CENG 384 - Signals and Systems for Computer Engineers
Spring 2018-2019
Written Assignment 4

Simsek, Halit
e2099760@ceng.metu.edu.tr

Yesilyurt, Yavuz Selim
e2259166@ceng.metu.edu.tr

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1. (a) Difference equation of the block diagram is:

   $y[n] = 2x[n] + \frac{3}{4}y[n-1] - \frac{1}{8}y[n-2]$

   (b) In order to find frequency response, let $x[n]$ to be e^{jwn} so:

   $x[n] = e^{jwn}$

   $y[n] = H(e^{jw}) \times e^{jwn}$

   $y[n-1] = H(e^{jw}) \times e^{jw(n-1)}$

   $y[n-2] = H(e^{jw}) \times e^{jw(n-2)}$

   Applying above equations to the difference equation:

   $8H(e^{jw})e^{jwn} + H(e^{jw})e^{jwn}e^{-2jw} - 6H(e^{jw})e^{jwn}e^{-jw} = 16e^{jwn}$

   $$H(e^{jw}) = \frac{16}{8 + e^{-2jw} - 6e^{-jw}}$$

   (c) Let $s = e^{-jw}$

   $$H(e^{jw}) = \frac{16}{s^2 - 6s + 8} = \frac{8}{s-4} - \frac{8}{s-2}$$

   Rearranging the above equation:

   $$H(e^{jw}) = -2\frac{1}{1 - \frac{1}{4}s} + 4\frac{1}{1 - \frac{1}{2}s}$$

   Using Fourier Transform lookup table, impulse response becomes:

   $h(n) = [-2(\frac{1}{4})^n + 4(\frac{1}{2})^n]u[n]$

   (d) Finding Fourier Transform of the given input:

   $$F\{x[n]\} = X(e^{jw}) = \frac{1}{1 - \frac{1}{4}e^{-jw}} = \frac{4}{4-s}$$

   Finding output:

   $Y(e^{jw}) = H(e^{jw})X(e^{jw})$

   $$Y(e^{jw}) = \left(\frac{8}{s-4} - \frac{8}{s-2}\right)\left(\frac{-4}{s-4}\right)$$

   Doing multiplication and using partial fraction expansion we have the following result:

   $$Y(e^{jw}) = \frac{16}{s-4} - \frac{32}{(s-4)^2} - \frac{16}{s-2}$$

Using inverse Fourier Transform we get:

$y[n] = \left(-4(\frac{1}{4})^n - 2(n+1)(\frac{1}{4})^n + 8(\frac{1}{2})^n\right) u[n]$

$y[n] = \left(-2(n+3)(\frac{1}{4})^n + 8(\frac{1}{2})^n\right) u[n]$

2. Overall impulse response can be represented as:

$$h[n] = h_1[n] + h_2[n]$$

Overall frequency response can be represented as:

$$H(e^{jw}) = H_1(e^{jw}) + H_2(e^{jw})$$

Finding frequency response of $h_2[n]$ and subtracting it from overall frequency response yields $h_1[n]$. So:

$$h_1[n] = \left(\frac{1}{3}\right)^n u[n]$$

$$F\{h_1[n]\} = H_1(e^{jw}) = \frac{1}{1 - \frac{1}{3}e^{-jw}}$$

Subtracting $H_1(e^{jw})$ from overall frequency response yields $H_2(e^{jw})$, so:

$$H_2(e^{jw}) = H(e^{jw}) - H_1(e^{jw})$$

$$= \frac{5e^{-jw} - 12}{e^{-2jw} - 7e^{-jw} + 12} - \frac{1}{1 - \frac{1}{3}e^{-jw}}$$

$$= \frac{5e^{-jw} - 12}{(e^{-jw} - 3)(e^{-jw} - 4)} + \frac{3}{(e^{-jw} - 3)}$$

$$= \frac{8}{(e^{-jw} - 4)}$$

Taking Inverse Fourier Transform of $H_2(e^{jw})$, yields $h_2[n]$. So:

$$H_2(e^{jw}) = \frac{8}{(e^{-jw} - 4)} = \frac{-2}{1 - \frac{1}{4}e^{-jw}}$$

$$F^{-1}\{H_2(e^{jw})\} = h_2[n] = -2\left(\frac{1}{4}\right)^n u[n]$$

3. (a) Finding Fourier Transform of given input:

$$x(t) = x_1(t) + x_2(t)$$

$$x_1(t) = \frac{sin(2\pi t)}{\pi t}$$

$$F\{x_1(t)\} = X_1(jw) = \begin{cases} 1, & \text{if } |w| < 2\pi \\ 0, & \text{otherwise} \end{cases}$$

$$x_2(t) = cos(3\pi t)$$

$$F\{x_2(t)\} = X_2(jw) = \pi(\delta(w - 3\pi) + \delta(w + 3\pi))$$

$$F\{x(t)\} = X(jw) = X_1(jw) + X_2(jw)$$

We can plot the Fourier transform of the given input as:

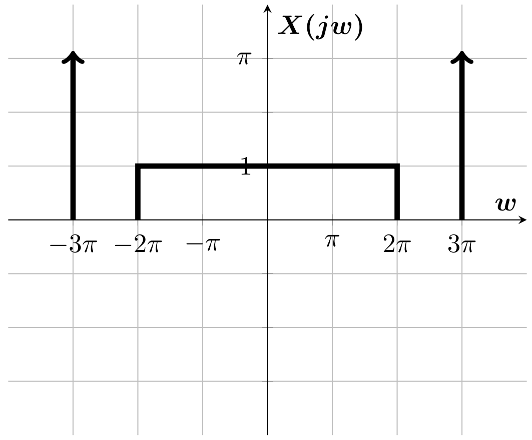

Figure 1: w vs. $X(jw)$.

