astronomy at last.

having reviewed most of the nescessary elements of physics involved in astrophysics (the exception being the pressure from a degenerate fermi gas) we will now start with the discussion of the basic building blocks of stars.

On a clear night some 3000 stars can be seen with the unaided eye, with a telescope there is essentially no limit. We will start the discussion by establishing some basic questions about stars ~~the~~ ~~answers to which~~ whose answers will define the study to some level. Clearly there are the physical parameters which should be established.

1. Mass.
2. Radius.
3. Temperature.
4. Luminosity (energy output)
5. Chemical composition.

In getting these parameters the distance to the stars will have to be established.

Theoretical modeling of stars will have the above list of parameters to explain and in so doing answer two basic questions

1. Stellar Structure.
2. power source.

The statistical distribution of the physical parameters ~~[illegible]~~ will be argued to define the limits of stellar stability ie if a given value of a single or set of physical parameters is never observed we will take that as implying that such a star could not exist because it would be unstable. Our modeling of stars should indicate this.

This brings us to a second type of question. The above discussion treats stars as static unchanging objects, yet we certainly know if only by the supernova of ~~last year~~ 1987 that this is certainly not true. So some other "physical" parameters (in quotes because they are more indirectly inferred)

1. Age.
2. life times
3. evolutionary sequence

This probably sums up the bulk of the information about the "average" star. There are some exceptions (cepheid variables, T tauri stars, x ray sources, etc) which will also be dealt with and presumably the list will be lengthened as our study defines itself.

Luminosities, radii, and temperatures,

It is an observed fact that the intensity distribution of light coming from stars has the ~~[illegible]~~ characteristic shape of the plank radiation law.

$$\frac{dI}{d\nu} = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT}-1}$$

or rather a shape proportional to

$$\frac{dI_{obs.}}{d\nu} = (\text{const}) \cdot \frac{\nu^3}{e^{c\nu}-1}$$

the determination of c and equating it to $\frac{h}{kT} = c$ is argued to establish the temperature of the star. This temperature is the same as what is found from wein's law which implies that the plank law gives an accurate description of the shape as wein's law can be derived from the Plank spectrum.

Having established that stars radiate like Black Bodies the stephan Boltzman law is known to ~~[illegible]~~ describe their ~~total~~ energy output / unit area.

$$\text{Flux} = \sigma T^4$$

$\Rightarrow$ the total energy output will be.

$$\text{Flux} \cdot \text{Area} = \sigma T^4 \cdot 4\pi R^2. = \text{Luminosity}.$$

$$L = 4\pi\sigma R^2 T^4$$

this establishes a relation which will hold for stars ie from the color (peak wavelength) we know the Temperature, if we know the Luminosity we can deduce the radius.

$$R = \sqrt{\frac{L}{4\pi\sigma T^4}}$$

We know that as light propagates it carries energy and energy is conserved. Consider a point light source in an isolated universe ie all by itself. At some radius R_1 there is a Flux

$$F(R_1)$$

the total energy/sec flowing through a ball of radius R_1 is.

$$F(R_1) \cdot 4\pi R_1^2$$

at a larger radius (R_2) there is a flux $F(R_2)$ and a total energy/sec flowing through the ball of radius R_2

$$F(R_2) \cdot 4\pi R_2^2$$

as energy is conserved and if there is nothing to absorb energy within the volume between the two spherical shells.

$$F(R_1) \cdot 4\pi R_1^2 = F(R_2) 4\pi R_2^2$$

in fact these two numbers by the same argument of conservation of energy

$$F(R_1) \cdot 4\pi R_1^2 = F(R_2) \cdot 4\pi R_2^2 = \text{Luminosity of the source}$$

as there is nowhere else for the energy to go.

$$\Rightarrow \boxed{F(R) = \frac{\text{Lum.}}{4\pi R^2}}$$

Thus if we measure the flux of light from a star and know the distance to the star we can establish it's

Luminosity.

$$\boxed{L = 4\pi(dist)^2 \cdot F_{obs.}}$$

from this and the color we can establish the size of a star.

There is one star we can easily check this system with: The Sun. Given the angular size of the sun we can calculate its diameter from the small angle approximation.

$$\cancel{D = 2.06265\times10^5 \cdot \alpha}$$

$$Diam = \frac{\alpha \cdot Dist}{2.06265\times10^5} \qquad \alpha \text{ (in sec of arc)}$$

$$= 1920'' \cdot 1.5\times10^{13} / 2.06265\times10^5 = 1.396\times10^{11}\,cm$$

$R = 6.98\times10^{10}$

given the apparent surface temperature of $5800°K$. and the solar constant $f = 1.38\times10^6$ ergs/sec-cm^2 we can also deduce the size of the sun.

* Book wrong.!

$$dist = 1\,A.U. = 1.5\times10^{13}\,cm.$$

$$L = f \cdot 4\pi R^2 = 1.38\times10^6 \cdot 4\pi \cdot (1.5\times10^{13})^2$$

(dist)

$$= 3.90\times10^{33}\ ergs/sec.$$

$$\Rightarrow \quad R = \sqrt{\frac{L}{4\pi\sigma T^4}} = \sqrt{\frac{\cancel{f}\,\cancel{4\pi}(dist)^2}{\cancel{4\pi}\,\sigma T^4}}$$

$$= \sqrt{\frac{1.38\times10^6 \cdot (1.5\times10^{13})^2}{5.67\times10^{-5} \cdot (5.8\times10^3)^4}}$$

$$= 6.96\times10^{10}\ cm.$$

the two methods of determining the Radius of the Sun agree. to 1 part in 350.

mass.

In the case of the son the mass can easily be extracted. Kelpers' third law as derived from newtons' laws of motion is.

$$P^2 = \frac{4\pi^2}{G(m_1+m_2)} a^3.$$

$m_1 + m_2$ being the sum of the two masses of the orbiting bodies. In the case of the son - earth system $M_\odot \gg M_\oplus$

$\Rightarrow$.

$$M_\odot = M_\odot + M_\oplus = \frac{4\pi^2}{G} \frac{a^3}{P^2}$$

$$= \frac{4\pi^2}{6.67 \times 10^{-8}} \cdot \frac{(1.5 \times 10^{13})^3}{(3.16 \times 10^7)^2}.$$

$$= 2.00 \times 10^{33} \text{ gm.}$$

This gives us the mass of one star. In fact we will need a statistical distribution of masses to check our ideas on stellar structure and the method for this determination (Study of Binary Stars) will be done next week. ~~(I'll be in switzerland V Hagopian will discuss ch 10).~~

The general point is that the study of orbit paramerers can give us the masses of stars.

Chemical compositions.

The spectra of stars yield an enormous amount of information. Beyond the temperature, the composition of a star and even its magnetic field strength (zeeman effect) can be deduced. Of course what we ~~measure~~ deduce from the relative intensities of emission and absorbtion lines from different elements is only the chemical composition at the surface and might not reflect the interior composition (consider a similar study of the earth seen from a rocket ship). The result of such a study reveals that the sun and all stars ~~is~~ are approximately.

73% hydrogen
25% helium
2% other stuff.

in fact the element helium named for the greek (roman?) god helios (god of the sun) was discovered by looking at the solar spectrum and finding emission lines which could not be accounted for by any known element. It will turn our that this "cosmic abundance" as it's called is any essential piece of data ~~[illegible]~~ in testing cosmological theories and stellar evolution theories.

The spectra of stars in fact give another independent method of determining the surface temperatures of stars. The states of atoms and molecules also obey the Boltzman ~~distribution~~ probability distribution law

$$P(\varepsilon_\ell) = e^{-\varepsilon_\ell / KT}$$

where the subscript l denotes the atomic state.
consider a heavy element like carbon with 6 electrons
the spectrum (energy levels) of neutral carbon (CI) is different
from the spectrum (energy levels) of singly ionized carbon CII
and different again from doubly ionized carbon CIII
and so on. The states of carbon (singly, doubly
ionized, neutral etc.) differ in their energies. The
ratio of their line strengths (intensities in emission and absorbtion) will
be of the form.

$$R \equiv \frac{I(CI)}{I(CII)} = \cancel{K} \cdot \frac{P(CI)}{P(CII)} \text{const.}$$

$$= \frac{e^{-\varepsilon(CI)/KT}}{e^{-\varepsilon(CII)/KT}} \cdot \text{const.}$$

$$= e^{(\varepsilon(CII)-\varepsilon(CI))/KT} \cdot \text{Const.}$$

$$= e^{\Delta\varepsilon/KT} \cdot \text{const.}$$

the constant depending on excitation probabilities, phase space etc. (ie completely calculable) as ΔE is known and the intensity ratio measured.

$$\boxed{\frac{1}{\Delta E} \ln\left(\frac{R}{\text{const}}\right) = 1/KT}$$

$$\boxed{T = \cancel{\frac{\Delta E}{K}/\ln} \ \frac{\Delta E}{K \ln(R/\text{const})}}$$

The gist of this is that the relative line strengths give the relative probabilities of the various occupancies of the atomic states. This and the Boltzman probabilities predicts the temperature.

start at Ch 9.

near by stars. Observational Hertzsprung Russell diagram Ch 9.

If we look at stars in the night sky one of the first things you notice is the range in brightness. for the purposes of discussion it is customary to use a logarithmic scale (magnitude scale) for the brightness of stars. five units on this magnitude scale correspond to a factor of 100 in observed flux i.e. for 2 stars (1, 2)

disq. see Ch 9 Pg 159-16

$$mag_1 - mag_2 = 2.5 \log_{10} (f_2/f_1)$$

also note the magnitude scale is inverted the larger the magnitude the dimmer the object. The brightest star Sirius has a magnitude of $\underline{-1.5}$ The sun, the brightest object in the sky has a magnitude of -27

M_{sum}	-26.85
M_{moon}	-12.5
M_{venus}	-4.4
$M_{jupiter}$	-2.7
M_{sirius}	-1.5
limit of unaided eye	$+6$
photographic limit	$+25$

While the apparent magnitude is very nice it's not truely fundamental as the flux is

$$F = \frac{Lum}{4\pi (dist)^2}$$

dependent on the distance to the star.

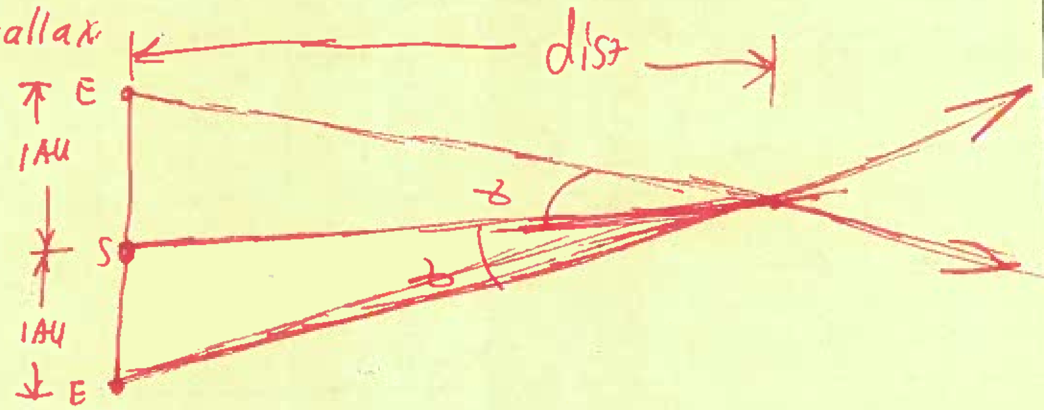
The <u>absolute</u> <u>magnitude</u> is defined as the apparent magnitude a star would have if it were at a distance of 10 parsecs

a note on distance scales.

the ~~astronomical unit~~ 1AU is the average (mean) distance between the earth and the center of the sun.

$$1\,AU \sim 1.5 \times 10^8 \text{ km.}$$

The parsec is based on parallax. as the earth moves around the sun nearby stars shift their positions with respect to very distant background stars due to parallax



(seconds of arc) $\alpha = 2.06265 \times 10^5 \dfrac{1AU}{dist.}$

if $\alpha = 1$ second.

dist $= 1$ parsec

$$1 = 2.06265 \times 10^5 \cdot \frac{1AU}{dist.}$$

$$1 \text{ parsec} = dist = 2.06265 \times 10^5 \text{ AU.}$$

$$= 2.06265 \times 10^5 \cdot 1.5 \times 10^8 \text{ km.}$$

$$= 3.09 \times 10^{13} \text{ km.}$$

~~Stars within 10 parsecs~~

note: a star at 1 parsec distance perpendicular to the plane of the orbit will move by 2 seconds of arc in position on the celestial sphere.

Stars within 10 parsecs can have their distances accurately determined by this method.

light years are defined to be the distance light will travel in one year.

$$dist = v \cdot t = ct$$
$$= 3 \times 10^{5} km/sec \cdot 3.16 \times 10^{7}$$
$$= 9.48 \times 10^{12} \text{ km.}$$

$$\frac{9.48 \times 10^{12}}{1.5 \times 10^{8}} = 6.32 \times 10^{4} \text{ AU.}$$

$$1 \text{ pc} = 1 \text{ parsec} = 3.26 \text{ ly}$$

nearest star centauri proxima is 1.23 pc

Back to magnitudes.

