Finite element methods

Wenchong Huang

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1 Introduction

In this month, my major work is learning the FEM. In the theoretical aspect, I learned how to prove the convergence of linear elements in the H^1 and the L^2 norm. In the technical aspect, things I learned are listed below.

1. Make the meshgrid.
2. A solver to elliptic equations.
3. The adaptive refinment technich.
4. The multigrid technich.
5. A solver to the heat equation.
6. A solver to the advection-diffusion equation.
7. A solver to the convection-diffusion equation.
8. A solver to the INSE with UPPE.

The codes are based on the deal.II liberary. I will choose some numerical tests to report in this article.

2 The reports

2.1 The meshgrid

I made a hexahedral meshgrid for our 3D ball-dragging problem. The meshgrid could be successively refined.

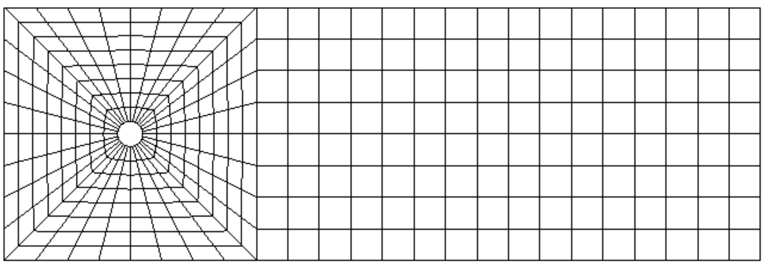


Figure 1: The snapshot of the meshgrid inside the plane $z = 5$.

2.2 The advection-diffusion equation

Write the advection-diffusion equation as

$$\frac{\partial \phi}{\partial t} = L(\phi, t) + D(\phi), \tag{1}$$

where

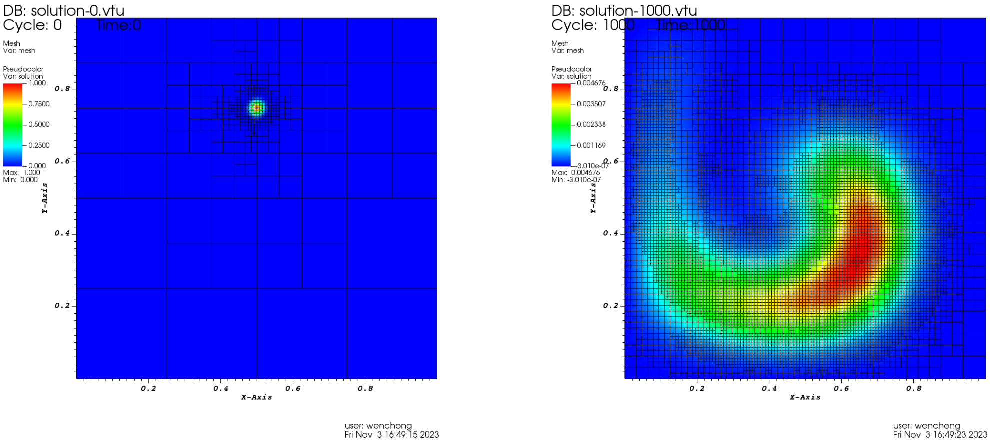
$$L(\phi, t) = -\nabla \cdot (\mathbf{u}\phi) + f(\mathbf{x}, t), \qquad D(\phi) = \nu \Delta \phi.$$

The time-discretization method is the IMEX-trapezoidal method:

$$\left(1 - \frac{k}{2}D\right)\phi^* = \phi^n + kL(\phi^n, t_n) + \frac{k}{2}D(\phi^n),$$

$$\phi^{n+1} = \phi^n + \frac{k}{2}L(\phi^n, t_{n+1}) + \frac{k}{2}D(\phi^n) + \frac{k}{2}L(\phi^*, t_{n+1}) + \frac{k}{2}D(\phi^*).$$

The space-discretization method is Q_1 elements. The test problem is **Gaussian Patch in Vortex Shear**, follows [6], except the boundary conditions, which are modified to the homogeneous Dirichlet.



(a) $t = 0$. (b) $t = 10$.

