Apply the Perfect Match Layer to the Time-Space Synchronized FDTD Algorithm

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Abstract

Applying PML to the Time-Space Synchronized FDTD algorithm is straightforward; there is not interpolating and averaging involved, as is the case for applying PML to the Yee algorithm. Here I am presenting numeric results of using 4-th order PML with and without PEMC boundary conditions, with different loss magnitudes and with different thicknesses of the absorbing layer.

Contents

PML Formulation for TSS FDTD

Make a Fourier transformation of the Maxwell's equations

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon} \nabla \times H \tag{1}$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu} \nabla \times E \tag{2}$$

We have

$$\nabla \times \mathbb{H}(\omega) = i\omega\varepsilon\mathbb{E}(\omega) \tag{3}$$

$$\nabla \times \mathbb{E}(\omega) = -i\omega\mu\mathbb{H}(\omega) \tag{4}$$

To apply PML is to change $\varepsilon\ and\ \mu$ in the desired layers, see [1] and [2].

For writing formulas concisely, I'll express a diagonal matrix as

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \langle \begin{bmatrix} a \\ b \\ c \end{bmatrix} \rangle$$

PML way of changing $\varepsilon\ and\ \mu$ in (3) and (4) can be expressed as

$$\nabla \times \mathbb{H}(\omega) = i\omega\varepsilon[s]\mathbb{E}(\omega) \tag{5}$$

$$\nabla \times \mathbb{E}(\omega) = -i\omega\mu[s]\mathbb{H}(\omega) \tag{6}$$

$$[s] = \langle \begin{bmatrix} \dfrac{\left(\alpha_y + \dfrac{\beta_y}{i\omega}\right)\left(\alpha_z + \dfrac{\beta_z}{i\omega}\right)}{\left(\alpha_x + \dfrac{\beta_x}{i\omega}\right)} \\ \dfrac{\left(\alpha_x + \dfrac{\beta_x}{i\omega}\right)\left(\alpha_z + \dfrac{\beta_z}{i\omega}\right)}{\left(\alpha_y + \dfrac{\beta_y}{i\omega}\right)} \\ \dfrac{\left(\alpha_x + \dfrac{\beta_x}{i\omega}\right)\left(\alpha_y + \dfrac{\beta_y}{i\omega}\right)}{\left(\alpha_z + \dfrac{\beta_z}{i\omega}\right)} \end{bmatrix} \rangle \tag{7}$$

$$\alpha_x = \begin{cases} 1 + a_{max} \cdot \left(\frac{x_{pml}}{L}\right)^p; 0 < x_{pml} \le L \\ 1; x\ outside\ of\ PML \end{cases};\ \ \beta_x = \begin{cases} \beta_{max} \left(\frac{x_{pml}}{L}\right)^p; 0 < x_{pml} \le L \\ 0; x\ outside\ of\ PML \end{cases}$$

$$\alpha_y = \begin{cases} 1 + a_{max} \cdot \left(\frac{y_{pml}}{L}\right)^p; 0 < y_{pml} \le L \\ 1; y\ outside\ of\ PML \end{cases};\ \ \beta_y = \begin{cases} \beta_{max} \left(\frac{y_{pml}}{L}\right)^p; 0 < y_{pml} \le L \\ 0; y\ outside\ of\ PML \end{cases}$$

$$\alpha_z = \begin{cases} 1 + a_{max} \cdot \left(\frac{z_{pml}}{L}\right)^p; 0 < z_{pml} \le L \\ 1; z\ outside\ of\ PML \end{cases};\ \ \beta_z = \begin{cases} \beta_{max} \left(\frac{z_{pml}}{L}\right)^p; 0 < z_{pml} \le L \\ 0; z\ outside\ of\ PML \end{cases}$$

$$\begin{gathered} when\ x\ is\ within\ the\ PML, x_{pml}\ is\ the\ depth\ into\ the\ layer \\ when\ y\ is\ within\ the\ PML, y_{pml}\ is\ the\ depth\ into\ the\ layer \\ when\ z\ is\ within\ the\ PML, z_{pml}\ is\ the\ depth\ into\ the\ layer \end{gathered} \tag{8}$$

$$a_{max} > 0$$

$$\beta_{max} > 0$$

$$p > 0$$

$$L > 0$$

In digitized expression,

$$x = i * \Delta_s, y = j * \Delta_s, z = k * \Delta_s$$

$$L = L_n * \Delta_s$$

$$L_n > 0, L_n \textit{ is the layer thickness expressed in an integer}$$

To derive the PML for the TSS [4], start with the Taylor's series:

$$E(t+\Delta_t) = \sum_{k=0}^{\infty} \frac{\Delta_t^k}{k!} \frac{\partial^k E(t)}{\partial t^k} \tag{9}$$

$$H(t+\Delta_t) = \sum_{k=0}^{\infty} \frac{\Delta_t^k}{k!} \frac{\partial^k H(t)}{\partial t^k} \tag{10}$$

