HOMEWORK #2 NONLINEAR SYSTEMS AND CONTROL

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Exercise #1

Point 1

We can consider $V(x) = \frac{1}{2}x^T x$ which is a positive definite function in Lyapunov sense and $\mathcal{C}^1(\mathbb{R}^n;\mathbb{R})$; we, also, need to verify if it is a control Lyapunov function for the system we are considering, i.e. $\forall\, x \in \mathcal{X}\{\bar{x} = 0\}\ \exists\, \mathcal{U}_x \neq \emptyset$ s.t. if $u \in \mathcal{U}_x$ then $\frac{\partial V}{\partial x}(x)f(x,u) < 0$; we can now verify that this happens for the $V(x)$ chosen.

$$\frac{\partial V}{\partial x}(x)f(x,u) = \sum_{i=1}^{n} x_i^2 \left(\sin(x_i) + u\right) < 0 \qquad \forall x \quad iff \quad u < -1$$

So if we define $\mathcal{U}_x = \{u \in \mathbb{R}\,|u < -1\}$ we verified that $V(x) = \frac{1}{2}x^T x$ is a valid control Lyapunov Function for our system.

Point 2

We can observe that the system is control affine i.e $\dot{x} = f(x) + g(x)u$ with:

$$f(x) = \begin{pmatrix} \sin(x_1)\, x_1 \\ \vdots \\ \sin(x_n)\, x_n \end{pmatrix} \qquad\qquad g(x) = x$$

We can apply Arnstein Theorem that grants us that a continuous $\psi(x)$ exists and moreover that

$$\dot{V}(x) = \frac{\partial V}{\partial x}(x)f(x) + \frac{\partial V}{\partial x}(x)g(x)u \tag{1}$$

Since we want to make $\bar{x} = 0$ AS we need to have $\dot{V}(x) < 0 \; \forall x \neq 0$ and $\dot{V}(x) = 0$ if $x = 0$. To calculate $\dot{V}(x)$, we observe what follows:

$$\frac{\partial V}{\partial x}(x) f(x) = \sum_{i=1}^{n} x_i^2 \left(\sin(x_i)\right)$$
$$\frac{\partial V}{\partial x}(x) g(x) u = u \sum_{i=1}^{n} x_i^2$$

If we now substitute what we have just obtained in eq.1 we obtain:

$$\dot{V}(x) = \sum_{i=1}^{n} x_i^2 \left(\sin(x_i)\right) + u \sum_{i=1}^{n} x_i^2$$

It is simple to verify that $\dot{V}(\bar{x}) = 0$ independently of the value of u. We have to find a control law $\psi(x) = u$ s.t $\dot{V}(x) < 0 \; \forall x \neq 0$, since $x \neq 0$ we can note that $\sum_{i=1}^{n} x_i^2 \neq 0$ and so it is invertible so we obtain:

$$\dot{V}(x) = \sum_{i=1}^{n} x_i^2 \left(\sin(x_i)\right) + u \sum_{i=1}^{n} x_i^2 \leq \sum_{i=1}^{n} x_i^2 + u \sum_{i=1}^{n} x_i^2 < 0$$
$$u < -\frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i^2} = -1$$

So we can consider $\psi(x) = c$ with $c < -1$ which being a constant is also continuous and as we saw makes $\bar{x}$ AS

Exercise #2

Point 1

In the first point of the exercise we are considering $\omega_1(x,t) \neq 0$ and $\omega_2(x,t) = 0$ so our system becomes

$$\dot{x}(t) = f\left(x(t)\right) + g\left(x(t)\right)\left(u(t) + \omega_1\left(x(t), t\right)\right)$$

We can now calculate $\dot{V}(x(t))$ and find $\gamma(x)$ s.t $\dot{V}(x(t)) < 0$

$$\begin{aligned}
\dot{V}(x) &= \frac{\partial V}{\partial x}(x)f(x) + \frac{\partial V}{\partial x}(x)g(x)\left(-\frac{\frac{\partial V}{\partial x}(x)g(x)}{\left\|\frac{\partial V}{\partial x}(x)g(x)\right\|}\gamma(x) + \omega_1(x,t)\right) && (2)\\
&\leq \frac{\partial V}{\partial x}(x)f(x) + \frac{\partial V}{\partial x}(x)g(x)\left(-\frac{\frac{\partial V}{\partial x}(x)g(x)}{\left\|\frac{\partial V}{\partial x}(x)g(x)\right\|}\gamma(x) + a\left\|x\right\| + b\right) && (3)\\
&= \frac{\partial V}{\partial x}(x)f(x) - \frac{\left(\frac{\partial V}{\partial x}(x)g(x)\right)^2}{\left\|\frac{\partial V}{\partial x}(x)g(x)\right\|}\gamma(x) + \frac{\partial V}{\partial x}(x)g(x)\left(a\left\|x\right\| + b\right)\\
&< 0
\end{aligned}$$

