Linear-Covariance of Maxwell’s Equations

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Abstract. It is discovered that covariance for Maxwell’s equations is a relation between rows of coordinate transform matrix and columns of field transform matrix. A necessary and sufficient condition for any full rank linear coordinate transform to be covariant is deduced and proved. A formula is given to construct all possible field transforms to achieve such covariance for any given coordinate transform. It is thus theoretically proved that Galilean transform is not covariant and that there is one and only one linearly independent field transform to make Lorentz transform covariant.

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Introduction

The meaning of Lorentz-covariance of Maxwell’s equations is ambiguous[1]. Mr. D V Redzic points out that this ambiguity can lead to incorrect implications.

Inspired by Mr. Redzic’s thoughts, I want to go one step further to ask such questions:

1. Is it possible to construct a field transformation to make Galilean coordinate transformation Maxwell’s equations covariant? People constructed some field transformations as negative examples [2]. Negative examples, no matter how many of them, cannot logically answer the question.

2. Are there other linearly independent field transformations, besides the one well known (see [1]), to make Lorentz coordinate transformation Maxell’s equations covariant? If this question cannot be answered then the role of covariance of Maxwell’s equations in the special relativity is questionable, because it is hard to image that one reality in one coordinate system should be mapped to multiple realities in a transformed coordinate system.

My approach to answer these questions is to try to construct covariant field transformations for all linear coordinate transformations. Mathematically it is to find a formula to link a 4 by 4 matrix representing a linear coordinate transformation to a 6 by 6 matrix representing a field transformation, using covariance of Maxwell’s equations as binding condition. My efforts reveal that covariance of Maxwell’s equations is equivallent to a relationship of rows of coordinate transform matrix and columns of field transform matrix. A necessary and sufficient condition is thus derived for a full rank linear coordinate transformations to be covariant for Maxwell’s equations. Using this condition, it can be theoretially proved that it is not possible to construct field transformation to make Galilean transformation Maxwell’s equations covariant.

A formula is derived to construct all possible field transformations for a full rank linear coordinate transformation to make it Maxwell’s equations covariant. By this formula, it is theoretically proved that there is one and only one linearly independent field transformation to make Lorentz coordinate transformation Maxwell’s quations covariant.

In this paper, “covariant” means Maxwell’s equations covariant; “covariance” means covariance of Maxwell’s equations.

In formulas, one 0 can be used to represent a matrix of appropriate dimensions with all its elements being 0s.

Maxwell’s equations without coefficients

Consider Maxwell’s equations in following form

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon}\nabla \times H - \frac{1}{\varepsilon}J_e \tag{1}$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu}\nabla \times E - \frac{1}{\mu}J_m \tag{2}$$

Where

$$E, H, J_e, J_m \in \mathcal{R}^3;\ \varepsilon, \mu\ \in \mathcal{R}^1$$

$$coordinates: x, y, z, t$$

Suppose $\varepsilon\ and\ \mu$ are constants. For convenience and without losing generality, use following scaling to absorb the material coefficents.

$$\tilde{E} = \sqrt{\varepsilon}E \tag{3}$$

$$\tilde{H} = \sqrt{\mu}H \tag{4}$$

$$\tilde{J_e} = \sqrt{\frac{\mu}{\varepsilon}}J_e \tag{5}$$

$$\widetilde{J_m} = \sqrt{\frac{\varepsilon}{\mu}}J_m \tag{6}$$

$$\tilde{t} = \frac{1}{\sqrt{\mu\varepsilon}}t \tag{7}$$

We get following Maxwell’s equations without coefficients.

$$\frac{\partial \tilde{E}}{\partial \tilde{t}} = \nabla \times \tilde{H} - \tilde{J_e} \tag{8}$$

$$\frac{\partial \tilde{H}}{\partial \tilde{t}} = -\nabla \times \tilde{E} - \widetilde{J_m} \tag{9}$$

For easier math manipulations, define following symbols. Note that by (10), t has different meaning from now on.

$$t \equiv \tilde{t} \tag{10}$$

$$F \equiv [f_1 \quad f_2 \quad f_3 \quad f_4 \quad f_5 \quad f_6]^T \equiv \begin{bmatrix} \tilde{E} \\ \tilde{H} \end{bmatrix} \tag{11}$$

$$J \equiv \begin{bmatrix} \tilde{J_e} \\ \widetilde{J_m} \end{bmatrix} \tag{12}$$

$$\nabla_{-2} \times F \equiv \begin{bmatrix} \nabla \times \begin{bmatrix} f_4 \\ f_5 \\ f_6 \end{bmatrix} \\ -\nabla \times \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \end{bmatrix} \tag{13}$$

$$\nabla_2 \cdot F \equiv \begin{bmatrix} \nabla \cdot \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \\ \nabla \cdot \begin{bmatrix} f_4 \\ f_5 \\ f_6 \end{bmatrix} \end{bmatrix} \tag{14}$$

