The metric tensor (General Relativity for real)

I will use Weinbergs notation and not Tolmans which differ slightly I'll show the difference now and explain it later.

Weinberg. $$g_{\mu\nu} \equiv \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}$$

Tolman never singles out $\eta_{\alpha\beta}$ which is the metric tensor for flat space and just points out that. $g_{\mu\nu}$ is a tensor $\Rightarrow$ ($g_{\mu\nu}$ is a covariant tensor of rank 2)

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$$

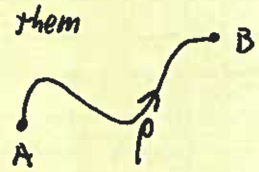
$$g'_{\mu\nu} = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta g_{\alpha\beta}$$

and the simplest form of $g_{\mu\nu}$ is $\eta_{\mu\nu}$

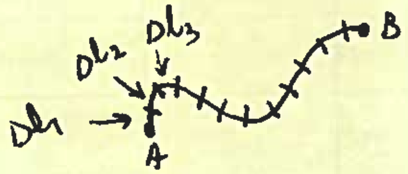
① Straight lines.

You <u>think</u> you know what a straight line is. Most likely you're wrong.

Consider two points A and B and some path ρ going between them



The straight line is the path which minimizes the length



$$L = \Sigma \Delta l_i = \int_A^B dl$$

in 2 dimensions. $$L = \int_A^B dl = \int_A^B (dx^2+dy^2)^{1/2}$$

let S be an arbitrary paramererization of the position along the curve ρ such that for example

$$S(0) = A$$
$$S(1) = B$$

the $L = \int_A^B dl = \int_0^1 \frac{dl}{ds}\, ds = \int \left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 \right)^{1/2} ds$

to find the shortest length path we vary the positions $x \to x + \delta x$ $\quad y \to y + \delta y$ until the variation on the length L due to these is zero always holding $\delta x = \delta y = 0$ at A and B

$$0 = \delta L = \int \delta \left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 \right)^{1/2} ds.$$

$$= \frac{1}{2} \int \left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 \right)^{-1/2} \cdot 2\left(\frac{d\delta x}{ds}\frac{dx}{ds} + \frac{d\delta y}{ds}\frac{dy}{ds} \right) ds$$

the first term is just $\frac{ds}{dl}$

$$= \frac{1}{2} \int \frac{ds}{dl} \cdot 2 \cdot \left(\frac{d\delta x}{ds}\frac{dx}{ds} + \frac{d\delta y}{ds}\frac{dy}{ds} \right) \frac{ds}{dl} \cdot dl \quad \left(\text{ie multiply by } \frac{dl}{dl} = 1 \right)$$

$$= \int \left(\frac{d\delta x}{dl}\frac{dx}{dl} + \frac{d\delta y}{dl}\frac{dy}{dl} \right) dl$$

integrate by parts and note $\delta x = \delta y = 0$ at A and b.

$$= -\int \left(\frac{d^2 x}{dl^2}\delta x + \frac{d^2 y}{dl^2} \cdot \delta y \right) dl.$$

since the δx and δy are arbitrary the only way this can be zero is if.

$$\frac{d^2 x}{dl^2} = \frac{d^2 y}{dl^2} = 0$$ ie no second derivatives along the path

$$\Rightarrow \left.\begin{aligned} \frac{dx}{dl} &= \text{const} \\ \frac{dy}{dl} &= \text{const}' \end{aligned}\right\}$$ <u>this</u> is a straight line!

Straight line in 4 space.

similarly a straight line in 4 dimensional space is a curve defined by the 4 second derivatives with the proper time being zero

$$\frac{d^2 x^\mu}{d\tau^2} = 0$$

where $d\tau^2 = -\eta_{\alpha\beta}\, dx^\alpha dx^\beta$

The derivation of this is identical to the one for 3 space with.

$$dx^2 + dy^2 \rightarrow -\eta_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$T_{AB} = \int_A^B d\tau = \int_A^B \frac{d\tau}{d\rho}\, d\rho = \int_A^B \left(-\eta_{\alpha\beta} \frac{dx^\alpha}{d\rho}\frac{dx^\beta}{d\rho}\right)^{1/2} d\rho.$$

vary the paths $x^\alpha \rightarrow x^\alpha + \delta x^\alpha$

$$0 = \delta T_{AB} = \frac{1}{2}\int_A^B \left(-\eta_{\alpha\beta}\frac{dx^\alpha}{d\rho}\frac{dx^\beta}{d\rho}\right)^{-1/2} \left(-2\eta_{\alpha\beta}\frac{d\delta x^\alpha}{d\rho}\frac{dx^\beta}{d\rho}\right) d\rho.$$

$$= \frac{1}{2}\int_A^B \frac{d\rho}{d\tau}\left(-2\eta_{\alpha\beta}\frac{d\delta x^\alpha}{d\rho}\frac{dx^\beta}{d\rho}\right) d\rho \cdot \frac{d\tau}{d\tau}.$$

$$= -\eta_{\alpha\beta}\int_A^B \left(\frac{d\delta x^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right) d\tau.$$

integrate by parts $\delta x^\alpha = 0$ at A, B.

$$0 = \delta T_{AB} = +\eta_{\alpha\beta}\int_A^B \frac{d^2 x^\beta}{d\tau^2}\,\delta x^\alpha\, d\tau$$

$$\Rightarrow \boxed{\frac{d^2 x^\beta}{d\tau^2} = 0}$$

Connection with gravitation, principle of equivalence.

The principle of equivalence states that we can define a coordinate system in the vicinity of a given point such that in an infintesmal region around that point space looks flat. ie in free fall the local coordinates of the falling observer look like euclidean (actually Minkowski) space

with
$$d\tau^2 = -\eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta$$

where ξ^α are the local flat coordinates.

the "true" cartesian coordinates of the "stationary" observer in the gravitational field are x^μ..

The ξ^α are functions of the "true" coordinates x^μ
$$\xi^\alpha = \xi(x^\mu) \qquad \mu \text{ goes from } 0 \text{ to } 3$$

The equations of motion for the freely falling system are those of a straight line ie.
$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \qquad (4 \text{ equations}),$$

which can be re written in terms of the x^μ as,
$$\frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) = 0$$

ie $$\frac{d\xi^\alpha}{d\tau} = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau} \qquad (\text{implied summation over } \mu)$$

$$\Rightarrow \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial}{\partial x^\mu}\frac{d\xi^\alpha}{d\tau}\frac{dx^\mu}{d\tau} = 0$$

$$\Rightarrow \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} = 0$$

multiplying the 4 equations in α by

$$\frac{\partial x^\lambda}{\partial \xi^\alpha}$$

and summing over α. This becomes.

$$\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2 x^\mu}{d\tau^2} + \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} = 0$$

but $\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial \xi^\alpha}{\partial x^\mu} = \delta_{\lambda\mu}$ ie coordinates are orthogonal

$$\Rightarrow \frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

where $\Gamma^\lambda_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}$

is called the <u>affine connection</u> or the <u>Christoffel symbol</u> sometimes denoted $\left\{ {\lambda \atop \mu\nu} \right\}$ or $\{\mu\nu,\lambda\}$ (the last from Tolman.)

Similarly

$$d\tau^2 = -\eta_{\alpha\beta}\frac{\partial \xi^\alpha}{\partial x^\mu}dx^\mu\frac{\partial \xi^\beta}{\partial x^\nu}dx^\nu.$$

$$= -\eta_{\alpha\beta}\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}dx^\mu dx^\nu$$

$$= -g_{\mu\nu}\,dx^\mu dx^\nu$$

where

$$g_{\mu\nu} \equiv \eta_{\alpha\beta}\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}$$

which has an inverse

$$g^{\nu K} \equiv \eta^{\alpha\beta}\frac{\partial x^\nu}{\partial \xi^\alpha}\frac{\partial x^K}{\partial \xi^\beta}$$

$$g^{\nu K}g_{\mu\nu} = \eta^{\alpha\beta}\frac{\partial x^\nu}{\partial \xi^\alpha}\frac{\partial x^K}{\partial \xi^\beta}\eta_{\gamma\delta}\frac{\partial \xi^\gamma}{\partial x^\mu}\frac{\partial \xi^\delta}{\partial x^\nu} = \eta^{\alpha\beta}\eta_{\gamma\delta}\frac{\partial x^\nu}{\partial \xi^\alpha}\frac{\partial x^K}{\partial \xi^\beta}\frac{\partial \xi^\gamma}{\partial x^\mu}\frac{\partial \xi^\delta}{\partial x^\nu}$$

contin. re ordering

$$= \eta^{\alpha\beta}\eta_{\gamma\delta}\underbrace{\frac{\partial x^\nu}{\partial \xi^\alpha}\frac{\partial \xi^\delta}{\partial x^\nu}}_{\delta^\delta_\alpha}\frac{\partial x^\kappa}{\partial \xi^\beta}\frac{\partial \xi^\gamma}{\partial x^\mu}$$

now $\eta^{\alpha\beta}\eta_{\gamma\alpha} = \delta^\beta_\gamma$

$$\Rightarrow g^{\nu\kappa} g_{\mu\nu} = \delta^\beta_\gamma \frac{\partial x^\kappa}{\partial \xi^\beta}\frac{\partial \xi^\gamma}{\partial x^\mu}$$

$$= \frac{\partial x^\kappa}{\partial \xi^\beta}\frac{\partial \xi^\beta}{\partial x^\mu} = \delta^\kappa_\mu$$

ie $g^{\nu\kappa}$ is the inverse of $g_{\mu\nu}$

now. consider the equation

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

in fact $\Gamma^\lambda_{\mu\nu}$ will be in the end the gravitational field which we will hopefully get to show??

The relation between $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ must be found.

so.

$$g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}$$

differentiate wrt x^λ

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial^2 \xi^\beta}{\partial x^\lambda \partial x^\nu}\eta_{\alpha\beta}$$

now.

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}$$

multiply by $\frac{\partial \xi^\beta}{\partial x^\lambda}$ and sum over λ

$$\frac{\partial \xi^\beta}{\partial x^\lambda}\Gamma^\lambda_{\mu\nu} = \frac{\partial \xi^\beta}{\partial x^\lambda}\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} = \delta_{\beta\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} = \frac{\partial^2 \xi^\beta}{\partial x^\mu \partial x^\nu}$$

$$\Rightarrow \frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial^2 \xi^\beta}{\partial x^\lambda \partial x^\nu}\eta_{\alpha\beta}$$

$$= \frac{\partial \xi^\alpha}{\partial x^\rho}\Gamma^\rho_{\lambda\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\rho}\Gamma^\rho_{\lambda\nu}\eta_{\alpha\beta}$$

but $\frac{\partial \xi^\alpha}{\partial x^\rho}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} = g_{\rho\nu}$

and $\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\rho}\eta_{\alpha\beta} = g_{\mu\rho}$

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = g_{\rho\nu}\Gamma^\rho_{\lambda\mu} + g_{\mu\rho}\Gamma^\rho_{\lambda\nu} \qquad \text{as } g \text{ is sym.}$$

$$= g_{\rho\nu}\Gamma^\rho_{\lambda\mu} + g_{\rho\mu}\Gamma^\rho_{\lambda\nu}$$

summing,

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} = g_{\rho\nu}\Gamma^\rho_{\lambda\mu} + g_{\rho\mu}\Gamma^\rho_{\lambda\nu} + g_{\rho\lambda}\Gamma^\rho_{\mu\nu} + g_{\rho\nu}\Gamma^\rho_{\mu\lambda} - g_{\rho\mu}\Gamma^\rho_{\nu\lambda} - g_{\rho\lambda}\Gamma^\rho_{\nu\mu}$$

as $g_{\mu\nu}$ and $\Gamma^\rho_{\mu\nu}$ are symmetric under interchange of μ and ν, the second term cancels the 5th and the 3rd cancels the 6th.

$$\Rightarrow \frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} = 2\, g_{\rho\nu}\Gamma^\rho_{\lambda\mu}$$

multiplying by $\frac{1}{2}g^{\nu\sigma}$ (the inverse of $2 \cdot g_{\rho\nu}$)

$$\frac{1}{2}g^{\nu\sigma}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\right) = g^{\nu\sigma}g_{\rho\nu}\Gamma^\rho_{\lambda\mu}$$

$$= \delta^\sigma_\rho \Gamma^\rho_{\lambda\mu} -$$

$$\boxed{\frac{1}{2}g^{\nu\sigma}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\right) = \Gamma^\sigma_{\lambda\mu}}$$

The Newtonian limit

In the case of slowly moving objects and weak stationary fields all the derivatives

$$\frac{dx^i}{d\tau} \quad i \text{ goes from one to } 3$$

can be neglected compared to

$$\frac{dt}{d\tau}$$

to see this include the value c

$$d\tau \to c\,dt.$$

$$\frac{dt}{d\tau} \to 1$$

$$\frac{dx}{d\tau} \to \frac{1}{c}\frac{dx}{dt} = \frac{v}{c} \ll 1.$$

so.

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

becomes.

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{00}\left(\frac{dt}{d\tau}\right)^2 = 0$$

$$\Gamma^\alpha_{00} = \frac{1}{2}g^{\nu\alpha}\left(\frac{\partial g_{0\nu}}{\partial x^0} + \frac{\partial g_{0\nu}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\nu}\right)$$

as the field is stationary all derivatives with respect to $x^0 = t$ vanish.

$$\Rightarrow \Gamma^\alpha_{00} = -\frac{1}{2}g^{\nu\alpha}\frac{\partial g_{00}}{\partial x^\nu}$$

if the field is weak $g^{\nu\alpha}$ is approximately $\eta^{\nu\alpha}$

ie

$$g^{\nu\alpha} = \eta^{\nu\alpha} + h^{\nu\alpha}$$

with $|h^{\nu\alpha}| \ll 1$

$\Rightarrow$ the equations of motion become to first order

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{00}\left(\frac{dt}{d\tau}\right)^2 = 0$$

$$\Gamma^0_{00} = 0 \quad = -\frac{1}{2} g^{\gamma 0} \frac{\partial g_{00}}{\partial x^0} = 0 \quad \text{field stationary.}$$

$$\Rightarrow \quad \frac{d^2x^0}{d\tau^2} = \frac{d^2t}{d\tau^2} = 0$$

~~and~~ $\cancel{\frac{d^2x^i}{d\tau^2} \div \frac{1}{2} g^{ri}}$

$$\Gamma^\alpha_{00} = -\frac{1}{2} g^{\gamma\alpha} \frac{\partial g_{00}}{\partial x^\gamma}$$

$$= -\frac{1}{2} \eta^{\alpha\gamma} \frac{\partial h_{00}}{\partial x^\gamma}$$

$$\Rightarrow \quad \frac{d^2x^\alpha}{d\tau^2} + \left(-\frac{1}{2}\eta^{\alpha\gamma}\frac{\partial h_{00}}{\partial x^\gamma}\right)\left(\frac{dt}{d\tau}\right)^2 = 0$$

$\alpha = 0$ reduces to.

$$\frac{d^2t}{d\tau^2} = 0$$

$\alpha = 1,2,3$

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\frac{\partial h_{00}}{\partial x^i}\left(\frac{dt}{d\tau}\right)^2$$

as $\frac{dt}{d\tau} = \text{const.}$

$$\left(\frac{d\tau}{dt}\right)^2 \cdot \frac{d^2x^i}{d\tau^2} = \frac{d^2x^i}{dt^2} = \frac{1}{2}\frac{\partial h_{00}}{\partial x^i}$$

$$\boxed{\frac{d^2\vec{x}}{dt^2} = \frac{1}{2}\vec{\nabla} h_{00}}$$

which corresponds to

$$\frac{d^2x}{dt^2} = -\nabla\phi$$

where ϕ = gravitational potential $= -\frac{GM}{R}.$

$\Rightarrow h_{00} = -2\phi + \text{constant}.$

at large distances $g^{\mu\nu} \to \eta^{\mu\nu}.$

$$\Rightarrow \boxed{g_{00} = -(1+2\phi)}$$

So we can clearly now see that the metric tensor is associated with the gravitational potential and the affine connection or Christoffel symbol is the gravitational field being a "gradient" of the metric tensor.

