Curs 3 ✓ 06.03.2024

Probabilități limită

Definiție: $(\Omega, \mathcal{F}, \mathbb{P}) \overset{not}{=}$ sp. de probabilitate

Șirul $(A_n)_{n\geq 1}$ de evenimente $(A_n \in \mathcal{F}, (\forall) n \in \mathbb{N}^*)$ se numește:

1) Monoton crescător dacă $A_n \subset A_{n+1}, (\forall) n \in \mathbb{N}^*$

$\lim\limits_{n\to\infty} A_n \overset{def.}{=} \bigcup\limits_{n=1}^{\infty} A_n \overset{not}{=} A \in \mathcal{F}$; Notăm $\boxed{A_n \uparrow A}$

2) Șirul s.n. monoton descrescător dacă $A_n \supset A_{n+1}, (\forall) n \in \mathbb{N}^*$

$\lim\limits_{n\to\infty} A_n = \bigcap\limits_{n=1}^{\infty} A_n \overset{not}{=} A \in \mathcal{F}$; Notăm $\boxed{A_n \downarrow A}$

3)
a) $\limsup\limits_{n\to\infty} A_n \overset{def}{=} \bigcap\limits_{n=1}^{\infty} \underbrace{\bigcup\limits_{k=n}^{\infty} A_k}_{B_n} = \lim\limits_{n\to\infty} B_n$

$\underset{A_n \cup B_{n+1}}{\overset{\shortparallel}{B_n}} \supset B_{n+1}, (\forall) n \in \mathbb{N}^* \Rightarrow B_n$ este monoton descrescător $\Rightarrow$
$\Rightarrow B_n \downarrow$

b) $\liminf\limits_{n\to\infty} A_n \overset{def}{=} \bigcup\limits_{n=1}^{\infty} \underbrace{\bigcap\limits_{k=n}^{\infty} A_k}_{B_n} = \lim\limits_{n\to\infty} B_n$

$\underset{A_n \cap B_{n+1}}{\overset{\shortparallel}{B_n}} \subset B_{n+1}, (\forall) n \in \mathbb{N}^* \Rightarrow B_n$ monoton crescător $\Rightarrow B_n \uparrow$

~~Teore~~

Teoremă: $(\Omega, \mathcal{F}, \mathbb{P})$, sp. de probabilitate

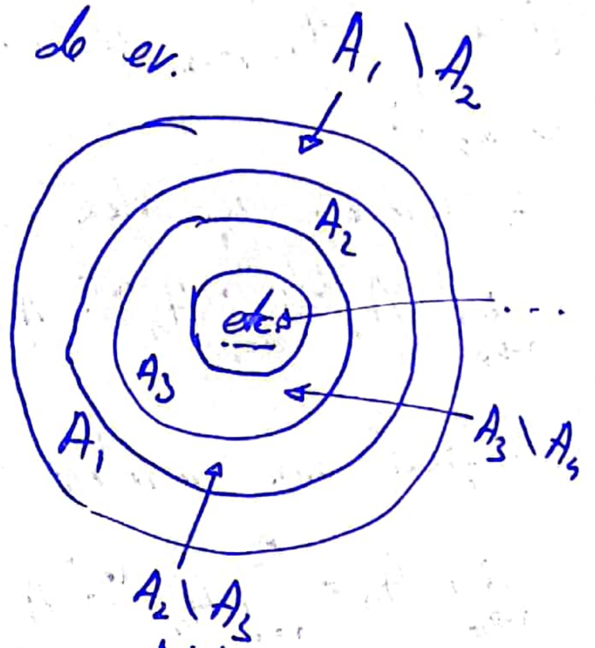
$\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(\lim_{n\to\infty} A_n)$, $(\forall)\,(A_n)_{n\geq 1}$ șir monoton de eveniment

Demonstrație:

§ Fie $(A_n)_{n\geq 1}$, un șir monoton de ev.

