3. CASUÍSTICA D'EDOS

Són edos de la forma $x' = f(t,x)$, $f: \mathcal{U} \subset \mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ contínua (a trossos).

- $x: I \subset \mathbb{R} \to \mathbb{R}^n$ és sol. si $x'(t) = f(t, x(t))$ i $(t, x(t)) \in \mathcal{U}$
- Edo separable: $x' = f(t,x)$ es pot escriure com a $h(x)x' = g(t)$.

  Ex: $t^2 + 2xx' = 0 \rightarrow 2xx' = -t^2 \rightarrow \int 2x\,dx = \int -t^2\,dt \Rightarrow \boxed{x^2 = -\frac{t^3}{3} + C}$

(Picard): $f: \mathcal{U} \subset \mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ cont., $(t_0, x_0) \in \mathcal{U}$.

$a, b > 0$: $\Omega = [-a+t_0, a+t_0] \times \{\|x - x_0\| < b\} \subset \mathcal{U}$, $\|\cdot\|$ qualsevol

$M = \max_{(t,x)\in\Omega} \|f(t,x)\|$, $\alpha = \min\{a, \frac{b}{M}\}$

f Lipschitz respecte x de Ω, $L > 0$ independent de t.

Aleshores,

i) $\exists!$ sol. del PVI

ii) $x(t)$ està definida a I_α

iii) $\forall t \in I_\alpha$, $x(t) \in \{\|x - x_0\| \le b\} =: B_b$

$\Big\}$ $\exists!\ x: I_\alpha \longrightarrow B_b$

- f cont i $D_x f$ cont $\Rightarrow$ f loc. Lipschitz al voltant de (t_0, x_0)

$$\|f(t,x_1) - f(t,x_2)\| \le \underbrace{\sup_{z\in\overline{x_1x_2}}}_{\text{TVM}} \underbrace{\|D_x f(t,z)\|}_{\le L \text{ si } (t,z)\in K \text{ cpt}} \|x_1 - x_2\|$$

(Peano): $f: \mathcal{U} \subset \mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}$ contínua, $(t_0, x_0) \in \mathcal{U}$

fixada $\|\cdot\|$: escollim $a, b > 0$: $\Omega \subset \mathcal{U}$ (Ω de Picard)

M, α de Picard.

Aleshores $\exists$ sol. $x: I_\alpha \longrightarrow B_b$, $x(t_0) = x_0$ (No unicitat!)

(Weierstrass): $K \subset \mathbb{R}^n$ cpt, $f: K \to \mathbb{R}^m$ cont.

$\forall \varepsilon\ \exists p: \|f - p\|_\infty = \sup \|f(x) - g(x)\| < \varepsilon$

↳ $\exists P_\ell$: i) $P_\ell \longrightarrow f$ unif

ii) $\|P_\ell\|_\infty = \|f\|_\infty$

(Ascoli - Arzelà): $K \subset \mathbb{R}^n$ cpt. Sigui $\Sigma \subset \mathcal{C}(K, \mathbb{R}^m)$ tq

i) $x \in K \Rightarrow \{h(x) : h \in \Sigma\}$ fitat

ii) Σ equicontinu

Llavors, $\exists \{h_n\} \in \Sigma$ unif. conv.

Solucions maximals

- La solució maximal d'una edo és $\varphi(t; t_0, x_0)$ definida a $I(t_0, x_0) = (\omega_-, \omega_+)$.

- $f: \mathcal{U} \subset \mathbb{R}\times\mathbb{R}^n$ cont, $(t_0, x_0) \in \mathcal{U}$, $\exists!$ sol. $x' = f(t,x)$, $x(t_0) = x_0$.

$\Rightarrow \exists!\ \varphi(\cdot\,; t_0, x_0)$ def. a $I(t_0, x_0)$ tq:

φ $\mathcal{C}^1$, $\varphi(t; t_0, x_0) \in \mathcal{U}$ i si $x: J \to \mathbb{R}^n$ sol., $\varphi|_J = x$.

* x sol. a I_x i $\mathcal{C}^1$ $\Rightarrow$ φ $\mathcal{C}^1$ a I_x

- $x' = f(t,x)$, $f: \mathcal{U} \subset \mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ cont. i $\exists!$ sol. del PVI
  $(t_0, x_0) \in \mathcal{U}$, $I(t_0,x_0)$ interval maximal i $\varphi(t;t_0,x_0)$ sol. maximal
  K cpt amb $(t_0,x_0) \in K$

  Aleshores, $\exists t \in I(t_0,x_0)$ tq $(t, \varphi(t;t_0,x_0)) \notin K$

  - Si $I(t_0,x_0) = (\omega_-, \omega_+)$: $\lim_{t\to\omega_\pm} (t; \varphi(t;t_0,x_0)) \in \partial\mathcal{U}$ $\leftarrow$ Potser $\nexists$ lim

$\mapsto$ $f: V \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ loc. Lipschitz (Picard), $x' = f(t,x)$

Aleshores, si $(t_0, x_0) \in \mathbb{R}\times V$:

i) Si ω_+ finit $\Rightarrow$ $\forall K \subset V$, $\exists t \in I(t_0,x_0)$ tq $\varphi(t;t_0,x_0) \notin K$
   Si $V = \mathbb{R}^n$ $\Rightarrow$ $\|\varphi(t;t_0,x_0)\| \xrightarrow[t\to\omega_+]{} \infty$

ii) $\varphi(t;t_0,x_0) \in \tilde{K}$, $\tilde{K}$ cpt i $\forall t \in [t_0, \omega_+)$ $\Rightarrow$ $\omega_+ = +\infty$.

- $f: \mathcal{U} \to \mathbb{R}$ contínua i $\exists$ sol. maximal $\Rightarrow$ $\forall K \subset \mathcal{U}$ $\exists t_* \in I(t_0,x_0) : (t_*, \varphi(t_*;t_0 x_0)) \notin K$

- (Gronwall): $u, v: [a,b) \longrightarrow [0,\infty)$ cont, $u(t) \leq c + \int_a^t v(s)u(s)ds$, $t \in [a,b)$

  Aleshores, $u(t) \leq c\, e^{\int_a^t v(s)ds}$

  $\mapsto$ $w(t) = c + \int_a^t v(s)u(s)ds$ $\Rightarrow$ $w'(t) = v(t)u(t) \leq v(t)\,w(t)$.

  - $c > 0$ i $w > 0$ $\Rightarrow$ $\dfrac{w'(t)}{w(t)} \leq v(t)$ $\Rightarrow$ $\log\left(\dfrac{|w(t)|}{|w(a)|}\right) \leq \int_a^t v(s)ds$

    $\Rightarrow$ $w(t) \leq w(a) \cdot e^{\int_a^t v(s)ds}$, $w(a) := c$

  - $c = 0$ $\Rightarrow$ $u(t) \leq \int_a^t v(s)u(s)\,ds$ $\Rightarrow$ $u(t) \equiv 0$.

3 Casuística d'edos

Edos: $x' = f(t,x)$, $f: U \subset \mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ contínua / a trossos.

Estudiarem si $\exists$ flux $\varphi(t; t_0, x_0)$ tq: $\frac{d}{dt}\varphi(t; t_0, x_0) = f(t; \varphi(t; t_0, x_0))$

$\varphi(t_0; t_0, x_0) = x_0$.

On està definit $\varphi(t; t_0, x_0)$?? (Regularitat del flux respecte $t; t_0, x_0$

$\exists$ flux $\Rightarrow$ $\exists!$ solució de $\begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases}$

Ex: $x' = x^2$, $x(0) = 0$

$x'(t) = x^2(t)$, $x \equiv 0$ ✓

Sup. que $x(t) \neq 0$, $t \in I$, $0 \in I$

$$\int \frac{x'(t)}{x^2(t)}\,dt = \int 1\,dt \Rightarrow \frac{-1}{x(t)} = t + k \Leftrightarrow \boxed{x(t) = \frac{-1}{t+k}}$$

$k \in \mathbb{R}$

Imposem que $x(0) = 0$:

$0 = x(0) = \frac{-1}{k}$!! $\Longrightarrow \boxed{x(t) \equiv 0}$

En qualsevol cas, si $x(0) = x_0$: $x(0) = \frac{-1}{k} = x_0 \Rightarrow k = \frac{-1}{x_0}$

$\Rightarrow x(t) = \frac{-1}{t - \frac{1}{x_0}} = \frac{-x_0}{1 - tx_0}$, i observem que $x(0) = 0$ ✓

Ex: $x' = x^{1/3}$, $x(0) = 0$

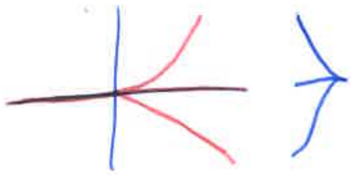
$x \equiv 0$ és sol.