Make Fourier transformation of (9) and (10), we have

$$\mathbb{E}(\omega) e^{i\omega\Delta_t} = \sum_{k=0}^{\infty} \frac{\Delta_t^k}{k!} (i\omega)^k \mathbb{E}(\omega) \tag{11}$$

$$\mathbb{H}(\omega) e^{i\omega\Delta_t} = \sum_{k=0}^{\infty} \frac{\Delta_t^k}{k!} (i\omega)^k \mathbb{H}(\omega) \tag{12}$$

Substitute (5), (6) and (7) into (11) and (12), we have

$$\begin{aligned} \mathbb{E}(\omega) e^{i\omega\Delta_t} = & \sum_{k=0}^{\infty} \left(\langle \begin{bmatrix} \mathrm{w_x} \\ \mathrm{w_y} \\ \mathrm{w_z} \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}\}} \times \mathbb{E}(\omega) + \langle \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}+1\}} \times \mathbb{H}(\omega) \right) \\ & + \frac{1}{i\omega} \sum_{k=0}^{\infty} \left(\langle \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}\}} \times \mathbb{E}(\omega) + \langle \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}+1\}} \times \mathbb{H}(\omega) \right) \end{aligned} \tag{13}$$

$$\begin{aligned} \mathbb{H}(\omega) e^{i\omega\Delta_t} = & \sum_{k=0}^{\infty} \left(\langle \begin{bmatrix} \mathrm{w_x} \\ \mathrm{w_y} \\ \mathrm{w_z} \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}\}} \times \mathbb{H}(\omega) + \langle \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}+1\}} \times \mathbb{E}(\omega) \right) \\ & + \frac{1}{i\omega} \sum_{k=0}^{\infty} \left(\langle \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \rangle (k) \nabla^{\{2\mathrm{k}\}} \times \mathbb{H}(\omega) + \langle \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}+1\}} \times \mathbb{E}(\omega) \right) \end{aligned} \tag{14}$$

Cut-off the summations to make estimations, we have

$$\begin{aligned} \mathbb{E}(\omega) e^{i\omega\Delta_t} \cong & \sum_{k=0}^{k_{max}} \left(\langle \begin{bmatrix} \mathrm{w_x} \\ \mathrm{w_y} \\ \mathrm{w_z} \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}\}} \times \mathbb{E}(\omega) + \langle \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}+1\}} \times \mathbb{H}(\omega) \right) \\ & + \frac{1}{i\omega} \sum_{k=0}^{k_{max}} \left(\langle \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}\}} \times \mathbb{E}(\omega) + \langle \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}+1\}} \times \mathbb{H}(\omega) \right) \end{aligned} \tag{15}$$

$$\mathbb{H}(\omega)e^{i\omega\Delta_t} \cong \sum_{k=0}^{k_{max}} \left(\langle \begin{bmatrix} \mathrm{w_x} \\ \mathrm{w_y} \\ \mathrm{w_z} \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}\}} \times \mathbb{H}(\omega) + \langle \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}+1\}} \times \mathbb{E}(\omega) \right) + \frac{1}{i\omega} \sum_{k=0}^{k_{max}} \left(\langle \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \rangle (k) \nabla^{\{2\mathrm{k}\}} \times \mathbb{H}(\omega) + \langle \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}+1\}} \times \mathbb{E}(\omega) \right) \tag{16}$$

Make inverse Fourier transformations, we have

$$E(t+\Delta_t) \cong \sum_{k=0}^{k_{max}} \left(\langle \begin{bmatrix} \mathrm{w_x} \\ \mathrm{w_y} \\ \mathrm{w_z} \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}\}} \times E(t) + \langle \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}+1\}} \times H(t) \right) + \int_{-\infty}^{t} \sum_{k=0}^{k_{max}} \left(\langle \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}\}} \times E(\tau) + \langle \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}+1\}} \times H(\tau) \right) d\tau \tag{17}$$

$$H(t+\Delta_t) \cong \sum_{k=0}^{k_{max}} \left(\langle \begin{bmatrix} \mathrm{w_x} \\ \mathrm{w_y} \\ \mathrm{w_z} \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}\}} \times H(t) + \langle \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}+1\}} \times E(t) \right) + \int_{-\infty}^{t} \sum_{k=0}^{k_{max}} \left(\langle \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \rangle (k) \nabla^{\{2\mathrm{k}\}} \times H(\tau) + \langle \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} (k) \rangle \nabla^{\{2\mathrm{k}+1\}} \times E(\tau) \right) d\tau \tag{18}$$

For

$$k_{max} = 1$$

The coefficients are given below

$$\langle \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix} \rangle (0) = \langle \begin{bmatrix} 1 - \Delta_t p_{01x} + \frac{\Delta_t^2}{2} p_{02x} + \frac{\Delta_t^3}{6} p_{03x} \\ 1 - \Delta_t p_{01y} + \frac{\Delta_t^2}{2} p_{02y} + \frac{\Delta_t^3}{6} p_{03y} \\ 1 - \Delta_t p_{01z} + \frac{\Delta_t^2}{2} p_{02z} + \frac{\Delta_t^3}{6} p_{03z} \end{bmatrix} \rangle$$

