Curs 2

28.02.2024

Probabilitatea

$(\Omega, \mathcal{F})$ - spațiu măsurabil
$\quad \llcorner$ σ-corp

$\mathcal{F} \overset{not}{=}$ mulțimea evenimentelor posibile

$\mathbb{P}: \mathcal{F} \to \mathbb{R} \overset{not}{=}$ funcția de probabilitate.

Obs! Numim funcție de probabilitate o funcție ce satisface următoarele axiome:

~~P_1: $\mathbb{P}(A) \geq 0$, $(\forall) A \in \mathcal{F}$ [orice eveniment a~~

~~P~~

P_1: Orice eveniment are o probabilitate pozitivă

$$\mathbb{P}(A) \geq 0, (\forall) A \in \mathcal{F}$$

P_2:

$$\mathbb{P}(\Omega) = 1$$

P_3: Dacă avem un șir de evenimente, ce nu se pot realiza simultan, atunci probabilitatea este suma probabilităților evenimentelor

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n), (\forall) \{A_n\}_{n \geq 1} \subset \mathcal{F} \text{ a.î.}$$

$$A_n \cap A_m = \emptyset, (\forall) m, n \in \mathbb{N}^*, m \neq n$$

Teoremă: Probabilitatea are următoarele proprietăți care decurg din definiție:

$P_4)$

$$\mathbb{P}(\emptyset)=0$$

Demonstrația:

Definim șirul $(A_n)_{n\geq 1}\subset \mathcal{F}$

$A_n=\emptyset, (\forall)\ n\in\mathbb{N}^*$

Presupunem că $\mathbb{P}(\emptyset)\neq 0 \overset{P_1}{\Longrightarrow} \mathbb{P}(\emptyset)>0$.

Fie $\mathbb{P}(\emptyset)=p>0$

Atunci $p=\mathbb{P}(\emptyset)=\mathbb{P}(\bigcup_{n=1}^{\infty} A_n)\overset{P_3}{=}\sum_{n=1}^{\infty}\underbrace{\mathbb{P}(A_n)}_{p}=\infty$, contradicție $\Rightarrow$

$\Rightarrow$ Presupunerea noastră este falsă $\Rightarrow \mathbb{P}(\emptyset)=0$

Obs! $\emptyset\cap\emptyset=\emptyset\ ,\ (\forall)\cap\emptyset=\emptyset$

$P_5)$

$\mathbb{P}(\bigcup_{k=1}^{n} A_k)=\sum_{k=1}^{n}\mathbb{P}(A_k), (\forall)\ A_1, A_2, \dots, A_n\in\mathcal{F}$ a.î. $A_i\cap A_j=\emptyset, (\forall)\ i\neq j$

Demonstrație:

Fie $A_1, A_2, \dots, A_n\in\mathcal{F}$, $n\geq 2$ a.î. $A_i\cap A_j=\emptyset, (\forall)\ i\neq j$

Definim $A_k=\emptyset, (\forall)\ k\in\mathbb{N}^*, k>n$

Șirul de evenimente $\{A_k\}_{k\geq 1}$ satisface condiția $A_i\cap A_j=\emptyset, (\forall)\ i\neq j$