The sun at a distance of 10 parsecs would

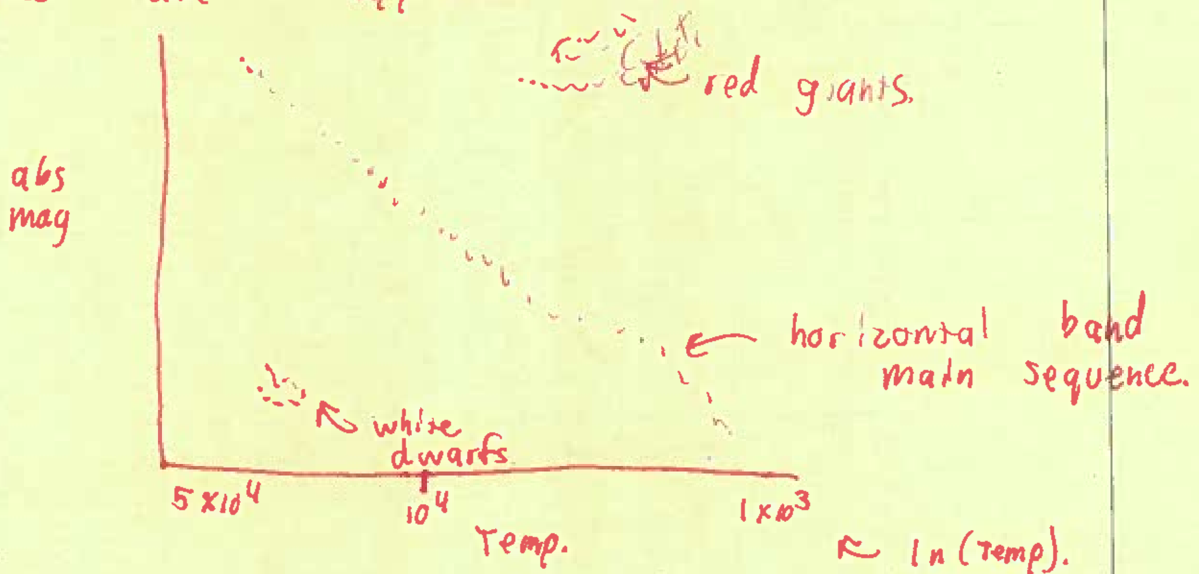
see problem 9.3

$$M = -26.85 - 5 \log_{10}\left(\frac{1 AU}{2.06265 \times 10^{6}}\right)$$
$$= 4.72.$$

ie correction factor of +31.57

The Hertzsprung Russell diagram.

If you take the stars with known luminosities ie known absolute magnitudes and Temperatures and make a scatter plot of absolute magnitude versus temperature strong correlations in these variables are apparent



There are basicly three populated regions
The most heavily populated area is the ~~horizontal~~ diagonal band called the main sequence. In the upper right hand corner are stars called red giants and in the lower left white dwarfs, Using the relation

$$Lum = \sigma T^4 \, 4\pi R^2.$$

and normalizing to the sun.

$$\frac{L}{L_\odot} = \left(\frac{T}{T_\odot}\right)^4 \cdot \left(\frac{R}{R_\odot}\right)^2.$$

the Red giants are typically 5 magnitudes brighter than the sun $(\text{abs Mag}(RG) \sim 0)$ $M_\odot \sim 4.7$

hence. $L/L_\odot \sim 100.$

~~as~~ the temperatures are around 3500° and the sun's ~~is~~ 5800

$$\frac{L}{L_\odot} \cdot \left(\frac{T_\odot}{T}\right)^4 = \left(\frac{R}{R_\odot}\right)^2.$$

$$R/R_\odot = \sqrt{\frac{L}{L_\odot}\left(\frac{T_\odot}{T}\right)^4}$$

$$= \sqrt{100 \cdot (5.8/3.5)^4} = 27.5$$

as $R_\odot = 6.96 \times 10^5$ km

$R = 1.9 \times 10^7$ km

about $1/3$ of the radius of mercury's orbit.

the white dwarfs have temperatures of $\sim 10^4$ °K but absolute magnitudes ~ 5 dimmer (abs mag ~ 10) ie Luminosities of 1/100 of the sun

$$\Rightarrow R_{wd} = R_\odot \sqrt{10^{-2} \cdot (5.8/10)^4} = R_\odot \cdot 3.36 \times 10^{-2}$$

$$= 2.34 \times 10^4 \text{ km.}$$

about twice that of the earth!

Stars.

intro the Sun

The sun is the nearest star to us and sufficiently close that all of its physical parameters can be easily measured. The radius from both it angular size and the Luminosity, temperature, radius relation $L = 4\pi\sigma T^4 R^2$. It's mass from Keplers ~~laws~~ 3rd law as derived by Newton

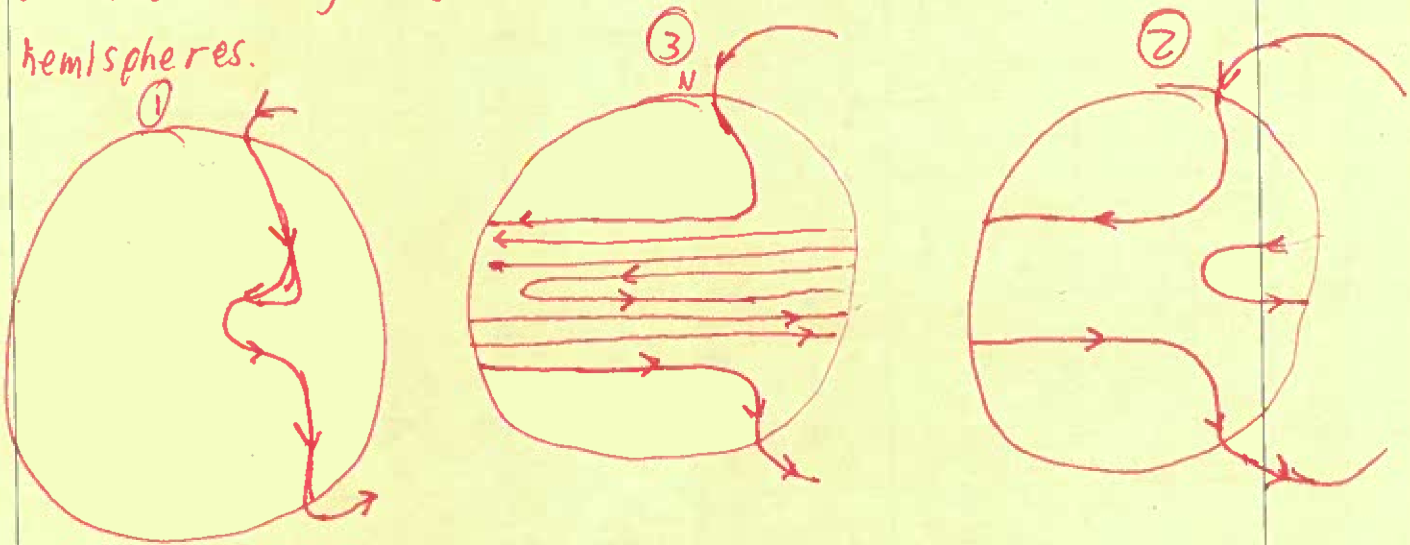
$$P^2 = \frac{4\pi^2}{GM_\odot} a^3$$

applied to the planets. Chemical composition can be found from the emission and absorbtion spectra. Its magnetic field can be found by the "Zeeman" splitting of emission and absorbtion lines (Due to the interaction of the atomic electrons orbital and spin angular momentum with the magnetic field sepparating states with otherwise identical energies)

The sun is close enough that surface features can be resolved. The most immediately seen feature (originally by Gallileo) are the sunspots. Since they were first observed these irregularities to the suns' surface have been monitored ~~regular~~ continuously. Sunspots appear on the sun and move across the surface of the disk in the same direction as the rest of the orbital and rotational motions in the solar system (all such motions are in the same direction except the rotation of venus and Uranus and the orbit of triton around neptune we use this rotational motion of the sunspots to determine the rotational period of the Sun. The sun rotates differentially with a slightly longer period at the poles.

Perhaps the most dramatic property of sunspots is the frequency of their appearance. If you plot the number of sunspots which appear each year vs the year a very apparent eleven year cycle is observed. The frequency of sunspots has been recorded since the time of gallileo. ~~Over time~~ there have been an approximately constant number of sunspots (correcting for the eleven year cycle) over most of this period. There are some noticeable exceptions like the maunder minimum ( ) which correspond to periods of extremely cold weather on earth. The period corresponding to the maunder minimum ~~was~~ is sometimes called a minor ice age as the lowest ~~recorded~~ temperatures ever recorded in northern europe were during that period. This pattern of sunspot inactivity ~~also~~ and low temperatures also applies to the other extended periods of ~~so~~ few sunspots.

In fact the eleven year sunspot cycle is a bit more complicated. Sunspots are known to be areas of intense magnetic activity from their effect on emission lines (zeeman effect). ~~[illegible]~~ They usually have the unusual property that if sunspots in the northern hemisphere have a North magnetic pole to the east and south magnetic pole on the western side of the spot in the southern hemisphere it is exactly reversed with the North magnetic pole toward the ~~east~~ west. Then every eleven years this reverses with the northern hemisphere having north magnetic poles on the western side of sunspots. This reversal corresponds to the reversal of the magnetic poles of the sun which also flip every eleven years. Hence the cycle really takes twenty two years until everything has reproduced. (both sunspots/year and magnetic poles) It must be kept in mind that the sun is made of a very conductive plasma. Therefore there is a strong interaction between the magnetic fields and the materials which make up the sun. A dynamo mechanism where the magnetic field lines get wound around the middle of the sun (equator) due to the differential rotation will produce the reversal of the leading magnetic poles between northern and southern hemispheres.



if these magnetic field lines in state 2 or 3 are pinched up through the surface you get a sunspot,

The stability of stars.

At this point we stop discussing the properties of stars as we observe them and ~~move to theoretically~~ move on to a theoretical discussion of the nature of stars. The two most pressing issues are the large energy output.

$$L_{\odot} \cong 4 \times 10^{26} \text{ watts.}$$

$$4 \times 10^{33} \text{ ergs/sec.}$$

$$1 \text{ watt} = 10^{7} \text{ ergs/sec}$$

and the obvious stability of stars. From geological records on earth we can conclude that the sun has been not only supplying a large power output but been doing so in excess of 4 billion years. The light we recieve from other stars is emitted over a distribution of times corresponding to their distances. We thus on any instant of viewing observe a statistical distribution of stars over the last several hundred million years using large telescopes. (Perhaps more conservatively only ~~over~~ out to 50 million light years can stars be well resolved in distant galaxies). So from these observations the role of a theoretical model of stellar structure is to explain the stability and enormous power outputs.

We will first discuss the stability. The set of equations which define stellar structure are completely defined just by the requirement of stability. These equations predict a rate of energy generation which is constrained by the need to create sufficient thermal energy to balance the force of gravity (the source of the prediction) and the total luminosity of stars. The full set of equations were first derived by Chandrasekhar & Eddington in the 1920's before any knowledge of nuclear reactions ~~was~~ existed.

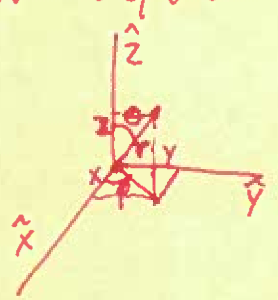
~~Hydrostatic equilibrium:~~ note: the following will not be discussed in class.

~~Consider a ball of gas where throughout the force of gravity is balanced by gas pressure.~~ ~~So~~ first a note on ~~gravity.~~ gravity, electrostatics and Gauss' law. The force of gravity as stated before between two point objects is.

$$F_g = -\frac{G M_1 M_2}{R_{12}^2} \quad \text{or, in vector notation} \quad \vec{F}_{g_{12}} = \frac{G M_1 M_2}{R_{12}^3}(\bar{R}_1 - \bar{R}_2)$$

force of 1 acting on 2

For an extended object it is a bit more complicated. Consider a ball with mass density $\rho(r)$ (i.e. only a function of the distance from the center) radius R total mass M. By spherical symmetry we know that the gravitational force must point toward the origin (the center of the ball) and cannot depend of the polar angles θ or ϕ



$$z = r\cos\theta$$
$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$\sqrt{x^2+y^2+z^2} = r$$

if we had a test mass m_0 at a position $\bar{r}_0$ we could compute the force on it $\vec{F}_g(r_0)$ we will define the gravitational field $(\bar{g})$ by the equation that the

the force on m_0 at r_0 $\vec{F}(r_0)$ can be written as.

$$\vec{F}(r_0) = \bar{g}(r_0) \cdot m_0$$

given the symmetry of the gravitational force in this problem and hence of $\bar{g}$ we can use some tricks in our 2 point object example. we could rewrite the force using the idea of the gravitational field g.