Maxwell’s equations (8) and (9) can be writtern as a single equation (15).

$$\frac{\partial F}{\partial t} = \nabla_{-2} \times F - J \tag{15}$$

Coordinate and Field Transforms

A linear coordinate transformation can be represented by a 4 by 4 matrix matrix V , called a coordinate transformation matrix.

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = V \begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix} \tag{16}$$

$$V \equiv \begin{bmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ v_{31} & v_{32} & v_{33} & v_{34} \\ v_{41} & v_{42} & v_{43} & v_{44} \end{bmatrix} \tag{17}$$

To investigate covariance properties of a linear coordinate transformation, a usefull tool is what I called Row Matrixes. A Row-Matrix is formed by one row of a coordinate transformation matrix as following.

$$R_i \equiv \begin{bmatrix} v_{i4} & 0 & 0 & 0 & v_{i3} & -v_{i2} \\ 0 & v_{i4} & 0 & -v_{i3} & 0 & v_{i1} \\ 0 & 0 & v_{i4} & v_{i2} & -v_{i1} & 0 \\ 0 & -v_{i3} & v_{i2} & v_{i4} & 0 & 0 \\ v_{i3} & 0 & -v_{i1} & 0 & v_{i4} & 0 \\ -v_{i2} & v_{i1} & 0 & 0 & 0 & v_{i4} \end{bmatrix}; i = 1,2,3,4 \tag{18}$$

We may also use coordinate name to identify each Row-Matrix as following

$$\begin{gathered} X \equiv \begin{bmatrix} v_{14} & 0 & 0 & 0 & v_{13} & -v_{12} \\ 0 & v_{14} & 0 & -v_{13} & 0 & v_{11} \\ 0 & 0 & v_{14} & v_{12} & -v_{11} & 0 \\ 0 & -v_{13} & v_{12} & v_{14} & 0 & 0 \\ v_{13} & 0 & -v_{11} & 0 & v_{14} & 0 \\ -v_{12} & v_{11} & 0 & 0 & 0 & v_{14} \end{bmatrix}; Y \equiv \begin{bmatrix} v_{24} & 0 & 0 & 0 & v_{23} & -v_{22} \\ 0 & v_{24} & 0 & -v_{23} & 0 & v_{21} \\ 0 & 0 & v_{24} & v_{22} & -v_{21} & 0 \\ 0 & -v_{23} & v_{22} & v_{24} & 0 & 0 \\ v_{23} & 0 & -v_{21} & 0 & v_{24} & 0 \\ -v_{22} & v_{21} & 0 & 0 & 0 & v_{24} \end{bmatrix} \\ Z \equiv \begin{bmatrix} v_{34} & 0 & 0 & 0 & v_{33} & -v_{32} \\ 0 & v_{34} & 0 & -v_{33} & 0 & v_{31} \\ 0 & 0 & v_{34} & v_{32} & -v_{31} & 0 \\ 0 & -v_{33} & v_{32} & v_{34} & 0 & 0 \\ v_{33} & 0 & -v_{31} & 0 & v_{34} & 0 \\ -v_{32} & v_{31} & 0 & 0 & 0 & v_{34} \end{bmatrix}; T \equiv \begin{bmatrix} v_{44} & 0 & 0 & 0 & v_{43} & -v_{42} \\ 0 & v_{44} & 0 & -v_{43} & 0 & v_{41} \\ 0 & 0 & v_{44} & v_{42} & -v_{41} & 0 \\ 0 & -v_{43} & v_{42} & v_{44} & 0 & 0 \\ v_{43} & 0 & -v_{41} & 0 & v_{44} & 0 \\ -v_{42} & v_{41} & 0 & 0 & 0 & v_{44} \end{bmatrix} \end{gathered} \tag{19}$$

Below I list some basic Row Matrix properties which may be usefull in current and future researches. For examples, we may use them to decompose a Row Matrix; to calculate inverse matrix of a Row Matrix, etc.

Property 1. Row Matrixes are symmetric.

Property 2. Each Row Matrix for a full rank linear coordinate transformation is not a 0 matrix.