(b) Nyquist period for sampling is equal to two times of given input's period. Let w_s is sampling period, T_s is sampling frequency and w_m is given input's period.

$w_m = 3\pi$
$w_s = 2w_m = 6\pi$

$$T_s = \frac{2\pi}{w_s} = \frac{1}{3}$$

(c) We can directly use the sampling formula which is:

$$X_p(jw) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(w - kw_s))$$
$$= X_p(jw) = 3\sum_k X(j(w - 6\pi k))$$

Let us plot $X_p(jw)$ and see what it corresponds to:

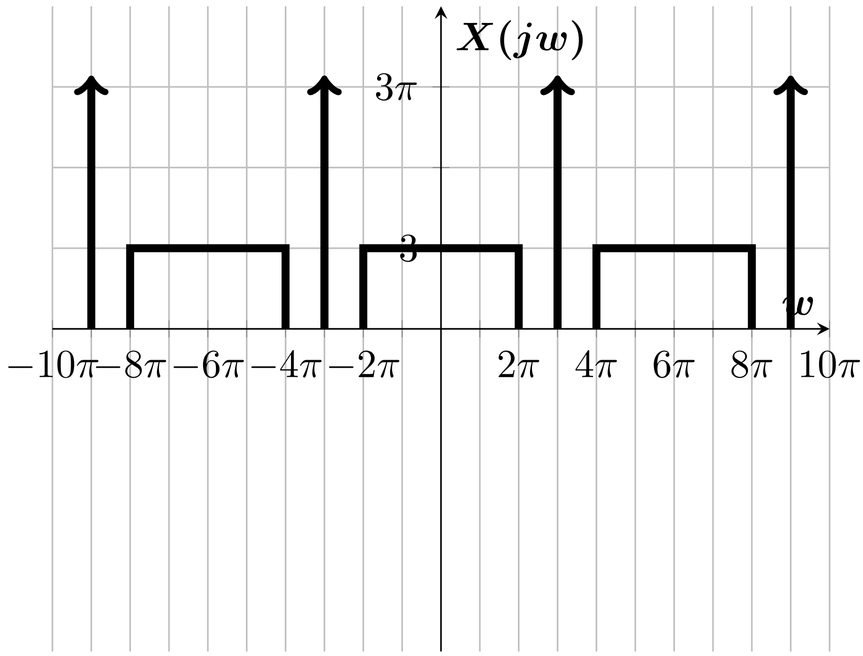

Figure 2: w vs. $X(j(w - 6\pi k))$.

4. (a) Let us first sample $X(jw)$ with an impulse train $p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$.

Sampling period of the input signal can be shown as:

$$T = \frac{2\pi}{w_s} = 2$$

Let X_p represent sampled version of $X(jw)$. So:

$$X_p(jw) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(w - kw_s))$$

Now we can normalize $X_p(jw)$ as $X_d(e^{jw})$ as the following:

$$\begin{aligned} X_d(e^{jw}) &= X_p(j\frac{w}{T}) \\ &= \begin{cases} \frac{2w}{\pi}, & \text{if } |w| < \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

where $X_d(e^{jw}) = X_d(e^{j(w+N)})$.

Let $X_d(e^{jw})$ represent discrete version of $X(jw)$. So:

$$X_d(e^{jw}) = \sum_{k=-\infty}^{\infty} X\left(\frac{j(w-2k\pi)}{T}\right)$$

(b) We know that $h[n] = cos(\pi n) = \frac{1}{2}(e^{j\pi n} + e^{-j\pi n})$ is periodic and we know that $F(e^{jwn}) \leftrightarrow 2\pi\sum_{-\infty}^{\infty}\delta(w - w_0 - 2\pi k)$, therefore we can proceed as the following:

$$\begin{aligned} H(e^{jw}) &= \frac{1}{2}\left(2\pi\sum_{k=-\infty}^{\infty}\delta(w-\pi-2\pi k) + 2\pi\sum_{k=-\infty}^{\infty}\delta(w+\pi-2\pi k)\right) \\ &= \pi\left(\sum_{k=-\infty}^{\infty}\delta(w-\pi-2\pi k) + \delta(w+\pi-2\pi k)\right) \end{aligned}$$

(c) We know that multiplication in time domain corresponds to convolution in frequency domain. Therefore we will convolute $X_d(e^{jw})$ and $H(e^{jw})$ over 1 period (which is 2π, so use $-\pi$ to π). This can be done with shifting $X_d(e^{jw})$ by π to both sides:

$$\begin{aligned} y_d[n] = x_d[n]h[n] \leftrightarrow Y_d(e^{jw}) &= \frac{1}{2\pi}X_d(e^{jw}) * H(e^{jw}) \\ &= \begin{cases} \frac{w}{\pi}, & \text{if } \frac{\pi}{2} \leq |w| \leq \frac{3\pi}{2} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

where $Y_d(e^{jw}) = X_d(e^{j(w+2\pi)})$.