_k)_{nk}| = b_1(-1)^{1+k}|\mathbf{A}_{1k}| + b_2(-1)^{2+k}|\mathbf{A}_{2k}| + \cdots + b_n(-1)^{n+k}|\mathbf{A}_{nk}| = b_1\mathcal{A}_{1k} + b_2\mathcal{A}_{2k} + \cdots + b_n\mathcal{A}_{nk}$.

(d) Theorem 3.13 claims that the kth entry of the solution $\mathbf{X}$ of $\mathbf{AX} = \mathbf{B}$ is $|\mathbf{A}_k|/|\mathbf{A}|$. Part (b) gives the expression $\frac{1}{|\mathbf{A}|}(b_1\mathcal{A}_{1k} + \cdots + b_n\mathcal{A}_{nk})$ for this kth entry, and part (c) replaces $(b_1\mathcal{A}_{1k} + \cdots + b_n\mathcal{A}_{nk})$ with $|\mathbf{A}_k|$.

Section 3.4

(1) In each part, let $\mathbf{A}$ represent the given matrix.

(a) $p_{\mathbf{A}}(x) = |x\mathbf{I}_2 - \mathbf{A}| = \begin{vmatrix} x-3 & -1 \\ 2 & x-4 \end{vmatrix} = (x-3)(x-4) - (2)(-1) = x^2 - 7x + 14.$

(c) $p_{\mathbf{A}}(x) = |x\mathbf{I}_3 - \mathbf{A}| = \begin{vmatrix} x-2 & -1 & 1 \\ 6 & x-6 & 0 \\ -3 & 0 & x \end{vmatrix}$. At this point, we could use basketweaving, but instead, we will use a cofactor expansion on the last column. Hence,
$p_{\mathbf{A}}(x) = (1)(-1)^{1+3}\begin{vmatrix} 6 & x-6 \\ -3 & 0 \end{vmatrix} + (0) + x(-1)^{3+3}\begin{vmatrix} x-2 & -1 \\ 6 & x-6 \end{vmatrix}$
$= (1)(0 + 3(x-6)) + x\Big((x-2)(x-6) + 6\Big) = x^3 - 8x^2 + 21x - 18.$

(e) $p_{\mathbf{A}}(x) = |x\mathbf{I}_4 - \mathbf{A}| = \begin{vmatrix} x & 1 & 0 & -1 \\ 5 & x-2 & 1 & -2 \\ 0 & -1 & x-1 & 0 \\ -4 & 1 & -3 & x \end{vmatrix}$. Following the hint, we do a cofactor expansion along the third row: $p_{\mathbf{A}}(x) = (0) + (-1)(-1)^{3+2}\begin{vmatrix} x & 0 & -1 \\ 5 & 1 & -2 \\ -4 & -3 & x \end{vmatrix}$
$+ (x-1)(-1)^{3+3}\begin{vmatrix} x & 1 & -1 \\ 5 & x-2 & -2 \\ -4 & 1 & x \end{vmatrix} + (0)$. Using basketweaving produces $p_{\mathbf{A}}(x) =$

homogeneous system $\left\{ \begin{array}{rcl} 0 & = & 0 \\ -x_2 & = & 0 \end{array} \right.$, which is $\{a[1,0] \mid a \in \mathbb{R}\}$, the set of all 2-vectors having a zero in the second coordinate. Hence, $[1,0] \in E_1$, but $[1,0]$ is not in the row space of $(1\mathbf{I}_2 - \mathbf{A})$. Therefore, E_1 and the row space of $1\mathbf{I}_2 - \mathbf{A}$ are different sets.

Section 4.4

(1) In each part, let S represent the given set.

(a) Because S contains only 1 element, S is linearly independent since that element is not the zero vector. (See the discussion in the textbook directly after the definition of linear independence.)

(b) Because S contains 2 elements, S is linearly dependent if and only if either vector in S is a linear combination of the other vector in S. But since neither vector in S is a scalar multiple of the other, S is linearly independent instead. (See the discussion in the textbook directly after Example 1.)

(c) Because S contains 2 elements, S is linearly dependent if and only if either vector in S is a linear combination of the other vector in S. Since $[-2,-4,10] = (-2)[1,2,-5]$, S is linearly dependent. (See the discussion in the textbook directly after Example 1.)

(d) S is linearly dependent because S contains the zero vector $[0,0,0]$, as explained in Example 4 in the textbook.