I. ~~$A_n \to \emptyset$~~ P.p. $A_n \searrow \emptyset$

Deci $\bigcap_{n=1}^{\infty} A_n = \emptyset$



Avem mulțimea $A_1 = \bigcup_{n=1}^{\infty} (A_n \setminus A_{n+1})$

$(A_n \setminus A_{n+1}) \cap (A_m \setminus A_{m+1}) = \emptyset$, $(\forall)$ $m, n \in \mathbb{N}^*$, $m \neq n$

Rezultă:

$$\mathbb{P}(A_1) = \mathbb{P}\Big(\bigcup_{n=1}^{\infty} (A_n \setminus A_{n+1})\Big) \overset{P_3}{=} \sum_{n=1}^{\infty} \mathbb{P}(A_n \setminus A_{n+1})$$

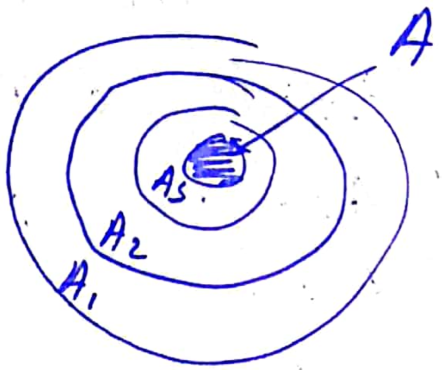
$\mathbb{P}(A_1) \in [0,1] \Rightarrow \sum_{n=1}^{\infty} \mathbb{P}(A_n \setminus A_{n+1})$ - convergentă $\Rightarrow$

$\Rightarrow \lim_{n\to\infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k \setminus A_{k+1}) = 0$

Obs! $\sum_{k=n}^{\infty} \mathbb{P}(A_k \setminus A_{k+1}) \overset{P_3}{=} \mathbb{P}\Big(\bigcup_{k=n}^{\infty} (A_k \setminus A_{k+1})\Big) = \mathbb{P}(A_n)$

Deci: $\lim_{n\to\infty} \mathbb{P}(A_n) = 0 = \mathbb{P}(\emptyset) = \mathbb{P}(\lim_{n\to\infty} A_n)$

II. pp. că $A_n \downarrow A$, $A=\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$

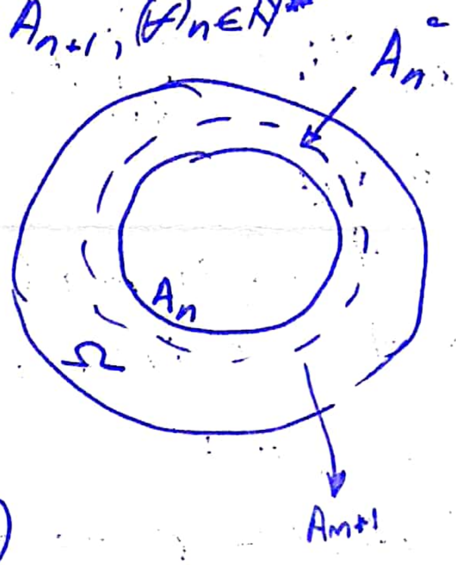


$A_n \downarrow A \Rightarrow A_n \setminus A \downarrow \emptyset \overset{\text{cf. I}}{\Longrightarrow}$

$$\Rightarrow \lim_{n\to\infty} \underbrace{\mathbb{P}(A_n \underset{\supset}{\setminus} A)}_{\mathbb{P}(A_n)-\mathbb{P}(A)} = \mathbb{P}(\emptyset)=0$$

<u>Rezultă</u>: $\lim\limits_{n\to\infty} \mathbb{P}(A_n)=\mathbb{P}(A)=\mathbb{P}(\lim\limits_{n\to\infty} A_n)$

III pp. că $A_n \uparrow A$, $A=\bigcup_{n=1}^{\infty} A_n$, $A_n \subset A_{n+1}$, $(\forall) n\in \mathbb{N}^*$



Atunci $A_n^c \downarrow A^c \overset{\text{II}}{\Longrightarrow}$

$\Rightarrow \lim\limits_{n\to\infty} \mathbb{P}(A_n^c)=\mathbb{P}(A^c) \Rightarrow$

$\Rightarrow \lim\limits_{n\to\infty} (1-\mathbb{P}(A_n)) = 1-\mathbb{P}(A) \Rightarrow$

$\Rightarrow \lim\limits_{n\to\infty} \mathbb{P}(A_n)=\mathbb{P}(A)=\mathbb{P}(\lim\limits_{n\to\infty} A_n)$

Σ - sumă

Π - produs

EVENIMENTE INDEPENDENTE

Definiție: $(\Omega, \mathcal{F}, \mathbb{P})$ - sp. prob.