Si $x \neq 0$, $t \in I$: $\int \frac{x'(t)}{x^{1/3}(t)}\,dt = \int 1\,dt \Rightarrow x^{2/3}(t) = \frac{2}{3}t + k$

$\Rightarrow x(t) = \pm\left(\frac{2}{3}t + k\right)^{3/2}$, $x(0) = 0 \Rightarrow k = 0$

$\Rightarrow$ obtenim $x(t) = \pm\left(\frac{2}{3}t\right)^{3/2}$ (definida a $t \geq 0$) $\hookrightarrow$

Aleshores, una sol. és $x(t) = \begin{cases} 0 & t \leq 0 \\ \left(\frac{2}{3}t\right)^{3/2} & t > 0 \end{cases}$



tres possibles solucions $\Rightarrow \nexists!$

<u>Ex:</u> $xx' = -t \quad , \quad x(0) = 0$

$$\int x(t)\,x'(t) = \int -t \Rightarrow \frac{x^2(t)}{2} = \frac{-t^2}{2} + C \Rightarrow x^2 + t^2 = C$$

$\nexists \ x(0) = 0$

<u>Def:</u> Solució de $x' = f(t,x)$

$f: U \subset \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ cont / a trossos

Diem que $x: I \subset \mathbb{R} \longrightarrow \mathbb{R}^n$ és sol. si

i) x derivable / a trossos , $x'(t) = f(t, x(t))$

ii) $(t, x(t)) \in U$

<u>Teorema</u> (Picard): Sigui $f: U \subset \mathbb{R} \times \mathbb{R}^n$ contínua. $(t_0, x_0) \in U$.

Siguin $a, b > 0$ tq $\Omega = [-a+t_0, t_0+a] \times \{\|x - x_0\| < b\} \subset U$

amb $\|\cdot\|$ norma de $\mathbb{R}^n$ fixada qualsevol.

Siguin $M = \max\limits_{(t,x)\in\Omega} \|f(t,x)\|$, $\alpha = \min\left\{a, \frac{b}{M}\right\}$

Suposem que f Lipschitz respecte la 2a variable de Ω:

$\|f(t,x_1) - f(t,x_2)\| \leq L \|x_1 - x_2\|$, $(t,x_1), (t,x_2) \in \Omega$

essent $L > 0$ (indep. de t)

Llavors: i) $\exists!$ sol. $\begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases}$

ii) $x(t)$ definida a $I_\alpha = [-\alpha + t_0, \alpha + t_0]$

iii) $\forall t \in I_\alpha$, $\underset{\text{sol.}}{\underline{x(t)}} \in \{\|x - x_0\| \leq b\} : B_b$ variable

<u>Dem:</u>

Obs: si f contínua i $D_x f$ contínua $\Rightarrow$ f Lipschitz al voltant de (t_0, x_0)

$$\|f(t,x_1) - f(t,x_2)\| \leq \overset{\text{tvm}}{\sup_{z \in \overline{x_1 x_2}}} \|D_x f(t,z)\| \, \|x_1 - x_2\|$$

sigui K cpt, $(t,z) \in K \Longrightarrow D_x f$ cont. $\Rightarrow \|D_x f(t,z)\| \leq L$ ✓

Obs: $x' = x^2$, $x(0) = 0$ Lipschitz ✓

$x' = x^{1/3}$, $x(0) = 0$ No lips.

$x' = \frac{-t}{x}$, $x(0) = 0$ No lips.

Dem (Teorema):

Fixem $(t_0, x_0) \in \mathcal{U}$, $\|\cdot\|$ a $\mathbb{R}^n$; agafem constants a, b, M, L

① Reformulem el problema:

$$\left.\begin{array}{l} x' = f(t,x) \\ x(t_0) = x_0 \end{array}\right\} \Longleftrightarrow x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

Volem:

i) $x(t)$ cont. $\Longrightarrow$ La sol. de l'equació integral serà $\mathcal{C}^1$

ii) $(t, x(t)) \in \mathcal{U}$

② Def. dels espais de Banach, $E = \mathcal{C}^0(I_\alpha, \mathbb{R}^n) = \{h : I_\alpha \to \mathbb{R}^n \ \mathcal{C}^0\}$

Escollim norma, $h \in E$:

$$\|h\|_\beta = \sup_{t \in I_\alpha} \|e^{\beta(t-t_0)} h(t)\|, \quad \beta < 0 \text{ a escollir}$$

Llavors, $(E, \|\cdot\|_\beta)$ és espai de Banach.

Definim $X \subset E$: $X = \{h \in E : \sup_{t \in I_\alpha} \|h(t) - x_0\| \leq b\}$

Volem definir $F : X \longrightarrow X$. Cal veure que X és tancat de $(E, \|\cdot\|_\beta)$

Ex $h_k \longrightarrow h$, $\|\cdot\|_\beta$, $h_k \in X \Rightarrow$ Cal veure $h \in X$

③ Definim el funcional $\mathcal{F}(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$

$\mathcal{F}: X \longrightarrow E$. Si $\mathcal{F}: X \to X$ i té un únic punt fix, el nostre problema està resolt?

i) $x = \mathcal{F}(x) \Rightarrow x$ sol. del PVI

ii) $x = \mathcal{F}(x) \Rightarrow x \in X \Rightarrow x \in \mathcal{C}^1$

iii) $x: I_\alpha \longrightarrow B_b$ per definició (si $x = \mathcal{F}(x)$)

iv) Comprovem que $\mathcal{F}(x) \in X$ $\forall x \in X$ i que $\mathcal{F}$ contractiva.

a) $\mathcal{F}(x) \in X$ si $x \in X$. $x: I_\alpha \longrightarrow B_b$

$t \in I_\alpha \Rightarrow s \in \overline{t\, t_0} \in I_\alpha \subset I_a$
A més, $x(s) \in B_b$ $\Big\} \Rightarrow (s, x(s)) \in I_a \times B_b = \Omega \in \mathcal{U}$ ⊛

Per tant, hem vist $\mathcal{F}(x) \in E$ $\forall x \in X$. Vegem que està a X:

$\mathcal{F}(x) \in X$, cal $\| \mathcal{F}(x)(t) - x_0 \| \leq b$ $\forall t \in I_\alpha$

En efecte: $\| \mathcal{F}(x)(t) - x_0 \| = \left\| \int_{t_0}^{t} f(s, x(s))\, ds \right\| \leq$

$\leq \left| \int_{t_0}^{t} \| f(s, x(s)) \|\, ds \right| \overset{⊛}{\leq} \left| \int_{t_0}^{t} M\, ds \right| = M\alpha \leq b$ $\quad \alpha = \min\{a, b/M\}$

on $M = \sup_{(t,x) \in \Omega} \| f(t,x) \|$

Per tant, $\mathcal{F}: X \longrightarrow X$ està ben definit.

b) $\mathcal{F}$ contractiva si $\forall x_1, x_2 \in X$:

$\| \mathcal{F}(x_1) - \mathcal{F}(x_2) \|_\beta \leq k \| x_1 - x_2 \|_\beta$, amb $k \in (0,1)$

$\mathcal{F}(x_1) - \mathcal{F}(x_2) = \int_{t_0}^{t} f(s, x_1(s)) - f(s, x_2(s))\, ds$

obs. que $x_1, x_2 \in X \Rightarrow x_1(s), x_2(s) \in B_b$, $s \in I_\alpha$

Llavors, $\| f(s, x_1(s)) - f(s, x_2(s)) \leq L \| x_1(s) - x_2(s) \|$, $\left.\begin{array}{l}(s, x_1(s)) \\ (s, x_2(s))\end{array}\right\} \in \Omega$

$\hookrightarrow$ (**)

$$\Rightarrow \| e^{\beta(t-t_0)} (\mathcal{F}(x_1)(t) - \mathcal{F}(x_2)(t)) \| \leq e^{\beta|t-t_0|} \left| \int_{t_0}^{t} \| f(s, x_1(s)) - f(s, x_2(s)) \| \, ds \right|$$

$$\leq e^{\beta|t-t_0|} L \left| \int_{t_0}^{t} \| x_1(s) - x_2(s) \| \, ds \right| \leq$$

$$\left[\| h \|_\beta = \sup_{t \in I_\alpha} \| e^{\beta|t-t_0|} h(t) \| \geq \| h(s) \| e^{\beta(s-t_0)} \quad \forall s \in I_\alpha \right]$$

$$\leq L e^{\beta(t-t_0)} \left| \int_{t_0}^{t} \| x_1 - x_2 \|_\beta \, e^{-\beta(s-t_0)} \, ds \right| \leq L \| x_1 - x_2 \|_\beta \left\{ e^{\beta(t-t_0)} \left| \int_{t_0}^{t} e^{-\beta(s-t_0)} \, ds \right| \right\}$$

$$\leq L \| x_1 - x_2 \|_\beta \cdot \frac{1}{|\beta|}, \quad \beta < 0 \quad = \frac{L}{|\beta|} \| x_1 - x_2 \|_\beta$$

Agafem $-\beta > L$, $K = \frac{L}{-\beta}$. Prenem $\sup_{t \in I_\alpha}$ a (**) i llavors,

$$\| \mathcal{F}(x_1) - \mathcal{F}(x_2) \|_\beta \leq K \| x_1 - x_2 \|_\beta$$

Teorema (Peano): $x' = f(t, x)$, $f: \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ contínua.