Figure 2: Meshgrid and solution plots of the Gaussian Patch in Vortex Shear test.

2.3 The convection-diffusion equation

Write the convection-diffusion equation as

$$\frac{\partial \mathbf{u}}{\partial t} = L(\mathbf{u}, t) + D(\mathbf{u}), \tag{2}$$

where

$$L(\mathbf{u}, t) = -(\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{g}(\mathbf{x}, t), \qquad D(\mathbf{u}) = \nu \Delta \mathbf{u}.$$

Still, use the IMEX-trapezoidal method and Q_1 elements. The test problem is **Vortex Shear**, follows [5], is derived from the exact solution

$$\mathbf{u}(x, y) = \cos\frac{\pi t}{T}(\sin^2(\pi x)\sin(2\pi y), -\sin(2\pi x)\sin^2(\pi y)), \tag{3}$$

where $T = 1$. To satisfy the CFL condition, we set the time-step $k = \frac{h}{31.25}$. The numerical test showed the 2nd-order convergence rate.

h	$\frac{1}{32}$	Rate	$\frac{1}{64}$	Rate	$\frac{1}{128}$
u_1, L_2, $t = \frac{T}{2}$	2.11e-03	1.99	5.31e-04	2.01	1.32e-04
u_2, L_2, $t = \frac{T}{2}$	2.11e-03	1.99	5.31e-04	2.01	1.32e-04
u_1, L_2, $t = T$	9.25e-04	1.99	2.33e-04	2.02	5.76e-05
u_2, L_2, $t = T$	9.25e-04	1.99	2.33e-04	2.02	5.76e-05

Table 1: Solution errors and convergence of the Vortex Shear test. $\nu = 0.01$.

2.4 The INSE

Write the Navier-Stokes equation as

$$\frac{\partial \mathbf{u}}{\partial t} = L(\mathbf{u}, t) + D(\mathbf{u}), \tag{4}$$

where

$$L(\mathbf{u}, t) = -\nabla p - (\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{f}(\mathbf{x}, t), \qquad D(\mathbf{u}) = \nu \Delta \mathbf{u},$$

with the Dirichlet boundary conditions

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega. \tag{5}$$

The pressure field p should ensure that

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega. \tag{6}$$

We ues the IMEX-trapezoidal method and Q_2 elements. After we extrapolated $\mathbf{u}$, we update the pressure by the UPPE formulation, see [2],

$$\int_\Omega \nabla p \cdot \nabla \psi \, \mathrm{d}\mathbf{x} = \int_{\partial\Omega} (\nu(\nabla \times \mathbf{u}) \cdot (\mathbf{n} \times \partial_t \mathbf{g})\psi) \, \mathrm{d}A + \int_\Omega (\mathbf{f} - (\mathbf{u} \cdot \nabla)\mathbf{u}) \cdot \nabla \psi \, \mathrm{d}\mathbf{x}, \tag{7}$$

for all test functions ψ.

2.4.1 Sinusoidal test

Follows [3], the **Sinusoidal test** is derived from the exact solution

$$\mathbf{u}(\mathbf{x}, t) = \pi \cos(t) \left(\sin^2(\pi x_1) \sin(2\pi x_2), -\sin(2\pi x_1) \sin^2(\pi x_2) \right), \tag{8}$$

$$q(\mathbf{x}, t) = -\cos(t) \cos(\pi x_1) \sin(\pi x_2). \tag{9}$$

The boundaries are set to be no-slip. We set a fixed time step $k = 0.0004$ to show the convergence rate of Q_2 elements.

h	$\frac{1}{64}$	Rate	$\frac{1}{128}$	Rate	$\frac{1}{256}$
u_1, L_2, $t = 0.1$	2.08e-05	3.72	1.58e-06	3.20	1.72e-07
u_2, L_2, $t = 0.1$	2.08e-05	3.72	1.58e-06	3.20	1.72e-07
p, L_2, $t = 0.1$	3.26e-05	3.88	2.22e-04	3.02	2.75e-07

Table 2: Solution errors and convergence of the Sinusoidal test. $\nu = 0.001$.

