Seminar 3 04.03.2024

Variabile aleatoare discrete

① Să se calculeze valoarea medie a următoarei variabile aleatoare discrete.

variabila ~~valoarea~~ aleatoare: $X: \begin{pmatrix} \frac{1}{1\cdot 2} & \frac{1}{2\cdot 3} & \dots & \frac{1}{n(n+1)} \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \end{pmatrix}$ ← valori / ← probabilități

Sumă telescopică!

nr. de elemente

$E(X) = \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)}\right)\frac{1}{n} = \left(\frac{2-1}{1\cdot 2} + \frac{3-2}{2\cdot 3} + \dots + \frac{n+1-n}{n(n+1)}\right)\frac{1}{n} =$

$= \left(\frac{\cancel{2}}{1\cdot \cancel{2}} - \frac{1}{1\cdot 2} + \frac{\cancel{3}}{2\cdot \cancel{3}} - \frac{2}{2\cdot 3} + \dots + \frac{\cancel{n+1}}{n\cancel{(n+1)}} - \frac{n}{n(n+1)}\right)\frac{1}{n} =$

$= \left(1 - \frac{1}{n+1}\right)\frac{1}{n} = \frac{\cancel{n}}{\cancel{n}}\frac{1}{n+1}$

② Se dau variabilele aleatoare independente X și Y ← De asta se pot înmulți factorii

având distribuțiile: $X: \begin{pmatrix} a & 1 & 2 \\ \frac{1}{3} & p & q \end{pmatrix} = \begin{pmatrix} a & 1 & 2 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

$Y: \begin{pmatrix} a+1 & 1 & 2 \\ \frac{1}{3} & \frac{2}{3}-q & p \end{pmatrix} = \begin{pmatrix} a+1 & 1 & 2 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

Să se determine nr. real a a.î. dispersia variabilei aleatoare $X-Y = \frac{4}{9}$.

$$\begin{cases} \frac{1}{3} + p + q = 1 \\ \frac{1}{3} + \frac{2}{3} - q + p = 1 \end{cases} \quad (+)$$

$\frac{2}{3} + \frac{2}{3} + 2p = 2 \Rightarrow 2p = \frac{1}{3},\ q = \frac{1}{3}$

$V \overset{not}{=}$ ~~dispe~~ dispersia, variația

$$V(X-Y)=\frac{4}{9}$$

$$X-Y: \begin{pmatrix} -1 & a-1 & a-2 & -a & 0 & -1 & 1-a & 1 & 0 \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{pmatrix}$$

$$X-Y: \begin{pmatrix} 0 & -1 & a-1 & a-2 & -a & 1-a & \cancel{0} & 1 \\ \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \cancel{\frac{2}{9}} & \frac{1}{9} \end{pmatrix}$$

v.a. $\overset{not.p.}{=}$ variabilă aleatorie

$$V(Z)=E(Z^2)-(E(Z))^2$$

$E \overset{not}{=}$ media

$$E(X-Y)= -1\cdot\frac{2}{9}+(a-1)\frac{1}{9}+(a-2)\frac{1}{9}+(-a)\frac{1}{9}+(1-a)\frac{1}{9}+\frac{1}{9}$$

$$E(X-Y)=\frac{1}{9}(-2+\cancel{a}\cancel{-1}+\cancel{a}-2\cancel{-a}+1\cancel{-a}\cancel{+1})$$

$$E(X-Y)=\frac{1}{\cancel{9}_3}(\cancel{-3})$$

$$E(X-Y)=\frac{-1}{3}$$

$$E(X-Y)^2=\frac{1}{9}\left(2+(a-1)^2+(a-2)^2+a^2+(1-a)^2+1\right)$$

$$E(X-Y)^2=\frac{1}{9}(2+a^2-2a+1+a^2-4a+2+a^2+1-2a+a^2+1)$$

$$E(X-Y)^2=\frac{1}{9}\left(4a^2-8a+\cancel{7}\cancel{8}9\right)$$

$$V(X-Y)=\frac{4}{\cancel{9}}-\frac{1}{\cancel{9}}(4a^2-8a+9)-\frac{1}{\cancel{9}}$$

