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Problem 1   I understand the course policies

Problem 2

a) $6n \cdot 2^n + n^{100} = O(3^n)$. 3^n outgrows $2^n \cdot 6n + n^{100}$ & never dips below it once it oes above f.

b) $\log(2n) = \Theta(\log 3n)$. $\log(3n)$ can be multiplied by a big or small C so $\log(2n)$ sits b/w it.

c) $\sqrt{n} = \Omega(\sqrt[3]{n})$. $n^{1/2}$ always outgrows $n^{1/3}$ no matter what $n^{1/3}$ is multiplied by.

d) $\frac{n^2}{\log n} = O(n(\log n)^4)$. when $n \geq 1$, $\frac{n^2}{\log n}$ is always greater than $n(\log n)^4$.

e) $n^2 + n\log n = \Theta(10n^2 + (\log n)^5)$. When $C=1$, g outgrows f & f outgrows g when $C=0.0001$.

f) $(\log_2 n)^{\log_2 n} = O(2^{\log_2 n^2})$. Exp functions outgrow other non-exp functions.

g) $n \cdot \log(n^{20}) = \Omega(\log(3n!))$. $n\log(n^{20}) = \log(n^{20n})$ & n^n functions always out grow factorials.

h) $\log(n^9 + \log n) = \Theta(\log(2n))$. A really big C can make g the upper bound. $C=0.01$ makes g the lower bound.

i) $8^n \cdot n^2 = \Omega((\lfloor\sqrt{n}\rfloor)!)$. $8^n n^2$ always outgrows $\lfloor\sqrt{n}\rfloor!$ no matter what C is in $C \cdot g$.

Problem 3 $\sum_{i=0}^{k} c^i = \begin{cases} \Theta(c^k) & c>1 \\ \Theta(k) & c=1 \\ \Theta(1) & 0<c<1 \end{cases}$

if c is b/w 0 & 1, then as i gets larger, c^i decreases & series converges to a number. Using the partial sum formula, we can prove $d_1 \cdot 1 \leq \frac{1-c^{k+1}}{1-c} \leq d_2 \cdot 1$.

→ starting w/ the upper bound, we have $d_2 \cdot 1 \geq \frac{1-c^{k+1}}{1-c}$

$$d_2 \geq \frac{1-(c^k \cdot c)}{1-c}$$

as $\lim_{k\to\infty} c^k$ converges to zero & thus, $d_2 \geq \frac{1}{1-c}$.

→ for the lower bound, we have $d_1 \cdot 1 \leq \frac{1-c^{k+1}}{1-c}$.

$$d_1 \leq \frac{1-c^k \cdot c}{1-c}$$

$d_1 \leq \frac{1}{1-c}$. The terms in the series are all positive, so we can remove a few terms to get an appropriate bound:

$$d_1 \leq \frac{1}{1-c} - c^0 - c^1 - c^2 \ldots$$

if $c=1$, then we have $d_1 \cdot k \leq \sum_{i=0}^{k} c^i \leq d_2 \cdot k$. $\sum_{i=0}^{k} c^i = 1 \cdot k$ when $c=1$. So we have:

$$d_1 \cdot k \leq 1 \cdot k \leq d_2 \cdot k$$

$$d_1 \leq 1 \leq d_2$$

Use any # for d_1 & d_2 to prove :)

if $c \geq 1$, then we have $d_1 \cdot c^k \leq \sum_{i=0}^{k} c^i \leq d_2 \cdot c^k$ for $c>1$.

$\sum_{i=0}^{k} c^k$ for $c>1$ diverges, & each term gets larger over time.

