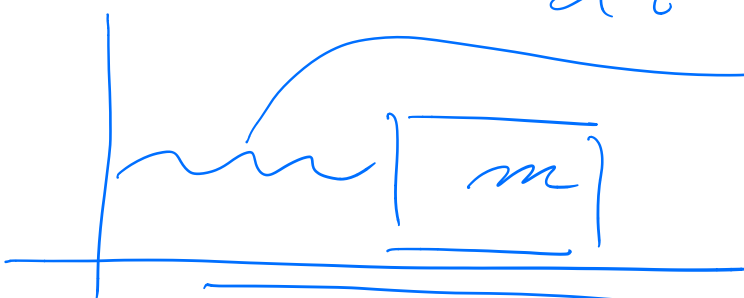
# Lecture october 1

Newton's law:

$$F(t) = m\frac{d^2x(t)}{dt^2} = -kx(t)$$



$$\boxed{\frac{d^2x}{dt^2} = -\frac{k}{m}x}$$

$$\frac{k}{m} = \omega^2 \Rightarrow \omega = \sqrt{k/m}$$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$\cancel{\omega^2}\frac{d^2x}{d\tau^2} = \frac{d}{dt}\left(\frac{d}{dt}\right)x = -\cancel{\omega^2}x$$

dim-less time

$$\tau = \omega \cdot t$$

$$\boxed{\frac{d^2x}{d\tau^2} = -x}$$

Define two first-order differential equations

$$v(\tau) = v = \frac{dx}{d\tau}$$

$$\frac{dv}{d\tau} = \frac{d^2x}{d\tau^2} = -x$$

$$\frac{dx}{d\tau} = v(\tau) \wedge \frac{dv}{d\tau} = -x(\tau) = a(\tau)$$

Discretize

$$\tau \in [\tau_0, \tau_n]$$

$$\tau \Rightarrow \tau_i = \tau_0 + ih$$

$$i = 0, 1, 2, \ldots n$$

$$h = \frac{\tau_n - \tau_0}{n}$$

$$x(\tau) = x(\tau_i) = x_i$$

$$v(\tau) = v(\tau_i) = v_i$$

Taylor-expand

$$x(\tau_i \pm h) = x(\tau_i) \pm h x'|_{\tau_i} + \frac{h^2}{2} x'' + O(h^3)$$

$$v(\tau_i \pm h) = v(\tau_i) \pm h v'|_{\tau_i} + \frac{h^2}{2} v'' + O(h^3)$$

Euler's forward method:

$$x_{i+1} = x_i + h x_i' \quad (+ O(h^2))$$
$$= x_i + h \underline{v_i}$$
$$v_{i+1} = v_i + h v_i'$$
$$= v_i + h a_i$$

Euler's method does not conserve energy.

Euler-Cromer's method

$$v_{i+1} = v_i + h a_i \quad (\underline{\text{error } O(h^2)})$$
$$x_{i+1} = x_i + h v_{i+1}$$

This is an energy conserving method.

Both methods have a local error $O(h^2)$. This repeated $n$ times, $n \sim 1/h \Rightarrow$ Global error $O(h)$.

Can we do better? $O(h^2)$? with simple additions

Verlet family

$$m \frac{d^2 x}{dt^2} = F(x,t)$$

$$\boxed{\begin{aligned} \frac{dx}{dt} &= v(x,t) \\ \frac{dv}{dt} &= a(x,t) = F(x,t)/m \end{aligned}}$$

Taylor expand

$$x(t \pm h) = x(t) \pm h x^{(1)}(t) + \frac{h^2}{2} x^{(2)}(t) + O(h^3)$$

$$x^{(2)}(t) = a(x,t)$$

Discretize

$$x_{i\pm1} = x_i \pm h v_i + \frac{h^2}{2} a_i \pm O(h^3)$$

$$x_{i+1} + x_{i-1} = 2x_i + h^2 a_i + O(h^4)$$

$$\frac{x_{i+1} - x_{i-1}}{2h} \simeq \frac{dx}{dt} = v$$

$$\boxed{\frac{x_{i+1} - x_{i-1}}{2h} = v_i}$$

$$x_{i+1} = 2x_i - x_{i-1} + h^2 a_i + O(h^4)$$

$$v_i = \frac{x_{i+1} - x_{i-1}}{2h} + O(h^2)$$

$x_{i-1}$ is not defined for

$i = 0$

Not a self-starting algorithm ?

