Matrices in Coordinate Geometry

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Overview

The Original Problem

JEE (Advanced)2018 - Paper 1 - Q.3

Let $P_1 : 2x + y - z = 3$ and $P_2 : x + 2y + z = 2$ be two planes. Then, which of the following statement(s) is(are) TRUE?

(A) The line of intersection of P_1 and P_2 has direction ratios $1, 2, -1$

(B) The line

$$\frac{3x-4}{9} = \frac{1-3y}{9} = \frac{z}{3}$$

is perpendicular to the line of intersection of P_1 and P_2

(C) The acute angle between P_1 and P_2 is 60°

(D) If P_3 is the plane passing through the point $(4, 2, -2)$ and perpendicular to the line of intersection of P_1 and P_2, then the distance of the point $(2, 1, 1)$ from the plane P_3 is $\frac{2}{\sqrt{3}}$

Visualization

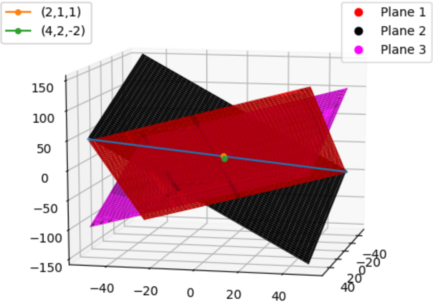

Figure: Question

Matrix Transformation

We can transform the problem into matrices in the following way

- The plane can be written in the form

$$\mathbf{n}^T \cdot \mathbf{x} = d$$

- The direction vector of a line in the form:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

is $\mathbf{L} : \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Solution

(A) To find the direction ratios of the line of intersection of P_1, P_2

- The two planes have normal vectors $\mathbf{n_1}, \mathbf{n_2}$
- The cross product, $\mathbf{n_1} \times \mathbf{n_2}$ will give us the direction vector of the line of intersection
- $$\mathbf{n_1} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \mathbf{n_2} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$
- $$\mathbf{n_1} \times \mathbf{n_2} = \begin{bmatrix} n_{12}n_{23} - n_{13}n_{22} & n_{13}n_{21} - n_{11}n_{23} & n_{11}n_{22} - n_{12}n_{21} \end{bmatrix}^T$$
- $$\mathbf{n_1} \times \mathbf{n_2} = \begin{bmatrix} 3 & -3 & 3 \end{bmatrix}^T$$
- The resultant ratio can be compared to the given direction ratios

Alternate Solution (RREF)

(A) To find the direction ratios of the line of intersection of $P_1 : 2x + y - z = 3, P_2 : x + 2y + z = 2$

- Let $M = \begin{bmatrix} \mathbf{n_1} \\ \mathbf{n_2} \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$

$$M\mathbf{x} = D$$

We take the combined matrix, and find the RREF.

$$[M \mid D] = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 1 & 2 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & \frac{4}{3} \\ 0 & 1 & 1 & \frac{1}{3} \end{array}\right]$$

Letting z be the parameter

$$\begin{bmatrix} x - z \\ y + z \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{1}{3} \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} + z \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{1}{3} \end{bmatrix}$$

..contd.

For the equation

$$\begin{bmatrix} x \\ y \end{bmatrix} - z \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

$$\frac{x-c}{a} = \frac{y-d}{b} = z$$

Hence, the direction ratios are $(1, -1, 1)$

Solution

(B) To test if line of intersection of P_1, P_2 is perpendicular to given line

- The two planes have normal vectors $\mathbf{n_1}, \mathbf{n_2}$ and let line of intersection have direction vector $\mathbf{L_1}(3, -3, 3)$
- The direction ratios of the line L_2

$$\frac{3x-4}{9} = \frac{1-3y}{9} = \frac{z}{3}$$

  are $d : (2, -2, 1)$
- If L_1 is $\perp$ to L_2, it is implied to be co-planar with P_1, P_2.
- Use Scalar Dot Product to check the above.

$$\mathbf{d}^T \cdot (\mathbf{n_1} \times \mathbf{n_2}) = \begin{bmatrix} 2 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 5 \neq 0$$

Solution

(C) Finding the angle between planes

- The angles between the planes is the same as the angle between normal vectors $\mathbf{n_1}, \mathbf{n_2}$
- The angle is given by

$$\mathbf{n_1}^T \cdot \mathbf{n_2} = \|\mathbf{n_1}\| \|\mathbf{n_2}\| \cos\theta$$

$$\theta = \arccos \frac{\begin{bmatrix} 2 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}{\sqrt{6} \cdot \sqrt{6}} = 60^\circ$$

Solution

(D) Distance Between Point and Plane

- The component of the vector joining $A(4,2,-2)$ and $P(2,1,1)$ along the normal n_3 is the distance between the point and plane.
- $$dist = \left| \frac{\mathbf{n^T} \cdot (\mathbf{A} - \mathbf{P})}{\|n\|} \right|$$
- $$dist = \left| \frac{\begin{bmatrix} 3 & -3 & 3 \end{bmatrix} \cdot \left(\begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right)}{\sqrt{27}} \right| = \frac{2}{\sqrt{3}}$$

