Seminar 5 18.03.2024

FUNCȚIA CARACTERISTICĂ

$i=(0,1)$, $i^2=-1$

1) $\varphi_X : \mathbb{R} \to \mathbb{C}$, $\varphi_X(t) = \mathbb{E}(e^{itX})$, $(\forall)\, t\in\mathbb{R}$, X v.a.

2) $E(X^k) = \dfrac{\varphi_X^{(k)}(0)}{i^k}$, $k\in\mathbb{N}^*$

$\boxed{\begin{array}{l}\text{Obs! pt. discretă} \\ X:\begin{pmatrix} x_k \\ p_k\end{pmatrix}_{k\in I}, \\ \varphi_X(t)=\sum_{k\in I} e^{i\cdot t\cdot x_k}\cdot p_k\end{array}}$

~~Funcția caracteristică~~

Aplicații:

① Se dă v.a. discretă $X:\begin{pmatrix} k \\ C_n^k\cdot p^k(1-p)^{n-k}\end{pmatrix}$, $p\in(0,1)$, $k=\overline{0,n}$

a) $\varphi_X(t)=?$

b) $\mathbb{V}(X)=?$

$\begin{cases} \mathbb{E}(X) \overset{\text{teorie}}{=} n\cdot p \\ \mathbb{V}(X) \overset{\text{teorie}}{=} n\cdot p(1-p)\end{cases}$

$\varphi_X(t) = \sum_{k=0}^{n} e^{i\cdot t\cdot k}\cdot C_n^k\cdot p^k(1-p)^{n-k} =$

$= \sum_{k=0}^{n} C_n^k (e^{it}p)^k\cdot(1-p)^{n-k} =$

$\underset{\text{lui Newton}}{\overset{\text{binomul}}{=}} (pe^{it}+1-p)^n \Rightarrow \varphi_X(t)=(pe^{it}+1-p)^n$

$\varphi_X'(t) = n(pe^{it}+1-p)^{n-1}\cdot(p\,i\,e^{it})$

$\mathbb{E}(X) = \dfrac{n(p+1-p)^{n-1}p\,i}{i} = np$

$$\Phi''(t) = \cancel{n(p+1-p)^{n-1} pi} = npi \cdot \left[(p \cdot e^{it} + 1 - p)^{n-1} \cdot e^{it}\right]' =$$

$$\boxed{\text{Obs.} = (u \cdot v)' = u'v + uv'} \qquad \boxed{(e^{w})' = e^{w} \cdot w'}$$

$$= npi \left\{ \left[(p \cdot e^{it} + 1 - p)^{n-1}\right]' e^{it} + (pe^{it} + 1 - p)^{n-1} \cdot e^{it} (it)' \right\} =$$

$$= npi \left\{ (n-1) \cdot (p \cdot e^{it} + 1 - p)^{n-2} \cdot e^{it} \cdot p \cdot i \cdot e^{it} + (pe^{it} + 1 - p)^{n-1} \cdot i \cdot e^{it} \right\} =$$

$$\boxed{(f^{n-1})' = (n-1) \cdot f^{n-2} \cdot f'} \qquad \boxed{(e^{it})' = e^{it} (it)' = i \cdot e^{it}}$$

$$\Rightarrow E(X^2) = \frac{\varphi''(0)}{i^2} = -\varphi''(0)$$

$$\varphi''(0) = -npi\left[(n-1) \cdot p \cdot i + i\right] =$$

$$= -npi(npi - pi + i) =$$

$$= np(np - p + 1) = n^2p^2 - np^2 + np$$

$$V(X) = E(X^2) - (E(X))^2$$

$$V(X) = n^2p^2 - np^2 + np - n^2p^2 = -np^2 + np =$$

$$= np(-p + 1) = np(1 - p)$$

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{S4}

② Fie v.a. discretă

$$X: \begin{pmatrix} n \\ \frac{2^n}{n!} \cdot e^{-2} \end{pmatrix}_{n \in \mathbb{N}}$$

