Generic FDTD for Full Maxwell Equations

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Abstract

A generic FDTD form is derived for the Maxwell equations with medium effects and electric and magnet sources. A time space relation is proposed and proved, together with the cascade curl theorem I proved earlier, they form the foundation for the FDTD algorithms. Other FDTD algorithms, i.e. Yee algorithm, high order algorithms, can be derived from this generic form by specifying parameters, adding other techniques, adding restrictions, etc. Several FDTD algorithms are derived from this generic FDTD, as examples.

Contents

Introduction

In [1] I derived a generic FDTD form for Maxwell equations in lossless medium. The targeted Maxwell equations are

$$\nabla \cdot E = \rho/\varepsilon \tag{1}$$

$$\nabla \cdot H = 0 \tag{2}$$

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon}\nabla \times H - \frac{1}{\varepsilon}J(t) \tag{3}$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu}\nabla \times E \tag{4}$$

Where t is time, ρ, ε *and* μ are time-invariants, $J(t)$ is a known 3D vector time function, E *and* H are 3D vectors in Cartesian coordinates (x, y, z), representing an electric field and a magnetic field, respectively, as

$$E(x,y,z,t) = \begin{bmatrix} E_x(x,y,z,t) \\ E_y(x,y,z,t) \\ E_z(x,y,z,t) \end{bmatrix}, H(x,y,z,t) = \begin{bmatrix} H_x(x,y,z,t) \\ H_y(x,y,z,t) \\ H_z(x,y,z,t) \end{bmatrix}$$

For a 3D vector V, its cascade curls are denoted as following.

$$\nabla^{\{0\}} \times V \equiv V, \nabla \times V \equiv \nabla^{\{1\}} \times V \equiv \begin{bmatrix} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \\ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{bmatrix}, \nabla^{\{k\}} \times V \equiv \underbrace{\nabla \times \nabla \times \dots \times \nabla \times}_{k} V, k > 0 \tag{5}$$

To consider medium effects, add conductivity constants σ and σ_m into to (3) and (4). Suppose electric source and magnetic source are time dependent, denoted by $J_e(t)$ and $J_m(t)$. We are dealing with the following Maxwell equations.

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon}\nabla \times H - \frac{\sigma}{\varepsilon}E - \frac{1}{\varepsilon}J_e(t) \tag{6}$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu}\nabla \times E - \frac{\sigma_m}{\mu}H - \frac{1}{\mu}J_m(t) \tag{7}$$

In this paper, the time space lemma proved in [1] is extended to support the above more generic Maxwell equations. Thus, the generic FDTD form derived in [1] is extended to support more generic scenarios.

Time Space Relations

Time Space Lemma. The following time space relations can be derived from equations (6) and (7).

$$\frac{\partial^{2k} E}{\partial t^{2k}} = p_{2k}^{\{2k\}}E + \sum_{i=1}^{k}\left(p_{2(k-i)}^{\{2k\}}\nabla^{\{2i\}} \times E + p_{2(k-i)+1}^{\{2k\}}\nabla^{\{2i-1\}} \times H\right) + S_e(2k) \tag{8}$$

$$\frac{\partial^{2k} H}{\partial t^{2k}} = q_{2k}^{\{2k\}}H + \sum_{i=1}^{k}\left(q_{2(k-i)}^{\{2k\}}\nabla^{\{2i\}} \times H + q_{2(k-i)+1}^{\{2k\}}\nabla^{\{2i-1\}} \times E\right) + S_m(2k) \tag{9}$$

$$\frac{\partial^{2k+1} E}{\partial t^{2k+1}} = \sum_{i=0}^{k}(p_{2i}^{\{2k+1\}}\nabla^{\{2(k-i)+1\}} \times H + p_{2i+1}^{\{2k+1\}}\nabla^{\{2(k-i)\}} \times E) + S_e(2k+1) \tag{10}$$

$$\frac{\partial^{2k+1} H}{\partial t^{2k+1}} = \sum_{i=0}^{k}(q_{2i}^{\{2k+1\}}\nabla^{\{2(k-i)+1\}} \times E + q_{2i+1}^{\{2k+1\}}\nabla^{\{2(k-i)\}} \times H) + S_m(2k+1) \tag{11}$$

Where S_e *and* S_m are formed by field sources J_e *and* J_m , and given by

$$S_e(2k)=\sum_{h=0}^{k-1}\left(\nabla^{\{2(k-1-h)\}}\times\sum_{i=1}^{2(h+1)}e_{i-1}^{\{2(k-1-h)\}}\frac{d^{2(h+1)-i}J_e}{dt^{2(h+1)-i}}+\nabla^{\{2(k-1-h)+1\}}\times\sum_{i=0}^{2h}m_i^{\{2(k-1-h)+1\}}\frac{d^{2h-i}J_m}{dt^{2h-i}}\right) \tag{12}$$

$$S_m(2k)=\sum_{h=0}^{k-1}\left(\nabla^{\{2(k-1-h)\}}\times\sum_{i=1}^{2(h+1)}m_{i-1}^{\{2(k-1-h)\}}\frac{d^{2(h+1)-i}J_m}{dt^{2(h+1)-i}}+\nabla^{\{2(k-1-h)+1\}}\times\sum_{i=0}^{2h}e_i^{\{2(k-1-h)+1\}}\frac{d^{2h-i}J_e}{dt^{2h-i}}\right) \tag{13}$$

$$S_e(2k+1)=\sum_{h=0}^{k}\nabla^{\{2(\mathrm{k-h})\}}\times\sum_{i=0}^{2h}e_i^{\{2(k-h)\}}\frac{d^{2h-i}J_e}{dt^{2h-i}}+\sum_{h=0}^{k-1}\nabla^{\{2(\mathrm{k-h})-1\}}\times\sum_{i=0}^{2h+1}m_i^{\{2(k-h)-1\}}\frac{d^{2h-i+1}J_m}{dt^{2h-i+1}} \tag{14}$$

$$S_m(2k+1)=\sum_{h=0}^{k}\nabla^{\{2(\mathrm{k-h})\}}\times\sum_{i=0}^{2h}m_i^{\{2(k-h)\}}\frac{d^{2h-i}J_m}{dt^{2h-i}}+\sum_{h=0}^{k-1}\nabla^{\{2(\mathrm{k-h})-1\}}\times\sum_{i=0}^{2h+1}e_i^{\{2(k-h)-1\}}\frac{d^{2h-i+1}J_e}{dt^{2h-i+1}} \tag{15}$$

Where the coefficients are determined by material properties, and given by

$$e_0^{\{0\}}=-\frac{1}{\varepsilon},p_0^{\{1\}}=\frac{1}{\varepsilon},p_1^{\{1\}}=-\frac{\sigma}{\varepsilon} \tag{16}$$

$$m_0^{\{0\}}=-\frac{1}{\mu},q_0^{\{1\}}=-\frac{1}{\mu},q_1^{\{1\}}=-\frac{\sigma_m}{\mu} \tag{17}$$

$$e_{2(k-i)}^{\{2i\}}=p_{2(k-i)}^{\{2k\}}e_0^{\{0\}},m_{2(k-i)}^{\{2i\}}=q_{2(k-i)}^{\{2k\}}m_0^{\{0\}};i=0,1,2,\dots,k \tag{18}$$

$$m_{2(k-i)+1}^{\{2i-1\}}=p_{2(k-i)+1}^{\{2k\}}m_0^{\{0\}},e_{2(k-i)+1}^{\{2i-1\}}=q_{2(k-i)+1}^{\{2k\}}e_0^{\{0\}};i=1,2,\dots,k \tag{19}$$

$$p_0^{\{2k+1\}}=p_0^{\{1\}}p_0^{\{2k\}},p_{2k+1}^{\{2k+1\}}=p_{2k}^{\{2k\}}p_1^{\{1\}} \tag{20}$$

$$q_0^{\{2k+1\}}=q_0^{\{1\}}q_0^{\{2k\}},q_{2k+1}^{\{2k+1\}}=q_{2k}^{\{2k\}}q_1^{\{1\}} \tag{21}$$

$$p_{2i-1}^{\{2k+1\}}=p_1^{\{1\}}p_{2i-2}^{\{2k\}}+q_0^{\{1\}}p_{2i-1}^{\{2k\}} \tag{22}$$

$$p_{2i}^{\{2k+1\}}=q_1^{\{1\}}p_{2i-1}^{\{2k\}}+p_0^{\{1\}}p_{2i}^{\{2k\}} \tag{23}$$

$$q_{2i-1}^{\{2k+1\}}=q_1^{\{1\}}q_{2i-2}^{\{2k\}}+p_0^{\{1\}}q_{2i-1}^{\{2k\}} \tag{24}$$

$$q_{2i}^{\{2k+1\}}=p_1^{\{1\}}q_{2i-1}^{\{2k\}}+q_0^{\{1\}}q_{2i}^{\{2k\}} \tag{25}$$

$$i=1,2,\dots,k;k\geq 1$$

$$e_{2(k-i)-1}^{\{2i\}}=p_{2(k-i)-1}^{\{2k-1\}}e_0^{\{0\}},m_{2(k-i)-1}^{\{2i\}}=q_{2(k-i)-1}^{\{2k-1\}}m_0^{\{0\}};i=0,1,2,\dots,k-1 \tag{26}$$

$$m_{2(k-i-1)}^{\{2i+1\}}=p_{2(k-i-1)}^{\{2k-1\}}m_0^{\{0\}},e_{2(k-i-1)}^{\{2i+1\}}=q_{2(k-i-1)}^{\{2k-1\}}e_0^{\{0\}};i=0,1,2,\dots,k-1 \tag{27}$$

$$\begin{gathered}p_0^{\{2k\}}=q_0^{\{1\}}p_0^{\{2k-1\}},p_{2k}^{\{2k\}}=p_{2k-1}^{\{2k-1\}}p_1^{\{1\}}\\ q_0^{\{2k\}}=p_0^{\{1\}}q_0^{\{2k-1\}},q_{2k}^{\{2k\}}=q_{2k-1}^{\{2k-1\}}q_1^{\{1\}}\end{gathered} \tag{28}$$

$$p_{2i+1}^{\{2k\}}=q_1^{\{1\}}p_{2i}^{\{2k-1\}}+p_0^{\{1\}}p_{2i+1}^{\{2k-1\}};i=0,1,2,\dots,k-1 \tag{29}$$

$$p_{2i+2}^{\{2k\}}=p_1^{\{1\}}p_{2i+1}^{\{2k-1\}}+q_0^{\{1\}}p_{2i+2}^{\{2k-1\}};i=1,2\dots,k-1;k>1 \tag{30}$$

The proof of this lemma is given in the Appendix.

Time Advancement Theorems

Using the above lemma to the time advancement theorems in [1], we have following new theorems.

