CENG 384 - Signals and Systems for Computer Engineers
Spring 2018-2019
Written Assignment 3

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1. (a) To find Fourier series coefficients of $x[n]$ we will use following formula for 1 period:

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jkw_0n}$$

And to be able to find them using this formula we need N and $w_0 = \frac{2\pi}{N}$. Checking the graph of $x[n]$ we see that $N = 4$ and $w_0 = \frac{\pi}{2}$. To find the Fourier Series coefficients let us use the period from 0 to 3:

$$\begin{aligned}
a_k &= \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-jk\frac{\pi}{2}n} \\
&= \frac{1}{4}(x[0] + x[1]e^{-jk\frac{\pi}{2}} + x[2]e^{-2jk\frac{\pi}{2}} + x[3]e^{-3jk\frac{\pi}{2}}) \\
&= 0 + \frac{1}{4}(cos(k\frac{\pi}{2}) - jsin(k\frac{\pi}{2})) + \frac{1}{2}(cos(k\pi) - jsin(k\pi)) + \frac{1}{4}(cos(3k\frac{\pi}{2}) - jsin(3k\frac{\pi}{2}))
\end{aligned}$$

From above we can find coefficients:

$$\begin{aligned}
a_0 &= \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1 \\
a_1 &= -\frac{j}{4} - \frac{1}{2} + \frac{j}{4} = -\frac{1}{2} \\
a_2 &= -\frac{1}{4} + \frac{1}{2} - \frac{1}{4} = 0 \\
a_3 &= \frac{j}{4} - \frac{1}{2} - \frac{j}{4} = -\frac{1}{2}
\end{aligned}$$

From periodicity we can simply say: $a_n = a_{n+4} = a_{n-4}$, so other coefficients can be found via this fact.

So the magnitude spectrum of the coefficients:

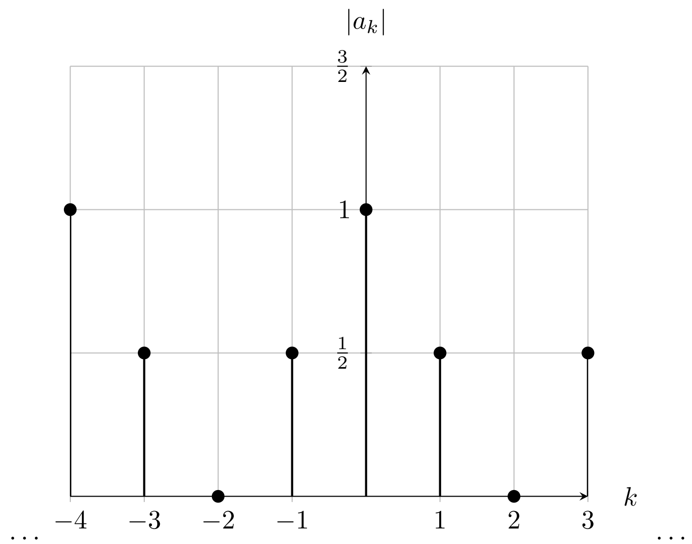

Figure 1: k vs. $|a_k|$.

So the phase spectrum of the coefficients:

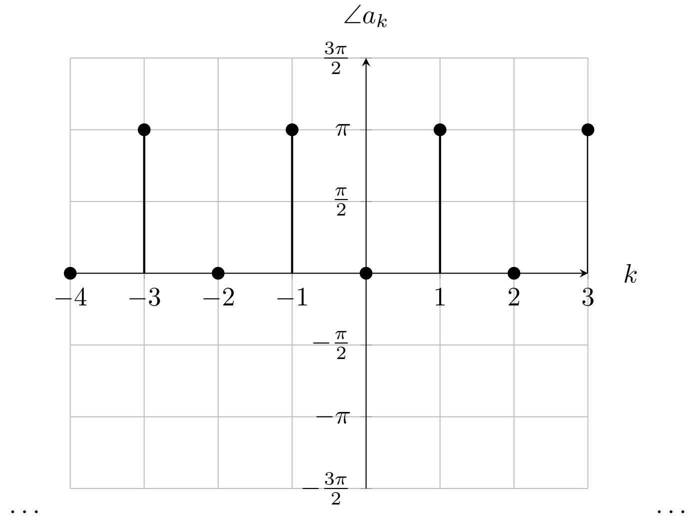

Figure 2: k vs. $\angle a_k$.

