Discrete Mathematics

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Contents

1 Logic

1.1 Propositional Logic—Zeroth Order Logic

In propositional logic, propositions are denoted by letters (p, q) and are formed by connecting other propositions using logical connectives. Propositions can either be true (T) or false (F).

Logical Connectives The logical connectives listed below are the basic connectives available in propositional logic in order of their precedence. Below are the truthtables corresponding to each of the connectives.

1. $\neg$, not
2. $\wedge$, and, $\bigwedge_{i=1}^{n} p_i$
3. $\vee$, or, $\bigvee_{i=1}^{n} p_i$
4. $\rightarrow$, $\Rightarrow$, implies (only if) defined as: $p \rightarrow q \equiv \neg p \vee q$
5. $\leftrightarrow$, $\Leftrightarrow$, is equivalent to (if and only if, iff) defined as: $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

Negation

p	¬ p
T	F T
F	T F

Logical Or

p q	p ∨ q
T T	T T T
T F	T T F
F T	F T T
F F	F F F

Equivalence

p q	p ↔ q
T T	T T T
T F	T F F
F T	F F T
F F	F T F

Logical And

p q	p ∧ q
T T	T T T
T F	T F F
F T	F F T
F F	F F F

Implication

p q	p → q
T T	T T T
T F	T F F
F T	F T T
F F	F T F

1.1.1 Definitions

Converse, Contrapositive, Inverse When given the proposition $p \rightarrow q$, $q \rightarrow p$ is its converse, $\neg q \rightarrow \neg p$ is its contrapositive and $\neg p \rightarrow \neg q$ is its inverse. The contrapositive is equivalent to the original proposition and the converse and inverse are also equivalent.

Tautology A proposition that is always true ($p \vee \neg p$).

Contradiction A proposition that is always false ($p \wedge \neg p$).

Contingency A proposition that is neither a tautology nor a contradiction.

Logical Equivalence p and q are logically equivalent if $p \leftrightarrow q$ is a tautology. The notation for equivalence is typically $\equiv$.

1.1.2 Properties

De Morgan's Laws

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$
$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

Identity Laws

$$p \vee F \equiv p$$
$$p \wedge T \equiv p$$

Domination Laws

$$p \vee T \equiv T$$
$$p \wedge F \equiv F$$

Idempotent Laws

$$p \vee p \equiv p$$
$$p \wedge p \equiv p$$

Negation Laws

$$p \vee \neg p \equiv T$$
$$p \wedge \neg p \equiv F$$

Commutative Laws

$$p \vee q \equiv q \vee p$$
$$p \wedge q \equiv q \wedge p$$

Associative Laws

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$
$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

Distributive Laws

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$
$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

Absorption Laws

$$p \vee (p \wedge q) \equiv p$$
$$p \wedge (p \vee q) \equiv p$$

1.1.3 Equivalence Proof

This is an example of how to perform an equivalence proof. The aim is to show that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $\neg p \wedge \neg q$. We can prove this by forming a series of logical equivalences.

$$\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{2}^{\text{nd}} \text{ De Morgan's law} \\
\neg p \wedge \neg(\neg p \wedge q) &\equiv \neg p \wedge (p \vee \neg q) && \text{1}^{\text{st}} \text{ De Morgan's law} \\
\neg p \wedge (p \vee \neg q) &\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{Associative law} \\
(\neg p \wedge p) \vee (\neg p \wedge \neg q) &\equiv F \vee (\neg p \wedge \neg q) && \text{Negation law} \\
F \vee (\neg p \wedge \neg q) &\equiv \neg p \wedge \neg q && \text{Identity law}
\end{aligned}$$

1.2 Predicate Logic —First Order Logic

Predicate logic uses quantified variables over non-logical objects and allows the use of sentences that contain variables. This allows a generalisation of propositions for a set of variables from a domain.

1.2.1 Definitions

Predicates A predicate is a generalisation of propositions when the variable x is replaced by a specific element from its domain. $P(x)$ becomes a proposition. When no other domain is specified the domain is U.

Quantifiers Quantifiers are used to express that a proposition is true for all elements of the domain and that there exists some element in the domain for which it is true. They also have the highest precedence among the logical operators.

