

# Generic FDTD for Full Maxwell Equations

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## Abstract

A generic FDTD form is derived for the Maxwell equations with medium effects and electric and magnet sources. A time space relation is proposed and proved, together with the cascade curl theorem I proved earlier, they form the foundation for this generic FDTD algorithm. Other FDTD algorithms, i.e. Yee algorithm, high order algorithms, etc., can be derived from this generic form by specifying parameters, adding other techniques, adding restrictions, etc. Several FDTD algorithms are derived from this generic FDTD, as examples. Among these derived algorithm examples, some of them keep the major benefit of the Yee's algorithm and remove the major limitation of it.

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## Introduction

In [1] I derived a generic FDTD form for Maxwell equations in lossless medium. The targeted Maxwell equations are

$$\nabla \cdot E = \rho/\varepsilon \quad (1)$$

$$\nabla \cdot H = 0 \quad (2)$$

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon} \nabla \times H - \frac{1}{\varepsilon} J(t) \quad (3)$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu} \nabla \times E \quad (4)$$

Where  $t$  is time,  $\rho$ ,  $\varepsilon$  and  $\mu$  are time-invariants,  $J(t)$  is a known 3D vector time function,  $E$  and  $H$  are 3D vectors in Cartesian coordinates  $(x, y, z)$ , representing an electric field and a magnetic field, respectively, as

$$E(x, y, z, t) = \begin{bmatrix} E_x(x, y, z, t) \\ E_y(x, y, z, t) \\ E_z(x, y, z, t) \end{bmatrix}, H(x, y, z, t) = \begin{bmatrix} H_x(x, y, z, t) \\ H_y(x, y, z, t) \\ H_z(x, y, z, t) \end{bmatrix}$$

For a 3D vector  $V$ , its cascade curls are denoted as following.

$\nabla^{(0)} \times V \equiv V, \nabla \times V \equiv \nabla^{(1)} \times V \equiv \begin{bmatrix} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \\ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{bmatrix}, \nabla^{(k)} \times V \equiv \underbrace{\nabla \times \nabla \times \dots \times \nabla \times}_k V, k > 0$	$(5)$
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To consider medium effects, add conductivity constants  $\sigma$  and  $\sigma_m$  into to (3) and (4). Suppose electric source and magnetic source are time dependent, denoted by  $J_e(t)$  and  $J_m(t)$ . We are dealing with the following Maxwell equations.

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon} \nabla \times H - \frac{\sigma}{\varepsilon} E - \frac{1}{\varepsilon} J_e(t) \quad (6)$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu} \nabla \times E - \frac{\sigma_m}{\mu} H - \frac{1}{\mu} J_m(t) \quad (7)$$

In this paper, the time space lemma proved in [1] is extended to support the above more generic Maxwell equations. Thus, the generic FDTD form derived in [1] is extended to support more generic scenarios.

## Time Space Relations

**Time Space Lemma.** The following time space relations can be derived from equations (6) and (7).

$\frac{\partial^{2k} E}{\partial t^{2k}} = p_{2k}^{\{2k\}} E + \sum_{i=1}^k (p_{2(k-i)}^{\{2k\}} \nabla^{[2i]} \times E + p_{2(k-i)+1}^{\{2k\}} \nabla^{[2i-1]} \times H) + S_e(2k)$	$(8)$
$\frac{\partial^{2k} H}{\partial t^{2k}} = q_{2k}^{\{2k\}} H + \sum_{i=1}^k (q_{2(k-i)}^{\{2k\}} \nabla^{[2i]} \times H + q_{2(k-i)+1}^{\{2k\}} \nabla^{[2i-1]} \times E) + S_m(2k)$	$(9)$

$\frac{\partial^{2k+1} E}{\partial t^{2k+1}} = \sum_{i=0}^k (p_{2i}^{\{2k+1\}} \nabla^{2(k-i)+1} \times H + p_{2i+1}^{\{2k+1\}} \nabla^{2(k-i)} \times E) + S_e(2k+1)$	(10)
$\frac{\partial^{2k+1} H}{\partial t^{2k+1}} = \sum_{i=0}^k (q_{2i}^{\{2k+1\}} \nabla^{2(k-i)+1} \times E + q_{2i+1}^{\{2k+1\}} \nabla^{2(k-i)} \times H) + S_m(2k+1)$	(11)

Where  $S_e$  and  $S_m$  are formed by field sources  $J_e$  and  $J_m$ , and given by

$S_e(2k) = \sum_{h=0}^{k-1} \left( \nabla^{2(k-1-h)} \times \sum_{i=1}^{2(h+1)} e_{i-1}^{\{2(k-1-h)\}} \frac{d^{2(h+1)-i} J_e}{dt^{2(h+1)-i}} + \nabla^{2(k-1-h)+1} \times \sum_{i=0}^{2h} m_i^{\{2(k-1-h)+1\}} \frac{d^{2h-i} J_m}{dt^{2h-i}} \right)$	(12)
$S_m(2k) = \sum_{h=0}^{k-1} \left( \nabla^{2(k-1-h)} \times \sum_{i=1}^{2(h+1)} m_{i-1}^{\{2(k-1-h)\}} \frac{d^{2(h+1)-i} J_m}{dt^{2(h+1)-i}} + \nabla^{2(k-1-h)+1} \times \sum_{i=0}^{2h} e_i^{\{2(k-1-h)+1\}} \frac{d^{2h-i} J_e}{dt^{2h-i}} \right)$	(13)
$S_e(2k+1) = \sum_{h=0}^k \nabla^{2(k-h)} \times \sum_{i=0}^{2h} e_i^{\{2(k-h)\}} \frac{d^{2h-i} J_e}{dt^{2h-i}} + \sum_{h=0}^{k-1} \nabla^{2(k-h)-1} \times \sum_{i=0}^{2h+1} m_i^{\{2(k-h)-1\}} \frac{d^{2h-i+1} J_m}{dt^{2h-i+1}}$	(14)
$S_m(2k+1) = \sum_{h=0}^k \nabla^{2(k-h)} \times \sum_{i=0}^{2h} m_i^{\{2(k-h)\}} \frac{d^{2h-i} J_m}{dt^{2h-i}} + \sum_{h=0}^{k-1} \nabla^{2(k-h)-1} \times \sum_{i=0}^{2h+1} e_i^{\{2(k-h)-1\}} \frac{d^{2h-i+1} J_e}{dt^{2h-i+1}}$	(15)

Where the coefficients are determined by material properties, and given by

$e_0^{\{0\}} = -\frac{1}{\varepsilon}, p_0^{\{1\}} = \frac{1}{\varepsilon}, p_1^{\{1\}} = -\frac{\sigma}{\varepsilon}$	(16)
$m_0^{\{0\}} = -\frac{1}{\mu}, q_0^{\{1\}} = -\frac{1}{\mu}, q_1^{\{1\}} = -\frac{\sigma_m}{\mu}$	(17)
$e_{2(k-i)}^{\{2i\}} = p_{2(k-i)}^{\{2k\}} e_0^{\{0\}}, m_{2(k-i)}^{\{2i\}} = q_{2(k-i)}^{\{2k\}} m_0^{\{0\}}, i = 0, 1, 2, \dots, k$	(18)
$m_{2(k-i)+1}^{\{2i-1\}} = p_{2(k-i)+1}^{\{2k\}} m_0^{\{0\}}, e_{2(k-i)+1}^{\{2i-1\}} = q_{2(k-i)+1}^{\{2k\}} e_0^{\{0\}}, i = 1, 2, \dots, k$	(19)
$p_0^{\{2k+1\}} = p_0^{\{1\}}, p_0^{\{2k\}}, p_{2k+1}^{\{2k+1\}} = p_{2k}^{\{2k\}} p_1^{\{1\}}$	(20)
$q_0^{\{2k+1\}} = q_0^{\{1\}}, q_0^{\{2k\}}, q_{2k+1}^{\{2k+1\}} = q_{2k}^{\{2k\}} q_1^{\{1\}}$	(21)
$p_{2i-1}^{\{2k+1\}} = p_1^{\{1\}} p_{2i-2}^{\{2k\}} + q_0^{\{1\}} p_{2i-1}^{\{2k\}}$	(22)
$p_{2i}^{\{2k+1\}} = q_1^{\{1\}} p_{2i-1}^{\{2k\}} + p_0^{\{1\}} p_{2i}^{\{2k\}}$	(23)
$q_{2i-1}^{\{2k+1\}} = q_1^{\{1\}} q_{2i-2}^{\{2k\}} + p_0^{\{1\}} q_{2i-1}^{\{2k\}}$	(24)
$q_{2i}^{\{2k+1\}} = p_1^{\{1\}} q_{2i-1}^{\{2k\}} + q_0^{\{1\}} q_{2i}^{\{2k\}}$	(25)
$i = 1, 2, \dots, k; k \geq 1$	
$e_{2(k-i)-1}^{\{2i\}} = p_{2(k-i)-1}^{\{2k-1\}} e_0^{\{0\}}, m_{2(k-i)-1}^{\{2i\}} = q_{2(k-i)-1}^{\{2k-1\}} m_0^{\{0\}}, i = 0, 1, 2, \dots, k-1$	(26)
$m_{2(k-i-1)}^{\{2i+1\}} = p_{2(k-i-1)}^{\{2k-1\}} m_0^{\{0\}}, e_{2(k-i-1)}^{\{2i+1\}} = q_{2(k-i-1)}^{\{2k-1\}} e_0^{\{0\}}, i = 0, 1, 2, \dots, k-1$	(27)
$p_0^{\{2k\}} = q_0^{\{1\}} p_0^{\{2k-1\}}, p_{2k}^{\{2k\}} = p_{2k-1}^{\{2k-1\}} p_1^{\{1\}}$	(28)
$q_0^{\{2k\}} = p_0^{\{1\}} q_0^{\{2k-1\}}, q_{2k}^{\{2k\}} = q_{2k-1}^{\{2k-1\}} q_1^{\{1\}}$	
$p_{2i+1}^{\{2k\}} = q_1^{\{1\}} p_{2i}^{\{2k-1\}} + p_0^{\{1\}} p_{2i+1}^{\{2k-1\}}, i = 0, 1, 2, \dots, k-1$	(29)
$p_{2i+2}^{\{2k\}} = p_1^{\{1\}} p_{2i+1}^{\{2k-1\}} + q_0^{\{1\}} p_{2i+2}^{\{2k-1\}}, i = 1, 2, \dots, k-1; k > 1$	
$q_{2i+1}^{\{2k\}} = p_1^{\{1\}} q_{2i}^{\{2k-1\}} + q_0^{\{1\}} q_{2i+1}^{\{2k-1\}}, i = 0, 1, \dots, k-1$	(30)
$q_{2i+2}^{\{2k\}} = q_1^{\{1\}} q_{2i+1}^{\{2k-1\}} + p_0^{\{1\}} q_{2i+2}^{\{2k-1\}}, i = 1, 2, \dots, k-1; k > 1$	

The proof of this lemma is given in the Appendix.

## Time Advancement Theorems

Applying the above lemma to the time advancement theorems in [1], we have following new theorems.

**Time Advancement Theorem H.** At a time  $t_h$ , suppose the fields and sources are  $E = E(t_h)$ ,  $H = H(t_h)$ ,  $J_e = J_e(t_h)$ , and  $J_m = J_m(t_h)$ , then for any time deviation  $\Delta_t$ ,  $H(t_h + \Delta_t)$  can be expressed by the following formula.

$H(t_h + \Delta_t) = C_0^h(\Delta_t) H + \sum_{k=0}^{\infty} \left( S^h(k, \Delta_t) + \left( \sum_{i=0}^k (C_e^h(k, i, \Delta_t) \nabla^{2i+1} \times E + C_h^h(k, i, \Delta_t) \nabla^{2(i+1)} \times H) \right) \right)$	(31)
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Where the coefficients are given by

$$\begin{aligned}
C_0^h(\Delta_t) &= 1 + \sum_{k=0}^{\infty} \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} q_{2k+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k+1)}^{\{2(k+1)\}} \right) \\
C_h^h(k, i, \Delta_t) &= \begin{cases} \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_0^{\{2(k+1)\}}, i = k \\ \frac{\Delta_t^{2k+1}}{(2k+1)!} q_{2(k-1-i)+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k-i)}^{\{2(k+1)\}}, 0 \leq i < k \end{cases} \\
C_e^h(k, i, \Delta_t) &= \frac{\Delta_t^{2k+1}}{(2k+1)!} q_{2(k-i)}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k-i)+1}^{\{2(k+1)\}} \\
S^h(k, \Delta_t) &= \frac{\Delta_t^{2k+1}}{(2k+1)!} S_m(2k+1) + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} S_m(2(k+1))
\end{aligned}$$

Where the related values are defined by (12) – (30).

Proof.

