Extend the Generic FDTD to Lossy Medium

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Abstract

By proving a time space lemma for lossy medium, the previously proved time advancement theorems are extended to lossy medium. Thus, the generic FDTD form previously proved is extended to lossy medium.

Contents

Introduction

In [1] I derived a generic FDTD form for Maxwell equations in lossless medium. The targeted Maxwell equations are

$$\nabla \cdot E = \rho/\varepsilon \tag{1}$$

$$\nabla \cdot H = 0 \tag{2}$$

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon} \nabla \times H + J(t) \tag{3}$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu} \nabla \times E \tag{4}$$

Where t is time, ρ, ε and μ are time-invariants, $J(t)$ is a known 3D vector time function, E and H are 3D vectors in Cartesian coordinates (x, y, z), representing an electric field and a magnetic field, respectively, as

$$E(x,y,z,t) = \begin{bmatrix} E_x(x,y,z,t) \\ E_y(x,y,z,t) \\ E_z(x,y,z,t) \end{bmatrix}, H(x,y,z,t) = \begin{bmatrix} H_x(x,y,z,t) \\ H_y(x,y,z,t) \\ H_z(x,y,z,t) \end{bmatrix} \tag{5}$$

For lossy medium, adding conductivity constants σ and σ_m into to (3) and (4), we get the following Maxwell equations.

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon} \nabla \times H + J(t) - \frac{\sigma}{\varepsilon} E \tag{6}$$

$$\frac{\partial H}{\partial t} = -\frac{1}{\mu} \nabla \times E - \frac{\sigma_m}{\mu} H \tag{7}$$

In this paper, the time space lemma proved in [1] is extended to support lossy medium. Thus, the generic FDTD form derived in [1] is extended to support lossy medium.

Time Space Relations

Time Space Lemma for Lossy Medium. The following time space relations can be derived from equations (6) and (7).

$$\frac{\partial^{2k} H}{\partial t^{2k}} = \frac{(-1)^k}{(\varepsilon\mu)^k} \nabla^{\{2k\}} \times H + \sum_{i=1}^{k} \left(\frac{1}{\mu} \frac{(-1)^i}{(\varepsilon\mu)^{i-1}} \nabla^{\{2i+1\}} \times \frac{d^{2(k-i)} J}{dt^{2(k-i)}} + q_{2i-1}^{\{2k\}} \nabla^{\{2k-2i+1\}} \times E + q_{2i}^{\{2k\}} \nabla^{\{2k-2i\}} \times H \right) \tag{8}$$

$$\frac{\partial^{2k+1} H}{\partial t^{2k+1}} = \frac{1}{\mu} \frac{(-1)^{k+1}}{(\varepsilon\mu)^k} \nabla^{\{2k+1\}} \times E + \sum_{i=1}^{k} \left(\frac{1}{\mu} \frac{(-1)^i}{(\varepsilon\mu)^{i-1}} \nabla^{\{2i-1\}} \times \frac{d^{2(k-i)+1} J}{dt^{2(k-i)+1}} + q_{2i-1}^{\{2k+1\}} \nabla^{\{2(k+1)-2i\}} \times H + q_{2i}^{\{2k+1\}} \nabla^{\{2k-2i+1\}} \times E \right) - \left(\frac{\sigma_m}{\mu}\right)^{2k+1} H \tag{9}$$

$$q_1^{\{2\}} = \frac{\sigma}{\varepsilon\mu} + \frac{\sigma_m}{\mu^2} \tag{10}$$

$$q_k^{\{k\}} = \left(-\frac{\sigma_m}{\mu}\right)^k \tag{11}$$

$$q_0^{\{2k\}} = \left(-\frac{1}{\varepsilon\mu}\right)^k \tag{12}$$

$$q_0^{\{2k+1\}} = -\frac{1}{\mu}\left(-\frac{1}{\varepsilon\mu}\right)^k \tag{13}$$

$$q_{2i-1}^{\{2k+1\}} = \frac{1}{\varepsilon} q_{2i-1}^{\{2k\}} - \frac{\sigma_m}{\mu} q_{2i-2}^{\{2k\}} \tag{14}$$

$$q_{2i}^{\{2k+1\}} = -\frac{\sigma}{\varepsilon} q_{2i-1}^{\{2k\}} - \frac{1}{\mu} q_{2i}^{\{2k\}} \tag{15}$$

$$q_{2i-1}^{\{2(k+1)\}} = -\frac{1}{\mu} q_{2i-1}^{\{2k+1\}} - \frac{\sigma}{\varepsilon} q_{2i-2}^{\{2k+1\}} \tag{16}$$

$$q_{2i}^{\{2(k+1)\}} = -\frac{\sigma_m}{\mu} q_{2i-1}^{\{2k+1\}} + \frac{1}{\varepsilon} q_{2i}^{\{2k+1\}} \tag{17}$$

$$\frac{\partial^{2k} E}{\partial t^{2k}} = (-1)^k \frac{1}{(\varepsilon\mu)^k} \nabla^{\{2k\}} \times E + \sum_{i=1}^{k} \left(\frac{(-1)^{i-1}}{(\varepsilon\mu)^{i-1}} \nabla^{\{2(i-1)\}} \times \frac{d^{2(k-i)+1} J}{dt^{2(k-i)+1}} + p_{2i-1}^{\{2k\}} \nabla^{\{2k-2i+1\}} \times H + p_{2i}^{\{2k\}} \nabla^{\{2k-2i\}} \times E \right) \tag{18}$$

$$\frac{\partial^{2k+1} E}{\partial t^{2k+1}} = (-1)^k \frac{1}{\varepsilon} \frac{1}{(\varepsilon\mu)^k} \nabla^{\{2k+1\}} \times H + \sum_{i=1}^{k} \left(\frac{(-1)^i}{(\varepsilon\mu)^i} \nabla^{\{2i\}} \times \frac{d^{2(k-i)} J}{dt^{2(k-i)}} + p_{2i-1}^{\{2k+1\}} \nabla^{\{2k-2i+2\}} \times E + p_{2i}^{\{2k+1\}} \nabla^{\{2k-2i+1\}} \times H \right) + \frac{d^{2k} J}{dt^{2k}} - \left(\frac{\sigma}{\varepsilon}\right)^{2k+1} E \tag{19}$$

$$p_1^{\{2\}} = -\frac{1}{\varepsilon} \frac{\sigma_m}{\mu} - \frac{\sigma}{\varepsilon^2} \tag{20}$$

$$p_k^{\{k\}} = \left(-\frac{\sigma}{\varepsilon}\right)^k \tag{21}$$

$$p_0^{\{2k\}} = \left(-\frac{1}{\varepsilon\mu}\right)^k \tag{22}$$

$$p_0^{\{2k+1\}} = \frac{1}{\varepsilon}\left(-\frac{1}{\varepsilon\mu}\right)^k \tag{23}$$

$$p_{2i-1}^{\{2k+1\}} = -\frac{\sigma}{\varepsilon} p_{2i-2}^{\{2k\}} - \frac{1}{\mu} p_{2i-1}^{\{2k\}} \tag{24}$$

$$p_{2i}^{\{2k+1\}} = -\frac{\sigma_m}{\mu} p_{2i-1}^{\{2k\}} + \frac{1}{\varepsilon} p_{2i}^{\{2k\}} \tag{25}$$

$$p_{2i-1}^{\{2k+2\}} = \frac{1}{\varepsilon} p_{2i-1}^{\{2k+1\}} - \frac{\sigma_m}{\mu} p_{2i-2}^{\{2k+1\}} \tag{26}$$

$$p_{2i}^{\{2k+2\}} = -\frac{\sigma}{\varepsilon} p_{2i-1}^{\{2k+1\}} - \frac{1}{\mu} p_{2i}^{\{2k+1\}} \tag{27}$$

$$i = 1,2, \dots, k;\; k = 1,2, \dots$$

Note that constant values $p_i^{\{j\}}, q_i^{\{j\}}$ can be calculated before a field simulation begins.