Time Advancement Theorem H. At a time t_h, suppose the fields and sources are $E=E(t_h),H=H(t_h),J_e=J_e(t_h), and\ J_m=J_m(t_h)$, then for any time deviation $\Delta_t, H(t_h+\Delta_t)$ can be expressed by the following formula.

$$\begin{aligned}H(t_h+\Delta_t)=H(t_h)&+\sum_{k=0}^{\infty}\frac{\Delta_t^{2k+1}}{(2k+1)!}\left(\sum_{i=0}^{k}(q_{2i}^{\{2k+1\}}\nabla^{\{2(\mathrm{k-i})+1\}}\times E+q_{2i+1}^{\{2k+1\}}\nabla^{\{2(\mathrm{k-i})\}}\times H)+S_m(2k+1)\right)\\&+\sum_{k=0}^{\infty}\frac{\Delta_t^{2(k+1)}}{(2(k+1))!}\left(q_{2(k+1)}^{\{2(k+1)\}}H+\sum_{i=1}^{k+1}\left(q_{2(k+1-i)}^{\{2(k+1)\}}\nabla^{\{2i\}}\times H+q_{2(k+1-i)+1}^{\{2(k+1)\}}\nabla^{\{2i-1\}}\times E\right)+S_m(2(k+1))\right)\end{aligned} \tag{31}$$

Where the related values are defined by (12) – (30).

Proof.

Apply the Taylor series, we have

$$H(t_h+\Delta_t)=\sum_{k=0}^{\infty}\frac{\Delta_t^k}{k!}\frac{\partial^k H(t_h)}{\partial t^k}$$

Rearrange the terms, we have

$$H(t_h+\Delta_t)=H(t_h)+\sum_{k=0}^{\infty}\frac{\Delta_t^{2k+1}}{(2k+1)!}\frac{\partial^{2k+1}H(t_h)}{\partial t^{2k+1}}+\sum_{k=0}^{\infty}\frac{\Delta_t^{2(k+1)}}{(2(k+1))!}\frac{\partial^{2(k+1)}H(t_h)}{\partial t^{2(k+1)}} \tag{32}$$

Substitute (9) and (11) into (32), we get (31).

QED.

Time Advancement Theorem E. At a time t_e, suppose the fields and sources are $E=E(t_e), H=H(t_e), J_e=J_e(t_e), and\ J_m=J_m(t_e)$, then for any time deviation $\Delta_t, E(t_e+\Delta_t)$ can be expressed by the following formula.

$$\begin{aligned}E(t_e+\Delta_t)=E(t_e)&+\sum_{k=0}^{\infty}\frac{\Delta_t^{2k+1}}{(2k+1)!}\left(\sum_{i=0}^{k}(p_{2i}^{\{2k+1\}}\nabla^{\{2(k-i)+1\}}\times H+p_{2i+1}^{\{2k+1\}}\nabla^{\{2(k-i)\}}\times E)+S_e(2k+1)\right)\\&+\sum_{k=0}^{\infty}\frac{\Delta_t^{2(k+1)}}{(2(k+1))!}\left(p_{2(k+1)}^{\{2(k+1)\}}E+\sum_{i=1}^{k+1}\left(p_{2(k+1-i)}^{\{2(k+1)\}}\nabla^{\{2i\}}\times E+p_{2(k+1-i)+1}^{\{2(k+1)\}}\nabla^{\{2i-1\}}\times H\right)+S_e(2(k+1))\right)\end{aligned} \tag{33}$$

Where the related values are defined by (12) – (30).

Proof.

Apply the Taylor series, we have

$$E(t_e+\Delta_t)=\sum_{k=0}^{\infty}\frac{\Delta_t^k}{k!}\frac{\partial^k E(t_e)}{\partial t^k}$$

Rearrange the terms, we have

$$E(t_e+\Delta_t)=E+\sum_{k=0}^{\infty}\frac{\Delta_t^{2k+1}}{(2k+1)!}\frac{\partial^{2k+1}E}{\partial t^{2k+1}}+\sum_{k=0}^{\infty}\frac{\Delta_t^{2(k+1)}}{(2(k+1))!}\frac{\partial^{2(k+1)}E}{\partial t^{2(k+1)}} \tag{34}$$

Substitute (8) and (10) into (34), we get (33).

QED.

Generic FDTD Form

In [1], I have proved a curl cascade theorem for providing high order FDTD algorithms. The theorem can be used in this paper without modifications. But I would like to point out that the space samplings can be different for different space derivative estimators. I'll describe it below.

In [1], the space derivative estimators are given by

$$\begin{aligned}D_x^h(V_u)=\frac{\partial^h V_u}{\partial x^h}\approx\mathfrak{D}_x^h(V_u)\equiv[a_{h1}\quad a_{h2}\quad\cdots\quad a_{hM}]\begin{bmatrix}V_u(x+\Delta_1,y,z)-V_u(x,y,z)\\V_u(x+\Delta_2,y,z)-V_u(x,y,z)\\\vdots\\V_u(x+\Delta_M,y,z)-V_u(x,y,z)\end{bmatrix}\\D_y^h(V_u)=\frac{\partial^h V_u}{\partial y^h}\approx\mathfrak{D}_y^h(V_u)\equiv[a_{h1}\quad a_{h2}\quad\cdots\quad a_{hM}]\begin{bmatrix}V_u(x,y+\Delta_1,z)-V_u(x,y,z)\\V_u(x,y+\Delta_2,z)-V_u(x,y,z)\\\vdots\\V_u(x,y+\Delta_M,z)-V_u(x,y,z)\end{bmatrix}\end{aligned} \tag{35}$$

$$D_z^h(V_u) = \frac{\partial^h V_u}{\partial z^h} \approx \mathfrak{D}_z^h(V_u) \equiv [a_{h1} \quad a_{h2} \quad \cdots \quad a_{hM}] \begin{bmatrix} V_u(x,y,z+\Delta_1) - V_u(x,y,z) \\ V_u(x,y,z+\Delta_2) - V_u(x,y,z) \\ \vdots \\ V_u(x,y,z+\Delta_M) - V_u(x,y,z) \end{bmatrix}$$

$$u \; can \; be \; x,y,z; h = 1,2,\ldots,M$$

Where the parameters are given by

$$\begin{bmatrix} \Delta_1 & \frac{1}{2!}\Delta_1^2 & \cdots & \frac{1}{M!}\Delta_1^M \\ \Delta_2 & \frac{1}{2!}\Delta_2^2 & \cdots & \frac{1}{M!}\Delta_2^M \\ \vdots & \vdots & \cdots & \vdots \\ \Delta_M & \frac{1}{2!}\Delta_M^2 & \cdots & \frac{1}{M!}\Delta_M^M \end{bmatrix}^{-1} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1M} \\ a_{21} & a_{22} & \cdots & a_{2M} \\ \vdots & \vdots & \cdots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MM} \end{bmatrix} \tag{36}$$

The above formulations gave an impression that space sampling $\Delta_1, \Delta_2, \ldots, \Delta_M$ are the same for the 3 axis. But it is not necessarily so. Let's change the formulations to show different samplings.

$$\begin{bmatrix} \Delta_{u1} & \frac{1}{2!}\Delta_{u1}^2 & \cdots & \frac{1}{M!}\Delta_{u1}^M \\ \Delta_{u2} & \frac{1}{2!}\Delta_{u2}^2 & \cdots & \frac{1}{M!}\Delta_{u2}^M \\ \vdots & \vdots & \cdots & \vdots \\ \Delta_{uM} & \frac{1}{2!}\Delta_{uM}^2 & \cdots & \frac{1}{M!}\Delta_{uM}^M \end{bmatrix}^{-1} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1M} \\ a_{21} & a_{22} & \cdots & a_{2M} \\ \vdots & \vdots & \cdots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MM} \end{bmatrix}_u = \begin{bmatrix} [a_{11} & a_{12} & \cdots & a_{1M}]_u \\ [a_{21} & a_{22} & \cdots & a_{2M}]_u \\ & \vdots & & \\ [a_{M1} & a_{M2} & \cdots & a_{MM}]_u \end{bmatrix} \tag{37}$$

$$D_x^h(V_u) = \frac{\partial^h V_u}{\partial x^h} \approx \mathfrak{D}_x^h(V_u) \equiv [a_{h1} \quad a_{h2} \quad \cdots \quad a_{hM}]_x \begin{bmatrix} V_u(x+\Delta_{x1},y,z) - V_u(x,y,z) \\ V_u(x+\Delta_{x2},y,z) - V_u(x,y,z) \\ \vdots \\ V_u(x+\Delta_{xM},y,z) - V_u(x,y,z) \end{bmatrix}$$
$$D_y^h(V_u) = \frac{\partial^h V_u}{\partial y^h} \approx \mathfrak{D}_y^h(V_u) \equiv [a_{h1} \quad a_{h2} \quad \cdots \quad a_{hM}]_y \begin{bmatrix} V_u(x,y+\Delta_{y1},z) - V_u(x,y,z) \\ V_u(x,y+\Delta_{y2},z) - V_u(x,y,z) \\ \vdots \\ V_u(x,y+\Delta_{yM},z) - V_u(x,y,z) \end{bmatrix}$$
$$D_z^h(V_u) = \frac{\partial^h V_u}{\partial z^h} \approx \mathfrak{D}_z^h(V_u) \equiv [a_{h1} \quad a_{h2} \quad \cdots \quad a_{hM}]_z \begin{bmatrix} V_u(x,y,z+\Delta_{z1}) - V_u(x,y,z) \\ V_u(x,y,z+\Delta_{z2}) - V_u(x,y,z) \\ \vdots \\ V_u(x,y,z+\Delta_{zM}) - V_u(x,y,z) \end{bmatrix}$$
$$u \; can \; be \; x,y,z; h = 1,2,\ldots,M \tag{38}$$

With the above space derivative estimators, we can get space curl estimators:

$$\overline{\nabla}^{\{k\}} \times E \approx \nabla^{\{k\}} \times E$$
$$\overline{\nabla}^{\{k\}} \times H \approx \nabla^{\{k\}} \times H$$
$$k = 1,2,\ldots \tag{39}$$

For details, see [1].

Substitute (39) into (31) and (33), we get the following generic FDTD forms.

Generic FDTD for magnetic field:

$$\begin{aligned} H(t_h + \Delta_t) \approx H(t_h) &+ \sum_{k=0}^{k_{max}} \frac{\Delta_t^{2k+1}}{(2k+1)!} \left(\sum_{i=0}^{k} (q_{2i}^{\{2k+1\}} \overline{\nabla}^{\{2(k-i)+1\}} \times E + q_{2i+1}^{\{2k+1\}} \overline{\nabla}^{\{2(k-i)\}} \times H) + \overline{S_m}(2k+1) \right) \\ &+ \sum_{k=0}^{k_{max}} \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} \left(q_{2(k+1)}^{\{2(k+1)\}} H + \sum_{i=1}^{k+1} \left(q_{2(k+1-i)}^{\{2(k+1)\}} \overline{\nabla}^{\{2i\}} \times H + q_{2(k+1-i)+1}^{\{2(k+1)\}} \overline{\nabla}^{\{2i-1\}} \times E \right) + \overline{S_m}(2(k+1)) \right) \end{aligned} \tag{40}$$

where $\overline{S_m}$ denotes S_m with curls replaced by curl estimators.