Apply the Taylor series, we have

$$H(t_h + \Delta_t) = \sum_{k=0}^{\infty} \frac{\Delta_t^k}{k!} \frac{\partial^k H(t_h)}{\partial t^k}$$

Rearrange the terms, we have

$H(t_h + \Delta_t) = H(t_h) + \sum_{k=0}^{\infty} \frac{\Delta_t^{2k+1}}{(2k+1)!} \frac{\partial^{2k+1} H(t_h)}{\partial t^{2k+1}} + \sum_{k=0}^{\infty} \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} \frac{\partial^{2(k+1)} H(t_h)}{\partial t^{2(k+1)}}$	(32)
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Substitute (9) and (11) into (32), we get

$$\begin{aligned}
H(t_h + \Delta_t) &= H(t_h) + \sum_{k=0}^{\infty} \frac{\Delta_t^{2k+1}}{(2k+1)!} \left( \sum_{i=0}^k (q_{2i}^{\{2k+1\}} \nabla^{\{2(k-i)+1\}} \times E + q_{2i+1}^{\{2k+1\}} \nabla^{\{2(k-i)\}} \times H) + S_m(2k+1) \right) \\
&\quad + \sum_{k=0}^{\infty} \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} \left( q_{2(k+1)}^{\{2(k+1)\}} H + \sum_{i=1}^{k+1} (q_{2(k+1-i)}^{\{2(k+1)\}} \nabla^{\{2i\}} \times H + q_{2(k+1-i)+1}^{\{2(k+1)\}} \nabla^{\{2i-1\}} \times E) + S_m(2(k+1)) \right) \\
&= H(t_h) + \sum_{k=0}^{\infty} \frac{\Delta_t^{2k+1}}{(2k+1)!} \left( \sum_{i=0}^k (q_{2(k-i)}^{\{2k+1\}} \nabla^{\{2i+1\}} \times E + q_{2(k-i)+1}^{\{2k+1\}} \nabla^{\{2i\}} \times H) + S_m(2k+1) \right) \\
&\quad + \sum_{k=0}^{\infty} \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} \left( q_{2(k+1)}^{\{2(k+1)\}} H + \sum_{i=1}^{k+1} (q_{2(k+1-i)}^{\{2(k+1)\}} \nabla^{\{2i\}} \times H + q_{2(k+1-i)+1}^{\{2(k+1)\}} \nabla^{\{2i-1\}} \times E) + S_m(2(k+1)) \right) \\
&= H(t_h) + \sum_{k=0}^{\infty} \left( \sum_{i=0}^k \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} q_{2(k-i)}^{\{2k+1\}} \nabla^{\{2i+1\}} \times E + \frac{\Delta_t^{2k+1}}{(2k+1)!} q_{2(k-i)+1}^{\{2k+1\}} \nabla^{\{2i\}} \times H \right) + \frac{\Delta_t^{2k+1}}{(2k+1)!} S_m(2k+1) \right) \\
&\quad + \sum_{k=0}^{\infty} \left( \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k+1)}^{\{2(k+1)\}} H + \sum_{i=1}^{k+1} \left( \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k+1-i)}^{\{2(k+1)\}} \nabla^{\{2i\}} \times H + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k+1-i)+1}^{\{2(k+1)\}} \nabla^{\{2i-1\}} \times E \right) \right. \\
&\quad \left. + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} S_m(2(k+1)) \right)
\end{aligned}$$

$$\begin{aligned}
&= H(t_h) + \sum_{k=0}^{\infty} \left( \sum_{i=0}^k \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} q_{2(k-i)}^{\{2k+1\}} \nabla^{\{2i+1\}} \times E + \frac{\Delta_t^{2k+1}}{(2k+1)!} q_{2(k-i)+1}^{\{2k+1\}} \nabla^{\{2i\}} \times H \right) + \frac{\Delta_t^{2k+1}}{(2k+1)!} S_m(2k+1) \right) \\
&\quad + \sum_{k=0}^{\infty} \left( \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k+1)}^{\{2(k+1)\}} H + \sum_{i=1}^{k+1} \left( \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k+1-i)}^{\{2(k+1)\}} \nabla^{\{2i\}} \times H \right) + \sum_{i=1}^{k+1} \left( \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k+1-i)+1}^{\{2(k+1)\}} \nabla^{\{2i-1\}} \times E \right) \right. \\
&\quad \left. + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} S_m(2(k+1)) \right) \\
&= H(t_h) + \sum_{k=0}^{\infty} \left( \sum_{i=0}^k \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} q_{2(k-i)}^{\{2k+1\}} \nabla^{\{2i+1\}} \times E + \frac{\Delta_t^{2k+1}}{(2k+1)!} q_{2(k-i)+1}^{\{2k+1\}} \nabla^{\{2i\}} \times H \right) + \frac{\Delta_t^{2k+1}}{(2k+1)!} S_m(2k+1) \right) \\
&\quad + \sum_{k=0}^{\infty} \left( \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_0^{\{2(k+1)\}} \nabla^{\{2(k+1)\}} \times H + \sum_{i=0}^k \left( \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k+1-i)}^{\{2(k+1)\}} \nabla^{\{2i\}} \times H \right) \right. \\
&\quad \left. + \sum_{i=0}^k \left( \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k-i)+1}^{\{2(k+1)\}} \nabla^{\{2(i+1)\}} \times E \right) + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} S_m(2(k+1)) \right) \\
&= H(t_h) + \sum_{k=0}^{\infty} \left( \sum_{i=0}^k \left( \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} q_{2(k-i)}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k-i)+1}^{\{2(k+1)\}} \right) \nabla^{\{2i+1\}} \times E + \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} q_{2(k-i)+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k+1-i)}^{\{2(k+1)\}} \right) \nabla^{\{2i\}} \times H \right) \right. \\
&\quad \left. + \frac{\Delta_t^{2k+1}}{(2k+1)!} S_m(2k+1) + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} S_m(2(k+1)) \right) + \sum_{k=0}^{\infty} \left( \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_0^{\{2(k+1)\}} \nabla^{\{2(k+1)\}} \times H \right) \\
&= H(t_h) + \sum_{k=0}^{\infty} \left( \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_0^{\{2(k+1)\}} \nabla^{\{2(k+1)\}} \times H + \sum_{i=0}^k \left( \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} q_{2(k-i)}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k-i)+1}^{\{2(k+1)\}} \right) \nabla^{\{2i+1\}} \times E \right) \right. \\
&\quad \left. + \sum_{i=0}^k \left( \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} q_{2(k-i)+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k+1-i)}^{\{2(k+1)\}} \right) \nabla^{\{2i\}} \times H \right) + \frac{\Delta_t^{2k+1}}{(2k+1)!} S_m(2k+1) + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} S_m(2(k+1)) \right) \\
&= H(t_h) + \sum_{k=0}^{\infty} \left( \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_0^{\{2(k+1)\}} \nabla^{\{2(k+1)\}} \times H + \sum_{i=0}^k \left( \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} q_{2(k-i)}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k-i)+1}^{\{2(k+1)\}} \right) \nabla^{\{2i+1\}} \times E \right) \right. \\
&\quad + \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} q_{2k+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k+1)}^{\{2(k+1)\}} \right) H + \sum_{i=1}^k \left( \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} q_{2(k-i)+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k+1-i)}^{\{2(k+1)\}} \right) \nabla^{\{2i\}} \times H \right) \\
&\quad \left. + \frac{\Delta_t^{2k+1}}{(2k+1)!} S_m(2k+1) + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} S_m(2(k+1)) \right) \\
&= \left( 1 + \sum_{k=0}^{\infty} \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} q_{2k+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k+1)}^{\{2(k+1)\}} \right) \right) H \\
&\quad + \sum_{k=0}^{\infty} \left( \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_0^{\{2(k+1)\}} \nabla^{\{2(k+1)\}} \times H + \sum_{i=0}^k \left( \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} q_{2(k-i)}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k-i)+1}^{\{2(k+1)\}} \right) \nabla^{\{2i+1\}} \times E \right) \right. \\
&\quad + \sum_{i=0}^{k-1} \left( \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} q_{2(k-1-i)+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k-i)}^{\{2(k+1)\}} \right) \nabla^{\{2(i+1)\}} \times H \right) + \frac{\Delta_t^{2k+1}}{(2k+1)!} S_m(2k+1) \\
&\quad \left. + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} S_m(2(k+1)) \right)
\end{aligned}$$

$$= C_0^h(\Delta_t)H + \sum_{k=0}^{\infty} \left( S^h(k, \Delta_t) + \left( \sum_{i=0}^k (C_e^h(k, i, \Delta_t) \nabla^{\{2i+1\}} \times E + C_h^h(k, i, \Delta_t) \nabla^{\{2(i+1)\}} \times H) \right) \right)$$

QED.

**Time Advancement Theorem E.** At a time  $t_e$ , suppose the fields and sources are  $E = E(t_e)$ ,  $H = H(t_e)$ ,  $J_e = J_e(t_e)$ , and  $J_m = J_m(t_e)$ , then for any time deviation  $\Delta_t$ ,  $E(t_e + \Delta_t)$  can be expressed by the following formula.

$E(t_e + \Delta_t) = C_0^e(\Delta_t)E + \sum_{k=0}^{\infty} \left( S^e(k, \Delta_t) + \sum_{i=0}^k (C_e^e(k, i, \Delta_t) \nabla^{\{2i+1\}} \times H + C_h^e(k, i, \Delta_t) \nabla^{\{2(i+1)\}} \times E) \right)$	(33)
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Where the coefficients are given by

$$C_0^e(\Delta_t) = 1 + \sum_{k=0}^{\infty} \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2k+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k+1)}^{\{2(k+1)\}} \right)$$

$$C_e^e(k, i, \Delta_t) = \begin{cases} \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_0^{\{2(k+1)\}}, i = k \\ \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2(k-i-1)+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k-i)}^{\{2(k+1)\}} \right), 0 \leq i < k \end{cases}$$

$$C_h^e(k, i, \Delta_t) = \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2(k-i)}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k-i)+1}^{\{2(k+1)\}}$$

$$S^e(k, \Delta_t) = \frac{\Delta_t^{2k+1}}{(2k+1)!} S_e(2k+1) + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} S_e(2(k+1))$$

Where the related values are defined by (12) – (30).

Proof.

Apply the Taylor series, we have

$$E(t_e + \Delta_t) = \sum_{k=0}^{\infty} \frac{\Delta_t^k}{k!} \frac{\partial^k E(t_e)}{\partial t^k}$$

Rearrange the terms, we have

$E(t_e + \Delta_t) = E + \sum_{k=0}^{\infty} \frac{\Delta_t^{2k+1}}{(2k+1)!} \frac{\partial^{2k+1} E}{\partial t^{2k+1}} + \sum_{k=0}^{\infty} \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} \frac{\partial^{2(k+1)} E}{\partial t^{2(k+1)}}$	(34)
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Substitute (8) and (10) into (34), we get

$$E(t_e + \Delta_t) = E(t_e) + \sum_{k=0}^{\infty} \frac{\Delta_t^{2k+1}}{(2k+1)!} \left( \sum_{i=0}^k (p_{2i}^{\{2k+1\}} \nabla^{\{2(k-i)+1\}} \times H + p_{2i+1}^{\{2k+1\}} \nabla^{\{2(k-i)\}} \times E) + S_e(2k+1) \right) \\ + \sum_{k=0}^{\infty} \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} \left( p_{2(k+1)}^{\{2(k+1)\}} E + \sum_{i=1}^{k+1} (p_{2(k+1-i)}^{\{2(k+1)\}} \nabla^{\{2i\}} \times E + p_{2(k+1-i)+1}^{\{2(k+1)\}} \nabla^{\{2i-1\}} \times H) + S_e(2(k+1)) \right)$$

$$\begin{aligned}
&= E(t_e) + \sum_{k=0}^{\infty} \frac{\Delta_t^{2k+1}}{(2k+1)!} \left( \sum_{i=0}^k (p_{2(k-i)}^{\{2k+1\}} \nabla^{\{2i+1\}} \times H + p_{2(k-i)+1}^{\{2k+1\}} \nabla^{\{2i\}} \times E) + S_e(2k+1) \right) \\
&\quad + \sum_{k=0}^{\infty} \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} \left( p_{2(k+1)}^{\{2(k+1)\}} E + \sum_{i=1}^{k+1} (p_{2(k+1-i)}^{\{2(k+1)\}} \nabla^{\{2i\}} \times E + p_{2(k+1-i)+1}^{\{2(k+1)\}} \nabla^{\{2i-1\}} \times H) + S_e(2(k+1)) \right) \\
&= E(t_e) + \sum_{k=0}^{\infty} \frac{\Delta_t^{2k+1}}{(2k+1)!} \left( \sum_{i=0}^k (p_{2(k-i)}^{\{2k+1\}} \nabla^{\{2i+1\}} \times H + p_{2(k-i)+1}^{\{2k+1\}} \nabla^{\{2i\}} \times E) + S_e(2k+1) \right) \\
&\quad + \sum_{k=0}^{\infty} \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} \left( \sum_{i=0}^{k+1} (p_{2(k+1-i)}^{\{2(k+1)\}} \nabla^{\{2i\}} \times E) + \sum_{i=1}^{k+1} (p_{2(k+1-i)+1}^{\{2(k+1)\}} \nabla^{\{2i-1\}} \times H) + S_e(2(k+1)) \right) \\
&= E(t_e) + \sum_{k=0}^{\infty} \frac{\Delta_t^{2k+1}}{(2k+1)!} \left( \sum_{i=0}^k (p_{2(k-i)}^{\{2k+1\}} \nabla^{\{2i+1\}} \times H + p_{2(k-i)+1}^{\{2k+1\}} \nabla^{\{2i\}} \times E) + S_e(2k+1) \right) \\
&\quad + \sum_{k=0}^{\infty} \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} \left( p_{20}^{\{2(k+1)\}} \nabla^{\{2(k+1)\}} \times E + \sum_{i=0}^k (p_{2(k+1-i)}^{\{2(k+1)\}} \nabla^{\{2i\}} \times E) + \sum_{i=1}^{k+1} (p_{2(k+1-i)+1}^{\{2(k+1)\}} \nabla^{\{2i-1\}} \times H) + S_e(2(k+1)) \right) \\
&= E(t_e) + \sum_{k=0}^{\infty} \left( \sum_{i=0}^k \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2(k-i)}^{\{2k+1\}} \nabla^{\{2i+1\}} \times H + \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2(k-i)+1}^{\{2k+1\}} \nabla^{\{2i\}} \times E \right) + \frac{\Delta_t^{2k+1}}{(2k+1)!} S_e(2k+1) \right) \\
&\quad + \sum_{k=0}^{\infty} \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} \left( p_{20}^{\{2(k+1)\}} \nabla^{\{2(k+1)\}} \times E + \sum_{i=0}^k (p_{2(k+1-i)}^{\{2(k+1)\}} \nabla^{\{2i\}} \times E) + \sum_{i=1}^{k+1} (p_{2(k+1-i)+1}^{\{2(k+1)\}} \nabla^{\{2i-1\}} \times H) + S_e(2(k+1)) \right) \\
&= E(t_e) + \sum_{k=0}^{\infty} \left( \sum_{i=0}^k \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2(k-i)}^{\{2k+1\}} \nabla^{\{2i+1\}} \times H + \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2(k-i)+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k+1-i)}^{\{2(k+1)\}} \right) \nabla^{\{2i\}} \times E \right) + \frac{\Delta_t^{2k+1}}{(2k+1)!} S_e(2k+1) \right) \\
&\quad + \sum_{k=0}^{\infty} \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} \left( p_{20}^{\{2(k+1)\}} \nabla^{\{2(k+1)\}} \times E + \sum_{i=0}^k (p_{2(k-i)+1}^{\{2(k+1)\}} \nabla^{\{2i+1\}} \times H + S_e(2(k+1))) \right) \\
&= E(t_e) + \sum_{k=0}^{\infty} \left( \sum_{i=0}^k \left( \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2(k-i)}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k-i)+1}^{\{2(k+1)\}} \right) \nabla^{\{2i+1\}} \times H + \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2(k-i)+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k+1-i)}^{\{2(k+1)\}} \right) \nabla^{\{2i\}} \times E \right) \right. \\
&\quad \left. + \frac{\Delta_t^{2k+1}}{(2k+1)!} S_e(2k+1) \right) + \sum_{k=0}^{\infty} \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} \left( p_{20}^{\{2(k+1)\}} \nabla^{\{2(k+1)\}} \times E + S_e(2(k+1)) \right) \\
&= E(t_e) + \sum_{k=0}^{\infty} \left( \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{20}^{\{2(k+1)\}} \nabla^{\{2(k+1)\}} \times E \right. \\
&\quad + \sum_{i=0}^k \left( \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2(k-i)}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k-i)+1}^{\{2(k+1)\}} \right) \nabla^{\{2i+1\}} \times H + \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2(k-i)+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k+1-i)}^{\{2(k+1)\}} \right) \nabla^{\{2i\}} \right. \\
&\quad \left. \times E \right) + \frac{\Delta_t^{2k+1}}{(2k+1)!} S_e(2k+1) + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} S_e(2(k+1)) \Bigg) \\
&= E(t_e) + \sum_{k=0}^{\infty} \left( \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{20}^{\{2(k+1)\}} \nabla^{\{2(k+1)\}} \times E + \sum_{i=0}^k \left( \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2(k-i)}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k+1-i)}^{\{2(k+1)\}} \right) \nabla^{\{2i\}} \times E \right) \right. \\
&\quad + \sum_{i=0}^k \left( \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2(k-i)}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k-i)+1}^{\{2(k+1)\}} \right) \nabla^{\{2i+1\}} \times H \right) + \frac{\Delta_t^{2k+1}}{(2k+1)!} S_e(2k+1) \\
&\quad \left. + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} S_e(2(k+1)) \right)
\end{aligned}$$

