3. Casuística de resolució d'edos

(PROBLEMES)

① $y' = f(x,y)$ separable si es pot escriure com $h(y)\, y' = g(x)$

Dem. que $H(y)$, $G(y)$ primitives de $h(y)$ i $g(y)$ $\Longrightarrow$ la sol. general és $H(y) = G(x) + C$, $C \in \mathbb{R}$

$$\underbrace{\int h(y(x))\, y'(x)\, dx}_{H(y(x))} = \underbrace{\int g(x)\, dx}_{G(x)} + C$$

sol. implícita

(explícita si $\partial_y H \neq 0$)

② $x^2 + 2yy' = 0$, $y(0) = 2$

$$x^2 + 2y\frac{dy}{dx} = 0 \Longrightarrow \int 2y\, dy = \int -x^2 dx + C \Longrightarrow$$

$$\Longrightarrow y^2 = -\frac{x^3}{3} + C$$, sol. general implícita

sol. general: $y = \pm\sqrt{-\frac{x^3}{3} + C}$

com $y(0) = 2 \Rightarrow 2 = \pm\sqrt{C} \overset{+}{\Rightarrow} C = 4$

$$\boxed{y = \sqrt{-\frac{x^3}{3} + 4}}$$

③ $\frac{dy}{dx} = y \sin x$, $y(\pi) = -3$

$$\int \frac{dy}{y} = \int \sin x\, dx \Rightarrow \ln y = -\cos x + C \rightarrow \boxed{y = k e^{-\cos x}}, \ k \neq 0$$

si $y = 0$: $y = 0$ és sol.

$$\Longrightarrow y = k e^{-\cos x},\ k \in \mathbb{R} \Longrightarrow y = -3e^{-1-\cos x}$$

④ $\frac{dy}{dx} = x^2(1+y) \quad , \quad x(0) = 3$

$$\int \frac{dy}{1+y} = \int x^2\,dx + C \Longrightarrow \begin{cases} \ln|1+y| = \frac{x^3}{3} + C \quad , \quad \text{si } y \neq -1 \\ y(x) = -1 \text{ és sol, } \quad \text{si } y = -1 \end{cases}$$

$$\Longrightarrow |1+y| = \underbrace{\pm e^{C}}_{c} e^{x^3/3} = c\, e^{x^3/3} \quad , \quad c \in \mathbb{R} \setminus \{0\}$$

$$\Longrightarrow \begin{cases} y = -1 + C_1 e^{x^3/3} \quad , \quad C_1 \neq 0 \\ y = -1 \qquad\qquad C_1 = 0 \end{cases} \Longrightarrow \boxed{y = -1 + c\, e^{x^3/3} \quad , \quad c \in \mathbb{R}}$$

⑤ $\frac{dy}{dx} = (x-3)(1+y)^{2/3}$

$$\begin{cases} y = -1 \\ \int \frac{dy}{(1+y)^{2/3}} = \int (x-3)\,dx + C \quad , \quad C \in \mathbb{R} \end{cases} \Longrightarrow$$

$$\Longrightarrow 3(1+y)^{1/3} = \frac{(x-3)^2}{2} + C \Longrightarrow (1+y)^{1/3} = \frac{(x-3)^2}{6} + C$$

$$\Longrightarrow \boxed{\begin{cases} y(x) = -1 + \left(\frac{(x-3)^2}{6} + C \right)^3 \quad , \quad C \in \mathbb{R} \\ y(x) = -1 \end{cases}}$$

Canvis de variable

$x' = f(t, x)$ — x: dep., t: Indep. Es pot fer el canvi a la variable indep. o a la dep.

(6) Sigui $s \in I \xrightarrow{\varphi} t = \varphi(s) \in J$, $I, J \subset \mathbb{R}$ oberts

φ difeom. $\mathcal{C}^k$, $k \geq 1$

$(t, x) \in \Omega \subset \mathbb{R} \times \mathbb{R}^n \longrightarrow f(t,x) \in \mathbb{R}^n$, $\mathcal{C}^r$, $r \geq 0$

Dem. que $t \longmapsto x = \alpha(t)$ sol. de $x'(t) = f(t, x)$ $\Longleftrightarrow$

$s \longmapsto x = \beta(s)$ sol. de $x'(s) = f(\varphi(s), x)\, \varphi'(s)$

$= \alpha(\varphi(s))$

$\boxed{\Longrightarrow}$ sabem α sol. $\Longrightarrow \frac{d\alpha}{dt}(t) = f(t, \alpha(t))$

$$\frac{d\beta}{ds} = \frac{d\alpha}{dt}(\varphi(s))\frac{d\varphi}{ds} = f(\varphi(s), \alpha(\varphi(s)))\, \varphi'(s) =$$
$$= f(\varphi(s), \beta(s))\, \varphi'(s)$$

$$= \frac{d\beta}{ds} = f(\varphi(s), \beta(s))\, \varphi'(s).$$

$\boxed{\Longleftarrow}$ Usem que β satisfà $\frac{dx}{ds} = g(s, x)$, $g(s,x) = f(\varphi(s), x\, \varphi'(s))$

Aplicar $\boxed{\Longrightarrow}$ amb $s = \varphi^{-1}(t)$

(8) Fer canvi $x = \operatorname{arctg}\sqrt{t}$ a $y' \cdot \cos^4 x - y \sin 2x = 0$

$\Longleftrightarrow y' = \frac{\sin 2x}{\cos^4 x}\, y$, $\tilde{y}(t) = y(\operatorname{arctg}\sqrt{t})$

$t \in (0, +\infty) \xrightarrow{\varphi} x \in (0, \frac{\pi}{2})$

φ difeom.

$$\frac{d\tilde{y}}{dt} = \frac{dy}{dx}(\arctan\sqrt{t}) \cdot \frac{1}{1+t} \cdot \frac{1}{2\sqrt{t}} =$$

$$= \frac{\sin(2x(t))}{\cos^4(x(t))}\, \tilde{y} \cdot \frac{1}{1+t} \cdot \frac{1}{2\sqrt{t}} \overset{*}{=}$$

Trobem quan val $\cos^4(\operatorname{arctg}\sqrt{t})$ i $\sin(2\operatorname{arctg}\sqrt{t})$:

- $\frac{1}{\cos^2(x(t))} = 1 + tg^2(x(t)) = 1 + (\sqrt{t})^2 = 1 + t \Longrightarrow \frac{1}{\cos^4(x(t))} = \frac{1}{(1+t)^2}$
- $\sin(2x(t)) = 2\sin(x(t))\cos x(t) = 2\, tg(x(t))\cos^2 x(t) = 2\sqrt{t}\,\frac{1}{1+t}$

$\hookrightarrow$

$$\overset{\circledast}{\Rightarrow} \frac{d\tilde{y}}{dt} = \frac{2\sqrt{t}}{1+t}(1+t)^2\tilde{y}\,\frac{1}{1+t}\,\frac{1}{2\sqrt{t}} = \tilde{y} \Rightarrow \boxed{\frac{d\tilde{y}}{dt} = \tilde{y}}$$

Sol. general per $\tilde{y}$: $\boxed{\tilde{y}(t) = ce^t \quad c\in\mathbb{R}}$

Sol. general per y: $x = \arctan\sqrt{t} \longleftrightarrow t = tg^2 x$

$$\boxed{y = c\,e^{tg^2 x},\quad c\in\mathbb{R}}, \quad \boxed{x\in(0,\pi/2)}$$

7 $(t,y)\in W\subset\mathbb{R}\times\mathbb{R}^n \longrightarrow x = \psi(t,y)\in\mathbb{R}^n$, $\mathcal{C}^r$, $r\geq 1$ tq

