A Generic FDTD Form for Maxwell Equations

David Ge (dge893@gmail.com)

October 21, 2020

Abstract

A generic FDTD form for the Maxwell equations is derived via the proving of time advancement theorems and curls cascade theorem. Other FDTD algorithms, i.e. Yee algorithm and its varieties, are special cases of this generic form by specifying data sampling schemes and time space estimation orders. The maximum time advancement estimation orders are determined by the maximum number of space curl estimations. The maximum space curl estimation orders are determined by the numbers of the sampling points.

Contents

Introduction

In trying to understand the essences of FDTD algorithms, I came up with this generic form of FDTD. Other FDTD algorithms, i.e. Yee algorithm [1] and its varieties, Time-Space-Synchronized FDTD [2], are special cases of this generic form by specifying data sampling schemes and time space estimation orders.

The generic form is for estimations based purely on Taylor's series. They do not cover those algorithms involving techniques other than the Taylor's series, i.e. Manry et al [3].

I believe this generic form not only reveals the essences of various FDTD algorithms, but it can also be used in theoretical researches, i.e. error analysis, comparing of algorithms. It can also be of practical use too. For example, it predicts that high time estimation order cannot be achieved via multiple sets of historical data; it has to be achieved via the number of space samplings. It can be applied to irregular geometry with high estimation orders both in time and space.

This paper does not deal with the boundary conditions. A FDTD algorithm can be independent of the boundary problems, at least for regions far away from the boundary and for a short time. I remind the readers of this fact because this paper does not deal with a well-posed problem. For a reader lack of software engineering background, he/she might immediately jump to a conclusion that my work is incorrect as soon as he/she sees that I am dealing with a not well-posed problem.

Notations

Consider the following Maxwell equations.

$$\nabla \cdot E = \rho/\varepsilon \tag{1}$$

$$\nabla \cdot H = 0 \tag{2}$$

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon} \nabla \times H - J(t) \tag{3}$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu} \nabla \times E \tag{4}$$

Where t is time, $\rho, \varepsilon\ and\ \mu$ are time-invariants, $J(t)$ is a known 3D vector time function, $E\ and\ H$ are 3D vectors in Cartesian coordinates (x, y, z), representing an electric field and a magnetic field, respectively, as

$$E(x,y,z,t) = \begin{bmatrix} E_x(x,y,z,t) \\ E_y(x,y,z,t) \\ E_z(x,y,z,t) \end{bmatrix}, H(x,y,z,t) = \begin{bmatrix} H_x(x,y,z,t) \\ H_y(x,y,z,t) \\ H_z(x,y,z,t) \end{bmatrix} \tag{5}$$

For a 3D vector V, its cascade curls is denoted as following.

$\nabla^{\{0\}} \times V \equiv V, \nabla \times V \equiv \nabla^{\{1\}} \times V \equiv \begin{bmatrix} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \\ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{bmatrix}, \nabla^{\{k\}} \times V \equiv \underbrace{\nabla \times \nabla \times \ldots \times \nabla \times}_{k} V, k > 0$	(6)

Based on the above definitions, we have

$\nabla^{\{k\}} \times \nabla^{\{h\}} \times V = \nabla^{\{k+h\}} \times V, k, h \geq 0$	(7)

In this paper, I assume that the order of temporal derivative and space curls is exchangeable on $E(t), H(t)\ and\ J(t)$, such that

$$\frac{\partial\left(\nabla^{\{k\}} \times E(t)\right)}{\partial t} = \nabla^{\{k\}} \times \frac{\partial E(t)}{\partial t}$$

$$\frac{\partial\left(\nabla^{\{k\}} \times H(t)\right)}{\partial t} = \nabla^{\{k\}} \times \frac{\partial H(t)}{\partial t}$$

$$\frac{\partial\left(\nabla^{\{k\}} \times J(t)\right)}{\partial t} = \nabla^{\{k\}} \times \frac{\partial J(t)}{\partial t}$$

Time Advancement

Time Advancement Theorems

Time Advancement Theorem H. Given a set of field data $E(t_h)$, H, and H may have one or more sets of data in different times as classified in 3 cases identified by an integer q_h in following way

$\begin{cases} H(t_h), q_h = 0 \\ H(t_h - \Delta_{t1}), \quad q_h = 1, \Delta_t = \Delta_{t1} \\ H(t_h), H\left(t_h - \Delta_{tq}\right), q_h > 0, q = 1,2, \ldots, q_h \end{cases}$ $\Delta_t, \Delta_{t1}, \Delta_{t2}, \ldots, \Delta_{tqh} > 0$	(8)

$H(t_h + \Delta_t)$ can be expressed by

$Case\ 1: H(t_h), q_h = 0 \rightarrow$ $H(t_h + \Delta_t) = H(t_h) + \sum_{k=0}^{\infty}\left(\frac{\Delta_t^{2k+1}}{(2k+1)!}\left[\frac{1}{\mu}\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\nabla^{\{2k+1\}} \times E(t_h) + J_h^{[2k+1]}(t_h)\right] + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!}\left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^{k+1}}\nabla^{\{2(k+1)\}} \times H(t_h) + J_h^{[2(k+1)]}(t_h)\right]\right)$	(9)
$Case\ 2: H(t_h - \Delta_{t1}), q_h = 1, \Delta_t = \Delta_{t1} \rightarrow$ $H(t_h + \Delta_t) = H(t_h - \Delta_t) + \sum_{k=0}^{\infty}\left(\frac{2\Delta_t^{2k+1}}{(2k+1)!}\left[\frac{1}{\mu}\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\nabla^{\{2k+1\}} \times E(t_h) + J_h^{[2k+1]}(t_h)\right]\right)$	
$Case\ 3: H(t_h), H\left(t_h - \Delta_{tq}\right), q_h > 0, q = 1,2, \ldots, q_h \rightarrow$	

$$H(t_h+\Delta_t)=\sum_{q=1}^{q_h}H(t_h-\Delta_{tq})+(1-q_h)H(t_h)$$
$$+\sum_{k=0}^{\infty}\left(\frac{\Delta_t^{2k+1}+\sum_{q=1}^{qh}(\Delta_{tq})^{2k+1}}{(2k+1)!}\left[\frac{1}{\mu}\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\nabla^{\{2k+1\}}\times E(t_h)+J_h^{[2k+1]}(t_h)\right]\right.$$
$$\left.+\frac{\Delta_t^{2(k+1)}-\sum_{q=1}^{qh}(\Delta_{tq})^{2(k+1)}}{(2(k+1))!}\left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^{k+1}}\nabla^{\{2(k+1)\}}\times H(t_h)+J_h^{[2(k+1)]}(t_h)\right]\right)$$

Where

$$J_h^{[2k+1]}(t_h)=\begin{cases}\vec{0},k=0\\ \frac{1}{\mu}\sum_{i=1}^{k}\frac{(-1)^{i-1}}{(\varepsilon\mu)^{i-1}}\nabla^{\{2i-1\}}\times\frac{d^{2(k-i)+1}J(t)}{dt^{2(k-i)+1}},k>0\end{cases}\tag{10}$$

$$J_h^{[2(k+1)]}(t_h)=\frac{1}{\mu}\sum_{i=0}^{k}\frac{(-1)^i}{(\varepsilon\mu)^i}\nabla^{\{2i+1\}}\times\frac{d^{2(k-i)}J(t)}{dt^{2(k-i)}},k\geq 0\tag{11}$$

Note that Case 1 and Case 2 are two special cases of Case 3. We can see that for Case 1, the term $\sum_{q=1}^{q_h}H(t_h-\Delta_{tq})$ disappears from Case 3; for Case 2, the term for $\nabla^{\{2(k+1)\}}\times H$ disappears from Case 3. To avoid confusions I list them separately.

Time Advancement Theorem E. Given a set of field data $E,H(t_e)$, and E may have one or more sets of data in different times as classified in 3 cases identified by an integer q_e in following way

$$\begin{cases}E(t_e),q_e=0\\ E(t_e-\Delta_{t1}),\quad q_e=1,\Delta_t=\Delta_{t1}\\ E(t_e),E(t_e-\Delta_{tq}),q_e>0,q=1,2,\dots,q_e\end{cases}\tag{12}$$
$$\Delta_t,\Delta_{t1},\Delta_{t2},\dots,\Delta_{tqe}>0$$

$E(t_e+\Delta_t)$ can be expressed by

Case 1: $E(t_e),q_e=0\rightarrow$

$$E(t_e+\Delta_t)=E(t_e)+\sum_{k=0}^{\infty}\left(\frac{\Delta_t^{(2k+1)}}{(2k+1)!}\left[\frac{(-1)^k}{\varepsilon(\varepsilon\mu)^k}\nabla^{\{2k+1\}}\times H(t_e)+J_e^{[2k+1]}(t_e)\right]+\frac{\Delta_t^{2(k+1)}}{(2(k+1))!}\left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\nabla^{\{2(k+1)\}}\times E(t_e)+J_e^{[2(k+1)]}(t_e)\right]\right)$$

$Case\ 2$: $E(t_e-\Delta_{t1}),q_e=1,\Delta_t=\Delta_{t1}\rightarrow$

$$E(t_e+\Delta_t)=E(t_e-\Delta_t)+\sum_{k=0}^{\infty}\left(\frac{2\Delta_t^{(2k+1)}}{(2k+1)!}\left[\frac{(-1)^k}{\varepsilon(\varepsilon\mu)^k}\nabla^{\{2k+1\}}\times H(t_e)+J_e^{[2k+1]}(t_e)\right]\right)$$

$E(t_e),E(t_e-\Delta_{tq}),q_e>0\rightarrow$

$$E(t_e+\Delta_t)=\sum_{q=1}^{q_e}E(t_e-\Delta_{tq})+(1-q_e)E(t_e)$$
$$+\sum_{k=0}^{\infty}\left(\frac{\Delta_t^{(2k+1)}+\sum_{q=1}^{qe}(\Delta_{tq})^{2k+1}}{(2k+1)!}\left[\frac{(-1)^k}{\varepsilon(\varepsilon\mu)^k}\nabla^{\{2k+1\}}\times H(t_e)+J_e^{[2k+1]}(t_e)\right]\right.$$
$$\left.+\frac{\Delta_t^{2(k+1)}-\sum_{q=1}^{qe}(\Delta_{tq})^{2(k+1)}}{(2(k+1))!}\left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\nabla^{\{2(k+1)\}}\times E(t_e)+J_e^{[2(k+1)]}(t_e)\right]\right)\tag{13}$$

Where

$$J_e^{[2k+1]}(t_e)=\sum_{i=0}^{k}\frac{(-1)^{i+1}}{(\varepsilon\mu)^i}\nabla^{\{2i\}}\times\frac{d^{2(k-i)}J(t)}{dt^{2(k-i)}},k\geq 0\tag{14}$$

$$J_e^{[2(k+1)]}(t_e) = \sum_{i=0}^{k} \frac{(-1)^{i+1}}{(\varepsilon\mu)^i} \nabla^{\{2i\}} \times \frac{d^{2(k-i)+1} J(t)}{dt^{2(k-i)+1}}, k \geq 0 \quad (15)$$

Note that Case 1 and Case 2 are two special cases of Case 3. To avoid confusions I list them separately.

To prove these two theorems, we need following lemma.