$$\langle \begin{bmatrix} \mathrm{e_x} \\ \mathrm{e_y} \\ \mathrm{e_z} \end{bmatrix} \rangle (0) = \langle \begin{bmatrix} \frac{\Delta_t}{\varepsilon_0} p_{11x} - \frac{\Delta_t^2}{2\varepsilon_0} p_{12x} + \frac{\Delta_t^3}{6\varepsilon_0} p_{13x} \\ \frac{\Delta_t}{\varepsilon_0} p_{11y} - \frac{\Delta_t^2}{2\varepsilon_0} p_{12y} + \frac{\Delta_t^3}{6\varepsilon_0} p_{13y} \\ \frac{\Delta_t}{\varepsilon_0} p_{11z} - \frac{\Delta_t^2}{2\varepsilon_0} p_{12z} + \frac{\Delta_t^3}{6\varepsilon_0} p_{13z} \end{bmatrix} \rangle$$

$$\langle\begin{bmatrix}\mathrm{w_x}\\ \mathrm{w_y}\\ \mathrm{w_z}\end{bmatrix}\rangle(1)=\langle\begin{bmatrix}-\frac{\Delta_t^2}{2\varepsilon_0\mu_0}p_{22x}+\frac{\Delta_t^3}{6\varepsilon_0\mu_0}p_{23x}\\ -\frac{\Delta_t^2}{2\varepsilon_0\mu_0}p_{22y}+\frac{\Delta_t^3}{6\varepsilon_0\mu_0}p_{23y}\\ -\frac{\Delta_t^2}{2\varepsilon_0\mu_0}p_{22z}+\frac{\Delta_t^3}{6\varepsilon_0\mu_0}p_{23z}\end{bmatrix}\rangle$$

$$\langle\begin{bmatrix}e_x\\ e_y\\ e_z\end{bmatrix}\rangle(1)=-\frac{\Delta_t^3}{6\varepsilon_0^2\mu_0}\langle\begin{bmatrix}p_{33x}\\ p_{33y}\\ p_{33z}\end{bmatrix}\rangle$$

$$\langle\begin{bmatrix}u_x\\ u_y\\ u_z\end{bmatrix}\rangle(0)=\langle\begin{bmatrix}-\Delta_t q_{01x}+\frac{\Delta_t^2}{2}q_{02x}-\frac{\Delta_t^3}{6}q_{03x}\\ -\Delta_t q_{01y}+\frac{\Delta_t^2}{2}q_{02y}-\frac{\Delta_t^3}{6}q_{03y}\\ -\Delta_t q_{01z}+\frac{\Delta_t^2}{2}q_{02z}-\frac{\Delta_t^3}{6}q_{03z}\end{bmatrix}\rangle$$

$$\langle\begin{bmatrix}\mathrm{f_x}\\ \mathrm{f_y}\\ \mathrm{f_z}\end{bmatrix}\rangle(0)=\langle\begin{bmatrix}\frac{\Delta_t}{\varepsilon_0}q_{11x}-\frac{\Delta_t^2}{2\varepsilon_0}q_{12x}+\frac{\Delta_t^3}{6\varepsilon_0}q_{13x}\\ \frac{\Delta_t}{\varepsilon_0}q_{11y}-\frac{\Delta_t^2}{2\varepsilon_0}q_{12y}+\frac{\Delta_t^3}{6\varepsilon_0}q_{13y}\\ \frac{\Delta_t}{\varepsilon_0}q_{11z}-\frac{\Delta_t^2}{2\varepsilon_0}q_{12z}+\frac{\Delta_t^3}{6\varepsilon_0}q_{13z}\end{bmatrix}\rangle$$

$$\langle\begin{bmatrix}u_x\\ u_y\\ u_z\end{bmatrix}\rangle(1)=\langle\begin{bmatrix}-\frac{\Delta_t^2}{2\varepsilon_0\mu_0}q_{22x}+\frac{\Delta_t^3}{6\varepsilon_0\mu_0}q_{23x}\\ -\frac{\Delta_t^2}{2\varepsilon_0\mu_0}q_{22y}+\frac{\Delta_t^3}{6\varepsilon_0\mu_0}q_{23y}\\ -\frac{\Delta_t^2}{2\varepsilon_0\mu_0}q_{22z}+\frac{\Delta_t^3}{6\varepsilon_0\mu_0}q_{23z}\end{bmatrix}\rangle$$

$$\langle\begin{bmatrix}f_x\\ f_y\\ f_z\end{bmatrix}\rangle(1)=-\frac{\Delta_t^3}{6\varepsilon_0^2\mu_0}\langle\begin{bmatrix}q_{33x}\\ q_{33y}\\ q_{33z}\end{bmatrix}\rangle$$

$$\langle\begin{bmatrix}\mathrm{h_x}\\ \mathrm{h_y}\\ \mathrm{h_z}\end{bmatrix}\rangle(0)=\langle\begin{bmatrix}-\frac{\Delta_t}{\mu_0}p_{11x}+\frac{\Delta_t^2}{2\mu_0}p_{12x}-\frac{\Delta_t^3}{6\mu_0}p_{13x}\\ -\frac{\Delta_t}{\mu_0}p_{11y}+\frac{\Delta_t^2}{2\mu_0}p_{12y}-\frac{\Delta_t^3}{6\mu_0}p_{13y}\\ -\frac{\Delta_t}{\mu_0}p_{11z}+\frac{\Delta_t^2}{2\mu_0}p_{12z}-\frac{\Delta_t^3}{6\mu_0}p_{13z}\end{bmatrix}\rangle$$