→ equals $c^0 + c^1 + c^2 \ldots + c^{k-1} + c^k$. So then:

$$d_1 \cdot c^k \leq c^0 + c^1 + c^2 \ldots + c^{k-1} + c^k \leq d_2 \cdot c^k$$

$$d_1 \cdot c^k \leq c^k\left(\frac{c^0}{c^k} + \frac{c^1}{c^k} + \ldots + \frac{c^{k-1}}{c^k} + 1\right) \leq d_2 \cdot c^k$$

$$d_1 \leq \left(\frac{c^0}{c^k} + \frac{c^1}{c^k} + \ldots + \frac{c^{k-1}}{c^k} + 1\right) \leq d_2$$

Use any # for d_1 less than the middle term & vice versa for d_2 to prove :)

Problem 4

a) $f(n)=\sum_{i=0}^{d} a_i\cdot n^i$ w/ $a_d\neq 0$. $f(n)=\begin{cases} O(n^k) & k\ge d, \\ \Omega(n^k) & k\le d, \\ \Theta(n^k) & k=d. \end{cases}$

For $k=d$, $\lim_{n\to\infty}\frac{f(n)}{n^k}=\lim_{n\to\infty}\frac{(a_0+a_1 n^1+\dots+a_d n^d)}{n^d}=\lim_{n\to\infty}\underbrace{\frac{a_0}{n^d}+\frac{a_1 n^1}{n^d}+\dots}_{\text{all go to zero}}+\frac{a_d n^d}{n^d}=\underbrace{a_d}_{\text{finite, non-zero}}$

can prove $\Theta(n^k)$ from this, Use constants greater & less than a_d.

For $O(n^k)$, $n^k\ge n^d$ when $k\ge d$.

For $\Omega(n^k)$, $n^k\le n^d$ when $k\le d$.

b) $\sum_{k=1}^{n} k^2=\Theta(n^3)$

$C_1\cdot n^3\le 1^2+2^2+\dots+(n-1)^2+(n)^2\le C_2 n^3$

$C_1\le\frac{1+2^2+\dots+(n-1)^2+n^2}{n^3}\le C_2$

LHS:

$n=2$: $1+2^2/2^3\ge C_1 \rightarrow \frac{5}{8}\ge C_1$

$n=3$: $\frac{1+4+9}{3^3}\ge C_1 \rightarrow \frac{14}{27}\ge C_1$

$n=4$: $\frac{1+4+9+16}{4^3}\ge C_1 \rightarrow \frac{30}{64}\ge C_1$

$n=5$: $\frac{1+4+9+16+25}{5^3}\ge C_1 \rightarrow \frac{55}{125}>C_1$

constantly decreasing

$C_1=\frac{5}{8}$, $n_0=1$

RHS:

$\lim_{n\to\infty}\frac{1+2^2+\dots+(n-1)^2+n^2}{n^3}\le C_2$

$0\le C_2$

C_2 can be any number but zero. n_0 can be anything.

c) $\sum_{k=1}^{n} k^j=\Theta(n^{j+1})$ $j>0$. $k\le n$ all terms

$k\ge\frac{n}{2}$ many terms

$C_1 n^{j+1}\le\sum_{k=1}^{n} k^j\le C_2 n^{j+1}$

RHS: $1^j+2^j+\dots+n^j\le\underbrace{n^j+n^j+\dots n^j}_{n\text{ times}}=n\cdot n^j\le C_2\cdot n^{j+1}$

LHS: remove 1st half of sum.

$1^j+\dots+\left(\frac{n}{2}\right)^j+\dots+n^j\ge\left(\frac{n}{2}\right)^j+\dots+n^j\ge\underbrace{\left(\frac{n}{2}\right)^j+\dots+\left(\frac{n}{2}\right)^j}_{n/2\text{ times}}=\frac{n}{2}\cdot\left(\frac{n}{2}\right)^j\ge C_1\cdot\left(\frac{n}{2}\right)^{j+1}$

d) $\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} ij = \Theta(n^4)$

$$C_1 \cdot n^4 \leq \left(\left[1\times 2+\ldots+1\times n\right]+\left[2\times 1+2\times 3+\ldots+2\times n\right]+\ldots+\left[n\times 1+\ldots+(n\times(n-1))\right]\right) \leq C_2 \cdot n^4$$