Universal quantifier	$\forall x P(x)$	$P(x)$ is true for every x in U
Existential quantifier	$\exists x P(x)$	$P(x)$ is true for some x in U

1.2.2 Properties

Uniqueness Quantifier The uniqueness quantifier is a commonly used quantifier to express that there is only one x for which $P(x)$ is true. It is usually written as $\exists!$ or $\exists_1$.

$$\exists_1 x P(x) \equiv \exists x(P(x) \land \forall y(P(y) \to y = x))$$

De Morgan's Laws De Morgan's laws for quantifiers state that $P(x)$ is not true for all x if and only if there exists an x for which $P(x)$ is false and furthermore that if $P(x)$ is false for all x if and only if there does not exist an x for which $P(x)$ is true.

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$
$$\forall x \neg P(x) \equiv \neg\exists x P(x)$$

1.3 Logical Proofs

Using logical inference it is possible to build an argument given a set of premises to reach a logical conclusion. An argument is valid if truth of all premises p_i implies that the conclusion q is also true.

$$\left(\bigwedge_{i=1}^{n} p_i\right) \to q \equiv T$$

1.3.1 Rules of Inference

Modus Ponens

$$\begin{array}{rl} & p \to q \\ & p \\ \hline \therefore & q \end{array}$$

Modus Tollens

$$\begin{array}{rl} & p \to q \\ & \neg q \\ \hline \therefore & \neg p \end{array}$$

Hypothetical Syllogism

$$\begin{array}{rl} & p \to q \\ & q \to r \\ \hline \therefore & p \to r \end{array}$$

Disjunctive Syllogism

$$\begin{array}{rl} & p \vee q \\ & \neg p \\ \hline \therefore & q \end{array}$$

Addition

$$\begin{array}{rl} & p \\ \hline \therefore & p \vee q \end{array}$$

Simplification

$$\begin{array}{rl} & p \wedge q \\ \hline \therefore & p \end{array} \qquad \begin{array}{rl} & p \wedge q \\ \hline \therefore & q \end{array}$$

Conjunction

$$\begin{array}{rl} & p \\ & q \\ \hline \therefore & p \wedge q \end{array}$$

Resolution

$$\begin{array}{rl} & \neg p \vee r \\ & p \vee q \\ \hline \therefore & q \vee r \end{array}$$

Universal Instantiation

$$\begin{array}{rl} & \forall x P(x) \\ \hline \therefore & P(c) \end{array}$$

Universal Generalisation

$$\begin{array}{rl} & P(c) \text{ for an arbitrary } c \\ \hline \therefore & \forall x P(x) \end{array}$$

Existential Instantiation

$$\begin{array}{rl} & \exists x P(x) \\ \hline \therefore & P(c) \text{ for some element } c \end{array}$$

Existential Generalisation

$$\begin{array}{rl} & P(c) \text{ for some element } c \\ \hline \therefore & \exists x P(x) \end{array}$$

Universal Modus Ponens

$$\begin{array}{rl} & \forall x (P(x) \to Q(x)) \\ & P(a) \text{ for a particular element } a \\ \hline \therefore & Q(a) \end{array}$$

1.3.2 Inference Proof

All men are mortal. Socrates is a man. Prove that Socrates is mortal.

$P(x) \equiv T$ means that x is a person. $M(x) \equiv T$ means that x is mortal. s is the particular element Socrates.

1.	$\forall x P(x) \to M(x)$	premise
2.	$P(s)$	premise
3.	$P(s) \to M(s)$	UI (1)
4.	$M(s)$	Modus Ponens, (2) & (3)

2 Set Theory

A set is a collection of distinct elements without duplicates and the order of elements is unimportant.

2.1 Definitions

Containment $a \in A$ denotes that a is an element of the set A, whereas $a \notin A$ denotes that a is not contained in A. A set may contain other sets.

2.1.1 Describing Sets

Roster Method List all elements contained in the set. $S = \{a, b, c, d\}$ is equivalent to both $S = \{d, c, a, b\}$ and also $S = \{a, a, b, c, d, d\}$. When the pattern is clear, (...) may be used as in $S = \{a, b, c, \ldots, z\}$.

Set-Builder Method Specify the proposition that all members must satisfy. $S = \{x|P(x)\}$ denotes that all elements x in S must satisfy the proposition $P(x)$.

Interval Notation The elements in a set can be constrained by an open or closed interval. $[a, b] = \{x|a \leq x \leq b\}$ represents the closed interval between a and b. $(a, b) = \{x|a < x < b\}$ represents the open interval between a and b. Additionally combinations like $[a, b)$ and $(a, b]$ are possible.