$(t,y)\longmapsto (t,\psi(t,y))$ és un difeo

$W\subset\mathbb{R}\times\mathbb{R}^n \longrightarrow U\subset\mathbb{R}\times\mathbb{R}^n$

Dem. que $x=\alpha(t)$ sol. de $\dot{x} = f(t,x) \iff y=\beta(t)$ sol. de $\dot{y} = g(t,y)$.

on $\begin{cases} \beta \text{ ve donada per } \alpha(t) = \psi(t,\beta(t)) \\ g \text{ ve donada per } g(t,y) = [D_2\psi(t,y)]^{-1}[f(t,\psi(t,y)) - D_1\psi(t,y)] \end{cases}$

$\frac{d}{dt}\psi(A(t),B(t)) = D_1\psi(\;)\frac{dA}{dt} + D_2\psi(\;)\frac{dB}{dt}$

$\boxed{\Rightarrow}$ Derivem $\alpha(t) = \psi(t,\beta(t))$:

$$f(t,\psi(t,\beta(t))) = f(t,\alpha(t)) = \frac{d\alpha}{dt} = D_1\psi(t,\beta(t)) + D_2\psi(t,\beta(t))\frac{d\beta}{dt}$$

$$\Rightarrow \frac{d\beta}{dt} = [D_2\psi(t,\beta(t))]^{-1}[f(t,\psi(t,\beta(t))) - D_1\psi(t,\beta(t))]$$

$\hookrightarrow$ D_2 invertible pq f difeo

$\boxed{\Leftarrow}$ $F: (t,y)\longmapsto (s,x) = (t,\psi(t,y))$ difeo

$F: (t,x)\longmapsto (t,\tilde{\psi}(t,x))$

$\psi(t,\tilde{\psi}(t,x)) = x\ \circledast$. $\quad \alpha(t) = \psi(t,\beta(t)) \longleftrightarrow \beta(t) = \tilde{\psi}(t,\alpha(t))$

Apliquem $\boxed{\Rightarrow}$; treballem amb les fórmules usant les derivades de $\circledast$ respecte D_1, D_2

(10) $\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$ Aplicar $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

1) Domini? 2) Expressió edo 3) $\begin{cases} r' = F(r,\theta) \\ \theta' = G(r,\theta) \end{cases}$

$$f(r\cos\theta, r\sin\theta) = x' = r'\cos\theta - r\sin\theta\,\theta'$$
$$g(r\cos\theta, r\sin\theta) = y' = r'\sin\theta + r\cos\theta\,\theta'$$

$$\begin{cases} r' = \cos\theta\, f(r\cos\theta, r\sin\theta) + \sin\theta\, g(r\cos\theta, r\sin\theta) \\ \theta' = \frac{1}{r}\left[\cos\theta\, g(r\cos\theta, r\sin\theta) - \sin\theta\, f(r\cos\theta, r\sin\theta)\right] \end{cases}$$

$$(r,\theta) \longmapsto (x,y) = (r\cos\theta, r\sin\theta)$$
$$(0,+\infty)\times(-\pi,\pi) \longmapsto \mathbb{R}^2 \setminus \{y=0,\ x\leq 0\}$$

Comentari: Canvi $\theta \in (-\pi,\pi) \longrightarrow$ L'edo té sentit per a $\theta\in\mathbb{R}$

sabem $x' = r'\cos\theta - r\sin\theta\,\theta' = F(r,\theta)\cos\theta - r\sin\theta\, G(r,\theta)$

$y' = \underbrace{F(r,\theta)\sin\theta + r\cos\theta\, G(r,\theta)}$

Ho expressem en (x,y):

Canvi invers de polars:

$$r = \sqrt{x^2+y^2}$$
$$\theta = \operatorname{arctg}\left(\tfrac{y}{x}\right) \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)\ !$$

Estudiem l'angle $\frac{\theta}{2} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$: $\operatorname{tg}\theta = \frac{y}{x}$

$$\operatorname{tg}\frac{\theta}{2} = \frac{-1 \pm \sqrt{1+\operatorname{tg}^2\theta}}{\operatorname{tg}\theta} = \frac{-1 \pm \sqrt{1+y^2/x^2}}{y/x} = \frac{-x \pm \sqrt{x^2+y^2}}{y}$$

Triem el signe: $\theta \in (-\pi,\pi)$, $\theta/2 \in \left(-\frac{\pi}{2}, \pi/2\right)$

Llavors, el signe de $\operatorname{tg}\theta/2 \equiv$ signe de y $\Rightarrow$ Triem $\oplus$

$$\Longrightarrow \theta = \theta(x,y) = 2\operatorname{arctg}\left(\frac{-x+\sqrt{x^2+y^2}}{y}\right) \in (-\pi,\pi)$$

$$\Longrightarrow \begin{cases} x' = F\left(\sqrt{x^2+y^2}, \theta(x,y)\right)\frac{x}{\sqrt{x^2+y^2}} - y\, G\left(\sqrt{x^2+y^2}, \theta(x,y)\right) \\ y' = F\left(\sqrt{x^2+y^2}, \theta(x,y)\right)\frac{y}{x^2+y^2} + x\, G\left(\sqrt{x^2+y^2}, \theta(x,y)\right) \end{cases}$$

11 $\begin{cases} x' = y + x(x^2+y^2) \\ y' = -x + y(x^2+y^2) \end{cases}$

Per 10 , $x' = r'\cos\theta - r\sin\theta\,\theta' = r\sin\theta + r^3\cos\theta$

$y' = r'\sin\theta + r\cos\theta\,\theta' = -r\cos\theta + r^3\sin\theta$

$$\Longrightarrow \begin{cases} r' = r^3 \\ r\theta' = -r \end{cases} \Longrightarrow \begin{cases} r' = r^3 \\ \theta' = -1 \end{cases} \quad = \frac{dr}{dt} = r^3 \quad = \frac{d\theta}{dt} = -1$$

$$\frac{dr}{dt} = r^3 \Longrightarrow \int \frac{dr}{r^3} = \int dt + C \Longrightarrow \frac{-1}{2r^2} = t + C_1 \Longrightarrow$$

$$\Longrightarrow \frac{1}{r^2} = -2t + C \longrightarrow \boxed{r = \frac{1}{\sqrt{-2t + C_1}}}$$

$$\frac{d\theta}{dt} = -1 \Longrightarrow \boxed{\theta = -t + C_2}$$

$C_1, C_2 \in \mathbb{R}$

Definida per $t < \frac{C_1}{2}$

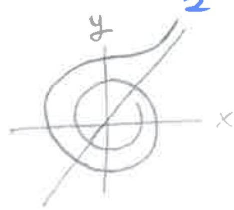
Ho expressem en cartesianes:

$$\begin{cases} x(t) = \dfrac{1}{\sqrt{-2t + C_1}} \cos(-t + C_2) \\ y(t) = \dfrac{1}{\sqrt{-2 + C_1}} \sin(-t + C_2) \end{cases}$$

En el límit:

$r(t) \longrightarrow +\infty$

$\theta(t) \longmapsto \frac{-C_1}{2} + C_2$



(13) $x \longrightarrow \varphi(x)$ solució de $\frac{dy}{dx} = f(x,y)$ tq $f(x,\varphi(x)) \neq 0$

Dem. $y \longrightarrow \varphi^{-1}(y)$ és sol. de $\frac{dx}{dy} = \frac{1}{f(x,y)}$

