

$$L = T - V$$

$$E = T + V$$

$$K = Gm_1 m_2$$

$$\vec{J} \equiv \text{ang. mom.}$$

$$T = \frac{1}{2} m v^2$$

$$V = -K/r \quad \Rightarrow \quad L = \frac{1}{2} m v^2 + K/r$$

in 2 d/m.

$$\vec{v} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta}$$

$$v^2 = \vec{v} \cdot \vec{v} = \dot{r}^2 + r^2 \dot{\theta}^2 \quad \Rightarrow \quad L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + K/r$$

Lagrange's eq.

$q \equiv$  generalized coord.

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0. \quad (m a - F = 0)$$

$$r \text{ eq.} \quad \frac{d}{dt} \left( \frac{m}{2} (2\dot{r}) \right) - \frac{\partial}{\partial r} \left( \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{K}{r} \right) = 0$$

$$\equiv m \ddot{r} - m r \dot{\theta}^2 - \frac{K}{r^2} = 0 \quad (1)$$

$$\theta \text{ eq.} \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt} \left( \frac{m}{2} (2r^2 \dot{\theta}) \right) = 0$$

$$J \equiv m r^2 \dot{\theta}$$

$$\frac{d}{dt} (J) = 0 \quad \Rightarrow \quad J = \text{const.}$$

$$m r \dot{\theta}^2 = \frac{J \dot{\theta}}{r} = \frac{J^2}{m r^3}$$

Insert into (1).

$$(2) \quad m \ddot{r} - \frac{J^2}{m r^3} + \frac{K}{r^2} = 0$$

$$m \ddot{r} - F'(r) = 0.$$

$$u = \frac{1}{r} \quad r = 1/u.$$

$$\dot{r} = \frac{dr}{du} \frac{du}{d\theta} \frac{d\theta}{dt} \quad \text{chain rule.}$$

$$= -\frac{1}{u^2} \frac{du}{d\theta} \dot{\theta} = -\dot{\theta} r^2 \frac{du}{d\theta} = -\frac{J}{m} \frac{du}{d\theta}.$$

$$\begin{aligned} \frac{d}{dt} \dot{r} = \ddot{r} &= -\frac{J}{m} \frac{d}{dt} \left( \frac{du}{d\theta} \right) = -\frac{J}{m} \frac{d}{d\theta} \left( \frac{du}{dt} \right) = -\frac{J}{m} \frac{d}{d\theta} \left( \frac{du}{d\theta} \frac{d\theta}{dt} \right) \\ &= -\frac{J}{m} \frac{d^2 u}{d\theta^2} \dot{\theta} \quad \left( \frac{d\dot{\theta}}{d\theta} = 0 \right) \\ &= -\frac{J^2}{m^2} u^2 \frac{d^2 u}{d\theta^2} \end{aligned}$$

$$\Rightarrow (2). \quad = -\frac{J^2}{m} u^2 \frac{d^2 u}{d\theta^2} - \frac{J^2}{m} u^3 + k u^2 = 0.$$

$$\boxed{\frac{d^2 u}{d\theta^2} + u = \frac{mk}{J^2} = 0.} \quad (3)$$

$$\frac{d^2 u}{d\theta^2} + u = 0.$$

has as solution.

$$u_0(\theta) = A \cos(\theta - \theta_0)$$

$$(3) \Rightarrow u(\theta) = u_0(\theta) + \frac{mk}{J^2} \quad \Leftarrow \text{this is a conic section.}$$

max min values.

$$\cos(\theta - \theta_0) = \pm 1$$

$$\frac{1}{r_1} = +\frac{mk}{J^2} + A \quad \frac{1}{r_2} = +\frac{mk}{J^2} - A$$

need to determine A

$$\begin{aligned} E = T + V &= \frac{1}{2} m v^2 - \frac{k}{r} = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{k}{r} \\ &= \frac{1}{2} m \dot{r}^2 + \frac{J^2}{2mr^2} - \frac{k}{r} \end{aligned}$$

(3).

at turning points.  $\min/\max r \quad \dot{r} = 0$

$$\Rightarrow E_{\text{at turning points}} = \frac{J^2}{2mr^2} - \frac{k}{r}$$

$$E = \frac{J^2}{2m} u^2 - k u_m$$

~~$$u = \frac{k}{J^2}$$~~

$$0 = \frac{u_m^2}{2} - \frac{mk}{J^2} u_m - \frac{mE}{J^2}$$

$$u_m = \frac{mk}{J^2} \pm \sqrt{\frac{m^2 k^2}{J^4} + \frac{2mE}{J^2}}$$

$$\Rightarrow A = \sqrt{\left(\frac{mk}{J^2}\right)^2 + \frac{2mE}{J^2}}$$

if  $E < 0$ . (bound orbit).

$$\left(\frac{mk}{J^2}\right)^2 > \left|\frac{2mE}{J^2}\right|$$

$$\frac{mk^2}{2J^2} > |E| \quad (E < 0)$$

note trivial solution

$$u = \text{const} = \frac{mk}{J^2}$$

$$\Rightarrow E = \frac{J^2}{2m} u^2 - k u$$

$$= \frac{J^2}{2m} \cdot \frac{m^2 k^2}{J^4} - \frac{mk^2}{J^2}$$

$$= \frac{mk^2}{2J} - \frac{mk^2}{J}$$

$$\boxed{T = -\frac{V}{2}}$$

Virial thm.

$$E=0.$$

parabola

$$\frac{J^2}{2m} U_m^2 - k U_m = 0.$$

$$\Rightarrow U_m = 0 ? \Rightarrow r = \infty$$

$$U_{\text{max}} \Rightarrow \frac{J^2}{2m} U - k = 0$$

$$U = \frac{2km}{J^2}$$

$$r_{\text{min.}} = \frac{J^2}{2km}$$

at min.

$$E=0 = \frac{J^2}{2mr^2} - \frac{k}{r} = \frac{J^2}{2m} \left( \frac{2km}{J^2} \right)^2 - \frac{2k^2m}{J^2} = 0.$$

$$T = v$$

$$E > 0$$

hyperbola



## Quantum mechanics

Quantum mechanics was started by the realization of the wave particle duality in the properties of light ~~realized~~ by Einstein in 1905 as a solution of the Photo electric effect. This phenomena is described as follows: a clean metal will "sputter off" or emit electrons when hit by light. This is not unreasonable as light waves have energy and when this energy is added to the electrons they can escape the atomic binding in the metal. The problem is the following: below a critical frequency the electrons cannot be emitted no matter how intense the light is but above the critical frequency electrons will be emitted no matter how faint the light is. In a wave description of light the energy density and flux increase with both the intensity and frequency thus this turn on effect absolutely cannot be explained. The solution proposed by Einstein was that the energy in light came in packets of energy

$$E = h \nu$$

and these packets ~~are~~ <sup>are</sup> quantized in the sense they always come in integer number. Thus the problem is solved because an electron can only absorb ~~one~~ a quantum. So unless the energy of a photon is above the binding energy it can't liberate an electron

(for reasons we'll see multiple absorptions are impossible)

2

The other key part to the development of Quantum mechanics was the observation of atomic emission and absorption light spectra. If a gas of an element is heated sufficiently (by say passing a current through it) it will emit light but not in a continuum (a rainbow) but only discrete colors or frequencies. These emission lines correspond to electrons in the atoms changing "states" and when they change they emit or absorb a photon having the energy equal to the difference of the energies of the states.

$$E_{\gamma} = h\nu = |E_i - E_j|$$

The spectra for each element and molecule is unique providing a technique for composition analysis. It had been found that the emission and absorption lines of hydrogen the lightest element could be described by the equation.

$$E_{\gamma_{ij}} = R \cdot \left( \frac{1}{n_i^2} - \frac{1}{n_j^2} \right) \quad n_i, n_j \text{ integers}$$

ie that the energy states followed a pattern

$$E_i = \frac{R}{n_i^2}$$

Niels Bohr explained this by postulating that angular momentum was intrinsically quantized coming in units of Planck's constant divided by  $2\pi$

$$J = n\hbar = \frac{nh}{2\pi} \quad n = 0, 1, 2, 3, \dots$$



This provides the following derivation of the energies of the hydrogen atom.

$$J = n\hbar = m_e v r$$

$$\frac{m_e v^2}{r} = \frac{k q_1 q_2}{r^2} \quad (f = m_e a)$$

$$\frac{J^2}{m_e^3 r^3} = \frac{k q_1 q_2}{r^2} \rightarrow m_e v^2 = \frac{k q^2}{r}$$

$$\boxed{\frac{J^2}{m_e k q_1 q_2} = r}$$

$$E = \frac{1}{2} m_e v^2 - \frac{k q_1 q_2}{r} = -\frac{1}{2} m_e v^2$$

$$= \frac{J^2}{2 m_e r^2} - \frac{k q_1 q_2}{r} = -\frac{J^2}{2 m_e r^2}$$

$$= \frac{-J^2}{2 m_e} \frac{1}{r^2}$$

$$= \frac{-J^2}{2 m_e} \left( \frac{m_e k q^2}{J^2} \right)^2 = -\frac{k^2 q^4 m_e}{2 J^2}$$

$$= -\frac{k^2 q^4 m_e}{2 (n\hbar)^2} = \boxed{-\frac{k^2 q^4}{\hbar^2} \frac{m_e}{2} \left( \frac{1}{n^2} \right)}$$

\* not completely correct  
as  
 $J=0$  states (s wave) exist.  
de Broglie hypothesis

This equation is completely determined in terms of known constants and not only does it have the  $\frac{1}{n^2}$  form but  $\frac{k^2 q^4}{\hbar^2} \frac{m}{2}$  is exactly equal to the constant  $R$  (the Rydberg)

The constant  $\frac{k^2 q^4}{\hbar^2} \frac{m}{2} = \frac{k^2 q^4}{\hbar^2 c^2} \cdot \frac{m c^2}{2} = \alpha^2 \frac{m c^2}{2}$

$$\boxed{= \frac{1}{137} \frac{m_e c^2}{2}}$$

## Schroedinger Heisenberg wave/matrix mechanics

In the 1920s two versions of quantum mechanics developed and it was some time until the two were shown to be identical. For the purpose of our discussions of stellar structure two results will be needed the uncertainty principle and the energy states of free particles confined to a volume. (particle in a box). So we'll derive and state them.