The force on mass 2 $\bar{F}_2 = \frac{G m_1 m_2}{r_{12}^3}(\bar{r}_1 - \bar{r}_2)$

put m_1 at the origin.

$$\bar{F}_2 = -\frac{G m_1 m_2}{r_2^2} = g(r_2) \cdot m_2.$$

$$g(r_2) = \frac{-G m_1}{r_2^2}.$$

this is identical to the form for electrostatic forces for which we can use gausses law.

$$\int_{\text{Surface}} \bar{E} \cdot d\bar{a} = \frac{1}{\varepsilon_0} q_{\text{inside}}.$$

ie the flux through a closed surface equals a constant $\left(\frac{1}{\varepsilon_0}\right)$ times the total charge inside the surface. The electric field $\bar{E}$ from a point charge at the origin is,

$$\bar{E}(r) = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$

$$\Rightarrow \int_{\text{ball of constant radius}} \bar{E} \cdot d\bar{a} = E(r) \int da = E(r) \cdot 4\pi r^2 = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} 4\pi r^2 = \underline{\underline{\frac{q}{\varepsilon_0}}}$$

E is constant as r is constant on the surface.

the function $\frac{1}{r}$ has the following property.

$$\left(\begin{array}{l} \bar{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \\ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \end{array} \right) \qquad \bar{\nabla}\left(\frac{1}{r}\right) = \frac{\hat{r}}{r^2}.$$

$$\bar{\nabla} \cdot \left(\frac{\hat{r}}{r^2}\right) = \nabla^2 \frac{1}{r} = \underline{\underline{4\pi \delta^3(r)}} = \bar{\nabla} \cdot \left(\bar{\nabla} \frac{1}{r}\right)$$

using this coulombs law

$$F_e = \bar{E}(r)\,q_2 = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\,q_2\,\hat{r}$$

$$\text{or} \quad E(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{r}$$

can be written in differential form.

$$\bar{\nabla}\cdot\bar{E} = \frac{1}{4\pi\epsilon_0}\,q.\,4\pi\delta^3(r) = \frac{q}{\epsilon_0}\,\delta^3(r)$$

integrating over ~~all space~~ a volume containing the origin.

(a). $$\int \bar{\nabla}\cdot\bar{E}\,d^3r = \int \frac{q}{\epsilon_0}\,\delta^3(r)\,d^3r = \frac{q}{\epsilon_0}$$

Gauss' law is in fact that the volume integral of a divergence is equal to a surface integral of a flux ie for any vector function $\bar{H}(r)$

$$\int_{\text{volume}} \bar{\nabla}\cdot\bar{H}(r)\,d^3r = \int_{\text{surface}} \bar{H}(r)\cdot d\bar{a}$$

a trivial example (originally shown by Gauss)

$$\Rightarrow \text{eq. (a).} \quad \int_{\text{volume}} \bar{\nabla}\cdot\bar{E}\cdot d^3r = \int_{\text{surface}} \bar{E}\cdot d\bar{a} = \frac{q}{\epsilon_0}$$

for an extended object which could be considered the sum of a large number of point charges

$$\bar{E}(r_0) = \sum \frac{1}{4\pi\epsilon_0}\frac{q_i}{|r_0-r_i|^3}(r_i-r_0) = \sum \bar{E}_i(r_0)$$

these point charges could be charge distributions smeared over a volume.

$$q_i(r_i) = \underbrace{\rho(r_i)}_{\text{charge density}} \cdot \underbrace{\Delta V_i}_{\text{volume element.}}$$

$$E(r_0) = \sum \bar{E}_i(r_0) \to \frac{1}{4\pi\epsilon_0}\int (r_0-r)\frac{\rho(r)\,d^3r}{|r_0-r|^3}$$

$$\cancel{\bar{\nabla}_r\cdot\bar{E}(r_0) = \frac{1}{4\pi\epsilon_0}\rho(r_0)\,4\pi\,\delta^3(r-r_0) = \frac{\rho(r_0)}{\epsilon_0}\,\delta^3(r-r_0)}$$

$$\cancel{= \frac{1}{4\pi\epsilon_0}\int \bar{\nabla}\cdot\left(\frac{\rho(r)}{|r_0-r|^3}(r_0-r)\right)d^3r}$$

$$E(r_0) = \frac{1}{4\pi\varepsilon_0} \int_{\text{~~all~~ space}} \frac{\rho(r)\,(\bar{r}_0 - \bar{r})}{|r_0 - r|^3}\, d^3r.$$

or for any value of r:

$$E(r) = \frac{1}{4\pi\varepsilon_0} \int_{\substack{\text{all}\\ \text{space}}} \frac{\rho(s)}{|\bar{r}-\bar{s}|^3} (\bar{r} - \bar{s})\, d^3s$$

$\bar{s}$ is ~~the~~ the coordinate we integrate over. ~~In order~~ a dummy variable

$$\bar{\nabla}\cdot E(r) = \frac{1}{4\pi\varepsilon_0} \int_{\substack{\text{all}\\ \text{space}}} \nabla\cdot\, \rho(s)\, \frac{\bar{r}-\bar{s}}{|r-s|^3}\, d^3s$$

the derivative is with respect to r (this explains my change of variables)

$$\bar{\nabla}\cdot\bar{E}(r) = \frac{1}{4\pi\varepsilon_0} \int_{\substack{\text{all}\\ \text{space}}} \rho(s)\, \bar{\nabla}\cdot\frac{(\bar{r}-\bar{s})}{|r-s|^3}\, d^3s$$

$$= \frac{1}{4\pi\varepsilon_0} \int_{\substack{\text{all}\\ \text{space}}} \rho(s)\, 4\pi\, \delta^3(\bar{r}-\bar{s})\, d^3s$$

$$\boxed{\bar{\nabla}\cdot E(r) = \frac{1}{\varepsilon_0}\, \rho(r).}$$

$$\Rightarrow \int_{\substack{\text{some}\\ \text{finite}\\ \text{volume}}} \nabla\cdot E(r)\, d^3r = \frac{1}{\varepsilon_0} \int_{\substack{\text{finite}\\ \text{volume}}} \rho(r)\, d^3r$$

$$\boxed{= \int_{\text{surface}} \bar{E}\cdot da = \frac{1}{\varepsilon_0} \int_{\text{volume}} \rho(r)\, d^3r = \frac{1}{\varepsilon_0} Q}$$

the total charge in the volume.

now everything that was done required a $\frac{1}{r^2}$ force or $\frac{1}{r^2}$ field to be more exact. This applies to Gravity as well as electrostatics the above equation becoming.

$$\int_{\text{surface}} \bar{g}\cdot d\bar{a} = 4\pi G \text{ Mass (inside the volume)}.$$

$$= 4\pi G\, M(r) \quad \text{(the interior mass).}$$

So back to our example of spherical symmetry.

$\rho(r)$ mass density.

R radius of the mass.

M total mass.

$$M = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho(r)\, d^3r = \int_{\text{volume}} \rho(r)\, d^3r$$

consider a spherical volume of radius r_0. and apply "Gauss' Law" as derived for gravity.

$$\bar{\nabla} \cdot g = 4\pi G \rho(r).$$

and integrate it over the spherical volume.

$$\int_0^{2\pi}\int_{-0}^{\pi}\int_0^{r_0} \nabla \cdot g \underbrace{(r^2 dr\, d\cos\theta\, d\phi)}_{d^3r} = \int_0^{2\pi}\int_0^{\pi}\int_0^{r_0} 4\pi G \rho(r)\, r^2 dr\, d\cos\theta\, d\phi.$$

$$= 4\pi G \int_{\substack{\text{volume}\\ r_0}} \rho(r)\, d^3r$$

$$= 4\pi G\, M(r_0)$$

mass interior to r_0

$$\int_{\substack{\text{Surface.}\\ \text{at } r_0}} \bar{g} \cdot d\bar{a}$$

due to the symmetry $\bar{g}$ can only depend on r hence is a constant over the surface. pointed inwards producing a minus sign

$$\Rightarrow \int \bar{g} \cdot d\bar{a} = -g(r_0) \cdot 4\pi r_0^2 = 4\pi G\, M(r_0)$$

if $r_0 > R$ $\quad M(r_0) = M.$

if $r_0 < R$ $\quad M(r_0) < M$

in any case.

$$\boxed{g(r_0) = -G \frac{M(r_0)}{r_0^2}\hat{r}_0}$$

Hydrostatic equilibrium.

Consider a ball of gas where the balls gravitational self interaction (ie the atoms attracting each other) is balanced by gas pressure. ~~As was just show~~ Assume that the ball has a spherical symmetry, the density (ρ) depending only on r. We know that the force of gravity on a test mass of mass m is just

$$F = g(r)\, m.$$

if the test mass is just the sum of the mass density in an infintesmally small volume. (dV)

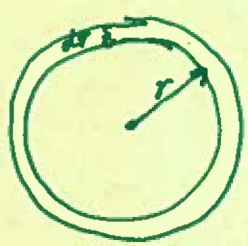
$$m = \rho(r)\, dV.$$

then

interior mass

$$F_g = g(r)\, \rho(r)\, dV = -\frac{G\, M(r)\, \rho(r)}{r^2}\, dV$$

consider say a volume made of a spherical shell with thickness dr.



$$dV = 4\pi r^2\, dr$$

on the inside of the shell there is a pressure $P_0(r)$ pushing outwards. On the outer surface there is a pressure $P_0(r)$ dP pushing inwards. the net force is then

$$F = P_0 \cdot A_{inside} - (P_0 - dP)\, A_{outside}$$

where A is the area of the surface.

$$F_{orce} = \text{Pressure} \cdot \text{Area}.$$

$$F_{press} = + dP \cdot 4\pi R^2 \qquad \text{outward.}$$

if $\quad F_{press} = - F_{grav.}$

then $\quad \sum_i F_i = F_{press} + F_{grav} = 0$

i.e. $\quad 4\pi R^2\, dP - \dfrac{G M(r)\, \rho(r)}{r^2}\, 4\pi r^2\, dr = 0$

$$\Rightarrow \boxed{\frac{dP}{dR} = \frac{G M(r)\, \rho(r)}{r^2}} \quad SS\, I$$

(stellar structure I)

This is the condition of hydrostatic equilibrium.

ie. Pressure balances gravity

This is the second of our equations the first being the definition of the interior mass

$$M(R) = 4\pi \int_0^R \rho(r) r^2 dr \qquad SS\, II$$

$$= \int_0^{2\pi} \int_{-1}^{1} \int_0^R \rho(r)\, r^2 dr\, d\cos\theta\, d\phi$$

To get a feeling for the implication of eq. SS I as a rough approximation we can do these differentials over the entire volume of the sun. ie

$$\frac{\Delta P}{\Delta R} = \frac{P_{center} - P_{surface}}{R} = -\frac{P_{center}}{R}$$

approximate.

$$\rho \frac{GM_r}{r^2}$$

at the value $R/2$ using $\bar{\rho}$ and $M(R/2) \sim M/2$.

$$\rho \frac{GM_r}{r} \sim 2\bar{\rho} \frac{GM}{R^2}.$$

or.

$$\frac{P_{center}}{R} = 2\bar{\rho} \frac{GM}{R^2}$$

for the sun

$$P_{center_\odot} = 2\bar{\rho} \frac{GM_\odot}{R_\odot}$$

$$\bar{\rho} = \frac{M}{\frac{4}{3}\pi R^3} = \frac{3 \times 2 \times 10^{33} gm}{4\pi \ (7 \times 10^{10} cm)^3} = 1.392 \ gm/cm^3$$

$$P_{center} = \frac{2 \times 1.39 \times 6.67 \times 10^{-8} gm^{-1} cm^3/s^2 \cdot 2 \times 10^{33} gm}{7 \times 10^{10} cm}$$

$$= 5.3 \times 10^{15} gm/cm\text{-}s^2 \qquad (dyne/cm^2 ?)$$

to normalize this atmospheric pressure is

$$1.01 \times 10^6 \ gm/cm\text{-}s^2$$

Or

$$P_{center} \sim 5 \times 10^9 \text{ atmospheres.}$$

a plasma (which is what the sun is made out of) is an ideal gas $\Rightarrow$ the equation of state

$$P = \frac{\rho}{m} kT = P = \frac{n}{V} kT$$

holds where ρ is the mass density and m is the average mass/particle as the sun is mainly ionized hydrogen the average mass/particle is

$$\frac{m_{proton} + m_{electron}}{2} \simeq \frac{m_{proton}}{2}$$

$$\Rightarrow \quad P = \frac{2\rho}{m_p} kT$$

at the halfway point if we assume $T(R/2) \simeq \frac{T_{center}}{2}$ and $P(R/2) \sim \frac{P_{center}}{2}$. $\rho(R/2) \sim \bar{\rho}$

$$T_{center} = P_{center} \cdot \frac{m_p}{2\bar{\rho}\, k}$$

$$= \frac{5.3 \times 10^{15}\ gm/cm\,s^2 \cdot 1.67 \times 10^{-24}\ gm}{2 \times 1.39\ gm/cm^3 \cdot 1.38 \times 10^{-16}\ ergs/K^\circ}$$

$$= 2.2 \times 10^{7}\ ^\circ K.$$

So the order of magnitude is 10 million degrees kelvin. Clearly ~~the~~ our assumption that the central region of the sun is very hot and is composed of completely ionized gas is correct. We will find this approximation to be a bit high which is obvious as

$$\rho_{center} > \bar{\rho}$$

$$\boxed{\frac{2\rho_{center}}{m_p} T_{center} = P_{center}}$$

is the ~~condition~~ eq. of state applied to the center.