$$\begin{aligned}
&= E(t_e) + \sum_{k=0}^{\infty} \left( \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{20}^{\{2(k+1)\}} \nabla^{\{2(k+1)\}} \times E + \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2k+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k+1)}^{\{2(k+1)\}} \right) E \right. \\
&\quad + \sum_{i=1}^k \left( \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2(k-i)+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k+1-i)}^{\{2(k+1)\}} \right) \nabla^{\{2i\}} \times E \right) \\
&\quad + \sum_{i=0}^k \left( \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2(k-i)}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k-i)+1}^{\{2(k+1)\}} \right) \nabla^{\{2i+1\}} \times H \right) + \frac{\Delta_t^{2k+1}}{(2k+1)!} S_e(2k+1) \\
&\quad \left. + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} S_e(2(k+1)) \right) \\
&= \left( 1 + \sum_{k=0}^{\infty} \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2k+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k+1)}^{\{2(k+1)\}} \right) \right) E \\
&\quad + \sum_{k=0}^{\infty} \left( \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{20}^{\{2(k+1)\}} \nabla^{\{2(k+1)\}} \times E + \sum_{i=1}^k \left( \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2(k-i)+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k+1-i)}^{\{2(k+1)\}} \right) \nabla^{\{2i\}} \times E \right) \right. \\
&\quad + \sum_{i=0}^k \left( \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2(k-i)}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k-i)+1}^{\{2(k+1)\}} \right) \nabla^{\{2i+1\}} \times H \right) + \frac{\Delta_t^{2k+1}}{(2k+1)!} S_e(2k+1) \\
&\quad \left. + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} S_e(2(k+1)) \right) \\
&= \left( 1 + \sum_{k=0}^{\infty} \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2k+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k+1)}^{\{2(k+1)\}} \right) \right) E \\
&\quad + \sum_{k=0}^{\infty} \left( \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{20}^{\{2(k+1)\}} \nabla^{\{2(k+1)\}} \times E + \sum_{i=0}^{k-1} \left( \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2(k-i-1)+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k-i)}^{\{2(k+1)\}} \right) \nabla^{\{2(i+1)\}} \times E \right) \right. \\
&\quad + \sum_{i=0}^k \left( \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2(k-i)}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k-i)+1}^{\{2(k+1)\}} \right) \nabla^{\{2i+1\}} \times H \right) + \frac{\Delta_t^{2k+1}}{(2k+1)!} S_e(2k+1) \\
&\quad \left. + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} S_e(2(k+1)) \right) \\
&= C_0^e E + \sum_{k=0}^{\infty} \left( S^e(k) + \sum_{i=0}^k (C_h^e(k, i) \nabla^{\{2i+1\}} \times H + C_e^e(k, i) \nabla^{\{2(i+1)\}} \times E) \right)
\end{aligned}$$

QED.

## Generic FDTD Form

In [1], I have proved a curl cascade theorem for providing high order FDTD algorithms. The theorem can be used in this paper without modifications. But I would like to point out that the space samplings can be different for different space derivative estimators. I'll describe it below.

In [1], the space derivative estimators are given by

$D_x^h(V_u) = \frac{\partial^h V_u}{\partial x^h} \approx \mathcal{D}_x^h(V_u) \equiv [a_{h1} \quad a_{h2} \quad \dots \quad a_{hM}] \begin{bmatrix} V_u(x + \Delta_1, y, z) - V_u(x, y, z) \\ V_u(x + \Delta_2, y, z) - V_u(x, y, z) \\ \vdots \\ V_u(x + \Delta_M, y, z) - V_u(x, y, z) \end{bmatrix}$	<b>(35)</b>
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$D_y^h(V_u) = \frac{\partial^h V_u}{\partial y^h} \approx \mathfrak{D}_y^h(V_u) \equiv [a_{h1} \quad a_{h2} \quad \dots \quad a_{hM}] \begin{bmatrix} V_u(x, y + \Delta_1, z) - V_u(x, y, z) \\ V_u(x, y + \Delta_2, z) - V_u(x, y, z) \\ \vdots \\ V_u(x, y + \Delta_M, z) - V_u(x, y, z) \end{bmatrix}$ $D_z^h(V_u) = \frac{\partial^h V_u}{\partial z^h} \approx \mathfrak{D}_z^h(V_u) \equiv [a_{h1} \quad a_{h2} \quad \dots \quad a_{hM}] \begin{bmatrix} V_u(x, y, z + \Delta_1) - V_u(x, y, z) \\ V_u(x, y, z + \Delta_2) - V_u(x, y, z) \\ \vdots \\ V_u(x, y, z + \Delta_M) - V_u(x, y, z) \end{bmatrix}$ <p style="text-align: center;"><i>u can be x, y, z; h = 1, 2, ..., M</i></p>	
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Where the parameters are given by

$\begin{bmatrix} \Delta_1 & \frac{1}{2!} \Delta_1^2 & \dots & \frac{1}{M!} \Delta_1^M \\ \Delta_2 & \frac{1}{2!} \Delta_2^2 & \dots & \frac{1}{M!} \Delta_2^M \\ \vdots & \vdots & \dots & \vdots \\ \Delta_M & \frac{1}{2!} \Delta_M^2 & \dots & \frac{1}{M!} \Delta_M^M \end{bmatrix}^{-1} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1M} \\ a_{21} & a_{22} & \dots & a_{2M} \\ \vdots & \vdots & \dots & \vdots \\ a_{M1} & a_{M2} & \dots & a_{MM} \end{bmatrix}$	(36)
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The above formulations gave an impression that space sampling  $\Delta_1, \Delta_2, \dots, \Delta_M$  are the same for the 3 axis. But it is not necessarily so. Let's change the formulations to show different samplings.

$\begin{bmatrix} \Delta_{u1} & \frac{1}{2!} \Delta_{u1}^2 & \dots & \frac{1}{M!} \Delta_{u1}^M \\ \Delta_{u2} & \frac{1}{2!} \Delta_{u2}^2 & \dots & \frac{1}{M!} \Delta_{u2}^M \\ \vdots & \vdots & \dots & \vdots \\ \Delta_{uM} & \frac{1}{2!} \Delta_{uM}^2 & \dots & \frac{1}{M!} \Delta_{uM}^M \end{bmatrix}^{-1} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1M} \\ a_{21} & a_{22} & \dots & a_{2M} \\ \vdots & \vdots & \dots & \vdots \\ a_{M1} & a_{M2} & \dots & a_{MM} \end{bmatrix}_u = \begin{bmatrix} [a_{11} & a_{12} & \dots & a_{1M}]_u \\ [a_{21} & a_{22} & \dots & a_{2M}]_u \\ \vdots \\ [a_{M1} & a_{M2} & \dots & a_{MM}]_u \end{bmatrix}$	(37)
$D_x^h(V_u) = \frac{\partial^h V_u}{\partial x^h} \approx \mathfrak{D}_x^h(V_u) \equiv [a_{h1} \quad a_{h2} \quad \dots \quad a_{hM}]_x \begin{bmatrix} V_u(x + \Delta_{x1}, y, z) - V_u(x, y, z) \\ V_u(x + \Delta_{x2}, y, z) - V_u(x, y, z) \\ \vdots \\ V_u(x + \Delta_{xM}, y, z) - V_u(x, y, z) \end{bmatrix}$ $D_y^h(V_u) = \frac{\partial^h V_u}{\partial y^h} \approx \mathfrak{D}_y^h(V_u) \equiv [a_{h1} \quad a_{h2} \quad \dots \quad a_{hM}]_y \begin{bmatrix} V_u(x, y + \Delta_{y1}, z) - V_u(x, y, z) \\ V_u(x, y + \Delta_{y2}, z) - V_u(x, y, z) \\ \vdots \\ V_u(x, y + \Delta_{yM}, z) - V_u(x, y, z) \end{bmatrix}$ $D_z^h(V_u) = \frac{\partial^h V_u}{\partial z^h} \approx \mathfrak{D}_z^h(V_u) \equiv [a_{h1} \quad a_{h2} \quad \dots \quad a_{hM}]_z \begin{bmatrix} V_u(x, y, z + \Delta_{z1}) - V_u(x, y, z) \\ V_u(x, y, z + \Delta_{z2}) - V_u(x, y, z) \\ \vdots \\ V_u(x, y, z + \Delta_{zM}) - V_u(x, y, z) \end{bmatrix}$ <p style="text-align: center;"><i>u can be x, y, z; h = 1, 2, ..., M</i></p>	(38)

With the above space derivative estimators, we can get space curl estimators:

$\begin{aligned} \overline{\nabla}^{(k)} \times E &\approx \nabla^{(k)} \times E \\ \overline{\nabla}^{(k)} \times H &\approx \nabla^{(k)} \times H \\ k &= 1, 2, \dots \end{aligned}$	(39)
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For details, see [1].

Substitute (39) into (31) and (33), we get the following generic FDTD forms.

**Generic FDTD for magnetic field:**

$H(t_h + \Delta_t) \approx \overline{C}_0^h(\Delta_t) H(t_h) + \sum_{i=0}^{k_{\max}} (S^h(i, \Delta_t) + G_e^h(i, \Delta_t) \overline{\nabla}^{(2i+1)} \times E + G_h^h(i, \Delta_t) \overline{\nabla}^{(2(i+1))} \times H)(t_h)$	(40)
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Where

$$\overline{C}_0^h(\Delta_t) = 1 + \sum_{k=0}^{k_{\max}} \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} q_{2k+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} q_{2(k+1)}^{\{2(k+1)\}} \right)$$

$$G_e^h(i, \Delta_t) = \sum_{k=i}^{k_{max}} C_e^h(k, i, \Delta_t)$$

$$G_h^h(i, \Delta_t) = \sum_{k=i}^{k_{max}} C_h^h(k, i, \Delta_t)$$

$$k_{max} \geq 0$$

**Generic FDTD for electric field:**

$E(t_e + \Delta_t) = \overline{C}_0^e(\Delta_t)E(t_e) + \sum_{i=0}^{k_{max}} (S^e(i, \Delta_t) + G_h^e(i, \Delta_t)\overline{V}^{\{2i+1\}} \times H + G_e^e(i, \Delta_t)\overline{V}^{\{2(i+1)\}} \times E)(t_e)$	(41)
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Where

$$\overline{C}_0^e(\Delta_t) = 1 + \sum_{k=0}^{k_{max}} \left( \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2k+1}^{\{2k+1\}} + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k+1)}^{\{2(k+1)\}} \right)$$

$$G_h^e(i, \Delta_t) = \sum_{k=i}^{k_{max}} C_h^e(k, i, \Delta_t)$$

$$G_e^e(i, \Delta_t) = \sum_{k=i}^{k_{max}} C_e^e(k, i, \Delta_t)$$

$$k_{max} \geq 0$$

## Deriving of FDTD Algorithms

Any Taylor's series based FDTD algorithms can be derived from the above generic FDTD forms. Below I'll derive some algorithms. We can see that each algorithm has its unique characteristics and conditions.