$$V(X-Y)=4=4a^2-8a+8 \quad /:4$$

$V(X-Y) = 1 = a^2 - 2a + 2$

$V(X-Y): \quad a^2 - 2a + 1 = 0$

$V(X-Y): \quad (a-1)^2 = 0 \Rightarrow a = 1$

③ Se dă v.a. $X: \begin{pmatrix} 1 & 2 & 3 & 4 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \end{pmatrix}$

Să se det. v.aloarea a, b a.î. $Y = aX + b$

$E(Y) = 0, \quad V(Y) = 1$

$E(Y) = E(aX+b) = a\,E(X) + E(\overset{b,\text{ constantă}}{b}) =$

$= a\,E(X) + b = a\left(\frac{1}{8} + 1 + \frac{3}{4} + \frac{1}{2}\right) + b =$

$= a\left(1 \cdot \frac{1}{8} + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8}\right) + b =$

$= a\,\frac{19}{8} + b = 0$

$V(Y) = E(Y^2) - (E(Y))^2 =$

$= E((aX+b)^2) - (E(aX+b))^2 =$

$= E(a^2X^2 + 2abX + b^2) - (a\,E(X) + b)^2$

$= a^2\,\cancel{E(X^2)} + 2ab \cdot E(X) + \cancel{b^2} - a^2 \cdot \cancel{E(X)^2} + 2ab\,E(X) \mp \cancel{b^2}$

$= a^2\, 2ab\,E(X)$

$= a^2\left(E(X^2) - E(X)^2\right)$

$\boxed{V(aX+b) = a^2\,V(X)}$

$\boxed{V_c = 0}$

$a^2 \cdot V(X) = 1$

$V(X) = E(X^2) - (E(X))^2 = \frac{51}{8} - \frac{19 \cdot 19}{8} < \begin{matrix} a_1, b_1 \\ a_2, b_2 \end{matrix}$

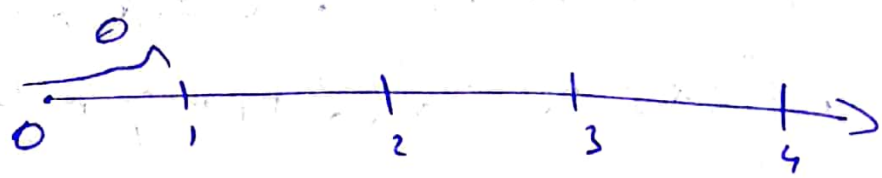
$E(X^2) = \frac{1}{8} + \frac{4}{2} + \frac{9}{4} + \frac{16}{8} = \frac{1+16+18+16}{8} = \frac{51}{8}$

④ Să se det. funcția de repartiție F.

pt $X: \begin{pmatrix} 1 & 2 & 3 & 4 \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \end{pmatrix}$

$X: \begin{pmatrix} x_k \\ p_k \end{pmatrix}_{k \in I}$; I - mulțime de indici cel mult numărabilă

$\sum_{k \in I} p_k = 1$



$F: \mathbb{R} \to [0, 1]$, $F(x) = \sum_{\substack{x_k \le x \\ k \in I}} p_k$

$F(x) \overset{not}{=}$ funcția de repartiție a lui X.

x, nr oarecare real

~~pt $x < 1$: $F(x) = 0$~~

~~pt $x \in (1, 2)$: $F(x) =$~~

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{8}, & 1 \le x < 2 \\ \frac{1}{8} + \frac{1}{2}, & 2 \le x < 3 \\ \frac{1}{8} + \frac{1}{2} + \frac{1}{4}, & 3 \le x < 4 \\ 1 = \frac{1}{8} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}, & x > 4 \end{cases}$$