Time Space Lemma. For the Maxwell equations given by (1) – (4), we have

$\frac{\partial^{2k+1} H}{\partial t^{2k+1}} = \frac{1}{\mu} \frac{(-1)^{k+1}}{(\varepsilon\mu)^k} \nabla^{\{2k+1\}} \times E + \begin{cases} \vec{0}, k = 0 \\ \frac{1}{\mu} \sum_{i=1}^{k} \frac{(-1)^{i-1}}{(\varepsilon\mu)^{i-1}} \nabla^{\{2i-1\}} \times \frac{d^{2(k-i)+1} J}{dt^{2(k-i)+1}}, k > 0 \end{cases}$	(16)
$\frac{\partial^{2(k+1)} H}{\partial t^{2(k+1)}} = \frac{(-1)^{k+1}}{(\varepsilon\mu)^{k+1}} \nabla^{\{2(k+1)\}} \times H + \frac{1}{\mu} \sum_{i=0}^{k} \frac{(-1)^i}{(\varepsilon\mu)^i} \nabla^{\{2i+1\}} \times \frac{d^{2(k-i)} J}{dt^{2(k-i)}}$	(17)
$\frac{\partial^{2k+1} E}{\partial t^{2k+1}} = \frac{1}{\varepsilon} \frac{(-1)^k}{(\varepsilon\mu)^k} \nabla^{\{2k+1\}} \times H + \sum_{i=0}^{k} \frac{(-1)^{i+1}}{(\varepsilon\mu)^i} \nabla^{\{2i\}} \times \frac{d^{2(k-i)} J}{dt^{2(k-i)}}$	(18)
$\frac{\partial^{2(k+1)} E}{\partial t^{2(k+1)}} = \frac{(-1)^{k+1}}{(\varepsilon\mu)^{k+1}} \nabla^{\{2(k+1)\}} \times E + \sum_{i=0}^{k} \frac{(-1)^{i+1}}{(\varepsilon\mu)^i} \nabla^{\{2i\}} \times \frac{d^{2(k-i)+1} J}{dt^{2(k-i)+1}}$	(19)
$k = 0,1,2, \ldots$	

For proof of this Lemma, see Appendix A.

With the above lemma, we may prove the time advancement theorems.

For a proof of the **Time Advancement Theorem H**, see Appendix B.

For a proof of the **Time Advancement Theorem E**, see Appendix C.

Time Advancement Estimations

With the above results, we can get generic forms of time advancement estimations.

From (9), we get time advancement estimations for *H* by cutting off the summation at $k_{max} \geq 0$:

$Case\ 1: H(t_h), q_h = 0 \rightarrow$ $H(t_h + \Delta_t) \approx H(t_h) + \sum_{k=0}^{k_{max}} \left(\frac{\Delta_t^{2k+1}}{(2k+1)!} \left[\frac{1}{\mu} \frac{(-1)^{k+1}}{(\varepsilon\mu)^k} \nabla^{\{2k+1\}} \times E(t_h) + J_h^{[2k+1]}(t_h) \right] + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} \left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^{k+1}} \nabla^{\{2(k+1)\}} \times H(t_h) + J_h^{[2(k+1)]}(t_h) \right] \right)$	(44)
$Case\ 2: H(t_h - \Delta_{t1}), q_h = 1, \Delta_t = \Delta_{t1} \rightarrow$ $H(t_h + \Delta_t) \approx H(t_h - \Delta_t) + \sum_{k=0}^{k_{max}} \left(\frac{2\Delta_t^{2k+1}}{(2k+1)!} \left[\frac{1}{\mu} \frac{(-1)^{k+1}}{(\varepsilon\mu)^k} \nabla^{\{2k+1\}} \times E(t_h) + J_h^{[2k+1]}(t_h) \right] \right)$	
$Case\ 3: H(t_h), H(t_h - \Delta_{tq}), q_h > 0, q = 1,2, \ldots, q_h \rightarrow$	

$$H(t_h+\Delta_t) \approx \sum_{q=1}^{q_h} H\left(t_h-\Delta_{tq}\right)+(1-q_h)H(t_h)$$
$$+\sum_{k=0}^{k_{max}}\left(\frac{\Delta_t^{2k+1}+\sum_{q=1}^{qh}\left(\Delta_{tq}\right)^{2k+1}}{(2k+1)!}\left[\frac{1}{\mu}\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\nabla^{\{2k+1\}}\times E(t_h)+J_h^{[2k+1]}(t_h)\right]\right.$$
$$\left.+\frac{\Delta_t^{2(k+1)}-\sum_{q=1}^{qh}\left(\Delta_{tq}\right)^{2(k+1)}}{(2(k+1))!}\left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^{k+1}}\nabla^{\{2(k+1)\}}\times H(t_h)+J_h^{[2(k+1)]}(t_h)\right]\right)$$

From (13), we get time advancement estimations for *E* by cutting off the summation at $k_{max} \geq 0$:

Case 1: $E(t_e), q_e = 0 \rightarrow$

$$E(t_e+\Delta_t)=E(t_e)+\sum_{k=0}^{k_{max}}\left(\frac{\Delta_t^{(2k+1)}}{(2k+1)!}\left[\frac{(-1)^k}{\varepsilon(\varepsilon\mu)^k}\nabla^{\{2k+1\}}\times H(t_e)+J_e^{[2k+1]}(t_e)\right]\right.$$
$$\left.+\frac{\Delta_t^{2(k+1)}}{(2(k+1))!}\left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\nabla^{\{2(k+1)\}}\times E(t_e)+J_e^{[2(k+1)]}(t_e)\right]\right)$$

$Case\ 2$: $E(t_e-\Delta_{t1}), q_e = 1, \Delta_t = \Delta_{t1} \rightarrow$

$$E(t_e+\Delta_t)=E(t_e-\Delta_t)+\sum_{k=0}^{k_{max}}\left(\frac{2\Delta_t^{(2k+1)}}{(2k+1)!}\left[\frac{(-1)^k}{\varepsilon(\varepsilon\mu)^k}\nabla^{\{2k+1\}}\times H(t_e)+J_e^{[2k+1]}(t_e)\right]\right)$$

$E(t_e), E\left(t_e-\Delta_{tq}\right), q_e > 0 \rightarrow$

$$E(t_e+\Delta_t)=\sum_{q=1}^{q_e} E\left(t_e-\Delta_{tq}\right)+(1-q_e)E(t_e)$$
$$+\sum_{k=0}^{k_{max}}\left(\frac{\Delta_t^{(2k+1)}+\sum_{q=1}^{qe}\left(\Delta_{tq}\right)^{2k+1}}{(2k+1)!}\left[\frac{(-1)^k}{\varepsilon(\varepsilon\mu)^k}\nabla^{\{2k+1\}}\times H(t_e)+J_e^{[2k+1]}(t_e)\right]\right.$$
$$\left.+\frac{\Delta_t^{2(k+1)}-\sum_{q=1}^{qe}\left(\Delta_{tq}\right)^{2(k+1)}}{(2(k+1))!}\left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\nabla^{\{2(k+1)\}}\times E(t_e)+J_e^{[2(k+1)]}(t_e)\right]\right) \quad (45)$$

Note that k_{max} in (44) and (45) do not have to be of the same value.

Some observations on (44) and (45):

- Estimation order for time advancement is $2(k_{max}+1)$ and is determined by the number of cascaded space curls used.
- Because the estimation errors are $O\left(\Delta_t^{(2k+1)}+\sum_{q=1}^{qe}\left(\Delta_{tq}\right)^{2k+1}\right)$ and $O\left(\Delta_t^{2(k+1)}-\sum_{q=1}^{qe}\left(\Delta_{tq}\right)^{2(k+1)}\right)$, more sets of historical data do not necessarily increase estimation accuracy. Most likely it might decrease estimation accuracy.

In the next section, we will give generic forms of curl estimations.

Space Curls

For clearly expressing space curl estimations, write space derivatives as

$$D_u^k(V_w) \equiv \frac{\partial^k V_w}{\partial u^k}, w, u\ can\ be\ x, y, z; k > 0;\ V\ is\ a\ 3D\ vector \quad (46)$$

For $k = 1$, the notation can be simplified as

$$D_u(V_w) \equiv D_u^1(V_w) \equiv \frac{\partial^1 V_w}{\partial u^1} \equiv \frac{\partial V_w}{\partial u}$$

For convenience, list some vector identities below.

$\nabla\times(\nabla(\nabla\cdot V))=\vec{0}$	(47)
$\nabla^{\{2\}}\times V=\nabla\times\nabla\times=\nabla(\nabla\cdot V)-\Delta V$	(48)

I'll first prove following theorem.

Curls Cascade Theorem

Curls Cascade Theorem. For a 3D vector V, its curls have following relationships

$\nabla^{\{2k+1\}}\times V=P^{\{2k+1\}}+(-1)^k\begin{bmatrix}D_y^{2k+1}(V_z)-D_z^{2k+1}(V_y)\\ D_z^{2k+1}(V_x)-D_x^{2k+1}(V_z)\\ D_x^{2k+1}(V_y)-D_y^{2k+1}(V_x)\end{bmatrix}$	(49)
$P^{\{2k+1\}}=\nabla\times P^{\{2k\}}+(-1)^k\begin{bmatrix}\frac{\partial}{\partial y}\left(D_x^{2k}(V_z)\right)-\frac{\partial}{\partial z}\left(D_x^{2k}(V_y)\right)\\ \frac{\partial}{\partial z}\left(D_y^{2k}(V_x)\right)-\frac{\partial}{\partial x}\left(D_y^{2k}(V_z)\right)\\ \frac{\partial}{\partial x}\left(D_z^{2k}(V_y)\right)-\frac{\partial}{\partial y}\left(D_z^{2k}(V_x)\right)\end{bmatrix}$	(50)
$\nabla^{\{2(k+1)\}}\times V=P^{\{2(k+1)\}}+(-1)^{k+1}\begin{bmatrix}D_y^{2(k+1)}(V_x)+D_z^{2(k+1)}(V_x)\\ D_z^{2(k+1)}(V_y)+D_x^{2(k+1)}(V_y)\\ D_x^{2(k+1)}(V_z)+D_y^{2(k+1)}(V_z)\end{bmatrix}$	(51)
$P^{\{2(k+1)\}}=\nabla\times P^{\{2k+1\}}+(-1)^k\begin{bmatrix}\frac{\partial}{\partial y}\left(D_x^{2k+1}(V_y)\right)+\frac{\partial}{\partial z}\left(D_x^{2k+1}(V_z)\right)\\ \frac{\partial}{\partial z}\left(D_y^{2k+1}(V_z)\right)+\frac{\partial}{\partial x}\left(D_y^{2k+1}(V_x)\right)\\ \frac{\partial}{\partial x}\left(D_z^{2k+1}(V_x)\right)+\frac{\partial}{\partial y}\left(D_z^{2k+1}(V_y)\right)\end{bmatrix}$	(52)
$P^{\{2\}}=-\begin{bmatrix}D_x^2(V_x)\\ D_y^2(V_y)\\ D_z^2(V_z)\end{bmatrix}$	(53)
$k=1,2,\ldots$	

For a proof of the Curl Cascade Theorem, see Appendix D.