Energy source / Kelvin Helmholtz contraction.

Consider the energy stored in a ball of gas. There is a total thermal or kinetic energy.

$$E_T = \int_0^R \left(\frac{3}{2} kT\right) \frac{\rho}{m} \, 4\pi r^2 dr \qquad (1)$$

this could be approximated by.

$$\frac{3}{2} \frac{k}{m} \bar{T} \cdot M \sim 5 \times 10^{48} \text{ ergs.}$$

and there is the gravitational potential energy.

$$E_G = \int_0^R V(r) \cdot \rho(r) \cdot 4\pi r^2 dr \qquad (2a)$$

where $V(r)$ is the potential defined by

$$g(r) = -\frac{\partial V(r)}{\partial r} = -\frac{G M(r)}{r^2}$$

$$\Rightarrow V(r) = -\frac{G M(r)}{r}$$

or

$$E_G = \int_0^R \left(-\frac{G M(r)}{r}\right) \rho(r) \, 4\pi r^2 dr \qquad (2b)$$

which can be approximated by.

$$-\left(\overline{\frac{G M(r)}{r}}\right) \cdot M \sim -4 \times 10^{49} \text{ ergs.}$$

it is actually not surprising that these two numbers are nearly equal in magnitude. Consider the equation for hydrostatic equilibrium.

$$\frac{dP}{dr} = -\frac{G M(r)}{r^2} \rho(r).$$

multiply this on both sides by $4\pi r^3$ and integrate from 0 to R.

$$\int_0^R \frac{dP}{dr} \cdot 4\pi r^3 \, dr = \int_0^R -\frac{GM(r)}{r^2} \rho(r) \, 4\pi r^3 dr.$$

Integrate the left hand side by parts

$$-\int_0^R P \, 4\pi (3r^2) \, dr = -\int_0^R \frac{GM(r)}{r} \rho(r) \, 4\pi r^2 dr$$

or

$$\int_0^R 3P \cdot 4\pi r^2 dr = \int_0^R \frac{GM(r)}{r} \rho(r) \, 4\pi r^2 dr.$$

$$P = \frac{\rho}{m} kT.$$

$$\Rightarrow \int_0^R \left(3 \frac{\rho}{m} kT\right) \cdot 4\pi r^2 dr = \int_0^R \frac{GM(r)}{r} \rho(r) \, 4\pi r^2 dr.$$

the left hand side is identically twice eq. 1
the right hand side is minus eq. 2b.

$$\Rightarrow \boxed{2E_T = -E_G}$$

which is the virial theorem in a sense.
What this tells us is that as a ball of gas collapses under its own gravity 50% of the energy goes into thermal heat content of the gas. Where does the rest go? The only place is in fact radiation. $\Rightarrow$ the Luminosity from this Kelvin - Helmholtz contraction.

$$L = \frac{dE}{dt} = -\frac{1}{2} \frac{dE_G}{dt}$$

or $$\boxed{L = \frac{1}{2}\frac{d}{dt}\left(\int_0^R \frac{GM(r)}{r}\,\rho(r)\,4\pi r^2\,dr\right)}$$

Could this mechanism account for the luminosity of the sun? well a typical time scale would be something like.

$$\frac{E_{T\odot}}{L_\odot} = -\frac{1}{2}\frac{E_{G\odot}}{L_\odot}$$

using our approximation for E_T.

$$\frac{E_{T\odot}}{L_\odot} \sim \frac{5\times10^{48}\text{ ergs}}{4\times10^{33}\text{ ergs/sec}} \sim 10^{15}\text{ sec.}$$

$$10^{15}\text{ sec} \sim 3\times10^{7}\text{ years}$$

which is much shorter than the age of the sun so this is <u>not</u> the energy source of the sun.

Thermal equilibrium

Assume in our ball of gas that there is an energy generation density/unit mass $\varepsilon(r)$ which accounts for the observed luminosity. Since we know that stars have stable radii

$$\frac{d}{dt}(E_G) \cong 0$$

ie the gravitational energy doesn't change further if the star is stable.

$$\frac{d}{dt}(E_T) = 0$$

by the definition of stability ~~ie time ch~~ ie there are no changes in time ie time derivatives are zero.

$\Rightarrow$ the only source of the luminosity observed is $\varepsilon(r)$.

$$\Rightarrow \quad L = \int_0^r \varepsilon(r)\, \rho(r)\, 4\pi r^2\, dr.$$

$\varepsilon(r) \cdot \rho(r)$ = energy generation density/unit mass · mass/unit volume.

= energy generation density/unit volume

this thermal stability must exist through out the volume of the star at all radii. Thus if L_r is the energy flux ~~[illegible]~~ through the surface of a sphere of radius r then

$$\frac{dL_r}{dr} = \varepsilon(r)\, \rho(r)\, 4\pi r^2 \qquad \text{SS III}$$

radiation transfer.

We know stars give off light ie heat and energy, from this alone and the zeroth law of thermodynamics we can deduce that stars are not isothermal and are hotter in their cores. In a system which is isothermal there is no net flux of radiation if their were the energy in a region of net outward flux would decrease and there would be a loss of energy. What causes the flow of radiation is the thermal gradient. which could be approximated by

$$T_{center}/R \sim \frac{1.5 \times 10^{7}\ {}^{\circ}K}{7 \times 10^{10} cm} \sim 2 \times 10^{-4}\ {}^{\circ}K/cm.$$

this is a <u>very</u> small number but the sun is extremely large. The rate at which energy can flow in the form of photons depends on how far a photon can go before interacting and changing directions this is the basic idea behind the concept of <u>opacity</u>. χ is the absorbtion /gm of material thus a length dl will have a net opacity (or blocking ability)

$$\chi \rho \, dl \quad \text{where } \rho \text{ is the mass density}$$

this would represent the energy attenuation of a beam of light trying to penetrate this material (In reality χ would depend on frequency but we will ignore this) In fact χ for stellar materials is of order 1.

Consider a cylinder of base area ds with its bottom at radius r, length dl inclined by an angle θ with respect to the radial direction at the bottom. Consider the gains and losses of the radiation field into a solid angle $(\Delta\phi\Delta\cos\theta)$ $d\omega$ per second ~~passing through~~ in this cylinder.

There is a positive increase through the bottom.

$$I(r,\theta)\cdot ds\cdot d\omega.$$

There is a negative decrease passing out the top.

$$-I(r+dr,\ \theta+d\theta)\ ds\ d\omega$$

Since the upper surface is at $r+dr$ and has an angle with respect to the radial direction which is slightly different due to the spherical geometry (ie definition of the radial direction.

There will be a loss due to absorbtion.

$$-I\ d\omega ds\cdot \kappa\rho\, dl$$

and a net gain due to emission by the hot gas but this will be isotropic so the fraction into the solid angle $d\omega$ will be.

$$j\rho\ ds\ dl\ \frac{d\omega}{4\pi}$$

where j represents the energy emitted / gm of material

for thermal equilibrium to hold the sum of these four terms must be zero.

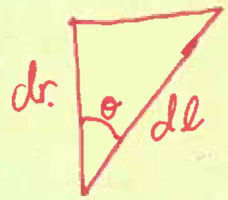
ie.

(a). $$I(r,\theta)\, ds\, d\omega - I(r+dr,\theta+d\theta)\, d\omega\, ds - I\, d\omega\, ds\, \chi\rho\, dl + j\rho\, ds\, dl \frac{d\omega}{4\pi} = 0.$$

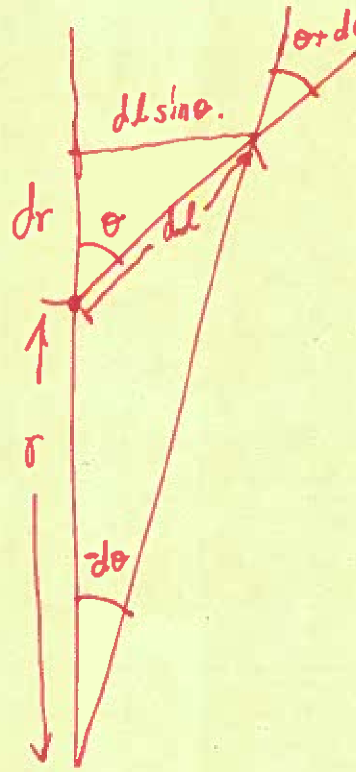
geometrically.

$dl \cos\theta = dr$

$dl = dr/\cos\theta.$



and.



$$d\theta \cong -\frac{dl \sin\theta}{r}.$$

with these (a) can be re written.

$$\left[I(r,\theta) - I(r+dr,\theta+d\theta)\right] ds\, d\omega = -dI\, ds\, d\omega$$

$$= -\left[+\frac{\partial I}{\partial r} dr + \frac{\partial I}{\partial \theta} d\theta\right] ds\, d\omega.$$

$$= -\left[\frac{\partial I}{\partial r} dl\cos\theta + \frac{\partial I}{\partial \theta}\left(\frac{-dl \sin\theta}{r}\right)\right] ds\, d\omega$$

$$= -\left[\frac{\partial I}{\partial r} \cos\theta - \frac{\partial I}{\partial \theta} \frac{\sin\theta}{r}\right] dl\, ds\, d\omega.$$

as the mean density of the sun is $1.3\ gm/cm^3$

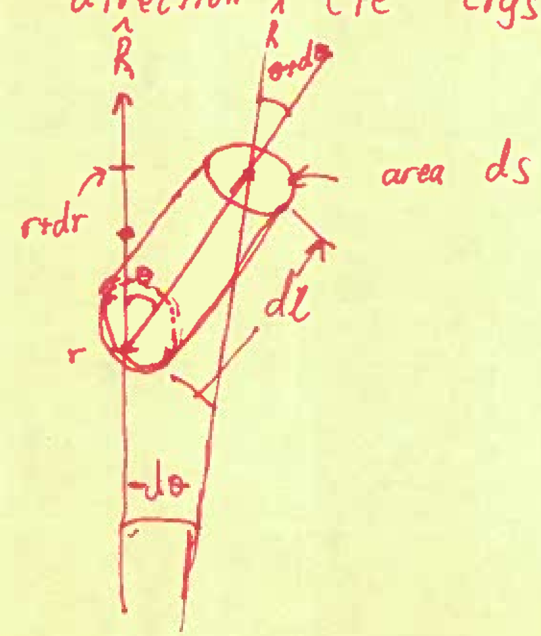
$$\chi\rho \sim 1/cm^3$$

$\Rightarrow$ as little as 1 cm of stellar material will completely block the light. This will lead to it taking an extremely long time ($\sim 10^{12}$ sec $\simeq 3\times10^4$ yrs) for light to propagate from the center to the surface.

~~Given the short length over which photons travel~~

Now over this short lenght which photons travel the temperature gradient is only a few parts in 10^4 which you would think you could neglect but if you did ~~you would be assuming that the~~ you would be neglecting ~~the~~ ~~only~~ precisely the thing which causes the net flow of radiation to the surface.

So. let us define a function $(I(r,\theta))$ which is the Intensity (flux) of the radiation at a given radius in a given direction defined by the angle θ with respect to the radial direction. (ie $ergs/cm^2/sec/$unit solid angle)



⇒ ⓐ. $-dI\, ds\, d\omega - I\chi\rho\, dl\, ds\, d\omega + \frac{j\rho}{4\pi}\, dl\, ds\, d\omega = 0$

$$= -\left[\frac{\partial I}{\partial r}\cos\theta - \frac{\partial I}{\partial \theta}\frac{\sin\theta}{r}\right] dl\, ds\, d\omega - I\chi\rho\, dl\, ds\, d\omega + \frac{j\rho}{4\pi}\, dl\, ds\, d\omega = 0$$

or

$$\boxed{\frac{\partial I}{\partial r}\cos\theta - \frac{\partial I}{\partial \theta}\frac{\sin\theta}{r} + I\chi\rho - \frac{j\rho}{4\pi} = 0} \quad \boxed{SS\_IV}\ ⓐ$$

This must hold every where.

now lets turn this into something connected with thermal gradients. The technique for doing this is rather than try to solve this differential equation to consider "moments" of the function $I(r,\theta)$

The three moments needed are the first three over the angle θ ie.