Where passing from eq. 2 to eq. 3 we used $\left\|\omega_1(x,t)\right\| \leq a\left\|x\right\| + b$.
Now we are going to find the desired $\gamma(x)$ but to do that we first note that $\frac{\partial V}{\partial x}(x)g(x) \neq 0$ and so $\frac{\left(\frac{\partial V}{\partial x}(x)g(x)\right)^2}{\left\|\frac{\partial V}{\partial x}(x)g(x)\right\|} > 0$ and invertible since we are considering dividing by $\left\|\frac{\partial V}{\partial x}(x)g(x)\right\|$ in the required $\psi(x)$.

$$\begin{aligned}
\frac{\partial V}{\partial x}(x)f(x) - \frac{\left(\frac{\partial V}{\partial x}(x)g(x)\right)^2}{\left\|\frac{\partial V}{\partial x}(x)g(x)\right\|}\gamma(x) + \frac{\partial V}{\partial x}(x)g(x)\left(a\left\|x\right\| + b\right) < 0\\
\frac{\partial V}{\partial x}(x)f(x) + \frac{\partial V}{\partial x}(x)g(x)\left(a\left\|x\right\| + b\right) < \frac{\left(\frac{\partial V}{\partial x}(x)g(x)\right)^2}{\left\|\frac{\partial V}{\partial x}(x)g(x)\right\|}\gamma(x)\\
\gamma(x) > \frac{\frac{\partial V}{\partial x}(x)f(x)\left\|\frac{\partial V}{\partial x}(x)g(x)\right\|}{\left(\frac{\partial V}{\partial x}(x)g(x)\right)^2} + \frac{\left(a\left\|x\right\| + b\right)\left\|\frac{\partial V}{\partial x}(x)g(x)\right\|}{\frac{\partial V}{\partial x}(x)g(x)} \qquad (4)
\end{aligned}$$

From eq. 4 we can derive that a possible choice of $\gamma(x)$ is:

$$\gamma(x) = \frac{\frac{\partial V}{\partial x}(x)f(x)\left\|\frac{\partial V}{\partial x}(x)g(x)\right\|}{\left(\frac{\partial V}{\partial x}(x)g(x)\right)^2} + \frac{\left(a\left\|x\right\| + b\right)\left\|\frac{\partial V}{\partial x}(x)g(x)\right\|}{\frac{\partial V}{\partial x}(x)g(x)} + W(x)$$

With $W(x) > 0$ if $x \neq 0$. We can verify that we obtain what we want by substituting what we have just obtained in $\dot{V}$ to get:

$$\dot{V}(x) \leq -W(x)\frac{\left(\frac{\partial V}{\partial x}(x)g(x)\right)^2}{\left\|\frac{\partial V}{\partial x}(x)g(x)\right\|} < 0$$

since both $W(x)$ and $\frac{\left(\frac{\partial V}{\partial x}(x)g(x)\right)^2}{\left\|\frac{\partial V}{\partial x}(x)g(x)\right\|}$ are greater than zero if $x \neq 0$.
In conclusion we verified the existence of $\gamma(x)$ that makes $\dot{V}(x) < 0$ if $x \neq 0$

Point 2

To show that the desired ε exists we can use Theorem 2.11 (pages 75-76) of the lecture notes and its extensions. First of all if we substitute ψ obtained in the previous point and the dynamics when $\omega_2(x,t) \neq 0$ into $\dot{V}$ we obtain:

$$\dot{V}(x) \leq -W(x)\frac{\left(\frac{\partial V}{\partial x}(x)g(x)\right)^2}{\left\|\frac{\partial V}{\partial x}(x)g(x)\right\|} + \frac{\partial V}{\partial x}(x)\omega_2(x,t) \tag{5}$$

Since in the previous point the only constraint on $W(x)$ was that it must be positive definite we can consider a specific choice of $W(x)$ that will make it easier to apply the theorem so we will use:

$$W(x) = \frac{\left\|\frac{\partial V}{\partial x}(x)g(x)\right\|}{\left(\frac{\partial V}{\partial x}(x)g(x)\right)^2}\left\|x\right\|\left\|\frac{\partial V}{\partial x}(x)\right\| \tag{6}$$

We can substitute eq. 6 in eq. 5 and we obtain:

$$\begin{aligned}\dot{V}(x) &\leq -\left\|x\right\|\left\|\frac{\partial V}{\partial x}(x)\right\| + \frac{\partial V}{\partial x}(x)\omega_2(x,t)\\ &\leq -\left\|x\right\|\left\|\frac{\partial V}{\partial x}(x)\right\| + \left\|\frac{\partial V}{\partial x}(x)\right\|c\\ &= \left\|\frac{\partial V}{\partial x}(x)\right\|(c - \left\|x\right\|)\end{aligned}$$

We now need to chose $\Omega_\delta^0 = \{x \in \Omega \mid V(x) < \delta\} \subset \mathcal{S}$ with δ chosen s.t $\|x\| > c$; with this choice of δ we obtain $\dot{V}(x) < 0$ if $x \in \Omega \backslash \Omega_\delta^0$ so now we satisfy the necessary conditions to apply the theorem mentioned before and conclude that $\mathcal{S}$ is UB for $x_0 \in \Omega$. We can use its extensions, first of all we define μ as the minimum δ s.t $\dot{V}(x) < 0 \; \forall \, x \in \Omega \backslash \Omega_\delta^0$ and this makes us conclude that $\mathcal{S} = \Omega_\mu$ is UUB.
Lastly, since $\Omega = \mathrm{R}^n$ and $V(x)$ is RU then $\mathcal{S} = \Omega_\mu$ is also globally UUB so now we just have to chose a neighborhood $\mathcal{N}_\varepsilon(0)$ s.t $\Omega_\mu \subset \mathcal{N}_\varepsilon(0)$ and we have proven what was requested.