$(t_0, x_0) \in \mathcal{U}$, fixada $\|\cdot\|$ a $\mathbb{R}^n$, escollim $a, b > 0$:

$$\Omega = \underbrace{[t_0 - a, t_0 + a]}_{I_a} \times \underbrace{\{ \| x - x_0 \| \leq b \}}_{B_b} \subset \mathcal{U}$$

$$M = \sup_{(t,x) \in \Omega} \| f(t, x) \|, \qquad \alpha = \min \left\{ \frac{b}{M}, a \right\}$$

Llavors, $\exists$ sol. $x: I_\alpha \longrightarrow B_b$ tq $x(t_0) = x_0$

Teorema (Weierstrass): $K \subset \mathbb{R}^n$ cpt, $f: K \longrightarrow \mathbb{R}$ contínua,

$\forall \varepsilon > 0 \;\; \exists \; p$ polinomi tq $\| f - p \|_\infty = \sup_{x \in K} \| f(x) - g(x) \| < \varepsilon$

Corol·lari: $K \subset \mathbb{R}^n$, $f: K \longrightarrow \mathbb{R}^m$ contínua, $\exists \, p_\ell: K \longrightarrow \mathbb{R}^m$ t.q:

i) $p_\ell \longrightarrow f$ unif.

ii) $\| p_\ell \|_\infty = \| f \|_\infty$

Dem (Corol·lari): i) ✓

ii) $q_\ell \longrightarrow f$ unif. Definim $p_\ell = \frac{\| f \|_\infty}{\| q_\ell \|_\infty} \cdot q_\ell$, $p_\ell \longrightarrow f$ unif

$\hookleftarrow$

$\longleftarrow\!\!\longrightarrow$

En efecte: No és cap restricció $\|q_\ell\|_\infty \neq 0$.

$$\|p_\ell - f\|_\infty \leq \|p_\ell - q_\ell\|_\infty + \|q_\ell - f\|_\infty = \left\| \frac{\|f\|_\infty}{\|q_\ell\|_\infty} q_\ell - q_\ell \right\|_\infty + \|q_\ell - f\|_\infty =$$

$$= \|q_\ell\|_\infty \left| \frac{\|f\|_\infty}{\|q_\ell\|_\infty} - 1 \right| + \|q_\ell - f\|_\infty \xrightarrow[\ell\to\infty]{} 0$$

Teorema (Ascoli - Arzelà): $K \subset \mathbb{R}^n$ cpt. Considerem $\mathcal{C}(K, \mathbb{R}^m)$.

Considerem un subcjt $\Sigma \subset \mathcal{C}(K, \mathbb{R}^m)$ satisfent:

i) $x \in K \Rightarrow \{h(x),\ h \in \Sigma\}$ fitat

ii) Equicontinu

Llavors, $\exists \{h_n\} \in \Sigma$ conv. unif.

Dem (T. Peano)

$x' = f(t,x)$ contínua, $(t_0, x_0) \in \mathcal{U}$. Considerem les constants a, b de l'enunciat ($\Omega \subset \mathcal{U}$), M i α.

1. Teorema de Weierstrass: $P_m(t,x)$ successió de polinomis unif. conv. a f (a Ω). A més, $\|P_m\|_\infty = \sup_{(t,x)\in\Omega} \|P_m(t,x)\| = \|f\|_\infty$

2. Teorema Picard: $\begin{cases} x' = P_m(t,x) \\ x(t_0) = x_0 \end{cases}$ Aleshores, $\exists!$ sol. del PVI

On està definida x_m? Recordem que tenim definida I_{α_m}, on $\alpha_m = \min\{a, b/M_m\}$, $M_m = \sup_{(t,x)\in\Omega} \|P_m(t,x)\| = \|f\|_\infty = M$

Així, $\alpha_m = \alpha$. Per tant, totes les solucions x_m són definides en el mateix interval. $x_m : I_\alpha \longrightarrow B_b$

Podem escriure que $x_m(t) = x_0 + \int_{t_0}^{t} P_m(s, x_m(s))\,ds$, $t \in I_\alpha$

3. Teorema Ascoli-Arzelà: Vegem que Σ satisfà les condicions:

i) $t \in I_\alpha$, $x_m(t) \in B_b$. Picard $\Longrightarrow$ Punt. acotada.

ii) Equicontínua. $\varepsilon > 0 \Longrightarrow$

$\longleftarrow\!\!\longrightarrow$

$\hookrightarrow \quad \varepsilon > 0: \quad x_m(t_1) - x_m(t_2) = \int_{t_2}^{t_1} P_m(s, x_m(s))\,ds$

Apliquem normes: $\| x_m(t_1) - x_m(t_2) \| = \left\| \int_{t_2}^{t_1} P_m(s, x(s))\,ds \right\| \leq$

$\leq \left| \int_{t_2}^{t_1} \| P_m(s, x(s))\,ds \| \right| \leq M\,|t_1 - t_2| \implies \{x_m\}$ equicontínua.

Per tant, $\exists \{x_{m_k}\}$ parcial convergent.

Sigui $x = \lim_{k\to\infty} x_{m_k}$. $\forall \varepsilon > 0 \ \exists k_0 : \forall k \geq k_0, \ \sup_{t \in I_\alpha} \| x_{m_k}(t) - x(t) \| < \varepsilon$

(4) La sol. del PVI com a límit uniforme. És a dir:

$$\left.\begin{array}{l} x_{m_k}(t) = x_0 + \int_{t_0}^{t} P_{m_k}(s, x_{m_k}(s))\,ds \\ x = \lim_{k\to\infty} x_{m_k}(t) \end{array}\right\} \implies x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds$$

$\hookrightarrow P_{m_k} \longrightarrow f$

Prenent límits, és suficient veure que:

$P_{m_k}(s, x_{m_k}(s)) \longrightarrow f(s, x(s))$ unif:

$P_{m_k}(s, x_{m_k}(s)) - f(s, x(s)) = P_{m_k}(s, x_{m_k}(s)) - f(s, x_{m_k}(s)) + f(s, x_{m_k}(s)) - f(s, x(s))$

Prenem $\varepsilon > 0$. f cont $\implies \exists \delta : \| x_1 - x_2 \| < \delta \implies \| f(s, x_1) - f(s, x_2) \| < \varepsilon/2$

Com $x_{m_k} \longrightarrow x$, prenem k_0 tq $\| x_{m_k} - x \|_\infty < \delta$ si $k \geq k_0$.

Si $k \geq k_0$: $\| f(s, x_{m_k}(s)) - f(s, x(s)) \| \leq \varepsilon/2$

$P_{m_k} \longrightarrow f$. $\forall s \in I_\alpha$, $x \in B_b$:

$\| P_{m_k}(s, x) - f(s, x) \| < \varepsilon/2$, $\forall (s, x) \in \Omega$.

Per tant, $\exists k_1 : \forall k \geq k_1$, $\| P_{m_k}(s, x_{m_k}(s)) - f(s, x_{m_k}(s)) \| < \varepsilon/2$.

Si prenem $k_2 = \max\{k_0, k_1\}$:

$\| f(s, x(s)) - P_{m_k}(s, x_{m_k}(s)) \| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Solucions maximals

Volem determinar l' interval màxim de definició.

<u>Ex</u>: $-x' = x^2$

$$\frac{-dx}{x^2} = dt \longrightarrow -\int \frac{1}{x^2}\,dx = \int_{t_0}^{t} dt \longrightarrow \frac{1}{x} = t\Big]_{t_0}^{t} + C$$

$$\longrightarrow x(t) = \frac{1}{(t-t_0)+C}, \quad x_0 = x(t_0) = \frac{1}{C} \Longrightarrow C = \frac{1}{x_0}$$

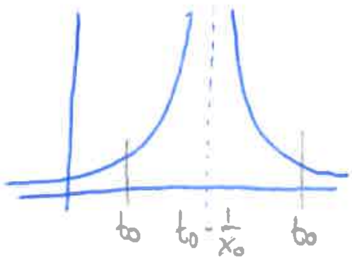
$$\Longrightarrow \boxed{x(t) = \frac{1}{(t-t_0)+\frac{1}{x_0}} = \frac{x_0}{1+x_0(t-t_0)}}$$

On estan definides les solucions?