The proof of this lemma is in the Appendix.

Time Advancement Theorems

Adding the above lemma to the time advancement theorems in [1], we have following theorems.

Time Advancement Theorem H. Given a set of field data $E(t_h)$, H, and H may have one or more sets of data in different times as classified in 3 cases identified by an integer q_h in following way

$$\begin{cases} H(t_h), q_h = 0 \\ H(t_h - \Delta_{t1}), \quad q_h = 1, \Delta_t = \Delta_{t1} \\ H(t_h), H(t_h - \Delta_{tq}), q_h > 0, q = 1,2, \dots, q_h \end{cases} \tag{28}$$

$$\Delta_t, \Delta_{t1}, \Delta_{t2}, \dots, \Delta_{tqh} > 0$$

$H(t_h + \Delta_t)$ can be expressed by

$$Case\ 1: H(t_h), q_h = 0 \rightarrow$$

$$H(t_h + \Delta_t) = H(t_h) + \sum_{k=0}^{\infty}\left(\frac{\Delta_t^{2k+1}}{(2k+1)!}\left[\frac{1}{\mu}\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\nabla^{\{2k+1\}} \times E(t_h) + J_h^{[2k+1]}(t_h)\right] + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!}\left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^{k+1}}\nabla^{\{2(k+1)\}} \times H(t_h) + J_h^{[2(k+1)]}(t_h)\right]\right) \tag{29}$$

$$Case\ 2: H(t_h - \Delta_{t1}), q_h = 1, \Delta_t = \Delta_{t1} \rightarrow$$

$$H(t_h + \Delta_t) = H(t_h - \Delta_t) + \sum_{k=0}^{\infty}\left(\frac{2\Delta_t^{2k+1}}{(2k+1)!}\left[\frac{1}{\mu}\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\nabla^{\{2k+1\}} \times E(t_h) + J_h^{[2k+1]}(t_h)\right]\right)$$

$$Case\ 3: H(t_h), H(t_h - \Delta_{tq}), q_h > 0, q = 1,2, \dots, q_h \rightarrow$$

$$H(t_h+\Delta_t)=\sum_{q=1}^{q_h}H\big(t_h-\Delta_{tq}\big)+(1-q_h)H(t_h)$$
$$+\sum_{k=0}^{\infty}\left(\frac{\Delta_t^{2k+1}+\sum_{q=1}^{qh}\big(\Delta_{tq}\big)^{2k+1}}{(2k+1)!}\left[\frac{1}{\mu}\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\nabla^{\{2k+1\}}\times E(t_h)+J_h^{[2k+1]}(t_h)\right]\right.$$
$$\left.+\frac{\Delta_t^{2(k+1)}-\sum_{q=1}^{qh}\big(\Delta_{tq}\big)^{2(k+1)}}{(2(k+1))!}\left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^{k+1}}\nabla^{\{2(k+1)\}}\times H(t_h)+J_h^{[2(k+1)]}(t_h)\right]\right)$$

Where

$$J_h^{[2k+1]}(t_h)=\begin{cases}\vec{0},k=0\\ \sum_{i=1}^{k}\left(\frac{1}{\mu}\frac{(-1)^i}{(\varepsilon\mu)^{i-1}}\nabla^{\{2i-1\}}\times\frac{d^{2(k-i)+1}J}{dt^{2(k-i)+1}}+q_{2i-1}^{\{2k+1\}}\nabla^{\{2(k+1)-2i\}}\times H+q_{2i}^{\{2k+1\}}\nabla^{\{2k-2i+1\}}\times E\right)-\left(\frac{\sigma_m}{\mu}\right)^{2k+1}H,k>0\end{cases}\qquad(30)$$

$$J_h^{[2(k+1)]}(t_h)=\sum_{i=1}^{k+1}\left(\frac{1}{\mu}\frac{(-1)^i}{(\varepsilon\mu)^{i-1}}\nabla^{\{2i-1\}}\times\frac{d^{2(k-i+1)}J(t)}{dt^{2(k-i+1)}}+q_{2i-1}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i+1\}}\times E+q_{2i}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i\}}\times H\right),k\ge 0\qquad(31)$$

Where $q_i^{\{j\}}$ are constants given by (10) – (17).

Note that Case 1 and Case 2 are two special cases of Case 3. We can see that for Case 1, the term $\sum_{q=1}^{q_h}H\big(t_h-\Delta_{tq}\big)$ disappears from Case 3; for Case 2, the term for $\nabla^{\{2(k+1)\}}\times H$ disappears from Case 3. To avoid confusions I list them separately.

Time Advancement Theorem E. Given a set of field data $E,H(t_e)$, and E may have one or more sets of data in different times as classified in 3 cases identified by an integer q_e in following way

$$\begin{cases}E(t_e),q_e=0\\ E(t_e-\Delta_{t1}),\quad q_e=1,\Delta_t=\Delta_{t1}\\ E(t_e),E\big(t_e-\Delta_{tq}\big),q_e>0,q=1,2,\dots,q_e\end{cases}\qquad(32)$$
$$\Delta_t,\Delta_{t1},\Delta_{t2},\dots,\Delta_{tqe}>0$$

$E(t_e+\Delta_t)$ can be expressed by

Case 1: $E(t_e),q_e=0\rightarrow$

$$E(t_e+\Delta_t)=E(t_e)+\sum_{k=0}^{\infty}\left(\frac{\Delta_t^{(2k+1)}}{(2k+1)!}\left[\frac{(-1)^k}{\varepsilon(\varepsilon\mu)^k}\nabla^{\{2k+1\}}\times H(t_e)+J_e^{[2k+1]}(t_e)\right]\right.$$
$$\left.+\frac{\Delta_t^{2(k+1)}}{(2(k+1))!}\left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\nabla^{\{2(k+1)\}}\times E(t_e)+J_e^{[2(k+1)]}(t_e)\right]\right)$$

(33)

Case 2: $E(t_e-\Delta_{t1}),q_e=1,\Delta_t=\Delta_{t1}\rightarrow$

$$E(t_e+\Delta_t)=E(t_e-\Delta_t)+\sum_{k=0}^{\infty}\left(\frac{2\Delta_t^{(2k+1)}}{(2k+1)!}\left[\frac{(-1)^k}{\varepsilon(\varepsilon\mu)^k}\nabla^{\{2k+1\}}\times H(t_e)+J_e^{[2k+1]}(t_e)\right]\right)$$

$E(t_e),E\big(t_e-\Delta_{tq}\big),q_e>0\rightarrow$

$$E(t_e+\Delta_t)=\sum_{q=1}^{q_e}E(t_e-\Delta_{tq})+(1-q_e)E(t_e)$$
$$+\sum_{k=0}^{\infty}\left(\frac{\Delta_t^{(2k+1)}+\sum_{q=1}^{qe}(\Delta_{tq})^{2k+1}}{(2k+1)!}\left[\frac{(-1)^k}{\varepsilon(\varepsilon\mu)^k}\nabla^{\{2k+1\}}\times H(t_e)+J_e^{[2k+1]}(t_e)\right]\right.$$
$$\left.+\frac{\Delta_t^{2(k+1)}-\sum_{q=1}^{qe}(\Delta_{tq})^{2(k+1)}}{(2(k+1))!}\left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^k}\nabla^{\{2(k+1)\}}\times E(t_e)+J_e^{[2(k+1)]}(t_e)\right]\right)$$

Where

$$J_e^{[2k+1]}(t_e)=\sum_{i=1}^{k}\left(\frac{(-1)^i}{(\varepsilon\mu)^i}\nabla^{\{2i\}}\times\frac{d^{2(k-i)}J}{dt^{2(k-i)}}+p_{2i-1}^{\{2k+1\}}\nabla^{\{2k-2i+2\}}\times E+p_{2i}^{\{2k+1\}}\nabla^{\{2k-2i+1\}}\times H\right)+\frac{d^{2k}J}{dt^{2k}}-\left(\frac{\sigma}{\varepsilon}\right)^{2k+1}E \qquad (34)$$
$$k\ge 0$$

$$J_e^{[2(k+1)]}(t_e)=\sum_{i=1}^{k+1}\left(\frac{(-1)^{i-1}}{(\varepsilon\mu)^{i-1}}\nabla^{\{2(i-1)\}}\times\frac{d^{2(k-i+1)+1}J(t)}{dt^{2(k-i+1)+1}}+p_{2i-1}^{\{2k+2\}}\nabla^{\{2(k+1)-2i+1\}}\times H+p_{2i}^{\{2k+2\}}\nabla^{\{2(k+1)-2i\}}\times E\right) \qquad (35)$$
$$k\ge 0$$

Where $p_i^{\{j\}}$ are constants given by (20) – (25).