### The Time-Space Synchronized FDTD (TSS)

Assume  $E_x, E_y, E_z, H_x, H_y$ , and  $H_z$  are all at the same space location  $(x, y, z)$ , and let

$$t_h = t_e$$

Then (40) and (41) give  $H(t_h + q\Delta_t), E(t_h + q\Delta_t), q = 1, 2, 3, \dots$

Such an algorithm allows us to calculate Poynting vector to study field energy transfer:

$$S = E \times H$$

It also allows us to calculate divergence as estimation errors:

$error_h = \mathfrak{D}_x(H_x) + \mathfrak{D}_y(H_y) + \mathfrak{D}_z(H_z)$	(a.1)
$error_e = \mathfrak{D}_x(E_x) + \mathfrak{D}_y(E_y) + \mathfrak{D}_z(E_z) - \rho/\epsilon$	(a.2)

These errors may give precise comparisons of different algorithms' accuracy. That is, for two algorithms, we can precisely say which one is more accurate.

### Time-Shifted Space-synchronized FDTD (TS-SS)

This is a half-Yee algorithm. The Yee's algorithm is both time-shifted and space-shifted. This algorithm only makes time shifted but keeps the space points synchronized.

By (40), we have

$H(t_h + \Delta_t) \approx H(t_h - \Delta_t) + T_0^h(\Delta_t)H(t_h) + \sum_{i=0}^{k_{max}} \left( \frac{2\Delta_t^{2i+1}}{(2i+1)!} S_m(2i+1) + T_e^h(i, \Delta_t) \bar{V}^{[2i+1]} \times E + T_h^h(i, \Delta_t) \bar{V}^{[2(i+1)]} \times H \right) (t_h)$	(a.3)
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Where

$$T_0^h(\Delta_t) = \bar{C}_0^h(\Delta_t) - \bar{C}_0^h(-\Delta_t) = \sum_{k=0}^{k_{max}} \left( \frac{2\Delta_t^{2k+1}}{(2k+1)!} q_{2k+1}^{\{2k+1\}} \right)$$

$$T_e^h(i, \Delta_t) = G_e^h(i, \Delta_t) - G_e^h(i, -\Delta_t) = \sum_{k=i}^{k_{max}} \frac{2\Delta_t^{2k+1}}{(2k+1)!} q_{2(k-i)}^{\{2k+1\}}$$

$$T_h^h(i, \Delta_t) = G_h^h(i, \Delta_t) - G_h^h(i, -\Delta_t) = \sum_{k=i+1}^{k_{max}} \frac{2\Delta_t^{2k+1}}{(2k+1)!} q_{2(k-i-1)+1}^{\{2k+1\}}$$

By (41), we have

$E(t_e + \Delta_t) \approx E(t_e - \Delta_t) + T_0^e(\Delta_t)E(t_e) + \sum_{i=0}^{k_{max}} \left( \frac{2\Delta_t^{2i+1}}{(2i+1)!} S_e(2i+1) + T_h^e(i, \Delta_t) \bar{V}^{[2i+1]} \times H + T_e^e(i, \Delta_t) \bar{V}^{[2(i+1)]} \times E \right) (t_e)$	(a.4)
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Where

$$T_0^e(\Delta_t) = \bar{C}_0^e(\Delta_t) - \bar{C}_0^e(-\Delta_t) = \sum_{k=0}^{k_{max}} \frac{\Delta_t^{2k+1}}{(2k+1)!} p_{2k+1}^{\{2k+1\}}$$

$$T_h^e(i, \Delta_t) = G_h^e(i, \Delta_t) - G_h^e(i, -\Delta_t) = \sum_{k=i}^{k_{max}} \frac{2\Delta_t^{2k+1}}{(2k+1)!} p_{2(k-i)}^{\{2k+1\}}$$

$$T_e^e(i, \Delta_t) = G_e^e(i, \Delta_t) - G_e^e(i, -\Delta_t) = \sum_{k=i+1}^{k_{max}} \frac{2\Delta_t^{2k+1}}{(2k+1)!} p_{2(k-i-1)+1}^{\{2k+1\}}$$

### Lossless Medium

For lossless medium,

$$\sigma = \sigma_m = 0 \rightarrow q_{2i+1}^{\{2k+1\}} = p_{2i+1}^{\{2k+1\}} = 0 \rightarrow T_0^e = T_0^h = T_e^e = T_h^h = 0$$

(a.3) and (a.4) become

$H(t_h + \Delta_t) \approx H(t_h - \Delta_t) + \sum_{i=0}^{k_{max}} \left( \frac{2\Delta_t^{2i+1}}{(2i+1)!} S_m(2i+1) + T_e^h(i, \Delta_t) \bar{V}^{[2i+1]} \times E \right) (t_h)$	
$E(t_e + \Delta_t) \approx E(t_e - \Delta_t) + \sum_{i=0}^{k_{max}} \left( \frac{2\Delta_t^{2i+1}}{(2i+1)!} S_e(2i+1) + T_h^e(i, \Delta_t) \bar{V}^{[2i+1]} \times H \right) (t_e)$	

Assume  $E_x, E_y, E_z, H_x, H_y$ , and  $H_z$  are all at the same space location  $(x, y, z)$ , and let

$$t_h = t_0 + (2q+1)\Delta_t$$

$$t_e = t_0 + (2q+2)\Delta_t$$

We have

$H(t_0 + (2q+2)\Delta_t) \approx H(t_0 + 2q\Delta_t) + \sum_{i=0}^{k_{max}} \left( \frac{2\Delta_t^{2i+1}}{(2i+1)!} S_m(2i+1) + T_e^h(i, \Delta_t) \bar{V}^{[2i+1]} \times E \right) (t_0 + (2q+1)\Delta_t)$	(a.5)
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$E(t_0 + (2q + 3)\Delta_t) \approx E(t_0 + (2q + 1)\Delta_t) + \sum_{i=0}^{k_{max}} \left( \frac{2\Delta_t^{2i+1}}{(2i+1)!} S_e(2i+1) + T_h^e(i, \Delta_t) \bar{V}^{[2i+1]} \times H \right) (t_0 + (2q + 2)\Delta_t)$	(a.6)
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(a.5) and (a.6) give  $H(t_0 + (2q + 2)\Delta_t), E(t_0 + (2q + 3)\Delta_t), q = 0, 1, 2, 3, \dots$

The advantage of this algorithm is that it uses a center-difference to increase the time advancement estimation order, which is the major benefit of the Yee's algorithm. But the Yee's algorithm is limited to the second order time advancement estimation, which is the major limitation of the Yee's algorithm. The time advancement estimation order of this algorithm is not limited, thus, it removes the major limitation of the Yee's algorithm while keeping the major benefit of the Yee's algorithm.

The limitation of this algorithm is that it only works for lossless medium. Note that this is also a limitation of the Yee's algorithm if the second order accuracy has to be kept. Therefore, this conclusion is still true that this algorithm keeps the major benefit of the Yee's algorithm and removes the major limitation of the Yee's algorithm.

### Lossy Medium

Let

$$k_{max} = 0$$

(a.3) and (a.4) become

$H\left(t_h + \frac{1}{2}\Delta_t\right) - H\left(t_h - \frac{1}{2}\Delta_t\right) \approx \Delta_t \left( q_0^{(1)} \bar{V}^{(1)} \times E + q_1^{(1)} H + S_m(1) \right) (t_h)$	(a.7)
$E\left(t_e + \frac{1}{2}\Delta_t\right) - E\left(t_e - \frac{1}{2}\Delta_t\right) \approx \Delta_t \left( p_0^{(1)} \bar{V}^{(1)} \times H + p_1^{(1)} E + S_e(1) \right) (t_e)$	(a.8)

But we have unavailable values

$$H(t_h) \text{ and } E(t_e) \text{ are unavailable}$$

One way to get around the problem is to use average values, as shown in [2]:

$H(t_h) \approx \frac{H\left(t_h + \frac{1}{2}\Delta_t\right) + H\left(t_h - \frac{1}{2}\Delta_t\right)}{2}$	(a.9)
$E(t_e) \approx \frac{E\left(t_e + \frac{1}{2}\Delta_t\right) + E\left(t_e - \frac{1}{2}\Delta_t\right)}{2}$	(a.10)

Insert (a.9) into (a.7), and insert (a.10) into (a.8), we have

$H\left(t_h + \frac{1}{2}\Delta_t\right) \approx \frac{\left(1 + \frac{\Delta_t q_1^{(1)}}{2}\right)}{\left(1 - \frac{\Delta_t q_1^{(1)}}{2}\right)} H\left(t_h - \frac{1}{2}\Delta_t\right) + \frac{\Delta_t \left( q_0^{(1)} \bar{V}^{(1)} \times E + S_m(1) \right) (t_h)}{\left(1 - \frac{\Delta_t q_1^{(1)}}{2}\right)}$	(a.11)
$E\left(t_e + \frac{1}{2}\Delta_t\right) \approx \frac{\left(1 + \frac{\Delta_t p_1^{(1)}}{2}\right)}{\left(1 - \frac{\Delta_t p_1^{(1)}}{2}\right)} E\left(t_e - \frac{1}{2}\Delta_t\right) + \frac{\Delta_t \left( p_0^{(1)} \bar{V}^{(1)} \times H + S_e(1) \right) (t_e)}{\left(1 - \frac{\Delta_t p_1^{(1)}}{2}\right)}$	(a.12)

Assume  $E_x, E_y, E_z, H_x, H_y$ , and  $H_z$  are all at the same space location  $(x, y, z)$ , and let

$$t_e = t_h + \frac{1}{2}\Delta t$$

(a.11) and (a.12) give  $H\left(t_h + \frac{1}{2}\Delta_t + q\Delta_t\right), E(t_h + q\Delta_t), q = 0,1,2,3, \dots$

The advantage of this algorithm is that it can handle lossy medium. The limitation is that its estimation order is limited.

Note that the above method of the field averaging can also be used in the Yee's algorithm. But by doing so we can no longer say the Yee's algorithm has the second order accuracy.

We can no longer calculate Poynting vector.

We can still calculate divergences to get precise estimation errors.

### Center-Difference Time-Space-Synchronized FDTD (CD-TSS)

The time-shifted-space-synchronized (TS-SS) borrows the idea of the center-difference of the Yee's algorithm. But using the center-difference does not have to be time shifted. It can still be time-synchronized. Here we present center-difference-time-space-synchronized (CD-TSS) algorithm.

We may alter (a.3) and (a.4) listed previously and get:

$H(t_0 + (q+1)\Delta_t) \approx H(t_0 + (q-1)\Delta_t) + T_0^h(\Delta_t)H(t_0 + q\Delta_t) + \sum_{i=0}^{k_{max}} \left( \frac{2\Delta_t^{2i+1}}{(2i+1)!} S_m(2i+1) + T_e^h(i, \Delta_t) \bar{\nabla}^{\{2i+1\}} \times E + T_h^h(i, \Delta_t) \bar{\nabla}^{\{2(i+1)\}} \times H \right) (t_0 + q\Delta_t)$	(a.13)
$E(t_0 + (q+1)\Delta_t) \approx E(t_0 + (q-1)\Delta_t) + T_0^e(\Delta_t)E(t_0 + q\Delta_t) + \sum_{i=0}^{k_{max}} \left( \frac{2\Delta_t^{2i+1}}{(2i+1)!} S_e(2i+1) + T_h^e(i, \Delta_t) \bar{\nabla}^{\{2i+1\}} \times H + T_e^e(i, \Delta_t) \bar{\nabla}^{\{2(i+1)\}} \times E \right) (t_0 + q\Delta_t)$	(a.14)

We may use TSS to get,

$$H(t_0 + \Delta_t)$$

$$E(t_0 + \Delta_t)$$

The above algorithm gives

$$\frac{H(t_0 + q\Delta_t)}{E(t_0 + q\Delta_t)} \quad q = 1,2,3, \dots$$

### Single Field Algorithm

By (41), we have

$E(t_e + \Delta_t) \approx -E(t_e - \Delta_t) + W_0(\Delta_t)E + \sum_{i=0}^{k_{max}} \left( \frac{2\Delta_t^{2(k+1)}}{(2(k+1))!} S_e(2(k+1)) + W_h(i, \Delta_t) \bar{\nabla}^{\{2i+1\}} \times H + W_e(i, \Delta_t) \bar{\nabla}^{\{2(i+1)\}} \times E \right) (t_e)$	(a.19)
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Where

$$W_0 = \bar{C}_0^e(\Delta_t) + \bar{C}_0^e(-\Delta_t) = 2 + \sum_{k=0}^{k_{ma}} \left( \frac{2\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k+1)}^{\{2(k+1)\}} \right)$$

$$W_e(i, \Delta_t) = G_e^e(i, \Delta_t) + G_e^e(i, \Delta_t) = \sum_{k=i}^{k_{max}} \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k-i)}^{\{2(k+1)\}}$$

$$W_h(i, \Delta_t) = G_h^e(i, \Delta_t) + G_h^e(i, \Delta_t) = \sum_{k=i}^{k_{max}} \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} p_{2(k-i)+1}^{\{2(k+1)\}}$$

If we are dealing with lossless medium, then

$$\sigma = \sigma_m = 0 \rightarrow p_{2(k-i)+1}^{\{2(k+1)\}} = 0 \rightarrow W_h(i, \Delta_t)$$

(a.19) becomes

$E(t_e + \Delta_t) \approx -E(t_e - \Delta_t) + W_0(\Delta_t)E + \sum_{i=0}^{k_{max}} \left( \frac{2\Delta_t^{2(k+1)}}{(2(k+1))!} S_e(2(k+1)) + W_e(i, \Delta_t) \bar{\nabla}^{\{2(i+1)\}} \times E \right) (t_e)$	(a.20)
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We can use two sets of history field data to get a new set of field data.

$$(E(t_e - \Delta_t), E(t_e)) \rightarrow E(t_e + \Delta_t)$$

Let

$$t_e = t_0 + q\Delta_t$$

We have

$E(t_0 + (q+1)\Delta_t) \approx -E(t_0 + (q-1)\Delta_t) + W_0(\Delta_t)E + \sum_{i=0}^{k_{max}} \left( \frac{2\Delta_t^{2(k+1)}}{(2(k+1))!} S_e(2(k+1)) + W_e(i, \Delta_t) \bar{\nabla}^{\{2(i+1)\}} \times E \right) (t_0 + q\Delta_t)$	(a.21)
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(a.21) gives  $E(t_0 + (q+1)\Delta_t), q = 1, 2, 3, \dots$ , starting from  $E(t_0), E(t_0 + \Delta_t)$

It has the same advantages of CD-TSS. Another advantage of this algorithm is that it only deals with one single field. Here we show the electric field. We can also only estimate the magnetic field.