Each element in a Row Matrix is 0 or an element in the same row of a linear coordinate transformation matrix. For a full rank matrix, there is at least one non-zero element for each row, therefore a Row Matrix must have at least one non-zero element.

Property 3. The eigenvalues of a Row Matrix are given by

$$\lambda_1 = \lambda_2 = v_{i4}$$

$$\lambda_3 = \lambda_4 = v_{i4} + \sqrt{v_{i1}^2 + v_{i2}^2 + v_{i3}^2}$$

$$\lambda_5 = \lambda_6 = v_{i4} - \sqrt{v_{i1}^2 + v_{i2}^2 + v_{i3}^2}$$

$$i = 1,2,3,4$$

This property can be proved by calculating the eigenvalues via $\det(R_i - \lambda I)$ and derive following equation

$$\det(R_i - \lambda I) = (\lambda - v_{i4})^2 \left((\lambda - v_{i4})^2 - (v_{i1}^2 + v_{i2}^2 + v_{i3}^2)\right)^2 = 0$$

$$i = 1,2,3,4$$

property 4. The rank of a Row Matrix can only be 4 or 6.

This property can be deduced from the previous properties.

Another usefull matrix can be constructed using Row Matrixes as following

$$\begin{gathered} C_T = \begin{bmatrix} X & -Y & 0 & 0 & 0 & 0 \\ 0 & Y & -Z & 0 & 0 & 0 \\ 0 & 0 & 0 & X & -Y & 0 \\ 0 & 0 & 0 & 0 & Y & -Z \end{bmatrix}; C_X = \begin{bmatrix} 0 & X & 0 & 0 & 0 & T \\ 0 & 0 & X & 0 & -T & 0 \\ 0 & 0 & -T & 0 & X & 0 \\ 0 & T & 0 & 0 & 0 & X \end{bmatrix} \\ C_Y = \begin{bmatrix} Y & 0 & 0 & 0 & 0 & -T \\ 0 & 0 & Y & T & 0 & 0 \\ 0 & 0 & T & Y & 0 & 0 \\ -T & 0 & 0 & 0 & 0 & Y \end{bmatrix}; C_Z = \begin{bmatrix} Z & 0 & 0 & 0 & T & 0 \\ 0 & Z & 0 & -T & 0 & 0 \\ 0 & -T & 0 & Z & 0 & 0 \\ T & 0 & 0 & 0 & Z & 0 \end{bmatrix} \\ C = \begin{bmatrix} C_T \\ C_X \\ C_Y \\ C_Z \end{bmatrix} \end{gathered} \tag{20}$$

I am calling matrix *C* "Construction Matrix" because it will be used to construct field transformations.

A field transformation is represented by a matrix as following.

$$\begin{gathered} F' = AF \\ A \neq 0;\ A \in \mathcal{R}^{6\times 6} \end{gathered} \tag{21}$$

Math Definition of Covariance

A strict mathematic definition of covariance is given below.

For a linear coordinate transformation given by (16) and (17), the coordinate transformation is covariant if and only if there is a field transformation defined by (21) such that Maxwell's equation (22) in coordinates (x', y', z', t') holds.

$$\frac{\partial F'}{\partial t'} = \nabla_{-2} \times F' - J' \tag{22}$$

Where J' is a field source to be constructed by fields and field source in (15). How to construct J' will be part of the covariance theory developed here.

Comparing (22) with (15) we can clearly see the meaning of "covariance". Simply put it, a 4 by 4 matrix *V* in (16) and (17) is covariant if and only if there is a 6 by 6 matrix *A* in (21) such that (22) holds.

Covariance Conditions

Linear-covariance Theorem 1. A full rank linear coordinate transformation represented by (16) and (17) is covariant if and only if the following condition is true, where matrix C is formed by (20), (17) and (19)

$$rank(C) < 36 \tag{23}$$

Linear-covariance Theorem 2. For a full rank linear coordinate transformation represented by (16) and (17), the number of linearly independent covariant field transformations is given by following value, where matrix C is formed by (20), (17) and (19)

$$36 - rank(C) \tag{24}$$

Linear-covariance Theorem 3. For a full rank linear coordinate transformation represented by (16) and (17), field transformations to make it covariant are given by the following formula and (21), where matrix C is formed by (20), (17) and (19),

$$C\vec{A} = 0 \tag{25}$$

$$\vec{A} \equiv \begin{bmatrix} a_i \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}; A \equiv [a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6]; a_i \in \mathcal{R}^{6\times 1}; i = 1,2,3,4,5,6 \tag{26}$$

Proof. Theorem 1 and Theorem 2 can be directly deduced from Theorem 3. So, only Theorem 3 needs to be proved.

