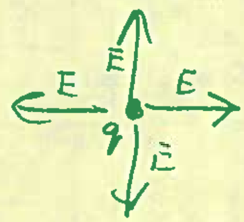
differential formulation of the laws of electromagnetism. (needed for wave eq.)

I. Gauss' law.

Charge is the source of electric fields. The field spreads out from a positively charged point source.



The mathematical way of describing this spreading out of a vector field is through the "Differential Operator", $\overline{\nabla}$ (del)

$$\overline{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Thus, for example, a scaler field $\phi(x,y,z)$ has a "Gradient", $\overline{\nabla}\phi$

$$\overline{\nabla}\,\phi(x,y,z) = \overline{V}(x,y,z) \quad \text{(a \underline{vector} field)}$$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Thus the $\hat{z}$ component of $\overline{V} = V_z = \frac{\partial\phi}{\partial z}$ etc.

In a way you can say $\overline{\nabla}$ (del) acts on the scaler, ϕ, producing a vector, $\overline{V}$.

$$\overline{V} = \overline{\nabla}\,\Phi$$

The $\bar{\nabla}$ operator can be treated like any vector and can be used to create "dot" products and "cross" products.

for a vector function $\bar{F}(x,y,z) =$ ~~$\hat{F}_x\hat{i}\hat{F}_y$~~

$$\bar{\nabla}\cdot\bar{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

as. $\bar{F}(x,y,z) = F_x(x,y,z)\,\hat{i} + F_y(x,y,z)\,\hat{j} + F_z(x,y,z)\,\hat{k}$

similarly

$$\bar{\nabla}\times\bar{F} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k} + \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{j}$$

1c) differential form of Gauss' law.

Gauss' law is

$$\phi_E = \int_{\text{Closed surface}} \bar{E}\cdot\bar{da} = \frac{1}{\varepsilon}\int_{\text{enclosed volume.}} \rho(x,y,z)\, dx\,dy\,dz$$

where ϕ_E is the electric flux out of a closed volume, ρ is the charge density.

thus the law states that the flux out of a closed surface is equal to $\frac{1}{\varepsilon} *$ (enclosed charge).

Gauss' theorem.

it can be shown that for <u>any</u> (continuous, differentiable etc ie reasonable) vector function $V(x,y,z)$

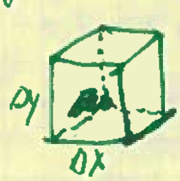
$$\int_{\text{closed Surface}} \bar{V}\cdot\bar{da} = \int_{\text{enclosed volume}} \bar{\nabla}\cdot\bar{V}\, dx\,dy\,dz$$

any volume can be made of a sum of infintesmal ~~volumes~~. cartesian volumes.

$$V_{olume} = \sum \Delta Volume_i$$

(not to be confused with $\bar{V}(x,y,$ sorry.

$$\Delta Volume_i = \bar{\Delta X}_i \cdot (\bar{\Delta Y}_i \times \bar{\Delta Z}_i)$$



origin in back corner right hand coord. system

consider $(\bar{\nabla}\cdot\bar{F})\, d\text{Volume}$. $\bar{F}$ some vector field.

$$\bar{\nabla}\cdot\bar{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

$$\bar{\nabla}\cdot\bar{F}\,\Delta x\,\Delta y\,\Delta z = \left(\frac{\partial F_x}{\partial x}\Delta x\right)\Delta y\,\Delta z + \left(\frac{\partial F_y}{\partial y}\Delta y\right)\Delta x\,\Delta z + \left(\frac{\partial F_z}{\partial z}\Delta z\right)\Delta x\,\Delta y$$

rearanging terms

let $F_{x_L} \equiv F_x(x_{Low})$ $\qquad x_{high} \equiv x_{low} + \Delta x$

$F_{x_h} = F_x(x_{high})$

$F_{y_L} = F_y(y_{low})$ etc.

$$(\bar{\nabla}\cdot\bar{F})\,\Delta x\,\Delta y\,\Delta z = (F_{x_h} - F_{x_L})\,\Delta y\,\Delta z + (F_{y_h} - F_{y_L})\,\Delta x\,\Delta z + (F_{z_h} - F_{z_L})\,\Delta x\,\Delta y.$$

as $\frac{\partial F_x}{\partial x}\Delta x = \Delta F_x = F_{x_h} - F_{x_L}$

$\Delta y\,\Delta z$ is the area of the sides at constant x

$\Delta x\,\Delta z$ is the area of the sides at constant y

$\Delta x\,\Delta y$ is the area of the sides at constant z.