2.1.2 Special Sets

Universal Set The universal set, denoted by U, contains everything currently under consideration.

Empty Set The empty set is the set that contains no elements. It is denoted by $\varnothing$. It is important to note that $\varnothing = \{\}$, but $\varnothing \neq \{\varnothing\}$. A set containing the empty set is not the equal to the empty set.

Singleton A singleton is a set containing exactly one element.

2.1.3 Set Equality

Two sets are equal if they contain the same elements. If A and B are sets then their equality is denoted as $A = B$.

$$\forall x(x \in A \equiv x \in B) \rightarrow A = B$$

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2.1.4 Subsets

Set A is a subset of B if all elements in A are also in B. This relation is denoted as $A \subseteq B$.

$$\forall x(x \in A \rightarrow x \in B) \equiv A \subseteq B$$

Every set is always a subset of itself and the empty set $\varnothing$ is a subset of every other set.

Proper Subset When $A \subseteq B$, but $A \neq B$ then A is a proper subset of B, which is denoted as $A \subset B$.

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A) \equiv A \subset B$$

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2.1.5 Set Cardinality

For finite sets the set cardinality describes the number of distinct elements in a set. It is denoted as $|S|$. The cardinality of the empty set is zero ($|\varnothing| = 0$). If A is the set containing all the letters of the english alphabet, then $|A| = 26$.

2.1.6 Power Sets

The set of all subsets of a set A is called the power set of A and is denoted as $\mathcal{P}(A)$. If $|A| = n$, then $|\mathcal{P}(A)| = 2^n$. For example, given the set $A = a, b$ we get the power set $\mathcal{P}(A) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$.

2.2 Tuples

Tuples are a collection of ordered elements. An n-tuple is written as $(a_1, a_2, a_3, \ldots, a_n)$. Two n-tuples are equal if and only if all of their corresponding elements are equal.

$$(a, b) = (c, d) \equiv (a = c) \wedge (b = d)$$

A subset R of $A \times B$ is called a relation from the set A to the set B.

2.2.1 Cartesian Product

The cartesian product $A \times B$ is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

The cartesian product of n sets is the set of tuples containing all possible combinations of elements as tuples.

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$$

The cardinality of the cartesian product is equal to the product of the cardinalities of the individual sets.

$$|A_1 \times A_2 \times \ldots \times A_n| = \prod_{i=1}^{n} |A_i|$$

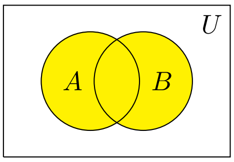
2.3 Operations

The set operators are based on boolean algebra and are analogous to the operators in propositional calculus. There must always be a universal set U and all sets are assumed to be a subset of U.

Union

The union of A and B is the set that contains those elements that are contained in A, B or in both.

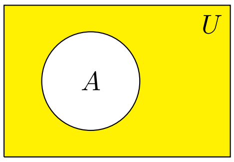
$$A \cup B = \{x | x \in A \vee x \in B\}$$



Complement

The complement of the set A contains all the elements in U that are not in A. It is equivalent to $U - A$.

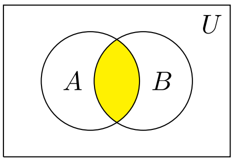
$$\bar{A} = \{x \in U | x \notin A\}$$



Intersection

The intersection of A and B is the set that contains those elements that are contained in both A and B.

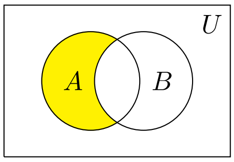
$$A \cap B = \{x | x \in A \wedge x \in B\}$$



Difference

The difference $A - B$ is the set that contains all the elements in A but not in B. It is the union of A with the complement of B.