Caz 1) $A, B \in \mathcal{F} \to$ evenimente independente dacă $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

Caz 2) $\{A_i, i \in I\} \subset \mathcal{F}$, I - mulțime cel mult numărabilă familie de evenimente independente dacă $(\forall)\, n \in \mathbb{N}, n \geq 2, n \leq card(I)$, $(\forall)\, \overbrace{i_1, i_2, \dots, i_n}^{\text{distincte}} \in I$ ↳ nr de elemente ale lui I

$$\mathbb{P}\left(\bigcap_{k=1}^{n} A_{i_k}\right) = \prod_{k=1}^{n} \mathbb{P}(A_{i_k})$$

Proprietăți:

a) $\begin{cases} \emptyset, A \text{ - ev. independente}, (\forall)\, A \in \mathcal{F} \\ \Omega, A \text{ - ev. independente}, (\forall)\, A \in \mathcal{F} \end{cases}$

Demonstrație:

$\mathbb{P}(\emptyset \cap A) = \mathbb{P}(\emptyset) = 0 = \mathbb{P}(\emptyset) \cdot \mathbb{P}(A) \Rightarrow \emptyset, A$ - indep.

$\mathbb{P}(\Omega \cap A) = \mathbb{P}(A) = \mathbb{P}(A) \cdot 1 = \mathbb{P}(A) \cdot \mathbb{P}(\Omega) \Rightarrow \Omega, A$ - indep.

b) A, B - ev. indep. $\Rightarrow \begin{cases} A, B^c \text{ - ev. indep.} \\ A^c, B^c \text{ - ev. indep.} \end{cases}$

Demonstrație:

$\mathbb{P}(A \cap B^c) = \mathbb{P}(A \setminus (A \cap B)) = \mathbb{P}(A) - \mathbb{P}(A \cap B) \overset{\text{ipoteză}}{=} \mathbb{P}(A) - \mathbb{P}(A) \cdot \mathbb{P}(B) = \mathbb{P}(A)(1 - \mathbb{P}(B)) = \mathbb{P}(A) \cdot \mathbb{P}(B^c) \Rightarrow$

$\Rightarrow A, B^c$ - ev. indep.

A, B^c - indep. $\Rightarrow A^c, B^c$ - ev. independente.

c) $A_1, \dots, A_n \in \mathcal{F}$ - ev. independente

Atunci $\mathbb{P}(\bigcup_{k=1}^{n} A_k) = 1 - \prod_{k=1}^{n} \mathbb{P}(A_k^c)$

Demonstrație:

$$\mathbb{P}(\bigcup_{k=1}^{n} A_k) = 1 - \mathbb{P}\left(\left(\bigcup_{k=1}^{n} A_k\right)^c\right) \overset{\text{de Morgan}}{=\!=} 1 - \mathbb{P}\left(\bigcap_{k=1}^{n} A_k^c\right) \overset{b)}{=} 1 - \prod_{k=1}^{n} \mathbb{P}(A_k^c).$$

Legea 0-1 (Legea lui Borel-Cantelli)

Fie $(\Omega, \mathcal{F}, \mathbb{P})$ - sp. prob.

