HOMEWORK #3 NONLINEAR SYSTEMS AND CONTROL

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Exercise #1

Consider the following system on $\mathcal{X} = \mathbb{R}^2 \times \mathbb{S}^1 \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$:

$$\begin{aligned}
\dot{x}_p &= v\cos(\theta) \\
\dot{y}_p &= v\sin(\theta) \\
\dot{\theta} &= \frac{1}{\ell}\tan(\alpha)v \\
\dot{\alpha} &= \omega
\end{aligned}$$

where ℓ is the "length" of the tricycle, θ the angle of the body with respect to the x axis, α the steering angle. Assume we can control the linear speed v and the steering speed $\omega = \dot{\alpha}$, choosing their value in $\mathbb{R}$.

Point a.

What type of system is it: TI/TV? driftless or with drift? Control-affine?
The system is:

- **Time Invariant**: since there is no explicit dependence on time but only a dependence through the state and the control.
- **Driftless**: since given $\mathbf{u} = [v\ \omega]^T$ and $\mathbf{x} = [x_p\ y_p\ \theta\ \alpha]^T$ we have:

$$f(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} v\cos(\theta) \\ v\sin(\theta) \\ \frac{1}{\ell}\tan(\alpha)v \\ \omega \end{bmatrix}$$

If we calculate it for $\mathbf{u} = 0$ namely $f_0(\mathbf{x}) = f(\mathbf{x}, 0)$ we obtain

$$f_0(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \forall\, \mathbf{x} \in \mathcal{X}$$

- **Control Affine**: since by using the same conventions introduced previously we have:

$$g(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} v\cos(\theta) \\ v\sin(\theta) \\ \frac{1}{\ell}\tan(\alpha)v \\ \omega \end{bmatrix}$$

we can rewrite $g(\mathbf{x}, \mathbf{u})$ as: $g(\mathbf{x}, \mathbf{u}) = g_1(\mathbf{x})v + g_2(\mathbf{x})\omega$ with:

$$g_1(\mathbf{x}) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ \frac{1}{\ell}\tan(\alpha) \\ 0 \end{bmatrix} \qquad g_2(\mathbf{x}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Point b.

Find the linearized system around $(0, 0, 0, 0)$, for $v = 0$, $\omega = 0$ and discuss if any controllability property can be assessed via linearization.

By using the same conventions as we did in previous points we need to linearize $f(\mathbf{x}, \mathbf{u})$ around the origin and with $\mathbf{u} = 0$, by doing so we obtain:

$$A = \frac{\partial f}{\partial \mathbf{x}}\bigg|_{\substack{\mathbf{x}=0 \\ \mathbf{u}=0}} = \left.\begin{bmatrix} 0 & 0 & -v\sin(\theta) & 0 \\ 0 & 0 & v\cos(\theta) & 0 \\ 0 & 0 & 0 & \frac{v}{\ell\cos^2(\alpha)} \\ 0 & 0 & 0 & 0 \end{bmatrix}\right|_{\substack{\mathbf{x}=0 \\ \mathbf{u}=0}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \frac{\partial f}{\partial \mathbf{u}}\bigg|_{\substack{\mathbf{x}=0 \\ \mathbf{u}=0}} = \left.\begin{bmatrix} \cos(\theta) & 0 \\ sin(\theta) & 0 \\ \frac{\tan(\alpha)}{\ell} & 0 \\ 0 & 1 \end{bmatrix}\right|_{\substack{\mathbf{x}=0 \\ \mathbf{u}=0}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We can now consider the reachability matrix of the linearized system $\mathcal{R} = [B|AB|A^2B|A^3B]$ and since $A = \mathbf{0}_{4\times4}$ then $rank(\mathcal{R}) = rank(B) = 2 < dim(\mathcal{X})$ hence we cannot conclude anything about accessibility though the linearized system.

Point c.

Discuss its accessibility, STLC at $(0,0,0,0)$ and global controllability using the nonlinear results we have seen in class.