ie $|da|$ (magnitude).

if I <u>define</u> $\bar{da}$ (the vector) to be equal

to magnitude $|da|$ in the direction pointed outward

$$\Rightarrow F_{x_h} |\Delta y \Delta z| = \bar{F}_{x_h} \cdot \overline{da}$$

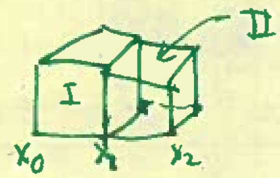
$$F_{x_L} \, \Delta y \Delta z = - \bar{F}_{x_L} \cdot \overline{da}$$

since da is pointed in the $-\hat{\iota}$ direction for this side.

$$\Rightarrow (\bar{\nabla} \cdot \bar{F})(\Delta x \Delta y \Delta z) = \sum_{1}^{6} \bar{F} \cdot \overline{da}_i$$

= Flux out of the infintesmal volume.

Consider two adjacent infintesmal volumes with a common surface at x_1



The (flux "out" of box I at x_1) =

− (flux "out" of box II at x_1)

because $\overline{da}_{I}$ (at x_1) $= -\overline{da}_{II}$ (at x_1)

being in opposite directions.

⇒ when I sum up the little boxes to make the big volume.

<u>All</u> <u>INTERIOR</u> <u>SURFACES</u> <u>CANCEL</u>

$$\sum \left(\bar{\nabla}\cdot\bar{F}\right)\left(\Delta y_i \Delta x_i \Delta z_i\right) = \sum \left(\sum_{j\in I}^{6} \bar{F}\cdot d\bar{a}_j\right) =$$

$$= \int_{\substack{\text{exterior}\\ \text{surface.}}} \bar{F}\cdot d\bar{a}$$

$$\int_{\substack{\text{entire}\\ \text{volume.}}} \bar{\nabla}\cdot\bar{F}\, dx\,dy\,dz = \int_{\substack{\text{exterior}\\ \text{surface}}} \bar{F}\cdot d\bar{a}$$

if this is true for <u>any</u> vector field. F it is true for the electric field. this is <u>Gauss' theorem</u>

$$\Rightarrow \int_{\substack{\text{closed}\\ \text{surface}}} \bar{E}\cdot d\bar{a} = \int_{\substack{\text{enclosed}\\ \text{volume}}} \bar{\nabla}\cdot E\, dx\,dy\,dz = \frac{1}{\varepsilon}\int \rho(x,y,z)\, dx\,dy\,dz.$$

$$\text{as} \int dx\,dy\,dz\,(\nabla\cdot E) = \frac{1}{\varepsilon}\int \rho\, dx\,dy\,dz$$

$$\Rightarrow \int dx\,dy\,dz\left(\bar{\nabla}\cdot\bar{E} - \frac{\rho}{\varepsilon}\right) = 0$$

$$\Rightarrow \boxed{\nabla\cdot E = \frac{\rho}{\varepsilon}}$$

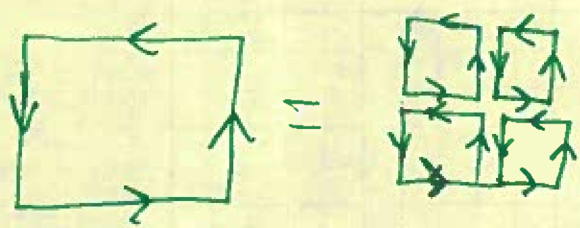
<u>local</u> (at every point) differential relation.

thus $\nabla\cdot E$ measures the spreading out of the field caused by sources.

II Ampère's law and Stokes' theorem.