The uncertainty principle is a basic postulate from which matrix mechanics is built. It states that the position and momentum (related to conjugate variables  $q, \dot{q}$ ) cannot be simultaneously known with infinite accuracy but that the product of their uncertainties must be  $\geq \hbar$

$$\Delta x \Delta p_x \geq \hbar$$

or the commutator of  $[x, p] = i\hbar$

$xp - px = i\hbar$  ie don't commute!

this defines a minimum volume in position-momentum space to which a particle can be confined.

wave mechanics/particle in a box

Schroedinger went at it a different way associating a particle with a wave function  $\psi$  that had the property that ~~the~~ was proportional to the probability density

$$|\psi(x_0)|^2$$

that the particle was at  $x_0$

### Operators

The values of physical quantities ~~are~~ must be extractable from the wave function. This



means physical quantities can be calculated by executing some operation on the wave function.  
for example

$$\hat{X} \psi(x) = x \psi(x)$$

where  $\hat{X}$  is the ~~momentum~~ position operation

and  $x$  is the value at the position

this works easily for position but momentum is a bit more complicated

given that:

$$[X, P] \psi(x) = i\hbar \psi(x).$$

$$X P \psi(x) - P X \psi(x) = i\hbar \psi(x)$$

$$x (P \psi(x)) - [(P x) \psi(x) + x P \psi(x)] = i\hbar \psi(x)$$

assuming the operation  $P$  is distributive.

$$\text{if } P = -i\hbar \frac{\partial}{\partial x} = \frac{\hbar}{i} \frac{\partial}{\partial x}$$

$$\text{then, } [X, P] \psi = x (-i\hbar) \frac{\partial \psi}{\partial x} + i\hbar \frac{\partial}{\partial x} (x \psi(x))$$

$$= -i\hbar x \frac{\partial \psi}{\partial x} + i\hbar \left( \frac{\partial x}{\partial x} \psi(x) + i\hbar x \frac{\partial \psi}{\partial x} \right)$$

$$= i\hbar \psi(x)$$

as desired.

~~The energy~~  
For a state described by a wave function to have a unique value associated with an operator it must be an Eigenfunction of that operator

The basic components to wave mechanics can be argued from the de Broglie hypothesis that for any particle of momentum  $p$  there is a wave of associated wavelength

$$\lambda = \frac{h}{p}$$

hence a plane wave  $e^{i2\pi(\frac{x}{\lambda} - \nu t)}$

$$e^{i\frac{2\pi \vec{x} \cdot \vec{p}}{h}} = e^{i\frac{\vec{p} \cdot \vec{x}}{\hbar}}$$

as for light one can associate

$$E = h\nu$$

the full wave can be written.

$$\Psi(x,t) \propto \exp\left[\frac{i}{\hbar}(\vec{p} \cdot \vec{x} - Et)\right]$$

consider the momentum operator  $\hat{p}$  defined by

$$\hat{p} \Psi(x,t) = p \Psi(x,t)$$

$$\hat{p} = \frac{-i\hbar}{\lambda} \frac{\partial}{\partial x}$$

$$\text{ie } p_x = \frac{-i\hbar}{\lambda} \frac{\partial}{\partial x}$$

similarly the energy operator (which is the Hamiltonian)

$$H \Psi(x,t) = E \Psi(x,t) = i\hbar \frac{\partial}{\partial t} \Psi(x,t)$$

$$\text{but } H(\Psi) = \left(\frac{p^2}{2m} + V(x)\right)\Psi = \left(\frac{-\hbar^2}{2m} \nabla^2 + V(x)\right)\Psi = i\hbar \frac{\partial}{\partial t} \Psi(x,t)$$



The commutators are connected to a classical mechanics construct called the Poisson bracket

$$\{A, B\} = \left\{ \frac{\partial A}{\partial p_k} \frac{\partial B}{\partial q_k} - \frac{\partial B}{\partial p_k} \frac{\partial A}{\partial q_k} \right\}$$

it can be shown in a system where you worry about the order that.

$$U_1 V_1 - V_1 U_1 = \text{const} \{U_1, V_1\}$$

The constant must be  $i\hbar$  if the uncertainty principle is to hold.

The Poisson bracket of a quantity with the Hamiltonian can be shown to yield the time derivative of the quantity.

$$\{H, A\} = \dot{A}$$

thus  $H$  can be shown to act as a time development operator. This led Schroedinger to ~~write~~ write the following wave eq.

$$H \Psi = +i\hbar \frac{\partial \Psi}{\partial t} \quad \text{Schroedinger eq.}$$

fuzzy source  
see Dirac  
Sec 2.1 → 2.8

$$H = KE + PE = \frac{p^2}{2m} + V$$

$$\Rightarrow H \Psi(x,t) = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \Psi(x,t) = +i\hbar \frac{\partial \Psi(x,t)}{\partial t}$$

this is an odd wave equation, while it is quadratic in the position derivative it is linear in the time derivative (as opposed to the usual wave equation like Maxwell's equations which are quadratic in both derivatives)



## eigenstates / eigenfunctions.

When a state (hydrogen atom in the ground level for ex.) is an eigenfunction of an operator, it ~~has~~ has a unique well defined value for that ~~operator~~ quantity. a trivial example is the ~~the~~ position operator

$$\hat{X} \psi(x) \Big|_{x_0} = x_0 \psi(x)$$

Our ground state of Hydrogen has a unique well defined energy and will therefore satisfy the eq.

$$H \bar{\psi}(x,t) = E_0 \bar{\psi}(x,t)$$

In this case the state is said to be an eigenstate of the operator ( $H$ ) with eigenvalue  $E_0$ .

The Schrodinger equation reduces to

$$H \bar{\psi}(x,t) = \left( \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \bar{\psi}(x,t) = E_0 \bar{\psi}(x,t) = \frac{i\hbar}{m} \frac{\partial}{\partial t} \bar{\psi}(x,t)$$

if  $\bar{\psi}(x,t) = \psi(x) \cdot \phi(t)$

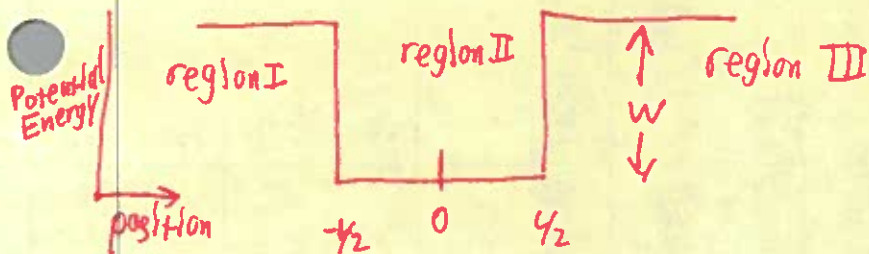
then  $\phi = \cancel{\psi(x)} e^{-i E_0 t / \hbar}$

and.  $\boxed{\frac{-\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + V(x) \psi(x) = E_0 \psi(x)}$

Why do we do all this? We will need to concern ourselves with what happens when a particle is constrained to a volume (particle in a box) It will become apparent that in certain cases the quantum mechanical nature of the universe can become manifest in objects the size of stars.

⑧

Particle in a box



region I  $x < -1/2$   
 region II  $-1/2 < x < 1/2$   
 region III  $1/2 < x$

in the box. (region II)

$E_0$  = lowest allowed energy state.

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} = E_0 \psi(x)$$

$$\text{or } \frac{\partial^2 \psi(x)}{\partial x^2} = \left( \frac{2mE_0}{\hbar^2} \right) \psi(x)$$

~~cost~~ solutions  $\psi(x) = A \cos\left(\sqrt{\frac{2mE_0}{\hbar^2}} x\right) + B \sin\left(\sqrt{\frac{2mE_0}{\hbar^2}} x\right)$

$$\frac{\partial}{\partial x} \cos(kx) = -k \sin(kx)$$

$$\frac{\partial}{\partial x} \sin(kx) = k \cos(kx)$$

outside the box. (regions I, III)

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + W \psi(x) = E_0 \psi(x)$$

$$\frac{\partial^2 \psi(x)}{\partial x^2} = \frac{2m(E_0 - W)}{\hbar^2} \psi(x)$$

Same if  $(E_0 - W) > 0$

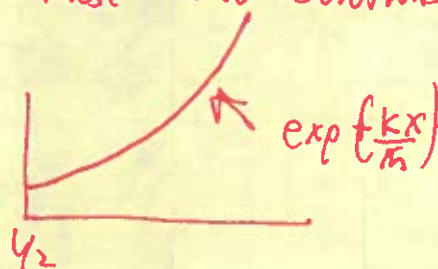
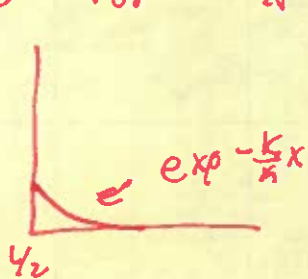
if  $E_0 - W < 0$  (as is the case)

Solutions  $k = \sqrt{\frac{2m(W - E_0)}{\hbar^2}}$

$$\psi(x) = A' \exp\left(-\frac{k}{\hbar} x\right) + B' \exp\left(\frac{k}{\hbar} x\right) \quad \text{for } x > 1/2$$

$$\frac{\partial}{\partial x} \exp(kx) = k \exp(kx)$$

Thus for  $x > 1/2$  these two solutions look like



(9)

The second solution has the problem that

$$|\psi(x)|^2 = \text{probability density.}$$

and  $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$  ie the particle is somewhere.

