SRNDNA "Model Fitting in RL" Workshop

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Topics:

1. trial-by-trial model fitting (morning; Yael)
2. model comparison (morning; Yael)
3. advanced topics (hierarchical fitting etc.) (afternoon; Nathaniel)
4. applications to fMRI (afternoon; Nathaniel)

Goals:

Understand how models are hypotheses that can be tested

General tool to apply to any time-varying data of interest

Practicalities – start here, know where you can go from there

Bandit tasks

a simple task often used in the laboratory:

- repeated choice between n options (n-armed bandit)
- ...whose properties (reward amounts, probabilities) are learned through trial-and-error

overall approach:

1. learn values for options
2. choose the best option

how do you suggest to model this learning process?

Bandit tasks

suppose we ran this experiment on a person

what are the data?

what does the model predict?

what can we conclude/infer from the data?

our models are basically detailed hypotheses about behavior and about the brain... we can test these hypotheses!

Writing down a full model of learning

what do we know?
what can we measure?
what do we not know?

Estimating model parameters

why estimate parameters?

1. May measure quantities of interest (learning rates in different populations, how variance in the task affects learning rate etc.)
2. have to use these to generate hidden variables of interest (eg. prediction errors) in order to look for these in the brain

how to estimate parameters?

we want: $P(\alpha,\beta | D,M)$

wwBd?

$P(\alpha,\beta | D,M) \propto P(D | \alpha,\beta,M)$ ⇐ This we know!

$P(D | \alpha,\beta,M) = \prod P(c_t | \alpha,\beta,M)$

Estimating parameters

$P(\alpha,\beta \mid D,M) \propto P(D \mid \alpha,\beta,M)$

...this is a probability distribution

we often consider a point estimate: the maximal likelihood point

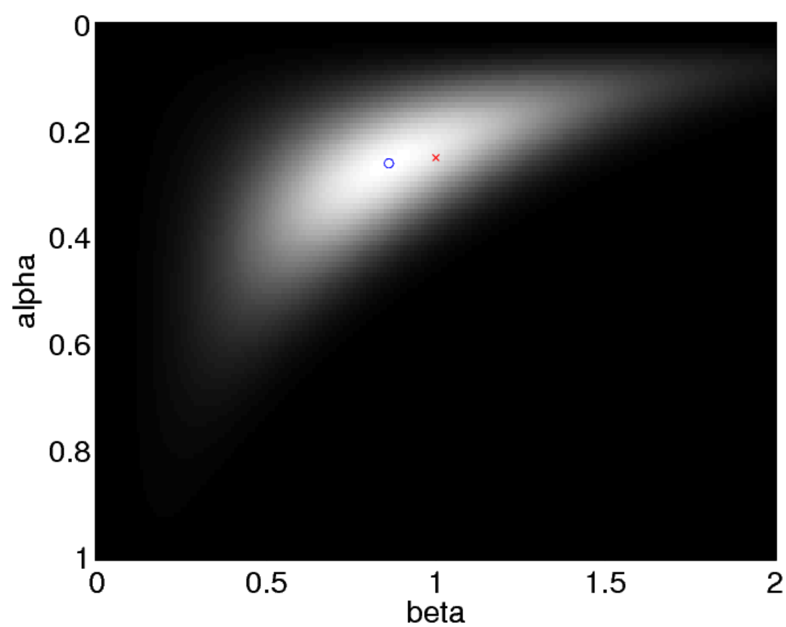
$\mathrm{argmax}_{\alpha,\beta}\, P(D \mid \alpha,\beta,M)$

(equivalently, can maximize the log likelihood)

Estimating parameters

$P(D \mid \alpha,\beta,M) = \prod P(c_t \mid \alpha,\beta,M)$

pragmatics:
fmincon, fminunc, fminsearch
minimize -LL
local minima: multiple starting points



simulated 1000 trials

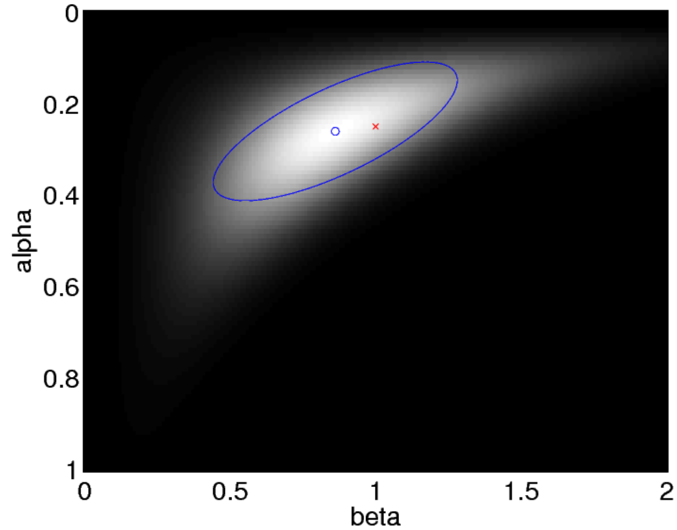
$\alpha=0.25, \beta=1$

recovered:

$\alpha=0.26, \beta=0.86$

Error bars on estimates

intuition: how fast likelihood is changing as you change the parameters

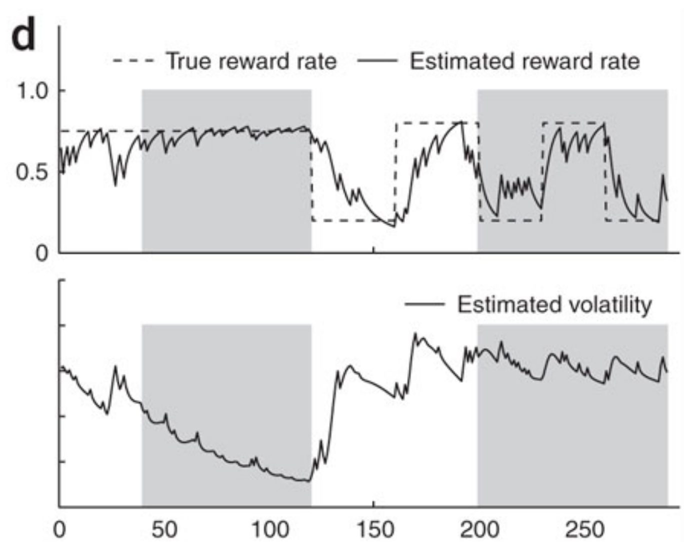


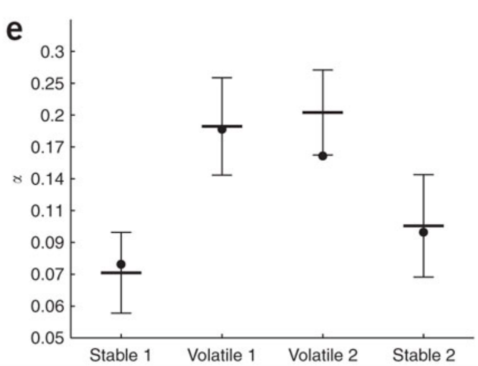
pragmatics: inverse Hessian (2nd derivative matrix; fmincon gives you this) of −LL estimates parameter covariance matrix

- error bars along the diagonal (sqrt), covariation off the diagonal
- can also look at variation in fits across subjects (more on this in the afternoon)

Example: do learning rates adapt to volatility?

Behrens et al. (2007)





dots = Bayesian

bar + SE = humans (estimated from choice data)

Summary so far

- Learning models are detailed hypotheses about trial-by-trial overt and covert variables
- these can be tested against the brain
- help understand what different brain regions/networks are doing/computing
- a whole host of interesting results so far, but many questions still unanswered (relatively new method!)

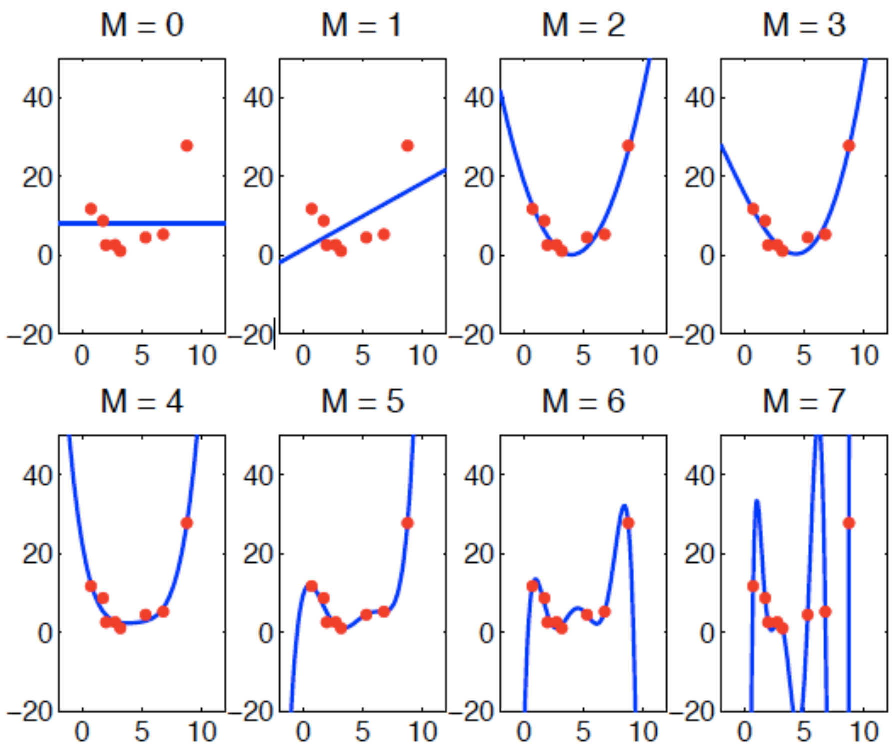
- the models help us learn about the brain... can we also use the brain (or behavior) to learn about the models??