Transformation of $\Gamma^{\lambda}_{\mu\nu}$

ie something <u>isn't</u> a tensor.

recall

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} = \frac{1}{2} g^{\lambda\sigma}\left[\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right]$$

in the coordinate system x'^{τ}

$$\Gamma'^{\rho}_{\kappa\tau} = \frac{\partial x'^{\rho}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x'^{\kappa} \partial x'^{\tau}}$$

by chain rule of partial derivatives rewrite this in x^{μ} coordinate system

$$\Gamma'^{\rho}_{\kappa\tau} = \frac{\partial x'^{\rho}}{\partial x^{\lambda}} \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \cdot \frac{\partial}{\partial x'^{\kappa}}\left(\frac{\partial x^{\mu}}{\partial x'^{\tau}} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}}\right)$$

$$= \frac{\partial x'^{\rho}}{\partial x^{\lambda}} \frac{\partial x^{\lambda}}{\partial \xi^{\alpha}}\left[\frac{\partial x^{\mu}}{\partial x'^{\tau}} \cdot \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x'^{\kappa}} + \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial^2 x^{\mu}}{\partial x'^{\tau} \partial x'^{\kappa}}\right].$$

Collecting things.

$$= \frac{\partial x'^{\rho}}{\partial x^{\lambda}} \frac{\partial x^{\mu}}{\partial x'^{\tau}} \frac{\partial x^{\nu}}{\partial x'^{\kappa}} \underbrace{\frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}} + \frac{\partial x'^{\rho}}{\partial x^{\lambda}} \underbrace{\frac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}}}_{\delta_{\lambda\mu}} \frac{\partial^2 x^{\mu}}{\partial x'^{\tau} \partial x'^{\kappa}}$$

$$= \frac{\partial x'^{\rho}}{\partial x^{\lambda}} \frac{\partial x^{\mu}}{\partial x'^{\tau}} \frac{\partial x^{\nu}}{\partial x'^{\kappa}} \Gamma^{\lambda}_{\mu\nu} + \frac{\partial x'^{\rho}}{\partial x^{\lambda}} \delta_{\lambda\mu} \frac{\partial^2 x^{\mu}}{\partial x'^{\tau} \partial x'^{\kappa}}$$

$$\Gamma'^{\rho}_{\kappa\tau} = \frac{\partial x'^{\rho}}{\partial x^{\lambda}} \frac{\partial x^{\mu}}{\partial x'^{\tau}} \frac{\partial x^{\nu}}{\partial x'^{\kappa}} \Gamma^{\lambda}_{\mu\nu} + \frac{\partial x'^{\rho}}{\partial x^{\lambda}} \frac{\partial^2 x^{\lambda}}{\partial x'^{\tau} \partial x'^{\kappa}}$$

the first term is what you would get if $\Gamma^{\lambda}_{\mu\nu}$ were a tensor the second term is a inhomogeneous one "messing" this up. note: the superscript λ in the second term is arbitrary as it is summed over in other words the λ in the second term has no relation to the λ in the first term (which is also summed over).

another way.

$$g'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\gamma}}\frac{\partial x'^{\nu}}{\partial x^{\delta}}\, g^{\gamma\delta} \qquad g'_{\mu\nu} = \frac{\partial x^{\gamma}}{\partial x'^{\mu}}\frac{\partial x^{\delta}}{\partial x'^{\nu}}\, g_{\gamma\delta}$$

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left[\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right].$$

$$= \frac{1}{2}\frac{\partial x^{\lambda}}{\partial x'^{\alpha}}\frac{\partial x^{\sigma}}{\partial x'^{\delta}} g'^{\alpha\delta}\left[\frac{\partial}{\partial x^{\nu}}\left[\frac{\partial x'^{\beta}}{\partial x^{\mu}}\frac{\partial x'^{\gamma}}{\partial x^{\sigma}} g'_{\beta\gamma}\right] + \frac{\partial}{\partial x^{\mu}}\left[\frac{\partial x'^{\beta}}{\partial x^{\nu}}\frac{\partial x'^{\gamma}}{\partial x^{\sigma}} g'_{\beta\gamma}\right] - \frac{\partial}{\partial x^{\sigma}}\left[\frac{\partial x'^{\beta}}{\partial x^{\mu}}\frac{\partial x'^{\gamma}}{\partial x^{\nu}} g'_{\beta\gamma}\right]\right]$$

$$= \frac{1}{2}\frac{\partial x^{\lambda}}{\partial x'^{\alpha}}\frac{\partial x^{\sigma}}{\partial x'^{\delta}} g'^{\alpha\delta}\Bigg[\left(\frac{\partial x'^{\beta}}{\partial x^{\mu}}\frac{\partial x'^{\gamma}}{\partial x^{\sigma}}\frac{\partial g'_{\beta\gamma}}{\partial x^{\nu}} + g'_{\beta\gamma}\frac{\partial x'^{\beta}}{\partial x^{\mu}}\frac{\partial^2 x'^{\gamma}}{\partial x^{\nu}\partial x^{\sigma}} + \cancel{g'_{\beta\gamma}\frac{\partial^2 x'^{\beta}}{\partial x^{\mu}\partial x^{\nu}}\frac{\partial x'^{\gamma}}{\partial x^{\sigma}}}\right).$$

$$+\left(\frac{\partial x'^{\beta}}{\partial x^{\nu}}\frac{\partial x'^{\gamma}}{\partial x^{\sigma}}\frac{\partial g'_{\beta\gamma}}{\partial x^{\mu}} + g'_{\beta\gamma}\frac{\partial x'^{\beta}}{\partial x^{\nu}}\frac{\partial^2 x'^{\gamma}}{\partial x^{\mu}\partial x^{\sigma}} + g'_{\beta\gamma}\frac{\partial x'^{\gamma}}{\partial x^{\sigma}}\frac{\partial^2 x'^{\beta}}{\partial x^{\nu}\partial x^{\mu}}\right)$$

$$-\left(\frac{\partial x'^{\beta}}{\partial x^{\mu}}\frac{\partial x'^{\gamma}}{\partial x^{\nu}}\frac{\partial g'_{\beta\gamma}}{\partial x^{\sigma}} + g'_{\beta\gamma}\frac{\partial x'^{\beta}}{\partial x^{\mu}}\frac{\partial^2 x'^{\gamma}}{\partial x^{\sigma}\partial x^{\nu}} + g'_{\beta\gamma}\frac{\partial x'^{\delta}}{\partial x^{\nu}}\frac{\partial^2 x'^{\beta}}{\partial x^{\sigma}\partial x^{\mu}}\right)\Bigg]$$

circled terms cancel.

$$\cancel{\frac{1}{2}\frac{\partial x^{\lambda}}{\partial x'^{\alpha}}\frac{\partial x^{\sigma}}{\partial x'^{\delta}} g'^{\alpha\delta}}$$

$$= \frac{1}{2}\frac{\partial x^{\lambda}}{\partial x'^{\alpha}} g'^{\alpha\delta}\left[\frac{\partial x^{\sigma}}{\partial x'^{\delta}}\frac{\partial x'^{\beta}}{\partial x^{\mu}}\frac{\partial x'^{\gamma}}{\partial x^{\sigma}}\frac{\partial g'_{\beta\gamma}}{\partial x^{\nu}} + \frac{\partial x^{\sigma}}{\partial x'^{\delta}}\frac{\partial x'^{\beta}}{\partial x^{\nu}}\frac{\partial x'^{\gamma}}{\partial x^{\sigma}}\frac{\partial g'_{\beta\gamma}}{\partial x^{\mu}} - \frac{\partial x^{\sigma}}{\partial x'^{\delta}}\frac{\partial x'^{\beta}}{\partial x^{\mu}}\frac{\partial x'^{\gamma}}{\partial x^{\nu}}\frac{\partial g'_{\beta\gamma}}{\partial x^{\sigma}}\right.$$

$$\left. + g'_{\beta\gamma}\frac{\partial x^{\sigma}}{\partial x'^{\delta}}\frac{\partial x'^{\gamma}}{\partial x^{\sigma}}\frac{\partial^2 x'^{\beta}}{\partial x^{\mu}\partial x^{\nu}} + g'_{\beta\gamma}\frac{\partial x^{\sigma}}{\partial x'^{\delta}}\frac{\partial x'^{\gamma}}{\partial x^{\sigma}}\frac{\partial^2 x'^{\beta}}{\partial x^{\nu}\partial x^{\mu}}\right]$$

$$= \frac{1}{2}\frac{\partial x^{\lambda}}{\partial x'^{\alpha}} g'^{\alpha\delta}\left[\frac{\partial x'^{\beta}}{\partial x^{\mu}}\frac{\partial g'_{\beta\delta}}{\partial x^{\nu}} + \frac{\partial x'^{\beta}}{\partial x^{\nu}}\frac{\partial g'_{\beta\delta}}{\partial x^{\mu}} - \frac{\partial x'^{\beta}}{\partial x^{\mu}}\frac{\partial x'^{\gamma}}{\partial x^{\nu}}\frac{\partial x^{\sigma}}{\partial x'^{\delta}}\frac{\partial g'_{\beta\gamma}}{\partial x^{\sigma}}\right.$$

$$\left. + g'_{\beta\gamma}\frac{\partial x^{\sigma}}{\partial x'^{\delta}}\frac{\partial x'^{\gamma}}{\partial x^{\sigma}}\frac{\partial^2 x'^{\beta}}{\partial x^{\mu}\partial x^{\nu}} + g'_{\beta\gamma}\frac{\partial x^{\sigma}}{\partial x'^{\delta}}\frac{\partial x'^{\gamma}}{\partial x^{\sigma}}\frac{\partial^2 x'^{\beta}}{\partial x^{\nu}\partial x^{\mu}}\right]$$

$$= \frac{1}{2}\frac{\partial x^{\lambda}}{\partial x'^{\alpha}} g'^{\alpha\delta}\left\{\frac{\partial x'^{\beta}}{\partial x^{\mu}}\frac{\partial x'^{\gamma}}{\partial x^{\nu}}\frac{\partial g'_{\beta\delta}}{\partial x'^{\gamma}} + \frac{\partial x'^{\beta}}{\partial x^{\nu}}\frac{\partial x'^{\gamma}}{\partial x^{\mu}}\frac{\partial g'_{\beta\delta}}{\partial x'^{\gamma}} - \frac{\partial x'^{\beta}}{\partial x^{\mu}}\frac{\partial x'^{\gamma}}{\partial x^{\nu}}\frac{\partial g'_{\beta\gamma}}{\partial x'^{\delta}}\right.$$

$$\left. + g'_{\beta\gamma}\frac{\partial x^{\sigma}}{\partial x'^{\delta}}\frac{\partial x'^{\beta}}{\partial x^{\nu}}\frac{\partial^2 x'^{\gamma}}{\partial x^{\mu}\partial x^{\sigma}} + g'_{\beta\gamma}\frac{\partial x^{\sigma}}{\partial x'^{\delta}}\frac{\partial x'^{\gamma}}{\partial x^{\sigma}}\frac{\partial^2 x'^{\beta}}{\partial x^{\nu}\partial x^{\mu}}\right]$$

finally interchanging names of indecies in second term (β, γ)

$$= \frac{1}{2} \frac{\partial x^\lambda}{\partial x'^\alpha} \frac{\partial x'^\beta}{\partial x^\mu} \frac{\partial x'^\gamma}{\partial x^\nu} g'^{\alpha\delta} \left[\frac{\partial g'_{\beta\delta}}{\partial x'^\gamma} + \frac{\partial g'_{\gamma\delta}}{\partial x'^\beta} - \frac{\partial g'_{\beta\gamma}}{\partial x'^\delta} \right].$$

$$+ \frac{1}{2} \frac{\partial x^\lambda}{\partial x'^\alpha} g'^{\alpha\delta} \left(g'_{\beta\gamma} \frac{\partial x^\sigma}{\partial x'^\delta} \frac{\partial x'^\gamma}{\partial x^\sigma} \frac{\partial^2 x'^\beta}{\partial x^\mu \partial x^\nu} + g'_{\beta\gamma} \frac{\partial x^\sigma}{\partial x'^\delta} \frac{\partial x'^\gamma}{\partial x^\sigma} \frac{\partial^2 x'^\beta}{\partial x^\nu \partial x^\mu} \right)$$

$$= \frac{\partial x^\lambda}{\partial x'^\alpha} \frac{\partial x'^\beta}{\partial x^\mu} \frac{\partial x'^\gamma}{\partial x^\nu} \Gamma'^\alpha_{\beta\gamma}$$

$$+ \frac{1}{2} \frac{\partial x^\lambda}{\partial x'^\alpha} g'^{\alpha\delta} \left(\cancel{g'_{\beta\gamma} \frac{\partial x^\sigma}{\partial x'^\delta} \frac{\partial x'^\beta}{\partial x^\nu} \frac{\partial^2 x'^\delta}{\partial x^\mu \partial x^\sigma}} \; 2 g'_{\beta\delta} \frac{\partial^2 x'^\beta}{\partial x^\nu \partial x^\mu} \right)$$

$$= \frac{1}{2} \frac{\partial x^\lambda}{\partial x'^\alpha} 2 \delta^\alpha_\beta \frac{\partial^2 x'^\beta}{\partial x^\nu \partial x^\mu} + \frac{\partial x^\lambda}{\partial x'^\alpha} \frac{\partial x'^\beta}{\partial x^\mu} \frac{\partial x'^\gamma}{\partial x^\nu} \Gamma'^\alpha_{\beta\gamma}$$

$$\boxed{\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial x'^\alpha} \frac{\partial x'^\beta}{\partial x^\mu} \frac{\partial x'^\gamma}{\partial x^\nu} \Gamma'^\alpha_{\beta\gamma} + \frac{\partial x^\lambda}{\partial x^\alpha} \frac{\partial^2 x^\alpha}{\partial x^\nu \partial x^\mu}}$$

The Curvature Tensor

The curvature tensor turns out to be the only independent tensor that can be constructed from the metric tensor it's first and second derivatives. As such it is an excellent candidate for constructing a covariant theory which associates mass and energy to a force which we have seen can reduce to Newtonian gravity. An extremely elegant derivation of the form of the tensor is as follows (Weinberg pg 132)

consider the eq.

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial x'^{\tau}} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \Gamma'^{\tau}_{\rho\sigma} + \frac{\partial x^{\lambda}}{\partial x'^{\tau}} \frac{\partial^2 x'^{\tau}}{\partial x^{\mu}\partial x^{\nu}}$$

$$\Rightarrow \frac{\partial^2 x'^{\tau}}{\partial x^{\mu}\partial x^{\nu}} = \frac{\partial x'^{\tau}}{\partial x^{\lambda}} \Gamma^{\lambda}_{\mu\nu} - \underbrace{\cancel{\frac{\partial x'^{\tau}}{\partial x^{\lambda}} \frac{\partial x^{\lambda}}{\partial x'^{\tau}}}}_{1} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \Gamma'^{\tau}_{\rho\sigma}$$

~~differentiate wrt~~ $\cancel{x^{\kappa}}$

$$\cancel{\frac{\partial^3 x'^{\tau}}{\partial x^{\kappa}\partial x^{\mu}\partial x^{\nu}} =}$$

ie.

$$\boxed{\frac{\partial^2 x'^{\tau}}{\partial x^{\mu}\partial x^{\nu}} = \frac{\partial x'^{\tau}}{\partial x^{\lambda}} \Gamma^{\lambda}_{\mu\nu} - \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \Gamma'^{\tau}_{\rho\sigma}} \quad (1)$$

differentiate (1) wrt x^{κ}.

$$\frac{\partial^3 x'^{\tau}}{\partial x^{\kappa}\partial x^{\mu}\partial x^{\nu}} = \frac{\partial^2 x'^{\tau}}{\partial x^{\kappa}\partial x^{\lambda}} \Gamma^{\lambda}_{\mu\nu} - \frac{\partial^2 x'^{\rho}}{\partial x^{\kappa}\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \Gamma'^{\tau}_{\rho\sigma} - \frac{\partial^2 x'^{\sigma}}{\partial x^{\kappa}\partial x^{\nu}} \frac{\partial x'^{\rho}}{\partial x^{\mu}} \Gamma'^{\tau}_{\rho\sigma}$$

$$+ \frac{\partial x'^{\tau}}{\partial x^{\lambda}} \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} \qquad - \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \frac{\partial}{\partial x^{\kappa}} \left(\Gamma'^{\tau}_{\rho\sigma} \right)$$

the first three terms can be replaced with eq (1)

the last term requires the chain rule be applied.

to become

$$- \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \frac{\partial x'^{\eta}}{\partial x^{\kappa}} \frac{\partial \Gamma'^{\tau}_{\rho\sigma}}{\partial x'^{\eta}}$$

thus.

