Sheaf Theory (II)

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§ Homological Algebra (I) §

Homological Algebra (I)

- Assume $\mathcal{M}$ is an abelain category with enough injective objects, and $F : \mathcal{M} \to \mathcal{N}$ an additive functor to another abelian category which is left exact. We can define RF^i as the following.
- For any object $M \in \mathcal{M}$, and fix an injective resolution $M \to I_M^\bullet$, define

$$RF^i(M) = H^i(F(I_M^\bullet)) = i\text{-th cohomology of } F(I_M^\bullet)$$

  For any morphism $\varphi : M \to N$ in $\mathcal{M}$, any choice of lifting $\hat{\varphi} : I_M^\bullet \to I_N^\bullet$, it induces a unique map $RF^i(M) \to RF^i(N)$.
- In particular, $RF^0 = F$, and for injective object I, $RF^{\geq i}(I) = 0$ for any such F.

The case of Sheaves

Theorem

In our case, the category of sheaves over X has enough injective objects.

- For any sheaf $\mathcal{F}$, define the Godment injective sheaf for $\mathcal{F}$

$$I_{\mathcal{F}} := \left[U \longmapsto \prod_{p \in U} \mathcal{F}_p \right].$$

- Formally, denote for each point $p \in X$, the inclusion $i_p : \{p\} \to X$, then

$$I_{\mathcal{F}} = \prod_{p \in X} (i_p)_* (i_p)^* \mathcal{F}.$$

The case of Sheaves

- Note that

$$\begin{aligned} \mathrm{Hom}_X(\mathcal{G}, I_{\mathcal{F}}) &= \prod_{p\in X} \mathrm{Hom}_X(\mathcal{G}, (i_p)_*(i_p)^*\mathcal{F}) \\ &= \prod_{p\in X} \mathrm{Hom}_p((i_p)^*\mathcal{G}, (i_p)^*\mathcal{F}) \\ &= \prod_{p\in X} \mathrm{Hom}_p(\mathcal{G}_p, \mathcal{F}_p). \end{aligned}$$

  So $\mathrm{Hom}_X(-, I_{\mathcal{F}})$ is exact, so $I_{\mathcal{F}}$ is an injective object.
- The natural map $\mathcal{F} \to I_{\mathcal{F}}$ is definitely injective morphism by checking at each stalk.
- Note that it is the set-theoretic section of $\mathbf{F} = \bigsqcup \mathcal{F}_x \to X$.
- The similar case for coherent case. See Hartshorn III. 2.2.

The case of Sheaves

- Remind for a continuous map $f : X \to Y$, the functor f_* and $f_!$ is left exact, so it defines Rf_*^i and $Rf_!^i$ from X-Sh to Y-Sh.
- In particular, taking global section Γ or Γ_c from X-Sh to pt-Sh define $R\Gamma^i$ and $R\Gamma_c^i$, which will be denoted by

$$H^i(X;\mathcal{F}) := R\Gamma(X;\mathcal{F}), \qquad H_c^i(X;\mathcal{F}) := R\Gamma_c(X;\mathcal{F}).$$

- Actually,

$$Rf_*^i\mathcal{F} = [U \mapsto H^i(f^{-1}(U);\mathcal{F}|_{f^{-1}(U)})]^\dagger.$$

$$Rf_!^i\mathcal{F}_y = H_c^i(f^{-1}(y);\mathcal{F}|_{f^{-1}(y)}).$$

By 5-lemma after constructing a morphism.

