Linear Buoyancy Wave (LBoW) theory

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1 Introduction

The aim of this document is to describe the theory and mathematical expressions behind the python package LBoW. The theory is based on linear theory of stratified flow, which can be found in many textbooks [1–4]. The general form of linear theory is derived in section 2 for the sake of completeness. Section 3 describes two solutions to linear theory assuming that there is one layer with a uniform background wind speed and buoyancy frequency.

2 Linear theory – General form

It is assumed that the velocity $(u_0, v_0, 0)$ and the Brunt–Väisälä or buoyancy frequency N of the background state are independent of height, and that the Boussinesq approximation can be made. In the absence of rotation and friction, the governing equations for small perturbations (u_1, v_1, w_1) in velocity, p_1 in pressure and θ_1 in potential temperature are

$$\frac{\mathrm{D}u_1}{\mathrm{D}t} = -\frac{1}{\rho_0}\frac{\partial p_1}{\partial x}, \tag{1a}$$

$$\frac{\mathrm{D}v_1}{\mathrm{D}t} = -\frac{1}{\rho_0}\frac{\partial p_1}{\partial y}, \tag{1b}$$

$$\frac{\mathrm{D}w_1}{\mathrm{D}t} = -\frac{1}{\rho_0}\frac{\partial p_1}{\partial z} + \frac{\theta_1}{\theta_0}g, \tag{1c}$$

$$\frac{\mathrm{D}\theta_1}{\mathrm{D}t} + w_1\frac{\mathrm{d}\theta_0}{\mathrm{d}z} = 0, \tag{1d}$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0, \tag{1e}$$

with the material derivative $\frac{\mathrm{D}}{\mathrm{D}t} = \left(\frac{\partial}{\partial t} + \boldsymbol{u}_0 \cdot \boldsymbol{\nabla}\right)$. Equations (1a)-(1e) can be reduced to a single equation in w_1 as follows. Taking the material derivative of the vertical momentum equation (1c) allows the substitution of the potential temperature equation (1d). The pressure is found by taking the divergence of the momentum equations (1a)-(1c) and applying the continuity equation (1e). This yields

$$\left(\frac{\mathrm{D}}{\mathrm{D}t}\right)^2 \nabla^2 w_1 + N^2 \nabla_H^2 w_1 = 0. \tag{2}$$

with the horizontal Laplacian operator and Brunt–Väisälä frequency defined as $\nabla_H^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $N^2 = \frac{g}{\theta_0}\frac{\mathrm{d}\theta_0}{\mathrm{d}z}$, respectively. Equation (2) is a simplified form of the more general Taylor-Goldstein equation for wave motions in a stably stratified shear flow.

Equation (2) can also be expressed in terms of the vertical displacement η_1 of a fluid parcel above its undisturbed level, which is related to the vertical wind speed perturbation via the kinematic condition [5]

$$w_1 = \frac{\mathrm{D}\eta_1}{\mathrm{D}t}. \tag{3}$$

Substituting equation (3) into equation (2) gives

$$\left(\frac{\mathrm{D}}{\mathrm{D}t}\right)^2 \nabla^2\eta_1 + N^2\nabla_H^2\eta_1 = 0. \tag{4}$$

Assuming a plane wave solution

$$\eta_1(x, y, z, t) = \hat{\eta}_1(z)\exp\left[j(kx + ly - \omega t)\right] \tag{5}$$

results in

$$\frac{\partial^2\hat{\eta}_1}{\partial z^2} + m^2\hat{\eta}_1 = 0 \tag{6}$$

with the vertical wave number m given by

$$m^2 = (k^2 + l^2)\left(\frac{N^2}{\Omega^2} - 1\right). \tag{7}$$

The intrinsic frequency is hereby defined as $\Omega = \omega - \mathbf{u_0}\cdot\mathbf{k}$, with $\mathbf{k} = (k, l, m)$ the wave vector and $\mathbf{u_0} = (u_0, v_0, 0)$ the background wind speed vector. It is important to remark that equation (7) gives rise to two possible solutions for m. For $\Omega^2 < N^2$, m is a real number and the planar wave is vertically propagating, while for $\Omega^2 > N^2$, m is imaginary and the wave becomes evanescent.