By using the same convention introduced before we have $\dot{\mathbf{x}} = g_1(\mathbf{x})v + g_2(\mathbf{x})\omega$. Since it is driftless control affine system and $v, \omega \in \mathbb{R}$ therefore the inputs can be set to zero then $\mathcal{L}_{\mathcal{F}} = Lie(\{g_1, g_2\})$. Since we only have two vector fields, and $\mathcal{L}_{\mathcal{F}}$ is closed by Lie brackets (i.e. if $a, b \in \mathcal{L}_{\mathcal{F}} \implies [a, b] \in \mathcal{L}_{\mathcal{F}}$), we will need to consider the Lie brackets, and subsequent iterations to check what type of accessibility, if any, we have:

$$
\begin{aligned}
[g_1, g_2](\mathbf{x}) &= \frac{\partial g_2}{\partial \mathbf{x}} g_1(\mathbf{x}) - \frac{\partial g_1}{\partial \mathbf{x}} g_2(\mathbf{x}) = -\frac{\partial g_1}{\partial \mathbf{x}} g_2(\mathbf{x}) \\
&= -\begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & \frac{1}{\ell \cos^2(\alpha)} \\ * & * & * & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{\ell \cos^2(\alpha)} \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
[g_1, [g_1, g_2]](\mathbf{x}) &= \frac{\partial [g_1, g_2]}{\partial \mathbf{x}} g_1 - \frac{\partial g_1}{\partial \mathbf{x}} [g_1, g_2] = -\frac{\partial [f_0, g_1]}{\partial \mathbf{x}} g_2(\mathbf{x}) \\
&= -\begin{bmatrix} 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ \frac{1}{\ell}\tan(\alpha) \\ 0 \end{bmatrix} - \begin{bmatrix} * & * & -\sin(\theta) & * \\ * & * & \cos(\theta) & * \\ * & * & 0 & * \\ * & * & 0 & * \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{\ell \cos^2(\alpha)} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{\sin(\theta)}{\ell \cos^2(\alpha)} \\ \frac{\cos(\theta)}{\ell \cos^2(\alpha)} \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

We can check if they are linearly independent by calculating the determinant of D:

$$
D = \begin{bmatrix} g_1(\mathbf{x}) & g_2(\mathbf{x}) & [g_1, g_2](\mathbf{x}) & [g_1, [g_1, g_2]](\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 & 0 & -\frac{\sin(\theta)}{\ell \cos^2(\alpha)} \\ \sin(\theta) & 0 & 0 & \frac{\cos(\theta)}{\ell \cos^2(\alpha)} \\ \frac{1}{\ell}\tan(\alpha) & 0 & -\frac{1}{\ell \cos^2(\alpha)} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
$$

$$
\implies det(D) = \frac{1}{\ell^2 \cos^4(\alpha)}
$$

Since $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ then $det(D) \neq 0 \ \forall \ \mathbf{x} \in \mathcal{X}$ this implies $dim(\mathcal{L}_{\mathcal{F}}(\mathbf{x})) = 4 = dim(\mathcal{X}) \ \forall \ \mathbf{x} \in \mathcal{X}$, via LARC we can conclude that the system is globally accessible, moreover since we have a control affine system, by Theorem 5.2 of the Lecture notes we can conclude that the system is globally strongly accessible.

Since the system is driftless and control affine and $\mathcal{U} = \mathbb{R}^2$ is symmetric then it is also

time-reversible via control.
Since LARC hold globally this means that it also holds in $\mathbf{x} = 0$ and the system is time-reversible via control, then using Theorem 5.8 of the Lecture notes we can conclude that the system is STLC at $\mathbf{x} = 0$.
Since LARC holds globally, $\mathcal{X} = \mathbb{R}^2 \times \mathbb{S}^1 \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is connected (cross product of connected sets) and the system is time reversible via control, we can apply Theorem 5.9 of the Lecture notes to conclude that the system is globally controllable.

Exercise #2

Consider the following dynamical system on the open unit ball in $\mathbb{R}^3$, namely $\mathcal{X} = \mathcal{N}_1(0) \subsetneq \mathbb{R}^3$:

$$\dot{x}_1 = a_1 x_2 x_3 + u_1$$
$$\dot{x}_2 = a_2 x_3 x_1 + u_2$$
$$\dot{x}_3 = a_3 x_1 x_2$$

where the controls are $u_{1,2} \in [-B, B]$ with $B > 2$, $0 < a_i < 1$.

Point a.

