Exercise 6

Machine Learning I

6A-1.

First, rearrange through Bayes theorem:

$$p(\mathbf{t}|\mathbf{X}) = \frac{p(\mathbf{t}|\mathbf{w},\mathbf{X})p(\mathbf{w}|\mathbf{X})}{p(\mathbf{w}|\mathbf{t},\mathbf{X})}$$

Decompose:

$$\log p(\mathbf{t}|\mathbf{X}) = \log p(\mathbf{t}|\mathbf{w},\mathbf{X}) + \log p(\mathbf{w}|\mathbf{X}) - \log p(\mathbf{w}|\mathbf{t},\mathbf{X})$$

We already determined the densities for the likelihood, posterior and prior in the previous exercises:

$$p(\mathbf{t}|\mathbf{w},\mathbf{X}) = N\left(\mathbf{t};\boldsymbol{\Phi}\mathbf{w},\tau_\epsilon^{-1}\mathbf{I}\right)$$

$$p(\mathbf{w}|\mathbf{X}) = N\left(\mathbf{w};\mathbf{0},\tau_0^{-1}\mathbf{I}\right)$$

$$p(\mathbf{w}|\mathbf{t},\mathbf{X}) = N\left(\mathbf{w};\tau_e\boldsymbol{\Sigma}_N\boldsymbol{\Phi}^T\mathbf{t},\left(\tau_0\mathbf{I}+\tau_e\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}\right)^{-1}\right)$$

No need to recompute everything again. Because above has to be valid for any $\mathbf{w}$, we pick $\mathbf{w} = \boldsymbol{\mu}_N$ for convenience. Let us first convert only the exponent

$$\begin{aligned}\log p(\mathbf{t}|\mathbf{X}) \quad &\propto -\frac{1}{2}(\mathbf{t}-\boldsymbol{\Phi}\mathbf{w})^T\tau_\epsilon\mathbf{I}(\mathbf{t}-\boldsymbol{\Phi}\mathbf{w}) - \frac{1}{2}\mathbf{w}^T\tau_0\mathbf{I}\mathbf{w} \\ &+\frac{1}{2}(\mathbf{w}-\tau_e\boldsymbol{\Sigma}_N\boldsymbol{\Phi}^T\mathbf{t})^T\left(\tau_0\mathbf{I}+\tau_e\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}\right)(\mathbf{w}-\tau_e\boldsymbol{\Sigma}_N\boldsymbol{\Phi}^T\mathbf{t}) \\ &= -\frac{1}{2}(\mathbf{t}-\boldsymbol{\Phi}\boldsymbol{\mu}_N)^T\tau_\epsilon\mathbf{I}(\mathbf{t}-\boldsymbol{\Phi}\boldsymbol{\mu}_N) - \frac{1}{2}\boldsymbol{\mu}_N{}^T\tau_0\mathbf{I}\boldsymbol{\mu}_N \qquad set\ \mathbf{w} = \boldsymbol{\mu}_N\end{aligned}$$

Now let us add the normalization constants:

$$\log p(\mathbf{t}|\mathbf{X}) = -\frac{1}{2}(\mathbf{t}-\boldsymbol{\Phi}\boldsymbol{\mu}_N)^T\tau_\epsilon\mathbf{I}(\mathbf{t}-\boldsymbol{\Phi}\boldsymbol{\mu}_N) - \frac{1}{2}\boldsymbol{\mu}_N{}^T\tau_0\mathbf{I}\boldsymbol{\mu}_N + \frac{1}{|r_0|}$$

$$\begin{aligned}\log p(\mathbf{t}|\mathbf{X}) \quad &= -\frac{1}{2}(\mathbf{t}-\boldsymbol{\Phi}\boldsymbol{\mu}_N)^T\tau_\epsilon\mathbf{I}(\mathbf{t}-\boldsymbol{\Phi}\boldsymbol{\mu}_N) - \frac{1}{2}\boldsymbol{\mu}_N{}^T\tau_0\boldsymbol{\mu}_N \\ &+\frac{d}{2}\log r_0 + \frac{n}{2}\log r_\epsilon - \frac{n}{2}\log 2\pi + \frac{1}{2}\log|\boldsymbol{\Sigma}_n|\end{aligned}$$

6A-2.

Let w_{MAX} be the (unregularized) maximum likelihood parameters. The lasso path illustrates the current selection of weights w_i given weight regularization $\|w_{current}\| = \frac{t}{\|w_{MAX}\|_1},\ t \in [0, \|w_{MAX}\|_1]$. For a very thorough and detailed explanation, please refer to page 69 of "Elements of statistical learning".

You can find the online version here:

https://web.stanford.edu/~hastie/ElemStatLearn//printings/ESLII_print10.pdf

This means that for low values on the x axis, the model is heavily regularized. Moving along the x axis then relaxes the regularization constraint until you get to the unregularized maximum likelihood solution $\|w_{MAX}\|$.