$$\frac{\partial^3 x'^\tau}{\partial x^\kappa \partial x^\mu \partial x^\nu} = \Gamma^\lambda_{\mu\nu}\left(\frac{\partial x'^\tau}{\partial x^\eta}\Gamma^\eta_{\kappa\lambda} - \frac{\partial x'^\rho}{\partial x^\kappa}\frac{\partial x'^\sigma}{\partial x^\lambda}\Gamma'^\tau_{\rho\sigma}\right)$$

$$-\frac{\partial x'^\sigma}{\partial x^\nu}\Gamma'^\tau_{\rho\sigma}\left(\frac{\partial x'^\rho}{\partial x^\eta}\Gamma^\eta_{\kappa\mu} - \frac{\partial x'^\eta}{\partial x^\kappa}\frac{\partial x'^\xi}{\partial x^\mu}\Gamma'^\rho_{\eta\xi}\right)$$

$$-\frac{\partial x'^\rho}{\partial x^\mu}\Gamma'^\tau_{\rho\sigma}\left(\frac{\partial x'^\sigma}{\partial x^\eta}\Gamma^\eta_{\kappa\nu} - \frac{\partial x'^\eta}{\partial x^\kappa}\frac{\partial x'^\xi}{\partial x^\nu}\Gamma'^\sigma_{\eta\xi}\right)$$

$$+\frac{\partial x'^\tau}{\partial x^\lambda}\frac{\partial\Gamma^\lambda_{\mu\nu}}{\partial x^\kappa} - \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\frac{\partial x'^\eta}{\partial x^\kappa}\frac{\partial\Gamma'^\tau_{\rho\sigma}}{\partial x'^\eta}$$

Collecting similar terms and interchanging repeated indecies.

term 1A $\eta \rightleftarrows \lambda$ term 2B $\sigma \to \lambda$, $\xi \to \sigma$ term 3B $\xi \to \rho$, $\rho \to \lambda$ terms 2A and 3A $\eta \to \lambda$ and ρ and σ are interchanged and use $\Gamma^\lambda_{\rho\sigma} = \Gamma^\lambda_{\sigma\rho}$ symmetry.

$$\frac{\partial^3 x'^\tau}{\partial x^\kappa\partial x^\mu\partial x^\nu} = \frac{\partial x'^\tau}{\partial x^\lambda}\left[\overbrace{\frac{\partial\Gamma^\lambda_{\mu\nu}}{\partial x^\kappa}}^{\text{term 4}} + \overbrace{\Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\kappa\eta}}^{\text{term 1A}}\right]$$

$$-\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\frac{\partial x'^\eta}{\partial x^\kappa}\left(\overbrace{\frac{\partial\Gamma'^\tau_{\rho\sigma}}{\partial x'^\eta}}^{\text{term 5}} - \overbrace{\Gamma'^\tau_{\lambda\sigma}\Gamma'^\lambda_{\eta\rho}}^{\text{term 3B}} - \overbrace{\Gamma'^\tau_{\rho\lambda}\Gamma'^\lambda_{\eta\sigma}}^{\text{term 2B}}\right)$$

$$-\Gamma'^\tau_{\rho\sigma}\frac{\partial x'^\sigma}{\partial x^\lambda}\left[\underbrace{\Gamma^\lambda_{\mu\nu}\frac{\partial x'^\rho}{\partial x^\kappa}}_{\text{term 1B}} + \underbrace{\Gamma^\lambda_{\kappa\nu}\frac{\partial x'^\rho}{\partial x^\mu}}_{\text{term 3A}} + \underbrace{\Gamma^\lambda_{\kappa\mu}\frac{\partial x'^\rho}{\partial x^\nu}}_{\text{term 2A}}\right].$$

Now the magic! interchange ν and κ (make a new eq). and subtract.

$$\frac{\partial^3 x'^\tau}{\partial x^\kappa\partial x^\mu\partial x^\nu} - \frac{\partial^3 x'^\tau}{\partial x^\nu\partial x^\mu\partial x^\kappa} = 0 = \frac{\partial x'^\tau}{\partial x^\lambda}\left[\frac{\partial\Gamma^\lambda_{\mu\nu}}{\partial x^\kappa} - \frac{\partial\Gamma^\lambda_{\mu\kappa}}{\partial x^\nu} + \Gamma^\lambda_{\kappa\eta}\Gamma^\eta_{\mu\nu} - \Gamma^\lambda_{\nu\eta}\Gamma^\eta_{\mu\kappa}\right]$$

$$-\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\frac{\partial x'^\eta}{\partial x^\kappa}\left[\frac{\partial\Gamma'^\tau_{\rho\sigma}}{\partial x'^\eta} - \frac{\partial\Gamma'^\tau_{\rho\eta}}{\partial x'^\sigma} - \Gamma'^\tau_{\lambda\sigma}\Gamma'^\lambda_{\eta\rho} + \Gamma'^\tau_{\lambda\eta}\Gamma'^\lambda_{\sigma\rho}\right]$$

the mixed terms in $\Gamma'\Gamma$ cancel due to the symmetry of $\Gamma_{\kappa\mu} = \Gamma_{\mu\kappa}$ and explicitly there is a cross cancelation in the $\Gamma'\Gamma'$ terms after renaming repeated indecies.

but therefore!

$$\left[\frac{\partial\Gamma^\lambda_{\mu\nu}}{\partial x^\kappa} - \frac{\partial\Gamma^\lambda_{\mu\kappa}}{\partial x^\nu} + \Gamma^\lambda_{\kappa\eta}\Gamma^\eta_{\mu\nu} - \Gamma^\lambda_{\nu\eta}\Gamma^\eta_{\mu\kappa}\right] = \frac{\partial x^\lambda}{\partial x'^\tau}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\frac{\partial x'^\eta}{\partial x^\kappa}\left[\frac{\partial\Gamma'^\tau_{\rho\sigma}}{\partial x'^\eta} - \frac{\partial\Gamma'^\tau_{\rho\eta}}{\partial x'^\sigma} + \Gamma'^\tau_{\lambda\eta}\Gamma'^\lambda_{\sigma\rho} - \Gamma'^\tau_{\lambda\sigma}\Gamma'^\lambda_{\eta\rho}\right]$$

$$\boxed{\text{ie } R^\lambda_{\mu\nu\kappa} \equiv \left[\frac{\partial\Gamma^\lambda_{\mu\nu}}{\partial x^\kappa} - \frac{\partial\Gamma^\lambda_{\mu\kappa}}{\partial x^\nu} + \Gamma^\lambda_{\kappa\eta}\Gamma^\eta_{\mu\nu} - \Gamma^\lambda_{\nu\eta}\Gamma^\eta_{\mu\kappa}\right] \text{ is a tensor!}}$$

ie. $R^{\lambda}_{\mu\nu\kappa} \equiv \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\lambda}_{\kappa\eta}\Gamma^{\eta}_{\mu\nu} - \Gamma^{\lambda}_{\nu\eta}\Gamma^{\eta}_{\mu\kappa}$

is a mixed tensor of rank 4.

in fact it is unique, But it can be used to construct all possible tensors,

Now we can make some useful things with this

In particular the Ricci tensor defined as.

$$R_{\mu\kappa} \equiv R^{\lambda}_{\mu\lambda\kappa}$$

and the scaler curvature.

$$R \equiv g^{\mu\kappa} R_{\mu\kappa}.$$

This process is known as contraction and is similar to a dot product,

$$R_{\mu\kappa} = \left[\frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\lambda}} + \Gamma^{\lambda}_{\kappa\eta}\Gamma^{\eta}_{\mu\lambda} - \Gamma^{\lambda}_{\lambda\eta}\Gamma^{\eta}_{\mu\kappa}\right]$$

The Energy Momentum tensor. (following weinberg).

Consider the electric curent density.

$$J^{\alpha}_{(\bar{x},\bar{t})} = \sum_i Q_i \frac{dx_i^{\alpha}(t)}{dt} \delta^3(x - x_i(t))$$

where $\frac{dx_i^0(t)}{dt} = 1$

we would like a similar quantity in assosiation with energy and momentum particularly, using a bit of premonition, as the energy density reduces to the mass density which is the source of newtonian gravity. We construct a similar quantity but our problem is a little more complicated because in the case of electric current densities we started with a scaler Charge in the case of energy momentum we start with a four vector hence we are led to a Tensor of rank 2.

$$T^{\mu\nu}_{(x,t)} \equiv \sum_i P_i^{\mu} \frac{dx_i^{\nu}}{dt} \delta^3(\bar{x}_4 - \bar{x}_i(t))$$

which can also be written as

$$T^{\mu\nu}(x,t) = \sum_i \frac{P_i^{\mu} P_{\nu}^{i}}{E_i} \delta^3(\bar{x} - \bar{x}_i(t))$$

this object is clearly a tensor by construction. in the case of a fluid or gas in the Co moving system the sum becomes an integral over a maxwellian distribution (the gas has no net velocity).

and

$$\text{(spatial)} \quad T^{ij} = p\,\delta_{ij}$$

$$T^{i0} = 0$$

$$T^{00} = \rho$$

with P and ρ being the pressure and mass density.

The Einstein Field equations.

General relativistic gravity will be an unusual beast if for no other reason simply due to the equality of energy and mass. As mass/energy is the source of the gravitational field and the field itself has energy the field is self coupling and therefore non linear.

Classically. $G = 1.67 \times 10^{-8}$ in cgs.

$$\Phi_g(r) = \int \frac{-G\rho_m(r')}{|r'-r|} d^3r' \quad \text{or} \quad \nabla^2(\Phi) = 4\pi G \rho_m$$

$$\Rightarrow PE = \int \Phi_g(r)\rho_m(r)\, d^3r = \frac{1}{4\pi G}\int \Phi_g \nabla^2(\Phi_g)\, d^3r$$

integrate by parts.

$$= \frac{-1}{4\pi G}\int (\nabla\phi_g)(\nabla\phi_g)\, d^3r$$

$$= \frac{-1}{4\pi G}\int F_g^2\, d^3r$$

ie classical field theory and $E=mc^2$ tell you you've got problems.

In the weak field limit we found that.

$$g_{00} \sim -(1 + 2\phi_g)$$

and $\nabla^2\phi = 4\pi G\rho_m$

a tensor can be constructed from the energy and momentum called the Energy momentum tensor.

$$T^{\mu\nu}_{E} = \sum_n \int d\tau\, p_n^\mu \frac{dx^\nu}{d\tau}\, \delta^4(x - x(\tau))$$

where the sum is over the individual particles this can be turned into a Tensor density through some magic. and clearly in the non relativistic low field limit the largest term will be.

$$T_{00} \hat{=} \rho_m$$

$$\Rightarrow \nabla^2 g_{00} \sim -8\pi G T_{00}$$

the general extension of this idea would lead to an equation of the form.

$$G_{\alpha\beta} = -8\pi G T_{\alpha\beta}$$

where $G_{\alpha\beta}$ is a linear combination of the metric tensor it's first and second derivatives (we are evaluating this in the falling frame where space is nearly flat and $g_{\alpha\beta} \simeq \eta_{\alpha\beta} + h_{\alpha\beta}$ $|h_{\alpha\beta}| \ll |\eta_{\alpha\beta}|$)

By the principle of equivalence Then the extension to the general coordinate system would be to simply go to the full tensor extension.

$$G_{\mu\nu} = -8\pi G T_{\mu\nu}.$$

we make the following assertions and assumptions.

1. $G_{\mu\nu}$ is a tensor. by definition.
2. $G_{\mu\nu}$ only consists of terms of 2 derivatives of the metric ie linear in the second derivative or quadratic in the first derivative.
3. $T_{\mu\nu}$ is symmetric $\Rightarrow$ $G_{\mu\nu}$ must be also
4. $T_{\mu\nu}$ is conserved ie $\frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0$

   (this must be extended to "covariant differentiation).

$$\Rightarrow \frac{\partial G^{\alpha\beta}}{\partial x^\beta} = 0.$$

(5). For a weak stationary field (const in time) the 00 component must reduce to.

$$G_{00} \cong \nabla^2 g_{00}$$

The second statement (an assumption) says we must start with the General Curvature tensor

$$R^{\lambda}{}_{\mu\nu\kappa}$$

The symmetry properties of $R_{\lambda\mu\nu\kappa} = g_{\lambda\eta} R^{\eta}{}_{\mu\nu\kappa}$ leave only <u>two</u> potential choices.

$$\cancel{R_{\mu\nu} = R_{\lambda\mu\nu}}$$

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$$

and $$R = g^{\nu\mu} R_{\mu\nu} = R^{\nu}{}_{\nu}$$

The other three requirements lead to only one possibility,

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = - 8\pi G T_{\mu\nu}}$$

this can be contracted to produce.

$$g^{\nu\mu} R_{\mu\nu} - \frac{1}{2} g^{\nu\mu} g_{\mu\nu} R = -8\pi G g^{\nu\mu} T_{\mu\nu}.$$

$$R - \frac{1}{2}(4) R = -8\pi G T^{\mu}{}_{\mu}$$

$$R = 8\pi G T^{\mu}{}_{\mu}$$

note. $T^{\mu}{}_{\mu}$ is the sum of the diagonal elements or the trace.

In empty space $T_{\mu\nu} = 0 \Rightarrow T^{\mu}{}_{\mu} = 0 \Rightarrow R = 0$

$$\Rightarrow \boxed{R_{\mu\nu} = 0}$$

this is a second order homogeneous differential eq. in the metric.

This equation has a solution found by Karl Schwarzschild in 1916 for being outside of a spherically symmetric mass M.

- $g_{\mu\nu}$ in such a case in spherical coordinates will be.

$$g_{00}(r) = \left[1 - \frac{2MG}{r}\right]$$
$$g_{rr}(r) = -\left[1 - \frac{2MG}{r}\right]^{-1}$$
$$g_{\theta\theta}(r) = -r^2$$
$$g_{\phi\phi}(r) = -r^2 \sin^2\theta$$

- all others are zero!

$$\Rightarrow \boxed{d\tau^2 = \left[1 - \frac{2MG}{r}\right] dt^2 - \left[1 - \frac{2MG}{r}\right]^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2.}$$

with a singularity occurring when.

$$\frac{2MG}{r} = 1 \quad \text{ie} \quad \boxed{r = \frac{2MG}{c^2}}$$

this solution is only correct when space is empty ie $T^\mu{}_\mu = R = 0 = T_{\mu\nu}$

$$\underline{R_{\mu\nu} = 0.}$$

$\Rightarrow$ inside of a low density <u>large</u> cloud it is irrelevant!

Cosmology.

In discussing the "standard Model" of cosmological evolution we will attempt to understand the properties of a maximally symmetric universe. If this "universe" should bear any resemblance to our own this would prove to be a valuable study beyond the mathematical understanding developed in the course of determining the evolution of the simplest system under the constraints of the laws of physics ie conservation of energy / momentum, General relativity and thermodynamics.

As it will turn out the real universe has as first approximation the properties of the maximally symmetric universe so we will start there.

It is observed, in the study of external galaxies, that the distribution of galaxies is expanding and with increasing rate ie the galaxies are receeding and the velocity of recession increases with distance This is formulated mathematically as Hubbles law.

$$z = H_0 \cdot \text{dist.}$$

where z is the red shift of light due to the doppler effect.

$$z = \frac{\lambda_{obs} - \lambda_{true}}{\lambda_{true}} = \frac{(1+v/c)^{1/2}}{(1-v/c)^{1/2}} - 1 \sim \frac{v}{c} \quad \text{for } z \ll 1$$

As we showed before such a law implies that the expansion of the universe appears the same to all observers, locally at rest, independent of their position. ~~So~~ For a distant observer at R_0 to us he has velocity. (for low redshifts)

$$\bar{V}_0 = c H_0 \cdot \bar{R}_0$$

In the distant coordinate system (Gallilean transformation)

$$\bar{R}' = \bar{R} - \bar{R}_0$$

and

$$\begin{aligned}\bar{V}' &= \bar{V} - V_0 \\ &= V - c H_0 R_0 \\ &= cH_0\bar{R} - cH_0\bar{R}_0 \\ &= cH_0(\bar{R} - \bar{R}_0) \\ &= cH_0 \bar{R}'\end{aligned}$$

or Hubbles law has ~~some~~ the same form to all observers. Further on a grand scale the universe is reasonably isotropic. In other words on distance scales large compared to clusters of galaxies their is no particularly singled out direction. These two arguements together imply that the Universe is Homogeneous and Isotropic

This is known as the Cosmological Principle →

It appears the same to every observer, in all directions, at every point, when evaluated at the same local time t_0 in a reference frame at rest with respect to the local mass density. These conditions in fact give a precise meaning to Machs' principle.