Prove that the system is globally accessible;

It is immediate to see that the system is control affine and can be rewritten as follows by considering $\mathbf{x} = [x_1 \; x_2 \; x_3]^T$:

$$\dot{\mathbf{x}} = f_0(\mathbf{x}) + g_1(\mathbf{x})u_1 + g_2(\mathbf{x})u_2$$

with:

$$f_0(\mathbf{x}) = \begin{bmatrix} a_1 x_2 x_3 \\ a_2 x_3 x_1 \\ a_3 x_1 x_2 \end{bmatrix} \qquad g_1(\mathbf{x}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad g_2(\mathbf{x}) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Since the system is control affine and $u_{1,2} \in [-B, B]$ (so the inputs can be set to zero), then $\mathcal{L}_{\mathcal{F}} = Lie(\{f_0, g_1, g_2\})$, so to check the dimension we need to check if the vector field are linearly independent, to do that we can consider the determinant of the matrix D and see when it is different from zero:

$$D = \begin{bmatrix} f_0(\mathbf{x}) & g_1(\mathbf{x}) & g_2(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} a_1 x_2 x_3 & 1 & 0 \\ a_2 x_3 x_1 & 0 & 1 \\ a_3 x_1 x_2 & 0 & 0 \end{bmatrix} \qquad \Longrightarrow \qquad det(D) = a_3 x_1 x_2$$

It is clear that $det(D) \neq 0$ iff $x_1 \neq 0$ and $x_2 \neq 0$, this implies $dim(\mathcal{L}_{\mathcal{F}}(\mathbf{x})) = 3 = dim(\mathcal{X})$ iff $x_1 \neq 0$ and $x_2 \neq 0$ so, by using LARC we can conclude that the system is accessible $\forall\, \mathbf{x} \in \mathcal{X}$ s.t. $x_1 \neq 0$ and $x_2 \neq 0$

We now need to study what happens in the cases where we cannot conclude anything about accessibility by using Lie brackets, we note that $\mathcal{L}_{\mathcal{F}}$ is closed by Lie brackets.

First of all we note that $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$ are constant so their jacobian will be null.

$$[f_0, g_1](\mathbf{x}) = \frac{\partial g_1}{\partial \mathbf{x}} f_0(\mathbf{x}) - \frac{\partial f_0}{\partial \mathbf{x}} g_1(\mathbf{x}) = -\frac{\partial f_0}{\partial \mathbf{x}} g_1(\mathbf{x}) = -\begin{bmatrix} 0 & * & * \\ a_2 x_3 & * & * \\ a_3 x_2 & * & * \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -a_2 x_3 \\ -a_3 x_2 \end{bmatrix}$$

which is linearly independent form $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$ iff $x_2 \neq 0$ this implies, by LARC, that we also have accessibility for $x_1 = 0$.
We can notice that by applying the Lie brackets we reduced the order of the vector field we can try by iterating, this time on $g_2(\mathbf{x})$ and check if we obtain a vector field which does not depend on $\mathbf{x}$ and is also linearly independent form $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$.

$$
\begin{aligned}
[[f_0, g_1], g_2](\mathbf{x}) &= \frac{\partial g_2}{\partial \mathbf{x}}[f_0, g_1] - \frac{\partial [f_0, g_1]}{\partial \mathbf{x}} g_2(\mathbf{x}) = -\frac{\partial [f_0, g_1]}{\partial \mathbf{x}} g_2(\mathbf{x}) \\
&= -\begin{bmatrix} * & 0 & * \\ * & 0 & * \\ * & -a_3 & * \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ a_3 \end{bmatrix}
\end{aligned}
$$

Since $a_3 > 0$, $[[f_0, g_1], g_2](\mathbf{x})$ is linearly independent form $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$ $\forall\, \mathbf{x} \in \mathcal{X}$, by LARC, we can conclude that the system is globally accessible.

Point b.

If the actuator of the control u_2 breaks down, imposing a saturated constant control $u_2 = B$, is the system still accessible?
In this second case the system is still control affine and can be rewritten, with the same conventions as before, as:

$$
\dot{\mathbf{x}} = f_0'(\mathbf{x}) + g_1(\mathbf{x})u_1
$$

with:

$$
f_0'(\mathbf{x}) = \begin{bmatrix} a_1 x_2 x_3 \\ a_2 x_3 x_1 + B \\ a_3 x_1 x_2 \end{bmatrix} \qquad g_1(\mathbf{x}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$

Since the system is control affine and $u_1 \in [-B, B]$ (so the input can be set to zero), then $\mathcal{L}_{\mathcal{F}} = Lie(\{f_0', g_1\})$, since we only have two vector fields we will need to consider the Lie brackets of the two to try and verify if we can still have global accessibility, by using the same reasoning as in Point a. and the fact that $\mathcal{L}_{\mathcal{F}}$ is closed by Lie brackets.

$$
[f_0', g_1](\mathbf{x}) = \frac{\partial g_1}{\partial \mathbf{x}} f_0'(\mathbf{x}) - \frac{\partial f_0'}{\partial \mathbf{x}} g_1(\mathbf{x}) = -\frac{\partial f_0'}{\partial \mathbf{x}} g_1(\mathbf{x}) = -\begin{bmatrix} 0 & * & * \\ a_2 x_3 & * & * \\ a_3 x_2 & * & * \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -a_2 x_3 \\ -a_3 x_2 \end{bmatrix}
$$