## Velocity Verlet method

Taylor expansion ;

$$x_{i+1} = x_i + h x_i^{(1)} + \frac{h^2}{2} x_i^{(2)} + O(h^3)$$

$$v_{i+1} = v_i + h v_i^{(1)} + \frac{h^2}{2} v_i^{(2)} + O(h^3)$$

$$v_i^{(1)} = a_i = F(x_i, t_i)/m$$

$$v_{i+1} = v_i + h a_i + \boxed{\frac{h^2}{2} v_i^{(2)}}$$

?

$$v_i^{(2)} = ?$$

$$v_i^{(2)} = \frac{v_{i+1}^{(1)} - v_i^{(1)}}{h}$$

$$\boxed{h v_i^{(2)}} = v_{i+1}^{(1)} - v_i^{(1)}$$

$$= a_{i+1} - \underline{a_i}$$

$$\Rightarrow v_{i+1} = v_i + \frac{h}{2}\left[v^{(1)}_{i+1} + v^{(1)}_i\right] + O(h^3)$$

$$= v_i + \frac{h}{2}\left[a_{i+1} + a_i\right] + O(h^3)$$

$$\Rightarrow x_{i+1} = x_i + h v_i + \frac{h^2}{2} a_i + O(h^3)$$

Evaluate first $x_{i+1}$, keep

$$a_i = F(x_i)/m$$

Need intermediate step

$$a_{i+1} = F(x_{i+1})/m$$

Then evaluate/update $v_{i+1}$

## Project 3

<u>part a</u> : Earth Sun

$$F_G = \frac{G M_\odot M_E}{r^2}$$

Earth

$|\vec{r}_i| = r = 1 AU$

x
Sun

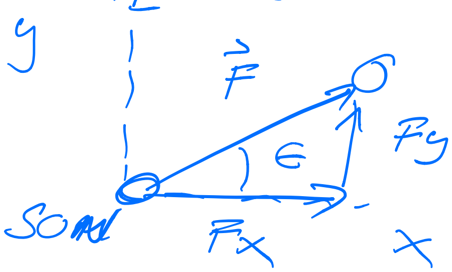
$= 1.5 \cdot 10^{11}\, m$

$M_\odot = 2 \cdot 10^{30}\, kg$

$M_E = 6 \cdot 10^{24}\, kg$

$G = 6.67 \cdot 10^{-11}\, Nm^2/kg$

$M_E \cdot \vec{a} = \vec{F}$




$a_x = F_x / M_E$

$a_y = F_y / M_E$

$r = \sqrt{x^2 + y^2} \quad \wedge \quad x = r \cdot \cos\theta$

$y = r \sin\theta$

1) $a_x = \dfrac{dv_x}{dt} = -\dfrac{GM_\odot}{r^3} x$

2) $a_y = \dfrac{dv_y}{dt} = -\dfrac{GM_\odot\, y}{r^3}$

$\left(\sqrt{x^2+y^2}\right)^3$

3) $v_x = \dfrac{dx}{dt}$

4) $v_y = \dfrac{dy}{dt}$

Get rid of $GM_\odot$

centrifugal acceleration:

$$a = v^2/r = F/M_E$$

$$= \frac{GM_\odot}{r^2} \Rightarrow$$

$$v^2 \cdot r = GM_\odot$$

Circular motion

$$v = 2\pi r / 1\,yr$$

$$= 2\pi \cdot 1\,AU / 1\,yr$$

$$= 2\pi\, AU/yr$$

$$GM_\odot = 4\pi^2 (AU)^3/(yr)^2$$

$$\boxed{\frac{dv_x}{dt} = a_x = -\frac{4\pi^2 x}{\sqrt{x^2+y^2}} \qquad v_x = \frac{dx}{dt}}$$

$$\boxed{\frac{dv_y}{dt} = a_y = -\frac{4\pi^2 y}{\sqrt{x^2+y^2}} \qquad v_y = \frac{dy}{dt}}$$

P3 part a

Euler's method algorithm

- Define $h$
- choose initial conditions

$x_0 \quad y_0 \qquad r_0 = \sqrt{x_0^2 + y_0^2} = 1\,AU$

$x_0 = 1\,AU \qquad y_0 = 0$

$v_0 = ? \qquad v_{x_0} = ? \qquad v_{y_0} = ?$

$v_0 = 2\pi r_0 / 1\,yr$

$$\boxed{\begin{aligned}
x_{i+1} &= x_i + h\, v_{x_i} \\
y_{i+1} &= y_i + h\, v_{y_i} \\
\underline{v_{x_{i+1}}} &= v_{x_i} - \boxed{\frac{h \cdot 4\pi^2 x_i}{\sqrt{x_i^2 + y_i^2}}} \\
\underline{v_{y_{i+1}}} &= v_{y_i} - \boxed{\frac{h\, 4\pi^2 y_i}{\sqrt{x_i^2 + y_i^2}}}
\end{aligned}}$$

$$\left( x_{i+1} = x_i + h\, x_i^{(1)} \right)$$