Which model is best? Model comparison

- P(Model | Data) = ?
- comparing two models: $\frac{P(M_1|D)}{P(M_2|D)} = \frac{P(D|M_1) \cdot P(M_1)}{P(D|M_2) \cdot P(M_2)}$

Bayes factor

Which model is best? Model comparison



Which model is best? Model comparison

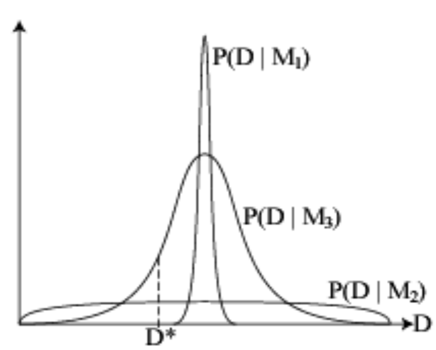
automatic Occam's razor: simple models tend to make precise predictions

can put in preference for simple models here, but don't need to...

- P(Model | Data) = ?
- comparing two models:

$$\frac{P(M_1|D)}{P(M_2|D)} = \boxed{\frac{P(D|M_1)}{P(D|M_2)}} \cdot \boxed{\frac{P(M_1)}{P(M_2)}}$$

- "Pluralitas non est ponenda sine necessitate"
  Plurality should not be posited without necessity – William of Ockham (1349)
- we should go for the simplest model that explains the data



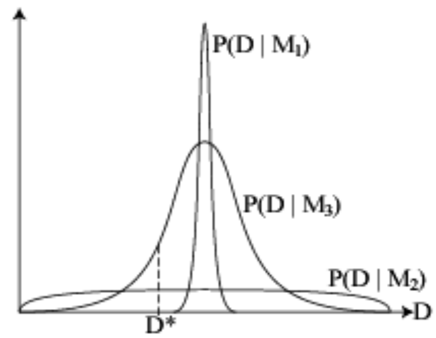
Which model is best? Model comparison

$$\frac{P(M_1|D)}{P(M_2|D)} = \frac{P(D|M_1) \cdot P(M_1)}{P(D|M_2) \cdot P(M_2)}$$

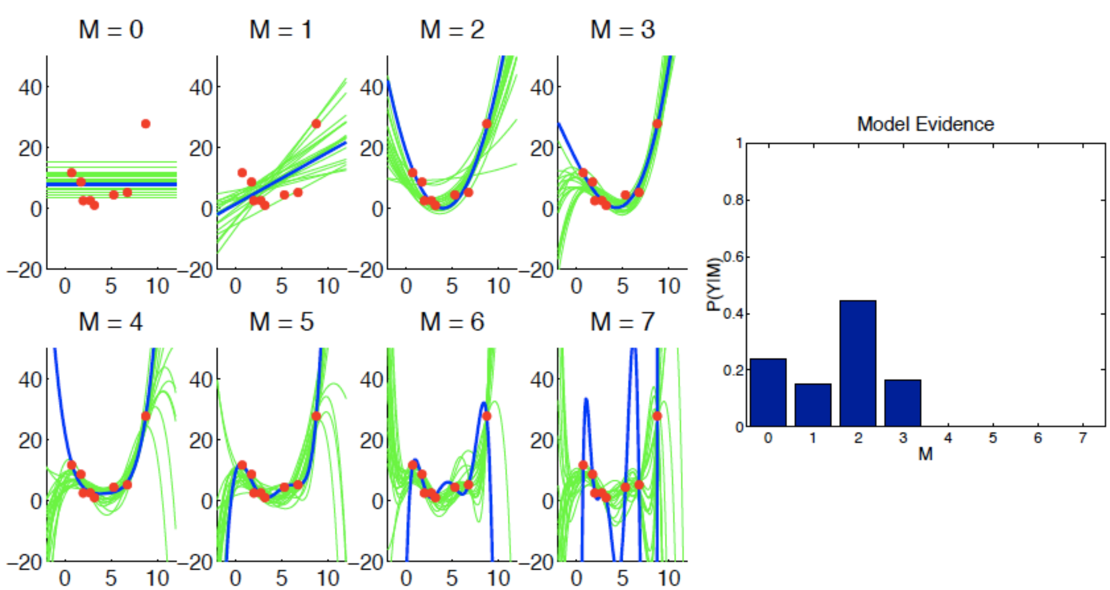
assuming uniform prior over models all we care about is P(D | M)

$$P(D|M) = \int d\theta P(D|M,\theta) \cdot P(\theta)$$

Bayesian evidence
for model M
(marginal likelihood)