Space Derivative and Curl Estimators

Suppose for a function $v(s)$ there are M sampling data available: $v(s+\Delta_i), i=1,2,\ldots,M$. Expressing them in Taylor series, we have

$$v(s+\Delta_i)=\sum_{k=0}^{\infty}\frac{\Delta_i^k}{k!}\frac{d^k v(s)}{d^k}, i=1,2,\ldots,M$$

Cut the series off at M, we can get derivatives estimations by solving a set of linear equations:

$$v(s+\Delta_i)\approx\sum_{k=0}^{M}\frac{\Delta_i^k}{k!}\frac{d^k v(s)}{d^k}, i=1,2,\ldots,M$$

$$\begin{bmatrix} v(s+\Delta_1)-v(s) \\ v(s+\Delta_2)-v(s) \\ \vdots \\ v(s+\Delta_M)-v(s) \end{bmatrix} \approx \begin{bmatrix} \Delta_1 & \frac{1}{2!}\Delta_1^2 & \cdots & \frac{1}{M!}\Delta_1^M \\ \Delta_2 & \frac{1}{2!}\Delta_2^2 & \cdots & \frac{1}{M!}\Delta_2^M \\ \vdots & \vdots & \cdots & \vdots \\ \Delta_M & \frac{1}{2!}\Delta_M^2 & \cdots & \frac{1}{M!}\Delta_M^M \end{bmatrix} \begin{bmatrix} \frac{dv(s)}{ds} \\ \frac{d^2v(s)}{ds^2} \\ \vdots \\ \frac{d^Mv(s)}{ds^M} \end{bmatrix} = A \begin{bmatrix} \frac{dv(s)}{ds} \\ \frac{d^2v(s)}{ds^2} \\ \vdots \\ \frac{d^Mv(s)}{ds^M} \end{bmatrix}$$

$$A = \begin{bmatrix} \Delta_1 & \frac{1}{2!}\Delta_1^2 & \cdots & \frac{1}{M!}\Delta_1^M \\ \Delta_2 & \frac{1}{2!}\Delta_2^2 & \cdots & \frac{1}{M!}\Delta_2^M \\ \vdots & \vdots & \cdots & \vdots \\ \Delta_M & \frac{1}{2!}\Delta_M^2 & \cdots & \frac{1}{M!}\Delta_M^M \end{bmatrix}, A^{-1} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1M} \\ a_{21} & a_{22} & \cdots & a_{2M} \\ \vdots & \vdots & \cdots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MM} \end{bmatrix}$$

$$\begin{bmatrix} \frac{dv(s)}{ds} \\ \frac{d^2v(s)}{ds^2} \\ \vdots \\ \frac{d^Mv(s)}{ds^M} \end{bmatrix} \approx A^{-1} \begin{bmatrix} v(s+\Delta_1)-v(s) \\ v(s+\Delta_2)-v(s) \\ \vdots \\ v(s+\Delta_M)-v(s) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1M} \\ a_{21} & a_{22} & \cdots & a_{2M} \\ \vdots & \vdots & \cdots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MM} \end{bmatrix} \begin{bmatrix} v(s+\Delta_1)-v(s) \\ v(s+\Delta_2)-v(s) \\ \vdots \\ v(s+\Delta_M)-v(s) \end{bmatrix}$$

Using notation (46) and introducing a new symbol $\mathfrak{D}_w^h(V_u)$ for derivative estimations, the above derivative estimations for a 3D vector can be written as

$D_x^h(V_u) = \frac{\partial^h V_u}{\partial x^h} \approx \mathfrak{D}_x^h(V_u) \equiv [a_{h1} \quad a_{h2} \quad \cdots \quad a_{hM}] \begin{bmatrix} V_u(x+\Delta_1,y,z)-V_u(x,y,z) \\ V_u(x+\Delta_2,y,z)-V_u(x,y,z) \\ \vdots \\ V_u(x+\Delta_M,y,z)-V_u(x,y,z) \end{bmatrix}$ $D_y^h(V_u) = \frac{\partial^h V_u}{\partial y^h} \approx \mathfrak{D}_y^h(V_u) \equiv [a_{h1} \quad a_{h2} \quad \cdots \quad a_{hM}] \begin{bmatrix} V_u(x,y+\Delta_1,z)-V_u(x,y,z) \\ V_u(x,y+\Delta_2,z)-V_u(x,y,z) \\ \vdots \\ V_u(x,y+\Delta_M,z)-V_u(x,y,z) \end{bmatrix}$ $D_z^h(V_u) = \frac{\partial^h V_u}{\partial z^h} \approx \mathfrak{D}_z^h(V_u) \equiv [a_{h1} \quad a_{h2} \quad \cdots \quad a_{hM}] \begin{bmatrix} V_u(x,y,z+\Delta_1)-V_u(x,y,z) \\ V_u(x,y,z+\Delta_2)-V_u(x,y,z) \\ \vdots \\ V_u(x,y,z+\Delta_M)-V_u(x,y,z) \end{bmatrix}$ $u \text{ can be } x,y,z; h = 1,2,\ldots,M$	(64)

Using the above derivative estimators and introducing a new symbol $\overline{\nabla} \times V$ for curl estimation, the curl estimations can be written as

$\nabla \times V = \begin{bmatrix} D_y(V_z) - D_z(V_y) \\ D_z(V_x) - D_x(V_z) \\ D_x(V_y) - D_y(V_x) \end{bmatrix} \approx \overline{\nabla} \times V \equiv \overline{\nabla}^{\{1\}} \times V \equiv \begin{bmatrix} \mathfrak{D}_y(V_z) - \mathfrak{D}_z(V_y) \\ \mathfrak{D}_z(V_x) - \mathfrak{D}_x(V_z) \\ \mathfrak{D}_x(V_y) - \mathfrak{D}_y(V_x) \end{bmatrix}$	(65)
$\nabla^{\{2\}} \times V = \nabla(\nabla \cdot V) - \Delta V \approx \overline{\nabla}^{\{2\}} \times V \equiv \begin{bmatrix} \mathfrak{D}_x(\nabla \cdot V) \\ \mathfrak{D}_y(\nabla \cdot V) \\ \mathfrak{D}_z(\nabla \cdot V) \end{bmatrix} - \begin{bmatrix} \mathfrak{D}_x^2(V_x) + \mathfrak{D}_y^2(V_x) + \mathfrak{D}_z^2(V_x) \\ \mathfrak{D}_x^2(V_y) + \mathfrak{D}_y^2(V_y) + \mathfrak{D}_z^2(V_y) \\ \mathfrak{D}_x^2(V_z) + \mathfrak{D}_y^2(V_z) + \mathfrak{D}_z^2(V_z) \end{bmatrix}$	(66)

Using above estimation symbols, the Curl Cascade Theorem can be written using estimators.

Curl Cascade Estimations:

$$P^{\{2\}} = -\begin{bmatrix} D_x^2(V_x) \\ D_y^2(V_y) \\ D_z^2(V_z) \end{bmatrix} \approx \overline{P}^{\{2\}} \equiv -\begin{bmatrix} \mathfrak{D}_x^2(V_x) \\ \mathfrak{D}_y^2(V_y) \\ \mathfrak{D}_z^2(V_z) \end{bmatrix} \tag{67}$$

$$P^{\{2k+1\}} \approx \overline{P}^{\{2k+1\}} \equiv \overline{\nabla} \times \overline{P}^{\{2k\}} + (-1)^k \begin{bmatrix} \mathfrak{D}_y\left(\mathfrak{D}_x^{2k}(V_z)\right) - \mathfrak{D}_z\left(\mathfrak{D}_x^{2k}(V_y)\right) \\ \mathfrak{D}_z\left(\mathfrak{D}_y^{2k}(V_x)\right) - \mathfrak{D}_x\left(\mathfrak{D}_y^{2k}(V_z)\right) \\ \mathfrak{D}_x\left(\mathfrak{D}_z^{2k}(V_y)\right) - \mathfrak{D}_y\left(\mathfrak{D}_z^{2k}(V_x)\right) \end{bmatrix} \tag{68}$$

$$\nabla^{\{2k+1\}} \times V \approx \overline{\nabla}^{\{2k+1\}} \times V \equiv \overline{P}^{\{2k+1\}} + (-1)^k \begin{bmatrix} \mathfrak{D}_y^{2k+1}(V_z) - \mathfrak{D}_z^{2k+1}(V_y) \\ \mathfrak{D}_z^{2k+1}(V_x) - \mathfrak{D}_x^{2k+1}(V_z) \\ \mathfrak{D}_x^{2k+1}(V_y) - \mathfrak{D}_y^{2k+1}(V_x) \end{bmatrix} \tag{69}$$

$$P^{\{2(k+1)\}} \approx \overline{P}^{\{2(k+1)\}} \equiv \overline{\nabla} \times \overline{P}^{\{2k+1\}} + (-1)^k \begin{bmatrix} \mathfrak{D}_y\left(\mathfrak{D}_x^{2k+1}(V_y)\right) + \mathfrak{D}_z\left(\mathfrak{D}_x^{2k+1}(V_z)\right) \\ \mathfrak{D}_z\left(\mathfrak{D}_y^{2k+1}(V_z)\right) + \mathfrak{D}_x\left(\mathfrak{D}_y^{2k+1}(V_x)\right) \\ \mathfrak{D}_x\left(\mathfrak{D}_z^{2k+1}(V_x)\right) + \mathfrak{D}_y\left(\mathfrak{D}_z^{2k+1}(V_y)\right) \end{bmatrix} \tag{70}$$

$$\nabla^{\{2(k+1)\}} \times V \approx \overline{\nabla}^{\{2(k+1)\}} \times V \equiv \overline{P}^{\{2(k+1)\}} + (-1)^{k+1} \begin{bmatrix} \mathfrak{D}_y^{2(k+1)}(V_x) + \mathfrak{D}_z^{2(k+1)}(V_x) \\ \mathfrak{D}_z^{2(k+1)}(V_y) + \mathfrak{D}_x^{2(k+1)}(V_y) \\ \mathfrak{D}_x^{2(k+1)}(V_z) + \mathfrak{D}_y^{2(k+1)}(V_z) \end{bmatrix} \tag{71}$$

$$k = 1,2,\ldots,\begin{cases} \dfrac{M-1}{2}, M \geq 3\ is\ odd \\ \dfrac{M}{2} - 1, M \geq 4\ is\ even \end{cases} \tag{72}$$

Formula (65) to (72) gives curl estimations from curl 1 to curl *M*:

$$\overline{\nabla}^{\{k\}} \times V, k = 1,2,\ldots,M$$

Apply the above results to the Maxwell equations (1) to (4), we get field curl estimations, as listed below.

$$\overline{\nabla}^{\{1\}} \times E = \begin{bmatrix} \mathfrak{D}_y(E_z) - \mathfrak{D}_z(E_y) \\ \mathfrak{D}_z(E_x) - \mathfrak{D}_x(E_z) \\ \mathfrak{D}_x(E_y) - \mathfrak{D}_y(E_x) \end{bmatrix}$$

$$\overline{\nabla}^{\{1\}} \times H = \begin{bmatrix} \mathfrak{D}_y(H_z) - \mathfrak{D}_z(H_y) \\ \mathfrak{D}_z(H_x) - \mathfrak{D}_x(H_z) \\ \mathfrak{D}_x(H_y) - \mathfrak{D}_y(H_x) \end{bmatrix} \tag{73}$$

$$\overline{\nabla}^{\{2\}} \times E = \begin{bmatrix} \mathfrak{D}_x(\rho/\varepsilon) \\ \mathfrak{D}_y(\rho/\varepsilon) \\ \mathfrak{D}_z(\rho/\varepsilon) \end{bmatrix} - \begin{bmatrix} \mathfrak{D}_x^2(E_x) + \mathfrak{D}_y^2(E_x) + \mathfrak{D}_z^2(E_x) \\ \mathfrak{D}_x^2(E_y) + \mathfrak{D}_y^2(E_y) + \mathfrak{D}_z^2(E_y) \\ \mathfrak{D}_x^2(E_z) + \mathfrak{D}_y^2(E_z) + \mathfrak{D}_z^2(E_z) \end{bmatrix} \tag{74}$$

$$\overline{\nabla}^{\{2\}} \times H = -\begin{bmatrix} \mathfrak{D}_x^2(H_x)+\mathfrak{D}_y^2(H_x)+\mathfrak{D}_z^2(H_x) \\ \mathfrak{D}_x^2(H_y)+\mathfrak{D}_y^2(H_y)+\mathfrak{D}_z^2(H_y) \\ \mathfrak{D}_x^2(H_z)+\mathfrak{D}_y^2(H_z)+\mathfrak{D}_z^2(H_z) \end{bmatrix}$$	
$\overline{\nabla}^{\{h\}} \times E\ and\ \overline{\nabla}^{\{h\}} \times H, h = 3,4,\dots,M\ are\ calculated\ by\ (65)\ to\ (71)$	(75)

Use $\overline{\nabla}^{\{h\}} \times E\ and\ \overline{\nabla}^{\{h\}} \times H$ to replace $\nabla^{\{h\}} \times E\ and\ \nabla^{\{h\}} \times H$ in (44) and (45), we finally get generic forms of FDTD.

