away until the rates are balanced.

now as all the rates of the reactions are equal when equilibrium is reached the rate of energy production / unit volume is just.

$$\varepsilon\rho = rE$$

where E is ~ 25 MeV ie the energy produced in the entire chain.

for the proton proton chain. the easiest choise is reaction ① $pp \rightarrow De^+\nu$ as it depends only on the hydroden density. for the CNO cycle Schwarzschild chose the N^{14} reaction because N^{14} is the most abundant element of the three (he claims).

$$\Rightarrow \varepsilon_{12} = \frac{rE}{\rho} = E\, C_{12}\, \rho\, X_1 X_2 \frac{1}{T^{2/3}} \exp\left[-3\left(\frac{2\pi^2 e^4 z_1^2 z_2^2 \mu}{h^2 kT}\right)^{1/3}\right].$$

So for the choices of T_{cen} & X etc.

$$\varepsilon_{pp} = 2.5\times10^6 \rho X^2 \left(\frac{10^6}{T}\right)^{2/3} \cdot \exp\left[-33.8\left(\frac{10^6}{T}\right)^{1/3}\right]$$

$$\varepsilon_{CNO} = 9.5\times10^{28} \rho X X_N \cdot \left(\frac{10^6}{T}\right)^{2/3} \exp\left[-152.3\left(\frac{10^6}{T}\right)^{1/3}\right]$$

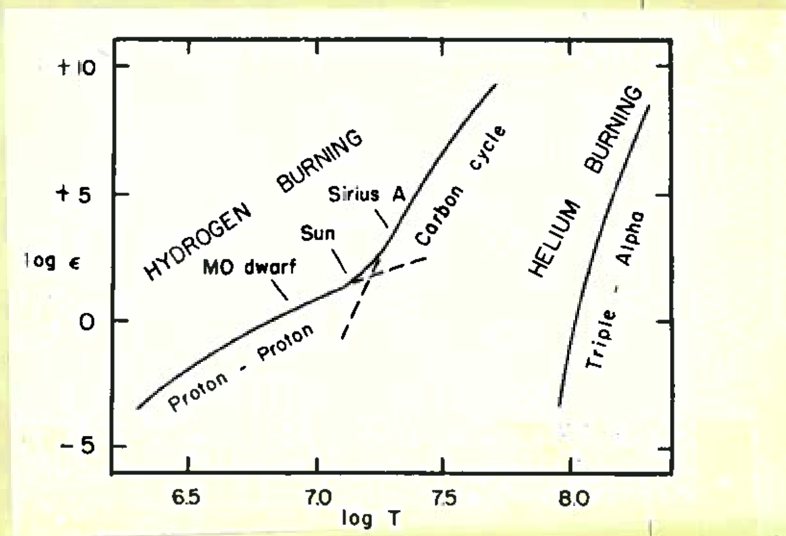


Fig. 10.1. Nuclear energy generation as a function of temperature (with $\rho X^2 = 100$ and $X_{CN} = 0.005X$ for the proton-proton reaction and the carbon cycle, but $\rho^2 Y^3 = 10^8$ for the triple-alpha process).

tripple α process.

To get past the gap at $A = 8$ $(Be^8 \; B^8)$ is a bit tricky and it was realized in the 1950s (Salpeter) that the only way this could happen is through a nearly 3 body reaction. The problem is that

$$Be^8 \rightarrow 2\, He^4 \quad \text{in} \quad 2.6 \times 10^{-16} \text{ sec}$$

$$\text{ie} \quad N(t) = N_0 \, e^{-t/2.6\times10^{-16}}$$

$$\frac{N(t)}{N_0} = 10^{-9} \Rightarrow \frac{t}{2.6\times10^{-16}} = 20.7$$

if the densities are sufficiently high and the Temperatures sufficiently high the reaction rate can become important. The continuous build up and break down of Be^8 can be treated the same way as excited atomic states ie by the so called Saha equation.

$$N(Be^8) = N^2(He^4) \cdot \frac{1}{T^{3/2}} \, e^{-DE/kT}$$

where DE is the energy released in the Be^8 decay

$$DE = 95 \text{ KeV} = 1.04 \times 10^{-7} \text{ ergs.}$$

on calculating this.

$$\frac{N(Be)}{N(he^4)} \sim 10^{-10} \quad \text{but finite.}$$

even so it was realized that given normal cross sections that there would be no way that heavy elements could be built up in stars (ie we're now trying to predict <u>all</u> nuclear abundances) that is the

A = 8 state could not be jumped unless for some reason the cross section for

$$Be^8 + He^4 \rightarrow C^{12} + \gamma$$

became anomoulosly large in the region of helium kinetic energies < 500 keV. Hoyle therefore concluded that such a resonance must exist. Some years later it was discovered to be true with a peak value at

$$E_{kin} = 310 \text{ keV.} = 5 \times 10^{-7} \text{ ergs.}$$

$$T = \frac{E_{kin}}{k} = \frac{5 \times 10^{-7}}{1.38 \times 10^{-16}} = 3.6 \times 10^{9} \,°K.$$

At this point we have established an overall understanding of the equations of stellar structure, the source of stellar energy (thermo nuclear reactions), and the methods of solution of the stellar structure equations (exact calculation for polytropic gas spheres, and numerical integration for a more exact solution using nuclear reaction rates to get $\epsilon(r)$). Having accomplished a detailed model for stable stars (which in fact works quite well though we haven't really shown this) we should determine what remaining ~~[illegible]~~ questions our study of stellar properties should entail. As a preliminary list we might include

limits.

$2\times10^{3}\,K < T_{star} < 4\times10^{4}$

$.1M_{\odot} < M_{star} < 30$

1. The limits of stellar distributions, ie the Hertzsprung-Russell main sequence.
2. How stars form?
3. Do stars change after very long periods and if so how?

The answer to the first question is intrinsicly tied to the answers to the other two but some general relations can be found defining the limits of the main sequence.

So a few relations.

Our radiative equilibrium equation.

$$L_r = -4\pi r^2 \frac{4ac}{3} \frac{T^3}{\varkappa\rho} \frac{dT}{dR}$$

Computed as differences. is.

$$L = 4\pi R^2 \frac{4ac}{3} \frac{T^4}{\varkappa\rho R} = \frac{16ac}{3} \frac{R\, T^4}{\varkappa\rho}$$

with the association of $\frac{1}{\varkappa\rho}$ being the mean distance a photon will travel. This gives the result asked for in question 5.12 (up to constants). In a previous lecture we outlined that the opacity $\varkappa\rho$ was

$\varkappa\rho \propto \rho^2/T^{3.5}$ for low mass stars
bound free translations in heavy atoms.

$\varkappa\rho \propto \rho$ for high mass high temp stars.
Thompson cross section.

$\Rightarrow$ for low mass stars

$$L \sim \frac{R T^4}{\rho^2/T^{3.5}}$$

for high mass stars,

$$L \sim \frac{R T^4}{\rho}$$

Clearly $\rho \sim M/R^3$. $\left(\bar{\rho} = \frac{M}{\frac{4\pi}{3} R^3} \right)$

and from

$$\frac{dP}{dr} = -\frac{G M(r)}{r^2}\rho$$

as differences.

$$\frac{P_{cent}}{R} \sim \frac{GM}{R^2}\rho \sim \frac{GM^2}{R^5}$$

$$P \sim M^2/R^4.$$

now. for low mass stars we find the central temperatures are not stagerlingly high so while the Pressure is always

$$P \simeq \frac{\rho}{m} kT + \frac{a}{3}T^4$$

the second term can be ignored.

In high mass stars the second term dominates

In fact it has been believed (but I've seen no proof) that the upper mass limit is due to the central pressure being dominated by radiation pressure. If one defines

$$\beta = \frac{P_g}{P_{total}} = \frac{P_g}{P_g + P_r}$$

$$\beta (P_g + P_r) = P_g.$$

$$\beta P_r = P_g (1-\beta)$$

$$\beta P_r = \beta P_{tot} (1-\beta)$$

$$P_r = P_{tot} (1-\beta) \quad \text{etc etc etc.}$$

ie β is the fraction of gas pressure

$1-\beta$ is the fraction of radiation Pressure

for valid stars (ie stars that are observed to exist).

$$.05 < \beta < .99$$

But back to what we were doing.

assume. for low mass.

$$P = \frac{\rho}{m} kT \sim M^2/R^4$$

$$\frac{M}{R^3} . T \sim M^2/R^4$$

$$T \sim M/R$$

for high mass.

$$P \sim T^4 \sim M^2/R^4$$

for low mass.

$$L \sim \frac{RT^4}{\rho^2/T^{3.5}} = \frac{R\ T^{7.5}}{\rho^2} \sim \frac{R\ (M/R)^{7.5}}{M^2/R^6} \sim \underline{M^{5.5}/\sqrt{R}}$$

for high mass (ie rad dominated).

$$L \sim \frac{RT^4}{\rho} \sim \frac{R\ M^2/R^4}{M/R^3} \sim M.$$

and for stars which are high mass but not enough to have pressures dominated by radiation ($2M_\odot < M < 10 M_\odot$?) but have opacities dominated by thompson scattering.

$$P \simeq \frac{\rho}{m} kT \Rightarrow T \sim M/R.$$

$$L \sim \frac{RT^4}{\rho} \sim \frac{R \; M^4/R^4}{M/R^3} \sim M^3 \quad \text{(as shown before)}$$

So as the mass changes we see the mass luminosity relation change from.

$$L \sim M^{5.5}/\sqrt{R} \qquad 2M_\odot > M$$

$$L \sim M^3 \qquad 10M_\odot > M > 2M_\odot$$

$$L \sim M \qquad M > 10 M_\odot$$

if we couple this with the relation that

$$L = 4\pi R^2 \sigma T_{eff}^4$$

T_{eff} is the effective surface temperature.

we could get the shape of the main sequence if we can eliminate R.

now we saw that the energy generation functions were.

$$\varepsilon_{pp} \sim \frac{\rho}{T_6^{2/3}} \exp\left[-33.8 \; T_6^{-1/3}\right]$$

$$\varepsilon_{CNO} \sim \frac{\rho}{T_6^{2/3}} \exp\left[-152.3 \; T_6^{-1/3}\right]$$

where $T_6 \equiv \frac{T}{10^6}$

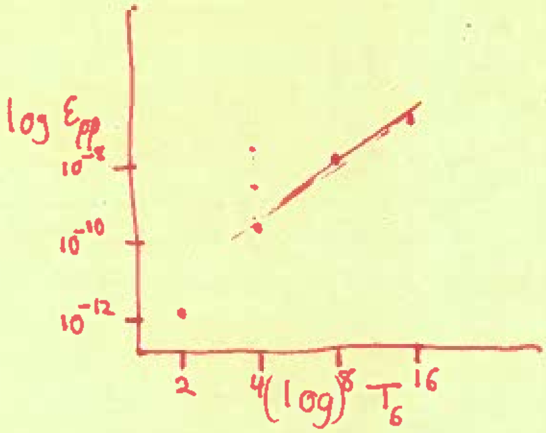
$T_6^{-1/3} = \left(10^6/T\right)^{1/3}$

these functions can be approximated in the regions where they dominate as.

$\varepsilon_{pp} \sim \rho T_6^4 \qquad T \sim 10^7 \,^\circ K \qquad$ maybe $\rho T^{4.5}$??

$\varepsilon_{CNO} \sim \rho T_6^{16} \qquad T \sim 2.5 \times 10^7 \,^\circ K.$

This can be seen by plotting $T^{-2/3} e^{-33.8\, T^{-1/3}}$



T_6	$T^{-2/3} e^{-33.8\, T^{-1/3}}$
2	1.4×10^{-12}
4	2.2×10^{10}
8	1.1×10^{-8}
16	5.9×10^{-7}

$\nu \quad \dfrac{3}{.6} \sim 5$

$T^{\nu} \sim 5$ by me.

the slope yielding the local power.

TABLE 10.1

Constants for interpolation equations (10.14) and (10.15) for various temperature ranges. (Bosman-Crespin, Fowler, Humblet, *Bull. Soc. Royale Sciences Liège*, No. 9–10, 327, 1954.)