$d\omega = d\phi\, d\cos\theta$

$\int_{\text{all angles}} d\omega = \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta = 4\pi$

$$\frac{1}{c} M_1 = \frac{1}{c}\int_{\text{all angles}} I\, d\omega \simeq$$ which is exactly Energy density $E(r)$

$$M_2 = \int I \cos\theta\, d\omega =$$ radiation flux $H(r)$ (think about it) ie how much flows in a net outward direction projection onto the radial direction

$$\frac{1}{c} M_3 = \frac{1}{c}\int I \cos^2\theta\, d\omega = P_R(r)$$ radiation pressure

---

So what do we do with these. ?

take SS IV and multiply by $1/c$ and integrate over all angles. It will still be zero

$$\frac{1}{c}\int_{\text{all angles}} \left[\frac{\partial I}{\partial r}\cos\theta - \frac{\partial I}{\partial \theta}\frac{\sin\theta}{r} - I\chi\rho + \frac{j\rho}{4\pi}\right] d\phi\, d\cos\theta = \frac{1}{c}\int (0)\, d\omega = 0.$$

this will be.

$$\frac{1}{c}\int \frac{\partial I}{\partial r}\cos\theta\, d\phi\, d\cos\theta - \frac{1}{c}\int \frac{\partial I}{\partial\theta}\frac{\sin\theta}{r}\, d\phi\, d\cos\theta + \frac{\chi\rho}{c}\int I\, d\phi\, d\cos\theta - \frac{j\rho}{4\pi c}\int d\phi\, d\cos\theta$$

$$= \frac{1}{c}\int \frac{\partial I}{\partial r}\cos\theta\, d\phi\, d\cos\theta - \frac{1}{c}\int \frac{\partial I}{\partial\theta}\frac{\sin\theta}{r}\, d\phi\, d\cos\theta + \chi\rho\cdot E(r) - \frac{j\rho}{c} = 0$$

$$= \frac{1}{c}\frac{\partial}{\partial r}\int I\cos\theta\, d\phi\, d\cos\theta \quad \cancel{\ldots}$$

$$-\frac{1}{c}\int \frac{\partial I}{\partial\theta}\frac{\sin\theta}{r}\, d\phi\, (-\sin\theta\, d\theta) + \chi\rho E(r) - \frac{j\rho}{c} = 0$$

integration is not over r. Can pull out derivative

integrate by parts, pull out $\frac{1}{r}$.

$$= \frac{1}{c}\frac{\partial H}{\partial r} + \frac{1}{cr} - \int I \frac{\partial \sin^2\theta}{\partial\theta}\, d\phi\, d\theta + \chi\rho E(r) - \frac{j\rho}{c} = 0$$

$$= \frac{1}{c}\frac{\partial H}{\partial r} + \frac{1}{cr}\, 2\cdot\int I \cos\theta\sin\theta\, d\phi\, d\theta + \chi\rho E(r) - \frac{j\rho}{c} = 0$$

$$+\int I\cos\theta\, d\phi\, d\cos\theta$$

or: $\frac{1}{c}\frac{\partial H}{\partial r} + \frac{2}{cr}H(r) + \chi\rho E(r) - \frac{j\rho}{c} = 0$

or $$\boxed{\frac{\partial H}{\partial r} + \frac{2H}{r} + c\chi\rho E - j\rho = 0}$$

and taking SS IV multiplying by $\frac{\cos\theta}{c}$ and integrating.

$$\frac{1}{c}\int \frac{\partial I}{\partial r}\cos^2\theta\, d\omega - \frac{1}{c}\int\frac{\partial I}{\partial\theta}\frac{\sin\theta\cos\theta}{r}\, d\omega + \frac{1}{c}\int I\chi\rho\cos\theta\, d\omega - \frac{1}{c}\int\frac{j\rho}{4\pi}\cos\theta\, d\omega = 0$$

$$= \frac{1}{c}\frac{\partial}{\partial r}\int I\cos^2\theta\, d\omega + \frac{1}{rc}\int\frac{\partial I}{\partial\theta}\sin^2\theta\cos\theta\, d\phi\, d\theta + \frac{\chi\rho}{c}\int I\cos\theta\, d\omega - \underbrace{\frac{j\rho}{4\pi c}\int\cos\theta\, d\omega}_{=0} = 0$$

integrate by parts

$$\frac{\partial}{\partial r}\left(P_R\right) + \frac{-1}{rc}\int I\frac{\partial}{\partial\theta}\left(\sin^2\theta\cos\theta\right) d\phi\, d\theta + \frac{\chi\rho}{c}H(r) = 0$$

$$= \frac{\partial}{\partial r}P_R + \frac{-1}{rc}\int I\left(2\sin\theta\cos^2\theta - \sin^3\theta\right) d\theta\, d\phi + \frac{\chi\rho}{c}H(r) = 0$$

$$\left(2\sin\theta\cos^2\theta - \sin^3\theta\right) = \sin\theta\left(2\cos^2\theta - \sin^2\theta\right) = \sin\theta\left(3\cos^2\theta - 1\right)$$

$$\Rightarrow \frac{\partial P_r}{\partial r} + \frac{-1}{rc}\int I\sin\theta\left(3\cos^2\theta - 1\right) d\phi\, d\theta + \frac{\chi\rho}{c}H(r) = 0$$

sign reabsorbed.

$$\frac{\partial P_R}{\partial r} + \frac{1}{r}\frac{1}{c}\int I(3\cos^2\theta - 1)\, d\phi\, d\cos\theta + \frac{\varkappa\rho}{c} H(r) = 0$$

$$\boxed{\frac{\partial P_R}{\partial r} + \frac{1}{r}\left[3P_R - E\right] + \frac{\varkappa\rho}{c} H(r) = 0}$$ rad trans (b)

and copying

$$\boxed{\frac{\partial H}{\partial r} + \frac{2H}{r} + c\varkappa\rho E - j\rho = 0}$$ rad trans (a)

now given that the temperature change is so small over the mean photon path length. ~~might~~ the radiation function $I(r,\theta)$ must be <u>very</u> close to isotropic.

ie $I(r,\theta) \sim I_0(r)$ doesn't depend on θ. at all

So a sensible thing to do is approximate $I(r,\theta)$ as a series in expansion over $\cos\theta$.

$$I(r,\theta) = I_0(r) + I_1(r)\cos\theta + I_2(r)\cdot\cos^2\theta + \ldots\ldots$$

if we ignore terms proportional to $\cos^2\theta$ & higher

then.

$$E(r) = \frac{1}{c}\int I(r,\theta)\, d\omega = \frac{1}{c}\int I_0\, d\omega + \frac{1}{c}\int I_1 \cos\theta\, d\omega$$
$$= \frac{I_0}{c}\int d\omega + \frac{I_1}{c}\underbrace{\int \cos\theta\, d\omega}_{0}$$
$$= \frac{I_0}{c}\cdot 4\pi$$

$$H(r) = \int I(r,\theta)\cos\theta\, d\omega = \int I_0\cos\theta\, d\omega + \int I_1 \cos^2\theta\, d\omega$$
$$= I_0\int \cos\theta\, d\omega + I_1\int \cos^2\theta\, d\omega.$$
$$= I_1\frac{4\pi}{3}$$

$$P(r) = \frac{1}{c}\int I(r,\theta)\cos^2\theta\, d\omega = \frac{1}{c}\int I_0\cos^2\theta\, d\omega + \frac{1}{c}\underbrace{\int I_1\cos^3\theta\, d\omega}_{0}$$
$$= \frac{4\pi I_0}{3c}$$

now a few points.

$$E = \frac{4\pi}{c} I_0 \qquad H = \frac{4\pi}{3} I_1 \qquad \boxed{P = \frac{4\pi}{3c} I_0 = \frac{1}{3} E}$$

and the flux H can be integrated over the spherical surface. and is by definition exactly L_r

$$H \cdot 4\pi r^2 = L_r$$

now consider the emission function "j" according to kirchhoff's law the normal thermal emission process is related to the absorbtion process and the Stephan Boltzman law by $(a = 4\sigma/c)$

$$j_1 = \kappa a c T^4$$

and a second term due to energy generation ε

ie $j = \kappa a c T^4 + \varepsilon$.

using these relations and rad trans ⓐ and ⓑ

$$\cancel{\frac{\partial}{\partial r}\left(\frac{4\pi I}{3}\right)}$$

$$\frac{\partial}{\partial r}\left(\frac{L_r}{4\pi r^2}\right) + \frac{2L_r}{4\pi r^3} + c\kappa\rho E - c\kappa\rho a T^4 - \varepsilon\rho = 0$$

and.

$$\frac{\partial}{\partial r}\left(\frac{E}{3}\right) + \frac{1}{r}\left[3\left(\frac{E}{3}\right) - E\right] + \frac{\kappa\rho}{c}\frac{L_r}{4\pi r^2} = 0$$

the first of these is.

$$\frac{1}{4\pi r^2}\frac{\partial L_R}{\partial r} - \frac{2L_r}{4\pi r^3} + \frac{2L_r}{4\pi r^3} + c\kappa\rho E - c\kappa\rho a T^4 - \varepsilon\rho = 0$$

$$\frac{1}{4\pi r^2}\frac{\partial L_r}{\partial r} - \varepsilon\rho + c\kappa\rho E - c\kappa\rho a T^4 = 0$$

but $\frac{dL_r}{dr} = 4\pi r^2 \varepsilon\rho$ $\Rightarrow$ the sum of the first two terms is zero

$$\Rightarrow c\chi\rho E - c\chi\rho a T^4 = 0$$

$$E = aT^4$$

which is just the result we derived in the discussion of the statistical mechanics of the photon distribution (Plank law)

$$a = \frac{k^4 \pi^2}{c^3 15 \hbar^3}$$

given the small gradient of the temperature

2×10^{-4} °K/cm

a thermal equilibrium result for the energy density is understandable.

the second equation.

$$\frac{\partial}{\partial r}\left(\frac{E}{3}\right) + \frac{1}{r}\left[3\left(\frac{E}{3}\right) - E\right] + \frac{\chi\rho}{c}\frac{L_r}{4\pi r^2} = 0$$

$$\frac{\partial}{\partial r}\frac{aT^4}{3} + \frac{\chi\rho}{c}\frac{L_r}{4\pi r^2} = 0$$

$$\frac{4}{3}aT^3\frac{\partial T}{\partial r} + \frac{\chi\rho}{c}\frac{L_r}{4\pi r^2} = 0$$

or.

$$\boxed{L_r = -4\pi r^2 \frac{4}{3} ac \frac{T^3}{\chi\rho}\frac{dT}{dr}} \qquad \boxed{SS\ \text{IV}}\ a$$

we now have the energy flux in terms of the temperature gradient.

as before we can use the equation relating the Luminosity to the temperature gradient to estimate the total luminosity. In doing this we should realize that the estimate is based on thermodynamics and gravitation and not on the details of the energy production source, an object this large in hydrostatic, thermal and radiative equilibrium made of gas has to have these properties. If we take equation SS IV a and evaluate it at the midpoint of the sun $(r = R/2)$ and use differences for the derivatives.

$$L \sim 4\pi\left(R/2\right)^2 \cdot \frac{4}{3} ac \frac{T^3}{\varkappa\rho} \frac{T_{cent}}{R.}$$

assume $T = 10^7\ {}^0K$ $\qquad T_{cent} = 2.2 \times 10^7\ {}^0K$ as before,

$\varkappa\rho = 1$ $\qquad ac = 2.268 \times 10^{-4}\ erg \cdot cm^{-2}\ s^{-1}\ {}^0K^{-4}$

$R = 7 \times 10^{10}$ cm.

$$L = 4\pi \cdot 7 \times 10^{10} \cdot \frac{4}{3} \times 2.3 \times 10^{-4} \cdot \frac{(10^7)^3}{1} \cdot 2.2 \times 10^7.$$

$$= 1.5 \times 10^{36}.$$

which is $\sim$ 400 times too large but really quite close. In fact the central temperature of the sun is closer to $5 \times 10^6\ {}^0K$ and the temperature at the half way point closer to $2 \times 10^6\ {}^0K$ or a factor of $\frac{1}{4} \cdot \left(\frac{1}{5}\right)^3 \sim \frac{1}{500}$

We can also use this relation to get an approximate idea of the dependence of the Luminosity of a star on the total mass.

realizing that

$$\rho \sim \frac{M}{R^3}$$

and

$$\frac{dP}{dR} = -G\frac{M(r)\,\rho(r)}{r^2}$$

$$\frac{P_{cent}}{R} \approx G\frac{M}{R^2} \cdot \frac{M}{R^3}$$

$$P \sim \frac{M^2}{R^4}$$

and $P = \frac{\rho}{m} kT$ ideal gas.

$$\Rightarrow \frac{mP}{\rho k} = T$$

$$\Rightarrow T \propto \frac{M^2/R^4}{M/R^3} = M/R.$$

$$\Rightarrow L = -4\pi R^2 \frac{4ac}{3} \frac{T^3}{\cancel{\chi}\rho} \frac{T}{R}$$

$$= 4\pi R^2 \frac{4ac}{3}\frac{1}{\chi} \frac{R^3}{M} \cdot \left(\frac{M}{R}\right)^3 \cdot \frac{M}{R} \frac{1}{R}$$

$$= 4\pi \frac{4ac}{3}\frac{1}{\chi} \frac{\cancel{R^5}\; M^4}{M\; \cancel{R^5}}$$

$L \propto M^3$ idependent of radius!

the analysis of binary stars shows a similar result. as can be seen on the graph of

$\frac{15}{2} = 7.5$ bolometric magnitudes vs $\log M/M_\odot$

as $M_{bol} \sim -2.5 \log L/L_\odot + 4.72$

if $L \propto M^3$.

then $M_{bol} \propto 2.5 \log M^3/M_\odot^3$

$\propto 7.5 \log (M/M_\odot)$

or the slope of the scatter plot would be 7.5

16 I. OBSERVATIONAL BASIS

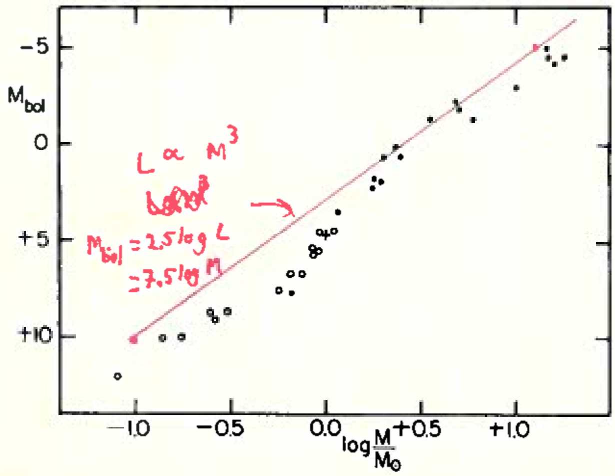


Fig. 2.1. Empirical mass-luminosity relation for main-sequence stars. Data from Tables 2.1 and 2.2. Dots represent spectroscopic binaries, circles visual binaries, and the cross the sun.

