Curs 3 06.03.2024

Probabilități Limită

Definiție:

Fie tripletul $(\Omega, \mathcal{F}, \mathbb{P}) \overset{not}{=}$ spațiu de probabilitate.

Șirul $(A_n)_{n\geq 1}$ de evenimente $(A_n \in \mathcal{F}, (\forall) n \in \mathbb{N}^*)$ se numește:

1) Monoton crescător dacă $A_n \subset A_{n+1}, (\forall) n \in \mathbb{N}^*$

$$\lim_{n\to\infty} A_n \overset{def}{=} \bigcup_{n=1}^{\infty} A_n \overset{not}{=} A \in \mathcal{F}$$

$A_n \uparrow A \overset{not}{=}$ monoton crescător

2) Monoton descrescător dacă $A_n \supset A_{n+1}, (\forall) n \in \mathbb{N}^*$

$$\lim_{n\to\infty} A_n \overset{def.}{=} \bigcap_{n=1}^{\infty} A_n \overset{not}{=} A \in \mathcal{F}.$$

$A_n \downarrow A \overset{not}{=}$ monoton descrescător

3) a) $$\limsup_{n\to\infty} A_n \overset{def}{=} \bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k=n}^{\infty} A_k}_{B_n} = \lim_{n\to\infty} B_n$$

$$\underset{A_n \cup B_{n+1}}{\underset{\parallel}{B_n}} \supset B_{n+1}, (\forall) n \in \mathbb{N}^* \Rightarrow B_n \text{, monoton descrescător} \Rightarrow B_n \downarrow$$

b) $$\liminf_{n\to\infty} A_n \overset{def}{=} \bigcup_{n=1}^{\infty} \underbrace{\bigcap_{k=n}^{\infty} A_k}_{B_n} = \lim_{n\to\infty} B_n$$

$$\underset{A_n \cap B_{n+1}}{\underset{\parallel}{B_n}} \subset B_{n+1}, (\forall) n \in \mathbb{N}^* \Rightarrow B_n \text{, monoton crescător} \Rightarrow B_n \downarrow$$

TEOREMĂ:

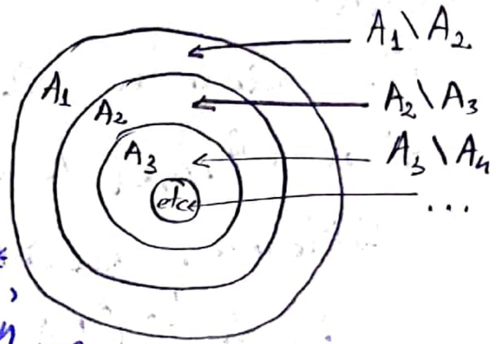
Tripletul $(\Omega, \mathcal{F}, \mathbb{P})$ se numește spațiu de probabilitate:

$\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(\lim_{n\to\infty} A_n), (\forall) (A_n)_{n\geq 1}$; șir monoton de evenimente

Demonstrație:

Fie $(A_n)_{n\geq 1}$, un șir monoton de evenimente

I. Presupunem că $A_n \downarrow \emptyset$



Deci $\bigcap\limits_{n=1}^{\infty} A_n = \emptyset$

Avem mulțimea $A_1 = \bigcup\limits_{n=1}^{\infty} (A_n \setminus A_{n+1})$

$(A_n \setminus A_{n+1}) \cap (A_m \setminus A_{m+1}) = \emptyset, (\forall) m,n \in \mathbb{N}^*, m \neq n$

Rezultă $\mathbb{P}(A_1) = \mathbb{P}\left(\bigcup\limits_{n=1}^{\infty} (A_n \setminus A_{n+1})\right) \overset{P_3}{=} \sum\limits_{n=1}^{\infty} \mathbb{P}(A_n \setminus A_{n+1})$

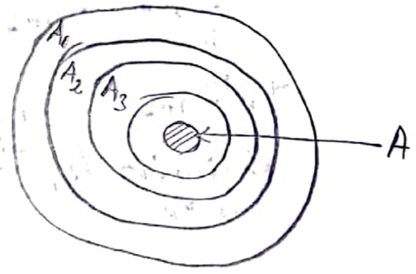
$\mathbb{P}(A_1) \in [0,1] \Rightarrow \sum\limits_{n=1}^{\infty} \mathbb{P}(A_n \setminus A_{n+1})$, convergentă $\Rightarrow$

$\Rightarrow \lim\limits_{n\to\infty} \sum\limits_{k=n}^{\infty} \mathbb{P}(A_k \setminus A_{k+1}) = 0$

$\boxed{\underline{\text{Obs}}: \sum\limits_{k=n}^{\infty} \mathbb{P}(A_k \setminus A_{k+1}) \overset{P_3}{=} \mathbb{P}\left(\bigcup\limits_{k=n}^{\infty} (A_k \setminus A_{k+1})\right) = \mathbb{P}(A_n)}$

