Seminar 4 11.03.2024

V.A. CONTINUE

f - funcție de probabilitate

V.A. DISCRETE	V.A. CONTINUE
$\sum_{k\in I} p_k = 1$, $p_k \geq 0, (\forall) k \in I$	$\int_{-\infty}^{+\infty} f(x)\, dx = 1$, $f(x) \geq 0, (\forall) x \in \mathbb{R}$
$E(X^n) = \sum_{k\in I} x_k^n \cdot p_k$	$E(X^n) = \int_{-\infty}^{+\infty} x^n f(x)\, dx$
$F(t) = \sum_{x_k \leq t} p_k$	$F(t) = \int_{-\infty}^{t} f(x)\, dx$, $f = F'$
$F(t) = \mathbb{P}(X \leq t)$ ~~Se dă~~	la fel
$V(X) = E(X^2) - (E(X))^2$	la fel

Se dă funcția

$$f_x = \begin{cases} -6a \cdot x^2, & x \in [-1, 0] \\ 6a^2 \cdot x, & x \in (0, 1] \\ 0, & x \in \mathbb{R} \setminus [-1, 1] \end{cases}$$

a) Să ~~S~~ se determine const. reală a a.î. f să fie o densitate.

b) Să se calculeze funcția de repartiție

c) Să se ~~det~~ noteze dispersia

d) Să să calc. probabilitatea $\mathbb{P}(X \in [-1, \frac{1}{2}] \mid X \geq 0)$

a) $$1=\int_{-1}^{0}-6ax^2\,dx+\int_{0}^{1}6a^2x\,dx=$$

$$=-6a\frac{x^3}{3}\Big|_{-1}^{0}+6a^2\frac{x^2}{2}\Big|_{0}^{1}=$$

$$=0+6a\frac{(-1)^3}{3}+3a^2=-2a+3a^2$$

$$+3a^2-2a-1=0$$

$$\Delta=2^2+4\cdot3\cdot1=\cancel{8}16$$

$$a_{1,2}=\frac{2\pm\sqrt{16}}{2\cdot3}=\frac{2\pm4}{6}\;\begin{cases}1 & \text{nu merge pe } x\in[-1,0]\Rightarrow\\ -\frac{1}{3}\end{cases}$$

$$\Rightarrow a=\frac{-1}{3}$$

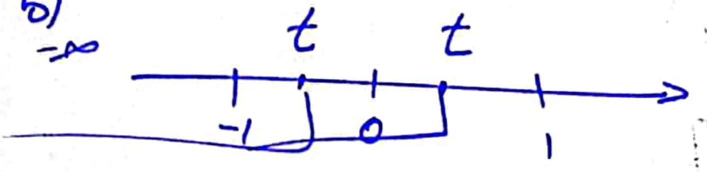
c)
$$f(x)=\begin{cases}2x^2 &, x\in[-1,0]\\ \frac{2}{3}x &, x\in(0,1]\\ 0 &, x\in\mathbb{R}\setminus[-1,1]\end{cases}$$

$$V(x)=E(x^2)-(E(x))^2$$

$$E(x^2)=\int_{-\infty}^{+\infty}x^2f(x)\,dx=\int_{-1}^{0}2x^4\,dx+\int_{0}^{1}\frac{2}{3}x^3\,dx=$$

$$=2\frac{x^5}{5}\Big|_{-1}^{0}+\frac{2}{3}\frac{x^4}{\cancel{4}\,2}\Big|_{0}^{1}=\frac{2}{5}+\frac{1}{6}=\frac{17}{30}$$

b)



$F(t)=0,\ (\forall)\ t \le -1$

$t \in [-1,0]$

$$F(t)=\int_{-\infty}^{-1} f(x)\,dx+\int_{-1}^{t} f(x)\,dx=\int_{-1}^{t} 2x^2\,dx=$$

$$=2\frac{x^3}{3}\Big|_{-1}^{t}=\frac{2t^3}{3}+\frac{2}{3}=\frac{2t^3+2}{3}$$

$t \in (0,1]$

$$F(t)=\int_{-\infty}^{-1} f(x)\,dx+\int_{-1}^{0} 2x^2\,dx+\int_{0}^{t}\frac{2}{3}x\,dx=$$

$$=\frac{2x^3}{3}\Big|_{-1}^{0}+\frac{2}{3}\frac{x^2}{2}\Big|_{0}^{t}=\frac{2}{3}+\frac{t^2}{3}=\frac{t^2+2}{3}$$