$\varphi'(x) = f(x,\varphi(x)) \neq 0 \overset{\text{TFInv}}{\Longrightarrow} y \longrightarrow \varphi^{-1}(y)$ ben definida i $\mathcal{C}^1$

$$D_y\, \varphi^{-1}(y) = \frac{1}{\varphi'(\varphi^{-1}(y))}$$

$$\left((\varphi^{-1}(y))\right)' = \frac{1}{f(\varphi^{-1}(y), \varphi(\varphi^{-1}(y)))} = \frac{1}{f(\varphi^{-1}(y), y)} \Longleftrightarrow x = \varphi^{-1}(y) \text{ és sol de } \frac{dx}{dy} = \frac{1}{f(x,y)}$$

Equacions de Bernoulli

(14) $y' = a(x)\,y + b(x)\,y^r$, $r \in \mathbb{R} \setminus \{0,1\}$

Dem $z = y^{1-r}$ transforma l'edo en lineal

$$z' = (1-r)\,y^{-r} \cdot y' = (1-r)\,y^{-r}\left(a(x)\,y + b(x)\,y^r\right) =$$

$$\boxed{z' = (1-r)\,a(x)\,z + (1-r)\,b(x)}$$ lineal en z.

Obs: si $r > 0$: perdem la solució $y = 0$ amb el canvi.

Aplicació: $2xy' + y + 3x^2y^2 = 0$

$y' = -\frac{1}{2x}\,y - \frac{3}{2}\,xy^2$, $x \neq 0$ Bernoulli, $r = 2$

Fem el canvi $z = y^{1-2} = y^{-1}$ ($y = 0$ és sol.)

$$\circledast \quad z' = \frac{-1}{y^2}\,y' = \frac{-1}{y^2}\left(\frac{-1}{2x}\,y - \frac{3}{2}\,xy^2\right) = \frac{1}{2x}\,z + \frac{3}{2}\,x$$

Resolem $\circledast$

- $z_h(x) = c\, e^{\int \frac{1}{2x}dx} = c\, e^{\frac{1}{2}\ln|x|} = c\sqrt{|x|}$, definida a $x \in \mathbb{R}$, derivable a $x \in \mathbb{R} \setminus \{0\}$

- $z_p(x) = \sqrt{|x|} \int \frac{1}{\sqrt{|x|}}\left(\frac{3}{2}x\right)dx =$
  - Si $x > 0 \Longrightarrow z_p(x) = \sqrt{x}\int \frac{1}{\sqrt{x}}\, 3/2\, x\, dx = \sqrt{x}\,\frac{3}{2}\int \sqrt{x}\,dx = \sqrt{x}\cdot x^{3/2}$
  - Si $x < 0 \Longrightarrow z_p(x) = \sqrt{-x}\int \frac{3}{2}\frac{x}{\sqrt{-x}} = \sqrt{-x}\int -\frac{3}{2}(-x)^{1/2}dx = \sqrt{-x}\,(-x)^{3/2}$

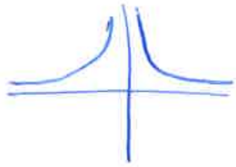
$\Longrightarrow z_p(x) = \sqrt{|x|}\,|x|^{3/2} = x^2$

$\hookrightarrow$

Per tant, la sol. general és de la forma $z(x) = C\sqrt{|x|} + x^2,\ C \in \mathbb{R}$

$$z = \frac{1}{y} \implies \boxed{\begin{cases} y = 0 \\ y = \dfrac{1}{C\sqrt{|x|} + x^2} \end{cases}}$$

Domini: $C > 0 \longrightarrow x \in \mathbb{R} \setminus \{0\}$



$c < 0$: $C\sqrt{|x|} + |x|^2 = 0$

$c + |x|^{3/2} = 0 \iff x = \pm(-c)^{2/3}$

$D = \mathbb{R} \setminus \{0;\ \pm(-c)^{2/3}\}$

(15) $y' = a(x)y + b(x)y^r$. Fer el canvi $y = u(x)z$ amb u tq la edo sigui separable.

$y' = u'(x)z + u(x)z' = a(x)u(x)z + b(x)u(x)^r z^r$

Triem u tq $u'(x) = a(x)u(x) \to$ sol. homogènia $\left(u(x) = e^{\int a(x)dx}\right)$

$\implies u(x)z' = b(x)u(x)^r z^r \implies z' = b(x)u(x)^{r-1}z^r$

separable ✓

$\cancel{\implies u(x) = e^{\int a(x)dx}}$

Aplicació: $y' + xy = x^3y^3 \to y' = -xy + x^3y^3$

Apliquem el canvi $y = u(x)z$:

$u'(x)z + u(x)z' = -xu(x)z + x^3u(x)^3z^3$

Busquem $u(x)$ tq $u'(x) = -xu(x)$: $u(x) = ce^{-x^2/2}$

$\implies z' \cdot ce^{-x^2/2} = x^3c^3e^{-\frac{3}{2}x^2}z^3 \longrightarrow z' = x^3c^2e^{-x^2}z^3$ separable.

$\to \displaystyle\int \frac{dz}{z^3} = c^2\int x^3e^{-x^2}dx \longrightarrow \frac{1}{\cancel{2}z^2} = c^2\frac{(x^2+1)e^{-x^2}}{\cancel{2}} + C_1 \longrightarrow$

$\to z^2 = \dfrac{e^{x^2}}{c^2(x^2+1)} + C_1 \longrightarrow z = \sqrt{\dfrac{e^{x^2}}{c^2(x^2+1)} + C_1} = \dfrac{1}{c}\sqrt{\dfrac{e^{x^2}}{x^2+1}}$

$\implies y = \cancel{c}e^{-x^2/2}\left(\dfrac{1}{\cancel{c}}\sqrt{\dfrac{e^{x^2}}{x^2+1}}\right) \Rightarrow \boxed{y = e^{-x^2/2}\sqrt{\dfrac{e^{x^2}}{x^2+1} + C_1}}$

(19) (Edo generalitzada de Riccati) $y' = a_0(x) + a_1(x)y + a_2(x)y^2$

En general no són resolubles, però si sabem una sol. particular, se'n pot calcular la general

Sigui $y_1(x)$ la sol. particular:

- Fem el canvi $y = y_1(x) + z$, que la transforma en Bernoulli

obé: Fem el canvi $y = y_1(x) + \frac{1}{u}$, que la transforma en lineal:

$$y' = a_0 + a_1 y + a_2 y^2 \Rightarrow y_1'(x) - \frac{1}{u^2}u' = a_0 + a_1\left(y_1 + \frac{1}{u}\right) + a_2\left(y_1 + \frac{1}{u}\right)^2$$

$$\Rightarrow \cancel{a_0} + \cancel{a_1 y_1} + \cancel{a_2 y_1^2} - \frac{u'}{u^2} = \cancel{a_0} + \cancel{a_1 y_1} + \frac{1}{u}a_1 + \cancel{a_2 y_1^2} + 2a_2 y_1 \frac{1}{u} + a_2 \frac{1}{u^2}$$

$$\Rightarrow \frac{-u'}{u^2} = \frac{a_1}{u} + \frac{2a_2 y_1}{u} + \frac{a_2}{u^2} \Rightarrow$$

$$\Rightarrow \boxed{u' = u(-a_1 - 2a_2 y_1) - a_2} \longrightarrow \text{Edo lineal en } u.$$

Aplicació

a) $(1-x^3)y' = y^2 - x^2 y - 2x$

1) Obs. que $y_1 = x+1$ és sol. particular:

$(1-x^3) = (x+1)^2 - x^2(x+1) - 2x$ ✓

Sigui l'edo de la forma $y' = \frac{-2x}{1-x^3} + \frac{-x^2}{1-x^3}y + \frac{1}{1-x^3}y^2$

Fent el canvi de variable $y = y_1 + \frac{1}{u} = x+1+\frac{1}{u}$ la convertim en una edo lineal:

$$\boxed{u' = u\left(\frac{x^2 - x - 1}{1-x^3}\right) - \frac{1}{1-x^3}}$$

2) Una altra sol particular és $y_2 = -x^2$.