Note that Case 1 and Case 2 are two special cases of Case 3. To avoid confusions I list them separately.

Generic FDTD Form

In [1], I have proved a curl cascade theorem for providing high order FDTD algorithms. The theorem can be used in this paper without modifications. But I would like to point out that the space samplings can be different for different space derivative estimators. I'll describe it below.

In [1], the space derivative estimators are given by

$$D_x^h(V_u)=\frac{\partial^h V_u}{\partial x^h}\approx\mathfrak{D}_x^h(V_u)\equiv[a_{h1}\quad a_{h2}\quad\cdots\quad a_{hM}]\begin{bmatrix}V_u(x+\Delta_1,y,z)-V_u(x,y,z)\\V_u(x+\Delta_2,y,z)-V_u(x,y,z)\\\vdots\\V_u(x+\Delta_M,y,z)-V_u(x,y,z)\end{bmatrix}$$
$$D_y^h(V_u)=\frac{\partial^h V_u}{\partial y^h}\approx\mathfrak{D}_y^h(V_u)\equiv[a_{h1}\quad a_{h2}\quad\cdots\quad a_{hM}]\begin{bmatrix}V_u(x,y+\Delta_1,z)-V_u(x,y,z)\\V_u(x,y+\Delta_2,z)-V_u(x,y,z)\\\vdots\\V_u(x,y+\Delta_M,z)-V_u(x,y,z)\end{bmatrix}$$
$$D_z^h(V_u)=\frac{\partial^h V_u}{\partial z^h}\approx\mathfrak{D}_z^h(V_u)\equiv[a_{h1}\quad a_{h2}\quad\cdots\quad a_{hM}]\begin{bmatrix}V_u(x,y,z+\Delta_1)-V_u(x,y,z)\\V_u(x,y,z+\Delta_2)-V_u(x,y,z)\\\vdots\\V_u(x,y,z+\Delta_M)-V_u(x,y,z)\end{bmatrix} \qquad (36)$$
$$u\ can\ be\ x,y,z;h=1,2,\ldots,M$$

Where the parameters are given by

$$\begin{bmatrix}\Delta_1&\frac{1}{2!}\Delta_1^2&\cdots&\frac{1}{M!}\Delta_1^M\\\Delta_2&\frac{1}{2!}\Delta_2^2&\cdots&\frac{1}{M!}\Delta_2^M\\\vdots&\vdots&\cdots&\vdots\\\Delta_M&\frac{1}{2!}\Delta_M^2&\cdots&\frac{1}{M!}\Delta_M^M\end{bmatrix}^{-1}=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1M}\\a_{21}&a_{22}&\cdots&a_{2M}\\\vdots&\vdots&\cdots&\vdots\\a_{M1}&a_{M2}&\cdots&a_{MM}\end{bmatrix}$$

The above definitions gave an impression that space sampling $\Delta_1,\Delta_2,\ldots,\Delta_M$ are the same for the 3 axis. But it is not necessarily so. Let's change the definitions to allow different samplings.

$$\begin{bmatrix} \Delta_{u1} & \frac{1}{2!}\Delta_{u1}^2 & \cdots & \frac{1}{M!}\Delta_{u1}^M \\ \Delta_{u2} & \frac{1}{2!}\Delta_{u2}^2 & \cdots & \frac{1}{M!}\Delta_{u2}^M \\ \vdots & \vdots & \cdots & \vdots \\ \Delta_{uM} & \frac{1}{2!}\Delta_{uM}^2 & \cdots & \frac{1}{M!}\Delta_{uM}^M \end{bmatrix}^{-1} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1M} \\ a_{21} & a_{22} & \cdots & a_{2M} \\ \vdots & \vdots & \cdots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MM} \end{bmatrix}_u = \begin{bmatrix} [a_{11} & a_{12} & \cdots & a_{1M}]_u \\ [a_{21} & a_{22} & \cdots & a_{2M}]_u \\ & & \vdots & \\ [a_{M1} & a_{M2} & \cdots & a_{MM}]_u \end{bmatrix} \tag{37}$$

$$D_x^h(V_u) = \frac{\partial^h V_u}{\partial x^h} \approx \mathfrak{D}_x^h(V_u) \equiv [a_{h1} \quad a_{h2} \quad \cdots \quad a_{hM}]_x \begin{bmatrix} V_u(x+\Delta_{x1},y,z) - V_u(x,y,z) \\ V_u(x+\Delta_{x2},y,z) - V_u(x,y,z) \\ \vdots \\ V_u(x+\Delta_{xM},y,z) - V_u(x,y,z) \end{bmatrix}$$

$$D_y^h(V_u) = \frac{\partial^h V_u}{\partial y^h} \approx \mathfrak{D}_y^h(V_u) \equiv [a_{h1} \quad a_{h2} \quad \cdots \quad a_{hM}]_y \begin{bmatrix} V_u\left(x,y+\Delta_{y1},z\right) - V_u(x,y,z) \\ V_u\left(x,y+\Delta_{y2},z\right) - V_u(x,y,z) \\ \vdots \\ V_u\left(x,y+\Delta_{yM},z\right) - V_u(x,y,z) \end{bmatrix} \tag{38}$$

$$D_z^h(V_u) = \frac{\partial^h V_u}{\partial z^h} \approx \mathfrak{D}_z^h(V_u) \equiv [a_{h1} \quad a_{h2} \quad \cdots \quad a_{hM}]_z \begin{bmatrix} V_u(x,y,z+\Delta_{z1}) - V_u(x,y,z) \\ V_u(x,y,z+\Delta_{z2}) - V_u(x,y,z) \\ \vdots \\ V_u(x,y,z+\Delta_{zM}) - V_u(x,y,z) \end{bmatrix}$$

$$u\ can\ be\ x,y,z;\ h = 1,2,\ldots,M$$

The generic FDTD form given by [1] is the same for lossy medium. I simply copy it below.

$Case\ 1: H(t_h), q_h = 0 \rightarrow$

$$H(t_h+\Delta_t) \approx H(t_h) + \sum_{k=0}^{k_{max}} \left(\frac{\Delta_t^{2k+1}}{(2k+1)!} \left[\frac{1}{\mu} \frac{(-1)^{k+1}}{(\varepsilon\mu)^k} \overline{\nabla}^{\{2k+1\}} \times E(t_h) + J_h^{[2k+1]}(t_h) \right] + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} \left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^{k+1}} \overline{\nabla}^{\{2(k+1)\}} \times H(t_h) + J_h^{[2(k+1)]}(t_h) \right] \right)$$

$Case\ 2: H(t_h - \Delta_{t1}), q_h = 1, \Delta_t = \Delta_{t1} \rightarrow$

$$H(t_h+\Delta_t) \approx H(t_h-\Delta_t) + \sum_{k=0}^{k_{max}} \left(\frac{2\Delta_t^{2k+1}}{(2k+1)!} \left[\frac{1}{\mu} \frac{(-1)^{k+1}}{(\varepsilon\mu)^k} \overline{\nabla}^{\{2k+1\}} \times E(t_h) + J_h^{[2k+1]}(t_h) \right] \right)$$