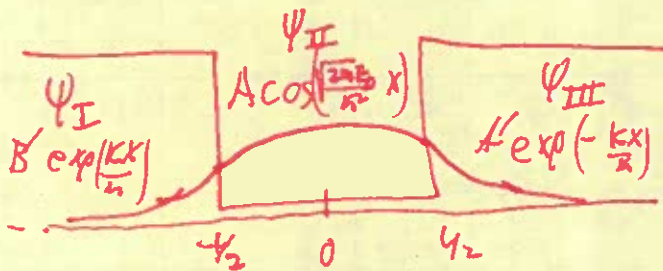
the increasing exponential diverges.

ie  $\int_{\frac{1}{2}}^{\infty} \left| \exp \frac{kx}{\hbar} \right|^2 dx = \infty$

$\Rightarrow$  in region III  $\psi(x) = A' \exp\left(-\frac{k}{\hbar}x\right)$   
 in region I  $\psi(x) = B' \exp\left(\frac{k}{\hbar}x\right)$

The wavefunction must be smooth and continuous.

thus if we think a little we can draw it.



the constants  $A, A', B'$  can be determined

by  $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$

and the continuity of  $\psi$  at the boundaries.

$$\psi_I(-1/2) = \psi_{II}(-1/2)$$

$$\frac{\partial \psi_I}{\partial x}(-1/2) = \frac{\partial \psi_{II}}{\partial x}(-1/2)$$

$$\psi_{II}(1/2) = \psi_{III}(1/2) \quad (1)$$

$$\frac{\partial \psi_{II}}{\partial x}(1/2) = \frac{\partial \psi_{III}}{\partial x}(1/2) \quad (2)$$

~~A cos~~



$$\text{let } P = \sqrt{2mE_0}$$

$$k = \sqrt{2m(W-E_0)}$$

$$\textcircled{1} \quad A \cos \frac{PL}{2\hbar} = A' \exp\left(-\frac{KL}{2\hbar}\right)$$

$$\textcircled{2} \quad -\frac{AP}{\hbar} \sin\left(\frac{PL}{2\hbar}\right) = -\frac{A'K}{\hbar} \exp\left(-\frac{KL}{2\hbar}\right)$$

taking the ratio  $\textcircled{2}/\textcircled{1}$

$$\frac{P}{\hbar} \tan\left(\frac{PL}{2\hbar}\right) = \frac{K}{\hbar}$$

$$P \tan\left(\frac{PL}{2\hbar}\right) = K$$

now let the walls of the box become infinitely hard i.e.  $W \rightarrow \infty \Rightarrow K \rightarrow \infty$

$$\Rightarrow \tan\left(\frac{PL}{2\hbar}\right) \rightarrow \infty$$

$$\Rightarrow \boxed{\frac{PL}{2\hbar} = \frac{\pi}{2} (2n+1)} \quad n=1, 2, 3, 4$$

ie  $\frac{PL}{2\hbar} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \dots$

$$\text{Or. } \frac{\sqrt{2mE_0} L}{2\hbar} = \pi/2$$

$$2mE_0 = \frac{\pi^2 \hbar^2}{L^2}$$

$$\boxed{E_0 = \frac{\pi^2 \hbar^2}{2mL^2}}$$

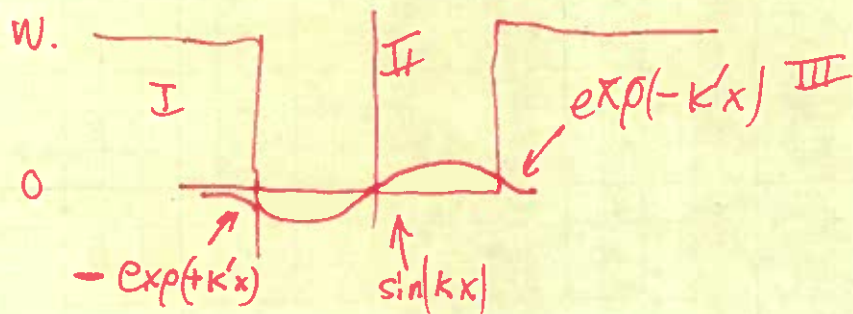
$$E_1 = \frac{3^2 \pi^2 \hbar^2}{2mL^2}$$

$$\boxed{E_n = \frac{(2n+1)^2 \pi^2 \hbar^2}{2mL^2}}$$

The advantage of this derivation is that we don't require  $\psi(\pm L/2) = 0$  at the boundaries, we derive it.

Supplement on solutions

first excited state is a sin function.



$$k = \sqrt{2mE_1}$$

$$k' = \sqrt{2m(W-E_1)}$$

Conditions 1/2.

$$\textcircled{1} \quad A \sin \frac{kL}{2\hbar} = A' \exp\left(-\frac{k'L}{2\hbar}\right)$$

$$\psi_{II} = \psi_{III}$$

$$\textcircled{2} \quad \frac{k}{\hbar} A \cos \frac{kL}{2\hbar} = \frac{-k'}{2\hbar} A' \exp\left(-\frac{k'L}{2\hbar}\right)$$

taking ratios

$$\frac{k}{\hbar} \cot \frac{kL}{2\hbar} = \frac{k'}{\hbar}$$

$$\text{walls} \rightarrow \infty \Rightarrow k' \rightarrow \infty$$

$$\cot \rightarrow \infty \quad n\pi \quad n = 0, 1, 2, 3$$

$$\Rightarrow \frac{kL}{2\hbar} = n\pi$$

$$k = \frac{2n\pi\hbar}{L}$$

$$E = \frac{4n^2\pi^2\hbar^2}{2mL^2} = \frac{(2n)^2\pi^2\hbar^2}{2mL^2}$$

SO. the cosine solutions had,

$$E_{\cos} = \frac{(2n+1)^2 \pi^2 \hbar^2}{2mL^2} \Rightarrow \text{series } 1, 3^2, 5^2, 7^2, 9^2$$

the sine solutions

$$E_{\sin} = \frac{(2n)^2 \pi^2 \hbar^2}{2mL^2} \Rightarrow \text{series } 2^2, 4^2, 6^2, 8^2, \dots$$

therefore the full seq.

$$E = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$$



The point of this is that there are only a discrete countable set of states that can be put into a given volume. with  $E_{\text{state}} < E_{\text{max}}$

Conclusions:

①. Quantum mechanics (+ EM) correctly derives the energy spectrum of light emitted by hydrogen, ie it explains the line spectra of atoms. which can be used to determine the chemical compositions of astronomical objects like stars and gas clouds.

②. Particle in a box.

an object constrained to move in one dimension within a boundary of separation  $L$  can only have energies.

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2} \quad \text{on} \quad 2mE = \frac{n^2 \pi^2 \hbar^2}{L^2}$$

$$\Rightarrow P = \sqrt{2mE} = \frac{n\pi\hbar}{L}$$

in 3 dimensions

$$E_{n_x, n_y, n_z} = \left( \frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2} \right) \frac{\pi^2 \hbar^2}{2m}$$

~~for a cubic box~~ for a cubic box  $L_x = L_y = L_z$ .

$$2mE = n^2 \cdot \frac{\pi^2 \hbar^2}{L^2} \quad n^2 = n_x^2 + n_y^2 + n_z^2$$

$$P = \sqrt{2mE} = \sqrt{n^2} \cdot \frac{\pi\hbar}{L}$$

$$\boxed{PL = \sqrt{n^2} \pi \hbar} \quad \text{like uncertainty principle.}$$

$\sim P \cdot (\text{Vol})^{1/3} \sim \sqrt{n^2} \pi \hbar$  ie can count states.

## Special relativity.

The postulates of relativity are (1). that the laws of physics as written down by any observer in their own local coordinate system with local variables will have the same form as those written down by another observer in another coordinate system as long as the two coordinate systems are only moving with respect to one another with a constant velocity.  $V$

(2). That the speed of light as measured by both these observers in their own coordinates will always have the value  $c = 2.997... \times 10^8 \text{ m/sec}$

The first postulate has always been assumed. To understand special relativity one must first be very clear about what ~~the~~ form of relativity existed before the Galilean relativity.

Consider two observers  $O, O'$  moving with respect to each other with a velocity  $\vec{V}$ . for example let  $O$  be a person on a sidewalk and  $O'$  a person moving by in a bus.

for the observer  $O'$  on the bus reading a book (the physical universe by F. Shu.) the location of the book is fixed 50 cm away from his face

$$x'(\tau) = +50 \text{ cm.} \equiv \text{const.} \equiv x'_0$$

for the observer O at time  $t=0$ . (starts the watch)  
the book was also 50 cm in the x direction.

$$x(t=0) = 50 \text{ cm.} \equiv x_0$$

but as time moves on and the bus and book do the position changes.

$$x(t) = x_0 + v \cdot t$$

as it moves a distance  $v \cdot t$  in time  $t$ .

Consider a force.

lets take our two observers now to have one moving at constant velocity upwards in an elevator.  
velocity =  $v_y$  and again our stationary book.

$$\Rightarrow y(t) = y_0 + v_y t$$

$$y'(t) = y_0$$

or now drops the book.