Keep in mind that to this point there has been no discussion of nuclear fusion. In fact even the energy generation density $\varepsilon(r)$ has not really played a role. We will only end up calculating what it has to be in order to conserve energy.

In fact knowing about nuclear processes will allow us to calculate the function ε from densities, maxwellian velocity distributions and measured cross sections so we have an over constrained system where two completely independent calculations must give exactly the same result if the model is correct.

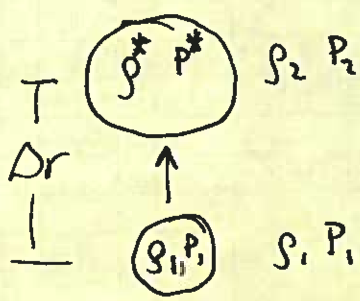
Convective energy transport.

In our discussion of radiative energy transport we did not consider the bulk motion of the material of the plasma in moving energy to the surface. Convection is precisely this mechanism. It will turn out that in the centers of stars convection is not usually relevant but near their surfaces it is. (this is the source of the granular surface appearance).

The mechanism of convection is a motion of the material where an adiabatic expansion of inner material rising to larger radius is unstable ie increases with the length of the disturbance so following Schwarzschild. consider a volume of the plasma with density ρ_1 and pressure P_1 the same as its' surroundings. assume it rises to a different layer where the surroundings have density and pressure ρ_2, P_2 respectively. when the volume rises to this layer it achieves a density and pressure ρ^*, P^* related to the original by the requirements of an adiabatic expansion (no change in heat content of the volume)

$$\rho^{-\gamma} P = \text{const.}$$

$$\gamma = \frac{C_p}{C_v} \quad \text{(ratio of specific heats)} = 5/3$$



$$\Rightarrow \boxed{\rho^* = \rho_1 \left(\frac{P^*}{P}\right)^{1/\gamma}}$$

but when it rises the Pressures always balance

$$\Rightarrow \quad P^* = P_2.$$

ie $\rho^* = \rho_1 \left(\frac{P_2}{P_1}\right)^{1/\gamma}$

now if $\boxed{\rho^* > \rho_2}$ it will be pulled back down by gravity and this determines the stability of the material in the layer against convective flow.

if. $\rho^* = \rho_1 \left(\frac{P_2}{P_1}\right)^{1/\gamma} > \rho_2$ this is stable.

now. $\left.\begin{array}{l} \rho_2 = \rho_1 + \dfrac{d\rho}{dr} \cdot \Delta r \\ \text{and} \quad P_2 = P_1 + \dfrac{dP}{dr} \cdot \Delta r \end{array}\right\}$ taylor expansion.

$$\Rightarrow \quad \rho_1 \left(\frac{P_1 + \frac{dP}{dr} \Delta r}{P_1} \right)^{1/\gamma} \overset{GT}{>} \rho_1 + \frac{d\rho}{dr} \Delta r.$$

$$\Rightarrow \left(1 + \frac{1}{P_1}\frac{dP}{dr}\Delta r\right)^{1/\gamma} > 1 + \frac{1}{\rho_1}\frac{d\rho}{dr}\Delta r.$$

dropping the subscript. and noting for $\Delta r \ll 1$.

$$(1+x)^\alpha = 1 + \alpha x \qquad x \ll 1$$

$$1 + \frac{1}{\gamma}\frac{1}{P}\frac{dP}{dr}\Delta r > 1 + \frac{1}{\rho}\frac{d\rho}{dr}\Delta r$$

$$\boxed{\frac{1}{\gamma}\frac{1}{P}\frac{dP}{dr} > \frac{1}{\rho}\frac{d\rho}{dr}} \qquad P = \frac{\rho}{m} kT$$

$$\rho = \frac{Pm}{kT}.$$

$$\frac{1}{\gamma}\frac{1}{P}\frac{dP}{dr} > \frac{kT}{Pm} \cdot \frac{m}{k}\frac{d}{dr}\left(P/T\right) = \frac{T}{P}\left[\frac{1}{T}\frac{dP}{dr} - \frac{P}{T^2}\frac{dT}{dr}\right]$$

$$\frac{1}{\gamma}\frac{1}{P}\frac{dP}{dr} > \frac{1}{P}\frac{dP}{dr} - \frac{1}{T}\frac{dT}{dr}$$

$$\left(\frac{1}{\gamma}-1\right)\frac{1}{P}\frac{dP}{dr} > -\frac{1}{T}\frac{dT}{dr}$$

or $$\boxed{-\left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} > -\frac{dT}{dr}}$$

as both the temperature and pressure gradients are less than zero and $\frac{1}{\gamma} = \frac{3}{5}$ with the minus signs both sides of the inequality are positive. The left hand side is called the adiabatic temperature gradient as this is the temperature gradient the layers would have if they were related to each other through the adiabatic relation.

$$\rho^{-\gamma} P = \text{Const.}$$

$$\text{or } \rho P^{-1/\gamma} = \text{Const}' \qquad \text{Const}' = (\text{Const})^{-1/\gamma}$$

in other words unless the temperature gradient is less than the adiabatic gradient the layers are unstable against convective flow driven by adiabatic expansions.

Now assume that a layer in a star is unstable ~~against~~ with respect to adiabatic convective flow. Hot material will flow upward expanding and moving heat up similarly cool material will flow down becoming denser than the surroundings and continue to flow inwards. This will result in heat being moved from the center outward. In doing this the temperature gradient will be decreased lowering the convective flow the situation will stabilize when the sum of the convective and radiative energy transports can account for the energy generation ie eq. SS. III

$$\frac{dL_r}{dr} = \epsilon(r)\rho(r)4\pi r^2$$

the thing that drives convection is the temperature excess beyond the adiabatic gradient. ie in a layer of thickness Δr.

$$\Delta T = \left(\left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} - \frac{dT}{dr}\right)\Delta r$$

The excess thermal energy (by thermodynamics) is just the temperature excess times the specific heat at constant pressure times the density.

$$\Delta \text{energy}/\text{cm}^3 = C_p \rho \Delta T$$

if Δr moves at velocity V there is a net flux of energy.

$$H_{con} = V C_p \rho \Delta T$$

In order to calculate V we need to know the effect of gravity on the density excess of the expanding/rising (or ~~contracting~~ sinking) volume of material. So using the stability relation we derived

$$\frac{1}{\gamma}\frac{1}{P}\frac{dP}{dr} > \frac{1}{\rho}\frac{d\rho}{dr} \Rightarrow -\frac{1}{\gamma}\frac{1}{P}\frac{dP}{dr} < -\frac{1}{\rho}\frac{d\rho}{dr}$$

ie unstable $\Rightarrow \frac{1}{\gamma}\frac{1}{P}\frac{dP}{dr} < \frac{1}{\rho}\frac{d\rho}{dr} \Rightarrow$

the excess will be

$$D\rho = \left[-\frac{1}{\gamma}\frac{\rho}{P}\frac{dP}{dr} + \frac{d\rho}{dr}\right]\Delta r = \frac{\rho}{T}\nabla\Delta T\,dr$$

The acceleration of gravity times this density excess will be the net force.

$$F = \Delta\rho \cdot \frac{G\,M(r)}{r^2}$$

this force is zero at the beginning of the displacement (as the density of a layer is uniform see the little picture) So the net result is the average over the displacement this introduces a factor of ½. The work done ~~(F dr)~~ (F dr) the change in kinetic energy

kinetic energy density $\rightarrow \frac{1}{2}\rho v^2 = \langle F\rangle dr = \frac{1}{2}\Delta\rho\,\frac{G M(r)}{r^2}\,\Delta r$

$$\frac{1}{2}\rho v^2 = \frac{\rho}{T}\,\frac{1}{2}\left[\left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} - \frac{dT}{dr}\right]\Delta r\,\frac{G\,M(r)}{r^2}\,\Delta r$$

$$= \frac{1}{2}\frac{\rho}{T}\left[\left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} - \frac{dT}{dr}\right]\frac{G\,M(r)}{r^2}\,\Delta r^2$$

note this is quadratic in both v and Δr and gives the result for both rising and falling material. we can use this to get v and eliminate v from the equation for the convective flux. If there is a distance scale ℓ over which the volumes form, move, and merge into their surroundings then ~~the~~ the average distance they move will be $l/2$.

$$\overline{\Delta r} = l/2.$$

So

$$V^2 = \frac{g}{T}\left[\qquad\right]\frac{GM(r)}{r^2}\,\overline{\Delta r^2}$$

$$V = \left(\frac{1}{T}\left[\left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} - \frac{dT}{dr}\right]\frac{GM(r)}{r^2}\right)^{1/2}\Delta r$$

and

$$H = V C_p \rho\, \Delta T$$

$$= \left(\left[\left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} - \frac{dT}{dr}\right]\left(\frac{GM(r)}{Tr^2}\right)\right)^{1/2} C_p \rho\, \Delta r \cdot \left[\left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} - \frac{dT}{dr}\right]\Delta r$$

$$\boxed{H_{conv} = C_p \rho \left(\frac{GM(r)}{Tr^2}\right)^{1/2}\left[\left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} - \frac{dT}{dr}\right]^{3/2} \cdot \frac{l^2}{4}}$$

~~if we use our estimates of T P $\frac{dP}{dr}$ and $\frac{dt}{dr}$ at the half way point we find that the temperature excess gradient.~~

$$\cancel{\left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{\Delta P}{R} - \frac{\Delta T}{R} \sim \left[\frac{2}{5}\frac{10^7}{5\times10^{15}}\cdot\frac{5\times10^{15}}{7\times10^{10}} - \frac{2\times10^7}{7\times10^{10}}\right]}$$

let's estimate the excess gradient.

$$\left[\left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} - \frac{dT}{dr}\right] \equiv \Delta\nabla T$$

by the following technique assume all the flux of the Sun is convective

$$L_r = 4\pi r^2 H_{conv.}$$

and solve for $\Delta\nabla T$

$$H_{conv.} = c_p \rho \left(\frac{G M(r)}{T r^2}\right)^{1/2} (\Delta\nabla T)^{3/2} \, \ell^2/4.$$

$$\left(\frac{L_r}{4\pi r^2} \, \frac{1}{c_p \rho} \left(\frac{T r^2}{G M(r)}\right)^{1/2} \cdot \frac{4}{\ell^2}\right)^{2/3} = \Delta\nabla T$$

~~and for ℓ~~

$$c_p = R(1+\gamma) = 8.3\times10^7 (4/3)$$

evaluate this at $R/2$ use $\ell \simeq R/10$

$$\left(\frac{L_r}{4\pi (R/2)^2} \cdot \frac{1}{1.4\cdot\frac{8}{3}\cdot 8.3\times10^7} \left[\frac{10^7\cdot(R/2)^2}{G \; M/2}\right]^{1/2} \cdot \frac{4}{(R/10)^2}\right)^{2/3}$$

$$\left(\frac{4\times10^{33}}{\pi\cdot(7\times10^{10})^{\cancel{2}}} \, \frac{\cancel{7}\cdot 10^{10}}{1.4\cdot\frac{8}{3}\cdot 8.3\times10^7} \left[\frac{10^7\cdot\cancel{(7\times10^{10})^2}}{2\cdot 6.6\times10^{-8}\cdot 2\times10^{33}}\right]^{1/2} \frac{400}{(7\times10^{10})^2}\right)^{2/3}$$

$$\left(\frac{1.6\times10^{36}}{3.34\times10^{41}} \quad 1.9\times10^{-10}\right)^{2/3} = \underline{\underline{9.39\times10^{-11}}}$$

agrees with Schwarzschild!

but the temperature gradient

$$\frac{T_c}{R} \sim 2\times10^{-4}$$

$\Rightarrow$ the excess gradient is one part in a million of the thermal gradient so one concludes that for most of the sun convection is irrelevant. In fact detailed calculations bear this out star has exhausted the hydrogen in its core. Only one observed type of star has thus far been identified fairly certainly with a helium-burning phase, namely the top of the red-giant branch in globular clusters. It appears highly likely, however, that helium burning provides the main energy source in some, if not most, of the most advanced phases of stellar evolution that are as yet poorly identified.