Generic FDTD for electric field:

$$E(t_e+\Delta_t)=E(t_e)+\sum_{k=0}^{k_{max}}\frac{\Delta_t^{2k+1}}{(2k+1)!}\left(\sum_{i=0}^{k}\left(p_{2i}^{\{2k+1\}}\overline{\nabla}^{\{2(k-i)+1\}}\times H+p_{2i+1}^{\{2k+1\}}\overline{\nabla}^{\{2(k-i)\}}\times E\right)+\overline{S}_e(2k+1)\right)$$ $$+\sum_{k=0}^{k_{max}}\frac{\Delta_t^{2(k+1)}}{(2(k+1))!}\left(p_{2(k+1)}^{\{2(k+1)\}}E+\sum_{i=1}^{k+1}\left(p_{2(k+1-i)}^{\{2(k+1)\}}\overline{\nabla}^{\{2i\}}\times E+p_{2(k+1-i)+1}^{\{2(k+1)\}}\overline{\nabla}^{\{2i-1\}}\times H\right)+\overline{S}_e(2(k+1))\right)$$	(41)

where $\overline{S_e}$ denotes S_e with curls replaced by curl estimators.

Deriving of FDTD Algorithms

Any Taylor's series based FDTD algorithms can be derived from the above generic FDTD forms. Below I'll derive some algorithms. We can see that each algorithm has its unique characteristics and conditions.

The Time-Space Synchronized FDTD (TSS)

Assume $E_x, E_y, E_z, H_x, H_y, and\ H_z\ are\ all\ at\ the\ same\ space\ location\ (x,y,z), and$ let

$$t_h = t_e$$

Then (40) and (41) give $H(t_h+q\Delta_t), E(t_h+q\Delta_t), q=1,2,3,\ldots$

Such an algorithm allows us to calculate Poynting vector to study field energy transfer:

$$S = E \times H$$

It also allows us to calculate divergence as estimation errors:

$error_h = \mathfrak{D}_x(H_x)+\mathfrak{D}_y(H_y)+\mathfrak{D}_z(H_z)$	(a.1)
$error_e = \mathfrak{D}_x(E_x)+\mathfrak{D}_y(E_y)+\mathfrak{D}_z(E_z)-\rho/\varepsilon$	(a.2)

These errors may give precise comparisons of different algorithms' accuracy. That is, for two algorithms, we can precisely say which one is more accurate.

Time-Leap-Frog Space-synchronized FDTD (TLFSS)

By (40), we have

$$H\left(t_h+\frac{1}{2}\Delta_t\right)-H\left(t_h-\frac{1}{2}\Delta_t\right)\approx\sum_{k=0}^{k_{max}}\frac{2\left(\frac{1}{2}\Delta_t\right)^{2k+1}}{(2k+1)!}\left(\sum_{i=0}^{k}\left(q_{2i}^{\{2k+1\}}\overline{\nabla}^{\{2(k-i)+1\}}\times E+q_{2i+1}^{\{2k+1\}}\overline{\nabla}^{\{2(k-i)\}}\times H\right)+\overline{S_m}(2k+1)\right)(t_h)$$	(a.3)

By (41), we have

$$E\left(t_e+\frac{1}{2}\Delta_t\right)-E\left(t_e-\frac{1}{2}\Delta_t\right)\approx\sum_{k=0}^{k_{max}}\frac{2\left(\frac{1}{2}\Delta_t\right)^{2k+1}}{(2k+1)!}\left(\sum_{i=0}^{k}\left(p_{2i}^{\{2k+1\}}\overline{\nabla}^{\{2(k-i)+1\}}\times H+p_{2i+1}^{\{2k+1\}}\overline{\nabla}^{\{2(k-i)\}}\times E\right)+\overline{S}_e(2k+1)\right)(t_e)$$	(a.4)

Lossless Medium

For lossless medium,

$$\sigma=\sigma_m=0\rightarrow q_{2i+1}^{\{2k+1\}}=p_{2i+1}^{\{2k+1\}}=0$$

(a.3) and (a.4) become

$$H\left(t_h+\frac{1}{2}\Delta_t\right)-H\left(t_h-\frac{1}{2}\Delta_t\right)\approx\sum_{k=0}^{k_{max}}\frac{2\left(\frac{1}{2}\Delta_t\right)^{2k+1}}{(2k+1)!}\left(\sum_{i=0}^{k}\left(q_{2i}^{\{2k+1\}}\overline{\nabla}^{\{2(k-i)+1\}}\times E\right)+\overline{S_m}(2k+1)\right)(t_h)$$	(a.5)

$E\left(t_e+\frac{1}{2}\Delta_t\right)-E\left(t_e-\frac{1}{2}\Delta_t\right)\approx\sum_{k=0}^{k_{max}}\frac{2\left(\frac{1}{2}\Delta_t\right)^{2k+1}}{(2k+1)!}\left(\sum_{i=0}^{k}\left(p_{2i}^{\{2k+1\}}\overline{\nabla}^{\{2(k-i)+1\}}\times H\right)+\overline{S_e}(2k+1)\right)(t_e)$	(a.6)

Assume $E_x, E_y, E_z, H_x, H_y, and\ H_z\ are\ all\ at\ the\ same\ space\ location\ (x, y, z), and$ let

$$t_e = t_h + \frac{1}{2}\Delta t$$

(a.5) and (a.6) give $H\left(t_h+\frac{1}{2}\Delta_t+q\Delta_t\right), E(t_h+q\Delta_t), q = 0,1,2,3, \ldots$

The advantage of this algorithm is that it increases the time advancement estimation order.

The limitation of this algorithm is that it only works for lossless medium.

Lossy Medium

Let

$$k_{max} = 0$$

(a.3) and (a.4) become

$H\left(t_h+\frac{1}{2}\Delta_t\right)-H\left(t_h-\frac{1}{2}\Delta_t\right)\approx\Delta_t\left(q_0^{\{1\}}\overline{\nabla}^{\{1\}}\times E+q_1^{\{1\}}H++\overline{S_m}(1)\right)(t_h)$	(a.7)
$E\left(t_e+\frac{1}{2}\Delta_t\right)-E\left(t_e-\frac{1}{2}\Delta_t\right)\approx\Delta_t\left(p_0^{\{1\}}\overline{\nabla}^{\{1\}}\times H+p_1^{\{1\}}E+\overline{S_e}(1)\right)(t_e)$	(a.8)

But we have unavailable values

$$H(t_h)\ and\ E(t_e)\ are\ unvailable$$

One way to get around the problem is to use average values:

$H(t_h)\approx\frac{\left(H\left(t_h+\frac{1}{2}\Delta_t\right)+H\left(t_h-\frac{1}{2}\Delta_t\right)\right)}{2}$	(a.9)
$E(t_e)\approx\frac{E\left(t_e+\frac{1}{2}\Delta_t\right)+E\left(t_e-\frac{1}{2}\Delta_t\right)}{2}$	(a.10)

Insert (a.9) into (a.7), and insert (a.10) into (a.8), we have

$H\left(t_h+\frac{1}{2}\Delta_t\right)\approx\frac{\left(1+\frac{\Delta_t q_1^{\{1\}}}{2}\right)}{\left(1-\frac{\Delta_t q_1^{\{1\}}}{2}\right)}H\left(t_h-\frac{1}{2}\Delta_t\right)+\frac{\Delta_t\left(q_0^{\{1\}}\overline{\nabla}^{\{1\}}\times E++\overline{S_m}(1)\right)(t_h)}{\left(1-\frac{\Delta_t q_1^{\{1\}}}{2}\right)}$	(a.11)
$E\left(t_e+\frac{1}{2}\Delta_t\right)\approx\frac{\left(1+\frac{\Delta_t p_1^{\{1\}}}{2}\right)}{\left(1-\frac{\Delta_t p_1^{\{1\}}}{2}\right)}E\left(t_e-\frac{1}{2}\Delta_t\right)+\frac{\Delta_t\left(p_0^{\{1\}}\overline{\nabla}^{\{1\}}\times H+\overline{S_e}(1)\right)(t_e)}{\left(1-\frac{\Delta_t p_1^{\{1\}}}{2}\right)}$	(a.12)

Assume $E_x, E_y, E_z, H_x, H_y, and\ H_z\ are\ all\ at\ the\ same\ space\ location\ (x, y, z), and$ let

$$t_e = t_h + \frac{1}{2}\Delta t$$

(a.11) and (a.12) give $H\left(t_h+\frac{1}{2}\Delta_t+q\Delta_t\right), E(t_h+q\Delta_t), q = 0,1,2,3, \ldots$

The advantage of this algorithm is that it can handle lossy medium. The limitation is that its estimation order is limited.

We can no longer calculate Poynting vector.

We can still calculate divergences to get precise estimation errors.

Single Field Algorithm

By (41), we have

$$E\left(t_e+\frac{1}{2}\Delta_t\right)+E\left(t_e-\frac{1}{2}\Delta_t\right) \approx \sum_{k=0}^{k_{max}}\frac{2\left(\frac{1}{2}\Delta_t\right)^{2k+1}}{(2(k+1))!}\left(p_{2(k+1)}^{\{2(k+1)\}}E+\sum_{i=1}^{k+1}\left(p_{2(k+1-i)}^{\{2(k+1)\}}\overline{\nabla}^{\{2i\}}\times E+p_{2(k+1-i)+1}^{\{2(k+1)\}}\overline{\nabla}^{\{2i-1\}}\times H\right)+\overline{S}_e(2(k+1))\right)(t_e) \quad \text{(a.13)}$$

If we are dealing with lossless medium, then

$$\sigma=\sigma_m=0\rightarrow p_{2(k+1-i)+1}^{\{2(k+1)\}}=0$$

(a.13) becomes

$$E\left(t_e+\frac{1}{2}\Delta_t\right)\approx -E\left(t_e-\frac{1}{2}\Delta_t\right)+\sum_{k=0}^{k_{max}}\frac{2\left(\frac{1}{2}\Delta_t\right)^{2k+1}}{(2(k+1))!}\left(p_{2(k+1)}^{\{2(k+1)\}}E+\sum_{i=1}^{k+1}\left(p_{2(k+1-i)}^{\{2(k+1)\}}\overline{\nabla}^{\{2i\}}\times E\right)+\overline{S}_e(2(k+1))\right)(t_e) \quad \text{(a.14)}$$

We can use two sets of history field data to get a new set of field data.

$$\left(E(t_e-\frac{1}{2}\Delta_t),E(t_e)\right)\rightarrow E\left(t_e+\frac{1}{2}\Delta_t\right)$$

(a.14) gives $E\left(t_e+\frac{1}{2}q\Delta_t\right), q=1,2,3,\ldots,$

The advantage of this algorithm is that it only deals with one single field. Here we show the electric field. We can also only estimate the magnetic field.

The limitation is that it only handles lossless medium.