(e) S is a subset of $\mathbb{R}^3$ that contains 4 distinct vectors. Hence, S is linearly dependent by Theorem 4.7.

(2) In each part, let S represent the given set of vectors.

(a) We follow the Independence Test Method.

Step 1: Form the matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 & -2 \\ 9 & 4 & 5 \\ -2 & 5 & -7 \end{bmatrix}$, whose columns are the vectors in S.

Step 2: $\mathbf{A}$ row reduces to $\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$.

Step 3: There is no pivot in column 3, so S is linearly dependent.

(b) We follow the Independence Test Method.

Step 1: Form the matrix $\mathbf{A} = \begin{bmatrix} 2 & 4 & -2 \\ -1 & -1 & 0 \\ 3 & 6 & 2 \end{bmatrix}$, whose columns are the vectors in S.

Step 2: $\mathbf{A}$ row reduces to $\mathbf{B} = \mathbf{I}_3$.

Step 3: Since every column in $\mathbf{I}_3$ is a pivot column, S is linearly independent.

(e) We follow the Independence Test Method.

Step 1: Form the matrix $\mathbf{A} = \begin{bmatrix} 2 & 4 & 1 \\ 5 & 3 & -1 \\ -1 & 1 & 1 \\ 6 & 4 & -1 \end{bmatrix}$, whose columns are the vectors in S.

Step 2: $\mathbf{A}$ row reduces to $\mathbf{B} = \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Step 3: There is no pivot in column 3, so S is linearly dependent.

(3) In each part, let S represent the given set of polynomials.

(a) First, we convert the polynomials in S into vectors in $\mathbb{R}^3$:
$x^2 + x + 1 \to [1, 1, 1]$, $x^2 - 1 \to [1, 0, -1]$, and $x^2 + 1 \to [1, 0, 1]$. Next, we use the Independence Test Method on this set of 3-vectors.
Step 1: Form the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$, whose columns are the vectors converted from the polynomials in S.
Step 2: $\mathbf{A}$ row reduces to $\mathbf{B} = \mathbf{I}_3$.
Step 3: Since every column in $\mathbf{I}_3$ is a pivot column, S is linearly independent.

(c) First, we convert the polynomials in S into vectors in $\mathbb{R}^2$:
$2x - 6 \to [2, -6]$, $7x + 2 \to [7, 2]$, and $12x - 7 \to [12, -7]$. Next, we use the Independence Test Method on this set of 2-vectors.
Step 1: Form the matrix $\mathbf{A} = \begin{bmatrix} 2 & 7 & 12 \\ -6 & 2 & -7 \end{bmatrix}$, whose columns are the polynomials in S converted to vectors.
Step 2: $\mathbf{A}$ row reduces to $\mathbf{B} = \begin{bmatrix} 1 & 0 & \frac{73}{46} \\ 0 & 1 & \frac{29}{23} \end{bmatrix}$.

Step 3: There is no pivot in column 3, so S is linearly dependent. (Note that, in this problem, once we converted the polynomials to vectors in $\mathbb{R}^2$, we could easily see from Theorem 4.7 that the vectors are linearly dependent, confirming the result we obtained using the Independence Test Method.)

(4) In each part, let S represent the given set of polynomials.

(a) First, we convert the polynomials in S into vectors in $\mathbb{R}^3$:
$x^2 - 1 \to [1, 0, -1]$, $x^2 + 1 \to [1, 0, 1]$, and $x^2 + x \to [1, 1, 0]$. We use the Independence Test Method on this set of 3-vectors.
Step 1: Form the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$, whose columns are the polynomials in S converted into vectors.
Step 2: $\mathbf{A}$ row reduces to $\mathbf{B} = \mathbf{I}_3$.
Step 3: Since every column in $\mathbf{I}_3$ is a pivot column, S is linearly independent.

(c) First, we convert the polynomials in S into vectors in $\mathbb{R}^3$:
$4x^2 + 2 \to [4, 0, 2]$, $x^2 + x - 1 \to [1, 1, -1]$, $x \to [0, 1, 0]$, and $x^2 - 5x - 3 \to [1, -5, -3]$. But this set of four vectors in $\mathbb{R}^3$ is linearly dependent by Theorem 4.7. Hence, S is linearly dependent.

(e) No vector in S can be expressed as a finite linear combination of the others since each polynomial in S has a distinct degree, and no other polynomial in S has any term of that degree. By the remarks before Example 15 in the textbook, the given set S is linearly independent.