$Case\ 3: H(t_h), H\left(t_h - \Delta_{tq}\right), q_h > 0, q = 1,2,\ldots,q_h \rightarrow$

$$H(t_h+\Delta_t) \approx \sum_{q=1}^{q_h} H\left(t_h - \Delta_{tq}\right) + (1-q_h)H(t_h) + \sum_{k=0}^{k_{max}} \left(\frac{\Delta_t^{2k+1} + \sum_{q=1}^{qh}\left(\Delta_{tq}\right)^{2k+1}}{(2k+1)!} \left[\frac{1}{\mu} \frac{(-1)^{k+1}}{(\varepsilon\mu)^k} \overline{\nabla}^{\{2k+1\}} \times E(t_h) + J_h^{[2k+1]}(t_h) \right] + \frac{\Delta_t^{2(k+1)} - \sum_{q=1}^{qh}\left(\Delta_{tq}\right)^{2(k+1)}}{(2(k+1))!} \left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^{k+1}} \overline{\nabla}^{\{2(k+1)\}} \times H(t_h) + J_h^{[2(k+1)]}(t_h) \right] \right)$$

$$k_{max} \geq 0 \tag{39}$$

Case 1: $E(t_e), q_e = 0 \rightarrow$

$$E(t_e+\Delta_t) = E(t_e) + \sum_{k=0}^{k_{max}} \left(\frac{\Delta_t^{(2k+1)}}{(2k+1)!} \left[\frac{(-1)^k}{\varepsilon(\varepsilon\mu)^k} \overline{\nabla}^{\{2k+1\}} \times H(t_e) + J_e^{[2k+1]}(t_e) \right] + \frac{\Delta_t^{2(k+1)}}{(2(k+1))!} \left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^k} \overline{\nabla}^{\{2(k+1)\}} \times E(t_e) + J_e^{[2(k+1)]}(t_e) \right] \right)$$

$Case\ 2: E(t_e - \Delta_{t1}), q_e = 1, \Delta_t = \Delta_{t1} \rightarrow$

$$E(t_e+\Delta_t) = E(t_e-\Delta_t) + \sum_{k=0}^{k_{max}} \left(\frac{2\Delta_t^{(2k+1)}}{(2k+1)!} \left[\frac{(-1)^k}{\varepsilon(\varepsilon\mu)^k} \overline{\nabla}^{\{2k+1\}} \times H(t_e) + J_e^{[2k+1]}(t_e) \right] \right) \tag{40}$$

$E(t_e), E\left(t_e - \Delta_{tq}\right), q_e > 0 \rightarrow$ $E(t_e + \Delta_t) = \sum_{q=1}^{q_e} E\left(t_e - \Delta_{tq}\right) + (1 - q_e)E(t_e) + \sum_{k=0}^{k_{max}} \left(\frac{\Delta_t^{(2k+1)} + \sum_{q=1}^{qe}\left(\Delta_{tq}\right)^{2k+1}}{(2k+1)!} \left[\frac{(-1)^k}{\varepsilon(\varepsilon\mu)^k} \overline{\nabla}^{\{2k+1\}} \times H(t_e) + J_e^{[2k+1]}(t_e) \right] + \frac{\Delta_t^{2(k+1)} - \sum_{q=1}^{qe}\left(\Delta_{tq}\right)^{2(k+1)}}{(2(k+1))!} \left[\frac{(-1)^{k+1}}{(\varepsilon\mu)^k} \overline{\nabla}^{\{2(k+1)\}} \times E(t_e) + J_e^{[2(k+1)]}(t_e) \right] \right)$	
$k_{max} \geq 0$	

Note that $J_h^{[2k+1]}, J_h^{[2(k+1)]}, J_e^{[2k+1]}, J_e^{[2(k+1)]}$ are defined in (30), (31), (34) and (35). They include effects of lossy medium.

See [1] for curl estimations

$$\overline{\nabla}^{\{k\}} \times E, \overline{\nabla}^{\{k\}} \times H$$

Reference

[1] A Generic FDTD Form for Maxwell Equations, https://www.researchgate.net/publication/344868091_A_Generic_FDTD_Form_for_Maxwell_Equations

Appendix

Time Space Lemma for Lossy Medium. The time space relations described by (8) – (27) can be derived from equations (6) and (7).

Proof. To simplify the notations, define following values.

$\alpha_e = -\frac{\sigma}{\varepsilon}$	(41)
$\alpha_h = -\frac{\sigma_m}{\mu}$	(42)
$\beta_e = \frac{1}{\varepsilon}$	(43)
$\beta_h = -\frac{1}{\mu}$	(44)

Using the above notations, (6) and (7) become

$\frac{\partial E}{\partial t} = \beta_e \nabla \times H + \alpha_e E$	(45)
$\frac{\partial H}{\partial t} = \beta_h \nabla \times E + \alpha_h H$	(46)

Because $J(t)$ in (6) does not involve E and H , due to linearity of the equations, we do not need to prove the following terms in (8), (9), (18) and (19) because I already proved them in [1].

$$\frac{1}{\mu}\frac{(-1)^i}{(\varepsilon\mu)^{i-1}} \nabla^{\{2i+1\}} \times \frac{d^{2(k-i)}J}{dt^{2(k-i)}}$$

$$\frac{1}{\mu}\frac{(-1)^i}{(\varepsilon\mu)^{i-1}} \nabla^{\{2i-1\}} \times \frac{d^{2(k-i)+1}J}{dt^{2(k-i)+1}}$$

$$\frac{(-1)^{i-1}}{(\varepsilon\mu)^{i-1}} \nabla^{\{2(i-1)\}} \times \frac{d^{2(k-i)+1}J}{dt^{2(k-i)+1}}$$

$$\frac{(-1)^i}{(\varepsilon\mu)^i} \nabla^{\{2i\}} \times \frac{d^{2(k-i)}J}{dt^{2(k-i)}}$$

By removing the above terms from (8), (9), (18) and (19), and using the symbols defined in (41) – (44), what we need to prove is the following equations based on (45) and (46).