$x_0 = 0 \Longrightarrow x(t) \equiv 0 \quad \forall t \in \mathbb{R}$

$x_0 \neq 0 \Longrightarrow 1 + x_0(t-t_0) = 0 \Longleftrightarrow t = -\frac{1}{x_0} + t_0$

Per tant, l' interval de definició ha de contenir t_0.



$x(t) = \dfrac{x_0}{1+x_0(t-t_0)}$

* <u>$x_0 < 0$</u> : $t_0 - \frac{1}{x_0} > t_0$

Per tant, $I(t_0, x_0) = \left(-\infty, t_0 - \frac{1}{x_0}\right)$

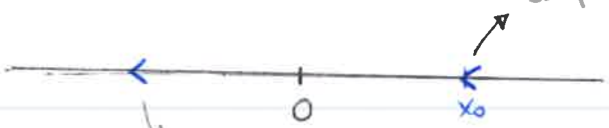
* <u>$x_0 > 0$</u> : $t_0 - \frac{1}{x_0} < t_0$

Per tant, $I(t_0, x_0) = \left(t_0 - \frac{1}{x_0}, +\infty\right)$

Interval de definició

Retrat de fase

es fa petit si $t \to \infty$, mai arriba a 0



es fa petit quan $t \to 0$ fins a $-\infty$

$x' = -x^2 \to$ decreixent

$x(t) = \dfrac{x_0}{1+x_0(t-t_0)}$

Prop: $f: \mathcal{U} \subset \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$. $\mathcal{U}$ obert, f contínua i $+$ q. fixats $(t_0, x_0) \in \mathcal{U}$, el PVI $x' = f(t,x)$, $x(t_0) = x_0$ té sol. única

Per tant, $\exists!$ funció $\varphi(\cdot\, ; t_0, x_0)$ definida a $I(t_0, x_0) =$
$= (\omega_-(t_0,x_0), \omega_+(t_0,x_0))$, on $t_0 \in I(t_0,x_0)$.

La funció φ satisfà:

i) És sol. de $x' = f(t,x)$, $x(t_0) = x_0$, $\mathcal{C}^1$

ii) Si $x: J \longrightarrow \mathcal{U}$ és sol. de $x' = f(t,x)$, $x(t_0) = x_0$ llavors, $\varphi|_J = x$.

iii) $\varphi(t; t_0, x_0) \in \mathcal{U}$

Dem:

Prenem $(t_0, x_0) \in \mathcal{U}$. Siguin $x_1: I_1 \longrightarrow \mathbb{R}^n$, $x_2: I_2 \longrightarrow \mathbb{R}^n$ solucions.

Definim $S = \{ x : I_x \longrightarrow \mathbb{R}^n,\ x \text{ sol. del PVI},\ I_x \text{ oberts} \}$

Agafem $I(t_0, x_0) = \bigcup_{x \in S} I_x$ interval obert. Considerem que I és interval maximal. Llavors $\varphi(t; t_0, x_0) = x(t)$ si $t \in I_x$

Vegem que $\varphi(t; t_0, x_0)$ està ben definida, és a dir:

$t \in I_{x_1} \cap I_{x_2}$, x_1, x_2 dues sol $\Longrightarrow x_1(t) = x_2(t)$

Sigui $C = \{ t \in I_{x_1} \cap I_{x_2},\ x_1(t) = x_2(t) \}$. Veurem que $C = I_{x_1} \cap I_{x_2}$, i d'aquesta manera, ja tindrem φ ben definida:

Sigui $I_{x_1} \cap I_{x_2}$ connex. Vegem que C és obert, tancat i no buit:

- $C = ((x_1 - x_2)^{-1}(0)$ tancat
- $C \neq \emptyset$ $(t_0 \in C)$
- C obert:

  $t_* \in C$ i $I_{x_1} \cap I_{x_2}$. $x_* = x_1(t_*) = x_2(t_*)$.

  Com que $\exists!$ sol. per (t_*, x_*) definida en $J \ni t_*$

  $\Longrightarrow x_1(t) = x_2(t)$, $t \in J \Rightarrow J \subset C. \Rightarrow$

$\Longrightarrow C = I_{x_1} \cap I_{x_2} \Rightarrow \varphi(t; t_0, x_0)$ està ben definida i és sol. de l'edo.

$\varphi(t; t_0, x_0) = x(t)$ si $t \in (-\alpha + t_*, \alpha + t_*)$.

Obs: Si $x(t)$ sol. a I_x i és $\mathcal{C}^1 \Rightarrow \varphi(t;t_0,x_0) \in \mathcal{C}^1$ a I_x

Resum: si estem en les condicions de Picard:

$f: U \longrightarrow \mathbb{R}^n$, llavors $\exists\, I(t_0,x_0)$ interval màxim de definició,

$(t, \varphi(t;t_0,x_0)) \in U \longrightarrow t \in I(t_0,x_0)$

$f: V \longrightarrow \mathbb{R}^n$, $V \subsetneq U$ $\qquad \#$

$(t, \varphi(t;t_0,x_0)) \in V \longrightarrow t \in \tilde{I}(t_0,x_0)$

Teorema: $x' = f(t,x)$, $f: U \subset \mathbb{R}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ contínua, $\exists!$ sol. del PVI. $(t_0,x_0) \in U$, $I(t_0,x_0)$ interval maximal i $\varphi(t;t_0,x_0)$ solució maximal.

Sigui K compacte amb $(t_0,x_0) \in K$.

Llavors, $\exists\, t \in I(t_0,x_0)$ tq $(t, \varphi(t;t_0,x_0)) \notin K$

Si $I(t_0,x_0) = (\omega_-(t_0,x_0), \omega_+(t_0,x_0))$, s'escriu:

$$\lim_{t\to\omega_\pm(t_0,x_0)} (t, \varphi(t;t_0,x_0)) \in \partial U \longrightarrow \text{Pot no } \exists \lim \text{ !!}$$

Ex: $x' = \frac{1}{t^2}\cos\frac{1}{t}$ $\qquad f: \underbrace{(0,+\infty)\times\mathbb{R}}_{U} \longrightarrow \mathbb{R}$

$x(t) = \sin\frac{1}{t}$, $I(t_0,x_0) = (0,+\infty)$

Però $\nexists \lim_{t\to 0} \sin\frac{1}{t}$!!

Corol·lari: $f: V \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ localment Lipschitz (Picard)

Considereu $x' = f(x)$. Llavors, sigui $(t_0,x_0) \in \mathbb{R}\times V$:

i) Si $\omega_+(t_0,x_0)$ finit. Llavors, $\forall K \subset V$, $\exists\, t \in I(t_0,x_0)$ tq $\varphi(t;t_0,x_0) \notin K$. Si $V = \mathbb{R}^n \Rightarrow \|\varphi(t;t_0,x_0)\| \xrightarrow{t\to\omega_+} +\infty$

ii) Si $\varphi(t;t_0,x_0) \in \tilde{K}$, $\tilde{K}$ cpt, $\forall t \in [t_0, \omega_+(t_0,x_0))$, llavors $\omega_+(t_0,x_0) = +\infty$

Prop: $f: U \to \mathbb{R}$ contínua. $\exists$ sol. maximals ($x' = f(t,x)$)

$\forall K \subset U \subset \mathbb{R}\times\mathbb{R}^n$, $\exists\, t_* \in I(t_0, x_0)$: $(t_*, \varphi(t_*; t_0, x_0)) \notin K$

Dem:

$(t_0, x_0) \in U$ i K cpt amb $(t_0, x_0) \in K$.

Sigui $d = \frac{1}{2}\,\mathrm{dist}(K, \partial U)$

Si $(t_1, x_1) \in K \Rightarrow \Omega(t_1, x_1) = \{|t-t_1| \le d\} \times \{\|x - x_1\| \le d\} \subset U$

$\tilde{K} = \bigcup_{(t_1,x_1)\in K} \Omega(t_1, x_1)$, que observem que és compacte.

Sigui $\tilde{M} = \sup_{(t,x)\in\tilde{K}} \|f(t,x)\|$

Per Peano, tenim sol. definida a $[-\alpha + t_0, \alpha + t_0]$, on $\alpha = \min\{a, b/M\}$

i $M = \sup_{(t,x)\in\Omega_{a,b}} \|f(t,x)\|$.

Com $a = b = d$: $M \le \tilde{M} \Longrightarrow \tilde{\alpha} = \min\{d, \frac{d}{\tilde{M}}\} \le \min\{d, \frac{d}{M}\}$

on $\tilde{\alpha}$ és el mateix per tots els $(t_1, x_1) \in K$

Llavors, $\exists\, \alpha > 0$ tq $\forall (t_1, x_1) \in K$, $\varphi(t; t_1, x_1)$ està definit a $[-\tilde{\alpha} + t_1, \tilde{\alpha} + t_1]$.

