Nonlinear Systems and Control,
Homework #2

Due April, 29 2023

Exercise #1: Control Lyapunov Functions

1. Find a control Lyapunov function $V(x)$, positive definite and $\mathcal{C}^1(\mathbb{R}^n;\mathbb{R})$, for $\dot{x} = f(x) + xu$, where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ and the components of $f(x) = [f_1(x), \ldots, f_n(x)]^T$ are

$$f_i(x) = \sin(x_i)x_i,\ i = 1, \ldots, n.$$

2. Find a continuous (static) feedback control law $\psi(x)$ that makes $\bar{x} = 0$ AS for the system described above.

Exercise #2: UUB for non-vanishing perturbations

Consider a controlled system

$$\dot{x}(t) = f(x(t)) + g(x(t))(u(t) + \omega_1(x(t), t)) + \omega_2(x(t), t)), \tag{1}$$

with $x \in \mathbb{R}^n$, where $u(t) : [0, +\infty) \to \mathbb{R}$ is the control, and $\omega(t) : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}$ is an *unknown* perturbation (or a disturbance) such that for all $t \geq 0$, $x \in \mathbb{R}^n$:

$$\|\omega_1(x, t)\| \leq a\|x\| + b, \quad \|\omega_1(x, t)\| \leq c \quad a, b, c > 0.$$

Assume that there exists a $V(x)$ which is a $\mathcal{C}^1$, RU, positive definite *control Lyapunov function* for the nominal system when $\omega_1(x, t) = \omega_2(x, t) = 0$.

1. Show that, when $\omega_1(x, t) \neq 0$, $\omega_2(x, t) = 0$, there exist $\gamma(x)$ such that:

$$\psi(x) = -\gamma(x)\frac{\frac{\partial V}{\partial x}(x)g(x)}{\|\frac{\partial V}{\partial x}(x)g(x)\|}$$

   makes $\dot{V}(x(t)) < 0$ for $x \neq 0$.

2. In the case where it may also be $\omega_2(x, t) \neq 0$, show that there exists ε such that $\mathcal{N}_\varepsilon(0)$ is globally UUB for the trajectories of the system controlled with the ψ of the previous point.

Exercise #3: **Stability and convergence**

Consider the following system on $\mathbb{R}^3$:

$$\begin{aligned} \dot{x}_1 &= \sin^2(x_3) - (x_1 - x_2)^3 \\ \dot{x}_2 &= -\sin^2(x_3) \\ \dot{x}_3 &= -5x_3 + 6x_3^3. \end{aligned}$$

a. Study the stability properties of $\bar{x} = 0 \in \mathbb{R}^3$ via center manifold analysis.

b. [optional] Find the union of the ω-limit sets $\mathcal{L}_x^+$ for initial conditions $x(0) = (x_1, x_2, x_3) \in \mathbb{R}^3$ such that $\|x_3\| < \sqrt{\frac{5}{6}}$.

Rules and honor system: The instructor will evaluate one or more exercises of his choice. The total of the homeworks will account for 60% of your final evaluation. If you skip one (or more) homeworks, that fraction of the 60% will be evaluated zero. The agreement is the following: you can work on the homework until the beginning of class on the due day. HWs that have not been handed over to the instructor at the beginning of class or emailed to him before the beginning of class will not be evaluated. The students are required to work on the HWs alone. The course lecture notes or other material can be consulted, but the solution should not be copied from another source. In case of suspect violation of the honor system, the instructor can suspend grading using HWs either for the concerned students or the whole class, to his discretion.