Fem el canvi $y = -x^2 + \frac{1}{u}$:

$$u' = u\left(\frac{x^2}{1-x^3} - 2\frac{1}{1-x^3}\cdot(-x^2)\right) - \frac{1}{1-x^3}$$

$$\boxed{u' = \frac{3x^2}{1-x^3}u - \frac{1}{1-x^3}}$$

La que s'ha utilitzat en el canvi s'ha d'afegir, ja que s'ha considerat $u \neq 0$

$$u_h(x) = c\,e^{-\ln|1-x^3|} = \frac{c}{|1-x^3|} \quad \dots \Rightarrow \boxed{u(x) = \frac{c}{|1-x^3|} - \frac{x}{1-x^3}} \quad c \in \mathbb{R}$$

$y = -x^2 + \frac{1}{u}$

$$\boxed{y(x) = -x^2 + \frac{1-x^3}{c-x}, \; c \in \mathbb{R}, \; y(x) = -x^2}$$

Corbes solució

(27), (28), (29)

Les corbes es poden donar de varies maneres:

- Paramètriques
- Explícites
- Implícites

(27) Diem que $C \in \mathbb{R}^2$ és corba regular si $\forall P_0 \in C$ $\exists\, U_0 \ni P_0$ i una parametrització $\Phi : I_0 \subset \mathbb{R} \longrightarrow U_0 \subset \mathbb{R}^2$ $\mathcal{C}^r$ i tq $\Phi'(t) \neq 0$ en I_0 tq $C \cap U = \operatorname{Im} \Phi(t)$, $t \in I_0$

Definim $P, Q : W \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$, contínua,

$(P(x,y), Q(x,y)) \neq (0,0)$ en W.

Diem que $C \subset W$ és sol. de $P(x,y)\,dx + Q(x,y)\,dy = 0$ sii $\Phi(t) = (\alpha(t), \beta(t))$ és sol. de $P(\Phi(t))\alpha'(t) + Q(\Phi(t))\beta'(t) = 0$

Cal veure que:

i) La definició de solució no depèn de Φ

ii) Interpretació geomètrica.

ii) $P \circ \Phi\, \alpha' + Q \circ \Phi\, \beta' = 0$ $\circledast$

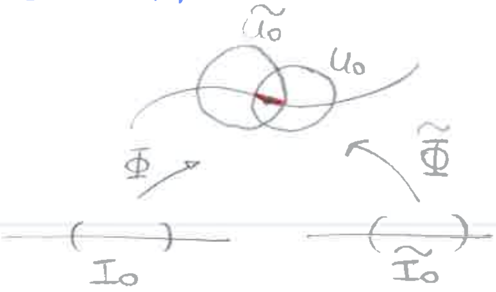
$\Phi' = (\alpha', \beta')$ vector tangent a C

(P, Q) vector normal a la corba

$P \circ \phi\, \alpha' + Q \circ \phi\, \beta' = 0 \iff C \perp a\ (P(x,y), Q(x,y))\ \forall (x,y)$

i) Sigui $C \ni p_0$. Prenem entorn U_0 de p_0. $U_0 \subset \mathbb{R}^2$, parametritzada per $\phi : I_0 \longrightarrow U_0 \subset \mathbb{R}$, sol. de $\circledast$

Agafem una altra $\tilde{\Phi} : \tilde{I}_0 \longrightarrow \tilde{U}_0 \ni p_0$



$\left.\begin{array}{l} \phi(t_0) = p_0 \\ \tilde{\phi}(s_0) = p_0 \end{array}\right\}$ $V_0 = U_0 \cap \tilde{U}_0$

$\phi, \tilde{\phi}$ regulars $\Rightarrow \phi'(t) \neq 0$, $\tilde{\phi}'(s) \neq 0$ en I_0 i $\tilde{I}_0$

Suposem $\alpha'(t) \neq 0$ $\forall t \in I_0$:

$\Rightarrow \left.\begin{array}{l} \phi \text{ bij. entre } I_0 \text{ i } C \cap V_0 \\ \tilde{\phi} \text{ bij. entre } \tilde{I}_0 \text{ i } C \cap V_0 \end{array}\right\} \Rightarrow \exists\, g : \tilde{I}_0 \longrightarrow I_0 : \phi(g(s)) = \tilde{\phi}(s)$

$\Rightarrow \begin{cases} \alpha(g(s)) = \tilde{\alpha}(s) : \mathbb{R} \longrightarrow \mathbb{R} \\ \alpha'(t) \neq 0 \quad \forall t \in I_0 \end{cases}$

$\Rightarrow$ TFInv en I_0 : $g(s) = \alpha^{-1}(\tilde{\alpha}(s))$. $\Rightarrow g$ és $\mathcal{C}^r$, $g'(s) \neq 0$ en $\tilde{I}_0$

Hipòtesi: ϕ sol $(P \circ \phi\, \alpha' + Q \circ \phi\, \beta' = 0)$. Vegem $\tilde{\phi}$ també:

$\tilde{\phi}$ sol. si $(P \circ \tilde{\phi})(s)\, \tilde{\alpha}'(s) + (Q \circ \tilde{\phi})(s)\, \tilde{\beta}'(s) = 0$

$$\underbrace{g'(s)}_{\neq 0} \Big[\underbrace{P(\phi(g(s)))\, \alpha'(g(s)) + Q(\phi(g(s)))\, \beta'(g(s))}_{} \Big] = 0$$

Per tant, $= 0 \iff ⊛ = 0$.

$\Rightarrow \phi$ solució $\iff \tilde{\phi}$ solució.

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a) C corba regular $\mathcal{C}^r \iff \forall p_0 \in C\ \exists U_0 : C|_{U_0}$ és gràfica de $\{y = f(x)\}$ o $\{x = g(y)\}$. $(\{(x,y) : x \in I_0,\ y = f(x)\})$, $f(x) \in \mathcal{C}^r$

b) Dem. que. si $A = \{(x,y) \in W \mid Q(x,y) \neq 0\}$, llavors

$$P(x,y)dx + Q(x,y)dy = 0 \quad \equiv \quad \frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)}$$

si $B = \{(x,y) \in W \mid P(x,y) \neq 0\}$, llavors

$$P(x,y)\,dx + Q(x,y)\,dy = 0 \quad \equiv \quad \frac{dx}{dy} = \frac{-Q(x,y)}{P(x,y)}$$

a) $\Leftarrow$) Suposem $C \cap U_0 = \{y = f(x)\}$. Es té una parametrització $\phi(t) = (t, f(t))$, $\phi'(t) = (t, f'(t)) \neq (0,0)$

$\Rightarrow$) Prenem $C \cap U_0$: Param. $\phi(t) = (\alpha(t), \beta(t))$, $\phi'(t) \neq 0$

Suposem $\alpha'(t) \neq 0$ per $t \in I_0$.

