FEM for the 2D convection-diffusion equation

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1 Formulas

We want to solve the 2D convection-diffusion equation

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{g} + \nu \Delta \mathbf{u}. \tag{1}$$

The IMEX-trapezoidal RK method gives the time discretization

$$\left(1 - \frac{1}{2}k\nu\Delta\right)\mathbf{u}^* = \mathbf{u}^n + \frac{1}{2}k\nu\Delta\mathbf{u}^n - k(\mathbf{u}^n \cdot \nabla)\mathbf{u}^n + k\mathbf{g}^n, \tag{2}$$

$$\mathbf{u}^{**} = \frac{1}{2}(\mathbf{u}^n + \mathbf{u}^*), \tag{3}$$

$$\mathbf{u}^{n+1} = \mathbf{u}^n + k\nu\Delta\mathbf{u}^{**} - k(\mathbf{u}^{**} \cdot \nabla)\mathbf{u}^{**} + \frac{k}{2}(\mathbf{g}^n + \mathbf{g}^{n+1}). \tag{4}$$

The only challenge is the convection terms $(\mathbf{u} \cdot \nabla)\mathbf{u}$. The FEM discretization gives that

$$u_1^n = \sum_{j=1}^{N} U_{1,j}^n \Phi_j, \tag{5}$$

$$u_2^n = \sum_{j=1}^{N} U_{2,j}^n \Phi_j. \tag{6}$$

Inner-product the convection terms with the test function Φ_i, we have, for example

$$\left(u_1^n \frac{\partial u_1^n}{\partial x}, \Phi_i\right) = \sum_{j,k=1}^{N} U_{1,j}^n U_{1,k}^n \left(\Phi_j \frac{\partial \Phi_k}{\partial x}, \Phi_i\right) \tag{7}$$

$$= \sum_{q=1}^{Q} \sum_{j,k=1}^{N} U_{1,j}^n U_{1,k}^n \Phi_j(v_q) \frac{\partial \Phi_k}{\partial x}(v_q) \Phi_i(v_q) w_q \tag{8}$$

$$= \sum_{q=1}^{Q} w_q \Phi_i(v_q) \left(\sum_{j=1}^{N} U_{1,j}^n \Phi_j(v_q)\right) \left(\sum_{k=1}^{N} U_{1,k}^n \frac{\partial \Phi_k}{\partial x}(v_q)\right). \tag{9}$$

Similar for other convection terms. So the (q, i, j, k)-loop is reduced to (q, i)-loops. The summation shall not write in hand. Instead, `deal.ii` provided functions `get_function_values` and `get_function_gradients` to do such work.

Note that if we use the Q_k element, a quadrature formula with algebraical accuracy of order at least $(3k - 1)$ is required. For example, the Q_1 element needs the 2-points Gauss quadrature formula.

2 Numerical experiments

In this sectoin, we test our program with the conditions derived from the exact solution, a divergence-free velocity field with a single vortex,

$$\mathbf{u}(x,y) = \cos\frac{\pi t}{T}(\sin^2(\pi x)\sin(2\pi y), -\sin(2\pi x)\sin^2(\pi y)), \tag{10}$$

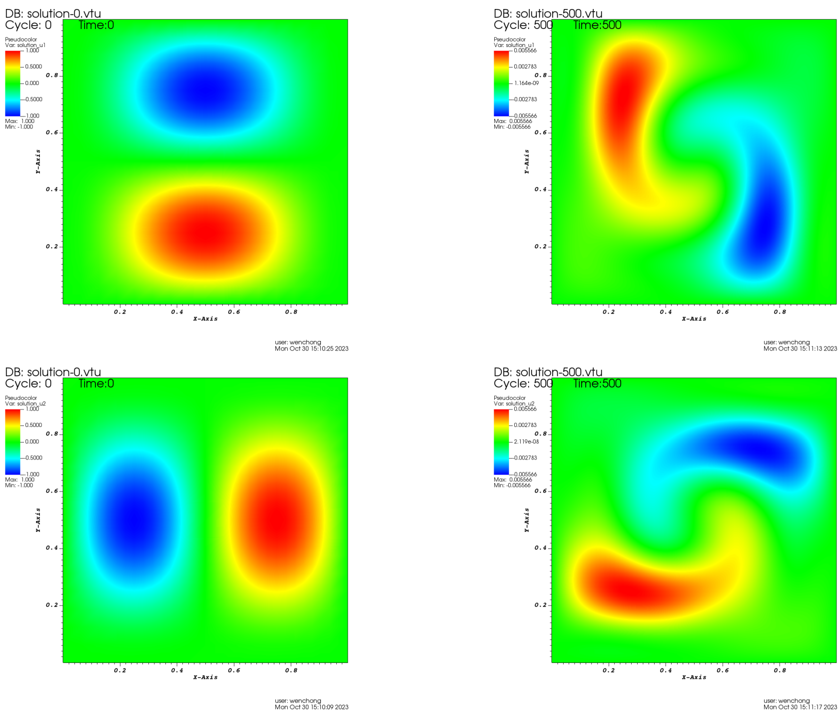
where T could be any positive real number, for example, 1.