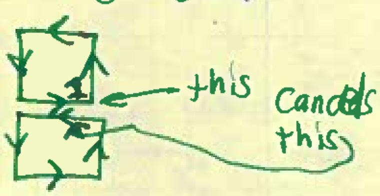
$$\oint_{\substack{\text{closed}\\\text{loop}}} \bar{B}\cdot d\bar{l} = \mu \int_{\substack{\text{encircled}\\\text{area}}} \bar{j}\cdot d\bar{a} = \mu\cdot\left(\begin{array}{c}\text{current through}\\\text{surface}\end{array}\right)$$

$\oint_{\substack{\text{closed}\\\text{loop}}} B\cdot dl$ can be made up from a large number of very small loops.

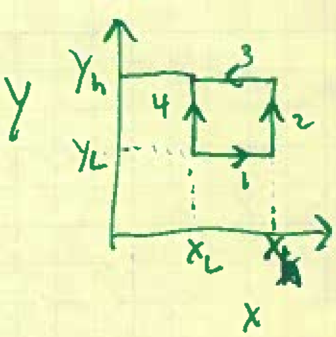


Sum over 4.

because the contributions over the interior line segments cancel.



What is the line integral around an infintesmal loop in the $x\,y$ plane?



$$\int \bar{B}\cdot d\bar{L} = B_x(Y_L)\cdot\Delta X + B_y(x_h)\,\Delta Y - B_x(Y_h)\,\Delta X - B_y(x_L)\,\Delta Y$$

where $\Delta X = X_h - X_L$
$\Delta Y = Y_h - Y_L$
$B_x(Y_L) = B_x$ evaluated at Y_L etc.

$$\int B\cdot dl = \left(B_x(y_L) - B_x(y_H)\right)\Delta x + \left(B_y(x_H) - B_y(x_L)\right)\Delta y.$$

$$= \left(-\frac{\partial B_x}{\partial y}\Delta y\right)\Delta x + \left(\frac{\partial B_y}{\partial x}\Delta x\right)\Delta y.$$

$$= \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\Delta x\Delta y$$

but $$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \left(\bar{\nabla}\times\bar{B}\right)_z$$

$$\Rightarrow \int \bar{B}\cdot d\bar{l} = \int(\bar{\nabla}\times\bar{B})\cdot d\bar{a} = \mu\int\bar{j}\cdot d\bar{a}$$

closed loop

enclosed surface.

This equality is true for all vector functions and is known as STOKES' Theorem.

$$\Rightarrow \int d\bar{a}\cdot\left(\bar{\nabla}\times\bar{B} - \bar{j}\mu\right) = 0$$

$$\boxed{\Rightarrow \bar{\nabla}\times\bar{B} = \mu\bar{j}}$$

In fact another term will be added.

no magnetic monopoles.

There are no magnetic monopoles.
This is a <u>FaCT</u> (at least none have ever been observed and so it is a FaCT until shown otherwise).
As there are no sources for magnetic fields as there are for electric fields.

$$\bar{\nabla} \cdot \bar{B} = 0$$

Faradays law.

$$\oint E \cdot dl = -\frac{d}{dt}\int B \cdot da.$$

but as shown in the discussion of Ampères' law and Stokes' theorem.

$$\oint_{\text{closed loop}} \bar{B} \cdot d\bar{l} = \int_{\text{enclosed area}} (\nabla \times B) \cdot da$$

$$\oint_{\text{closed loop}} \bar{E} \cdot d\bar{l} = \int_{\text{enclosed area}} (\bar{\nabla} \times \bar{E}) \cdot d\bar{a}$$

$$\Rightarrow \oint E \cdot d\bar{l} = \int (\bar{\nabla} \times \bar{E}) \cdot d\bar{a} = \int da \cdot \left(-\frac{\partial B}{\partial t}\right)$$

$$\Rightarrow \int d\bar{a} \cdot \left(\nabla \times E - \frac{\partial B}{\partial t}\right) = 0$$

$$\Rightarrow \bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

Something is missing.