Generic Forms of FDTD

$Case\ 1: H(t_h), q_h = 0 \rightarrow$

$$H(t_h+\Delta_t) \approx H(t_h) + \sum_{k=0}^{k_{max}}\left(\frac{\Delta_t^{2k+1}}{(2k+1)!}\left[\frac{1}{\mu}\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\overline{\nabla}^{\{2k+1\}} \times E(t_h) + J_h^{[2k+1]}(t_h)\right] + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!}\left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^{k+1}}\overline{\nabla}^{\{2(k+1)\}} \times H(t_h) + J_h^{[2(k+1)]}(t_h)\right]\right)$$

$Case\ 2: H(t_h-\Delta_{t1}), q_h = 1, \Delta_t = \Delta_{t1} \rightarrow$

$$H(t_h+\Delta_t) \approx H(t_h-\Delta_t) + \sum_{k=0}^{k_{max}}\left(\frac{2\Delta_t^{2k+1}}{(2k+1)!}\left[\frac{1}{\mu}\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\overline{\nabla}^{\{2k+1\}} \times E(t_h) + J_h^{[2k+1]}(t_h)\right]\right)$$

$Case\ 3: H(t_h), H(t_h-\Delta_{tq}), q_h > 0, q = 1,2,\dots,q_h \rightarrow$

$$H(t_h+\Delta_t) \approx \sum_{q=1}^{q_h} H(t_h-\Delta_{tq}) + (1-q_h)H(t_h) + \sum_{k=0}^{k_{max}}\left(\frac{\Delta_t^{2k+1} + \sum_{q=1}^{qh}(\Delta_{tq})^{2k+1}}{(2k+1)!}\left[\frac{1}{\mu}\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\overline{\nabla}^{\{2k+1\}} \times E(t_h) + J_h^{[2k+1]}(t_h)\right] + \frac{\Delta_t^{2(k+1)} - \sum_{q=1}^{qh}(\Delta_{tq})^{2(k+1)}}{(2(k+1))!}\left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^{k+1}}\overline{\nabla}^{\{2(k+1)\}} \times H(t_h) + J_h^{[2(k+1)]}(t_h)\right]\right)$$

$k_{max} \geq 0$ (76)

Case 1: $E(t_e), q_e = 0 \rightarrow$

$$E(t_e+\Delta_t) = E(t_e) + \sum_{k=0}^{k_{max}}\left(\frac{\Delta_t^{(2k+1)}}{(2k+1)!}\left[\frac{(-1)^k}{\varepsilon(\varepsilon\mu)^k}\overline{\nabla}^{\{2k+1\}} \times H(t_e) + J_e^{[2k+1]}(t_e)\right] + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!}\left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\overline{\nabla}^{\{2(k+1)\}} \times E(t_e) + J_e^{[2(k+1)]}(t_e)\right]\right)$$

$Case\ 2: E(t_e-\Delta_{t1}), q_e = 1, \Delta_t = \Delta_{t1} \rightarrow$

$$E(t_e+\Delta_t) = E(t_e-\Delta_t) + \sum_{k=0}^{k_{max}}\left(\frac{2\Delta_t^{(2k+1)}}{(2k+1)!}\left[\frac{(-1)^k}{\varepsilon(\varepsilon\mu)^k}\overline{\nabla}^{\{2k+1\}} \times H(t_e) + J_e^{[2k+1]}(t_e)\right]\right)$$

$E(t_e), E(t_e-\Delta_{tq}), q_e > 0 \rightarrow$ (77)

$$E(t_e+\Delta_t)=\sum_{q=1}^{q_e}E(t_e-\Delta_{tq})+(1-q_e)E(t_e) + \sum_{k=0}^{k_{max}}\left(\frac{\Delta_t^{(2k+1)}+\sum_{q=1}^{qe}(\Delta_{tq})^{2k+1}}{(2k+1)!}\left[\frac{(-1)^k}{\varepsilon(\varepsilon\mu)^k}\overline{\nabla}^{\{2k+1\}}\times H(t_e)+J_e^{[2k+1]}(t_e)\right] + \frac{\Delta_t^{2(k+1)}-\sum_{q=1}^{qe}(\Delta_{tq})^{2(k+1)}}{(2(k+1))!}\left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\overline{\nabla}^{\{2(k+1)\}}\times E(t_e)+J_e^{[2(k+1)]}(t_e)\right]\right)$$	
$k_{max}\ge 0$	

Note that $k_{max}, \Delta_t, \Delta_{tq}$ in (76) and (77) do not have to be of the same values. The matrix A , which defines the derivative estimator (64), can also be different for (76) and (77). Actually when we are handling boundary conditions, we may choose different values for these parameters.

Deriving of Special Algorithms

The Time-Space Synchronized FDTD

Assume $E_x, E_y, E_z, H_x, H_y, and\ H_z\ are\ all\ at\ the\ same\ space\ location\ (x,y,z), and$ let

$$t_h = t_e$$

(76) and (77) give $H(t_h+q\Delta_t), E(t_h+q\Delta_t), q=1,2,3,\ldots$

Such an algorithm allows us to calculate Poynting vector to study field energy transfer:

$$S = E\times H$$

It also allows us to calculate divergence as estimation errors:

$error_h = \mathfrak{D}_x(H_x)+\mathfrak{D}_y(H_y)+\mathfrak{D}_z(H_z)$	(78)
$error_e = \mathfrak{D}_x(E_x)+\mathfrak{D}_y(E_y)+\mathfrak{D}_z(E_z)-\rho/\varepsilon$	(79)

These errors may give precise comparisons of different algorithms' accuracy. That is, for two algorithms, we can precisely say which one is more accurate.

Time-shifted Space-synchronized FDTD

Assume $E_x, E_y, E_z, H_x, H_y, and\ H_z\ are\ all\ at\ the\ same\ space\ location\ (x,y,z), and$ let

$$t_e = t_h + \frac{1}{2}\Delta t$$

(76) and (77) give $H\left(t_h+\frac{1}{2}\Delta_t+q\Delta_t\right), E(t_h+q\Delta_t), q=0,1,2,3,\ldots$

It gives a higher accuracy than the Time-Space Synchronized schemes.

But we can no longer calculate Poynting vector.

We can still calculate divergences to get precise estimation errors.

The Yee algorithm

In (76) and (77), choose following parameters:

$$k_{max} = 0\ for\ both\ (76) and\ (77)$$

$$\Delta t \ is \ the \ same \ for \ both \ (76) and \ (77), but \ use \frac{1}{2}\Delta t$$

$$t_e = t_h + \frac{1}{2}\Delta t$$

We get Yee style time advancement:

$H\left(t_h + \frac{1}{2}\Delta_t\right) \approx H\left(t_h - \frac{1}{2}\Delta_t\right) + \Delta_t\left(\frac{1}{\mu}\overline{\overline{\nabla}}^{\{1\}} \times E(t_h) + J_h^{[2k+1]}(t_h)\right)$ $E(t_h + \Delta_t) = E(t_h) + \Delta_t\left(\frac{1}{\varepsilon}\overline{\overline{\nabla}}^{\{2k+1\}} \times H(t_h + \frac{1}{2}\Delta t) + J_e^{[2k+1]}\left(t_h + \frac{1}{2}\Delta t\right)\right)$	(78)

For curl estimation, choose *M=1*, the inverse matrix is simply $\frac{1}{\Delta s}$. The derivative estimator is $\frac{v(s+\Delta s)-v(s)}{\Delta s}$. The curl estimations become

$$\overline{\overline{\nabla}}^{\{1\}} \times E = \begin{bmatrix} \mathfrak{D}_y(E_z) - \mathfrak{D}_z(E_y) \\ \mathfrak{D}_z(E_x) - \mathfrak{D}_x(E_z) \\ \mathfrak{D}_x(E_y) - \mathfrak{D}_y(E_x) \end{bmatrix} = \frac{1}{\Delta s}\begin{bmatrix} E_z(x, y + \Delta s, z) - E_z(x, y, z) - E_y(x, y, z + \Delta s) + E_y(x, y, z) \\ E_x(x, y, z + \Delta s) - E_x(x, y, z) - E_z(x + \Delta s, y, z) + E_z(x, y, z) \\ E_y(x + \Delta s, y, z) - E_y(x, y, z) - E_x(x, y + \Delta s, z) + E_x(x, y, z) \end{bmatrix}$$

$$\overline{\overline{\nabla}}^{\{1\}} \times H = \begin{bmatrix} \mathfrak{D}_y(H_z) - \mathfrak{D}_z(H_y) \\ \mathfrak{D}_z(H_x) - \mathfrak{D}_x(H_z) \\ \mathfrak{D}_x(H_y) - \mathfrak{D}_y(H_x) \end{bmatrix} = \frac{1}{\Delta s}\begin{bmatrix} H_z(x, y, z) - H_z(x, y - \Delta s, z) - H_y(x, y, z) + H_y(x, y, z - \Delta s) \\ H_x(x, y, z) - H_x(x, y, z - \Delta s) - H_z(x, y, z) + H_z(x - \Delta s, y, z) \\ H_y(x, y, z) - H_y(x - \Delta s, y, z) - H_x(x, y, z) + H_x(x, y - \Delta s, z) \end{bmatrix}$$

Before substituting the above estimations into (78), we notice that to get a derivative at $v(s)$, $\frac{v\left(s+\frac{1}{2}\Delta s\right)-v(s-\frac{1}{2}\Delta s)}{\Delta s}$ is one order more accurate than $\frac{v(s+\Delta s)-v(s)}{\Delta s}$. Yee gave a clever way to do it by defining

$$E \rightarrow \begin{bmatrix} E_x(x + \frac{1}{2}\Delta s, y, z) \\ E_y(x, y + \frac{1}{2}\Delta s, z) \\ E_z(x, y, z + \frac{1}{2}\Delta s) \end{bmatrix}, H \rightarrow \begin{bmatrix} H_x(x, y + \frac{1}{2}\Delta s, z + \frac{1}{2}\Delta s) \\ H_y(x + \frac{1}{2}\Delta s, y, z + \frac{1}{2}\Delta s) \\ H_z(x + \frac{1}{2}\Delta s, y + \frac{1}{2}\Delta s, z) \end{bmatrix}$$

The curl estimations become

$$\overline{\overline{\nabla}}^{\{1\}} \times E = \frac{1}{\Delta s}\begin{bmatrix} E_z\left(x, y + \Delta s, z + \frac{1}{2}\Delta s\right) - E_z\left(x, y, z + \frac{1}{2}\Delta s\right) - E_y\left(x, y + \frac{1}{2}\Delta s, z + \Delta s\right) + E_y(x, y + \frac{1}{2}\Delta s, z) \\ E_x\left(x + \frac{1}{2}\Delta s, y, z + \Delta s\right) - E_x\left(x + \frac{1}{2}\Delta s, y, z\right) - E_z\left(x + \Delta s, y, z + \frac{1}{2}\Delta s\right) + E_z(x, y, z + \frac{1}{2}\Delta s) \\ E_y\left(x + \Delta s, y + \frac{1}{2}\Delta s, z\right) - E_y\left(x, y + \frac{1}{2}\Delta s, z\right) - E_x\left(x + \frac{1}{2}\Delta s, y + \Delta s, z\right) + E_x(x + \frac{1}{2}\Delta s, y, z) \end{bmatrix}$$

$$\overline{\overline{\nabla}}^{\{1\}} \times H = \frac{1}{\Delta s}\begin{bmatrix} H_z\left(x + \frac{1}{2}\Delta s, y + \frac{1}{2}\Delta s, z\right) - H_z\left(x + \frac{1}{2}\Delta s, y - \frac{1}{2}\Delta s, z\right) - H_y\left(x + \frac{1}{2}\Delta s, y, z + \frac{1}{2}\Delta s\right) + H_y(x + \frac{1}{2}\Delta s, y, z - \frac{1}{2}\Delta s) \\ H_x\left(x, y + \frac{1}{2}\Delta s, z + \frac{1}{2}\Delta s\right) - H_x\left(x, y + \frac{1}{2}\Delta s, z - \frac{1}{2}\Delta s\right) - H_z\left(x + \frac{1}{2}\Delta s, y + \frac{1}{2}\Delta s, z\right) + H_z(x - \frac{1}{2}\Delta s, y + \frac{1}{2}\Delta s, z) \\ H_y\left(x + \frac{1}{2}\Delta s, y, z + \frac{1}{2}\Delta s\right) - H_y\left(x - \frac{1}{2}\Delta s, y, z + \frac{1}{2}\Delta s\right) - H_x\left(x, y + \frac{1}{2}\Delta s, z + \frac{1}{2}\Delta s\right) + H_x(x, y - \frac{1}{2}\Delta s, z + \frac{1}{2}\Delta s) \end{bmatrix}$$