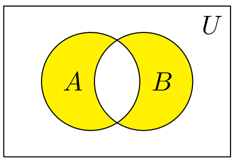
$$A - B = \{x | x \in A \wedge x \notin B\} = A \cap \bar{B}$$



Symmetric Difference

The symmetric difference is equivalent to a logical exclusive or meaning that the symmetric difference between A and B contains the elements that are in A or in B, but not those that are in both A and B.

$$A \oplus B = (A - B) \cup (B - A)$$



2.4 Properties

Inclusion-Exclusion Principle The inclusion-exclusion principle helps to calculate the cardinality of unions of sets. The principle states that the cardinality of the union of A and B is equal to the cardinality of A plus the cardinality of B minus the cardinality of the intersection of A and B.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

De Morgan's Laws

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

Identity Laws

$$A \cup \varnothing = A$$

$$A \cap U = A$$

Domination Laws

$$A \cup U = U$$

$$A \cap \varnothing = \varnothing$$

Idempotent Laws

$$A \cup A = A$$

$$A \cap A = A$$

Commutative Laws

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Associative Laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive Laws

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Absorption Laws

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

Complement Laws

$$A \cup \bar{A} = U$$

$$A \cap \bar{A} = \varnothing$$

Double Negation Law

$$\overline{(\bar{A})} = A$$

3 Relations

3.1 Definitions

A relation is the subset of the cartesian product of a number of sets. The simplest relation is a binary relation such as R from the set A to the set B.

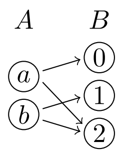
$$R \subseteq A \times B$$

Given are the following sets A and B and the relation R from A to B.

$$A = \{a, b\}, B = \{0, 1, 2\}, R = \{(a, 0), (a, 2), (b, 1), (b, 2)\}$$

This relation can be visualised in the following ways.

R	a	b
0	X	
1		X
2	X	X



The following notation can be used to simply show whether or not the pair (a, b) is in R.

$$a \; R \; b = (a, b) \in R$$

$$a \not R \; b = (a, b) \notin R$$

The number of relations on a set A is equal to 2 to the power of of the cardinality of its cartesian product.

$$2^{|A \times A|} = 2^{|A|^2}$$

3.2 Properties

Relations are sets so the regular set operations can be used on them. Additionally it is possible to create a composition of two relations. If R_1 is a relation from A to B and R_2 is relation from B to C then $R_2 \circ R_1$ is a relation from A to C.

Power of a relation

$$R^n = \left\{ \begin{array}{ll} R, & \text{for } n = 1 \\ R^{n-1} \circ R, & \text{for } n > 1 \end{array} \right\}$$

3.2.1 Reflexivity

A relation is reflexive if every element is related to itself.

$$\forall a[a \in U \rightarrow (a, a) \in R]$$

This property can be seen in a matrix if there is a diagonal line through the matrix.

R	a	b	c
a	1		
b		1	
c			1

If this property is negated then a relation is irreflexive.

3.2.2 Symmetry

In a symmetric relation for every pair (a, b) in R there must also be a pair b, a.

$$\forall a \forall b[(a, b) \in R \to (b, a) \in R]$$

The resulting matrix will also be a symmetric matrix.

R	a	b	c
a	0	1	0
b	1	0	1
c	0	1	0

In an antisymmetric relation for every pair which has a symmetric counterpart the elements must be related to themselves.

$$\forall a \forall y[(a, b) \in R \wedge (b, a) \in R \to x = y]$$

3.2.3 Transitivity

In a transitive relation if $(a, b) \in R$ and $(b, c \in R$ then $a, c \in R$ must also be true.

$$\forall a \forall b \forall c[(a, b) \in R \wedge (b, c) \in R \to (a, c) \in R]$$

3.3 Equivalence Classes and Partitions

3.3.1 Equivalence Relations

If a relation is reflexive, symmetric and transitive then it is an equivalence relation meaning that for a related pair (a, b), a and b are equivalent.

$$a \sim b$$

Examples of equivalence relations are $=, \equiv$.

3.3.2 Equivalence Classes

When elements of a set have some notion of equivalence they can be partitioned into a set of equivalence classes where all elements in the same equivalence class have an equivalence relation. The equivalent class of the element a from the set S, which has an equivalence relation $\sim$, is the set in which every element is equivalent to a.

$$[a] = \{x \in S | x \sim a\}$$

In set partitioned into equivalent classes x is equivalent to y if and only if x and y are in the same equivalence class.

$$x \sim y \leftrightarrow [x] = [y] \leftrightarrow [x] \cap [y] \neq \varnothing$$

3.3.3 Partition of a set

A set can be partitioned into a many disjoint non-empty subsets. The union of these subsets is the original set.

$$S = \bigcup_i A_i$$

Properties None of the subsets can be an empty set and no two subsets have any common elements.

$$A_i \neq \varnothing$$
$$i \neq j \rightarrow A_i \cap A_j = \varnothing$$

Partitioning a set into equivalent classes follows the same principle.

$$\bigcup_{a \in A} [a] = A$$

3.3.4 Partially Ordered Sets

A partially ordered set, also known as a poset, consists of a set together with a binary relation indicating that, for certain pairs of the set, one element precedes the other. Not every pair must be comparable in a partial order. In order to be a partial order, the relation must be reflexive, antisymmetric and transitive. A relation which is a partial order is denoted by $\preceq$. Given the poset $(S, \preceq)$ and the elements $a \in S$ and $b \in S$, then $a \preceq b$ means that a precedes b.

