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Answer 1

a) Using membership table to prove $A \cap B \subseteq (A \cup \overline{B}) \cap (\overline{A} \cup B)$:

A	B	$\overline{A}$	$\overline{B}$	$A \cap B$	$A \cup \overline{B}$	$\overline{A} \cup B$	$(A \cup \overline{B}) \cap (\overline{A} \cup B)$
1	1	0	0	1	1	1	1
1	0	0	1	0	1	0	0
0	1	1	0	0	0	1	0
0	0	1	1	0	1	1	1

b) Using membership table to prove $\overline{A} \cap \overline{B} \subseteq (A \cup \overline{B}) \cap (\overline{A} \cup B)$:

A	B	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$	$A \cup \overline{B}$	$\overline{A} \cup B$	$(A \cup \overline{B}) \cap (\overline{A} \cup B)$
1	1	0	0	0	1	1	1
1	0	0	1	0	1	0	0
0	1	1	0	0	0	1	0
0	0	1	1	1	1	1	1

Answer 2

$$f^{-1}((A \cap B) \times C) = f^{-1}(A \times B) \cap f^{-1}(B \times C)$$
$$f^{-1}((A \times C) \cap (B \times C)) = f^{-1}(A \times C) \cap f^{-1}(B \times C)$$

Part 1, show that $f^{-1}((A \times C) \cap (B \times C)) \subseteq f^{-1}(A \times C) \cap f^{-1}(B \times C)$;
Assume that $x \in f^{-1}((A \times C) \cap (B \times C))$, then $f(x) \in ((A \times C) \cap (B \times C))$,
so $f(x) \in (A \times C)$ and $f(x) \in (B \times C)$, therefore $x \in f^{-1}(A \times C)$ and $x \in f^{-1}(B \times C)$,
so $x \in (f^{-1}(A \times C) \cap f^{-1}(B \times C))$

Part 2, show that $f^{-1}(A \times C) \cap f^{-1}(B \times C) \subseteq f^{-1}((A \times C) \cap (B \times C))$;
Assume that $x \in f^{-1}(A \times C) \cap f^{-1}(B \times C)$, then $x \in f^{-1}(A \times C)$ and $x \in f^{-1}(B \times C)$,
so $f(x) \in (A \times C)$ and $f(x) \in (B \times C)$, which implies $f(x) \in ((A \times C) \cap (B \times C))$,
therefore $x \in f^{-1}((A \times C) \cap (B \times C))$

Hence by part1 and part2 $f^{-1}((A \times C) \cap (B \times C)) = f^{-1}(A \times C) \cap f^{-1}(B \times C)$
Therefore $f^{-1}((A \cap B) \times C) = f^{-1}(A \times B) \cap f^{-1}(B \times C)$

Answer 3

a) From the definition of one to one functions, if a function is one to one then for all x and for all y; if $f(x) = f(y)$ then $x = y$. We can clearly see this is not the case for this function. Since the function has x^2 term in its interior, it maps the negative items from R to their positive equivalents in R (i.e $(-x)^2 = x^2$ while $x \neq -x$, even function). So we see that there is at least one situation which contradicts with our assumption, therefore $f(x)$ can not be one-to-one. We know that $f : A \to B$ is onto if $f(A) = B$. Which means if a function f is onto then every element in its codomain is being mapped. This function is defined from R to R so if $f(x)$ is onto then every item on R must be mapped by $f(x)$. But negative values on R will not be mapped by $f(x)$. Therefore $f(x)$ can not be onto.

b) From the definitions of one-to-one functions and onto functions on part a, $f(x)$ is one-to-one, since the function maps every distinct item from R to a distinct item in R. But $f(x)$ is not onto, since the 0 in R will remain unmapped from $f(x)$. Therefore $f(x)$ can not be onto.

Answer 4

a) Let A and/or B countably infinite sets such that $A = \{a_1, a_2 ...\}$ and $B = \{b_1, b_2 ...\}$ we can write the elements of $A \times B$ as;

$A \setminus B$	b_1	b_2	b_3	...
a_1	(a_1, b_1)	(a_1, b_2)	(a_1, b_3)	...
a_2	(a_2, b_1)	(a_2, b_2)	(a_2, b_3)	...
a_3	(a_3, b_1)	(a_3, b_2)	(a_3, b_3)	...
...	...	...	...	...