ε_{pp}			ε_{cc}		
$T/10^6$	$\log \varepsilon_1$	ν	$T/10^6$	$\log \varepsilon_1$	ν
4–6	−6.84	6	12–16	−22.2	20
6–10	−6.04	5	16–24	−19.8	18
9–13	−5.56	4.5	21–31	−17.1	16
11–17	−5.02	4	24–36	−15.6	15
16–24	−4.40	3.5	36–50	−12.5	13

this evaluation is ~~done~~ taken from schwarzschild

Shu claims that these representations of ϵ lead to the radius of the star being proportional to the mass to a power.

$$R \propto M^{\alpha}$$

where $\alpha \sim 1$ for low mass stars.
and $\alpha \sim .6$ for high mass stars.

the calculation should go as,

$$L \sim \int \epsilon \rho \, r^2 dr \sim \overline{\epsilon \rho} \cdot Vol \sim \epsilon M.$$

$$\epsilon \propto \rho T^{\gamma}$$

$$\Rightarrow L \propto \rho M T^{\gamma}$$

$$\propto \frac{M^2}{R^3} T^{\gamma}$$

for high mass stars $\gamma \sim 16$ and $T^4 \sim M^2/R^4$

$$\Rightarrow L \sim \frac{M^2}{R^3}\left(\frac{M^2}{R^4}\right)^4 \sim \frac{M^{10}}{R^{17}}$$

and for high mass stars as we've seen,

$$L \propto M$$

thus

$$M \propto \frac{M^{10}}{R^{17}}$$

$$R^{17} \propto M^9$$

$$R \propto M^{9/17} \sim M^{.53}$$

now in fact in this calculation we made some assumptions which were not correct. This error would in the case of low mass stars produce a gravely wrong result. If we followed the same calculation for the luminosity from the pp chain in low mass stars we would find.

$$L \sim \varepsilon_{n} M \sim \rho T^4 M \sim \frac{M^2}{R^3}\left(\frac{M}{R}\right)^4$$

$$L \sim \frac{M^6}{R^7}$$

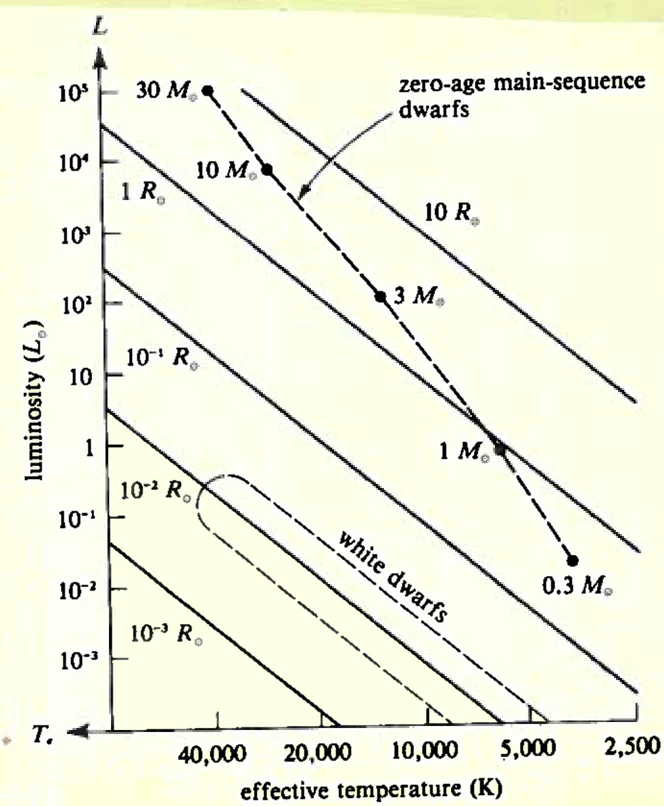
In the lower main sequence.

$$L \sim M^4$$

$$L \sim M^4 \sim \frac{M^6}{R^7} \Rightarrow R^7 \propto M^2$$

$$R \propto M^{2/7} \sim M^{.29}$$

this is clearly wrong as can be seen from the following arguement. From the main sequence it is clear that $L \propto T_{eff}^{7}$



and from the data on binaries we know that

$$M_{bol} \propto M^{8} \sim M^{9}.$$

$$M_{bol} \sim L^{2.5}$$

$$\Rightarrow L \sim M^{8/2.5} = M^{3.2} \rightarrow M^{9/2.5} = M^{3.6}$$

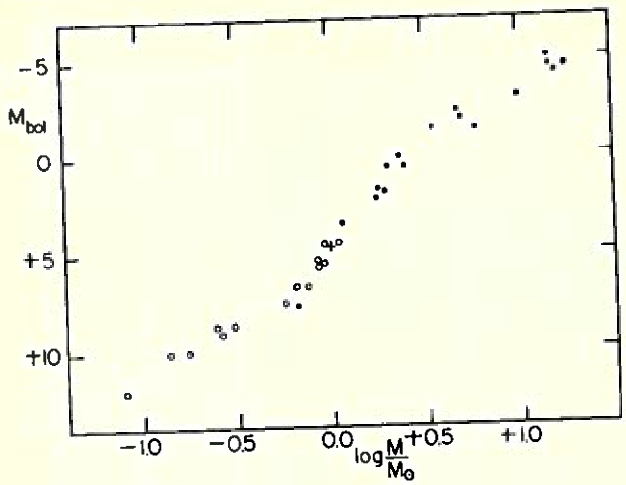


Fig. 2.1. Empirical mass-luminosity relation for main-sequence stars. Data from Tables 2.1 and 2.2. Dots represent spectroscopic binaries, circles visual binaries, and the cross the sun.

from these two experimental relations, we can conclude.

$$T_{eff}^{4} \sim L^{4/7} \sim L^{.57}$$

$$\Rightarrow R^{2} \propto \frac{L}{T^{4}} \sim L^{.43}$$

$$R \propto L^{.21}$$

$$L^{.21} \sim \left(M^{3.5}\right)^{.21} \sim M^{.74} \sim R.$$

So ~~doing~~ comparing this with our result $R \propto M^{.29}$ clearly something is wrong.

The solution is to be a little more careful in what we are actually doing, from hydrostatic equilibrium

$$\frac{dP}{dr} \approx -\frac{G M(r)}{r^2}\rho$$

we evaluated this at R/2 and therefor $\bar{\rho}$ made good sense similarly the radiative equilibrium was done at the midpoint.

$$L \approx -4\pi r^2 \frac{4ac}{3} \frac{T^3}{\kappa\rho} \frac{dT}{dr}$$

and the opacities should be done as before and $\bar{\rho}$ can be substituted for ρ.

<u>But</u> for the energy generation.

$$\varepsilon = \rho T^{\gamma}$$

we <u>cannot</u> substitute $\bar{\rho}$ because the energy generation is all done in the center of the star, and what we need is not $\bar{\rho}$ but ρ_{center}. The conclusion that

$L \propto \rho_c \varepsilon \cdot R^3$ was fine as the fraction of a stars volume that produces the energy doesn't change very much. This can be found by looking at a table for low mass stars from schwarzschild for the sun (a G2 star $T \approx 5800°K$) and Castor (an M0 $\cdot T \sim 3300°K$). From the HR diagram we see that

$$\frac{L_{castor}}{L_\odot} \sim 10^{-2}.$$

and from the mass luminosity graph we

see that $\Delta \log \frac{M}{M_\odot} \sim .5$

$$\Rightarrow \quad \frac{M_{castor}}{M_\odot} \sim .32$$

if $\rho_c \propto \frac{M}{R^3}$

and $R \propto M^{.7}$ as the HR diagram Indicates.

then ~~ρ_c castor~~
$$\rho_c \sim \frac{M}{M^{2.1}} \sim M^{-1}$$

and $\frac{\rho_c \text{ castor}}{\rho \text{ sun}} \sim 3.2$

this is clearly nonsense.

In fact ρ_c is almost Independent of mass.

TABLE 16.2

Physical properties of the sun and Castor C as deduced from lower main-sequence models for various assumed hydrogen contents. (See Table 23.2 for improved solar data.)

	Sun (G2)			Castor C (M0)		
X	0.6	0.7	0.8	0.7	0.8	0.9
Y	0.344	0.276	0.197	0.271	0.184	0.091
Z	0.056	0.024	0.003	0.029	0.016	0.009
E	1.02	0.86	0.68	19.9	19.4	18.7
x_f	0.887	0.891	0.896	0.663	0.666	0.669
q_f	0.9997	0.9998	1.0000	0.888	0.894	0.900
T_f	0.8×10^6	0.7×10^6	0.6×10^6	2.6×10^6	2.4×10^6	2.2×10^6
ρ_f	0.0068	0.0058	0.0051	1.94	1.88	1.77
T_c	15.0×10^6	13.8×10^6	12.9×10^6	8.9×10^6	8.3×10^6	7.8×10^6
ρ_c	87	88	90	76	79	81

$$\Rightarrow \quad L = \int \varepsilon \rho \, 4\pi r^2 dr \sim \varepsilon \rho_c R^3$$

$\varepsilon_{pp} \sim \rho_c T^{4.5}$ for low mass.

$\Rightarrow \quad L \sim T^{4.5} R^3 \rho_c^2$ but ρ_c is a constant

$$\Rightarrow \quad L \sim T^{4.5} R^3 \sim \left(\frac{M}{R}\right)^{4.5} \cdot R^3 .$$

but $L \sim M^3$

$$\Rightarrow M^3 \sim \frac{M^4}{R^4} R^3$$

$$\boxed{R \sim M}$$

for low mass $T_c \to$ down,

$$L \sim T^{5.5} R^3 \sim \left(\frac{M}{R}\right)^{5.5} R^3 \sim \frac{M^{5.5}}{R^{.5}}$$

or the radius goes to a constant!
(for low mass stars convection becomes important).

for high mass stars.

$$L \sim T^{16} R^3 \sim \left(\frac{M^2}{R^4}\right)^4 R^3 \sim \frac{M^8}{R^{13}}$$

but $L \sim M$.

$$\Rightarrow M^7 \sim R^{13}$$

$$\boxed{R \sim M^{.54}}$$

and we in fact have an H R diagram.
ie.

$$L \propto R^2 T_{ef.}^4$$

if $R \propto M$
and $L \propto M^4$ ~~(over a wider range)~~ over a wider range,

$R^2 \propto L^{.5}$

$$\Rightarrow L \propto L^{.5} T^4$$

or $L^{.5} \propto T^4$

$$\boxed{L \propto T^8}$$ <u>H R diagram.</u>

a more detailed check from high mass region is

TABLE 15.2

Results for upper main-sequence models. (Kushwaha, *Ap.J.* *125*, 242, 1957.)

$M =$	$10M_\odot$	$5M_\odot$	$2.5M_\odot$
$\log C$	-6.579	-6.140	-5.793
U_f	2.404	2.478	2.547
V_f	1.791	1.551	1.333
x_f	0.232	0.192	0.155
q_f	0.244	0.201	0.162
$\log p_f$	$+1.439$	$+1.677$	$+1.942$
$\log t_f$	-0.242	-0.174	-0.106
$1-\beta_f$	0.025	0.007	0.002
δ_f	4.41	2.51	1.69
x^*_f	1.432	1.338	1.244
t^*_f	0.705	0.738	0.770
$\log p_c$	$+1.818$	$+2.007$	$+2.227$
$\log t_c$	-0.090	-0.042	$+0.008$
$\log D$	$+1.124$	$+0.198$	-0.780
$\log L/L_\odot$	$+3.477$	$+2.463$	$+1.327$
$\log R/R_\odot$	$+0.559$	$+0.376$	$+0.202$
$\log T_e$	$+4.350$	$+4.188$	$+3.991$
Sp. T.	B1	B5	A2
T_f	1.95×10^7	1.74×10^7	1.52×10^7
ρ_f	4.62	12.3	32.4
T_c	2.76×10^7	2.36×10^7	1.98×10^7
ρ_c	7.80	19.5	48.3

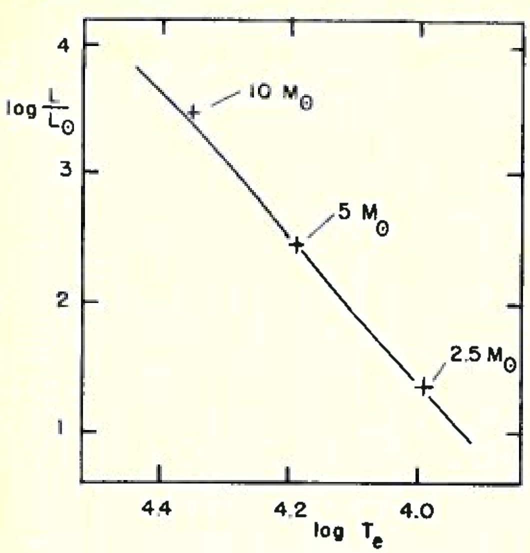


Fig. 15.3. Hertzsprung-Russell diagram for upper main-sequence models (crosses) compared with observations (line, see Table 1.2).

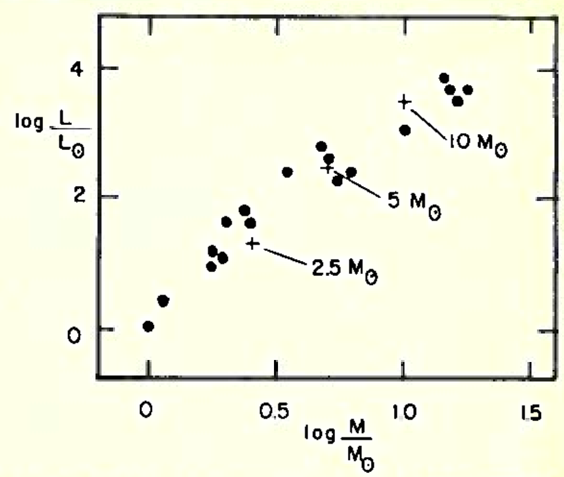


Fig. 15.2. Mass-luminosity relation for the upper main-sequence models (crosses) compared with observations (dots, see Fig. 2.1)

So now we have the shape of the HR diagram main sequence and can even get it right.