Totally Ordered Set If all elements in a set are comparable, then it is known as a totally ordered set.

Well Ordered Set If the set contains a least element then it is known as a well ordered set.

3.4 Functions

A special kind of relation is a function. A function is a subset of $A \times B$ with the restriction that no two elements may have the same first element.

$$\forall x\, [x \in A \rightarrow \exists y\, (y \in B \wedge (x, y) \in f)] \wedge \forall x, y_1, y_2\, [((x, y_1) \in f \wedge (x, y_2) \in f) \rightarrow y_1 = y_2]$$

3.4.1 Definitions

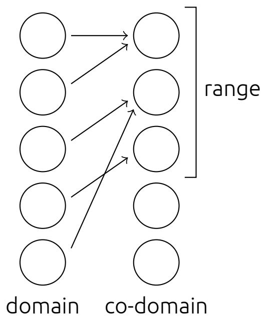
A function is a mapping from one set to another. A and B are two non-empty sets and the function f is a mapping from A to B.

$$f : A \rightarrow B$$

Each element of A is assigned to exactly to exactly one element B.

$$f(a) = b$$

In this case the set A is the domain of the function and the set B is the co-domain. The range is a subset of the co-domain and contains all the elements of B that f actually maps to.



3.4.2 Properties

Addition & Multiplication

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$(f_1 f_2)(x) = f_1(x) f_2(x)$$

Increasing & Decreasing

Increasing

$$\forall a, b \in A\,(a < b \to f(a) \leq f(b))$$

Strictly Increasing

$$\forall a, b \in A\,(a < b \to f(a) < f(b))$$

Decreasing

$$\forall a, b \in A\,(a < b \to f(a) \geq f(b))$$

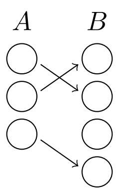
Strictly Decreasing

$$\forall a, b \in A\,(a < b \to f(a) > f(b))$$

3.4.3 Types of Functions

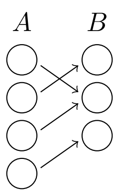
Injection Functions that are injective have a one-to-one mapping from A to B. That means that every a is assigned to one b and every b is the image of at most one pre-image a.

$$\forall a, b(f(a) = f(b) \to a = b)$$

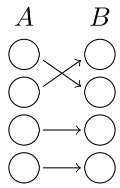


Surjection A surjective function maps A onto B, which means that for every element in B has a pre-image.

$$\forall b \in B\,[\exists a \in A(f(a) = b)]$$

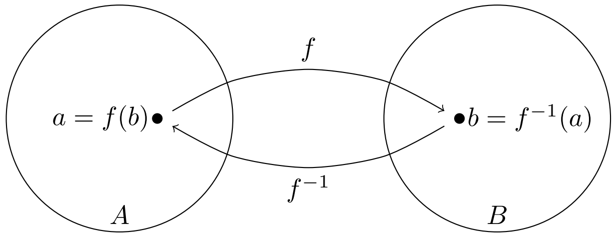


Bijection A bijection is a function, which is both injective and surjective.



3.4.4 Inverse Function

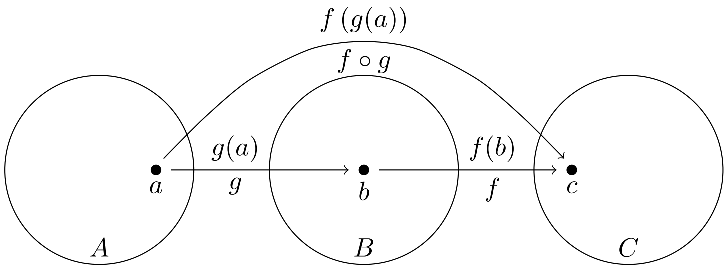
If the function f is bijective then it has an inverse. The inverse of f is denoted as f^{-1}.