Under these symmetric assumptions the metric will take the following form for the comoving reference frame (The one at rest with respect to the local mass density)

$$d\tau^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\right]$$

where $R(t)$ is an unknown (for now) function of the local time t and K can have the values $+1$, 0, or -1. The three values corresponding to a "closed", "flat" or "open" universe. These terms being defined later. The basic form of this metric is quite plausible given the symmetry of the assumptions. Clearly something of the form "$d\sigma^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$" is demanded by spherical symmetry and for $K=0$ this is exactly what is gotten with the overall scale being determined by a function of time $R(t)$. This is known as the Robertson - Walker metric and is a requirement of what is called a "space of maximally symmetric subspaces" (see ch 13 Weinberg).

The metric tensor is

$$g_{tt} = 1$$

$$g_{rr} = \frac{R^2(t)}{1-Kr^2}$$

$$g_{\theta\theta} = R^2(t)\, r^2$$

$$g_{\phi\phi} = R^2(t)\, r^2 \sin^2\theta$$

recalling that

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma}\left(\frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right)$$

we can rather easily evaluate all of the Γ which we will do shortly. But first let's consider the trajectory of light in such a metric.

Cosmological Red Shift

Consider an electromagnetic wave emitted from a distant galaxy in our direction. For this discussion we will place ourselves at the origin of our coordinate system, the distant galaxy at r_1, θ_1, ϕ_1. The trajectory of light is defined by,

$$0 = d\tau^2 = dt^2 - R^2(t)\frac{dr^2}{1-kr^2}$$

if it travels from the distant galaxy to us ie $\theta = \theta_1$ $\phi = \phi_1$ are constants $d\theta = d\phi = 0$

$$\Rightarrow \quad 0 = \frac{dt^2}{R^2(t)} - \frac{dr^2}{1-kr^2}$$

or $$\frac{dt^2}{R^2(t)} = \frac{dr^2}{1-kr^2}$$

if the wave crest leaves at time t_1 and arrives at time t_0 and travels from r_1 to 0 (us) then.

$$① \quad \int_{t_1}^{t_0}\frac{dt}{R(t)} = -\int_0^{r_1}\frac{dr}{\sqrt{1-kr^2}} \equiv -f(r_1)$$

$$f(r_1) = \begin{cases} \sin^{-1}(r_1) & k=1 \\ r_1 & k=0 \\ \sinh^{-1}(r_1) & k=-1 \end{cases}$$

if the next wave crest leaves at $t_1 + \delta t_1$ and arrives at $t_0 + \delta t_0$ then

② $\displaystyle\int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{R(t)} = -\int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}} = -f(r_1)$

subtracting ② from ① ~~and as~~

$$\int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{R(t)} - \int_{t_1}^{t_0} \frac{dt}{R(t)} = 0$$

$$\cancel{\int_{t_1+\delta t_1}^{t_0} \frac{dt}{R(t)}} + \int_{t_0}^{t_0+\delta t_0} \frac{dt}{R(t)} - \int_{t_1}^{t_1+\delta t_1} \frac{dt}{R(t)} - \cancel{\int_{t_1+\delta t_1}^{t_0} \frac{dt}{R(t)}} = 0$$

assuming $R(t)$ is ~ constant over a single period (10^{-14} sec for visible light)

$$\frac{\delta t_0}{R(t_0)} = \frac{\delta t_1}{R(t_1)}$$

$$\frac{\delta t_0}{\delta t_1} = \frac{R(t_0)}{R(t_1)} = \frac{\nu_1}{\nu_0} = \frac{\lambda_0}{\lambda_1}$$

or $\displaystyle\frac{\nu_0}{\nu_1} = \frac{R(t_1)}{R(t_0)} = \frac{\lambda_1}{\lambda_0}$

note: If the laws of physics are invariant λ_1 = true wavelength

$$\Rightarrow \boxed{z = \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{\lambda_0}{\lambda_1} - 1 = \frac{R(t_0)}{R(t_1)} - 1}$$

If the universe is expanding then $R(t_0) > R(t_1)$ and the light is red shifted. If $R(t_0) < R(t_1)$ then the universe is contracting and the light is blue shifted.

The proper distance might best be defined as the distance integral evaluated at constant local time. This could be imagined as having a

String of local observers each reasonably near each other, simultaneously (same <u>local</u> t ie the universe appears the same to all of them at the time they each do their measurements) measuring the distance to their neighboring observer. The distance so defined would be.

$$d_{prop} = \int_0^{r_1} \sqrt{g_{rr}}\, dr \Big|_{t=t_1} = \int_0^{r_1} \frac{R(t_1)\, dr}{\sqrt{1-kr^2}}$$

$$= R(t_1) \int_0^{r_1} \frac{dr}{\sqrt{1-kr^2}}$$

$$= R(t_1) \cdot f(r_1)$$

This is very nice but what we really need is a physically determined distance related to observable the choice is the "distance" of the apparent luminosity

$$l = \frac{L}{4\pi d_L^2}$$

$$d_L = \left(\frac{L}{4\pi l}\right)^{1/2}$$

In fact for reasonably near ($d < 10^9$ lt. yrs) all of these are the same.

If we expand $R(t)$ in a taylor expansion.

$$R(t) = R(t_0)\left\{1 + \frac{\dot{R}(t_0)}{R(t_0)}(t-t_0) + \frac{1}{2}\frac{\ddot{R}(t_0)}{R(t_0)}(t-t_0)^2 + \cdots\right.$$

and define.

$$c\, H_0 = \frac{\dot{R}(t_0)}{R(t_0)}$$

and $$q_0 = -\frac{\ddot{R}(t_0)}{1} \cdot \frac{R(t_0)}{\dot{R}(t_0)}$$

then.

$$R(t) = R(t_0)\left[1 + cH_0(t-t_0) - \tfrac{1}{2} q_0 H_0^2 c^2 (t-t_0)^2 + \cdots\right.$$

or.

$$\frac{R(t)}{R(t_0)} = 1 + cH_0(t-t_0) - \tfrac{1}{2} q_0 H_0^2 c^2 (t-t_0)^2 + \cdots$$

$$\text{if} \quad H_0 c(t-t_0) - \tfrac{1}{2} q_0 H_0^2 c^2 (t-t_0)^2 \cdots \ll 1$$

then.

$$\frac{R(t_0)}{R(t)} = \frac{1}{1 + H_0 c(t-t_0) - \frac{1}{2} q_0 H_0^2 c^2 (t-t_0)^2 + \cdots}$$

$$= 1 - \left[H_0 c(t-t_0) - \tfrac{1}{2} q_0 H_0^2 c^2 (t-t_0)^2 + \cdots\right] + \left[H_0 c(t-t_0) - \tfrac{1}{2} q_0 H_0^2 c^2 (t-t_0)^2 \cdots\right]^2 - \left[\quad\right]^3 \cdots$$

collecting terms to $(t-t_0)^2$.

$$= 1 - H_0 c(t-t_0) + \tfrac{1}{2} q_0 H_0^2 c^2 (t-t_0)^2 + H_0^2 c^2 (t-t_0)^2 + \cdots$$

$$\frac{R(t_0)}{R(t)} - 1 = z = H_0 c(t_0-t) + \left(1 + \frac{q_0}{2}\right) H_0^2 c^2 (t-t_0)^2 + \cdots$$

as $d \sim c(t_0 - t)$

$$\boxed{z = H_0 d + \left(1 + \frac{q_0}{2}\right) H_0^2 d^2 + \cdots}$$

which is Hubble law with the deceleration parameter q_0 Included.

The expansion equation

as stated before

$g_{tt} = -1 \qquad g_{\theta\theta} = R^2(t)\, r^2$

$g_{rr} = \dfrac{R^2(t)}{1-kr^2} \qquad g_{\phi\phi} = R^2(t)\, r^2 \sin^2\theta$ all others zero

for convenience define $\tilde{g}$ 3×3

$\tilde{g}_{rr} = \dfrac{1}{1-kr^2} \qquad \tilde{g}_{\theta\theta} = r^2 \qquad \tilde{g}_{\phi\phi} = r^2\sin^2\theta$ all others zero

$g_{ij} = R^2(t)\, \tilde{g}_{ij}$

and

$g^{ij} g_{ki} = \delta^j_k$ and $\tilde{g}^{ij} \tilde{g}_{ki} = \delta^j_k \qquad i,j,k \quad 1 \to 3$

in general $g^{\mu\nu} g_{\gamma\mu} = \delta^\nu_\gamma$ and $g_{\mu\nu} = g_{\nu\mu}$

The affine connection $\Gamma^\alpha_{\mu\nu}$ is

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma}\left(\frac{\partial g_{\sigma\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right)$$

start evaluating

$$\Gamma^\alpha_{tt} = \frac{1}{2} g^{\alpha\sigma}\left(\frac{\partial g_{\sigma t}}{\partial t} + \frac{\partial g_{t\sigma}}{\partial t} - \frac{\partial g_{tt}}{\partial x^\sigma}\right) \qquad \begin{matrix} g_{tt} = -1 \\ g_{ti} = 0 \end{matrix}$$

$= 0 \qquad$ in gen $g_{t\alpha} = \text{const} \Rightarrow \dfrac{\partial g_{t\alpha}}{\partial x^\mu} = 0$

$i = 1 \to 3$
$$\Gamma^\alpha_{ti} = \frac{1}{2} g^{\alpha\sigma}\left(\frac{\partial g_{\sigma i}}{\partial t} + \frac{\partial g_{t\sigma}}{\partial x^i} - \frac{\partial g_{ti}}{\partial x^\sigma}\right)$$

$$= \frac{1}{2} g^{\alpha\sigma} \frac{\partial g_{\sigma i}}{\partial t}$$

$$= \frac{1}{2} \frac{1}{R^2(t)} \tilde{g}^{jk} \frac{\partial g_{ki}}{\partial t} \qquad \text{as } \frac{\partial g_{ti}}{\partial t} = 0 \Rightarrow \underline{\underline{\Gamma^t_{ti} = 0}}$$

$$= \frac{1}{2} \frac{1}{R^2(t)} \tilde{g}^{jk} \tilde{g}_{ki} \frac{\partial R^2(t)}{\partial t} \qquad \boxed{\dot{R}(t) \equiv \frac{\partial R(t)}{\partial t}}$$

$$= \frac{1}{2} \frac{1}{R^2(t)} \delta^j_i \, 2R(t)\, \dot{R}(t)$$

$$\boxed{= \frac{\dot{R}}{R(t)} \delta^j_i = \Gamma^j_{ti}} \qquad \boxed{\Gamma^t_{ti} = 0}$$

$$\Gamma^t_{ij} = \frac{1}{2} g^{t\sigma}\left(\frac{\partial g_{\sigma i}}{\partial x^j} + \frac{\partial g_{j\sigma}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\sigma}\right) \qquad g^{t\sigma} = -\delta_{t\sigma}$$

$$= -\frac{1}{2}\left(\frac{\partial g_{ti}}{\partial x^j} + \frac{\partial g_{jt}}{\partial x^i} - \frac{\partial g_{ij}}{\partial t}\right)$$

$$= \cdot\frac{1}{2}\,\tilde{g}_{ij}\cdot 2R(t)\,\dot{R}(t) \qquad \text{(only last term is non zero)}$$

$$\boxed{\Gamma^0_{ij} = R(t)\,\dot{R}(t)\,\tilde{g}_{ij}}$$

and finally,

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\sigma}\left(\frac{\partial g_{\sigma k}}{\partial x^j} + \frac{\partial g_{j\sigma}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^\sigma}\right)$$

if $\sigma = 0$, $x^\sigma = t$; g^{i0} or $g^{it} = 0$

$$= \frac{1}{2} g^{il}\left(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)$$

$i\,j\,k\,l \quad 1 \to 3$

$$= \frac{1}{2}\frac{1}{R^2(t)}\tilde{g}^{il}\left(R^2(t)\left(\frac{\partial \tilde{g}_{lk}}{\partial x^j} + \frac{\partial \tilde{g}_{jl}}{\partial x^k} - \frac{\partial \tilde{g}_{jk}}{\partial x^l}\right)\right)$$

$$= \frac{1}{2}\tilde{g}^{il}\left(\frac{\partial \tilde{g}_{lk}}{\partial x^j} + \frac{\partial \tilde{g}_{jl}}{\partial x^k} - \frac{\partial \tilde{g}_{jk}}{\partial x^l}\right) \equiv \tilde{\Gamma}^i_{jk}$$

only spatial dependence

~~same for flat space spherical coords?~~ no!

Curvature tensor.

$$\boxed{R^\lambda_{\mu\kappa\nu} = \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\nu} - \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\kappa} + \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\nu\eta} - \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\kappa\eta}}$$

Ricci tensor contracted curvature tensor

$$\boxed{R_{\mu\nu} \equiv R^\lambda_{\mu\lambda\nu} = \underline{\frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\nu} - \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\nu\eta} - \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\lambda\eta}}}$$

we now have all the Γs so we can evaluate $R_{\mu\nu}$ by being careful

$$R_{tt} = \frac{\partial \Gamma^\lambda_{t\lambda}}{\partial t} - \frac{\partial \Gamma^\lambda_{tt}}{\partial x^\lambda} + \Gamma^\eta_{t\lambda}\Gamma^\lambda_{t\eta} - \Gamma^\eta_{tt}\Gamma^\lambda_{\lambda\eta}$$

$\Gamma^\lambda_{tt} = 0$

only non vanishing is $\Gamma^j_{ti} = \frac{\dot{R}}{R}\delta^j_i$

$$= \frac{\partial \Gamma^j_{tj}}{\partial t} + \Gamma^i_{tj}\Gamma^j_{ti}$$

$$= 3\frac{\partial}{\partial t}\left(\frac{\dot{R}}{R}\right) + 3\left(\frac{\dot{R}}{R}\right)^2 \qquad \text{(factor 3 comes from implied summation)}$$

$$= 3\frac{\ddot{R}}{R} - 3\frac{\dot{R}}{R^2}\cdot\dot{R} + 3\left(\frac{\dot{R}}{R}\right)^2$$

$$\boxed{R_{tt} = \frac{3\ddot{R}(t)}{R(t)}}$$

$$\ddot{R}(t) = \ddot{R} = \frac{\partial^2 R(t)}{\partial t^2} = \frac{\partial \dot{R}(t)}{\partial t} \quad \text{etc.}$$

$$R_{ti} = \frac{\partial \Gamma^{\lambda}_{t\lambda}}{\partial x^i} - \frac{\partial \Gamma^{\lambda}_{ti}}{\partial x^{\lambda}} + \Gamma^{\eta}_{t\lambda}\Gamma^{\lambda}_{i\eta} - \Gamma^{\eta}_{ti}\Gamma^{\lambda}_{\lambda\eta}$$

$$= \frac{\partial \Gamma^{k}_{tk}}{\partial x^i} - \frac{\partial \Gamma^{k}_{tk}}{\partial x^{k}} + \Gamma^{k}_{tk}\Gamma^{k}_{ik} - \Gamma^{i}_{ti}\Gamma^{\lambda}_{\lambda i}$$

as $\Gamma^{\lambda}_{t\alpha} = \Gamma^{j}_{ti}\,\delta^{i}_{i}$ etc. $= \frac{\dot R(t)}{R(t)}\delta^i_j$ no spatial depend.