What happens as time evolves?

naively the total amount of energy that can be converted during a main sequence life time is approximately fixed

$$E_{tot} \sim f\,M.$$

as $L \propto M^4.$

$$T \sim E/L \sim \frac{fM}{M^4} \sim \frac{1}{M^3}$$

or the lifetime drops like $1/M^3$

if a one solar mass star burns for 1×10^{10} yrs.

a 10 solar mass star will last $\sim 10^7$ yrs

and a 25 solar mass star will last $\sim 6.5 \times 10^5$ yrs!

if nothing else perhaps we don't see stars with masses $> 50 M_\odot$ because the don't last long compared with formation times ($T_{Lifetime}(50 M_\odot) \sim 8 \times 10^4$ yrs!)

ie we now have at least one understanding of the high mass cut off if not two (radiation pressure dominance is probably unstable)

The shape of the main sequence.

We can use the relations at our disposal to estimate the shape of the main sequence.

$$\frac{dP}{dr} = -\frac{G\,M(r)\,\rho(r)}{r^2} \qquad P = \frac{\rho}{m}kT + \frac{a}{3}T^4$$

$$L_r = -\frac{4ac}{3}\,4\pi r^2 \frac{T^3(r)}{\kappa(r)\,\rho(r)}\frac{dT}{dr}.$$

The equations for the energy generation

$$\varepsilon_{pp} \sim \frac{\rho}{T_6^{2/3}}\, e^{\left(-33.8\, T_6^{-1/3}\right)} \sim \rho\, T_6^{4} = \rho\, T^{\nu_1}$$

$$\varepsilon_{CNO} = \frac{\rho}{T_6^{2/3}}\, e^{\left(-152.3\, T_6^{-1/3}\right)} \sim \rho\, T_6^{16} \qquad \rho\, T^{\nu_2}$$

for T_6 going from 4 to 24 ν_1 goes from 6 to 3.5

and for T_6 going from 12 to 50 ν_2 goes from 20 to 13.

and $L_r = \int_0^r 4\pi r^2 \rho\varepsilon\, dr.$

and finally the black body relation

$$L = 4\pi\sigma R^2 T_{surf}^4.$$

The objective here is to use the last relation to establish a relation of the form

$$L \propto T_{surf}^{\alpha}$$

ie eliminating R by finding a relation between $L = L(R)$ and inverting.

for stars on the main sequence we have 3 basic cases describing the equation of state and opacity.

$\varkappa\rho$	P	M
$\rho^2/T^{3.5}$ bound free	$\frac{\rho}{m} kT$	$M < 2M_\odot$
ρ thompson	$\frac{\rho}{m} kT$	$2M_\odot < M < 10M_\odot$
ρ thompson	$\frac{a}{3} T^4$ radiation dominated.	$M > 10M_\odot$

for a matter dominated pressure.
from hydrostatic equilibrium

$$\frac{P}{R} \sim \frac{M}{R^2} \frac{M}{R^3} \Rightarrow P_c \sim M^2/R^4.$$

from ideal gas evaluated at $R/2$.

$$P_{(r/2)} \sim \frac{M}{R^3} T_{(r/2)}$$

and <u>assume</u> $P(R/2) \sim P_c/2$

$$\Rightarrow \boxed{T_c = \frac{P R^3}{M} \sim M/R}$$

from radiative equilibrium. we have

$$L \sim \frac{R^2 T^4}{R \varkappa\rho}$$

and the 2 cases.

$$L_{bf} \sim \frac{R T^{7.5}}{\rho^2} \qquad L_t \sim \frac{R T^4}{\rho} \sim \frac{R^4 T^4}{M.}$$

$$\sim \frac{R^7 T^{7.5}}{M^2}$$

for the very high mass case

$$P \propto T^4 \sim M^2/R^4$$

$$L \sim \frac{R^4 T^4}{M} \sim M$$

---

for the intermediate case.

$$T \sim M/R$$

$$L \sim \frac{R^4 T^4}{M} \sim M^3$$

---

and the low mass case

$$T \sim M/R$$

$$L \sim \frac{R^7 T^{7.5}}{M^2} \sim M^{5.5}/\sqrt{R}$$

---

we now need a relation between M and R. this can then be used to get $L(R)$ and from that and the blackbody curve

$$L(T_{surf})$$

To do this we use the energy generation equation to give us another relation.

$$L \simeq \int \varepsilon \rho \, 4\pi r^2 dr$$

In fact detailed numerical integration yields that $\rho_c \sim$ constant ~ 90 gm/cm^3 and <u>drops</u> for

Very high mass stars. see table 15.2 / 16.2

M	.3	1	2.5	5	10
ρ	80	90	50	20	8

in all of these stars only around the central third of the radius is involved in the energy generation and the central density is roughly constant therefore

$$L = \int_0^{R/3} 4\pi \rho_c \varepsilon r^2 dr$$

$$\sim \rho_c^2 T^\nu 4\pi \int_0^{R/3} r^2 dr$$

$$\boxed{L \sim R^3 T^\nu.}$$

thus for

low mass	medium	high.
$L \sim M^{5.5}/\sqrt{r}$	$L \sim M^3$	$L \sim M.$
$T \sim M/R$	$T \sim M/R$	$T^4 \sim M^2/R^4$

electron degeneracy pressure

Before we start on our discussion of the final states of stars and their leaving the main sequence we must point out a third term needed for the equation of state.

$$P = f(\rho, T).$$
$$\simeq \rho kT + \frac{aT^4}{3}$$

When the densities approach the phenomenally high levels which are found in the final states of stars ($\rho \gg 10^5\ gm/cm^3$) the same effect which keeps all the electrons from falling into the lowest orbital and making all chemistry look like hydrogen - hydrogen bonding, becomes a major effect in stars ie the Pauli exclusion principle.

by the denumeration of states that we discussed in the early part of the semester the total number of states in a volume V with momentum less than ρ_f (standing for P_{fermi}) is.

$$N = (2s+1) \cdot \frac{4\pi}{3} \frac{\rho_f^3 V}{h^3}$$

$$= 2s+1 \iiint_{p=0}^{p_f} \frac{V}{h^3}\, d^3p = \frac{8\pi}{3} \frac{\rho_f^3 V}{h^3}$$

$$\rho_E = \frac{N}{V} = \frac{8\pi}{3} \frac{P_f^3}{h^3}$$

ie if every state is filled up to P_f then the density in terms of this upper fermi momentum is

$$\rho_e = \frac{8\pi}{3} P_f^3/h^3$$

a more sensible as phyical statement is if an electron gas is completely cold $(T=0)$ and has a number density ρ_E, if an electron is added it must have a momentum greater than.

$$p_f = \sqrt[3]{\frac{3\rho_e}{8\pi} h^3} \qquad \rho_e = \frac{\#\ \text{electrons}}{cm^3}$$

as all the states below this level are filled.

This acts as a chemical potential. thus for the eq. 4.4 in Shu.

$$dn_f(p) = \frac{2S+1}{\exp[(E-C)/kT] + 1} \frac{4\pi p^2 dp}{h^3}$$

$$C = KE(p_f)$$
$$= \sqrt{p_f^2c^2 + m_e^2c^4} - m_e c^2 \sim \frac{p_0^2}{2m_e} \quad \left(\text{non relativistic limit}\right)$$

define $C = \mu$ ie normal chemical potential

Thus for $T \neq 0$.

$$\frac{N}{V} = n_e \equiv \rho_e = \frac{4\pi(2S+1)}{h^3} \int_0^\infty \frac{p^2\, dp}{e^{(\epsilon-\mu)/kT} + 1}$$

non relativ. ie $\epsilon = p^2/2m$.

define $x = \frac{\epsilon}{kT} = \frac{p^2}{2m_ekT} \Rightarrow \frac{dx}{dp} = \frac{p}{m_ekT}$

$$\boxed{p^2dp = 2m_ekT x \cdot \sqrt{\frac{m_ekT}{2x}}\, dx} \Rightarrow dx = \frac{p\,dp}{m_ekT}.$$

$$\boxed{p^2dp = \frac{1}{2}\left(\sqrt{2m_ekT}\right)^3 x^{1/2} dx} \quad \text{or} \quad dp = \frac{m_ekT}{p} dx.$$

$$p = \sqrt{2m_ekTx}$$
$$dp = \sqrt{\frac{m_ekT}{2x}}\, dx$$

$$\Rightarrow n_e = 2\pi(2S+1)(2m_ekT)^{3/2} \int_0^\infty \frac{x^{1/2}\, dx}{e^{(x-\mu/kT)} + 1}$$

$$\frac{E_k}{V} = \text{energy density} = \int_0^\infty \epsilon(p)\, dn_f(p)$$

$$= \frac{4\pi}{h^3}(2S+1) \int_0^\infty \frac{\epsilon\, p^2\, dp}{e^{(\epsilon-\mu)/kT} + 1}$$

in the non relativistic limit. the energy density.

$$\frac{E_{th}}{V} = \frac{4\pi(2s+1)}{h^3}\int \frac{\frac{p^4}{2m_e}\,dp}{e^{(\varepsilon-\mu)/kT}-1}$$

$$x = \frac{p^2}{2m_e kT} \Rightarrow dp = \sqrt{\frac{m_e kT}{2x}}\,dx \quad \text{as before.}$$

$$\frac{E_{th}}{V} = \frac{4\pi(2s+1)}{h^3\;2m_e}(2m_e kT)^2\cdot\sqrt{\frac{m_e kT}{2}}\int_0^\infty \frac{x^{3/2}dx}{e^{(x-\mu/kT)}+1}$$

or ~~and~~ as $T\to 0$ this can be redone as.

$$\frac{E_{th}}{V} = \frac{4\pi(2s+1)}{h^3\,2m_e}\int\frac{p^4\,dp}{e^{(\varepsilon-\mu)/kT}+1} \xrightarrow[\lim T\to 0]{} \frac{4\pi(2s+1)}{h^3\,2m_e}\int_0^{p_f} p^4\,dp$$

$$= \frac{4\pi(2s+1)}{10\,m_e h^3}\,p_f^5$$

for exactness $s=\frac{1}{2}$. $2s+1=2$.

$$\frac{E_{th}}{V} = \frac{4\pi}{5m_e h^3}\cdot\left(\frac{h}{2}\left(\frac{3\rho_e}{\pi}\right)^{1/3}\right)^5$$

$$= \frac{\pi h^2}{5\cdot 8\cdot m_e}\left(\frac{3}{\pi}\rho\right)^{5/3} \propto \rho^{5/3}.$$

all very well but let's face it. what <u>we</u> are interested in is <u>pressure</u>.

$$P \equiv \int_0^\infty \bar{p}\cdot\bar{v}_x\,dn_f(p)$$ ie the rate of momentum flow through a surface. (the yz surface in this case)

$\boxed{\text{let } 2s+1=2}$

$$= \frac{8\pi}{h^3}\int_0^\infty\frac{\bar{p}\cdot\bar{v}_x\,p^2dp}{e^{(\varepsilon-\mu)/kT}+1} = \frac{8\pi}{3h^3 m_e}\int_0^\infty\frac{p^4\,dp}{e^{(\varepsilon-\mu)/kT}+1}$$ ($p=mv$ non relativistic)

a little sloppy

$$\iint p\cdot v_x p^2 dp\,d\phi\,d\cos\theta$$

$\langle p\cdot v_x\rangle = ~~p\cos\theta~~\ \frac{1}{3}\langle pv\rangle.$

$$\langle p\cdot v_x\rangle = \frac{1}{3}\langle\bar{p}\cdot\bar{v}\rangle = \frac{1}{3}\left\langle\frac{p^2}{m_e}\right\rangle.$$

as $T\to 0$ $$= \frac{8\pi}{3h^3 m_e}\int_0^{p_f} p^4\,dp = \frac{8\pi}{15h^3 m_e}\,p_f^5$$

$$\boxed{P_{T=0} = \frac{8\pi}{15\,h^3 m_e}\left(\frac{h}{2}\right)^5\left(\frac{3}{\pi}\rho_e\right)^{5/3}}$$

and in the extreme relativistic limit.

ie. $p_f c = \varepsilon_f$ $\quad v_f = c$.

$$P = \frac{8\pi c}{3h^3}\int_0^{p_f} p^3\,dp = \frac{8\pi c}{3h^3}\,\frac{1}{4}\,p_f^4$$

$$= \frac{2\pi c}{3h^3}\left(\frac{h}{2}\right)^4\left(\frac{3}{\pi}\rho_e\right)^{4/3} \quad \propto \rho_e^{4/3}.$$

In fact this can all be done exactly. see chandrasekhar pgs 375 → 400

(sommerfeld) $\quad \varepsilon = \sqrt{p^2c^2 + m^2c^4}$ $\quad \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$

$$\frac{p}{m_e c} \equiv \sinh\theta \qquad \frac{\varepsilon}{m_e c^2} = \cosh\theta \qquad \cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$$

$$(\cosh\theta)^2 - (\sinh\theta)^2 = \frac{\varepsilon^2}{m_e^2c^4} - \frac{p^2}{m_e^2c^2} = \frac{\varepsilon^2 - p^2c^2}{m_e^2c^4} = \frac{m_e^2c^4}{m_e^2c^4} = 1$$

$$\frac{N}{V} = \rho_e = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2\,dp}{e^{(\varepsilon-\mu)/kT}+1} = \frac{8\pi}{h^3}(m_e c)^3\int_0 \frac{\sinh^2\theta\, d(\sinh\theta)}{\exp[(mc^2\cosh\theta - \mu)/kT]+1}$$

$$= \frac{8\pi m_e^3c^3}{h^3}\int_0^\infty \frac{\sinh^2\theta\cosh\theta\,d\theta}{\exp[(mc^2\cosh\theta - \mu)/kT]+1}$$

and $\quad$ exact kinetic energy ↓

$$\frac{U}{V} = \frac{8\pi}{h^3}\int \frac{p^2(\varepsilon - m_ec^2)\,dp}{e^{(\varepsilon-\mu)/kT}+1}$$

note the exponential is really.