(7) It is useful to have a simplified description of the vectors in span(S). Using the Simplified Span Method, we row reduce $\begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ to obtain $\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$. Hence, $\text{span}(S) = \{[a, b, (-\frac{1}{2}a+\frac{1}{2}b)] \mid a, b \in \mathbb{R}\}$.

(b) We want to choose a vector $\mathbf{v}$ outside of span(S), because otherwise, $S \cup \{\mathbf{v}\}$ would be linearly dependent, by the second "boxed" alternate characterization of linear dependence after Example 11 in the textbook. Let $\mathbf{v} = [0, 1, 0]$, which, using the simplified form of span(S), is easily seen not to be in span(S). Then $S \cup \{\mathbf{v}\} = \{[1, 1, 0], [-2, 0, 1], [0, 1, 0]\}$. We now use the Independence Test Method.

Step 1: Form the matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, whose columns are the vectors in $S \cup \{\mathbf{v}\}$.

Step 2: $\mathbf{A}$ row reduces to $\mathbf{B} = \mathbf{I}_3$.

Step 3: Since every column in $\mathbf{I}_3$ is a pivot column, $S \cup \{\mathbf{v}\}$ is linearly independent.

(c) Many other choices for $\mathbf{v}$ will work. In fact any vector in $\mathbb{R}^3$ that is outside of span(S) can be chosen here. In particular, consider $\mathbf{v} = [0, 0, 1]$, which does not have the proper form to be in span(S). You can verify that $S \cup \{\mathbf{v}\} = \{[1, 1, 0], [-2, 0, 1], [0, 0, 1]\}$ is linearly independent using the Independence Test Method in a manner similar to that used in part (b).

(d) Any nonzero vector $\mathbf{u} \in \text{span}(S)$ other than $[1, 1, 0]$ or $[-2, 0, 1]$ will provide the example we need. In particular, consider $\mathbf{u} = [1, 1, 0] + [-2, 0, 1] = [-1, 1, 1]$. Then $S \cup \{\mathbf{u}\} = \{[1, 1, 0], [-2, 0, 1], [-1, 1, 1]\}$. You can verify that $S \cup \{\mathbf{u}\}$ is linearly dependent by using the Independence Test Method.
(Also, since $\mathbf{u} \in \text{span}(S)$, $S \cup \{\mathbf{u}\}$ is linearly dependent by the alternate characterization of linear dependence mentioned in part (b).)

(11) In each part, there are many different correct answers, but only one possibility is given here.

(a) Let $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$. We will use the definition of linear independence to verify that S is linearly independent. Now $a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_4\mathbf{e}_4 = [a_1, a_2, a_3, a_4]$, which clearly equals the zero vector $[0, 0, 0, 0]$ if and only if $a_1 = a_2 = a_3 = a_4 = 0$. Therefore, S is linearly independent.

(c) Let $S = \{1, x, x^2, x^3\}$. We will use the definition of linear independence to verify that S is linearly independent. Now $a_1(1) + a_2x + a_3x^2 + a_4x^3$ clearly equals the zero polynomial if and only if $a_1 = a_2 = a_3 = a_4 = 0$. Therefore, S is linearly independent.

(e) Let $S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\}$.

Notice that each matrix in S is symmetric. We will use the definition of linear independence to verify that S is linearly independent. Now

$$a_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + a_4 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & a_4 & 0 \\ a_3 & 0 & 0 \end{bmatrix},$$

which clearly equals the zero matrix $\mathbf{O}_3$ if and only if $a_1 = a_2 = a_3 = a_4 = 0$. Therefore, S is linearly independent.

(13) (b) Let S be the given set of three vectors. We prove that $\mathbf{v} = [0, 0, -6, 0]$ is redundant. To do this, we need to show that $\text{span}(S - \{\mathbf{v}\}) = \text{span}(S)$. We will do this by applying the Simplified

Span Method to both $S - \{\mathbf{v}\}$ and S. If the method leads to the same reduced row echelon form (except, perhaps, with an extra row of zeroes) with or without $\mathbf{v}$ as one of the rows, then the two spans are equal.

For span$(S - \{\mathbf{v}\})$:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \text{ row reduces to } \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

For span(S):

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix} \text{ row reduces to } \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the reduced row echelon form matrices are the same, except for the extra row of zeroes, the two spans are equal, and $\mathbf{v} = [0, 0, -6, 0]$ is a redundant vector.