The book feels the force of gravity (which is essentially constant.  $\frac{1}{R_e^2}$  doesn't change by much in

1 or 2 seconds in an elevator)

$$F_g' = m_{\text{book}} \cdot g$$

$$g = 9.8 \text{ m/s}^2 \cdot$$

= accel. of gravity.

$$\Rightarrow m a_y' = m \frac{dv_y'}{dt} = m \frac{d^2 y'}{dt^2} = m g.$$

$$\Rightarrow \frac{d^2 y'}{dt^2} = g.$$

$$\Rightarrow \frac{d^2 y(t)}{dt^2} = \frac{d^2 (y(t) - v_y t)}{dt^2} = \frac{d^2 (v_y t)}{dt^2} = \frac{d^2 y(t)}{dt^2} - \frac{dv_y}{dt} = \frac{d^2 y(t)}{dt^2}$$



~~$$\ddot{x} = g = \ddot{y}$$~~

~~and we~~

now if observer O had written down

$$F = ma$$

$$m \frac{d^2 y}{dt^2} = m g.$$

$$\frac{d^2 y}{dt^2} = g.$$

we get the same result

i.e. both observers can write down Newton's laws and get the same mathematical formula and even consistent answers but do the problem in ~~some~~ completely different coordinates and in fact if instead we use the transformations.

$$Y(t) = Y'(t) + V_y t$$

$$\begin{aligned} g = \frac{d^2 Y(t)}{dt^2} &= \frac{d^2 Y'(t)}{dt^2} + \frac{d^2 (V_y t)}{dt^2} \\ &= \frac{d^2 Y'(t)}{dt^2} + \frac{d V_y}{dt} \\ &= \frac{d^2 Y'(t)}{dt^2}. \end{aligned}$$

we can also derive these same equations i.e. everything is consistent no problems but. how do velocities transform?

$$Y(t) = Y'(t) + V_y t$$

$$\begin{aligned} V_y(t) &= \frac{dY(t)}{dt} = \frac{dY'(t)}{dt} + \frac{d(V_y t)}{dt} \\ &= V_y' + V_y \end{aligned}$$

ie velocities transform by adding a constant.  
so consider a light beam shot upward

if O says the speed of light is  
 $C = 2.997 \times 10^8 \text{ m/sec.}$

he predicts that O' will say.

$$C' = 2.997 \times 10^8 \text{ m/sec} + V_y$$

$$C' \neq C$$

this was experimentally proven wrong by  
michelson - morley experiment of 1881

H.A. Lorentz had found the set of transformations  
which left ~~the~~ Maxwells equations (the laws of  
electromagnetism) in the same form when written in the  
new coordinates or transforming the equations from the  
old to the new.

for a velocity in the x direction  $V_x$

$$X' = \frac{X}{\sqrt{1 - V_x^2/c^2}} - \frac{V_x t}{\sqrt{1 - V_x^2/c^2}}$$

$$Y' = Y$$

$$Z' = Z$$

$$t' = \frac{t}{\sqrt{1 - V_x^2/c^2}} - \frac{V_x/c^2 \cdot X}{\sqrt{1 - V_x^2/c^2}}$$

So not only do the spatial coordinates change but time does also

rewriting using  $\gamma \equiv \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-\beta^2}}$

$$\beta = v/c.$$

$$x' = \gamma x - \gamma \beta c t$$

$$y' = y$$

$$z' = z$$

$$c t' = \gamma c t - \gamma \beta x$$

for a light wave moving away from the origin starting at time = 0.

$$x^2 + y^2 + z^2 - c^2 t^2.$$

along the x axis  $\frac{\text{dist}^2}{x^2} - c^2 t^2 = 0.$  as  $c = \frac{\text{dist}}{t}$   
~~rewriting this~~

In the prime frame the velocity of light is also c.

$$x'^2 + y'^2 + z'^2 - (c t')^2 = 0$$

along the x axis.

$$x'^2 - (c t')^2 = 0$$

$$(\gamma x - \gamma \beta c t)^2 - (\gamma c t - \gamma \beta x)^2 = 0$$

$$= \gamma^2 x^2 + \gamma^2 \beta^2 (c t)^2 - 2 \gamma^2 \beta x c t - [\gamma^2 (c t)^2 + \gamma^2 \beta^2 x^2 - 2 \gamma^2 \beta x c t] = 0$$

$$= \gamma^2 x^2 + \gamma^2 \beta^2 (c t)^2 - 2 \gamma^2 \beta x c t - \gamma^2 (c t)^2 - \gamma^2 \beta^2 x^2 + 2 \gamma^2 \beta x c t = 0$$

$$= (\gamma^2 - \gamma^2 \beta^2) x^2 + (\gamma^2 \beta^2 - \gamma^2) (c t)^2 = 0$$

$$\gamma^2 (1 - \beta^2) = \frac{1 - \beta^2}{1 - \beta^2} = 1$$

$$\Rightarrow \boxed{x^2 - c^2 t^2 = 0}$$

ie the  $O'$  observer predicts that  $O$  will observe light = c. Velocity



With this transformation I predict that all observers measure the same velocity of light.

if  $v/c \ll 1$

then  $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = 1$

$$\delta\beta C = V$$

$\gamma_B \sim 0$

$$x' = x - vt$$

$$Y' = Y$$

$$z' = z$$

$$Ct' = Ct - 0$$

galilean  
transformations.

## time dilation/length contraction

Interval of two events (an event being described by a 4-vector  $(\vec{x}, t)$ ) ~~at a constant position and to measure a length~~ ~~you measure the time~~ ~~you must do it~~ ~~change~~ ~~to measure a time change~~

You must measure the position of the two ends at the same time.

the first is quite easy

Say two things happen separated by a time  $t$  at  $x=0$ . Start the clocks when the first happens.

$$Ct' = \gamma Ct - \gamma \beta x \quad \text{but } x=0.$$

$$\Rightarrow t' = \gamma t = \frac{t}{\sqrt{1-v^2/c^2}}$$

times get longer:

but these events now do not occur at the same place

now say we have a rod of length  $L$  in the rest frame to measure the length in the moving frame we have to do it in a tricky way set  $x=0$  at  $t=0$ .

$$\Rightarrow x'=0 \text{ at } t'=0.$$

We need to measure the position  $x=L$  at  $t'=0$ .

$$ct' = \gamma ct - \gamma \beta x = 0$$

$$\Rightarrow ct = \beta x$$

$$x' = \gamma x - \gamma \beta ct.$$

$$\begin{aligned}\Rightarrow L' &= \gamma L - \gamma \beta ct \\ &= \gamma L - \gamma \beta \beta L \\ &= \gamma L - \gamma \beta^2 L \\ &= \gamma (1 - \beta^2) L \\ &= \sqrt{1 - \beta^2} L.\end{aligned}$$

ie. the lengths get shorter.

## Doppler shift.

When either the source or observer of light are moving (with respect to each other) there is a shift in the observed wavelength from the true wavelength. This is the Doppler shift. The effect is most well known when discussing sound. In the case of sound the wave is a physical disturbance of a medium so in that case one finds

moving observer  $f' = f \left(1 \pm \frac{v}{V}\right)$   $V = \text{speed of sound.}$

moving source  $f' = \frac{f}{\left(1 \pm \frac{v}{V}\right)} \sim f \left(1 \mp \frac{v}{V}\right)$  neglecting terms of order  $v^2/V^2$

$$\frac{f' - f}{f} = \frac{v}{V} = \frac{\lambda' - \lambda}{\lambda}$$

In the case of light there is no medium. so intrinsically the only thing that can count is the relative motion. so the equations must be identical for moving source or moving observer. The result from special relativity is of the form.

$$\frac{\lambda}{\lambda_0} = \left(\frac{1+v/c}{1-v/c}\right)^{1/2}.$$

$$\text{or } \frac{\lambda - \lambda_0}{\lambda_0} = \left(\frac{1+v/c}{1-v/c}\right)^{1/2} - 1. \quad \equiv z$$

now to understand this a little better it is useful to point out that the frequency and wave number. ( $\omega, k$ ) form a 4 vector i.e. they transform just like  $\vec{x}$  and  $t$ . In other words.

$$\begin{aligned} \omega' &= \gamma(\omega - v k_x) \\ k'_x &= \gamma(k_x - \frac{v}{c^2} \omega) \\ k'_y &= k_y \\ k'_z &= k_z \end{aligned}$$



$$\text{define } \frac{p}{h} = \frac{1}{\lambda} \equiv k = \sqrt{k_x^2 + k_y^2 + k_z^2} = \gamma/c$$

$$k_x = k \cdot \cos \theta$$

$\theta$  = angle between  
direction of wave  
and x axis

$$\gamma'/c = \gamma \gamma/c - \gamma \beta k_x$$

$$k'_x = \gamma k_x - \gamma \beta \gamma/c$$

$$k'_y = k_y$$

$$k'_z = k_z$$

$$\begin{aligned} \gamma'/c &= \gamma \left( \gamma/c - \beta k \cos \theta \right) \\ &= \gamma \gamma/c (1 - \beta \cos \theta). \end{aligned}$$

this is almost the same but is in fact exactly correct. The difference is subtle. Classically (non relativistically) the doppler shift only applies to velocities along the line between the source and observer ie  $\theta = 0$

in that case.

$$\begin{aligned} \gamma' &= \gamma \gamma (1 - \beta) \\ &= \gamma \frac{1 - \beta}{\sqrt{1 - \beta^2}} \\ &= \gamma \frac{1 - \beta}{\left( (1 + \beta)(1 - \beta) \right)^{1/2}} \\ &= \gamma \left( \frac{1 - \beta}{1 + \beta} \right)^{1/2}. \end{aligned}$$

$$\Rightarrow \lambda' = \lambda_0 \left( \frac{1 + \beta}{1 - \beta} \right)^{1/2}$$

but there is even a doppler shift if the velocity is perpendicular to the direction

the light is travelling.

$$\gamma' = \gamma \gamma = \frac{\gamma}{\sqrt{1-\beta^2}}$$

this transverse doppler shift is very small but was first observed in 1938 by Ives and Stillwell.

for low velocities

$$\beta = v/c \ll 1$$

$$\gamma = 1.$$

$$\frac{\lambda - \lambda_0}{\lambda_0} = z = v/c.$$

mass, energy and momentum.

kinetic energy and momentum also make a 4 vector and transform as.

$$E' = \gamma E - \gamma \beta P_x$$

$$P'_x = \gamma P_x - \gamma \beta E$$

$$P'_y = P_y$$

$$P'_z = P_z$$

relative of observers  
for motion along the  
x axis

however we have to change our definition of momentum and kinetic energy. For a derivation see Berkeley physics course Vol 1 mechanics, (elastic scattering) our new definitions are.

$$P = \gamma m v = \gamma m \beta c. \quad \text{instead of } m v$$

$$E = \gamma m c^2 \quad \text{instead of } \frac{1}{2} m v^2.$$

$$\text{but. } E = \frac{m c^2}{\sqrt{1-v^2/c^2}} = m c^2 \left( -1 + \frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \frac{v^4}{c^4} + \dots \right)$$

$$= m c^2 + \frac{1}{2} m v^2 + \frac{3}{8} m \frac{v^4}{c^2}$$

$$\text{if } v/c \ll 1 = m c^2 + \frac{1}{2} m v^2.$$

One can consider a relativistic mass.

$$M = \gamma m_0$$

$m_0$  being the mass evaluated in the frame of reference where it is at rest. this is sometimes a useful quantity.