$\boxed{\text{Obs!} \quad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x, (\forall)\, x \in \mathbb{R}}$

a) $\varphi_X = ?$

b) $\mathbb{E}(X) = ?$

$$\varphi_X(t) = \sum_{\substack{\cancel{n \in \mathbb{N}} \\ n=0}}^{\infty} e^{itn} \cdot \frac{2^n}{n!} \cdot e^{-2} = e^{-2} \sum_{n=0}^{\infty} (e^{it})^n \cdot \frac{2^n}{n!} =$$

$$= e^{-2} \sum_{n=0}^{\infty} \frac{(2 \cdot e^{it})^n}{n!} = e^{-2} \cdot e^{2 \cdot e^{it}} = e^{2(e^{it}-1)}$$

Deci: $\varphi_X(t) = e^{2(e^{it}-1)}$

$$\varphi_X(t)' = e^{2(e^{it}-1)} \cdot (2(e^{it}-1))' = e^{2(e^{it}-1)} \cdot 2e^{it} \cdot i$$

$$\mathbb{E}(X) = \frac{\varphi_X'(0)}{i} = \frac{2i}{i} = 2$$

(5) Să se det. f. de repartiție a unei v.a. discrete știind că

$$\varphi_X(t) = \frac{\cancel{2}\cdot e^{it} + 4e^{-it} + 1}{6}, \quad t \in \mathbb{R}$$

$$\cancel{F_X(t) = \sum_{\substack{x_k \le t \\ k \in I}} p_k}$$

$$\varphi_X(t) \overset{def}{=} e^{it(1)}\frac{1}{6} + e^{it(-1)}\cdot\frac{4}{6} + e^{it0}\cdot\frac{1}{6} \Rightarrow$$

$$\Rightarrow X : \begin{pmatrix} -1 & 0 & 1 \\ \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

$$F(t) = \begin{cases} 0, & t < -1 \\ \frac{4}{6}, & -1 \le t < 0 \\ \frac{5}{6}, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$$

Obs:

$$\varphi_X(t) = \int_{\mathbb{R}} e^{itx}\cdot f(x)\,dx$$

(4) Se dă funcția $f(x)=\begin{cases} |x|, & x\in[-1,1] \\ 0, & x\in\mathbb{R}\setminus[-1,1] \end{cases}$, $\varphi_X(t)=?$

$\boxed{\text{Obs! } e^{iut}=\cos(tu)+i\cdot\sin(tu)}$

$\boxed{\text{Obs! } \varphi_X(0)=e^{it0}=1}$

$$\varphi_X(t)=\int_{-1}^{1} e^{itx}|x|\,dx=\int_{-1}^{0} e^{itx}\underbrace{(-x)}_{u}\,dx+\int_{0}^{1} e^{itx}x\,dx=$$

$$=-\int_{-1}^{0} x e^{itx}\,dx+\int_{0}^{1} x e^{itx}\,dx=+\int_{0}^{+1} e^{-itu}u\,du+\int_{0}^{1} e^{itu}u\,du=$$

(in the first integral: not. $x=-u$; in the second: not. $x=u$)

$$=\int_{0}^{1} e^{-itu}u+e^{itu}u\,du=$$

$$=\int_{0}^{1} u\left(e^{-itu}+e^{itu}\right)du=2\int_{0}^{1} u\cos(tu)\,du=\cancel{\tfrac{2}{t}}$$

$$=\frac{2}{t}\left(u\cdot\sin(tu)\Big|_0^1-\int_0^1 \sin(tu)\,du\right)=$$

$$\cancel{=\frac{2}{t}\int_0^1 u\cdot\sin(tu)'\,du}$$

$$=\frac{2}{t}\left(\sin t+\frac{1}{t}\cos(tu)\Big|_0^1\right)=\frac{2}{t}\left(\sin t+\frac{1}{t}\cdot\cos t-\frac{1}{t}\right),\ (\forall)\, t\in\mathbb{R}^*$$

$\boxed{\text{Obs! } f(x)=\frac{1}{2\pi}\cdot\int_{-\infty}^{+\infty} e^{-itx}\cdot\varphi_X(t)\,dt,\ (\forall)\, x\in\mathbb{R}}$