Homological Algebra (II)

- For a left exact functor $F : \mathcal{M} \to \mathcal{N}$. Assume $M \to I^\bullet$ with all $RF^{\geq 1}(I) = 0$, then the cohomology group of $F(I^\bullet)$ coincides with $RF^i(M)$.
- Actually, this follows from spectral sequence of double complex. We can resolve $I^\bullet$ by a double complex J.

$$J^{pq} \xrightarrow{\text{next page}} \delta_{q=0} I^p \xrightarrow{\text{next page}} \delta_{(p,q)=(0,0)} M \overset{\text{converges to}}{\Longrightarrow} \delta_{n=0} M$$

So $H^n(\operatorname{Tot} J) = \delta_{n=0} M$, $\operatorname{Tot} J$ is a resolution of M. Then

$$F(J^{pq}) \xrightarrow{\text{next page}} \begin{array}{c} RF^q(I^p) \\ = \delta_{q=0} F(I^p) \end{array} \overset{\text{converges to}}{\Longrightarrow} H^n(F(I^\bullet))$$

As a result $H^n(F(I^\bullet)) = H^n(\operatorname{Tot} F(J)) = H^n(F(\operatorname{Tot} J))$.

Soft Sheaves

- We call a sheave $\mathcal{F}$ is **flappy** if any open subset U, $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.
- We call a sheave $\mathcal{F}$ is **soft** if any closed subset K, $\mathcal{F}(X) \to \mathcal{F}(K)$ is surjective.
- We call a sheave $\mathcal{F}$ is **fine** if there exists a partition of unity over $\mathcal{F}$.
- Clearly,

$$\boxed{\text{flappy}} \Rightarrow \boxed{\text{soft}} \Leftarrow \boxed{\text{fine}}.$$

Theorem

For soft sheaves $\mathcal{F}$ over paracompact spaces, if X is locally compact $H^{\geq 0}(X, \mathcal{F}) = 0$. So the soft sheaves are enough to compute Γ.

Soft Sheaves

- It suffices to show that when $\mathcal{F}$ is soft the global section Γ is exact at the following short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \xrightarrow{\varphi} \mathcal{H} \to 0.$$

- For any $s \in \Gamma(\mathcal{H})$, we can use Zorn Lemma to pick a maximal open subset U and $\hat{s} \in \mathcal{G}(U)$ with the property $\varphi(\hat{s}) = s|_U$. Then for any point x outside of U, we can lift some $\hat{s}'$ in a neighborhood V of x. Then $t = \hat{s}|_{U\cap V} - \hat{s}'|_{U\cap V} \in \mathcal{F}(U \cap V)$. Then think t in $\mathcal{F}(K \setminus U)$ where $K \subseteq V$ is some closed neighborhood of x. By assumption it can be lift to $\hat{t} \in \mathcal{F}(X)$, then $t = \hat{t}$ in a neighborhood of $K \setminus U$, replace this neighborhood by V. We can exchange $\hat{s}'$ by $\hat{s}' + t$, then $\hat{s}'$ and s can be successfully glued.
- Here we use the assumption of paracompact to ensure that $\mathcal{F}(K) = \varinjlim_{W \supseteq K} \mathcal{F}(W)$.

Homological Algebra (III)

- Assume we have bifunctor $B(-,-)$ which is left exact indices-wise. Then

$$H^i(B(I_M^\bullet, N)) = H^i(B(M, I_N^\bullet)) = H^i(\mathrm{Tot}\, B(I_M^\bullet, I_N^\bullet)).$$

  This follows easily from spectral sequences.

- Actually,

$$B(I_M^p, I_N^q) \xrightarrow{\text{next page}} \delta_{q=0} B(I_M^p, N) \overset{\text{converges to}}{\Longrightarrow} H^i(B(I_M^\bullet, N))$$

$$B(I_M^p, I_N^q) \xrightarrow{\text{next page}} \delta_{p=0} B(M, I_N^p) \overset{\text{converges to}}{\Longrightarrow} H^i(B(M, I_N^\bullet))$$

Tor Functor

- In the category of coherent sheaves, there is no enough projective objects in general. But it has enough flat objects.
- For coherent sheaf $\mathcal{F}$, any open subset U and section $s \in \mathcal{F}(U)$, denote j_U the inclusion $U \subseteq X$. Then $\mathcal{O}_U \to (j_U) * \mathcal{F}$ with $1 \mapsto s$ defines $(j_U)_! \mathcal{O}_U \to \mathcal{F}$.
- The map is definitely a surjection

$$\bigoplus_{s \in \mathcal{F}(U)} (j_U)_! \mathcal{O}_U \longrightarrow \mathcal{F}.$$

  And $(j_U)_! \mathcal{O}_U$ is flat follows from local criteria.
- See stack project 05NI.