# Thermodynamics and Statistical mechanics.

The study of the flow of heat (thermodynamics) is a very old subject and quite remarkable in what it was able to do from the basis of very general arguments and a few measurements. This subject succeeded in establishing 4 basic laws which the Universe appears to obey

zeroth law.

Heat flows from hot areas to cold areas until the Temperatures are equal i.e thermal equilibrium is reached.

First law. (Conservation of energy).

the change in internal energy of a state is equal to the heat added minus the work the state does.

$$dU = dQ - dW$$
$$= dQ - P dV$$

$dU$  change in internal energy  
 $dQ$  change in heat  
 $dW$  work done.  
 $= P dV$

$P$  = pressure

$dV$  = change in volume

Second law. (Law of entropy)

Kelvin's form.   
No physical process can transform heat from one source into work with no other change i.e. can't make a 100% efficient machine.   
in any real phenomena the entropy increases for an isolated system.   
~~at best~~ i.e. Entropy is a measure of the disorder of a system and the disorder always increases.   
Third law.

As the temperature goes to zero the entropy goes to zero i.e. there is an ~~zero~~ absolute zero to the temperature scale.

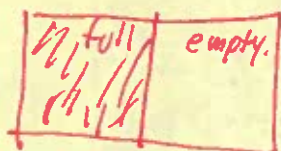
The laws  $1 \Rightarrow 3$  can sometimes humourously be stated as.

- ① you can't win.
- ② you can't break even (unless you're lucky)  
or things go from bad to worse
- ③ you can't stop playing the game.

but on a more serious note. In the second half of the 19<sup>th</sup> century it was found that the results of Thermodynamics could be derived from a statistical discussion of the state of a system. This statistical argument could be very exact when the number of degrees of freedom of the system was very large (ex a cubic centimeter of gas has  $\sim 3 \times 10^{20}$  molecules and  $\sim 10^{21}$  degrees of freedom)

In general we are interested in how a system (a box of gas) behaves as a function of time. In the statistical discussion we are interested in the average <sup>or most probable</sup> state  $\bar{\alpha}$  and the fluctuations around the average. This average is then an average over time. This however is an almost unmanageable mathematical problem. The difficulty is partially philosophical, in that the laws of physics are deterministic. Once the state of the system is specified (the positions and momenta of every gas molecule in the box) its' future time development is found by simply integrating the equations of motion. Similarly the system can be reversed by simply reversing the directions

of all the velocities. This leads to considerable confusion (ex. consider a box divided in half the right half full of gas the left half evacuated



with a wall in the middle to keep it that way, at  $t = 0$  the wall is removed and in no time at all the entire box is full of gas



$t = \epsilon \rightarrow$  very small number.

now if at  $t = \epsilon$  all the velocities <sup>of the gas atoms</sup> reverse ~~the~~ in a short time ( $\epsilon$ ) the left hand side of the box will again be empty! )

The alternative was to consider the Gibbs Ensemble. This is a large set of identical systems and average over the set or ensemble of systems rather than averaging over time. The idea being that these averages are equal and problems of reversability are conceptually less full of pitfalls and apparent paradoxes.

Quantum mechanics simplified the philosophical problems further because at least conceptually an exact count of the number of accessible states could be made.



The study of thermodynamics had yielded a series of mathematical relations which described how matter behaved as heat (energy) and work (mechanical energy) were added or extracted from an isolated system. As can be seen just by reading the second and third laws the quantity called the ENTROPY played a central role. In classical thermodynamics (the dynamics of heat) this could be shown to be, for the state A

$$S(A) = \int_0^A \frac{da}{T}$$

Where 0 is a standard state which can be selected arbitrarily and the change is done by what is called a reversible transformation.

From this function all other <sup>macroscopic</sup> quantities could be defined.

$$dS = \frac{da}{T}$$

$$da = du + PdV \Rightarrow dS = \frac{da}{T} = \frac{1}{T}(du + PdV)$$

$$\Rightarrow \frac{1}{T} = \left( \frac{\partial S}{\partial u} \right)_{PV}$$

$$\frac{P}{T} = \left( \frac{\partial S}{\partial V} \right)_T$$

P = pressure

T = temperature

u = internal energy

V = volume

S = Entropy

For a macroscopic system like a box of gas the state is defined by P, T, V and the number of gas molecule

So the entropy is the key thing. One important property of the entropy is that it is additive. If I have two systems A and B and add them the entropy of the combined system.

$$S(A+B) = S(A) + S(B).$$

Statistical mechanics.

back to our ensembles and gases.

Consider an ensemble of systems and the distribution of their ~~possible~~ conditions at a given time. With a sufficiently large sample we can define a probability for each state.

$P_\alpha$

Boltzman among other things showed that the entropy of ~~one~~ ~~one~~ one of our subsystems when it was in the state  $\alpha$  was.

$$S(\alpha) = k \ln(P_\alpha)$$

where  $k$  was the constant which now bears his name

"Proof" that  $S(\alpha)$  is proportional to  $\ln P_\alpha$  (Fermi thermodynamics)

Consider a system composed of two parts A and B.

the combined entropy, is.

$$S(A+B) = S(A) + S(B).$$

further assume that  $S(\alpha) = f(P(\alpha))$  i.e.  $S$  is some function of the probability (any function).

$P(A+B) = P(A) \cdot P(B)$  probabilities are multiplicative

$$S(A+B) = f(P(A+B)) = f(P(A) \cdot P(B))$$

true for all states.

$$\Rightarrow f(xy) = f(x) + f(y)$$

take  $y = 1 + \epsilon$   $\epsilon$  infinitesimally small.

$$f(x \cdot y) = f(x \cdot (1 + \epsilon)) = f(x + \epsilon x) = f(x) + f(1 + \epsilon)$$

do a Taylor expansion,

$$f(x_0 + \epsilon) = f(x_0) + \epsilon \frac{d f(x_0)}{dx} + \frac{1}{2} \epsilon^2 \frac{d^2 f(x_0)}{dx^2} + \dots$$

$$f(x + \epsilon x) = f(x) + \epsilon x f'(x) = f(x) + f(1) + \epsilon f'(1)$$

$$f'(x) \equiv \frac{d f(x)}{dx}$$

Set  $\epsilon = 0$ .

$$f(x) \equiv f(x) + f(1)$$

$$\Rightarrow f(1) = 0$$

$$\Rightarrow f(x) + \epsilon x f'(x) = f(x) + \epsilon f'(1)$$

$$\Rightarrow x f'(x) = f'(1) = \text{const} \equiv C$$

$$f'(x) = \frac{C}{x}$$

Integrating

$$f(x) = C \ln(x) + \text{const}'$$

$$\Rightarrow S(\alpha) = f(P(\alpha)) = \cancel{\text{const}} C \ln(P(\alpha)) + \text{const}'$$

and  $\text{const}'$  is irrelevant.

now this is where quantum mechanics comes in because we can count all of the possible states therefore we can compute their probabilities. From this we can compute the macroscopic properties of the system.



Ideal gas.

What is the point of this in astronomy?

A star we will find out is made of completely ionized hydrogen (a proton and electron) and helium (a helium nucleus or  $\alpha$  particle and two electrons) for the most part. These particles behave in this plasma (a sort of ionized soup) as infinitely hard objects which bounce off each other through their electromagnetic interaction. This is exactly the essential property of a theoretical construct from statistical mechanics called the ideal gas.

The ideal gas can be visualized as a set of hard billiard balls all bouncing off one another. The atoms/molecules have a distribution of velocities and in the course of their collisions the velocity of any given atom will change. Maxwell showed that for round hard balls, (monatomic gas, <sup>neon</sup>helium, <sup>Argon</sup>noble gas) that in the course of these collisions there was one stable velocity distribution ~~and that is~~ one distribution which would not change in time. Further if you start with a different distribution you will quickly arrive at the Maxwellian velocity distribution.

$$\text{Prob}(v_x, v_y, v_z) \propto \exp\left(-\frac{v_x^2 + v_y^2 + v_z^2}{2KT}\right) dv_x dv_y dv_z$$

Equation of state for the ideal gas.