$\alpha : I_0 \longrightarrow I_1$, $\alpha'(t) \neq 0$ en I_0

TFInv. $\Rightarrow \exists \alpha^{-1} : I_1 \longrightarrow I_0$, $\mathcal{C}^r$

$\hookrightarrow$

Aleshores, podem definir una nova parametrització de $C \cap U_0$:

$$\tilde{\phi} = \phi \circ \alpha^{-1} = (\underbrace{\alpha(\alpha^{-1}(t))}_{t}, \beta(\alpha^{-1}(t))) \quad , \quad \text{gràfic.}$$

b) $P(x,y)dx + Q(x,y)dy = 0 \longleftrightarrow (P \circ \phi)\alpha' + (Q \circ \phi)\beta' = 0$

$A = \{(x,y) \in W : Q(x,y) \neq 0\}$

$Q \neq 0 \implies Q \circ \phi \neq 0 \implies \alpha' \neq 0$

$\hookrightarrow \alpha' = 0 \implies (Q \circ \phi)\beta' = 0 \implies \beta' = 0$ (!!)

~~$\alpha: I_0 \longrightarrow I_1$, $\alpha' \neq 0$ a I_0~~

~~$\overset{TFInv}{\implies} \exists \alpha^{-1}: I_1 \longrightarrow I_0$, $\mathcal{C}^r$.~~

~~Podem~~

Per a), $\alpha'(t) \neq 0 \implies C \cap U_0$ es pot escriure com $y = f(x)$

$C \cap U_0 = \{(x, f(x)) : x \in I_0\}$.

La solució de $(*)$ és indep. de la parametrització ((27))

$\implies$ s'ha de complir per la param. $\phi(t) = (t, f(t))$

$\implies P(x, f(x)) \cdot 1 + Q(x, f(x)) f'(x) = 0 \implies f'(x) = \dfrac{-P(x, f(x))}{Q(x, f(x))}$

Com que $y = f(x)$, y és sol. de $\dfrac{dy}{dx} = \dfrac{-P(x,y)}{Q(x,y)}$.

$\implies C$ sol. de $Pdx + Qdy = 0 \iff$ sol. de $\dfrac{dy}{dx} = \dfrac{-P(x,y)}{Q(x,y)}$

(29) a) $C \subset W \subset \mathbb{R}^2$ corba regular $\mathcal{C}^r \iff \forall p_0 \in C \ \exists$ entorn $U_0 \subset W$:

$C|_{U_0}$ ve donada de forma implícita. $\exists h: U_0 \longrightarrow \mathbb{R}$, $\mathcal{C}^r$

$C|_{U_0} = \{h(x,y) = 0\}$, $D_h(p) \neq 0 \quad \forall p \in U_0$

b) Dem. que la família $h(x,y) = a$, $a \in \mathbb{R}$, $h: U_0 \subset W \longrightarrow \mathbb{R}$,

$D_h(x,y) \neq 0$ satisfà l'edo $\iff \begin{vmatrix} P & \partial_x h \\ Q & \partial_y h \end{vmatrix} = 0$ en U_0

a) C regular en $U_0 \iff C \cap U_0 = \{y = f(x)\}$ o $\{x = g(y)\}$

$\implies \exists h: U_0 \rightarrow \mathbb{R}$: $D_h(p) \neq 0$ en U_0 : $C \cap U_0 = \{h(x,y) = 0\}$

$\hookrightarrow$

b) Suposem que la família $h(x,y) = a$, $a \in \mathbb{R}$ resol $P dx + Q dy = 0$

$\iff \forall\, \phi(t) : I_1 \longrightarrow U$ amb $h \circ \phi = ct$, es compleix

$(P \circ \phi)(t)\, \alpha'(t) + (Q \circ \phi)(t)\, \beta'(t) = 0$.

Derivant: $(\partial_x h)(\phi(t))\, \alpha'(t) + (\partial_y h)(\phi(t))\, \beta'(t) = 0$.

$\forall \phi$ param, $h \circ \phi = ct$:

$$\left.\begin{array}{l} (P \circ \phi ; Q \circ \phi)(\alpha', \beta') = 0 \\ (\partial_x h \circ \phi, \partial_y h \circ \phi)(\alpha', \beta') = 0 \\ (\alpha', \beta') \neq (0,0) \end{array}\right\} \iff (P \circ \phi, Q \circ \phi) \text{ és paral·lel a } (\partial_x h \circ \phi, \partial_y h \circ \phi) \ \forall \phi : h \circ \phi\ ct$$

$\iff (P(x,y), Q(x,y)) \mathbin{\overset{\text{paral·lel}}{/\!/}} (\partial_x h(x,y), \partial_y h(x,y))$ en U_0

$\iff \begin{vmatrix} P & \partial_x h \\ Q & \partial_y h \end{vmatrix} = 0$ en U_0

(30) Una equació $P(x,y)\,dx + Q(x,y)\,dy = 0$, amb P i $Q \in \mathcal{C}^1$ a W, s'anomena <u>exacta</u> si $\partial_y P = \partial_x Q$

Dem que: a) $\forall\, V \subset W$ rectangle $\exists\, u : V \longrightarrow \mathbb{R}$ $\mathcal{C}^2$: $\partial_x u = P$, $\partial_y u = Q$

b) La sol. ve donada per $u(x,y) = ct$

b) Sup. que tenim u: $\partial_x u = P$, $\partial_y u = Q$. Aleshores,

$$\begin{vmatrix} P & \partial_x u \\ Q & \partial_y u \end{vmatrix} = \begin{vmatrix} P & P \\ Q & Q \end{vmatrix} = 0 \iff u(x,y) = ct \text{ és sol.}$$

$u(x,y) = \int_a^x P(s,y)\,ds + \int_c^y Q(a,s)\,ds$

a) Prenem $V = [a,b] \times [c,d] \subset W_0$.

$$\left.\begin{array}{l} \partial_x u = P \\ \partial_y u = Q \end{array}\right\} \iff u(x,y) = \int_a^x P(s,y)\,ds + h(y) \qquad h(y) = \int_c^y Q(a,s)\,ds$$

Imposem $\partial_y u = Q$: $\partial_y u(x,y) = \int_a^x \partial_y P(s,y)\,ds + h'(y) =$ $\qquad h'(y) = Q(a,y)$

$= \int_a^x \partial_x Q(s,y) + h'(y) = Q(x,y) - Q(a,y) + h'(y) \overset{\text{Imposem}}{=} Q(x,y)$

(31) $\underbrace{(3y+e^x)}_{P}dx + \underbrace{(3x+\cos y)}_{Q}dy = 0$

$\partial_y P = 3, \quad \partial_x Q = 3 \implies$ Edo exacta

Busquem u : $\partial_x u = P$
$\partial_y u = Q$

$\partial_x u = P = 3y + e^x \implies u(x,y) = \int (3y+e^x)\,dx + h(y)$
$= 3xy + e^x + h(y)$

$\partial_y u = Q = 3x + \cos y \implies \partial_y u = 3x + h'(y) = 3x + \cos y$

$\iff h'(y) = \cos y \iff h(y) = \sin y + C$

$\implies u(x,y) = 3xy + e^x + \sin y$

(32) Resoldre: $\begin{cases} 4t^3e^{t+y} + t^4e^{t+y} + 2t + (t^4e^{t+y} + 2y)\dfrac{dy}{dt} = 0 \\ y(0) = 1 \end{cases}$

$\underbrace{(4t^3e^{t+y} + t^4e^{t+y} + 2t)}_{P}dt + \underbrace{(t^4e^{t+y} + 2y)}_{Q}dy = 0$

$\partial_y P = \partial_t Q$? ✓ $\implies$ Exacta

Busquem u : $\begin{cases} \partial_t u = P \\ \partial_y u = Q \end{cases}$

$t^4e^{t+y} + y^2 + h(t)$
"
$\partial_y u = Q = t^4e^{t+y} + 2y \implies u(t,y) = \int (t^4e^{t+y} + 2y)\,dy + h(t)$

$\partial_t u = P$. $\quad \partial_t u = 4t^3e^{t+y} + t^4e^{t+y} + h'(t) =$
$P = 4t^3e^{t+y} + t^4e^{t+y} + 2t$

$\implies h'(y) = 2t \iff h(t) = t^2$.