Charge is conserved. For charge to stream out of a volume ~~it must be~~ the charge density must decrease this is mathematically written as.

$$\bar{\nabla}\cdot\bar{J} \quad -\frac{\partial\rho}{\partial t}$$

(divergence of the corent) = - time derivative of the current.

to see this integrate it over a fixed volume.

$$\int_{\text{Volume}} \bar{\nabla}\cdot\bar{J}\, d^3x = \int_{\substack{\text{closed}\\\text{surface}}} \bar{J}\cdot d\bar{a} \qquad \text{by gauss' theorem.}$$

but the flux of current out of a volume is clearly $-\frac{\partial Q}{\partial t}$ where Q is the charge left.

$$-\frac{\partial Q}{\partial t} = -\frac{\partial}{\partial t}\int \rho\, d^3x = \int(\bar{\nabla}\cdot\bar{J})\, d^3x$$

hence $\nabla\cdot j = -\frac{\partial\rho}{\partial t}.$

but

$$\nabla \times B = \mu\bar{j}$$

$$\Rightarrow \bar{\nabla}\cdot(\bar{\nabla}\times\bar{B}) = \mu(\bar{\nabla}\cdot\bar{J})$$

but

$$\bar{\nabla}\cdot(\bar{\nabla}\times\bar{B}) = \bar{\nabla}\cdot\left(\left[\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right]\hat{i} + \left[\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right]\hat{j} + \left[\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right]\hat{k}\right)$$

cancel

$$= \frac{\partial^2 B_z}{\partial x \partial y} - \frac{\partial^2 B_y}{\partial x \partial z} + \frac{\partial^2 B_x}{\partial y \partial z} - \frac{\partial^2 B_z}{\partial y \partial x} + \frac{\partial^2 B_y}{\partial z \partial x} - \frac{\partial^2 B_x}{\partial z \partial y}$$

Cancel. cancel

$$= 0.$$

note $A\cdot(\bar{A}\times\bar{B}) = 0$ as $\bar{A}\times\bar{B}$ is $\perp$ to $\bar{A}$.

$$\Rightarrow \bar{\nabla}\cdot(\bar{\nabla}\times\bar{B}) = \mu(\bar{\nabla}\cdot\bar{J}) = 0$$

but this is wrong!

The realization of this inconsistency in the laws of electromagnetism is perhaps J. Maxwells sign of genius.

but lets start again.

$$\bar{\nabla}\times\bar{B} = \mu j + ?$$

$$\bar{\nabla}\cdot\bar{J} + \frac{\partial \rho}{\partial t} = 0$$ Charge conservation.

but $\frac{\rho}{\varepsilon} = \bar{\nabla}\cdot E$?? hmmmm.

$$\Rightarrow \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}(\varepsilon\, \bar{\nabla}\cdot E) = \varepsilon\left(\bar{\nabla}\cdot \frac{\partial \bar{E}}{\partial t}\right)$$

$$\Rightarrow \bar{\nabla}\cdot(\bar{\nabla}\times\bar{B}) = \nabla\cdot j + ? = 0$$

$$\text{but} \quad \nabla\cdot j + \frac{\partial \rho}{\partial t} = 0$$

$$= \nabla\cdot j + \varepsilon\, \bar{\nabla}\cdot\frac{\partial \bar{E}}{\partial t} = 0$$

$$= \bar{\nabla}\cdot\left(\bar{j} + \varepsilon\frac{\partial \bar{E}}{\partial t}\right) = 0$$

$$\Rightarrow \boxed{\bar{\nabla}\times\bar{B} = \mu\bar{j} + \varepsilon\mu\frac{\partial \bar{E}}{\partial t}}$$

where $\varepsilon \frac{\partial \bar{E}}{\partial t}$ is called the Displacement Current and is added to the electrical current

$$\bar{\nabla}\times\bar{B} = \mu\bar{J} + \varepsilon\mu\frac{\partial \bar{E}}{\partial t}$$

now the fun begins.!

Maxwells equations.

I $\quad \bar{\nabla}\cdot\bar{E} = \dfrac{\rho}{\varepsilon}$

II $\quad \bar{\nabla}\cdot\bar{B} = 0$

III $\quad \bar{\nabla}\times\bar{E} = -\dfrac{\partial B}{\partial t}$

IV $\quad \bar{\nabla}\times\bar{B} = \mu\bar{j} + \varepsilon\mu\dfrac{\partial E}{\partial t}.$