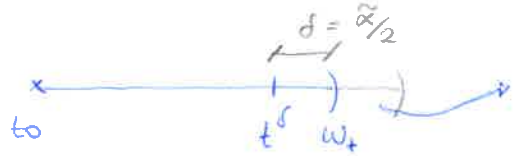
$I(t_0, x_0) = (\omega_-(t_0, x_0), \omega_+(t_0, x_0))$.

- Si $\omega_+(t_0, x_0) = +\infty$ ✓
- Si $\omega_+(t_0, x_0) < +\infty$:

  $\forall \delta > 0$ $\exists\, t^\delta$ tq $t^\delta \in (\omega_+ - \delta, \omega_+)$: $\varphi(t^\delta; t_0, x_0) \in K$.

  Sigui $\delta = \tilde{\alpha}/2$. $t^\delta \in (\omega_+ - \frac{\tilde{\alpha}}{2}, \omega_+)$

  Per tant, $x^\delta = \varphi(t^\delta, t_0, x_0) \in K$



ω_+ no maximal (!!)

$\Longrightarrow \exists\, t \notin (\omega_-, \omega_+)$

$K \ni (t^\delta, x^\delta) \xrightarrow{\tilde{\alpha}}$

La podem allargar $\tilde{\alpha}$ unitats

Considerem $\varphi(t) = \begin{cases} \varphi(t; t_0, x_0) & t \in (\omega_-, \omega_+) \\ \varphi(t; t^d, x^d) & t \in [t^d, t^d + \tilde{\alpha}] \end{cases}$

si $t \in (\omega_-, \omega_+) \cap [t^d, t^d + \tilde{\alpha}]$, φ està ben definit per ! de solucions.

$t^d + \tilde{\alpha} > \omega_+$ per def $\Longrightarrow$ ω_+ no maximal !!

Lema (Gronwall)

$u, v \geq 0$ contínues, $u, v : [a, b) \longrightarrow [0, \infty)$.

Suposem que $u(t) \leq c + \int_a^t v(s) u(s) ds$, $t \in [a, b)$, $c \geq 0$

Llavors, $u(t) \leq c \cdot \exp\left(\int_a^t v(s) ds\right)$

Obs: si $c = 0$ $\left(u(t) \leq 0 + \int_a^t v(s) u(s) ds\right) \Longrightarrow u(t) \equiv 0$

Dem: $\omega(t) = c + \int_a^t v(s) u(s) ds$, $\omega'(t) = v(t) u(t) \leq v(t) \omega(t)$

Si $c > 0$, $\omega > 0$: $\frac{\omega'(t)}{\omega(t)} \leq v(t) \Longrightarrow \int_a^t \frac{\omega'(s)}{\omega(s)} ds \leq \int_a^t v(s) ds$

$\Longrightarrow \log\left(\frac{|\omega(t)|}{\omega(a)}\right) \leq \int_a^t v(s) ds \Longrightarrow \boxed{\omega(t) \leq \underset{\overset{\parallel}{c}}{\omega(a)} \cdot e^{\int_a^t v(s) ds}}$

Corol·lari: $u, v : (a, b) \longrightarrow [0, +\infty)$, $a, b \in \mathbb{R} \cup \{\infty\}$, $t_0 \in (a, b)$

Si $\exists\ c \geq 0$ i $u(t) \leq c + \left|\int_{t_0}^t u(s) v(s) ds\right|$. Llavors,

$u(t) \leq c\, e^{\left|\int_{t_0}^t v(s) ds\right|}$

Continuïtat del flux respecte (t, t_0, x_0)

Teorema: $U \subset \mathbb{R} \times \mathbb{R}^n$ obert, $f : U \longrightarrow \mathbb{R}^n$ lipschitz respecte la 2a variable.

$\|f(t, x) - f(t, \bar{x})\| \leq L\|x - \bar{x}\|$.

Considerem l'edo $x' = f(t, x)$, $(t_0, x_0) \in U$.

Sigui $J = [a, b] \subset I(t_0, x_0)$ tq $t_0 \in J$, i I és interval maximal.

Fixem $K \subset \mathbb{R}^n$ cpt qualsevol tq $\{\varphi(t; t_0, x_0), t \in J\} \subset \mathring{K}$

Llavors, $\exists\ W_{(t_0, x_0)}$ entorn tq $\forall (t_1, x_1) \in W$, $\varphi(t; t_1, x_1)$ està definit $\forall t \in J$ i $\varphi(t; t_1, x_1) \in K$. A més, si $\sup \|f(t, x)\| =: M$:

$\|\varphi(t; t_0, x_0) - \varphi(\bar{t}; t_1, x_1)\| \leq M|t - \bar{t}| + M|t_1 - t_0| e^{L(t - t_0)} + \|x_1 - x_0\| e^{L|t - t_0|}$

CoroL·lari: En les hipòtesis anteriors:

i) $D = \{(t; t_0, x_0) \in I(t; t_0, x_0) \times \mathcal{U}\}$ és un obert

ii) φ és contínua a D

Obs: Si apliquem el teorema a una equació amb paràmetres:

$x' = g(t, x, \lambda)$, $\lambda \in \Lambda \subset \mathbb{R}^k$. Es pot considerar

$$\begin{cases} x' = g(t, x, \lambda) \\ \lambda' = 0 \end{cases} \longrightarrow \varphi(t; t_0, x_0, \lambda).$$ Pel teorema, es

té continuïtat respecte λ també.

Dem (Teorema):

Lema: Hipòtesis del teorema. $(t_0, x_0) \in \mathcal{U}$, $(t_0, x_1) \in \mathcal{U}$

Llavors, $t \in I(t_0, x_0) \cap I(t_0, x_1)$ i

$$\| \varphi(t; t_0, x_0) - \varphi(t; t_0, x_1) \| \leq \| x_0 - x_1 \| e^{L|t-t_0|}$$

Obs: $I(t_0, x_0) \cap I(t_0, x_1) \neq \emptyset$, però no sabem si

$\bigcap I(t_0, x_i) \neq \{t_0\}$

Dem (Lema):

Sigui $\varphi_0(t) = \varphi(t; t_0, x_0)$, $\varphi_1(t) = \varphi(t; t_0, x_1)$

$$\varphi_0(t) - \varphi_1(t) = \left[x_0 + \int_{t_0}^{t} f(s, \varphi_0(s))\, ds \right] - \left[x_1 + \int_{t_0}^{t} f(s, \varphi_1(s))\, ds \right]$$

$$\| \varphi_0(t) - \varphi_1(t) \| \leq \| x_0 - x_1 \| + \left\| \int_{t_0}^{t} [f(s, \varphi_0(s)) - f(s, \varphi_1(s))] ds \right\| \leq$$

$$\leq \| x_1 - x_0 \| + \left| \int_{t_0}^{t} \| f(s, \varphi_0(s)) - f(s, \varphi_1(s)) \| ds \right| \leq$$

$$\leq \| x_1 - x_0 \| + L \left| \int_{t_0}^{t} \| \varphi_0(s) - \varphi_1(s) \| ds \right|$$

Pel Lema de Gronwall:

$C = \| x_1 - x_0 \|$, $u(s) = \| \varphi_0(s) - \varphi_1(s) \|$, $v(s) = L$

$$\Rightarrow \| \varphi_0(s) - \varphi_1(s) \| \leq \| x_1 - x_0 \| e^{L|t-t_0|} \quad \square$$

Lema: En la notació i hip. anteriors:

$(t_0, x_0) \in U$, $\quad J = [a,b] \subset I(t_0, x_0)$

$M_J = \sup_{s \in J} \| f(s, \varphi(s, t_0, x_0)) \|$. Llavors:

i) $t, \bar{t} \in J$: $\| \varphi(t; t_0, x_0) - \varphi(\bar{t}, t_0, x_0) \| \leq M_J |t - \bar{t}|$

ii) $t_1 \in J$, $\quad t \in I(t_0, x_0) \cap I(t_1, x_0) \ni t_1$

$$\| \varphi(t; t_1, x_0) - \varphi(t; t_0, x_0) \| \leq M_J |t_1 - t_0| \, e^{L|t - t_1|}$$

Dem:

i) $\varphi(t; t_0, x_0) - \varphi(\bar{t}, t_0, x_0) = \left[x_0 + \int_{t_0}^{t} f(s, \varphi(s; t_0, x_0)) \, ds \right] -$

$- \left[x_0 + \int_{t_0}^{\bar{t}} f(s; \varphi(s; t_0, x_0)) \, ds \right] = \int_{\bar{t}}^{t} f(s, \varphi(s, t_0, x_0)) \, ds$

Així:

$\| \varphi(t; t_0, x_0) - \varphi(\bar{t}; t_0, x_0) \| \leq \left| \int_{\bar{t}}^{t} \overbrace{\| f(s, \varphi(s, t_0, x_0)) \|}^{\leq M_J} ds \right|$

$\leq M_J |t - t_0|$

ii) Observem que $\underbrace{\varphi(t; t_0, \varphi(t_0, t_1, x_0))}_{\varphi_1(t)} = \underbrace{\varphi(t; t_1, x_0)}_{\varphi_2(t)}$.