$$\langle\begin{bmatrix}h_x\\ h_y\\ h_z\end{bmatrix}\rangle(1)=\frac{\Delta_t^3}{6\varepsilon_0\mu_0^2}\langle\begin{bmatrix}p_{33x}\\ p_{33y}\\ p_{33z}\end{bmatrix}\rangle$$

$$\langle\begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix}\rangle(0) = \langle\begin{bmatrix} -\frac{\Delta_t}{\mu_0}q_{11x} + \frac{\Delta_t^2}{2\mu_0}q_{12x} - \frac{\Delta_t^3}{6\mu_0}q_{13x} \\ -\frac{\Delta_t}{\mu_0}q_{11y} + \frac{\Delta_t^2}{2\mu_0}q_{12y} - \frac{\Delta_t^3}{6\mu_0}q_{13y} \\ -\frac{\Delta_t}{\mu_0}q_{11z} + \frac{\Delta_t^2}{2\mu_0}q_{12z} - \frac{\Delta_t^3}{6\mu_0}q_{13z} \end{bmatrix}\rangle$$

$$\langle\begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix}\rangle(1) = \frac{\Delta_t^3}{6\varepsilon_0\mu_0^2}\langle\begin{bmatrix} q_{33x} \\ q_{33y} \\ q_{33z} \end{bmatrix}\rangle$$

$$p_{01x} = \frac{(\alpha_z\beta_y + \alpha_y\beta_z)}{\alpha_y\alpha_z}; p_{02x} = \left(\frac{(\alpha_z\beta_y + \alpha_y\beta_z)}{\alpha_y\alpha_z}\right)^2 - \frac{\beta_y\beta_z}{\alpha_y\alpha_z}; p_{03x} = \frac{2\beta_y\beta_z(\alpha_z\beta_y + \alpha_y\beta_z)}{(\alpha_y\alpha_z)^2} - \left(\frac{(\alpha_z\beta_y + \alpha_y\beta_z)}{\alpha_y\alpha_z}\right)^3$$

$$p_{01y} = \frac{(\alpha_x\beta_z + \alpha_z\beta_x)}{\alpha_z\alpha_x}; p_{02y} = \left(\frac{(\alpha_x\beta_z + \alpha_z\beta_x)}{\alpha_z\alpha_x}\right)^2 - \frac{\beta_z\beta_x}{\alpha_z\alpha_x}; p_{03y} = \frac{2\beta_z\beta_x(\alpha_x\beta_z + \alpha_z\beta_x)}{(\alpha_z\alpha_x)^2} - \left(\frac{(\alpha_x\beta_z + \alpha_z\beta_x)}{\alpha_z\alpha_x}\right)^3$$

$$p_{01z} = \frac{(\alpha_y\beta_x + \alpha_x\beta_y)}{\alpha_x\alpha_y}; p_{02z} = \left(\frac{(\alpha_y\beta_x + \alpha_x\beta_y)}{\alpha_x\alpha_y}\right)^2 - \frac{\beta_x\beta_y}{\alpha_x\alpha_y}; p_{03z} = \frac{2\beta_x\beta_y(\alpha_y\beta_x + \alpha_x\beta_y)}{(\alpha_x\alpha_y)^2} - \left(\frac{(\alpha_y\beta_x + \alpha_x\beta_y)}{\alpha_x\alpha_y}\right)^3$$

$$p_{11x} = \frac{\alpha_x}{\alpha_y\alpha_z}; p_{12x} = \frac{2\alpha_x(\alpha_z\beta_y + \alpha_y\beta_z)}{(\alpha_y\alpha_z)^2} - \frac{\beta_x}{\alpha_y\alpha_z}; p_{13x} = \frac{3\alpha_x(\alpha_z\beta_y + \alpha_y\beta_z)^2}{(\alpha_y\alpha_z)^3} - 2\frac{\beta_x(\alpha_z\beta_y + \alpha_y\beta_z) + \alpha_x\beta_y\beta_z}{(\alpha_y\alpha_z)^2}$$

$$p_{11y} = \frac{\alpha_y}{\alpha_z\alpha_x}; p_{12y} = \frac{2\alpha_y(\alpha_x\beta_z + \alpha_z\beta_x)}{(\alpha_z\alpha_x)^2} - \frac{\beta_y}{\alpha_z\alpha_x}; p_{13y} = \frac{3\alpha_y(\alpha_x\beta_z + \alpha_z\beta_x)^2}{(\alpha_z\alpha_x)^3} - 2\frac{\beta_y(\alpha_x\beta_z + \alpha_z\beta_x) + \alpha_y\beta_z\beta_x}{(\alpha_z\alpha_x)^2}$$

$$p_{11z} = \frac{\alpha_z}{\alpha_x\alpha_y}; p_{12z} = \frac{2\alpha_z(\alpha_y\beta_x + \alpha_x\beta_y)}{(\alpha_x\alpha_y)^2} - \frac{\beta_z}{\alpha_x\alpha_y}; p_{13z} = \frac{3\alpha_z(\alpha_y\beta_x + \alpha_x\beta_y)^2}{(\alpha_x\alpha_y)^3} - 2\frac{\beta_z(\alpha_y\beta_x + \alpha_x\beta_y) + \alpha_z\beta_x\beta_y}{(\alpha_x\alpha_y)^2}$$