(b) i. We need to add a negative impulse (i.e $-\delta(n)$) at every $n+1$ th point to ensure periodicity of $y[n]$. So we would end up something such as:

$$y[n] = x[n] - \sum_{k=-\infty}^{\infty} \delta(n+1-4k)$$

ii. To find Fourier series coefficients of $y[n]$ we will use following formula for 1 period:

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jkw_0 n}$$

And to be able to find them using this formula we need N and $w_0 = \frac{2\pi}{N}$. Checking the graph of $y[n]$ we see that $N = 4$ and $w_0 = \frac{\pi}{2}$. To find the Fourier Series coefficients let us again use the period from 0 to 3:

$$\begin{aligned} a_k &= \frac{1}{4} \sum_{n=0}^{3} x[n] e^{-jk\frac{\pi}{2}n} \\ &= \frac{1}{4}(x[0] + x[1]e^{-jk\frac{\pi}{2}} + x[2]e^{-2jk\frac{\pi}{2}} + x[3]) \\ &= 0 + \frac{1}{4}(cos(k\frac{\pi}{2}) - jsin(k\frac{\pi}{2})) + \frac{1}{2}(cos(k\pi) - jsin(k\pi)) + 0 \end{aligned}$$

From above we can find coefficients:

$$\begin{aligned} a_0 &= \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \\ a_1 &= -\frac{j}{4} - \frac{1}{2} = -\frac{1}{4}(j+2) \\ a_2 &= -\frac{1}{4} + \frac{1}{2} = \frac{1}{4} \\ a_3 &= \frac{j}{4} - \frac{1}{2} = \frac{1}{4}(j-2) \end{aligned}$$

From periodicity we can simply say: $a_n = a_{n+4} = a_{n-4}$, so other coefficients can be found via this fact.

So the magnitude spectrum of the coefficients:

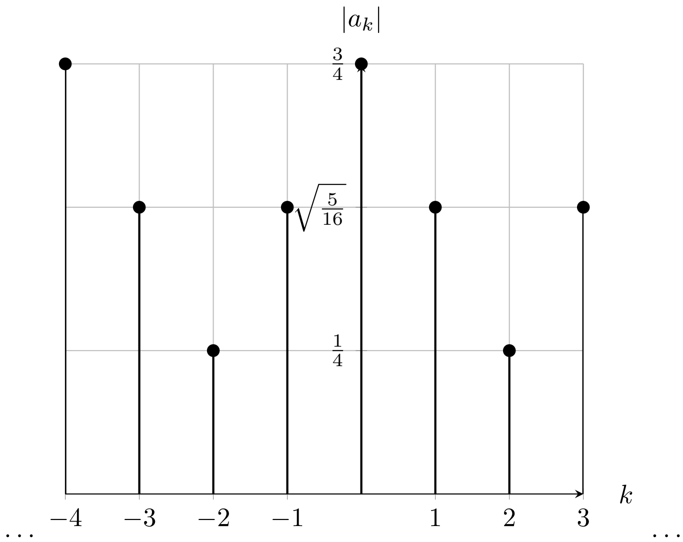

Figure 3: k vs. $|a_k|$.

So the phase spectrum of the coefficients:

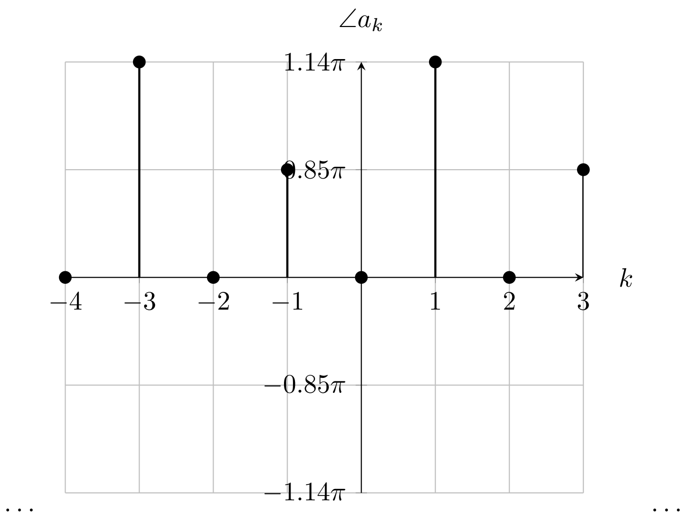

Figure 4: k vs. $\angle a_k$.