In free empty space $\rho = \bar{j} = 0$

$\varepsilon = \varepsilon_0 \qquad \mu = \mu_0$

Consider

$$\bar{\nabla}\times(\bar{\nabla}\times\bar{F}) = ?$$

$$\bar{A}\times(\bar{B}\times\bar{C}) = ?$$

let $\bar{D} = \bar{B}\times\bar{C}$

$$\bar{A}\times\bar{D} = (A_y D_z - A_z D_y)\hat{i} + (A_z D_x - A_x D_z)\hat{j} + (A_x D_y - A_y D_x)\hat{k}$$

$$D_x = (B_y C_z - B_z C_x)$$
$$D_y = (B_z C_x - B_x C_z)$$
$$D_z = (B_x C_y - B_y C_x)$$

$\Rightarrow A\times(B\times C) = \bar{B}(\bar{A}\cdot\bar{C}) - \bar{C}(\bar{A}\cdot\bar{B})$ (bac - cab)

$$\Rightarrow \boxed{\bar{\nabla}\times\bar{\nabla}\times\bar{F} = \bar{\nabla}(\bar{\nabla}\cdot\bar{F}) - (\bar{\nabla}\cdot\bar{\nabla})\bar{F}}$$

In free space.

$$\bar{\nabla}\cdot\bar{E} = 0$$
$$\bar{\nabla}\cdot\bar{B} = 0$$
$$\nabla \times E = -\frac{\partial B}{\partial t}$$
$$\bar{\nabla}\times\bar{B} = \varepsilon_0 \mu_0 \frac{\partial E}{\partial t}.$$

$$\nabla \times (\bar{\nabla}\times\bar{B}) = \varepsilon_0 \mu_0 \; \nabla \times \frac{\partial E}{\partial t} = \varepsilon_0 \mu_0 \frac{\partial}{\partial t}\left(\bar{\nabla}\times\bar{E}\right)$$
$$= -\varepsilon_0 \mu_0 \frac{\partial}{\partial t}\left(\frac{\partial B}{\partial t}\right) = -\varepsilon_0 \mu_0 \frac{\partial^2 B}{\partial t^2}.$$
$$\bar{\nabla}\times(\bar{\nabla}\times\bar{B}) = \bar{\nabla}(\bar{\nabla}\cdot\bar{B}) - \nabla^2 B = -\nabla^2 B \quad \text{as} (\nabla\cdot B = 0)$$
$$\Rightarrow \; -\nabla^2 B = -\varepsilon_0 \mu_0 \frac{\partial^2 B}{\partial t^2}$$
$$\nabla^2 B = \varepsilon_0 \mu_0 \frac{\partial^2 B}{\partial t^2}.$$

similarly

$$\nabla \times (\bar{\nabla}\times\bar{E}) = -\frac{\partial}{\partial t}\left(\bar{\nabla}\times\bar{B}\right) = -\frac{\partial}{\partial t}\left(\varepsilon_0 \mu_0 \frac{\partial E}{\partial t}\right) = -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}.$$
$$\nabla(\bar{\nabla}\cdot\bar{E}) - \nabla^2 E = -\nabla^2 E \quad \text{as } \nabla\cdot E = 0 \text{ in empty space.}$$
$$\Rightarrow \; \nabla^2 \bar{E} = \varepsilon_0 \mu_0 \frac{\partial^2 \bar{B}}{\partial t^2}.$$
$$\nabla^2 \bar{B} = \varepsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2}$$

this is a wave equation with velocity $= \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

$$\nabla^2 W = \frac{1}{V^2}\frac{\partial^2 W}{\partial t^2} \qquad \boxed{\frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3\times 10^8 \, m/s. = c}$$

truely Cultural.

Potentials.

$$\bar{\nabla}\cdot\bar{E} = \frac{\rho}{\varepsilon}$$

$$\bar{\nabla}\cdot\bar{B} = 0$$

if $\bar{\nabla}\cdot\bar{B} = 0 \Rightarrow \bar{B} = \bar{\nabla}\times\bar{A}$ where

$\bar{A}$ is the vector potential as.