We can see that

$$center\ of\ \overline{\nabla}^{\{1\}} \times E\ is\ at \begin{bmatrix} E_x(x + \frac{1}{2}\Delta s, y, z) \\ E_y(x, y + \frac{1}{2}\Delta s, z) \\ E_z(x, y + \frac{1}{2}\Delta s, z) \end{bmatrix}$$

$$center\ of\ \overline{\nabla}^{\{1\}} \times H\ is\ at \begin{bmatrix} H_x(x, y + \frac{1}{2}\Delta s, z + \frac{1}{2}\Delta s) \\ H_y(x + \frac{1}{2}\Delta s, y, z + \frac{1}{2}\Delta s) \\ H_z(x + \frac{1}{2}\Delta s, y + \frac{1}{2}\Delta s, z) \end{bmatrix}$$

Now we get the standard Yee algorithm.

By deriving the Yee algorithm from the generic forms, we can clearly see the advantages and limitations of it.

Advantages:

- Use calculation amount of a first order estimation to get a second order precision
- Higher order curl estimations can be used to increase accuracy of $\overline{\nabla}^{\{1\}} \times$, but not $\overline{\nabla}^{\{h\}} \times, h > 1$

Limitations

- Because not more than one curl can be used, that is, $\overline{\nabla}^{\{h\}} \times, h > 1$, cannot be used, the time advancement order cannot be higher than 2
- Cannot calculate Poynting vector
- Cannot calculate divergences

Conclusion

For estimations purely based on Taylor's series, this paper shows that there is a generic form of FDTD algorithm for the Maxwell equations. By "generic" I mean that all other forms of the algorithms are special cases of this generic form. But it does not cover those algorithms involving techniques other than the Taylor series. However, almost all estimation algorithms involve the use of Taylor series in one way or the other. Therefore, this generic form may be used as a base for developing new algorithms, or to improve the existing algorithms.

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Appendixes

Appendix A. Proof of Time Space Lemma

Proof.

Consider the case when $k = 0$.

For $k = 0$ (16) becomes

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu}\nabla \times E$$

It is (4). So (16) holds.

For $k = 0$ (17) becomes

$$\frac{\partial^2 H}{\partial t^2} = -\frac{1}{\varepsilon\mu}\nabla^{\{2\}} \times H + \frac{1}{\mu}\nabla \times J \tag{20}$$

By taking temporal derivative of (4) and substituting (3) into it, we have

$$\frac{\partial^2 H}{\partial t^2} = -\frac{1}{\mu}\nabla \times \left(\frac{1}{\varepsilon}\nabla \times H - J(t)\right) = -\frac{1}{\varepsilon\mu}\nabla^{\{2\}} \times H + \frac{1}{\mu}\nabla \times J(t)$$

It is the same as (20). So, (17) holds.

For $k = 0$ (18) becomes

$$\frac{\partial^1 E}{\partial t^1} = \frac{1}{\varepsilon}\nabla^{\{1\}} \times H - J$$

It is (3). So, (18) holds.

For $k = 0$ (19) becomes

$$\frac{\partial^2 E}{\partial t^2} = (-1)^1 \frac{1}{\varepsilon\mu}\nabla^{\{2\}} \times E - \frac{dJ}{dt} \tag{21}$$

By taking temporal derivative of (3) and substituting (4) into it, we have

$$\frac{\partial^2 E}{\partial t^2} = \frac{1}{\varepsilon}\nabla \times \left(-\frac{1}{\mu}\nabla \times E\right) - \frac{dJ(t)}{dt} = -\frac{1}{\varepsilon\mu}\nabla^{\{2\}} \times E - \frac{dJ(t)}{dt}$$

It is the same as (21). So, (19) holds.

So, (16) – (19) hold for $k = 0$.

Consider the case when $k = 1$.

For $k = 1$ (16) becomes

$$\frac{\partial^3 H}{\partial t^3} = \frac{1}{\mu}\frac{1}{(\varepsilon\mu)^1}\nabla^{\{3\}} \times E + \frac{1}{\mu}\nabla^{\{1\}} \times \frac{d^1 J}{dt^1} \tag{22}$$

Taking the temporal derivative on (20) and substituting (4) into it, we have

$$\frac{\partial^3 H}{\partial t^3} = -\frac{1}{\varepsilon\mu}\nabla^{\{2\}} \times \left(-\frac{1}{\mu}\nabla \times E\right) + \frac{1}{\mu}\nabla \times \frac{dJ}{dt} = \frac{1}{\mu}\frac{1}{\varepsilon\mu}\nabla^{\{3\}} \times E + \frac{1}{\mu}\nabla \times \frac{dJ}{dt}$$

It is the same as (22). So, (16) holds.

For $k = 1$ (17) becomes

$$\frac{\partial^4 H}{\partial t^4} = \frac{1}{(\varepsilon\mu)^2}\nabla^{\{4\}} \times H + \frac{1}{\mu}\left(\nabla \times \frac{d^2 J}{dt^2} - \frac{1}{\varepsilon\mu}\nabla^{\{3\}} \times J\right) \tag{23}$$

Taking the temporal derivative on (22) and substituting (3) into it, we have

$$\frac{\partial^4 H}{\partial t^4} = \frac{1}{\mu}\frac{1}{(\varepsilon\mu)^1}\nabla^{\{3\}} \times \left(\frac{1}{\varepsilon}\ \nabla \times H - J\right) + \frac{1}{\mu}\nabla^{\{1\}} \times \frac{d^2 J}{dt^2} = \frac{1}{(\varepsilon\mu)^2}\nabla^{\{4\}} \times H - \frac{1}{\mu}\frac{1}{(\varepsilon\mu)^1}\nabla^{\{3\}} \times J + \frac{1}{\mu}\nabla^{\{1\}} \times \frac{d^2 J}{dt^2}$$

It is the same as (23). So, (17) holds.

For $k = 1$ (18) becomes

$$\frac{\partial^3 E}{\partial t^3} = -\frac{1}{\varepsilon}\frac{1}{(\varepsilon\mu)^1}\nabla^{\{3\}} \times H - \frac{d^2 J}{dt^2} + \frac{1}{(\varepsilon\mu)^1}\nabla^{\{2\}} \times J \tag{24}$$

Taking the temporal derivative on (21) and substituting (3) into it, we have

$$\frac{\partial^3 E}{\partial t^3} = (-1)^1\frac{1}{\varepsilon\mu}\nabla^{\{2\}} \times \left(\frac{1}{\varepsilon}\ \nabla \times H - J\right) - \frac{d^2 J}{dt^2} = -\frac{1}{\varepsilon}\frac{1}{\varepsilon\mu}\nabla^{\{3\}} \times H + \frac{1}{\varepsilon\mu}\nabla^{\{2\}} \times J - \frac{d^2 J}{dt^2}$$

It is the same as (24). So, (18) holds.

For $k = 1$ (19) becomes

$$\frac{\partial^4 E}{\partial t^4} = \frac{1}{(\varepsilon\mu)^2}\nabla^{\{4\}} \times E - \frac{d^3 J}{dt^3} + \frac{1}{\varepsilon\mu}\nabla^{\{2\}} \times \frac{d^1 J}{dt^1} \tag{25}$$

Taking the temporal derivative on (24) and substituting (4) into it, we have

$$\begin{aligned}\frac{\partial^4 E}{\partial t^4} &= -\frac{1}{\varepsilon}\frac{1}{(\varepsilon\mu)^1}\nabla^{\{3\}} \times \left(-\frac{1}{\mu}\ \nabla \times E\right) - \frac{d^3 J}{dt^3} + \frac{1}{(\varepsilon\mu)^1}\nabla^{\{2\}} \times \frac{dJ}{dt} \\ &= \frac{1}{(\varepsilon\mu)^2}\nabla^{\{4\}} \times E - \frac{d^3 J}{dt^3} + \frac{1}{(\varepsilon\mu)^1}\nabla^{\{2\}} \times \frac{dJ}{dt}\end{aligned}$$

It is the same as (25). So, (19) holds.

So, (16) – (19) hold for $k = 1$.

Assume for a $k \geq 1$, (16) – (19) hold. Consider the case of $k + 1$.

For $k + 1$, (16) becomes

$$\frac{\partial^{2(k+1)+1} H}{\partial t^{2(k+1)+1}} = \frac{1}{\mu}\frac{(-1)^{k+2}}{(\varepsilon\mu)^{k+1}}\nabla^{\{2k+3\}} \times E + \frac{1}{\mu}\sum_{i=1}^{k+1}\frac{(-1)^{i-1}}{(\varepsilon\mu)^{i-1}}\nabla^{\{2i-1\}} \times \frac{d^{2(k+1-i)+1} J}{dt^{2(k+1-i)+1}} \tag{26}$$

Taking the temporal derivative on (17) and substituting (4) into it, we have

$$\frac{\partial^{2(k+1)+1}H}{\partial t^{2(k+1)+1}} = \frac{(-1)^{k+1}}{(\varepsilon\mu)^{k+1}}\nabla^{\{2(k+1)\}}\times\left(-\frac{1}{\mu}\nabla\times E\right)+\frac{1}{\mu}\sum_{i=0}^{k}\frac{(-1)^{i}}{(\varepsilon\mu)^{i}}\nabla^{\{2i+1\}}\times\frac{d^{2(k-i)+1}J}{dt^{2(k-i)+1}}$$
$$=\frac{1}{\mu}\frac{(-1)^{k+2}}{(\varepsilon\mu)^{k+1}}\nabla^{\{2(k+1)+1\}}\times E+\frac{1}{\mu}\sum_{i=1}^{k+1}\frac{(-1)^{i-1}}{(\varepsilon\mu)^{i-1}}\nabla^{\{2(i-1)+1\}}\times\frac{d^{2(k-(i-1))+1}J}{dt^{2(k-(i-1))+1}}$$

It is the same as (26). So, (16) holds for $k+1$.