$$p_{21x} = 0; p_{22x} = \left(\frac{\alpha_x}{\alpha_y\alpha_z}\right)^2; p_{23x} = \frac{3\alpha_x^2(\alpha_z\beta_y + \alpha_y\beta_z)}{(\alpha_y\alpha_z)^3} - \frac{2\alpha_x\beta_x}{(\alpha_y\alpha_z)^2}$$

$$p_{21y} = 0; p_{22y} = \left(\frac{\alpha_y}{\alpha_z\alpha_x}\right)^2; p_{23y} = \frac{3\alpha_y^2(\alpha_x\beta_z + \alpha_z\beta_x)}{(\alpha_z\alpha_x)^3} - \frac{2\alpha_y\beta_y}{(\alpha_z\alpha_x)^2}$$

$$p_{21z} = 0; p_{22z} = \left(\frac{\alpha_z}{\alpha_x\alpha_y}\right)^2; p_{23z} = \frac{3\alpha_z^2(\alpha_y\beta_x + \alpha_x\beta_y)}{(\alpha_x\alpha_y)^3} - \frac{2\alpha_z\beta_z}{(\alpha_x\alpha_y)^2}$$

$$p_{31x} = 0; p_{32x} = 0; p_{33x} = \left(\frac{\alpha_x}{\alpha_y\alpha_z}\right)^3$$

$$p_{31y} = 0; p_{32y} = 0; p_{33y} = \left(\frac{\alpha_y}{\alpha_z\alpha_x}\right)^3$$

$$p_{31z} = 0; p_{32z} = 0; p_{33z} = \left(\frac{\alpha_z}{\alpha_x\alpha_y}\right)^3$$

$$q_{01x} = \frac{\beta_y\beta_z}{\alpha_y\alpha_z}; q_{02x} = \frac{\beta_y\beta_z(\alpha_z\beta_y + \alpha_y\beta_z)}{(\alpha_y\alpha_z)^2}; q_{03x} = \frac{\beta_y\beta_z}{\alpha_y\alpha_z}\left(\frac{(\alpha_z\beta_y + \alpha_y\beta_z)^2}{(\alpha_y\alpha_z)^2} - \frac{\beta_y\beta_z}{\alpha_y\alpha_z}\right)$$

$$q_{01y} = \frac{\beta_z\beta_x}{\alpha_z\alpha_x}; q_{02y} = \frac{\beta_z\beta_x(\alpha_x\beta_z + \alpha_z\beta_x)}{(\alpha_z\alpha_x)^2}; q_{03y} = \frac{\beta_z\beta_x}{\alpha_z\alpha_x}\left(\frac{(\alpha_x\beta_z + \alpha_z\beta_x)^2}{(\alpha_z\alpha_x)^2} - \frac{\beta_z\beta_x}{\alpha_z\alpha_x}\right)$$

$$q_{01z} = \frac{\beta_x\beta_y}{\alpha_x\alpha_y}; q_{02z} = \frac{\beta_x\beta_y(\alpha_y\beta_x + \alpha_x\beta_y)}{(\alpha_x\alpha_y)^2}; q_{03z} = \frac{\beta_x\beta_y}{\alpha_x\alpha_y}\left(\frac{(\alpha_y\beta_x + \alpha_x\beta_y)^2}{(\alpha_x\alpha_y)^2} - \frac{\beta_x\beta_y}{\alpha_x\alpha_y}\right)$$

$$q_{11x} = \frac{\beta_x}{\alpha_y\alpha_z}; q_{12x} = \frac{\beta_x(\alpha_z\beta_y + \alpha_y\beta_z) + \alpha_x\beta_y\beta_z}{(\alpha_y\alpha_z)^2}; q_{13x} = \frac{\left(2\alpha_x\beta_y\beta_z + \beta_x(\alpha_z\beta_y + \alpha_y\beta_z)\right)(\alpha_z\beta_y + \alpha_y\beta_z)}{(\alpha_y\alpha_z)^3} - \frac{2\beta_x\beta_y\beta_z}{(\alpha_y\alpha_z)^2}$$

$$q_{11y} = \frac{\beta_y}{\alpha_z\alpha_x}; q_{12y} = \frac{\beta_y(\alpha_x\beta_z + \alpha_z\beta_x) + \alpha_y\beta_z\beta_x}{(\alpha_z\alpha_x)^2}; q_{13y} = \frac{\left(2\alpha_y\beta_z\beta_x + \beta_y(\alpha_x\beta_z + \alpha_z\beta_x)\right)(\alpha_x\beta_z + \alpha_z\beta_x)}{(\alpha_z\alpha_x)^3} - \frac{2\beta_x\beta_y\beta_z}{(\alpha_z\alpha_x)^2}$$