RHS

$$\underbrace{\left[1\times 2+\ldots+1\times n\right]+\left[2\times 1+2\times 3+\ldots+2\times n\right]+\ldots+\left[n\times 1+\ldots+(n\times(n-1))\right]}\leq (\overbrace{n^2+\ldots+n^2}^{n \text{ times added}})\,\underbrace{n} = n^3 \cdot n \leq C_2 \cdot n^4$$

LHS Cut out 1st half of terms...

$$\left[1\times 2+\ldots+1\times n\right]+\left[2\times 1+2\times 3+\ldots+2\times n\right]+\ldots+\left[n\times 1+\ldots+(n\times(n-1))\right] \geq \left[\frac{n}{2}\times 1+\ldots+\frac{n}{2}\times n\right]+\ldots+\left[n\times 1+\ldots+(n\times(n-1))\right] \geq \overbrace{\left[\frac{n}{2}\times 1+\ldots+\frac{n}{2}\times n\right]}^{n \text{ times}}+\ldots+\overbrace{\left[n\times 1+\ldots+n\times(n-1)\right]}$$

Cut $\frac{1}{2}$ terms again

$$\geq \left[\frac{n}{2}\cdot\left(\frac{n}{2}-1\right)+\ldots+\frac{n}{2}\cdot n\right]+\ldots+\left[n\cdot\frac{n}{2}+\ldots+n(n-1)\right]$$

$$\geq \left[\frac{n}{2}\cdot\frac{n}{2}+\frac{n}{2}\cdot\frac{n}{2}+\ldots+\frac{n}{2}\cdot\frac{n}{2}\right]+\ldots+\left[\frac{n}{2}\cdot\frac{n}{2}+\ldots+\frac{n}{2}\cdot\frac{n}{2}\right]$$

$$= \left(\left(\frac{n}{2}\right)^2\cdot\frac{n}{2}\right)\frac{n}{2}$$

$$= \frac{n^4}{16}$$

$$\geq C_1 \frac{n^4}{16}$$

Problem 5

a) $f(n) = O(g(n)) \longleftrightarrow g(n) = \Omega(f(n))$

Part 1: $f(n) = O(g(n)) \longrightarrow g(n) = \Omega(f(n))$

$f(n) = O(g(n))$

$f(n) \leq C_1 \cdot g(n)$

$f(n) \cdot \frac{1}{C_1} \leq g(n)$

$f(n) \cdot C_2 \leq g(n)$

$\Omega(f(n)) = g(n)$ ✓

Part 2: $g(n) = \Omega(f(n)) \longrightarrow f(n) = O(g(n))$

$g(n) \geq C_1 \cdot f(n)$

$\frac{1}{C_1} g(n) \geq f(n)$

$C_2\, g(n) \geq f(n)$

$O(g(n)) = f(n)$ ✓

b) $C_1 \cdot (f(n) + g(n)) \leq \max(f(n), g(n)) \leq C_2 \cdot (f(n) + g(n))$

$\hookrightarrow = 1$

by intuition

$f(n) + g(n) \geq \max(f(n), g(n))$

$f(n), g(n) \leq \max(f(n), g(n))$

$f(n) + g(n) \leq 2 \max(f(n), g(n))$

$\frac{1}{2}(f(n) + g(n)) \leq \max(f(n), g(n))$

c) $\log_a n = \Theta(\log_b n) \quad \forall a, b > 1$

$C_1 \cdot \log_b n \leq \log_a n \leq C_2 \cdot \log_b n$

$C_1 \cdot \frac{\cancel{\log_a n}}{\log_a b} \leq \cancel{\log_a n} \leq C_2 \cdot \frac{\cancel{\log_a n}}{\log_a b}$ $\quad \log_b(n) = \frac{\log_a(n)}{\log_a(b)}$

constants found. ✓