⑤ Într-o cutie se află a bile albe și b bile negre. Din cutie se extrag n bile fără repunere după fiecare extragere. Să se calculeze valoarea medie a v.a. ce indică nr. de bile albe existente printre cele n bile extrase.

$$X : \begin{pmatrix} 0 & 1 & 2 & 3 & \dots & n \\ & & & & & \end{pmatrix}$$

k

Schema hypergeometrică = schema nerepunerii

$$X : \begin{pmatrix} k \\ \dfrac{C_a^k \cdot C_b^{n-k}}{C_{a+b}^n} \end{pmatrix}_{k=\overline{0,n}}$$

$$E(X) = \sum_{k=\cancel{0}1}^{n} k \cdot \frac{C_a^k \cdot C_b^{n-k}}{C_{a+b}^n} = \frac{1}{C_{a+b}^n} \cdot \sum_{k=1}^{n} k \cdot C_a^k \cdot C_b^{n-k} \overset{*}{=}$$

$$* \; C_a^k = \frac{a!}{(a-k)! \cdot k!} = \frac{1}{k} \cdot \frac{a!}{(a-k)!(k-1)!} = \frac{a}{k} \cdot \frac{(a-1)!}{(a-k)!(k-1)!} =$$

$$= \frac{a}{k} \cdot C_{a-1}^{k-1}$$

$$E(X) = \frac{a}{C_{a+b}^n} \sum_{k=1}^{n} C_{a-1}^{k-1} \cdot C_b^{n-k}$$

$$\sum_{k=0}^{n} \frac{C_a^k \cdot C_b^{n-k}}{C_{a+b}^n} = 1 \iff \sum_{k=0}^{n} C_a^k \cdot C_b^{n-k} = C_{a+b}^n$$

$$E(X) = C_{a+b-1}^{n-1} \cdot C_{a+b}^{n} = \left(\frac{a+b}{n} C_{a+b-1}^{n-1}\right) \cdot C_{a+b-1}^{n-1} =$$

$$C_{a+b}^{n} = \frac{a+b}{n} C_{a+b-1}^{n-1}$$

$$\Rightarrow E(X) = \frac{an}{a+b} = n \frac{a}{a+b} = p.$$ ca prima bilă extrasă să fie albă

⑥ Să se determine o v.a. discretă ce nu admite medie ($E(X)$ nu se poate calcula).

Obs! Trebuie să aibe un nr. infinit de valori

$X : \begin{pmatrix} 1 & 2 & \dots & n & \dots \end{pmatrix}$, Trebuie să fie o serie ~~divergent~~ divergentă

$$X : \begin{pmatrix} n \\ p_n \end{pmatrix}_{n \geq 1}$$

1) $p_n \in (0,1)$

2) $\sum_{n=1}^{\infty} p_n = 1$

3) $E(X) = \sum_{n=1}^{\infty} n \cdot p_n$ să fie divergentă

$p_n = \frac{1}{n}$

2) $\sum_{n=1}^{\infty} \frac{1}{n}$, seria armonică e divergentă $\Rightarrow$ nu dă 1

$p_n = \frac{1}{2^n}$

2) $\sum_{n=1}^{\infty} \frac{1}{2^n} =$, sumă geometrică

$= \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} = 1$

3) $E(X) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot n$, seria

Criteriul raportului:

$\lim_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2} < 1$, este convergentă, nu e bună

$p_n = \frac{1}{\ln(n)}$, $n > 1$

2) $\sum_{n=2}^{\infty} \frac{1}{\ln(n)} =$

$p_n = \frac{1}{n^2}$ ✓ $\boxed{p_n = \frac{1}{n^2} \cdot \frac{6}{\pi^2}}$ acesta e răspunsul

2) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ~~convergent~~ $= \frac{\pi^2}{6}$

$E(X) = \sum_{n=1}^{\infty} \frac{1}{n}$, divergent ✓