$$q_{11z}=\frac{\beta_z}{\alpha_x\alpha_y};q_{12z}=\frac{\beta_z(\alpha_y\beta_x+\alpha_x\beta_y)+\alpha_z\beta_x\beta_y}{(\alpha_x\alpha_y)^2};q_{13z}=\frac{\left(2\alpha_z\beta_x\beta_y+\beta_z(\alpha_y\beta_x+\alpha_x\beta_y)\right)(\alpha_y\beta_x+\alpha_x\beta_y)}{(\alpha_x\alpha_y)^3}-\frac{2\beta_x\beta_y\beta_z}{(\alpha_x\alpha_y)^2}$$

$$q_{21x}=0;q_{22x}=\frac{\alpha_x\beta_x}{(\alpha_y\alpha_z)^2};q_{23x}=\frac{2\alpha_x\beta_x(\alpha_z\beta_y+\alpha_y\beta_z)+\alpha_x^2\beta_y\beta_z}{(\alpha_y\alpha_z)^3}-\left(\frac{\beta_x}{\alpha_y\alpha_z}\right)^2$$

$$q_{21y}=0;q_{22y}=\frac{\alpha_y\beta_y}{(\alpha_z\alpha_x)^2};q_{23y}=\frac{2\alpha_y\beta_y(\alpha_x\beta_z+\alpha_z\beta_x)+\alpha_y^2\beta_z\beta_x}{(\alpha_z\alpha_x)^3}-\left(\frac{\beta_y}{\alpha_z\alpha_x}\right)^2$$

$$q_{21z}=0;q_{22z}=\frac{\alpha_z\beta_z}{(\alpha_x\alpha_y)^2};q_{23z}=\frac{2\alpha_z\beta_z(\alpha_y\beta_x+\alpha_x\beta_y)+\alpha_z^2\beta_x\beta_y}{(\alpha_x\alpha_y)^3}-\left(\frac{\beta_z}{\alpha_x\alpha_y}\right)^2$$

$$q_{31x}=0;q_{32x}=0;q_{33x}=\frac{\alpha_x^2\beta_x}{(\alpha_y\alpha_z)^3}$$

$$q_{31y}=0;q_{32y}=0;q_{33y}=\frac{\alpha_y^2\beta_y}{(\alpha_z\alpha_x)^3}$$

$$q_{31z}=0;q_{32z}=0;q_{33z}=\frac{\alpha_z^2\beta_z}{(\alpha_x\alpha_y)^3}$$

Use simple summation to estimate the integration, (17) and (18) can be expressed as

$$\boxed{\begin{aligned}
E_s(q\Delta_t+\Delta_t)&=E_s(q\Delta_t)\\
&\quad+\Delta_t\left(\langle\begin{bmatrix}u_x\\u_y\\u_z\end{bmatrix}(0)\rangle E(q\Delta_t)+\langle\begin{bmatrix}f_x\\f_y\\f_z\end{bmatrix}(0)\rangle\nabla\times H(q\Delta_t)+\langle\begin{bmatrix}u_x\\u_y\\u_z\end{bmatrix}(1)\rangle\nabla^{\{2\}}\times E(q\Delta_t)\right.\\
&\quad\left.+\langle\begin{bmatrix}f_x\\f_y\\f_z\end{bmatrix}(1)\rangle\nabla^{\{3\}}\times H(q\Delta_t)\right)\\
H_s(q\Delta_t+\Delta_t)&=H_s(q\Delta_t)\\
&\quad+\Delta_t\left(\langle\begin{bmatrix}u_x\\u_y\\u_z\end{bmatrix}\rangle(0)H(q\Delta_t)+\langle\begin{bmatrix}g_x\\g_y\\g_z\end{bmatrix}(0)\rangle\nabla\times E(q\Delta_t)+\langle\begin{bmatrix}u_x\\u_y\\u_z\end{bmatrix}\rangle(k)\nabla^{\{2\}}\times H(q\Delta_t)\right.\\
&\quad\left.+\langle\begin{bmatrix}g_x\\g_y\\g_z\end{bmatrix}(k)\rangle\nabla^{\{3\}}\times E(q\Delta_t)\right)
\end{aligned}}$$

$$E(t+\Delta_t) \cong \langle \begin{bmatrix} \mathrm{w_x} \\ \mathrm{w_y} \\ \mathrm{w_z} \end{bmatrix} (0) \rangle\, E(q\Delta_t) + \langle \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} (0) \rangle\, \nabla \times H(q\Delta_t) + \langle \begin{bmatrix} \mathrm{w_x} \\ \mathrm{w_y} \\ \mathrm{w_z} \end{bmatrix} (1) \rangle\, \nabla^{\{2\}} \times E(q\Delta_t)$$
$$+ \langle \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} (1) \rangle\, \nabla^{\{3\}} \times H(q\Delta_t) + E_s(q\Delta_t + \Delta_t)$$

$$H(t+\Delta_t) \cong \langle \begin{bmatrix} \mathrm{w_x} \\ \mathrm{w_y} \\ \mathrm{w_z} \end{bmatrix} (0) \rangle\, H(q\Delta_t) + \langle \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} (0) \rangle\, \nabla \times E(q\Delta_t) + \langle \begin{bmatrix} \mathrm{w_x} \\ \mathrm{w_y} \\ \mathrm{w_z} \end{bmatrix} (1) \rangle\, \nabla^{\{2\}} \times H(q\Delta_t)$$
$$+ \langle \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} (1) \rangle\, \nabla^{\{3\}} \times E(q\Delta_t) + H_s(q\Delta_t + \Delta_t)$$

$$E_s(0) = 0$$

$$H_s(0) = 0$$

$$q = 0,1,2, \ldots$$

The above are the field update equations. Next, let's see some numeric results produced by the above formulas.