The Yee algorithm

We start from the above Time-Leap-Frog-Space-Synchronized algorithm. But we limit the estimation order:

$$k_{max}=0$$

For curl estimation, choose *M=1*, the inverse matrix is simply $\frac{1}{\Delta s}$. The derivative estimator is $\frac{v(s+\Delta s)-v(s)}{\Delta s}$. The curl estimations become

$$\overline{\nabla}^{\{1\}}\times E=\begin{bmatrix}\mathfrak{D}_y(E_z)-\mathfrak{D}_z(E_y)\\ \mathfrak{D}_z(E_x)-\mathfrak{D}_x(E_z)\\ \mathfrak{D}_x(E_y)-\mathfrak{D}_y(E_x)\end{bmatrix}=\frac{1}{\Delta s}\begin{bmatrix}E_z(x,y+\Delta s,z)-E_z(x,y,z)-E_y(x,y,z+\Delta s)+E_y(x,y,z)\\ E_x(x,y,z+\Delta s)-E_x(x,y,z)-E_z(x+\Delta s,y,z)+E_z(x,y,z)\\ E_y(x+\Delta s,y,z)-E_y(x,y,z)-E_x(x,y+\Delta s,z)+E_x(x,y,z)\end{bmatrix}$$

$$\overline{\nabla}^{\{1\}}\times H=\begin{bmatrix}\mathfrak{D}_y(H_z)-\mathfrak{D}_z(H_y)\\ \mathfrak{D}_z(H_x)-\mathfrak{D}_x(H_z)\\ \mathfrak{D}_x(H_y)-\mathfrak{D}_y(H_x)\end{bmatrix}=\frac{1}{\Delta s}\begin{bmatrix}H_z(x,y,z)-H_z(x,y-\Delta s,z)-H_y(x,y,z)+H_y(x,y,z-\Delta s)\\ H_x(x,y,z)-H_x(x,y,z-\Delta s)-H_z(x,y,z)+H_z(x-\Delta s,y,z)\\ H_y(x,y,z)-H_y(x-\Delta s,y,z)-H_x(x,y,z)+H_x(x,y-\Delta s,z)\end{bmatrix}$$

Before substituting the above estimations into the estimation formula, we notice that to get a derivative at $v(s)$, $\frac{v\left(s+\frac{1}{2}\Delta s\right)-v\left(s-\frac{1}{2}\Delta s\right)}{\Delta s}$ is one order more accurate than $\frac{v(s+\Delta s)-v(s)}{\Delta s}$. Yee gave a clever way to do it by defining

$$E \rightarrow \begin{bmatrix} E_x(x+\frac{1}{2}\Delta s, y, z) \\ E_y(x, y+\frac{1}{2}\Delta s, z) \\ E_z(x, y, z+\frac{1}{2}\Delta s) \end{bmatrix}, H \rightarrow \begin{bmatrix} H_x(x, y+\frac{1}{2}\Delta s, z+\frac{1}{2}\Delta s) \\ H_y(x+\frac{1}{2}\Delta s, y, z+\frac{1}{2}\Delta s) \\ H_z(x+\frac{1}{2}\Delta s, y+\frac{1}{2}\Delta s, z) \end{bmatrix}$$

The curl estimations become

$$\overline{\nabla}^{\{1\}} \times E = \frac{1}{\Delta s}\begin{bmatrix} E_z\left(x, y+\Delta s, z+\frac{1}{2}\Delta s\right) - E_z\left(x, y, z+\frac{1}{2}\Delta s\right) - E_y\left(x, y+\frac{1}{2}\Delta s, z+\Delta s\right) + E_y(x, y+\frac{1}{2}\Delta s, z) \\ E_x\left(x+\frac{1}{2}\Delta s, y, z+\Delta s\right) - E_x\left(x+\frac{1}{2}\Delta s, y, z\right) - E_z\left(x+\Delta s, y, z+\frac{1}{2}\Delta s\right) + E_z(x, y, z+\frac{1}{2}\Delta s) \\ E_y\left(x+\Delta s, y+\frac{1}{2}\Delta s, z\right) - E_y\left(x, y+\frac{1}{2}\Delta s, z\right) - E_x\left(x+\frac{1}{2}\Delta s, y+\Delta s, z\right) + E_x(x+\frac{1}{2}\Delta s, y, z) \end{bmatrix}$$

$$\overline{\nabla}^{\{1\}} \times H = \frac{1}{\Delta s}\begin{bmatrix} H_z\left(x+\frac{1}{2}\Delta s, y+\frac{1}{2}\Delta s, z\right) - H_z\left(x+\frac{1}{2}\Delta s, y-\frac{1}{2}\Delta s, z\right) - H_y\left(x+\frac{1}{2}\Delta s, y, z+\frac{1}{2}\Delta s\right) + H_y(x+\frac{1}{2}\Delta s, y, z-\frac{1}{2}\Delta s) \\ H_x\left(x, y+\frac{1}{2}\Delta s, z+\frac{1}{2}\Delta s\right) - H_x\left(x, y+\frac{1}{2}\Delta s, z-\frac{1}{2}\Delta s\right) - H_z\left(x+\frac{1}{2}\Delta s, y+\frac{1}{2}\Delta s, z\right) + H_z(x-\frac{1}{2}\Delta s, y+\frac{1}{2}\Delta s, z) \\ H_y\left(x+\frac{1}{2}\Delta s, y, z+\frac{1}{2}\Delta s\right) - H_y\left(x-\frac{1}{2}\Delta s, y, z+\frac{1}{2}\Delta s\right) - H_x\left(x, y+\frac{1}{2}\Delta s, z+\frac{1}{2}\Delta s\right) + H_x(x, y-\frac{1}{2}\Delta s, z+\frac{1}{2}\Delta s) \end{bmatrix}$$

We can see that

$$center\ of\ \overline{\nabla}^{\{1\}} \times E\ is\ at \begin{bmatrix} E_x(x+\frac{1}{2}\Delta s, y, z) \\ E_y(x, y+\frac{1}{2}\Delta s, z) \\ E_z(x, y+\frac{1}{2}\Delta s, z) \end{bmatrix}$$

$$center\ of\ \overline{\nabla}^{\{1\}} \times H\ is\ at \begin{bmatrix} H_x(x, y+\frac{1}{2}\Delta s, z+\frac{1}{2}\Delta s) \\ H_y(x+\frac{1}{2}\Delta s, y, z+\frac{1}{2}\Delta s) \\ H_z(x+\frac{1}{2}\Delta s, y+\frac{1}{2}\Delta s, z) \end{bmatrix}$$

Now we get the standard Yee algorithm.

By deriving the Yee algorithm from the generic forms, we can clearly see the advantages and limitations of it.

Advantages:

- Use calculation amount of a first order estimation to get a second order precision

- Higher order curl estimations can still be used to increase accuracy of $\overline{\overline{\nabla}}^{\{1\}}\times$, but not $\overline{\overline{\nabla}}^{\{\mathrm{h}\}}\times, h > 1$

Limitations

- Because not more than one curl can be used, that is, $\overline{\overline{\nabla}}^{\{\mathrm{h}\}}\times, h > 1$, cannot be used, the time advancement order cannot be higher than 2. For the Time-Leap-Frog-Space-Synchronized algorithm (TLFSS), this limitation is only for the lossy medium. For Yee algorithm, this limitation applies also to lossless medium because the Yee algorithm is no longer space synchronized. This is the biggest limitation of the Yee algorithm and there is no way to get around it.
- Cannot calculate Poynting vector
- Cannot calculate divergences

Conclusion

For estimations purely based on Taylor's series, this paper shows that there is a generic form of FDTD algorithm for the Maxwell equations. By "generic" I mean that all other forms of the algorithms can be derived from this generic form. But it does not cover those algorithms involving techniques other than the Taylor series. However, almost all estimation algorithms involve the use of Taylor series in one way or the other. Therefore, this generic form may be used as a base for developing new algorithms, or to improve the existing algorithms.

Reference

[1] A Generic FDTD Form for Maxwell Equations, https://www.researchgate.net/publication/344868091_A_Generic_FDTD_Form_for_Maxwell_Equations

Appendix

Proof of the Time Space Lemma

Note that (16) – (30) are constants. We do not need to prove them. We just use them. What we need to prove are (8) - (15).

By (16) and (17), (6) and (7) become

$\frac{\partial E}{\partial t} = p_0^{\{1\}}\nabla\times H + p_1^{\{1\}}E + e_0^{\{0\}}J_e$	(42)
$\frac{\partial H}{\partial t} = q_0^{\{1\}}\nabla\times E + q_1^{\{1\}}H + m_0^{\{0\}}J_m$	(43)

We need to prove (8) – (15) based on (42) and (43). Due to complete symmetry of E and H shown in (8)-(15), (42) and (43), we only need to prove (8), (10), (12) and (14).

For $k = 1$, (8), (10), (12) and (14) become

$\frac{\partial^2 E}{\partial t^2} = p_2^{\{2\}}E + p_0^{\{2\}}\nabla^{\{2\}}\times E + p_1^{\{2\}}\nabla^{\{1\}}\times H + S_e(2)$	(44)
$S_e(2) = \nabla^{\{0\}}\times\left(e_0^{\{0\}}\frac{d^1J_e}{dt^1} + e_1^{\{0\}}\frac{d^0J_e}{dt^0}\right) + \nabla^{\{1\}}\times m_0^{\{1\}}\frac{d^0J_m}{dt^0}$	(45)

$\frac{\partial^3 E}{\partial t^3} = p_0^{\{3\}}\nabla^{\{3\}} \times H + p_1^{\{3\}}\nabla^{\{2\}} \times E + p_2^{\{3\}}\nabla^{\{1\}} \times H + p_3^{\{3\}}\nabla^{\{0\}} \times E + S_e(3)$	(46)
$S_e(3) = \nabla^{\{2\}} \times e_0^{\{2\}}\frac{d^0 J_e}{dt^0} + \nabla^{\{0\}\}} \times (e_0^{\{0\}}\frac{d^2 J_e}{dt^2} + e_1^{\{0\}}\frac{d^1 J_e}{dt^1} + e_2^{\{0\}}\frac{d^0 J_e}{dt^0}) + \nabla^{\{1\}} \times (m_0^{\{1\}}\frac{d^1 J_m}{dt^1} + m_1^{\{1\}}\frac{d^0 J_m}{dt^0})$	(47)

Taking temporal derivative on (42) and substitute (42) and (43) into it, we get

$$\frac{\partial^2 E}{\partial t^2} = q_0^{\{1\}}p_0^{\{1\}}\nabla^{\{2\}} \times E + (q_1^{\{1\}}p_0^{\{1\}} + p_1^{\{1\}}p_0^{\{1\}})\nabla \times H + p_1^{\{1\}}p_1^{\{1\}}E + \nabla \times p_0^{\{1\}}m_0^{\{0\}}J_m + p_1^{\{1\}}e_0^{\{0\}}J_e + e_0^{\{0\}}\frac{dJ_e}{dt}$$

By (28)

$$q_0^{\{1\}}p_0^{\{1\}} = p_0^{\{2\}}$$

$$p_2^{\{2\}} = p_1^{\{1\}}p_1^{\{1\}}$$

By (29)

$$p_1^{\{2\}} = q_1^{\{1\}}p_0^{\{1\}} + p_0^{\{1\}}p_1^{\{1\}}$$

By (26)

$$e_1^{\{0\}} = p_1^{\{1\}}e_0^{\{0\}}$$

By (27)

$$m_0^{\{1\}} = p_0^{\{1\}}m_0^{\{0\}}$$

Combine the above, we have

$$\frac{\partial^2 E}{\partial t^2} = p_0^{\{2\}}\nabla^{\{2\}} \times E + p_1^{\{2\}}\nabla \times H + p_2^{\{2\}}E + \nabla \times m_0^{\{1\}}J_m + e_1^{\{0\}}J_e + e_0^{\{0\}}\frac{dJ_e}{dt}$$

The above is the same as (44) and (45). Thus (8) and (12) hold for $k = 1$.