For an alternate approach to proving that $\text{span}(S - \{\mathbf{v}\}) = \text{span}(S)$, first note that $S - \{\mathbf{v}\} \subseteq S$, and so $\text{span}(S - \{\mathbf{v}\}) \subseteq \text{span}(S)$ by Corollary 4.6. Next, $[1, 1, 0, 0]$ and $[1, 1, 1, 0]$ are clearly in $\text{span}(S - \{\mathbf{v}\})$ by part (1) of Theorem 4.5. But $\mathbf{v} \in \text{span}(S - \{\mathbf{v}\})$ as well, because $[0, 0, -6, 0] = (6)[1, 1, 0, 0] + (-6)[1, 1, 1, 0]$. Hence, S is a subset of the subspace $\text{span}(S - \{\mathbf{v}\})$. Therefore, by part (3) of Theorem 4.5, $\text{span}(S) \subseteq \text{span}(S - \{\mathbf{v}\})$. Thus, since we have proven subset inclusion in both directions, $\text{span}(S - \{\mathbf{v}\}) = \text{span}(S)$.

(16) Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$, with $n < k$, and let $\mathbf{A}$ be the $n \times k$ matrix having the vectors in S as columns. Then the system $\mathbf{AX} = \mathbf{O}$ has nontrivial solutions by Corollary 2.6. But, if $\mathbf{X} = [x_1, \ldots, x_k]$, then $\mathbf{AX} = x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k$, and so, by the definition of linear dependence, the existence of a nontrivial solution to $\mathbf{AX} = \mathbf{0}$ implies that S is linearly dependent.

(19) (b) The converse to the statement is: If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent subset of $\mathbb{R}^m$, then $T = \{\mathbf{Av}_1, \ldots, \mathbf{Av}_k\}$ is a linearly independent subset of $\mathbb{R}^n$ with $\mathbf{Av}_1, \ldots, \mathbf{Av}_k$ distinct. We can construct specific counterexamples that contradict either (or both) of the conclusions of the converse. For a specific counterexample in which $\mathbf{Av}_1, \ldots, \mathbf{Av}_k$ are distinct but T is linearly dependent, let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $S = \{[1, 0], [1, 2]\}$. Then $T = \{[1, 1], [3, 3]\}$. Note that S is linearly independent, but the vectors $\mathbf{Av}_1$ and $\mathbf{Av}_2$ are distinct and T is linearly dependent. For a specific counterexample in which $\mathbf{Av}_1, \ldots, \mathbf{Av}_k$ are not distinct but T is linearly independent, use $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $S = \{[1, 2], [2, 1]\}$. Then $T = \{[3, 3]\}$. Note that $\mathbf{Av}_1$ and $\mathbf{Av}_2$ both equal $[3, 3]$, and that both S and T are linearly independent. For a final counterexample in which both parts of the conclusion are false, let $\mathbf{A} = \mathbf{O}_{22}$ and $S = \{\mathbf{e}_1, \mathbf{e}_2\}$. Then $\mathbf{Av}_1$ and $\mathbf{Av}_2$ both equal $\mathbf{0}$, S is linearly independent, but $T = \{\mathbf{0}\}$ is linearly dependent.

(27) Suppose that S is linearly independent, and suppose $\mathbf{v} \in \text{span}(S)$ can be expressed both as $\mathbf{v} = a_1\mathbf{u}_1 + \cdots + a_k\mathbf{u}_k$ and $\mathbf{v} = b_1\mathbf{v}_1 + \cdots + b_l\mathbf{v}_l$, for distinct $\mathbf{u}_1, \ldots, \mathbf{u}_k \in S$ and distinct $\mathbf{v}_1, \ldots, \mathbf{v}_l \in S$, where these expressions differ in at least one nonzero term. Since the $\mathbf{u}_i$'s might not be distinct from the $\mathbf{v}_i$'s, we consider the set $X = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \cup \{\mathbf{v}_1, \ldots, \mathbf{v}_l\}$ and label the distinct vectors in X as $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$. Then we can express $\mathbf{v} = a_1\mathbf{u}_1 + \cdots + a_k\mathbf{u}_k$ as $\mathbf{v} = c_1\mathbf{w}_1 + \cdots + c_m\mathbf{w}_m$, and also $\mathbf{v} = b_1\mathbf{v}_1 + \cdots + b_l\mathbf{v}_l$ as $\mathbf{v} = d_1\mathbf{w}_1 + \cdots + d_m\mathbf{w}_m$, choosing the scalars c_i, d_i, $1 \leq i \leq m$ with $c_i = a_j$ if $\mathbf{w}_i = \mathbf{u}_j$, $c_i = 0$ otherwise, and $d_i = b_j$ if $\mathbf{w}_i = \mathbf{v}_j$, $d_i = 0$ otherwise. Since the original linear combinations for $\mathbf{v}$ are distinct, we know that $c_i \neq d_i$ for some i. Now, $\mathbf{v} - \mathbf{v} = \mathbf{0} = (c_1 - d_1)\mathbf{w}_1 + \cdots + (c_m - d_m)\mathbf{w}_m$. Since $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\} \subseteq S$, a linearly independent set, each