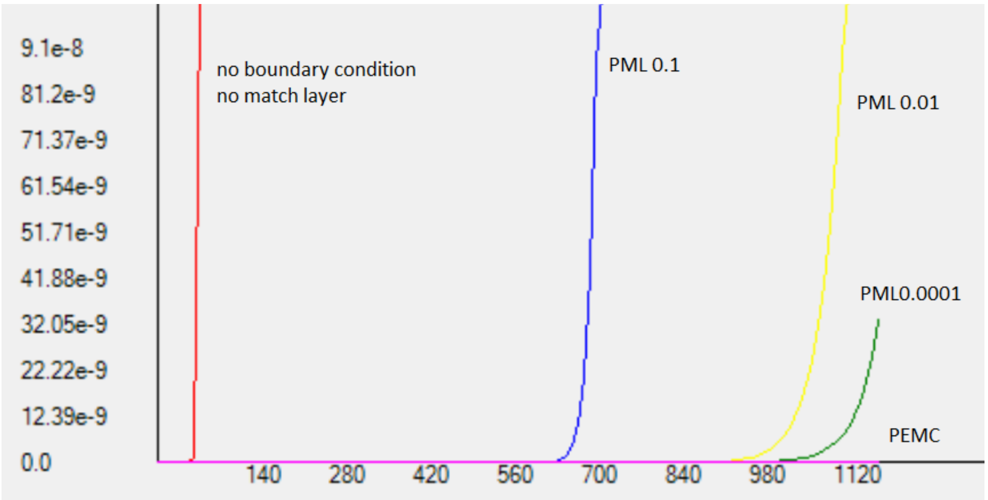
Numeric Experiments

I was using Schneider's following source code in ricker.c for field source [3]:

```
ricker.c:
        double ezInc(double time, double location) {
                double arg;
                arg = M_PI * ((cdtds * time - location) / ppw - 1.0);
                arg = arg * arg;
                return (1.0 - 2.0 * arg) * exp(-arg);
        }
```

The computing domain is a cube of (-1, 1), space grids are 85x85x85. Space step size is 0.02; time step size is 3.8e-11. The field source is located in the center of the computing domain. There are 42 grids between the center space and the boundary. Therefore, after 42 time steps, the effects of the field source will reach the boundary.

Effects of loss magnitudes



The horizontal axis shows time steps simulated; the vertical axis shows the energy within the computing domain.

The red line shows the energy-time when there is not a boundary condition is applied and there is not an absorbing layer applied. It shows that after the effects of the field source hiting the boundary, the field energy quickly grow to infinity.

The line marked by "PML 0.1" shows the energy-time when PML is applied and the loss magnitudes are 0.1:

$$\alpha_{max} = 0.1$$

$$\beta_{max} = 0.1$$

Comparing with the red line, "PML 0.1" shows the effects of PML. It does not prevent the energy from growing, but it does much delayed the energy grow, showing that the layer absorbed lots of energy.

The line marked by "PML 0.01" shows the energy-time when PML is applied and the loss magnitudes are 0.01:

$$\alpha_{max} = 0.01$$

$$\beta_{max} = 0.01$$

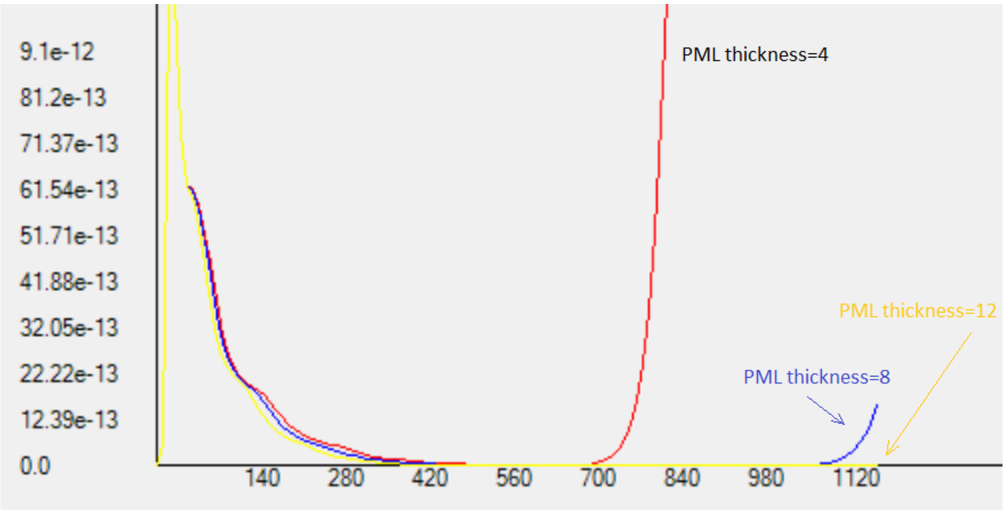
Comparing with "PML 0.1", "PML 0.01" further delayed the energy grow, showing that it absorbs more energy but not enough to prevent eventual energy grow.