Tor Functor

- So let us fix a choice of flat resolution $P_\bullet^{\mathcal{F}}$ for each coherent sheaf $\mathcal{F}$.
- For two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$, we can define

$$\mathrm{Tor}_i^X(\mathcal{F},\mathcal{G}) = H_i(P_\bullet^{\mathcal{F}} \otimes_{\mathcal{O}} \mathcal{G}) = H_i(\mathcal{F} \otimes_{\mathcal{O}} P_\bullet^{\mathcal{G}}) = H_i(\mathrm{Tot}\, P_\bullet^{\mathcal{F}} \otimes_{\mathcal{O}} P_\bullet^{\mathcal{G}}).$$

- It is a fortune that localization is exact, and commutes with tensor product, so to make it a functor, we can simply glue the unique induced map locally.
- It is the universal derived functor.

Ext Functor

- To define Ext over coherent sheaves, we define

$$\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) = R^i\big[\,\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},-)\big](\mathcal{G}).$$

$$\mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) = R^i\big[\,\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{F},-)\big](\mathcal{G}).$$

- But for $i \geq 1$ it is not generally true that

$$\mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) = [U \longmapsto \mathrm{Ext}^i_{\mathcal{O}_U}(\mathcal{F}|_U,\mathcal{G}|_U)],$$

but

$$\mathcal{E}\mathrm{xt}^i_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})|_U = \mathcal{E}\mathrm{xt}^i_{\mathcal{O}_U}(\mathcal{F}|_U,\mathcal{G}|_U).$$

Ext Functor

- The main defect is due to the non-exactness of $\Gamma(X,-)$.

$$\begin{tikzcd} \mathcal{O}_X\text{-}\mathrm{Sh} \arrow[rr, "\mathrm{Hom}(\mathcal{F}{,}-)"] \arrow[dr, "\mathcal{H}om(\mathcal{F}{,}-)"'] & & \mathrm{Abel} \\ & \mathcal{O}_X\text{-}\mathrm{Sh} \arrow[ur, "\Gamma(X{,}-)"'] & \end{tikzcd}$$

- Then it satisfies the condition of Grothendieck spectral sequences.

$\gg$ *Questions?* $\ll$

§ Realization of Topology (I) §

Singular Cohomology

Theorem

The cohomology group

$$H^i(X;\mathbb{Z}) = R^i\Gamma(X;\mathbb{Z})$$

with $\mathbb{Z}$ the constant sheaf coincides with singular cohomology X when X is locally contractible.

- Let $\mathcal{S}^n(U)$ be the n-dimensional singular cochain inside open set U. Then it is clear that $[U \mapsto \mathcal{S}(U)]$ is a presheaf.
- Unfortunately, it is not a sheaf, so we consider the associated sheaf to $\mathcal{S}^n$, and show that

$$H^i(\mathcal{S}^\bullet(X)) = H^i((\mathcal{S}^\bullet)^\dagger(X)).$$

Singular Cohomology

- Note that

$$\mathcal{F}^{\dagger}(U_0) = \varinjlim_{\mathcal{U}\text{ cover }U_0} \ker\Big[\prod_{U\in\mathcal{U}} \mathcal{F}(U) \to \prod_{U,V\in\mathcal{U}} \mathcal{F}(U\cap V)\Big].$$

- Algebraic topology tells us

$$\ker\Big[\prod_{U\in\mathcal{U}} \mathcal{S}^{\bullet}(U) \to \prod_{U,V\in\mathcal{U}} \mathcal{S}^{\bullet}(U\cap V)\Big] = \mathrm{Hom}\Big(\sum_{U\in\mathcal{U}} \mathcal{S}^{\bullet}(U), \mathbb{Z}\Big)$$

computes the same cohomology with $\mathcal{S}^{\bullet}(U_0)$.

- The injective limit commutes with taking cohomology since it is filtered.