$\bar{\nabla}\cdot(\bar{\nabla}\times\bar{A}) = 0$ ~~by definition~~ ~~and~~ shown

$$\bar{\nabla}\times\bar{B} = \nabla\times(\bar{\nabla}\times\bar{A}) = \bar{\nabla}(\bar{\nabla}\cdot\bar{A}) - \nabla^2 A$$

$E = -\nabla\phi + ?$ where ϕ is the electric potential

$$\bar{\nabla}\times\bar{E} = -\frac{\partial B}{\partial t} = -\frac{\partial}{\partial t}(\bar{\nabla}\times\bar{A}) = -\bar{\nabla}\times\frac{\partial\bar{A}}{\partial t}$$

$$= \bar{\nabla}\times(-\bar{\nabla}\phi + \bar{F})$$

$\bar{\nabla}\times\bar{\nabla}\phi = 0$ a vector cross itself is 0

$$\Rightarrow \bar{\nabla}\times\bar{F} = \nabla\times\left(-\frac{\partial A}{\partial t}\right) \Rightarrow \bar{F} = -\frac{\partial\bar{A}}{\partial t}$$

$$\bar{E} = -\bar{\nabla}\phi - \frac{\partial\bar{A}}{\partial t}$$

$$B = \bar{\nabla}\times\bar{A}$$

Can I change ϕ and $\bar{A}$ in such a way that $\bar{E}$ and $\bar{B}$ (which can be observed) will not be changed so that Physics is not changed?

Gauge Symmetry.

minus

let $\phi' = \phi - \frac{\partial \Lambda}{\partial t}$

Where Λ is a function any function of x y z and t

let $A' \to A + \bar{\nabla}\Lambda$

$$\nabla \times A' = \bar{\nabla} \times \bar{A} + \bar{\nabla} \times (\bar{\nabla}\Lambda)$$

$$= \bar{\nabla} \times \bar{A} = B \checkmark \text{ unchanged}$$

$$-\bar{\nabla}\phi' - \frac{\partial A'}{\partial t} = \left(-\nabla\phi + \bar{\nabla}\left(\frac{\partial\Lambda}{\partial t}\right)\right) - \frac{\partial}{\partial t}\left(\bar{A} + \bar{\nabla}\Lambda\right)$$

$$= -\nabla\phi + \bar{\nabla}\left(\frac{\partial\Lambda}{\partial t}\right) - \frac{\partial\bar{A}}{\partial t} - \frac{\partial}{\partial t}(\nabla\Lambda)$$

$$= -\nabla\phi - \frac{\partial\bar{A}}{\partial t} + \bar{\nabla}\left(\frac{\partial\Lambda}{\partial t}\right) - \frac{\partial}{\partial t}(\nabla\Lambda)$$

$$= -\nabla\phi - \frac{\partial A}{\partial t} = E \checkmark \text{ unchanged}$$

thus the potentials can be changed by the time and space derivatives of an absolutely arbitrary function $\Lambda(x,y,z,t)$ such that

$$\phi' \to \phi - \frac{\partial \Lambda}{\partial t}$$

$$\bar{A}' \to \bar{A} + \bar{\nabla}\Lambda$$

and there are <u>no</u> consequences.

It can be shown that this symmetry is due to conservation of charge.

Gauge invariance symmetry is the basis for all modern theories of.

electromagnetism

weak nuclear interactions

Strong nuclear interactions,

and gravitation.

Ie all the forces of nature.

Calculation of $\bar{\Phi}$ and $\bar{A}$

$$\nabla \cdot E = \frac{\rho}{\varepsilon} = -\nabla^2 \phi \quad \text{laplaces' eq.}$$

the solution of this equation is known

$$\phi(r) = \frac{1}{4\pi\varepsilon} \int d^3r' \frac{\rho(r')}{|\bar{r} - \bar{r}'|}$$

as you have seen from the Biot - Savart law.

$$d\bar{B} = \frac{\mu_0}{4\pi} \frac{j\,d\bar{l} \times \hat{r}}{r^2}$$

now we know.

$$\bar{\nabla} \times \bar{B} = \mu_0 \bar{J}$$

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{A}) = \mu_0 \bar{J}$$

$$\bar{\nabla}(\bar{\nabla} \cdot \bar{A}) - \nabla^2 \bar{A} = \mu_0 \bar{J}$$

we can set $\bar{\nabla} \cdot \bar{A} = 0$ by a gauge transform.

$$\bar{A}' = \bar{A} - \bar{\nabla}\Lambda$$

$$\bar{\nabla} \cdot \bar{A}' = \bar{\nabla} \cdot \bar{A} - \nabla^2 \Lambda$$

and this can be set to zero and will not affect $\bar{B}$ as we have shown

$$\Rightarrow -\nabla^2 \bar{A} = \mu_0 \bar{J} \quad \text{(3 Laplace equations!}$$

$$\Rightarrow \boxed{\bar{A}(\bar{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\bar{J}(r')}{|\bar{r}-\bar{r}'|}}$$