We have finished our review of the main physical processes characteristic of the gases in a star. What now is the resulting over-all structure of a star?

Density Distribution

The structure of a star may best be symbolized by its density distribution throughout the interior from the center to the surface. Fig. 28.2 gives the density distribution (normalized to 1 at the center) for four stars: three main sequence stars (of 10, 1.0, and 0.6 solar masses) and a white dwarf. The striking feature of this figure is the similarity in the density distributions of these four very different types of stars. Their internal structure differs little from that of the "standard model" of two

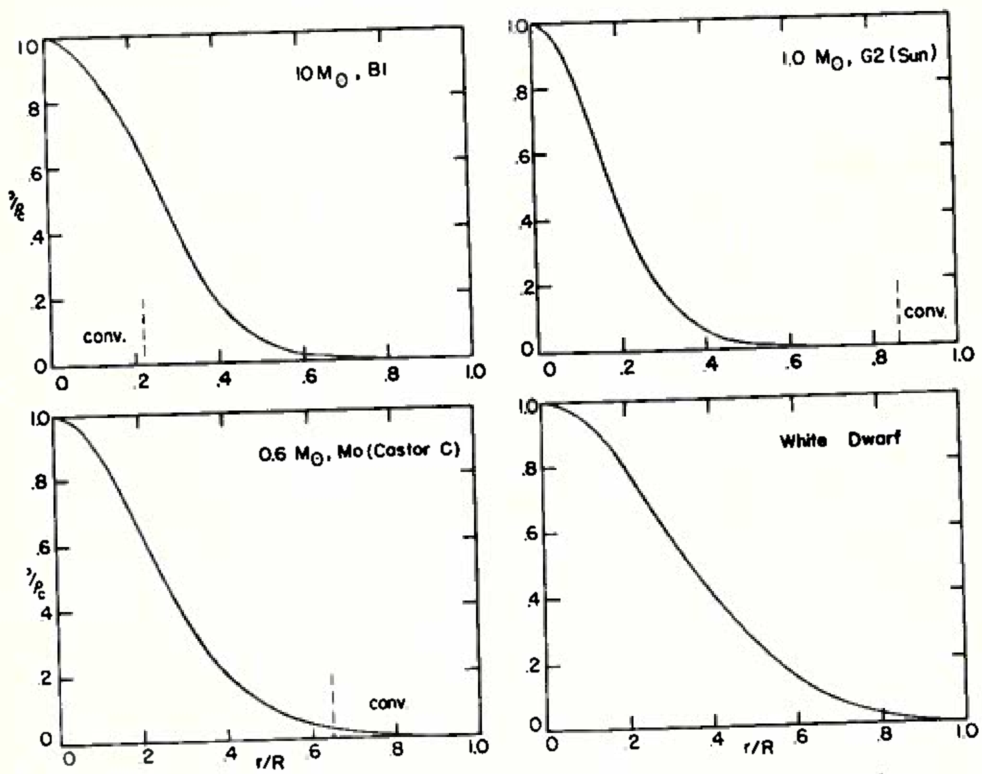


Fig. 28.2. Density distributions in three main-sequence stars and a white dwarf (data from Tables 28.1, 28.3, 28.4, and 28.8).

We have now derived a set of equations which we hope will allow the construction of a theoretical star.

Hydrostatic equilibrium $\frac{dP}{dr} = -\frac{G M(r)}{r^2} \rho(r)$

definition of density $\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$

energy generation $\frac{dL_r}{dr} = 4\pi r^2 \rho(r)\, \epsilon(r)$

radiative energy transport $\frac{dT}{dr} = -\frac{3}{4ac} \frac{\chi(r)\rho(r)}{T^3(r)} \frac{L_r}{4\pi r^2}$

Convective energy transport $\frac{dT}{dr} = \left(1 - \frac{1}{\gamma}\right) \frac{T}{P} \frac{dP}{dr}$

For most stars the last of these equations can be ignored for the central region where the energy is generated and the most interesting phenomena occur (an editorial)

To these equations we need to define 2 things the opacity $\chi(r) \cdot \rho(r)$, and the "equation of state" or the relation between P, ρ, and T. The opacity is dominated by the interactions with incompletely ionized heavy elements (> 2% by mass of the total) and has the form.

$$\chi \sim 1.5 \times 10^{+24} \frac{\rho}{T^{3.5}}$$

thus $\chi\rho \propto \frac{\rho^2}{T^{3.5}}$ (see pg 146 of text.)

for high mass stars where the temperatures are very great the $\frac{1}{T^{3.5}}$ factor suppresses the so called bound free and free-free transitions (interactions with highly but not completely ionized heavy elements (bound-free) and interactions with a free electron in the presence of the nucleus (free-free)). The remaining term is due to the thompson cross section with a free electron and has

$$\sigma_{thomp} = \frac{8\pi e^2}{3c^4 m_e^2}$$

$$\rho \cdot \chi_{thomp} = \sigma_{thomp} \cdot \frac{\rho}{m_p} = \sigma_{thomp} \cdot \rho_{electrons}$$

thus $\chi\rho \propto \rho$ (see pg 146)

So much for opacity, well a quick comment if the opacity were much higher than it is the intense radiation would blow a star to pieces.

The equation of state:

for an ideal gas.

$$P_{gas} = \frac{\rho}{m} kT = \frac{N}{V} kT$$

while this is always true under almost any quasi static change a relation of the form.

$$P_{gas} = C \rho^{\gamma^*}$$

will hold. This is called a polytropic condition. Adiabatic processes fall into this category with.

$$\gamma^* = \gamma = \frac{c_p}{c_v} = \frac{5}{3} \text{ (for a monotonic gas)}$$

a short note on polytropic changes (cosmology)

consider a uniform change of distance scales such that

$$R_1 \to y R_0 \quad \text{where } y \text{ is a constant}$$

$$\Rightarrow \quad Vol_1 \to y^3 \, Volume_0$$

and

$$\rho_1 \to y^{-3} \rho_0$$

by SSI

$$\frac{dP_0}{dr_0} = -\frac{G M(r)}{r_0^2} \rho(r)_0 \quad \text{and} \quad \frac{dP_1}{dr_1} = \frac{-G M(r) \rho(r)_1}{r_1^2}$$

~~the left hand~~

the right hand side goes to.

$$-\frac{G M(r)}{y^2 r_0^2} y^{-3} \rho(r)_0 = -y^{-5} \frac{G M(r) \rho(r)_0}{r_0^2}$$

and

$$\frac{dP_0}{dr_0} \to \frac{dP_0}{y\, dr_1}$$

$$\Rightarrow \frac{dP_0}{dr_1} = y^{+4} \frac{G M(r)}{r_1^2} \rho(r)_1$$

$$\Rightarrow dP_0 \to y^{+4} dP_1$$

$$\Rightarrow \quad P_0 \to y^{+4} P_1$$

$$\frac{-G M(r) \rho_0}{r_0^2} = \frac{-G M(r_1) \, y^3 \rho_1}{y^{-2} r_1^2} = y^5 \left(-\frac{G M(r) \rho}{r_1^2}\right)$$

$$\frac{dP_0}{dr_0} = y \frac{dP_0}{dr_1} = y^5 \frac{G M \rho_1}{r_1^2}$$

$$= y^5 \frac{dP_1}{dr_1}$$

$$dP_0 = y^4 dP_1$$

$$P_0 = y^4 P_1$$

$$P_1 = y^{-4} P_0$$

from

$$P_1 = \frac{\rho_1}{m} k T_1$$

$$T_1 = \frac{m P_1}{\rho_1 k} \Rightarrow \frac{m \, y^{-4} P_0}{k \, y^{-3} \rho_0} = y^{-1} \frac{m P_0}{k \rho_0}$$

$$T_1 \to y^{-1} T_0$$

$$\frac{R_0}{R_1} = y^{-1}$$

$$\frac{P_1}{P_0} = y^{-4} = \left(\frac{R_0}{R_1}\right)^4$$

$$\frac{T_1}{T_0} = y^{-1} = \left(\frac{R_0}{R_1}\right)$$

$$\Rightarrow \frac{\rho_1}{\rho_0} = \left(\frac{R_0}{R_1}\right)^3, \quad \frac{P_1}{P_0} = \left(\frac{R_0}{R_1}\right)^4, \quad \left(\frac{T_1}{T_0}\right) = \left(\frac{R_0}{R_1}\right)$$

or.

$$\frac{P_1}{P_0} = \left(\frac{\rho_1}{\rho_0}\right)^{4/3}$$

$$\Rightarrow P_1 \rho_1^{-4/3} = P_0 \rho_0^{-4/3}.$$

ie a polytropic change with $\gamma = \frac{4}{3}$ $n = \frac{1}{\gamma - 1} = 3$

a scale change is precisely what happens in cosmological models.

$\gamma = \frac{1}{n-1}$

$n-1 = \frac{1}{\gamma}$

$n = \frac{\gamma+1}{\gamma}$

note:

$$\frac{T_1}{T_0} = \frac{R_0}{R_1}$$

$\frac{\frac{4}{3}+1}{4/3}$ $\frac{\frac{7}{3}}{\frac{4}{3}} = 7/4$

ie as the size decreases the temperature goes up. this is known as Lane's theorem

$\gamma = 1 + \frac{1}{n}.$

$\frac{1}{\gamma - 1} = n.$

$P\rho$

$PV^{\gamma} \rightarrow P\rho$

$P^{-\frac{1}{\gamma}}\rho$

such relations are derived from the condition.

$$\frac{da}{dT} = \text{const} =$$

for adiabatic changes $da=0 \Rightarrow$ the constant $=0$

for isothermal changes the constant is infinite.

If one assumes a polytropic gas a rather straight forward solution can by found. This was actually first done by Emden in ~~the~~ ~~early~~ 1907 ~~1900's~~ following work done by Lane in ~~the~~ ~~1870's~~ 1870.

SSI $$\frac{dP}{dr} = -\frac{G M(r)}{r^2} \rho(r)$$

SSII $$\frac{d M(r)}{dr} = 4\pi r^2 \rho(r).$$

rewriting SSI as.

$$\frac{r^2}{\rho} \frac{dP}{dr} = -G M(r).$$

diferentiating.

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G \frac{d M(r)}{dr}$$

using SS II

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G r^2 \rho(r)$$

if $P = C\rho^{\gamma}.$

then $\frac{dP}{dr} = C\gamma\rho^{\gamma-1}\frac{d\rho}{dr}$

and. $$\boxed{\frac{d}{dr}\left(\frac{r^2}{\rho} C\gamma\rho^{\gamma-1}\frac{d\rho}{dr}\right) = -G 4\pi r^2 \rho(r)}$$

and we now have a single second order differential equation in ρ. The solutions were first tabulated by Emden for various values of γ. (In fact what is frequently used in $n = \frac{1}{\gamma-1}$ for $n > 5$ the solution extends to infinite radius and $M \to \infty$).

Real ~~solutions~~ solutions of ~~these~~ these system of stellar structure equations do not rely on the polytropic assumption. The general solutions also have to include the radiation pressure as at the center it ranges from 10% to 95% of the total pressure thus.

$$\boxed{\text{SS} \qquad P = P_g + P_r = \frac{\rho}{m} kT + \frac{a}{3} T^4}$$

equation of state

one usually uses

$$\beta = P_g/P = \frac{P - P_r}{P} = 1 - P_r/P$$

then $$P = (1-\beta)\frac{a}{3} T^4$$

or $$\rho = \frac{m\beta P}{kT}$$

in any event there are now 5 eq. (ignoring convection) There are also some constraints. established by the overall properties of the star.

$$M(R) \simeq \text{total mass.}$$

$$L(R) \simeq 4\pi\sigma R^2 T_{(R)}^4 = \text{total Luminosity.}$$

$$P(R) = 0 \quad \text{(or very small at most.}$$

This is quite different from the case of the polytropic gas sphere. In this case ρ could be found explicitly. From ρ P could be calculated and then T

from $\frac{dJ}{dr}$ the Luminosity could be found without reference to the energy generation equation. The reason that this can be done is because we have introduce another equation ($P = c\rho^{\gamma}$). Without this all 5 equations must be used and integrated numerically to get a solution (see M. Schwarzschild).

So with that discussion we will start on thermonuclear processes and the energy generation function ε.

Even as early as the mid 1920's when Eddington was first studying radiative transfer (see Internal Constitution of Stars A. Eddington) it was clear that given the Einstein relation

$$E = mc^2$$

and the small measured mass differences

$$\Delta M = ZM_p + (A-Z)M_n - M_{nuc}(A, Z)$$

(where Z = number of protons and $A = Z$ + number of neutrons
M_p = mass of the proton M_n = mass of the neutron.)

that this nuclear binding energy was likely to be the source of the stellar Luminosity (see Eddington!) This was 8 years before fission reactions were observed. The problem was originally solved by Bethe in the late 1930s (1938 → 1939)

The chain of reactions is shown diagramatically as follows.