2.4.2 Cylindrical turbulence

For the **cylindrical turbulence** test, we follow the setup in [1]. Then the domain is $[0.3, 2.5] \times [0, 0.41] \setminus B((0.2, 0.2), 0.05)$. The time-dependent inflow and outflow profile

$$\mathbf{u}(0.3, x_2, t) = \mathbf{u}(2.5, x_2, t) = 0.41^{-2} \sin(\pi t/8)(6x_2(0.41 - x_2), 0) \tag{10}$$

is prescribed. Other boundaries are set to be no-slip. ν is chosen to be 0.001. The numerical test is running on a meshgrid similar as Fig. 1, where we have 8704 cells and 35360 DoFs. We set the time step $k = 0.0004$. The numerical results are shown in Fig. 3.

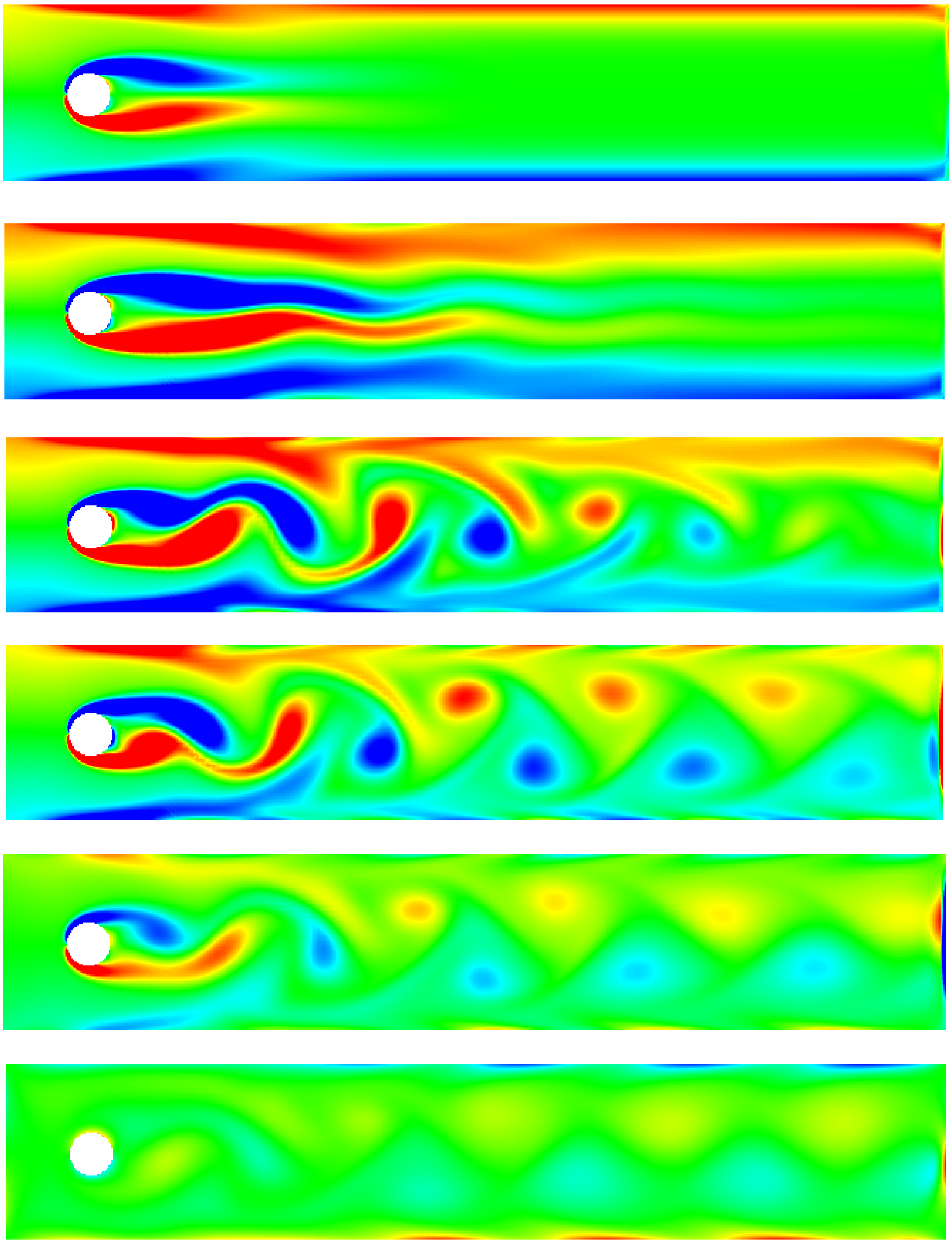
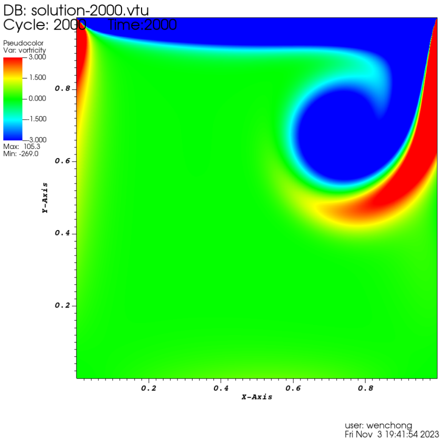