$c_j - d_j = 0$ for every j with $1 \le j \le m$. But this is a contradiction since $c_i \ne d_i$.

Conversely, assume every vector in span(S) can be uniquely expressed as a linear combination of elements of S. Since $\mathbf{0} \in \text{span}(S)$, there is exactly one linear combination of elements of S that equals $\mathbf{0}$. Now, if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is any finite subset of S, we have $\mathbf{0} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n$. Because this representation is unique, it means that in any linear combination of $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ that equals $\mathbf{0}$, the only possible coefficients are zeroes. Thus, by definition, S is linearly independent.

(28) (a) False. $[-8, 12, -4] = (-4)[2, -3, 1]$. Thus, one vector is a scalar multiple (hence a linear combination) of the other vector. Therefore, the set is linearly dependent by the remarks after Example 1 in the textbook.

(b) True. This follows directly from Theorem 4.8.

(c) True. See the discussion in the textbook directly after the definition of linear independence.

(d) False. This statement directly contradicts the first "boxed" alternate characterization for linear independence following Example 11 in the textbook.

(e) True. This is the definition of linear independence for a finite nonempty set.

(f) True. This follows directly from Theorem 4.7.

(g) False. In order to determine linear independence, we need to row reduce the matrix whose *columns* are the vectors in S, not the matrix whose *rows* are the vectors in S. (The matrix $\mathbf{A}$ whose rows are the vectors in S is used, instead, in the Simplified Span Method to find a simplified form for span(S).) For a specific counterexample to the given statement, consider $S = \{\mathbf{i}, \mathbf{j}, \mathbf{0}\}$ in $\mathbb{R}^2$. Then $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, which row reduces to $\mathbf{A}$ itself. Note that there is a pivot in both columns of $\mathbf{A}$. However, S is linearly dependent since it contains the zero vector (see Example 4 in the textbook).

(h) True. This follows directly from Theorem 4.9 (or from Theorem 4.10).

(i) True. This follows directly from Theorem 4.8, since the given statement indicates that $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. (Note: In the special cases in which either $\mathbf{v}_3 = \mathbf{v}_1$ or $\mathbf{v}_3 = \mathbf{v}_2$, it follows that $\mathbf{v}_1 = \mathbf{v}_2$, which is not allowed.)

Section 4.5

(4) (a) The set S is not a basis because S cannot span $\mathbb{R}^4$. For if S spans $\mathbb{R}^4$, then by part(1) of Theorem 4.13, $4 = \dim(\mathbb{R}^4) \le |S| = 3$, a contradiction.

(c) The set S is a basis. To prove this, we will show that S spans $\mathbb{R}^4$ and then apply Theorem 4.13. We show that S spans $\mathbb{R}^4$ using the Simplified Span Method. The matrix $\mathbf{A} = \begin{bmatrix} 7 & 1 & 2 & 0 \\ 8 & 0 & 1 & -1 \\ 1 & 0 & 0 & -2 \\ 3 & 0 & 1 & -1 \end{bmatrix}$ whose rows are the vectors in S reduces to $\mathbf{I}_4$. Hence, $\text{span}(S) = \text{span}(\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}) = \mathbb{R}^4$. Now, $|S| = 4 = \dim(\mathbb{R}^4)$, and so part (1) of Theorem 4.13 implies that S is a basis for $\mathbb{R}^4$.

(e) The set S is not a basis since S is not linearly independent. For if S is linearly independent, then by part (2) of Theorem 4.13, $5 = |S| \le \dim(\mathbb{R}^4) = 4$, a contradiction.