Singular Cohomology

- Now, $(\mathcal{S}^n)^\dagger$ is clearly flappy.
- When X is locally contractible, $(\mathcal{S}^\bullet)^\dagger$ is a resolution of constant sheaf $\mathbb{Z}$. So by the discussion above,

$$H^i(X;\mathbb{Z}) = R\Gamma(X;\mathbb{Z}) = H^i((\mathcal{S}^\bullet)^\dagger(X)) = H^i(\mathcal{S}^\bullet(X)) = H^i_{\text{singular}}(X;\mathbb{Z}).$$

- The same method, we can show that

$$H^i_c(X;\mathbb{Z}) = R\Gamma_c(X;\mathbb{Z}) = H^i_c(X).$$

De Rham Theorem

- Another example of resolution is de Rham complex. By the Poincaré lemma, it is a resolution of the constant sheaf $\mathbb{R}$. So

$$H^i(X;\mathbb{R}) = R\Gamma(X;\mathbb{R}) = H^i(\Omega^\bullet(X)(X)) = H^i_{\text{de Rham}}(X).$$

  In particular, $H^i_{\text{de Rham}}(X) = H^i_{\text{singular}}(X;\mathbb{R})$.
- The same method, we can show that

$$H^i_c(X;\mathbb{R}) = H^i_{\text{compact supp de Rham}}(X).$$

Cup Product

- The cup product is induced by $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \to \mathbb{Z}$. More generally, if we have a map $\mathcal{F} \otimes_{\mathbb{Z}} \mathcal{G} \to \mathcal{H}$, it induces

$$I^\bullet_{\mathcal{F}} \otimes_{\mathbb{Z}} I^\bullet_{\mathcal{G}} \xrightarrow{*} I^\bullet_{\mathcal{H}}.$$

We can pick $\mathbb{Z}$-flat resolution, so left hand side is a resolution of $\mathcal{F} \otimes_{\mathbb{Z}} \mathcal{G}$.

$$\Longrightarrow \Gamma(I^\bullet_{\mathcal{F}}) \otimes \Gamma(I^\bullet_{\mathcal{G}}) \longrightarrow \Gamma(I^\bullet_{\mathcal{F}} \otimes_{\mathbb{Z}} I^\bullet_{\mathcal{G}}) \longrightarrow \Gamma(I^\bullet_{\mathcal{H}}).$$

$$\Longrightarrow H^i\Gamma(I^\bullet_{\mathcal{F}}) \otimes H^j\Gamma(I^\bullet_{\mathcal{G}}) \longrightarrow H^{i+j}(\Gamma(I^\bullet_{\mathcal{F}}) \otimes \Gamma(I^\bullet_{\mathcal{G}})) \longrightarrow H^i(\Gamma(I^\bullet_{\mathcal{H}})).$$

- Why cup product of H_{singular} and $H_{\text{de Rham}}$ coincides? It is merely because that $*$ is a morphism of complex (we only need this!).

Singular Homology

Now turn to homology.

- Denote $\mathcal{C}_\bullet(U)$ the set of sum of simplex which is locally finite. We view the simplex the same to its subdivision. Then we can well-define the restriction
$$\mathcal{C}_\bullet^{BM}(U) \to \mathcal{C}_\bullet^{BM}(V)$$
by $\Delta \mapsto \Delta \cap V$, where $\Delta \cap V$ is a locally sum over V.
- For paracompact space, it is a sheaf. We define the homology of $\mathcal{C}_\bullet^{BM}(X)$ to be the **Borel–Moore homology** $H_*^{BM}(X)$.

Singular Homology

Theorem

For compact space X, $H_*^{BM}(X) = H_*(X)$.

- The first assertion is trivial, since locally finite is just finite. Then $\mathcal{C}_\bullet^{BM}(X) = \varinjlim_{\text{subdivision}} \mathcal{C}_\bullet^{\text{ordinary}}$ of course computes the same cohomology group.