Llavors, $u(t,y) = t^4e^{t+y} + y^2 + t^2 = C_1$

$y(0) = 1 \implies (t,y) = (0,1) \implies \boxed{u(0,1) = 1} \implies C_1 = 1$

(36) Una família de corbes $U(x,y)=a$, $U: W \longrightarrow \mathbb{R}^2$ $\mathcal{C}^1$, $DU(p)\neq 0$ en W satisfà $P\,dx + Q\,dy = 0$ sii

$\exists \mu : W \longrightarrow \mathbb{R}$ cont, $\mu(P)\neq 0$ i $\forall p \in W$: $\partial_x U = \mu P$, $\partial_y U = \mu Q$

$$\begin{vmatrix} P & \partial_x U \\ Q & \partial_y U \end{vmatrix} = 0 \text{ en } W \iff \exists \mu : W \to \mathbb{R} : (\partial_x U, \partial_y U) = \mu(P,Q)$$

amb $\mu \neq 0$. μ és contínua on $P\neq 0$: $\mu = \dfrac{\partial_x U}{P}$, $\mu = \dfrac{\partial_y U}{Q}$

$$P\,dx + Q\,dy = 0 \iff (\mu P)\,dx + (\mu Q)\,dy = 0$$

derivades creuades iguals

$$\left.\begin{array}{l} \partial_x U = \mu P \\ \partial_y U = \mu Q \\ U \in \mathcal{C}^2 \end{array}\right\} \Longrightarrow \partial_y(\mu P) = \partial_x(\mu Q) \Longrightarrow \text{exacta.}$$

(37) Diem $\mu : W \to \mathbb{R}$, $\mathcal{C}^1$, $\mu(p)\neq 0$ $\forall p \in W$ és un factor integrant de $P\,dx + Q\,dy = 0$ si $(\mu P)\,dx + (\mu Q)\,dy = 0$ és exacta

$$\mu \text{ factor integrant} \iff \partial_y(\mu P) + \mu\,\partial_y P = \partial_x \mu Q + \mu\,\partial_x Q$$

$$\boxed{\partial_x \mu Q - \partial_y \mu P = \mu(\partial_y P - \partial_x Q)} \quad \text{EDP}$$

(39)
a) $\left(\frac{y^2}{2} + 2ye^x\right)dx + (y+e^x)\,dy = 0$

$\partial_y P = y + 2e^x$, $\partial_x Q = e^x \Longrightarrow$ No exacta

Busquem μ que compleixi:

$$\partial_x \mu Q - \partial_y \mu P = \mu(\partial_y P - \partial_x Q) = \mu(y+e^x) = \mu Q$$

Si prenem una μ que només depengui de x:

$$\partial_x \mu Q = \mu Q \Longrightarrow \partial_x \mu(x) = \mu(x) \iff \boxed{\mu(x) = e^x > 0}$$

$$\Longrightarrow e^x\left(\frac{y^2}{2} + 2ye^x\right)dx + e^x(y+e^x)\,dy = 0 \quad \text{És exacta}$$

Ex Trobar la solució $\longrightarrow U(x,y) = \frac{y^2}{2}e^x + ye^{2x} = ct$

46 Prod. químics A, B formen C

- velocitat reacció directament proporcional al producte de quantitats de A i B
- Reacció requereix 2kg de A per cada kg de B
- Inicialment, 10 kg (A) i 20 kg (B).
- 20 min → tenim 6 kg de C. ⟹ $C(t)$?

$A(t)$, $B(t)$, $C(t)$ quantitats en temps t.

$$C'(t) = K\,A(t)\,B(t) \quad , \qquad A(0) = A(t) + \tfrac{2}{3}C(t)$$
$$B(0) = B(t) + \tfrac{1}{3}C(t)$$

$$\Rightarrow C'(t) = K\left(A(0) - \tfrac{2}{3}C(t)\right)\left(B(0) - \tfrac{1}{3}C(t)\right) =$$
$$= K\Big(\underbrace{\tfrac{3}{2}A(0)} - C(t)\Big)\Big(\underbrace{3B(0)} - C(t)\Big) =$$
$$\frac{dC}{dt} = K(a_0 - C)(b_0 - C) \qquad \text{edo separable.}$$

$$\frac{dC}{dt} = K(a_0 - C)(b_0 - C) \Rightarrow K\int dt = \int \frac{dC}{(a_0 - C)(b_0 - C)} \Rightarrow$$ Fracc simples

$$\Rightarrow Kt + K_0 = \frac{1}{b_0 - a_0}\left(\int \frac{dC}{C - b_0} - \int \frac{dC}{C - a_0}\right) = \frac{1}{b_0 - a_0}\ln\left|\frac{C - b_0}{C - a_0}\right|$$

$$\Rightarrow Kt + K_0 = \ln\left(\frac{C - b_0}{C - a_0}\right) \Rightarrow \frac{C - b_0}{C - a_0} = e^{Kt + K_0} \Rightarrow$$

$$\Rightarrow C(t) = \frac{b_0 - a_0 e^{Kt + K_0}}{1 - e^{Kt + K_0}} \xrightarrow[b_0 = 3B(0)]{a_0 = \frac{3}{2}A(0)} C(t) = \frac{60 - 15e^{Kt + K_0}}{1 - e^{Kt + K_0}}$$

Condicions inicials:

$C(0) = 0 \longrightarrow b_0 - a_0 e^{K_0} = 0 \longrightarrow e^{K_0} = b_0/a_0$

$C(20) = 6 \longrightarrow \ldots$

$$\Rightarrow \boxed{C(t) = \frac{60\left(1 - e^{\frac{\log 3/2}{20}t}\right)}{1 - 4e^{\frac{\log 3/2}{20}t}}}$$

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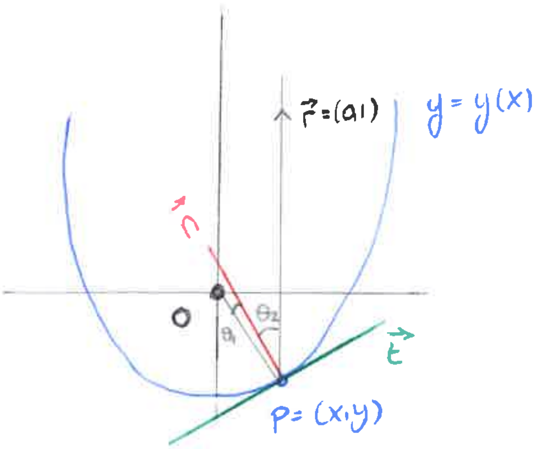