① $P + P \rightarrow D^2 + e^+ + \nu_e$ $+ 1.44$ MeV ($\tau = 1.4 \times 10^{10}$ yrs)
② $D^2 + P \rightarrow He^3_e + \gamma$ $+ 5.49$ MeV ($\tau = 6$ sec)
③ $He^3_e + He^3_e \rightarrow He^4_e + P + P$ $+ 12.85$ MeV ($\tau = 10^6$ yrs)

The first two reactions have to occur twice to make two He^3 nuclei and on average the neutrino (ν) takes away .26 MeV, as the neutrino does not interact this energy is lost to the stellar interior. Thus for each helium nucleus made the following energy is released.

$$2*(1.44 - .26) + 2* 5.49 + 12.85 = 26.2 \text{ MeV}.$$
$$= 4.2 \times 10^{-5} \text{ erg}$$

In extremely heavy stars with much higher temperatures another set of reactions takes place known as the CNO cycle. This is essentially a catylitic chain with the heavier carbon nucleus acting as a nuclear catylist.

$$C^{12} + P \rightarrow N^{13} + \gamma \qquad 1.95 \text{ MeV} \quad (1.3 \times 10^7 \text{ yrs})$$
$$N^{13} \rightarrow C^{13} + e^+ + \nu \qquad 2.2 \text{ MeV} \quad (7 \text{ min})$$
$$C^{13} + P \rightarrow N^{14} + \gamma \qquad 7.54 \text{ MeV} \quad (2.7 \times 10^6 \text{ yrs})$$
$$N^{14} + P \rightarrow O^{15} + \gamma \qquad 7.35 \text{ MeV} \quad (3.2 \times 10^8 \text{ yrs})$$
$$O^{15} \rightarrow N^{15} + e^+ + \nu \qquad 2.71 \text{ MeV} \quad (82 \text{ sec})$$
$$N^{15} + P \rightarrow C^{12} + He^4 \qquad 4.96 \text{ MeV} \quad (1.1 \times 10^5 \text{ yrs})$$

in the N^{13} decay the neutrino removes .76 MeV and in the O^{15} decay the neutrino removes .98 MeV.
$\Rightarrow$ the net energy released is.

$$1.95 + (2.2 - .76) + 7.54 + 7.35 + (2.71 - .98) + 4.96 \approx 25.0$$
$$= 4 \times 10^{-5} \text{ ergs}.$$

the difference being due to the different neutrino energies.

The electrostatic repulsion in the CNO chain between the protons (hydrogen Nuclei) and the heavy Nuclei is typically 1.5 → 8 times that involved in the PP chain. But the interaction probabilities are much greater. The limiting part is that in low mass stars the PP chain dominates (in spite of the poor probability of reaction 1 $PP \to D + e^+ + \gamma$ $\tau = 1.4 \times 10^{10}$ yrs) is that the temperatures are lower and there just isn't as much carbon in the stellar core.

To get a feeling for what's going on consider a ~~few~~ few numbers.

assume $T = 10^7\ {}^\circ K$.

$$\langle E_{therm} \rangle = \frac{3}{2} kT = \frac{3}{2} \cdot 1.38 \times 10^{-16} \cdot 10^7 = 2.07 \times 10^{-9}\ \text{ergs}$$

$$\sim 1.3 \times 10^3\ \text{eV}$$

$$1.3\ \text{KeV}$$

The range of the strong nuclear force which will bind a proton and neutron is of the order of 10^{-13} cm. and is the compton wavelength of the pion. $m_\pi = 140$ MeV.

$$\lambda_c = \frac{h}{mc} = \cancel{\frac{6.63 \times 10^{-27}}{(1.4 \times 10^8 \cdot 1.6 \times 10^{-12}) \cdot 3 \times 10^{10}}} \qquad \cancel{1.6 \times 10^{-12}\ \text{erg/eV}}$$

$$m_\pi = 145\ \text{MeV}/c^2 ! \Rightarrow$$

$$\lambda_\pi = \frac{h}{m_\pi c} = \frac{h}{\left(\frac{1.45 \times 10^8 \cdot 1.6 \times 10^{-12}}{c^2}\right) \cdot c}$$

$$= \frac{hc}{1.45 \times 10^8 \cdot 1.6 \times 10^{-12}} = \frac{6.63 \times 10^{-27} \cdot 3 \times 10^{10}}{1.45 \times 10^8 \cdot 1.6 \times 10^{-12}}$$

$$= 8.57 \times 10^{-13}\ \text{cm} ! \sim 10^{-12}\ \text{cm}$$

The electro static energy between two protons at this distance $(10^{-13} cm)$ is.

$$PE = \underset{\substack{\uparrow \\ \cancel{joul} \\ ergs/joule}}{10^{+7}} \times \frac{9\times10^{9} \cdot (1.6\times10^{-19})^2}{10^{-15} m.} = 2.3\times10^{-6} \text{ ergs.}$$

~ 1.44 MeV.

1440 KeV.

ie $\boxed{PE(10^{-13} cm) \sim 10^3 (E_{therm})}$

So this looks a little tough. but quantum mechanics will come to the rescue. It is possible in quantum mechanics for a wave function to "tunnel" through a barrier. The probability for this being proportional to.

$$P_{rob}(v) \propto \exp -2\int_{r_p}^{r_c} \left[\frac{2M\left(\frac{e^2}{r} - \frac{mv^2}{2}\right)}{\hbar}\right]^{1/2} dr. \qquad \text{(WKB approx.)}$$

where r_p is the proton radius or pion compton wavelength.

and r_c is defined by.

$$\tfrac{1}{2}mv^2 = \frac{e^2}{r_c}$$

$$r_c = \frac{e^2}{E_{th.}} \approx \frac{(4.8\times10^{-10} esu)^2}{2\times10^{-9} ergs} \sim 1.2\times10^{-10} cm.$$

The overall rate will be a convolution ~~over the~~ of the Probabilities over the relative velocity distribution in the collision or more precisely

$$r_{12} = \int_0^\infty N_1 \cdot N_2 \cdot V \cdot q(v) \cdot P_{rob}(v) \cdot P_{12} \cdot D(T,v)\, dv.$$

where these factors are as follows.

N_1 is the number density of particle type 1 $\frac{\rho}{m_1} X$
N_2 is the number density of particle type 2. $\frac{\rho}{m_2} Y$

X, Y fraction by mas[s]

V is the relative velocity

$q(v)$ is the reaction cross section dependence on velocity

$$q(v) \quad \alpha \quad \frac{1}{v^2} \qquad \frac{K_{12}}{V_2^2}$$

Prob(v) is the tunnel effect probability according to schwarzschild is

$$P_{rob}(v) \quad \alpha \quad exp\left[-\frac{4\pi^2 Z_1 Z_2 e^2}{h} \frac{1}{v}\right)$$

Z_1e, Z_2e are the respective charges.

P_{12} is an overall constant of the reaction

$D(T, v)$ is the maxwellian velocity distribution for <u>relative</u> velocities.

remember.
$$P_{max} = \frac{}{(\sqrt{2\pi KT})^3} e^{-mv^2/2KT} 4\pi v^2 dv$$

$$4\pi \frac{e^{-mv^2/2KT}}{(\sqrt{2\pi KT})^3} v^2 dv.$$

or $$\frac{v^2}{T^{3/2}} e^{-\frac{mv^2}{2KT}} dv.$$

for relative velocities the distribution only gets changed by using the reduced mass. μ.

$$\mu = \frac{M_1 M_2}{M_1 + M_2} = m_p \cdot \frac{A_1 A_2}{A_1 + A_2}$$

evaluation of WKB Integral.

$$\frac{z_1 z_2 e^2}{r_c} = \frac{mv^2}{2}$$

$$\frac{1}{r_c} = \frac{mv^2}{2 z_1 z_2 e^2}$$

$$\text{Prob} \propto \exp\left[-2 \int_{r_p}^{r_c} \left[\frac{2M\sqrt{V-E}}{\hbar}\right]^{1/2} dr\right.$$

$$V = \frac{z_1 z_2 e^2}{r} \quad (\text{cgs.}$$

$$E = \tfrac{1}{2} M v^2$$

$$\text{Prob} \propto \exp\left[-\frac{(8M)^{1/2}}{\hbar} \int_{r_p}^{r_c} \sqrt{\frac{z_1 z_2 e^2}{r} - \frac{mv^2}{2}}\, dr\right.$$

$$= \exp\left[-\frac{(8M)^{1/2}}{\hbar} \int_{r_p}^{r_c} \left(\sqrt{\frac{1}{r} - \frac{mv^2}{2 z_1 z_2 e^2}}\right) \cdot \left(z_1 z_2 e^2\right)^{1/2} dr\right.$$

define $\frac{mv^2}{2 z_1 z_2 e^2} = a.$

$$x = \frac{1}{r}$$

$$\frac{dx}{dr} = -\frac{1}{r^2} \Rightarrow dr = -r^2 dx = -\frac{dx}{x^2}.$$

$$\text{Prob} \propto \exp\left[-\frac{(8M z_1 z_2 e^2)^{1/2}}{\hbar} \int_{1/r_c}^{r_p} \frac{\sqrt{x-a}}{x^2}\, dx\right.$$

See Gradshteyn & Ryzhick Pg 73 2.225/2.224.

$$\int \frac{\sqrt{x-a}}{x^2} dx = \frac{\sqrt{x-a}}{x} + \frac{1}{2} \frac{2}{\sqrt{a}} \operatorname{arctg}\left(\frac{\sqrt{x-a}}{\sqrt{a}}\right) \Bigg|_{\text{lim.}}$$

$$\text{Prob} \propto \exp\left[-\frac{(8M z_1 z_2 e^2)^{1/2}}{\hbar}\left[r_p \sqrt{\frac{1}{r_p} - a} + \left(\frac{2 z_1 z_2 e^2}{mv^2}\right)^{1/2} \operatorname{arctg}\left[\frac{\sqrt{\frac{1}{r_p} - a}}{\sqrt{a}}\right]\right)\right.$$

$$\boxed{x - a = 0 \quad x = \frac{1}{r_c}}$$

$$r_p \to 0$$

$$\operatorname{arctg}(\infty) = \frac{\pi}{2}$$

$$r_p \sqrt{\tfrac{1}{r_p} - a} \to 0$$

$$\Rightarrow \text{Prob} \propto \exp\left[-\frac{(8M z_1 z_2 e^2)^{1/2}}{\hbar} \cdot \left(\frac{2 z_1 z_2 e^2}{m v^2}\right)^{1/2} \frac{\pi}{2}\right]$$

$$\hbar = \frac{h}{2\pi}$$

$$\boxed{= \exp\left(-\frac{4\pi^2}{h} \frac{z_1 z_2 e^2}{v}\right)}$$

so: the overall probability is proportional to:

$$r_{12} \propto \int \frac{\rho^2}{m_1 m_2} XY \; v \frac{K_{12}}{v^2} \exp\left[-\frac{4\pi^2 z_1 z_2 e^2}{h} \frac{1}{v}\right] P_{12} \cdot \frac{v^2}{T^{3/2}} \exp\left[-\frac{\mu v^2}{2KT}\right] dv$$

$$\propto \int \rho^2 \frac{v}{T^{3/2}} \exp\left[-\left(\frac{\mu v^2}{2KT} + \frac{4\pi^2 e^2 z_1 z_2}{h} \frac{1}{v}\right)\right] dv.$$

$$\boxed{\propto \frac{\rho^2 XY}{m_1 m_2 T^{3/2}} \int v \, \exp\left[-\left(\frac{\mu v^2}{2KT} + \frac{4\pi^2 e^2 z_1 z_2}{h} \frac{1}{v}\right)\right] dv.}$$

$$F(v) = v \exp\left[-\left(\frac{\mu v^2}{2KT} + \frac{4\pi^2 e^2 z_1 z_2}{h v}\right)\right]$$

$\frac{dF(v)}{dv} = 0$ at peak find peak.

according to Schwarzschild the integral is dominated by the peak and is approximately

$$r = C_{12}' \frac{\rho^2 X Y}{T^{2/3}} \exp\left[-3\left(2\pi^4 e^4 \frac{\mu}{h^2 K} \frac{z_1^2 z_2^2}{T}\right)^{1/3}\right]$$

I suspect this is a typo and should be $3/2$

I'll check this and find out what K is. (Boltzmans const!)

the times given for the various reactions can be defined as.

$$\tau = \frac{N_1}{r} = \frac{\rho X}{m A_1} \cdot \frac{1}{r}$$

the assumed central values wer.

$$T_c = 1.3 \times 10^7 \,{}^\circ K \quad \rho X(\text{hyd}) = 100 \quad \rho X(He^3) = .01$$

The relative abundances of the various nuclear states will reach an equilibrium when the rates of all the reactions are equal. If one ~~density~~ fractional density is too high it will quickly react