Singular Homology

Theorem

For an open subset $U \subseteq X$, we have long exact sequence

$$\cdots \longrightarrow H_*^{BM}(X \setminus U) \longrightarrow H_*^{BM}(X) \longrightarrow H_*^{BM}(U) \longrightarrow \cdots$$

(Note that this is not true for ordinary homology which cannot realize as a sheaf)

- We have $\mathcal{C}_\bullet(X) \to \mathcal{C}_\bullet(U)$, which is of course surjective. By definition, the kernel is just the local finite sum of simplex over $X \setminus U$.

Singular Homology

Theorem

For a locally compact X, $H_*^{BM}(X) = H_*^{ordinary}(X_\infty, \infty)$.

Theorem

The Borel–Moore "cohomology", cohomology group of $\mathrm{Hom}(\mathcal{C}_\bullet^{BM}(X), \mathbb{Q})$ *coincides with $H_c^*(X; \mathbb{Q})$ cohomology of compact support for locally compact X.*

- Since $H_c^* X; \mathbb{Q} = H^*(X_{\cup\infty}, \infty)$ dual to $H_*(X_{\cup\infty}, \infty) = \mathcal{C}_\bullet^{BM}(X)$ over $\mathbb{Q}$.

PoincarĎuality

Theorem (Poincaré duality)

Assume X is locally compact, and embedded in some oriented m-dimensional manifold M, we have $H_*^{BM}(X) = H^{m-*}(M, M \setminus X)$.

- Actually, this theorem is a version of Poincaré duality itself. We can find $[M] \in H_m^{BM}(M)$ by orientation, and define cap product

$$H^{m-*}(M, M \setminus X) \to H_*^{BM}(M).$$

- The main techinque is the Bootstrap Lemma (see Broden) that any closed set in Euclidean space is intersection of finite union of convex closed sets.
- The main step is $\mathcal{C}_*^{BM}(\bigcap X_i) = \varprojlim \mathcal{C}_*^{BM}(X_i)$, since any element from the right hand side restricted to be 0 over the complement of $\bigcap X_i$.

$\gg$ *Questions?* $\ll$

§ Homological Algebra (II) §

Motivation

- Derived category we will define is an analogy to the category of CW-complexes

CW-complex	$\leftrightarrow$	complex
cellular map	$\leftrightarrow$	complex morphism
homotopy	$\leftrightarrow$	homotopy
homotopy group	$\leftrightarrow$	homology
weak homotopy equvalence	$\leftrightarrow$	quasi-isomorphism
suspension	$\leftrightarrow$	dimension shift

Motivation

- In topology, cohomology group is relatively superficial with respect to homotopy. We may lose important information after taking cohomology. So the idea is to express everything before "taking cohomology".

Language of cohomology	ESSENCE
cohomology group	complex
always exact = split	homotopy
inducing isomorphism	quasi-isomorphism
long exact sequence	exact triangle

Definition

- For an abelian category $\mathcal{M}$, define the derived category $\mathcal{D}$ of M the category with

$$\begin{cases} \textbf{Obj}: & \text{complexes in } \mathcal{M} \\ \textbf{Mor}: & \text{complex morphisms} \\ & + \begin{cases} \varphi = \psi \text{ if they are Homotopy} \\ \varphi \text{ is invertible if it is quasi-isomorphism} \end{cases} \end{cases}$$

Quasi-isomorphism $=$ inducing isomorphism in cohomology.

Some Examples

- For any $M \in \mathcal{M}$, consider M as a complex centralized at 0. Then a resolution $M \to I^\bullet$ is an quasi-isomorphism, thus invertible in derived category.

$$\begin{array}{ccccccccccc} \cdots \to & 0 & \to & 0 & \to & M & \to & 0 & \to & 0 & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & 0 & \to & 0 & \to & I^0 & \to & I^1 & \to & I^2 & \to \cdots \end{array}$$

- Actually, if we consider the full subcategory of complex bounded below, then it is equivalent to the category of complexes of injective objects bounded below by taking hyper-resolution.
- Note that, between complexes injective objects bounded below, quasi-isomorphism is invertible already, up to homotopy.