Maxwells eq. with potentials.

$$\bar{\nabla}\cdot\bar{E} = \frac{\rho}{\varepsilon_0}$$

$$\bar{\nabla}\cdot\bar{B} = 0$$

$$\bar{\nabla}\times\bar{E} = -\frac{\partial B}{\partial t}$$

$$\bar{\nabla}\times\bar{B} = \mu_0\bar{J} + \varepsilon\mu\frac{\partial E}{\partial t}$$

$$\bar{\nabla}\cdot\bar{E} = -\nabla^2\phi - \bar{\nabla}\cdot\frac{\partial\bar{A}}{\partial t} = -\nabla^2\phi - \frac{\partial}{\partial t}(\bar{\nabla}\cdot\bar{A})$$

and as we have shown we can <u>choose</u> a gauge where $\bar{\nabla}\cdot\bar{A} = 0$

$$\Rightarrow \boxed{\nabla\cdot E = -\nabla^2\phi = \frac{\rho}{\varepsilon_0}}$$

$\bar{\nabla}\cdot\bar{B} = 0 \Rightarrow \bar{B} = \bar{\nabla}\times\bar{A}$ but we knew this

$$\bar{\nabla}\times\bar{E} = \bar{\nabla}\times\left(-\bar{\nabla}\phi - \frac{\partial A}{\partial t}\right) = -\nabla\times\bar{\nabla}\phi - \bar{\nabla}\times\frac{\partial\bar{A}}{\partial t}$$

$$= -\frac{\partial}{\partial t}(\nabla\times\bar{A}) = -\frac{\partial\bar{B}}{\partial t} \text{ by definition } \checkmark$$

so no new information! this is in fact held within the definition of the potentials!

last one.

$$\nabla \times B = \mu j + \varepsilon \mu \frac{\partial E}{\partial t}$$

in free empty space $\varepsilon = \varepsilon_0$, $\mu = \mu_0$, $\bar{J} = 0$, $\rho = 0$

$$\nabla \times B = \varepsilon_0 \mu_0 \frac{\partial E}{\partial t}.$$

$$\bar{\nabla} \times (\bar{\nabla} \times \bar{A}) = \varepsilon_0 \mu_0 \frac{\partial}{\partial t}\left(-\bar{\nabla}\phi - \frac{\partial \bar{A}}{\partial t}\right)$$

$$\bar{\nabla}(\bar{\nabla} \cdot \bar{A}) - \bar{\nabla}^2 A = \varepsilon_0 \mu_0 \left[-\bar{\nabla}\left(\frac{\partial \phi}{\partial t}\right) - \frac{\partial^2 \bar{A}}{\partial t^2}\right].$$

rewrite.

$$\nabla^2 \bar{A} - \varepsilon_0 \mu_0 \frac{\partial^2 \bar{A}}{\partial t^2} = \bar{\nabla}(\bar{\nabla} \cdot \bar{A}) + \bar{\nabla}\left(\varepsilon_0 \mu_0 \frac{\partial \phi}{\partial t}\right)$$

$$= \bar{\nabla}\left(\bar{\nabla} \cdot \bar{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t}\right).$$

just as we made $\bar{\nabla} \cdot \bar{A} = 0$ by a gauge transformation we can make

$\frac{\partial \phi}{\partial t} = 0$ ~~by setting~~ $\cancel{A = \int dt\, \phi}$ actually better just to explicitly $\rightarrow \frac{\partial^2 \lambda}{\partial t^2} = \frac{\partial \phi}{\partial t}$

$$\Rightarrow \nabla^2 \bar{A} - \varepsilon_0 \mu_0 \frac{\partial^2 \bar{A}}{\partial t^2} = 0$$

$$\boxed{\begin{aligned} &\nabla^2 \bar{A} - \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} = 0 \quad \text{3 eq.} \\ &\text{but } \bar{\nabla} \cdot \bar{A} = 0 \quad \Rightarrow \underline{\underline{\text{only 2 free eq.}}} \end{aligned}}$$