$$\quad \overset{\Vert}{0} \qquad \overset{\Vert}{0} \qquad +3\frac{\dot R}{R}\tilde\Gamma^{k}_{ik} - 3\frac{\dot R}{R}\left(\tilde\Gamma^{k}_{ki}\right).$$

$$\tilde\Gamma^{k}_{ik} = \tilde\Gamma^{k}_{ki}$$

$$= 0 \ !$$

and finally

$$R_{ij} = \frac{\partial \Gamma^{\lambda}_{i\lambda}}{\partial x^j} - \frac{\partial \Gamma^{\lambda}_{ij}}{\partial x^{\lambda}} + \Gamma^{\eta}_{i\lambda}\Gamma^{\lambda}_{j\eta} - \Gamma^{\eta}_{ij}\Gamma^{\lambda}_{\lambda\eta} \qquad \Gamma^{t}_{t\alpha} = 0 = \Gamma^{t}_{\alpha t}$$

$$= \frac{\partial \Gamma^{k}_{ik}}{\partial x^j} - \frac{\partial \Gamma^{k}_{ij}}{\partial x^{k}} - \frac{\partial \Gamma^{t}_{ij}}{\partial t} + \Gamma^{l}_{it}\Gamma^{t}_{jl} + \Gamma^{t}_{il}\Gamma^{l}_{jt} + \Gamma^{l}_{ik}\Gamma^{k}_{jl} - \Gamma^{t}_{ij}\Gamma^{l}_{lt} - \Gamma^{k}_{ij}\Gamma^{l}_{lk}$$

$$= \left(\frac{\partial \Gamma^{k}_{ik}}{\partial x^j} - \frac{\partial \Gamma^{k}_{ij}}{\partial x^{k}} + \Gamma^{l}_{ik}\Gamma^{k}_{jl} - \Gamma^{k}_{ij}\Gamma^{l}_{lk}\right) \quad \text{purely spatial}$$

$$-\frac{\partial}{\partial t}(R\dot R\tilde g_{ij}) + \frac{\dot R}{R}\delta^{l}_{i}\Gamma^{t}_{jl} + \Gamma^{t}_{il}\frac{\dot R}{R}\delta^{l}_{j} - \left(R(t)\dot R(t)\tilde g_{ij}\right)\left(3\frac{\dot R}{R}\right)$$

$$= \left(\frac{\partial \tilde\Gamma^{k}_{ik}}{\partial x^j} - \frac{\partial \tilde\Gamma^{k}_{ij}}{\partial x^{k}} + \tilde\Gamma^{l}_{ik}\tilde\Gamma^{k}_{jl} - \tilde\Gamma^{k}_{ij}\tilde\Gamma^{l}_{lk}\right)$$

$$-\tilde g_{ij}\left(R\ddot R + \dot R^2\right) + \frac{\dot R}{\cancel{R}}(\cancel{R}\dot R)\tilde g_{ij} + \cancel{R}\dot R\tilde g_{ij}\frac{\dot R}{\cancel{R}} - 3\tilde g_{ij}\dot R^2$$

$$= \tilde R_{ij} - \tilde g_{ij}\left(R\ddot R + \dot R^2\right) + 2\tilde g_{ij}\dot R^2 - 3\tilde g_{ij}\dot R^2$$

$$\boxed{R_{ij} = \tilde R_{ij} - \tilde g_{ij}\left(R\ddot R + 2\dot R^2\right)}$$

collecting

$$\boxed{\begin{aligned} R_{tt} &= \frac{3\ddot R}{R} \\ R_{ti} &= 0 \\ R_{ij} &= \tilde R_{ij} - \left(R\ddot R + 2\dot R^2\right)\tilde g_{ij} \end{aligned}}$$

in fact $\tilde R_{ij}$ must equal a constant $\cdot\ \tilde g_{ij}$ (see weinberg Ch 13)

$$\tilde R_{ij} = C\,\tilde g_{ij} = \underline{-2k\,\tilde g_{ij}}$$

but let's verify this. $\tilde\Gamma^i_{jk} = \tilde\Gamma^i_{kj} \Rightarrow \underline{\underline{18}}\ \tilde\Gamma^i_{jk}$

$$\tilde\Gamma^1_{11} = \frac12 \tilde g^{1\ell}\left(2\frac{\partial \tilde g_{1\ell}}{\partial r} - \frac{\partial \tilde g_{rr}}{\partial x^\ell}\right) = \frac12 \tilde g^{rr}\left(2\frac{\partial \tilde g_{rr}}{\partial r} - \frac{\partial \tilde g_{rr}}{\partial r}\right)$$

$$= \frac12 \tilde g^{rr}\frac{\partial \tilde g_{rr}}{\partial r} = \frac{1-kr^2}{2}\cdot\frac{1}{(1-kr^2)^2}\cdot 2kr$$

$$\boxed{= \frac{kr}{1-kr^2}} = \tilde\Gamma^1_{11}$$

$$\tilde\Gamma^1_{12} = \frac12 \tilde g^{1\ell}\left(\frac{\partial \tilde g_{1\ell}}{\partial\theta} + \frac{\partial \tilde g_{\ell\theta}}{\partial r} - \frac{\partial \tilde g_{r\theta}}{\partial x^\ell}\right)$$

$$= 0$$

$$\tilde\Gamma^1_{13} = \frac12 \tilde g^{1\ell}\left(\frac{\partial \tilde g_{1\ell}}{\partial\phi} + \frac{\partial \tilde g_{\ell\phi}}{\partial r} - \frac{\tilde g_{r\phi}}{\partial x^\ell}\right)$$

$$= 0$$

$$\tilde\Gamma^2_{11} = \frac12 \tilde g^{2\ell}\left(2\frac{\partial \tilde g_{1\ell}}{\partial r} - \frac{\partial \tilde g_{11}}{\partial x^\ell}\right) = \frac12 \tilde g^{22}\left(2\frac{\partial \tilde g_{12}}{\partial r} - \frac{\partial g_{11}}{\partial\theta}\right) = 0$$

$$\tilde\Gamma^2_{22} = \frac12 \tilde g^{2\ell}\left(2\frac{\partial \tilde g_{2\ell}}{\partial\theta} - \frac{\partial \tilde g_{22}}{\partial x^\ell}\right) = \frac12 \tilde g^{22}\left(\frac{\partial \tilde g_{22}}{\partial\theta}\right) = 0 \quad \frac12\frac{1}{r^2}\left(\frac{\partial r^2}{\partial\theta}\right)$$

$$\tilde\Gamma^2_{23} = \frac12 \tilde g^{2\ell}\left(\frac{\partial \tilde g_{2\ell}}{\partial\phi} + \frac{\partial \tilde g_{\phi\ell}}{\partial\theta} - \frac{\partial \tilde g_{\phi\theta}}{\partial x^\ell}\right) = 0 \qquad (\ell = 2)$$

$$\tilde\Gamma^1_{22} = \frac12 \tilde g^{1\ell}\left(2\frac{\tilde g_{\theta\ell}}{\partial\theta} - \frac{\partial \tilde g_{\theta\theta}}{\partial x^\ell}\right) = \frac12 \tilde g^{11}\left(-\frac{\partial \tilde g_{\theta\theta}}{\partial r}\right) = \cancel{\frac12\frac{1}{r^2}(-3r)}\ \cancel{= -\frac1r}\quad \frac12 (1-kr^2)(-2r) = \boxed{-r(1-kr^2)}$$

$$\tilde\Gamma^1_{23} = \frac12 \tilde g^{1\ell}\left(\frac{\partial g_{2\ell}}{\partial\phi} + \frac{\partial g_{\phi\ell}}{\partial\theta} - \frac{\partial g_{\theta\phi}}{\partial x^\ell}\right) = 0 \quad (\ell = 1)$$

$$\tilde\Gamma^1_{33} = \frac12 \tilde g^{1\ell}\left(2\frac{\partial g_{3\ell}}{\partial\phi} - \frac{\partial g_{\phi\phi}}{\partial x^\ell}\right) = \frac12 g^{11}\left(-\frac{\partial g_{\phi\phi}}{\partial r}\right) = \frac12(1-kr^2)\cdot(-2r\sin^2\theta)$$

$$= \boxed{-(1-kr^2)(r\sin^2\theta)}$$

redone

$$\tilde\Gamma^2_{11} = \frac12 g^{2\ell}\left(2\frac{\partial g_{1\ell}}{\partial\theta} - \frac{\partial g_{rr}}{\partial x^\ell}\right) = 0$$

$$\tilde\Gamma^2_{12} = \frac12 g^{2\ell}\left(\frac{\partial g_{r\ell}}{\partial\theta} + \frac{\partial g_{\theta\ell}}{\partial r} - \frac{\partial g_{r\theta}}{\partial x^\ell}\right) = \frac12 g^{22}\left(\frac{\partial g_{\theta\theta}}{\partial r}\right) = \frac12\frac{1}{r^2}\cdot 2r = \boxed{\frac1r}$$

$$\tilde\Gamma^2_{13}$$

~~So far~~ no zeros

$$\boxed{\begin{aligned}&\tilde{\Gamma}^1_{11} = \frac{kr}{1-kr^2}, \quad \hat{\Gamma}^1_{22} = -r(1-kr^2) \quad \tilde{\Gamma}^1_{33} = -r\sin^2\theta(1-kr^2)\\ &\tilde{\Gamma}^2_{12} = \frac{1}{r} \quad \Gamma^2_{33} = -\sin\theta\cos\theta \quad \Gamma^3_{13} = \frac{1}{r} \quad \Gamma^3_{23} = \cancel{\sin\theta}\,\frac{\cos\theta}{\sin\theta}\end{aligned}}$$

$$\hat{\Gamma}^2_{13} = \frac{1}{2}\tilde{g}^{2l}\left(\frac{\partial \tilde{g}_{rl}}{\partial\phi} + \frac{\partial \tilde{g}_{l\phi}}{\partial r} - \frac{\partial \tilde{g}_{r\phi}}{\partial x^l}\right) = 0 \quad (l=2)$$

$$\tilde{\Gamma}^2_{33} = \frac{1}{2}\tilde{g}^{2l}\left(2\frac{\partial \tilde{g}_{\phi l}}{\partial\phi} - \frac{\partial \tilde{g}_{\phi\phi}}{\partial x^l}\right) = \frac{1}{2}\frac{1}{r^2}\left(-\frac{\partial r^2\sin^2\theta}{\partial\theta}\right) = \boxed{-\sin\theta\cos\theta} \quad \Gamma^2_{33}$$

$$\hat{\Gamma}^3_{11} = \frac{1}{2}\tilde{g}^{3l}\left(2\frac{\partial \tilde{g}_{rl}}{\partial r} - \frac{\partial \tilde{g}_{rr}}{\partial x^l}\right) = 0 \quad l=3$$

$$\hat{\Gamma}^3_{12} = \frac{1}{2}\tilde{g}^{3l}\left(\frac{\partial \tilde{g}_{rl}}{\partial\theta} + \frac{\partial \tilde{g}_{l\theta}}{\partial r} - \frac{\partial \tilde{g}_{r\theta}}{\partial x^l}\right) = 0 \quad l=3$$

$$\tilde{\Gamma}^3_{13} = \frac{1}{2}\tilde{g}^{3l}\left(\frac{\partial \tilde{g}_{rl}}{\partial\phi} + \frac{\partial \tilde{g}_{l\phi}}{\partial r} - \frac{\partial \tilde{g}_{r\phi}}{\partial x^l}\right) = \frac{1}{2}\frac{1}{r^2\sin^2\theta}\left(\frac{\partial \tilde{g}_{\phi\phi}}{\partial r}\right) = \frac{1}{2}\frac{1}{r^2\sin^2\theta}\frac{\partial r^2\sin^2\theta}{\partial r} = \boxed{\frac{1}{r}}$$

$$\tilde{\Gamma}^3_{23} = \frac{1}{2}g^{3l}\left(\frac{\partial \hat{g}_{\theta l}}{\partial\phi} + \frac{\partial \tilde{g}_{l\phi}}{\partial\theta} - \frac{\partial \tilde{g}_{\theta\phi}}{\partial x^l}\right) = \frac{1}{2}\frac{1}{r^2\sin^2\theta}\frac{\partial \tilde{g}_{\phi\phi}}{\partial\theta} = \frac{1}{2r^2\sin^2\theta}\frac{\partial r^2\sin^2\theta}{\partial\theta} = \boxed{\cancel{\sin\theta}\,\frac{\cos\theta}{\sin\theta}}$$

$$\Gamma^3_{22} = \frac{1}{2}\hat{g}^{3l}\left(2\frac{\partial \hat{g}_{\theta l}}{\partial\theta} - \frac{\partial \tilde{g}_{\theta\theta}}{\partial x^l}\right) = 0 \quad (l=3)$$

$$\tilde{\Gamma}^3_{33} = \frac{1}{2}g^{3l}\left(2\frac{\partial \tilde{g}_{\phi l}}{\partial\phi} - \frac{\partial g_{\phi\phi}}{\partial x^l}\right) = \frac{1}{2}\tilde{g}^{33}\left(2\frac{\partial g_{\phi\phi}}{\partial\phi} - \frac{\partial g_{\phi\phi}}{\partial\phi}\right) = 0 \quad \frac{\partial g_{\phi\phi}}{\partial\phi} = 0$$

$$\tilde{R}_{ij} = \left(\frac{\partial \hat{\Gamma}^k_{ik}}{\partial x^j} - \frac{\partial \hat{\Gamma}^k_{ij}}{\partial x^k} + \tilde{\Gamma}^l_{ik}\tilde{\Gamma}^k_{jl} - \hat{\Gamma}^k_{ij}\tilde{\Gamma}^l_{kl}\right)$$

$$\tilde{R}_{11} = \left(\frac{\partial \tilde{\Gamma}^k_{ik}}{\partial r} - \frac{\partial \tilde{\Gamma}^k_{11}}{\partial x^k} + \bar{\Gamma}^l_{1k}\bar{\Gamma}^k_{1l} - \tilde{\Gamma}^k_{11}\tilde{\Gamma}^l_{kl}\right)$$

$$= \frac{\partial}{\partial r}\left(\frac{kr}{1-kr^2} + \frac{1}{r} + \frac{1}{r}\right) - \frac{\partial}{\partial r}\Gamma^1_{11} + \Gamma^1_{11}\Gamma^1_{11} + \Gamma^2_{12}\Gamma^2_{12} + \Gamma^3_{13}\Gamma^3_{13} - \Gamma^1_{11}\left(\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}\right)$$

$$= \frac{\partial}{\partial r}\left(\frac{2}{r}\right) + \cancel{(\Gamma^1_{11})^2} + (\Gamma^2_{12})^2 + (\Gamma^3_{13})^2 - \cancel{(\Gamma^1_{11})^2} - \Gamma^1_{11}\Gamma^2_{12} - \Gamma^1_{11}\Gamma^3_{13}$$

$$= -\frac{2}{r^2} + \frac{1}{r^2} + \frac{1}{r^2} - \left(\frac{kr}{1-kr^2}\cdot\frac{2}{r}\right)$$

$$\boxed{= \frac{-2k}{1-kr^2}.}$$

$$\tilde{R}_{12} = \frac{\partial \Gamma^k_{1k}}{\partial\theta} - \frac{\partial \Gamma^k_{12}}{\partial x^k} + \Gamma^l_{1k}\Gamma^k_{2l} - \Gamma^k_{12}\Gamma^l_{kl}$$

$$\underset{0}{\shortparallel} \quad - \underset{\underset{0}{\shortparallel}}{\frac{\partial \Gamma^2_{12}}{\partial\theta}} + \Gamma^3_{13}\Gamma^3_{23} - \Gamma^2_{12}\Gamma^3_{23} = \frac{1}{r}\Gamma^3_{23} - \frac{1}{r}\Gamma^3_{23} = \underline{0}$$

$$\tilde R_{13} = \frac{\partial \tilde\Gamma^k_{1k}}{\partial\phi} - \frac{\partial\tilde\Gamma^k_{13}}{\partial x^k} + \tilde\Gamma^l_{1k}\tilde\Gamma^k_{3l} - \tilde\Gamma^k_{13}\tilde\Gamma^l_{kl}$$

$$\overset{\shortparallel}{0} \quad - \frac{\partial\tilde\Gamma^3_{13}}{\partial\phi} = 0 \quad + \underbrace{\tilde\Gamma^3_{13}\tilde\Gamma^3_{33}}_{0} + \underbrace{\tilde\Gamma^3_{13}\Gamma_3}_{0}$$

no non zero terms on inspection. ✓

note R is not symmetric??