2. First let us classify and order the given conditions for $x[n]$.
   (a) Says that $N = 4$ and $w_0 = \frac{\pi}{2}$
   (b) Says that the sum of $x[n]$ for 2 periods is equal to 8. Then we would have:
   $$x[0] + x[1] + x[2] + x[3] = 4$$
   (c) Says that $a_{-3} = a_{15}^*$ which also means $a_1 = a_3^*$, also it says that $|a_1 - a_{11}| = 1$ which also means $|a_1 - a_3| = 1$
   (d) Says that one of the coefficients is zero.
   (e) For this condition, if we would expand the e terms inside the sum we would have the following:
   $$e^{\frac{-j\pi k}{2}} = cos(\frac{\pi k}{2}) - jsin(\frac{\pi k}{2}) \quad (1)$$
   $$e^{\frac{-j\pi 3k}{2}} = cos(\frac{\pi 3k}{2}) - jsin(\frac{\pi 3k}{2}) \quad (2)$$
   Now subtract 2π from both cos and sin terms in equation (2) and add (1) and (2), result will be as follows:
   $$2cos(\tfrac{\pi k}{2})$$
   Therefore this condition actually says $\sum_{k=0}^{3} x[k](2cos(\frac{\pi k}{2}))$, Furthermore calculation of this sum results in:
   $$2x[0] - 2x[2] = 4 \text{ namely } x[0] - x[2] = 2$$

So first let us find Fourier series coefficients for $x[n]$ using $a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jkw_0 n}$ formula:

$$\begin{aligned}
a_k &= \frac{1}{4}(x[0] + x[1]e^{-jk\frac{\pi}{2}} + x[2]e^{-jk\pi} + x[3]e^{-3jk\frac{\pi}{2}}) \\
a_0 &= \frac{1}{4}(x[0] + x[1] + x[2] + x[3]) \\
a_1 &= \frac{1}{4}(x[0] + x[1](cos(\frac{\pi}{2}) - jsin(\frac{\pi}{2})) + x[2](cos(\pi) - jsin(\pi)) + x[3](cos(\frac{3\pi}{2}) - jsin(\frac{3\pi}{2})) \\
a_2 &= \frac{1}{4}(x[0] + x[1](cos(\pi) - jsin(\pi)) + x[2](cos(2\pi) - jsin(2\pi)) + x[3](cos(3\pi) - jsin(3\pi)) \\
a_3 &= \frac{1}{4}(x[0] + x[1](cos(\frac{3\pi}{2}) - jsin(\frac{3\pi}{2})) + x[2](cos(3\pi) - jsin(3\pi)) + x[3](cos(\frac{9\pi}{2}) - jsin(\frac{9\pi}{2}))
\end{aligned}$$

Each of them yield as:

$$\begin{aligned}
a_0 &= \frac{1}{4}(x[0] + x[1] + x[2] + x[3]) \\
a_1 &= \frac{1}{4}(x[0] - jx[1] - x[2] + jx[3]) \\
a_2 &= \frac{1}{4}(x[0] - x[1] + x[2] - x[3]) \\
a_3 &= \frac{1}{4}(x[0] + jx[1] - x[2] - jx[3])
\end{aligned}$$

From condition a we see that $a_0 = \frac{1}{4}4 = 1$.
From condition c we know that if $a_1 = a_3^*$ then they are conjugates of each other and they would be in a form such as:

$$\begin{aligned}
a_1 &= a + bj \\
a_3 &= a - bj
\end{aligned}$$

and since $|a_1 - a_3| = |2b| = 1$ then $y \neq 0$ also $a_1 \neq 0$ and $a_3 \neq 0$. Since we have condition d which says that one of the coefficients is zero, we deduce that $a_2 = 0$.
For a_1 and a_3, let us subtract a_3 from a_1 and compare with condition c:

$$\begin{aligned}
|a_1 - a_3| &= 1 \\
\frac{1}{4}|2jx[3] - 2jx[1]| &= 1 \\
|jx[3] - jx[1]| &= 2 \\
x[3] - x[1] = 2 \text{ or } x[1] - x[3] &= 2 \ \text{ So we need to choose one of them}
\end{aligned}$$

To be able to use in a_3 expansion let us choose $x[1] - x[3] = 2$, multiply this equation by j, $jx[1] - jx[3] = 2j$ and also, from condition e we had $x[0] - x[2] = 2$, add these up and we would end up with a_3 expansion, namely:

We had $a_3 = \frac{1}{4}(x[0] + jx[1] - x[2] - jx[3])$
Now we know that $jx[1] - jx[3] = 2j$ and $x[0] - x[2] = 2$
So $a_3 = \frac{1}{2} + \frac{1}{2}j$
and since a_1 was the conjugate of a_3 so $a_1 = \frac{1}{2} - \frac{1}{2}j$

Now to determine the signal $x[n]$ we need to find its values for 1 period, using condition b, e and value of a_2 we can find $x[0]$ and $x[2]$ as follows:

$$\begin{aligned}
x[0] + x[1] + x[2] + x[3] &= 4 \ \text{ from cond b} \\
x[0] - x[1] + x[2] - x[3] &= 0 = a_2 \\
x[0] + x[2] &= 2 \\
x[0] - x[2] &= 2 \ \text{ from cond e} \\
\text{So } \ x[0] = 2 \ &\text{ and } \ x[2] = 0
\end{aligned}$$

Using a_0 and a_1 we can also determine $x[1]$ and $x[3]$ as follows:

$$\begin{aligned}
\frac{1}{4}(x[0] + x[1] + x[2] + x[3]) &= 1 = a_0 \\
2 + x[1] + 0 + x[3] &= 4 \\
x[1] + x[3] &= 2 \ \ (1)
\end{aligned}$$

Also for a_1:

$$\begin{aligned}
\frac{1}{4}(x[0] - jx[1] + x[2] + jx[3]) &= \frac{1}{2} - \frac{1}{2}j = a_1 \\
2 - jx[1] - 0 + jx[3] &= 2 - 2j \\
j(-x[1] + x[3]) &= -2j \\
x[3] - x[1] &= -2 \ \ (2)
\end{aligned}$$

Equating (1) and (2) we get:

$$x[1] + x[3] = 2 \quad (1)$$
$$x[3] - x[1] = -2 \quad (2)$$
$$\text{So} \quad x[1] = 2 \quad \text{and} \quad x[3] = 0$$

So in the and we have an $x[n]$ with $N = 4$, $w_0 = \frac{\pi}{2}$ and $x[0] = 2$, $x[1] = 2$, $x[2] = 0$ and $x[3] = 0$. Its graph is:

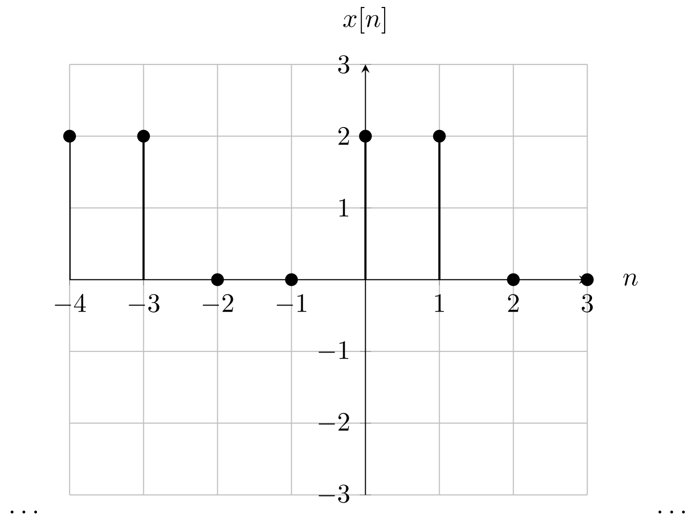

Figure 5: n vs. $x[n]$.

3. We are needed to find $h(t)$ where $x(t) = h(t) * (x(t) + r(t))$. If we would take the Fourier Transform of this equation, we will end up with:

$$X(jw) = H(jw) \times (X(jw) + R(jw))$$
$$X(jw) = H(jw) \times X(jw) + H(jw) \times R(jw)$$
$$X(jw) = H(jw) \times X(jw) \quad (\text{since } R(jw) = 0)$$
$$H(jw) = 1$$

Since we know $H(jw)$, we will now use $h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(jw)e^{jwt}dw$ to find $h(t)$:

$$\begin{aligned} h(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(jw)e^{jwt}dw \\ &= \frac{1}{2\pi}\int_{-\frac{K2\pi}{T}}^{\frac{K2\pi}{T}} 1e^{jwt}dw \\ &= \frac{1}{2\pi}\Big|_{-\frac{K2\pi}{T}}^{\frac{K2\pi}{T}} \frac{e^{jwt}}{jt} \\ &= \frac{1}{2\pi}\left(\frac{e^{\frac{jK2\pi t}{T}}}{jt} - \frac{e^{-\frac{jK2\pi t}{T}}}{jt}\right) \\ &= \frac{1}{2\pi jt}\left(cos\left(\frac{K2\pi t}{T}\right) + jsin\left(\frac{K2\pi t}{T}\right) - cos\left(\frac{K2\pi t}{T}\right) + jsin\left(\frac{K2\pi t}{T}\right)\right) \\ &= \frac{sin\left(\frac{K2\pi t}{T}\right)}{\pi t} \end{aligned}$$