Taking temporal derivative on (44) and (45), and substitute (42) and (43) into it, we get

$$\frac{\partial^3 E}{\partial t^3} = p_0^{\{1\}}p_0^{\{2\}}\nabla^{\{3\}} \times H + (p_2^{\{2\}}p_0^{\{1\}} + q_1^{\{1\}}p_1^{\{2\}})\nabla^{\{1\}} \times H + (p_1^{\{1\}}p_0^{\{2\}} + q_0^{\{1\}}p_1^{\{2\}})\nabla^{\{2\}} \times E + p_2^{\{2\}}p_1^{\{1\}}E$$

$$+\nabla^{\{2\}} \times p_0^{\{2\}}e_0^{\{0\}}J_e + \nabla^{\{1\}} \times (p_1^{\{2\}}m_0^{\{0\}}J_m + m_0^{\{1\}}\frac{d^1 J_m}{dt^1}) + \nabla^{\{0\}} \times (e_0^{\{0\}}\frac{d^2 J_e}{dt^2} + e_1^{\{0\}}\frac{d^1 J_e}{dt^1} + p_2^{\{2\}}e_0^{\{0\}}J_e)$$

By (20)

$$p_0^{\{3\}} = p_0^{\{1\}}p_0^{\{2\}}$$

By (23)

$$p_2^{\{3\}} = q_1^{\{1\}}p_1^{\{2\}} + p_0^{\{1\}}p_2^{\{2\}}$$

By (22)

$$p_1^{\{3\}} = p_1^{\{1\}}p_0^{\{2\}} + q_0^{\{1\}}p_1^{\{2\}}$$

By (20)

$$p_3^{\{3\}} = p_2^{\{2\}}p_1^{\{1\}}$$

By (18)

$$e_0^{\{2\}} = p_0^{\{2\}}e_0^{\{0\}}$$

$$e_2^{\{0\}} = p_2^{\{2\}}e_0^{\{0\}}$$

By (19)

$$m_1^{\{1\}} = p_1^{\{2\}} m_0^{\{0\}}$$

Combine the above, we have

$$\frac{\partial^3 E}{\partial t^3} = p_0^{\{3\}} \nabla^{\{3\}} \times H + p_2^{\{3\}} \nabla^{\{1\}} \times H + p_1^{\{3\}} \nabla^{\{2\}} \times E + p_3^{\{3\}} E$$

$$+\nabla^{\{2\}} \times e_0^{\{2\}} J_e + \nabla^{\{1\}} \times (m_1^{\{1\}} J_m + m_0^{\{1\}} \frac{d^1 J_m}{dt^1}) + \nabla^{\{0\}} \times (e_0^{\{0\}} \frac{d^2 J_e}{dt^2} + e_1^{\{0\}} \frac{d^1 J_e}{dt^1} + e_2^{\{0\}} J_e)$$

It is the same as (46) and (47). Thus (10) and (14) hold for $k = 1$.

Thus, (8), (10), (12) and (14) hold for $k = 1$.

Suppose for an integer $k \geq 1$, (8), (10), (12) and (14) hold.

Consider a case of $k + 1$.

For $k + 1$ (8), (10), (12) and (14) become

$\frac{\partial^{2(k+1)} E}{\partial t^{2(k+1)}} = p_{2(k+1)}^{\{2(k+1)\}} E + \sum_{i=1}^{k+1} \left(p_{2(k+1-i)}^{\{2(k+1)\}} \nabla^{\{2i\}} \times E + p_{2(k+1-i)+1}^{\{2(k+1)\}} \nabla^{\{2i-1\}} \times H \right) + S_e(2(k+1))$	(48)
$\frac{\partial^{2(k+1)+1} E}{\partial t^{2(k+1)+1}} = \sum_{i=0}^{k+1} (p_{2i}^{\{2(k+1)+1\}} \nabla^{\{2(k+1-i)+1\}} \times H + p_{2i+1}^{\{2(k+1)+1\}} \nabla^{\{2(k+1-i)\}} \times E) + S_e(2(k+1)+1)$	(49)

$S_e(2(k+1)) = \sum_{h=0}^{k} \left(\nabla^{\{2(k-h)\}} \times \sum_{i=1}^{2(h+1)} e_{i-1}^{\{2(k-h)\}} \frac{d^{2(h+1)-i} J_e}{dt^{2(h+1)-i}} + \nabla^{\{2(k-h)+1\}} \times \sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i} J_m}{dt^{2h-i}} \right)$	(50)
$S_e(2(k+1)+1) = \sum_{h=0}^{k+1} \nabla^{\{2(k+1-h)\}} \times \sum_{i=0}^{2h} e_i^{\{2(k+1-h)\}} \frac{d^{2h-i} J_e}{dt^{2h-i}} + \sum_{h=0}^{k} \nabla^{\{2(k+1-h)-1\}} \times \sum_{i=0}^{2h+1} m_i^{\{2(k+1-h)-1\}} \frac{d^{2h-i+1} J_m}{dt^{2h-i+1}}$	(51)

Taking temporal derivatives on (10) and (14), and substitute (42) and (43) into them, we have

$$\frac{\partial^{2k+2} E}{\partial t^{2k+2}} = \sum_{i=0}^{k} (q_0^{\{1\}} p_{2i}^{\{2k+1\}} \nabla^{\{2(k-i)+2\}} \times E + \nabla^{\{2(k-i)\}} \times E + (q_1^{\{1\}} p_{2i}^{\{2k+1\}} + p_1^{\{1\}} p_{2i+1}^{\{2k+1\}}) \nabla^{\{2(k-i)+1\}} \times H + \nabla^{\{2(k-i)+1\}} \times p_{2i}^{\{2k+1\}} m_0^{\{0\}} J_m + \nabla^{\{2(k-i)\}}$$

$$\times p_{2i+1}^{\{2k+1\}} e_0^{\{0\}} J_e) + \frac{dS_e(2k+1)}{dt}$$

$$= \sum_{i=0}^{k} (q_0^{\{1\}} p_{2i}^{\{2k+1\}} \nabla^{\{2(k-i)+2\}} \times E + p_1^{\{1\}} p_{2i+1}^{\{2k+1\}} \nabla^{\{2(k-i)\}} \times E) +$$

$$\sum_{i=0}^{k} ((q_1^{\{1\}} p_{2i}^{\{2k+1\}} + p_0^{\{1\}} p_{2i+1}^{\{2k+1\}}) \nabla^{\{2(k-i)+1\}} \times H) +$$

$$\sum_{i=0}^{k} (\nabla^{\{2(k-i)+1\}} \times p_{2i}^{\{2k+1\}} m_0^{\{0\}} J_m + \nabla^{\{2(k-i)\}} \times p_{2i+1}^{\{2k+1\}} e_0^{\{0\}} J_e) + \frac{dS_e(2k+1)}{dt}$$

$$\frac{dS_e(2k+1)}{dt} = \sum_{h=0}^{k} \nabla^{\{2(k-h)\}} \times \sum_{i=0}^{2h} e_i^{\{2(k-h)\}} \frac{d^{2h-i+} J_e}{dt^{2h-i+1}} + \sum_{h=0}^{k-1} \nabla^{\{2(k-h)-1\}} \times \sum_{i=0}^{2h+1} m_i^{\{2(k-h)-1\}} \frac{d^{2h-i+2} J_m}{dt^{2h-i+}}$$

By (30),

$$q_1^{\{1\}} p_{2i}^{\{2k+1\}} + p_0^{\{1\}} p_{2i+1}^{\{2k+1\}} = p_{2i+1}^{\{2(k+1)\}}$$

We have

$$\sum_{i=0}^{k}((q_1^{\{1\}}p_{2i}^{\{2k+1\}}+p_0^{\{1\}}p_{2i+1}^{\{2k+1\}})\nabla^{\{2(\mathrm{k-i})+1\}}\times H)=\sum_{i=0}^{k}(p_{2i+1}^{\{2(k+1)\}}\nabla^{\{2(\mathrm{k-i})+1\}}\times H)=\sum_{i=1}^{k+1}(p_{2i-1}^{\{2(k+1)\}}\nabla^{\{2(\mathrm{k+1-i})+1\}}\times H)$$
$$=\sum_{i=1}^{k+1}(p_{2(\mathrm{k+1-i})+1}^{\{2(k+1)\}}\nabla^{\{2i-1\}}\times H)$$

Thus,

$$\frac{\partial^{2k+2}E}{\partial t^{2k+2}}=\sum_{i=0}^{k}(q_0^{\{1\}}p_{2i}^{\{2k+1\}}\nabla^{\{2(\mathrm{k-i})+2\}}\times E+p_1^{\{1\}}p_{2i+1}^{\{2k+1\}}\nabla^{\{2(\mathrm{k-i})\}}\times E)+$$
$$\sum_{i=1}^{k+1}(p_{2(\mathrm{k+1-i})+1}^{\{2(k+1)\}}\nabla^{\{2i-1\}}\times H)+$$
$$\sum_{i=0}^{k}(\nabla^{\{2(\mathrm{k-i})+1\}}\times p_{2i}^{\{2k+1\}}m_0^{\{0\}}J_m+\nabla^{\{2(\mathrm{k-i})\}}\times p_{2i+1}^{\{2k+1\}}e_0^{\{0\}}J_e)+\frac{dS_e(2k+1)}{dt}$$

By (28)

$$p_{2k}^{\{2k\}}=p_{2k-1}^{\{2k-1\}}p_1^{\{1\}}\rightarrow p_1^{\{1\}}p_{2k+1}^{\{2k+1\}}=p_{2(k+1)}^{\{2(k+1)\}}$$
$$p_0^{\{2k\}}=q_0^{\{1\}}p_0^{\{2k-1\}}\rightarrow p_0^{\{2(k+1)\}}=q_0^{\{1\}}p_0^{\{2k+1\}}$$

By (30)

$$p_{2i+2}^{\{2k\}}=p_1^{\{1\}}p_{2i+1}^{\{2k-1\}}+q_0^{\{1\}}p_{2i+2}^{\{2k-1\}}\rightarrow p_{2i}^{\{2(k+1)\}}=p_1^{\{1\}}p_{2i-1}^{\{2k+1\}}+q_0^{\{1\}}p_{2i}^{\{2k+1\}}$$