$\frac{\partial^{2k}H}{\partial t^{2k}} = (\beta_e\beta_h)^k\nabla^{\{2k\}}\times H + \sum_{i=1}^{k} q_{2i-1}^{\{2k\}}\nabla^{\{2k-2i+1\}}\times E + \sum_{i=1}^{k} q_{2i}^{\{2k\}}\nabla^{\{2k-2i\}}\times H$	(47)
$\frac{\partial^{2k+1}H}{\partial t^{2k+1}} = \beta_h(\beta_e\beta_h)^k\nabla^{\{2k+1\}}\times E + \sum_{i=1}^{k+1} q_{2i-1}^{\{2k+1\}}\nabla^{\{2(k+1)-2i\}}\times H + \sum_{i=1}^{k} q_{2i}^{\{2k+1\}}\nabla^{\{2k-2i+1\}}\times E$	(48)
$q_1^{\{2\}} = \frac{\sigma}{\varepsilon\mu} + \frac{\sigma_m}{\mu^2} = \alpha_e\beta_h + \alpha_h\beta_h$	(49)
$q_k^{\{k\}} = \left(-\frac{\sigma_m}{\mu}\right)^k = \alpha_h^k$	(50)
$q_0^{\{2k\}} = \left(-\frac{1}{\varepsilon\mu}\right)^k = (\beta_e\beta_h)^k$	(51)
$q_0^{\{2k+1\}} = -\frac{1}{\mu}\left(-\frac{1}{\varepsilon\mu}\right)^k = \beta_h(\beta_e\beta_h)^k$	(52)
$q_{2i-1}^{\{2k+1\}} = \frac{1}{\varepsilon}q_{2i-1}^{\{2k\}} - \frac{\sigma_m}{\mu}q_{2i-2}^{\{2k\}} = \beta_e q_{2i-1}^{\{2k\}} + \alpha_h q_{2i-2}^{\{2k\}}$	(53)
$q_{2i}^{\{2k+1\}} = -\frac{\sigma}{\varepsilon}q_{2i-1}^{\{2k\}} - \frac{1}{\mu}q_{2i}^{\{2k\}} = \alpha_e q_{2i-1}^{\{2k\}} + \beta_h q_{2i}^{\{2k\}}$	(54)
$q_{2i-1}^{\{2(k+1)\}} = -\frac{1}{\mu}q_{2i-1}^{\{2k+1\}} - \frac{\sigma}{\varepsilon}q_{2i-2}^{\{2k+1\}} = \beta_h q_{2i-1}^{\{2k+1\}} + \alpha_e q_{2i-2}^{\{2k+1\}}$	(55)
$q_{2i}^{\{2(k+1)\}} = -\frac{\sigma_m}{\mu}q_{2i-1}^{\{2k+1\}} + \frac{1}{\varepsilon}q_{2i}^{\{2k+1\}} = \alpha_h q_{2i-1}^{\{2k+1\}} + \beta_e q_{2i}^{\{2k+1\}}$	(56)
$\frac{\partial^{2k}E}{\partial t^{2k}} = (\beta_e\beta_h)^k\nabla^{\{2k\}}\times E + \sum_{i=1}^{k} p_{2i-1}^{\{2k\}}\nabla^{\{2k-2i+1\}}\times H + \sum_{i=1}^{k} p_{2i}^{\{2k\}}\nabla^{\{2k-2i\}}\times E$	(57)
$\frac{\partial^{2k+1}E}{\partial t^{2k+1}} = \beta_e(\beta_e\beta_h)^k\nabla^{\{2k+1\}}\times H + \sum_{i=1}^{k+1} p_{2i-1}^{\{2k+1\}}\nabla^{\{2k-2i+2\}}\times E + \sum_{i=1}^{k} p_{2i}^{\{2k+1\}}\nabla^{\{2k-2i+1\}}\times H$	(58)
$p_1^{\{2\}} = -\frac{1}{\varepsilon}\frac{\sigma_m}{\mu} - \frac{\sigma}{\varepsilon^2} = \beta_e\alpha_h + \alpha_e\beta_e$	(59)
$p_k^{\{k\}} = \left(-\frac{\sigma}{\varepsilon}\right)^k = \alpha_e^k$	(60)
$p_0^{\{2k\}} = \left(-\frac{1}{\varepsilon\mu}\right)^k = (\beta_e\beta_h)^k$	(61)
$p_0^{\{2k+1\}} = \frac{1}{\varepsilon}\left(-\frac{1}{\varepsilon\mu}\right)^k = \beta_e(\beta_e\beta_h)^k$	(62)
$p_{2i-1}^{\{2k+1\}} = -\frac{\sigma}{\varepsilon}p_{2i-2}^{\{2k\}} - \frac{1}{\mu}p_{2i-1}^{\{2k\}} = \alpha_e p_{2i-2}^{\{2k\}} + \beta_h p_{2i-1}^{\{2k\}}$	(63)
$p_{2i}^{\{2k+1\}} = -\frac{\sigma_m}{\mu}p_{2i-1}^{\{2k\}} + \frac{1}{\varepsilon}p_{2i}^{\{2k\}} = \alpha_h p_{2i-1}^{\{2k\}} + \beta_e p_{2i}^{\{2k\}}$	(64)
$p_{2i-1}^{\{2k+2\}} = \frac{1}{\varepsilon}p_{2i-1}^{\{2k+1\}} - \frac{\sigma_m}{\mu}p_{2i-2}^{\{2k+1\}} = \beta_e p_{2i-1}^{\{2k+1\}} + \alpha_h p_{2i-2}^{\{2k+1\}}$	(65)
$p_{2i}^{\{2k+2\}} = -\frac{\sigma}{\varepsilon}p_{2i-1}^{\{2k+1\}} - \frac{1}{\mu}p_{2i}^{\{2k+1\}} = \alpha_e p_{2i-1}^{\{2k+1\}} + \beta_h p_{2i}^{\{2k+1\}}$	(66)
$k > 0, i = 1,2,\ldots,k$	

Note that $p_j^{\{i\}}$ *and* $q_j^{\{i\}}$ are constants. We do not need to prove them. What we need to prove are 4 equations (47), (48), (57) and (58).

Let

$$k = 1$$

(47) becomes

$$\frac{\partial^2 H}{\partial t^2} = \beta_e\beta_h\nabla^{\{2\}}\times H + q_1^{\{2\}}\nabla^{\{1\}}\times E + q_2^{\{2\}}\nabla^{\{0\}}\times H$$

By (49) and (50), we have

$$\frac{\partial^2 H}{\partial t^2} = \beta_e\beta_h \nabla^{\{2\}} \times H + (\alpha_e\beta_h + \alpha_h\beta_h)\nabla \times E + \alpha_h^2 H \tag{67}$$

Taking temporal derivative on (46) and substituting (45) and (46) into it, we have

$$\frac{\partial^2 H}{\partial t^2} = \beta_h \nabla \times (\beta_e \nabla \times H + \alpha_e E) + \alpha_h(\beta_h \nabla \times E + \alpha_h H) = \beta_e\beta_h \nabla^{\{2\}} \times H + \alpha_e\beta_h \nabla \times E + \alpha_h\beta_h \nabla \times E + \alpha_h^2 H$$
$$= \beta_e\beta_h \nabla^{\{2\}} \times H + (\alpha_e\beta_h + \alpha_h\beta_h)\nabla \times E + \alpha_h^2 H$$

It is the same as (67). Thus, from (45) and (46), we can get (47).

So, (47) holds for $k = 1$.

For $k = 1$, (48) becomes

$$\frac{\partial^3 H}{\partial t^3} = \beta_h\beta_e\beta_h \nabla^{\{3\}} \times E + q_1^{\{3\}} \nabla^{\{2\}} \times H + q_3^{\{3\}} \nabla^{\{0\}} \times H + q_2^{\{3\}} \nabla^{\{1\}} \times E \tag{68}$$

Taking temporal derivative on (67) and substituting (45) and (46) into it, we have

$$\frac{\partial^3 H}{\partial t^3} = \beta_e\beta_h \nabla^{\{2\}} \times (\beta_h \nabla \times E + \alpha_h H) + (\alpha_e\beta_h + \alpha_h\beta_h)\nabla \times (\beta_e \nabla \times H + \alpha_e E) + \alpha_h^2(\beta_h \nabla \times E + \alpha_h H)$$
$$= \beta_h\beta_e\beta_h \nabla^{\{3\}} \times E + \alpha_h\beta_e\beta_h \nabla^{\{2\}} \times H + \beta_e(\alpha_e\beta_h + \alpha_h\beta_h)\nabla^{\{2\}} \times H + \alpha_e(\alpha_e\beta_h + \alpha_h\beta_h)\nabla \times E + \alpha_h^2\beta_h \nabla \times E + \alpha_h^3 H$$

$$\frac{\partial^3 H}{\partial t^3} = \beta_h\beta_e\beta_h \nabla^{\{3\}} \times E + \big(\alpha_h\beta_e\beta_h + \beta_e(\alpha_e\beta_h + \alpha_h\beta_h)\big)\nabla^{\{2\}} \times H + (\alpha_e(\alpha_e\beta_h + \alpha_h\beta_h) + \alpha_h^2\beta_h)\nabla \times E + \alpha_h^3 H \tag{69}$$

By (53), (49) and (51) we have

$$q_1^{\{3\}} = \beta_e q_1^{\{2\}} + \alpha_h q_0^{\{2\}} = \beta_e(\alpha_e\beta_h + \alpha_h\beta_h) + \alpha_h\beta_e\beta_h$$

By (54), (49) and (50) we have

$$q_2^{\{3\}} = \alpha_e q_1^{\{2\}} + \beta_h q_2^{\{2\}} = \alpha_e(\alpha_e\beta_h + \alpha_h\beta_h) + \beta_h a_h^2$$

Substituting the above results into (69), we have

$$\frac{\partial^3 H}{\partial t^3} = \beta_h\beta_e\beta_h \nabla^{\{3\}} \times E + q_1^{\{3\}} \nabla^{\{2\}} \times H + q_2^{\{3\}} \nabla^{\{1\}} \times E + q_3^{\{3\}} \nabla^{\{0\}} \times H$$

It is the same as (68). Thus, (48) holds for $k = 1$.