Equation (6) is a linear, homogeneous, ordinary differential equation of second order, for which the general solution can be written as

$$\hat{\eta}_1(k, l, z, \omega) = A(k, l, \omega)e^{jm_1(k,l,\omega)z} + B(k, l, \omega)e^{jm_2(k,l,\omega)z}, \tag{8}$$

with m_1 and m_2 the positive and negative roots of equation (7), respectively. The coefficients A and B are determined by the boundary conditions.

3 One Layer Models (1D)

In this section, the solution to equation (6) is derived for various one layer models, i.e., models consisting of one layer with uniform background wind speed and buoyancy frequency. The flow perturbation is triggered by the bottom boundary condition, which represents a topographical feature. The difference between the models stems from the upper boundary condition. All models assume the flow perturbation is one-dimensional, like a ridge line, for which there is no variation in the spanwise direction and $l = 0$.

Topographical features are usually included in linear theory by requiring that the vertical displacement of the flow at the surface matches the surface elevation [5], giving rise to the linearized boundary condition for one-dimensional topographical features[1]

$$\eta_1(x, 0, t) = h(x, t). \tag{9}$$

[1]Note that for the time scales relevant to wind flow over hills and mountains, the shape of the topography is normally not time dependent. However, time is kept here as an independent variable to allow the study of theoretical cases where the surface perturbation is time-dependent. Such a case is for example useful to visualize how buoyancy waves start to develop in numerical simulations before a steady state is reached.

In general, $h(x,t)$ will excite a spectrum of plane waves, so the solution to equation (4) will be of the form

$$\eta_1(x,z,t) = \iint_{-\infty}^{\infty} \hat{\eta}_1(k,z,\omega) \exp\left[j(kx - \omega t)\right] \mathrm{d}k \, \mathrm{d}\omega \tag{10}$$

To obtain a boundary condition for $\hat{\eta}_1(k,\omega,z)$, the surface elevation $h(x,t)$ is decomposed into planar waves as well (in other words, the two-dimensional Fourier transform is applied)

$$\hat{h}(k,\omega) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} h(x,t) \exp\left[-j(kx - \omega t)\right] \mathrm{d}x \, \mathrm{d}t \tag{11}$$

The boundary condition in real space (equation (9))) then translates into a boundary condition per wavenumber–frequency pair:

$$\hat{\eta}_1(k,0,\omega) = \hat{h}(k,\omega). \tag{12}$$

Next to the surface boundary condition, an upper boundary condition is needed. Two cases are considered in the subsections below.

3.1 Half Plane Model

In the Half Plane Model, the flow is solved on an infinite half-plane above the ground, and the upper boundary condition effectively applies at infinity. Different boundary conditions apply for propagating and evanescent waves.

For evanescent waves ($\Omega^2 > N^2$), the negative imaginary root

$$m_2 = -j|k|\sqrt{1 - N^2/\Omega^2} \tag{13}$$

will give rise to unphysical growth of the perturbation amplitude with height, so the coefficient B needs to be set to zero in order for the solution to remain bounded.

For propagating waves ($\Omega^2 < N^2$), a radiation condition applies, which states that as waves are only excited at the surface, the vertical group velocity of the waves at infinity should be directed upward. The group velocity relative to the flow is defined as $\mathbf{c}_g = (\partial\Omega/\partial k, \partial\Omega/\partial l, \partial\Omega/\partial m)$. The group velocity relative to the ground can be found by taking the derivative of the apparent frequency ω rather than the intrinsic frequency, but since $\omega = \Omega + \mathbf{u}_0 \cdot \mathbf{k}$, this comes down to a simple vector summation. As the background vertical velocity is assumed to be zero, $w_g = \partial\Omega/\partial m$. The partial derivative of Ω to m can be found by reformulating the expression of the vertical wavenumber m (eq. 7) in terms of Ω, which gives $\Omega = \pm N|k|/\sqrt{k^2 + m^2}$. Then, it can be shown that $w_g = -\Omega m/(k^2 + m^2)$. Hence, for w_g to be positive, the vertical wave number must be chosen so that $\mathrm{sign}(m) = -\mathrm{sign}(\Omega)$.