The limitation is that it only handles lossless medium.

### The Yee's algorithm

We start from the above Time-Leap-Frog-Space-Synchronized algorithm. But we limit the estimation order:

$$k_{max} = 0$$

For curl estimation, choose  $M=1$ , the inverse matrix is simply  $\frac{1}{\Delta s}$ . The derivative estimator is  $\frac{v(s+\Delta s) - v(s)}{\Delta s}$ .

The curl estimations become

$$\begin{aligned} \bar{\nabla}^{(1)} \times E &= \begin{bmatrix} \mathfrak{D}_y(E_z) - \mathfrak{D}_z(E_y) \\ \mathfrak{D}_z(E_x) - \mathfrak{D}_x(E_z) \\ \mathfrak{D}_x(E_y) - \mathfrak{D}_y(E_x) \end{bmatrix} = \frac{1}{\Delta s} \begin{bmatrix} E_z(x, y + \Delta s, z) - E_z(x, y, z) - E_y(x, y, z + \Delta s) + E_y(x, y, z) \\ E_x(x, y, z + \Delta s) - E_x(x, y, z) - E_z(x + \Delta s, y, z) + E_z(x, y, z) \\ E_y(x + \Delta s, y, z) - E_y(x, y, z) - E_x(x, y + \Delta s, z) + E_x(x, y, z) \end{bmatrix} \\ \bar{\nabla}^{(1)} \times H &= \begin{bmatrix} \mathfrak{D}_y(H_z) - \mathfrak{D}_z(H_y) \\ \mathfrak{D}_z(H_x) - \mathfrak{D}_x(H_z) \\ \mathfrak{D}_x(H_y) - \mathfrak{D}_y(H_x) \end{bmatrix} = \frac{1}{\Delta s} \begin{bmatrix} H_z(x, y, z) - H_z(x, y - \Delta s, z) - H_y(x, y, z) + H_y(x, y, z - \Delta s) \\ H_x(x, y, z) - H_x(x, y, z - \Delta s) - H_z(x, y, z) + H_z(x - \Delta s, y, z) \\ H_y(x, y, z) - H_y(x - \Delta s, y, z) - H_x(x, y, z) + H_x(x, y - \Delta s, z) \end{bmatrix} \end{aligned}$$

Before substituting the above estimations into the estimation formula, we notice that to get a derivative at  $v(s)$ ,  $\frac{v(s+\frac{1}{2}\Delta s) - v(s-\frac{1}{2}\Delta s)}{\Delta s}$  is one order more accurate than  $\frac{v(s+\Delta s) - v(s)}{\Delta s}$ . Yee gave a clever way to do it by defining

$$E \rightarrow \begin{bmatrix} E_x(x + \frac{1}{2}\Delta s, y, z) \\ E_y(x, y + \frac{1}{2}\Delta s, z) \\ E_z(x, y, z + \frac{1}{2}\Delta s) \end{bmatrix}, H \rightarrow \begin{bmatrix} H_x(x, y + \frac{1}{2}\Delta s, z + \frac{1}{2}\Delta s) \\ H_y(x + \frac{1}{2}\Delta s, y, z + \frac{1}{2}\Delta s) \\ H_z(x + \frac{1}{2}\Delta s, y + \frac{1}{2}\Delta s, z) \end{bmatrix}$$

The curl estimations become

$$\begin{aligned} \bar{\nabla}^{(1)} \times E &= \frac{1}{\Delta s} \begin{bmatrix} E_z(x, y + \Delta s, z + \frac{1}{2}\Delta s) - E_z(x, y, z + \frac{1}{2}\Delta s) - E_y(x, y + \frac{1}{2}\Delta s, z + \Delta s) + E_y(x, y + \frac{1}{2}\Delta s, z) \\ E_x(x + \frac{1}{2}\Delta s, y, z + \Delta s) - E_x(x + \frac{1}{2}\Delta s, y, z) - E_z(x + \Delta s, y, z + \frac{1}{2}\Delta s) + E_z(x, y, z + \frac{1}{2}\Delta s) \\ E_y(x + \Delta s, y + \frac{1}{2}\Delta s, z) - E_y(x, y + \frac{1}{2}\Delta s, z) - E_x(x + \frac{1}{2}\Delta s, y + \Delta s, z) + E_x(x + \frac{1}{2}\Delta s, y, z) \end{bmatrix} \\ \bar{\nabla}^{(1)} \times H &= \frac{1}{\Delta s} \begin{bmatrix} H_z(x + \frac{1}{2}\Delta s, y + \frac{1}{2}\Delta s, z) - H_z(x + \frac{1}{2}\Delta s, y - \frac{1}{2}\Delta s, z) - H_y(x + \frac{1}{2}\Delta s, y, z + \frac{1}{2}\Delta s) + H_y(x + \frac{1}{2}\Delta s, y, z - \frac{1}{2}\Delta s) \\ H_x(x, y + \frac{1}{2}\Delta s, z + \frac{1}{2}\Delta s) - H_x(x, y + \frac{1}{2}\Delta s, z - \frac{1}{2}\Delta s) - H_z(x + \frac{1}{2}\Delta s, y + \frac{1}{2}\Delta s, z) + H_z(x - \frac{1}{2}\Delta s, y + \frac{1}{2}\Delta s, z) \\ H_y(x + \frac{1}{2}\Delta s, y, z + \frac{1}{2}\Delta s) - H_y(x - \frac{1}{2}\Delta s, y, z + \frac{1}{2}\Delta s) - H_x(x, y + \frac{1}{2}\Delta s, z + \frac{1}{2}\Delta s) + H_x(x, y - \frac{1}{2}\Delta s, z + \frac{1}{2}\Delta s) \end{bmatrix} \end{aligned}$$

We can see that

$$\begin{aligned} \text{center of } \bar{\nabla}^{(1)} \times E \text{ is at } & \begin{bmatrix} E_x(x + \frac{1}{2}\Delta s, y, z) \\ E_y(x, y + \frac{1}{2}\Delta s, z) \\ E_z(x, y + \frac{1}{2}\Delta s, z) \end{bmatrix} \\ \text{center of } \bar{\nabla}^{(1)} \times H \text{ is at } & \begin{bmatrix} H_x(x, y + \frac{1}{2}\Delta s, z + \frac{1}{2}\Delta s) \\ H_y(x + \frac{1}{2}\Delta s, y, z + \frac{1}{2}\Delta s) \\ H_z(x + \frac{1}{2}\Delta s, y + \frac{1}{2}\Delta s, z) \end{bmatrix} \end{aligned}$$

Now we get the standard Yee algorithm.

By deriving the Yee algorithm from the generic forms, we can clearly see the advantages and limitations of it.

Advantages:

- Use calculation amount of a first order estimation to get a second order precision
- Higher order curl estimations can still be used to increase accuracy of  $\bar{\nabla}^{\{1\}} \times$ , but not  $\bar{\nabla}^{\{h\}} \times, h > 1$

Limitations

- Because not more than one curl can be used, that is,  $\bar{\nabla}^{\{h\}} \times, h > 1$ , cannot be used, the time advancement order cannot be higher than 2. This is the biggest limitation of the Yee algorithm and there is no way to get around it as long as the Yee style space shifting is used, because the space shifting damages the Time Space Lemma.
- Cannot calculate Poynting vector
- Cannot calculate divergences

## Conclusion

For estimations purely based on Taylor's series, this paper shows that there is a generic form of FDTD algorithm for the Maxwell equations. By "generic" I mean that all other forms of the algorithms can be derived from this generic form, such as the Yee's algorithm. But it does not cover those algorithms involving techniques other than the Taylor series. However, almost all estimation algorithms involve the use of Taylor series in one way or the other. Therefore, this generic form may be used as a base for developing new algorithms, or to improve the existing algorithms.

This generic FDTD form gives a time-space-synchronized (TSS) algorithm. The time advancement estimation order is unlimited. The space curl estimation order is only limited to the number of space samplings used.

One variation of TSS is the time-shifted-space-synchronized (TS-SS) algorithm. TS-SS keeps the major benefit of the Yee's algorithm, that is, increased time advancement estimation accuracy. At the same time TS-SS removes the major limitation of the Yee's algorithm, that is, its time advancement estimation order is unlimited.

## Reference

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[https://www.researchgate.net/publication/344868091\\_A\\_Generic\\_FDTD\\_Form\\_for\\_Maxwell\\_Equations](https://www.researchgate.net/publication/344868091_A_Generic_FDTD_Form_for_Maxwell_Equations)
- [2] *Understanding the Finite-Difference Time-Domain Method*, John B. Schneider, [www.eecs.wsu.edu/~schneidj/ufdtd](http://www.eecs.wsu.edu/~schneidj/ufdtd), 2010.

## Appendix

### Proof of the Time Space Lemma

Note that (16) – (30) are constants. We do not need to prove them. We just use them. What we need to prove are (8) - (15) based on (6) and (7).

By (16) and (17), (6) and (7) become

$\frac{\partial E}{\partial t} = p_0^{(1)} \nabla \times H + p_1^{(1)} E + e_0^{(0)} J_e$	(42)
$\frac{\partial H}{\partial t} = q_0^{(1)} \nabla \times E + q_1^{(1)} H + m_0^{(0)} J_m$	(43)

We need to prove (8) – (15) based on (42) and (43). Due to complete symmetry of E and H shown in (8)-(15), (42) and (43), we only need to prove (8), (10), (12) and (14).

For  $k = 1$ , (8), (10), (12) and (14) become

$\frac{\partial^2 E}{\partial t^2} = p_2^{(2)} E + p_0^{(2)} \nabla^{(2)} \times E + p_1^{(2)} \nabla^{(1)} \times H + S_e(2)$	(44)
$S_e(2) = \nabla^{(0)} \times (e_0^{(0)} \frac{d^1 J_e}{dt^1} + e_1^{(0)} \frac{d^0 J_e}{dt^0}) + \nabla^{(1)} \times m_0^{(1)} \frac{d^0 J_m}{dt^0}$	(45)
$\frac{\partial^3 E}{\partial t^3} = p_0^{(3)} \nabla^{(3)} \times H + p_1^{(3)} \nabla^{(2)} \times E + p_2^{(3)} \nabla^{(1)} \times H + p_3^{(3)} \nabla^{(0)} \times E + S_e(3)$	(46)
$S_e(3) = \nabla^{(2)} \times e_0^{(2)} \frac{d^0 J_e}{dt^0} + \nabla^{(0)} \times (e_0^{(0)} \frac{d^2 J_e}{dt^2} + e_1^{(0)} \frac{d^1 J_e}{dt^1} + e_2^{(0)} \frac{d^0 J_e}{dt^0}) + \nabla^{(1)} \times (m_0^{(1)} \frac{d^1 J_m}{dt^1} + m_1^{(1)} \frac{d^0 J_m}{dt^0})$	(47)

Taking temporal derivative on (42) and substitute (42) and (43) into it, we get



$$\frac{\partial^2 E}{\partial t^2} = q_0^{\{1\}} p_0^{\{1\}} \nabla^{\{2\}} \times E + (q_1^{\{1\}} p_0^{\{1\}} + p_1^{\{1\}} p_0^{\{1\}}) \nabla \times H + p_1^{\{1\}} p_1^{\{1\}} E + \nabla \times p_0^{\{1\}} m_0^{\{0\}} J_m + p_1^{\{1\}} e_0^{\{0\}} J_e + e_0^{\{0\}} \frac{dJ_e}{dt}$$

By (28)

$$q_0^{\{1\}} p_0^{\{1\}} = p_0^{\{2\}}$$

$$p_2^{\{2\}} = p_1^{\{1\}} p_1^{\{1\}}$$

By (29)

$$p_1^{\{2\}} = q_1^{\{1\}} p_0^{\{1\}} + p_0^{\{1\}} p_1^{\{1\}}$$

By (26)

$$e_1^{\{0\}} = p_1^{\{1\}} e_0^{\{0\}}$$

By (27)

$$m_0^{\{1\}} = p_0^{\{1\}} m_0^{\{0\}}$$

Combine the above, we have

$$\frac{\partial^2 E}{\partial t^2} = p_0^{\{2\}} \nabla^{\{2\}} \times E + p_1^{\{2\}} \nabla \times H + p_2^{\{2\}} E + \nabla \times m_0^{\{1\}} J_m + e_1^{\{0\}} J_e + e_0^{\{0\}} \frac{dJ_e}{dt}$$

The above is the same as (44) and (45). Thus (8) and (12) hold for  $k = 1$ .