For $k = 1$, (57) becomes

$$\frac{\partial^2 E}{\partial t^2} = \beta_e\beta_h \nabla^{\{2\}} \times E + p_1^{\{2\}} \nabla^{\{1\}} \times H + p_2^{\{2\}} E \tag{70}$$

Taking temporal derivatives on (45), and substituting (45) and (46) into it, we have

$$\frac{\partial^2 E}{\partial t^2} = \beta_e \nabla \times (\beta_h \nabla \times E + \alpha_h H) + \alpha_e(\beta_e \nabla \times H + \alpha_e E) = \beta_e\beta_h \nabla^{\{2\}} \times E + \alpha_h\beta_e \nabla \times H + \alpha_e\beta_e \nabla \times H + \alpha_e^2 E$$
$$= \beta_e\beta_h \nabla^{\{2\}} \times E + (\alpha_h\beta_e + \alpha_e\beta_e)\nabla \times H + \alpha_e^2 E$$

By (59) and (60), we have

$$\frac{\partial^2 E}{\partial t^2} = \beta_e\beta_h \nabla^{\{2\}} \times E + p_1^{\{2\}} \nabla \times H + p_2^{\{2\}} E$$

We arrive at (70). Thus, (57) holds for $k = 1$.

For $k = 1$, (58) becomes

$$\frac{\partial^3 E}{\partial t^3} = \beta_e\beta_e\beta_h \nabla^{\{3\}} \times H + p_1^{\{3\}} \nabla^{\{2\}} \times E + p_3^{\{3\}} \nabla^{\{0\}} \times E + p_2^{\{3\}} \nabla^{\{1\}} \times H \tag{71}$$

Taking temporal derivative on (70) and substituting (45) and (46) into it, we have

$$\frac{\partial^3 E}{\partial t^3} = \beta_e\beta_h \nabla^{\{2\}} \times (\beta_e \nabla \times H + \alpha_e E) + p_1^{\{2\}} \nabla^{\{1\}} \times (\beta_h \nabla \times E + \alpha_h H) + p_2^{\{2\}}(\beta_e \nabla \times H + \alpha_e E)$$
$$= \beta_e\beta_e\beta_h \nabla^{\{3\}} \times H + \alpha_e\beta_e\beta_h \nabla^{\{2\}} \times E + \beta_h p_1^{\{2\}} \nabla^{\{2\}} \times E + \alpha_h p_1^{\{2\}} \nabla^{\{1\}} \times H + \beta_e p_2^{\{2\}} \nabla \times H + \alpha_e p_2^{\{2\}} E$$

We have

$$\frac{\partial^3 E}{\partial t^3} = \beta_e\beta_e\beta_h\nabla^{\{3\}}\times H + (\alpha_e\beta_e\beta_h + \beta_h p_1^{\{2\}})\nabla^{\{2\}}\times E + (\alpha_h p_1^{\{2\}} + \beta_e p_2^{\{2\}})\nabla\times H + \alpha_e p_2^{\{2\}} E \quad (72)$$

By (63) and (61) we have

$$p_1^{\{3\}} = \alpha_e p_0^{\{2\}} + \beta_h p_1^{\{2\}} = \alpha_e\beta_e\beta_h + \beta_h p_1^{\{2\}}$$

By (64), (49) and (50) we have

$$p_2^{\{3\}} = \alpha_h p_1^{\{2\}} + \beta_e p_2^{\{2\}}$$

Substituting the above results into (72), we have

$$\frac{\partial^3 E}{\partial t^3} = \beta_e\beta_e\beta_h\nabla^{\{3\}}\times H + p_1^{\{3\}}\nabla^{\{2\}}\times E + p_2^{\{3\}}\nabla\times H + \alpha_e p_2^{\{2\}} E$$

It is the same as (71). Thus, (58) holds for $k = 1$.

Thus, the lemma holds for $k = 1$.

Suppose for an integer $k, k > 0$, the lemma holds.

For $k + 1$, (47) becomes

$$\frac{\partial^{2(k+1)} H}{\partial t^{2(k+1)}} = (\beta_e\beta_h)^{k+1}\nabla^{\{2(k+1)\}}\times H + \sum_{i=1}^{k+1} q_{2i-1}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i+1\}}\times E + \sum_{i=1}^{k+1} q_{2i}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i\}}\times H \quad (73)$$

Taking temporal derivatives on (48) and substituting (45) and (46) into it, we have

$$\begin{aligned}\frac{\partial^{2k+2} H}{\partial t^{2k+2}} &= \beta_h(\beta_e\beta_h)^k\nabla^{\{2k+1\}}\times(\beta_e\nabla\times H + \alpha_e E) + \sum_{i=1}^{k+1} q_{2i-1}^{\{2k+1\}}\nabla^{\{2(k+1)-2i\}}\times(\beta_h\nabla\times E + \alpha_h H) + \sum_{i=1}^{k} q_{2i}^{\{2k+1\}}\nabla^{\{2k-2i+1\}}\times(\beta_e\nabla\times H + \alpha_e E)\\
&= (\beta_e\beta_h)^{k+1}\nabla^{\{2k+2\}}\times H + \alpha_e\beta_h(\beta_e\beta_h)^k\nabla^{\{2k+1\}}\times E + \sum_{i=1}^{k+1}\beta_h q_{2i-1}^{\{2k+1\}}\nabla^{\{2(k+1)-2i+1\}}\times E + \sum_{i=1}^{k+1}\alpha_h q_{2i-1}^{\{2k+1\}}\nabla^{\{2(k+1)-2i\}}\times H\\
&\quad + \sum_{i=1}^{k}\beta_e q_{2i}^{\{2k+1\}}\nabla^{\{2k-2i+2\}}\times H + \sum_{i=1}^{k}\alpha_e q_{2i}^{\{2k+1\}}\nabla^{\{2k-2i+1\}}\times E\\
&= (\beta_e\beta_h)^{k+1}\nabla^{\{2k+2\}}\times H + \alpha_e\beta_h(\beta_e\beta_h)^k\nabla^{\{2k+1\}}\times E + \sum_{i=1}^{k+1}\beta_h q_{2i-1}^{\{2k+1\}}\nabla^{\{2(k+1)-2i+1\}}\times E + \sum_{i=1}^{k+1}\alpha_h q_{2i-1}^{\{2k+1\}}\nabla^{\{2(k+1)-2i\}}\times H\\
&\quad + \sum_{i=1}^{k}\beta_e q_{2i}^{\{2k+1\}}\nabla^{\{2k-2i+2\}}\times H + \sum_{i=2}^{k+1}\alpha_e q_{2i-2}^{\{2k+1\}}\nabla^{\{2(k+1)-2i+1\}}\times E\end{aligned}$$

By (52), the above becomes

$$\begin{aligned}\frac{\partial^{2k+2} H}{\partial t^{2k+2}} = (\beta_e\beta_h)^{k+1}\nabla^{\{2k+2\}}\times H + \sum_{i=1}^{k+1}(\beta_h q_{2i-1}^{\{2k+1\}} + \alpha_e q_{2i-2}^{\{2k+1\}})\nabla^{\{2(k+1)-2i+1\}}\times E + \alpha_h q_{2k+1}^{\{2k+1\}}\nabla^{\{0\}}\times H\\
+ \sum_{i=1}^{k}(\alpha_h q_{2i-1}^{\{2k+1\}} + \beta_e q_{2i}^{\{2k+1\}})\nabla^{\{2(k+1)-2i\}}\times H\end{aligned}$$

Substituting (55) and (56) into the above, we have

$$\frac{\partial^{2k+2} H}{\partial t^{2k+2}} = (\beta_e\beta_h)^{k+1}\nabla^{\{2k+2\}}\times H + \sum_{i=1}^{k+1} q_{2i-1}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i+1\}}\times E + \alpha_h q_{2k+1}^{\{2k+1\}}\nabla^{\{0\}}\times H + \sum_{i=1}^{k} q_{2i}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i\}}\times H$$