$$\tilde R_{21} = \frac{\partial\tilde\Gamma^k_{2k}}{\partial r} - \frac{\partial\tilde\Gamma^k_{21}}{\partial x^k} + \tilde\Gamma^l_{2k}\tilde\Gamma^k_{1l} - \tilde\Gamma^k_{21}\tilde\Gamma^l_{kl}$$

$$= \underbrace{\frac{\partial\tilde\Gamma^3_{23}}{\partial r}}_{0} - \underbrace{\frac{\partial\tilde\Gamma^2_{21}}{\partial\theta}}_{0} + \tilde\Gamma^3_{23}\tilde\Gamma^3_{13} - \tilde\Gamma^2_{21}\tilde\Gamma^3_{23} = \frac{\cot\theta}{r} - \frac{\cot\theta}{r} = 0 \checkmark$$

$$\tilde R_{31} = \frac{\partial\tilde\Gamma^k_{3k}}{\partial r} - \frac{\partial\tilde\Gamma^k_{31}}{\partial x^k} + \underbrace{\tilde\Gamma^l_{3k}\tilde\Gamma^k_{1l}}_{0} - \underbrace{\tilde\Gamma^k_{31}\tilde\Gamma^l_{kl}}_{0}$$

$$= 0 - \frac{\partial\tilde\Gamma^3_{31}}{\partial\phi} = 0 \checkmark$$

$$\tilde R_{23} = \frac{\partial\tilde\Gamma^k_{2k}}{\partial\phi} - \frac{\partial\tilde\Gamma^k_{23}}{\partial x^k} + \tilde\Gamma^l_{2k}\tilde\Gamma^k_{3l} - \Gamma^k_{23}\Gamma^l_{kl}$$

$$\overset{\shortparallel}{0} \quad - \frac{\partial\Gamma^3_{23}}{\partial\phi} = 0 \quad + \underbrace{\underbrace{\tilde\Gamma^1_{22}\tilde\Gamma^2_{31}}_{0} + \underbrace{\Gamma^2_{21}\Gamma^1_{32}}_{\frac{1}{r}\cdot 0} + \underbrace{\Gamma^3_{23}\Gamma^3_{33}}_{0} + \Gamma^3_{23}\Gamma^?_{3?}}_{0} = 0 \text{ no terms}$$

$$= 0 \checkmark$$

$$\tilde R_{32} = \frac{\partial\Gamma^k_{3k}}{\partial\theta} - \frac{\partial\Gamma^k_{32}}{\partial x^k} + \text{The same as } \tilde R_{23}$$

$$= 0 - \frac{\partial\Gamma^3_{32}}{\partial\phi} = 0 \checkmark$$

$$\tilde R_{22} = \frac{\partial\Gamma^k_{2k}}{\partial\theta} - \frac{\partial\Gamma^k_{22}}{\partial x^k} + \Gamma^l_{2k}\Gamma^k_{2l} - \Gamma^k_{22}\Gamma^l_{kl}$$

$$\frac{\partial \cot\theta}{\partial\theta} - \frac{\partial\Gamma^1_{22}}{\partial r} + \Gamma^1_{22}\Gamma^2_{21} + \Gamma^2_{21}\Gamma^1_{22} + \Gamma^3_{23}\Gamma^3_{23} - \Gamma^1_{22}\left(\Gamma^1_{11} + \Gamma^2_{12} + \Gamma^3_{13}\right)$$

$$\cancel{\cos^2\theta - \sin^2\theta} + \frac{\partial r(1-kr^2)}{\partial r} + \frac{2}{r}r(1-kr^2) + \cot^2\theta + r(1-kr^2)\left(\frac{kr}{1-kr^2} + \frac{2}{r}\right)$$

$$\cancel{\cos^2\theta - \sin^2\theta} + (1-kr^2) - 2kr^2 - 2(1-kr^2) + \cot^2\theta + kr^2 + 2(1-kr^2)$$

$$\cancel{\cos^2\theta - \sin^2\theta} + \cot^2\theta + 1 - 2kr^2$$

$$\tilde R_{22} = \frac{\partial \tilde\Gamma^k_{2k}}{\partial\theta} - \frac{\partial \tilde\Gamma^k_{22}}{\partial x^k} + \tilde\Gamma^l_{2k}\tilde\Gamma^k_{2l} - \tilde\Gamma^k_{22}\tilde\Gamma^l_{kl}$$

$$= \frac{\partial\tilde\Gamma^3_{23}}{\partial\theta} - \frac{\partial\tilde\Gamma^1_{22}}{\partial r} + \tilde\Gamma^1_{22}\tilde\Gamma^2_{21} + \tilde\Gamma^2_{21}\tilde\Gamma^1_{22} + \tilde\Gamma^3_{23}\tilde\Gamma^3_{23} - \tilde\Gamma^1_{22}\left(\tilde\Gamma^1_{11}+\tilde\Gamma^2_{12}+\tilde\Gamma^3_{13}\right)$$

$$= \frac{\partial\tilde\Gamma^3_{23}}{\partial\theta} - \frac{\partial\tilde\Gamma^1_{22}}{\partial r} + \left(\tilde\Gamma^3_{23}\right)^2 - \tilde\Gamma^1_{22}\tilde\Gamma^1_{11} \qquad \left(\text{as } \tilde\Gamma^2_{21} = \tilde\Gamma^2_{12} = \tilde\Gamma^3_{13} = \frac{1}{r}\right)$$

$$= \frac{\partial \cot\theta}{\partial\theta} + \frac{\partial\, r(1-kr^2)}{\partial r} + \cot^2\theta + r(1-kr^2)\cdot\frac{kr}{1-kr^2}$$

$$= \left(\frac{-\sin\theta}{\sin\theta} - \frac{\cos\theta}{\sin^2\theta}\cdot\cos\theta\right) + \left((1-kr^2) - 2kr^2\right) + \cot^2\theta + kr^2$$

$$= \left(\cancel{-1} - \cancel{\cot^2\theta}\right) + \left(\cancel{1} - 3kr^2\right) + \cancel{\cot^2\theta} + kr^2$$

$$\boxed{= -2kr^2 \checkmark}$$

$$\tilde R_{33} = \underbrace{\frac{\partial\tilde\Gamma^k_{3k}}{\partial\phi}}_{=0} - \frac{\partial\Gamma^k_{33}}{\partial x^k} + \Gamma^l_{3k}\Gamma^k_{3l} - \Gamma^k_{33}\Gamma^l_{kl}$$

$$\cancel{\frac{\partial\left(r\sin^2\theta(1-kr^2)\right)}{\partial r}}$$

$$= -\frac{\partial\Gamma^1_{33}}{\partial r} - \frac{\partial\Gamma^2_{33}}{\partial\theta} + \Gamma^1_{33}\Gamma^3_{31} + \Gamma^2_{33}\Gamma^3_{32} + \cancel{\Gamma^3_{31}\Gamma^1_{33}} + \cancel{\Gamma^3_{32}\Gamma^2_{33}}$$
$$- \Gamma^1_{33}\left(\Gamma^1_{11} + \Gamma^2_{12} + \cancel{\Gamma^3_{13}}\right)$$
$$- \cancel{\Gamma^2_{33}\Gamma^3_{23}}$$

$$= \frac{\partial\left(r\sin^2\theta(1-kr^2)\right)}{\partial r} + \frac{\partial(\sin\theta\cos\theta)}{\partial\theta} - \sin^2\theta(1-kr^2) - \sin\theta\cos\theta\cdot\cot\theta$$
$$+ r\sin^2\theta(1-kr^2)\left(\frac{kr}{1-kr^2} + \frac{1}{r}\right)$$

$$= \left(\sin^2\theta(1-kr^2) - (2kr^2\sin^2\theta)\right) + \left(\cos^2\theta - \sin^2\theta\right) - \cos^2\theta + kr^2\sin^2\theta$$

$$= \left(\sin^2\theta - 3kr^2\sin^2\theta\right) + \cos^2\theta - \sin^2\theta - \cos^2\theta + kr^2\sin^2\theta$$

$$= \boxed{-2kr^2\sin^2\theta \checkmark}$$

or. $\boxed{\tilde R_{ij} = -2k\,\tilde g_{ij}}$!!!

Symmetry of $R_{\mu\nu}$.

$$R_{\mu\nu} = \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\nu} - \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\nu\eta} - \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\eta\lambda} \quad (1)$$

$$R_{\nu\mu} = \frac{\partial \Gamma^\lambda_{\nu\lambda}}{\partial x^\mu} - \frac{\partial \Gamma^\lambda_{\nu\mu}}{\partial x^\lambda} + \Gamma^\eta_{\nu\lambda}\Gamma^\lambda_{\mu\eta} - \Gamma^\eta_{\nu\mu}\Gamma^\lambda_{\eta\lambda}$$

as $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}$

$$R_{\nu\mu} = \frac{\partial \Gamma^\lambda_{\nu\lambda}}{\partial x^\mu} - \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + \Gamma^\lambda_{\mu\eta}\Gamma^\eta_{\nu\lambda} - \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\eta\lambda}$$

interchange η and λ in third term.

$$\frac{\partial \Gamma^\lambda_{\nu\lambda}}{\partial x^\mu} - \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\nu\eta} - \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\eta\lambda} \quad (2)$$

subtract (2) from (1).

$$R_{\mu\nu} - R_{\nu\mu} = \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\nu} - \frac{\partial \Gamma^\lambda_{\nu\lambda}}{\partial x^\mu}$$

$$\Gamma^\lambda_{\mu\lambda} = \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\lambda}$$

$$\frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\nu} = \frac{\partial}{\partial x^\nu}\left(\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\lambda}\right) = \left(\frac{\partial}{\partial \xi^\alpha}\left(\frac{\partial x^\lambda}{\partial x^\nu}\right)\right)\cdot \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\lambda} + \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^3 \xi^\alpha}{\partial x^\mu \partial x^\nu \partial x^\lambda}$$

$$\frac{1}{2}\frac{\partial}{\partial x^\nu}\left(g^{\lambda\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\lambda} + \frac{\partial g_{\lambda\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\sigma}\right)\right) = \frac{1}{2}\frac{\partial}{\partial x^\nu} g^{\lambda\sigma}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu}$$

$$\Rightarrow R_{\mu\nu} - R_{\nu\mu} = \frac{1}{2}\left(\frac{\partial g^{\lambda\sigma}}{\partial x^\nu}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu} + g^{\lambda\sigma}\frac{\partial^2 g_{\lambda\sigma}}{\partial x^\nu \partial x^\mu}\right) - \frac{1}{2}\left(\frac{\partial g^{\lambda\sigma}}{\partial x^\mu}\frac{\partial g_{\lambda\sigma}}{\partial x^\nu} + g^{\lambda\sigma}\frac{\partial^2 g_{\lambda\sigma}}{\partial x^\mu \partial x^\nu}\right)$$

$$= \frac{1}{2}\left(\frac{\partial g^{\lambda\sigma}}{\partial x^\nu}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu} - \frac{\partial g^{\lambda\sigma}}{\partial x^\mu}\frac{\partial g_{\lambda\sigma}}{\partial x^\nu}\right)$$

now $g_{\lambda\sigma}\frac{\partial g^{\sigma\rho}}{\partial x^\kappa} = -g^{\sigma\rho}\frac{\partial g_{\lambda\sigma}}{\partial x^\kappa}$

$$\Rightarrow \quad = \frac{1}{2}\left(\frac{g^{\rho\lambda}g_{\rho\lambda}}{4}\frac{\partial g^{\lambda\sigma}}{\partial x^\nu}\frac{\partial g_{\lambda\sigma}}{\partial x^\mu} - \frac{g^{\rho\lambda}g_{\rho\lambda}}{4}\frac{\partial g^{\lambda\sigma}}{\partial x^\mu}\frac{\partial g_{\lambda\sigma}}{\partial x^\nu}\right)$$

$$= \frac{1}{8}\left(g^{\rho\lambda}\left(-g^{\lambda\sigma} \frac{\partial g_{\rho\lambda}}{\partial x^{\nu}} \frac{\partial g_{\lambda\sigma}}{\partial x^{\mu}} \right) - g^{\rho\lambda}\left(-g^{\lambda\sigma} \frac{\partial g_{\rho\lambda}}{\partial x^{\mu}} \frac{\partial g_{\lambda\sigma}}{\partial x^{\nu}} \right)\right)$$

$$= \frac{-1}{8}\left(+\left(g^{\rho\lambda} \frac{\partial g_{\rho\lambda}}{\partial x^{\nu}}\right)\left(g^{\lambda\sigma} \frac{\partial g_{\lambda\sigma}}{\partial x^{\mu}}\right) - \left(g^{\rho\lambda} \frac{\partial g_{\rho\lambda}}{\partial x^{\mu}}\right)\left(g^{\lambda\sigma} \frac{\partial g_{\lambda\sigma}}{\partial x^{\nu}}\right)\right)$$

as all indecies are summed over this is clearly <u>zero</u> Interchange $\rho\lambda \to \lambda\sigma$

$$\Rightarrow \underline{\underline{R_{\mu\nu} = R_{\nu\mu}}}$$

$$\boxed{\begin{aligned} \underline{\text{so}} \quad R_{tt} &= R_{00} = \frac{3\ddot{R}(t)}{R(t)} \\ R_{ti} &= R_{0i} = 0 \\ R_{ij} &= -\left(\ddot{R}(t)R(t) + 2\dot{R}^2(t) + 2K\right)\tilde{g}_{ij} \end{aligned}}$$

$$\boxed{\underline{\text{note}} \quad \underline{R(t) \neq R \equiv g^{\nu\mu}R_{\mu\nu} = R^{\mu}{}_{\mu}}}$$

So the energy momentum tensor for an ideal fluid is.

$$T_{\mu\nu} = P g_{\mu\nu} + (P+\rho)U_\mu U_\nu$$

P = pressure

ρ = mass density

where $U_\mu = g_{\nu\mu}U^\nu$

$$\left.\begin{aligned} U^0 &= 1 \\ U^i &= 0 \end{aligned}\right\} \text{co moving} \Rightarrow \text{vel.} = \frac{dx^i}{d\tau} = U^i$$

$$\Rightarrow T_{\mu\nu} = P g_{\mu\nu} + (P+\rho)g_{\mu\alpha}g_{\nu\beta}\delta^\alpha_0\delta^\beta_0 = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & g_{rr}P & 0 & 0 \\ 0 & 0 & g_{\theta\theta}P & 0 \\ 0 & 0 & 0 & g_{\phi\phi}P \end{pmatrix}$$

The Einstein equation is.

$$\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = -8\pi G T_{\mu\nu}$$

$$\text{and} \quad \left(g^{\nu\mu}R_{\mu\nu} - \frac{1}{2}g^{\nu\mu}g_{\mu\nu}R\right) = -8\pi G\, g^{\nu\mu}T_{\mu\nu}$$

$$= R - 2R = -8\pi G T^\mu{}_\mu$$

$$\Rightarrow R = +8\pi G T^\mu{}_\mu$$

$$\Rightarrow R_{\mu\nu} = -8\pi G T_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R$$

$$= -8\pi G T_{\mu\nu} + \frac{1}{2}g_{\mu\nu}(8\pi G T^\lambda{}_\lambda)$$

$$= -8\pi G S_{\mu\nu}.$$

$$\text{where} \quad \boxed{S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\lambda{}_\lambda}$$

$$T^\lambda{}_\lambda = \overset{\Delta\nu T.}{g^{\mu\lambda}T_{\lambda\mu}} = 4P - (P+\rho) = 3P - \rho$$

$\Rightarrow S_{\mu\nu}$ is

$$S_{tt} = \rho - \frac{1}{2} g_{00} (3P-\rho) = \rho + \frac{1}{2}(3P-\rho) = \frac{1}{2}(3P+\rho)$$

$$S_{ti} = 0 + 0(3P-\rho) = 0$$

$$S_{ij} = P \cdot R^2(t)\, \tilde{g}_{ij} - \frac{1}{2} R^2(t)\tilde{g}_{ij}\,(3P-\rho)$$
$$= \frac{1}{2}(\rho - P) R^2(t) \tilde{g}_{ij}$$

$$\Rightarrow R_{tt} = \frac{3\ddot{R}(t)}{R(t)} = -8\pi G\, S_{tt} = -8\pi G \cdot \frac{1}{2}(3P+\rho)$$

$$\boxed{3\ddot{R}(t) = -4\pi G(3P+\rho)\, R(t)} \quad (1).$$

$$R_{ij} = -\left(\ddot{R}(t)\, R(t) + 2\dot{R}^2(t) + 2k\right)\tilde{g}_{ij} = -8\pi G S_{ij}$$
$$= -\left(\ddot{R}(t)\, R(t) + 2\dot{R}^2(t) + 2k\right)\tilde{g}_{ij} = -8\pi G \cdot \frac{1}{2}(\rho - P) R^2(t)\tilde{g}_{ij}$$

$$\Rightarrow \boxed{\ddot{R}(t)\, R(t) + 2\dot{R}^2(t) + 2k = +4\pi G(\rho - P)\, R^2(t)} \quad (2).$$

$$\Rightarrow -\frac{4}{3}\pi G(3P+\rho) R^2(t) + 2\dot{R}^2(t) + 2k = 4\pi G(\rho - P)\, R^2(t)$$ inserting (1) for R̈

$$\cancel{-4\pi G P R^2(t)} - \frac{4}{3}\pi G \rho R^2(t) + 2\dot{R}^2(t) + 2k = 4\pi G \rho R^2(t) \cancel{- 4\pi G P R^2(t)}$$

$$\Rightarrow 2\dot{R}^2(t) + 2k = \frac{16}{3}\pi G \rho\, R^2(t)$$

$$\boxed{\dot{R}^2(t) + k = \frac{8\pi G}{3}\rho\, R^2(t)} \quad (3)$$

Which is eq (1) on Page 375 of Shu!

with $k = -c^2$ ie. $k = -1$ open universe.