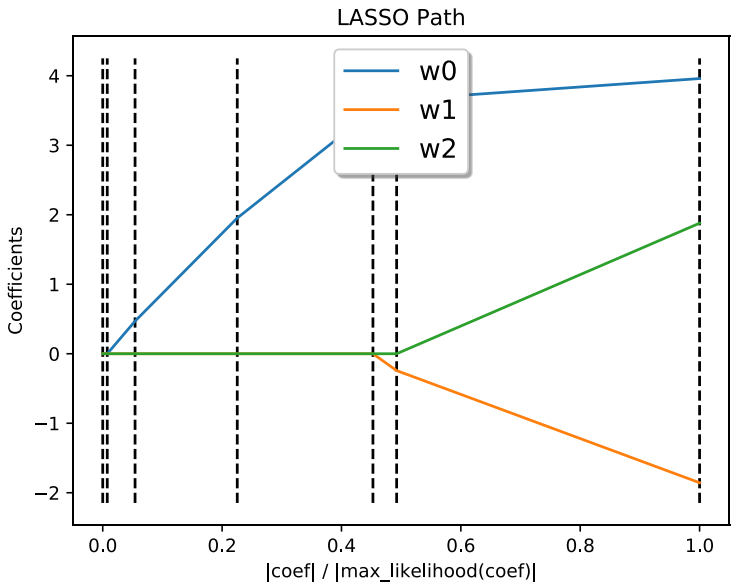
In order to study what happens, we are now using only 3 features. Using a design matrix, we can model each degree x^i of the regression polynomial $y(x) = \sum_{i=0}^{d} w_i x^i$ as a separate dimension for the lasso path. Let us have

$$f(x) = 4 - 2x + 2x^2$$

Given enough samples, the maximum likelihood solution should set our regressor weights as such:

$$w_0 = 4, w_1 = -2, w_2 = 2,$$

This can be seen, if we look at the right end of our lasso path:



The less we regularize our weights (we move towards 1 on the abscissa) the more we obtain the maximum likelihood solution for our weights. Model was created with n=3000 samples and sample interval $I = (0,2)$.

Now something should be noted: In the above image, weight w_0 was picked up first, then w_1 and lastly w_2. This means, in this case, relaxing the lasso translates to gradually picking up model complexity.

Now we are ready for eight features. Our polynomial is

$$g(x) = 2 - 3x + 4x^2 - 5x^3 + 4x^4 - 4x^5 + 6x^6 - 5x^7$$

The order of weight pick up is very much dependent on the sample interval I. Shown are two pictures of the same function $g(x)$, one time with a sample interval of $I = (0,25)$ and one time with $I = (-1,1)$:

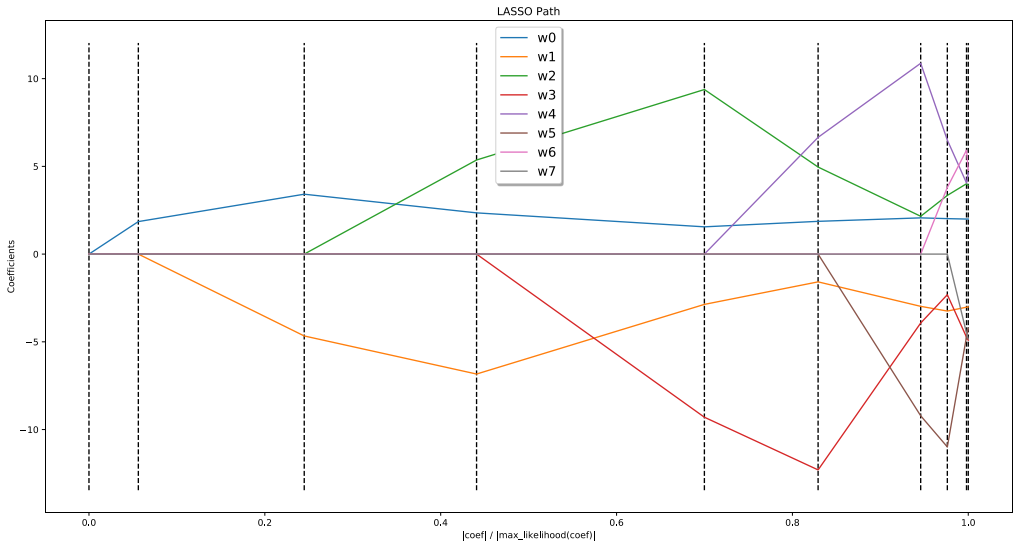


Figure 1 n=70000, I=(-1,1). The weights are picked up in ascending order of degree.