For $k+1$, (17) becomes

$$\frac{\partial^{2(k+2)}H}{\partial t^{2(k+2)}} = \frac{(-1)^{k+2}}{(\varepsilon\mu)^{k+2}}\nabla^{\{2(k+2)\}}\times H+\frac{1}{\mu}\sum_{i=0}^{k+1}\frac{(-1)^{i}}{(\varepsilon\mu)^{i}}\nabla^{\{2i+1\}}\times\frac{d^{2(k+1-i)}J}{dt^{2(k+1-i)}} \tag{27}$$

Taking the temporal derivative on (26) and substituting (3) into it, we have

$$\frac{\partial^{2(k+1)+2}H}{\partial t^{2(k+1)+2}} = \frac{1}{\mu}\frac{(-1)^{k+2}}{(\varepsilon\mu)^{k+1}}\nabla^{\{2k+3\}}\times\left(\frac{1}{\varepsilon}\nabla\times H-J\right)+\frac{1}{\mu}\sum_{i=1}^{k+1}\frac{(-1)^{i-1}}{(\varepsilon\mu)^{i-1}}\nabla^{\{2i-1\}}\times\frac{d^{2(k+1-i)+2}J}{dt^{2(k+1-i)+2}}$$
$$=\frac{(-1)^{k+2}}{(\varepsilon\mu)^{k+2}}\nabla^{\{2k+4\}}\times H-\frac{1}{\mu}\frac{(-1)^{k+2}}{(\varepsilon\mu)^{k+1}}\nabla^{\{2k+3\}}\times J+\frac{1}{\mu}\sum_{i=0}^{k}\frac{(-1)^{i}}{(\varepsilon\mu)^{i}}\nabla^{\{2i+1\}}\times\frac{d^{2(k-i)+2}J}{dt^{2(k-i)+2}}$$
$$=\frac{(-1)^{k+2}}{(\varepsilon\mu)^{k+2}}\nabla^{\{2k+4\}}\times H+\frac{1}{\mu}\sum_{i=0}^{k+1}\frac{(-1)^{i}}{(\varepsilon\mu)^{i}}\nabla^{\{2i+1\}}\times\frac{d^{2(k-i+1)}J}{dt^{2(k-i+1)}}$$

It is the same as (27). So, (17) holds for $k+1$.

For $k+1$, (18) becomes

$$\frac{\partial^{2k+3}E}{\partial t^{2k+3}} = (-1)^{k+1}\frac{1}{\varepsilon}\frac{1}{(\varepsilon\mu)^{k+1}}\nabla^{\{2k+3\}}\times H+\sum_{i=0}^{k+1}\frac{(-1)^{i+1}}{(\varepsilon\mu)^{i}}\nabla^{\{2i\}}\times\frac{d^{2(k+1-i)}J}{dt^{2(k+1-i)}} \tag{28}$$

Taking the temporal derivative on (19) and substituting (3) into it, we have

$$\frac{\partial^{2(k+1)+1}E}{\partial t^{2(k+1)+1}} = \frac{(-1)^{k+1}}{(\varepsilon\mu)^{k+1}}\nabla^{\{2(k+1)\}}\times\left(\frac{1}{\varepsilon}\nabla\times H-J\right)+\sum_{i=0}^{k}\frac{(-1)^{i+1}}{(\varepsilon\mu)^{i}}\nabla^{\{2i\}}\times\frac{d^{2(k-i)+2}J}{dt^{2(k-i)+2}}$$
$$=\frac{1}{\varepsilon}\frac{(-1)^{k+1}}{(\varepsilon\mu)^{k+1}}\nabla^{\{2(k+1)+1\}}\times H-\frac{(-1)^{k+1}}{(\varepsilon\mu)^{k+1}}\nabla^{\{2(k+1)\}}\times J+\sum_{i=0}^{k}\frac{(-1)^{i+1}}{(\varepsilon\mu)^{i}}\nabla^{\{2i\}}\times\frac{d^{2(k-i+1)}J}{dt^{2(k-i+1)}}$$
$$=\frac{1}{\varepsilon}(-1)^{k+1}\frac{1}{(\varepsilon\mu)^{k+1}}\nabla^{\{2(k+1)+1\}}\times H+\sum_{i=0}^{k+1}\frac{(-1)^{i+1}}{(\varepsilon\mu)^{i}}\nabla^{\{2i\}}\times\frac{d^{2(k-i+1)}J}{dt^{2(k-i+1)}}$$

It is the same as (28). So, (18) holds for $k+1$.

For $k+1$, (19) becomes,

$$\frac{\partial^{2(k+2)}E}{\partial t^{2(k+2)}} = \frac{(-1)^{k+2}}{(\varepsilon\mu)^{k+2}}\nabla^{\{2(k+2)\}}\times E+\sum_{i=0}^{k+1}\frac{(-1)^{i+1}}{(\varepsilon\mu)^{i}}\nabla^{\{2i\}}\times\frac{d^{2(k+1-i)+1}J}{dt^{2(k+1-i)+1}} \tag{29}$$

Taking the temporal derivative on (28) and substituting (4) into it, we have

$$\frac{\partial^{2k+4}E}{\partial t^{2k+4}} = (-1)^{k+1}\frac{1}{\varepsilon}\frac{1}{(\varepsilon\mu)^{k+1}}\nabla^{\{2k+3\}} \times \left(-\frac{1}{\mu}\nabla \times E\right) + \sum_{i=0}^{k+1}\frac{(-1)^{i+1}}{(\varepsilon\mu)^{i}}\nabla^{\{2i\}} \times \frac{d^{2(k+1-i)+1}J}{dt^{2(k+1-i)+1}}$$
$$= (-1)^{k+2}\frac{1}{(\varepsilon\mu)^{k+2}}\nabla^{\{2k+4\}} \times E + \sum_{i=0}^{k+1}\frac{(-1)^{i+1}}{(\varepsilon\mu)^{i}}\nabla^{\{2i\}} \times \frac{d^{2(k+1-i)+1}J}{dt^{2(k+1-i)+1}}$$

It is the same as (29). So, (19) holds for $k+1$.

So, (16) – (19) hold for $k+1$.

Thus, (16) – (19) hold for $k \geq 0$.

QED.

Appendix B. Proof of Time Advancement Theorem H

Proof.

Apply the Taylor series, we have

$$H(t_h + \Delta_t) = \sum_{k=0}^{\infty}\frac{\Delta_t^k}{k!}\frac{\partial^k H(t_h)}{\partial t^k}$$

Rearrange the terms, we have

$H(t_h + \Delta_t) = H(t_h) + \sum_{k=0}^{\infty}\frac{\Delta_t^{2k+1}}{(2k+1)!}\frac{\partial^{2k+1}H(t_h)}{\partial t^{2k+1}} + \sum_{k=0}^{\infty}\frac{\Delta_t^{2(k+1)}}{(2(k+1))!}\frac{\partial^{2(k+1)}H(t_h)}{\partial t^{2(k+1)}}$	(30)

Combine (10) and (16), (11) and (17), we have

$\frac{\partial^{2k+1}H}{\partial t^{2k+1}} = \frac{1}{\mu}\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\nabla^{\{2k+1\}} \times E + J_h^{[2k+1]}(t_h)$	(31)
$\frac{\partial^{2(k+1)}H}{\partial t^{2(k+1)}} = \frac{(-1)^{k+1}}{(\varepsilon\mu)^{k+1}}\nabla^{\{2(k+1)\}} \times H + J_h^{[2(k+1)]}(t_h)$	(32)

Substitute (31) and (32) into (30), we have

$H(t_h + \Delta_t) = H(t_h) + \sum_{k=0}^{\infty}\frac{\Delta_t^{2k+1}}{(2k+1)!}\left(\frac{1}{\mu}\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\nabla^{\{2k+1\}} \times E + J_h^{[2k+1]}(t_h)\right) + \sum_{k=0}^{\infty}\frac{\Delta_t^{2(k+1)}}{(2(k+1))!}\left(\frac{(-1)^{k+1}}{(\varepsilon\mu)^{k+1}}\nabla^{\{2(k+1)\}} \times H + J_h^{[2(k+1)]}(t_h)\right)$	(33)

(33) is Case 1 of (9).

For a series of historical data

$$H(t_h), H\left(t_h - \Delta_{tq}\right), q_h > 0, q = 1,2, \dots, q_h$$

We have

$$H(t_h + \Delta_{tq}) = H(t_h) + \sum_{k=0}^{\infty} \frac{\Delta_{tq}^{2k+1}}{(2k+1)!} \left(\frac{1}{\mu} \frac{(-1)^{k+1}}{(\varepsilon\mu)^k} \nabla^{\{2k+1\}} \times E + J_h^{[2k+1]}(t_h) \right) + \sum_{k=0}^{\infty} \frac{\Delta_{tq}^{2(k+1)}}{(2(k+1))!} \left(\frac{(-1)^{k+1}}{(\varepsilon\mu)^{k+1}} \nabla^{\{2(k+1)\}} \times H + J_h^{[2(k+1)]}(t_h) \right),$$
$$q = 1,2,\dots,q_h \quad (34)$$

Minus (34) from (33), we have

$$H(t_h + \Delta_t) = \sum_{q=1}^{q_h} H(t_h - \Delta_{tq}) + (1 - q_h)H(t_h) + \sum_{k=0}^{\infty} \left(\frac{\Delta_t^{2k+1} + \sum_{q=1}^{qh} (\Delta_{tq})^{2k+1}}{(2k+1)!} \left[\frac{1}{\mu} \frac{(-1)^{k+1}}{(\varepsilon\mu)^k} \nabla^{\{2k+1\}} \times E(t_h) + J_h^{[2k+1]}(t_h) \right] + \frac{\Delta_t^{2(k+1)} - \sum_{q=1}^{qh} (\Delta_{tq})^{2(k+1)}}{(2(k+1))!} \left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^{k+1}} \nabla^{\{2(k+1)\}} \times H(t_h) + J_h^{[2(k+1)]}(t_h) \right] \right) \quad (35)$$

(35) is Case 3 of (9).

Let $q_h = 1 \; and \; \Delta_t = \Delta_{t1}$, (35) becomes

$$H(t_h + \Delta_t) = H(t_h - \Delta_t) + \sum_{k=0}^{\infty} \left(\frac{2\Delta_t^{2k+1}}{(2k+1)!} \left[\frac{1}{\mu} \frac{(-1)^{k+1}}{(\varepsilon\mu)^k} \nabla^{\{2k+1\}} \times E(t_h) + J_h^{[2k+1]}(t_h) \right] \right) \quad (36)$$

(36) is Case 2 of (9).

Thus, (9) is proved.

QED.

Appendix C. Proof of Time Advancement Theorem E

Proof.

Apply the Taylor series, we have

$$E(t_e + \Delta_t) = \sum_{k=0}^{\infty} \frac{\Delta_t^k}{k!} \frac{\partial^k E(t_e)}{\partial t^k}$$

Rearrange the terms, we have

$$E(t_e + \Delta_t) = E(t_e) + \sum_{k=0}^{\infty} \frac{\Delta_t^{2k+1}}{(2k+1)!} \frac{\partial^{2k+1} E(t_e)}{\partial t^{2k+1}} + \sum_{k=0}^{\infty} \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} \frac{\partial^{2(k+1)} E(t_e)}{\partial t^{2(k+1)}} \quad (37)$$

Combine (14) and (18), (15) and (19), we have

$$\frac{\partial^{2k+1} E}{\partial t^{2k+1}} = (-1)^k \frac{1}{\varepsilon} \frac{1}{(\varepsilon\mu)^k} \nabla^{\{2k+1\}} \times H + J_e^{[2k+1]}(t_e) \quad (38)$$

$$\frac{\partial^{2(k+1)} E}{\partial t^{2(k+1)}} = (-1)^{k+1} \frac{1}{(\varepsilon\mu)^{k+1}} \nabla^{\{2(k+1)\}} \times E + J_e^{[2(k+1)]}(t_e) \quad (39)$$

Substitute (38) and (39) into (37), we have

$$
\begin{aligned}
E(t_e+\Delta_t) = E(t_e) &+ \sum_{k=0}^{\infty} \frac{\Delta_t^{2k+1}}{(2k+1)!}\left((-1)^k \frac{1}{\varepsilon}\frac{1}{(\varepsilon\mu)^k}\nabla^{\{2k+1\}} \times H + J_e^{[2k+1]}(t_e)\right) \\
&+ \sum_{k=0}^{\infty} \frac{\Delta_t^{2(k+1)}}{(2(k+1))!}\left((-1)^{k+1} \frac{1}{(\varepsilon\mu)^{k+1}}\nabla^{\{2(k+1)\}} \times E + J_e^{[2(k+1)]}(t_e)\right)
\end{aligned}
\tag{40}
$$

(40) is Case 1 of (13).