Figures

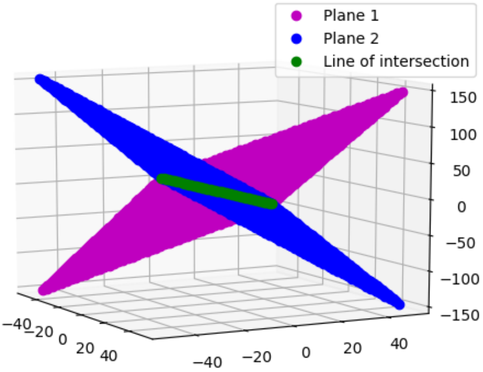

Figure: A - Planes & Line Of Intersection

Visualization

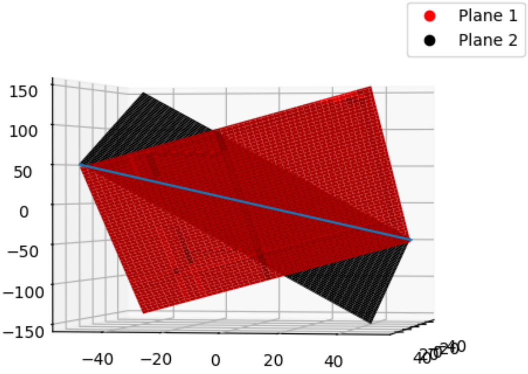

Figure: A - Planes & Line Of Intersection

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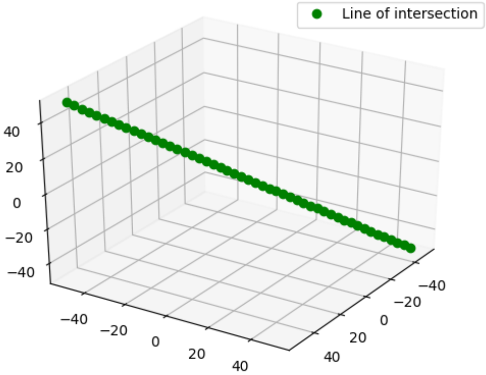


Figure: A - Line Of Intersection

Figures

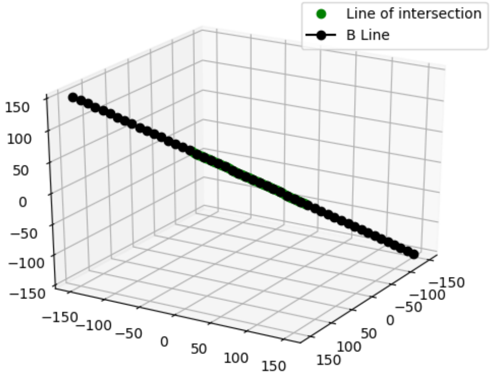


Figure: B- Line In Question

Figures

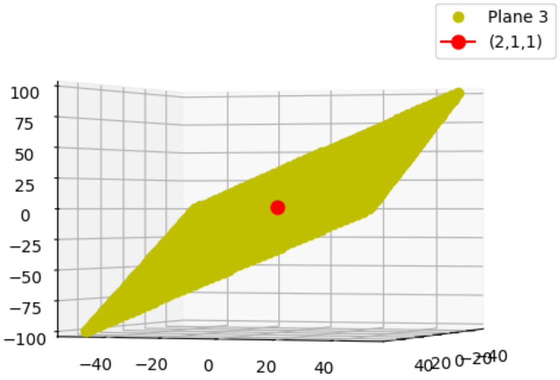


Figure: D- Point and Plane

Figures

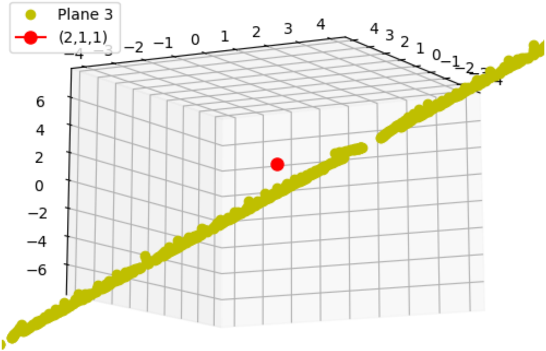
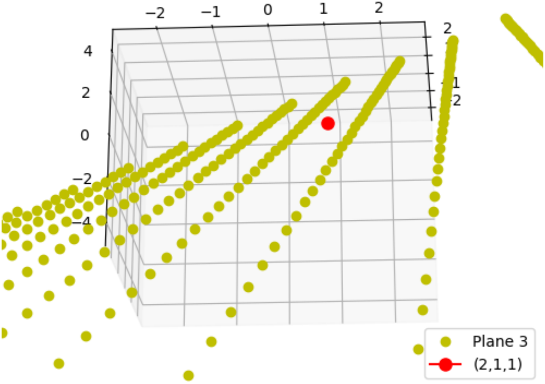


Figure: D- Point and Plane

Figures



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Figures

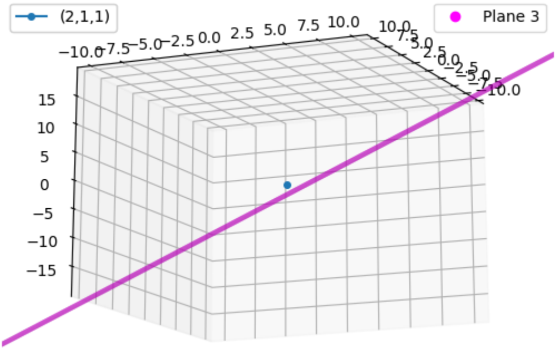

Figure: D- Point and Plane

The locus obtained was: $G = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{3} \begin{bmatrix} a\cos t + b\sin t + 1 \\ a\sin t - b\cos t \end{bmatrix}$

$$\mathbf{G} = \frac{1}{3} \begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{G}' = \begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\mathbf{G}' = R(t)\mathbf{r}, 0 < t \leq 2\pi$$

$$\mathbf{G}' = e^{it}\mathbf{r}, 0 < t \leq 2\pi$$

$$\left\| G' \right\| = \|r\| = \sqrt{a^2 + b^2}, 0 < t \leq 2\pi$$

The End