The domain boundary conditions are homogeneous Dirichlet. The forcing terms are

$$\begin{aligned} g_1(x,y) = &- \frac{\pi}{T}\sin\frac{\pi t}{T}\sin^2(\pi x)\sin(2\pi y) \\ &+ \pi\cos^2\frac{\pi t}{T}\sin^2(\pi x)\sin(2\pi x)\sin^2(2\pi y) \\ &- 2\pi\cos^2\frac{\pi t}{T}\sin(2\pi x)\sin^2(\pi x)\sin^2(\pi y)\cos(2\pi y) \\ &- 2\pi^2\nu\cos\frac{\pi t}{T}\cos(2\pi x)\sin(2\pi y) \\ &+ 4\pi^2\nu\cos\frac{\pi t}{T}\sin^2(\pi x)\sin(2\pi y), \end{aligned} \tag{11}$$

$$\begin{aligned} g_2(x,y) = &\frac{\pi}{T}\sin\frac{\pi t}{T}\sin(2\pi x)\sin^2(\pi y) \\ &+ \pi\cos^2\frac{\pi t}{T}\sin^2(\pi y)\sin(2\pi y)\sin^2(2\pi x) \\ &- 2\pi\cos^2\frac{\pi t}{T}\sin(2\pi y)\sin^2(\pi y)\sin^2(\pi x)\cos(2\pi x) \\ &+ 2\pi^2\nu\cos\frac{\pi t}{T}\cos(2\pi y)\sin(2\pi x) \\ &- 4\pi^2\nu\cos\frac{\pi t}{T}\sin^2(\pi y)\sin(2\pi x). \end{aligned} \tag{12}$$

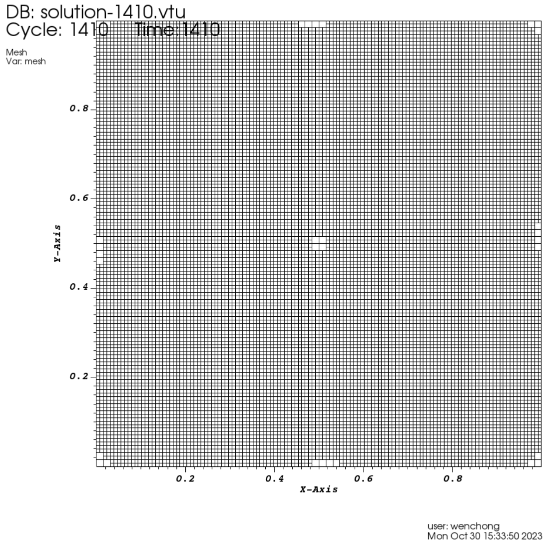
The vorticity changes its sign at $t = \frac{T}{2}$, see the following figures.



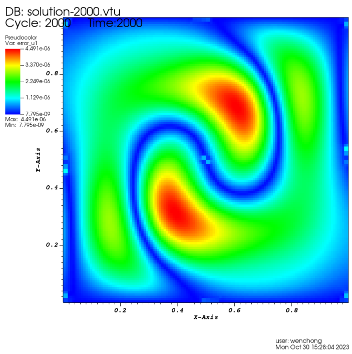
u_1, u_2 at $t = 0$.

u_1, u_2 at $t = \frac{T}{2}$.

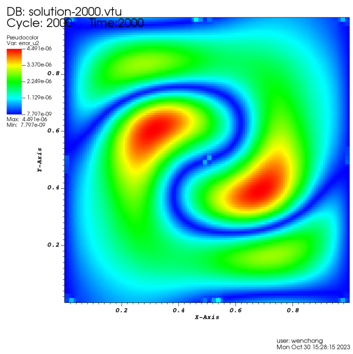
We use an adaptive mesh with width $h, \frac{h}{2}, \frac{h}{4}, \frac{h}{8}$. But the error distribution is so uniform that the adaptive mesh looks useless.



Adaptive mesh, $h = \frac{1}{8}$.



Error distribution of u_1, $t = T$, $h = \frac{1}{8}$.



Error distribution of u_1, $t = T$, $h = \frac{1}{8}$.

We choose the Q_1 element. And the time step is $\frac{h}{250}$. Here is the convergence table. The 2-nd order convergence rate of the L_2 norm is valid. It is interesting that the errors decrease over time.

h	$\frac{1}{4}$	Rate	$\frac{1}{8}$	Rate	$\frac{1}{16}$
$\|\|u_1^h - u_1\|\|_{L_2}, t = \frac{T}{2}$	0.00211047	1.99	0.000531287	2.01	0.000132154
$\|\|u_2^h - u_2\|\|_{L_2}, t = \frac{T}{2}$	0.00211048	1.99	0.000531294	2.01	0.000132157
$\|\|u_1^h - u_1\|\|_{L_2}, t = T$	0.000925229	1.99	0.000232991	2.02	5.76028e-05
$\|\|u_2^h - u_2\|\|_{L_2}, t = T$	0.000925245	1.99	0.000232999	2.02	5.76034e-05