$$\exp\left[\left((\varepsilon - m_ec^2) - (\varepsilon_f)\right)/kT\right].$$

$$= \exp[(\varepsilon - \mu')/kT] \qquad \mu' \equiv \mu + m_ec^2.$$

$$= \frac{8\pi m_e^4c^5}{h^3}\int \frac{\sinh^2\theta\cosh\theta(\cosh\theta - 1)\,d\theta}{\exp[(m_ec^2\cosh\theta - \mu)/kT]+1}$$

and

$$P = \frac{8\pi m_e^4c^5}{3h^3}\int \frac{\sinh^4\theta\,d\theta}{\exp[(m_ec^2\cosh\theta - \mu)/kT]+1}$$

Sommerfelds solution.

$$\int_0^\infty \frac{du}{\frac{1}{\lambda}e^u + 1}\frac{d\phi(u)}{du} = \phi(u_0) + 2\left[c_2\left.\frac{d^2\phi}{du^2}\right|_{u_0} + c_4\left.\frac{d^4\phi(u)}{du^4}\right|_{u_0} + c_6\left.\frac{d^6\phi(u)}{du^6}\right|_{u_0} + \dots\right.$$

$$c_2 = \frac{\pi^2}{12} \qquad c_4 = \frac{7\pi^4}{720} \qquad c_6 = \frac{3\pi^6}{30240}$$

$$u_0 \equiv \log\lambda = \frac{\mu}{KT} = \frac{\varepsilon_f}{KT} = \frac{m_e c^2 \cosh\theta_0}{KT}$$

defining $\quad x = \sinh\theta_0 = p_f/m_e c$

$$f(x) \equiv x(2x^2-3)(x^2+1)^{1/2} + 3\ \mathrm{arcsinh}(x)$$

$$= x(2x^2-3)(x^2+1)^{1/2} + 3\sinh^{-1}(x)$$

$$\Rightarrow P = \frac{8\pi m_e^4 c^5}{3h^3}\int \frac{\sinh^4\theta\, d\theta}{\exp((m_e c^2\cosh\theta - \mu)/KT)+1}$$

$$u = m_e c^2 \cosh\theta / KT$$

$$\frac{du}{d\theta} = +m_e c^2 \sinh\theta / KT$$

$$\frac{KT}{m_e c^2}\, du = \sinh\theta\, d\theta.$$

$$\Rightarrow P = \frac{8\pi m_e^4 c^5}{3h^3}\,\frac{KT}{mc^2}\int_0^{\Lambda} \frac{du}{\frac{1}{\Lambda}e^u + 1}\,\frac{d\phi(u)}{du}$$

with $\quad \dfrac{d\phi(u)}{du} = \sinh^3\theta$

$$\Rightarrow \phi(u_0) = \int_0^{u_0} \sinh^3\theta\, du = \frac{mc^2}{KT}\int \sinh^4\theta\, d\theta$$

$$= \left[\frac{\sinh^3\theta_0\cosh\theta_0}{4} - \frac{3\sinh 2\theta_0}{16} + \frac{3}{8}\theta_0\right]\frac{mc^2}{KT}$$

or $\quad x = p_0/mc$

$$\phi(x) = \frac{1}{8}\left[x(2x^2-3)(x^2+1)^{1/2} + 3\sinh^{-1}x\right]\cdot\frac{m_e c^2}{KT} = \frac{mc^2}{8KT}f(x)$$

and.

$$P = \frac{\pi m_e^4 c^5}{3h^3} f(x)\left[1 + 4\pi^2\left(\frac{KT}{mc^2}\right)^2\cdot\frac{x(x^2+1)^{1/2}}{f(x)} + \frac{7\pi^4}{15}\left(\frac{KT}{mc^2}\right)^4\frac{(x^2+1)^{1/2}(2x^2-1)}{x^3 f(x)} + \cdots\right.$$

for this to converge.

$$4\pi^2\left(\frac{KT}{mc^2}\right)^2\frac{x(x^2+1)^{1/2}}{f(x)} \ll 1.$$

the two cases easy to evaluate are,

$x \to 0$ extreme non relativistic.

$$\frac{4\pi^2}{}\left(\frac{kT}{m_e c^2}\right)^2 \frac{x(x^2+1)^{1/2}}{\left[x(2x^2-3)(x^2+1)^{1/2}+3\sinh^{-1}(x)\right]}$$

$$f(x)\ (x\to 0) = \frac{8}{5}x^5 - \frac{4}{7}x^7 + \frac{1}{3}x^9 - \frac{5}{22}x^{11}$$

$$f(x)\ (x\to\infty) = 2x^4 - 3x^2 + \dots$$

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$$\Rightarrow 4\pi^2\left(\frac{kT}{mc^2}\right)^2 \cdot \frac{x(x^2+1)^{1/2}}{f(x)}$$

$$x\to 0 \quad = 4\pi^2\left(\frac{kT}{m_e c^2}\right)^2 \frac{x}{\frac{8}{5}x^5} = \frac{5\pi^2}{2x^4}\frac{k^2T^2}{m_e^2c^4}$$

$$= \frac{5\pi^2}{2}\frac{k^2T^2\, m_e^2}{p_f^4} = \frac{5\pi^2}{8}\left(\frac{kT}{\varepsilon_f}\right)^2$$

$$x\to\infty \quad 4\pi^2\left(\frac{kT}{m_e c^2}\right)^2 \frac{x^2}{2x^4} = \frac{2\pi^2}{x^2}\frac{k^2T^2}{m_e^2c^4}$$

$$= 2\pi^2\frac{k^2T^2}{p_f^2c^2} = 2\pi^2\left(\frac{kT}{\varepsilon_f}\right)^2$$

in both cases. if.

$$4\pi^2\left(\frac{kT}{m_e c^2}\right)^2 \frac{x(x^2+1)}{f(x)} \ll 1$$

then. $\frac{kT}{\varepsilon_f} \ll 1$ and the gas is degenerate.

ie. all the orbitals are filled.

In such a case.

$$P \sim \frac{\pi m_e^4 c^5}{3h^3} f(x)$$

and. in the degenerate cases.

$$P(x \to 0) = \frac{\pi m_e^4 c^5}{3 h^3} \frac{8}{5} x^5 = \frac{8\pi}{15 h^3} m_e^4 c^5 \left(\frac{P_f}{m_e c}\right)^5$$

$$= \frac{8\pi}{15 m_e h^3} P_f^5 \quad \checkmark$$

$$P(x \to \infty) = \frac{\pi m_e^4 c^5}{3 h^3} 2x^4 = \frac{2\pi}{3 h^3} m^4 c^5 \left(\frac{P_f}{m_e c}\right)^4$$

$$= \frac{2\pi}{3 h^3} c \, P_f^4 \quad \checkmark$$

confirming our earlier calculation.

in either case we have polytropic type gas laws.

$$P = k \rho^\gamma$$

In the non relativistic case.

$$P = \frac{8\pi}{15 m_e h^3} \left(\frac{h}{2}\right)^5 \left(\frac{3\rho}{\pi m}\right)^{5/3} = k_1 \rho^{5/3}$$

not m_e

in the relativistic case.

$$P = \frac{2\pi c}{3 h^3} \left(\frac{h}{2}\right)^4 \left(\frac{3\rho}{\pi m}\right)^{4/3} = k_2 \rho^{4/3}$$

not m_e

note $\frac{1}{2}\frac{\rho}{m} = \frac{1}{2}\frac{N}{V} = \rho_e$ (ie half the particles are electrons)

this is true in a hydrogen star.

however in the cases that we are interested in.

if we start with pure hydrogen and produce helium.

then there are 3 particle $(\alpha + 2e^-)$ and all the mass is in the α. If the triple α process has produced a lot of C^{12} then there are 7 particles 6 of which are electrons in other words

$\rho_e = \gamma \frac{\rho}{m}$ $\qquad \frac{1}{2} < \gamma < 1$

This degeneracy pressure will dominate in certain realms. In the non relativistic case.

$$P = \frac{\rho}{m} kT + \frac{a}{3} T^4 + \frac{h^2}{15 \cdot 4 \cdot m_e} \left(\frac{3}{\pi m}\right)^{5/3} \rho^{5/3}$$

$m = 2m_p = 2 * 1.67 \times 10^{-24}$ gm $\qquad k = 1.38 \times 10^{-16}$ erg/K° $\qquad a = 7.56 \times 10^{-15}$ erg/cm^3(K)^4

~~assume~~ $h = 6.63 \times 10^{-27}$ erg-sec $\qquad m_e = 9.11 \times 10^{-28}$ $\qquad G = 6.67 \times 10^{-8}$

$$P = \underset{A}{4.13 \times 10^{7} \rho T} + \underset{B}{2.52 \times 10^{-15} T^4} + \underset{C}{9.92 \times 10^{11} \rho^{5/3}}$$

$\rho = 1$ gm/cm^3, $T = 10^6$: $\quad 4.13 \times 10^{13} + 2.52 \times 10^{9} + 9.92 \times 10^{11}$ $\cancel{1.00 \times 10^{13}}$

however. term c is only accurate if.

$$\frac{5\pi^2}{8} \left(\frac{kT}{\varepsilon_F}\right)^2 << 1.$$

$$\varepsilon_F = \frac{p_F^2}{2m_e} = \frac{h^2}{8m_e} \left(\frac{3}{\pi} \rho_e\right)^{2/3} = \frac{h^2}{8m_e} \left(\frac{3}{\pi} \frac{\rho}{2m_p}\right)^{2/3}$$
$$= 2.6 \times 10^{-11}$$

$$\Rightarrow \frac{5\pi^2}{8} \left(\frac{1.38 \times 10^{-16} \cdot 10^6}{2.6 \times 10^{-11}}\right)^2 = 17 \not< 1$$

$\Rightarrow$ term c is not accurate! gas not degenerate ideal gas law is accu

---

$\rho = 10^5$ gm/cm^3 $\qquad P = 1.65 \times 10^{8} \rho T + 2.52 \times 10^{-15} T^4 + 1.0 \times 10^{13} \rho^{5/3}.$

$T = 10^8$ °K

$$\varepsilon_F = \frac{h^2}{8m_e} \left(\frac{3}{\pi} \frac{\rho}{2m_p}\right)^{2/3} = \cancel{1.42 \times 10^{-7}}\ 5.68 \times 10^{-8}$$

$$\frac{5\pi^2}{8} \left(\frac{kT}{\varepsilon_F}\right)^2 = .37 < 1. \Rightarrow C \text{ is accurate. (?)}$$

$$P = 4.13 \times 10^{20} + 2.52 \times 10^{17} + 2.14 \times 10^{20}$$

ie term C is ~~the largest~~. half the size of A.

mass limits and mass radius relations. (see Pathria Stat. Mech)

define.

$$x = \frac{p_F}{m_e c} = \frac{h}{2m_e c}\left(\frac{3}{\pi}\rho_e\right)^{1/3} = \sinh\theta$$

and recall that.

$$P = \frac{8\pi m_e^4 c^5}{3h^3}\int_0^{\theta_F} \sinh^4\theta \, d\theta$$

$$= \frac{\pi m_e^4 c^5}{3h^3} f(x)$$

$$f(x) = x(x^2+1)^{1/2}(2x-3) + 3\sinh^{-1}x.$$

$$= \frac{8}{5}x^5 - \frac{4}{7}x^7 + \frac{1}{3}x^9 - \frac{5}{22}x^{11} \qquad x \ll 1 \quad \text{non rel}$$

$$= 2x^4 - 2x^2 + 3\left(\ln 2x - \frac{7}{12}\right) + \frac{5}{4}x^{-2} \qquad x \gg 1 \quad \text{extreme rel.}$$

$$\rho_e = \frac{\rho}{2m_p}$$ and for ease use. $\bar\rho = \frac{M}{\frac{4}{3}\pi R^3}$

$$\Rightarrow \bar\rho_e = \frac{\bar\rho}{2m_p} = \frac{3M}{8\pi m_p R^3}$$

$$\Rightarrow x = \frac{h}{2m_e c}\left(\frac{3}{\pi}\,\frac{3M}{8\pi m_p R^3}\right)^{1/3} = \frac{\hbar}{m_e c R}\left(\frac{9\pi M}{8 m_p}\right)^{1/3}$$

$$= \frac{\lambda_{e\,comp}}{R}\left(\frac{9\pi M}{8 m_p}\right)^{1/3}$$

from hydrostatic equilibrium.