Thus, we have

$$\sum_{i=0}^{k}(q_0^{\{1\}}p_{2i}^{\{2k+1\}}\nabla^{\{2(\mathrm{k-i})+2\}}\times E+p_1^{\{1\}}p_{2i+1}^{\{2k+1\}}\nabla^{\{2(\mathrm{k-i})\}}\times E)$$
$$=q_0^{\{1\}}p_0^{\{2k+1\}}\nabla^{\{2(\mathrm{k+1})\}}\times E+\sum_{i=1}^{k}(q_0^{\{1\}}p_{2i}^{\{2k+1\}}\nabla^{\{2(\mathrm{k-i})+2\}}\times E)+\sum_{i=0}^{k}(p_1^{\{1\}}p_{2i+1}^{\{2k+1\}}\nabla^{\{2(\mathrm{k-i})\}}\times E)$$
$$=q_0^{\{1\}}p_0^{\{2k+1\}}\nabla^{\{2(\mathrm{k+1})\}}\times E+\sum_{i=0}^{k-1}(q_0^{\{1\}}p_{2i+2}^{\{2k+1\}}\nabla^{\{2(\mathrm{k-i})\}}\times E)+\sum_{i=0}^{k}(p_1^{\{1\}}p_{2i+1}^{\{2k+1\}}\nabla^{\{2(\mathrm{k-i})\}}\times E)$$
$$=q_0^{\{1\}}p_0^{\{2k+1\}}\nabla^{\{2(\mathrm{k+1})\}}\times E+\sum_{i=0}^{k-1}((p_1^{\{1\}}p_{2i+1}^{\{2k+1\}}+q_0^{\{1\}}p_{2i+2}^{\{2k+1\}})\nabla^{\{2(\mathrm{k-i})\}}\times E)+p_1^{\{1\}}p_{2k+1}^{\{2k+1\}}\nabla^{\{0\}}\times E$$
$$=q_0^{\{1\}}p_0^{\{2k+1\}}\nabla^{\{2(\mathrm{k+1})\}}\times E+\sum_{i=1}^{k}((p_1^{\{1\}}p_{2i-1}^{\{2k+1\}}+q_0^{\{1\}}p_{2i}^{\{2k+1\}})\nabla^{\{2(\mathrm{k+1-i})\}}\times E)+p_1^{\{1\}}p_{2k+1}^{\{2k+1\}}\nabla^{\{0\}}\times E$$
$$=p_0^{\{2(k+1)\}}\nabla^{\{2(\mathrm{k+1})\}}\times E+\sum_{i=1}^{k}(p_{2i}^{\{2(k+1)\}}\nabla^{\{2(\mathrm{k+1-i})\}}\times E)+p_{2(k+1)}^{\{2(k+1)\}}E$$
$$=\sum_{i=0}^{k}(p_{2i}^{\{2(k+1)\}}\nabla^{\{2(\mathrm{k+1-i})\}}\times E)+p_{2(k+1)}^{\{2(k+1)\}}E$$
$$=\sum_{i=0}^{k}(p_{2(k-i)}^{\{2(k+1)\}}\nabla^{\{2(\mathrm{i+1})\}}\times E)+p_{2(k+1)}^{\{2(k+1)\}}E$$
$$=\sum_{i=1}^{k+1}(p_{2(k-i+1)}^{\{2(k+1)\}}\nabla^{\{2(\mathrm{i})\}}\times E)+p_{2(k+1)}^{\{2(k+1)\}}E$$

Thus,

$$\frac{\partial^{2k+2}E}{\partial t^{2k+2}} = p_{2(k+1)}^{\{2(k+1)\}}E + \sum_{i=1}^{k+1}(p_{2(k-i+1)}^{\{2(k+1)\}}\nabla^{\{2i\}} \times E + p_{2(\mathrm{k}+1-\mathrm{i})+1}^{\{2(k+1)\}}\nabla^{\{2i-1\}} \times H) +$$

$$\sum_{i=0}^{k}(\nabla^{\{2(\mathrm{k-i})+1\}} \times p_{2i}^{\{2k+1\}}m_0^{\{0\}}J_m + \nabla^{\{2(\mathrm{k-i})\}} \times p_{2i+1}^{\{2k+1\}}e_0^{\{0\}}J_e) + \frac{dS_e(2k+1)}{dt}$$

By (14),

$$\sum_{i=0}^{k}(\nabla^{\{2(\mathrm{k-i})+1\}} \times p_{2i}^{\{2k+1\}}m_0^{\{0\}}J_m + \nabla^{\{2(\mathrm{k-i})\}} \times p_{2i+1}^{\{2k+1\}}e_0^{\{0\}}J_e) + \frac{dS_e(2k+1)}{dt}$$

$$= \sum_{i=0}^{k}(\nabla^{\{2(\mathrm{k-i})+1\}} \times p_{2i}^{\{2k+1\}}m_0^{\{0\}}J_m + \nabla^{\{2(\mathrm{k-i})\}} \times p_{2i+1}^{\{2k+1\}}e_0^{\{0\}}J_e) + \sum_{h=0}^{k}\nabla^{\{2(\mathrm{k-h})\}} \times \sum_{i=0}^{2h} e_i^{\{2(k-h)\}}\frac{d^{2h-i+1}J_e}{dt^{2h-i+}}$$
$$+ \sum_{h=0}^{k-1}\nabla^{\{2(\mathrm{k-h})-1\}} \times \sum_{i=0}^{2h+1} m_i^{\{2(k-h)-1\}}\frac{d^{2h-i+2}J_m}{dt^{2h-i+2}}$$

$$= \sum_{h=0}^{k}\nabla^{\{2(\mathrm{k-h})\}} \times (p_{2h+1}^{\{2k+1\}}e_0^{\{0\}}J_e + \sum_{i=0}^{2h} e_i^{\{2(k-h)\}}\frac{d^{2h-i+1}J_e}{dt^{2h-i+1}}) + \sum_{i=0}^{k}(\nabla^{\{2(\mathrm{k-i})+1\}} \times p_{2i}^{\{2k+1\}}m_0^{\{0\}}J_m) + \sum_{h=0}^{k-1}\nabla^{\{2(\mathrm{k-h})-1\}} \times \sum_{i=0}^{2h+1} m_i^{\{2(k-h)-1\}}\frac{d^{2h-i+2}J_m}{dt^{2h-i+2}}$$

By (18),

$$e_{2(k-i)-1}^{\{2i\}} = p_{2(k-i)-1}^{\{2k-1\}}e_0^{\{0\}} \to e_{2(k-i)+1}^{\{2i\}} = p_{2(k-i)+1}^{\{2k+1\}}e_0^{\{0\}} \to e_{2h+1}^{\{2(k-h)\}} = p_{2h+1}^{\{2k+1\}}e_0^{\{0\}}$$

We have

$$\sum_{h=0}^{k}\nabla^{\{2(\mathrm{k-h})\}} \times (p_{2h+1}^{\{2k+1\}}e_0^{\{0\}}J_e + \sum_{i=0}^{2h} e_i^{\{2(k-h)\}}\frac{d^{2h-i+1}J_e}{dt^{2h-i+}}) = \sum_{h=0}^{k}\nabla^{\{2(\mathrm{k-h})\}} \times (p_{2h+1}^{\{2k+1\}}e_0^{\{0\}}J_e + \sum_{i=1}^{2h+1} e_{i-1}^{\{2(k-h)\}}\frac{d^{2h-i+}\ J_e}{dt^{2h-i+}})$$

$$= \sum_{h=0}^{k}\nabla^{\{2(\mathrm{k-h})\}} \times (\sum_{i=1}^{2h+2} e_{i-1}^{\{2(k-h)\}}\frac{d^{2h-i+}\ J_e}{dt^{2h-i+2}})$$

We have

$$\frac{\partial^{2k+2}E}{\partial t^{2k+2}} = p_{2(k+1)}^{\{2(k+1)\}}E + \sum_{i=1}^{k+1}(p_{2(k-i+1)}^{\{2(k+1)\}}\nabla^{\{2i\}} \times E + p_{2(\mathrm{k}+1-\mathrm{i})+1}^{\{2(k+1)\}}\nabla^{\{2i-1\}} \times H)$$

$$+ \sum_{h=0}^{k}\nabla^{\{2(\mathrm{k-h})\}} \times \left(\sum_{i=1}^{2h+2} e_{i-1}^{\{2(k-h)\}}\frac{d^{2h-i+}\ J_e}{dt^{2h-i+2}}\right)$$

$$+ \sum_{i=0}^{k}(\nabla^{\{2(\mathrm{k-i})+1\}} \times p_{2i}^{\{2k+1\}}m_0^{\{0\}}J_m) + \sum_{h=0}^{k-1}\nabla^{\{2(\mathrm{k-h})-1\}} \times \sum_{i=0}^{2h+1} m_i^{\{2(k-h)-1\}}\frac{d^{2h-i+}\ J_m}{dt^{2h-i+2}}$$

By (27)

$$m_{2(k-i-1)}^{\{2i+1\}} = p_{2(k-i-1)}^{\{2k-1\}}m_0^{\{0\}} \to m_{2(k-i)}^{\{2i+1\}} = p_{2(k-i)}^{\{2k+1\}}m_0^{\{0\}} \to m_{2h}^{\{2(k-h)+1\}} = p_{2h}^{\{2k+1\}}m_0^{\{0\}} \to m_0^{\{2k+1\}} = p_0^{\{2k+1\}}m_0^{\{0\}}$$