Substituting (50) into the above, we have

$$\frac{\partial^{2k+2} H}{\partial t^{2k+2}} = (\beta_e\beta_h)^{k+1}\nabla^{\{2k+2\}}\times H + \sum_{i=1}^{k+1} q_{2i-1}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i+1\}}\times E + \sum_{i=1}^{k+1} q_{2i}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i\}}\times H$$

It is the same as (73). Thus, (47) holds for the case of $k + 1$

For $k + 1$, (48) becomes

$$\frac{\partial^{2(k+1)+1}H}{\partial t^{2(k+1)+1}} = \beta_h(\beta_e\beta_h)^{k+1}\nabla^{\{2(k+1)+1\}} \times E + \sum_{i=1}^{k+2} q_{2i-1}^{\{2(k+1)+1\}}\nabla^{\{2(k+2)-2i\}} \times H + \sum_{i=1}^{k+1} q_{2i}^{\{2k+3\}}\nabla^{\{2k-2i+3\}} \times E \qquad (74)$$

Taking temporal derivatives on (73) and substituting (45) and (46) into it, we have

$$\begin{aligned}\frac{\partial^{2(k+1)+1}H}{\partial t^{2(k+1)+1}} &= (\beta_e\beta_h)^{k+1}\nabla^{\{2(k+1)\}} \times (\beta_h\nabla \times E + \alpha_h H) + \sum_{i=1}^{k+1} q_{2i-1}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i+1\}} \times (\beta_e\nabla \times H + \alpha_e E) \\ &+ \sum_{i=1}^{k+1} q_{2i}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i\}} \times (\beta_h\nabla \times E + \alpha_h H) \\ &= \beta_h(\beta_e\beta_h)^{k+1}\nabla^{\{2(k+1)+1\}} \times E + \alpha_h(\beta_e\beta_h)^{k+1}\nabla^{\{2(k+1)\}} \times H + \sum_{i=1}^{k+1} \beta_e q_{2i-1}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i+2\}} \times H \\ &+ \sum_{i=1}^{k+1} \alpha_e q_{2i-1}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i+1\}} \times E + \sum_{i=1}^{k+1} \beta_h q_{2i}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i+1\}} \times E + \sum_{i=1}^{k+1} \alpha_h q_{2i}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i\}} \times H\end{aligned}$$

By (51), we have

$$\begin{aligned}\frac{\partial^{2(k+1)+1}H}{\partial t^{2(k+1)+1}} &= \beta_h(\beta_e\beta_h)^{k+1}\nabla^{\{2(k+1)+1\}} \times E + \sum_{i=1}^{k+1} \beta_e q_{2i-1}^{\{2(k+1)\}}\nabla^{\{2(k+2)-2i\}} \times H + \sum_{i=1}^{k+1} (\alpha_e q_{2i-1}^{\{2(k+1)\}} + \beta_h q_{2i}^{\{2(k+1)\}})\nabla^{\{2(k+1)-2i+1\}} \times E \\ &+ \sum_{i=0}^{k+1} \alpha_h q_{2i}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i\}} \times H \\ &= \beta_h(\beta_e\beta_h)^{k+1}\nabla^{\{2(k+1)+1\}} \times E + \sum_{i=1}^{k+1} \beta_e q_{2i-1}^{\{2(k+1)\}}\nabla^{\{2(k+2)-2i\}} \times H + \sum_{i=1}^{k+1} (\alpha_e q_{2i-1}^{\{2(k+1)\}} + \beta_h q_{2i}^{\{2(k+1)\}})\nabla^{\{2(k+1)-2i+1\}} \times E \\ &+ \sum_{i=1}^{k+2} \alpha_h q_{2i-2}^{\{2(k+1)\}}\nabla^{\{2(k+2)-2i\}} \times H\end{aligned}$$

$$\begin{aligned}\frac{\partial^{2(k+1)+1}H}{\partial t^{2(k+1)+1}} &= \beta_h(\beta_e\beta_h)^{k+1}\nabla^{\{2(k+1)+1\}} \times E + \sum_{i=1}^{k+1} (\beta_e q_{2i-1}^{\{2(k+1)\}} + \alpha_h q_{2i-2}^{\{2(k+1)\}})\nabla^{\{2(k+2)-2i\}} \times H + \sum_{i=1}^{k+1} (\alpha_e q_{2i-1}^{\{2(k+1)\}} + \beta_h q_{2i}^{\{2(k+1)\}})\nabla^{\{2(k+1)-2i+1\}} \times E \\ &+ \alpha_h q_{2(k+1)}^{\{2(k+1)\}}\nabla^{\{0\}} \times H\end{aligned}$$

By (53), we have

$$q_{2i-1}^{\{2(k+1)+1\}} = \beta_e q_{2i-1}^{\{2(k+1)\}} + \alpha_h q_{2i-2}^{\{2(k+1)\}}$$

By (54), we have

$$q_{2i}^{\{2(k+1)+1\}} = \alpha_e q_{2i-1}^{\{2(k+1)\}} + \beta_h q_{2i}^{\{2(k+1)\}}$$

By (50), we have

$$(q_{2i-1}^{\{2(k+1)+1\}})_{i=k+2} = q_{2(k+1)+1}^{\{2(k+1)+1\}} = \alpha_h q_{2(k+1)}^{\{2(k+1)\}}$$

Thus,

$$\frac{\partial^{2(k+1)+1}H}{\partial t^{2(k+1)+1}} = \beta_h(\beta_e\beta_h)^{k+1}\nabla^{\{2(k+1)+1\}} \times E + \sum_{i=1}^{k+2} q_{2i-1}^{\{2(k+1)+1\}}\nabla^{\{2(k+2)-2i\}} \times H + \sum_{i=1}^{k+1} q_{2i}^{\{2(k+1)+1\}}\nabla^{\{2(k+1)-2i+1\}} \times E$$

It is the same as (74). Thus, (48) holds for the case of $k+1$.

For $k+1$, (57) becomes

$$\frac{\partial^{2(k+1)}E}{\partial t^{2(k+1)}} = (\beta_e\beta_h)^{k+1}\nabla^{\{2(k+1)\}} \times E + \sum_{i=1}^{k+1} p_{2i-1}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i+1\}} \times H + \sum_{i=1}^{k+1} p_{2i}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i\}} \times E \qquad (75)$$

Taking temporal derivatives on (58) and substituting (45) and (46) into it, we have

$$\frac{\partial^{2k+2}E}{\partial t^{2k+2}} = \beta_e(\beta_e\beta_h)^k\nabla^{\{2k+1\}}\times(\beta_h\nabla\times E+\alpha_h H)+\sum_{i=1}^{k+1}p_{2i-1}^{\{2k+1\}}\nabla^{\{2k-2i+2\}}\times(\beta_e\nabla\times H+\alpha_e E)+\sum_{i=1}^{k}p_{2i}^{\{2k+1\}}\nabla^{\{2k-2i+1\}}\times(\beta_h\nabla\times E+\alpha_h H)$$

$$= (\beta_e\beta_h)^{k+1}\nabla^{\{2k+2\}}\times E+\alpha_h\beta_e(\beta_e\beta_h)^k\nabla^{\{2k+1\}}\times H+\sum_{i=1}^{k+1}\beta_e p_{2i-1}^{\{2k+1\}}\nabla^{\{2k-2i+3\}}\times H+\sum_{i=1}^{k+1}\alpha_e p_{2i-1}^{\{2k+1\}}\nabla^{\{2k-2i+2\}}\times E$$

$$+\sum_{i=1}^{k}\beta_h p_{2i}^{\{2k+1\}}\nabla^{\{2k-2i+2\}}\times E+\sum_{i=1}^{k}\alpha_h p_{2i}^{\{2k+1\}}\nabla^{\{2k-2i+1\}}\times H$$

$$= (\beta_e\beta_h)^{k+1}\nabla^{\{2k+2\}}\times E+\alpha_h\beta_e(\beta_e\beta_h)^k\nabla^{\{2k+1\}}\times H+\sum_{i=1}^{k+1}\beta_e p_{2i-1}^{\{2k+1\}}\nabla^{\{2k-2i+3\}}\times H+\sum_{i=1}^{k+1}\alpha_e p_{2i-1}^{\{2k+1\}}\nabla^{\{2k-2i+2\}}\times E$$

$$+\sum_{i=1}^{k}\beta_h p_{2i}^{\{2k+1\}}\nabla^{\{2k-2i+2\}}\times E+\sum_{i=2}^{k+1}\alpha_h p_{2i-2}^{\{2k+1\}}\nabla^{\{2k-2i+3\}}\times H$$