Construct the source as

$$J' \equiv TAJ - Xa_1\nabla \cdot \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} - Xa_4\nabla \cdot \begin{bmatrix} f_4 \\ f_5 \\ f_6 \end{bmatrix} \tag{27}$$

Combining equations (15), (17), (19), (21) and (27), we have

$$\begin{gathered} \frac{\partial F'}{\partial t'} - \left(\nabla_{-2} \times F' - J'\right) = -Xa_1\nabla \cdot \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} - Xa_4\nabla \cdot \begin{bmatrix} f_4 \\ f_5 \\ f_6 \end{bmatrix} + \\ Xa_1\frac{\partial f_1}{\partial p_1} + Ya_2\frac{\partial f_2}{\partial p_2} + Za_3\frac{\partial f_3}{\partial p_3} + Xa_4\frac{\partial f_4}{\partial p_1} + Ya_5\frac{\partial f_5}{\partial p_2} + Za_6\frac{\partial f_6}{\partial p_3} + \\ (Xa_2 + Ta_6)\frac{\partial f_2}{\partial p_1} + (Xa_3 - Ta_5)\frac{\partial f_3}{\partial p_1} + (Xa_5 - Ta_3)\frac{\partial f_5}{\partial p_1} + (Xa_6 + Ta_2)\frac{\partial f_6}{\partial p_1} + \\ (Ya_1 - Ta_6)\frac{\partial f_1}{\partial p_2} + (Ya_3 + Ta_4)\frac{\partial f_3}{\partial p_2} + (Ya_4 + Ta_3)\frac{\partial f_4}{\partial p_2} + (Ya_6 - Ta_1)\frac{\partial f_6}{\partial p_2} + \\ (Za_1 + Ta_5)\frac{\partial f_1}{\partial p_3} + (Za_2 - Ta_4)\frac{\partial f_2}{\partial p_3} + (Za_4 - Ta_2)\frac{\partial f_4}{\partial p_3} + (Za_5 + Ta_1)\frac{\partial f_5}{\partial p_3} \end{gathered} \tag{28}$$

By definition, the transformation is covariant if and only if (22) holds. Combining (20), (26) and (27) and (28), we can see that (22) holds if and only if (25) holds.

QED

Linear-covariance Theorem 4. For a full rank linear coordinate transformation represented by (16) and (17), if its Row Matrix *T* defined in (19) is full rank then the transformation is covariant if and only if the following condition holds.

$$rank(C_t) < 18 \tag{29}$$

Where

$$C_{t1} = \begin{bmatrix} T - ZZ_t & 0 & 0 \\ 0 & T - XX_t & 0 \\ 0 & 0 & T - YY_t \end{bmatrix}; C_{t2} = \begin{bmatrix} Z & 0 & X \\ Y & X & 0 \\ X & -Y & 0 \\ 0 & Z & Y \\ 0 & -Y & Z \end{bmatrix}; C_{t3} = \begin{bmatrix} YZ_t & 0 & -XY_t \\ 0 & -XX_t & -ZY_t \\ -XZ_t & 0 & -YY_t \\ -ZZ_t & -YX_t & 0 \\ YZ_t & -ZX_t & 0 \end{bmatrix} \tag{30}$$
$$C_t = \begin{bmatrix} C_{t1} \\ C_{t2} \\ C_{t3} \end{bmatrix}$$
$$X_t = T^{-1}X; Y_t = T^{-1}Y; Z_t = T^{-1}Z$$

Proof. Because T is full rank, there are full rank matrixes $P\ and\ Q$ such that

$$PCQ = \begin{bmatrix} \begin{matrix} T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{matrix} & 0 \\ 0 & C_t \end{bmatrix} \tag{31}$$

Combining (23) and (31) we get (29).