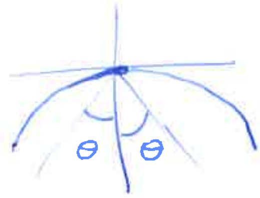

$\Rightarrow \theta_1 = \theta_2$

$\Phi(x) = (x, y(x))$, $\boxed{\vec t = \left(1, \frac{dy}{dx}\right)}$

sabem que la bisectriu de $\vec r$ i $\vec{PO}$ són $\perp$ a $\vec t$:

$$\begin{cases} \vec r = (0,1) \\ \vec{PO} = (-x,-y) \end{cases}$$

$$\vec b \text{ bisectriu} = (0,1) + \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}} \right) = \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{\sqrt{x^2+y^2} - y}{\sqrt{x^2+y^2}} \right)$$

$$\vec b \cdot \vec t = 0 \Rightarrow \frac{-x}{\sqrt{x^2+y^2}} + \frac{\sqrt{x^2+y^2} - y}{\sqrt{x^2+y^2}} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{\sqrt{x^2+y^2} - y} = x \, \frac{\sqrt{x^2+y^2} + y}{x^2} = \frac{\sqrt{x^2+y^2} + y}{x} =$$

$= \sqrt{1 + (y/x)^2} + y/x = g(y/x)$. (Edo homogènia $y' = g(y/x)$)

Fem el canvi de variable $z = y/x \longmapsto y = zx$

$y' = \sqrt{1 + \left(\frac{y}{x}\right)^2} + \frac{y}{x} \Rightarrow z'x + \cancel{z} = \sqrt{1+z^2} + \cancel{z}$ Edo separable

$$\Rightarrow \frac{dz}{dx} = \frac{\sqrt{1+z^2}}{x} \Rightarrow \int \frac{1}{x} dx = \int \frac{1}{\sqrt{1+z^2}} dz \Rightarrow$$

$\Rightarrow \ln x + C = \operatorname{arcsinh}(z) + C \Rightarrow z = \sinh(\ln(kx))$

$= \ln(kx)$

$$= \frac{e^{\ln(kx)} - e^{-\ln(kx)}}{2} = \frac{kx - \frac{1}{kx}}{2} = \frac{k^2x^2 - 1}{2kx} = z$$

$\Rightarrow \boxed{y = \frac{kx^2}{2} - \frac{1}{2k}}$ → paràboles amb focus $(0,0)$ i directriu horitzontal

4. Teoremes Fonamentals

(PROBLEMES)

Ex 2 final 2018

$\varepsilon \in \mathbb{R}$, $\begin{cases} x' = y + \varepsilon(x^2+y^2-1)x \\ y' = -x + \varepsilon(x^2+y^2-1)y \end{cases}$

1) Trobar sol. tq $x(0) = 2$, $y(0) = 0$

$$\begin{Bmatrix} x = r\cos\theta \\ y = r\sin\theta \end{Bmatrix} \Longrightarrow \dots \longrightarrow \begin{cases} r' = \varepsilon r(r^2-1) \\ \theta' = -1 \end{cases} + \begin{array}{l} r(0) = 2 \\ \theta(0) = 0 \end{array}$$

$$\theta(t) = \int -1\, dt = -t + C, \quad \xrightarrow{\theta(0)=0} \theta(t) = -t$$

$$r' = \varepsilon r(r^2-1) \Longrightarrow \int \frac{dr}{r(r^2-1)} = \int \varepsilon\, dt + C \Rightarrow$$

Fraccions simples

$$\Rightarrow \frac{1}{2} \ln\left|\frac{r^2-1}{r^2}\right| = \varepsilon t + C \Longrightarrow \ln\left|\frac{r^2-1}{r^2}\right| = 2\varepsilon t + C$$

$$\Longrightarrow \begin{cases} \left|\frac{r^2-1}{r^2}\right| = ke^{2\varepsilon t} \\ r(0) = 2 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{r^2-1}{r^2} = ke^{2t\varepsilon} \\ r(0) = 2 \end{cases} \longrightarrow k = \frac{3}{4} \longrightarrow r^2 - 1 = r^2 \frac{3}{4} e^{2\varepsilon t}$$

$$r^2 = \frac{1}{1 - \frac{3}{4} e^{2\varepsilon t}} \Longrightarrow$$

$$\Longrightarrow \boxed{\begin{cases} r = \frac{1}{\sqrt{1 - \frac{3}{4} e^{2\varepsilon t}}} \\ \theta = -t \end{cases}}$$

2) Interval maximal de definició

$\varepsilon = 0 \longrightarrow t \in \mathbb{R}$

$\varepsilon \neq 0$: $1 - \frac{3}{4} e^{2\varepsilon t} > 0 \longrightarrow \ln\left(\frac{4}{3}\right) > 2\varepsilon t$

$\underline{\underline{\varepsilon > 0}}$: $t < \frac{1}{2\varepsilon} \ln(4/3) \Longrightarrow t \in \left(-\infty, \frac{1}{2\varepsilon} \ln(4/3)\right)$

$\underline{\underline{\varepsilon < 0}}$: $t > \frac{1}{2\varepsilon} \ln(4/3) \Longrightarrow t \in \left(\frac{1}{2\varepsilon} \ln(4/3), +\infty\right)$

(6) $f: \mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ cont i loc. Lipschitz resp x

$(t,x) \longmapsto f(t,x)$

$|f(t,x)| \leq a(t)\,|x| + b(t)$, $|x| \geq R$, $a,b: \mathbb{R} \to \mathbb{R}^+$ cont

Dem. que les solucions es poden definir $\forall t \in \mathbb{R}$

Sigui $\varphi(t)$ solució. L'interval maximal de definició (ω_-, ω_+)

Volem veure $\omega_+ = +\infty$ (i $\omega_- = -\infty$)

! Suposem $\omega_+ < +\infty$:

Pel teorema de prolongació de solucions, sigui $\Omega = \mathbb{R}\times\mathbb{R}^n$.

es té que $(t, \varphi(t)) \xrightarrow[t \to \omega_+]{} \partial\Omega$

si sup $\omega_+ < +\infty$ $\Longrightarrow$ $\varphi(t)$ no fitada quan $t \to \omega_+$

Sigui $t_0 \in (\omega_-, \omega_+)$ i estudiem $\varphi(t)$ per $t \in [t_0, \omega_+)$:

$\varphi(t)$ no fitada quan $t \longrightarrow \omega_+$

$$|f(t,x)| \leq \begin{cases} M & |x| \leq R,\ t \in [t_0, \omega_+] \\ a(t)\,|x| + b(t), & |x| \geq R \end{cases}$$

$\to$ cpt i contínua $\Rightarrow$ fitada

Sigui $\tilde{b}(t) = \max\{b(t), M\}$

$|f(t,x)| \leq a(t)\,|x| + \tilde{b}(t)$ $\quad \forall t \in [t_0, \omega_+]$ i $\forall x \in \mathbb{R}^n$

$|f(t,\varphi(t))| \leq a(t)\,|\varphi(t)| + \tilde{b}(t)$ $\quad \forall t \in [t_0, \omega_+)$ obert pq $\varphi(t)$ no def a ω_+

Lema de Gronwall: $u, v : [a,b) \longrightarrow \mathbb{R}^+$, $C > 0$ tq

$u(t) \leq C + \int_a^t u(s)v(s)\,ds$, $t \in [a,b)$

$\Longrightarrow u(t) \leq C e^{\int_a^t v(s)\,ds}$ $\quad \forall t \in [a,b)$

$\forall t \in [t_0, \omega_+)$

Nosaltres tenim: $x' = f(t,x) \longrightarrow \varphi(t) = \varphi(t_0) + \int_{t_0}^t f(\ ,\varphi(\ ))\,ds$

$\Rightarrow |\varphi(t)| \leq |\varphi(t_0)| + \int_{t_0}^t |f(s,\varphi(s))|\,ds$

$\leq |\varphi(t_0)| + \int_{t_0}^t [a(s)\,|\varphi(s)| + \tilde{b}(s)]\,ds =$

$= \underbrace{|\varphi(t_0)| + \int_{t_0}^t \tilde{b}(s)\,ds}_{\leq C} + \int_{t_0}^t a(s)\,|\varphi(s)|\,ds$

$\to$ usant $t \in [t_0, \omega_+]$ cont en cpt

$\hookrightarrow$

$$|\varphi(t)| \leq C + \int_{t_0}^{t} A|\varphi(s)|\,ds \quad , \quad \text{on} \quad A = \max_{t\in[t_0,\omega_+]}\{a(t)\} \geq 0$$

$t \in [t_0, \omega_+)$

Per Gronwall: $|\varphi(t)| \leq c\, e^{\int_{t_0}^{t} A\,ds} = c\, e^{A(t-t_0)} \quad \forall t \in [t_0, \omega_+]$.