3.4.5 Composition

Given the functions $f : B \to C$ and $g : A \to B$, the composition $f \circ g$ is the function $A \to C$.

$$f \circ g(x) = f\,(g(x))$$



3.5 Relational Alegebra

Relational algebra defines additional operators for operations between two relations. This is used in relational databases, where a relation is a table and the tables attributes are the relation's sets. Each row in the below tables is a tuple of the relations.

Employee

Name	EmployeeId	DepartmentName
Harry	3415	Finance
Sally	2241	Sales
George	3401	Finance
Harriet	2202	Sales

Department

DepartmentName	Manager
Finance	George
Sales	Harriet
Production	Charles

3.5.1 Projection

$$\Pi_{a_1,\ldots,a_n}(R)$$

$a_1, \ldots, a_n$ are names of attributes in the relation R. The result of the projection is a new relation containing only the attributes specified in the projection.

3.5.2 Selection

$$\sigma_\varphi(R)$$

φ is a propositional formula. The selection returns a relation containing only those tuples for which φ is true.

3.5.3 Rename

$$\rho_{a/b}(R)$$

The result of renaming is identical to the original relation, but the the attribute a is renamed to b.

3.5.4 Join

$$R \bowtie S$$

The natural join merges two relations on a common attribute. If there is no common attribute then the natural join becomes the cartesian product combining all possible attributes. Joining the two relations from earlier on the common attribute DepartmentName, would yield the below relation.

Employee $\bowtie$ Department

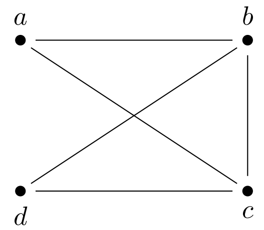
Name	EmployeeId	DepartmentName	Manager
Harry	3415	Finance	George
Sally	2241	Sales	Harriet
George	3401	Finance	George
Harriet	2202	Sales	Harriet

4 Graphs

4.1 Definitions

A graph G is defined by its vertices V and edges E. The set of vertices must be non-empty. Every edge in E has one or two endpoints.

$$
\begin{aligned}
&G = (V, E) \\
&V = \{a, b, c, d\} \\
&E = \{(a, b), (a, c), (b, c), (b, d), (c, d)\}
\end{aligned}
$$



Two vertices are adjacent or neighbouring if they have a connecting edge. An edge is incident with the vertices it connects.

4.1.1 Types of Graphs

Simple Graph In a simple graph there are no loops and no multiple edges allowed.

Multigraph In a multigraph it is allowed to have multiple edges between two vertices.

Pseudograph In a pseudograph there can be multiple edges as well as loops.

Digraph A directed graph has directed edges also known as arcs, so every edge is an ordered pair of vertices.

4.1.2 Neighbourhood

The neighbourhood of a vertex $N(v)$ is the set of all neighbours of that vertex. It is a subset of V.

If A is a subset of V, then $N(A)$ is the set of all vertices that are adjacent to at least one vertice in A.

$$N(A) = \bigcup_{v \in A} N(v)$$

4.1.3 Degree of a Vertex

Undirected Graphs In an undirected graph, the degree of a vertex deg(v) denotes the number of edges incident with the vertice. Loops contribute 2 to the degree.

Handshaking Theorem If $G = (V, E)$ is an undirected graph with m edges, then every new edge increases the degree of two vertices by one.

$$2m = \sum_{v \in V} \deg(v)$$

Directed Graphs A directed graph's vertices have an in-degree $\deg^{-}(v)$, which corresponds to the number of edges terminating at v and an out-degree $\deg^{+}(v)$, which corresponds to the number of edges originating from v. The sum of in-degrees equals the number of out-degrees and the number of edges.

$$|E| = \sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v)$$

4.2 Special Types of Graphs

Complete Graphs

Cycles

Wheels

n-Cubes

4.2.1 Computer Network Architecture

4.2.2 Bipartite Graphs

4.3 Subgraphs

4.4 Representation of Graphs

4.5 Isomorphism of Graphs

4.6 Paths

4.6.1 Connectedness

4.6.2 Vertex Connectivity

4.6.3 Edge Connectivity

4.6.4 Euler Paths and Circuits

4.6.5 Hamiltonian Paths and Circuits

4.7 Planar Graphs

4.8 Trees