Taking temporal derivative on (44) and (45), and substitute (42) and (43) into it, we get

$$\begin{aligned} \frac{\partial^3 E}{\partial t^3} &= p_0^{\{1\}} p_0^{\{2\}} \nabla^{\{3\}} \times H + (p_2^{\{2\}} p_0^{\{1\}} + q_1^{\{1\}} p_1^{\{2\}}) \nabla^{\{1\}} \times H + (p_1^{\{1\}} p_0^{\{2\}} + q_0^{\{1\}} p_1^{\{2\}}) \nabla^{\{2\}} \times E + p_2^{\{2\}} p_1^{\{1\}} E \\ &+ \nabla^{\{2\}} \times p_0^{\{2\}} e_0^{\{0\}} J_e + \nabla^{\{1\}} \times (p_1^{\{2\}} m_0^{\{0\}} J_m + m_0^{\{1\}} \frac{d^1 J_m}{dt^1}) + \nabla^{\{0\}} \times (e_0^{\{0\}} \frac{d^2 J_e}{dt^2} + e_1^{\{0\}} \frac{d^1 J_e}{dt^1} + p_2^{\{2\}} e_0^{\{0\}} J_e) \end{aligned}$$

By (20)

$$p_0^{\{3\}} = p_0^{\{1\}} p_0^{\{2\}}$$

By (23)

$$p_2^{\{3\}} = q_1^{\{1\}} p_1^{\{2\}} + p_0^{\{1\}} p_2^{\{2\}}$$

By (22)

$$p_1^{\{3\}} = p_1^{\{1\}} p_0^{\{2\}} + q_0^{\{1\}} p_1^{\{2\}}$$

By (20)

$$p_3^{\{3\}} = p_2^{\{2\}} p_1^{\{1\}}$$

By (18)

$$e_0^{\{2\}} = p_0^{\{2\}} e_0^{\{0\}}$$

$$e_2^{\{0\}} = p_2^{\{2\}} e_0^{\{0\}}$$

By (19)

$$m_1^{\{1\}} = p_1^{\{2\}} m_0^{\{0\}}$$

Combine the above, we have

$$\begin{aligned}\frac{\partial^3 E}{\partial t^3} &= p_0^{[3]} \nabla^{[3]} \times H + p_2^{[3]} \nabla^{[1]} \times H + p_1^{[3]} \nabla^{[2]} \times E + p_3^{[3]} E \\ &+ \nabla^{[2]} \times e_0^{[2]} J_e + \nabla^{[1]} \times (m_1^{[1]} J_m + m_0^{[1]} \frac{d^1 J_m}{dt^1}) + \nabla^{[0]} \times (e_0^{[0]} \frac{d^2 J_e}{dt^2} + e_1^{[0]} \frac{d^1 J_e}{dt^1} + e_2^{[0]} J_e)\end{aligned}$$

It is the same as (46) and (47). Thus (10) and (14) hold for  $k = 1$ .

Thus, (8), (10), (12) and (14) hold for  $k = 1$ .

Suppose for an integer  $k \geq 1$ , (8), (10), (12) and (14) hold.

Consider a case of  $k + 1$ .

For  $k + 1$  (8), (10), (12) and (14) become

$\frac{\partial^{2(k+1)} E}{\partial t^{2(k+1)}} = p_{2(k+1)}^{[2(k+1)]} E + \sum_{i=1}^{k+1} \left( p_{2(k+1-i)}^{[2(k+1)]} \nabla^{[2i]} \times E + p_{2(k+1-i)+1}^{[2(k+1)]} \nabla^{[2i-1]} \times H \right) + S_e(2(k+1))$	(48)
$\frac{\partial^{2(k+1)+1} E}{\partial t^{2(k+1)+1}} = \sum_{i=0}^{k+1} \left( p_{2i}^{[2(k+1)+1]} \nabla^{[2(k+1-i)+1]} \times H + p_{2i+1}^{[2(k+1)+1]} \nabla^{[2(k+1-i)]} \times E \right) + S_e(2(k+1)+1)$	(49)

$S_e(2(k+1)) = \sum_{h=0}^k \left( \nabla^{[2(k-h)]} \times \sum_{i=1}^{2(h+1)} e_{i-1}^{[2(k-h)]} \frac{d^{2(h+1)-i} J_e}{dt^{2(h+1)-i}} + \nabla^{[2(k-h)+1]} \times \sum_{i=0}^{2h} m_i^{[2(k-h)+1]} \frac{d^{2h-i} J_m}{dt^{2h-i}} \right)$	(50)
$S_e(2(k+1)+1) = \sum_{h=0}^{k+1} \nabla^{[2(k+1-h)]} \times \sum_{i=0}^{2h} e_i^{[2(k+1-h)]} \frac{d^{2h-i} J_e}{dt^{2h-i}} + \sum_{h=0}^k \nabla^{[2(k+1-h)-1]} \times \sum_{i=0}^{2h+1} m_i^{[2(k+1-h)-1]} \frac{d^{2h-i+1} J_m}{dt^{2h-i+1}}$	(51)

Taking temporal derivatives on (10) and (14), and substitute (42) and (43) into them, we have

$$\begin{aligned}\frac{\partial^{2k+2} E}{\partial t^{2k+2}} &= \sum_{i=0}^k (q_0^{[1]} p_{2i}^{[2k+1]} \nabla^{[2(k-i)+2]} \times E + \nabla^{[2(k-i)]} \times E + (q_1^{[1]} p_{2i}^{[2k+1]} + p_1^{[1]} p_{2i+1}^{[2k+1]}) \nabla^{[2(k-i)+1]} \times H + \nabla^{[2(k-i)+1]} \times p_{2i}^{[2k+1]} m_0^{[0]} J_m + \nabla^{[2(k-i)]} \\ &\quad \times p_{2i+1}^{[2k+1]} e_0^{[0]} J_e) + \frac{dS_e(2k+1)}{dt} \\ &= \sum_{i=0}^k (q_0^{[1]} p_{2i}^{[2k+1]} \nabla^{[2(k-i)+2]} \times E + p_1^{[1]} p_{2i+1}^{[2k+1]} \nabla^{[2(k-i)]} \times E) + \\ &\quad \sum_{i=0}^k ((q_1^{[1]} p_{2i}^{[2k+1]} + p_0^{[1]} p_{2i+1}^{[2k+1]}) \nabla^{[2(k-i)+1]} \times H) + \\ &\quad \sum_{i=0}^k (\nabla^{[2(k-i)+1]} \times p_{2i}^{[2k+1]} m_0^{[0]} J_m + \nabla^{[2(k-i)]} \times p_{2i+1}^{[2k+1]} e_0^{[0]} J_e) + \frac{dS_e(2k+1)}{dt} \\ \frac{dS_e(2k+1)}{dt} &= \sum_{h=0}^k \nabla^{[2(k-h)]} \times \sum_{i=0}^{2h} e_i^{[2(k-h)]} \frac{d^{2h-i+1} J_e}{dt^{2h-i+1}} + \sum_{h=0}^{k-1} \nabla^{[2(k-h)-1]} \times \sum_{i=0}^{2h+1} m_i^{[2(k-h)-1]} \frac{d^{2h-i+2} J_m}{dt^{2h-i+2}}\end{aligned}$$

By (29),

$$q_1^{[1]} p_{2i}^{[2k+1]} + p_0^{[1]} p_{2i+1}^{[2k+1]} = p_{2i+1}^{[2(k+1)]}$$

We have

$$\begin{aligned}
\sum_{i=0}^k ((q_1^{\{1\}} p_{2i}^{\{2k+1\}} + p_0^{\{1\}} p_{2i+1}^{\{2k+1\}}) \nabla^{2(k-i)+1} \times H) &= \sum_{i=0}^k (p_{2i+1}^{\{2(k+1)\}} \nabla^{2(k-i)+1} \times H) = \sum_{i=1}^{k+1} (p_{2i-1}^{\{2(k+1)\}} \nabla^{2(k+1-i)+1} \times H) \\
&= \sum_{i=1}^{k+1} (p_{2(k+1-i)+1}^{\{2(k+1)\}} \nabla^{2i-1} \times H)
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\partial^{2k+2} E}{\partial t^{2k+2}} &= \sum_{i=0}^k (q_0^{\{1\}} p_{2i}^{\{2k+1\}} \nabla^{2(k-i)+2} \times E + p_1^{\{1\}} p_{2i+1}^{\{2k+1\}} \nabla^{2(k-i)} \times E) + \\
&\quad \sum_{i=1}^{k+1} (p_{2(k+1-i)+1}^{\{2(k+1)\}} \nabla^{2i-1} \times H) + \\
&\quad \sum_{i=0}^k (\nabla^{2(k-i)+1} \times p_{2i}^{\{2k+1\}} m_0^{\{0\}} J_m + \nabla^{2(k-i)} \times p_{2i+1}^{\{2k+1\}} e_0^{\{0\}} J_e) + \frac{dS_e(2k+1)}{dt}
\end{aligned}$$

By (28)

$$\begin{aligned}
p_{2k}^{\{2k\}} &= p_{2k-1}^{\{2k-1\}} p_1^{\{1\}} \rightarrow p_1^{\{1\}} p_{2k+1}^{\{2k+1\}} = p_{2(k+1)}^{\{2(k+1)\}} \\
p_0^{\{2k\}} &= q_0^{\{1\}} p_0^{\{2k-1\}} \rightarrow p_0^{\{2(k+1)\}} = q_0^{\{1\}} p_0^{\{2k+1\}}
\end{aligned}$$

By (29)

$$p_{2i+2}^{\{2k\}} = p_1^{\{1\}} p_{2i+1}^{\{2k-1\}} + q_0^{\{1\}} p_{2i+2}^{\{2k-1\}} \rightarrow p_{2i}^{\{2(k+1)\}} = p_1^{\{1\}} p_{2i-1}^{\{2k+1\}} + q_0^{\{1\}} p_{2i}^{\{2k+1\}}$$

Thus, we have

$$\begin{aligned}
&\sum_{i=0}^k (q_0^{\{1\}} p_{2i}^{\{2k+1\}} \nabla^{2(k-i)+2} \times E + p_1^{\{1\}} p_{2i+1}^{\{2k+1\}} \nabla^{2(k-i)} \times E) \\
&= q_0^{\{1\}} p_0^{\{2k+1\}} \nabla^{2(k+1)} \times E + \sum_{i=1}^k (q_0^{\{1\}} p_{2i}^{\{2k+1\}} \nabla^{2(k-i)+2} \times E) + \sum_{i=0}^k (p_1^{\{1\}} p_{2i+1}^{\{2k+1\}} \nabla^{2(k-i)} \times E) \\
&= q_0^{\{1\}} p_0^{\{2k+1\}} \nabla^{2(k+1)} \times E + \sum_{i=0}^{k-1} (q_0^{\{1\}} p_{2i+2}^{\{2k+1\}} \nabla^{2(k-i)} \times E) + \sum_{i=0}^k (p_1^{\{1\}} p_{2i+1}^{\{2k+1\}} \nabla^{2(k-i)} \times E) \\
&= q_0^{\{1\}} p_0^{\{2k+1\}} \nabla^{2(k+1)} \times E + \sum_{i=0}^{k-1} ((p_1^{\{1\}} p_{2i+1}^{\{2k+1\}} + q_0^{\{1\}} p_{2i+2}^{\{2k+1\}}) \nabla^{2(k-i)} \times E) + p_1^{\{1\}} p_{2k+1}^{\{2k+1\}} \nabla^{(0)} \times E \\
&= q_0^{\{1\}} p_0^{\{2k+1\}} \nabla^{2(k+1)} \times E + \sum_{i=1}^k ((p_1^{\{1\}} p_{2i-1}^{\{2k+1\}} + q_0^{\{1\}} p_{2i}^{\{2k+1\}}) \nabla^{2(k+1-i)} \times E) + p_1^{\{1\}} p_{2k+1}^{\{2k+1\}} \nabla^{(0)} \times E \\
&= p_0^{\{2(k+1)\}} \nabla^{2(k+1)} \times E + \sum_{i=1}^k (p_{2i}^{\{2(k+1)\}} \nabla^{2(k+1-i)} \times E) + p_{2(k+1)}^{\{2(k+1)\}} E \\
&= \sum_{i=0}^k (p_{2i}^{\{2(k+1)\}} \nabla^{2(k+1-i)} \times E) + p_{2(k+1)}^{\{2(k+1)\}} E \\
&= \sum_{i=0}^k (p_{2(k-i)}^{\{2(k+1)\}} \nabla^{2(i+1)} \times E) + p_{2(k+1)}^{\{2(k+1)\}} E \\
&= \sum_{i=1}^{k+1} (p_{2(k-i+1)}^{\{2(k+1)\}} \nabla^{2(i)} \times E) + p_{2(k+1)}^{\{2(k+1)\}} E
\end{aligned}$$

Thus,

$$\frac{\partial^{2k+2} E}{\partial t^{2k+2}} = p_{2(k+1)}^{\{2(k+1)\}} E + \sum_{i=1}^{k+1} (p_{2(k-i+1)}^{\{2(k+1)\}} \nabla^{\{2i\}} \times E + p_{2(k+1-i)+1}^{\{2(k+1)\}} \nabla^{\{2i-1\}} \times H) +$$

$$\sum_{i=0}^k (\nabla^{\{2(k-i)+1\}} \times p_{2i}^{\{2k+1\}} m_0^{\{0\}} J_m + \nabla^{\{2(k-i)\}} \times p_{2i+1}^{\{2k+1\}} e_0^{\{0\}} J_e) + \frac{dS_e(2k+1)}{dt}$$