En efecte:

$\varphi_1(t_0) = \varphi(t_0; t_1, x_0)$

$\varphi_2(t_0) = \varphi(t_0; t_1, x_0)$ $\quad \|$ $\quad$ (per ! de solucions)

Fem servir $\varphi(t; t_0, x_0) = \varphi(t; t_1, \underbrace{\varphi(t_1; t_0, x_0)}_{x_1}) =$

$= \varphi(t; t_1, x_1)$

$t \in J \cap I(t_1, x_0)$

$\| \varphi(t; t_0, x_0) - \overbrace{\varphi(t; t_1, x_0)}^{x_1} \| = \| \varphi(t; t_1, \varphi(t_1; t_0, x_0)) - \varphi(t; t_1, x_0) \|$

$= \| \varphi(t; t_1, x_1) - \varphi(t; t_1, x_0) \| \leq \| x_1 - x_0 \| \, e^{L|t - t_1|} =$

$= \| \varphi(t_1; t_0, x_0) - \varphi(t_0, t_0, x_0) \| \, e^{L|t - t_1|} \leq$

$\leq M_J |t_1 - t_0| \, e^{L|t - t_1|}$

Lema: $f: \mathcal{U} \longrightarrow \mathbb{R}^n$ contínua, Lipschitz resp. x, $(t_0, u_0) \in \mathcal{U}$ i $I(t_0, x_0)$ interval maximal de def.

Llavors, $\forall J = [a,b] \subset I(t_0, x_0)$ $\exists V_{(t_0,x_0)}$ entorn de (t_0, x_0) : $\forall (t_1, x_1) \in V_{(t_0,x_0)}$,

$\varphi(t; t_1, x_1)$ està def. $\forall t \in J$. A més, $\varphi(t; t_1, x_1) \in K$ cpt.

Dem:

De moment només canviem x_0:

Fixat $(t_0, x_0) \in \mathcal{U}$, $\exists \delta > 0$: $\|x_0 - x_1\| < \delta \implies$

$\implies \varphi(t; t_0, x_1)$ està def. a $t \in J$.

Definim $d = \frac{1}{2} \sup_{t \in J} \operatorname{dist}\Big((t, \varphi(t; t_0, x_0)), \partial \mathcal{U}\Big)$.

Sigui $\tilde{K} = [a,b] \times K := [a,b] \times \bigcup_{t \in J} \{\|\varphi(t; t_0, x_0) - x\| \leq d\}$ cpt. Sigui x_1 qualsevol.

Si $t \in I(t_0, x_1) \cap I(t_0, x_0)$: $\|\varphi(t; t_0, x_0) - \varphi(t; t_0, x_1)\| \leq$

$\leq \|x_0 - x_1\| e^{L|t-t_0|}$

Ens interessa el cas que $t = I(t_0, x_1) \cap J$:

$\|\varphi(t; t_0, x_0) - \varphi(t; t_0, x_1)\| \leq \|x_1 - x_0\| e^{L|t-t_0|} \leq$

$\leq \|x_1 - x_0\| e^{L(b-a)}$

Agafem $\delta e^{L(b-a)} \leq d$. Llavors, $\|\varphi(t; t_0, x_0) - \varphi(t; t_0, x_1)\| \leq d$.

$\implies (t, \varphi(t; t_0, x_1)) \in \tilde{K}$

D'altra banda, sabem que $\exists t_* \in I(t_0, x_1)$ i $(t_*, \varphi(t_*, t_0, x_1)) \notin \tilde{K}$

Volem veure que $J \subset I(t_0, x_1)$. Vegem-ho per contradicció:

Sigui $b > \omega_+$: Llavors $a < t_0 < t_* < \omega_+ < b \implies t_* \in J$

$\implies (t_*, \varphi(t_*; t_0, x_1)) \in \tilde{K}$ (!!)

Anàlogament es demostra amb $a < \omega_-$.

Amb aquests lemes queda demostrat el teorema.

Corol·lari: En les mateixes condicions,

i) D obert

ii) $\varphi : D \longrightarrow \mathbb{R}^n$ contínua

Dem: Sigui $(t; t_0, x_0) \in D$.

$J \subset I(t_0, x_0)$, $t, t_0 \in \mathring{J} = \overset{\circ}{\overline{[a,b]}}$

Llavors, $W_{(t_0,x_0)} \ni (t_0, x_0)$: $(t; t, x_1) \in \mathring{J} \times W_{(t_0,x_0)} \subset D$

Def: $f : U \subset \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ localment Lipschitz si $\forall (t,x) \in U$ $\exists V_x$ i L_x tq. $\|f(t,x) - f(t,y)\| \leq L_x \|x-y\|$ $\forall y \in V_x$.

Lema: $f : U \longrightarrow \mathbb{R}^n$ loc. Lipschitz $\iff$ $f|_K$ és Lipschitz $\forall K \subset U$ cpt

Corol·lari: $f : U \longrightarrow \mathbb{R}^n$ loc. Lipschitz. Llavors:

i) D obert

ii) $\varphi : D \longrightarrow \mathbb{R}^n$ contínua.

Dem: Quan f és loc. Lipschitz també tenim $\exists!$ solucions.

En efecte, podem restringir-nos al domini

$\Omega = \{ |t - t_0| \leq a \} \times \{ \|x - x_0\| \leq b \}$

on f és Lipschitz $\Longrightarrow$ $\exists!$ solucions.

Diferenciabilitat

$f : U \times \Lambda \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k \longrightarrow \mathbb{R}^n$ $\mathcal{C}^r$, $r \geq 1$

$\varphi : D \longrightarrow \mathbb{R}^n$

Sigui $D = \{ (t; t_0, x_0, \lambda) ; (t_0, x_0, \lambda) \in U \times \Lambda, t \in I(t_0, x_0, \lambda) \}$

Teorema: $\varphi : D \longrightarrow \mathbb{R}^n$ és $\mathcal{C}^r$

Dem: Molt llarga, però ens ho creiem

suposarem a partir d' ara que $r \geq 2$.

Recordem:

$$\frac{\partial}{\partial t} \varphi(t; t_0, x_0, \lambda) = f(t_0, \varphi(t; t_0, x_0, \lambda), \lambda)$$

A més: $\varphi(t_0, t_0, x_0, \lambda) = x_0$:

- $\partial_\lambda \varphi(t_0, t_0, x_0, \lambda) = 0 \quad \forall \lambda, t_0, x_0$
- $\partial_{x_0} \varphi(t_0, t_0, x_0, \lambda) = Id \quad \forall \lambda, t_0, x_0$
- $\partial_t \varphi(t_0, t_0, x_0, \lambda) + \partial_{t_0} \varphi(t_0, t_0, x_0, \lambda) = 0$

podem suposar $\varphi(t_0, t_0, x_0, \lambda) =: \psi(t_0, x_0)$

Càlcul de $D\varphi(t; t_0, x_0, \lambda)$

Notació: $D_i \varphi(t; t_0, x_0, \lambda)$, derivada resp. la i-èsima variable, $i = 1 \div 4$

Aquestes derivades les anomenem variacionals

Prop: $x' = f(t, x, \lambda)$. f és $\mathcal{C}^r$ i suposem $\varphi(t; t_0, x_0, \lambda)$ $\mathcal{C}^r$. A més $r \geq 2$. Llavors:

i) $z(t) = D_3 \varphi(t; t_0, x_0, \lambda) \Longrightarrow z'(t) = z(t) \left[D_2 f(t, \varphi(t; t_0, x_0, \lambda), \lambda) \right]$; $z(t_0) = Id$

ii) $z(t) = D_2 \varphi(t; t_0, x_0, \lambda) \Longrightarrow z'(t) = z(t) \left[D_2 f(t, \varphi(t; t_0, x_0, \lambda), \lambda) \right]$, $z(t_0) = -f(t_0, x_0, \lambda)$

iii) $z(t) = D_4 \varphi(t; t_0, x_0, \lambda) \Longrightarrow z'(t) = D_2 f(t, \varphi(t; t_0, x_0, \lambda), \lambda)\, z(t) + D_3 f(t, \varphi(t; t_0, x_0, \lambda), \lambda)$ $\quad z(t_0) = 0$

Ex: $x' = \lambda^2 x - x^2 + t \sin \lambda x$

i) $D_3 \varphi(t, t_0, x_0, \lambda)$, $z'(t) = [\lambda^2 - 2\varphi + t\lambda \cos \lambda \varphi]\, z(t)$ i $x(0) = 0$