It had been observed experimentally that a volume of gas had the following relation between its macroscopic parameters.

$$PV = NKT$$

$P$  = pressure

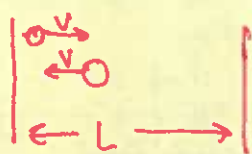
$V$  = volume

$N$  = number of atoms/molecules in the volume

$K$  = Boltzmann's constant

$T$  = Temperature on the absolute scale of Kelvin

Our model of the ideal gas can produce a result which is rather similar to this. Consider a one dimensional box with a ball bouncing between the walls elastically.



the rate of collisions with a given wall is

$$\frac{1}{\text{Period}} = \frac{v}{2L}$$

with each collision the change in momentum is.

$$\Delta P = \bar{P}_f - P_i = 2mv \quad P \text{ is momentum}$$

$$F = \frac{\Delta P}{\Delta T} = 2mv \cdot \frac{v}{2L} = \frac{mv^2}{L}$$

$$\text{Pressure} = F/\text{Area} = \frac{mv^2}{L \cdot \text{Area}} = \frac{mv^2}{\text{Volume}}$$

if there were more balls each would do the same thing hence the pressure is proportional to the number of balls.

$$\text{Press.} \cdot \text{Volume} = N \cdot (mv^2)$$

or

$$PV = N \cdot (2 KE)$$

KE kinetic energy,

if ~~the~~ I say that the kinetic energy/ball/degree of freedom is equal to  $\frac{1}{2}$  (Boltzmann's const)  $\cdot$  Temp. then.

$$PV = NKT$$

the ideal gas law.



Quantum mechanics and occupation densities.

In the course of his discussions of ideal gasses Boltzman realized that the probability of a state being occupied was of the form,

$$P(\epsilon) = \exp(-\epsilon/kT)$$

where  $\epsilon$  is the energy of the states. This is apparent from the velocity distribution.

$$P(v_x, v_y, v_z) \sim \exp\left(-\frac{m(v_x^2 + v_y^2 + v_z^2)}{2kT}\right) dv_x dv_y dv_z.$$

We retain the above form but now include ~~the~~ some results from quantum mechanics. ~~There are~~ Particles

can have intrinsic angular momentum like a top has spin. This intrinsic angular momentum is called spin and is quantized but now not in units of  $\hbar$

$$J = n \hbar$$

but in units of  $\hbar/2$

$$S = n \hbar/2 \quad n = 0, 1, 2, 3, 4, \dots$$

Thus spin can have the values.

$$S = \hbar/2, \hbar, 3\hbar/2, 2\hbar, 5\hbar/2, \dots$$

the particles with half integer spin ( $\hbar/2, 3\hbar/2, 5\hbar/2$ ) are called fermions (named after Enrico Fermi)

The particles with integer spin ( $0, \hbar, 2\hbar, 3\hbar, \dots$ ) are called bosons (named after Bose).

~~Examples of these are things~~ This spin angular momentum is added to orbital angular momentum to get the total. While the direction of a particles spin can be changed its magnitude cannot.

examples of these particles are.

Fermions.

electron (positron)  
proton  
neutron  
 $\text{He}^3$  nucleus  
 $\Lambda^0$  (lambda)  
 $\Sigma^*$  (sigma star)

$$S = \frac{h}{2}$$

$$S = \frac{3}{2}h$$

Bosons.

photon.  $S = h$   
 $W^\pm, Z$  (weak intermediate vector bosons)  $S = h$   
 $\pi^0 \pi^+ \pi^-$   $S = 0$  pions  
 $\rho^+ \rho^- \rho^0$   $S = h$  rho mesons  
Deuteron  $S = h$   
alpha particle =  $\text{He}^4$   $S = 0$

it turns out these particles have very different properties based on this classification. This is due to what is called the Pauli exclusion principle, which states:

Only one fermion can occupy a defined state  $\alpha$   
Or Fermion wave functions must be antisymmetric to interchange of any two ~~fermions~~ particles.

These two are equivalent

This requirement does not apply to Bosons hence as many particles as you like can be put into a given energy state.

Boson wave functions for multiparticle states must be symmetric under particle interchange fermion multiparticle wave functions must be antisymmetric.

The antisymmetry <sup>for fermions</sup> clearly implies you can't put 2 particles in the same state (interchange would leave you with the original state). The converse (symmetry for bosons) does not require 2 particles in the same state, it just allows it.



Boltzman probability factor / fermi-Dirac, Bose-Einstein distributions.

a), fermi Dirac.

Boltzman tells us that the probability a state  $l$  is occupied is proportional to.

$$P(\epsilon_l) \propto e^{-\epsilon_l/kT}$$

where  $\epsilon_l$  is the energy of the state  $l$ .

For fermions there are only two cases for a given state it is occupied or unoccupied as it must be one of these two the sum of the two probabilities must be 1. The energy of an unoccupied state is 0. therefore the boltzman factor is.

$$e^{-0/kT} = 1.$$

the sum of the probabilities

$$1 = \text{Const} [1 + e^{-\epsilon_l/kT}]$$

$$\text{Const} = \frac{1}{1 + e^{-\epsilon_l/kT}}$$

the form is the book sort of.

now to be slightly more exact there is in fact an extra term in all of this discussion called the chemical potential and this has to do with the energy required to add or subtract a particle the first law is changed to

$$dQ = dU - \mu dN + PdV.$$

where  $\mu$  is the chemical potential and  $dN$  is the change in the number.



the Gibbs factor includes this extra energy in the exponential.

$$P_{\text{Gibbs}} = \exp((N\mu - E_l)/kT) = e^{N\mu/kT} \cdot e^{-E_l(N)/kT}$$

where  $E_l(N)$  is the energy of state  $l$  when it has  $N$  particles.

the sum of these probabilities for fermions

$$1 = \text{const} \left[ 1 + e^{\mu/kT} e^{-E_l(1)/kT} \right] \quad \text{with no other terms } N = 0, 1.$$

$$\text{const} = \frac{1}{(1 + e^{(\mu - E_l)/kT})}$$

which is the factor from eq. 4.4.

let's consider the number density as a function of momentum  $p_f$  (momentum of the fermion)

In the problems we will be considering the energy of a state does not depend on the direction of the spin. As angular momentum is quantized the value of the  $z$  component (for example) can have values between  $+S\hbar$  to  $-S\hbar$  in steps of  $\hbar$

hence there will be  $2S+1$  ~~same~~ different states with the same energy but distinguishable by the values of the spin. (you can only measure one component of the spin at a time i.e.  $[S_x, S_y] = i\hbar S_z$ .)

hence the occupation density between  $\bar{p}$  and  $\bar{p} + d\bar{p}$  will be.

$$dn_f(p) = g(p) \frac{dp_x dp_y dp_z}{(h)^3}$$

with  $f(p)$  being just proportional to.

$$f(p) \propto \frac{2s+1}{1 + \exp[(\mu - \epsilon)/kT]}$$

with a normalization of.

$$\int dn(p) = \frac{N}{V} \text{ the total number density.}$$

The factor  $\frac{dp_x dp_y dp_z}{h^3}$

comes from the following argument.

The density to which you can pack in states is of the form

$$\Delta x \Delta p_x \approx h$$

hence

$$\Delta x \Delta y \Delta z \Delta p_x \Delta p_y \Delta p_z = h^3$$

$$= V \cdot \Delta p_x \Delta p_y \Delta p_z = h^3.$$

~~the total number of states~~  
the total number of states up to some upper limit of volume and momentum will be.

$$N = \frac{1}{(h)^3} \int_0^{\text{Vol}} dV \int dp_x dp_y dp_z.$$

$$\frac{N}{V} = \frac{1}{(h)^3} \int dp_x dp_y dp_z$$

but this must include the occupation ~~den~~ probability,  $f(p)$ .

hence.

$$\frac{N}{V} = \int \frac{2s+1}{e^{(\mu - \epsilon)/kT} + 1} \frac{d^3 p}{h^3} = \int \frac{2s+1}{e^{(\mu - \epsilon)/kT} + 1} \frac{4\pi p^2 dp}{(h)^3} = \int dn(p).$$

Going back to.

$$\text{Const} = \frac{1}{1 + e^{(\mu - \epsilon)/kT}}$$

$$\begin{aligned}\langle n(\epsilon_0) \rangle &= \frac{\sum n \cdot \text{Prob}(n)}{\sum \text{Prob}(n)} \\ &= \frac{0 + 1 \cdot e^{(\mu - \epsilon)/kT}}{1 + e^{(\mu - \epsilon)/kT}} \\ &= \frac{e^{(\mu - \epsilon)/kT}}{1 + e^{(\mu - \epsilon)/kT}}\end{aligned}$$

$$\langle n(\epsilon_0) \rangle = \frac{1}{e^{(\epsilon - \mu)/kT} + 1}$$

$$\begin{aligned}dn_f(p) &= \frac{2S+1}{e^{(\epsilon - \mu)/kT} + 1} \frac{dp^3}{(h)^3} \\ &= \frac{2S+1}{e^{(\epsilon - \mu)/kT} + 1} \frac{4\pi p^2 dp}{(h^3)}\end{aligned}$$



## Bosons / Bose Einstein Statistics.

with bosons you can put as many particles into a state as you like hence the sum of the probabilities ~~becomes~~ can be derived as follows for  $n$  particles.

$$\epsilon_l(n) = n \epsilon_l(1) \equiv n \epsilon_l$$

hence.

$$\begin{aligned} \sum_{n=0}^{\infty} P_{\text{Gibbs}}(n) &= \sum_{n=0}^{\infty} e^{n(\mu - \epsilon_l)/kT} \\ &= \sum_{n=0}^{\infty} \left( e^{(\mu - \epsilon_l)/kT} \right)^n \end{aligned}$$

define  $x \equiv e^{(\mu - \epsilon_l)/kT}$

the sum becomes

$$\begin{aligned} &= \sum_{n=0}^{\infty} x^n \\ &= \frac{1}{1-x} \end{aligned}$$

taylor expansion.