By (14),

$$\begin{aligned} & \sum_{i=0}^k (\nabla^{\{2(k-i)+1\}} \times p_{2i}^{\{2k+1\}} m_0^{\{0\}} J_m + \nabla^{\{2(k-i)\}} \times p_{2i+1}^{\{2k+1\}} e_0^{\{0\}} J_e) + \frac{dS_e(2k+1)}{dt} \\ &= \sum_{i=0}^k (\nabla^{\{2(k-i)+1\}} \times p_{2i}^{\{2k+1\}} m_0^{\{0\}} J_m + \nabla^{\{2(k-i)\}} \times p_{2i+1}^{\{2k+1\}} e_0^{\{0\}} J_e) + \sum_{h=0}^k \nabla^{\{2(k-h)\}} \times \sum_{i=0}^{2h} e_i^{\{2(k-h)\}} \frac{d^{2h-i+} J_e}{dt^{2h-i+1}} \\ & \quad + \sum_{h=0}^{k-1} \nabla^{\{2(k-h)-1\}} \times \sum_{i=0}^{2h+1} m_i^{\{2(k-h)-1\}} \frac{d^{2h-i+2} J_m}{dt^{2h-i+2}} \\ &= \sum_{h=0}^k \nabla^{\{2(k-h)\}} \times (p_{2h+1}^{\{2k+1\}} e_0^{\{0\}} J_e + \sum_{i=0}^{2h} e_i^{\{2(k-h)\}} \frac{d^{2h-i+1} J_e}{dt^{2h-i+1}}) + \sum_{i=0}^k (\nabla^{\{2(k-i)+1\}} \times p_{2i}^{\{2k+1\}} m_0^{\{0\}} J_m) + \sum_{h=0}^{k-1} \nabla^{\{2(k-h)-1\}} \times \sum_{i=0}^{2h+1} m_i^{\{2(k-h)-1\}} \frac{d^{2h-i+2} J_m}{dt^{2h-i+2}} \end{aligned}$$

By (18),

$$e_{2(k-i)-1}^{\{2i\}} = p_{2(k-i)-1}^{\{2k-1\}} e_0^{\{0\}} \rightarrow e_{2(k-i)+1}^{\{2i\}} = p_{2(k-i)+1}^{\{2k+1\}} e_0^{\{0\}} \rightarrow e_{2h+1}^{\{2(k-h)\}} = p_{2h+1}^{\{2k+1\}} e_0^{\{0\}}$$

We have

$$\begin{aligned} & \sum_{h=0}^k \nabla^{\{2(k-h)\}} \times (p_{2h+1}^{\{2k+1\}} e_0^{\{0\}} J_e + \sum_{i=0}^{2h} e_i^{\{2(k-h)\}} \frac{d^{2h-i+1} J_e}{dt^{2h-i+1}}) = \sum_{h=0}^k \nabla^{\{2(k-h)\}} \times (p_{2h+1}^{\{2k+1\}} e_0^{\{0\}} J_e + \sum_{i=1}^{2h+1} e_{i-1}^{\{2(k-h)\}} \frac{d^{2h-i+2} J_e}{dt^{2h-i+2}}) \\ &= \sum_{h=0}^k \nabla^{\{2(k-h)\}} \times \left( \sum_{i=1}^{2h+2} e_{i-1}^{\{2(k-h)\}} \frac{d^{2h-i+} J_e}{dt^{2h-i+}} \right) \end{aligned}$$

We have

$$\begin{aligned} & \frac{\partial^{2k+2} E}{\partial t^{2k+2}} = p_{2(k+1)}^{\{2(k+1)\}} E + \sum_{i=1}^{k+1} (p_{2(k-i+1)}^{\{2(k+1)\}} \nabla^{\{2i\}} \times E + p_{2(k+1-i)+1}^{\{2(k+1)\}} \nabla^{\{2i-1\}} \times H) \\ & \quad + \sum_{h=0}^k \nabla^{\{2(k-h)\}} \times \left( \sum_{i=1}^{2h+2} e_{i-1}^{\{2(k-h)\}} \frac{d^{2h-i+} J_e}{dt^{2h-i+}} \right) \\ & \quad + \sum_{i=0}^k (\nabla^{\{2(k-i)+1\}} \times p_{2i}^{\{2k+1\}} m_0^{\{0\}} J_m) + \sum_{h=0}^{k-1} \nabla^{\{2(k-h)-1\}} \times \sum_{i=0}^{2h+1} m_i^{\{2(k-h)-1\}} \frac{d^{2h-i+2} J_m}{dt^{2h-i+2}} \end{aligned}$$

By (27)

$$m_{2(k-i)-1}^{\{2i+1\}} = p_{2(k-i)-1}^{\{2k-1\}} m_0^{\{0\}} \rightarrow m_{2(k-i)}^{\{2i+1\}} = p_{2(k-i)}^{\{2k+1\}} m_0^{\{0\}} \rightarrow m_{2h}^{\{2(k-h)+1\}} = p_{2h}^{\{2k+1\}} m_0^{\{0\}} \rightarrow m_0^{\{2k+1\}} = p_0^{\{2k+1\}} m_0^{\{0\}}$$

We have

$$\begin{aligned} & \sum_{i=0}^k (\nabla^{\{2(k-i)+1\}} \times p_{2i}^{\{2k+1\}} m_0^{\{0\}} J_m) + \sum_{h=0}^{k-1} \nabla^{\{2(k-h)-1\}} \times \sum_{i=0}^{2h+1} m_i^{\{2(k-h)-1\}} \frac{d^{2h-i+} J_m}{dt^{2h-i+2}} = \\ &= \sum_{i=0}^k (\nabla^{\{2(k-i)+1\}} \times p_{2i}^{\{2k+1\}} m_0^{\{0\}} J_m) + \sum_{h=1}^k \nabla^{\{2(k-h)+1\}} \times \sum_{i=0}^{2h-1} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i} J_m}{dt^{2h-i}} \\ &= \sum_{h=0}^k (\nabla^{\{2(k-h)+1\}} \times p_{2h}^{\{2k+1\}} m_0^{\{0\}} J_m) + \sum_{h=1}^k \nabla^{\{2(k-h)+1\}} \times \sum_{i=0}^{2h-1} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i} J_m}{dt^{2h-i}} \end{aligned}$$

$$\begin{aligned}
&= \nabla^{\{2k+1\}} \times p_0^{\{2k+1\}} m_0^{\{0\}} J_m + \sum_{h=1}^k (\nabla^{\{2(k-h)+1\}} \times p_{2h}^{\{2k+1\}} m_0^{\{0\}} J_m) + \sum_{h=1}^k \nabla^{\{2(k-h)+1\}} \times \sum_{i=0}^{2h-1} m_i^{\{2(k-h)-1\}} \frac{d^{2h-i} J_m}{dt^{2h-i}} \\
&= \nabla^{\{2k+1\}} \times p_0^{\{2k+1\}} m_0^{\{0\}} J_m + \sum_{h=1}^k \nabla^{\{2(k-h)+1\}} \times (p_{2h}^{\{2k+1\}} m_0^{\{0\}} J_m + \sum_{i=0}^{2h-1} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i} J_m}{dt^{2h-i}}) \\
&= \nabla^{\{2k+1\}} \times p_0^{\{2k+1\}} m_0^{\{0\}} J_m + \sum_{h=1}^k \nabla^{\{2(k-h)+1\}} \times (p_{2h}^{\{2k+1\}} m_0^{\{0\}} J_m + \sum_{i=0}^{2h-1} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i} J_m}{dt^{2h-i}}) \\
&= \nabla^{\{2k+1\}} \times p_0^{\{2k+1\}} m_0^{\{0\}} J_m + \sum_{h=1}^k \nabla^{\{2(k-h)+1\}} \times (\sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i} J_m}{dt^{2h-i}}) \\
&= \sum_{h=0}^k \nabla^{\{2(k-h)+1\}} \times (\sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i} J_m}{dt^{2h-i}})
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^{2k+2} E}{\partial t^{2k+2}} &= p_{2(k+1)}^{\{2(k+1)\}} E + \sum_{i=1}^{k+1} (p_{2(k-i+1)}^{\{2(k+1)\}} \nabla^{\{2i\}} \times E + p_{2(k+1-i)+1}^{\{2(k+1)\}} \nabla^{\{2i-1\}} \times H) \\
&+ \sum_{h=0}^k \nabla^{\{2(k-h)\}} \times \left( \sum_{i=1}^{2h+2} e_{i-1}^{\{2(k-h)\}} \frac{d^{2h-i+1} J_e}{dt^{2h-i+2}} \right) + \sum_{h=0}^k \nabla^{\{2(k-h)+1\}} \times \left( \sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i} J_m}{dt^{2h-i}} \right)
\end{aligned}$$

The above is the same as (48) and (50). Thus, (8) and (12) hold for  $k + 1$ .

Taking temporal derivatives on (48) and (50), and substitute (42) and (43) into them, we have

$$\begin{aligned}
\frac{\partial^{2(k+1)+1} E}{\partial t^{2(k+1)+1}} &= p_{2(k+1)}^{\{2(k+1)\}} p_0^{\{1\}} \nabla \times H + \sum_{i=1}^{k+1} (p_0^{\{1\}} p_{2(k+1-i)}^{\{2(k+1)\}} \nabla^{\{2i+1\}} \times H + q_1^{\{1\}} p_{2(k+1-i)+1}^{\{2(k+1)\}} \nabla^{\{2i-1\}} \times H) \\
&+ p_{2(k+1)}^{\{2(k+1)\}} p_1^{\{1\}} E + \sum_{i=1}^{k+1} ((p_1^{\{1\}} p_{2(k+1-i)}^{\{2(k+1)\}} + q_0^{\{1\}} p_{2(k+1-i)+1}^{\{2(k+1)\}}) \nabla^{\{2i\}} \times E) \\
&+ p_{2(k+1)}^{\{2(k+1)\}} e_0^{\{0\}} J_e + \sum_{i=1}^{k+1} (\nabla^{\{2i\}} \times p_{2(k+1-i)}^{\{2(k+1)\}} e_0^{\{0\}} J_e + \nabla^{\{2i-1\}} \times p_{2(k+1-i)+1}^{\{2(k+1)\}} m_0^{\{0\}} J_m) + \frac{dS_e(2(k+1))}{dt} \\
\frac{dS_e(2(k+1))}{dt} &= \sum_{h=0}^k \left( \nabla^{\{2(k-h)\}} \times \sum_{i=1}^{2(h+1)} e_{i-1}^{\{2(k-h)\}} \frac{d^{2(h+1)-i+1} J_e}{dt^{2(h+1)-i+1}} + \nabla^{\{2(k-h)+1\}} \times \sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i+1} J_m}{dt^{2h-i+1}} \right)
\end{aligned}$$

By (20)

$$p_{2k+1}^{\{2k+1\}} = p_{2k}^{\{2k\}} p_1^{\{1\}} \rightarrow p_{2(k+1)+1}^{\{2(k+1)+1\}} = p_{2(k+1)}^{\{2(k+1)\}} p_1^{\{1\}}$$

By (22)

$$p_{2(k-i)+1}^{\{2k+1\}} = p_1^{\{1\}} p_{2(k-i)}^{\{2k\}} + q_0^{\{1\}} p_{2(k-i)+1}^{\{2k\}} \rightarrow p_{2((k+1)-i)+1}^{\{2(k+1)+1\}} = p_1^{\{1\}} p_{2((k+1)-i)}^{\{2(k+1)\}} + q_0^{\{1\}} p_{2(k+1-i)+1}^{\{2(k+1)\}}$$

We have

$$\begin{aligned}
p_{2(k+1)}^{\{2(k+1)\}} p_1^{\{1\}} E + \sum_{i=1}^{k+1} ((p_1^{\{1\}} p_{2(k+1-i)}^{\{2(k+1)\}} + q_0^{\{1\}} p_{2(k+1-i)+1}^{\{2(k+1)\}}) \nabla^{\{2i\}} \times E) &= p_{2(k+1)+1}^{\{2(k+1)+1\}} E + \sum_{i=1}^{k+1} (p_{2(k+1-i)+1}^{\{2(k+1)+1\}} \nabla^{\{2i\}} \times E) = \sum_{i=0}^{k+1} (p_{2(k+1-i)+1}^{\{2(k+1)+1\}} \nabla^{\{2i\}} \times E) \\
&= \sum_{i=0}^{k+1} (p_{2i+1}^{\{2(k+1)+1\}} \nabla^{\{2(k+1-i)\}} \times E)
\end{aligned}$$

By (20),

$$p_0^{\{2k+1\}} = p_0^{\{1\}} p_0^{\{2k\}} \rightarrow p_0^{\{2(k+1)+1\}} = p_0^{\{1\}} p_0^{\{2(k+1)\}}$$

By (23),

$$p_{2i}^{\{2k+1\}} = q_1^{\{1\}} p_{2i-1}^{\{2k\}} + p_0^{\{1\}} p_{2i}^{\{2k\}} \rightarrow p_{2i}^{\{2(k+1)+1\}} = q_1^{\{1\}} p_{2i-1}^{\{2(k+1)\}} + p_0^{\{1\}} p_{2i}^{\{2(k+1)\}}$$