$\Longrightarrow \varphi = 0$

$z'(t) = (\lambda^2 + t\lambda)\, z(t)$, $z(0) = 1$

Comportament dominant

$$\varphi = \varphi(t, t_0, 0, \lambda) + D_3 \varphi(t, t_0, 0, \lambda) x_0 + O(\|x_0\|^2) = \underbrace{D_3 \varphi(t, t_0, 0, \lambda) x_0}_{} + O(\|x_0\|^2)$$

Dem: $\circledast$ $\frac{d}{dt} \varphi(t; t_0, x_0, \lambda) = f(t, \varphi(t; t_0, x_0, \lambda), \lambda)$, $\varphi \in \mathcal{C}^2$

(i) Derivem ⊛ respecte x_0 a les dues bandes i intercanviem D_3 i $\frac{d}{dt} = D_1$

$$\frac{d}{dt} \underbrace{D_3 \varphi(t; t_0, x_0, \lambda)}_{z(t)} = D_2 f(t, \varphi(t, t_0, x_0, \lambda), \lambda) \cdot \underbrace{D_3 \varphi(t, t_0, x_0, \lambda)}_{z(t)}$$

c.i.: $x_0 = \varphi(t_0, t_0, x_0, \lambda)$ $\forall (t_0, x_0, \lambda)$, $I = D_3 \varphi(t_0, t_0, x_0, \lambda) = z(t_0)$

(ii) Derivem ⊛ respecte t_0:

$$\frac{d}{dt} \underbrace{D_2 \varphi(t; t_0, x_0, \lambda)}_{z(t)} = D_2 f(t, \varphi(t, t_0, x_0, \lambda), \lambda) \underbrace{D_2 \varphi(t, t_0, x_0, \lambda)}_{z(t)}$$

$$x_0 = \varphi(t_0, t_0, x_0, \lambda) \xrightarrow{\partial_{t_0}} 0 = D_1 \varphi(t_0, t_0, x_0, \lambda) + D_2 \varphi(t_0, t_0, x_0, \lambda)$$

$$\Longrightarrow D_2 \varphi(t_0, t_0, x_0, \lambda) = -D_1 \varphi(t_0, t_0, x_0, \lambda)$$

Per ⊛ tenim que $D_1 \varphi(t, t_0, x_0, \lambda) = f(t, \varphi(t, t_0, x_0, \lambda), \lambda)$ $\longmapsto t = t_0$

$$D_2 \varphi(t_0, t_0, x_0, \lambda) = -f(t_0, \varphi(t, t_0, x_0, \lambda), \lambda) = -f(t_0, x_0, \lambda)$$

Obs: $D_2 \varphi(t; t_0, x_0, \lambda) = -D_3 \varphi(t; t_0, x_0, \lambda) f(t_0, x_0, \lambda)$

(iii) Derivem ⊛ Respecte λ:

$$\frac{d}{dt} D_4 \varphi(t; t_0, x_0, \lambda) = D_2 f(t, \varphi, \lambda) D_4 \varphi + D_3 f(t, \varphi, \lambda)$$

Lineal, no homogeni. La primera part homogènia és comuna a les de $D_2 \varphi$ i $D_3 \varphi$.

Tenim $\varphi(t_0, t_0, x_0, \lambda) = x_0 \xrightarrow{\partial_\lambda} D_4 \varphi(t_0, t_0, x_0, \lambda) = 0$

Rec: Suposem que $x' = f(t, x, \lambda)$ i $\varphi(t, t_0, x_0, \lambda)$ és $\mathcal{C}^r$, $r \geq 2$.
Volem estudiar les equacions que satisfan $D_{1,2,3,4}\, \varphi(t, t_0, x_0, \lambda)$

Aplicacions:

(1) Variacionals al voltant d'un punt fix.

Suposem $x' = f(x)$ i $f(p) = 0$. Llavors, $\varphi(t; t_0, p, \lambda) = p$

Obs: Com que estem en el cas autònom : $\varphi(t; t_0, x_0) = \varphi(t - t_0, 0, x_0)$

Per tant treiem el t_0 de la notació. Així, el flux es $\varphi(t; x_0)$

Taylor $x_0 \sim p$, de $\varphi(t; x_0)$

$$\varphi(t; x_0) = \overbrace{\varphi(t; p)}^{p} + \underbrace{D_2 \varphi(t; p)}_{??}(x_0 - p) + \frac{1}{2} D_2^2 \varphi(t; p)(x_0 - p)^2 + $$
$$+ \mathcal{O}(\|x_0 - p\|^3)$$

Vegem què és $D_2 \varphi(t, p)$:

$\varphi(t, x_0)$ $\quad p + D_2 \varphi(t, p)(x_0 - p)$. Calculem $D_2 \varphi(t, p)$:

Recordem: $\frac{d}{dt} \varphi(t; x_0) = f(\varphi(t; x_0)) \implies$

$$\implies \boxed{\frac{d}{dt} D_2 \varphi(t, p) = Df(\overbrace{\varphi(t; p)}^{p}) \cdot D_2 \varphi(t, p)}$$

Lineal homogènia a coeficients constants

D'aquí: $D_2 \varphi(t; p) = e^{Df(p)\, t}$. Com $t_0 = 0$:

$D_2 \varphi(0, p) = Id$, $\quad \varphi(0, x_0) = x_0$

Ex: $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x + x^2 + x^3 \\ 2y + xy \end{pmatrix}$

Obs que $p = (0,0)$ és punt fix. Calculem $D_2 \varphi(t, p)$

$$\frac{d}{dt}\left(D_2 \varphi(t, x_0)\right) = \begin{pmatrix} -1 + 2\varphi_1(t, x_0) + 3\varphi_1^2(t, x_0) & 0 \\ \varphi_2(t, x_0) & 2 + \varphi_1(t, x_0) \end{pmatrix} \cdot D_2 \varphi_{(t, x_0)}$$

prenem $x_0 = p = (0,0)$: $\quad \varphi(t,p) = (0,0) \implies$

$$\implies \frac{d}{dt} D_2\varphi(t,p) = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \cdot D_2\varphi(t,p) \implies D_2\varphi(t,p) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix}$$

$$D_2^2 \varphi(t,x_0)\Big|_{x_0=p} \quad ??$$

Denotem per $\partial_i D_2\varphi(t,x_0)$ per la derivada parcial respecte x_0^i

$x_0 = (x_0^1, \dots, x_0^n)$. Llavors:

o bé ∂_j

$$\frac{d}{dt}\big(\partial_i D_2\varphi(t,x_0)\big) = D^2 f(\varphi(t,x_0))\big(\partial_i\varphi(t,x_0), D_2\varphi(t,x_0)\big) +$$

o bé ∂_j

$$+ Df(\varphi(t,x_0)) \cdot \partial_i D_2\varphi(t,x_0)$$

$\hookrightarrow$ o bé ∂_j

Si $x_0 = p$:

$$\boxed{\frac{d}{dt}\big(\partial_i D_2\varphi(t,p)\big) = D^2 f(p)\big[\partial_i\varphi(t,p), D_2\varphi(t,p)\big] + Df(p)\, \partial_i D_2\varphi(t,p)}$$

En el nostre exemple:

$$\begin{cases} x' = -x + x^2 + x^3 \\ y' = 2y + yx \end{cases} \implies \begin{cases} \dot\varphi_1 = -\varphi_1 + \varphi_1^2 + \varphi_1^3 \\ \dot\varphi_2 = 2\varphi_2 + \varphi_1\varphi_2 \end{cases}$$

$$\begin{cases} \frac{d}{dt}(\partial_1\varphi_1) = -\partial_1\varphi_1 + 2\varphi_1\partial_1\varphi_1 + 3\varphi_1^2\partial_1\varphi_1 \\ \frac{d}{dt}(\partial_1\varphi_2) = 2\partial_1\varphi_2 + \partial_1\varphi_1\cdot\varphi_2 + \varphi_1\cdot\partial_1\varphi_2 \end{cases}$$

Agafem c.i. $p(0,0)$:

$$\frac{d}{dt}\big(\partial_1\varphi_1(t,p)\big) = \partial_1\varphi_1(t,p)$$

$$\frac{d}{dt}\big(\partial_1\varphi_2(t,p)\big) = 2\partial_1\varphi_2(t,p)$$

$\partial_1 \partial_1 \varphi(t,p)$?

$$\begin{cases} \frac{d}{dt}\left(\partial_1^2 \varphi_1(t,p)\right) = -\partial_1^2 \varphi_1(t,p) + 2\left[\partial_1 \varphi_1(t,p)\right]^2 \\ \frac{d}{dt}\left(\partial_1^2 \varphi_2(t,p)\right) = 2\,\partial_1^2 \varphi_2(t,p) + 2\left[\partial_1 \varphi_1(t,p)\cdot \partial_1 \varphi_2(t,p)\right] \end{cases}$$

Cond. inicial ? $\partial_1^2 \varphi(0,p)$? $= 0$, ja que $D\varphi(0,x_0) = Id$

Aplicacions variacionals

* **Variacions al voltant d'un punt fix**

$\dot{x} = f(x)$, $f(p) = 0$ i $\varphi(t;p) = p$

$\varphi(t;x_0) = \varphi(t,p) + D_2(\varphi(t,p))(x_0 - p) + \cdots$, $x_0 \sim p$

* **Variacionals respecte paràmetres**

$\dot{x} = f(x,\varepsilon)$, $\varepsilon \in \mathbb{R}^k$

Suposem coneguda una sol. per $\varepsilon = 0$ i $x^0(t)$.