Deci $\lim\limits_{n\to\infty} \mathbb{P}(A_n) = 0 = \mathbb{P}(\emptyset) = \mathbb{P}(\lim\limits_{n\to\infty} A_n)$

II Presupunem că $A_n \downarrow A$, $A = \bigcap\limits_{n=1}^{\infty} A_n \in \mathcal{F}$



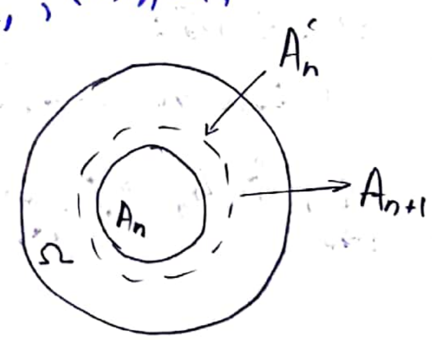
$A_n \downarrow A \Rightarrow A_n \setminus A \downarrow \emptyset \overset{I}{\Rightarrow}$

$\overset{I}{\Rightarrow} \lim\limits_{n\to\infty} \underset{\displaystyle \mathbb{P}(A_n) - \mathbb{P}(A)}{\underset{\parallel}{\mathbb{P}(A_n \setminus A)}} = \mathbb{P}(\emptyset) = 0$

Rezultă $\lim\limits_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A) = \mathbb{P}(\lim\limits_{n\to\infty} A_n)$

III Presupunem că $A_n \uparrow A$, $A = \bigcup\limits_{n=1}^{\infty} A_n$, $A_n \subset A_{n+1}$, $(\forall) n \in \mathbb{N}^*$

Atunci $A_n^c \downarrow A^c \overset{II}{\Rightarrow} \lim\limits_{n\to\infty} \mathbb{P}(A_n^c) = \mathbb{P}(A^c) \Rightarrow$



$\Rightarrow \lim\limits_{n\to\infty} (1 - \mathbb{P}(A_n)) = 1 - \mathbb{P}(A) \Rightarrow$

$\Rightarrow \lim\limits_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A) = \mathbb{P}(\lim\limits_{n\to\infty} A_n)$

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Evenimente Independente

Definiție:

Fie tripletul $(\Omega, \mathcal{F}, \mathbb{P})$ un spațiu de probabilitate. Evenimentele A și B se numesc independente dacă $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$. Mai general, o mulțime cel mult numărabilă $\{A_i, i \in I\} \subset \mathcal{F}$ reprezintă o familie de evenimente independente dacă pentru oricare evenimente distincte $A_1, A_2, \dots, A_n$ avem:

$$\mathbb{P}\left(\bigcap_{k=1}^{n} A_{i_k}\right) = \prod_{k=1}^{n} \mathbb{P}(A_{i_k})$$

Proprietăți:

a) $\emptyset, A$ - evenimente independente, $(\forall)\ A \in \mathcal{F}$
Ω, A - evenimente independente, $(\forall)\ A \in \mathcal{F}$

$\mathbb{P}(\emptyset \cap A) = \mathbb{P}(\emptyset) = 0 = \mathbb{P}(\emptyset) \cdot \mathbb{P}(A) \Rightarrow \emptyset, A$ - ev. independente

$\mathbb{P}(\Omega \cap A) = \mathbb{P}(A) = \mathbb{P}(A) \cdot 1 = \mathbb{P}(A) \cdot \mathbb{P}(\Omega) \Rightarrow \Omega, A$ - ev. independente

b) A, B - evenimente independente $\Rightarrow \begin{cases} A, B^c \\ A^c, B^c \end{cases}$

$\mathbb{P}(A \cap B^c) = \mathbb{P}(A \setminus (A \cap B)) = \mathbb{P}(A) - \mathbb{P}(A \cap B) \overset{\text{ipoteză}}{=} \mathbb{P}(A) - \mathbb{P}(A) \cdot \mathbb{P}(B) =$
$= \mathbb{P}(A)(1 - \mathbb{P}(B)) = \mathbb{P}(A) \cdot \mathbb{P}(B^c) \Rightarrow A, B^c$ - ev. independente

A, B^c - ev. indipendondente $\Rightarrow A^c, B^c$ - ev. independente

$$\boxed{c)\ A_1, \dots, A_n \in \mathcal{F} \text{ - ev. indep.} \Rightarrow \mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) = 1 - \prod_{k=1}^{n} \mathbb{P}(A_k^c)}$$

$$\mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) = 1 - \mathbb{P}\left(\left(\bigcup_{k=1}^{n} A_k\right)^c\right) \overset{\text{de Morgan}}{=\!=} 1 - \mathbb{P}\left(\bigcap_{k=1}^{n} A_k^c\right) \overset{b)}{=}$$

$$\overset{b)}{=} 1 - \prod_{k=1}^{n} \mathbb{P}(A_k^c)$$

Legea 0-1 (Legea lui Borel-Cantelli)