$t \in (1,+\infty),\ F(t)=$

$$\int_{-\infty}^{t} f(x)\,dx=0+\int_{-1}^{0} 2x^2\,dx+\int_{0}^{1}\frac{2x}{3}\,dx+\int_{1}^{t} 0\,dx=$$

$$=\frac{2}{3}+\frac{2}{3}\frac{x^2}{2}\Big|_{0}^{1}=\frac{2}{3}+\frac{1}{3}=1$$

$$F(t)=\begin{cases} 0, & t\in(-\infty,1) \\ \frac{2(t^3+1)}{3}, & t\in[-1,0) \\ \frac{t^2+2}{3}, & t\in[0,1) \\ 1, & t\in[1,\infty) \end{cases}$$

Obs! $\angle$ se închid în partea stângă

d) $\mathbb{P}(X\in[-1,\frac{1}{2}]\,|\,X\geq 0)=?$

$$=\frac{\mathbb{P}(X\in[0,\frac{1}{2}])}{\mathbb{P}(X\geq 0)}=\frac{F(\frac{1}{2})-F(0)}{1-\mathbb{P}(X<0)}=$$

$$=\frac{F(\frac{1}{2})-F(0)}{1-F(0)}=\frac{\frac{\frac{1}{4}+\frac{8}{4}}{3}-\frac{2}{3}}{1-\frac{2}{3}}=\frac{\frac{\frac{9}{4}}{3}-\frac{2}{3}}{1-\frac{2}{3}}=$$

$$=\frac{\frac{9}{12}-\frac{8}{12}}{\frac{3-2}{3}}=\frac{1}{12}\cdot\frac{3}{1}=\frac{1}{4}$$

② Să ~~să~~ se calc. disp. V și funct. de repartiție F a ~~o~~ v.a. continue X când densitatea de ~~prob~~ prob.:

$f(x) = \left\{ a \cdot e^{-|x|}, x \in \mathbb{R} \right.$

$\boxed{\text{Obs!} \quad |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}}$

$1 = \int_{-\infty}^{+\infty} f(x)\, dx = a \int_{-\infty}^{+\infty} e^{-|x|} dx = a\left(\int_{-\infty}^{0} e^{+x} dx + \int_{0}^{+\infty} e^{-x} dx \right) =$

$= a \left(e^{x} \Big|_{-\infty}^{0} - e^{-x} \Big|_{0}^{\infty} \right) =$

$= a \; (1 - 0 + 1 - 0) = 2a$

$1 = 2a \Rightarrow a = \frac{1}{2}$

$f(x) = \left\{ \frac{1}{2} e^{-|x|}, x \in \mathbb{R} \right.$

$E(x) = \int_{-\infty}^{+\infty} x f(x)\, dx = \int_{-\infty}^{0} x \frac{1}{2} e^{+x} dx + \int_{0}^{\infty} x \frac{1}{2} e^{-x} dx =$

$= \frac{x^2}{2} \frac{1}{2} e^{x} \Big|_{-\infty}^{0} - \frac{x^2}{2} \frac{1}{2} e^{x} \Big|_{0}^{\infty} = \ldots = 0$ pt că este o funcție impară

$E(x^2) = \int_{-\infty}^{+\infty} x^2 f(x)\, dx = 2 \int_{0}^{\infty} x^2 e^{-x} dx =$

(integrare prin părți de 2ori)

gama mare = Γ ~~Euler~~ Oiler?

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx, \; p > 0$$

$$= 2\Gamma(3) = 2! \cdot 2 = 4$$

$$V(x) = E(x^2) - (E(x))^2 =$$

$$= 4 - 0^2 = 4$$

~~$F(x) =$~~

$t < 0$

$$F(t) = \frac{1}{2}\int_{-\infty}^{t} e^{x}\, dx = \frac{1}{2}\left(e^{x}\Big|_{-\infty}^{t}\right) =$$

$$= \frac{1}{2} e^{t}$$

$t \geq 0$

$$F(t) = \frac{1}{2}\int_{-\infty}^{0} e^{x}\, dx + \frac{1}{2}\int_{0}^{+\infty} e^{-x}\, dx =$$

$$= \frac{1}{2} e^{0} + \frac{1}{2}\int_0^t e^{-x}\, dx = \frac{1}{2} - \frac{1}{2} e^{-t}$$

$$= \frac{1}{2}(e^{0} - e^{-\infty}) - \frac{1}{2}(e^{-t} - e^{0}) = \frac{1}{2} - \frac{1}{2}(e^{-t} - 1) =$$

$$= 1 - \frac{e^{-t}}{2}$$

$$F(t) = \begin{cases} \frac{1}{2} e^{t}, & t < 0 \\ \frac{2 - e^{-t}}{2}, & t \geq 0 \end{cases}$$