Some Examples

Theorem

Assume $I^\bullet$ is a complex of injective objects bounded below, then

$$\mathrm{Hom}_{\mathcal{D}}(A^\bullet, I^\bullet) = \mathrm{Hom}_{complex}(A^\bullet, I^\bullet)/homotopy.$$

- By construction, a morphism $A^\bullet \to I^\bullet$ is presented by

$$A^\bullet \xrightarrow{\varphi} B^\bullet \xrightarrow{\psi^{-1}} I^\bullet$$

Let us pick a hyper-injective resolution $J^\bullet$ of $B^\bullet$, from the below picture, we can exchange $B^\bullet$ by $J^\bullet$.

$$\begin{tikzcd} A^\bullet \arrow[r, "\varphi"] \arrow[dr] & B^\bullet \arrow[d] & I^\bullet \arrow[l, "\psi"'] \arrow[dl] \\ & J^\bullet & \end{tikzcd}$$

We can simply shrink the right triangle to a point, since they differ only by a homotopy.

Some Examples

- For any $N \in \mathcal{M}$, think it at position n, denote by $N[-n]$. Then

$$\mathrm{Hom}_{\mathcal{D}}(N[-n], M) = \mathrm{Ext}^n_{\mathcal{M}}(N, M).$$

Since

$$\mathrm{Hom}_{\mathcal{D}}(N[-n], M) = \mathrm{Hom}_{\mathcal{D}}(N[-n], I^\bullet) \overset{\text{definition}}{=} \mathrm{Ext}^n_{\mathcal{M}}(N, M)$$

$$\begin{array}{ccccccccccc}
\cdots \to & 0 & \to & M & \to & 0 & \to \cdots \to & 0 & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & 0 & \to & I^0 & \to & I^1 & \to \cdots \to & I^n & \to \cdots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\cdots \to & 0 & \to & 0 & \to & 0 & \to \cdots \to & N & \to \cdots
\end{array}$$

Exact Triangles

- For complex $A^\bullet$, denote $A^\bullet[n] = A^{\bullet+n}$ the dimension shift.
- For a complex morphism $f : X^\bullet \to Y^\bullet$, define its mapping cone

$$\operatorname{cone} f^\bullet = X^{\bullet+1} \oplus Y^\bullet, \qquad d\binom{x}{y} = \begin{pmatrix} -d & \\ 1 & d \end{pmatrix} = (-dx, f(x) + dy).$$

Then it defines a **triangle**

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{y \mapsto (0,y)} \operatorname{cone} f^\bullet \xrightarrow{(x,y) \mapsto -x} X^{\bullet+1}$$

Any triangle homotopy equivalent to above is called an **exact triangle**.

$\gg$ *Questions?* $\ll$

§ Realization of Topology (II) §

Derived Functors

- Since we want everything before cohomology, we need to define a more general derived functor.
- Let $F : \mathcal{M} \to \mathcal{N}$ be a left exact functor. Denote $\mathcal{D}(\mathcal{M})$ and $\mathcal{D}(\mathcal{N})$ the derived category of lower bounded complexes. Then they are equivalent to the derived category of lower bounded injective complexes (again, quasi-isomorphisms are already invertible). For $A^\bullet$, pick a hyper injective resolution $I_A^\bullet$, define

$$RF^\bullet = F(I_A^\bullet) \in \mathcal{D}(\mathcal{N}).$$

  The morphism are defined automatically.
- Note that the usually RF are nothing but the composition

$$\mathcal{M} \stackrel{\text{concentrated at } 0}{\longrightarrow} \mathcal{D}(\mathcal{M}) \xrightarrow{RF} \mathcal{D}(\mathcal{N}) \stackrel{\text{taking cohomology}}{\longrightarrow} \mathcal{N}$$

Sheaves

- Return our case of sheaves. Denote $\mathcal{D}(X) = \mathcal{D}^{>-\infty}(X\text{-}\mathrm{Sh})$. Then for any continuous map $f : X \to Y$, we have
$$Rf_* : \mathcal{D}(X) \to \mathcal{D}(Y), \qquad Rf_! : \mathcal{D}(X) \to \mathcal{D}(Y)$$
$$f^* : \mathcal{D}(Y) \to \mathcal{D}(X).$$
We do not write Rf^* because f^* is exact.
- Then all conclusions before can be lifted to the derived category. $Rg_* \circ Rf_* = R(g \circ f)_*$ since f send injective to injective. The fact $Rg_! \circ Rf_! = R(g \circ f)_!$ is not trivial, but some fine sheaf theory.