$$\frac{dP}{dr} = -\frac{G M(r)\rho(r)}{R^2}$$

and for demonstration purposes assume $\rho = \text{const} = \bar\rho$

$$M(r) = \frac{4}{3}\pi r^3 \bar\rho$$

at the mid point. $M(R/2) = \frac{M}{8}$

$$\frac{P_c}{R} \sim G\frac{\frac{4}{3}\pi\left(\frac{R}{2}\right)^3(\bar\rho)^2}{R^2} = G\frac{M}{8}\cdot\frac{M}{\frac{4}{3}\pi R^3}\frac{1}{R^2}$$

$$= \frac{GM^2}{R^5}\,\frac{3}{32\pi}$$

$$= \frac{GM}{R^5}\,\frac{\alpha}{4\pi} \qquad \alpha = \frac{3}{8}$$

$$P_c = \frac{GM^2}{R^4}\frac{\alpha}{4\pi}$$

for a general solution $\alpha \sim 1$ ie $(.1 < \alpha < 1)$

$$P_c = \frac{GM^2}{R^4}\frac{\alpha}{4\pi} = \frac{\pi m_e^4 c^5}{3h^3} f(x)$$

$$\text{or } f(x) = \frac{3h^3\alpha}{4\pi^2 m_e^4 c^5}\frac{GM^2}{R^4} = 6\pi\alpha\left(\frac{\hbar}{m_e c}\frac{1}{R}\right)^3 \cdot \frac{GM^2}{R m_e c^2}$$

if $x \ll 1$ $\quad f(x) = \frac{8}{5}x^5 = \frac{8}{5}\left(\frac{P_f}{m_e c}\right)^5$

using our assumption that $\rho_e = \bar{\rho}_e$

$$\cancel{\frac{8}{5}\left(\frac{1}{m_e c}\right)^5}$$

$$x = \left(\frac{\hbar}{m_e c}\frac{1}{R}\right)\cdot\left(\frac{9\pi M}{8 m_p}\right)^{1/3}$$

$$\Rightarrow \frac{8}{5}\left(\frac{\hbar}{m_e c}\frac{1}{R}\right)^5 \cdot \left(\frac{9\pi M}{8 m_p}\right)^{5/3} = 6\pi\alpha\left(\frac{\hbar}{m_e c}\frac{1}{R}\right)^3 \frac{GM^2}{R m_e c^2}$$

$$\Rightarrow \frac{8}{5}\left(\frac{\hbar}{m_e c}\right)^2\left(\frac{9\pi}{8m_p}\right)^{5/3} M^{5/3} = 6\pi\alpha \frac{GM^2}{m_e c^2} R$$

$$\boxed{\frac{8}{30\pi\alpha}\left(\frac{\hbar}{m_e c}\right)^2\left(\frac{9\pi}{8m_p}\right)^{5/3}\frac{m_e c^2}{G} M^{-1/3} = R.}$$

$$\boxed{\frac{3(9\pi)^{2/3}}{40\alpha}\cdot\frac{\hbar^2}{G m_e m_p^{5/3}} M^{-1/3} = R}$$

in the relativistic limit. $x \gg 1$ we must carry our $f(x)$ to two terms. to see this I'll do it to one term.

$$f(x) \sim 2x^4 = \cancel{2\left(\frac{\hbar}{m_e c}\right)^4}\ 2\cdot\left(\frac{\hbar}{m_e c}\frac{1}{R}\right)^4\cdot\left(\frac{9\pi M}{8 m_p}\right)^{4/3} = 6\pi\alpha\left(\frac{\hbar}{m_e c}\frac{1}{R}\right)^3\frac{GM^2}{R m_e c^2}$$

$$\text{or } \boxed{2\cdot\left(\frac{\hbar}{m_e c}\right)\left(\frac{9\pi}{8m_p}\right)^{4/3}\cdot\frac{m_e c^2}{G}\cdot\frac{1}{6\alpha} = M^{2/3}}$$

which makes no sense. ie there is no Radius dependence

~~rewriting M =~~

so.

$$f(x) = 2x^4 - 2x^2 = 6\pi\alpha\left(\frac{\hbar}{m_e c}\frac{1}{R}\right)^3 \frac{GM^2}{R m_e c^2}$$

$$= 2\left[\left(\frac{\hbar}{m_e c}\frac{1}{R}\right)^4\left(\frac{9\pi M}{8 m_p}\right)^{4/3} - \left(\frac{\hbar}{m_e c}\frac{1}{R}\right)^2\left(\frac{9\pi M}{8 m_p}\right)^{2/3}\right] = 6\pi\alpha\left(\frac{\hbar}{m_e c}\right)^3 \frac{1}{R^4}\frac{GM^2}{m_e c^2}$$

$$\left(\frac{\hbar}{m_e c}\frac{1}{R}\right)^4\left(\frac{9\pi M}{8 m_p}\right)^{4/3} - 3\pi\alpha\left(\frac{\hbar}{m_e c}\right)^3\frac{1}{R^4}\frac{GM^2}{m_e c^2} = \left(\frac{\hbar}{m_e c}\right)^2\frac{1}{R^2}\left(\frac{9\pi M}{8 m_p}\right)^{2/3}$$

define $A = \frac{9\pi}{8 m_p}$, $\lambda = \frac{\hbar}{m_e c}$

$$A^{4/3} M^{4/3} - \frac{3\pi\alpha}{\cancel{\alpha}}\frac{1}{\lambda}\frac{GM^2}{m_e c^2} = \frac{R^2}{\lambda^2}\cdot A^{2/3} M^{2/3}$$

$$\left[\lambda^2 A^{2/3} M^{2/3} - \frac{3\pi\alpha}{\cancel{\alpha}}\lambda\frac{G}{m_e c^2}\frac{M^{4/3}}{A^{2/3}}\right] = R^2$$

$$\lambda^2 M^{2/3} A^{2/3}\left[1 - \frac{3\pi\alpha G}{\cancel{\alpha}\, m_e c^2}\frac{1}{\lambda}\frac{1}{A^{4/3}}\cdot M^{2/3}\right] = R^2.$$

$$\boxed{\lambda M^{1/3} A^{1/3}\left[1 - \left(\frac{M}{M_0}\right)^{2/3}\right]^{1/2} = R.}$$

$$M_0 = \left(\frac{\cancel{\alpha} m_e c^2}{3\pi G \alpha m_e c}\hbar\right)^{3/2}\cdot\left(\frac{9\pi}{8 m_p}\right)^2 = \left(\frac{9\pi}{8}\right)^2\left(\frac{1}{3\pi}\right)^2\cdot\left(\frac{\hbar c}{G}\right)^{3/2}\cdot\left(\frac{3\pi}{\alpha^3}\right)^{1/2}\frac{1}{m_p^2}$$

$$= \frac{9}{64}\cdot\left(\frac{\hbar c}{G}\right)^{3/2}\cdot\left(\frac{3\pi}{\alpha^3}\right)^{1/2}\frac{1}{m_p^2}.$$

$$= \frac{9}{64}\cdot\left(4.75\times 10^{-10}\right)^{3/2}\cdot\frac{\sqrt{3\pi}}{m_p^2}\left(\frac{1}{\alpha}\right)^{3/2}$$

$$= \cancel{9.42\times 10^{40}\ gm}$$

$$\frac{1.6\times 10^{33}}{\alpha^{3/2}}$$

$$= \frac{1.25}{\alpha^{3/2}} M_\odot$$

detailed calculations show $1.44\ M_\odot$ or.

$$\alpha = \left(\frac{1.25}{1.44}\right)^{2/3} = .91$$

rewriting the solution.

$$\frac{\hbar}{m_e c}\cdot\left(\frac{9\pi}{8}\right)^{1/3}\left(\frac{M}{m_p}\right)^{1/3}\left[1-\left(\frac{M}{M_0}\right)^{2/3}\right]=R.$$

$$M_0=\frac{9}{64}\frac{\sqrt{3\pi}}{\alpha^{3/2}}\left(\frac{\hbar c}{G}\right)^{3/2}\cdot\frac{1}{m_p^2}=\frac{1.25}{\alpha^{3/2}}M_\odot\simeq 1.44\,M_\odot$$

as $M\to M_0$ $R\to 0$ hence M_0 is the limiting mass beyond which the thing shrinks out of sight. This limiting value is known as the chandrasekhar mass limit.

How large is a one solar mass system.

$$3.86\times10^{-11}\cdot 1.28\times10^{8}\ M^{1/3}\left[1-\left(\frac{M}{1.44M_\odot}\right)^{2/3}\right]=R.$$

$$4.96\times10^{-3}\ M^{1/3}\left[1-\left(\frac{M}{1.44M_\odot}\right)^{2/3}\right]=R$$

So a one solar mass ball $M=2\times10^{33}$ gm

$$4.96\times10^{-3}\,(2)^{1/3}\cdot10^{11}\left[\cdot\,1-\left(\frac{1}{1.44}\right)^{2/3}\right]=R.$$

$$1.35\times10^{8}\ \text{cm}$$

$$=1.35\times10^{3}\ \text{km}.$$

$$\sim\frac{1}{4}\,R(\text{earth}).$$

note this radius only depends on the electron mass through the first factor the electron compton wavelength $\Rightarrow$ if we had built the system with neutrons (also fermions) everything would be the same except for the overall size which would be reduced by $\frac{m_e}{m_n}\sim\frac{1}{2000}$

now is this a reasonable statement.?

The result we used was derived with Newtonian gravity

$$F_G = \frac{G M_1 M_2}{r^2}$$

to get a measure of the accuracy of this or rather to determine if relativistic effects might be important a useful number is the escape velocity determined by.

$$E_{tot} = 0 = K.E. + P.E.$$

$$= \frac{1}{2} m v_{esc}^2 - \frac{G M m}{r}$$ ie enough KE to escape.

$$\Rightarrow V_{esc} = \sqrt{\frac{2 G M}{r}}$$

for a $1\ M_\odot$ white dwarf

$$V_{esc.} = \sqrt{\frac{2 \cdot 6.63 \times 10^{-8} \cdot 2 \times 10^{33}\ gm}{1.35 \times 10^{8}\ cm}}$$

$$= 1.4 \times 10^{9}\ cm/sec$$

$$.047\ C\ !$$

if $r_{ns} = \frac{r_{wd}}{2000}$

$$V_{esc\,ns} = \sqrt{2000} \cdot V_{esc\,wd} = 2.09\ c\ !!!$$

So one might conclude that general relativistic effects might be important.

let's recheck a few things.

for the 1 solar mass white dwarf we used the relativistic form. is this accurate. we approximated

$$x = \frac{p_f}{mc} = \frac{\hbar}{m_e c} \frac{1}{R} \left(\frac{9\pi}{8} \frac{M}{M_p} \right)^{1/3}$$

putting in numbers

$$x = 3.86 \times 10^{-11} \frac{1}{1.35 \times 10^8} \cdot \left(4.23 \times 10^{57}\right)^{1/3} = 4.6$$

so its ok.?

how much larger or smaller would it have been if we had used the non relativistic form.

$$R = \frac{1}{\alpha} \frac{3}{40} (9\pi)^{2/3} \frac{\hbar^2}{G m_e m_p^{5/3}} \quad M^{-1/3}$$

$$\frac{1}{\alpha} \frac{1.27 \times 10^{-20}}{m_p^{5/3}} \quad M^{-1/3}$$

$$= \frac{1}{\alpha} \; 5.42 \times 10^{19} \quad M^{-1/3}$$

$$= \frac{1}{\alpha} \quad 4.31 \times 10^{8} \text{ cm.}$$

$$x = \frac{3.86 \times 10^{-11}}{4.31 \times 10^{8}} \cdot \left(4.23 \times 10^{57}\right)^{1/3} = 1.45$$

still relativistic but **barely**

$$\Rightarrow \boxed{1.35 \times 10^8 \text{cm} < R_{wd}(M = 1 M_\odot) < 4.3 \times 10^8 \text{ cm}}$$

We have now determined in some detail the structure of stars supported by electron degeneracy pressure. These are rather naturally occuring objects, ~~and~~ within the framework of stellar evolution, and appear to have the properties of ~~[illegible]~~ white dwarf stars. A similar type of object can be proposed where the electron degeneracy pressure is replaced by a neutron degeneracy pressure. The electron star solutions cannot be trivially replaced by the corresponding solutions with the electron mass replaced by the neutron mass because as we saw you end up with escape velocities greater than c and things like that. So two basic theoretical questions are presented.

1. How would a star of pure neutrons develop? (even if only in the core).
2. How do general relativistic effects change the answers?