Exercise #3

Point a

Given the system to analyze the stability of $\bar{x} = 0$, which is clearly an equilibrium, via center manifold analysis we first need to linearize the system by calculating the jacobian and then find the matrix of the linearized system by calculating the jacobian at the equilibrium:

$$J = \begin{bmatrix} -3(x_1 - x_2)^2 & 3(x_1 - x_2)^2 & 2\cos(x_3)\sin(x_3) \\ 0 & 0 & -2\cos(x_3)\sin(x_3) \\ 0 & 0 & 18x_3^2 - 5 \end{bmatrix} \qquad A = J\Big|_{x=0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

As we can note A is already block diagonal so we don't need to do any change of basis to obtain $A_S = -5$ and $A_C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ we can now rewrite the system considering $x_C = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $x_S = x_3$ and taking into account the higher order terms:

$$\begin{aligned} \dot{x}_1 &= 0x_1 + g_{C_1}(x_C, x_S) = \sin^2(x_S) - (x_1 - x_2)^3 \\ \dot{x}_2 &= 0x_1 + g_{C_2}(x_C, x_S) = -\sin^2(x_S) \\ \dot{x}_S &= -5x_S + g_S(x_C, x_S) = -5x_S + 6x_S^3 \end{aligned}$$

We can now note that $g_S(x_C, x_S) = g_S(x_S)$ i.e. g_S is independent from x_C so we know that $x_S = h(x_C) = 0$ solves the CME so we can now use the reduction principle and substitute what we have just learnt to obtain:

$$\begin{aligned} \dot{y}_1 &= g_{C_1}(x_C, h(x_C)) = \sin^2(h(x_C)) - (y_1 - y_2)^3 = -(y_1 - y_2)^3 \\ \dot{y}_2 &= g_{C_2}(x_C, h(x_C)) = -\sin^2(h(x_C)) = 0 \end{aligned}$$

We can immediately note that since $\dot{y}_2 = 0$ then y_2 will be a constant fixed its starting value so we can immediately rule out AS, we now have to see if given the dynamics of y_1 will explode or converge to the initial value of y_2 which will be our new equilibrium; to simplify calculations we will consider $y_2(0) = 0$ but the same can be shown for every $y_2(0)$ by translating y_1 by $y_2(0)$ and repeat thew same reasoning. We obtain:

$$\dot{y}_1 = -y_1^3$$

We can now consider the Lyapunov function $V(y_1) = \frac{1}{2}y_1^2$ which is positive definite in Lyapunov sense and so obtain $\dot{V}(y_1) = -y_1^4$ which is positive definite in Lyapunov sense so y_1 will converge to the equilibrium so we can conclude that $y = 0$ for the reduced system is simply stable and by the reduction principle this implies that also $\bar{x} = 0$ is simply stable for the original system.

Point b [optional]

Requiring $||x_3|| < \sqrt{\frac{5}{6}}$ means that $x_3 \in \left(-\sqrt{\frac{5}{6}}; \sqrt{\frac{5}{6}}\right)$ if we analyze the behaviour of $\dot{x_3}$ in this interval we obtain: $\dot{x_3} < 0$ if $x_3 > 0$ and $\dot{x_3} > 0$ if $x_3 < 0$ this implies that x_3 will converge to zero and that $|x_3|$ will always be decreasing in the interval of interest.

We can now consider the dynamics of x_2 and note that $\forall\ x_3 \in \left(-\sqrt{\frac{5}{6}}; \sqrt{\frac{5}{6}}\right)$ then $\dot{x_2} < 0$, this implies that the value of x_2 will decrease until $x_3 \approx 0$ and then will stabilize to its final asymptotic value say x_{2f}. Depending on the initial value of x_2 and x_3 we will have a different x_{2f} so if we consider the union of all the $\omega - limit$ sets of the coordinate x_2 we will obtain $\mathbb{R}$.

Lastly we will consider the dynamics of x_1 as we have shown in the first point of the exercise when $x_3 = 0$ then x_1 will converge to the value of x_2 expanding this reasoning to the case we are considering, after a transient in which we cannot say anything about the behaviour of x_1 we will asymptotically reach $x_3 = 0$ and so $x_2 \to x_{2f}$ consequently $x_1 \to x_2 \to x_{2f}$.

If we now put everything together we can conclude that the union of all $\omega - limit$ set for the desired set of initial conditions will be:

$$\mathcal{L}_x^+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = x_2,\ x_3 = 0\}$$