Sheaves

- The adjoint

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}(X)}(Rf^*\mathcal{G}^\bullet, \mathcal{F}^\bullet) &= \mathrm{Hom}(f^*I^\bullet_{\mathcal{G}}, I^\bullet_{\mathcal{F}})/\text{homotopy}\\ &= \mathrm{Hom}(I^\bullet_{\mathcal{G}}, f_*I^\bullet_{\mathcal{F}})/\text{homotopy}\\ &= \mathrm{Hom}_{\mathcal{D}(Y)}(\mathcal{G}^\bullet, f_*\mathcal{F}^\bullet)\end{aligned}$$

- For a Cartesian square $\begin{array}{ccc} W & \xrightarrow{F} & Y \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[f]{} & X \end{array}$, then $f^* \circ Rg_! = RG_! \circ F^*$. Since

$$f^* \circ RG_!(\mathcal{F}^\bullet) = f^* \circ G_!(I^\bullet_{\mathcal{F}}) = G_! \circ F^*(I^\bullet_{\mathcal{F}}) = RG_! \circ F^*(\mathcal{F}^\bullet).$$

An Exact Triangle

- Consider closed immersion $i : F \to X$ and open immersion $j : U \to X$ with $F \sqcup U = X$. Then there is an exact triangle for each complex $\mathcal{F}^\bullet$

$$Rj_! j^* \mathcal{F}^\bullet \longrightarrow \mathcal{F}^\bullet \longrightarrow Ri_* i^* \mathcal{F}^\bullet \xrightarrow{+1},$$

  since it is a short exact sequence of complex, which is homotopy to an exact triangle. Actually, by the universal property of mapping cone, any sequence inducing long exact sequence is from some exact triangle.
- Taking in $\mathcal{F}$ be the constand sheaf $\mathbb{Z}$ over X concentred at 0 gives long exact sequence

$$\ldots \to H^*_c(U) \to H^*(X) \to H^*(F) \to \cdots .$$

Poincaré Duality

- Recall we proved Poincaré duality for Borel–Moore homology and cohomology, which can be restated that

$$\begin{array}{ccccccccccc} \cdots & \to 0 \to & \mathbb{Z} & \to \cdots \to & 0 & \to \cdots \\ \vdots & \downarrow & \downarrow & \downarrow & \downarrow & \vdots \\ \cdots & \to 0 \to & \mathcal{C}_n^{BM} & \to \cdots \to & \mathcal{C}_0^{BM} & \to \cdots \end{array}$$

  is a quasi-equivalence. The map is given by $1 \mapsto [M]$. This follows from the local computation and the easy fact $\mathcal{C}^{BM}$ is flappy.
- So it essentially does not need the technique Bootstrap Lemma — framework of sheaves does all the work.

Duality

- For a sheaf, $\mathcal{F}$, $U \mapsto \Gamma_c(U;\mathcal{F})$ is not a sheaf, but when $\mathcal{F}$ is c-soft (soft for compact subset), it is a "cosheaf". Say, for any cover $\mathcal{U}$ of U_0

$$\bigoplus \Gamma_c(U \cap V;\mathcal{F}) \longrightarrow \bigoplus \Gamma_c(U;\mathcal{F}) \longrightarrow \Gamma_c(U_0;\mathcal{F}) \longrightarrow 0.$$

  So $\mathrm{Hom}(\Gamma_c(U;\mathcal{F}), \mathbb{Q})$ forms a sheaf, which can be understood as a "dual".
- Actually, from the discussion above, $\mathbb{Q}$ is "dual" to $\mathcal{C}_\bullet^{BM}$.
- How to generalize the notation of dual is important, this is the Verdier duality.