The answer to the first question can be found as follows: Consider a high mass stellar core ($M > 3 M_\odot$!) Under such conditions the core mass is larger than the Chandrasekhar mass limit ($1.41\ M_\odot$) and therefore the electron degeneracy pressure is insufficient to counter act gravity. The core will condense, growing hotter and the rate of inverse β decay reactions

$$e^- + p \rightarrow n + \nu$$

will increase within the nuclei suspended in the degenerate electron soup. As the electrons dissappear due to the "neutronization" process the degeneracy pressure decreases allowing faster contraction. It is hypothesized that a solid neutron core will develop supported by the neutron degeneracy pressure.

in the non relativistic limit.

$$P_{ress}(nr) = \frac{8\pi}{15 h^3} \frac{P_f^5}{m_n}$$

$$P_f = \frac{h}{2}\left(\frac{3\rho_n}{\pi}\right)^{1/3} \qquad \rho_n = \frac{\rho}{m_n} \quad \left(\text{all particles are now neutrons}\right)$$

$$\text{or.} \quad P_{ress} = \frac{\pi h^2}{60\, m_n^{8/3}}\left(\frac{3}{\pi}\rho\right)^{5/3}$$

the density is essentially nuclear density at this point.

$$\rho_{nucl.} \sim \frac{m_p}{\frac{4}{3}\pi R_p^3} \sim \frac{1.67\times10^{-24}\,gm}{\frac{4}{3}\pi\,(10^{-13}\,cm)^3}$$

$$\sim \underline{4\times10^{14}\,gm/cm^3} \quad \left(\text{ie } 10^8\,\rho_{wd.}\right)$$

$$P_f = \frac{h}{2}\left(\frac{3\rho_n}{\pi}\right)^{1/3} \sim \frac{6.63\times10^{-27}}{2}\left(\frac{3}{\pi}\,\frac{4\times10^{14}}{m_n}\right)^{1/3}$$

$$\sim \cancel{2.4\times10^{-23}}\ 2.03\times10^{-14}$$

is in relativistic? $\quad V \sim \frac{P_f}{m} \simeq 1.2\times10^{10}\ cm/s$ 40% of c.!

$\Rightarrow$ Can use nonrelativistic form for pressure.

$$P = \frac{8\pi}{15 h^3}\frac{P_f^5}{m_n} = 1.2\times10^{34}$$

$$= \cancel{3.66\times10^{30}}\ 1.2\times10^{34}$$

$$\sim 10^{19}\ P_{center\ of\ the\ Sun}$$

$$\sim 2\times10^{12}\ \text{Pressure in a W.D.}$$

At this point you must realize we have extrapolated far beyond our understanding of nuclear physics but we continue undaunted. If we assume a uniform density. $\rho \simeq \rho_{nucl} = 4\times10^{14}\,gm/cm^3$ (ie 4×10^{8} tonnes/cm^3!)

a one solar mass object would have a size of.

$$R = \left(\frac{M}{\frac{4}{3}\pi\rho}\right)^{1/3} \sim \left(\frac{2\times10^{33}}{\frac{4}{3}\pi\cdot 4\times10^{14}}\right)^{1/3} \sim 1\times10^{6}\,cm \quad \sim \underline{10\,km}!$$

$$V_{esc} \sim \sqrt{\frac{2GM}{R}} \sim \sqrt{\frac{2\times 6.67\times10^{-8}\cdot 2\times10^{33}}{1\times10^{6}}}$$

$$\sim 1.6\times10^{10} \quad \sim 50\% \text{ of } c.!$$

This means that material falling onto the solid nuclear core ~~at~~ from the nuclear burning shells. (si → fe mg → si etc) will hit the nuclear core at 10 → 30% of the speed of light. If 5% of the mass of a star were to do this the energy released would be. (assume $v = .1c \Rightarrow$ non rel ok)

$$E \cong \frac{1}{2}\,.05\times 2\times10^{33}\cdot(3\times10^{9})^{2}.$$

$$\sim 4.5\times10^{50}\ ergs$$

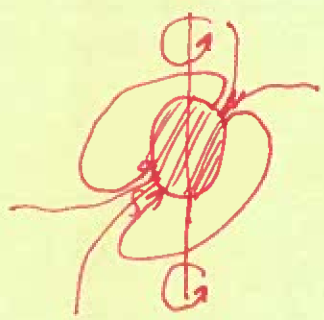
roughly the equivalent of all the energy released by a one solar mass star in 10^{10} yrs. This is a super nova and the resulting explosion will blow off all of the exterior stellar material.

This is all very nice but is there anything to justify this rampant speculation. The crab nebula is almost certainly a super nova remnant. If the gas cloud light is analyzed the doppler shift can be used to estimate how long the cloud has been expanding.

The result is $\sim$ 900 yrs. In 1054 chinese astronomers noted a new bright star in the constellation taurus which lasted $\sim$ 2 years. (There seem to be no western records of this event). The total energy output over all frequencies is $\sim 3 \times 10^{38}$ ergs/sec or $\sim 10^5\ L_\odot$!

Pulsars.

In 1967 Bell and Hewish observed compact radio pulsation sources (with periods $\sim$ 1msec) which were dubbed pulsars. After several years of study including some in binary systems it was concluded that these objects were small ($R \sim 10 \to 30$ km) massive ($m \sim 1 M_\odot$) magnetic dipoles with enormous field strengths. If a magnetic dipole rotates about an axis which is not parallel to the magnetic polar axis the system will radiate.



Calculations of the intensity of radiation yield numbers in the order of 10^{38} erg/sec. Now are high magnetic fields and fast rotations reasonably associated with neutron stars??

If we conserve the angular momentum of a star as it contracts.

$$= L\,(\text{begin}) \qquad = L\,(\text{final})$$

$$L = I \cdot w$$

w angular veloc.

I = moment of Inertia.

$$\sim \tfrac{3}{5} m r^2 w$$

$$= \int \rho\,(\bar{r}\bar{r} - \bar{r}\cdot\bar{r})\,d^3x$$

$$w = \frac{2\pi}{\text{Period}}$$

$$= \tfrac{3}{5} m r^2 \quad \text{if } \rho = \text{const}$$

$$\Rightarrow L \sim \frac{m r^2}{\text{Period}}$$

$$\Rightarrow \frac{\tfrac{3}{5} m r_1^2}{P_1} = \frac{\tfrac{3}{5} m r_2^2}{P_2}$$

$$\left(\frac{r_1}{r_2}\right)^2 = \left(\frac{P_1}{P_2}\right)$$

$$P_2 = P_1 \left(\frac{r_2}{r_1}\right)^2$$

$r_1 \sim 7\times10^5$ km (star)

$r_2 \sim 10$ km.

$P_1 \sim 10$ days? (star)

$$= 8.64\times10^5 \cdot \left(\frac{1}{7\times10^4}\right)^2$$

$$= .176 \text{ msec}$$

furthermore if we assume magnetic flux is conserved (which is not too unreasonable.)

$$B_1 \cdot \text{Area}_1 = B_2 \cdot \text{Area}_2$$

$$B_2 = B_1 \cdot \left(\frac{r_1}{r_2}\right)^2$$

$$= B_1 \cdot 5\times10^9$$

ie the fields are enormous!

So a star which shrunk to a radius of 10 km would have its rotational period reduce from days to milliseconds and its' field increase by some factor of $\sim 10^9$. This has all the nescessary features to produce a pulsar. In fact a rather remarkable observation of pulsars is that their periods are not absolutely constant but slow down very slightly with time. This turns out to have a simple explanation. The kinetic energy of a rotating object is

$$KE_{rot} = \frac{1}{2} I \omega^2 = \frac{L^2}{2I}$$

$$\omega = \frac{2\pi}{P}$$

$$\frac{d\omega}{dt} = \frac{-2\pi}{P^2}\frac{dP}{dt} = \frac{-\omega^2}{2\pi}\frac{dP}{dt}$$

$$\Rightarrow \frac{d\,KE_{rot}}{dt} = I\omega \frac{d\omega}{dt} = \cancel{-2\pi L \frac{d\,\text{Period}}{dt}} = -\frac{I\,4\pi^2}{P^3}\frac{dP}{dt}$$

now assume that pulsars are supernova remnants in fact assume they are neutron stars. To give added credence to this there is a pulsar in the center of the Crab nebula. So the energy emitted by the crab nebula is $\sim 3\times 10^{38}$ ergs/sec. If you make reasonable assumptions about the size and mass of a neutron star.

$$R = 20\text{ km.} \qquad M = 2 M_\odot$$

$$I = \frac{3}{5} m r^2 = \frac{3}{5} \times 4\times 10^{33}\,\text{gm}\cdot(2\times 10^6\,\text{cm})^2$$

$$\sim 9\times 10^{45}\,\text{gm cm}^2.$$

For the Crab pulsar the period is 33 msec and the rate of change of the period is $\sim 3\times 10^{-8}$ sec/day

or $\frac{dP}{dt} = -3.47\times 10^{-13}$.

$$\exists\ \frac{dKE_{rot}}{dt} \sim \cancel{2\pi \cdot I\omega \frac{dP}{dt}} = \frac{4\pi^2 I}{P^3}\left|\frac{dP}{dt}\right| = \frac{4\pi^2 \cdot 9\times10^{45}}{(.83)^3} \cdot 3.47\times10^{-13}$$

$$= \cancel{2\pi \times 8\times10^{45} \cdot \frac{2\pi}{3.3\times10^{-3}} \cdot 3.47\times10^{-13}} \qquad = 3.43\times10^{39}$$

$$\sim \cancel{3.38 \times 10^{37}\ \text{ergs/sec}}$$

or within a factor 10 which is not too bad considering how crude this was.

To date there have been some 300 pulsars observed.

X ray sources.

In the ~~early~~ late 1960's rockets were used to lift x ray detection devices above the atmosphere and started x Ray astronomy. In 1970 the Uhuru (explorer 42) was put in orbit and a team headed by Riccardo Giacconi (now in charge of the space telescope project) analyzed the data. By 1973 339 x ray sources had been identified. Two of them Centaurus X3 (third X ray source in centaurus) and Hercules x1 (first X ray source in hercules) were found to pulse with periods of 4.84 sec and 1.24 sec respectively. Further these two sources turn off on a periodic basis Centaurus for ~12 hrs every 2.087 days and Hercules for ~6 hrs every 1.7 days. To add to this hercules X1 has a periodic doppler shift with a period also equal to 1.7 days. The

Standard explanation for these effects are that these systems are binary stars where one of the "stars" is infact a neutron star. The strong gravitational field causes the "normal" star to fill its Roche lobe and matter is accreted onto the neutron star, well not exactly... in fact a gas accretion disk forms around the neutron star (angular momentum is conserved) and as the material has fallen into a very deep gravitational potential energy well the kinetic energies of the atoms ie temperature of the gas in the accretion disk becomes very high. This ionizes the gas. Charged particles will follow the field lines so they will move around and finally fall onto the magnetic poles. At the point of impact a very high energy will be released in the form of photons in the x ray region of the spectrum. If the rotation axis is not aligned with the magnetic axis these hot points will rotate around only being visible for a fraction of the rotation period.

Black holes.

Presumably a chandrasekhar type of mass limit occurs for neutron stars even in view of ~~special~~ general relativity, but it is also clear that some very unusual properties also exist probably even before this occurs and certainly for higher masses. We have seen that the escape velocities are becoming significant fractions of the speed of light.

$$V_{esc} \sim .5 C.$$

As nothing can go faster than C some curious situations must occur when.

$$V_{esc} = \sqrt{\frac{2GM}{R}} = C$$

$$\text{or} \quad R = \frac{2GM}{c^2} \equiv \text{Schwarzschild radius}$$

General relativity is a geometric interpretation of gravity where an energy density distorts the "shape" of space producing a "curvature". What this means in a more mathematical sense is that distances and rulers get distorted. The proper distance or lorentz Invariant distance would be defined as.

$$(\Delta\tau)^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2$$

which could be written in matrix notation as

$$\Delta\tau^2 = (\Delta x, \Delta y, \Delta z, c\Delta t)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \\ c\Delta t \end{pmatrix}$$

or $g_{\mu\nu}\Delta x^{\mu}\Delta x^{\nu}$

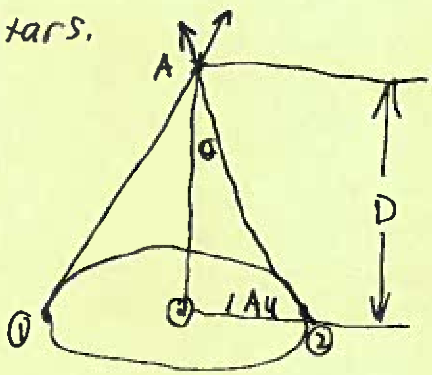
where $g_{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

This matrix is called the metric tensor (tensor is a generalized term for matrix and can have more than 2 dimensions). A curvature is found by ~~off diagonal and non unit values, or rather~~ differential properties of this metric tensor which will be discussed later.

Distance scales.

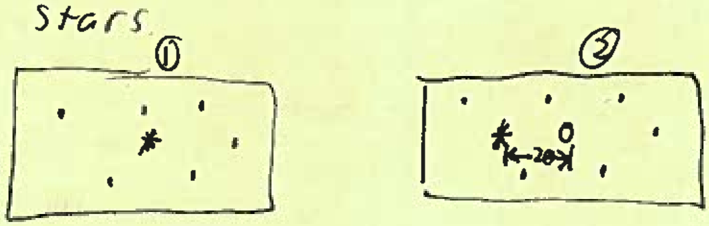
①. Parallax.

Clearly distances are a major problem in astronomy. The universe is three dimensional in nature and while two of the coordinates of any object in the sky can be found easilly (θ and ϕ or declination and right ascension, ie the angles in a spherical coordinate system) the distance to the object cannot. Only since the mid 1800's have telescopes and optics been good enough to detect the parallax motion of nearby stars.



$$\theta\,(\text{in sec}) = \frac{206265 \cdot 1\,\text{AU}}{D\ (\text{in AU})}$$

on a photo graph of stars the star A would appear to move with respect to the more common distant stars.