For a series of historical data

$$
E(t_e), E(t_e-\Delta_{tq}), q_e > 0, q = 1,2,\ldots,q_e
$$

We have

$$
\begin{aligned}
E(t_e+\Delta_{tq}) = E(t_e) &+ \sum_{k=0}^{\infty} \frac{\Delta_{tq}^{2k+1}}{(2k+1)!}\left((-1)^k \frac{1}{\varepsilon}\frac{1}{(\varepsilon\mu)^k}\nabla^{\{2k+1\}} \times H + J_e^{[2k+1]}(t_e)\right) \\
&+ \sum_{k=0}^{\infty} \frac{\Delta_{tq}^{2(k+1)}}{(2(k+1))!}\left((-1)^{k+1} \frac{1}{(\varepsilon\mu)^{k+1}}\nabla^{\{2(k+1)\}} \times E + J_e^{[2(k+1)]}(t_e)\right), \\
&q = 1,2,\ldots,q_e
\end{aligned}
\tag{41}
$$

Minus (41) from (40), we have

$$
\begin{aligned}
E(t_e+\Delta_t) = \sum_{q=1}^{q_e} E(t_e-\Delta_{tq}) &+ (1-q_e)E(t_e) \\
&+ \sum_{k=0}^{\infty}\left(\frac{\Delta_t^{(2k+1)} + \sum_{q=1}^{qe}(\Delta_{tq})^{2k+1}}{(2k+1)!}\left[\frac{(-1)^k}{\varepsilon(\varepsilon\mu)^k}\nabla^{\{2k+1\}} \times H(t_e) + J_e^{[2k+1]}(t_e)\right]\right. \\
&\left. + \frac{\Delta_t^{2(k+1)} - \sum_{q=1}^{qe}(\Delta_{tq})^{2(k+1)}}{(2(k+1))!}\left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\nabla^{\{2(k+1)\}} \times E(t_e) + J_e^{[2(k+1)]}(t_e)\right]\right)
\end{aligned}
\tag{42}
$$

(42) is Case 3 of (13).

Let $q_e = 1 \; and \; \Delta_t = \Delta_{t1}$, (42) becomes

$$
E(t_e+\Delta_t) = E(t_e-\Delta_t) + \sum_{k=0}^{\infty}\left(\frac{2\Delta_t^{(2k+1)}}{(2k+1)!}\left[\frac{(-1)^k}{\varepsilon(\varepsilon\mu)^k}\nabla^{\{2k+1\}} \times H(t_e) + J_e^{[2k+1]}(t_e)\right]\right)
\tag{43}
$$

(43) is Case 2 of (13).

Thus, (13) is proved.

QED.

Appendix D. Proof of Curls Cascade Theorem.

Proof. For $k = 1$, by (53) and (50), we have

$$
\begin{aligned}
P^{\{3\}} &= -\nabla \times \begin{bmatrix} D_x^2(V_x) \\ D_y^2(V_y) \\ D_z^2(V_z) \end{bmatrix} - \begin{bmatrix} \frac{\partial}{\partial y}\left(D_x^2(V_z)\right) - \frac{\partial}{\partial z}\left(D_x^2(V_y)\right) \\ \frac{\partial}{\partial z}\left(D_y^2(V_x)\right) - \frac{\partial}{\partial x}\left(D_y^2(V_z)\right) \\ \frac{\partial}{\partial x}\left(D_z^2(V_y)\right) - \frac{\partial}{\partial y}\left(D_z^2(V_x)\right) \end{bmatrix} = -\left(\begin{bmatrix} \frac{\partial}{\partial y}\left(D_z^2(V_z)\right) - \frac{\partial}{\partial z}\left(D_y^2(V_y)\right) \\ \frac{\partial}{\partial z}\left(D_x^2(V_x)\right) - \frac{\partial}{\partial x}\left(D_z^2(V_z)\right) \\ \frac{\partial}{\partial x}\left(D_y^2(V_y)\right) - \frac{\partial}{\partial y}\left(D_x^2(V_x)\right) \end{bmatrix} + \begin{bmatrix} \frac{\partial}{\partial y}\left(D_x^2(V_z)\right) - \frac{\partial}{\partial z}\left(D_x^2(V_y)\right) \\ \frac{\partial}{\partial z}\left(D_y^2(V_x)\right) - \frac{\partial}{\partial x}\left(D_y^2(V_z)\right) \\ \frac{\partial}{\partial x}\left(D_z^2(V_y)\right) - \frac{\partial}{\partial y}\left(D_z^2(V_x)\right) \end{bmatrix} \right) \\
&= -\begin{bmatrix} \frac{\partial}{\partial y}\left(D_z^2(V_z) + D_x^2(V_z)\right) - \frac{\partial}{\partial z}\left(D_y^2(V_y) + D_x^2(V_y)\right) \\ \frac{\partial}{\partial z}\left(D_x^2(V_x) + D_y^2(V_x)\right) - \frac{\partial}{\partial x}\left(D_z^2(V_z) + D_y^2(V_z)\right) \\ \frac{\partial}{\partial x}\left(D_y^2(V_y) + D_z^2(V_y)\right) - \frac{\partial}{\partial y}\left(D_x^2(V_x) + D_z^2(V_x)\right) \end{bmatrix}
\end{aligned} \tag{54}
$$

Insert the above result into (49), we have

$$
\begin{aligned}
\nabla^{\{3\}} \times V &= P^{\{3\}} - \begin{bmatrix} D_y^3(V_z) - D_z^3(V_y) \\ D_z^3(V_x) - D_x^3(V_z) \\ D_x^3(V_y) - D_y^3(V_x) \end{bmatrix} \\
&= -\begin{bmatrix} \frac{\partial}{\partial y}\left(D_z^2(V_z) + D_x^2(V_z) + D_y^2(V_z)\right) - \frac{\partial}{\partial z}\left(D_y^2(V_y) + D_x^2(V_y) + D_z^2(V_y)\right) \\ \frac{\partial}{\partial z}\left(D_x^2(V_x) + D_y^2(V_x) + D_z^2(V_x)\right) - \frac{\partial}{\partial x}\left(D_z^2(V_z) + D_y^2(V_z) + D_x^2(V_z)\right) \\ \frac{\partial}{\partial x}\left(D_y^2(V_y) + D_z^2(V_y) + D_x^2(V_y)\right) - \frac{\partial}{\partial y}\left(D_x^2(V_x) + D_z^2(V_x) + D_y^2(V_x)\right) \end{bmatrix} \\
&= -\nabla \times \Delta V
\end{aligned} \tag{55}
$$

Calculating $\nabla^{\{3\}} \times V$ by (47) and (48) gives us

$$
\nabla^{\{3\}} \times V = \nabla \times \nabla^{\{2\}} \times V = \nabla \times (\nabla(\nabla \cdot V) - \Delta V) = -\nabla \times \Delta V
$$

It is the same as (55). So, (49) holds for $k = 1$.

For $k = 1$, (52) becomes

$$
P^{\{4\}} = \nabla \times P^{\{3\}} - \begin{bmatrix} \frac{\partial}{\partial y}\left(D_x^3(V_y)\right) + \frac{\partial}{\partial z}\left(D_x^3(V_z)\right) \\ \frac{\partial}{\partial z}\left(D_y^3(V_z)\right) + \frac{\partial}{\partial x}\left(D_y^3(V_x)\right) \\ \frac{\partial}{\partial x}\left(D_z^3(V_x)\right) + \frac{\partial}{\partial y}\left(D_z^3(V_y)\right) \end{bmatrix} \tag{56}
$$

Insert (56) into (51), we have

$$
\begin{aligned}
\nabla^{\{4\}} \times V &= P^{\{4\}} + \begin{bmatrix} D_y^4(V_x) + D_z^4(V_x) \\ D_z^4(V_y) + D_x^4(V_y) \\ D_x^4(V_z) + D_y^4(V_z) \end{bmatrix} \\
&= \nabla \times P^{\{3\}} - \begin{bmatrix} \frac{\partial}{\partial y}\left(D_x^3(V_y)\right) + \frac{\partial}{\partial z}\left(D_x^3(V_z)\right) \\ \frac{\partial}{\partial z}\left(D_y^3(V_z)\right) + \frac{\partial}{\partial x}\left(D_y^3(V_x)\right) \\ \frac{\partial}{\partial x}\left(D_z^3(V_x)\right) + \frac{\partial}{\partial y}\left(D_z^3(V_y)\right) \end{bmatrix} + \begin{bmatrix} D_y^4(V_x) + D_z^4(V_x) \\ D_z^4(V_y) + D_x^4(V_y) \\ D_x^4(V_z) + D_y^4(V_z) \end{bmatrix} \\
&= \nabla \times P^{\{3\}} - \begin{bmatrix} \frac{\partial}{\partial y}\left(D_x^3(V_y) - D_y^3(V_x)\right) + \frac{\partial}{\partial z}\left(D_x^3(V_z) - D_z^3(V_x)\right) \\ \frac{\partial}{\partial z}\left(D_y^3(V_z) - D_z^3(V_y)\right) + \frac{\partial}{\partial x}\left(D_y^3(V_x) - D_x^3(V_y)\right) \\ \frac{\partial}{\partial x}\left(D_z^3(V_x) - D_x^3(V_z)\right) + \frac{\partial}{\partial y}\left(D_z^3(V_y) - D_y^3(V_z)\right) \end{bmatrix} \\
&= \nabla \times P^{\{3\}} - \nabla \times \begin{bmatrix} D_y^3(V_z) - D_z^3(V_y) \\ D_z^3(V_x) - D_x^3(V_z) \\ D_x^3(V_y) - D_y^3(V_x) \end{bmatrix}
\end{aligned} \tag{57}
$$

By (49), we have

$$
\nabla^{\{4\}} \times V = \nabla \times \nabla^{\{3\}} \times V = \nabla \times \left(P^{\{3\}} - \begin{bmatrix} D_y^3(V_z) - D_z^3(V_y) \\ D_z^3(V_x) - D_x^3(V_z) \\ D_x^3(V_y) - D_y^3(V_x) \end{bmatrix} \right)
$$

It is the same as (57). So, (51) holds for $k = 1$.

So, (49) and (51) hold for $k = 1$.

Suppose for a $k \geq 1$, (49) and (51) hold. Consider the case of $k + 1$.