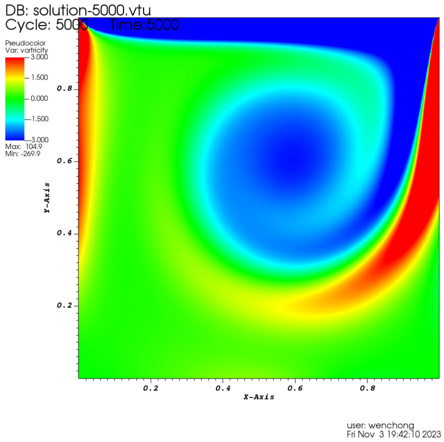
Figure 3: Vortricity plots of the cylindrical turbulence test. $t = [2, 4, 5, 6, 7, 8]$.

2.4.3 Lip-driven cavity

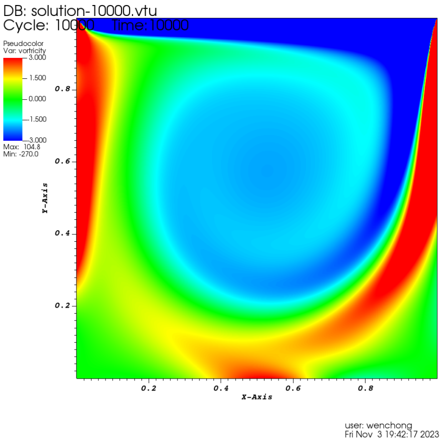
For the **lip-driven cavity** test, we set the condition at the top boundary $y = 1$ to be $\mathbf{u} = (1, 0)$. Other boundaries are set to be no-slip. The numerical test is running on a uniform grid with $h = \frac{1}{128}$. We set the time step $k = 0.002$. The results are shown in Fig. 4.



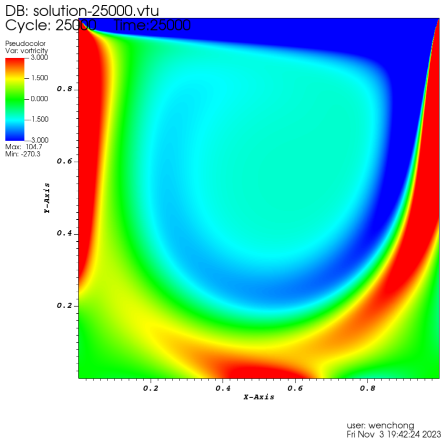
(a) $t = 4$.



(b) $t = 10$.



(c) $t = 20$.



(d) $t = 50$.

Figure 4: Vortricity plots of the lip-driven cavity test.

3 Future work

In my graduation thesis, things I want to do are listed below.

1. A solver to the INSE with GePUP, see [4].
2. A solver to the Boussinesq equations.
3. Extend all finished works to 3D.
4. Supplement relative theoretical analysis.

In the future, I want to learn DG methods. And implement the GePUP with DG methods.

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