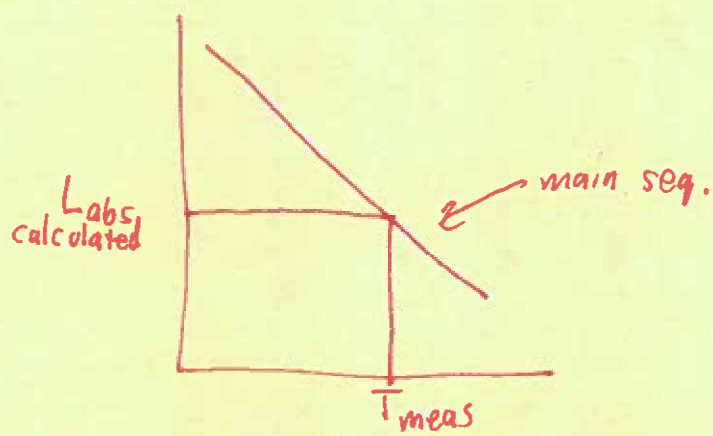
By this technique we have determined the distances to stars out to 10 – 15 parsecs. These stars are the basis for the normalized Hertzsprung - Russell diagram as given the distance the absolute magnitude can be determined.

$$M_{abs} = M_{app} - 5 \log\left(\frac{dist}{10\,pc}\right)$$

this technique however only works for a very small number of stars, ie those closer than 10 to 15 pc. and so far greater distances we need other techniques.

Main sequence fitting.

In principle we can evaluate the distance to a star if it is on the main sequence. From the nearby stars we can determine the relation between the effective (surface) temperature and the absolute luminosity. If we determine the temperature of a distant star (by the peak intensity wavelength, wien's law and by the relative intensities of absorbtion and emission lines) we can invert the procedure. Assume the star is on the main sequence then given the temperature we can determine the absolute luminosity and from the apparent luminosity determine the distance.



This method has the drawback that not all stars are on the main sequence and the scatter around the main sequence caused by differing ages and chemical compositions make the luminosity determination somewhat uncertain.

Cepheid variables. (mechanism).

Eddington was one of the first proponents of the idea that cepheid variables were pulsating stars and not eclipsing binaries. In such a model the star is veiwed as a thermodynamic engine. Simply stated such an engine works as follows:

When the gas is at or near maximum compression heat is added causing expansion.

When the gas is at or near maximum expansion heat is removed causing contraction or compression. This can be accomplished by having a "leaky" system ie one which is not well insulated. In this manner heat is constantly lost. Due to this we only need add more heat than the leak rate at compression and less heat than the leak rate at expansion for the system to idle. Such a system with oscillating heat addition (and subtraction) can be looked at as a "valve". Eddington also proposed another method in which this might work. Instead of varying the rate at which energy is added one holds that constant and varies the rate of the Leak. Ie at maximum compression the leak rate decreases and at maximum expansion the leak rate increases. In a star where a significant amount of the pressure is provided by the radiation pressure the opacity can act as the valve. If at maximum compression and temperature the opacity were to increase blocking the flow of heat and at maximum expansion the opacity were to decrease and the rate of energy flow or flux increase thereby cooling the star the system could pulsate. This is the currently believed mechanism.

The method by which this happens is believed to be due to the singly ionized state of helium, He II. The idea is that a layer of the star a small distance below the surface is in a temperature range of 30,000 °K → 55,000 °K and has a 10→15% by number helium fraction ⇒ by mass.

$$\frac{4*10}{40+90} = \frac{40}{130} = f_{min} \sim 31\%$$

$$\frac{60}{60+85} = \frac{60}{145} = f_{max} \sim 41\%$$

assume an adiabatic oscilation of density and volume and temperature. As the gas compresses the temperature rises and the fraction of helium which is ionized increases. This in turn increases the opacity trapping the radiant energy in the layer. ~~preventing~~ The layer then expands and cools. As it cools the helium recombines to become neutral and this causes the opacity to drop allowing the radiant energy to flow out. This will act as a positive feedback mechanism and will drive the oscillation. The oscillation does not grow indeffinitely as the opacity will reach a limiting value. hence once an oscillation starts it will grow to this limiting value.

To get a feeling for the luminosity period relation one can realize each atom obeying keplers laws and pretend it is in a bound orbit of extreme eccentricity. Such an orbit requires that the particle spend most of its time at the largest radius (keplers second law)

$$\Rightarrow \quad MP^2 \propto R^3.$$

$$P^2 \propto R^3/M \qquad\qquad M = \rho/\tfrac{4}{3}\pi R^3$$

$$P^2 \propto \frac{1}{\rho}$$

$$P\sqrt{\rho} = \text{const.}$$

as we saw in our discussions of stellar structure

$$M \propto R. \qquad\qquad L \propto R^2 T_{eff}^4$$

and $\quad L \propto M$

such a set of relations would lead to something like

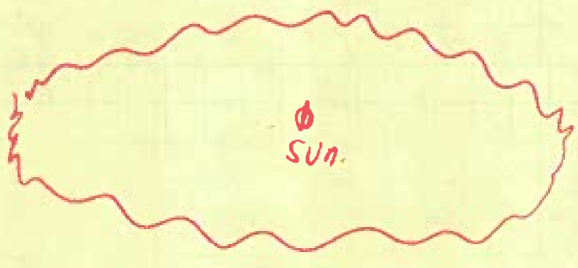
$$P^2 \propto R^3/M \propto R^2 \propto M^2 \propto L^2$$

$$\underline{P \propto L.}$$

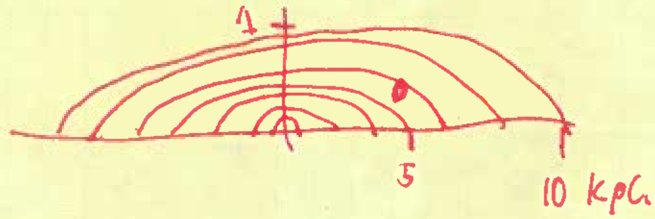
Interstellar dust. and the size of the galaxy.

Using the Period - luminosity relation for cepheids Shapley realized a novel method for determining the size and shape of the milky way galaxy. Up until that time the standard technique had been basicly star counting. To illustrate this we'll discuss two attempts first one by Herschel (discovered uranus)

Herschel assumed that all stars were the same with their brightness falling like $1/Dist^2$. Thus for each direction in the sky he scaled the distribution of stars by $1/\sqrt{brightness}$ the result from this observations were that the sun was at the center of the galaxy and it had a flattened shape something like

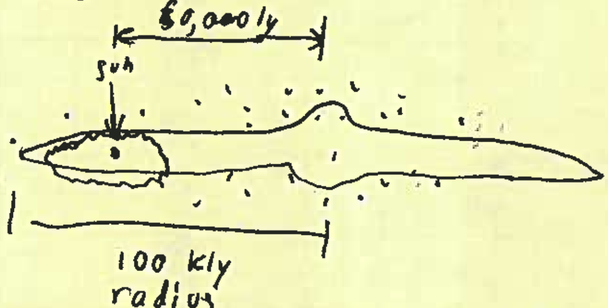


At the end of the 19^{th} century a more elaborate effort was made by Kapsteyn and Van Rhijn. The photographic plate had only recently been added to the arsenal of astronometrical tools, Kapsteyn enlisted the help of a large number of people to evaluate the distribution in brightness vs. proper motion ie change in $(\theta\ \phi)$, (right ascension, declination) over the years) with some considerable effort a model of the ~~solar system~~ galaxy known as the Kapsteyn universe evolved



and a galaxy with a radius of around 10 kpc, developed,

Shapely Shaply after this had noted that most of the globular clusters that surround the milky way and are near enough to resolve stars (as opposed to external galaxies like andromeda) and concluded they were part of the milky way galaxy. It turns out that all of the globular glusters have cepheid variables in them allowing their distances to be determined. Using these objects Shapely concluded that the were distributed like



In fact his estimate was roughly a factor of two too large due to the effect of interstellar dust attenuating the light this has since been understood.

External galaxies.

There are several kind of Nebula, the true differences between them ~~was~~ were not allways apparent. The great Nebula in Orion is a gas cloud, through even our 8" telescope this is apparent. There ~~are other~~ is another type of nebula which is clearly quite different, the "spiral" nebula typified by ~~the~~ andromeda nebula. Diffuse light sources have been observed ~~for~~ a long time. The nebulous light sources have turned out to be gas clouds in our own galaxy (Orion), gravitationally bound balls of stars globular clusters (Hercules), and these funny things the spirals. The spiral structure has been very clear for a long time, Lord Rosse using his 72" reflector clearly saw the spiral structure of M 51 the "wirlpool" galaxy (Pg 282), ~~For a~~ It was only extremely recently (1923) that the true nature of the spiral nebula became clear. Prior to that time almost everything was confused. Approximately 15 galaxies had had redshifts determined and of these ~ 2/3 (11) were found to be receeding with what at the time seemed to be very large velocities. In addition individual stars and main sequence fitting was not a well understood bussiness. The distances to galaxies had been estimated by a variety of techniques the most believed had been the intensity or brilliance of stars going Nova. (white dwarf in a ~~contact~~ semi detatched contact binary accreting matter and igniting), These estimates of looking at nova in our galaxy and comparing them to external nova led to a certain distance estimate for the external galaxies but not that good. In fact it was grotesquely wrong as what was being compared was supernova explosions in external galaxies with galactic nova explosions.

During the Curtis-Shapley debate Shapely took the more conservative but entirely wrong viewpoint that the spirals were comparatively close using as his evidence the distances estimated by Nova (Supernova) and arguing that they had to be close so that our galaxy could effect them. The idea being that since they are all receeding our galaxy must repell them and for this to be reasonable they must be smaller than our galaxy. If they are smaller they must be close as they have a known angular size. The alternative that the external spirals were as large as the milky way and that they were all moving at these enormous red shifts ($\sim c \times 10^{-3}$) just seemed a little far fetched. (mention zone of avoidance).

In 1923 the debate was settled by Hubble using the new 100" mount palomar telescope. He found two cepheid variables in andromeda and concluded a distance of 2,000,000 Ly to it. It's angular size then implied a radius of something like 60 ~~kpc~~ KLy. Hubble went on in his study of external galaxies classifying them by type (hubble classification system) and attempting to define the large scale structure of the universe (hubbles law), Before going into this we should discuss a bit the structure of the spirals as we now understand them.

The milky way is very similar in appearance to andromeda or other spiral galaxies. This can be deduced from 21 cm maps of the galaxy. Due to this it is better to analyze the structure of spirals by looking at external galaxies rather than our own even though it's closer! The reason is the externals can be seen at all angles (statistically) and as we only can see them out of the plane of our own galaxy we don't have a problem with light attenuation by interstellar dust.

A few comments on General Relativity.

General Relativity is a geometric interpretation of gravity based on the Principle of Equivalence. The principle of equivalence is a formal statement of the equality of gravitational and inertial mass. What this means is the following: in the low velocity (non relativistic) limit.

$$F = m_i a$$

where m_i is the inertial mass the measure of an objects ability to resist acceleration. Gravitation is a force whose source is the gravitational mass

$$F_{g_{12}} = \frac{-G\, M_{g_1} M_{g_2}}{r_{12}^2}\, \hat{r}_{12}$$

where M_{g_1}, M_{g_2} are the gravitational masses of objects one and two. Equating these to relations

$$F = F_g.$$

$$m_{i_2} a = \frac{-G\, M_{g_1}\, M_{g_2}}{r^2}\, \hat{r}$$

$$a = \frac{-G\, M_{g_2}}{r^2}\, \hat{r} \cdot \left[\frac{m_{g_1}}{m_{i_2}}\right]$$

The term in brakets is one by all measurements made to date

Another way of thinking about this is to consider a transformation to a coordinate system (an elevator for example) in free fall. ie consider two systems; one "stationary" in empty space where the laws of physics hold and can be written down in the local coordinates.

$$F = m\frac{d^2x}{dt^2} \qquad \text{etc.}$$

and one (our elevator) in free fall under the influence of a gravitational field. If we do the same experiment in both coordinate systems like apply a force to a mass and determine the acceleration, in the inertial frame (in empty space) we find.

$$\boxed{F = m\frac{d^2x}{dt^2}}$$

in the elevator there are two forces on the mass our test force and the force of the gravitational field.

$$F'' = F + F_g.$$

and to a stationary observer ~~who~~ watching the proceedings. ($X \equiv$ coord of stationary) ($X' \equiv$ coord. in elevator)

$$X = x' - \tfrac{1}{2}gt^2$$

$$V = \frac{dx}{dt} = \frac{dx'}{dt} - gt.$$

$$a = \frac{d^2x}{dt^2} = \frac{d^2x'}{dt^2} - g.$$

$\Rightarrow$ when F'' is applied in the elevator. (as seen by the stationary observer)

$$F'' = F + F_g = \cancel{F + mg}\ F - mg \qquad \text{(gravity points down)}$$

$$= ma = m\frac{d^2x'}{dt^2} - mg.$$

$$\Rightarrow \boxed{F = m\frac{d^2x'}{dt^2}}$$

ie the observer in empty space and the observer in the elevator see the same thing.