We have

$$\begin{aligned} & p_{2(k+1)}^{\{2(k+1)\}} p_0^{\{1\}} \nabla \times H + \sum_{i=1}^{k+1} \left( p_0^{\{1\}} p_{2(k+1-i)}^{\{2(k+1)\}} \nabla^{\{2i+1\}} \times H + q_1^{\{1\}} p_{2(k+1-i)+1}^{\{2(k+1)\}} \nabla^{\{2i-1\}} \times H \right) \\ &= p_{2(k+1)}^{\{2(k+1)\}} p_0^{\{1\}} \nabla \times H + p_0^{\{1\}} p_0^{\{2(k+1)\}} \nabla^{\{2(k+1)+1\}} \times H + q_1^{\{1\}} p_{2k+1}^{\{2(k+1)\}} \nabla^{\{1\}} \times H + \sum_{i=1}^{k+1} \left( p_0^{\{1\}} p_{2(k+1-i)}^{\{2(k+1)\}} \nabla^{\{2i+1\}} \times H + q_1^{\{1\}} p_{2(k+1-i)-1}^{\{2(k+1)\}} \nabla^{\{2i+1\}} \times H \right) \\ &= p_0^{\{2(k+1)+1\}} \nabla^{\{2(k+1)+1\}} \times H + \left( p_0^{\{1\}} p_{2(k+1)}^{\{2(k+1)\}} + q_1^{\{1\}} p_{2k+1}^{\{2(k+1)\}} \right) \nabla^{\{1\}} \times H + \sum_{i=1}^{k+1} \left( p_0^{\{1\}} p_{2(k+1-i)}^{\{2(k+1)\}} + q_1^{\{1\}} p_{2(k+1-i)-1}^{\{2(k+1)\}} \right) \nabla^{\{2i+1\}} \times H \\ &= p_0^{\{2(k+1)+1\}} \nabla^{\{2(k+1)+1\}} \times H + p_{2(k+1)}^{\{2(k+1)+1\}} \nabla^{\{1\}} \times H + \sum_{i=1}^{k+1} p_{2(k+1-i)}^{\{2(k+1)+1\}} \nabla^{\{2i+1\}} \times H = \sum_{i=0}^{k+1} p_{2(k+1-i)}^{\{2(k+1)+1\}} \nabla^{\{2i+1\}} \times H \\ &= \sum_{i=0}^{k+1} p_{2i}^{\{2(k+1)+1\}} \nabla^{\{2(k+1-i)+1\}} \times H \end{aligned}$$

Combine the above results, we have

$$\begin{aligned} \frac{\partial^{2(k+1)+1} E}{\partial t^{2(k+1)+1}} &= \sum_{i=0}^{k+1} p_{2i}^{\{2(k+1)+1\}} \nabla^{\{2(k+1-i)+1\}} \times H + \sum_{i=0}^{k+1} \left( p_{2i+1}^{\{2(k+1)+1\}} \nabla^{\{2(k+1-i)\}} \times E \right) \\ &+ p_{2(k+1)}^{\{2(k+1)\}} e_0^{\{0\}} J_e + \sum_{i=1}^{k+1} \left( \nabla^{\{2i\}} \times p_{2(k+1-i)}^{\{2(k+1)\}} e_0^{\{0\}} J_e + \nabla^{\{2i-1\}} \times p_{2(k+1-i)+1}^{\{2(k+1)\}} m_0^{\{0\}} J_m \right) \\ &+ \sum_{h=0}^k \left( \nabla^{\{2(k-h)\}} \times \sum_{i=1}^{2(h+1)} e_{i-1}^{\{2(k-h)\}} \frac{d^{2(h+1)-i+1} J_e}{dt^{2(h+1)-i+1}} + \nabla^{\{2(k-h)+1\}} \times \sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i+1} J_m}{dt^{2h-i+1}} \right) \end{aligned}$$

For  $J_m$  terms, we have

$$\begin{aligned} & \sum_{i=1}^{k+1} \left( \nabla^{\{2i-1\}} \times p_{2(k+1-i)+1}^{\{2(k+1)\}} m_0^{\{0\}} J_m \right) + \sum_{h=0}^k \left( \nabla^{\{2(k-h)+1\}} \times \sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i+1} J_m}{dt^{2h-i+1}} \right) \\ &= \sum_{i=0}^k \left( \nabla^{\{2i+1\}} \times p_{2(k-i)+1}^{\{2(k+1)\}} m_0^{\{0\}} J_m \right) + \sum_{h=0}^k \left( \nabla^{\{2(k-h)+1\}} \times \sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i+1} J_m}{dt^{2h-i+1}} \right) \\ &= \sum_{h=0}^k \left( \nabla^{\{2(k-h)+1\}} \times p_{2h+1}^{\{2(k+1)\}} m_0^{\{0\}} J_m \right) + \sum_{h=0}^k \left( \nabla^{\{2(k-h)+1\}} \times \sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i+1} J_m}{dt^{2h-i+1}} \right) \\ &= \sum_{h=0}^k \left( \left( \nabla^{\{2(k-h)+1\}} \times p_{2h+1}^{\{2(k+1)\}} m_0^{\{0\}} J_m \right) + \nabla^{\{2(k-h)+1\}} \times \sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i+1} J_m}{dt^{2h-i+1}} \right) \\ &= \sum_{h=0}^k \left( \nabla^{\{2(k-h)+1\}} \times \left( p_{2h+1}^{\{2(k+1)\}} m_0^{\{0\}} J_m + \sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i+1} J_m}{dt^{2h-i+1}} \right) \right) \end{aligned}$$

By (19),

$$m_{2(k-i)+1}^{\{2i-1\}} = p_{2(k-i)+1}^{\{2k\}} m_0^{\{0\}} \rightarrow m_{2h+1}^{\{2(k-h)+1\}} = p_{2h+1}^{\{2(k+1)\}} m_0^{\{0\}}$$

We have

$$\sum_{h=0}^k \left( \nabla^{\{2(k-h)+1\}} \times \left( p_{2h+1}^{\{2(k+1)\}} m_0^{\{0\}} J_m + \sum_{i=0}^{2h} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i+1} J_m}{dt^{2h-i+1}} \right) \right) = \sum_{h=0}^k \left( \nabla^{\{2(k-h)+1\}} \times \sum_{i=0}^{2h+1} m_i^{\{2(k-h)+1\}} \frac{d^{2h-i+1} J_m}{dt^{2h-i+1}} \right)$$

The above is the  $J_m$  terms in  $S_e(2(k+1)+1)$  as defined in (14).

For  $J_e$  terms, we have

$$\begin{aligned}
& p_{2(k+1)}^{\{2(k+1)\}} e_0^{\{0\}} J_e + \sum_{i=1}^{k+1} \left( \nabla^{\{2i\}} \times p_{2(k+1-i)}^{\{2(k+1)\}} e_0^{\{0\}} J_e \right) + \sum_{h=0}^k \left( \nabla^{\{2(k-h)\}} \times \sum_{i=1}^{2(h+1)} e_{i-1}^{\{2(k-h)\}} \frac{d^{2(h+1)-i+1} J_e}{dt^{2(h+1)-i+1}} \right) \\
&= p_{2(k+1)}^{\{2(k+1)\}} e_0^{\{0\}} J_e + \sum_{i=0}^k \left( \nabla^{\{2i+2\}} \times p_{2(k-i)}^{\{2(k+1)\}} e_0^{\{0\}} J_e \right) + \sum_{h=0}^k \left( \nabla^{\{2(k-h)\}} \times \sum_{i=1}^{2(h+1)} e_{i-1}^{\{2(k-h)\}} \frac{d^{2(h+1)-i+1} J_e}{dt^{2(h+1)-i+1}} \right) \\
&= p_{2(k+1)}^{\{2(k+1)\}} e_0^{\{0\}} J_e + \sum_{h=0}^k \left( \nabla^{\{2(k+1-h)\}} \times p_{2h}^{\{2(k+1)\}} e_0^{\{0\}} J_e \right) + \sum_{h=0}^k \left( \nabla^{\{2(k-h)\}} \times \sum_{i=1}^{2(h+1)} e_{i-1}^{\{2(k-h)\}} \frac{d^{2(h+1)-i+1} J_e}{dt^{2(h+1)-i+1}} \right) \\
&= p_{2(k+1)}^{\{2(k+1)\}} e_0^{\{0\}} J_e + \sum_{h=0}^k \left( \nabla^{\{2(k+1-h)\}} \times p_{2h}^{\{2(k+1)\}} e_0^{\{0\}} J_e \right) + \sum_{h=0}^k \left( \nabla^{\{2(k-h)\}} \times \sum_{i=0}^{2h+1} e_i^{\{2(k-h)\}} \frac{d^{2(h+1)-i} J_e}{dt^{2(h+1)-i}} \right) \\
&= p_{2(k+1)}^{\{2(k+1)\}} e_0^{\{0\}} J_e + \sum_{h=0}^k \left( \nabla^{\{2(k+1-h)\}} \times p_{2h}^{\{2(k+1)\}} e_0^{\{0\}} J_e \right) + \sum_{i=0}^{2k+1} e_i^{\{0\}} \frac{d^{2(k+1)-i} J_e}{dt^{2(k+1)-i}} + \sum_{h=0}^k \left( \nabla^{\{2(k-h)\}} \times \sum_{i=0}^{2h+1} e_i^{\{2(k-h)\}} \frac{d^{2(h+1)-i} J_e}{dt^{2(h+1)-i}} \right) \\
&= p_{2(k+1)}^{\{2(k+1)\}} e_0^{\{0\}} J_e + \nabla^{\{2(k+1)\}} \times p_0^{\{2(k+1)\}} e_0^{\{0\}} J_e + \sum_{h=1}^k \left( \nabla^{\{2(k+1-h)\}} \times p_{2h}^{\{2(k+1)\}} e_0^{\{0\}} J_e \right) + \sum_{i=0}^{2(k+1)-1} e_i^{\{0\}} \frac{d^{2(k+1)-i} J_e}{dt^{2(k+1)-i}} \\
&\quad + \sum_{h=1}^k \left( \nabla^{\{2(k+1-h)\}} \times \sum_{i=0}^{2h-1} e_i^{\{2(k+1-h)\}} \frac{d^{2(h)-i} J_e}{dt^{2(h)-i}} \right) \\
&= p_{2(k+1)}^{\{2(k+1)\}} e_0^{\{0\}} J_e + \nabla^{\{2(k+1)\}} \times p_0^{\{2(k+1)\}} e_0^{\{0\}} J_e + \sum_{i=0}^{2(k+1)-1} e_i^{\{0\}} \frac{d^{2(k+1)-i} J_e}{dt^{2(k+1)-i}} + \sum_{h=1}^k \left( \nabla^{\{2(k+1-h)\}} \times \left( p_{2h}^{\{2(k+1)\}} e_0^{\{0\}} J_e + \sum_{i=0}^{2h-1} e_i^{\{2(k+1-h)\}} \frac{d^{2(h)-i} J_e}{dt^{2(h)-i}} \right) \right) \\
&\text{by (18): } e_{2(k-i)}^{\{2i\}} = p_{2(k-i)}^{\{2k\}} e_0^{\{0\}}; i = k-h \rightarrow p_{2h}^{\{2k\}} e_0^{\{0\}} = e_{2h}^{\{2(k-h)\}} \rightarrow p_{2h}^{\{2(k+1)\}} e_0^{\{0\}} = e_{2h}^{\{2(k+1-h)\}}
\end{aligned}$$

We have

$$\begin{aligned}
&= p_{2(k+1)}^{\{2(k+1)\}} e_0^{\{0\}} J_e + \nabla^{\{2(k+1)\}} \times p_0^{\{2(k+1)\}} e_0^{\{0\}} J_e + \sum_{i=0}^{2(k+1)-1} e_i^{\{0\}} \frac{d^{2(k+1)-i} J_e}{dt^{2(k+1)-i}} + \sum_{h=1}^k \left( \nabla^{\{2(k+1-h)\}} \times \left( \sum_{i=0}^{2h} e_i^{\{2(k+1-h)\}} \frac{d^{2h-i} J_e}{dt^{2h-i}} \right) \right) \\
&\text{by (18): } e_{2(k-i)}^{\{2i\}} = p_{2(k-i)}^{\{2k\}} e_0^{\{0\}}; i = k \rightarrow e_0^{\{2k\}} = p_0^{\{2k\}} e_0^{\{0\}} \rightarrow e_0^{\{2(k+1)\}} = p_0^{\{2(k+1)\}} e_0^{\{0\}} \\
&= p_{2(k+1)}^{\{2(k+1)\}} e_0^{\{0\}} J_e + \sum_{i=0}^{2(k+1)-1} e_i^{\{0\}} \frac{d^{2(k+1)-i} J_e}{dt^{2(k+1)-i}} + \sum_{h=0}^k \left( \nabla^{\{2(k+1-h)\}} \times \left( \sum_{i=0}^{2h} e_i^{\{2(k+1-h)\}} \frac{d^{2h-i} J_e}{dt^{2h-i}} \right) \right) \\
&\text{by (18): } e_{2(k-i)}^{\{2i\}} = p_{2(k-i)}^{\{2k\}} e_0^{\{0\}}; i = 0 \rightarrow e_{2k}^{\{0\}} = p_{2k}^{\{2k\}} e_0^{\{0\}} \rightarrow e_{2(k+1)}^{\{0\}} = p_{2(k+1)}^{\{2(k+1)\}} e_0^{\{0\}} \\
&= \sum_{i=0}^{2(k+1)} e_i^{\{0\}} \frac{d^{2(k+1)-i} J_e}{dt^{2(k+1)-i}} + \sum_{h=0}^k \left( \nabla^{\{2(k+1-h)\}} \times \left( \sum_{i=0}^{2h} e_i^{\{2(k+1-h)\}} \frac{d^{2h-i} J_e}{dt^{2h-i}} \right) \right) \\
&= \sum_{h=0}^{k+1} \left( \nabla^{\{2(k+1-h)\}} \times \left( \sum_{i=0}^{2h} e_i^{\{2(k+1-h)\}} \frac{d^{2h-i} J_e}{dt^{2h-i}} \right) \right)
\end{aligned}$$

The above is the  $J_e$  terms for  $k+1$  as defined in (14). Thus we get (51).

Combine the above results, we have

$$\frac{\partial^{2(k+1)+1} E}{\partial t^{2(k+1)+1}} = \sum_{i=0}^{k+1} \left( p_{2i}^{\{2(k+1)+1\}} \nabla^{\{2(k+1-i)+1\}} \times H + p_{2i+1}^{\{2(k+1)+1\}} \nabla^{\{2(k+1-i)\}} \times E \right) + S_e(2(k+1)+1)$$

The above is the same as (49). Thus (10) and (14) hold for  $k + 1$ .

Thus, the Time Space Lemma hold for  $k + 1$ .

QED.