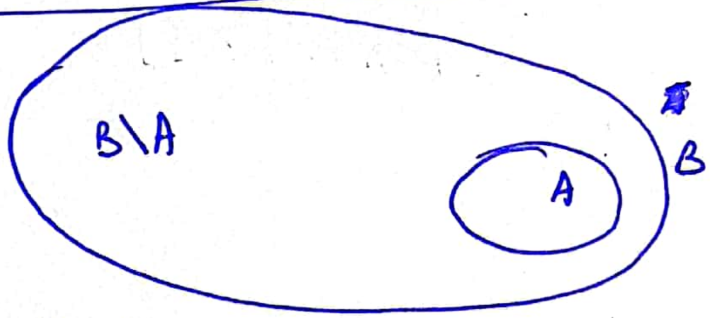
$$\mathbb{P}(\bigcup_{k=1}^{n} A_k)=\mathbb{P}(\bigcup_{k=1}^{\infty} A_k)\overset{P_3}{=}\underbrace{\sum_{k=1}^{\infty}\mathbb{P}(A_k)}_{\mathbb{P}(\bigcup_{k=1}^{n} A_k)}=\underbrace{\sum_{k=1}^{n}\mathbb{P}(A_k)+\sum_{k=n+1}^{\infty}\mathbb{P}(A_k)}_{\text{de la 1 la } n \text{ și de la } n+1 \text{ la infinit}}\overset{P_4}{=}\sum_{k=1}^{n}\mathbb{P}(A_k)$$

$P_6)$

$$\mathbb{P}(B\setminus A) = \mathbb{P}(B) - \mathbb{P}(A),\ (\forall)\ A, B \in \mathcal{F},\ A \subset B$$

Obs! $A \subset B \overset{not}{=}$ dacă se produce A, atunci se produce şi B

Desen:



Demonstraţie:

Fie $A, B \in \mathcal{F}$, $A \subset B$

$B = A \cup (B \setminus A)$

$A \cup (B\setminus A)$, disjuncte $\Rightarrow A \cup (B \setminus A) = \emptyset$ $\Big\} \overset{P_5}{\Longrightarrow} \mathbb{P}(B) = \mathbb{P}(A \cup (B\setminus A)) =$

$= \mathbb{P}(A) + \mathbb{P}(B \setminus A)$

Deci $\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B\setminus A) \Rightarrow \mathbb{P}(B\setminus A) = \mathbb{P}(B) - \mathbb{P}(A)$

$P_7)$ Funcţia de probabilitate este o funcţie monoton crescătoare

$(\forall)\ A, B \in \mathcal{F},\ A \subset B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

Demonstraţie:

Fie $A, B \in \mathcal{F}$, $A \subset B$

$\mathbb{P}(B) - \mathbb{P}(A) \overset{P_6}{=} \mathbb{P}(B\setminus A) \overset{P_1}{\geq} 0 \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

P_8) Probabilitatea este cuprinsă între 0 și 1

$\mathbb{P}(A) \in [0,1]$, $(\forall) A \in \mathcal{F}$

Demonstrație:

Fie $A \in \mathcal{F}$

$$\left.\begin{array}{l} \mathbb{P}(A) \overset{P_1}{\geq} 0 \\ A \subset \Omega \overset{P_7}{\Rightarrow} \mathbb{P}(A) \leq \mathbb{P}(\Omega) \overset{P_2}{=} 1 \end{array}\right\} \Rightarrow \mathbb{P}(A) \in [0,1]$$

P_9)

$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$, $(\forall) A \in \mathcal{F}$

Demonstrație:

Fie $A \in \mathcal{F}$

$$\left.\begin{array}{l} A \cup A^c = \Omega \\ A \cap A^c = \emptyset \end{array}\right\} \overset{P_5}{\Rightarrow} \mathbb{P}(A) + \mathbb{P}(A^c) = \mathbb{P}(A \cup A^c) = \mathbb{P}(\Omega) = 1 \Rightarrow$$

$$\Rightarrow \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$$

P_{10}) Formula lui Poincaré / Principiul includerii-excluderii

$(\forall) A_1, A_2, \dots, A_n \in \mathcal{F}$:

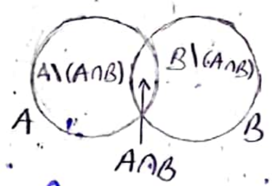
$$\mathbb{P}\left(\bigcup_{k=1}^{n} A_k\right) = \overbrace{\sum_{k=1}^{n} \mathbb{P}(A_k)}^{C_n^1 \text{ termeni}} - \overbrace{\sum_{1 \leq k_1 < k_2 \leq n} \mathbb{P}(A_{k_1} \cap A_{k_2})}^{C_n^2 \text{ termeni}} + \overbrace{\sum_{1 \leq k_1 < k_2 < k_3 \leq n} (\mathbb{P}(A_{k_1} \cap A_{k_2} \cap A_{k_3})}^{C_n^3 \text{ termeni}}$$

$$\dots + \overbrace{(-1)^{n-1} \mathbb{P}\left(\bigcap_{k=1}^{n} A_k\right)}^{C_n^n \text{ termeni}} = \sum_{i=1}^{n} (-1)^{i-1} \sum_{1 \leq k_1 < k_2 < \dots < k_i \leq n} (\mathbb{P}(A_{k_1} \cap A_{k_2} \cap \dots \cap A_{k_i})$$

Demonstrație:

Fie $n = 2 \Rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$, $(\forall) A, B \in \mathcal{F}$

$$A\cup B=\underbrace{(A\setminus(A\cap B))\cup(A\cap B)\cup(B\setminus(A\cap B))}_{\text{disjuncte, incompatibile}}$$



$$P(A\cup B)\overset{P_5}{=}\underbrace{P(A\setminus(A\cap B))}_{\supset A}+P(A\cap B)+\underbrace{P(B\setminus(A\cap B))}_{\supset B}\overset{P_6}{=}$$

$$\overset{P_6}{=}P(A)-\cancel{P(A\cap B)}+\cancel{P(A\cap B)}+P(B)-P(A\cap B)=P(A)+P(B)-P(A\cap B)$$

$n=2\Rightarrow 2^2-1=3$ termeni

P_{11}) Subadedivitate

$$\boxed{P\left(\bigcup_{k=1}^{n}A_k\right)\le\sum_{k=1}^{n}P(A_k),\ (\forall)\ A_1,\dots,A_n\in\mathcal{F}}$$

Demonstrație:

Fie $A_1,\dots,A_n\in\mathcal{F}$, $n\ge 2$

Definim $B_1,B_2,\dots,B_n\in\mathcal{F}$ prin $\begin{cases}B_1=A_1\\ B_k=A_k\setminus\left(\bigcup_{i=1}^{k}A_i\right), & k=2,\dots,n\end{cases}$

Au loc proprietățile:

1) $B_i\cap B_j=\emptyset,\ (\forall)\ i,j\in\{1,\dots,n\},\ i<j$

2) $\bigcup_{k=1}^{n}B_k=\bigcup_{k=1}^{n}A_k$

3) $B_k\subset A_k$, $k=\overline{1,n}$

Atunci $P\left(\bigcup_{k=1}^{n}A_k\right)\overset{2)}{=}P\left(\bigcup_{k=1}^{n}B_k\right)\underset{P_5}{\overset{1)}{=}}\sum_{k=1}^{n}P(B_k)\underset{P_7}{\overset{3)}{\le}}\sum_{k=1}^{n}P(A_k)$

Consecință:

P_{12}) Inegalitatea lui Boole

$$\boxed{P\left(\bigcap_{k=1}^{n}A_k\right)\ge 1-\sum_{k=1}^{n}P(A_k^c),\ (\forall)\ A_1,\dots,A_n\in\mathcal{F}}$$

Demonstrație:

Fie $A_1, \dots, A_n \in \mathcal{F}$

$$\mathbb{P}\left(\bigcap_{k=1}^{n} A_k\right) \overset{P_3}{=} 1 - \mathbb{P}\left(\left(\bigcap_{k=1}^{n} A_k\right)^c\right) \overset{\text{de Morgan}}{=} 1 - \mathbb{P}\left(\bigcup_{k=1}^{n} A_k^c\right) \overset{P_{11}}{\geq}$$

$$\overset{P_{11}}{\geq} 1 - \sum_{k=1}^{n} \mathbb{P}(A_k^c)$$