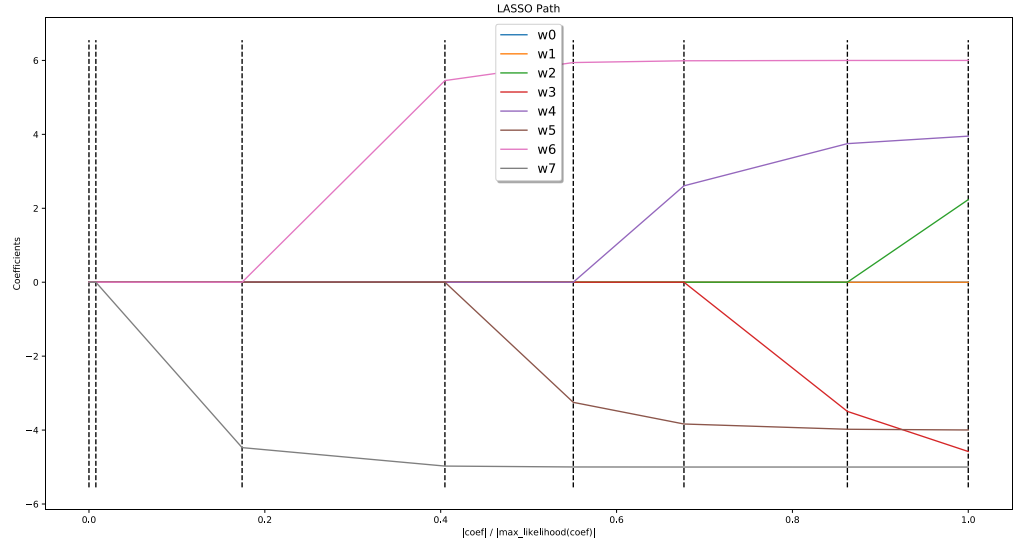


Figure 2 n=70000, I=(0,25). The weights are mostly picked up in descending order.

Think about why this happens (Tip: With polynomial basis functions, we have $x^i \geq x^j, i < j, x \in [-1,1]$. What does this mean for our regularized error?).

If we put the length of our interval between the ascending (I=(-1,1)) and descending (I=(0,25)) cases, then the following more chaotic weight pick up happens:

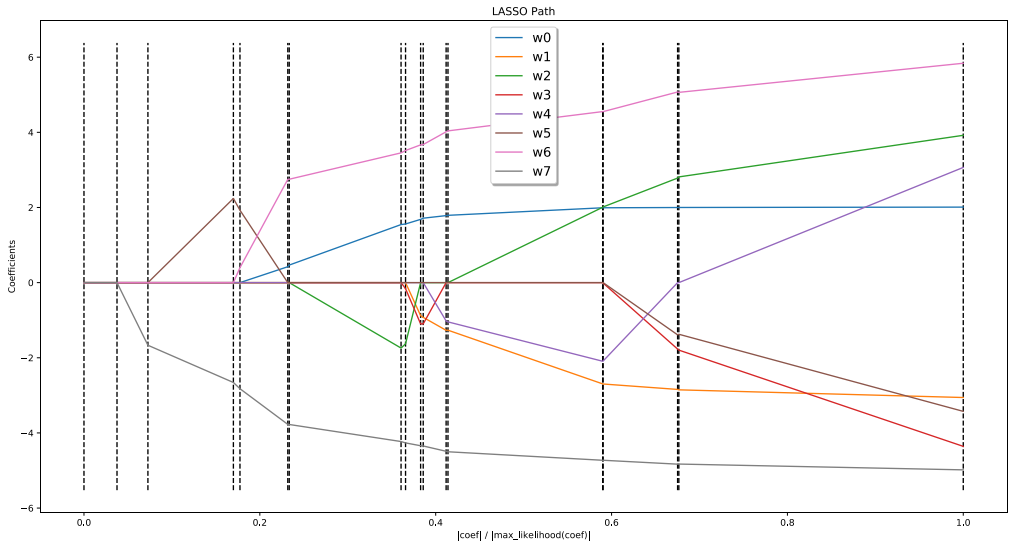


Figure 3 n=70000, I=(0,2) Chaotic pick up of weights

6A-3.