Sigui $x_0 = x^0(0)$ una cond. inicial

$$\varphi(t,x_0,\varepsilon) = \varphi(t,x_0,0) + D_3\varphi(t;x_0,0)\cdot\varepsilon + O(\varepsilon^2), \quad \varepsilon \sim 0$$

$\frac{d}{dt}\varphi(t,x_0,\varepsilon) = f(\varphi(t,x_0,\varepsilon),\varepsilon)$. Eq. $D_3\varphi(t,x_0,0)$?

$$\frac{d}{dt} D_3\varphi(t;x_0,0) = D_1 f(\varphi(t,x_0,\varepsilon),\varepsilon)\, D_3\varphi(t,x_0,\varepsilon) + D_2 f(\varphi(t,x_0,\varepsilon),\varepsilon)$$

$\xrightarrow[\varepsilon=0]{}$ $\Longrightarrow$ $= D_1 f(x^0(t),0)\, D_3\varphi(t,x_0,0) + D_2 f(x^0(t),0)$

$\longrightarrow$ EDO lineal no homogènia.

$\underline{Ex}$: $x' = f(t,x,\lambda) := ax(1-x) + \lambda(1-\sin t)$, $a>0$

- $\lambda = 0$: Aplicació logística, mesura poblacions aïllades:

  $x' = ax(1-x)$

- λ petit. $\lambda = 0 \implies x' = ax(1-x)$, $\left.\begin{array}{l} x\equiv 0 \\ x \equiv 1 \end{array}\right\}$ solucions fàcils

Volem estudiar les solucions prop d'aquestes:

$\varphi(t, x_0, \lambda)$ quan $x_0 \sim 0$, $x_0 \sim 1$ i $\lambda \sim 0$

$$\varphi(t,x_0,\lambda) = \varphi(t,0,0) + D_2\varphi(t,0,0)(x_0 - 0) + D_3\varphi(t,0,0)\lambda + O(\ )$$

$\varphi(t,0,0) \equiv 0$, $\underline{D_2\varphi(t,0,0)}$?

$\underline{Obs}$: Podem avaluar primer a $\lambda = 0$ i després derivar resp. x_0.

$$\frac{d}{dt} D_2\varphi(t,0,0) = [a - 2a\varphi(t,0,0)]D_2\varphi(t,0,0)$$
$$= a D_2\varphi(t,0,0)$$

$\implies D_2\varphi(t,0,0) = e^{at}$, ja que $D_2\varphi(0,0,0) = Id\ (=1)$

$\underline{D_3\varphi(t,0,0)}$?

$$\frac{d}{dt} D_3\varphi(t,0,0) = a D_3\varphi(t,0,0) - (1-\sin t)$$

Eq lineal no homogènia.

Recordem: $D_3\varphi(0,0,0) = 0 \implies$

$$\implies D_3\varphi(t,0,0) = e^{at}\left[\cancel{0} + \int_0^t e^{as}(\sin s - 1)\,ds\right]$$
$$= e^{at}\int_0^t e^{-as}(\sin s - 1)\,ds$$

$$\varphi(t,x_0,\lambda) = e^{at}x_0 + \left[e^{at}\int_0^t e^{-as}(\sin s - 1)\,ds\right]\lambda + O(\|(x_0,\lambda)\|^2)$$

⊛ Òrbites periòdiques a partir de punts fixos

Context: $x' = f(x) + \varepsilon g(x,t,\varepsilon)$, $x \in \mathbb{R}^n$, $\varepsilon \in \mathbb{R}$

Totes les funcions $\mathcal{C}^r$, $r \geq 1$ i $g(x, t+T, \varepsilon) = g(x,t,\varepsilon)$

Suposem $f(p) = 0$ punt fix. i.e: $\varphi(t,p,0) = p$

A més, suposem que 0 no és vap de $Df(p)$.

Llavors, si ε prou petit, $\exists$ sol. periòdica $\gamma_\varepsilon(t)$, $\mathcal{C}^r$,

$\gamma_\varepsilon(t+T) = \gamma_\varepsilon(t)$ tq $\|\gamma_\varepsilon(t) - p\| \leq C \cdot |\varepsilon|$

Dem: Recordem $x(t)$ solució. Llavors; $x(t+T)$ és sol.

En efecte, denominem $F(x,t,\varepsilon) = f(x) + \varepsilon g(t,x,\varepsilon)$
$y(t) = x(t+T)$

$y'(t) = F(x(t+T), t+T, \varepsilon) = F(y(t), t, \varepsilon)$

Llavors, $x(t)$ és sol. periòdica $\Longleftrightarrow$ $x(T) = x(0)$

En efecte:

$\Rightarrow$ | ✓

$\Leftarrow$ | x_0 : $\varphi(T, 0, x_0) = x_0$

Considerem $\left.\begin{array}{l} x(t) = \varphi(t, 0, x_0) \\ x(t+T) = y(t) = \varphi(t+T, 0, x_0) \end{array}\right\}$ dues solucions

i es té que $x(0) = x(T) = y(0)$ $\Rightarrow x = y$
$\Rightarrow x(t) = x(t+T)$

Aleshores es té $\gamma(t+T) = \gamma(t) \Longleftrightarrow \gamma(0) = \gamma(T)$

Per tant l'únic que cal és trobar un punt $q(\varepsilon)$ tq:

$$\boxed{\varphi(T; 0, q(\varepsilon), \varepsilon) = q(\varepsilon)}$$

Defini $F(q,\varepsilon) = \varphi(T; 0, q, \varepsilon) - q$. Volem aplicar TFI:

- $F \in \mathcal{C}^r$, $r \geq 1$
- $F(p, 0) = 0$
- $D_q F(q,\varepsilon)\big|_{\substack{q=p \\ \varepsilon=0}} = D_3\varphi(T, 0, q, \varepsilon) - Id = D_3\varphi(T, 0, p, 0) - Id$ ↪

$\dot{x} = f(x) + \varepsilon g(x,t,\varepsilon)$

Calculem $\frac{d}{dt} D_3\varphi(t,0,q,\varepsilon)\Big|_{\substack{q=p\\ \varepsilon=0}} = Df(\varphi(t,0,q,\varepsilon))\, D_3\varphi(t,0,q,\varepsilon) + \varepsilon[\quad\quad]$

Avaluant $q=p$ i $\varepsilon=0$:

$$\left.\begin{array}{l} \frac{d}{dt} D_3\varphi(t,0,p,0) = Df(p)\, D_3\varphi(t,0,p,0) \\ \quad\quad \text{Edo a coefs constants} \\ \text{Recordem } D_3\varphi(0,0,p,0) = Id \end{array}\right\} \Rightarrow \boxed{D_3\varphi(t,0,p,0) = e^{tDf(p)}}$$

Volem que $D_3\varphi(T,0,p,0) - Id$ fos invertible.

Això es complirà sii $D_3\varphi(T,0,p,0)$ no té vap 1

En efecte, ja que $D_3\varphi(T,0,p,0) = e^{tD(f)}$

Llavors, $D_3\varphi(T,0,p,0)$ no té vap 1 $\Longleftrightarrow$ $Df(p)$ no té vap 0 (Hipòtesi ✓)

Aleshores, aplicant TFI: $F(q,\varepsilon) = 0 \Longleftrightarrow q = q(\varepsilon)$

$\Longrightarrow q(0) = p + O(\varepsilon) \Longrightarrow \gamma_\varepsilon(t) = \varphi(t,0,q(\varepsilon), \;)$

Teoria de pertorbacions

$\dot{x} = f_0(x)t) + \varepsilon f_1(x,t) + \varepsilon^2 f_2(x,t) + O(\varepsilon^3)$

$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + O(\varepsilon^3)$

amb c.i. $x^0 = x(t_0)$. Llavors:

$x_0(t_0) = x^0, \quad x_1(t_0) = x_2(t_0) = \dots = 0$

Trobem les equacions per $x_0, \dots$ igualant ordres a l'equació

$x_0' + \varepsilon x_1' + \varepsilon_2 x_2' + \dots = f_0(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots, t) + \varepsilon f_1(x_0 + \varepsilon x_1 + \dots) + \dots$

Dem diferenciabilitat a Atenea

POT SORTIR AIXÒ A L'EXAMEN :)