$t \longrightarrow \omega_+ \implies \varphi(t)$ fitada !! $\implies \omega_+ = +\infty$

(10) $X: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $\dot{x} = X(x)$, X loc. lipschitz, cont. i U obert

$W: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, $\mathcal{C}^1$: $\dot{W}(x) = DW(x)X(x) \leq 0$ si $x \in W^{-1}([c,+\infty))$

i $W^{-1}((-\infty, c])$ compacte de U.

Dem. que $\forall$ sol. Φ de $x' = X(x)$: $W(\Phi(t_0)) \leq C$ es pot prolongar sobre $[t_0, +\infty)$ i que $W(\Phi(t)) \leq c \quad \forall t \in [t_0, +\infty)$

Sigui $\Phi(t)$ una solució tq $\Phi(t_0) \in K$.

(ω_-, ω_+) interval maximal

! Suposem $\omega_+ < +\infty$

Pel teorema de prolongació de solucions, si tenim l'edo definida a $\Omega = \mathbb{R} \times U$: $(t, \Phi(t)) \xrightarrow[t\to\omega_+]{} \partial\Omega \implies \Phi(t) \xrightarrow[t\to\omega_+]{} \partial U$

$\exists \delta > 0 : \forall t \in (\omega_+ - \delta, \omega_+)$, $\Phi(t) \notin K = W^{-1}((-\infty, c])$

Considerem $t_1 > t_0$ tq $\Phi(t) \notin K \quad \forall t \in (t_1, \omega_+)$

Per $t \in (t_1, \omega_+)$ estudiem $r(t) = W(\phi(t))$.

$$\frac{dr(t)}{dt} = DW(\Phi(t))\,\dot{\Phi}(t) = DW(\Phi(t))X(\Phi(t))$$

$\forall t > t_1$, $\Phi(t) \in W^{-1}(c, +\infty)$; $\dfrac{dr(t)}{dt} \leq 0$ hipòtesi

$\implies W$ decreix al llarg de $\Phi(t)$

Triem t_1 tq $t_1 = \sup\{t : W(\Phi(t)) \leq c\}$. Aleshores,

~~$\forall t \in (t_1, +\infty)$. $\Phi(t) \notin K \implies \boxed{W(\Phi(t)) > c}$~~

$\implies t > t_1$, $r(t)$ decreix, i $\implies$

$\implies r(t) \leq r(t_1) \implies W(\Phi(t)) \leq |W(\Phi(t_1)) \leq c|$ $\longrightarrow$!!

8) Sigui $X : (a_1, a_2) \subset \mathbb{R} \longrightarrow \mathbb{R}$ cont : $X(x) = 0 \iff x = x_0$

Dem. que $\begin{cases} x' = X(x) \\ x(t_0) = x_0 \end{cases}$ té sol. única $\iff \int_{x_0^+} \frac{ds}{X(s)} ds$ i $\int_{x_0^-} \frac{ds}{X(s)} ds$ són divergents.

Comentari:

Si X fos Lipschitz: $\underset{\overset{\parallel}{|X(x)|}}{|X(x) - X(x_0)|} \leq L|x - x_0|$

$\Longrightarrow \int_{x_0^+} \frac{1}{|X(x)|} \geq \int_{x_0^+} \frac{1}{L|x-x_0|} = +\infty \Longrightarrow$ div. per comp. directa.

Exemples:

$\begin{cases} x' = |x|^\alpha, \ \alpha > 0 \\ x(t_0) = 0 \end{cases}$ $\quad \int_0^x \frac{1}{|s|^\alpha} ds$ conv. si $\alpha < 1$; div si $\alpha \geq 1$

- $\alpha = 1/2 \Longrightarrow \begin{cases} x(t) = 0 \\ \int_0^x \frac{ds}{\sqrt{|s|}} = \int_{t_0}^t dz = t - t_0 \end{cases} \Longrightarrow \begin{cases} 2\sqrt{x} & \text{si } x > 0 \\ -2\sqrt{-x} & \text{si } x < 0 \end{cases}$

$\Longrightarrow \begin{cases} 2\sqrt{x} = t - t_0 & \text{si } t - t_0 \geq 0 \Rightarrow \\ -2\sqrt{-x} = t - t_0 & \text{si } t - t_0 \leq 0 \Rightarrow \end{cases}$

$\Longrightarrow x(t) = \begin{cases} 1/4\,(t - t_0)^2 & \text{si } t \geq t_0 \\ -1/4\,(t - t_0)^2 & \text{si } t \geq t_0 \end{cases}$

Més d'una solució $\Longrightarrow \int \frac{1}{\sqrt{|x|}} dx$ convergent.

- $\alpha = 2 \Longrightarrow \begin{cases} x(t) = 0 \\ \int \frac{dx}{x^2} = \int dt + C \end{cases} \Longrightarrow x(t) = \frac{-1}{t + C} \neq 0 \ \forall t$

$\Longrightarrow x(t)$ és la única

Dem: $\Longleftarrow$ Suposem $\int_{x_0^\pm} \frac{1}{X(s)} ds$ divergents, i veiem que la sol. és única.

- $X(x)$ cont. $\Longrightarrow$ Peano implica $\exists$ sol. de $\begin{cases} x' = X(x) \\ x(t_0) = x_0 \end{cases}$
- Obs. que $x(t) = x_0$ és sol. Vegem si n'hi ha d'altres:

$\Longleftarrow$

$!$ Sup. que $\exists$ sol $\varphi(t)$ del PVI diferent a $x(t) = x_0$.

$\Longrightarrow \exists t_1 : \varphi(t_1) \neq x_0$. Suposem $t_1 > t_0$ i $\varphi(t_1) > x_0$

$x > x_0 \Longrightarrow X(x) \neq 0$. Suposem $X(x) > 0$:

Sigui $t_2 = \sup\{t : \varphi(t) = x_0\}$.

φ sol. $\Longrightarrow \varphi'(t) = X(\varphi(t))$. $\quad x > x_0 \Rightarrow \varphi(t)$ creixent $\Rightarrow t_2 < t_1$

Estudiem $\varphi(t)$ per $t > t_2$:

$\varphi(t) > x_0$. $\quad \varphi'(t) = X(\varphi(t)) > 0 \Longrightarrow \displaystyle\int_{t_2}^{t} \frac{\varphi'(t)}{X(\varphi(t))} = \int_{t_2}^{t} 1$

hipòtesi

$$\Longrightarrow t - t_2 = \int_{t_2}^{t} \frac{\varphi'(s)}{X(\varphi(s))}\,ds = \left\{ \begin{array}{l} r = \varphi(s) \\ dr = \varphi'(s)\,ds \end{array} \right\} = \int_{\varphi(t_2)}^{\varphi(t)} \frac{dr}{X(r)} = +\infty$$

$\Longrightarrow t - t_2 = +\infty$!!