In table, first row, above the horizontal line contains the countably infinite elements of set B, first column, left of vertical line contains the countably infinite elements of set A. We can write $A \times B$ on this table and we can count the elements of $A \times B$ diagonally (i.e. 1st element: (a_1, b_1), 2nd element: (a_2, b_1), 3rd element (a_1, b_2), 4th element: (a_1, b_3), 5th element: (a_2, b_2), 6th element: (a_3, b_1) and so on.). With this way of counting we will eventually get all elements in $A \times B$, then we can index these elements with unique numbers from $\mathbb{N}$, therefore we can get the one-to-one correspondence between $A \times B$ and $\mathbb{N}$. Therefore we can infer that $A \times B$ is countable.

b) Prove by contradiction, suppose that B were countable, say with elements $b1, b2, ...$ .Then since $A \subseteq B$, we can list the elements of A using the order in which they appear in this listing of B. Therefore A is countable. But this contradicts with the hypothesis (A is uncountable). Thus B is not countable.

c) If B is countable with the elements $b1, b2, ...$ and $A \subseteq B$, then we can list the elements of A using the order in which they appear in this listing of B. Therefore A is countable.

Answer 5

a) Let $f_1(x) = 7x^2$ and $f_2(x) = x^3$ ($f_1(x)$ is $\mathcal{O}(f_2(x))$ and they are increasing functions). Prove by the Big-Oh definition, $ln|f_1(x)|$ is $\mathcal{O}(ln|f_2(x)|)$ if $ln|f_1(x)| \leq c.(ln|f_2(x)|)$ for some $x > k$. Let us check this condition: If $ln|7x^2|$ is $\mathcal{O}(ln|x^3|)$ then $ln|7x^2| \leq c.ln|x^3|$. Exponential both sides $e^{ln|7x^2|} \leq e^{c.ln|x^3|}$, eliminate the terms $7x^2 \leq e^c.x^3$, note that when $x \geq k = 7$ and $c \geq 1$, we have $7x^2 \leq e^c.x^3$, consequently, we can take $c = 1$ and $k = 7$ as witnesses to establish the relationship $ln|7x^2|$ is $\mathcal{O}(ln|x^3|)$. Therefore $ln|f_1(x)|$ is $\mathcal{O}(ln|f_2(x)|)$.

b) Let $f_1(x) = 7x^2$ and $f_2(x) = x^3$ ($f_1(x)$ is $\mathcal{O}(f_2(x))$ and they are increasing functions). Prove by the Big-Oh definition, $3^{f_1(x)}$ is $\mathcal{O}(3^{f_2(x)})$ if $3^{f_1(x)} \leq c.3^{f_2(x)}$ for some $x > k$. Let us check this condition: If 3^{7x^2} is $\mathcal{O}(3^{x^3})$ then $3^{7x^2} \leq c.3^{x^3}$. Take the log of both sides $7x^2.log3 \leq x^3.log3 + logc$, divide log3 both sides $7x^2 \leq x^3 + log(c-3)$, note that when $x \geq k = 7$ and $c \geq 4$, we have $7x^2 \leq x^3 + log(c-3)$, consequently, we can take $c = 4$ and $k = 7$ as witnesses to establish the relationship 3^{7x^2} is $\mathcal{O}(3^{x^3})$. Therefore $3^{f_1(x)}$ is $\mathcal{O}(3^{f_2(x)})$.

Answer 6

a)

$$\begin{aligned}
(3^x - 1)\ mod(3^y - 1) &= 3^{(x\ mod\ y)} - 1 \\
3^x - 1 &= k.(3^y - 1) + 3^{(x\ mod\ y)} - 1 \qquad where\ k \in \mathbb{Z} \\
3^x &= k.(3^y - 1) + 3^{(x\ mod\ y)} \\
log_3(3^x) &= log_3(k.(3^y - 1) + 3^{(x\ mod\ y)}) \\
x &= log_3(k.(3^y - 1) + 3^{(x\ mod\ y)})
\end{aligned}$$

Now substitute x in original equation:

$$\begin{aligned}
(3^{log_3(k.(3^y-1)+3^{(x\ mod\ y)})} - 1)\ mod(3^y - 1) &= 3^{(x\ mod\ y)} - 1 \\
k.(3^y - 1) + 3^{(x\ mod\ y)} - 1 &= k.(3^y - 1) + 3^{(x\ mod\ y)} - 1 \\
0 &= 0 \qquad (1)
\end{aligned}$$

Since we get (1) in the last step, $(3^x - 1)\ mod(3^y - 1) = 3^{(x\ mod\ y)} - 1$ is true.

b)

$$
\begin{aligned}
gcd(123, 277) &= gcd(277, 123 \ mod \ 277) \\
&= gcd(277, 123) \\
&= gcd(123, 277 \ mod \ 123) \\
&= gcd(123, 31) \\
&= gcd(31, 123 \ mod \ 31) \\
&= gcd(31, 30) \\
&= gcd(30, 31 \ mod \ 30) \\
&= gcd(30, 1) \\
&= gcd(1, 30 \ mod \ 1) \\
&= gcd(1, 0) \\
&= 1
\end{aligned}
$$