QED

Linear-covariance Theorem 5. For a full rank linear coordinate transformation represented by (16) and (17), if its Row Matrixes *T* and *X* defined in (19) are full rank then the number of linearly independent covariant field transformations is given by the following value.

$$6 - rank(C_{tx}) \tag{32}$$

Where

$$C_{tx1} = \begin{bmatrix} T - XX_t \\ -(T - YY_t)Z_xY_x \\ (T - ZZ_t)Y_x \end{bmatrix}; C_{tx2} = \begin{bmatrix} X + YY_x \\ -Y - ZZ_xY_x \\ Z - YZ_xY_x \end{bmatrix}; C_{tx3} = \begin{bmatrix} -XX_t + ZY_tZ_xY_x \\ -YX_t - ZZ_tY_x \\ ZX_t - YZ_tY_x \\ (-XZ_t + YY_tZ_x)Y_x \\ (YZ_t + XY_tZ_x)Y_x \end{bmatrix} \tag{33}$$
$$C_{tx} = \begin{bmatrix} C_{tx1} \\ C_{tx2} \\ C_{tx3} \end{bmatrix}$$
$$Y_x = X^{-1}Y; Z_x = X^{-1}Z$$

Proof. Because $T\ and\ X$ are full rank, there are full rank matrixes $P\ and\ Q$ such that

$$PCQ = \begin{bmatrix} \begin{matrix} T & 0 & 0 & 0 & 0 \\ 0 & T & 0 & 0 & 0 \\ 0 & 0 & T & 0 & 0 \\ 0 & 0 & 0 & X & 0 \\ 0 & 0 & 0 & 0 & X \end{matrix} & 0 \\ 0 & C_{tx} \end{bmatrix} \tag{34}$$

Combining (23) and (34) we get (32).

QED

Due to space symmetry of the related equations, for Row-Matrixes *Y* and *Z*, we have theorems similar to the above theorem.

Galilean-covariance Theorem. Galilean coordinate transformation, as defined by (35), is not covariant.

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix} \tag{35}$$
$$0 < v < 1$$

Proof.

From (35) we know that $T = I$, and $T - XX_t = T - X^2$. Then $T - XX_t$ is full rank, thus by (33),

$$rank(C_{tx}) = 6$$

By (32), the number of covariant field transformation is 0.

QED.

Note that due to the scaling used, speed range of $(0,1)$ is the same range of $(0,c)$ for unscaled coordinates.

Lorentz-covariance Theorem. Lorentz coordinate transformation, as defined by (36), has one and only one linearly independent covariant field transformation.

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & v\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix} \tag{36}$$
$$0 < v < 1; \gamma = \frac{1}{\sqrt{1-v^2}}$$

Proof.

From (36) and (19) we can verify that the inverse matrixes of $T\ and\ X$ are as following (these inverse matrixes can be derived from the Row-Matrix properties listed previously).

$$T^{-1} = \begin{bmatrix} \gamma^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 & 0 & -v\gamma \\ 0 & 0 & \gamma & 0 & v\gamma & 0 \\ 0 & 0 & 0 & \gamma^{-1} & 0 & 0 \\ 0 & 0 & v\gamma & 0 & \gamma & 0 \\ 0 & -v\gamma & 0 & 0 & 0 & \gamma \end{bmatrix}$$

$$X^{-1} = \begin{bmatrix} (v\gamma)^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & -v\gamma & 0 & 0 & 0 & \gamma \\ 0 & 0 & -v\gamma & 0 & -\gamma & 0 \\ 0 & 0 & 0 & (v\gamma)^{-1} & 0 & 0 \\ 0 & 0 & -\gamma & 0 & -v\gamma & 0 \\ 0 & \gamma & 0 & 0 & 0 & -v\gamma \end{bmatrix}$$

Apply these inverse matrixes to (33) we know that

$$rank(C_{tx}) = 5$$

By (32), the number of covariant field transformation is 1.

QED.

Summary and Future Research

Discovery of equation (28) shows that covariance is a relation between rows of coordinate transform matrix and columns of field transform matrix.

Discovery of equation (27) shows how a field source is transformed between coordinate systems.

These discoveries pave a way to theoretically prove that: 1. Galilean transform is not covariant; 2. Lorentz transform has one and only one covariant field transform. These conclusions had been taken for granted without proving.

The theorems derived from these discoveries can also be used to answer other covariance-related questions. For example, are there covariant transforms other than Lorentz transform? If the answer is positive then is it true that for any covariant transform there is only one linearly independent field transform? We know that the rank of a Row Matrix can only be 4 or 6; is it true that the rank of a construction matrix defined by (20) can only be 35 or 36?

Making use of these discoveries, other research subjects are possible, for example, try to get a coordinate transform to match a desired field transform.

Reference

[1] D V Redžić, Are Maxwell's equations Lorentz-covariant?, Eur. J. Phys. 38 (2017) 015602 (4pp)

[2] https://en.wikipedia.org/wiki/Moving_magnet_and_conductor_problem