Bayesian model comparison: Occam's Razor at work



Computing P(D | M)

$$P(D|M) = \int d\theta P(D|M,\theta) \cdot P(\theta)$$

- Integrating over all settings of the parameters is too hard...
- Approximate solutions:
  - sample posterior at many places to approximate integral and compute Bayes factor directly
  - Laplace approximation: make Gaussian approximation around MAP parameter estimate
  - BIC: asymptotic approximation to above in limit of infinite data
- Non-Bayesian methods:
  - Likelihood ratio test
  - AIC
  - hold-out set to verify model fits

Laplace approximation

$$P(D|M) = \int d\theta P(D|M,\theta) \cdot P(\theta)$$

- For large amounts of data (compared to # of parms d) the posterior is approximately Gaussian around the maximum a postriori (MAP) estimate $\hat{\theta}$

$$P(\theta|D,M) \approx 2\pi^{-\frac{d}{2}}|A|^{\frac{1}{2}} exp\left\{-\frac{1}{2}(\theta-\hat{\theta})^T A(\theta-\hat{\theta})\right\}$$

- and we also know that

$$P(D|M) = \frac{P(\theta,D|M)}{P(\theta|D,M)} = \frac{P(\theta|M)P(D|\theta,M)}{P(\theta|D,M)}$$

- so we can compute around the MAP estimate:

$$lnP(D|M) \approx lnP(\hat{\theta}|M) + lnP(D|\hat{\theta},M) + \frac{d}{2}ln(2\pi) - \frac{1}{2}ln|A|$$

- where -A is the Hessian matrix of lnP(θ | D,M)

$$A_{kl} = -\frac{\partial^2}{\partial\theta_{mk}\partial\theta_{ml}} lnP(\theta|D,M)|_{\hat{\theta}}$$

BIC approximation

$$lnP(D|M) \approx \boxed{lnP(\hat{\theta}|M)} + \boxed{lnP(D|\hat{\theta}, M)} + \boxed{\frac{d}{2}ln(2\pi)} - \boxed{\frac{1}{2}ln|A|}$$

prior on θ　　data log likelihood　　easy

- In the limit of LOTS of data ($N \rightarrow \infty$) A grows as NA_0 (for fixed A_0) so $\ln |A| = \ln |NA_0| = \ln N^d |A_0| = d\ln N + \ln|A_0|$.
- Retaining only terms that grow with N, we can approximate further:

$$lnP(D|M) \approx lnP(D|\hat{\theta}, M) - \frac{d}{2}ln(N)$$

(so, for each model we compute the log likelihood for the ML parameters and then add to that a penalty that depends on d (# of parameters), and then we compare the results between the models)

- Advantages: easy to compute; can use ML rather than MAP estimate
- Disadvantage: hard to determine d (only identifiable parameters) and N (only samples used to fit parameters; what if not same for diff parms?)

Non-Bayesian alternatives

- Likelihood ratio test: for nested models (one is a special case of the other; compares hypothesis $\mathcal{H}_1$ to one where some parameters are fixed, $\mathcal{H}_0$).
  - Statistical test on the likelihood differences: compare 2*difference in log likelihood (ML) to χ^2 statistic with df=#additional parameters
- AIC (Akaike's information criterion, 1974): measures goodness of a model based on bias and variance of the estimated model and measures of entropy.
  - Not statistical test, only ranks models.
  - Penalize log likelihood (ML) by adding # of parameters
- Fit models on training set and validate fit on hold-out set.
  - Problem: often hard to find two i.i.d. sets in a learning setting

Summary so far...

- Learning models are detailed hypotheses about trial-by-trial overt and covert variables

- trial-by-trial model fitting lets us test these hypotheses
- ...and compare alternatives

- premium on detailed model fitting when considering learning: non-stationary, can't use traditional averaging techniques
- a lot of leverage to pinpoint the neural correlates of learning and decision making in the brain

Additional reading

- Daw (2011) - Trial by trial data analysis using computational models

- Hare et al. (2008) - Dissociating the role of the orbitofrontal cortex and the striatum in the computation of goal values and prediction errors

- Niv (2009) - Reinforcement learning in the brain