For both propagating and evanescent waves, the boundary condition implies that only one root of equation (7) should be used. Hence, the solution can be summarised as follows:

$$\hat{\eta}_1(k,z,\omega) = A(k,\omega) e^{jm(k,\omega)z}, \tag{14}$$

with $A(k,\omega) = \hat{h}(k,\omega)$ following from equation (12), and the vertical wave number given by

$$m(k,\omega) = \begin{cases} j|k|\sqrt{1 - N^2/\Omega^2} & \text{for } \Omega^2 > N^2 \\ -\text{sign}(\Omega)\,|k|\sqrt{N^2/\Omega^2 - 1} & \text{for } \Omega^2 < N^2 \end{cases} \tag{15}$$

With the vertical displacement given by equations (14) and (15), other flow variables like u_1, w_1, and p_1 are easily related to the vertical displacement $\hat{\eta}_1(k,z,\omega)$. The vertical velocity w_1 follows from the kinematic condition (3):

$$\hat{w}_1(k,z,\omega) = -j\Omega\hat{\eta}_1(k,z,\omega) \tag{16}$$

The streamwise velocity component u_1 is obtained from the continuity equation (1e):

$$\hat{u}_1(k,z,\omega) = -\frac{m}{k}\hat{w}_1(k,z,\omega) = \frac{jm\Omega}{k}\hat{\eta}_1(k,z,\omega) \tag{17}$$

The pressure follows from the momentum equation (1a):

$$\hat{p}_1(k,z,\omega) = \rho_0\frac{\Omega}{k}\hat{u}_1(k,z,\omega) = \rho_0\frac{jm\Omega^2}{k^2}\hat{\eta}_1(k,z,\omega) \tag{18}$$

Finally, the full flow solution for any flow variable is found by applying the 2D inverse Fourier transform[2]:

$$\phi_1(x,z,t) = \iint_{-\infty}^{\infty} \hat{\phi}_1(k,z,\omega)\exp\left[j(kx - \omega t)\right]\,\mathrm{d}k\,\mathrm{d}\omega \tag{19}$$

with ϕ_1 representing any variable in (η_1, u_1, w_1, p_1).

Steady state solution

For steady state flow solutions, equations (14) to (18) can be further simplified since $\omega = 0$ and hence $\Omega = -u_0k$. For completeness, the simplified solution is given below:

$$\hat{\eta}_1(k,z) = \hat{h}(k)e^{jm(k)z}, \tag{20}$$

with

$$\hat{h}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(x)\exp\left(-jkx\right)\,\mathrm{d}x \tag{21}$$

and

$$m(k) = \begin{cases} j|k|\sqrt{1 - N^2/(u_0^2k^2)} & \text{for } u_0^2k^2 > N^2, \\ -\text{sign}(-u_0k)\,|k|\sqrt{N^2/(u_0^2k^2) - 1} & \text{for } u_0^2k^2 < N^2, \end{cases} \tag{22}$$

Other flow variables are given by

$$\hat{w}_1(k,z) = j\,u_0k\,\hat{\eta}_1(k,z), \tag{23}$$

[2]Note that the solution is developed for a plane wave of the form $\sim \exp\left[j(kx - \omega t)\right]$ whereas the 2D Fourier transform is normally defined using $\sim \exp\left[j(kx + \omega t)\right]$. This needs to be taken into account in the numerical implementation.