We define the matrix D as done before and checking its determinant, we obtain:

$$D = \begin{bmatrix} f_0'(\mathbf{x}) & g_1(\mathbf{x}) & [f_0', g_1](\mathbf{x}) \end{bmatrix} = \begin{bmatrix} a_1x_2x_3 & 1 & 0 \\ a_2x_3x_1 + B & 0 & -a_2x_3 \\ a_3x_1x_2 & 0 & -a_3x_2 \end{bmatrix} \implies$$

$$det(D) \;=\; (a_3x_3x_1 + B)(a_3x_2 - a_2x_3)$$

We can observe that $(a_3x_3x_1+B) > 0\,\forall x_1, x_3$, since $\mathcal{X} = \mathcal{N}_1(0) \subsetneq \mathbb{R}^3$, $0 < a_1 < 1$ and $B > 2$, so $det(D) = 0$ iff $x_2 = \frac{a_2}{a_3}x_3$ which implies that $dim(\mathcal{L}_{\mathcal{F}}(\mathbf{x})) = 3 = dim(\mathcal{X})$ iff $x_2 \neq \frac{a_2}{a_3}x_3$ this lets is conclude by LARC that the system is accessible $\forall$ $\mathbf{x} \in \mathcal{X}$ s.t. $x_2 \neq \frac{a_2}{a_3}x_3$.
We can now check what happens for $x_2 = \frac{a_2}{a_3}x_3$ by nesting Lie brackets and using the closeness of $\mathcal{L}_{\mathcal{F}}$ by Lie brackets.

$$\begin{aligned}
[f_0', [f_0', g_1]](\mathbf{x}) &= \frac{\partial\,[f_0', g_1]}{\partial \mathbf{x}} f_0'(\mathbf{x}) - \frac{\partial f_0'}{\partial \mathbf{x}} [f_0', g_1](\mathbf{x}) \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -a_2 \\ 0 & -a_3 & 0 \end{bmatrix} \begin{bmatrix} a_1x_2x_3 \\ a_2x_3x_1 + B \\ a_3x_1x_2 \end{bmatrix} - \begin{bmatrix} * & a_1x_3 & a_1x_2 \\ * & 0 & a_2x_1 \\ * & a_3x_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -a_2x_3 \\ -a_3x_2 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ -a_2a_3x_1x_2 \\ -a_2a_3x_1x_3 - a_3B \end{bmatrix} - \begin{bmatrix} -a_1a_2x_3^2 - a_1a_3x_2^2 \\ -a_2a_3x_1x_2 \\ -a_2a_3x_1x_3 \end{bmatrix} \\
&= \begin{bmatrix} a_1a_2x_3^2 + a_1a_3x_2^2 \\ 0 \\ -a_3B \end{bmatrix}
\end{aligned}$$

If we substitute $x_2 = \frac{a_2}{a_3}x_3$ in $[f_0', [f_0', g_1]](\mathbf{x})$ we obtain:

$$[f_0', [f_0', g_1]](\mathbf{x}) = \begin{bmatrix} a_1a_2(1 + a_2)x_3^2 \\ 0 \\ -a_3B \end{bmatrix}$$

as done before we check $dim(D')$ by calculating its determinant:

$$D' = \begin{bmatrix} f_0'(\mathbf{x}) & g_1(\mathbf{x}) & [f_0', g_1](\mathbf{x}) \end{bmatrix} = \begin{bmatrix} a_1x_2x_3 & 1 & a_1a_2(1 + a_2)x_3^2 \\ a_2x_3x_1 + B & 0 & 0 \\ a_3x_1x_2 & 0 & -a_3B \end{bmatrix} \implies$$

$$det(D') \;=\; a_3B(a_3x_3x_1 + B)$$

As discussed previously $a_3x_3x_1 + B > 0 \,\forall\, \mathbf{x} \in \mathcal{X}$ and since $B > 2$ and $0 < a_3 < 1$ then $a_3B \neq 0$ this implies $det(D') \neq 0 \,\forall\, \mathbf{x} \in \mathcal{X}$ so $dim(\mathcal{L}_{\mathcal{F}}(\mathbf{x})) = 3 = dim(\mathcal{X}) \,\forall\, \mathbf{x} \in \mathcal{X}$ so using LARC we can conclude that the system is still globally accessible even after the actuator breaks down.