4. (a) First let us find the system equation from block diagram:

$$y^{'}(t) = -5y(t) + 4x(t) + \int x(t) - 6y(t)dt$$
$$\text{namely:} \quad y^{''}(t) = -5y^{'}(t) + 4x^{'}(t) + x(t) - 6y(t)$$

We will find frequency response $H(jw)$ by giving input $x(t) = e^{jwt}$, so we would have $y(t) = H(jw) \times e^{jwt}$ as the output. So plugging in the values to the system:

$$(jw)^2 e^{jwt} H(jw) = -5H(jw)jwe^{jwt} + 4jwe^{jwt} + e^{jwt} - 6H(jw)e^{jwt}$$
$$H(jw)e^{jwt}(j^2w^2 + 5jw + 6) = 4jwe^{jwt} + e^{jwt}$$
$$H(jw) = \frac{4jw + 1}{j^2w^2 + 5jw + 6}$$

(b) We know that $H(jw) \xrightarrow{\mathfrak{F}^{-1}} h(t)$, so all we need to do is to take inverse Fourier transform of $H(jw)$. It can be done as follows:

$$\begin{aligned}
H(jw) &= \frac{4jw+1}{j^2w^2+5jw+6} \\
&= \frac{4jw+1}{(jw+3)(jw+2)} \\
\frac{4jw+1}{(jw+3)(jw+2)} &= \frac{A}{jw+3} + \frac{B}{jw+2} \\
4jw+1 &= Ajw + 2A + Bjw + 3B
\end{aligned}$$

Equating both sides according to their Re and Im parts:

$$\begin{aligned}
A + B &= 4 \\
2A + 3B &= 1 \\
\text{we get: } B &= -7 \text{ and } A = 11
\end{aligned}$$

$$\text{So } H(jw) = \tfrac{11}{jw+3} - \tfrac{7}{jw+2}$$

Now take the $\mathfrak{F}^{-1}$ of $H(jw)$ (using $\mathfrak{F}$ table):

$$H(jw) = \tfrac{11}{3+jw} - \tfrac{7}{2+jw} \xrightarrow{\mathfrak{F}^{-1}} h(t) = 11e^{-3t}u(t) - 7e^{-2t}u(t)$$

(c) To be able to find $y(t)$ for $x(t) = \frac{1}{4}e^{-\frac{t}{4}}u(t)$ using $H(jw)$, we need to first take $\mathfrak{F}$ of $x(t)$ and get $X(jw)$, then using multiplication (instead of convolution) find $Y(jw)$ and then again take $\mathfrak{F}^{-1}$ of $Y(jw)$ and find $y(t)$. This can be done as follows:

$$\begin{aligned}
\text{From } \mathfrak{F} \text{ table we know that: } e^{-|a|t}u(t) &\xrightarrow{\mathfrak{F}} \frac{1}{|a|+jw} \\
\text{So } \frac{1}{4}e^{-\frac{t}{4}}u(t) &\xrightarrow{\mathfrak{F}} \frac{1}{4} \times \frac{1}{\frac{1}{4}+jw} = X(jw)
\end{aligned}$$

Now find $Y(jw)$ using multiplication with $H(jw)$:

$$\begin{aligned}
Y(jw) &= H(jw) \times X(jw) \\
&= \frac{4jw+1}{(jw+3)(jw+2)} \times \frac{1}{1+4jw} \\
&= \frac{1}{(jw+3)(jw+2)}
\end{aligned}$$

Now take the $\mathfrak{F}^{-1}$ of $Y(jw)$ (again using $\mathfrak{F}$ table):

$$\begin{aligned}
Y(jw) &= \frac{1}{jw+3} - \frac{1}{jw+2} \\
\frac{1}{3+jw} - \frac{1}{2+jw} &\xrightarrow{\mathfrak{F}^{-1}} y(t) = e^{-3t}u(t) - e^{-2t}u(t) = (e^{-2t} - e^{-3t})u(t)
\end{aligned}$$