Fie $(\Omega, \mathcal{F}, \mathbb{P})$ - spaţiu de probabilitate

Fie $(A_n)_{n\geq 1}$, şir de evenimente, cu $\limsup\limits_{n\to\infty} A_n = A \in \mathcal{F}$

$$\boxed{1)\ \sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty \Rightarrow \mathbb{P}(A) = 0}$$

• Să pp. că $\sum\limits_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ (este convergentă) $\Rightarrow$

$$\Rightarrow \sum_{k=n}^{\infty} \mathbb{P}(A_k) \xrightarrow{n\to\infty} 0$$

Avem: $0 \leq \mathbb{P}\left(\bigcup\limits_{k=n}^{\infty} A_k\right) \leq \sum\limits_{k=n}^{\infty} \mathbb{P}(A_k), (\forall) n \in \mathbb{N}^*$

Rezultă: $\lim\limits_{n\to\infty} \mathbb{P}(\underbrace{\bigcup\limits_{k=n}^{\infty} A_k}_{B_n}) = 0$

$A_n \cup B_{n+1} = B_n \supset B_{n+1}, (\forall) n \in \mathbb{N} \Rightarrow B_n$, şir descrescător $\Rightarrow B_n \downarrow$

Deci:

$$\lim_{n\to\infty} \mathbb{P}(B_n) = \mathbb{P}(\lim_{n\to\infty} B_n) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n\right) = \mathbb{P}(\limsup_{n\to\infty} A_n)$$

Obţinem: $\mathbb{P}(A) = \mathbb{P}(\limsup\limits_{n\to\infty} A_n) = 0$

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$$\boxed{\left.\begin{array}{l} 2)\ \sum\limits_{n=1}^{\infty} \mathbb{P}(A_n)=\infty \\ (A_n)_{n\geq 1} \text{ - ev. independent} \end{array}\right\} \Rightarrow \mathbb{P}(A)=1}$$

$$\begin{cases} \sum\limits_{n=1}^{\infty} \mathbb{P}(A_n)=\infty \\ (A_n)_{n\in\mathbb{N}} \text{ - ev. indepen.} \end{cases}$$

$\mathbb{P}(A)=1 \Leftrightarrow \mathbb{P}(A^c)=0$

$$A^c=\Big(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k\Big)^c=\bigcup_{n=1}^{\infty}\underbrace{\bigcap_{k=n}^{\infty} A_k^c}_{B_n}=\liminf_{n\to\infty} A_n^c=\lim_{n\to\infty} B_n$$

şir crescător, $B_n \uparrow$

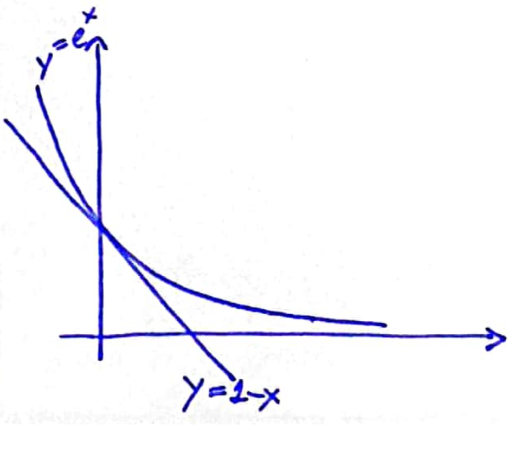
$$\mathbb{P}(A^c)=\mathbb{P}\big(\lim_{n\to\infty}(B_n)\big)=\lim_{n\to\infty}\mathbb{P}(B_n)=\lim_{n\to\infty}\mathbb{P}\Big(\bigcap_{k=n}^{\infty} A_k^c\Big)$$

$(A_k)_{k\geq 1}$ - ev. indep. $\Rightarrow (A_k^c)_{k\geq 1}$ - şir de ev. indep.

$$\mathbb{P}\Big(\bigcap_{k=n}^{\infty} A_k^c\Big)=\lim_{p\to\infty}\mathbb{P}\Big(\bigcap_{k=n}^{n+p} A_k^c\Big)\overset{\text{indep.}}{=}\lim_{p\to\infty}\prod_{k=n}^{n+p}\mathbb{P}(A_k^c)=$$

$$\left.\begin{array}{l} =\lim\limits_{p\to\infty}\prod\limits_{k=n}^{n+p}\big(1-\mathbb{P}(A_k)\big) \\ 1-x\leq e^{-x},\ (\forall)x\in\mathbb{R} \end{array}\right\} \Rightarrow \prod_{k=n}^{n+p}\big(1-\mathbb{P}(A_k)\big)\leq\prod_{k=n}^{n+p} e^{-\mathbb{P}(A_k)}=$$

$$=e^{\underbrace{-\sum_{k=n}^{n+p}\mathbb{P}(A_k)}_{\to-\infty}}\to e^{-\infty}=0$$



$\mathbb{P}(A^c)=0$