$$f(x) = f(x_0) + f'(x_0) \cdot x + \frac{1}{2} f''(x_0) x^2 + \dots + \frac{1}{n!} f^{(n)}(x_0) x^n$$

$$\begin{aligned} f(x) = \frac{1}{1-x} &= 1 + \frac{1 \cdot x}{(1)^2} + \frac{1}{2} \cdot \frac{2}{(1)^3} x^2 + \frac{1}{3!} \frac{2 \cdot 3}{(1)^4} x^3 \dots \dots \\ &= \sum_{n=0}^{\infty} x^n. \end{aligned}$$

$$\langle n(\epsilon) \rangle = \frac{\sum n x^n}{\sum x^n} = \frac{\sum n \text{ Prob}(n)}{\sum \text{ Prob}(n)}$$

$$= \frac{\sum_{n=0}^{\infty} x \frac{d}{dx} (x^n)}{\sum x^n}$$

$$= \frac{x \sum \frac{d}{dx}(x^n)}{\sum x^n}$$

$$= \frac{x \frac{d}{dx} \sum x^n}{\sum x^n}$$

$$\frac{x \frac{d}{dx} \frac{1}{(1-x)}}{(1-x)^{-1}} = \frac{x \frac{d}{dx} (1-x)^{-1}}{(1-x)^{-1}}$$

$$= \frac{x (1-x)^{-2}}{(1-x)^{-1}}$$

$$= \frac{x}{1-x}$$

$$= \frac{e^{(\mu - \epsilon_i)/kT}}{1 - e^{(\mu - \epsilon_i)/kT}}$$

$$\boxed{\langle n(\epsilon_i) \rangle = \frac{1}{e^{(\epsilon_i - \mu)/kT} - 1}}$$

and by the same argument shown before.

$$dn_b(p) = \frac{2s+1}{e^{(\epsilon(p) - \mu)/kT} - 1} \frac{4\pi p^2 dp}{h^3}$$

Enumeration of orbitals.

for a 1 dimensional box we saw

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2m L^2}.$$

for a cube  $V = L^3$

in 3 dimensions the problem can be factorized into 3 problems in 1 dimension and the wave function is the product of the 3 wave functions,

$$\Psi(x, y, z, t) = \phi_x(x) \phi_y(y) \cdot \phi_z(z) \psi(t)$$

$$E = (E_x + E_y + E_z)$$

$$\psi(t) = e^{-i(E/\hbar)t}$$

and 3 equations of the form, (one for  $x, y$  and  $z$ )

~~$$\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \phi_x(x) + V(x) \phi_x(x) = E \phi_x(x) - E_y - E_z = E_x$$~~

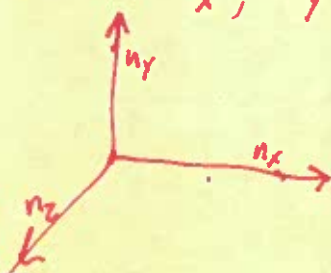
$$\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \phi_x(x) + V(x) \phi_x(x) = (E - E_y - E_z) \phi_x(x) = E_x \phi_x(x)$$

hence.

$$E = \frac{\hbar^2 \pi^2}{2m L^2} (n_x^2 + n_y^2 + n_z^2)$$

the number of states ~~between E and E+dE~~ below energy  $E$  can be approximated by an integral in the limit that  $(n_x^2 + n_y^2 + n_z^2)$  is large and becomes an almost continuous distribution

a state is defined by the unique value of  $(n_x, n_y, n_z)$  so consider a cartesian set of coordinates with the axes labeled  $n_x, n_y, n_z$ .





define  $R^2 \equiv n_x^2 + n_y^2 + n_z^2 \equiv n^2$

the volume of a ball of radius  $R$  will be very close to 8 times the number of points in the octant  $n_x > 0$   $n_y > 0$   $n_z > 0$  and

$$R^2 < E \cdot \frac{2mL^2}{\pi^2 \hbar^2}$$

$$\Rightarrow \text{number of states} = \frac{1}{8} \cdot \frac{4}{3} \pi R^3$$

$$= \frac{1}{6} \pi \cdot (n_x^2 + n_y^2 + n_z^2)^{3/2}$$

~~the~~ Including the effect of spin this would become

$$N = \frac{(2S+1)}{6} \pi (n_x^2 + n_y^2 + n_z^2)^{3/2} \equiv \frac{2S+1}{6} \pi n^3$$

the number between  $E$  and  $E + dE$

will be

$$\frac{dN}{dE} = \frac{(2S+1)}{6} \pi 3n^2 \frac{dn}{dE}$$

$$= \frac{(2S+1)}{2} \pi n^2 \frac{dn}{dE}$$

## Planck radiation Law.

for photons. which can have two states from spin (even though they are spin 1)

$$\text{the factor } \frac{dN}{dE} = \frac{2}{2} \pi n^2 \frac{dn}{dE}$$

$$E = \hbar \omega = h \nu$$

$$\omega = \nu \cdot 2\pi \quad (\text{angular frequency radians/sec})$$

light is ~~represented~~ <sup>manifested</sup> by a massless particle the photon hence the schrodinger equation <sup>(non relativistic)</sup> is not the right thing to use but one should use maxwells wave equation. (relativistic)

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Psi(x,y,z,t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \Psi(x,y,z,t)$$

if the radiation is constrained to a metal box of length L. the wave function must be zero at the boundaries one concludes that

$$\Psi(x,y,z,t) = (\text{Const}) \cdot e^{-i\omega t} \sin\left(\frac{n_x \pi x}{L}\right) \sin\left(\frac{n_y \pi y}{L}\right) \sin\left(\frac{n_z \pi z}{L}\right)$$

and the wave equation becomes.

$$\frac{\omega^2}{c^2} = \frac{\pi^2}{L^2} (n_x^2 + n_y^2 + n_z^2)$$

or

$$\hbar \omega = E = \hbar \frac{c\pi}{L} \sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{\hbar c \pi}{L} n$$

$$n = \frac{EL}{\hbar c \pi}$$

$$\frac{dn}{dE} = \frac{L}{\hbar c \pi}$$

or

$$\frac{dn}{d\omega} = \frac{L}{c \pi}$$

number / unit frequency

In calculating the intensity of light you usually discuss the energy emitted per unit frequency that is what you mean by the brightness.

$$\frac{dE}{d\omega} \equiv \mathcal{I}(\omega) = \langle n(\omega) \rangle \hbar \omega \cdot \frac{dN}{d\omega}$$

$$= \langle n(\omega) \rangle \hbar \omega \cdot \pi n^2 \frac{dn}{d\omega}$$

$$n^2 = \left( \frac{EL}{\hbar c \pi} \right)^2 = \frac{\omega^2 L^2}{c^2 \pi^2}$$

$$\Rightarrow \mathcal{I}(\omega) = \langle n(\omega) \rangle \hbar \omega \pi \frac{\omega^2 L^3}{c^3 \pi^3}$$

$$\boxed{\mathcal{I}(\omega) = \frac{1}{e^{\hbar \omega / kT} - 1} \cdot \frac{\hbar \omega^3}{c^3 \pi^2} \cdot \text{Volume.}}$$

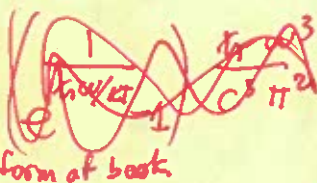
or in freq.  $\nu = \omega/2\pi$ .

$$\frac{dE}{d\nu} = 2\pi \frac{dE}{d\omega} = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{\hbar \nu / kT} - 1}$$

In fact we need the density of photons

$$\frac{dE}{d\nu} = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{\hbar \nu / kT} - 1}$$

$$\frac{dE}{d\nu d\lambda} = \frac{1}{4\pi} \frac{d\nu}{d\lambda} \frac{dE}{d\nu} = \frac{c}{4\pi \lambda^2} \frac{8\pi h \nu^3}{c^3} \left( \frac{1}{\lambda} \right)$$



$$\frac{dE}{dV d\omega} = \frac{\hbar \omega^3}{c^3 \pi^2} \frac{1}{e^{\hbar \omega / kT} - 1}$$

The total energy inside of the box is the integral over all frequencies, and the volume.

$$E = \int \frac{dE}{dV d\omega} dV d\omega = (Vol) \frac{\hbar}{c^3 \pi^2} \int_0^\infty \frac{\omega^3}{e^{\hbar \omega / kT} - 1} d\omega.$$

$$x = \frac{\hbar \omega}{kT}$$

$$dx = \frac{\hbar}{kT} d\omega$$

$$E = \frac{V \hbar}{c^3 \pi^2} \left( \frac{kT}{\hbar} \right)^4 \int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{V \hbar}{c^3 \pi^2} \left( \frac{kT}{\hbar} \right)^4 \cdot \frac{\pi^4}{15}$$



$$E = \left( \frac{V k^4 \pi^2}{c^3 \cdot 15 h^3} \right) T^4 \quad = \text{total energy in the box}$$

Now if we put a hole in the side of the box how much energy would come out and at what rate?

lets make the hole have unit area so we will be calculating the rate of energy emission/unit area this will be proportional to the energy density

$$\text{Flux} = \frac{E}{V} \cdot (\text{times a geometrical factor}) \cdot (\text{times the speed of light})$$

↑  
rate the energy move

$$\text{Flux} = \frac{Ec}{V} \cdot (?)$$

the Energy density is light travelling in random directions.

$$\Rightarrow \frac{dF}{d\Omega} = \frac{dF}{d\phi d\cos\theta} = \frac{F}{4\pi} \quad \text{rate of Flux/unit solid angle.}$$

the rate directed outward (along the z axis  $\perp$  to hole).

$$\begin{aligned} & \frac{F}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta d\cos\theta \\ &= \frac{F}{4\pi} \cdot 2\pi \cdot \frac{1}{2} = \frac{F}{4} \end{aligned}$$

$$\begin{aligned} \Rightarrow \text{Flux} &= \frac{Ec}{V} \cdot \frac{1}{4} = \left( \frac{k^4 \pi^2}{c^2 \cdot 60 h^3} \right) T^4 = \left( \frac{8\pi^5 k^4}{c^2 60 h^3} \right) T^4. \quad \rightarrow \text{see appendix A} \\ &= \sigma T^4 \quad \text{stephen-boltzmann law} \end{aligned}$$

Wien's law.

