Lecture Thursday Sept 10

Jacobi's iterative scheme:

$A \in \mathbb{R}^{4\times 4} \qquad x, b \in \mathbb{R}^4$

$$Ax = b$$

$$\boxed{a_{11}x_1} + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1$$

$$a_{21}x_1 + \boxed{a_{22}x_2} + a_{23}x_3 + a_{24}x_4 = b_2$$

$$a_{31}x_1 + \quad - \quad \boxed{a_{33}x_3} \qquad = b_3$$

$$a_{41}x_1 + \quad - \quad - \quad \boxed{a_{44}x_4} = b_4$$

$x^{(k)} \qquad k = \#$ iterations

$$x_1^{(k+1)} = \left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - a_{14}x_4^{(k)}\right]/a_{11}$$

$$\boxed{x_2^{(k+1)}} = \left[b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)} - a_{24}x_4^{(k)}\right]/a_{22}$$

$$\boxed{x_3^{(k+1)}} = \left[b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)} - a_{34}x_4^{(k)}\right]/a_{33}$$

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Jacobi iteration

$$L = \begin{bmatrix} 0 & & & \\ a_{21} & 0 & & 0 \\ a_{31} & \ddots & \ddots & \\ a_{41} & \cdots & \cdots & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ & \ddots & \ddots & \vdots \\ 0 & & \ddots & \vdots \\ & & & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & a_{33} & \\ 0 & & & a_{44} \end{bmatrix}$$

$$x^{(k+1)} = D^{-1}\left(b - (L+U)x^{(k)}\right)$$

continue iterations till

$$\|x^{(k+1)} - x^{(k)}\|_2 \leq \varepsilon \sim 10^{-10}$$

Gauss-Seidel

$x_1^{(k)}$ from previous

calc first

$$x_1^{(k+1)} = \left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} - a_{14}x_4^{(k)}\right]/a_{11}$$

$$x_2^{(k+1)} = \left[b_2 - a_{21}x_1^{(k+1)} - \boxed{a_{23}x_3^{(k)}} - \boxed{a_{24}x_4^{(k)}}\right]/a_{22}$$

$$x_3^{(k+1)} = \left[b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)} - a_{34}x_4^{(k)}\right]/a_{33}$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\Big[b_i - \underbrace{\sum_{j>i} a_{ij}x_j^{(k)}}_{U} - \underbrace{\sum_{j<i} a_{ij}x_j^{(k+1)}}_{L}\Big]$$

$i = 1, 2, \ldots, n$

$$X^{(k+1)} = D^{-1}\left[b - \underline{L\,x^{(k+1)}} - U\,x^{(k)}\right]$$

Fitting of functions

- Least squares (matrix inv)
- Cubic spline (Thomas algo)

Basic problem: Data set

$$y = \{y_1\ y_2 \ldots\ y_n\}$$

$$x = \{x_1\ x_2 \ldots\ x_n\}$$

a functional relation

$$y_i = f(x_i) \simeq P_{n-1} = \sum^{n-1} a_i x_j^i$$

$$j = 1, 2, \dots n$$

$$y_1 = a_0 + a_1 x_1^1 + a_2 x_1^2 + \dots a_{n-1} x_1^{n-1}$$

$$y_2 = a_0 + a_1 x_2^1 + a_2 x_2^2 + \dots a_{n-1} x_2^{n-1}$$

$$\vdots$$

$$\underline{y_n} = a_0 + a_1 x_n^1 + \dots a_{n-1} x_n^{n-1}$$

unknown coefficients a_i

$$a = [a_0, a_1 \dots a_{n-1}]^T$$

$$X = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & & & \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{bmatrix}$$

VanderMonde Matrix

$$\underset{\text{known}}{y} = \underset{\text{known}}{X} \cdot \underset{\text{unknown}}{a}$$

$X = LU$ and solve.

Least square:
optimize Mean-squared error $= \frac{1}{n}\sum_{i=1}^{n}(y_i - \tilde{y}_i)^2$

$\tilde{y} = X \cdot a$ Model

$$a = (X^T X)^{-1} X^T \cdot y \quad \Big| \quad \frac{\partial C}{\partial a} = 0$$

$$C = C(a) = \frac{1}{n}\sum_{i=1}^{n}(y_i - \tilde{y}_i)^2$$

Eigenvalue problems and project 2.

- project 2

$$A \cdot x = \lambda x$$

 - write own solver using Jacobi rotations.
 - A is similar to project 1

$$\frac{d^2u}{dx^2} = \lambda u(x)$$

$x \in [0,1]$

- reuse all matrices/vectors from project 1

$u(x) \rightarrow u(x_i) = u_i$

$$\frac{d^2u}{dx^2} \rightarrow \frac{u_{i+1}+u_{i-1}-2u_i}{h^2}$$

$$h = \frac{1-0}{n} = \frac{1}{n} \qquad \boxed{x_i = x_0 + i\cdot h}$$

$i = 0, 1, \ldots n$

$u(0) = u(1) = 0$

- Scaling of the equations

First case:

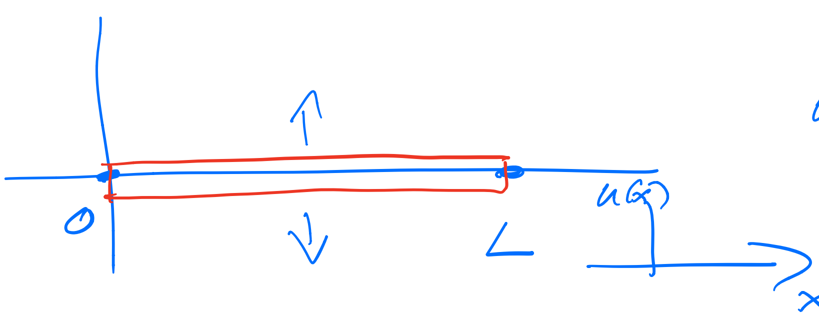
Buckling Beam

$\boxed{x_n = x_0 + i\frac{1}{n}}$

$i = n$

$x_0 = 0$

$x_n = x_n$



u(x) is the vertical displacement.

1-Dim eq, From Newton's

$$R\frac{d^2u}{dx^2} = -Fu(x)$$

$-\frac{q^2}{2m}$

x has dimension length

R a material specific physical constant.

F = applied force (could be known)

Scaling of equations:

Scaling: introduce a dimensionless length

$\xi = x \cdot \alpha \Rightarrow$

$$R \frac{d^2 u(x)}{dx^2} = -F\, u(x)$$

$$R\alpha^2 \frac{d^2 u(\xi)}{d\xi^2} = -F\, u(\xi)$$

$\xi \in [0,1] \quad x \in [0, L]$

$\alpha = L^{-1}$

$$\frac{R}{L^2} \frac{d^2 u(\xi)}{d\xi^2} = -F\, u(\xi)$$

$$\frac{d^2 u(\xi)}{}$$

$$\boxed{-\frac{d^2 u(\rho)}{d\rho^2} = +\lambda\, u(\rho)}$$

$$\lambda = \frac{FL^2}{R}$$

Discretize for the unknown

$u(0) = u(1) = 0$

$u_1, u_2 \ldots u_{m-1}$

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m-1} \end{bmatrix} = \lambda \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{m-1} \end{bmatrix}$$