$(A_n)_{n\geq 1}$, șir de evenimente, cu $\limsup_{n\to\infty} A_n = A \in \mathcal{F}$

1) $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty \Rightarrow \mathbb{P}(A) = 0$

2) $\left.\begin{array}{l} \sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty \\ (A_n)_{n\geq 1} \text{ - ev. indep.} \end{array}\right\} \Rightarrow \mathbb{P}(A) = 1$

Demonstrație:

1) Să pp. că $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ (este convergentă) $\Rightarrow$

$\Rightarrow \sum_{k=n}^{\infty} \mathbb{P}(A_k) \xrightarrow{n\to\infty} 0$

Avem:

$$\mathbb{P}\left(\bigcup_{k=n}^{\infty} A_k\right) \le \sum_{k=n}^{\infty} \mathbb{P}(A_k), (\forall)\, n \in \mathbb{N}^*$$

Rezultă

$$\lim_{n\to\infty} \mathbb{P}\Big(\underbrace{\bigcup_{k=n}^{\infty} A_k}_{B_n}\Big) = 0$$

$\underbrace{A_n \cup B_{n+1}}$ $= B_n \supset B_{n+1}$, $(\forall)\, n \in \mathbb{N} \Rightarrow B_n$ șir descrescător $\Rightarrow B_n \downarrow$

Deci

$$\lim_{n\to\infty} \mathbb{P}(B_n) = \mathbb{P}(\lim_{n\to\infty} B_n) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n\right) = \mathbb{P}(\limsup_{n\to\infty} A_n)$$

Obținem $\mathbb{P}(A) = \mathbb{P}(\limsup_{n\to\infty} A_n) = 0$

2. $\begin{cases} \sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty \\ (A_n)_{n\in\mathbb{N}} \text{ - ev. indep} \end{cases}$

$\mathbb{P}(A) = 1 \Leftrightarrow \mathbb{P}(A^c) = 0$

$$A^c = \left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right)^c = \bigcup_{n=1}^{\infty} \underbrace{\bigcap_{k=n}^{\infty} A_k^c}_{B_n} = \liminf_{n\to\infty} A_n^c =$$

șir crescător, $B_n \uparrow$

$$= \lim_{n\to\infty} B_n$$

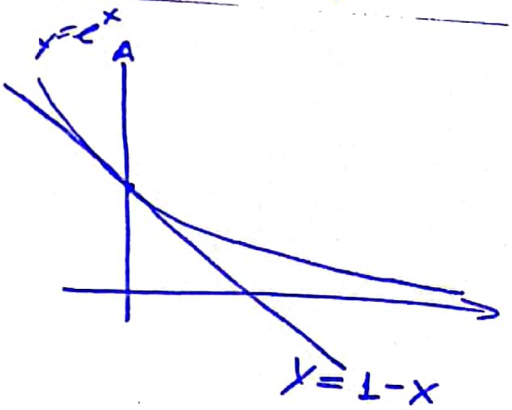
$$\mathbb{P}(A^c) = \mathbb{P}(\lim_{n\to\infty}(B_n)) = \lim_{n\to\infty} \mathbb{P}(B_n) = \lim_{n\to\infty} \mathbb{P}\left(\bigcap_{k=n}^{\infty} A_k^c\right)$$

$(A_k)_{k\ge1}$ - ev. indep. $\Rightarrow (A_k^c)_{k\ge1}$ - șir de ev. indep.

$$\mathbb{P}\left(\bigcap_{k=n}^{\infty} A_k^c\right) = \lim_{p\to\infty} \mathbb{P}\left(\bigcap_{k=n}^{n+p} A_k^c\right) \overset{\text{indep.}}{=} \lim_{p\to\infty}\prod_{k=n}^{n+p}\mathbb{P}(A_k^c) =$$

$$= \lim_{p\to\infty}\prod_{k=n}^{n+p}\left(1-\mathbb{P}(A_k)\right)$$

$1 - x \le e^{-x}, (\forall) x \in \mathbb{R}$



$$\Rightarrow \prod_{k=n}^{n+p}\left(1-\mathbb{P}(A_k)\right) \le \prod_{k=n}^{n+p} e^{-\mathbb{P}(A_k)} =$$

$$= e^{-\underbrace{\sum_{k=n}^{n+p}\mathbb{P}(A_k)}_{\to\infty}} \to e^{-\infty} = 0$$

R

$\mathbb{P}(A^c) = 0$