What is the wavelength at which light is most intensely emitted

$$\frac{dE}{d\lambda} = \frac{c}{\lambda^2} \frac{dE}{d\nu} = \frac{c}{\lambda^2} \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

$$= \frac{8\pi hc}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda kT}} - 1}$$

at peak  $\frac{d^2E}{d\lambda^2} = 0 = -\frac{5 \cdot 8\pi hc}{\lambda^6} \frac{1}{(e^{\frac{hc}{\lambda kT}} - 1)} + \frac{8\pi hc}{\lambda^5} \frac{e^{\frac{hc}{\lambda kT}}}{(e^{\frac{hc}{\lambda kT}} - 1)^2} \cdot \frac{hc}{kT} \frac{1}{\lambda^2}$

$$= \frac{40\pi hc}{\lambda^6} \frac{1}{(e^{\frac{hc}{\lambda kT}} - 1)} = \frac{8\pi hc}{\lambda^7} \cdot \frac{hc}{kT} \frac{e^{\frac{hc}{\lambda kT}}}{(e^{\frac{hc}{\lambda kT}} - 1)^2}$$

$$1 = \frac{1}{5} \cdot \left(\frac{hc}{\lambda kT}\right) \cdot \frac{e^{\frac{hc}{\lambda kT}}}{(e^{\frac{hc}{\lambda kT}} - 1)}$$

$$x = \frac{hc}{\lambda kT}$$

$$5 = x \frac{e^x}{e^x - 1}$$

$$5 = \frac{x}{1 - e^{-x}}$$

$$5(1 - e^{-x}) = x$$

$$5 = x + 5e^{-x}$$

transcendental equation (I think)

$$x \quad x + 5e^{-x}$$

$$1 \quad 2.84$$

$$2 \quad 2.677$$

$$0.1 \quad 4.624$$

$$0.001 \quad 4.996$$

$$\Rightarrow \text{first term is } 5 \quad 5.033$$

$$\boxed{4.97 \quad 5.0047}$$

$$4.97 = \frac{hc}{\lambda_{\max} kT}$$

$$\lambda_{\max} = \frac{hc}{4.97 k T}$$

$$h = 6.63 \times 10^{-27} \text{ ergs}$$

$$k = 1.38 \times 10^{-16} \text{ erg/K}$$

$$c = 3 \times 10^{10} \text{ cm/sec}$$

$$\frac{6.63 \times 10^{-27} \cdot 3 \times 10^{10}}{4.97 \cdot 1.38 \times 10^{-16}} \left(\frac{\text{cm}}{\text{deg}}\right) \frac{1}{T}$$

$$\boxed{\lambda_{\max} = \frac{0.29}{T}}$$

# Sackur - Tetrode equation

entropy of a mono atomic ideal gas.

The total number of particles  $N$  in a box of gas is just

$$N = \text{const} \sum_E \langle n(E) \rangle.$$

The constant normalizing the whole thing (the probabilities),

$$\langle n(E) \rangle = \frac{2s+1}{e^{(E-\mu)/kT} \pm 1}$$

In the classical regime <sup>(room temp)</sup> we can ignore the  $\pm 1$  and just get the ~~Boltzmann~~ <sup>Gibbs</sup> factor

$$\langle n(E) \rangle = (2s+1) \cdot e^{(\mu-E)/KT}$$

now the sum over energy states becomes an integral of the type.

$$N = \int \frac{1}{2} \pi n^2 dn \cdot \text{Prob}(E)$$

using the denumeration of states.

$$E = \frac{\hbar^2}{2M} \frac{\pi^2 n^2}{L^2} \Rightarrow n^2 = \frac{2ML^2 E}{\hbar^2 \pi^2}$$

$$N = \frac{1}{2} \pi (2s+1) \int dn n^2 \exp\left(\frac{\mu}{KT}\right) \cdot \exp\left[-\frac{\hbar^2 \pi^2 n^2}{2ML^2 KT}\right]$$

for  $s=0$

$$N = \frac{\pi}{2} \left(e^{\mu/KT}\right) \cdot \int dn n^2 \exp\left[-\frac{\hbar^2 \pi^2 n^2}{2ML^2 KT}\right]$$

$$x^2 = \frac{\hbar^2 \pi^2 n^2}{2ML^2 KT} \quad dx = \frac{\hbar \pi}{L \sqrt{2MKT}} dn$$

$$N = \frac{\pi}{2} e^{\mu/KT} \cdot \left(\frac{2ML^2 KT}{\pi^2 \hbar^2}\right)^{3/2} \int dx x^2 e^{-x^2}$$



the integral is equal to  $\frac{\sqrt{\pi}}{4}$

$$\Rightarrow N = \frac{e^{\mu/KT} \cdot L^3}{(2\pi\hbar^2/MKT)^{3/2}} = e^{\mu/KT} \frac{V}{V_Q}$$

the quantum volume  $V_Q = \left(\frac{2\pi\hbar^2}{MKT}\right)^{3/2}$

$$\text{the density } \rho \approx \frac{N}{V} = \frac{1}{V_Q} e^{\mu/KT}$$

$$\Rightarrow \rho V_Q = e^{\mu/KT}$$

$$\ln(\rho V_Q) = \mu/KT$$

$$\boxed{KT \ln(\rho V_Q) = \mu}$$

$$dQ = dU + PdV - \mu dN$$

first law

(entropy)

$$dS = \frac{dQ}{T} = \frac{dU}{T} + \frac{P}{T} dV - \frac{\mu}{T} dN$$

$$S = - \int_0^N \frac{\mu}{T} dN \quad \text{holding all else fixed.}$$

$$\frac{\mu}{T} = K \ln \frac{N}{V} + \frac{3}{2} K \ln \left( \frac{2\pi\hbar^2}{MKT} \right)$$

$$\begin{aligned} S &= + \int \frac{\mu}{T} dN = - \int K \ln \left( N \left( \frac{V_Q}{V} \right) \right) dN = -K \int (\ln V_Q + \ln N) dN \\ &= -K N \ln \left( \frac{V_Q}{V} \right) - K N \ln N + K N \end{aligned}$$

$\int \ln(N) dN = N \ln(N) - N$

We now use our result about the ideal gas connecting the temperature to the kinetic energy

$$\langle KE \rangle = \frac{3}{2} KT$$

The total internal energy  $U$  is just

$$U = N \langle KE \rangle = \frac{3}{2} NKT \Rightarrow KT = \frac{2U}{3N}$$

$$\Rightarrow \frac{U}{T} = k \ln \frac{N}{V} + \frac{3}{2} k \ln \left( \frac{2\pi\hbar^2}{m} \frac{3N}{2U} \right)$$

$$= k \ln \frac{N}{V} + \frac{3}{2} k \ln \left( \frac{3\pi\hbar^2 N}{m U} \right) = \frac{3}{2} k \ln \left[ \frac{\frac{3\pi\hbar^2}{m U V^{2/3}}}{\cancel{N^{1/3}}} \right] + \frac{5}{2} k \ln N$$

$$S = \int ds = - \int \frac{U}{T} dN \quad \text{holding all else fixed}$$

(this is the reason for the substitution of  $KT$  for  $\left(\frac{2}{3} \frac{U}{N}\right)$ )

$$S = - \int \left( \frac{3}{2} k \ln \left( \frac{3\pi\hbar^2}{m U V^{2/3}} \right) + \frac{5}{2} k \ln(N) \right) dN.$$

$$= - \left[ \frac{3}{2} k \ln \left( \frac{3\pi\hbar^2}{m U V^{2/3}} \right) N + \frac{5}{2} k N \ln N - \frac{5}{2} k N \right]$$

$$= - \left[ k N \ln \left( \frac{3\pi\hbar^2}{m U V^{2/3}} \right)^{3/2} + \cancel{N} N \ln(N)^{5/2} - \frac{5}{2} k N \right]$$

$$= - \left[ k N \ln \left[ \frac{3\pi\hbar^2}{m} \cdot \frac{2}{3 NKT} \cdot \frac{1}{V^{2/3}} \right]^{3/2} + k N \ln(N)^{5/2} - \frac{5}{2} k N \right]$$

$$= - \left[ k N \ln \left( \frac{2\pi\hbar^2}{mKT} \right)^{3/2} + k N \ln \left( \frac{1}{V} \right) - \frac{3}{2} k N \ln(N) + \frac{5}{2} k N \ln(N) - \frac{5}{2} k N \right]$$

$$= k N \ln \left( \frac{mKT}{2\pi\hbar^2} \right)^{3/2} + k N \ln V - k N \ln(N) + \frac{5}{2} k N$$

$$= k N \ln \left[ \left( \frac{mKT}{2\pi\hbar^2} \right)^{3/2} \left( \frac{V}{N} \right) \right] + \frac{5}{2} N.$$

## Tests.

The entropy of a gas at a given temperature Volume and density can be measured it is the sum of 5 parts

- ① warming the solid to the melting point
- ② melting the solid.
- ③ warming the liquid to the boiling point
- ④ boiling the liquid to gas
- ⑤ warming the gas

all 5 parts require heat adding to  
The integral

$$S = \int \frac{dq}{T}$$

Gas	Temp.	Experimentally determined Integral	Calculated from Sackur-Tetrode
Ne	27.2	96.4	96.45
Ar	87.29	129.75	129.24
Kr	119.93	144.56	145.06

$$\text{Pressure} \quad \frac{P}{T} = \left( \frac{\partial S}{\partial V} \right)_{N,u} = \frac{d}{dV} \left( kN \ln V + kN \ln \left[ \left( \frac{MKT}{2\pi k^2} \right)^{3/2} \frac{1}{N} \right] + \frac{5}{2} N \right)$$

$$= \frac{kN}{V}$$

$$PV = NKT \quad \text{Ideal gas law.}$$