The formal way of stating this is to say that "at every space time point in an arbitrary gravitational field it is possible to choose a locally inertial coordinate system such that within a sufficiently small region of the point in question the laws of nature take the same form as in an unaccelerated cartesian coordinate system in the absence of gravitation" (Weinberg) This should be considered in parallel with the basis of non-euclidean geometry as first really understood by Gauss. Here the discussion was centered on Euclids fifth postulate

"If a straight line intersects one of two parallels it will intersect the other"

or.

~~Through a given point,~~

"For a given line, through a given point not on the line there is one and only one line parallel to the given line"

or.

"The sum of the angles of a triangle is 180°"

These three statements are equivalent. In fact it turns out to be possible to construct a logical geometry in which the fifth postulate is false. (which is why nobody could prove the fifth postulate) This is called non euclidean geometry. Gauss realized that such a geometry could be constructed and that it would have the local property that in the vicinity of a given point within a sufficiently small region the law of Pythagoras will be valid. This is very similar to the principle of equivalence.

Special relativity revisited.

Following Weinberg we will use the Minkowski metric.

ie. $d\tau^2 = dt^2 - [dx^2 + dy^2 + dz^2]\frac{1}{c^2}$

$\Rightarrow$ velocities become $v' \to v/c$

everything is scaled to $\frac{1}{c}$ (distances are in seconds)

define.

$$\eta_{\alpha\beta} \equiv \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{d\tau^2 = -\eta_{\alpha\beta}\, dx^\alpha dx^\beta}$$

implied summation over repeated ~~dx~~ index.

~~$\alpha \to 0 \to 3$~~

α goes from 0 to 3.

lorentz transformations.

$$dx'^\alpha = \Lambda^\alpha{}_\beta\, dx^\beta$$

$$\Lambda^0{}_0 = \gamma = \frac{1}{\sqrt{1-v^2}}$$

$$\Lambda^i{}_0 = v_i \Lambda^0{}_0 = v_i \gamma$$

$$\Lambda^i{}_j = \delta_{ij} + v_i v_j \frac{\gamma - 1}{v^2}$$

$\Big\}$ i goes from 1 to 3
j goes from 1 to 3

$\delta_{ij} = 1 \quad j = i$
$\quad\;\; = 0 \quad j \neq i$

$\Lambda(v)$ is the boost transformation it can be multiplied by an arbitrary rotation. ie taking a stationary particle to a frame where it is moving with velocity v it can be rotated arbitrarily before the boost.

in general vectors (4 vectors) will transform as

$$V'^{\alpha} = \Lambda^{\alpha}{}_{\beta} V^{\beta}$$

because $dx'^{\alpha} = \Lambda^{\alpha}{}_{\beta}\, dx^{\beta}$

$$\Rightarrow \quad \Lambda^{\alpha}{}_{\beta} = \frac{dx'^{\alpha}}{dx^{\beta}} \quad \text{or rather} \quad \frac{\partial x'^{\alpha}}{\partial x^{\beta}} = \Lambda^{\alpha}{}_{\beta}$$

There are two kinds of vectors covariant and contravariant and they are distinguished by their transformation properties. These can be written in two ways

note Λ has an inverse.

$$\Lambda_{\gamma}{}^{\beta} \Lambda^{\alpha}{}_{\beta} = \delta_{\gamma\alpha} \quad \text{ie unit matrix}$$

~~a contravariant vector~~

$$\frac{\partial x^{\beta}}{\partial x'^{\gamma}} \cdot \frac{\partial x'^{\alpha}}{\partial x^{\beta}} = \delta_{\gamma\alpha}$$ (a property of orthogonal coordinates)

a contravariant vector transforms as.

$$V'^{\alpha} = \Lambda^{\alpha}{}_{\beta} V^{\beta}$$

$$= \frac{\partial x'^{\alpha}}{\partial x^{\beta}} V^{\beta}$$

a covariant vector transforms as.

$$V'_{\alpha} = \Lambda_{\alpha}{}^{\beta} V_{\beta}$$

$$= \frac{\partial x^{\beta}}{\partial x'^{\alpha}} V_{\beta}$$

The second Important principle in general relativity is the principle of covariance. This means that the laws of physics must be form invariant or that the mathematical form of the laws of physics must be the same in any coordinate system. A further part of this is that not only must the form be the same but the numerical values on evaluation must not depend on any velocities of transformation such that the ~~answer~~ numerical value would indicate an absolute motion. It is only when comparisons with other frames of reference are made that the transformation velocities can be deduced.

The way the principle of covariance is used is the following:

If the laws of physics are formulated in vectors and Tensor form then by construction the laws of physics will be form invariant ~~to~~ different reference frames due to the transformation properties of Tensors (or vectors = tensors of rank 1)

Schematically this would be.

let. $\Lambda^{\alpha\beta\gamma}{}_{\delta\rho\phi}$ be the transformation of a third rank tensor (3 indecies).

and $T^{\delta\rho\phi}$ and $F^{\delta\rho\phi}$ be third rank tensors of physical parameters which satisfy the law of physics

$$F^{\delta\rho\phi} = (\text{const})\cdot T^{\delta\rho\phi}$$

This would correspond to $(4)^3 = (4 \text{ dimensions})^{3 \text{ Indecies}}$

$(4)^3 = 64$ equations.

by the transformation properties.

$$F'^{\alpha\beta\gamma} = \Lambda^{\alpha\beta\gamma}{}_{\delta\rho\psi} F^{\delta\rho\psi}$$

(repeated indecies summed over ie matrix multiplication)

and

$$T'^{\alpha\beta\gamma} = \Lambda^{\alpha\beta\gamma}{}_{\delta\rho\psi} T^{\delta\rho\psi}$$

$$\Rightarrow \quad F^{\delta\rho\psi} = \text{const } T^{\delta\rho\psi}$$

$$\Rightarrow \quad \Lambda^{\alpha\beta\gamma}{}_{\delta\rho\psi} F^{\delta\rho\psi} = \text{const} \cdot \Lambda^{\alpha\beta\gamma}{}_{\delta\rho\psi} T^{\delta\rho\psi}$$

$$F'^{\alpha\beta\gamma} = \text{const } T'^{\alpha\beta\gamma}$$

I doubt this means much to you but lets try an example : electrodynamics.

maxwell's equations. (Heaviside units). $C = 1$.

$$\nabla \cdot E = \rho$$

$$\nabla \times B = \frac{\partial E}{\partial t} + J$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

define

$$F^{\mu\nu} = \left(\frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}} \right)$$

note the derivative. $\frac{\partial}{\partial x^{\mu}}$ transforms as a covariant.

$\frac{\partial}{\partial x_{\mu}}$ transforms as a contravariant.

$$F^{12} = B_3 \quad F^{23} = B_1 \quad F^{31} = B_2$$

$$F^{01} = E_1 \quad F^{02} = E_2 \quad F^{03} = E_3$$

$$F^{\alpha\beta} = -F^{\beta\alpha}$$

the first two equations

$$\nabla \cdot E = \rho$$

$$\nabla \times B = \frac{\partial E}{\partial t} + j$$

can be written,

$$\frac{\partial}{\partial x^\alpha} F^{\alpha\beta} = -j^\beta$$

or $$\partial_\alpha F^{\alpha\beta} = -j^\beta$$

with $j^0 = \rho$

Consider the eq.

$$\partial_\alpha F^{\alpha 0} = -j^0 \qquad \text{time or 0 component.}$$

$$+\frac{\partial}{\partial x} F^{10} + \frac{\partial}{\partial y} F^{20} + \frac{\partial}{\partial z} F^{30} + \frac{\partial}{\partial t} F^{40} = -\rho.$$

$$F^{10} = -F^{01} = -E_1 \qquad F^{40} = 0$$

$$\Rightarrow \partial_\alpha F^{\alpha 0} = -\bar{\nabla} \cdot \bar{E} = -\rho$$

$$\partial_\alpha F^{\alpha 1} = -j^1 \qquad x \text{ component}$$

$$\frac{\partial}{\partial x} F^{11} + \frac{\partial}{\partial y} F^{21} + \frac{\partial}{\partial z} F^{31} + \frac{\partial}{\partial t} F^{01} = -j_x$$

$$0 \; \underbrace{- \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z}} + \frac{\partial E_x}{\partial t} = -j_x$$

$$-(\nabla \times B)_x + \frac{\partial E_x}{\partial t} = -j_x$$

$$\frac{\partial E_x}{\partial t} + j_x = (\nabla \times B)_x$$

now how does this transform. ?

$$\frac{\partial}{\partial x^\alpha} \rightarrow \Lambda_\beta{}^\alpha \frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial x'^\beta}$$

$$F^{\mu\gamma} \rightarrow \Lambda^\alpha{}_\mu \Lambda^\beta{}_\gamma F^{\mu\gamma} = F'^{\alpha\beta}$$

$$j^\beta \to \Lambda^\alpha{}_\beta\, j^\beta = j'^\alpha$$

now.

$$\frac{\partial F^{\alpha\beta}}{\partial x^\alpha} = -j^\beta$$

~~multiply~~ and sum eq

multiply each of 4 eq. by appropriate number in transformation matrix and sum.

$$\Lambda^\mu{}_\beta \frac{\partial F^{\alpha\beta}}{\partial x^\alpha} = -\Lambda^\mu{}_\beta\, j^\beta = -j'^\mu$$

$$\Lambda^\gamma{}_\alpha \cdot \Lambda_\gamma{}^\alpha = \delta_{\gamma\gamma}$$

$$\Rightarrow \Lambda^\mu{}_\beta \frac{\partial F^{\alpha\beta}}{\partial x^\alpha} = \Lambda^\mu{}_\beta \underbrace{\Lambda^\gamma{}_\alpha \Lambda_\gamma{}^\alpha}_{\delta_{\gamma\gamma}=1} \frac{\partial F^{\alpha\beta}}{\partial x^\alpha} = \frac{\partial F'^{\gamma\mu}}{\partial x'^\gamma}$$

$$\Rightarrow \boxed{\frac{\partial F'^{\gamma\mu}}{\partial x'^\gamma} = -j'^\mu}$$ same form in primed system.

another way, (slightly more rigorous)

$\frac{\partial}{\partial x^\alpha} F^{\beta\gamma}$ is a mixed tensor $T_\alpha{}^{\beta\gamma}$

and $\Rightarrow$ transforms as.

$$T'_\mu{}^{\gamma\delta} = \Lambda_\mu{}^\alpha \Lambda^\gamma{}_\beta \Lambda^\delta{}_\gamma\, T_\alpha{}^{\beta\gamma}$$

if we start with the eq. in the primed system.

$$\frac{\partial F'^{\gamma\mu}}{\partial x'^\gamma} = -j'^\mu$$

and write it in the unprimed variables.

$$\cancel{\Lambda_\delta{}^\gamma \Lambda^\alpha{}_\gamma \Lambda^\beta{}_\mu \frac{\partial}{\partial x}}$$

$$\Lambda_\gamma{}^\delta \Lambda^\gamma{}_\varepsilon \Lambda^\mu{}_\beta \frac{\partial F^{\delta\beta}}{\partial x^\delta} = -\Lambda^\mu{}_\beta\, j^\beta$$

$$\delta_{\gamma\delta}\, \Lambda^\mu{}_\beta \frac{\partial F^{\delta\beta}}{\partial x^\delta} = -\Lambda^\mu{}_\beta\, j^\beta$$

$$\Lambda^{\mu}{}_{\beta} \frac{\partial F^{\gamma\beta}}{\partial x^{\gamma}} = - \Lambda^{\mu}{}_{\beta}\, j^{\beta}$$

multiply both sides by $\Lambda_{\mu}{}^{\alpha}$ and sum 4 eq. in μ.

$$\Lambda_{\mu}{}^{\alpha} \Lambda^{\mu}{}_{\beta} \frac{\partial F^{\gamma\beta}}{\partial x^{\gamma}} = - \Lambda_{\mu}{}^{\alpha} \Lambda^{\mu}{}_{\beta}\, j^{\beta}$$

$$\delta_{\alpha\beta} \frac{\partial F^{\gamma\beta}}{\partial x^{\gamma}} = - \delta_{\alpha\beta}\, j^{\beta}$$

$$\boxed{\frac{\partial F^{\gamma\alpha}}{\partial x^{\gamma}} = - j^{\alpha}}$$

in fact the product of a covariant and contravariant summed over the index is a scaler hence

$\frac{\partial F^{\alpha\beta}}{\partial x^{\alpha}}$ is a Tensor of rank 1

as is j^{α} the 4 component current.

and $\partial_{\alpha} F^{\alpha\beta} = - j^{\beta}$ is a tensor eq. of rank 1 and is $\Rightarrow$ form invariant under relativistic (Lorentz) transformations.

theorem if two tensors, with ~~an eq~~ the same upper and lower indecies, are equal in one coordinate system then they are equal in any other coordinate system connected to the first by a lorentz transformation and in particular if a tensor vanishes ($= 0$) in any coordinate system it is zero in all coordinate systems. (weinberg' Pg 39).