"PML 0.0001" further delayed energy grow but not much comparing to "PML 0.01". Obviously we cannot keep lowering the magnitude because when the magnitude is 0, we return to the case of not using PML shown by the red line.

Note the line marked by "PEMC", it almost aligns with the x-axis, showing that the energy is always very low. It shows that the PEMC boundary condition works perfectly without any absorbing layer. For applying PEMC to the TSS FDTD algorithm, see [5].

For all these PML tests, the layer thickness is 4. Next, let's see the effects of different layer thicknesses.

Effects of Layer Thickness

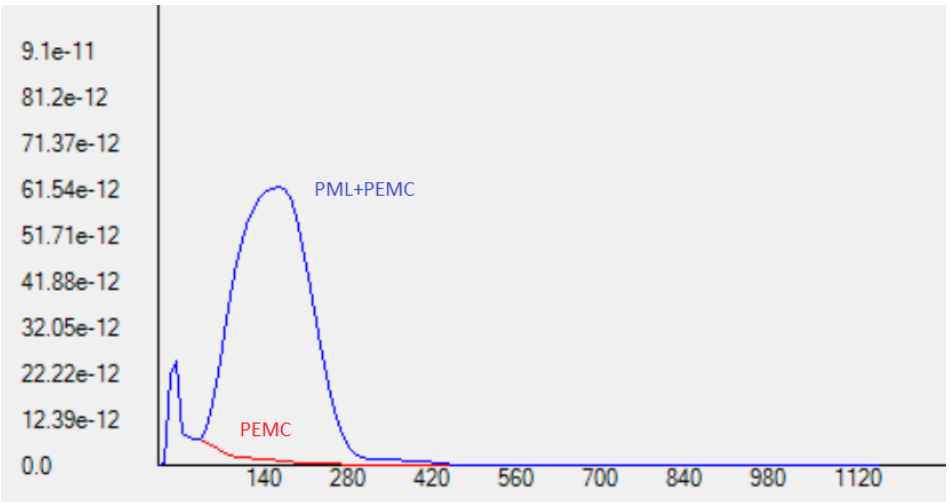


Before the time step 420, the curves are almost identical; actually these curves show the energy generated by the field source. After time step 700, we start to see the energy generated at the boundary. The PML is supposed to suppress such energy.

Comparing the lines of "PML thickness=4", "PML thickness=8" and "PML thickness=12", we can see that the thicker the layer the lower the energy. This is the expected behavior.

We see that the PML can work if the layer is thick enough.

PML with PEMC



The line of "PEMC" is produced with PEMC boundary condition applied but without any absorbing layer; the line of "PML+PEMC" is produced with PEMC boundary condition together with PML absorbing layer. PML uses magnitude 0.1 and layer thickness 4.

From the previous tests we know that thickness 4 does not produce good absorbing. From the line of "PML+PEMC" we see that the boundary condition of PEMC fixes the problem.

Summary

The test data show that 1) the loss magnitude plays a big role in the PML performance; 2) for the PML to work well enough, the layer must be thick enough.

For introducing the loss gradually, I was using a power function:

$$\alpha = 1 + a_{max} \cdot \left(\frac{i}{L_n}\right)^p$$

$$\beta = \beta_{max} \cdot \left(\frac{i}{L_n}\right)^p$$

$$i = 1,2, \dots, L_n$$

For all the tests, I was using $p = 3$, and $\alpha_{max} = \beta_{max}$. We may test the effects of other p values and use $\alpha_{max} \neq \beta_{max}$. We may also try other functions, for example in [2], sine function is used:

$$\alpha = 1 + a_{max} \cdot \sin\left(\frac{i}{L_n}\frac{\pi}{2}\right)$$

$$\beta = 1 + \beta_{max} \cdot \sin\left(\frac{i}{L_n}\frac{\pi}{2}\right)$$

For this sample simulation, the PEMC boundary condition works perfectly without using PML. Since applying PEMC to the TSS is extremely simple [5] and it works perfectly without any absorbing layer, the efforts of applying PML is unnecessary in this case.

References

[1] J. P. Berenger, A Perfectly Matched Layer for the Absorption of Electromagnetic Waves, J. Comput. Phys., No. 114, 1994, pp. 185-200.

[2]. Lecture 13 (FDTD) -- The Perfectly Matched Layer, https://www.youtube.com/watch?v=w_NnRZlNuAA

[3]. John B. Schneider, Understanding the Finite-Difference Time-Domain Method, www.eecs.wsu.edu/~schneidj/ufdtd, 2010.

[4]. David Ge, Time-Space Synchronized FDTD Method, https://github.com/DavidGeUSA/TSS/blob/master/TSS.pdf

[5]. David Ge, Apply the Perfect Electromagnetic Conducting Boundary to the Time-Space Synchronized FDTD Algorithm, https://www.researchgate.net/publication/345348206_Apply_the_Perfect_Electromagnetic_Conducting_Boundary_to_the_Time-Space_Synchronized_FDTD_Algorithm