~~rewith~~ re writing this.

$$\left(\frac{\dot{R}(t)}{R(t)}\right)^2 + \frac{k}{R^2(t)} = \frac{8\pi G \rho}{3}$$

or in terms of Hubbles constant

$$\boxed{c^2 H_0^2 + \frac{k}{R^2(t)} = \frac{8\pi G \rho}{3}} \quad (4)$$

and the deceleration parameter.

~~and~~ $$\frac{\ddot{R}(t)}{R(t)} = \frac{-4\pi G}{3}(3P+\rho)$$

$$= -q_0 H_0^2 c^2 = \frac{-4\pi G}{3}(3P+\rho)$$

$$\Rightarrow \boxed{q_0 H_0^2 c^2 = \frac{4\pi G}{3}(3P+\rho)} \quad (5)$$

$$q_0 = -\ddot{R}\frac{R}{\dot{R}^2}$$

$$q_0 H_0^2 c^2 = -\ddot{R}\frac{R}{\dot{R}^2}\cdot\left(\frac{\dot{R}}{R}\right)^2$$

$$= -\frac{\ddot{R}}{R}$$

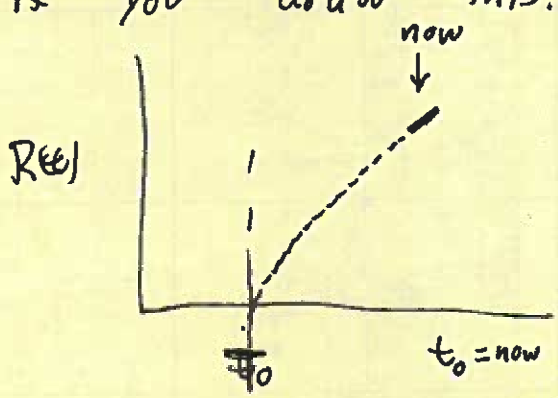
from eq (1) it is seen that as long as P and ρ are positive $\frac{\ddot{R}(t)}{R(t)}$ will be negative

Because we observe red shifts in the galaxy

$\frac{\dot{R}}{R} = H_0 > 0$

$\Rightarrow$ as we go back in time the slope of R ie $\frac{\dot{R}}{R}$ gets steeper the further back we go (In fact as $R(t)$ gets smaller as we go back in time ρ and P must increase as $\rho R^3 \sim$ total mass. etc.

$\Rightarrow$ if you draw this.



at some time $(\bar{t}_0)$ $\underline{R(\bar{t}_0) = 0} = R(0)$ ie $\underline{\bar{t}_0 = 0}$

Covariant differentiation.

$$V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}$$

differentiating wrt x'^{λ}

$$\frac{\partial}{\partial x'^{\lambda}} V'^{\mu} = \frac{\partial}{\partial x'^{\lambda}} \left(\frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu} \right) = \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial}{\partial x^{\rho}} \left(\frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu} \right)$$

$$= \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial V^{\nu}}{\partial x^{\rho}} + \frac{\partial^2 x'^{\mu}}{\partial x^{\nu} \partial x^{\rho}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\nu}$$

the second term spoiling the tensor transformation

noting that.

$$\Gamma'^{\mu}_{\lambda\kappa} V'^{\kappa} = \left[\frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial x^{\sigma}}{\partial x'^{\kappa}} \Gamma^{\nu}_{\rho\sigma} - \frac{\partial^2 x'^{\mu}}{\partial x^{\rho} \partial x^{\sigma}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \frac{\partial x^{\sigma}}{\partial x'^{\kappa}} \right] \frac{\partial x'^{\kappa}}{\partial x^{\eta}} V^{\eta}$$

$$= \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \Gamma^{\nu}_{\rho\sigma} V^{\sigma} - \frac{\partial^2 x'^{\mu}}{\partial x^{\rho} \partial x^{\sigma}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} V^{\sigma}$$

$$\Rightarrow \frac{\partial V'^{\mu}}{\partial x'^{\lambda}} + \Gamma'^{\mu}_{\lambda\kappa} V'^{\kappa} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \frac{\partial x^{\rho}}{\partial x'^{\lambda}} \left[\frac{\partial V^{\nu}}{\partial x^{\rho}} + \Gamma^{\nu}_{\rho\sigma} V^{\sigma} \right].$$

$\Rightarrow$ covariant derivative

$$\boxed{D_{\lambda} V^{\mu} \equiv \frac{\partial V^{\mu}}{\partial x^{\lambda}} + \Gamma^{\mu}_{\lambda\kappa} V^{\kappa}}$$

for covariant vectors.

$$D_{\lambda} V_{\mu} \equiv \frac{\partial V_{\mu}}{\partial x^{\lambda}} - \Gamma^{\kappa}_{\mu\lambda} V_{\kappa}.$$

For a general tensor

ex. $T^{\alpha\beta\gamma}{}_{\delta\rho}$

$$D_{\lambda} T^{\alpha\beta\gamma}{}_{\delta\rho} = \frac{\partial T^{\alpha\beta\gamma}{}_{\delta\rho}}{\partial x^{\lambda}} + \Gamma^{\alpha}_{\lambda\kappa} T^{\kappa\beta\gamma}{}_{\delta\rho} + \Gamma^{\beta}_{\lambda\kappa} T^{\alpha\kappa\gamma}{}_{\delta\rho} + \Gamma^{\gamma}_{\lambda\kappa} T^{\alpha\beta\kappa}{}_{\delta\rho} - \Gamma^{\kappa}_{\lambda\delta} T^{\alpha\beta\gamma}{}_{\kappa\rho} - \Gamma^{\kappa}_{\lambda\rho} T^{\alpha\beta}$$

Conservation of energy momentum.

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0 \rightarrow D_\nu T^{\mu\nu} = 0$$

$$D_\nu \left[p g^{\mu\nu} + (p+\rho) u^\mu u^\nu \right]$$

note
$$\Gamma^\nu_{\nu\sigma} = \frac{1}{2} g^{\nu\rho} \left(\frac{\partial g_{\rho\nu}}{\partial x^\sigma} + \frac{\partial g_{\sigma\rho}}{\partial x^\nu} - \frac{\partial g_{\nu\sigma}}{\partial x^\rho} \right)$$

$$= \frac{1}{2} g^{\nu\rho} \frac{\partial g_{\rho\nu}}{\partial x^\sigma} + \frac{1}{2} g^{\nu\rho} \frac{\partial g_{\sigma\rho}}{\partial x^\nu} - \frac{1}{2} g^{\rho\nu} \frac{\partial g_{\rho\sigma}}{\partial x^\nu}$$ (interchanging summed indecies $\rho\nu$ in last term)

$$= \frac{1}{2} g^{\nu\rho} \frac{\partial g_{\rho\nu}}{\partial x^\sigma}$$

$$= \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^\sigma} \qquad g \equiv -\operatorname{Det} g_{\mu\nu}.$$

$$\Rightarrow D_\nu \left(p g^{\mu\nu} + (p+\rho) u^\mu u^\nu \right) \qquad \begin{matrix} u^0 = 1 \\ u^i = 0 \end{matrix}$$

$$\frac{\partial}{\partial x^\nu} p g^{\mu\nu} + \Gamma^\mu_{\nu\rho} p g^{\rho\nu} + \Gamma^\nu_{\nu\rho} p g^{\mu\rho}$$
$$+ \frac{\partial}{\partial x^\nu} (p+\rho) u^\mu u^\nu + (p+\rho) \left[\Gamma^\mu_{\nu\rho} u^\rho u^\nu + \Gamma^\nu_{\nu\rho} u^\mu u^\rho \right]$$

but

$$D_\nu T^{\mu\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu_{\nu k} T^{k\nu} + \Gamma^\nu_{\nu k} T^{\mu k}$$

$$= \frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu_{\nu k} T^{k\nu} + \frac{1}{\sqrt{g}} \left(\frac{\partial \sqrt{g}}{\partial x^k} \right) T^{\mu k}$$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\nu} \left(\sqrt{g}\, T^{\mu\nu} \right) + \Gamma^\mu_{\nu k} T^{k\nu}$$

$$D_\nu T^{\mu\nu} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\nu}\left(\sqrt{g}\,P g^{\mu\nu}\right) + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\nu}\sqrt{g}\,(P+\rho)u^\mu u^\nu$$

$$+\Gamma^\mu_{\nu k}\left[P g^{k\nu} + (P+\rho)u^k u^\nu\right].$$

$$= \frac{\partial P}{\partial x^\nu} g^{\mu\nu} + P\frac{\partial g^{\mu\nu}}{\partial x^\nu} + \frac{P g^{\mu\nu}}{\sqrt{g}}\frac{\partial \sqrt{g}}{\partial x^\nu} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\nu}\left(\sqrt{g}(P+\rho)u^\mu u^\nu\right)$$

$$+ P\,\Gamma^\mu_{\nu k} g^{k\nu} + \Gamma^\mu_{\nu k}(P+\rho)u^k u^\nu.$$

note. $D_\nu g^{\mu\nu}$ is a tensor therefore is equal to the value in the local free fall frame.

$$\frac{\partial}{\partial x^\alpha}\eta^{\beta\alpha} = 0$$

if zero in one frame must be zero in all frames.

$$\Rightarrow \quad D_\nu T^{\mu\nu} = D_\nu P g^{\mu\nu} + D_\nu\left((P+\rho)u^\mu u^\nu\right).$$

$$= g^{\mu\nu} D_\nu P + D_\nu\left((P+\rho)u^\mu u^\nu\right)$$

$$= g^{\mu\nu}\frac{\partial P}{\partial x^\nu} + g^{-1/2}\frac{\partial}{\partial x^\nu}\left(\sqrt{g}(P+\rho)u^\mu u^\nu\right) + \Gamma^\mu_{\lambda k}(P+\rho)u^\lambda u^k.$$

$u^0 = 1$
$u^i = 0$

$$\Rightarrow \boxed{D_\nu T^{\mu\nu} = g^{\mu\nu}\frac{\partial P}{\partial x^\nu} + g^{-1/2}\frac{\partial}{\partial t}\sqrt{g}\,(P+\rho)u^\mu + \Gamma^\mu_{tt}(P+\rho)}$$

$\Gamma^\mu_{tt} = 0$ already shown

$$\Rightarrow \boxed{D_\nu T^{\mu\nu} = 0 = g^{\mu\nu}\frac{\partial P}{\partial x^\nu} + \frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}\,(P+\rho)u^\mu\right)}$$

$\mu = 1, 2, 3$

$$g^{i\mu} \frac{\partial P}{\partial x^{\mu}} = 0$$

which is trivial as we assumed a homogeneous isotropic universe.

$\mu = 0$

$$(-1) \frac{\partial P}{\partial t} + \frac{1}{\sqrt{g}} \frac{\partial}{\partial t}\left(\sqrt{g}\,(P+\rho)\right) = 0$$

$$\sqrt{g} = \frac{R^3(t)\, r^4 \sin^2\theta}{1-Kr^2}$$

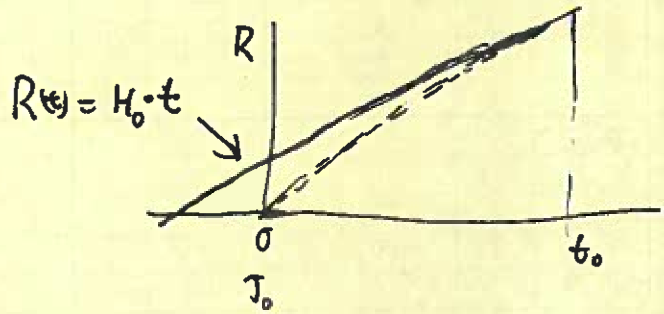
$$\Rightarrow \boxed{R^3(t) \frac{\partial P}{\partial t} = \frac{\partial}{\partial t}\left(R^3(t)(P+\rho)\right)}$$

$$= R^3(t) \frac{d P}{dt} = \frac{d}{dt}\left(R^3(t)(P+\rho)\right)$$

as $v^i = 0$

if you extrapolate back.

$\dot{R}$ is negative for $0 < t < \underset{\text{now}}{t_0}$



$$\Rightarrow \quad t_0 < \frac{1}{H_0}$$

Conservation of energy is a relation something like

$$0 = \frac{\partial}{\partial x^\nu} T^{\mu\nu} + (\text{some terms which are zero in special frames})$$

this leads to

$$R^3(t)\frac{dP}{dt} = \frac{d}{dt}\left(R^3(t)(\rho + P)\right)$$

$$= \frac{d}{dt}(\rho R^3) + P\frac{dR^3}{dt} + R^3\frac{dP}{dt}.$$

$$\Rightarrow \frac{d}{dt}(\rho R^3) = -P\frac{dR^3}{dt}$$

Change $\frac{d}{dt} \to \frac{dR}{dt}\frac{d}{dR}$ as $R = R(t)$ only.

$$\Rightarrow \frac{dR}{dt}\frac{d}{dR}(\rho R^3) = -P\frac{dR}{dt}\frac{dR^3}{dR}$$

$$\boxed{\Rightarrow \quad \frac{d}{dR}(\rho R^3) = -3PR^2} \qquad -3PR^2$$

if P is always positive then $\frac{d}{dR}(\rho R^3) < 0$

$\Rightarrow$ ρ must decrease with increasing R at least as fast as R^{-3} so that $\rho R^3 = kR^{-\alpha}$ $\underline{\underline{\alpha > 0}}$

going back to eq 3

$$\dot{R}^2 + k = \frac{8\pi G}{3} \rho R^2$$

as $R \to \infty$ ρR^2 must go to zero

as $\rho \propto R^{-(3+\alpha)}$ $\alpha \geq 0$

$\Rightarrow$ if $k = -1$

$R(t) \to t$ as $R \to \infty$ $\Rightarrow$ $\frac{dR}{dt} \to 1$.

universe keeps on expanding.

if $k = 0$

$R(t)$ will continue to increase but not as quickly as t.

if $k = +1$

at some point ρR^2 will decrease to $\frac{3}{8\pi G}$

at that point.

$$\dot{R}^2 + 1 = \frac{8\pi G}{3} \cdot \frac{3}{8\pi G} = 1$$

$$\Rightarrow \dot{R}^2 = 0$$

at this point the expansion stops and the universe will start to collapse

$\Rightarrow$ $k = -1$ or zero the universe expands forever

$k = +1$ the universe stops expanding and will contract.

relation of $R = R^{\mu}_{\ \mu}$ and $R(t)$

$$R_{\mu\nu} = \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} + \Gamma^{\eta}_{\mu\lambda}\Gamma^{\lambda}_{\nu\eta} - \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\eta\lambda}$$

$$g^{\mu\nu} R_{\nu\mu} = R^{\mu}_{\ \mu}$$

but think about it ..