Theorem (Verdier)

For any continuous map $f: X \to Y$, there is an $f^! : \mathcal{D}^{>-\infty}(Y) \to \mathcal{D}^{>-\infty}(X)$, such that

$$\mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet) = \mathrm{Hom}_{\mathcal{D}(Y)}(Rf_!\mathcal{F}^\bullet, \mathcal{G}^\bullet)$$

Proof Sketch

- Denote $\mathrm{Hom}^n(A^\bullet, B^\bullet) = \mathrm{Hom}^n(A^\bullet, B^\bullet[-n])$, and $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ the tri-complex of sheaves.
- Then let us define

$$f^!\mathcal{G}^\bullet := \left[U \mapsto \mathrm{Hom}^\bullet(f_!(j_U)_! I^\bullet_{\mathbb{Z}}|_U, \mathcal{G}^\bullet)\right]$$

One can check that it forms a sheaf, and maps quasi-isomorphisms to quasi-isomorphisms. Then one can prove the adjointness holds, firstly for $(j_U)_!\mathbb{Z}_U$, then all by a finite resolution (finiteness from paracompactness). See Verdier.

Dualizing Complexes

- For $\pi : X \to \mathrm{pt}$, denote $\omega_X^\bullet = \pi^! \mathbb{Z}$, and call it **dualizing complex**.

Theorem

Assume X is locally compact, then the dualizing complex ω_X is quasi-isomorphic to $\mathcal{C}_\bullet^{BM}$.

- For an open subset U and $j : U \to X$,

$$\mathrm{Hom}_{\mathcal{D}(X)}(j_! \mathbb{Z}|_U, \pi^! \mathbb{Z}) = \mathrm{Hom}_{\mathcal{D}(\mathrm{pt})}(\pi_! j_! \mathbb{Z}|_U, \mathbb{Z}).$$

  The left hand side is exactly $\omega_X(U)$, the right hand is the dual of complex $\mathcal{S}_c^\bullet(U)$ of compact support cochains over U.
- There is a map $\mathcal{C}_\bullet^{BM} \to \mathrm{Hom}_{\mathcal{D}(\mathrm{pt})}(\mathcal{S}_c^\bullet, \mathbb{Z})$, by a long exact sequence argument over $U_{\cup\infty}$, their cohomologies are the same.

Theorem

For closed embedding $i : F \to X$, i^* coincides the functor induced by

$$i^! \mathcal{F} := \left[U \longmapsto \lim_{V \supseteq U} \{ s \in \mathcal{F}(V) : \operatorname{supp} s \subseteq F \} \right]^\dagger .$$

For open embedding $j : U \to X$, $j^* = j^!$.

- Actually, it suffices to show the adjointness. But the adjointness holds in the category of sheaves.

Exact Triangle

- Recall that

$$Rj_! j^* \mathcal{F}^\bullet \longrightarrow \mathcal{F}^\bullet \longrightarrow Ri_* i^* \mathcal{F}^\bullet \xrightarrow{+1},$$

  gives

$$\cdots \to H_c^*(U) \to H^*(X) \to H^*(X \setminus U) \to \cdots .$$

- It has the following dual form, see Hartshorne Ex.II.1.20

$$Ri_* i^! \mathcal{F}^\bullet \longrightarrow \mathcal{F}^\bullet \longrightarrow Rj_* j^* \mathcal{F}^\bullet \xrightarrow{+1}.$$

  which gives long exact sequence

$$\cdots \to H_*^{BM}(X \setminus U) \to H_*^{BM}(X) \to H_*^{BM}(U) \to \cdots .$$

$$\cdots \to H^*(X, U) \to H^*(X) \to H^*(U) \to \cdots .$$

$\gg$ *Questions?* $\ll$

§ <u>THANKS</u> §

References

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- Borel, Moore. Homology theory for locally compact spaces. [pdf]