For $k + 1$, (50) becomes

$$
P^{\{2(k+1)+1\}} = \nabla \times P^{\{2(k+1)\}} + (-1)^{k+1} \begin{bmatrix} \frac{\partial}{\partial y}\left(D_x^{2(k+1)}(V_z)\right) - \frac{\partial}{\partial z}\left(D_x^{2(k+1)}(V_y)\right) \\ \frac{\partial}{\partial z}\left(D_y^{2(k+1)}(V_x)\right) - \frac{\partial}{\partial x}\left(D_y^{2(k+1)}(V_z)\right) \\ \frac{\partial}{\partial x}\left(D_z^{2(k+1)}(V_y)\right) - \frac{\partial}{\partial y}\left(D_z^{2(k+1)}(V_x)\right) \end{bmatrix} \tag{58}
$$

Because (51) holds for $k \geq 1$, we have

$$\nabla^{\{2(k+1)+1\}} \times V = \nabla \times (\nabla^{\{2(k+1)\}} \times V) = \nabla \times \left(P^{\{2(k+1)\}} + (-1)^{k+1} \begin{bmatrix} D_y^{2(k+1)}(V_x) + D_z^{2(k+1)}(V_x) \\ D_z^{2(k+1)}(V_y) + D_x^{2(k+1)}(V_y) \\ D_x^{2(k+1)}(V_z) + D_y^{2(k+1)}(V_z) \end{bmatrix} \right)$$

$$= \nabla \times P^{\{2(k+1)\}} + (-1)^{k+1} \nabla \times \begin{bmatrix} D_y^{2(k+1)}(V_x) + D_z^{2(k+1)}(V_x) \\ D_z^{2(k+1)}(V_y) + D_x^{2(k+1)}(V_y) \\ D_x^{2(k+1)}(V_z) + D_y^{2(k+1)}(V_z) \end{bmatrix}$$

$$= \nabla \times P^{\{2(k+1)\}} + (-1)^{k+1} \begin{bmatrix} \frac{\partial}{\partial y}\left(D_x^{2(k+1)}(V_z) + D_y^{2(k+1)}(V_z)\right) - \frac{\partial}{\partial z}\left(D_z^{2(k+1)}(V_y) + D_x^{2(k+1)}(V_y)\right) \\ \frac{\partial}{\partial z}\left(D_y^{2(k+1)}(V_x) + D_z^{2(k+1)}(V_x)\right) - \frac{\partial}{\partial x}\left(D_x^{2(k+1)}(V_z) + D_y^{2(k+1)}(V_z)\right) \\ \frac{\partial}{\partial x}\left(D_z^{2(k+1)}(V_y) + D_x^{2(k+1)}(V_y)\right) - \frac{\partial}{\partial y}\left(D_y^{2(k+1)}(V_x) + D_z^{2(k+1)}(V_x)\right) \end{bmatrix}$$

$$= \nabla \times P^{\{2(k+1)\}} + (-1)^{k+1} \begin{bmatrix} \frac{\partial}{\partial y}\left(D_x^{2(k+1)}(V_z)\right) - \frac{\partial}{\partial z}\left(D_x^{2(k+1)}(V_y)\right) \\ \frac{\partial}{\partial z}\left(D_y^{2(k+1)}(V_x)\right) - \frac{\partial}{\partial x}\left(D_y^{2(k+1)}(V_z)\right) \\ \frac{\partial}{\partial x}\left(D_z^{2(k+1)}(V_y)\right) - \frac{\partial}{\partial y}\left(D_z^{2(k+1)}(V_x)\right) \end{bmatrix} + (-1)^{k+1} \begin{bmatrix} \frac{\partial}{\partial y}\left(D_y^{2(k+1)}(V_z)\right) - \frac{\partial}{\partial z}\left(D_z^{2(k+1)}(V_y)\right) \\ \frac{\partial}{\partial z}\left(D_z^{2(k+1)}(V_x)\right) - \frac{\partial}{\partial x}\left(D_x^{2(k+1)}(V_z)\right) \\ \frac{\partial}{\partial x}\left(D_x^{2(k+1)}(V_y)\right) - \frac{\partial}{\partial y}\left(D_y^{2(k+1)}(V_x)\right) \end{bmatrix}$$

We get

$$\nabla^{\{2(k+1)+1\}} \times V = \nabla \times P^{\{2(k+1)\}} + (-1)^{k+1} \begin{bmatrix} \frac{\partial}{\partial y}\left(D_x^{2(k+1)}(V_z)\right) - \frac{\partial}{\partial z}\left(D_x^{2(k+1)}(V_y)\right) \\ \frac{\partial}{\partial z}\left(D_y^{2(k+1)}(V_x)\right) - \frac{\partial}{\partial x}\left(D_y^{2(k+1)}(V_z)\right) \\ \frac{\partial}{\partial x}\left(D_z^{2(k+1)}(V_y)\right) - \frac{\partial}{\partial y}\left(D_z^{2(k+1)}(V_x)\right) \end{bmatrix} + (-1)^{k+1} \begin{bmatrix} D_y^{2(k+1)+1}(V_z) - D_z^{2(k+1)+1}(V_y) \\ D_z^{2(k+1)+1}(V_x) - D_x^{2(k+1)+1}(V_z) \\ D_x^{2(k+1)+1}(V_y) - D_y^{2(k+1)+1}(V_x) \end{bmatrix} \qquad (59)$$

Insert (58) into (59), we have

$$\nabla^{\{2(k+1)+1\}} \times V = P^{\{2(k+1)+1\}} + (-1)^{k+1} \begin{bmatrix} D_y^{2(k+1)+1}(V_z) - D_z^{2(k+1)+1}(V_y) \\ D_z^{2(k+1)+1}(V_x) - D_x^{2(k+1)+1}(V_z) \\ D_x^{2(k+1)+1}(V_y) - D_y^{2(k+1)+1}(V_x) \end{bmatrix} \qquad (60)$$

For $k+1$, (49) becomes

$$\nabla^{\{2(k+1)+1\}} \times V = P^{\{2(k+1)+1\}} + (-1)^{k+1} \begin{bmatrix} D_y^{2(k+1)+1}(V_z) - D_z^{2(k+1)+1}(V_y) \\ D_z^{2(k+1)+1}(V_x) - D_x^{2(k+1)+1}(V_z) \\ D_x^{2(k+1)+1}(V_y) - D_y^{2(k+1)+1}(V_x) \end{bmatrix}$$

It is the same as (60). So, (49) holds for $k+1$.

For $k+1$, (52) becomes

$$P^{\{2(k+2)\}} = \nabla \times P^{\{2(k+1)+1\}} + (-1)^{k+1} \begin{bmatrix} \frac{\partial}{\partial y}\left(D_x^{2(k+1)+1}(V_y)\right) + \frac{\partial}{\partial z}\left(D_x^{2(k+1)+1}(V_z)\right) \\ \frac{\partial}{\partial z}\left(D_y^{2(k+1)+1}(V_z)\right) + \frac{\partial}{\partial x}\left(D_y^{2(k+1)+1}(V_x)\right) \\ \frac{\partial}{\partial x}\left(D_z^{2(k+1)+1}(V_x)\right) + \frac{\partial}{\partial y}\left(D_z^{2(k+1)+1}(V_y)\right) \end{bmatrix} \tag{61}$$

Because for $k+1$, (49) holds, we have

$$\nabla^{\{2(k+2)\}} \times V = \nabla \times \nabla^{\{2(k+1)+1\}} \times V = \nabla \times \left(P^{\{2(k+1)+1\}} + (-1)^{k+1} \begin{bmatrix} D_y^{2(k+1)+1}(V_z) - D_z^{2(k+1)+1}(V_y) \\ D_z^{2(k+1)+1}(V_x) - D_x^{2(k+1)+1}(V_z) \\ D_x^{2(k+1)+1}(V_y) - D_y^{2(k+1)+1}(V_x) \end{bmatrix} \right)$$

$$= \nabla \times P^{\{2(k+1)+1\}} + (-1)^{k+1} \nabla \times \begin{bmatrix} D_y^{2(k+1)+1}(V_z) - D_z^{2(k+1)+1}(V_y) \\ D_z^{2(k+1)+1}(V_x) - D_x^{2(k+1)+1}(V_z) \\ D_x^{2(k+1)+1}(V_y) - D_y^{2(k+1)+1}(V_x) \end{bmatrix}$$

$$= \nabla \times P^{\{2(k+1)+1\}} + (-1)^{k+1} \begin{bmatrix} \frac{\partial}{\partial y}\left(D_x^{2(k+1)+1}(V_y) - D_y^{2(k+1)+1}(V_x)\right) - \frac{\partial}{\partial z}\left(D_z^{2(k+1)+1}(V_x) - D_x^{2(k+1)+1}(V_z)\right) \\ \frac{\partial}{\partial z}\left(D_y^{2(k+1)+1}(V_z) - D_z^{2(k+1)+1}(V_y)\right) - \frac{\partial}{\partial x}\left(D_x^{2(k+1)+1}(V_y) - D_y^{2(k+1)+1}(V_x)\right) \\ \frac{\partial}{\partial x}\left(D_z^{2(k+1)+1}(V_x) - D_x^{2(k+1)+1}(V_z)\right) - \frac{\partial}{\partial y}\left(D_y^{2(k+1)+1}(V_z) - D_z^{2(k+1)+1}(V_y)\right) \end{bmatrix}$$

$$= \nabla \times P^{\{2(k+1)+1\}} + (-1)^{k+1} \begin{bmatrix} \frac{\partial}{\partial y}\left(D_x^{2(k+1)+1}(V_y)\right) + \frac{\partial}{\partial z}\left(D_x^{2(k+1)+1}(V_z)\right) \\ \frac{\partial}{\partial z}\left(D_y^{2(k+1)+1}(V_z)\right) + \frac{\partial}{\partial x}\left(D_y^{2(k+1)+1}(V_x)\right) \\ \frac{\partial}{\partial x}\left(D_z^{2(k+1)+1}(V_x)\right) + \frac{\partial}{\partial y}\left(D_z^{2(k+1)+1}(V_y)\right) \end{bmatrix} + (-1)^{k+2} \begin{bmatrix} \frac{\partial}{\partial y}\left(D_y^{2(k+1)+1}(V_x)\right) + \frac{\partial}{\partial z}\left(D_z^{2(k+1)+1}(V_x)\right) \\ \frac{\partial}{\partial z}\left(D_z^{2(k+1)+1}(V_y)\right) + \frac{\partial}{\partial x}\left(D_x^{2(k+1)+1}(V_y)\right) \\ \frac{\partial}{\partial x}\left(D_x^{2(k+1)+1}(V_z)\right) + \frac{\partial}{\partial y}\left(D_y^{2(k+1)+1}(V_z)\right) \end{bmatrix}$$

We get

$$\begin{aligned} \nabla^{\{2(k+2)\}} \times V = \nabla \times P^{\{2(k+1)+1\}} &+ (-1)^{k+1} \begin{bmatrix} \frac{\partial}{\partial y}\left(D_x^{2(k+1)+1}(V_y)\right) + \frac{\partial}{\partial z}\left(D_x^{2(k+1)+1}(V_z)\right) \\ \frac{\partial}{\partial z}\left(D_y^{2(k+1)+1}(V_z)\right) + \frac{\partial}{\partial x}\left(D_y^{2(k+1)+1}(V_x)\right) \\ \frac{\partial}{\partial x}\left(D_z^{2(k+1)+1}(V_x)\right) + \frac{\partial}{\partial y}\left(D_z^{2(k+1)+1}(V_y)\right) \end{bmatrix} \\ &+ (-1)^{k+2} \begin{bmatrix} \frac{\partial}{\partial y}\left(D_y^{2(k+1)+1}(V_x)\right) + \frac{\partial}{\partial z}\left(D_z^{2(k+1)+1}(V_x)\right) \\ \frac{\partial}{\partial z}\left(D_z^{2(k+1)+1}(V_y)\right) + \frac{\partial}{\partial x}\left(D_x^{2(k+1)+1}(V_y)\right) \\ \frac{\partial}{\partial x}\left(D_x^{2(k+1)+1}(V_z)\right) + \frac{\partial}{\partial y}\left(D_y^{2(k+1)+1}(V_z)\right) \end{bmatrix} \end{aligned} \tag{62}$$

Insert (61) into (62), we have

$$\nabla^{\{2(k+2)\}} \times V = P^{\{2(k+2)\}} + (-1)^{k+2} \begin{bmatrix} D_y^{2(k+2)}(V_x) + D_z^{2(k+2)}(V_x) \\ D_z^{2(k+2)}(V_y) + D_x^{2(k+2)}(V_y) \\ D_x^{2(k+2)}(V_z) + D_y^{2(k+2)}(V_z) \end{bmatrix} \tag{63}$$

For $k+1$, (51) becomes

$$\nabla^{\{2(k+2)\}} \times V = P^{\{2(k+2)\}} + (-1)^{k+2} \begin{bmatrix} D_y^{2(k+2)}(V_x) + D_z^{2(k+2)}(V_x) \\ D_z^{2(k+2)}(V_y) + D_x^{2(k+2)}(V_y) \\ D_x^{2(k+2)}(V_z) + D_y^{2(k+2)}(V_z) \end{bmatrix}$$

It is the same as (63). So, (51) holds for $k+1$.

Thus, (49) and (51) hold for $k+1$.

QED.