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1-kr^2}{R^2(t)} & 0 & 0 \\ 0 & 0 & \frac{1}{R^2(t)r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{R^2(t)r^2\sin^2\theta} \end{pmatrix}$$

$\Rightarrow$ only diagonal elements.

$$\Rightarrow R^{\mu}_{\ \mu} = -1\left(R_{tt}\right) + \frac{1-kr^2}{R^2(t)} R_{rr} + \frac{1}{R^2(t) r^2} R_{\theta\theta} + \frac{1}{R^2(t) r^2 \sin^2\theta} R_{\phi\phi}$$

or
$$= -1\left(R_{tt}\right) + \frac{1}{R^2(t)} \cdot \tilde{g}^{ii}\left(R_{ii}\right)$$

$$= -1\left(R_{tt}\right) + \frac{1}{R^2(t)} \tilde{g}^{ii}\left(-\left(\ddot{R}(t)R(t) + 2\dot{R}^2(t) + 2k\right)\tilde{g}_{ii}\right)$$

$$= -\frac{3\ddot{R}(t)}{R(t)} - \frac{3}{R^2(t)}\left(\ddot{R}(t)R(t) + 2\dot{R}^2(t) + 2k\right) \qquad 3 = \tilde{g}^{ii}\cdot\tilde{g}_{ii}$$

$$= -6\frac{\ddot{R}(t)}{R(t)} - 6\frac{\dot{R}^2(t)}{R^2(t)} - \frac{6k}{R^2(t)} = -\frac{eq①}{R(t)} - 3 * eq②/R^2(t)$$

using eq. ① and ②
$$= 4\pi G(3P+\rho) - 3 * 4\pi G(\rho - P)$$

$$= 8\pi G(3P - \rho) = T^{\mu}_{\ \mu}$$ so I did it right.

but there is no trivial relation between $R^{\mu}_{\ \mu}$ and $R(t)$.

evaluation of the curvature from data.

let's look at equations ④ and ⑤. evaluated now.

$R(t_0) = R_0$

$$c^2 H_0^2 + \frac{K}{R_0^2} = \frac{8\pi G}{3} \rho_0 \quad ④$$

$$q_0 H_0^2 c^2 = + \frac{4\pi G}{3}(3P_0) + \frac{4\pi G \rho_0}{3} \quad ⑤.$$

$$= +4\pi G P_0 + \frac{1}{2} c^2 H_0^2 + \frac{1 K}{2 R_0^2}$$

$$\Rightarrow 2 q_0 H_0^2 c^2 = +8\pi G P_0 + H_0^2 c^2 + \frac{K}{R_0^2}$$

$$\Rightarrow 8\pi G P_0 = -\left(\frac{K}{R_0^2} + H_0^2 c^2 (1 - 2q_0) \right)$$

$$\boxed{P_0 = \frac{-1}{8\pi G}\left[\frac{K}{R_0^2} + H_0^2 c^2 (1 - 2q_0) \right]} \quad ⑥$$

and

$$\boxed{\rho_0 = \frac{3}{8\pi G}\left(H_0^2 c^2 + \frac{K}{R_0^2} \right)} \quad ⑦$$

K will be positive or negative depending on whether ρ is greater or less than ρ_c.

$$\boxed{\rho_c = \frac{3 H_0^2 c^2}{8\pi G}} = 1.1 \times 10^{-29} \left(\frac{H_0}{75\ \text{km/sec/Mpc}} \right)^2 \ \text{gm/c}$$

using ⑦ in ⑥.

$$P_0 = \frac{-1}{8\pi G}\left[\frac{8\pi G}{3}\rho_0 - 2 q_0 H_0^2 c^2 \right]$$

for non relativistic matter $P_0 \ll \rho_0$

$$\Rightarrow 0 \sim \left[\frac{8\pi G \rho_0}{3} - 2 q_0 H_0^2 c^2 \right)$$

or.

$$\boxed{\frac{K}{R_0^2} \approx (2q_0 - 1) H_0^2 c^2}$$

$$\frac{\rho_0}{\rho_c} = \frac{8\pi G}{3 H_0^2 c^2}\left[\frac{3}{8\pi G}\left[H_0^2 c^2 + \frac{k}{R_0^2}\right]\right].$$

$$= 1 + \frac{k}{H_0^2 c^2 R_0^2} = 1 + (2q_0 - 1)$$

$$= 2q_0$$

if $2q_0 > 1$ $\quad k > 0$

if $2q_0 < 1$ $\quad k < 0$

as $\frac{k}{R_0^2} = (2q_0 - 1) H_0^2 c^2$

$$\Rightarrow \boxed{\begin{array}{ll} \text{if } \rho > \rho_c & k > 0 \\ \text{if } \rho \leq \rho_c & k \leq 0 \end{array}} \quad \begin{array}{l} \text{closed} \\ \text{open} \end{array}$$

Radiation dominated universe

$$\frac{\rho}{3} = P \qquad = \frac{4}{3} a T^4.$$

the energy conservation equation is then.

$$\frac{d(\rho R^3)}{dR} = -3\rho R^2$$

$$4a \frac{d(T^4 R^3)}{dR} = -4a T^4 R^2.$$

$$\frac{d(T^4 R^3)}{dR} = -T^4 R^2$$

if $T = K R^n$

$T^4 = K^4 R^{4n}$

$$\frac{d\, T^4 R^3}{dR} = K^4 \frac{d\, R^{4n+3}}{dR} = K^4 (4n+3)\, R^{4n+2}$$

$$= -K^4 R^{4n+2}$$

$\Rightarrow 4n+3 = -1$

$\underline{n = -1}$

$$T(t) = \frac{K}{R(t)}$$

or $\boxed{T(t) = T(t_0) \cdot \frac{R_0}{R(t)}}$

General case.

In the more general case where the energy density is a mixture of the energy due to radiation and gas the energy conservation equation is slightly different. The total pressure is

$$P = nkT + \frac{1}{3}aT^4$$

where n is the number density of gas atoms

and the energy density is

$$\rho = nm + \frac{nkT}{(\gamma-1)} + aT^4$$

where m is the mass/particle and γ is the ratio of specific heats $c_P/c_V = 5/3$ for monatomic atoms

Then the energy conservation equation is

$$\frac{d(\rho R^3)}{dR} = -3PR^2$$

$$\boxed{\frac{d}{dR}\left(nmR^3 + \frac{nkT}{(\gamma-1)}R^3 + aT^4R^3\right) = -3nkTR^2 - aT^4R^2}$$

as the particle number is conserved

$$nR^3 = n_0R_0^3 = \text{Const}$$

and therefore the derivative of the first term is zero

$$\Rightarrow \frac{nk}{(\gamma-1)}\frac{d(TR^3)}{dR} + a\frac{d(T^4R^3)}{dR} = -3nkTR^2 - aT^4R^2.$$

$$\cancel{\frac{nk}{(\gamma-1)}3TR^2} + \frac{nk}{\gamma-1}R^3\frac{dT}{dR} + 3aT^4R^2 + 4aT^3R^3\frac{dT}{dR} = -3nkTR^2 - aT^4R^2$$

$$\frac{kT}{\gamma-1}\frac{d\,nR^3}{dR} +$$

$$R^3\left(4aT^3 + \frac{nk}{\gamma-1}\right)\frac{dT}{dR} = -3nkTR^2 - 4aT^4R^2$$

$$\frac{R}{T}\frac{dT}{dR} = -\left[\frac{3nk + 4aT^3}{4aT^3 + \frac{nk}{\gamma-1}}\right].$$

let $\sigma = \dfrac{4aT^3}{nk}$ = photon entropy / gas atom

$$\frac{R}{T}\frac{dT}{dR} = -\left[\frac{1 + \sigma}{\sigma + \frac{1}{3}(\gamma-1)^{-1}}\right]$$

if $\sigma \gg 1$ then. (radiation dominated.)

$$\frac{R}{T}\frac{dT}{dR} = -1$$

$$T = \frac{\text{const}}{R}.$$

if $\sigma \ll 1$ then.

$$\frac{R}{T}\frac{dT}{dR} = 3(\gamma-1)$$

$$T = \frac{\text{const}}{R^{3(\gamma-1)}}$$

for $\gamma = 5/3$ $\quad \gamma - 1 = \frac{2}{3}$

$$T = \frac{\text{const}}{R^2}$$

Astronomy 3033 final.

(1). Given the radius of the sun, its' luminosity and the average opacity ($\overline{\varkappa\rho}$) estimate the central temperature assuming all energy transport is due to radiative transfer.

$R_\odot = 7 \times 10^5$ km $= 7 \times 10^{10}$ cm.

$L_\odot = 4 \times 10^{33}$ ergs/sec.

$\overline{\varkappa\rho} = 1$.

$a = 7.56 \times 10^{-15}$ ergs/(sec-cm^3-°K^4) (radiation constant).

$c = 3 \times 10^{10}$ cm/sec.

(2). Given in addition the mass of the sun estimate the central pressure of the sun from hydrostatic equilibrium. (assume 1/2 of the mass is within 1/2 of $R_\odot$) (estimate the average density).

$M_\odot = 2 \times 10^{33}$ gm.

$G = 6.67 \times 10^{-8}$ gm^{-1} cm^3 sec^{-1}

(3). Using the above two results calculate the central density

k = Boltzmans const = 1.38×10^{-16} ergs/°K.

$$m = \frac{m_p}{2} = \frac{1.67 \times 10^{-24}\ \text{gm}}{2} = 8.35 \times 10^{-25}\ \text{gm}.$$

30 33 final.

1. The covariant derivative is defined as

$$D_\gamma T^{\alpha\beta} = \frac{\partial T^{\alpha\beta}}{\partial x^\gamma} + \Gamma^\alpha_{\gamma\mu} T^{\mu\beta} + \Gamma^\beta_{\gamma\mu} T^{\alpha\mu}$$

for a second rand tensor with

$$\Gamma^\alpha_{\mu\gamma} = \frac{1}{2} g^{\alpha\sigma} \left(\frac{\partial g_{\mu\sigma}}{\partial x^\gamma} + \frac{\partial g_{\gamma\sigma}}{\partial x^\mu} - \frac{g_{\mu\gamma}}{\partial x^\sigma} \right)$$

Show explicity that the covariant divergence of the metric tensor

$$D_\gamma g^{\mu\nu} = \frac{\partial g^{\mu\gamma}}{\partial x^\gamma} + \Gamma^\mu_{\gamma k} g^{k\gamma} + \Gamma^\gamma_{\gamma k} g^{\mu k} = 0$$

for the schwarzschild metric defined by

$$d\tau^2 = \left(1 - \frac{2MG}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2MG}{r}\right)} - r^2 d\theta^2 - r^2 \sin^2\theta \, d\varphi^2$$

4. If the absolute magnitude of the sun is +4.72 and its' Luminosity is 4×10^{33} ergs/sec, what is the luminosity of a star with a magnitude (absolute) of -5.28?

5. If that star has a surface (or effective) temperature of $38{,}000\ ^\circ K$ what is its' radius?

$\sigma = 5.67 \times 10^{-5}\ erg/cm^2\text{-}s\text{-}K^4$

and now a couple of tough ones for challenge.

6. a) Given the luminosity from part 4 the radius from 5 and assume $\overline{\chi\rho} = 1$ calculate the central temperature of the hypothetical star of questions 4 and 5 (assume all energy is transported by radiative transfer).

b). Using the graph on the next page what is the energy source of this star. (note: you <u>must</u> have a ~~ten~~ central temperature to do this)

c). Using the same graph and the answer to question 1. What is the energy source for the sun?

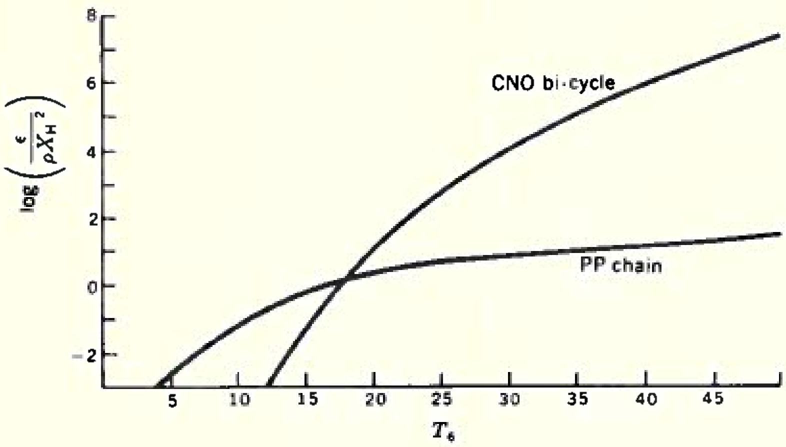


Fig. 5-16 A comparison of thermonuclear power from the PP chains and the CNO cycle. Both chains are assumed to be operating in equilibrium. The calculation was made for the choice $X_{CN}/X_H = 0.02$, which is representative of population I composition.

7. In concluding that the limiting value of the deceleration parameter, q_0, would be ½ for an open or closed universe ($k=+1$, $k=-1$) we assumed that the matter in the universe was non-relativistic ie $\rho \gg P$

If we assumed that the universe was radiation dominated ie all the energy density were in photons (or in relativistic particles). the $P = \rho/3$.

How would this change the limiting value of q_0?

Some useful equations. (?)

Stars.

$$\frac{dP}{dR} = -\frac{G\,M(r)}{r^2}\,\rho(r)$$

$$M(R) = 4\pi\int_0^R r^2 \rho(r)\,dr. \quad \text{or} \quad \frac{dM(r)}{d(r)} = 4\pi r^2 \rho(r)$$

$$\frac{dL_r}{dr} = 4\pi r^2 \rho(r)\ \varepsilon(r)$$

radiative transport.

$$\frac{dT}{dr} = -\frac{3}{4ac}\,\frac{\chi(r)\rho(r)}{T^3(r)}\,\frac{L_r}{4\pi r^2}$$

or

$$L_r = -\frac{4ac}{3}\,4\pi r^2\,\frac{T^3(r)}{\chi(r)\rho(r)}\,\frac{dT}{dr}$$

Convective transport

$$\frac{dT}{dr} = \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\,\frac{dP}{dr}$$

$$\bar{\rho} = \frac{M}{Vol} = \frac{M}{\frac{4}{3}\pi R^3}$$

$$m_1 - m_2 = 2.5\log\left(L_2/L_1\right)$$

$$P = \frac{\rho}{m}kT$$

Cosmology.

$$\ddot{R}(t) = \frac{4\pi G}{3}\left(3P+\rho\right)R(t)$$

$$\dot{R}^2(t) + K = \frac{8\pi G}{3}\rho\,R^2(t)$$

$$q_0 = -\frac{\ddot{R}(t_0)\cdot R(t_0)}{\dot{R}^2(t_0)} \Rightarrow q_0 H_0^2 c^2 = \frac{4\pi G}{3}\left(3P_0+\rho_0\right)$$

$$H_0^2 c^2 = \frac{8\pi G}{3}\rho_0 - \frac{K}{R^2(t_0)}$$

exam solution

$$L_r = -\frac{4ac}{3} 4\pi R^2 \frac{T^3(R)}{\varkappa(R)\rho(R)} \left.\frac{dT}{dR}\right|_R.$$

$$-\left.\frac{dT}{dr}\right|_{R/2} = \left.\frac{3 L_r \varkappa\rho}{16\pi ac\, R^2 T^3(r)}\right|_{R/2}.$$

$$L(R/2) = L \qquad T(R/2) = T_c/2$$

$$\varkappa\rho|_{R/2} = \overline{\varkappa\rho} = 1$$

$$-\left.\frac{dT}{dr}\right|_{R/2} \sim \frac{T_c}{R}$$

$$\Rightarrow \frac{T_c}{R}\cdot\left(\frac{T_c}{2}\right)^3 = \frac{3 L \cdot 1}{16\pi ac\,(R/2)^2}$$

$$\Rightarrow T_c^4 = \frac{6L}{\pi acR}$$

$$T_c = \sqrt[4]{\frac{6L}{\pi acR}} = \sqrt[4]{\frac{6\cdot 4\times10^{33}}{3.14\cdot 7.56\times10^{-15}\cdot 3\times10^{10}\; 7\times10^{10}}}$$

$$\sqrt[4]{4.812\times10^{26}} = \underline{\underline{4.68\times10^{6}\ {}^\circ K}}$$

low by about 2

2.

$$\frac{dP}{dr} = -\frac{G\, M(r)\, \rho(r)}{r^2}$$

evaluate at $R/2$ assume $\rho(R/2) = \bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3} = 1.39$

assume $M(R/2) = M/2$.

$$-\left.\frac{dP}{dr}\right|_{R/2} = \frac{P_c}{R} = \frac{G\, M\, \bar{\rho}}{2\,(R/2)^2}$$

$$\Rightarrow P_c = \frac{2GM\bar{\rho}}{R} = 5.3 \times 10^{15} \text{ dynes/cm}^2$$

$$= 5.24 \times 10^{9} \text{ atm.}$$

(5) $P = \frac{\rho}{m} KT$

$$\Rightarrow \rho_c = \frac{\bar{m} P_c}{K T_c} = \frac{8.35 \times 10^{-25} \cdot 5.3 \times 10^{15}}{1.38 \times 10^{-16} \cdot 4.68 \times 10^{6}} = 6.8 \text{ gm.}$$

note low by ~ 10.