By (62), the above becomes

$$\frac{\partial^{2k+2}E}{\partial t^{2k+2}} = (\beta_e\beta_h)^{k+1}\nabla^{\{2k+2\}}\times E+\sum_{i=1}^{k+1}(\beta_e p_{2i-1}^{\{2k+1\}}+\alpha_h p_{2i-2}^{\{2k+1\}})\nabla^{\{2k-2i+3\}}\times H+\sum_{i=1}^{k+1}\alpha_e p_{2i-1}^{\{2k+1\}}\nabla^{\{2k-2i+2\}}\times E+\sum_{i=1}^{k}\beta_h p_{2i}^{\{2k+1\}}\nabla^{\{2k-2i+2\}}\times E$$

Substituting (65) and (66) into the above, we have

$$\frac{\partial^{2k+2}E}{\partial t^{2k+2}} = (\beta_e\beta_h)^{k+1}\nabla^{\{2k+2\}}\times E+\sum_{i=1}^{k+1}p_{2i-1}^{\{2k+2\}}\nabla^{\{2k-2i+3\}}\times H+\alpha_e p_{2k+1}^{\{2k+1\}}+\sum_{i=1}^{k}p_{2i}^{\{2k+2\}}\nabla^{\{2k-2i+2\}}\times E$$

Substituting (60) into the above, we have

$$\frac{\partial^{2k+2}E}{\partial t^{2k+2}} = (\beta_e\beta_h)^{k+1}\nabla^{\{2k+2\}}\times E+\sum_{i=1}^{k+1}p_{2i-1}^{\{2k+2\}}\nabla^{\{2k-2i+3\}}\times H+\sum_{i=1}^{k+1}p_{2i}^{\{2k+2\}}\nabla^{\{2k-2i+2\}}\times E$$

It is the same as (75). Thus, (57) holds for the case of $k+1$

For $k+1$, (58) becomes

$$\frac{\partial^{2(k+1)+1}E}{\partial t^{2(k+1)+1}} = \beta_e(\beta_e\beta_h)^{k+1}\nabla^{\{2(k+1)+1\}}\times H+\sum_{i=1}^{k+2}p_{2i-1}^{\{2(k+1)+1\}}\nabla^{\{2(k+1)-2i+2\}}\times E+\sum_{i=1}^{k+1}p_{2i}^{\{2(k+1)+1\}}\nabla^{\{2(k+1)-2i+1\}}\times H \qquad (76)$$

Taking temporal derivatives on (75) and substituting (45) and (46) into it, we have

$$\frac{\partial^{2(k+1)+1}E}{\partial t^{2(k+1)+1}} = (\beta_e\beta_h)^{k+1}\nabla^{\{2(k+1)\}}\times(\beta_e\nabla\times H+\alpha_e E)+\sum_{i=1}^{k+1}p_{2i-1}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i+1\}}\times(\beta_h\nabla\times E+\alpha_h H)$$

$$+\sum_{i=1}^{k+1}p_{2i}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i\}}\times(\beta_e\nabla\times H+\alpha_e E)$$

$$= \beta_e(\beta_e\beta_h)^{k+1}\nabla^{\{2(k+1)+1\}}\times H+\alpha_e(\beta_e\beta_h)^{k+1}\nabla^{\{2(k+1)\}}\times E+\sum_{i=1}^{k+1}\beta_h p_{2i-1}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i+2\}}\times E$$

$$+\sum_{i=1}^{k+1}\alpha_h p_{2i-1}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i+1\}}\times H+\sum_{i=1}^{k+1}\beta_e p_{2i}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i+1\}}\times H+\sum_{i=1}^{k+1}\alpha_e p_{2i}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i\}}\times E$$

By (61), we have

$$\frac{\partial^{2(k+1)+1}E}{\partial t^{2(k+1)+1}} = \beta_e(\beta_e\beta_h)^{k+1}\nabla^{\{2(k+1)+1\}}\times H+\sum_{i=1}^{k+1}(\alpha_h p_{2i-1}^{\{2(k+1)\}}+\beta_e p_{2i}^{\{2(k+1)\}})\nabla^{\{2(k+1)-2i+1\}}\times H+\sum_{i=1}^{k+1}\beta_h p_{2i-1}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i+2\}}\times E$$

$$+\sum_{i=0}^{k+1}\alpha_e p_{2i}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i\}}\times E$$

$$= \beta_e(\beta_e\beta_h)^{k+1}\nabla^{\{2(k+1)+1\}}\times H+\sum_{i=1}^{k+1}(\alpha_h p_{2i-1}^{\{2(k+1)\}}+\beta_e p_{2i}^{\{2(k+1)\}})\nabla^{\{2(k+1)-2i+1\}}\times H+\sum_{i=1}^{k+1}\beta_h p_{2i-1}^{\{2(k+1)\}}\nabla^{\{2(k+1)-2i+2\}}\times E$$

$$+\sum_{i=1}^{k+2}\alpha_e p_{2i-2}^{\{2(k+1)\}}\nabla^{\{2(k+2)-2i\}}\times E$$

$$\frac{\partial^{2(k+1)+1}E}{\partial t^{2(k+1)+1}} = \beta_e(\beta_e\beta_h)^{k+1}\nabla^{\{2(k+1)+1\}} \times H + \sum_{i=1}^{k+1}(\alpha_h p_{2i-1}^{\{2(k+1)\}} + \beta_e p_{2i}^{\{2(k+1)\}})\nabla^{\{2(k+1)-2i+1\}} \times H$$
$$+ \sum_{i=1}^{k+1}(\beta_h p_{2i-1}^{\{2(k+1)\}} + \alpha_e p_{2i-2}^{\{2(k+1)\}})\nabla^{\{2(k+1)-2i+2\}} \times E + \alpha_e p_{2(k+1)}^{\{2(k+1)\}} E$$

By (63), we have

$$p_{2i-1}^{\{2(k+1)+1\}} = \alpha_e p_{2i-2}^{\{2(k+1)\}} + \beta_h p_{2i-1}^{\{2(k+1)\}}$$

By (64), we have

$$p_{2i}^{\{2(k+1)+1\}} = \alpha_h p_{2i-1}^{\{2(k+1)\}} + \beta_e p_{2i}^{\{2(k+1)\}}$$

By (60), we have

$$(p_{2i-1}^{\{2(k+1)+1\}})_{i=k+2} = p_{2(k+1)+1}^{\{2(k+1)+1\}} = \alpha_e p_{2(k+1)}^{\{2(k+1)\}}$$

Thus,

$$\frac{\partial^{2(k+1)+1}E}{\partial t^{2(k+1)+1}} = \beta_e(\beta_e\beta_h)^{k+1}\nabla^{\{2(k+1)+1\}} \times H + \sum_{i=1}^{k+1} p_{2i}^{\{2(k+1)+1\}}\nabla^{\{2(k+1)-2i+1\}} \times H + \sum_{i=1}^{k+2} p_{2i-1}^{\{2(k+1)+1\}}\nabla^{\{2(k+1)-2i+2\}} \times E$$

It is the same as (76). Thus, (58) holds for the case of $k+1$.

Thus, the lemma holds for the case of $k+1$.

QED