$$\hat{u}_1(k,z) = -j\, u_0 m\, \hat{\eta}_1(k,z), \tag{24}$$

$$\hat{p}_1(k,z) = j\, \rho_0 u_0^2 m\, \hat{\eta}_1(k,z). \tag{25}$$

The full flow solution is given by the 1D inverse Fourier transform

$$\phi_1(x,z) = \int_{-\infty}^{\infty} \hat{\phi}_1(k,z) \exp(jkx)\, \mathrm{d}k \tag{26}$$

with ϕ_1 representing any variable in (η_1, u_1, w_1, p_1)

3.2 Channel Model

The Channel Model assumes that the flow is bounded above by a rigid lid at a certain height H, causing downward wave reflections. This situation is reminiscent of numerical simulations of buoyancy waves with inadequate reflective boundary conditions at the top of the numerical domain. The rigid lid condition translates to $\eta_1(x,H,t) = 0$, and hence $\hat{\eta}_1(k,H,\omega) = 0$. Combined with the surface boundary condition (12), this results in a system of equations in terms of the unknowns A and B:

$$\begin{cases} A + B & = \hat{h}(k,\omega) \\ Ae^{jmH} + Be^{-jmH} & = 0 \end{cases} \tag{27}$$

where the same definition of vertical wave number is used as before (equation (15)). The solution to (27) can be written as

$$A = \hat{h}(k,\omega)/(1 - e^{2jmH}) \quad \text{and} \quad B = -\hat{h}(k,\omega)e^{2jmH}/(1 - e^{2jmH}) \tag{28}$$

The flow solution is then given by

$$\hat{\eta}_1(k,z,\omega) = \frac{e^{jmz} - e^{jm(2H-z)}}{(1 - e^{2jmH})}\hat{h}(k,\omega), \tag{29}$$

and

$$\hat{w}_1(k,z,\omega) = -j\Omega\hat{\eta}_1(k,z,\omega) = -j\Omega\frac{e^{jmz} - e^{jm(2H-z)}}{(1 - e^{2jmH})}\hat{h}(k,\omega), \tag{30}$$

$$\hat{u}_1(k,z,\omega) = -\frac{1}{jk}\frac{\partial}{\partial z}\hat{w}_1(k,z,\omega) = \frac{jm\Omega}{k}\frac{e^{jmz} + e^{jm(2H-z)}}{(1 - e^{2jmH})}\hat{h}(k,\omega), \tag{31}$$

$$\hat{p}_1(k,z,\omega) = \rho_0\frac{\Omega}{k}\hat{u}_1(k,z,\omega) = \rho_0\frac{jm\Omega^2}{k^2}\frac{e^{jmz} + e^{jm(2H-z)}}{(1 - e^{2jmH})}\hat{h}(k,\omega). \tag{32}$$

Steady state solution

As before, the general solution can be simplified under steady state conditions since $\omega = 0$ and hence $\Omega = -u_0 k$. The solution is

$$\hat{\eta}_1(k,z) = \frac{e^{jmz} - e^{jm(2H-z)}}{(1 - e^{2jmH})}\hat{h}(k), \tag{33}$$

with $\hat{h}(k)$ and $m(k)$ defined by equations (21) and (22), respectively. The solution for other flow variables is given by

$$\hat{w}_1(k,z) = ju_0k\hat{\eta}_1(k,z) \qquad = ju_0k\frac{e^{jmz} - e^{jm(2H-z)}}{(1-e^{2jmH})}\hat{h}(k), \tag{34}$$

$$\hat{u}_1(k,z) = -\frac{1}{jk}\frac{\partial}{\partial z}\hat{w}_1(k,z) \qquad = -jmu_0\frac{e^{jmz} + e^{jm(2H-z)}}{(1-e^{2jmH})}\hat{h}(k), \tag{35}$$

$$\hat{p}_1(k,z) = -\rho_0u_0\hat{u}_1(k,z) \qquad = \rho_0jmu_0^2\frac{e^{jmz} + e^{jm(2H-z)}}{(1-e^{2jmH})}\hat{h}(k). \tag{36}$$

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