We have

$$\sum_{i=0}^{k}(\nabla^{\{2(\mathrm{k-i})+1\}} \times p_{2i}^{\{2k+1\}}m_0^{\{0\}}J_m) + \sum_{h=0}^{k-1}\nabla^{\{2(\mathrm{k-h})-1\}} \times \sum_{i=0}^{2h+1} m_i^{\{2(k-h)-1\}}\frac{d^{2h-i+2}J_m}{dt^{2h-i+}} =$$

$$= \sum_{i=0}^{k}(\nabla^{\{2(\mathrm{k-i})+1\}} \times p_{2i}^{\{2k+1\}}m_0^{\{0\}}J_m) + \sum_{h=1}^{k}\nabla^{\{2(\mathrm{k-h}+1)-1\}} \times \sum_{i=0}^{2h-1} m_i^{\{2(k-h)+1\}}\frac{d^{2h-i}J_m}{dt^{2h-i}}$$

$$= \sum_{h=0}^{k} (\nabla^{\{2(\mathrm{k-h})+1\}} \times p_{2h}^{\{2k+1\}} m_0^{\{0\}} J_m) + \sum_{h=1}^{k} \nabla^{\{2(\mathrm{k-h})+1\}} \times \sum_{i=0}^{2h-1} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i} J_m}{dt^{2h-i}}$$

$$= \nabla^{\{2\mathrm{k}+1\}} \times p_0^{\{2k+1\}} m_0^{\{0\}} J_m + \sum_{h=1}^{k} (\nabla^{\{2(\mathrm{k-h})+1\}} \times p_{2h}^{\{2k+1\}} m_0^{\{0\}} J_m) + \sum_{h=1}^{k} \nabla^{\{2(\mathrm{k-h})+1\}} \times \sum_{i=0}^{2h-1} m_i^{\{2(k-h)-1\}} \frac{d^{2h-i} J_m}{dt^{2h-i}}$$

$$= \nabla^{\{2\mathrm{k}+1\}} \times p_0^{\{2k+1\}} m_0^{\{0\}} J_m + \sum_{h=1}^{k} \nabla^{\{2(\mathrm{k-h})+1\}} \times (p_{2h}^{\{2k+1\}} m_0^{\{0\}} J_m + \sum_{i=0}^{2h-1} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i} J_m}{dt^{2h-i}})$$

$$= \nabla^{\{2\mathrm{k}+1\}} \times p_0^{\{2k+1\}} m_0^{\{0\}} J_m + \sum_{h=1}^{k} \nabla^{\{2(\mathrm{k-h})+1\}} \times (p_{2h}^{\{2k+1\}} m_0^{\{0\}} J_m + \sum_{i=0}^{2h-1} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i} J_m}{dt^{2h-i}})$$

$$= \nabla^{\{2\mathrm{k}+1\}} \times p_0^{\{2k+1\}} m_0^{\{0\}} J_m + \sum_{h=1}^{k} \nabla^{\{2(\mathrm{k-h})+1\}} \times (\sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i} J_m}{dt^{2h-i}})$$

$$= \sum_{h=0}^{k} \nabla^{\{2(\mathrm{k-h})+1\}} \times (\sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i} J_m}{dt^{2h-i}})$$

$$\frac{\partial^{2k+2} E}{\partial t^{2k+2}} = p_{2(k+1)}^{\{2(k+1)\}} E + \sum_{i=1}^{k+1} (p_{2(k-i+1)}^{\{2(k+1)\}} \nabla^{\{2\mathrm{i}\}} \times E + p_{2(\mathrm{k}+1-\mathrm{i})+1}^{\{2(k+1)\}} \nabla^{\{2i-1\}} \times H)$$

$$+ \sum_{h=0}^{k} \nabla^{\{2(\mathrm{k-h})\}} \times \left(\sum_{i=1}^{2h+2} e_{i-1}^{\{2(k-h)\}} \frac{d^{2h-i+} J_e}{dt^{2h-i+}} \right) + \sum_{h=0}^{k} \nabla^{\{2(\mathrm{k-h})+1\}} \times (\sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i} J_m}{dt^{2h-i}})$$

The above is the same as (48) and (50). Thus, (8) and (12) hold for $k+1$.

Taking temporal derivatives on (48) and (50), and substitute (42) and (43) into them, we have

$$\frac{\partial^{2(k+1)+1} E}{\partial t^{2(k+1)+1}} = p_{2(k+1)}^{\{2(k+1)\}} p_0^{\{1\}} \nabla \times H + \sum_{i=1}^{k+1} \left(p_0^{\{1\}} p_{2(k+1-i)}^{\{2(k+1)\}} \nabla^{\{2i+1\}} \times H + q_1^{\{1\}} p_{2(k+1-i)+1}^{\{2(k+1)\}} \nabla^{\{2i-1\}} \times H \right)$$

$$+ p_{2(k+1)}^{\{2(k+1)\}} p_1^{\{1\}} E + \sum_{i=1}^{k+1} \left((p_1^{\{1\}} p_{2(k+1-i)}^{\{2(k+1)\}} + q_0^{\{1\}} p_{2(k+1-i)+1}^{\{2(k+1)\}}) \nabla^{\{2i\}} \times E \right)$$

$$+ p_{2(k+1)}^{\{2(k+1)\}} e_0^{\{0\}} J_e + \sum_{i=1}^{k+1} (\nabla^{\{2i\}} \times p_{2(k+1-i)}^{\{2(k+1)\}} e_0^{\{0\}} J_e + \nabla^{\{2i-1\}} \times p_{2(k+1-i)+1}^{\{2(k+1)\}} m_0^{\{0\}} J_m) + \frac{dS_e\big(2(k+1)\big)}{dt}$$

$$\frac{dS_e\big(2(k+1)\big)}{dt} = \sum_{h=0}^{k} \left(\nabla^{\{2(k-h)\}} \times \sum_{i=1}^{2(h+1)} e_{i-1}^{\{2(k-h)\}} \frac{d^{2(h+1)-i+1} J_e}{dt^{2(h+1)-i+1}} + \nabla^{\{2(k-h)+1\}} \times \sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i+1} J_m}{dt^{2h-i+1}} \right)$$

By (20)

$$p_{2k+1}^{\{2k+1\}} = p_{2k}^{\{2k\}} p_1^{\{1\}} \rightarrow p_{2(k+1)+1}^{\{2(k+1)+1\}} = p_{2(k+1)}^{\{2(k+1)\}} p_1^{\{1\}}$$

By (22)

$$p_{2(k-i)+1}^{\{2k+1\}} = p_1^{\{1\}} p_{2(k-i)}^{\{2k\}} + q_0^{\{1\}} p_{2(k-i)+1}^{\{2k\}} \rightarrow p_{2((k+1)-i)+1}^{\{2(k+1)+1\}} = p_1^{\{1\}} p_{2((k+1)-i)}^{\{2(k+1)\}} + q_0^{\{1\}} p_{2(k+1-i)+1}^{\{2(k+1)\}}$$

We have

$$p_{2(k+1)}^{\{2(k+1)\}} p_1^{\{1\}} E + \sum_{i=1}^{k+1} \left(\left(p_1^{\{1\}} p_{2(k+1-i)}^{\{2(k+1)\}} + q_0^{\{1\}} p_{2(k+1-i)+1}^{\{2(k+1)\}} \right) \nabla^{\{2i\}} \times E \right) = p_{2(k+1)+1}^{\{2(k+1)+1\}} E + \sum_{i=1}^{k+1} \left(p_{2(k+1-i)+1}^{\{2(k+1)+1\}} \nabla^{\{2i\}} \times E \right) = \sum_{i=0}^{k+1} \left(p_{2(k+1-i)+1}^{\{2(k+1)+1\}} \nabla^{\{2i\}} \times E \right)$$

$$= \sum_{i=0}^{k+1} \left(p_{2i+1}^{\{2(k+1)+1\}} \nabla^{\{2(k+1-i)\}} \times E \right)$$

By (20),

$$p_0^{\{2k+1\}} = p_0^{\{1\}} p_0^{\{2k\}} \rightarrow p_0^{\{2(k+1)+1\}} = p_0^{\{1\}} p_0^{\{2(k+1)\}}$$

By (23),

$$p_{2i}^{\{2k+1\}} = q_1^{\{1\}} p_{2i-1}^{\{2k\}} + p_0^{\{1\}} p_{2i}^{\{2k\}} \rightarrow p_{2i}^{\{2(k+1)+1\}} = q_1^{\{1\}} p_{2i-1}^{\{2(k+1)\}} + p_0^{\{1\}} p_{2i}^{\{2(k+1)\}}$$

We have

$$p_{2(k+1)}^{\{2(k+1)\}} p_0^{\{1\}} \nabla \times H + \sum_{i=1}^{k+1} \left(p_0^{\{1\}} p_{2(k+1-i)}^{\{2(k+1)\}} \nabla^{\{2i+1\}} \times H + q_1^{\{1\}} p_{2(k+1-i)+1}^{\{2(k+1)\}} \nabla^{\{2i-1\}} \times H \right)$$

$$= p_{2(k+1)}^{\{2(k+1)\}} p_0^{\{1\}} \nabla \times H + p_0^{\{1\}} p_0^{\{2(k+1)\}} \nabla^{\{2(k+1)+1\}} \times H + q_1^{\{1\}} p_{2k+1}^{\{2(k+1)\}} \nabla^{\{1\}} \times H + \sum_{i=1}^{k+1} \left(p_0^{\{1\}} p_{2(k+1-i)}^{\{2(k+1)\}} \nabla^{\{2i+1\}} \times H + q_1^{\{1\}} p_{2(k+1-i)-1}^{\{2(k+1)\}} \nabla^{\{2i+1\}} \times H \right)$$

$$= p_0^{\{2(k+1)+1\}} \nabla^{\{2(k+1)+1\}} \times H + \left(p_0^{\{1\}} p_{2(k+1)}^{\{2(k+1)\}} + q_1^{\{1\}} p_{2k+1}^{\{2(k+1)\}} \right) \nabla^{\{1\}} \times H + \sum_{i=1}^{k+1} \left(p_0^{\{1\}} p_{2(k+1-i)}^{\{2(k+1)\}} + q_1^{\{1\}} p_{2(k+1-i)-1}^{\{2(k+1)\}} \right) \nabla^{\{2i+1\}} \times H$$

$$= p_0^{\{2(k+1)+1\}} \nabla^{\{2(k+1)+1\}} \times H + p_{2(k+1)}^{\{2(k+1)+1\}} \nabla^{\{1\}} \times H + \sum_{i=1}^{k+1} p_{2(k+1-i)}^{\{2(k+1)+1\}} \nabla^{\{2i+1\}} \times H = \sum_{i=0}^{k+1} p_{2(k+1-i)}^{\{2(k+1)+1\}} \nabla^{\{2i+1\}} \times H$$

$$= \sum_{i=0}^{k+1} p_{2i}^{\{2(k+1)+1\}} \nabla^{\{2(\mathrm{k}+1-\mathrm{i})+1\}} \times H$$

Combine the above results, we have

$$\frac{\partial^{2(k+1)+1} E}{\partial t^{2(k+1)+1}} = \sum_{i=0}^{k+1} p_{2i}^{\{2(k+1)+1\}} \nabla^{\{2(\mathrm{k}+1-\mathrm{i})+1\}} \times H + \sum_{i=0}^{k+1} \left(p_{2i+1}^{\{2(k+1)+1\}} \nabla^{\{2(k+1-i)\}} \times E \right)$$

$$+ p_{2(k+1)}^{\{2(k+1)\}} e_0^{\{0\}} J_e + \sum_{i=1}^{k+1} \left(\nabla^{\{2i\}} \times p_{2(k+1-i)}^{\{2(k+1)\}} e_0^{\{0\}} J_e + \nabla^{\{2i-1\}} \times p_{2(k+1-i)+1}^{\{2(k+1)\}} m_0^{\{0\}} J_m \right)$$

$$+ \sum_{h=0}^{k} \left(\nabla^{\{2(k-h)\}} \times \sum_{i=1}^{2(h+1)} e_{i-1}^{\{2(k-h)\}} \frac{d^{2(h+1)-i+1} J_e}{dt^{2(h+1)-i+1}} + \nabla^{\{2(k-h)+1\}} \times \sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i+} J_m}{dt^{2h-i+}} \right)$$