The noise ϵ is symmetric around t. So by the same way we derived $t \sim N(t; \mathbf{w}^T\mathbf{x}, \epsilon)$ when the noise was normal, we can now say

$$t \sim Stud\_t(t; \nu, \mathbf{w}^T\mathbf{x}, \sigma_\epsilon^2)$$

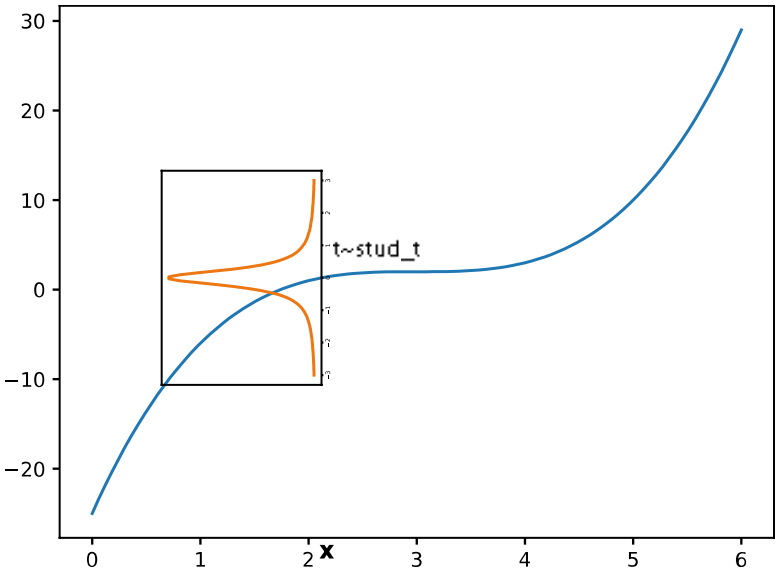


Figure 4 Because the noise is symmetrically distributed around t and the hidden function f(x) is non-random (p(t|x)=1 for one value of t),t is identically distributed to its noise with adjusted mean.

This leads to the log likelihood:

$$\begin{aligned}\log p(\mathbf{t}|\mathbf{x},\nu,\sigma_\epsilon) &= \log\prod_{i=1}^{n} Stud\_t(t_i;\nu,\mathbf{w}^T\mathbf{x}_i,\sigma_\epsilon^2)\\ &= n\left[\log\Gamma\left(\frac{\nu+1}{2}\right)-\log\left(\sqrt{\nu\pi\sigma_\epsilon^2}\,\Gamma\left(\frac{\nu}{2}\right)\right)\right]\\ &\quad -\frac{(\nu+1)}{2}\sum_{i=1}^{n}\log\left(1+\frac{1}{\nu}\frac{(t_i-\mathbf{w}^T\mathbf{x}_i)}{\sigma_\epsilon^2}\right)\end{aligned}$$

Let us set $d=(t_1-\mathbf{w}^T\mathbf{x}_1)$. We now plot the error of one datapoint in comparison to a log normal pdf:

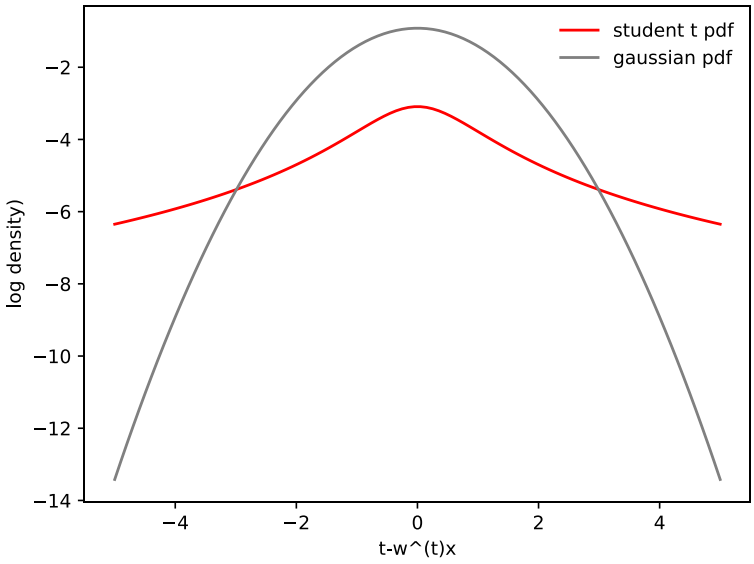


Figure 5 Comparison of the test error between a log normal pdf and a student t pdf. Image was attained with the parameters $\sigma_\epsilon = 1, \nu = 1$.

For small errors $\mathbf{w}^T\mathbf{x} \approx t$, the student t distribution appears to be less forgiving, as less density is concentrated around small errors ($\log Stud\_t(t_1; 1, d, 1) < \log N(t_1; d, 1)$). Larger errors $|t - \mathbf{w}^T\mathbf{x}| \gg 1$ are less penalized however, as $\log N(t_1; d, 1) < \log Stud\_t(t_1; 1, d, 1)$.

Generally speaking, this means if we try to maximize our weights $\mathbf{w}$ with respect to $e \sim N(0, \sigma^2)$, then our fit is trying to mitigate large errors while being mostly indifferent to smaller ones.
If we maximize $\mathbf{w}$ with respect to $t \sim Stud\_t(t; \nu, \mathbf{w}^T\mathbf{x}, \sigma_\epsilon^2)$, we get a more balanced fit.