For J_m terms, we have

$$\sum_{i=1}^{k+1} \left(\nabla^{\{2i-1\}} \times p_{2(k+1-i)+1}^{\{2(k+1)\}} m_0^{\{0\}} J_m \right) + \sum_{h=0}^{k} \left(\nabla^{\{2(k-h)+1\}} \times \sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i+} J_m}{dt^{2h-i+1}} \right)$$

$$= \sum_{i=0}^{k} \left(\nabla^{\{2i+1\}} \times p_{2(k-i)+1}^{\{2(k+1)\}} m_0^{\{0\}} J_m \right) + \sum_{h=0}^{k} \left(\nabla^{\{2(k-h)+1\}} \times \sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i+} J_m}{dt^{2h-i+1}} \right)$$

$$= \sum_{h=0}^{k} \left(\nabla^{\{2(k-h)+1\}} \times p_{2h+}^{\{2(k+1)\}} m_0^{\{0\}} J_m \right) + \sum_{h=0}^{k} \left(\nabla^{\{2(k-h)+1\}} \times \sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i+1} J_m}{dt^{2h-i+1}} \right)$$

$$= \sum_{h=0}^{k} \left(\left(\nabla^{\{2(k-h)+1\}} \times p_{2h+1}^{\{2(k+1)\}} m_0^{\{0\}} J_m \right) + \nabla^{\{2(k-h)+1\}} \times \sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i+} J_m}{dt^{2h-i+}} \right)$$

$$= \sum_{h=0}^{k} \left(\nabla^{\{2(k-h)+1\}} \times (p_{2h+1}^{\{2(k+1)\}} m_0^{\{0\}} J_m + \sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i+} J_m}{dt^{2h-i+1}}) \right)$$

By (19),

$$m_{2(k-i)+1}^{\{2i-1\}} = p_{2(k-i)+1}^{\{2k\}} m_0^{\{0\}} \rightarrow m_{2h+1}^{\{2(k-h)+1\}} = p_{2h+1}^{\{2(k+1)\}} m_0^{\{0\}}$$

We have

$$\sum_{h=0}^{k}\left(\nabla^{\{2(k-h)+1\}}\times(p_{2h+1}^{\{2(k+1)\}}m_0^{\{0\}}J_m+\sum_{i=0}^{2h}m_i^{\{2(k-h)+1\}}\frac{d^{2h-i+1}J_m}{dt^{2h-i+1}})\right)=\sum_{h=0}^{k}\left(\nabla^{\{2(k-h)+1\}}\times\sum_{i=0}^{2h+1}m_i^{\{2(k-h)+1\}}\frac{d^{2h-i+1}J_m}{dt^{2h-i+1}})\right)$$

The above is the J_m terms in $S_e(2(k+1)+1)$ as defined in (14).

For J_e terms, we have

$$p_{2(k+1)}^{\{2(k+1)\}}e_0^{\{0\}}J_e+\sum_{i=1}^{k+1}\left(\nabla^{\{2i\}}\times p_{2(k+1-i)}^{\{2(k+1)\}}e_0^{\{0\}}J_e\right)+\sum_{h=0}^{k}\left(\nabla^{\{2(k-h)\}}\times\sum_{i=1}^{2(h+1)}e_{i-1}^{\{2(k-h)\}}\frac{d^{2(h+1)-i+1}J_e}{dt^{2(h+1)-i+1}}\right)$$

$$=p_{2(k+1)}^{\{2(k+1)\}}e_0^{\{0\}}J_e+\sum_{i=0}^{k}\left(\nabla^{\{2i+2\}}\times p_{2(k-i)}^{\{2(k+1)\}}e_0^{\{0\}}J_e\right)+\sum_{h=0}^{k}\left(\nabla^{\{2(k-h)\}}\times\sum_{i=1}^{2(h+1)}e_{i-1}^{\{2(k-h)\}}\frac{d^{2(h+1)-i+1}J_e}{dt^{2(h+1)-i+1}}\right)$$

$$=p_{2(k+1)}^{\{2(k+1)\}}e_0^{\{0\}}J_e+\sum_{h=0}^{k}\left(\nabla^{\{2(k+1-h)\}}\times p_{2h}^{\{2(k+1)\}}e_0^{\{0\}}J_e\right)+\sum_{h=0}^{k}\left(\nabla^{\{2(k-h)\}}\times\sum_{i=1}^{2(h+1)}e_{i-1}^{\{2(k-h)\}}\frac{d^{2(h+1)-i+1}J_e}{dt^{2(h+1)-i+1}}\right)$$

$$=p_{2(k+1)}^{\{2(k+1)\}}e_0^{\{0\}}J_e+\sum_{h=0}^{k}\left(\nabla^{\{2(k+1-h)\}}\times p_{2h}^{\{2(k+1)\}}e_0^{\{0\}}J_e\right)+\sum_{h=0}^{k}\left(\nabla^{\{2(k-h)\}}\times\sum_{i=0}^{2h+1}e_{i}^{\{2(k-h)\}}\frac{d^{2(h+1)-i}J_e}{dt^{2(h+1)-i}}\right)$$

$$=p_{2(k+1)}^{\{2(k+1)\}}e_0^{\{0\}}J_e+\sum_{h=0}^{k}\left(\nabla^{\{2(k+1-h)\}}\times p_{2h}^{\{2(k+1)\}}e_0^{\{0\}}J_e\right)+\sum_{i=0}^{2k+1}e_i^{\{0\}}\frac{d^{2(k+1)-i}J_e}{dt^{2(k+1)-i}}+\sum_{h=0}^{k-1}\left(\nabla^{\{2(k-h)\}}\times\sum_{i=0}^{2h+1}e_{i}^{\{2(k-h)\}}\frac{d^{2(h+1)-i}J_e}{dt^{2(h+1)-i}}\right)$$

$$=p_{2(k+1)}^{\{2(k+1)\}}e_0^{\{0\}}J_e+\nabla^{\{2(k+1)\}}\times p_0^{\{2(k+1)\}}e_0^{\{0\}}J_e+\sum_{h=1}^{k}\left(\nabla^{\{2(k+1-h)\}}\times p_{2h}^{\{2(k+1)\}}e_0^{\{0\}}J_e\right)+\sum_{i=0}^{2(k+1)-1}e_i^{\{0\}}\frac{d^{2(k+1)-i}J_e}{dt^{2(k+1)-i}}$$
$$+\sum_{h=1}^{k}\left(\nabla^{\{2(k+1-h)\}}\times\sum_{i=0}^{2h-1}e_i^{\{2(k+1-h)\}}\frac{d^{2(h)-i}J_e}{dt^{2(h)-i}}\right)$$

$$=p_{2(k+1)}^{\{2(k+1)\}}e_0^{\{0\}}J_e+\nabla^{\{2(k+1)\}}\times p_0^{\{2(k+1)\}}e_0^{\{0\}}J_e+\sum_{i=0}^{2(k+1)-1}e_i^{\{0\}}\frac{d^{2(k+1)-i}J_e}{dt^{2(k+1)-i}}+\sum_{h=1}^{k}\left(\nabla^{\{2(k+1-h)\}}\times\left(p_{2h}^{\{2(k+1)\}}e_0^{\{0\}}J_e+\sum_{i=0}^{2h-1}e_i^{\{2(k+1-h)\}}\frac{d^{2(h)-i}J_e}{dt^{2(h)-i}}\right)\right)$$

$$by\ (18): e_{2(k-i)}^{\{2i\}}=p_{2(k-i)}^{\{2k\}}e_0^{\{0\}}; i=k-h\rightarrow p_{2h}^{\{2k\}}e_0^{\{0\}}=e_{2h}^{\{2(k-h)\}}\rightarrow p_{2h}^{\{2(k+1)\}}e_0^{\{0\}}=e_{2h}^{\{2(k+1-h)\}}$$

We have

$$=p_{2(k+1)}^{\{2(k+1)\}}e_0^{\{0\}}J_e+\nabla^{\{2(k+1)\}}\times p_0^{\{2(k+1)\}}e_0^{\{0\}}J_e+\sum_{i=0}^{2(k+1)-1}e_i^{\{0\}}\frac{d^{2(k+1)-i}J_e}{dt^{2(k+1)-i}}+\sum_{h=1}^{k}\left(\nabla^{\{2(k+1-h)\}}\times\left(\sum_{i=0}^{2h}e_i^{\{2(k+1-h)\}}\frac{d^{2h-i}J_e}{dt^{2h-i}}\right)\right)$$

$$by\ (18): e_{2(k-i)}^{\{2i\}}=p_{2(k-i)}^{\{2k\}}e_0^{\{0\}}; i=k\rightarrow e_0^{\{2k\}}=p_0^{\{2k\}}e_0^{\{0\}}\rightarrow e_0^{\{2(k+1)\}}=p_0^{\{2(k+1)\}}e_0^{\{0\}}$$

$$=p_{2(k+1)}^{\{2(k+1)\}}e_0^{\{0\}}J_e+\sum_{i=0}^{2(k+1)-1}e_i^{\{0\}}\frac{d^{2(k+1)-i}J_e}{dt^{2(k+1)-i}}+\sum_{h=0}^{k}\left(\nabla^{\{2(k+1-h)\}}\times\left(\sum_{i=0}^{2h}e_i^{\{2(k+1-h)\}}\frac{d^{2h-i}J_e}{dt^{2h-i}}\right)\right)$$

$$by\ (18): e_{2(k-i)}^{\{2i\}}=p_{2(k-i)}^{\{2k\}}e_0^{\{0\}}; i=0\rightarrow e_{2k}^{\{0\}}=p_{2k}^{\{2k\}}e_0^{\{0\}}\rightarrow e_{2(k+1)}^{\{0\}}=p_{2(k+1)}^{\{2(k+1)\}}e_0^{\{0\}}$$

$$=\sum_{i=0}^{2(k+1)}e_i^{\{0\}}\frac{d^{2(k+1)-i}J_e}{dt^{2(k+1)-i}}+\sum_{h=0}^{k}\left(\nabla^{\{2(k+1-h)\}}\times\left(\sum_{i=0}^{2h}e_i^{\{2(k+1-h)\}}\frac{d^{2h-i}J_e}{dt^{2h-i}}\right)\right)$$

$$=\sum_{h=0}^{k+1}\left(\nabla^{\{2(k+1-h)\}}\times\left(\sum_{i=0}^{2h}e_i^{\{2(k+1-h)\}}\frac{d^{2h-i}J_e}{dt^{2h-i}}\right)\right)$$

The above is the J_e terms for$k + 1$ as defined in (14). Thus we get (51).

Combine the above results, we have

$$\frac{\partial^{2(k+1)+1}E}{\partial t^{2(k+1)+1}} = \sum_{i=0}^{k+1}\left(p_{2i}^{\{2(k+1)+1\}}\nabla^{\{2(\mathrm{k}+1-\mathrm{i})+1\}} \times H + p_{2i+1}^{\{2(k+1)+1\}}\nabla^{\{2(k+1-i)\}} \times E\right) + S_e(2(k+1)+1)$$

The above is the same as (49). Thus (10) and (14) hold for $k + 1$.

Thus, the Time Space Lemma hold for $k + 1$.

QED.